ON THE ALGEBRAIC UNKNOTTING NUMBER

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Abstract. The algebraic unknotting number $u_a(K)$ of a knot K was introduced by Hitoshi Murakami. It equals the minimal number of crossing changes needed to turn K into an Alexander polynomial one knot. In a previous paper the authors used the Blanchfield form of a knot K to define an invariant $n(K)$ and proved that $n(K) \leq u_a(K)$. They also showed that $n(K)$ subsumes all previous classical lower bounds on the (algebraic) unknotting number. In this paper we prove that $n(K) = u_a(K)$.

1. Introduction

Let K be a knot. The *unknotting number* $u(K)$ is defined to be the minimal number of crossing changes needed to turn K into the trivial knot. The unknotting number is one of the most basic but also most intractable invariants of a knot. Hitoshi Murakami [Muk90] introduced a more accessible invariant, namely the *algebraic unknotting number* $u_a(K)$ which is defined to be the minimal number of crossing changes needed to turn K into a knot with Alexander polynomial equal to one. (The definition we gave above was shown by Fogel [Fo93, Theorem 1.4], see also [Sa99], to be equivalent to Murakami's original definition which was given in terms of certain operations on Seifert matrices.)

It is obvious that the algebraic unknotting number is a lower bound on the unknotting number $u(K)$ of a knot. It is furthermore well–known that the 'classical' lower bounds on the unknotting number, i.e. the lower bounds which can be described in terms of the Seifert matrix of a knot, like the Nakanishi index [Na81], the Levine–Tristram signatures [Mus65, Le69, Tr69, Ta79, BF14], the Lickorish obstruction [Li85, CL86], the Murakami obstruction [Muk90] and the Jabuka obstruction [Ja09] give in fact lower bounds on the algebraic unknotting number.

In [BF11] the authors introduced a new invariant $n(K)$ of a knot K as follows. We write $X(K) = S^3 \setminus \nu K$ and we consider the Blanchfield form

$$Bl(K)\colon H_1(X(K);\mathbb{Z}[t^{\pm 1}]) \times H_1(X(K);\mathbb{Z}[t^{\pm 1}]) \longrightarrow \mathbb{Q}(t)/\mathbb{Z}[t^{\pm 1}].$$

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(We refer to Section 2.4 for the definition.) Furthermore, given a hermitian $n \times n$-matrix A over $\mathbb{Z}[t^{\pm 1}]$ with $\det(A) \neq 0$ we denote by $\lambda(A)$ the form

$$\begin{array}{rcl} \mathbb{Z}[t^{\pm1}]^n/A\mathbb{Z}[t^{\pm1}]^n \times \mathbb{Z}[t^{\pm1}]^n/A\mathbb{Z}[t^{\pm1}]^n & \to & \mathbb{Q}(t)/\mathbb{Z}[t^{\pm1}] \\ (a,b) & \mapsto & \overline{a}^t A^{-1} b, \end{array}$$

where we view a, b as represented by column vectors in $\mathbb{Z}[t^{\pm1}]^n$. In [BF11] we defined

$$n(K) := \min \left\{ n \;\middle|\; \begin{array}{l} \text{there exists a hermitian } n \times n\text{–matrix } A(t) \text{ over } \mathbb{Z}[t^{\pm1}] \\ \text{such that } \lambda(A(t))) \cong Bl(K) \\ \text{and such that } A(1) \text{ is diagonalizable over } \mathbb{Z} \end{array} \right\}.$$

In [BF11] we proved that such a matrix A exists, i.e. $n(K)$ is defined, and in fact we showed that $n(K) \leq \deg \Delta_K(t) + 1$. We also proved that $n(K)$ is a lower bound on the algebraic unknotting number, i.e. $n(K) \leq u_a(K)$. We furthermore showed that $n(K)$ subsumes all the previous classical lower bounds on the unknotting number mentioned above. In this paper we will now prove that $n(K)$ agrees with the algebraic unknotting number, that is we will show the following theorem:

Theorem 1.1. *Let $K \subset S^3$ be a knot, then*

$$n(K) = u_a(K).$$

In fact in Section 5 we will state and prove a slightly stronger statement which takes into account positive and negative crossing changes.

We have now a following characterization of the algebraic unknotting number.

Proposition 1.2. *Let $K \subset S^3$ be a knot. Then the following numbers are equal.*

1. *The algebraic unknotting number, that is the minimal number of crossing needed to turn K into an Alexander polynomial one knot.*
2. *The minimal number of algebraic unknotting moves, see* [Muk90, Sa99]*, needed to change the Seifert matrix of K into the trivial matrix.*
3. *The minimal second Betti number of a topological 4–manifold that strictly cobounds $M(K)$, the zero framed surgery along K, see* [BF11, Definition 2.5].
4. *The invariant $n(K)$.*

Proof. Saeki [Sa99, Theorem 1.1] showed that (1) = (2). In [BF11] it was shown that (4) $\leq$ (3) $\leq$ (1). By Theorem 1.1 we have actually (4) = (1). □

Note that (2) and (4) are purely algebraic quantities. It would be interesting to find a direct algebraic proof that (2) = (4).

The paper is organized as follows. In Section 2 we recall the definition of the Alexander module and of the Blanchfield form using Poincaré duality. In Section 3 we then give a more geometric interpretation of the Blanchfield form.

Convention 1.3. All manifolds are assumed to be oriented and compact, unless it says specifically otherwise.

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2. The Blanchfield form

2.1. **Homologies of complexes over $\mathbb{Z}[t^{\pm 1}]$.** Let C_* be any chain complex of finitely generated free $\mathbb{Z}[t^{\pm 1}]$–modules and let M be any $\mathbb{Z}[t^{\pm 1}]$-module. We can then consider the corresponding homology and cohomology modules:

$$\begin{array}{rcl} H_*(C;M) & := & H_*(C_* \otimes_{\mathbb{Z}[t^{\pm 1}]} M), \text{ and} \\ H^*(C;M) & := & H_*(\operatorname{Hom}_{\mathbb{Z}[t^{\pm 1}]}(C_*, M)). \end{array} \tag{2.1}$$

By [Lev77, Theorem 2.3] there is a spectral sequence $E^r_{p,q}$ with

$$E^2_{p,q} = \operatorname{Ext}^p_{\mathbb{Z}[t^{\pm 1}]}(H_q(C), M)$$

and which converges to $H^*(C, M)$. This spectral sequence is called the Universal Coefficient Spectral Sequence, or UCSS for short. We note that for any two $\mathbb{Z}[t^{\pm 1}]$-modules H and M the module $\operatorname{Ext}^0_{\mathbb{Z}[t^{\pm 1}]}(H, M)$ is canonically isomorphic to $\operatorname{Hom}_{\mathbb{Z}[t^{\pm 1}]}(H, M)$.

Also note that

$$\operatorname{Ext}^p_{\mathbb{Z}[t^{\pm 1}]}(H, M) = 0$$

for any $p > 2$ since $\mathbb{Z}[t^{\pm 1}]$ has cohomological dimension 2. Finally note, that if $\mathbb{Z}$ is considered as a $\mathbb{Z}[t^{\pm 1}]$ module with trivial t–action, then $\mathbb{Z}$ admits a resolution of length 1, in particular

$$\operatorname{Ext}^p_{\mathbb{Z}[t^{\pm 1}]}(\mathbb{Z}, M) = 0$$

for any $p > 1$.

For later use we also record the following lemma.

Lemma 2.1. *Let H be a finitely generated $\mathbb{Z}[t^{\pm 1}]$–module, then $\operatorname{Ext}^0_{\mathbb{Z}[t^{\pm 1}]}(H, \mathbb{Z}[t^{\pm 1}])$ is a free $\mathbb{Z}[t^{\pm 1}]$–module.*

The lemma is well-known but we are not aware of a good reference. We thus provide a short proof, whose key idea was supplied to us by Jonathan Hillman.

Proof. Let H be a finitely generated $\mathbb{Z}[t^{\pm 1}]$–module. Since $\mathbb{Z}[t^{\pm 1}]$ is Noetherian there exists an exact sequence of the form $\mathbb{Z}[t^{\pm 1}]^r \xrightarrow{\varphi} \mathbb{Z}[t^{\pm 1}]^s \to H$. Since the Hom-functor

$M \mapsto \operatorname{Hom}_{\mathbb{Z}[t^{\pm 1}]}(M, \mathbb{Z}[t^{\pm 1}])$ is left-exact the above exact sequence gives rise to an exact sequence

$$\begin{array}{rlrl} 0 & \to \operatorname{Hom}(H, \mathbb{Z}[t^{\pm 1}]) & \to & \operatorname{Hom}(\mathbb{Z}[t^{\pm 1}]^s, \mathbb{Z}[t^{\pm 1}]) \xrightarrow{\varphi^*} \operatorname{Hom}(\mathbb{Z}[t^{\pm 1}]^r, \mathbb{Z}[t^{\pm 1}]) \\ & \to \operatorname{coker}(\varphi^*) & \to & 0. \end{array}$$

Note that $\mathbb{Z}[t^{\pm 1}]$ is a ring of homological dimension 2. (This is for example a straightforward consequence of the fact that the ring $\mathbb{Z}[t]$ has homological dimension 2 which is proved in [La06, Theorem 5.36].) We can therefore find a projective resolution

$$0 \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow \operatorname{coker}(\varphi^*) \longrightarrow 0$$

for $\operatorname{coker}(\varphi^*)$ of length two. Comparing these two resolutions for $\operatorname{coker}(\varphi^*)$ and noting that $\operatorname{Hom}(\mathbb{Z}[t^{\pm 1}]^s, \mathbb{Z}[t^{\pm 1}])$ and $\operatorname{Hom}(\mathbb{Z}[t^{\pm 1}]^r, \mathbb{Z}[t^{\pm 1}])$ are free $\mathbb{Z}[t^{\pm 1}]$-modules implies by Schanuel's lemma (see [La99, Corollary 5.5]) that $\operatorname{Hom}(H, \mathbb{Z}[t^{\pm 1}])$ is projective. Finally, it is a special case of the Serre Conjecture, see e.g. [La06, Corollary 4.12], that a finitely generated projective $\mathbb{Z}[t^{\pm 1}]$-module is in fact free. This concludes the proof that $\operatorname{Hom}(H, \mathbb{Z}[t^{\pm 1}])$ is a free $\mathbb{Z}[t^{\pm 1}]$-module. □

2.2. **Twisted homology, cohomology groups and Poincaré duality.** Let X be a topological space and let $\phi\colon \pi_1(X) \to \langle t \rangle$ be an epimorphism onto the infinite cyclic group generated by t. We denote by $\pi\colon \widetilde{X} \to X$ the corresponding infinite cyclic covering of X. Given a subspace $Y \subset X$ we write $\widetilde{Y} := \pi^{-1}(Y)$.

The deck transformation induces a canonical $\mathbb{Z}[t^{\pm 1}]$–action on $C_*(\widetilde{X}, \widetilde{Y}; \mathbb{Z})$ and we can thus view $C_*(\widetilde{X}, \widetilde{Y}; \mathbb{Z})$ as a chain complex of free $\mathbb{Z}[t^{\pm 1}]$–modules. Now let M be a module over $\mathbb{Z}[t^{\pm 1}]$. We then consider homologies $H_*(X, Y; M)$ and $H^*(X, Y; M)$ as defined in (2.1). The most important instance will be $M = \mathbb{Z}[t^{\pm 1}]$.

If $K \subset S^3$ is an oriented knot, then we denote by $\phi\colon \pi_1(X(K)) \to \langle t \rangle$ the epimorphism given by sending the oriented meridian to t. Furthermore, if X is a space with $H_1(X; \mathbb{Z}) \cong \mathbb{Z}$, then we pick either epimorphism from $\pi_1(X)$ onto $\langle t \rangle$. For different choices of epimorphisms the resulting modules $H_*(X, Y; \mathbb{Z}[t^{\pm 1}])$ and $H^*(X, Y; \mathbb{Z}[t^{\pm 1}])$ will be anti–isomorphic, i.e. multiplication by t in one module corresponds to multiplication by t^{-1} in the other module. Since this does not affect any of the arguments we will usually not record the choice of ϕ in our notation.

Finally suppose that X is an orientable n-manifold and that W is a union of components of ∂X. Then for any $\mathbb{Z}[t^{\pm 1}]$-module M, Poincaré duality (see e.g. [Wa99, Chapter 2]) defines isomorphisms of $\mathbb{Z}[t^{\pm 1}]$-modules

$$H_i(X, W; M) \cong \overline{H^{n-i}(X, \partial X \setminus W; M)},$$

in particular if $W = \emptyset$, then we get a canonical isomorphism

$$H_i(X; M) \cong \overline{H^{n-i}(X, \partial X; M)}.$$

Here, given a $\mathbb{Z}[t^{\pm 1}]$–module N we denote by $\overline{N}$ the same abelian group as N but with the involuted $\mathbb{Z}[t^{\pm 1}]$–action, i.e. multiplication by t on $\overline{N}$ corresponds to multiplication by t^{-1} on N.

2.3. **Orders of $\mathbb{Z}[t^{\pm 1}]$–modules.** Let H be a finitely generated $\mathbb{Z}[t^{\pm 1}]$–module. Since $\mathbb{Z}[t^{\pm 1}]$ is Noetherian it follows that H is also finitely presented, i.e. we can find a resolution

$$\mathbb{Z}[t^{\pm 1}]^m \xrightarrow{A} \mathbb{Z}[t^{\pm 1}]^n \longrightarrow H,$$

where we can assume that $m \geq n$. We then define $\operatorname{order}(H) \in \mathbb{Z}[t^{\pm 1}]$ to be the greatest common divisor of the $n \times n$–minors of A. It is well–known that, up to multiplication by a unit in $\mathbb{Z}[t^{\pm 1}]$, i.e. up to multiplication by an element of the form $\pm t^k, k \in \mathbb{Z}$, the invariant $\operatorname{order}(H)$ is independent of the choice of A. We refer to [Hi02] for details. In the following, given $f, g \in \mathbb{Z}[t^{\pm 1}]$ we write $f \doteq g$ if f and g agree up to multiplication by a unit in $\mathbb{Z}[t^{\pm 1}]$.

Example 2.2. If H admits a square presentation matrix A over $\mathbb{Z}[t^{\pm 1}]$ of size n, then it follows immediately from the definition that the order of H equals $\det(A)$.

Example 2.3. The Alexander polynomial of a knot K is defined to be the order of the Alexander module $H_1(X(K); \mathbb{Z}[t^{\pm 1}])$. Throughout this paper we will normalize the Alexander polynomial such that $\Delta_K(1) = 1$ and $\Delta_K(t^{-1}) = \Delta_K(t)$.

The following result is standard (see e.g. [Hi02, Section 3]), we will use it often in the future.

Lemma 2.4. *The order of any $\mathbb{Z}[t^{\pm 1}]$–module is also an annihilator, i.e.* $\operatorname{order}(H) \cdot v = 0$ *for any $v \in H$. In particular if K is knot, then for any $c \in H_1(X(K); \mathbb{Z}[t^{\pm 1}])$, we have $\Delta_K(t) \cdot c = 0$.*

Remark 2.5. Given $p = p(t) \in \mathbb{Z}[t^{\pm 1}]$ we define $\overline{p} := p(t^{-1})$. Note that for any $\mathbb{Z}[t^{\pm 1}]$–module we have

$$\operatorname{order}(\overline{M}) \doteq \overline{\operatorname{order}(M)}.$$

We will later make use of the following lemma (see again [Hi02] for details).

Lemma 2.6. *Let*

$$0 \longrightarrow H \longrightarrow H' \longrightarrow H'' \longrightarrow 0$$

be a short exact sequence of $\mathbb{Z}[t^{\pm 1}]$–modules, then

$$\operatorname{order}(H') \doteq \operatorname{order}(H) \cdot \operatorname{order}(H'').$$

2.4. **The homological definition of the Blanchfield form.** Let $K \subset S^3$ be a knot. We consider the following sequence of maps:

$$\begin{array}{rcl}\Phi\colon H_1(X(K);\mathbb{Z}[t^{\pm 1}]) & \to & H_1(X(K),\partial X(K);\mathbb{Z}[t^{\pm 1}]) \\ & \to & \overline{H^2(X(K);\mathbb{Z}[t^{\pm 1}])} \xleftarrow{\cong} \overline{H^1(X(K);\mathbb{Q}(t)/\mathbb{Z}[t^{\pm 1}])} \\ & \to & \overline{\operatorname{Hom}_{\mathbb{Z}[t^{\pm 1}]}(H_1(X(K);\mathbb{Z}[t^{\pm 1}]),\mathbb{Q}(t)/\mathbb{Z}[t^{\pm 1}])}.\end{array}$$

Here the first map is the inclusion induced map, the second map is Poincaré duality, the third map comes from the long exact sequence in cohomology corresponding to the coefficients $0 \to \mathbb{Z}[t^{\pm 1}] \to \mathbb{Q}(t) \to \mathbb{Q}(t)/\mathbb{Z}[t^{\pm 1}] \to 0$, and the last map is the evaluation map. All these maps are isomorphisms, and hence define a non-singular form

$$\begin{array}{rcl}Bl(K)\colon H_1(X(K);\mathbb{Z}[t^{\pm 1}]) \times H_1(X(K);\mathbb{Z}[t^{\pm 1}]) & \to & \mathbb{Q}(t)/\mathbb{Z}[t^{\pm 1}] \\ (a,b) & \mapsto & \Phi(a)(b),\end{array}$$

called the *Blanchfield form of* K. This form is well-known to be hermitian, in particular $Bl(K)(a_1,a_2) = \overline{Bl(K)(a_2,a_1)}$ and $Bl(K)(\mu_1 a_1,\mu_2 a_2) = \overline{\mu_1} Bl(K)(a_1,a_2)\mu_2$ for $\mu_i \in \mathbb{Z}[t^{\pm 1}], a_i \in H_1(X(K);\mathbb{Z}[t^{\pm 1}])$. The Blanchfield form was initially introduced by Blanchfield [Bl57]. We will give a more geometric definition in the next section.

Remark 2.7. By Lemma 2.4 the polynomial $\Delta_K(t)$ annihilates $H_1(X(K);\mathbb{Z}[t^{\pm 1}])$, it follows easily from the definitions that $Bl(K)$ takes in fact values in $\Delta_K(t)^{-1}\mathbb{Z}[t^{\pm 1}]/\mathbb{Z}[t^{\pm 1}] \subset \mathbb{Q}(t)/\mathbb{Z}[t^{\pm 1}]$.

3. The twisted linking form

3.1. **Pairings on infinite cyclic covers.** Let $K \subset S^3$ be an oriented knot. We write $X = X(K)$, which we endow with the orientation coming from S^3, and we denote by Δ the Alexander polynomial of K. Recall that $\phi\colon \pi_1(X) \to \langle t\rangle$ is the unique epimorphism which sends the oriented meridian of K to t. Then $\langle t\rangle$ acts on $\widetilde{X}$, the corresponding infinite cyclic cover of X; we can thus view $H_i(\widetilde{X})$ as a $\mathbb{Z}[t^{\pm 1}]$–module. This module is by definition precisely the Alexander module $H_i(X;\mathbb{Z}[t^{\pm 1}])$ as defined above.

We say that a simple closed curve $c \subset \widetilde{X}$ is in *general position* if $t^i c$ and c are disjoint for any $i \in \mathbb{Z}$. Furthermore we say that a pair of simple closed oriented curves c,d is in general position in $\widetilde{X}$, if $t^i c$ and d are disjoint for any $i \in \mathbb{Z}$. Finally, if c is a simple closed curve and F an embedded surface in $\widetilde{X}$, then we say that they are in general position if for any $i \in \mathbb{Z}$ the curve $t^i c$ intersects F transversely.

If c is a simple closed oriented curve in $\widetilde{X}$ and $n \in \mathbb{N}$, then we denote by nc the union of n parallel copies of c. We can and will assume that these parallel copies are in general position to each other. If $-n \in \mathbb{N}$, then we denote by nc the union of $-n$ parallel copies of $-c$, i.e. of c with opposite orientation. Finally if $p(t) = \sum_{i=k}^{l} a_i t^i \in \mathbb{Z}[t^{\pm 1}]$, then we denote by $p(t)c$ the union of $a_k t^k c \cup \dots \cup a_l t^l c$.

The following definition is now a variation on the equivariant intersection number in a covering space (see e.g. [COT03, p. 495]).

Definition 3.1. Let $c, d \subset \widetilde{X}$ be simple closed oriented curves in general position. By Lemma 2.4 there exists an embedded oriented surface $F \subset \widetilde{X}$ such that $\partial F = \Delta \cdot c$. We can arrange that F and d are in general position. The *twisted linking number* of c and d is defined as

$$\widetilde{\mathrm{lk}}(c,d) := \frac{1}{\Delta} \sum_{i \in \mathbb{Z}} (F \cdot t^i d) \cdot t^{-i} \in \frac{1}{\Delta} \mathbb{Z}[t^{\pm 1}]. \tag{3.1}$$

Here $F \cdot t^i d$ denotes the ordinary intersection number of the oriented submanifolds F and $t^i d$ in $\widetilde{X}$.

Lemma 3.2. *The twisted linking form* $\widetilde{\mathrm{lk}}(c,d)$ *is independent of the choice of* F.

Proof. By Poincaré duality we have

$$H_2(X;\mathbb{Z}[t^{\pm 1}]) \cong \overline{H^1(X,\partial X;\mathbb{Z}[t^{\pm 1}])},$$

but $H_1(X,\partial X;\mathbb{Z}[t^{\pm 1}])$ is $\mathbb{Z}[t^{\pm 1}]$–torsion and $H_0(X,\partial X;\mathbb{Z}[t^{\pm 1}]) = 0$. It now follows from the UCSS that $H_2(\widetilde{X};\mathbb{Z}) = H_2(X;\mathbb{Z}[t^{\pm 1}]) = 0$. Now let F' be any other surface cobounding $\Delta \cdot c$, then $F \cup -F'$ forms a closed oriented surface in $\widetilde{X}$, in particular it represents an element in $H_2(X;\mathbb{Z}[t^{\pm 1}])$. But since $H_2(X;\mathbb{Z}[t^{\pm 1}]) = 0$ it now follows that $(F \cup -F') \cdot d = 0$. This concludes the proof of the lemma. □

Lemma 3.3.

$$\widetilde{\mathrm{lk}}(d,c) = \overline{\widetilde{\mathrm{lk}}(c,d)}.$$

Proof. Let $F, G \subset \widetilde{X}$ be embedded oriented surfaces such that $\partial F = \Delta \cdot c$ and $\partial G = \Delta \cdot d$. We can assume that $t^i F$ intersects G transversely for any i. For any i the 1–manifold $t^i F \cap G$ defines a cobordism between $t^i F \cap d$ and $G \cap t^i c$. It thus follows that

$$\begin{aligned} \Delta \cdot \widetilde{\mathrm{lk}}(d,c) &= \textstyle\sum_{i\in\mathbb{Z}} (G \cdot t^i c) t^{-i} = \sum_{i\in\mathbb{Z}} (t^i F \cdot d) t^{-i} = \\ &= \textstyle\sum_{i\in\mathbb{Z}} (F \cdot t^{-i} d) t^{-i} = \sum_{i\in\mathbb{Z}} (F \cdot t^i d) t^{i} = \overline{\sum_{i\in\mathbb{Z}} (F \cdot t^i d) t^{-i}} = \\ &= \overline{\Delta \cdot \widetilde{\mathrm{lk}}(c,d)} = \overline{\Delta} \cdot \overline{\widetilde{\mathrm{lk}}(c,d)} = \Delta \cdot \overline{\widetilde{\mathrm{lk}}(c,d)}. \end{aligned}$$

□

In general, if c and c' are homologous curves in $\widetilde{X}$, the linking form $\widetilde{\mathrm{lk}}(c,d)$ and $\widetilde{\mathrm{lk}}(c',d)$ will be different (unless c and c' are homologous in $\widetilde{X} \setminus d$). Nevertheless, $\widetilde{\mathrm{lk}}(c,d) \bmod \mathbb{Z}[t^{\pm 1}]$ is homology invariant. Therefore, $\widetilde{\mathrm{lk}}(c,d)$ descends to a form

$$H_1(X;\mathbb{Z}[t^{\pm 1}]) \times H_1(X;\mathbb{Z}[t^{\pm 1}]) \longrightarrow \frac{1}{\Delta}\mathbb{Z}[t^{\pm 1}]/\mathbb{Z}[t^{\pm 1}],$$

which by definition is precisely the Blanchfield form $Bl(K)$. We refer to [Bl57] for details.

3.2. **Based curves and surfaces.** In this section we will take a point of view which differs from the discussion in the previous section: instead of studying objects in the infinite cyclic cover of $X(K)$ we will now consider based objects in $X(K)$.

Let $K \subset S^3$ be an oriented knot. As above we write $X = X(K)$ and we denote the infinite cyclic cover of X by $\widetilde{X}$. In this section we will define an invariant lk_t which will turn out to capture the same information as $\widetilde{\operatorname{lk}}$ in the previous section, but instead of considering curves in $\widetilde{X}$ we will now work with based curves in X.

We fix once and for all a base point $*$ in X. We now need several definitions:

(1) By a *surface* in X we always mean an immersed surface. By a *smooth curve* on the immersed surface we mean the image of a smooth curve on the original surface under the immersion.
(2) A *based* curve (respectively surface) in X is an oriented curve (respectively oriented surface) in X together with a path, called *basing* connecting it to the base point $*$. We assume that the basing intersects the curve (respectively the surface) in only one point.
(3) By an orientation of a based curve (respectively surface) we mean an orientation of the unbased curve (respectively surface).
(4) A curve c in X is called *homologically trivial* if c is trivial in $H_1(X;\mathbb{Z})$.
(5) A surface F in X is *homologically invisible* if any smooth curve on F is null–homologous in X. Note that a curve (respectively surface) is homologically trivial (respectively invisible) if and only if it lifts to $\widetilde{X}$.
(6) We say that two based homologically trivial curves are *equivalent* if the unbased curves agree and if the basings are homologous relative to the base point and relative to a path connecting the end points on the curve. (This condition does not depend on the path since the curve is assumed to be homologically trivial.) Similarly we define equivalence of based homologically invisible surfaces.
(7) We say that two based objects are disjoint if the corresponding unbased objects are disjoint.
(8) We say that a based curve c and a based surface F in X are in *general position* if the unbased curve and the unbased surface are in general position and if furthermore the basings are embedded and disjoint from c and from F.

Let c be a homologically trivial based curve in X and let F be a homologically invisible based surface in X such that F and c are in general position. Any intersection point P of the (unbased) curve and the (unbased) surface comes with a sign $\epsilon_P \in \{-1, 1\}$. To any intersection point P we can also associate a loop l_P in X in the following way. We go from the base point $*$ via a smooth curve on the based surface F to the intersection P, and then we go back to $*$ along the curve c. Since F is homologically invisible and c is homologically trivial, it follows that $\phi(l_P)$ is independent of the choices. Following [COT03, p. 499] we now define

$$F \cdot c := \sum_{P \in c \cap F} \epsilon_P \, \phi(l_P) \in \mathbb{Z}[t^{\pm 1}].$$

Note that $F \cdot c$ only depends on the equivalence classes of F and c. We will thus in the following mostly consider based curves and surfaces up to equivalence.

Given a based curve c and $k \in \mathbb{Z}$ we now denote by $t^k c$ the based curve which is given by precomposing the basing with a closed loop l which satisfies $\phi(l) = t^k$. Note that the equivalence class of $t^k c$ is well–defined. Furthermore, given $n \in \mathbb{Z}$ we denote by nc the union of $|n|$ parallel copies of c, with opposite orientation if $n < 0$. For any Laurent polynomial $p(t) \in \mathbb{Z}[t^{\pm 1}]$ we define $p(t)c$ in the obvious way. Obviously

$$F \cdot p(t)c = p(t)(F \cdot c).$$

Let F be a based homologically invisible surface. Its boundary components inherit basings which are well–defined up to equivalence. We can thus view ∂F as a union of based curves.

We denote the infinite cyclic covering map of X by $\pi\colon \widetilde{X} \to X$ and we pick a base point $\widetilde{*}$ in $\widetilde{X}$ lying over $*$. With these choices there is a one-to-one correspondence

$$\begin{array}{c}\text{equivalence classes of}\\ \text{based curves (surfaces) in } X\end{array} \Leftrightarrow \text{ curves (surfaces) in } \widetilde{X}.$$

Now let c, d be based curves which only intersect at $*$. Then the corresponding closed curves $\widetilde{c}, \widetilde{d}$ in $\widetilde{X}$ are in general position.

By Lemma 2.4, there exists a surface $\widetilde{F} \subset \widetilde{X}$ such that $\partial \widetilde{F} = \Delta \widetilde{c}$. Let us choose a curve $\widetilde{\gamma}$ connecting $\widetilde{*}$ to a point on $\widetilde{F}$. The projection of $\widetilde{F}$ to X yields an immersed surface $F \subset X$. Then F is a based surface, the basing is γ, a projection of $\widetilde{\gamma}$ to X.

Any smooth curve on F is an image of a curve on $\widetilde{F}$ by definition. In particular, any smooth curve on F lifts to $\widetilde{X}$, which means that F is homologically invisible. By construction $\partial F = \Delta c$. We can now define

$$\mathrm{lk}_t(c, d) := \frac{1}{\Delta} F \cdot d \in \frac{1}{\Delta}\mathbb{Z}[t^{\pm 1}].$$

It is straightforward to see that

$$\mathrm{lk}_t(c, d) = \widetilde{\mathrm{lk}}(\widetilde{c}, \widetilde{d}) \in \frac{1}{\Delta}\mathbb{Z}[t^{\pm 1}].$$

It thus follows from the previous section that $\mathrm{lk}_t(c, d)$ is well–defined and that it satisfies $\mathrm{lk}_t(d, c) = \overline{\mathrm{lk}_t(c, d)}$. It also follows easily from the definitions that

$$\mathrm{lk}_t(c, d)|_{t=1} \; = \mathrm{lk}(c, d),$$

i.e. the evaluation of $\mathrm{lk}_t(c, d)$ at $t = 1$ equals the linking number of the unbased curves c and d. Finally note, that $\mathrm{lk}_t(c, d)$ is an invariant of the isotopy class of $c \cup d$. This follows from the definitions and the fact that any isotopy of $c \cup d$ extends to an isotopy of S^3.

From now on we shall use only the notation $\mathrm{lk}_t(c,d)$.

By a *framed curve* in X we mean a pair (c,m) where c is a based simple closed curve and $m\in\mathbb{Z}$. Given such (c,m) we define

$$\mathrm{lk}_t((c,m),(c,m)) := \mathrm{lk}_t(c,c'),$$

where c' is a longitude of c with the property that $\mathrm{lk}(c,c')=m$. It follows immediately from the above that

$$\mathrm{lk}_t((c,m),(c,m))|_{t=1} \;=\; \mathrm{lk}_t(c,c')|_{t=1} \;=\; \mathrm{lk}(c,c')=m.$$

If $n\neq m$, then we define

$$\mathrm{lk}_t((c,n),(c,m)) := \mathrm{lk}_t((c,m),(c,m))+n-m.$$

In the following we will often suppress m and we will just say that c is a based simple closed curve with framing m. In particular if the framing is understood, then we will just write $\mathrm{lk}_t(c,c)$. Also, if $c=(c,m)$ and $d=(d,n)$ are framed curves, such that c and d are disjoint, then we define

$$\mathrm{lk}_t((c,m),(d,n)) := \mathrm{lk}_t(c,d).$$

4. 4–MANIFOLDS AND INTERSECTION FORMS

4.1. **The twisted intersection form.** In the following let W be a 4–manifold, possibly with boundary, with the following properties:

1. $H_1(W;\mathbb{Z})\cong\mathbb{Z}$,
2. $H_1(W;\mathbb{Z}[t^{\pm1}])=0$,
3. $FH_2(W;\mathbb{Z}[t^{\pm1}]) := H_2(W;\mathbb{Z}[t^{\pm1}])/\{\mathbb{Z}[t^{\pm1}]\text{–torsion}\}$ is a free $\mathbb{Z}[t^{\pm1}]$–module.

We now define the intersection form Q_W on $FH_2(W;\mathbb{Z}[t^{\pm1}])$. First consider the sequence of maps

$$\begin{aligned}\Psi\colon H_2(W;\mathbb{Z}[t^{\pm1}]) \;&\to\; H_2(W,\partial W;\mathbb{Z}[t^{\pm1}]) \xrightarrow{\cong} \overline{H^2(W;\mathbb{Z}[t^{\pm1}])}\\ &\to\; \overline{\mathrm{Hom}_{\mathbb{Z}[t^{\pm1}]}(H_2(W;\mathbb{Z}[t^{\pm1}]),\mathbb{Z}[t^{\pm1}])},\end{aligned} \tag{4.1}$$

where the first map is the inclusion induced map, the second map is Poincaré duality and the third map is the evaluation map. The second map is evidently an isomorphism. The third map is also an isomorphism, indeed, since $H_1(W;\mathbb{Z}[t^{\pm1}])=0$ and since $\mathrm{Ext}^i_{\mathbb{Z}[t^{\pm1}]}(\mathbb{Z},\mathbb{Z}[t^{\pm1}])=0$ for $i>1$ we see that the UCSS for $H^2(W;\mathbb{Z}[t^{\pm1}])$ collapses, i.e. the evaluation map

$$H^2(W;\mathbb{Z}[t^{\pm1}]) \longrightarrow \mathrm{Hom}_{\mathbb{Z}[t^{\pm1}]}(H_2(W;\mathbb{Z}[t^{\pm1}]),\mathbb{Z}[t^{\pm1}])$$

is in fact an isomorphism. In constrast, the first map in (4.1) is in general not an isomorphism.

From (4.1) we now obtain a form

$$\begin{aligned}H_2(W;\mathbb{Z}[t^{\pm1}])\times H_2(W;\mathbb{Z}[t^{\pm1}]) \;&\to\; \mathbb{Z}[t^{\pm1}]\\ (a,b)\;&\mapsto\; \Psi(a)(b)\end{aligned}$$

but this clearly descends to a form

$$FH_2(W;\mathbb{Z}[t^{\pm 1}]) \times FH_2(W;\mathbb{Z}[t^{\pm 1}]) \longrightarrow \mathbb{Z}[t^{\pm 1}],$$

which we denote by Q_W. The form Q_W can also be defined more geometrically using equivariant intersection numbers of immersed based surfaces. This interpretation then quickly shows that Q_W is hermitian. We refer to [Wa99, Chapter 5] for details.

We now pick a basis for the free $\mathbb{Z}[t^{\pm 1}]$–module $FH_2(W;\mathbb{Z}[t^{\pm 1}])$ and we denote by $\det(Q_W)$ the matrix of the intersection form Q_W with respect to this basis. Note that the determinant is in fact well–defined, that is, up to a unit in $\mathbb{Z}[t^{\pm 1}]$ it does not depend on the choice of basis for $FH_2(W;\mathbb{Z}[t^{\pm 1}])$. The following lemma shows, that one can also determine $\det(Q_W)$ using any maximal set of linearly independent vectors in $FH_2(W;\mathbb{Z}[t^{\pm 1}])$, not necessarily a basis.

Lemma 4.1. *Let $v_1,\dots,v_n \in FH_2(W;\mathbb{Z}[t^{\pm 1}])$ be a maximal set of linearly independent vectors in $FH_2(W;\mathbb{Z}[t^{\pm 1}])$. We denote by $f \in \mathbb{Z}[t^{\pm 1}]$ the order of the $\mathbb{Z}[t^{\pm 1}]$–module*

$$FH_2(W;\mathbb{Z}[t^{\pm 1}])/(v_1,\dots,v_n),$$

then

$$\det(Q_W)\cdot f\cdot \overline{f} \doteq \det\left(\{Q_W(v_i,v_j)\}_{ij}\right).$$

Proof. Since $FH_2(W;\mathbb{Z}[t^{\pm 1}])$ is free, there is a basis $w_1,\dots,w_n$. The vectors $v_1,\dots,v_n$ can be expressed in terms of $w_1,\dots,w_n$. Let P be an $n\times n$ matrix over $\mathbb{Z}[t^{\pm 1}]$, such that $Pv_j = w_j$ for any $j=1,\dots,n$. We have

$$\det(Q_W(v_i,v_j)_{ij}) = \overline{\det(P)}\det(Q_W(w_i\cdot w_j)_{ij})\det(P) \doteq \det(Q_W)\cdot\det(P)\cdot\overline{\det(P)}.$$

We claim that $f \doteq \det(P)$. Indeed, P can be regarded as a map $\mathbb{Z}[t^{\pm 1}]^n \to \mathbb{Z}[t^{\pm 1}]^n$. On the one hand, $\det P$ is the order of the cokernel (see Example 2.2). On the other hand, the cokernel of P is $FH_2(W;\mathbb{Z}[t^{\pm 1}])/(v_1,\dots,v_n)$. □

4.2. **$\mathbb{Z}[t^{\pm 1}]$–cobordisms.** We say that a 3–manifold M is a *homology* $S^1\times S^2$ if M is closed, if $H_1(M;\mathbb{Z}) = \mathbb{Z}$ and if M comes equipped with a choice of an isomorphism $H_1(M;\mathbb{Z})\to\mathbb{Z}$. Given a 3–manifold M which is a homology $S^1\times S^2$ we can consider the module $H_1(M;\mathbb{Z}[t^{\pm 1}])$, and we can define a Blanchfield form on $H_1(M;\mathbb{Z}[t^{\pm 1}])$ in the same fashion as for $X(K)$. We denote by $\Delta_M = \Delta_M(t)$ the order of $H_1(M;\mathbb{Z}[t^{\pm 1}])$. Note that $H_1(M;\mathbb{Z}) = \mathbb{Z}$ implies that $\Delta_M(1) = 1$, in particular $\Delta_M(t)$ is non-zero. The standard arguments already employed for $X(K)$ show that

$$H_2(M;\mathbb{Z}[t^{\pm 1}]) \cong \overline{H^1(M;\mathbb{Z}[t^{\pm 1}])} \cong \overline{\operatorname{Ext}^0_{\mathbb{Z}[t^{\pm 1}]}(\mathbb{Z},\mathbb{Z}[t^{\pm 1}])} \cong \mathbb{Z}$$

is in fact isomorphic to the trivial $\mathbb{Z}[t^{\pm 1}]$–module $\mathbb{Z}$.

Example 4.2. Let K be a knot. We denote by $M(K)$ the zero–framed surgery on K. The inclusion map $X(K)\to M(K)$ induces an isomorphism $H_1(X(K);\mathbb{Z})\to H_1(M(K);\mathbb{Z})$. Together with the isomorphism $H_1(X(K);\mathbb{Z})\to\mathbb{Z}$ sending an oriented meridian to one we get a preferred isomorphism $H_1(M(K);\mathbb{Z})\to\mathbb{Z}$. It follows that

$M(K)$ is a homology $S^1 \times S^2$. It is well–known that the inclusion $X(K) \to M(K)$ induces an isomorphism $H_1(X(K); \mathbb{Z}[t^{\pm 1}]) \to H_1(M(K); \mathbb{Z}[t^{\pm 1}])$, which is in fact an isometry of the Blanchfield forms.

Definition 4.3. Let M and M' be 3–manifolds which are homology $S^1 \times S^2$'s. By a $\mathbb{Z}[t^{\pm 1}]$*–cobordism between* M *and* M' we understand an orientable, compact 4–manifold W with the following properties:

1. $\partial W = M \cup -M'$,
2. $H_1(M; \mathbb{Z}) \to H_1(W; \mathbb{Z})$ and $H_1(M'; \mathbb{Z}) \to H_1(W; \mathbb{Z})$ are isomorphisms, and the following diagram given by the inclusions and the preferred isomorphisms commutes:
$$\begin{tikzcd}
H_1(M;\mathbb{Z}) \arrow[r] \arrow[dr] & H_1(W;\mathbb{Z}) & H_1(M';\mathbb{Z}) \arrow[l] \arrow[dl] \\
& \mathbb{Z}, &
\end{tikzcd}$$
3. $H_1(W; \mathbb{Z}[t^{\pm 1}]) = 0$.

We now have the following lemma:

Lemma 4.4. *Let* M *and* M' *be 3–manifolds which are homology* $S^1 \times S^2$*'s. Let* W *be a* $\mathbb{Z}[t^{\pm 1}]$*–cobordism between* M *and* M'*, then the following* $\mathbb{Z}[t^{\pm 1}]$*–modules are free:*

1. $H_2(W, M; \mathbb{Z}[t^{\pm 1}])$ *and* $H_2(W, M'; \mathbb{Z}[t^{\pm 1}])$*,*
2. $H_2(W, \partial W; \mathbb{Z}[t^{\pm 1}])$*, and*
3. $FH_2(W; \mathbb{Z}[t^{\pm 1}]) = H_2(W; \mathbb{Z}[t^{\pm 1}])/\mathbb{Z}[t^{\pm 1}]$*–torsion.*

Proof. (1) We first consider $H_2(W, M; \mathbb{Z}[t^{\pm 1}])$. By Poincaré duality this is isomorphic to $\overline{H^2(W, M'; \mathbb{Z}[t^{\pm 1}])}$. The long exact sequence in $\mathbb{Z}[t^{\pm 1}]$–homology of the pair (W, M') yields:

$$\begin{array}{ccccccc} & & H_1(W; \mathbb{Z}[t^{\pm 1}]) & \to & H_1(W, M'; \mathbb{Z}[t^{\pm 1}]) & \to & \\ H_0(M'; \mathbb{Z}[t^{\pm 1}]) & \to & H_0(W; \mathbb{Z}[t^{\pm 1}]) & \to & H_0(W, M'; \mathbb{Z}[t^{\pm 1}]) & \to & 0. \end{array}$$

Our assumptions on W imply that $H_1(W; \mathbb{Z}[t^{\pm 1}]) = 0$ and that $H_0(M'; \mathbb{Z}[t^{\pm 1}]) \to H_0(W; \mathbb{Z}[t^{\pm 1}])$ is an isomorphism. We thus conclude that

$$H_1(W, M'; \mathbb{Z}[t^{\pm 1}]) = H_0(W, M'; \mathbb{Z}[t^{\pm 1}]) = 0.$$

The UCSS implies that

$$H^2(W, M'; \mathbb{Z}[t^{\pm 1}]) \cong \operatorname{Hom}_{\mathbb{Z}[t^{\pm 1}]}(H_2(W, M'; \mathbb{Z}[t^{\pm 1}]), \mathbb{Z}[t^{\pm 1}]),$$

but from Lemma 2.1 it follows that $\operatorname{Hom}_{\mathbb{Z}[t^{\pm 1}]}(H_2(W, M'; \mathbb{Z}[t^{\pm 1}]), \mathbb{Z}[t^{\pm 1}])$ is a free $\mathbb{Z}[t^{\pm 1}]$–module. We infer that $H_2(W, M; \mathbb{Z}[t^{\pm 1}])$ is a free $\mathbb{Z}[t^{\pm 1}]$–module. The same argument shows of course that $H_2(W, M'; \mathbb{Z}[t^{\pm 1}])$ is also free.

(2) By Poincaré duality we have an isomorphism

$$H_2(W, \partial W; \mathbb{Z}[t^{\pm 1}]) \cong \overline{H^2(W; \mathbb{Z}[t^{\pm 1}])}.$$

Since $H_1(W;\mathbb{Z}[t^{\pm 1}])=0$ by assumption and since $\operatorname{Ext}^i_{\mathbb{Z}[t^{\pm 1}]}(\mathbb{Z},\mathbb{Z}[t^{\pm 1}])=0$ for $i>1$ it follows from the UCSS, that $H^2(W;\mathbb{Z}[t^{\pm 1}])\cong \operatorname{Hom}_{\mathbb{Z}[t^{\pm 1}]}(H_2(W;\mathbb{Z}[t^{\pm 1}]),\mathbb{Z}[t^{\pm 1}])$, which is free by Lemma 2.1.

(3) Finally we want to show that $FH_2(W;\mathbb{Z}[t^{\pm 1}])$ is also free. Recall that by assumption $H_1(W;\mathbb{Z}[t^{\pm 1}])=0$. We obtain the following exact sequence

$$\begin{array}{rcccccc} & H_2(M;\mathbb{Z}[t^{\pm 1}]) & \to & H_2(W;\mathbb{Z}[t^{\pm 1}]) & \to & H_2(W,M;\mathbb{Z}[t^{\pm 1}]) & \to \\ \to & H_1(M;\mathbb{Z}[t^{\pm 1}]) & \to & 0. & & & \end{array}$$

Note that $H_2(M;\mathbb{Z}[t^{\pm 1}])$ is $\mathbb{Z}[t^{\pm 1}]$–torsion and $H_2(W,M;\mathbb{Z}[t^{\pm 1}])$ is a free $\mathbb{Z}[t^{\pm 1}]$–module by the above, in particular the module $H_2(W,M;\mathbb{Z}[t^{\pm 1}])$ is $\mathbb{Z}[t^{\pm 1}]$–torsion free. The above exact sequence thus descends to the following short exact sequence

$$0\longrightarrow FH_2(W;\mathbb{Z}[t^{\pm 1}])\longrightarrow H_2(W,M;\mathbb{Z}[t^{\pm 1}])\longrightarrow H_1(M;\mathbb{Z}[t^{\pm 1}])\longrightarrow 0. \tag{4.2}$$

Since $H_2(W,M;\mathbb{Z}[t^{\pm 1}])$ is free we can find an isomorphism

$$\Phi\colon \mathbb{Z}[t^{\pm 1}]^n\longrightarrow H_2(W,M;\mathbb{Z}[t^{\pm 1}])$$

for some appropriate n.

Now let $v_1,\dots,v_m$ be a minimal generating set for $FH_2(W;\mathbb{Z}[t^{\pm 1}])$. We thus obtain the following commutative diagram of exact sequences:

$$\begin{tikzcd}
 & \mathbb{Z}[t^{\pm 1}]^m \arrow[r, "A"] \arrow[d, "\Psi"] & \mathbb{Z}[t^{\pm 1}]^n \arrow[r] \arrow[d, "\Phi"', "\cong"] & H_1(M;\mathbb{Z}[t^{\pm 1}]) \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & FH_2(W;\mathbb{Z}[t^{\pm 1}]) \arrow[r, "d"] & H_2(W,M;\mathbb{Z}[t^{\pm 1}]) \arrow[r] & H_1(M;\mathbb{Z}[t^{\pm 1}]) \arrow[r] & 0,
\end{tikzcd}$$

where Ψ sends the i-th standard basis vector of $\mathbb{Z}[t^{\pm 1}]^m$ to v_i and where A is given by $\Phi^{-1}\circ d\circ\Psi$. The $n\times m$–matrix A over $\mathbb{Z}[t^{\pm 1}]$ is thus a presentation matrix for $H_1(M;\mathbb{Z}[t^{\pm 1}])$. It is well–known that $H_1(M;\mathbb{Z}[t^{\pm 1}])$ admits a square presentation matrix B, e.g. we can take $B=Vt-V^t$, where V denotes a Seifert matrix. Note that $\det(B)=\Delta_K(t)$ is non–zero, i.e. the columns of B are linearly independent over $\mathbb{Z}[t^{\pm 1}]$.

It now follows from [Li97, Theorem 6.1], that

$$\begin{array}{rl} & \text{minimal number of generators of column space of } A \;-\; \text{number of rows of } A \\ = & \text{minimal number of generators of column space of } B \;-\; \text{number of rows of } B. \end{array}$$

The latter is zero by the above, so we see that $m=n$. Since A is therefore a square matrix we see that $\det(A)=\Delta_K(t)$, in particular the map given by the matrix A is injective.

We thus obtain the following commutative diagram of short exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}[t^{\pm 1}]^n \arrow[r] \arrow[d] & \mathbb{Z}[t^{\pm 1}]^n \arrow[r] \arrow[d, "\Phi"', "\cong"] & H_1(M;\mathbb{Z}[t^{\pm 1}]) \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & FH_2(W;\mathbb{Z}[t^{\pm 1}]) \arrow[r] & H_2(W,M;\mathbb{Z}[t^{\pm 1}]) \arrow[r] & H_1(M;\mathbb{Z}[t^{\pm 1}]) \arrow[r] & 0.
\end{tikzcd}$$

It now follows from the 5–lemma that the vertical map $\mathbb{Z}[t^{\pm 1}]^n \to FH_2(W;\mathbb{Z}[t^{\pm 1}])$ is an isomorphism, in particular $FH_2(W;\mathbb{Z}[t^{\pm 1}])$ is free. □

The following result is one of the two homological ingredients in the proof of Theorem 1.1.

Proposition 4.5. *Let K and J be knots in S^3 and let W be a $\mathbb{Z}[t^{\pm 1}]$–cobordism between $M(K)$ and $M(J)$, then*

$$\det(Q_W) \doteq \Delta_K(t)\cdot\Delta_J(t).$$

Proof. Recall that the last two maps in the definition of the intersection form Q_W, (4.1), are isomorphisms. On the other hand the first map fits into the long exact sequence

$$\begin{array}{rlllll} & H_2(\partial W;\mathbb{Z}[t^{\pm 1}]) & \to & H_2(W;\mathbb{Z}[t^{\pm 1}]) & \to & H_2(W,\partial W;\mathbb{Z}[t^{\pm 1}]) \to \\ \to & H_1(\partial W;\mathbb{Z}[t^{\pm 1}]) & \to & H_1(W;\mathbb{Z}[t^{\pm 1}]) & \to & \end{array}$$

In our case $\partial W = M(K)\sqcup M(J)$, it thus follows that

$$H_i(\partial W;\mathbb{Z}[t^{\pm 1}]) = H_i(M(K);\mathbb{Z}[t^{\pm 1}])\oplus H_i(M(J);\mathbb{Z}[t^{\pm 1}]) \text{ for } i=1,2,$$

which is $\mathbb{Z}[t^{\pm 1}]$–torsion. Since $H_2(W,\partial W;\mathbb{Z}[t^{\pm 1}])$ is free and since $H_1(W;\mathbb{Z}[t^{\pm 1}])=0$ we now see that the above long exact sequence descends to the following short exact sequence:

$$\begin{array}{rcl} 0 & \to & FH_2(W;\mathbb{Z}[t^{\pm 1}]) \to H_2(W,\partial W;\mathbb{Z}[t^{\pm 1}]) \to \\ H_1(M(K);\mathbb{Z}[t^{\pm 1}])\oplus H_1(M(J);\mathbb{Z}[t^{\pm 1}]) & \to & 0. \end{array}$$

Let A be a matrix representing Q_W for a basis for $FH_2(W;\mathbb{Z}[t^{\pm 1}])$. It follows from the definition of Q_W that the matrix A also represents the map $FH_2(W;\mathbb{Z}[t^{\pm 1}]) \to H_2(W,\partial W;\mathbb{Z}[t^{\pm 1}])$ for some appropriate bases. We thus see that A is a presentation matrix for the $\mathbb{Z}[t^{\pm 1}]$–module

$$H_1(M(K);\mathbb{Z}[t^{\pm 1}])\oplus H_1(M(J);\mathbb{Z}[t^{\pm 1}]),$$

which by the definition of the Alexander polynomials implies that

$$\det(A) \doteq \Delta_K(t)\cdot\Delta_J(t).$$

□

4.3. **Surgeries and intersection forms.** Let M be a 3–manifold which is a homology $S^1 \times S^2$. Let $(c_1, \epsilon_1), \dots, (c_n, \epsilon_n)$ be framed oriented curves in M with the following properties:

1. the framings are either -1 or 1,
2. $c_1, \dots, c_n$ are homologically trivial in M.

We then consider the 4–manifold W which is given by attaching 2–handles $h_1, \dots, h_n$ with framings $\epsilon_1, \dots, \epsilon_n$ to $M \times [0,1]$ along $c_1 \times \{1\}, \dots, c_n \times \{1\} \subset M \times \{1\}$. We identify M with $M \times \{0\}$ and we denote by M' the other boundary component of W.

It follows from (2) that $H_1(W;\mathbb{Z}) = \mathbb{Z}$ and that the maps $H_1(M;\mathbb{Z}) \to H_1(W;\mathbb{Z})$ and $H_1(M';\mathbb{Z}) \to H_1(W;\mathbb{Z})$ are isomorphisms. It furthermore follows from (2) that $c_1, \dots, c_n$ define elements of $H_1(M;\mathbb{Z}[t^{\pm1}])$, which are well–defined up to a power of t. It is straightforward to see that

$$H_1(W;\mathbb{Z}[t^{\pm1}]) \cong H_1(M;\mathbb{Z}[t^{\pm1}])/(c_1, \dots, c_n).$$

Next result is the second homological ingredient needed in the proof of Theorem 1.1.

Proposition 4.6. *If $c_1, \dots, c_n$ generate $H_1(M(K);\mathbb{Z}[t^{\pm1}])$, then W is a $\mathbb{Z}[t^{\pm1}]$–cobordism between M and M', and*

$$\det(Q_W) \doteq \det\left(\{\mathrm{lk}_t(c_i, c_j)\}_{ij}\right) \cdot \Delta_M(t)^2.$$

Proof. Throughout the proof we write $\Delta = \Delta_M(t)$. It follows from the definitions and the discussion preceding the lemma that W is indeed a $\mathbb{Z}[t^{\pm1}]$–cobordism between M and M'. We consider the short exact sequence (4.2)

$$0 \longrightarrow FH_2(W;\mathbb{Z}[t^{\pm1}]) \xrightarrow{\iota} H_2(W,M;\mathbb{Z}[t^{\pm1}]) \longrightarrow H_1(M;\mathbb{Z}[t^{\pm1}]) \longrightarrow 0.$$

It is clear that the cores of the 2–handles $h_1, \dots, h_n$ give rise to a generating set for $H_2(W,M;\mathbb{Z}[t^{\pm1}])$. By a slight abuse of notation we denote the cores of the 2–handles by $h_1, \dots, h_n$ as well. Note that each h_i then naturally defines an element $[h_i] \in H_2(W,M;\mathbb{Z}[t^{\pm1}])$. By Lemma 2.4, there exist $k_1, \dots, k_n \in FH_2(W;\mathbb{Z}[t^{\pm1}])$, such that $\iota(k_i) = \Delta \cdot [h_i] \in H_2(W,M;\mathbb{Z}[t^{\pm1}])$, $i = 1, \dots, n$.

Lemma 4.7.

$$k_i \cdot k_j = \Delta^2 \cdot \mathrm{lk}_t(c_i, c_j).$$

Proof. We denote the infinite cyclic covers of M and $X = X(K)$ by $\widetilde{M}$ and $\widetilde{X}$. By Lemma 2.4 we can find surfaces $F_1, \dots, F_n$ in $\widetilde{M}$ such that $\partial F_i = \Delta c_i$. We can arrange the surfaces such that F_i and $t^k c_j$ are in general position for any i, j, k.

We first consider the case $i \neq j$. We then consider the surface

$$T_i := \Delta \cdot h_i \ \cup \ (\Delta \cdot c_i \times [0, \frac{1}{2}]) \ \cup \ (F_i \times \frac{1}{2})$$

in $\widetilde{W}$ where we think of $\Delta \cdot h_i$ and $\Delta \cdot c_i$ as a disjoint union of appropriate translates of the surface h_i respectively the curve c_i. Note that the surface T_i is closed and the image of $[T_i]$ in $H_2(W,M;\mathbb{Z}[t^{\pm1}])$ is the same as the image of $\Delta[h_i]$ in $H_2(W,M;\mathbb{Z}[t^{\pm1}])$.

Since $FH_2(W;\mathbb{Z}[t^{\pm 1}]) \xrightarrow{\iota} H_2(W,M;\mathbb{Z}[t^{\pm 1}])$ is injective it now follows that T_i represents the class k_i. Similarly we consider the surface

$$T_j := \Delta \cdot h_j \ \cup\ (\Delta \cdot c_j \times [0,1]) \ \cup\ (F_j \times 1),$$

where F_j is a surface in $\widetilde{M}$ which has boundary $\Delta \cdot c_j$. Note that the surface T_j is closed and represents the class k_j.

We can thus use the surfaces T_i and T_j to calculate $k_i \cdot k_j$. But it is clear from the definitions that

$$T_i \cdot T_j = (\Delta F_i \times \frac{1}{2}) \cdot (c_j \times \frac{1}{2}),$$

but this clearly equals $\Delta \cdot (F_i \cdot c_j) = \Delta^2 \cdot \mathrm{lk}_t(c_i, c_j)$.

The case $i = j$ can be proved completely analogously by constructing an appropriate surface T_i' using the longitude of c_i with framing ϵ_i which connects up with the core of the 2–handle which we had attached to c_i with framing ϵ_i. We leave the details to the reader. This concludes the proof of the lemma. □

Lemma 4.8. *The order of the* $\mathbb{Z}[t^{\pm 1}]$*–module*

$$FH_2(W;\mathbb{Z}[t^{\pm 1}])/(k_1,\dots,k_n)$$

equals Δ^{n-1}.

Proof. It follows from (4.2) and from the definitions that we have the following commutative diagram of maps where the horizontal sequences are exact:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \bigoplus_{i=1}^n k_i\mathbb{Z}[t^{\pm 1}] \arrow[r, "\iota"] \arrow[d] & \bigoplus_{i=1}^n \Delta h_i\mathbb{Z}[t^{\pm 1}] \arrow[rr] \arrow[d] & & 0 \arrow[d] \\
0 \arrow[r] & FH_2(W;\mathbb{Z}[t^{\pm 1}]) \arrow[r, "\iota"] & H_2(W,M;\mathbb{Z}[t^{\pm 1}]) \arrow[r] & H_1(M;\mathbb{Z}[t^{\pm 1}]) \arrow[r] & 0.
\end{tikzcd}$$

It then follows that the following sequence of maps

$$\begin{array}{rl} 0 & \to\ FH_2(W;\mathbb{Z}[t^{\pm 1}])/(k_1,\dots,k_n) \to H_2(W,M;\mathbb{Z}[t^{\pm 1}])/(\Delta h_1,\dots,\Delta h_n) \\ & \to\ H_1(M;\mathbb{Z}[t^{\pm 1}]) \to 0. \end{array}$$

is well–defined and exact. By the multiplicativity of orders (see Lemma 2.6) it follows that

$$\begin{array}{l} \mathrm{order}(H_2(W,M;\mathbb{Z}[t^{\pm 1}])/(\Delta h_1,\dots,\Delta h_n)) \\ \qquad \doteq \mathrm{order}(FH_2(W;\mathbb{Z}[t^{\pm 1}])/(k_1,\dots,k_n)) \ \cdot\ \mathrm{order}(H_1(M;\mathbb{Z}[t^{\pm 1}])). \end{array}$$

But the order on the left is clearly Δ^n and the order of $H_1(M;\mathbb{Z}[t^{\pm 1}])$ equals Δ by the definition of Δ. This concludes the proof of the lemma. □

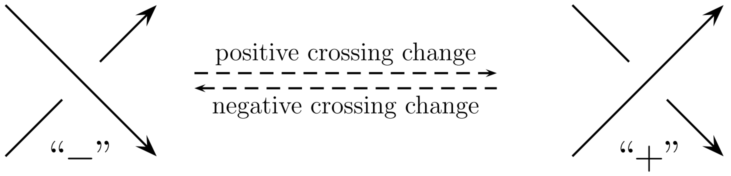
Using Lemma 4.1 we now see that

$$\begin{array}{rcl} \det(Q_W) & \doteq & \det(\{\Delta^2\,\mathrm{lk}_t(c_i,c_j)\}_{ij}) \cdot \Delta^{-2(n-1)} = \det(\{\mathrm{lk}_t(c_i,c_j)\}_{ij}) \cdot \Delta^{2n} \cdot \Delta^{-2(n-1)} \\ & = & \det(\{\mathrm{lk}_t(c_i,c_j)\}_{ij}) \cdot \Delta^2. \end{array}$$

□

5. The main theorem

5.1. **Statement of the main theorem.** In this section we will state a slightly stronger version of our main theorem. In order to state the theorem we first have to recall the following definition: A crossing change is a *positive crossing change* if it turns a negative crossing into a positive crossing. Otherwise we refer to the crossing change as a *negative crossing change.*



The following theorem is now our main result, it clearly implies Theorem 1.1 from the introduction.

Theorem 5.1. *Let K be a knot and let $A = A(t)$ be an $n \times n$–matrix over $\mathbb{Z}[t^{\pm 1}]$ such that $Bl(K) \cong \lambda(A)$ and such that $A(1)$ is diagonalizable over $\mathbb{Z}$. We denote the number of positive eigenvalues of $A(1)$ by n_+ and we denote the number of negative eigenvalues by n_-. Then K can be turned into a knot with Alexander polynomial one using n_+ negative crossing changes and n_- positive crossing changes.*

There are two ingredients in the proof of Theorem. The homological part was given in Proposition 4.5 and Proposition 4.6. The main topological tool will be Lemma 5.5 which we will state in the following section.

Remark 5.2. The theorem applies also to knots in $\mathbb{Z}$–homology spheres. In general, such a knot can not be unknotted using ‘crossing changes’ (i.e. using surgeries along curves which bound nice disks) since the knot might not even be null-homotopic. But any knot can be turned into Alexander polynomial one knots, using $n(K)$ unknotting moves.

5.2. **The main technical lemma.** In order to state our main technical lemma we need a few more definitions:

Definition 5.3. Let $K \subset S^3$ be a knot. A (based) disk $D \subset S^3$ is called *nice* if the disk is embedded (that is the unbased disk is embedded), if it intersects K transversely and if it intersects K exactly twice with opposite signs.

Definition 5.4. Let D, D' be embedded disks in S^3. We say that the disk D *precedes* the disk D' if D' and D intersect transversely and if $D' \cap \partial D = \emptyset$.

As an example, consider the disks in Figure 1, then the blue (dashed) disk precedes the green (solid) disk, but not vice versa.

We can now state our main technical lemma. It will be proved in Section 6.

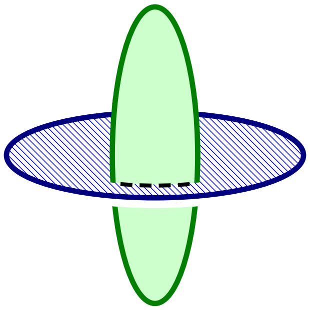


FIGURE 1. The blue (the dashed one) and green disks.

Lemma 5.5. *Let K be a knot and let $x_1, \dots, x_n$ be elements in $H_1(X(K); \mathbb{Z}[t^{\pm 1}])$. Let $p_{ij}(t) \in \mathbb{Z}[t^{\pm 1}]$, $i, j \in \{1, \dots, n\}$ be such that*

$$Bl(x_i, x_j) = \frac{p_{ij}(t)}{\Delta_K(t)} \in \mathbb{Q}(t)/\mathbb{Z}[t^{\pm 1}] \text{ and } p_{ij}(t) = p_{ji}(t^{-1})$$

for any i and j. Then there exists an ordered set $\{D_1, \dots, D_n\}$ of based nice disks with the following properties:

1. *for $i < j$ the disk D_i precedes D_j,*
2. *for any i the based curve $c_i := \partial D_i$ represents x_i,*
3. *if for $i = 1, \dots, n$ we equip $c_i = \partial D_i$ with the framing $p_{ii}(1)$, then*

$$\operatorname{lk}_t(c_i, c_j) = \frac{p_{ij}(t)}{\Delta_K(t)} \in \mathbb{Q}(t),$$

for any i and j.

5.3. **Proof of Theorem 5.1 assuming Lemma 5.5.** We will now prove Theorem 5.1 using Lemma 5.5. Let K be a knot. We write $\Delta = \Delta_K(t)$. Let $A = A(t)$ be an $n \times n$–matrix over $\mathbb{Z}[t^{\pm 1}]$ such that $Bl(K) \cong \lambda(A)$ and such that $A(1)$ is diagonalizable over $\mathbb{Z}$. We denote the number of positive eigenvalues of $A(1)$ by n_+ and we denote the number of negative eigenvalues by n_-.

Note that since $A(1)$ is diagonalizable over $\mathbb{Z}$ we can find an invertible matrix P over $\mathbb{Z}$ such that $PA(1)P^t$ is diagonal over $\mathbb{Z}$. We can thus, without loss of generality assume, that $A(1)$ is diagonal.

The matrix $A(t)$ is in particular a presentation matrix for the Alexander module. It follows that $\det(A(t)) = \pm\Delta_K(t)$ and in particular $\det(A(1)) = \pm 1$. The entries on the diagonal of $A(1)$ are therefore either $+1$ or -1. We now denote by $\epsilon_1, \dots, \epsilon_n$ the diagonal entries. Given $i, j \in \{1, \dots, n\}$ we denote by $b_{ij}(t) \in \mathbb{Z}[t^{\pm 1}]$ the polynomial which satisfies

$$ij\text{–entry of } A(t)^{-1} = \frac{b_{ij}(t)}{\Delta}.$$

We denote by $e_1, \dots, e_n$ the canonical generating set of $\mathbb{Z}[t^{\pm 1}]^n / A\mathbb{Z}[t^{\pm 1}]^n$ and we denote by $x_1, \dots, x_n$ the images of $e_1, \dots, e_n$ under the isometry $\lambda(A) \to Bl(K)$. By Lemma 5.5 there exists an ordered set $\{D_1, \dots, D_n\}$ of based nice disks with the following properties:

1. for any $i < j$ the disk D_i precedes D_j,
2. for any i the based curve $c_i := \partial D_i$ represents x_i,
3. if for $i = 1, \dots, n$ we equip $c_i = \partial D_i$ with the framing $b_{ii}(1)$, then
$$\operatorname{lk}_t(c_i, c_j) = \frac{b_{ij}(t)}{\Delta},$$
for any i and j.

We now consider the disk D_1. After an isotopy of S^3 we can assume that it is 'standard' as in Figure 2 on the left. We now perform ϵ_1–surgery on the unknot $c_1 = \partial D_1$. The resulting 3–manifold is again S^3. Furthermore the knot K_1, which is defined as the image of K in the surgery S^3, is obtained from $K_0 := K$ through adding a full ϵ_1–twist along the disk (see Figure 2). Adding a full ϵ_1–twist corresponds to a $(-\epsilon_1)$–crossing change in an appropriate diagram of K. The fact that D_1 precedes $D_2, \dots, D_n$ implies that the disks $D_2, \dots, D_n$ are 'unaffected' by the surgery, in particular for $j = 2, \dots, n$, ∂D_j is again an unknot and for $2 \le i < j$, D_i precedes D_j. We can therefore iterate this process, and perform ϵ_i–surgery along the unknots $c_i = \partial D_i$ for $i = 2, \dots, n$. As given $i < j$ the disk D_i precedes D_j, the consecutive surgeries do not affect the remaining disks, in particular at each step the remaining curves are unknots in the 3–sphere.

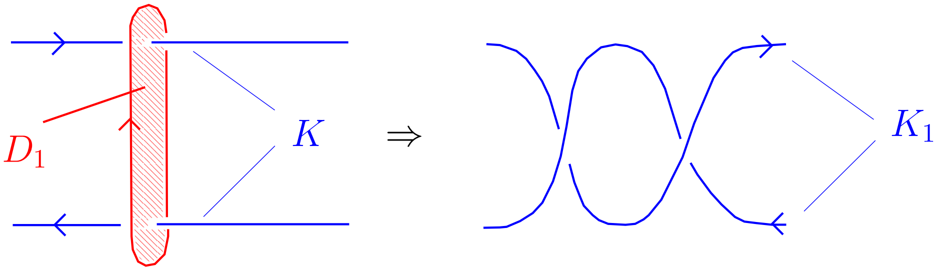


FIGURE 2. A nice disk in standard position and the result of adding a full +1–twist along the disk.

We denote the resulting knots by $K_2, \dots, K_n$. As above, for each $i = 2, \dots, n$ the knot K_i is obtained from K_{i-1} by doing an ϵ_i–crossing change. In particular $K = K_0$ can be turned into the knot $J := K_n$ using n_+ negative crossing changes and n_- positive crossing changes. It remains to show that $\Delta_J(t) = 1$.

For $i = 0, \dots, n-1$ we now denote by W_i the result of adding 2–handles along c_{i+1} to $M(K_i) \times [0,1]$ with framing ϵ_{i+1}. Adding a 2–handle gives a cobordism between the original manifold and the surgered 3–manifold. In particular we see that $\partial W_i =$

$-M(K_i) \sqcup M(K_{i+1})$. We can also add all the 2–handles simultaneously along $c_1, \dots, c_n$ with framings $\epsilon_1, \dots, \epsilon_n$ and we thus obtain a 4–manifold W which is diffeomorphic to the union $W_1, \dots, W_n$ along the corresponding boundaries. Note that $\partial W = -M(K) \sqcup M(J)$. By the discussion of Section 4.3 the manifold W has furthermore the following properties:

1. $H_1(W; \mathbb{Z}) = \mathbb{Z}$,
2. $H_1(M(K); \mathbb{Z}) \to H_1(W; \mathbb{Z})$ and $H_1(M(J); \mathbb{Z}) \to H_1(W; \mathbb{Z})$ are isomorphisms,
3. $H_1(W; \mathbb{Z}[t^{\pm 1}]) \cong H_1(M(K); \mathbb{Z}[t^{\pm 1}])/(c_1, \dots, c_n) = 0$.

Furthermore, by Proposition 4.6 we see that

$$\det(Q_W) \doteq \det\left(\{\mathrm{lk}_t(c_i, c_j)\}_{ij}\right) \cdot \Delta^2 \doteq \det(A(t)^{-1}) \cdot \Delta^2 \doteq \Delta^{-1} \cdot \Delta^2 = \Delta.$$

It now follows from Proposition 4.5 that the knot $J = K_n$ has trivial Alexander polynomial. This concludes the proof of Theorem 5.1, modulo the proof of Lemma 5.5 which will be given in the next section.

6. Proof of Lemma 5.5

In this section we shall prove Lemma 5.5. The proof is given in a couple of steps. First, we find pairwise disjoint nice disks $D_1, \dots, D_n$, with $c_j = \partial D_j$, such that for any $i, j = 1, \dots, n$ we have $Bl(c_i, c_j) = \frac{p_{ij}(t)}{\Delta(t)} \in \mathbb{Q}(t)/\mathbb{Z}[t^{\pm 1}]$. This is an adaptation of Fogel's argument [Fo94, p. 287] and is done in Section 6.1. The property that $Bl(c_j, c_j) = \frac{p_{ij}(t)}{\Delta(t)} \in \mathbb{Q}(t)/\mathbb{Z}[t^{\pm 1}]$ is weaker than that $\mathrm{lk}_t(c_i, c_j) = \frac{p_{ij}(t)}{\Delta(t)} \in \mathbb{Q}(t)$, it only means that $\mathrm{lk}_t(c_i, c_j) - \frac{p_{ij}(t)}{\Delta(t)}$ is an element of $\mathbb{Z}[t^{\pm 1}]$.

To ensure that $\mathrm{lk}_t(c_i, c_j) - \frac{p_{ij}(t)}{\Delta(t)} = 0$ we need to perform several moves on the disks. We introduce four types of moves in Section 6.3 and one type in Section 6.4. These moves potentially introduce intersections among disks $D_1, \dots, D_n$, therefore an analysis must be careful and take into account the ordering of disks. In our proof we perform only the moves that preserve the ordering of the disks. The details are given in Section 6.5.

6.1. **Finding nice based disks.**

In this section we prove the following lemma.

Lemma 6.1. *Let K be a knot and let $x_1, \dots, x_n \in H_1(X(K); \mathbb{Z}[t^{\pm 1}])$. Then there exist n disjoint nice based disks $D_1, \dots, D_n$ such that for $i = 1, \dots, n$ the curve $c_i := \partial D_i$ represents x_i.*

This lemma is a slight generalization of a result by Fogel [Fo94, p. 287]. The proof we give is also basically due to Fogel.

Proof. Let $x_1, \dots, x_n \in H_1(X(K); \mathbb{Z}[t^{\pm 1}])$. The multiplication by $t - 1$ is an isomorphism of $H_1(X(K); \mathbb{Z}[t^{\pm 1}])$ (see e.g. [Lev77]). We can therefore find $y_1, \dots, y_n \in H_1(X(K); \mathbb{Z}[t^{\pm 1}])$ such that $(t-1)y_i = x_i$, $i = 1, \dots, n$. We now represent $y_1, \dots, y_n$ by disjoint based curves $d_1, \dots, d_n$. (By doing crossing changes on the curves $d_1, \dots, d_n$, we can without loss of generality assume that the unbased curves are unknotted in S^3, this justifies the illustration below, but is not necessary for the argument.) We also pick disjoint embedded oriented disks $S_1, \dots, S_n$ with the following properties:

(1) for $i = 1, \dots, n$ the disk S_i intersects K precisely once with positive intersection number,
(2) for $i = 1, \dots, n$ the curve $m_i := \partial S_i$ intersects d_i in precisely one point,
(3) for $i \neq j$ the curves m_i and d_j are disjoint.

We refer to Figure 3 for a schematic picture. Now note that for each i the unbased curve $m_i d_i m_i^{-1} d_i^{-1}$ bounds a nice disk D_i which can be placed in a small neighborhood around the disk S_i and the unbased curve d_i. (We again refer to Figure 3 for a schematic picture.) By construction the disks $D_1, \dots, D_n$ are disjoint. On the other hand, since m_i is a meridian we see that $m_i d_i m_i^{-1} d_i^{-1} = (m_i d_i m_i^{-1}) \cdot d_i^{-1}$ represents $ty_i - y_i = (t-1)y_i = x_i$ in the Alexander module. If we equip $D_1, \dots, D_n$ with the basings of the based curves $d_1, \dots, d_n$ we thus obtain the required based disks. □

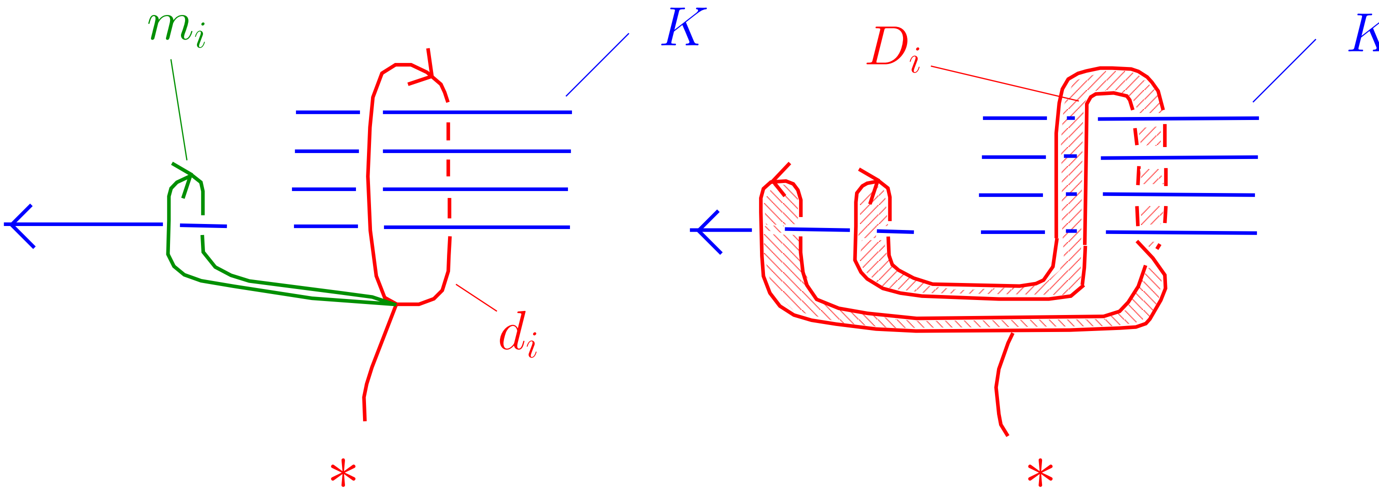


Figure 3. Construction of nice disks.

6.2. **Properly arranged disks.** The following discussion will be essential in the remainder of the proof.

Definition 6.2. Let $K \subset S^3$ be a knot and let $D_1, \dots, D_n$ be nice based disks. We say that they are *properly arranged* if the following conditions hold:

(1) the segment $S := [0,1] \times 0 \times 0 \subset \mathbb{R}^3 \subset S^3$ is part of the knot K, and the orientation of K agrees with the canonical orientation on that segment,
(2) all intersection points of the disks with the knot K lie on S,
(3) for $i < j$ the disk D_i precedes the disk D_j.

Remark 6.3. If $D_1, \ldots, D_n$ are nice based disks that are disjoint, then it is straightforward to see that a segment $S \subset K$ exists which satisfies Conditions (1) and (2) from Definiton 6.2.

Note that if the disks $D_1, \ldots, D_n$ are properly arranged then we can find a tubular neighborhood of the segment S of the knot K which is isotopic to the picture shown in Figure 4. We call such a neighborhood of S a *standard segment*. We refer to each of the $2n$ components of the disks as a *piece*. The orientation on the disks endows each piece with an orientation, which we refer to as positive or negative depending on the intersection with the oriented S. Finally each cube in S which contains precisely two pieces is called a *subsegment*. In the following we will furthermore use the expressions 'adjacent pieces' and 'piece to the left' and 'piece to the right' with the obvious meanings.

We henceforth equip the set of points S with the canonical ordering coming from the ordering on the interval $[0, 1]$. If the disjoint nice disks $D_1, \ldots, D_n$ are properly arranged then the intersection points are of the form $z_1 < z_2 < \cdots < z_{2n}$. Given $i \in \{1, \ldots, 2n\}$ we denote by $\sigma(i) \in \{1, \ldots, n\}$ the integer which has the property that the disk $D_{\sigma(i)}$ intersects S in the point z_i. We refer to the ordered set

$$\{\sigma(1), \ldots, \sigma(2n)\}$$

as the *arrangement* of the properly arranged disks $D_1, \ldots, D_n$. We refer to Figure 4 for an illustration.

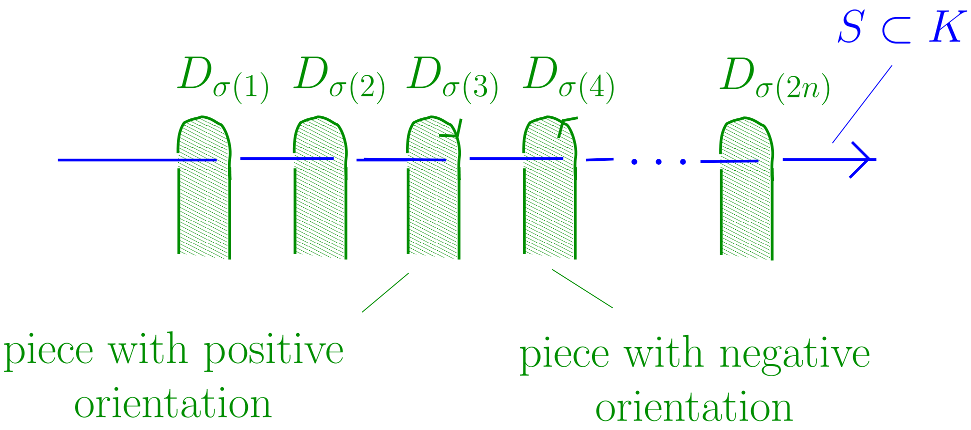


FIGURE 4. Properly arranged disks.

6.3. **Type R and F moves.** Given properly arranged nice disks $D_1, \ldots, D_n$ we consider the following local moves which produce new sets of properly arranged nice disks $D'_1, \ldots, D'_n$. The subsequent figures show moves on sets of properly arranged disks which take place in subsegments, in particular no other disks and no basings are allowed in these subsets of S^3.

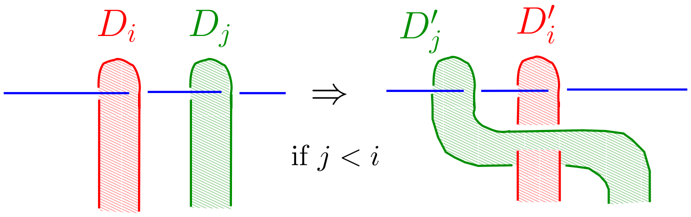


FIGURE 5. A type R_1 move.

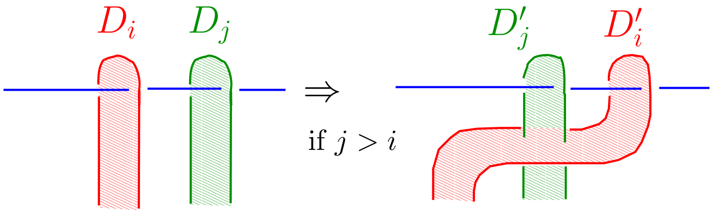


FIGURE 6. A type R_2 move.

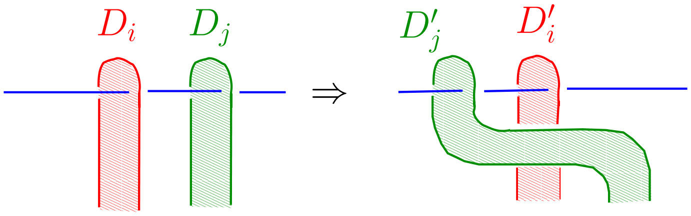


FIGURE 7. Type F_1 move.

If $j < i$, then a **type R_1 move** consists of the change as drawn on Figure 5, i.e. we push the disk D_j 'on the right' over the disk D_i 'on the left'. Note that the isotopy types of the boundary curves are unchanged, so the twisted linking numbers of the new boundary curves agree with the twisted linking numbers of the old boundary curves. The resulting disk D'_j precedes D'_i, because D_j precedes D_i.

If $j > i$, then a **type R_2 move** consists of the move shown in Figure 6, which is almost of the same form as the type R_1 move, except that we now push the disk D_i 'on the left' over the disk D_j 'on the right'. Note that $D'_1, \dots, D'_n$ are again properly arranged.

A **type F_1 move** consists of applying the move shown in Figure 7 to two adjacent pieces with opposite orientations.

A **type F_2 move** consists of applying the move shown in Figure 8 to two adjacent pieces with opposite orientations.

Remark 6.4. One could define F moves for two adjacent pieces with the same orientation, but we will not need that.

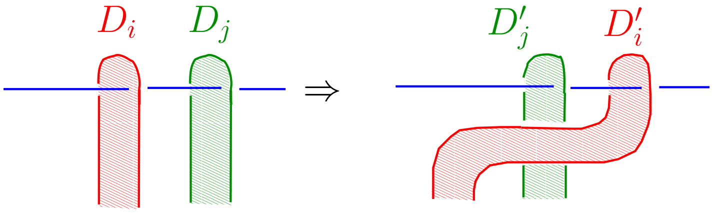


FIGURE 8. Type F_2 move.

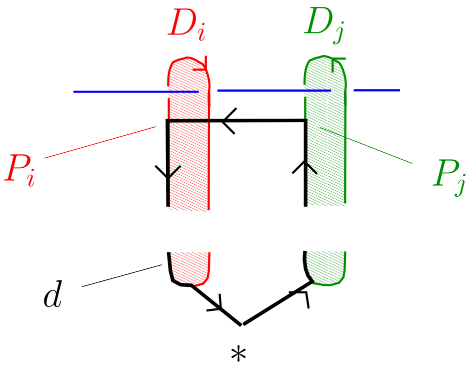


FIGURE 9. Curve d for different orientations of the pieces.

We denote by $D'_1, \dots, D'_n$ the disks resulting from applying a type F_1 move or a type F_2 move to disks $D_1, \dots, D_n$. We write $c_l := \partial D_l$ and $c'_l := \partial D'_l$ for $l = 1, \dots, n$. Note that neither move creates any new intersections between the disks. In particular $D'_1, \dots, D'_n$ are again properly arranged. On the other hand the isotopy type of the boundary curves changes. To state how the twisted linking numbers change we consider the curve d which is given by concatenation of the following paths:

1. a path from the base point $*$ along the based curve $c_j = \partial D_j$ to a point P_j on the piece of D_j involved in the type F moves,
2. a horizontal path to the corresponding point P_i on $c_i = \partial D_i$,
3. a path from the point P_i on c_i to the base point $*$ along the based curve c_i.

We refer to Figure 9 for an illustration. We then denote by

$$k := k(D_i, D_j) \tag{6.1}$$

the image of d under the epimorphism $\pi_1(X_K) \to \mathbb{Z}$ given by sending the oriented meridian of K to 1. It is straightforward to see that k is independent of the choice of P_j made.

Lemma 6.5. *For any r, s with $\{r, s\} \neq \{i, j\}$ we have*

$$\mathrm{lk}_t(c'_r, c'_s) = \mathrm{lk}_t(c_r, c_s),$$

furthermore

(1) *if* $i \neq j$, *then*

$$\mathrm{lk}_t(c_i', c_j') = \mathrm{lk}_t(c_i, c_j) + \epsilon t^k (t^\eta - 1) \text{ and } \mathrm{lk}_t(c_j', c_i') = \mathrm{lk}_t(c_j, c_i) + \epsilon t^{-k}(t^{-\eta} - 1), \tag{6.2}$$

(2) *if* $i = j$, *then*

$$\mathrm{lk}_t(c_i', c_i') = \mathrm{lk}_t(c_i, c_j) + \epsilon t^k (t^\eta - 1) + \epsilon t^{-k}(t^{-\eta} - 1), \tag{6.3}$$

where $\epsilon = -1$ *if we apply a type* F_1 *move and* $\epsilon = 1$ *if we apply a type* F_2 *move, furthermore* $\eta = -1$ *if the piece on the left has positive orientation and* $\eta = 1$ *if the piece on the left has negative orientation.*

We will first consider the case of a type F_1 move such that the piece on the left has positive orientation.

Case 1. $i \neq j$**.** It is clear, that for $\{r, s\} \neq \{i, j\}$ we have $\mathrm{lk}_t(c_k', c_l') = \mathrm{lk}_t(c_k, c_l)$. We will now show that

$$\mathrm{lk}_t(c_j', c_i') = \mathrm{lk}_t(c_j, c_i) + t^k(t - 1).$$

The claim regarding $\mathrm{lk}_t(c_i', c_j')$ then follows from the antisymmetry of the twisted linking number.

First recall that the twisted linking numbers only depend on isotopy invariants of the curves. We can therefore ignore the disks and we can also first apply a type R_1 move, which is an isotopy. We therefore have to compare the twisted linking numbers of the two sets of curves shown in Figure 10.

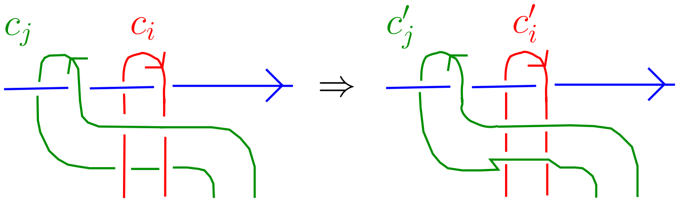


FIGURE 10. Composition of the inverse of a type R_1 move and a type F move.

We pick a based immersed surface F such that $\partial F = \Delta_K(t) \cdot c_j$. In the subsegment we can and will assume that the surface F is orthogonal to the plane which contains the diagram and that it points 'upwards'. We now obtain a surface F' with $\partial F' = \Delta_K(t) \cdot c_j'$ by cutting out a small rectangle of F around the modification. The surfaces F and F' in the neighborhood of the modification are sketched in Figure 11. Note that in Figure 11 we only show one sheet of the surfaces F and F', in reality each sheet which is drawn should be considered $\Delta_K(t)$–times.

We are now interested in the difference between $F \cdot c_i$ and $F' \cdot c_i'$. In the subsequent discussion we will continue with the notation in the definition of $F \cdot c_i$ (see Section 3.2). We consider the intersection points P and Q of F and c_i as shown in Figure 11

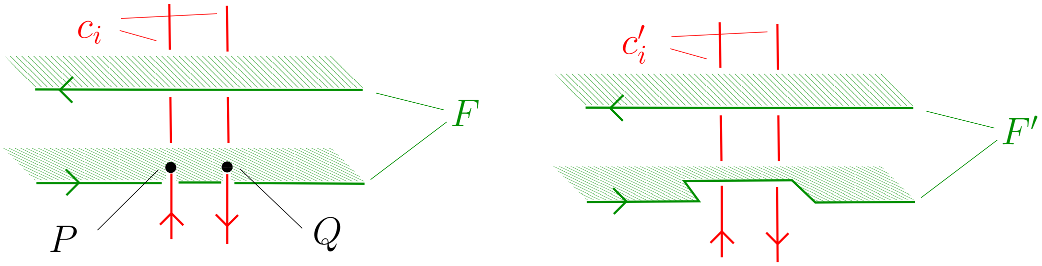


FIGURE 11. One sheet of F respectively F' in the subsegment glued to c_i as in Figure 10. The surfaces go 'vertically out of the plane' in the direction of the reader. On the left, the lower vertical sheet thus intersects c_i in two points P and Q. On the right, we pushed the surface across c_i, and thus removed the intersection points.

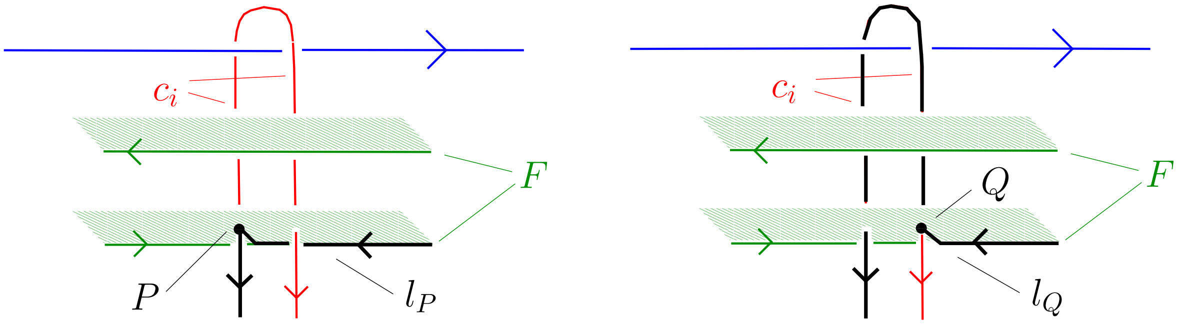


FIGURE 12. The curves l_P and l_Q in the definition of $F \cdot c$. Here the sheets are again pointing outwards toward the reader. The upper sheet lies above c_i and thus has no intersections with c_i, whereas the lower sheet intersects c_i in two points P and Q.

on the left. It is clear that $\epsilon_P = -1$ and $\epsilon_Q = +1$. It furthermore follows easily from the definitions (see also Figure 12) that

$$\phi(l_P) = t^k \text{ and } \phi(l_Q) = t^{k-1}.$$

(The point is that in the definition of $\phi(l_P)$ the curve l_P wraps around the knot once more in the negative direction.) Now recall that F and F' consist of $\Delta_K(t)$ copies of the sheets indicated in the diagrams. It now follows that

$$\begin{array}{rcl} \operatorname{lk}_t(c_j', c_i') & = & F' \cdot c_i' \\ & = & F \cdot c_i - \Delta_K(t) \cdot \epsilon_P \, \phi(l_P) - \Delta_K(t) \cdot \epsilon_Q \, \phi(l_Q) \\ & = & F \cdot c_i - \Delta_K(t)(t^{k-1} - t^k) \\ & = & \operatorname{lk}_t(c_j, c_i) - \Delta_K(t) t^k (t^{-1} - 1). \end{array}$$

This concludes the proof in the case that $i \neq j$.

Case 2. $i = j$. We again pick a based immersed surface F such that $\partial F = \Delta_K(t) \cdot c_i$. In a neighborhood of the modification we can and will assume that the surface F is orthogonal to the plane which contains the diagram and that it points 'upwards'. We again obtain a surface F' with $\partial F' = \Delta_K(t) \cdot c'_j$ by cutting out a small rectangle of F around the modification. The surfaces F and F' in the neighborhood of c_i and the modification are sketched in Figure 11. Note that $F \cap c_i$ contains two intersection points, P and Q, which do not appear in $F' \cap c'_i$, in turn $F' \cap c'_i$ contains two new intersection points, namely P' and Q'. We refer to Figure 14 for an illustration. Note that in Figure 14 we now only indicate the parts of the sheets of F and F' which contain the extra intersection points. A careful consideration of the intersection points now shows that

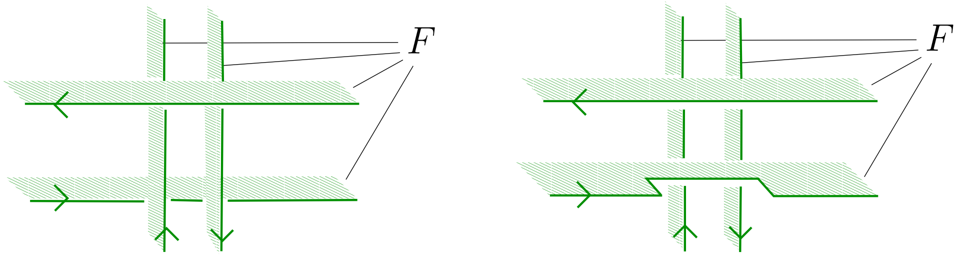


FIGURE 13. One sheet of F respectively F' in the subsegment. The surfaces F and F' point vertically outwards towards the reader. They are indicated only in a small neighborhood of the curves and they have to be extended in the direction of the reader beyond what is shown. In particular on the left the lower horizontal vertical sheet of F intersects the two vertical sheets of F. On the other hand, on the right the two vertical sheets of F' intersect the lower horizontal sheet of F'.

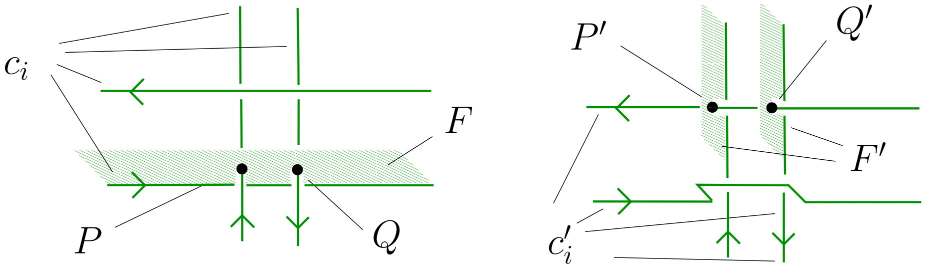


FIGURE 14. Extra intersection points of F and c_i respectively of F' and c'_i. Here we show only parts of the surfaces F and F', the two parts are again meant to point outwards towards the reader.

$$\mathrm{lk}_t(c'_i, c'_i) = \mathrm{lk}_t(c_i, c_i) - t^k(t^{-1} - 1) - t^{-k}(t - 1).$$

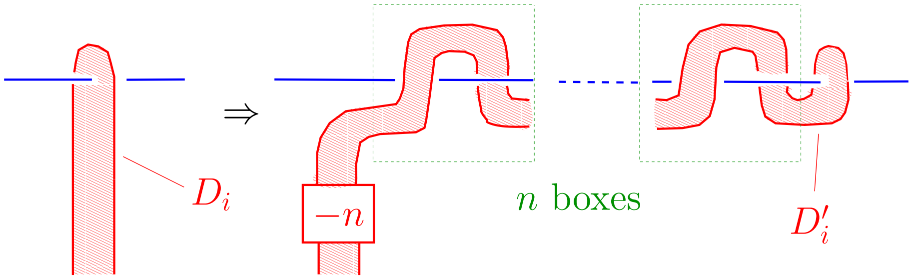


FIGURE 15. Type $T(n)$ move.

We leave the details to the reader.

This concludes the proof of Lemma 6.5 in the case of a type F_1 move such that the piece on the left has positive orientation. It is straightforward to verify that the other cases of Lemma 6.5 can be proved completely analogously. We again leave the details to the reader.

6.4. **The type $T(n)$ move.** A *type $T(n)$ move* consists of applying the move shown in Figure 15 to the based disk D_i. This move is in fact an isotopy of the disk D_i as will be shown later in Lemma 6.6. In particular this move leaves all twisted linking numbers unchanged. The move is important because it allows us to modify the term $k(D_i, D_j)$ which appears in the F–moves, see (6.1). More precisely, suppose we have two adjacent pieces of D_i and D_j, with the piece corresponding to D_i to the left. Let $k \in \mathbb{Z}$ be the integer which is defined as in the discussion of the type F moves. If we first apply a type $T(n)$ move to D_i, then

$$k(D'_i, D_j) = k(D_i, D_j) + n. \tag{6.4}$$

We will prove the following lemma which shows that the type $T(n)$ move does not change the isotopy type of the disk involved.

Lemma 6.6. *The two disks in Figure 16 are isotopic relative to the boundary of the cube which contains the figures.*

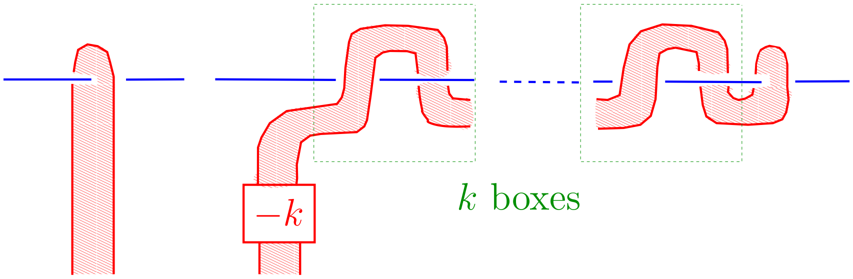


FIGURE 16. Isotopic disks in the statement of Lemma 6.6.

Proof. We first consider the set of isotopies (relative to the boundary of the cube) in Figures 17 and 18. We then iterate this process k times. The lemma now follows from first adding a canceling pair of a full k twist and a full $-k$ twist to the disk on the left hand side of Figure 16. ☐

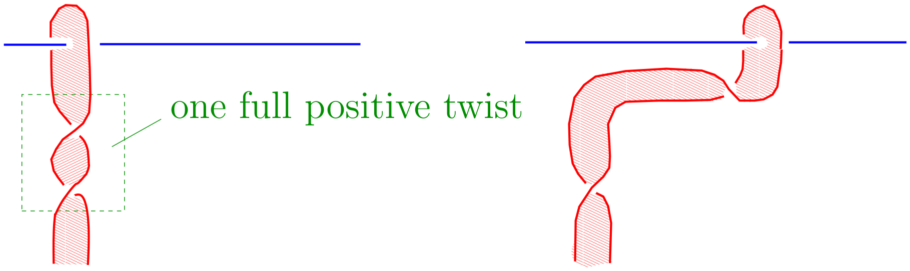


FIGURE 17. First isotopy in the proof of Lemma 6.6.

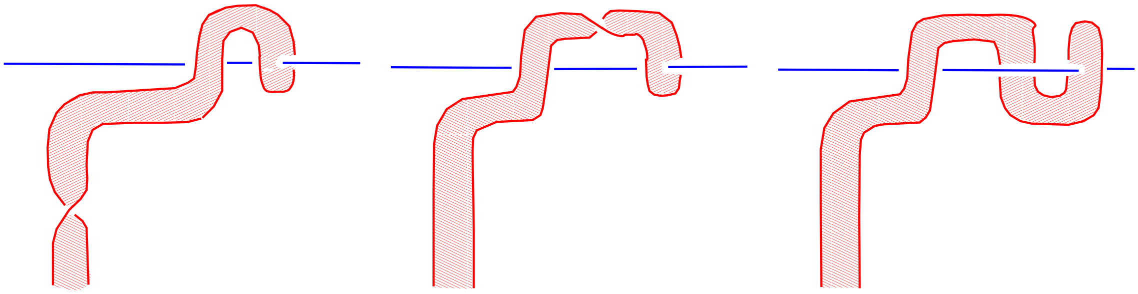


FIGURE 18. Second set of isotopies in the proof of Lemma 6.6.

6.5. **Proof of Lemma 5.5.** We are now in a position to prove Lemma 5.5.

Proof of Lemma 5.5. Let K be a knot and let $x_1, \dots, x_n$ be elements in $H_1(X(K); \mathbb{Z}[t^{\pm 1}])$. Let $p_{ij}(t) \in \mathbb{Z}[t^{\pm 1}]$, $i, j \in \{1, \dots, n\}$ be such that

$$Bl(x_i, x_j) = \frac{p_{ij}(t)}{\Delta_K(t)} \in \mathbb{Q}(t)/\mathbb{Z}[t^{\pm 1}] \text{ and } p_{ij}(t) = p_{ji}(t^{-1})$$

for any i and j. We will prove the following claim.

Claim. Let $l \in \{1, \dots, n\}$. Then there exist based nice disks $D_1, \dots, D_n$ with the following properties:

(1) the disks $D_1, \dots, D_n$ are properly arranged,
(2) for any i the based curve $c_i := \partial D_i$ represents x_i,
(3) the disks $D_{l+1}, \dots, D_n$ are disjoint,
(4) the first $2l$ entries of the arrangement of $D_1, \dots, D_n$ are

$$\{1, 1, 2, 2, \dots, l, l\},$$

(5) if for $i = 1, \dots, n$ we equip $c_i = \partial D_i$ with the framing $p_{ii}(1)$, then

$$\operatorname{lk}_t(c_i, c_j) = \frac{p_{ij}(t)}{\Delta_K(t)} \in \mathbb{Q}(t),$$

for any $i \in \{1, \dots, l\}$ and $j \in \{1, \dots, n\}$.

It is clear that the statement of the claim for $l = n$ is precisely the statement of Lemma 5.5.

We will prove the claim by induction on l. We begin with $l = 0$. First note that Lemma 6.1 allows us to find disjoint disks $D_1, \dots, D_n$ such that (2) and (3) are satisfied. The remark after Definition 6.2 shows that $D_1, \dots, D_n$ are properly arranged. Conditions (4) and (5) for $l = 0$ are empty.

Now suppose that the statement of the claim holds for $l - 1$. We thus pick based nice disks $D_1, \dots, D_n$ which satisfy the statement of the claim for $l-1$. We first apply the type R_1 move several times to the 'left most' intersection point of D_l so that the first $2(l-1)+1$ entries of the arrangement of $D_1, \dots, D_n$ are

$$\{1, 1, 2, 2, \dots, l-1, l-1, l\}.$$

We repeat this procedure with the 'right most' intersection point of D_l so that after several further type R_1 moves the first $2l$ entries of the arrangement of the resulting disks $D_1, \dots, D_n$ are

$$\{1, 1, 2, 2, \dots, l-1, l-1, l, l\}.$$

Since we applied type R_1 moves it follows that the disks are properly arranged.

For $i = 1, \dots, n$ we equip $c_i := \partial D_i$ with the framing $p_{ii}(1)$. We denote by $q_{ij}(t)$, $i, j \in \{l, \dots, n\}$ the polynomials which satisfy

$$\operatorname{lk}_t(c_i, c_j) = \frac{q_{ij}(t)}{\Delta_K(t)}.$$

Given $s \in \{1, \dots, n\}$ we now also consider the following property:

($\mathbf{5_s}$) for any $i \in \{1, \dots, l\}$ and $j \in \{1, \dots, s\}$ we have

$$\operatorname{lk}_t(c_i, c_j) = \frac{p_{ij}(t)}{\Delta_K(t)} \in \mathbb{Q}(t).$$

Note that (5_{l-1}) holds since the disks satisfy Property (5) for $l-1$ and since the p_{ij} and q_{ij} are both antisymmetric in i and j. We now proceed with two steps, first we will arrange the disks such that (5_l) holds, and then we will furthermore modify the disks such that (5_s) holds for any $s > l$.

(a) Recall that by the discussion in Section 3.1 we have

$$q_{ll}(1) = \operatorname{lk}_t(c_l, c_l)|_{t=1} = \text{framing of } c_l = p_{ll}(1).$$

It thus follows that $q_{ll}(1) - p_{ll}(1) = 0$. Note that furthermore $p_{ll}(t) = p_{ll}(t^{-1})$ by assumption and that $q_{ll}(t) = q_{ll}(t^{-1})$ by the symmetry of l. It now follows that we can write

$$q_{ll}(t) - p_{ll}(t) = \sum_{i=0}^{k} a_i(t^i + t^{-i})$$

for some $a_0, \dots, a_k \in \mathbb{Z}$ with $\sum_{i=0}^{k} a_i = 0$. Put differently, we can write

$$q_{ll}(t) - p_{ll}(t) = \sum_{i=1}^{k} b_i(t^i - t^{i-1} - t^{-(i-1)} + t^{-i})$$

for some $b_1, \dots, b_k \in \mathbb{Z}$.

Considering (6.3) and (6.4) it follows easily that for $i = 1, \dots, k$ we can now apply $|b_i|$ times an appropriate combination of a type $T(n)$ move together with either a type F_1 move or a type F_2 move to arrange that

$$\mathrm{lk}_t(c_l, c_l) = \frac{p_{ll}(t)}{\Delta_K(t)} \in \mathbb{Q}(t).$$

This concludes the proof of (5_l).

(b) We now suppose that we have disks which satisfy Properties (1)…(4) and (5_{s-1}) for some $s - 1 \geq l$. It follows from the discussion in Section 3.1 that

$$q_{sl}(1) = \mathrm{lk}_t(c_s, c_l)|_{t=1} = \mathrm{lk}(c_s, c_l) = 0 = p_{sl}(1).$$

It thus follows that $q_{sl}(1) - p_{sl}(1) = 0$. We can therefore write

$$q_{sl}(t) - p_{sl}(t) = \sum_{i=-k}^{k} b_i(t^i - t^{i-1})$$

for some $b_{-k}, \dots, b_k \in \mathbb{Z}$. We now apply the type R_2 moves several times so that the right hand piece of D_l is adjacent to the piece of D_s with the opposite orientation. Considering (6.2) and (6.4) it follows easily that for $i = -k, \dots, k$ we can now apply $|b_i|$ times an appropriate combination of a type $T(n)$ move together with either a type F_1 move or a type F_2 move to arrange that

$$\mathrm{lk}_t(c_s, c_l) = \frac{p_{sl}(t)}{\Delta_K(t)} \in \mathbb{Q}(t).$$

Finally we conclude with several type R_1 moves so that the arrangement is unchanged. Note that the resulting disks are again properly arranged.

After Steps (a) and (b) the resulting disks clearly have the required properties. This concludes the proof of the claim and thus of Lemma 5.5. □

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