ON VIRTUAL PROPERTIES OF KÄHLER GROUPS

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Abstract. This paper stems from the observation (arising from work of T. Delzant) that "most" Kähler groups virtually algebraically fiber, i.e. admit a finite index subgroup that maps onto $\mathbb{Z}$ with finitely generated kernel. For the remaining ones, the Albanese dimension of all finite index subgroups is at most one, i.e. they have virtual Albanese dimension one. We show that the existence of (virtual) algebraic fibrations has implications in the study of coherence and of higher BNSR invariants of the fundamental group of aspherical Kähler surfaces. The class of Kähler groups of virtual Albanese dimension one contains groups commensurable to surface groups. It is not hard to give further (albeit unsophisticated) examples; however, groups of this class exhibit strong similarities with surface groups. In fact, we show that its only virtually residually finite $\mathbb{Q}$–solvable (vRFRS) elements are commensurable to surface groups, and we show that their Green–Lazarsfeld sets (virtually) coincide with those of surface groups.

1. Introduction

This paper is devoted to the study of some virtual properties of Kähler groups, i.e. fundamental groups of compact Kähler manifolds. Recall that if $\mathcal{P}$ is a property of groups, we say that a group G is *virtually* $\mathcal{P}$ if a finite index subgroup $H \leq_f G$ is $\mathcal{P}$.

This paper stems from the desire to understand if some of the virtual properties of fundamental groups of irreducible 3–manifolds with empty or toroidal boundary, that have recently emerged from the work of Agol, Wise [Ag13, Wi09, Wi12] and their collaborators, have a counterpart for Kähler groups. Admittedly, there is no *a priori* geometric reason to expect any analogy. However this viewpoint seems to be not completely fruitless: for example in [FV16] we investigated consequences of the fact that both classes of groups satisfy a sort of "relative largeness" property, namely that any epimomorphism $\phi\colon G \to \mathbb{Z}$ with *infinitely generated* kernel virtually factorizes through an epimorphism to a free nonabelian group. (This is a property that is nontrivial to prove for both classes.)

In this paper we study, in a sense, the opposite phenomenon, namely the existence of epimorphisms $\phi\colon G \to \mathbb{Z}$ with *finitely generated* kernel. Or, stated otherwise, whether G is an extension of $\mathbb{Z}$ by a finitely generated subgroup. This condition has been conveniently referred to in [JNW17] by saying that G *algebraically fibers* and here we will adhere to that terminology. Again, thanks to the recent results of

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Agol, Wise and collaborators (see [AFW15] for accurate statements and references) the emerging picture is that "most" freely indecomposable 3–manifold groups (e.g. hyperbolic groups) virtually algebraically fiber. (By Stallings Theorem [Sta62], this is actually equivalent to the fact that the underlying 3–manifold is virtually a surface bundle over S^1.) This result has triggered recent interest in the study of (virtual) algebraic fibration for various classes of groups, and relevant results have appeared, including during the preparations of this manuscript, see [FGK17, JNW17].

We have tasked ourselves with the purpose of understanding virtual algebraic fibrations in the realm of Kähler groups.

The first result is little more than a rephrasing of Delzant's results on the Bieri–Neumann–Strebel invariants of Kähler groups ([De10]), and is possibly known at least implicitly to those familiar with that result. It asserts that the "generic" Kähler group virtually algebraically fibers, and more importantly gives a geometric meaning to latter notion. To state this result, recall that the Albanese dimension $a(X) \geq 0$ of a Kähler manifold is defined as the complex dimension of the image of X under the Albanese map Alb. We define the *virtual* Albanese dimension $va(X)$ to be the supremum of the Albanese dimension of all finite covers of X. (This definition replicates that of virtual first Betti number vb_1, that will be as well of use in what follows.) The property of having Albanese dimension equal to zero or equal to one is determined by the fundamental group $G = \pi_1(X)$ alone (see e.g. Proposition 2.1); because of that, it makes sense to talk of (virtual) Albanese dimension of a Kähler group as an element of $\{0, 1, > 1\}$. With this in mind we have the following:

Theorem A. *Let G be a Kähler group. Then either G virtually algebraically fibers, or $va(G) \leq 1$.*

This statement is, in essence, an alternative: the only intersection are groups G that have a finite index subgroup $H \leq_f G$ with $b_1(H) = vb_1(G) = 2$ such that the commutator subgroup $[H, H]$ is finitely generated. (Such a H appears as fundamental group of a genus 1 Albanese pencil without multiple fibers.) We could phrase this theorem as an alternative, but the form above comes naturally from its proof, and fits well with what follows.

Theorem A kindles some interest in identifying the class of Kähler groups of virtual Albanese dimension at most one, and in what follow we summarize what we know about this class.

To start, a Kähler group has $va(G) = 0$ if and only if all its finite index subgroup have finite abelianization, or equivalently $vb_1(G) = 0$. All finite groups fall in this class, but importantly for us, there exist infinite examples as well: a noteworthy one is $Sp(2n, \mathbb{Z})$ for $n \geq 2$ (see [To90]).

The class of Kähler groups with $va(G) = 1$ contains some obvious examples, namely all surface groups (i.e. fundamental groups of compact Riemann surfaces of positive genus); a moment's thought shows that the same is true for all groups that are commensurable (in the sense of Gromov, see [Gr89] or Section 3) to surface groups. These groups do not exhaust the class of groups with $va(G) = 1$: an easy method to build further examples comes by taking the product of a surface group with $Sp(2n, \mathbb{Z})$, $n \geq 2$, by which we obtain Kähler groups with $va(G) = 1$ but not commensurable with surface groups. That said, we are not aware of any subtler construction, that does not hinge on the existence of Kähler groups with $vb_1 = 0$, and it would interesting to decide if such constructions exist. In particular we analyze in Section 3 the implications of the existence of (virtual) algebraic fibrations in the context of aspherical Kähler surfaces. This allows us to partly recast and refine some results about coherence of their fundamental group, which appear in [Ka98, Ka13, Py16]. (Recall that a group is coherent if all its finitely generated subgroups are finitely presented.) Combining these results with ours yields the following:

Theorem B. *Let G be a group with $b_1(G) > 0$ which is the fundamental group of an aspherical Kähler surface X; then G is not coherent, except for the case where it is virtually the product of $\mathbb{Z}^2$ by a surface group, and perhaps for the case where X is finitely covered by a Kodaira fibration of virtual Albanese dimension one.*

(A *Kodaira fibration* is a smooth non-isotrivial pencil of curves.) We are not aware of the existence of Kodaira fibrations of virtual Albanese dimension one (Question 3.2).

The proof of the theorem above actually entails the existence of Kähler groups whose second Bieri-Neumann-Strebel-Renz invariant is *strictly* contained in the first (see Lemma 3.4) and are not a direct product.

In fact, the properties of the Albanese map give a tight relation between Kähler groups with $va(G) = 1$ and surface groups. In fact, the underlying Kähler manifold (virtually) admits the following form: if a Kähler manifold X has Albanese dimension one, the Albanese map $f\colon X \to \Sigma$ (after restricting its codomain to the image) is a genus $g = q(X)$ pencil. (Here $q(X) := \frac{1}{2}b_1(X)$ denotes the irregularity of a Kähler manifold X.) When $va(G) = 1$, the Albanese pencil lifts to an Albanese pencil $\tilde{f}\colon \widetilde{X} \to \widetilde{\Sigma}$ for all finite covers $\widetilde{X} \to X$.

If we impose to $G = \pi_1(X)$ residual properties that mirror those of 3–manifold groups, we obtain a refinement to Theorem A that can be thought of as an analogue (with much less work on our side) to Agol's virtual fiberability result ([Ag08]) for 3–manifold groups that are virtually RFRS (see Section 3 for the definition).

Theorem C. *Let G be a Kähler group that is virtually RFRS. Then either G virtually algebraically fibers, or is commensurable to a surface group.*

Turning this theorem on its head, examples of Kähler groups with $va(G) \leq 1$ that are not commensurable to surface groups are certified not to be virtually RFRS. To date, the infinite Kähler groups known to be RFRS are subgroups of the direct product of surface groups and abelian groups. This is a remarkable but not yet completely understood class of Kähler groups: in [DG05, 7.5 sbc–Corollary] some conditions for a Kähler group to be virtually of this type are presented.

In general, the relation between X and Σ induced by the Albanese pencil $f\colon X\to\Sigma$ turns out to be much stronger than the isomorphism of the first cohomology groups. In fact, up to going to a finite index subgroup if necessary, the Green–Lazarsfeld sets of their fundamental groups coincide. Given an Albanese pencil, we refer to the induced map in homotopy $f\colon G\to\Gamma$ (where $G:=\pi_1(X)$ and $\Gamma:=\pi_1(\Sigma)$) as Albanese map as well. We have the following:

Theorem D. *Let G be a group with $va(G)=1$. After going to a finite index normal subgroup if necessary, the Albanese map $f\colon G\to\Gamma$ induces an isomorphism of the Green–Lazarsfeld sets*

$$\widehat{f}\colon W_i(\Gamma)\xrightarrow{\cong}W_i(G).$$

Structure of the paper. Section 2 discusses some preliminary results on the Albanese dimension of Kähler manifolds and groups, as well as the proof of Theorem A. Section 3 is devoted to the study of groups of virtual Albanese dimension one, and contains the proofs of Theorems B, C and D.

In order to keep the presentation reasonably self–contained, we included some fairly classical results, for which we could not find a formulation in the literature suitable for our purposes. For the same reason, we describe – hopefully with accurate and appropriate attribution – more recent work that is germane to the purposes of this paper.

Conventions. All manifolds are assumed to be compact, connected and orientable, unless we say otherwise.

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2. Albanese dimension and algebraic fibrations

We start with some generalities on pencils on Kähler manifolds and the properties of their fundamental group. The reader can find in the monograph [ABCKT96] a detailed discussion of Kähler groups, and the rôle they play in determining pencils on the underlying Kähler manifolds. Given a genus g pencil on X (i.e. a surjective holomorphic map with connected fibers $f\colon X\to\Sigma$ to a surface with $g=g(\Sigma)$) we can consider the homotopy–induced epimorphism $f\colon\pi_1(X)\to\pi_1(\Sigma)$. In presence of multiple fibers we have a factorization $f\colon\pi_1(X)\to\pi_1^{\text{orb}}(\Sigma)\to\pi_1(\Sigma)$ through a further epimorphism onto $\pi_1^{\text{orb}}(\Sigma)$, the orbifold fundamental group of Σ associated to the pencil f, with orbifold points and multiplicities corresponding to the multiple fibers of the pencil. Throughout the paper we write $G:=\pi_1(X)$ and $\Gamma:=\pi_1^{\text{orb}}(\Sigma)$. The factor epimorphism, that by slight abuse of notation we denote as well as $f\colon G\to\Gamma$, has finitely generated kernel so we have the short exact sequence of finitely generated groups

$$1 \to K \to G \xrightarrow{f} \Gamma \to 1$$

(see e.g. [Cat03] for details of the above).

We have the following two results about Kähler manifolds of (virtual) Albanese dimension one. These are certainly well–known to the experts (at least implicitly), and we provide proofs for completeness.

Proposition 2.1. *Let X be a Kähler manifold and let $G = \pi_1(X)$ be its fundamental group. If X has Albanese dimension $a(X) \leq 1$, any Kähler manifold with isomorphic fundamental group has the same Albanese dimension as X.*

Proof. The case where $a(X) = 0$ corresponds to manifolds with vanishing irregularity $q(X) = \frac{1}{2}b_1(X)$ so it is determined by the fundamental group alone. Next, consider a Kähler manifold X with positive irregularity. For any such X, the *genus* $g(X)$ is defined as the maximal rank of submodules of $H^1(X)$ isotropic with respect to the cup product. In [ABCKT96, Chapter 2] it is shown that $g(X)$ is in fact an invariant of the fundamental group alone. As the cup product is nondegenerate, there is a bound

$$g(X) \leq q(X) = \frac{1}{2}b_1(X) = \frac{1}{2}b_1(G).$$

By Catanese's version of the Castelnuovo–de Franchis theorem ([Cat91]), the case where X has Albanese dimension one occurs exactly for $g(X) = q(X)$. By the above, this equality is determined by the fundamental group alone. □

Based on the observations above, we will refer to the Albanese dimension of a Kähler group as an element of the set $\{0, 1, > 1\}$.

Whenever G has Albanese dimension one, the kernel of the map $f_*\colon H_1(G) \to H_1(\Gamma)$ induced by the (homotopy) Albanese map, identified by the Hochschild–Serre spectral sequence with a quotient of the coinvariant homology $H_1(K)_\Gamma$ by a torsion group, is torsion (or equivalently $f^*\colon H^1(\Gamma) \to H^1(G)$ is an isomorphism). Note that, by universality of the Albanese map, whenever a Kähler manifold X has a pencil $f\colon X \to \Sigma$ such that the kernel of $f_*\colon H_1(G) \to H_1(\Gamma)$ is torsion, the pencil is Albanese.

Let $\pi\colon \widetilde{X} \to X$ be a finite cover of X. Denote by $H \leq_f G$ the subgroup associated to this cover; the regular cover of X determined by the normal core $N_G(H) = \bigcap_{g \in G} g^{-1}Hg \leq_f H$ is a finite cover of $\widetilde{X}$ as well. By universality of the Albanese map, the Albanese dimension is nondecreasing when we pass to finite covers. Therefore (as happens with virtual Betti numbers) we can define the virtual Albanese dimension of X in terms of finite *regular* covers. Given an epimorphism onto a finite group

$\alpha\colon \pi_1(X) \to S$ we have the commutative diagram (with self–defining notation)

(1)
$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \Delta & \longrightarrow & H & \longrightarrow & \Lambda & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & K & \longrightarrow & G & \longrightarrow & \Gamma & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\alpha} & & \downarrow & & \\
1 & \longrightarrow & \alpha(K) & \longrightarrow & S & \longrightarrow & S/\alpha(K) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1 & &
\end{array}$$

Denote by $\widetilde{X}$ and $\widetilde{\Sigma}$ the induced covers of X and Σ respectively (so that $\pi_1(\widetilde{X}) = H$ and $\pi_1(\widetilde{\Sigma}) = \Lambda$). There exists a pencil $\tilde{f}\colon \widetilde{X} \to \widetilde{\Sigma}$, which is a lift of $f\colon X \to \Sigma$; in homotopy, this corresponds to the epimorphism $\widetilde{f}\colon H \to \Lambda := \pi_1^{\text{orb}}(\widetilde{\Sigma})$ appearing in (1) above.

In the next proposition, we illustrate the fact that when X has Albanese dimension one, its Albanese pencil $f\colon X \to \Sigma$ is the only irrational pencil of X, up to holomorphic automorphisms of Σ.

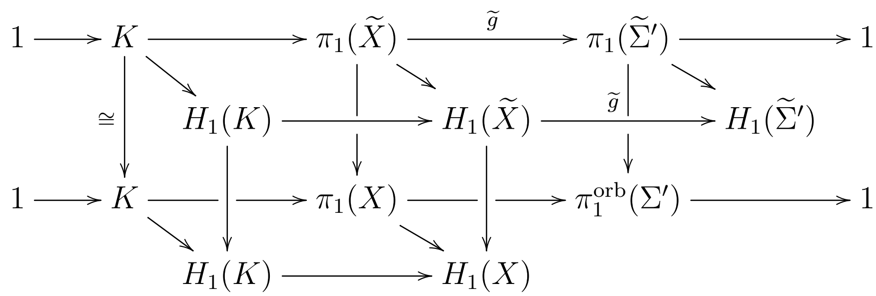
Proposition 2.2. *Let X be a Kähler manifold with $a(X) = 1$. Then the Albanese pencil $f\colon X \to \Sigma$ is the unique irrational pencil on X up to holomorphic automorphism of the base. Moreover, if X satisfies $va(X) = 1$, any rational pencil has orbifold base with finite orbifold fundamental group.*

Proof. By assumption the Albanese pencil $f\colon X \to \Sigma$ factorizes the Albanese map Alb. Let $g\colon X \to \Sigma'$ be an irrational pencil, and compose it with the Jacobian map $j\colon \Sigma' \to \text{Jac}(\Sigma')$. By universality of the Albanese map we have the commutative diagram

$$\begin{array}{ccccc}
X & \xrightarrow{f} & \Sigma & \longrightarrow & \text{Alb}(X) \\
 & {\scriptstyle g}\searrow & \downarrow{\scriptstyle h} & & \downarrow \\
 & & \Sigma' & \xrightarrow{j} & \text{Jac}(\Sigma').
\end{array}$$

The map $h\colon \Sigma \to \Sigma'$ is well defined by injectivity of the Jacobian map, and is a holomorphic surjection by universality of the Albanese map. Holomorphic surjections of Riemann surfaces are ramified covers; however, unless the cover is one–sheeted, i.e. h is a holomorphic isomorphism, the fibers of $g\colon X \to \Sigma'$ will fail to be connected.

This argument above does not prevent X from having rational pencils. However, if X has also virtual Albanese dimension one, this imposes constraints on the multiple fibers of those pencils. Recall that orbifolds with infinite $\pi_1^{\text{orb}}(\Sigma)$ are those that are flat or hyperbolic, hence admit a finite index normal subgroup that is a surface group with positive b_1 (see [Sc83] for this result and a characterization of these orbifolds in terms of the singular points). Therefore, if X were to admit a rational pencil $g\colon X \to \Sigma'$ with infinite $\pi_1^{\text{orb}}(\Sigma')$, there would exist an irrational pencil without multiple fibers $\widetilde{g}\colon \widetilde{X} \to \widetilde{\Sigma}'$ covering $g\colon X \to \Sigma'$. We claim that the pencil $\widetilde{g}\colon \widetilde{X} \to \widetilde{\Sigma}'$ cannot be Albanese: building from the commutative diagram in (1) we have the commutative diagram



The subgroup $\text{im}(H_1(K) \to H_1(X)) \leq H_1(X)$ has positive rank (it is a finite index subgroup), hence by commutativity the image $\text{im}(H_1(K) \to H_1(\widetilde{X})) \leq H_1(\widetilde{X})$ has positive rank. It follows that the kernel of the epimorphism $\widetilde{g}\colon H_1(\widetilde{X}) \to H_1(\widetilde{\Sigma}')$ is not torsion, so $\widetilde{g}\colon \widetilde{X} \to \widetilde{\Sigma}'$ is not Albanese. As $\widetilde{X}$ has Albanese dimension one, this is inconsistent with the first part of the statement. $\square$

We are now in a position to prove Theorem A. In order to do so, it is both practical and insightful to use the Bieri–Neumann–Strebel invariant of a finitely presented group G (henceforth BNS), for which we refer to [BNS87] for definitions and properties used here. This invariant is an open subset $\Sigma^1(G)$ of the character sphere $S(G) := (H^1(G;\mathbb{R}) \setminus \{0\})/\mathbb{R}_{>0}$ of $H^1(G;\mathbb{R})$. For rational rays, the invariant can be described as follows: a rational ray in $S(G)$ is determined by a primitive class $\phi \in H^1(G)$. Given such ϕ, we can write G as HNN extension $G = \langle A, t | t^{-1}Bt = C\rangle$ for some finitely generated subgroups $B, C \leq A \leq \text{Ker } \phi$ with $\phi(t) = 1$. The extension is called ascending (descending) if $A = B$ (resp. $A = C$). By [BNS87, Proposition 4.4] the extension is ascending (resp. descending) if and only if the rational ray determined by ϕ (resp. $-\phi$) is contained in $\Sigma^1(G)$.

We have the following theorem, which applied to the collection of finite index subgroups of G, implies Theorem A:

Theorem 2.3. *Let G be a Kähler group. The following are equivalent:*

(1) *G algebraically fibers;*
(2) *The BNS invariant $\Sigma^1(G) \subseteq S(G)$ is nonempty;*
(3) *For any compact Kähler manifold X such that $\pi_1(X) = G$ either the Albanese map is a genus 1 pencil without multiple fibers, or X has Albanese dimension greater than one.*

Before proving this proposition, let us mention that (with varying degree of complexity) it is possible to verify directly that groups commensurable to nonabelian surface groups cannot satisfy any of the three cases above.

Proof. We will first show (1) $\Leftrightarrow$ (2), and then $\neg$(2) $\Leftrightarrow$ $\neg$(3).

(1) $\Rightarrow$ (2): If (1) holds we can write the short exact sequence

$$1 \to \text{Ker } \phi \to G \xrightarrow{\phi} \mathbb{Z} \to 1 \tag{2}$$

with Ker ϕ finitely generated. Hence G is both an ascending and descending HNN extension. It follows that the rational rays determined by both $\pm\phi \in H^1(G)$ are contained in $\Sigma^1(G)$.

(2) $\Rightarrow$ (1): Let us assume that $\Sigma^1(G)$ is nonempty. As $\Sigma^1(G)$ is open, we can assume that there exists a primitive class $\phi \in H^1(G)$ whose projective class is determined by a rational ray in $\Sigma^1(G)$. A remarkable fact at this point is that Kähler groups cannot be written as *properly* ascending or descending extensions, i.e. ascending extensions are also descending and viceversa. (This was first proven in [NR08]; see also [FV16] for a proof much in the spirit of the present paper.) But this is to say that G has the form of Equation (2) with Ker ϕ finitely generated.

To show the equivalence of $\neg$(2) and $\neg$(3), we start by recalling Delzant's description of the BNS invariant of a Kähler group G. Let X be a Kähler manifold with $G = \pi_1(X)$. The collection of irrational pencils $f_\alpha \colon X \to \Sigma_\alpha$ such that the orbifold fundamental group $\Gamma_\alpha := \pi_1^{\text{orb}}(\Sigma)$ is a cocompact Fuchsian group, is finite up to holomorphic automorphisms of the base (see [De08, Theorem 2]). (In the language of orbifolds, these are the holomorphic orbifold maps with connected fibers from X to hyperbolic Riemann orbisurfaces.) The pencil maps give, in homotopy, a finite collection of epimorphisms with finitely generated kernel $f_\alpha \colon G \to \Gamma_a$. Then [De10, Théorème 1.1] asserts that the complement of $\Sigma^1(G)$ in $S(G)$ (i.e. the set of so–called exceptional characters) is given by

$$S(G) \setminus \Sigma^1(G) = \textstyle\bigcup_\alpha [f_\alpha^* H^1(\Gamma_\alpha; \mathbb{R}) - \{0\}], \tag{3}$$

where we use the brackets $[\cdot]$ to denote the image of a subset of $(H^1(G;\mathbb{R}) \setminus \{0\})$ in $(H^1(G;\mathbb{R}) \setminus \{0\})/\mathbb{R}_{>0}$. (Note, instead, that genus 1 pencils without multiple fibers do not induce exceptional characters.)

$\neg$(3) $\Rightarrow$ $\neg$(2): The negation of (3) asserts that X has either Albanese dimension zero, in which case $S(G)$ is empty, or it has an Albanese pencil $f \colon X \to \Sigma$ with $\Gamma := \pi_1^{\text{orb}}(\Sigma)$ cocompact Fuchsian, in which case $f^* H^1(\Gamma; \mathbb{R}) = H^1(G; \mathbb{R})$. In either case $\Sigma^1(G)$ is empty, i.e. $\neg$(2) holds.

$\neg$(2) $\Rightarrow$ $\neg$(3): If the set $\Sigma^1(G) \subseteq S(G)$ is empty, either $S(G)$ is empty (i.e. $b_1(G) = 0$) whence G has Albanese dimension zero, or by Equation (3) there exists an irrational pencil $f \colon X \to \Sigma$, with $\Gamma := \pi_1^{\text{orb}}(\Sigma)$ cocompact Fuchsian, inducing an isomorphism $f^* \colon H^1(\Gamma; \mathbb{R}) \to H^1(G; \mathbb{R})$. (The union of finitely many *proper* vector subspaces of $H^1(G; \mathbb{R})$ cannot equal $H^1(G; \mathbb{R})$.) Such a pencil is then the Albanese pencil of X. $\square$

In our understanding, Kähler manifolds whose Albanese dimension is smaller than their dimension are "nongeneric", and their study should reduce, through a sort of dimensional reduction induced by the Albanese map, to the study of lower dimensional spaces (see e.g. [Cat91]). In that sense, we think of groups that, together with their finite index subgroups, fail to satisfy the equivalent conditions (1) to (3) of Theorem 2.3 as nongeneric.

Examples of Kähler groups with $a(G) > 1$ abound. It is less obvious to provide examples of Kähler groups that have a jump in Albanese dimension, i.e. $a(G) = 1$ but $va(G) > 1$. Before doing so, we can make an observation about the geometric meaning of such occurrence. Given an irrational pencil $f \colon X \to \Sigma$, its *relative irregularity* is defined as

$$q_f = q(X) - q(\Sigma)$$

(where the irregularity of Σ equals its genus). The Albanese pencil occurs exactly when $q_f = 0$. The notion of relative irregularity allows us to tie the notion of virtual Albanese dimension larger than one with the more familiar notion of virtual positive Betti number (or more properly, in Kähler context, virtual irregularity): a Kähler group with $a(G) = 1$ has $va(G) > 1$ if and only if there is a lift $\tilde{f} \colon \widetilde{X} \to \widetilde{\Sigma}$ of the Albanese pencil which is *irregularly fibered*, i.e. $q_{\tilde{f}} > 0$.

A fairly simple class of examples comes from groups of type $G = \pi_g \times K$ where π_g is the fundamental group of a genus $g > 1$ surface and K the fundamental group of a hyperbolic orbisurface of genus 0, so that $b_1(K) = 0$. The group K is Kähler (e.g. it is the fundamental group of an elliptic surface with enough multiple fibers and multiplicity), hence so is G. As $H^1(G; \mathbb{Z}) = H^1(\pi_g; \mathbb{Z})$, the Albanese dimension of G must be one. On the other hand, K has a finite index subgroup that is the fundamental group of a genus $h > 1$ surface, hence G is virtually $\pi_g \times \pi_h$. The algebraic surface $\Sigma_g \times \Sigma_g$ has Albanese dimension 2, so by Proposition 2.1 the virtual Albanese dimension of G is greater than one. (Group theoretically, this can be seen as consequence of [BNS87, Theorem 7.4], which asserts that for cartesian products of groups with positive b_1 the BNS invariant is nonempty.)

Less trivial examples with $a(G) = 1$ but $va(G) > 1$ come from bielliptic surfaces. These possess an Albanese pencil of genus 1 without multiple fibers, but their fundamental groups are virtually $\mathbb{Z}^4$, hence have virtual Albanese dimension 2 (see [BHPV04, Section V.5]). More sophisticated examples of Kähler surfaces (hence groups) with $a(G) = 1$ that are finitely covered by the product of curves of genera bigger than one are discussed in [Cat00, Theorem F].

The most interesting class of examples we are aware of, however, comes from the recent paper of Stover ([Sto15, Theorem 3]) that gives examples of Kähler groups that arise as cocompact arithmetic lattices in $PU(n, 1)$ for which $a(G) = 1$; Stover proves that those groups are virtually extensions of $\mathbb{Z}$ by a finitely generated group, and application of Theorem 2.3 yields to the conclusion that $va(G) > 1$.

3. Groups with Virtual Albanese Dimension One

In this section, we will discuss groups with $va(G) = 1$. Familiar examples of Kähler groups with $va(G) = 1$ are given by surface groups, and other simple examples arise as follows. Following Gromov ([Gr89]) we say that a group G is *commensurable* to a surface group if it admits a (normal) finite index subgroup $H \leq_f G$ that admits an epimorphism $f\colon H \to \Gamma$ to a surface group Γ with finite kernel, namely for which there exists an exact sequence

$$1 \to F \to H \to \Gamma \to 1$$

with Γ a surface group and F finite. The map $f\colon H_1(H) \to H_1(\Gamma)$ is then an epimorphism with torsion kernel, i.e. $f\colon H \to \Gamma$ represents, in homotopy, an Albanese map. Quite obviously, each finite index subgroup of G will also be commensurable to a surface group, hence the virtual Albanese dimension of G equals one.

We want to analyze the picture so far in comparison with the situation for 3–manifold groups. The class of irreducible 3–manifolds that are not virtually fibered is limited (it is composed entirely by graph manifolds). One may contemplate that, similarly, the Kähler counterpart to that class contains only the obvious candidates, namely manifolds whose fundamental group is commensurable to a surface group. Proposition 3.1 below guarantees that this is not quite the case. The starting point is the existence of infinite Kähler groups (such as $Sp(2n, \mathbb{Z})$, $n > 1$) with $vb_1(G)$, hence $va(G)$, equal to zero. These do not have a counterpart in dimension 3.

Proposition 3.1. *Let Γ be the fundamental group of a genus $g > 0$ surface and let K be an infinite Kähler groups with $va(K) = 0$; then $va(K \times \Gamma) = a(K \times \Gamma) = 1$ and $K \times \Gamma$ is not commensurable to a surface group.*

Proof. The projection map $f\colon K \times \Gamma \to \Gamma$ is the Albanese map, hence $a(K \times \Gamma) = 1$. We claim that as $vb_1(K) = 0$, the virtual Albanese dimension of $K \times \Gamma$ is one. In fact, for any normal subgroup $H \leq_f K \times \Gamma$ we have from (1)

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Delta & \longrightarrow & H & \xrightarrow{\widetilde{f}} & \Lambda & \longrightarrow & 1 \\
 & & \downarrow & \searrow & \downarrow & \searrow & \downarrow & \searrow & \\
 & & & H_1(\Delta) & \longrightarrow & H_1(H) & \longrightarrow & H_1(\Lambda) & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & K & \longrightarrow & K \times \Gamma & \xrightarrow{f} & \Gamma & \longrightarrow & 1.
\end{array}$$

As $vb_1(K) = 0$, $H_1(\Delta)$ is torsion, hence $\operatorname{Ker}(H_1(H) \to H_1(\Lambda)) = \operatorname{Im}(H_1(\Delta) \to H_1(H))$ is torsion as well. It follows that the map $\widetilde{f}\colon H \to \Lambda$ is, in homotopy, the Albanese map, hence $a(H) = 1$.

Assume by contradiction that $K \times \Gamma$ is commensurable to a surface group, and denote by $H \leq_f K \times \Gamma$ a normal subgroup such that

$$1 \to F \to H \to \pi \to 1$$

with F finite and π a surface group. As F is finite, $\mathrm{Ker}(H_1(H) \to H_1(\pi))$ is torsion, hence the epimorphism $H \to \pi$ is, in homotopy, the Albanese map. But then it must coincide with the epimorphism $\widetilde{f}\colon H \to \Lambda$ as described above (perhaps up to an automorphism of Λ). This is not possible, as the former has a finite kernel F, while the latter has kernel Δ, which is infinite as $\Delta \leq_f K$ and K is infinite by assumption. □

Proposition 3.1 guarantees that the class of Kähler groups that do not virtually admit an epimorphism to $\mathbb{Z}$ with finitely generated kernel is more variegated than its counterpart in the 3–manifold world.

We should, however, qualify this result. The examples of Proposition 3.1 build on the existence of infinite Kähler groups with $vb_1 = 0$, and leverage on the fact that we can take products of finitely presented groups, as the class of Kähler manifolds is closed under cartesian product (and, more generally, holomorphic fiber bundles). Neither of these phenomena has a counterpart in the realm of 3–manifolds. It is perhaps not too greedy to ask for examples of Kähler groups with $va(G) = 1$ in a realm where simple constructions as the one of Proposition 3.1 are tuned out.

As we are about to see, an instance of this occurs in the case of aspherical surfaces:

Question 3.2. *Does there exist a group G with $b_1(G) > 0$ that is the fundamental group of an aspherical Kähler surface and does not virtually algebraically fiber?*

The reason why an example like the one we are after in Question 3.2 would be appealing comes from the fact that, perhaps going to a finite index subgroup, the fundamental group G of an aspherical Kähler surface with $va(G) = 1$ is a 4–dimensional Poincaré duality group whose Albanese pencil $f\colon X \to \Sigma$ determines a short exact sequence of finitely generated groups

$$1 \to K \to G \to \Gamma \to 1, \tag{4}$$

with Γ a surface group and K finitely generated. As first remarked by Kapovich in [Ka98], it is a theorem of Hillman that either K is itself a (nontrivial) surface group, or it is *not* finitely presented (see e.g. [Hil02, Theorem 1.19]). In either case, a construction like the one in Proposition 3.1 (or even a twist thereof) is excluded. Constructions of this type may not be easy to find. As we mentioned before, the examples of Stover virtually algebraically fiber. The same is true for the Cartwright–Steger surface ([CS10]), another ball quotient with $b_1(G) = 2$ and whose Albanese map (as shown in [CKY15, Corollary 5.3]) has no multiple fiber: by the discussion in the proof of Theorem 2.3 G itself has BNS invariant $\Sigma^1(G)$ equal to the entire $S(G)$ and all epimorphisms to $\mathbb{Z}$ have finitely generated kernel. (According to the introduction to [Sto15], this fact was known also to Stover and collaborators.)

We want now to show how the question above ties with the study of coherence of fundamental groups of aspherical Kähler surfaces, which was initiated in [Ka98, Ka13] using the aforementioned result of Hillman, and further pursued in [Py16]. The outcome of these articles is that, with the obvious exceptions, most aspherical

Kähler surfaces can be shown to have non–coherent fundamental group. (See [Py16, Theorem 4] for a detailed statement, which uses a notation slightly different from ours.) Drawing from the same circle of ideas of these references (as well as a minor extension of the work in [Ko99]) we can prove the following result, which improves on the existing results insofar as it further narrows possible coherent fundamental groups to finite index subgroups of the fundamental group of Kodaira fibrations with virtual Albanese dimension one. (A pencil on a Kähler surface is called a Kodaira fibration if it is smooth and not isotrivial.)

Theorem 3.3. *Let G be a group with $b_1(G) > 0$ which is the fundamental group of an aspherical Kähler surface X; then G is not coherent, except for the case where it is virtually the product of $\mathbb{Z}^2$ by a surface group, or perhaps for the case where X is finitely covered by a Kodaira fibration of virtual Albanese dimension one.*

Proof. If G has $va(G) > 1$, let $H \leq_f G$ be a subgroup, corresponding to a finite n–cover $\widetilde{X}$ of X, which algebraically fibers. Let

$$1 \to \text{Ker } \phi \to H \xrightarrow{\phi} \mathbb{Z} \to 1 \tag{5}$$

with Ker ϕ finitely generated represent an algebraic fibration. By [Hil02, Theorem 4.5(4)] the finitely generated group Ker ϕ has type FP_2 if and only if the Euler characteristic $e(\widetilde{X}) = ne(X) = 0$. A finitely presented group has type FP_2. It follows that Ker $\phi \leq G$ is finitely generated but not finitely presented, hence G is not coherent, unless $e(X) = 0$. If $e(X) = 0$ the classification of compact complex surfaces entails that X admits an irrational pencil with elliptic fibers. As $e(X) = 0$ the Zeuthen–Segre formula (see e.g. [BHPV04, Proposition III.11.4]) implies that the only singular fibers can be multiple covers of an elliptic fiber, hence X is finitely covered by a torus bundle. An holomorphic fibration with smooth fibers of genus 1 is also isotrivial (i.e. all fibers are isomorphic), namely a holomorphic fiber bundle, see [BHPV04, Section V.14]. We can invoke then [BHPV04, Sections V.5 and V.6] to deduce that some finite cover of X is a product $T^2 \times \Sigma_g$. In this case, the fundamental group is virtually $\mathbb{Z}^2 \times \Gamma$, with Γ a surface group. By Theorem B of [BHMS02] a finitely generated subgroup $L \leq \mathbb{Z}^2 \times \Gamma$ has a finite index subgroup that is the product of finitely generated subgroups of each factor. As surface groups are coherent, L must be finitely presented, hence in this case the fundamental group of X is coherent.

If G has $va(G) = 1$, perhaps going to a finite index subgroup, G is a 4–dimensional Poincaré duality group whose Albanese pencil $f\colon X \to \Sigma$ determines as usual the short exact sequence of finitely generated groups $1 \to K \to G \to \Gamma \to 1$, with Γ a surface group and K finitely generated. To deal with this case, we relay on the strategy of [Ka98, Ka13]: by the aforementioned theorem of Hillman [Hil02, Theorem 1.19] either K is not finitely presented (and G is not coherent) or it is itself a nontrivial surface group. In the latter case, we can follow the path of [Ko99] (see also [Hil00]) to complete the proof of the statement. We first observe that the surface X, being aspherical, is homotopy equivalent to a smooth 4–manifold M^4, a surface bundle over a surface $F \hookrightarrow M \to \Sigma$, where $\pi_1(F) = K$, $\pi_1(\Sigma) = \Gamma$. Note that M is uniquely determined by the short exact sequence of its fundamental group, see [Hil02, Theorem 5.2]. Next, as X and M are homotopy equivalent, they have the same Euler characteristic $e(X) = e(M) = e(F) \cdot e(\Sigma)$. At this point, the Zeuthen–Segre formula entails that the only nonsmooth fibers of the Albanese pencil could be multiple covers of an elliptic fiber (in particular, $g(F) = 1$). The assumption that K is finitely generated excludes the presence of multiple fibers ([Cat03, Lemma 4.2]). This means that Albanese pencil $f: X \to \Sigma$ is smooth (i.e. a holomorphic fibration of maximal rank), namely X is actually a surface bundle over a surface, in particular it is diffeomorphic to M. If the pencil was isotrivial (i.e. all fibers isomorphic), it would be a holomorphic fiber bundle, and we would conclude as above that some finite cover of X is a product, whence $va(X) = 2$ and $va(G) > 1$. The statement follows. □

In summary, the existence of non-obvious examples of coherent fundamental groups of aspherical Kähler surfaces hinges on an affirmative answer to Question 3.2.

Remarks.

(1) Note that the proof of the first case of Theorem 3.3 applies *verbatim* also in the case of aspherical surfaces with $a(G) = 1$ that algebraically fiber; in particular, this entails the Cartwright–Steger surface and the surfaces described in [Sto15, Theorem 2], [DiCS17, Theorem 1.2] have non–coherent fundamental groups. Those groups are torsion–free lattices $G \leq PU(2, 1)$, with the surfaces appearing as ball quotient $B^2_{\mathbb{C}}/G$, i.e. complex hyperbolic surfaces. This was implicitly known (for slightly different reasons) also from [Ka98]; we point out that for the argument above we don't need to invoke [Liu96].

(2) The first examples of Kodaira fibrations, due to Kodaira and Atiyah, actually carry two inequivalent structures of Kodaira fibrations, hence are guaranteed to have Albanese dimension two. By the above, their fundamental group is not coherent. The same result applies for the doubly fibered Kodaira fibrations constructed in [CR09]. It is not difficult to prove the existence of Kodaira fibrations of Albanese dimension one (which, by the Hochschild–Serre spectral sequence, are surface bundles whose coinvariant homology of the fiber $H_1(K; \mathbb{Z})_\Gamma$ has rank zero): in fact, the "generic" Kodaira fibration arising from a holomorphic curve in a moduli space of curves has Albanese dimension one. However, this seems to have no obvious consequences for the discussion above: we ignore if there exist Kodaira fibration of virtual Albanese dimension one. We can add that, even if such surfaces did exist, we cannot decide if their fundamental groups are coherent. For instance, it is not obvious whether they may contain $F_2 \times F_2$ as subgroup.

It is perhaps interesting to flesh out one consequence of the proof of Theorem 3.2 that gives some information on higher BNS–type invariants of some Kähler groups. Precisely, we will consider the homotopical BNSR invariant $\Sigma^2(G) \subseteq \Sigma^1(G) \subseteq S(G)$

introduced in [BR88]. We will not need the definition of these invariant and we will limit ourselves to mention the well–known facts (see e.g. [BGK10, Section 1.3]) that, using the notation preceding Theorem 2.3, given a primitive class $\phi \in H^1(G)$, the kernel Ker $\phi \leq G$ is of type F_2 (namely, finitely presented) if and only if the rational rays determined by both $\pm\phi \in H^1(G)$ are contained in $\Sigma^2(G)$.

While we know no way to get complete information on the full invariant $\Sigma^2(G)$, the ingredients of the proof of Theorem 3.2 is sufficient to entail the following lemma, that *per se* refines the previous result of non–coherence, and is possibly one of the first results on higher invariants of Kähler groups, besides the case of direct products:

Lemma 3.4. *The fundamental group G of an aspherical Kähler surface X of strictly positive Euler characteristic with Albanese dimension two has BNSR invariants satisfying the inclusions*

$$\Sigma^2(G) \subsetneq \Sigma^1(G) \subseteq S(G).$$

Proof. The point of this statement is that the first inclusion is strict. For sake of clarity, we review the argument we used in the proof of Theorem 3.2: the condition on the Albanese dimension implies that G algebraically fibers, for some primitive class $\phi \in H^1(G)$. As the Euler characteristic of X is strictly positive, Hillman's Theorem ([Hil02, Theorem 4.5(4)]) entails that Ker ϕ is not FP_2, nor *a fortiori* finitely presented. □

Note that the same conclusion of the lemma holds, even when $a(G) = 1$, as long as the fundamental group group algebraically fibers, e.g. for the Cartwright–Steger surface. Perhaps more importantly, the corollary applies to the aforementioned Kodaira fibrations defined by Kodaira and Atiyah. Topologically, these are surface bundles over a surface, so that their fundamental groups are nontrivial extension of a surface group by a surface group. Higher BNSR invariants of direct products of surfaces groups are (to an extent) well understood by purely group theoretical reasons. Instead, Lemma 3.4 seems to be the first result of that type for nontrivial extensions, and uses in crucial manner the fact that the group is Kähler: we are not aware of any other means to show that $\Sigma^1(G)$ is nonempty.

Remark. The reader familiar with BNSR invariants may notice that we are just shy of being able to conclude that the fundamental group of aspherical Kähler surfaces of positive Euler characteristic has empty $\Sigma^2(G)$. We conjecture that this is true. (The conjecture holds true whenever $\Sigma^2(G) = -\Sigma^2(G)$.) We mention also that the statement of Lemma 3.4 remains true if we consider the homological BNSR invariant $\Sigma^2(G;\mathbb{Z})$ of [BR88].

We want to discuss now a result that may prevent the existence of new simple examples of Kähler groups with $va(G) = 1$, under the assumption that the group satisfies residual properties akin to those holding for most irreducible 3–manifold groups, namely being virtually RFRS (in particular, that holds for the fundamental group of irreducible manifolds that have hyperbolic pieces in their geometric decomposition). This class of groups was first introduced by Agol in the study of virtual fibrations of 3–manifold groups: A group G is RFRS if there exists a filtration $\{G_i | i \geq 0\}$ of finite index normal subgroups $G_i \trianglelefteq_f G_0 = G$ with $\bigcap_i G_i = \{1\}$ whose successive quotient maps $\alpha_i \colon G_i \to G_i/G_{i+1}$ factorize through the maximal free abelian quotient:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & G_{i+1} & \longrightarrow & G_i & \xrightarrow{\qquad \alpha_i \qquad} & G_i/G_{i+1} & \longrightarrow & 1 \\ & & & & & \searrow \quad \nearrow & & & \\ & & & & & H_1(G_i)/\mathrm{Tor} & & & \end{array}$$

Subgroups of the direct product of surface groups and abelian groups are virtually RFRS. The largest source of virtually RFRS group we are aware of is given by subgroups of right–angled Artin groups (RAAGs), that are virtually RFRS by [Ag08]. However, this class does not give us new examples, as Py proved in [Py13, Theorem A] that all Kähler groups that are subgroups of RAAGs are in fact virtually subgroups of the product of surface groups and abelian groups.

We have the following, that combined with Theorem 2.3 gives Theorem C:

Theorem 3.5. *Let G be a virtually RFRS Kähler group. Then for any Kähler manifold X such that $\pi_1(X) = G$ either there exists a finite cover $\widetilde{X}$ of X with Albanese dimension greater than one, or G is virtually a surface group.*

Proof. After going to a suitable finite cover can assume that X has RFRS fundamental group G, with associated sequence $\{G_i\}$. A nontrivial RFRS group has positive first Betti number. In light of this, it is sufficient to show that if G is a RFRS group with Albanese map $f \colon G \to \Gamma$ and virtual Albanese dimension one, then it is a surface group. (This implies, because of the initial cover to get G RFRS, the theorem as stated.) Recall that we have a short exact sequence

$$1 \longrightarrow K \longrightarrow G \xrightarrow{f} \Gamma \longrightarrow 1$$

where Γ can be assumed to be a surface group and K is finitely generated. We claim that if X has virtual Albanese dimension one, then K is actually trivial, i.e. f is injective. Let $\gamma \in G$ be a nontrivial element; the assumption that $\bigcap_i G_i = \{1\}$ implies that there exist an index j such that $\gamma \in G_j \setminus G_{j+1}$. Consider now the diagram

$$\begin{array}{ccccccc} 1 \longrightarrow K \cap G_j & \longrightarrow & G_j & \xrightarrow{\qquad f_j \qquad} & \Gamma_j & \longrightarrow & 1 \\ & & {\scriptstyle \alpha_j}\big\downarrow & \searrow \qquad \swarrow & & & \\ & & & H_1(G_j)/\mathrm{Tor} & & & \\ & & & \swarrow & & & \\ & & G_j/G_{j+1} & & & & \\ & & \big\downarrow & & & & \\ & & 1 & & & & \end{array} \tag{6}$$

where $f_j \colon G_j \to \Gamma_j$ is the restriction epimorphism between finite index subgroups of G and Γ respectively determined by $G_j \trianglelefteq G$ as in the commutative diagram of

(1). This map represents, in homotopy, the pencil $f_j\colon X_j \to \Sigma_j$ of the cover of X associated to G_j. By assumption, this pencil is Albanese. This entails that there is an isomorphism

$$(f_j)_*\colon H_1(G_j)/\mathrm{Tor} \xrightarrow{\cong} H_1(\Gamma_j).$$

Composing the inverse of this isomorphism with the maximal free abelian quotient of Γ_j gives the map denoted with a dashed arrow in the diagram of Equation (6). By the commutativity of that diagram we deduce that the quotient map $\alpha_j\colon G_j \to G_j/G_{j+1}$ factors through $f_j\colon G_j \to \Gamma_j$. As $\gamma \in G_j \setminus G_{j+1}$, the image $\alpha_j(\gamma) \in G_j/G_{j+1}$ is nontrivial, hence so is $f_j(\gamma)$. This implies that $f(\gamma) = f_j(\gamma) \in \Gamma$ is nontrivial, i.e. $f\colon G \to \Gamma$ is injective. □

The conclusion of this theorem asserts that virtually RFRS Kähler groups of virtual Albanese dimension one are virtually (and not just commensurable to) surface groups. In fact, it is a simple exercise to verify that a residually finite group commensurable to a surface group is virtually a surface group, hence the result is exactly what we should expect.

We will finish this section with a result that further ties groups with virtual Albanese dimension one and surface groups, asserting that the Green–Lazarsfeld sets of such groups coincide (up to going to a finite index subgroup) with those of their Albanese image.

The Green–Lazarsfeld sets of a Kähler manifold X (and, by extension, of its fundamental group G) are subsets of the *character variety* of G, the complex algebraic group defined as $\widehat{G} := H^1(G;\mathbb{C}^*)$. The Green–Lazarsfeld sets $W_i(G)$ are defined as the collection of cohomology jumping loci of the character variety, namely

$$W_i(G) = \{\xi \in \widehat{G} \mid \operatorname{rk} H^1(G;\mathbb{C}_\xi) \geq i\},$$

nested by the *depth* i: $W_i(G) \subseteq W_{i-1}(G) \subseteq \ldots \subseteq W_0(G) = \widehat{G}$.

For Kähler groups the structure of $W_1(G)$ is well–understood. The projective case appeared in [Si93] (that refined previous results of [GL87, GL91]); this result was then extended to the Kähler case in [Cam01] (see also [De08]). Briefly, $W_1(G)$ is the union of a finite set of isolated torsion characters and the inverse image of the Green–Lazarsfeld set of hyperbolic orbisurfaces under the finite collection of pencils of X with hyperbolic base.

If X has Albanese dimension one, the Albanese map $f\colon G \to \Gamma$ induces an epimorphism $f_*\colon H_1(G) \to H_1(\Gamma)$. Therefore, we have an induced isomorphism of the connected components of the character varieties containing the trivial character

$$\widehat{f}\colon \widehat{\Gamma}_{\hat{1}} \xrightarrow{\cong} \widehat{G}_{\hat{1}} \tag{7}$$

where for a group G, we denote the connected component of the character variety containing the trivial character $\hat{1}\colon G \to \mathbb{C}^*$ as $\widehat{G}_{\hat{1}}$.

The next theorem, a restatement of Theorem D, shows that if X has virtual Albanese dimension one, then after perhaps going to a cover, the map $\widehat{f}$ restricts to an isomorphism of the Green–Lazarsfeld sets.

Theorem 3.6. *Let X be a Kähler manifold with $va(X) = 1$. Up to going to a finite index normal subgroup if necessary, the Albanese map $f\colon G \to \Gamma$ induces an isomorphism*

$$\widehat{f} : W_i(\Gamma) \xrightarrow{\cong} W_i(G)$$

of the Green–Lazarsfeld sets.

Proof. Up to going to a finite cover, we can assume that X admits an Albanese pencil $f\colon X \to \Sigma$. Moreover, as every cocompact Fuchsian group of positive genus admits a finite index normal subgroup which is a honest surface group, we can also assume that, after going to a further finite cover if necessary, the Albanese pencil doesn't contain any multiple fibers. In particular, $H_1(\Sigma)$ will be torsion–free. Without loss of generality, by going to the normal core of the associated finite index subgroup, we can always assume that the cover is regular. Summing up, after possibly going to a finite cover the Albanese map, in homotopy, is an epimorphism $f\colon G \to \Gamma$ where $\Gamma = \pi_1(\Sigma)$ is a genus $g(\Gamma)$ surface group.

The Green–Lazarsfeld sets $W_i(\Gamma)$ for a surface group are determined in [Hir97], and are given by

$$W_i(\Gamma) \;=\; \begin{cases} \widehat{\Gamma} & \text{if } 1 \leq i \leq 2g(\Gamma) - 2, \\ \hat{1} & \text{if } 2g(\Gamma) - 1 \leq i \leq 2g(\Gamma), \\ \emptyset & \text{if } i \geq 2g(\Gamma) + 1. \end{cases} \tag{8}$$

Given $\rho \in \widehat{\Gamma}$, surjectivity of $f\colon G \to \Gamma$ implies by general arguments (see e.g. [Hir97, Proposition 3.1.3]) that $f^*\colon H^1(\Gamma; \mathbb{C}_\rho) \to H^1(G; \mathbb{C}_{f^*(\rho)})$ is a monomorphism. But we will actually need more, namely that by [Br02, Theorem 1.1] or [Br03, Theorem 1.8] f^* is an isomorphism, except perhaps when $\rho \in \widehat{\Gamma}$ is a torsion character.

This implies that $\widehat{f}\colon W_i(\Gamma) \to W_i(G)$ is an injective map, and it will fail to preserve the depth (i.e. dimension of the twisted homology) only for torsion characters.

Consider the short exact sequence of groups

$$1 \longrightarrow \widehat{G}_{\hat{1}} \longrightarrow \widehat{G} \xrightarrow{t} \mathrm{Hom}(\mathrm{Tor} H_1(G); \mathbb{C}^*) \longrightarrow 1,$$

where $\widehat{G}_{\hat{1}}$ refers as above to the component of $\widehat{G}$ connected to the trivial character $\hat{1}$.

By [GL91, Theorem 0.1] all irreducible positive dimensional components of $W_i(G)$ are inverse images of the Green–Lazarsfeld set of the hyperbolic orbisurfaces. By Proposition 2.2, the Albanese pencil is unique, hence $\widehat{f}(W_i(\Gamma)) = \widehat{G}_{\hat{1}} \subseteq W_i(G)$ (for $1 \leq i \leq 2q(G) - 2$) is the only positive dimensional component, that will occur if (and only if) $q(G) = g(\Gamma) \geq 2$.

We now have the following claim.

Claim. For all $i \geq 1$, $W_i(G) \setminus \widehat{f}(W_i(\Gamma))$ is composed of torsion characters.

If $q(G) = 1$, this follows immediately from [Cam01, Théorème 1.3], as in this case $W_1(\Gamma)$ is torsion. If $q(G) \geq 2$, we need a bit more work: again by [Cam01, Théorème 1.3] and the above, we have that

$$W_1(G) = \widehat{G}_{\hat{1}} \bigcup Z$$

where Z is a finite collection of torsion characters, that we will assume to be disjoint from $\widehat{G}_{\hat{1}}$. By definition $W_i(G) \subseteq W_1(G)$. This implies, by the aforementioned result of Brudnyi, that:

– if $1 \leq i \leq 2q(G) - 2$ the isolated points of $W_i(G)$ are contained in Z;

– if $i \geq 2q(G) - 1$, they are either contained in Z, or are torsion characters in $\widehat{G}_{\hat{1}}$.

In either case, $W_i(G) \setminus \widehat{f}(W_i(\Gamma))$ is composed of torsion characters as claimed. This concludes the proof of the claim.

All this, so far, is a consequence of the fact that X has Albanese dimension one. Now we will make use of the assumption on the virtual Albanese dimension to show that $W_i(G) \setminus \widehat{f}(W_i(\Gamma))$ is actually empty.

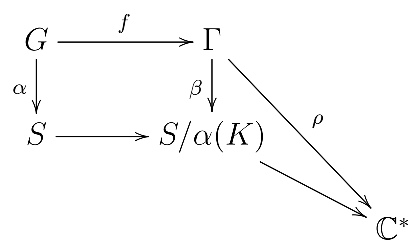
In order to prove this, recall the formula for the first Betti number for finite regular abelian covers of X, as determined in [Hir97]: Given an epimorphism $\alpha\colon G \to S$ to a finite abelian group, and following the notation from the diagram in (1), the finite cover H of X determined by α has first Betti number

$$b_1(H) = \sum_{i \geq 1} |W_i(G) \cap \widehat{\alpha}(\widehat{S})|. \tag{9}$$

This formula says that a character $\xi\colon G \to \mathbb{C}^*$ such that $\xi \in W_i(G)$ contributes with multiplicity equal to its depth to the Betti number of the cover defined by $\alpha\colon G \to S$ whenever it factorizes via α. Similarly, the corresponding cover Λ of Σ has first Betti number

$$b_1(\Lambda) = \sum_{i \geq 1} |W_i(\Gamma) \cap \widehat{\beta}(\widehat{S/\alpha(K)})|. \tag{10}$$

Consider a character $\rho \in W_i(\Gamma) \cap \widehat{\beta}(\widehat{S/\alpha(K)})$. We have a commutative diagram



whence $f^*(\rho)$ factorizes via α. As $\widehat{f}(W_i(\Gamma)) \subseteq W_i(G)$ this implies that $f^*(\rho) \in W_i(G) \cap \widehat{\alpha}(\widehat{S})$. We deduce that for any $\alpha : G \to S$ any contribution to the right hand side of Equation (10) is matched by an equal contribution to the right-hand-side of Equation (9).

Now assume that, for some $i \geq 1$, there is a *nontrivial* character $\xi\colon G \to \mathbb{C}^*$ such that $\xi \in W_i(G) \setminus \widehat{f}(W_i(\Gamma))$. (The case of the *trivial* character $\hat{1}\colon G \to \mathbb{C}^*$ is dealt with in the same way; we omit the details to avoid repetition.) As shown in the claim above, such character must be torsion. Because of that, its image is a finite abelian subgroup $S \leq \mathbb{C}^*$, i.e. ξ factors through an epimorphism $\alpha_\xi : G \to S$. This character either lies in Z or it is of the form $\xi = f^*(\rho)$ for $\rho \in W_{2g(\Gamma)-2}(\Gamma)$ (in which case $i > 2g(\Gamma) - 2$). These two cases are treated in slightly different ways:

– If the case $\xi \in Z$ holds, ξ will give a positive contribution to $b_1(H_\xi)$ that is not matched by any term in the right-hand-side of Equation (10).

– If the case $\xi = f^*(\rho)$ holds, it is immediate to verify that ρ factors through $\beta_\xi\colon \Gamma \to S/\alpha_\xi(K) \cong S$. However, the contribution of ρ to $b_1(\Lambda_\xi)$ is at least $i - 2g(\Gamma) + 2 > 0$ short of the contribution of $\xi = f^*(\rho)$ to $b_1(H_\xi)$, as

$$\dim\, H^1(G; \mathbb{C}_\xi) \geq i > \dim\, H^1(\Gamma; \mathbb{C}_\rho) = 2g(\Gamma) - 2.$$

In either case, the outcome is that $b_1(H_\xi) > b_1(\Lambda_\xi)$, which violates the assumption that X has virtual Albanese dimension one.

Summing up, $\widehat{f}\colon W_i(\Gamma) \to W_i(G)$ is a bijection. These sets coincide therefore with the connected components of the respective character variety (for $i \leq i \leq 2g(\Gamma) - 2$), the trivial character (for $2g(\Gamma) - 1 \leq i \leq 2g(\Gamma)$), and are empty otherwise. Whenever nonempty, both sets inherit a group structure as subsets of the respective character varieties. The map $\widehat{f}$ is the restriction of an homomorphism between these character varieties, hence an isomorphism as stated. □

Example. The bielliptic surfaces mentioned in Section 2 are a clean example of the fact that we need more than Albanese dimension one to get the isomorphism of Theorem 3.6. These surfaces admit genus one Albanese pencils without multiple fibers, and there exists an epimorphism $\alpha\colon G \to S$ with S abelian and $H = \text{Ker}\ \alpha \cong \mathbb{Z}^4$ (see [BHPV04, Section V.5]). This entails that, for some $i \geq 1$, $W_i(G)$ is strictly larger than $\widehat{f}(W_i(\mathbb{Z}^2))$ (the difference being torsion characters, contributing to the first Betti number of H).

By Theorem 3.6 the only characters of G that contribute to the rational homology of finite abelian covers of X are those that descend to Γ, i.e. restrict trivially to $\text{im}(H_1(K) \to H_1(G))$. In particular, the finite abelian cover of X determined by the image, in $H_1(G)$, of the homology of the fiber of the Albanese map has the same first Betti number as X. Note that such an image can be nonzero: infinitely many examples arise from Proposition 3.1 by taking K equal to any finite index subgroups of $Sp(2n, \mathbb{Z})$, $n \geq 2$.

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