CLARIFICATION TO 'NEW TOPOLOGICALLY SLICE KNOTS'

Abstract. In [FT05] we claimed that the three figures of [FT05, Figure 7.1] represent the Stevedore knot 6_1. In fact the middle knot is 9_{46}. In this note we clarify the situation and the ensuing examples.

Consider the knot $K(n)$ in Figure 1. The left most band is twisted by n twists.

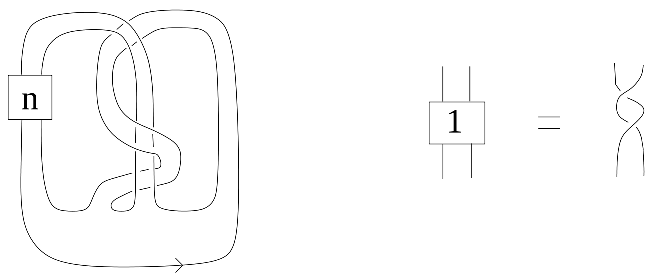

Figure 1. The knot $K(n)$.

We summarize the properties of the knots $K(n)$:

Lemma 1.
(1) *$K(n)$ is a ribbon knot with a ribbon disk D such that $\pi_1(D) \cong SR = \mathbb{Z} \ltimes \mathbb{Z}[1/2]$.*
(2) *A Seifert matrix of $K(n)$ is given by*
$$\begin{pmatrix} n & 2 \\ 1 & 0 \end{pmatrix}.$$
(3) *The knot $K(-2)$ is 6_1.*
(4) *The knot $K(0)$ is 9_{46}.*

Proof. Figure 2 shows that the knot $K(n)$ is formed by band connected sum of two trivial knots. In particular $K(n)$ is a ribbon knot. We refer to [GS99, p. 210–212] for the computation of the fundamental group of a ribbon disk complement. The argument in [GS99, p. 210–212] also shows immediately that the fundamental group is independent of n.

Now consider the Seifert surface for $K(n)$ given in Figure 3 with the curves a, b representing a basis for H_1. It is clear that with respect to this choice the Seifert matrix is given by
$$\begin{pmatrix} n & 2 \\ 1 & 0 \end{pmatrix}.$$

Date: December 13, 2007.
2000 *Mathematics Subject Classification.* Primary 57M25.

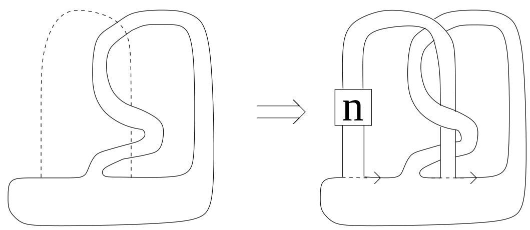

FIGURE 2. $K(n)$ as band connected sum.

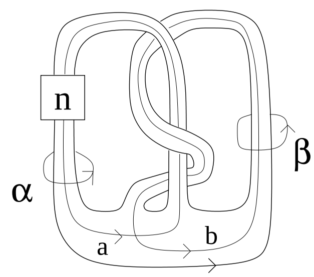

FIGURE 3. A Seifert surface for the knot $K(n)$.

Now consider the isotopies given in Figure 4. Clearly for $n = -2$ the resulting knot equals the Stevedore knot 6_1 given in Figure 5.

Finally we turn to $K(0)$. Note that $K(0)$ has a diagram with 12 crossings. A direct computation shows that the Alexander polynomial equals $2t^2 - 5t + 2$ and that the Jones polynomial equals $t^{-6} - t^{-5} + t^{-4} - 2t^{-3} + t^{-2} - t^{-1} + 2$. The knot tables show that the only knot with 12 crossings or less with these polynomials is 9_{46}. □

In [FT05, Section 7] we incorrectly thought that $K(0) = 6_1$. On pages 2153 and 2155 it should therefore say $K(0)$ instead of 6_1. The proof of [FT05, Proposition 7.7] is written for $K(0)$.

In fact, as we will show now, a version of [FT05, Proposition 7.7] holds for all knots $K(n)$, in particular for $K(-2) = 6_1$.

Indeed, consider the knot $K(n)$ together with curves a, b as in Figure 3. For given knots C_α, C_β consider the knot $S = S(K(n), \alpha, \beta, C_\alpha, C_\beta)$ which is the result of tieing the knots C_α and C_β into the bands α and β.

Proposition 2. *If one of the following holds:*

(1) $\Delta_{C_\alpha}(t) \neq 1$ *and* $\Delta_{C_\beta}(t) \neq 1$ *or*
(2) $\Delta_{C_\beta}(t) \neq 1$ *and* $n \neq 0$,

then S has no h–ribbon with fundamental group SR.

Proof. Let $S = S(K(n), \alpha, \beta, C_\alpha, C_\beta)$ be such a satellite knot for which (1) or (2) holds. Assume that S has in fact a h–ribbon D with fundamental group $G :=$

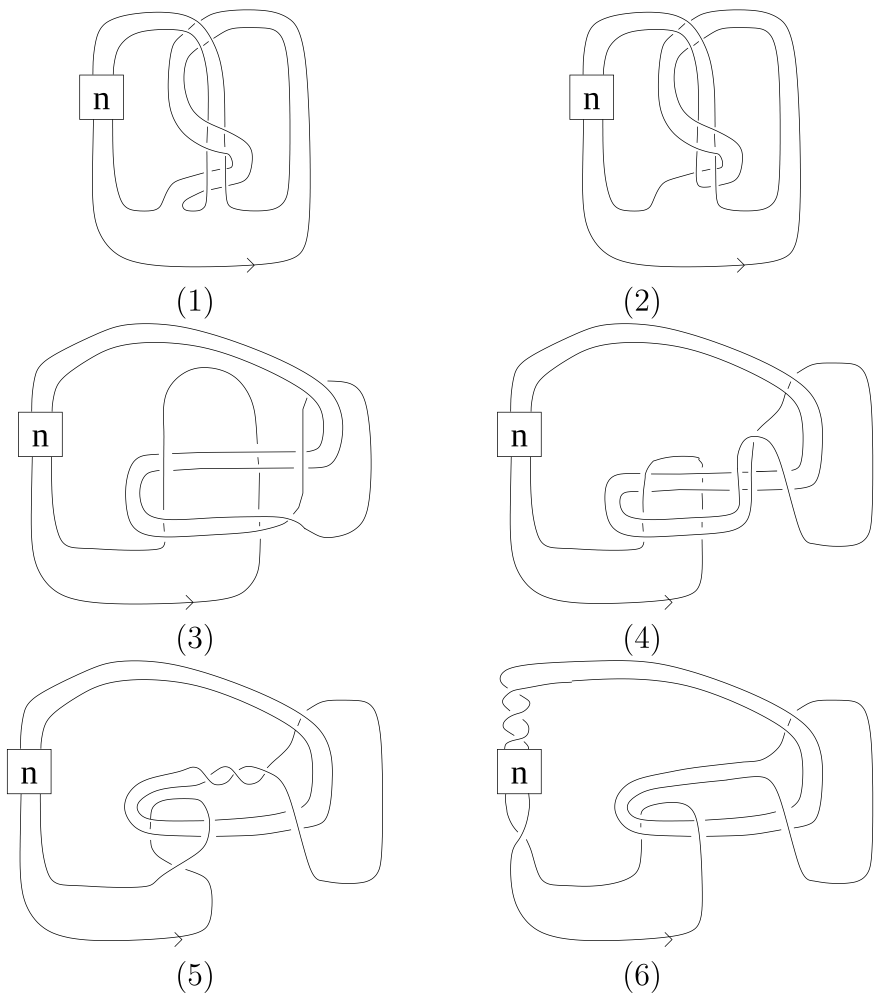

FIGURE 4. Isotopies of the knot $K(n)$.

$SR = \mathbb{Z} \ltimes \Lambda/(t-2)$. We denote the 0–framed surgery on S by M_S and we write $\Lambda := \mathbb{Z}[t, t^{-1}] \cong \mathbb{Z}[\mathbb{Z}]$. We also write $K = K(n)$. We write $N_D = M_S \setminus \nu D$. Then $\operatorname{Ker}\{H_1(M_S; \mathbb{Z}[\mathbb{Z}]) \to H_1(N_D; \mathbb{Z}[\mathbb{Z}])\}$ is a metabolizer for $B\ell(\mathbb{Z})$ (cf. e.g. [Fr04]).

Note that α, β in Figure 3 lift to elements $\tilde{\alpha}, \tilde{\beta}$ in $H_1(M_S; \Lambda)$, in fact

$$H_1(M_S; \Lambda) \cong (\Lambda\tilde{\alpha} \oplus \Lambda\tilde{\beta})/(At - A^t).$$

Furthermore the Blanchfield pairing $B\ell(\mathbb{Z})$ with respect to the generators $\tilde{\alpha}$ and $\tilde{\beta}$ is given by the matrix $(t-1)(At - A^t)^{-1}$.

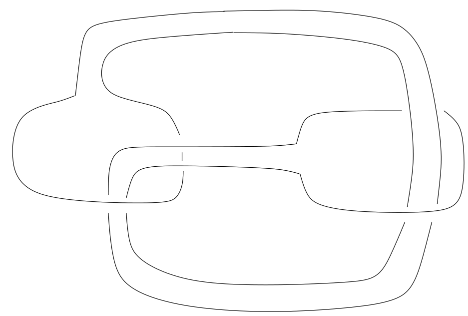

FIGURE 5. The Stevedore knot 6_1.

First assume that $n = 3k$ for some k. Then for $P = \begin{pmatrix} 1 & 0 \\ -k & 1 \end{pmatrix}$ we have

$$P^t A P = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} = A'.$$

Note that A' is also a Seifert matrix for K. We get a commutative diagram

$$\begin{array}{rccccc}
B\ell(\mathbb{Z}): & H_1(M_S;\Lambda) & \times & H_1(M_S;\Lambda) & \to & \mathbb{Q}(t)/\Lambda \\
 & \uparrow\cong & & \uparrow\cong & & \| \\
(t-1)(At-A^t)^{-1}: & \Lambda^2/(At-A^t) & \times & \Lambda^2/(At-A^t) & \to & \mathbb{Q}(t)/\Lambda \\
 & \downarrow\cong & & \downarrow\cong & & \| \\
(t-1)(A't-A'^t)^{-1}: & \Lambda^2/(A't-A'^t) & \times & \Lambda^2/(A't-A'^t) & \to & \mathbb{Q}(t)/\Lambda.
\end{array}$$

Here the top vertical map is given by $(1,0) \to \tilde{\alpha}, (0,1) \to \tilde{\beta}$ and the bottom vertical map is given by $w \mapsto P^t w$.

We see immediately that $B\ell(\mathbb{Z})$ has two metabolizers, which are generated by $\tilde{\alpha}' = \tilde{\alpha}$ and $\tilde{\beta}' = \tilde{\beta} + k\tilde{\alpha}$.

In particular the map $\pi := \pi_1(M_S) \to \pi_1(N_D)$ is up to automorphism of G either of the form

$$\varphi_{\tilde{\alpha}'} : \pi_1(M_S) \to \pi/\pi^{(2)} \cong \mathbb{Z} \ltimes H_1(M_S;\Lambda) \to \mathbb{Z} \ltimes \big(H_1(M_S;\Lambda)/\tilde{\alpha}'\Lambda\big) \xrightarrow{\cong} SR$$

or it is of the same form with $\tilde{\alpha}'$ replaced by $\tilde{\beta}'$. We denote this homomorphism by $\varphi_{\tilde{\beta}'}$. By Theorem [FT05, Theorem 1.3] we get $\mathrm{Ext}^1_{\mathbb{Z}[G]}(H_1(M_S;\mathbb{Z}[G]),\mathbb{Z}[G]) = 0$ with G–coefficients induced by either $\varphi_{\tilde{\alpha}'}$ or by $\varphi_{\tilde{\beta}'}$. Now consider coefficients induced by $\varphi_{\tilde{\alpha}'}$. Note that $\varphi_{\tilde{\alpha}'}(\alpha) = 0$ and $\varphi_{\tilde{\alpha}'}(\beta) \neq 0$. It therefore follows from [FT05, Lemma 6.2] that

$$H_1(M_S;\mathbb{Z}[G]) \cong H_1(M_K;\mathbb{Z}[G]) \oplus H_1(M_{C_\beta};\mathbb{Z}[\mathbb{Z}]) \otimes_{\mathbb{Z}[\mathbb{Z}]} \mathbb{Z}[G].$$

We compute

$$\begin{aligned} & \operatorname{Ext}^1_{\mathbb{Z}[G]}(H_1(M_S;\mathbb{Z}[G]),\mathbb{Z}[G]) \\ \cong\ & \operatorname{Ext}^1_{\mathbb{Z}[G]}\big(H_1(M_K;\mathbb{Z}[G]),\mathbb{Z}[G]) \oplus H_1(M_{C_\beta};\mathbb{Z}[\mathbb{Z}]) \otimes_{\mathbb{Z}[\mathbb{Z}]} \mathbb{Z}[G],\mathbb{Z}[G]\big) \\ \cong\ & \operatorname{Ext}^1_{\mathbb{Z}[G]}(H_1(M_K;\mathbb{Z}[G]),\mathbb{Z}[G]) \oplus \operatorname{Ext}^1_{\mathbb{Z}[G]}(H_1(M_{C_\beta};\mathbb{Z}[\mathbb{Z}]) \otimes_{\mathbb{Z}[\mathbb{Z}]} \mathbb{Z}[G],\mathbb{Z}[G]) \\ \cong\ & \operatorname{Ext}^1_{\mathbb{Z}[G]}(H_1(M_K;\mathbb{Z}[G]),\mathbb{Z}[G]) \oplus \operatorname{Ext}^1_{\mathbb{Z}[\mathbb{Z}]}(H_1(M_{C_\beta};\mathbb{Z}[\mathbb{Z}]),\mathbb{Z}[\mathbb{Z}]). \end{aligned}$$

Note that $H_1(M_{C_\beta};\mathbb{Z}[\mathbb{Z}]) \cong H_1(S^3 \setminus C_\beta;\mathbb{Z}[\mathbb{Z}])$, in particular it is $\mathbb{Z}$–torsion free. It follows from [Le77, Theorem 3.4] that $\operatorname{Ext}^1_{\mathbb{Z}[\mathbb{Z}]}(H_1(M_{C_\beta};\mathbb{Z}[\mathbb{Z}]),\mathbb{Z}[\mathbb{Z}]\big) \cong H_1(M_{C_\beta};\mathbb{Z}[\mathbb{Z}])$, which is not possible since by assumption $\Delta_{C_\beta}(t) \neq 1$. The only other possibility is therefore that $\operatorname{Ext}^1_{\mathbb{Z}[G]}(H_1(M_S;\mathbb{Z}[G]),\mathbb{Z}[G]) = 0$ with G–coefficients induced by $\varphi_{\tilde{\beta}'}$. If $n = 0$, we then have $\varphi_{\tilde{\alpha}'}(\alpha) \neq 0$ and $\varphi_{\tilde{\alpha}'}(\beta) = 0$ and

$$H_1(M_S;\mathbb{Z}[G]) \cong H_1(M_K;\mathbb{Z}[G]) \oplus H_1(M_{C_\alpha};\mathbb{Z}[\mathbb{Z}]) \otimes_{\mathbb{Z}[\mathbb{Z}]} \mathbb{Z}[G].$$

If $n \neq 0$, then we have $\varphi_{\tilde{\alpha}'}(\alpha) \neq 0$ and $\varphi_{\tilde{\alpha}'}(\beta) \neq 0$ and

$$H_1(M_S;\mathbb{Z}[G]) \cong H_1(M_K;\mathbb{Z}[G]) \oplus H_1(M_{C_\alpha};\mathbb{Z}[\mathbb{Z}]) \otimes_{\mathbb{Z}[\mathbb{Z}]} \mathbb{Z}[G] \oplus H_1(M_{C_\beta};\mathbb{Z}[\mathbb{Z}]) \otimes_{\mathbb{Z}[\mathbb{Z}]} \mathbb{Z}[G].$$

But in both cases the same calculation as above shows that we get a contradiction to either $n \neq 0$ or $\Delta_{C_\alpha}(t) \neq 0$.

Now assume that $n \not\equiv 0\,(3)$. We claim that $\Lambda^2/(At - A^t)$ is cyclic. Indeed, using simultaneous row and column operations the presentation matrix $At - A^t$ can be turned into

$$\begin{pmatrix} k(t-1) & 2t-1 \\ 1-2t & 0 \end{pmatrix}$$

where $k \in \{1,2\}$ and $k \equiv n\,(3)$. In the case $k = 1$ we can do the following row and column operations

$$\begin{pmatrix} t-1 & 2t-1 \\ t-2 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} -t & 2t-1 \\ t-2 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} -t & 2t-1 \\ 0 & (2t-1)(1-2t^{-1}) \end{pmatrix} \Rightarrow \begin{pmatrix} t & 0 \\ 0 & (2t-1)(1-2t^{-1}) \end{pmatrix}.$$

This shows that $\Lambda^2/(At - A^t)$ is cyclic. A similar sequence of row and column operations proves the claim for $k = 2$. This shows that the Blanchfield form has a unique metabolizer. It is clear that this metabolizer is generated by $\tilde{\alpha}$. We can now conclude the proof as in the case $n \equiv 0\,(3)$.

□

Acknowledgment. We would like to thank Tim Cochran for communicating this error to us.

References

[Fr04] S. Friedl, *Eta invariants as sliceness obstructions and their relation to Casson-Gordon invariants*, Algebraic and Geometric Topology, Vol. 4 (2004) 893–934.

[FT05] S. Friedl, P. Teichner, *New topologically slice knots*, Geometry and Topology, Volume 9 (2005) Paper no. 48, 2129–2158.

[GS99] R. Gompf, A Stipsicz, *4-manifolds and Kirby calculus*, Graduate Studies in Mathematics, 20, AMS (1999)

[Le77] J. Levine, *Knot modules. I*, Trans. Amer. Math. Soc. 229 (1977) 1–50.