The (non–) sliceness of satellite links

Stefan Friedl
Université du Québec à Montréal

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Slice knots

We consider knots $K \subset S^3$ throughout. By definition a knot is equivalent to the trivial knot if and only if K bounds an embedded disk in S^3.

A knot $K \subset S^3$ is called *smoothly (resp. topologically)* if K bounds a smooth (resp. locally flat) embedded disk in D^4.

Example. Consider the knot 6_1. It bounds an immersed disk which can be turned into a (smoothly) embedded disk in D^4 by resolving the singularities.

Sliceness obstructions

Question. So is every knot slice?

Theorem. Let $K \subset S^3$ be a knot, A a $2g \times 2g$–Seifert matrix for K. If K is slice then A is conjugate to a matrix of the form

$$\begin{pmatrix} 0 & P \\ Q & R \end{pmatrix}$$

where $0, P, Q, R$ are $g \times g$–matrices.

Corollary. Let $K \subset S^3$ slice. Let $z \in S^1$ (prime power root of unity), then the Levine–Tristram signature vanishes, i.e.

$$\sigma_z(K) := \mathrm{sign}(A(1-z) + A^t(1-\overline{z})) = 0.$$

Corollary. The trefoil has $\sigma(K) := \sigma_{-1}(K) = 2$, hence is not slice.

More examples of slice knots

We get more examples of slice knots by ‘tying a knot into the top band’. This means we start out with

and turn it into

Note that the singularities of the immersed disk did not change, so we can still resolve the singularities to get an embedded disk in D^4.

The satellite construction

The above construction of more examples of slice knots can be generalized: Let
(1) $K \subset S^3$ a knot or link,
(2) $\eta \in\subset S^3 \setminus K$ a curve which is the trivial knot in S^3, (the 'axis')
(3) $J \subset S^3$ another knot (the 'companion').
Now we form a new knot $S(K, J, \eta)$ out of K by
(1) cutting K along the disk bounding η,
(2) tying J into K,
(3) reglue.

Picture.

Satellite construction and sliceness

Question. Is the satellite knot of a slice knot always slice?

Example. No, take K the unknot, η a curve linking K once, J the trefoil. Then $S(K, J, \eta)$ is the trefoil and not slice.

Question. So when Is the satellite knot of a slice knot slice?

The plot thickens

Now consider the case where K is the unknot, η as follows:

Then $S(K, J, \eta)$ is the Whitehead double of J. Algebraically it looks like the unknot:

$$\Delta_{S(K,J,\eta)} = \Delta_K = 1.$$

Theorem.
(1) Any knot with $\Delta = 1$ is topologically slice (Freedman)
(2) There are Whitehead doubles which are not smoothly slice (Donaldson, Casson, Gompf, etc.).

We now concentrate on the question when are satellite knots topologically slice.

The Seifert matrix of satellite knots

Assume that $\eta \in S^3 \setminus K$ is homologically trivial, i.e. $[\eta] = H_1(S^3 \setminus K)$. In that case the Seifert matrix of $S(K, J, \eta)$ equals the Seifert matrix of K for any J.

In particular if K is slice, then the Levine–Tristram signatures of $S(K, J, \eta)$ vanish as well.

The Casson–Gordon invariants
Given K and $z \in S^1$ the Levine–Tristram signature signatures can be viewed as a twisted signature corresponding to a unitary representation

$$\pi_1(M_K) \to H_1(M_K) = \mathbb{Z} \to U(1)$$

factoring through the abelianization of $\pi_1(M_K)$ (M_K being the 0–framed surgery on K).

There are more sophisticated sliceness obstructions (‘Casson–Gordon invariants’) given by ‘twisted signatures’ corresponding to unitary representations

$$\pi_1(M_K) \to U(k)$$

factoring through metabelian quotients of $\pi_1(M_K)$.

The Casson–Gordon invariants of satellite knots

If K is slice, $[\eta] = 0 \in H_1(S^3 \setminus K)$ then the Casson–Gordon invariants of $S(K, J, \eta)$ are certain Levine–Tristram signatures of J.

Theorem. Assume
(1) K is slice with slice disk D,
(2) $\eta \subset S^3 \setminus K$ with $\eta = 0 \in \pi_1(D^4 \setminus D)$,
(3) J any knot,
then all Casson–Gordon invariants of $S(K, J, \eta)$ vanish.

Example. Take K the unknot, with slice disk D the obvious disk in S^3. Then

$$\pi_1(D^4 \setminus D) = \mathbb{Z} = H_1(S^3 \setminus K),$$

i.e. $\eta = 0 \in \pi_1(D^4 \setminus D)$ if its linking number with K vanishes. In particular the construction for the Whitehead double satisfies (1), (2) and (3). Which is good because we saw that the Whitehead double is slice.

The Main Theorem

Theorem (Cochran–F–Teichner). Assume
(1) K is slice with slice disk D,
(2) $\eta \subset S^3 \setminus K$ with $\eta = 0 \in \pi_1(D^4 \setminus D)$,
(3) J any knot,
then $S(K, J, \eta)$ is topologically slice.

Remark. This also works for K a slice link.

Example. Consider again the case of $K = 6_1$. The theorem applies to η as above, but also to any η homotopic (and not necessarily isotopic) to η in $S^3 \setminus K$.

The main ideas of the proof I

The following is a folklore theorem.

Theorem. $K \subset S^3$ is topologically slice if and only if there exists a 4–manifold W with
(1) $\partial W = M_K$,
(2) $H_1(M_K) \to H_1(W)$ is an isomorphism,
(3) $\pi_1(W)$ is normally generated by the meridian of K,
(4) $H_2(W) = 0$.

Proof. If K is slice, then $W = D^4 \setminus \nu D$.
Given W show that W with a 2–handle attached to the meridian is a homotopy 4–ball, hence topologically a 4–ball by Freedman.

Remark. One can always find W with properties (1), (2) and (3), the sticky point is killing $H_2(W)$.

The main ideas of the proof II

Now assume we have
(1) K (with slice disk D),
(2) η with $\eta = 0 \in \pi_1(D^4 \setminus D)$,
(3) J and we take any V bounding M_J with properties (1), (2), (3) and such that $\text{sign}(V) = 0$.

We then consider

$$W = (D^4 \setminus \nu D) \cup V$$

where we glue a tubular neighborhood of η in $M_K = \partial(D^4 \setminus \nu D)$ to a tubular neighborhood of the meridian of J in $M_J = \partial V$. Then $H_2(V) \to H_2(W)$ is an isomorphism. We have to kill $H_2(V)$ by doing surgery. We can *not* apply Freedman's sphere embedding theorem though since $\pi_1(W)$ is big and certainly not good.

Since $\eta = 0 \in \pi_1(D^4 \setminus D)$ we also have that $\pi_1(V) \to \pi_1(W)$ is trivial. We can now apply a theorem of Freedman and Teichner to kill $H_2(V) = H_2(W)$.