MINIMAL GENUS SEIFERT SURFACES ARE ROBUST

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on joint work with Dan Silver, Stefano Vidussi and Susan Williams.

Caveat: As is customary in talks some of the technical details in the note below might be imprecise or slightly incorect. For precise statements see the references.

1. Definitions and previous results

1.1. **The Thurston norm and fibered classes.** Let $K \subset S^3$ be a knot. The *genus* of K is the minimal genus of a Seifert surface of K. The following inequality often makes it possible to determine the genus of a given knot K:

$$\text{genus}(K) \geq \frac{1}{2} \deg \Delta_K(t),$$

where $\Delta_K(t)$ denotes the Alexander polynomial of K. For example if K is an alternating knot or a fibered knot, then the inequality is in fact an equality, i.e. we have

$$\text{genus}(K) = \frac{1}{2} \deg \Delta_K(t).$$

In the following we will also consider a generalization of the knot genus, namely the Thurston norm on 3–manifolds. More precisely, let N be a 3–manifold (we will always assume that N is oriented, connected, irreducible, compact with empty or toroidal boundary) and let $\phi \in H^1(N;\mathbb{Z}) = H_2(N, \partial N; \mathbb{Z})$. The *Thurston norm* (see [Th86]) of ϕ is defined as

$$x_N(\phi) = \min\{\chi^-(S) \,|\, S \subset N \text{ embedded surface dual to } \phi\},$$

where given a surface S with connected components $S_1, \ldots, S_k$ we define its complexity to be

$$\chi^-(S) = \sum_{i=1}^{k} \max\{-\chi(S_i), 0\}.$$

For example, if K is a non–trivial knot, $E_K = S^3 \setminus \nu K$ the exterior of K and $\phi_K \in H^1(E_K;\mathbb{Z})$ a generator, then it follows easily that

$$x_{E_K}(\phi_K) = 2\,\text{genus}(K) - 1.$$

Thurston showed that x_N defines a seminorm on $H^1(N;\mathbb{Z})$, in particular x_N can be uniquely extended to a seminorm on $H^1(N;\mathbb{Q})$. Furthermore Thurston showed that the norm ball

$$\{\psi \in H^1(N;\mathbb{Q}) \,|\, x_N(\psi) \le 1\}$$

is a convex polyhedron with finitely many rational (or infinite) vertices.

We say that $\phi \in H^1(N;\mathbb{Q})$ is a *fibered class* if there exists a fibration $p : N \to S^1$ such that $\phi \in H^1(N;\mathbb{Q}) = \text{Hom}(\pi_1(N);\mathbb{Q})$ factors through $p_* : \pi_1(N) \to \pi_1(S^1) = \mathbb{Z}$. Thurston showed that the set of all fibered classes are unions of cones on top dimensional faces of the Thurston norm ball.

1.2. **Previous results by Gabai, Calegari and Kronheimer.** There are various results which show that 'being a minimal genus Seifert surface' and 'being a Thurston norm minimizing surface' is quite robust. For example Gabai [Ga87] proved the following theorem:

Theorem A. (Gabai) *Let $K \subset S^3$ be a knot. Let $S \subset S^3$ be a minimal genus Seifert surface. Then*

$$S \cup \textit{disk} \;\subset\; N_K := \textit{0-framed surgery on } K,$$

is a Thurston norm minimizing surface.

Gabai [Ga83, p. 484] also proved the following result:

Theorem B. (Gabai) *Let N be a 3–manifold and $\phi \in H^1(N;\mathbb{Z})$, then*

$$x_N(\phi) = \min\{\chi^-(S) \,|\, S \subset N \textit{ immersed surface dual to } \phi\}.$$

Before we state the next result, recall that for a group π and $g \in \pi^{(1)} = [\pi,\pi]$, the *commutator length* of g is defined as

$$cl(g) := \min\{n \,|\, \text{ there exist } a_1, b_1, \dots, a_n, b_n \text{ with } g = [a_1,b_1][a_2,b_2]\dots[a_n,b_n]\},$$

and the *stable commutator length* of g is defined as

$$scl(g) := \lim_{n\to\infty} \frac{cl(g^n)}{n}.$$

Now let $K \subset S^3$ be a knot and denote by λ a longitude. If F is a Seifert surface, then we obviously have

$$scl(\lambda) \le cl(\lambda) \le \text{genus}(F).$$

Calegari (see [Ca09, Proof of Proposition 4.4]) proved the following theorem:

Theorem C. (Calegari) *Let $K \subset S^3$ be a knot. Denote by λ a longitude, then*

$$scl(\lambda) = genus(K).$$

Finally we turn to the study of complexities in 4–manifolds. Given any closed smooth 4–manifold W and given $\psi \in H_2(W;\mathbb{Z})$ we define

$$x_W(\psi) = \min\{\chi^-(S) \,|\, S \subset W \text{ smoothly embedded surface representing } \psi\}.$$

Now let N be a closed 3–manifold and let $\psi \in H_2(S^1 \times N;\mathbb{Z})$. We consider the Künneth isomorphism

$$\begin{array}{rcl} H_2(S^1 \times N;\mathbb{Z}) & \to & H_2(N;\mathbb{Z}) \oplus H_1(N;\mathbb{Z}) \\ \psi & \mapsto & (\phi, \gamma). \end{array}$$

Note that $\psi^2 = \phi \cdot \gamma$. Suppose ϕ can be represented by a connected Thurston norm minimizing surface S. Then we can find a curve c representing γ such that the geometric intersection number of S and c equals the absolute value of the algebraic intersection number of S and c. Then the immersed surface

$$S \,\cup\, c \times S^1$$

represents $\psi \in H_2(S^1 \times N;\mathbb{Z})$. We can remove the $|S \cdot c| = |\phi \cdot \gamma| = |\psi^2|$ self–intersection points by taking out two disks and gluing in an annulus at each intersection point. We thus obtain an embedded surface F representing ψ of complexity

$$\chi_-(F) = \chi_-(S \,\cup\, c \times S^1) + 2 \cdot |\psi^2| = x_N(\phi) + 2 \cdot |\psi^2|.$$

Put differently, we showed that

$$x_{S^1 \times N}(\psi) \le x_N(\phi) + 2 \cdot |\psi^2|.$$

Kronheimer [Kr99] proved the following result:

Theorem D. (Kronheimer) *Let N be a closed 3–manifold and let $\psi \in H_2(S^1 \times N;\mathbb{Z})$. We denote by $\phi \in H_2(N;\mathbb{Z})$ the Künneth component of ψ. Suppose that $x_N(\phi) > 0$ and suppose that ϕ can be represented by a connected Thurston norm minimizing surface, then*

$$x_{S^1 \times N}(\psi) = x_N(\phi) + 2 \cdot |\psi^2|.$$

2. The main results of this talk

Our first result is that we can extend Kronheimer's theorem to most S^1–bundles over hyperbolic 3–manifolds. More precisely in [FV11] we prove the following:

Theorem E. (F–Vidussi) *Let N be a closed hyperbolic 3–manifold. Then for all but finitely many S^1–bundles $p : M \to N$ the following holds: If $\psi \in H_2(M;\mathbb{Z})$ and if*

$\phi := p_*(\psi) \in H_2(N;\mathbb{Z})$ can be represented by a connected Thurston norm minimizing surface, then

$$x_M(\psi) = x_N(\phi) + 2 \cdot |\psi^2|.$$

In the product case we thus recover Kronheimer's result in the case of hyperbolic 3–manifolds. Our proof though is very different from Kronheimer's proof.

For the second result we go back to the study of minimal genus Seifert surfaces of knots. If S is an incompressible Seifert surface for K, then we can write $\pi_1(S^3 \setminus K)$ as an HNN extension with 'edge group' $\pi_1(S)$, i.e. there exists an isomorphism

$$\pi_1(S^3 \setminus K) = \langle \pi_1(S^3 \setminus S \times (-1,1)), t \,|\, t\iota_-\pi_1(S)t^{-1} = \iota_+\pi_1(S)\rangle.$$

Given any group π with $H_1(\pi;\mathbb{Z}) = \mathbb{Z}$ we define

$$F(\pi) := \min\{n \,|\, \text{ there exists an isomorphism } \pi \cong \langle A, t \,|\, tF_nt^{-1} = \varphi(F_n)\rangle\},$$

where F_n denotes the free group on n generators. It thus follows from the above that

$$F(\pi_1(S^3 \setminus K)) \le \operatorname{genus}(K).$$

In [FSW12] we will prove the following theorem.

Theorem F. (F–Silver–Williams) *Let K be a hyperbolic knot, then*

$$F(\pi_1(S^3 \setminus K)) = \mathit{genus}(K).$$

Given a 2–complex Y Turaev [Tu02] introduces a seminorm $x_Y : H^1(Y;\mathbb{Q}) \to \mathbb{Q}_{\ge 0}$, the definition of which mimics the Thurston norm. If Y is a deformation retract of $S^3 \setminus \nu K$ for some non–trivial knot K, and if $\phi \in H^1(Y;\mathbb{Z}) = H^1(S^3 \setminus \nu K;\mathbb{Z})$ denotes a generator, then it follows easily from the definitions, that

$$x_Y(\phi) \le 2\operatorname{genus}(K) - 1.$$

In [FSW12] we will also prove the following theorem.

Theorem G. (F–Silver–Williams) *Let K be a hyperbolic knot, then*

$$x_Y(\phi) = 2\,\mathit{genus}(K) - 1.$$

3. The proofs

3.1. **The Alexander norm.** As we mentioned earlier, the Alexander polynomial of a knot gives a lower bound on the knot genus, more precisely, the following inequality holds:

$$\operatorname{genus}(K) \ge \frac{1}{2}\deg \Delta_K(t),$$

and this is an equality if K is a fibered knot.

We now need to generalize the above results to general 3–manifolds. Let N be a 3–manifold. We write $H := H_1(N;\mathbb{Z})/\text{torsion}$. We can then consider the Alexander module

$$H_1(N;\mathbb{Z}[H]) = H_1(\text{universal torsion-free abelian cover of } N),$$

which is a $\mathbb{Z}[H]$–module via the action of the deck transformation group. The order of this $\mathbb{Z}[H]$–module is called the Alexander polynomial $\Delta_N \in \mathbb{Z}[H]$. We write

$$\Delta_N = \sum_{h\in H} a_h \cdot h,$$

and given $\phi \in H^1(N;\mathbb{Q})$ we define the *Alexander norm* of ϕ to be

$$a_N(\phi) := \max\{\phi(a_h) - \phi(a_g) \,|\, a_g, a_h \neq 0\}.$$

McMullen [Mc02] showed that the Alexander norm gives a lower bound on the Thurston norm. More precisely, if $b_1(N) \geq 2$, then

$$a_N(\phi) \leq x_N(\phi) \text{ for any } \phi \in H^1(N;\mathbb{Q}).$$

Furthermore this is an equality for fibered classes, and by continuity it is an equality for any class in the boundary of a fibered cone of the Thurston norm ball of N.

3.2. **First observation.** Let π be a group with $H_1(\pi;\mathbb{Z}) = \mathbb{Z} = \langle t\rangle$, then we can define an Alexander polynomial $\Delta_\pi(t) \in \mathbb{Z}[t^{\pm 1}]$ as the order of the Alexander module $H_1(\pi;\mathbb{Z}[t^{\pm 1}])$. Imitating the classical proof that $\deg \Delta_K(t) \leq 2\,\text{genus}(K)$ one can similarly show, that

$$\deg \Delta_\pi(t) \leq F(\pi_1(S^3 \setminus K)). \tag{1}$$

Furthermore, if Y is a 2–complex with $H_1(Y;\mathbb{Z}) = \mathbb{Z}$, then as above we can define an Alexander polynomial $\Delta_Y(t)$, and the following inequality holds:

$$\deg \Delta_Y(t) - 1 \leq x_Y(\phi), \tag{2}$$

where $\phi \in H^1(Y;\mathbb{Z})$ denotes a generator.

It follows, that if K is a knot such that $\deg \Delta_K(t) = 2\,\text{genus}(K)$, then we get the following inequalities

$$2\,\text{genus}(K) = \deg \Delta_K(t) = \deg \Delta_\pi(t) \leq F(\pi_1(S^3 \setminus K)) \leq 2\,\text{genus}(K),$$

and similarly

$$2\,\text{genus}(K) = \deg \Delta_K(t) = \deg \Delta_Y(t) \leq x_Y(\phi) + 1 \leq 2\,\text{genus}(K),$$

i.e. the conclusions of Theorems F and G hold.

Also, given any closed smooth 4–manifold W we obtain lower bounds on the complexity of surfaces representing homology classes from the 'adjunction inequality' coming from Seiberg–Witten invariants. By Meng–Taubes [MT96] the Seiberg–Witten invariants of $W = S^1 \times N$ are equivalent to the Alexander polynomial of N. Given $\psi \in H_2(S^1 \times N; \mathbb{Z})$ the adjunction inequality then says that

$$x_{S^1\times N}(\psi) \geq a_N(\phi) + 2 \cdot |\psi^2|, \tag{3}$$

where $\phi \in H_2(N; \mathbb{Z})$ denotes again the Künneth component of ψ. We thus see that the conclusion of Kronheimer's theorem holds for any ψ with $a_N(\phi) = x_N(\phi)$.

The case of a circle bundle M over a closed 3–manifold N is technically somewhat more involved. In this case the Euler class e of the circle bundle and the Alexander polynomial of N determine the Seiberg–Witten invariants of M (see [Ba01]), but 'some information can get lost'. We will not elaborate on this technical point in this talk.

3.3. **The results of Agol and Wise.** In order to state the results of Agol and Wise we need two more definitions: A finite graph Γ with vertex set V gives rise to a group presentation as follows:

$$\langle \{g_v\}_{v\in V} \,|\, [g_u, g_v] = 1 \text{ if } u \text{ and } v \text{ are connected by an edge}\rangle.$$

Any group which is isomorphic to such a group is called a *right-angled Artin group (RAAG).* Furthermore, if π is a group, then we say that π virtually has a a certain property if π admits a finite index subgroup which has this property. Agol [Ag08] proved the following theorem:

Theorem. (Agol) *Let N be an irreducible 3–manifold such that $\pi_1(N)$ is virtually a subgroup of a RAAG. Then given any $\phi \in H^1(N; \mathbb{Z})$ there exists a finite cover $p\colon \widetilde{N} \to N$ such that $p^*(\phi)$ is contained in the closure of a fibered cone of the Thurston norm ball of $\widetilde{N}$.*

This result of Agol fits in beautifully with the following recent result of Wise [Wi09, Wi11a, Wi11b]:

Theorem. (Wise) *Let N be a hyperbolic 3–manifold such that one of the following holds: either*

1. $b_1(N) \geq 2$, *or*
2. $b_1(N) = 1$ *and N is not fibered, or*
3. *N has boundary,*

then $\pi_1(N)$ is virtually a subgroup of a RAAG.

Remark.
1. The conditions (1) and (2) can be replaced by the condition 'N admits a geometrically finite surface',
2. the full details of the proof are available in [Wi11a], but the proof has not been officially verified yet,
3. it is expected that the conclusion of the theorem holds for any irreducible 3–manifold with at least one hyperbolic piece in its JSJ decomposition.

3.4. **Summary of the proofs of Theorems E, F and G.** Let N be a hyperbolic 3–manifold and let $\phi \in H^1(N;\mathbb{Z})$. Then the following hold:

(i) By the results of Agol and Wise there exists a finite cover $p : \widetilde{N} \to N$ such that $\widetilde{\phi} := p^*\phi$ sits on the closure of a fibered cone of $\widetilde{N}$.

(ii) It follows from the continuity of the Alexander norm and the Thurston norm that
$$a_{\widetilde{N}}(\widetilde{\phi}) = x_{\widetilde{N}}(\widetilde{\phi}).$$

(iii) By 'pushing up' the problem we can use (3) to reprove Kronheimer's Theorem for products $S^1 \times N$ with N hyperbolic. After some extra effort coming from the technical problems for S^1–bundles one can similarly prove Theorem E.

(iv) It follows from (2) and some fairly elementary algebraic arguments (see [FV12]) that there exists a representation $\alpha : \pi_1(N) \to \mathrm{GL}(n,\mathbb{C})$, such that the corresponding twisted Alexander polynomial detects the Thurston norm of ϕ.

(v) The lower bounds of (1) and (2) can be extended to the twisted case, which allows us to prove Theorems F and G.

References

[Ag08] I. Agol, *Criteria for virtual fibering*, Journal of Topology 1: 269-284 (2008)

[Ba01] S. Baldridge, *Seiberg-Witten invariants of 4-manifolds with free circle actions*, Commun. Contemp. Math. 3: 341–353 (2001).

[Ca09] D. Calegari, *scl*, MSJ Memoirs, 20. Mathematical Society of Japan, Tokyo, 2009.

[FK06] S. Friedl and T. Kim, *The Thurston norm, fibered manifolds and twisted Alexander polynomials*, Topology 45, 929–953 (2006)

[FV11] S. Friedl and S. Vidussi, *Minimal Genus on 4-manifolds with a Free Circle Action*, Preprint (2011)

[FV12] S. Friedl and S. Vidussi, *The Thurston norm and twisted Alexander polynomials*, in preparation (2012)

[FSW12] S. Friedl, D. Silver and S. Williams, *Complexities in groups, 2–complexes and 3–manifolds*, in preparation (2012)

[Ga83] D. Gabai, *Foliations and the topology of 3–manifolds*, J. Differential Geometry 18, no. 3: 445–503 (1983).

[Ga87] D. Gabai, *Foliations and the topology of 3-manifolds. II*, J. Differential Geom. 26 (1987) 461–478.

[Kr99] P. Kronheimer, *Minimal genus in $S^1 \times M^3$*, Invent. Math. 135, no. 1: 45–61 (1999).

[Mc02] C. T. McMullen, *The Alexander polynomial of a 3–manifold and the Thurston norm on cohomology*, Ann. Sci. Ecole Norm. Sup. (4) 35, no. 2: 153–171 (2002)

[MT96] G. Meng and C. H. Taubes, *SW = Milnor torsion*, Math. Res. Lett. 3: 661–674 (1996).

[Th86] W. P. Thurston, *A norm for the homology of 3–manifolds*, Mem. Amer. Math. Soc. 339: 99–130 (1986).

[Tu02] V. Turaev, *A norm for the cohomology of 2-complexes*, Algebraic & Geometric Topology 2 (2002) 137155

[Wi09] D. Wise, *The structure of groups with a quasi-convex hierarchy*, Electronic Res. Ann. Math. Sci., Volume 16, pp. 44-55 (2009)

[Wi11a] D. Wise, *The structure of groups with a quasi-convex hierarchy*, 186 pages, Preprint (2011) downloaded on September 9, 2011 from the conference webpage for the NSF-CBMS conference

'3-Manifolds, Artin Groups and Cubical Geometry', from August 1-5, 2011 held at the CUNY Graduate Center

[Wi11b] D. Wise, *From riches to RAAGs: 3-manifolds, right–angled Artin groups, and cubical geometry*, 78 pages, lecture notes (2011)
lecture notes for the NSF-CBMS conference '3-Manifolds, Artin Groups and Cubical Geometry' from August 1-5, 2011 held at the CUNY Graduate Center

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