Twisted Alexander polynomials, the Thurston norm and symplectic manifolds

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Abstract. We show that the degrees of twisted Alexander polynomials give lower bounds on the Thurston norm which are easy to compute and very strong at the same time. We also show that twisted Alexander polynomials are remarkably successful at detecting non–fibered manifolds and non–symplectic manifolds.

What is the Thurston norm?

Let M be a 3–manifold and $\phi \in H^1(M;\mathbb{Z})$. Loosely speaking it measures the minimal 'complexity' of an embedded submanifold dual to ϕ.

More precisely: For a surface S we define $\chi_-(S) = \sum -\chi(S_i)$ where S_i are all components of S with negative Euler characteristic. Then

$$||\phi||_M := \min\{\chi_-(S)\},$$

where we take the minimum over all properly embedded S dual to ϕ.

Let $K \subset S^3$ be a non–trivial knot. Let $\phi \in H^1(S^3 \setminus K)$ a generator. Then

$$||\phi||_T = 2\text{genus}(K) - 1.$$

Thurston showed that the function $||-||_T$ on $H^1(M;\mathbb{Z})$ defines a seminorm. In particular it can be extended to a seminorm on $H^1(M;\mathbb{R})$.

Thurston showed that the norm ball is a (possibly non–compact) polyhedron with finitely many faces.

History

1. For a knot
$$||\phi||_T = 2\text{genus}(K) - 1 \geq \deg(\Delta_K(t)) - 1.$$

2. McMullen generalized this to general 3–manifolds.

3. Vidussi used Seiberg–Witten theory to get Mc-Mullen's bounds.

4. Cochran and Harvey used non–commutative algebra to find lower bounds on the Thurston norm.

5. Turaev generalized Harvey's results.

McMullen's result

M will always be a compact, oriented, connected 3–manifold and $\partial(M)$ is either empty or a collection of tori. $\phi \in H^1(M;\mathbb{Z})$ will be assumed indivisible.

Let $\mathbb{F}$ be a (commutative) field. Since

$$H^1(M;\mathbb{Z}) \cong \text{Hom}(\pi_1(M), \langle t \rangle)$$

this defines

$$H_1(M;\mathbb{F}[t^{\pm 1}]) := H_1(C_*(M) \otimes_{\mathbb{F}[\pi_1(M)]} \mathbb{F}[t^{\pm 1}]).$$

Since $\mathbb{F}[t^{\pm 1}]$ is a PID we have

$$H_1(M;\mathbb{F}[t^{\pm 1}]) \cong \oplus \mathbb{F}[t^{\pm 1}]/p_i(t) \text{ for } p_i(t) \in \mathbb{F}[t^{\pm 1}].$$

Define $\Delta_\phi(t) = \prod p_i(t) \in \mathbb{F}[t^{\pm 1}]$.

Theorem 1 (McMullen) *If $\Delta_\phi(t) \neq 0$, then*

$$||\phi||_T \geq \deg(\Delta_\phi(t)) - 1 - b_3(M).$$

Idea of proof: Let S be dual to ϕ with minimal $\chi_-(S)$. $\Delta_\phi(t) \neq 0$ implies S connected. Now consider

$$H_1(S)\otimes\mathbb{F}[t^{\pm 1}] \xrightarrow{ti_- - i_+} H_1(M\setminus S)\otimes\mathbb{F}[t^{\pm 1}] \twoheadrightarrow H_1(M;\mathbb{F}[t^{\pm 1}])$$

and continue as for knots.

Motivating example

Consider the Conway knot $K = 11^n_{34}$. Its Alexander polynomial is one. In particular McMullen's (and the Cochran–Harvey) invariants are zero.

Gabai found a (minimal) Seifert surface of genus 3, hence $||\phi|| = 5$. We will show (algebraically) that $||\phi|| \geq 5$.

Twisted Alexander polynomials

Let $\mathbb{F}$ be a field and $\alpha : \pi_1(M) \to \mathrm{GL}(\mathbb{F}, k)$ a representation. Then

$$H_1(M; \mathbb{F}^k \otimes \mathbb{F}[t^{\pm 1}]) := H_1(C_*(\tilde{M}) \otimes_{\mathbb{F}[\pi_1(M)]} \mathbb{F}^k \otimes \mathbb{F}[t^{\pm 1}])$$

is an $\mathbb{F}[t^{\pm 1}]$–module and we can define $\Delta_\phi^\alpha(t)$ as before.

Example: Let $\alpha : \pi_1(M) \twoheadrightarrow G$ be an epimorphism to a finite group G, then $\pi_1(M)$ acts on the group ring $\mathbb{F}[G]$ and we can define $\Delta_\phi^G(t)$.

Main idea

Consider $\alpha : \pi_1(M) \twoheadrightarrow G$, G a finite group. Denote the G–cover of M by M_G and $H_1(M_G) \to H_1(M) \xrightarrow{\phi} \mathbb{Z}$ by ϕ_G.

Gabai showed

$$|G| \cdot ||\phi||_{T,M} = ||\phi_G||_{T,M_G}.$$

Define $n := n(\phi, \alpha) \in \mathbb{N}$ by

$$\phi(\mathrm{Ker}(\alpha)) = \phi(\pi_1(M_G)) = n(\phi, \alpha)\mathbb{Z} \subset \mathbb{Z}.$$

Then $\frac{\phi_G}{n} \in H^1(M_G; \mathbb{Z})$ is primitive.

Therefore

$$\begin{aligned} |G| \cdot ||\phi||_{T,M} &= ||\phi_G||_{T,M_G} \\ &= n||\tfrac{1}{n}\phi_G||_{T,M_G} \\ &\geq n\Big(\deg(\Delta_{\frac{1}{n}\phi_G,M_G}(t)) - (1 + b_3(M_G))\Big) \\ &= \deg(\Delta_{\phi_G,M_G}(t)) - n(1 + b_3(M)). \end{aligned}$$

The definitions show that

$$\Delta^G_{\phi,M}(t) = \Delta_{\phi_G,M_G}(t).$$

Main theorem

Theorem 2 (F–Taehee Kim) *Let* $\alpha : \pi_1(M) \twoheadrightarrow G$ *be an epimorphism to a finite group. If* $\Delta^G_\phi(t) \neq 0$, *then*

$$||\phi||_T \geq \frac{1}{|G|} \deg(\Delta^G_\phi(t)) - \frac{n(\phi,\alpha)}{|G|}(1 + b_3(M)).$$

1. $n(\phi,\alpha)$ can be computed efficiently.

2. $\Delta^G_\phi(t)$ can be computed using Fox calculus.

3. Similar statement holds for any representation $\pi_1(M) \to \mathrm{GL}(\mathbb{F}, k)$. These tend to be much easier to compute. For abelian representations this was shown by Turaev.

Examples

Consider again the Conway knot 11^n_{34}.

We found an epimorphism $\pi_1(S^3 \setminus K) \to A_5$. Then $n(\phi,\alpha) = 1$. We compute $\Delta_K^{A_5}(t) \in \mathbb{F}_7[t^{\pm 1}]$ to be of degree 209. Therefore

$$||\phi||_T \geq \frac{209}{60} - \frac{1}{60} > 3.4$$

Since $||\phi||_T = 2\text{genus}(K) - 1$ is odd, $||\phi||_T \geq 5$.

There are 36 knots with 12 crossings or less for which $||\phi||_T > \deg(\Delta_K(t)) - 1$. For all these knots we found representations such that our theorem gives the correct bound on $||\phi||_T$.

Let K be a knot. Denote the zero framed surgery along K by M_K. Then $H^1(M_K;\mathbb{Z}) = \mathbb{Z}$ and by Gabai $||\phi||_{T,M_K} = 2\text{genus}(K) - 2$. For all the above knots we also found representations giving the correct bound on $||\phi||_{T,M_K}$.

How good are our lower bounds?

The examples show that this approach works very well for manifolds with $b_1(M) = 1$. We therefore propose the following conjecture:

Conjecture 3 *Let M be a manifold with $b_1(M) = 1$. Then there exists an epimorphism $\alpha : \pi_1(M) \to G$ to a finite group such that $\Delta_\phi^G(t) \neq 0$ and*

$$||\phi||_T - 1 < \frac{1}{|G|}\deg(\Delta_\phi^G(t)) - \frac{n(\phi,\alpha)}{|G|}(1 + b_3(M)).$$

This does not hold for general M. For example if L is a boundary link, $M = S^3 \setminus L$, then $\Delta_\phi^G(t) = 0$ for any ϕ and any α.

Twisted Alexander norms McMullen defined a norm on $H^1(M;\mathbb{R})$ which he called *Alexander norm* and which is a lower bound for the Thurston norm. For many link complements he could show that the Alexander norm equals the Thurston norm. In particular this method made it possible to completely determine the Thurston norm ball of many links.

We define a twisted Alexander norm and show that it also gives a lower bound on the Thurston norm. This generalizes work of McMullen and Turaev. This allows us to compute the Thurston norm ball of many links for which the Alexander norm and the Thurston norm differ. For example Dunfield's link.

Fibered manifolds

Let M be a 3–manifold and $\phi \in H^1(M)$ primitive. We say (M,ϕ) fibers over S^1 if the homotopy class of maps $M \to S^1$ induced by ϕ contains a representative that is a fiber bundle over S^1.

Theorem 4 (F – Taehee Kim) *Assume* (M,ϕ) *fibers over* S^1. *Let* $\alpha : \pi_1(M) \to G$ *be an epimorphism to a finite group, then for any field* $\mathbb{F}$

$$||\phi||_T = \frac{1}{|G|}\deg(\Delta_\phi^G(t)) - \frac{n(\phi,\alpha)}{|G|}(1+b_3(M))$$

with $\Delta_\phi^G(t) \in \mathbb{F}[t^{\pm 1}]$.

This theorem has been known for a long time for the untwisted Alexander polynomial of fibered knots. McMullen, Cochran, Harvey and Turaev prove similar theorems. The theorem follows from the fact that

$$i_-, i_+ : H_1(S) \to H_1(M \setminus S)$$

are isomorphisms.

Corollary 5 *Assume* (M, ϕ) *fibers over* S^1. *Let* $\alpha : \pi_1(M) \to G$ *be an epimorphism. For a prime* p *denote* $\Delta_\phi^G(t) \in \mathbb{F}_p[t^{\pm 1}]$ *by* $\Delta_p^G(t)$. *Then*

$$\deg(\Delta_p^G(t)) = \deg(\Delta_q^G(t))$$

for any primes p *and* q.

This is a generalization of the fact that if a knot K is fibered then $\Delta_K(t)$ is monic.

Jae Choon Cha proves a similar result for knots.

Examples

There exist 52 12–crossing knots with monic Alexander polynomial such that genus$(K) = \Delta_K(t)$. This means that the Alexander polynomial and the genus can not be used to determine whether or not these knots are fibered.

Using our theorem we showed that 13 of these knots are not fibered. Using Gabai's methods Stoimenow and Hirasawa then showed that the remaining 39 12–crossing knots are fibered.

Conjecture 6 *Let $\phi \in H^1(M;\mathbb{Z})$. Then (M,ϕ) fibers over S^1 if and only if the conclusion of the theorem holds for all $\pi_1(M) \to G$, G finite group.*

************* outline proof ***************

Symplectic manifolds

Conjecture 7 (Taubes) *Let M be a 3–manifold. Then $S^1 \times M$ is symplectic if and only if (M, ϕ) fibers over S^1 for some ϕ.*

If $S^1 \times M$ is symplectic then by Kronheimer there exists a $\phi \in H^1(M; \mathbb{Z})$ such that $||\phi||_T = \deg(\Delta_\phi(t)) - 2$ and by Fintushel–Stern $\Delta_\phi(t)$ is monic if $b_1(M) = 1$. Both results are based on work of Taubes.

Put differently, the abelian invariants of M look like the invariants of a fibered manifold.

Theorem 8 (F – Stefano Vidussi) *Let M be a 3–manifold such that $S^1 \times M$ is symplectic. Then there exists a primitive $\phi \in H^1(M;\mathbb{Z})$ such that for any epimorphism $\pi_1(M) \to G$ to a finite group G we have*

$$||\phi||_T = \frac{1}{|G|}\deg(\Delta_\phi^G(t)) - 2\frac{n(\phi,\alpha)}{|G|}$$

and $\Delta_\phi^G(t) \in \mathbb{Z}[t^{\pm 1}]$ is monic.

The idea is that the cover of M corresponding to $\pi_1(M) \to G$ is symplectic again. A generalization of Fintushel–Stern's theorem to manifolds with $b_1(N) > 1$ and a little algebra relating multivariable and onevariable Alexander polynomials proves the theorem.

Put differently, the G–twisted invariants of M look like the invariants of a fibered manifold.

Examples

Let K be one of the 13 non–fibered 12–crossing knots with monic Alexander polynomial such that genus$(K) = \Delta_K(t)$. Denote the zero framed surgery along K by M_K. Then the abelian invariants are inconclusive.

Using our theorem we can show that in all cases $S^1 \times M_K$ has non–monic $\Delta_\phi^G(t)$, hence in all cases $S^1 \times M_K$ is not symplectic.

Note that the fiberedness conjecture (below) implies Taubes' conjecture.

Conjecture 9 *Let $\phi \in H^1(M;\mathbb{Z})$. Then (M,ϕ) fibers over S^1 if and only if the conclusion of the theorem holds for all $\pi_1(M) \to G$, G finite group.*