THE POSSIBLE SHAPES OF THE UNIVERSE AND THE VIRTUAL FIBERING THEOREM

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1. Introduction

It is natural to wonder what 'shape' our universe has. From our personal perspective we know that wherever we go, the universe looks three-dimensional. It is thus a reasonable assumption that the universe should look three-dimensional at every point. The following question thus arises: what can we say about the shape of the universe, if we only know, or assume, that at every point the universe looks three-dimensional.

Before we try to address this question we will first consider the case of zero-dimensional, one-dimensional and two-dimensional universes. We will then again turn to the three-dimensional setting and we will report on the recent work of Agol, Perelman, Przytcki–Wise and Wise.

2. Zero-dimensional universes

Let us first look at zero-dimensional universes. They are not very interesting, namely they consist of precisely one point. Pretty dull. So we immediately move on to the one-dimensional universes.

3. One-dimensional universes

In this section we suppose that we have a one-dimensional universe. This means that at each given point the area around the point looks like a line. We can imagine this like the universe an electron sees inside a cable.

There are now several possibilities.

(1) When we travel along our universe we eventually come back to the starting point. We sketch three such universes in Figure 1. Even though these three universes look different to us, this is an illusion, since we are looking at the universe from the outsider. For the person living within these universes they all look the same. Put differently, traveling *within* these one-dimensional universes we can not tell a difference. [1]

(2) We can travel for an infinite time forward and backward, without coming back to the starting point. We again sketch three such universes in Figure 2, and once

[1]The precise mathematical term is that these three universes are homeomorphic.

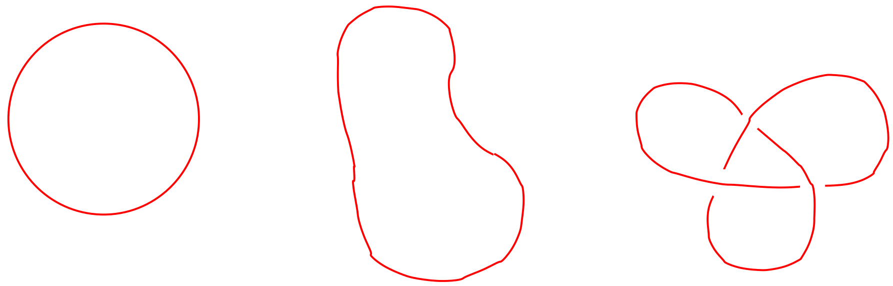

FIGURE 1. A one-dimensional universe which repeats itself.

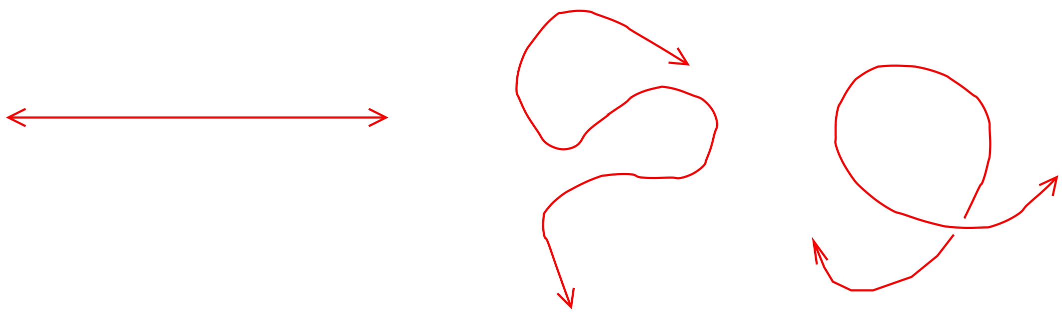

FIGURE 2. A one-dimensional universe which goes on indefinitely in both directions.

again, these three universes look different from the outside but they look the same when living within.

(3) The universe comes to and end if we travel forward and/or backward. We show

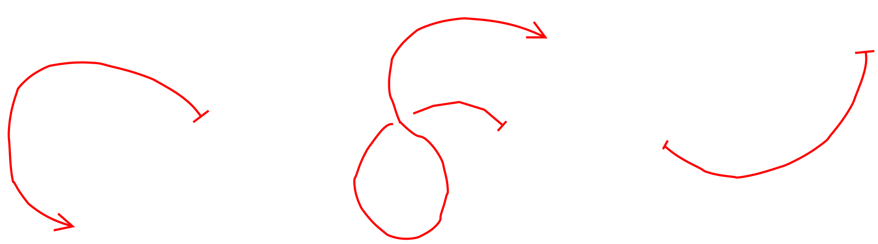

FIGURE 3. One-dimensional universes with at least one end.

some universes in Figure 3, The first two universes again look the same from the inside, each comes to an end in one direction but goes on indefinitely in the other direction. The third universe is different, it comes to an end in both directions.

4. Two-dimensional universes

We now turn to two-dimensional universes. Life within such a universe has been described beautifully in [Abbot1884]. In order to simplify the discussion a little bit we now assume that our two-dimensional universe is finite[2]

What shapes can such a universe have? The first answer which comes to mind is a sphere. Once again we will not make a distinction between spheres which look different 'from the outside', in particular we view the two spaces in Figure 4 as the same space.

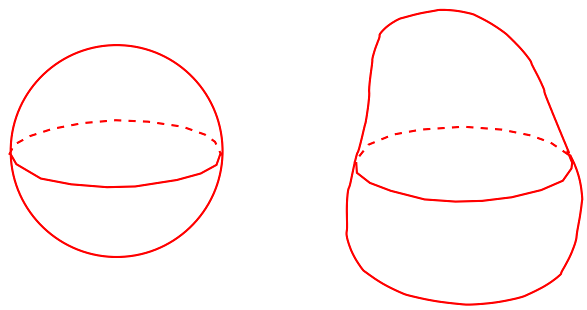

Figure 4. Two-dimensional universes which are spheres.

A little more thought shows that a two-dimensional universe could also have the shape of a torus or the 'surface of a donut' which is illustrated in Figure 5.

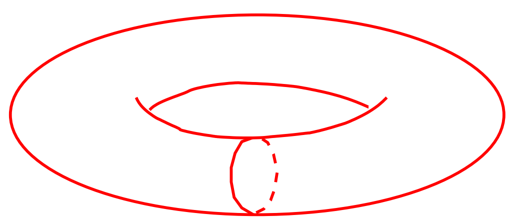

Figure 5. The torus.

From the outside the sphere and the torus look quite different, but do they look different for a person moving around in these universes? There is indeed a big qualitative difference which can be seen from the inside. As we can see in Figure 6, any closed curve (or 'lasso') in the sphere can be pulled back to a point, whereas in the torus not every lasso can be pulled back to a point.

One can now produce more surfaces by 'making more holes'. In Figure 7 we show a surface with 'three holes', which is referred to as a surface of genus three. Using lassos in somewhat more sophisticated way[3] one can show that surfaces with different genus are indeed 'different from the inside'.

[2]The technical term is that we assume our space to be compact.

[3]To be precise, using the 'fundamental group' of a space, which is defined using 'lassos' starting at fixed point.

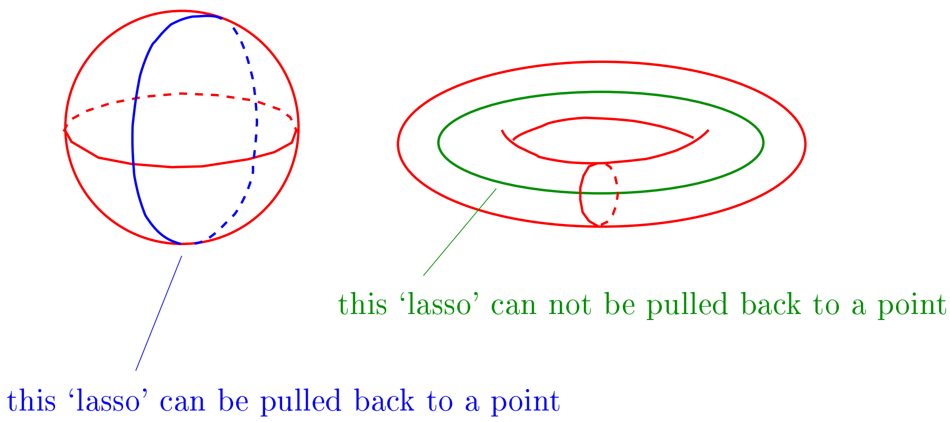

FIGURE 6. 'Lassos' in the sphere and in the torus behave differently.

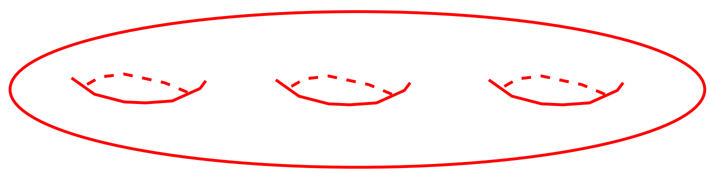

FIGURE 7. Surface of genus three.

Are there any other possible shapes? So far we have assumed that our two-dimensional universe has 'no end', or put differently, has 'no boundary'. Before we turn back to surfaces without boundary it is instructive to look at the case of surfaces with boundary. The disk has one boundary curve and the annulus is a surface with two boundary curves. One can

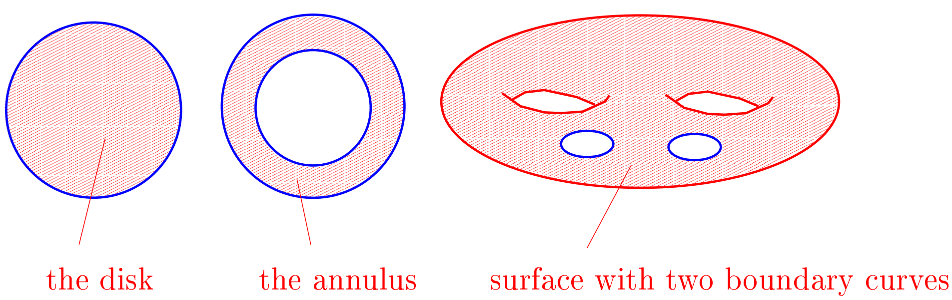

FIGURE 8. Surfaces which have a boundary.

get more examples of surfaces with boundary by taking any surface and cutting out holes.

Once again the question is, are there more examples? To answer this question let us think about how one could actually 'build' an annulus. One way of doing it is to take a rectangular strip of paper, then bend it, so that the sides on the left and right lie opposite to each other, and then glue the two opposite sides together.

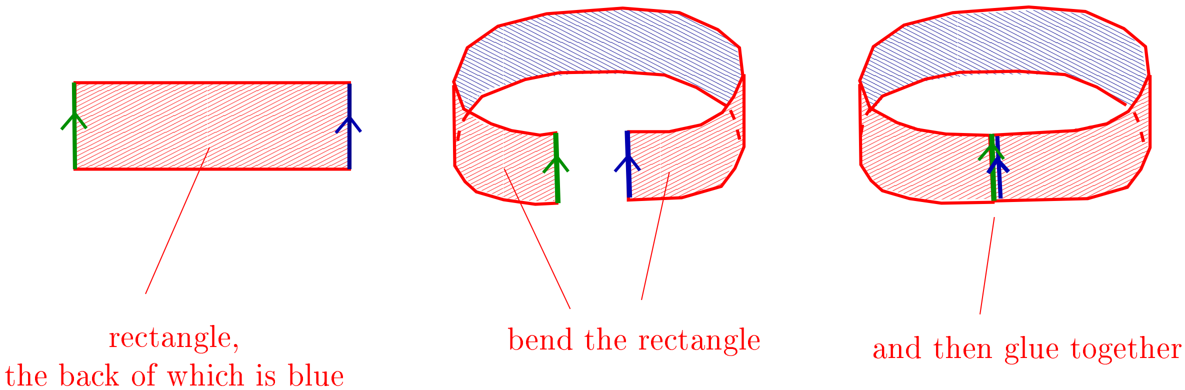

FIGURE 9. Building an annulus out of a rectangle.

When we glued the green and the blue side together we matched up the orientations, as indicated by the arrows. But we could also have glued up the two sides with opposite orientations. To picture this one first needs to bend the rectangle, add a twist, and then glue the two opposite sides together. The resulting surface is called a Möbius band.

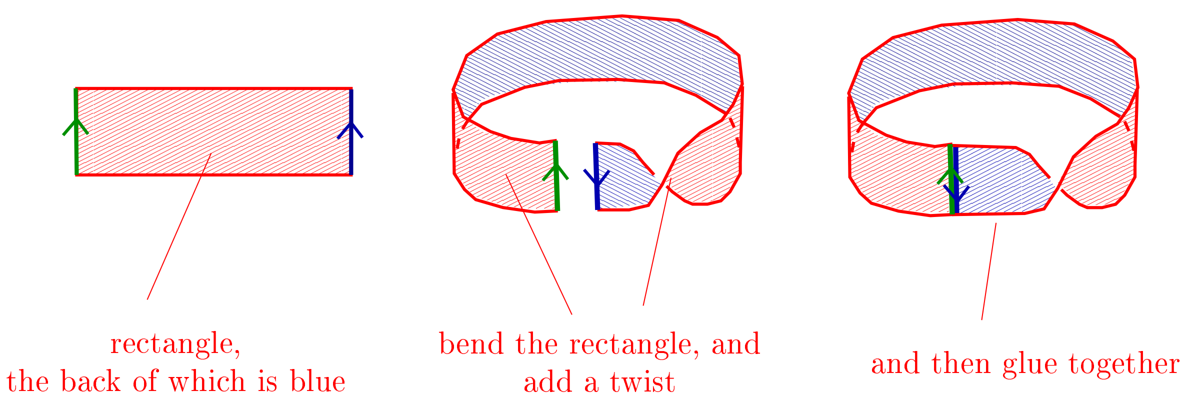

FIGURE 10. Building a Möbius band out of a rectangle.

Note that when we glue together the twisted band, then the colours do not match. This shows that the Möbius band does not have two sides, it only has one side. It is thus clear that from the outside the Möbius band is different from the annulus.

But it is also different from the inside? Let us say we start somewhere on a Möbiusband, holding a bouquet of blue flowers. We walk once around the Möbiusband, and suddenly the bouquet appears on the other side! This shows that there is no notion of 'left' and 'right' in a Möbius band. The Möbius band is thus qualitatively different from the annulus.

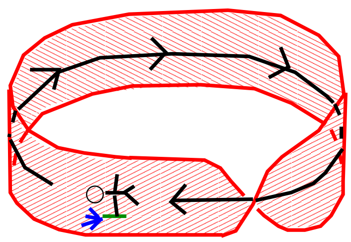

FIGURE 11. Walking once around the Möebius band swaps the two sides.

5. Excursion: Fibered spaces

We now want to reflect a little bit on how we built the annulus and the Möbius band. One way of describing the process is as follows:

(1) We take a space X (in our case X is an interval),
(2) we take an intervals worth of such X's, written as $X \times [0,1]$,
(3) we glue together the two copies of X_0 and X_1 on the left and on the right.

When we glue the two copies of X we have to decide how we match up the points of X_0 and X_1. AS we saw with the annulus and the Möbius band, different matchings can give rise to different spaces. In mathematics a space obtained through such a construction is called a *fibred space*.

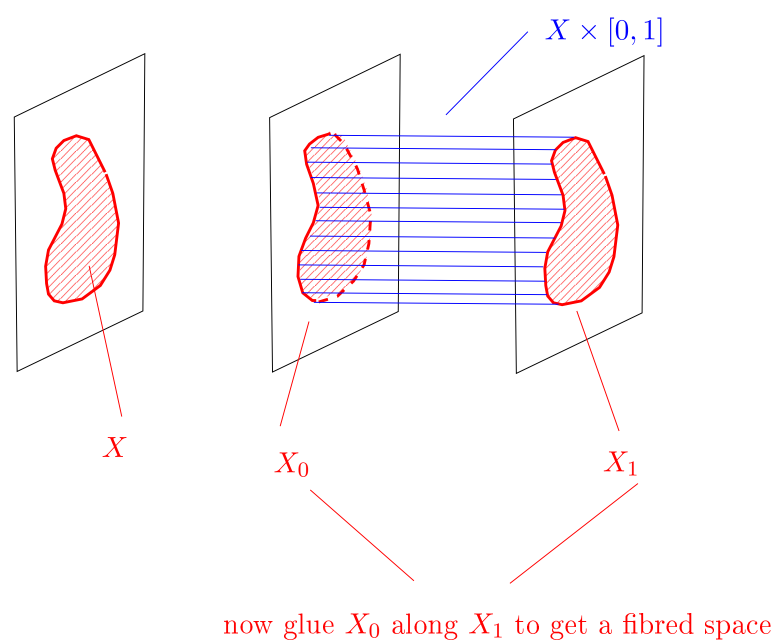

FIGURE 12. Construction of a fibred space.

We have already seen that the annulus and the Möbius band are fibred spaces. In fact we have already seen more fibred spaces. For example, if we start out with X a point and apply the above construction, then we see in Figure 13 that the result is a closed circle.

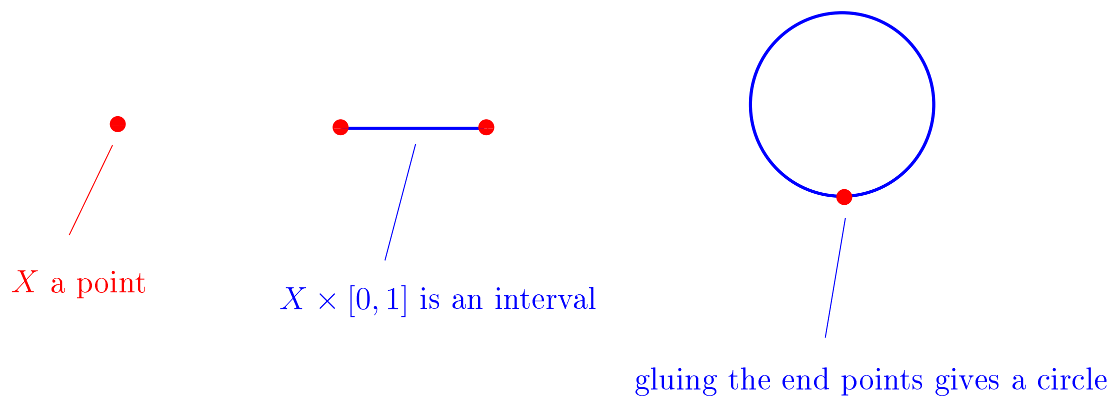

FIGURE 13. The circle viewed as a fibred space.

We now take X to be a circle. If we then take an intervals worth of circles we obtain an annulus. Now we have a choice for how we glue the circles together. Either we preserve the orientation or we reverse it. Let us first look at the case where we preserve the orientation. As we see in Figure 14 this results in a torus.

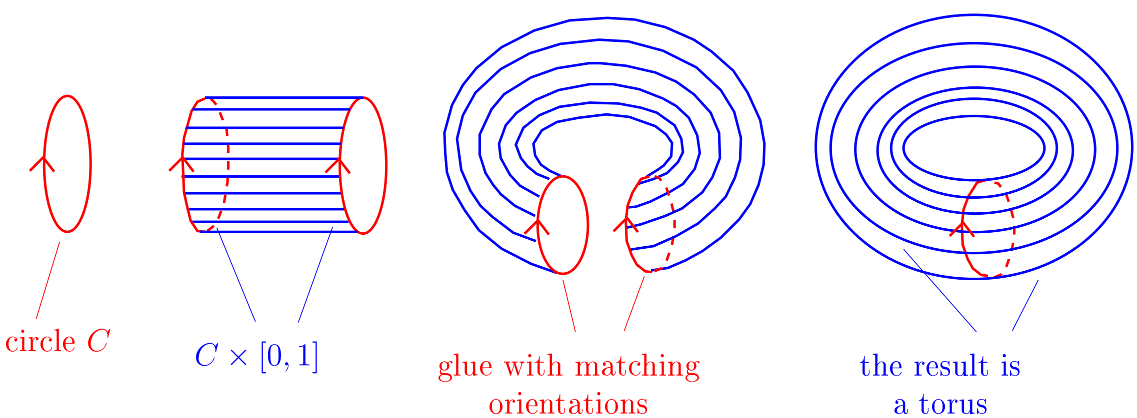

FIGURE 14. Gluing the circles with matching orientations results in a torus.

We will now try to glue the two circles together, but with opposite orientations. This construction is sketched in Figure 15. It is not possible to do this gluing in three-dimensional space without creating a self-intersection. But this self-intersection can be resolved by pushing one sheet slightly into the fourth dimension. Put differently, in $\mathbb{R}^4$ we can glue the two boundary circles of a cylinder with opposite orientations to obtain a two-dimensional space. This surface has the same property as the Möbiusband: there exists a circle on the

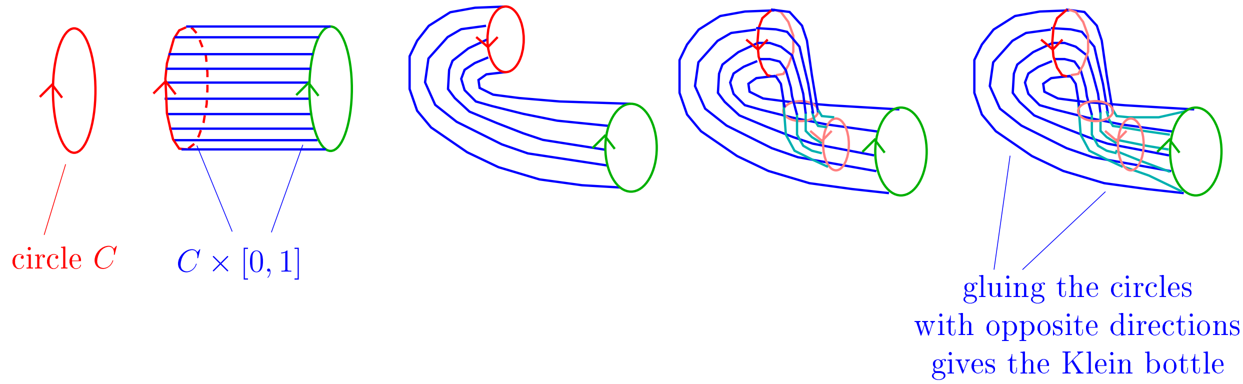

FIGURE 15. Gluing the circles with opposite orientations results in the Klein bottle.

surface, such that if we walk once along the circle we come back as the mirror image of ourselves.

In conclusion we have seen that a few two-dimensional spaces are fibred spaces, but most two-dimensional spaces, namely all surfaces of genus greater than one, are in fact not fibred spaces. Put differently, almost all surfaces are not fibred spaces.

6. Three-dimensional universes

We now finally turn to three-dimensional universes. We again restrict ourselves to 'finite' universes, and this time we only consider spaces with no boundary. One example of a finite three-dimensional space without boundary is given by the three-dimensional sphere. This space can be described rigorously as

$$\{(w,x,y,z) \in \mathbb{R}^4 \,|\, w^2 + x^2 + y^2 + z^2 = 1\}.$$

At first glance it is very hard to come up with more examples. But we have already seen a technique for producing more three-dimensional spaces: We start out with a surface S, we consider $S \times [0,1]$, and we then glue the surface $S \times 0$ to the surface $S \times 1$. Here again we have to decide how we glue the two surfaces together, and in fact this time there are infinitely many different ways of gluing, that then give rise to infinitely many different three-dimensional spaces. These spaces can not be pictured in $\mathbb{R}^3$ or even in $\mathbb{R}^4$, but they can be viewed as subspaces of $\mathbb{R}^5$ or $\mathbb{R}^6$.

Over the last decades mathematicians have been quite inventive when it comes to constructing more three-dimensional spaces. For example, if we take two solid pretzels P and Q, then the boundary in each case is surface of genus three. There are again infinitely many ways how we can glue together these two surfaces, and we obtain infinitely many three-dimensional spaces. It turns out that for almost all gluings the resulting three-dimensional space is not a fibred space.

In general it is quite hard to distinguish two three-dimensional spaces from the inside. The following theorem was known as the Poincaré Conjecture and finally proved by Perelman in 2003.

Theorem 6.1. *The three-dimensional sphere is the only three-dimensional space in which every 'lasso' can be pulled tight.*

This theorem thus gives a nice characterization of the three-dimensional sphere, but in general there is no classification of three-dimensional spaces. There is no 'complete list' of three-dimensional spaces, where one could try to look up what universe we are in.

7. Excursion: Covering spaces

Before we can state the recent theorem of Ian Agol, Piotr Przytycki and Dani Wise we need one more notion: We say that a space $\widetilde{X}$ is an n-fold cover of a space X if 'the space $\widetilde{X}$ can be put on top of X, such that precisely n points in $\widetilde{X}$ correspond to a single point in X'. [4] This concept is not so easy to grasp. Let us therefore look at two examples.

In Figure 16 both the space $\widetilde{X}$ and X are a closed circle. But these are arranged in such

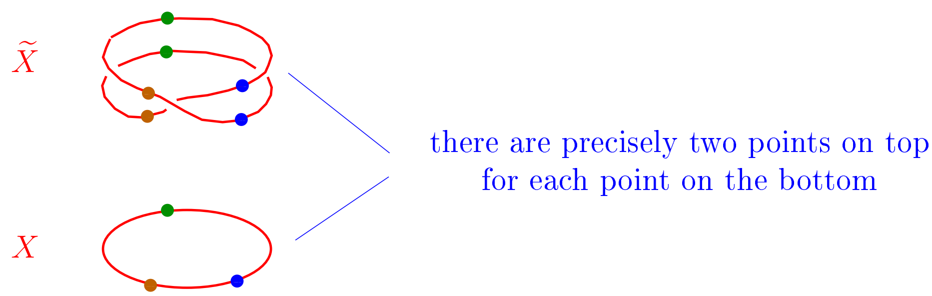

Figure 16. The circle viewed as a 2-fold cover of another circle.

a way, that there are precisely two points on top for one point on the bottom.

Another example of a covering is given in Figure 17. The space on top is a twice twisted band, whereas the space on the bottom is a one-twisted band, i.e. the Möbiusband. Note that a twice twisted band, from the inside, is just the annulus. We thus see that the annulus is a 2-fold cover of the Möbius band.

8. The virtual fibering theorem of Agol and Wise

The following theorem was proved in 2012 by Ian Agol, Piotr Przytycki and Dani Wise in several papers [Agol2008, Agol2013, PrzytyckiWise2012, Wise2012].

Theorem 8.1. *Almost all three-dimensional spaces are covered by a fibred space.*

[4]The mathematical precise way of saying this is that there exists a continous map $f\colon \widetilde{X} \to X$, such that the preimage of point in X is a union of k points in X.

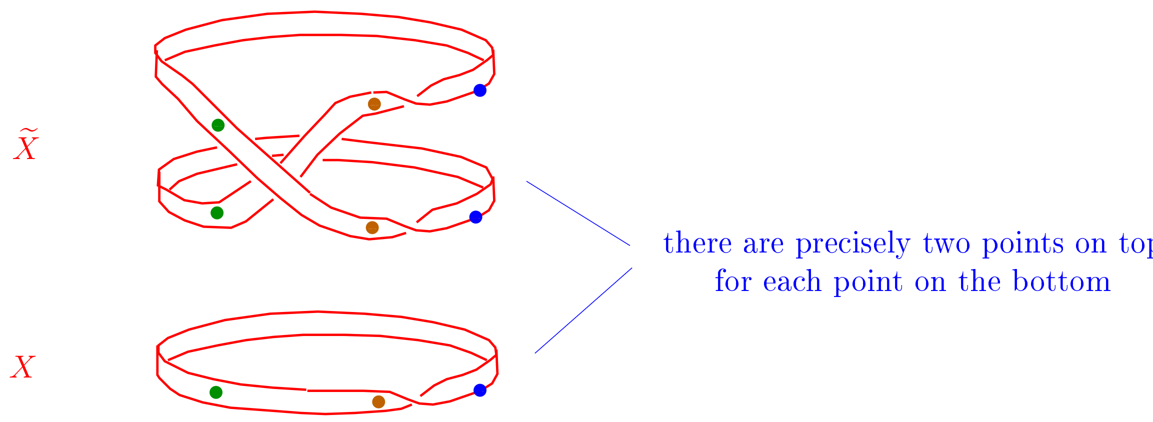

Figure 17. The twice twisted band is a 2-cover of the Möbius band.

The term 'almost all' is obviously somewhat imprecise, but there is a precise formulation, but which requires a little bit more mathematical theory. [5]

This theorem is a big step forward in our attempt to classify three-dimensional spaces. It is easily the most important result since Perelman's proof of the Poincaré Conjecture.

[5] To be precise, the theorem applies to all irreducible 3-manifolds with at least one hyperbolic piece in its JSJ decomposition. It follows from the work of Perelman that 'almost all' three-dimensional spaces are of that form.

References

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