ERRATUM TO "TAUT SUTURED MANIFOLDS AND TWISTED HOMOLOGY"

STEFAN FRIEDL AND TAEHEE KIM

ABSTRACT. We give a sufficient criterion for a sutured manifold (M, γ) to be taut in terms of the twisted homology of the pair (M, R_-). This fixes an error in the proof of Theorem 1.1 in the paper [FK13] of the authors.

1. Introduction

A sutured manifold (M, γ) is a compact, connected, oriented 3-manifold M together with a set of disjoint annuli γ on ∂M which turns M naturally into a cobordism between oriented surfaces $R_- = R_-(\gamma)$ and $R_+ = R_+(\gamma)$ with boundary. We refer to Section 2.1 for the precise definition.

We say that a sutured manifold (M, γ) is *balanced* if $\chi(R_+) = \chi(R_-)$. Balanced sutured manifolds arise in many different contexts. For example 3-manifolds cut along non-separating surfaces naturally give rise to balanced sutured manifolds.

Given a sutured manifold (M, γ) we say that a surface S is *properly embedded in* (M, γ) if $\partial S = S \cap \gamma$. Furthermore, given a surface S with connected components $S_1 \cup \dots \cup S_k$ we define its *complexity* to be $\chi_-(S) = \sum_{i=1}^k \max\{-\chi(S_i), 0\}$. Following Gabai [Ga83, Definition 2.10] we say that a balanced sutured manifold (M, γ) is *taut* if M is irreducible and if R_- and R_+ have minimal complexity among all properly embedded surfaces representing the homology class $[R_-] = [R_+] \in H_2(M, \gamma; \mathbb{Z})$.

Given a representation $\alpha \colon \pi_1(M) \to \mathrm{GL}(k, \mathbb{F})$ over a field $\mathbb{F}$ we can consider the twisted homology groups $H_*^\alpha(M, R_-; \mathbb{F}^k)$. In our paper [FK13] we gave the following characterization of taut balanced sutured manifold (M, γ) in terms of the twisted homology of the pair (M, R_-).

Theorem 1.1. *Let (M, γ) be an irreducible balanced sutured manifold with $M \neq S^1 \times D^2$ and $M \neq D^3$. Then (M, γ) is taut if and only if $H_1^\alpha(M, R_-; \mathbb{C}^k) = 0$ for some unitary representation $\alpha \colon \pi_1(M) \to U(k)$.*

The "only if" direction uses the recent revolutionary work by Agol [Ag08], Liu [Liu13], Przytycki-Wise [PW12] and Wise [Wi12]. In [FK13] the proof of the "if" direction relied on the following statement, that was [FK13, Theorem 3.1].

Theorem 1.2. *Let (M, γ) be an irreducible sutured manifold such that $R_\pm$ have no disk components. Let $\alpha \colon \pi_1(M) \to \mathrm{GL}(k, \mathbb{F})$ be a representation. Then the following inequality holds:*

$$\dim H_1(M, R_-; \mathbb{F}^k) + \dim H_1(M, R_+; \mathbb{F}^k) \geq k\big(\chi_-(R_+) + \chi_-(R_-) - 2x(M, \gamma)\big).$$

Date: March 27, 2016.

2010 *Mathematics Subject Classification.* 57M27.

Key words and phrases. sutured manifold, twisted homology.

The proof relied on the combination of several Mayer–Vietoris sequences together with elementary arguments using Euler characteristics. Unfortunately we lost track of signs and on top of page 298 we wrote $\chi(M_\pm, R_\pm; \mathbb{F}^k)$ instead of $-\chi(M_\pm, R_\pm; \mathbb{F}^k)$, which invalidates the proof of Theorem 1.2 and thus also of the "if" direction of Theorem 1.1.

In the following, given a representation $\alpha\colon \pi \to \mathrm{GL}(k,\mathbb{F})$ over a field $\mathbb{F}$ with (possibly trivial) involution we denote by $\alpha^\dagger$ the representation given by $\alpha(g) = \overline{\alpha(g^{-1})^t}$. Furthermore we say that two representations $\alpha, \beta\colon \pi \to \mathrm{GL}(k,\mathbb{F})$ are *conjugate* if there exists an $A \in \mathrm{GL}(k,\mathbb{F})$ such that $\alpha(g) = A\beta(g)A^{-1}$ for all $g \in \pi$.

In this erratum we prove the following statement.

Theorem 1.3. *Let (M,γ) be an irreducible balanced sutured manifold such that $R_\pm$ have no disk components. Let $\alpha\colon \pi_1(M) \to \mathrm{GL}(k,\mathbb{F})$ be a representation over a field with (possibly trivial) involution such that α and $\alpha^\dagger$ are conjugate. If $H_1^\alpha(M, R_-; \mathbb{F}^k) = 0$, then (M,γ) is taut.*

The condition on α is satisfied by any unitary representation and also by any representation over $\mathrm{SL}(2,\mathbb{C})$, see e.g. [HSW10, Section 3] for details. In the case that $R_\pm$ have no disk components, the "if" direction of Theorem 1.1 is now a special case of Theorem 1.3. In the case that $R_\pm$ have components that are disks the "if" direction of Theorem 1.1 is proved on page 295 of [FK13]. In particular Theorem 1.1 is correct as stated. Note also that Agol–Dunfield [AD15, Section 3] have given a proof of Theorem 1.3 under the slightly stronger assumption that $H_*^\alpha(M, R_\pm; \mathbb{F}^k) = 0$. The proof we provide is based on the ideas of the proof of Agol–Dunfield.

At the moment we can not prove Theorem 1.2 as stated, and in fact we suspect that in this generality it is incorrect. For example we expect that there are counterexamples for representations α which are not conjugate to $\alpha^\dagger$.

Conventions and notations. All 3-manifolds are assumed to be oriented, compact and connected, unless it says explicitly otherwise. By $\mathbb{F}$ we will always mean a field with (possibly trivial) involution.

Acknowledgment. We are very grateful to Gerrit Herrmann for carefully reading [FK13] and for pointing out our mistake.

2. Proof of Theorem 1.3

2.1. Sutured manifolds.

A *sutured manifold* is a 3-manifold M with non-trivial boundary and together with a decomposition of its boundary

$$\partial M = (-R_-) \cup (s \times [-1,1]) \cup R_+$$

into oriented submanifolds where the following conditions hold:

(1) s consists of oriented simple closed curves,
(2) $\partial R_- = R_- \cap (s \times [-1,1]) = s \times \{-1\}$ as oriented curves,
(3) $\partial R_+ = R_+ \cap (s \times [-1,1]) = s \times \{+1\}$ as oriented curves,
(4) R_- and R_+ are disjoint.

We denote by γ the union of the annuli $s \times [-1,1]$ together with an orientation of the 'sutures' $s = s \times 0$. Note that R_+ and R_- are determined by γ, following Gabai [Ga83] we therefore usually denote a sutured manifold by (M,γ) and we write $R_\pm(\gamma) = R_\pm$.

2.2. **Preliminaries.** We recall the following well-known duality theorem (see e.g. [CF13, Theorem 2.1] and [FK06, Lemma 2.3] for a proof).

Proposition 2.1. *Let M be an oriented n-dimensional manifold and let $\partial M = A \cup B$ be a decomposition of the boundary in two submanifolds A and B such that $A \cap B = \partial A = \partial B$. Let $\alpha\colon \pi_1(M) \to \mathrm{GL}(k, \mathbb{F})$ be a representation over a field with (possibly trivial) involution. Then for any i*

$$H_i^{\alpha}(M, A; \mathbb{F}^k) \cong H_{n-i}^{\alpha^\dagger}(M, B; \mathbb{F}^k).$$

The following lemma is also well-known.

Lemma 2.2. *Let (X, Y) be a pair of spaces with X path connected and with $Y \neq \emptyset$. Let $\alpha\colon \pi_1(X) \to \mathrm{GL}(k, \mathbb{F})$ be a representation. Then*

$$H_0(Y; \mathbb{F}^k) \to H_0(X; \mathbb{F}^k)$$

is surjective.

A standard argument (see e.g. [FK06]) shows the following lemma.

Lemma 2.3. *Let X be a manifold and let $\alpha\colon \pi_1(X) \to \mathrm{GL}(k, \mathbb{F})$ be a representation. Let $b_i^{\alpha}(X; \mathbb{F}^k) := \dim_{\mathbb{F}} H_i(X; \mathbb{F}^k)$, and let*

$$\chi^{\alpha}(X) \;=\; \sum_i (-1)^i b_i^{\alpha}(X; \mathbb{F}^k).$$

Then

$$\chi^{\alpha}(X) \;=\; k \cdot \chi(X).$$

2.3. **Proof of Theorem 1.3.**

Proof. Let (M, γ) be an irreducible balanced sutured manifold such that $R_\pm$ have no disk components. Let $\alpha\colon \pi_1(M) \to \mathrm{GL}(k, \mathbb{F})$ be a representation over a field with (possibly trivial) involution such that α and $\alpha^\dagger$ are conjugate. We suppose that $H_1^{\alpha}(M, R_-; \mathbb{F}^k) = 0$.

Note that, as for any 3-manifold, we have $2\chi(M) = \chi(\partial M)$. In our case we have $\chi(\partial M) = \chi(R_-) + \chi(R_+) = 2\chi(R_-)$. Thus $\chi(M, R_-) = 0$. From $H_1^{\alpha}(M, R_-; \mathbb{F}^k) = 0$ and from Lemmas 2.2 and 2.3 it follows that $H_*^{\alpha}(M, R_-; \mathbb{F}^k) = 0$. By Proposition 2.1 we also have $H_*^{\alpha}(M, R_+; \mathbb{F}^k) = 0$.

Let S be a properly embedded surface in (M, γ) that is homologous to $[R_-] = [R_+] \in H_2(M, \gamma; \mathbb{Z})$ and which has minimal complexity among all such surfaces. We need to show that $\chi_-(S) \geq \chi_-(R_-) = \chi_-(R_+)$. As shown in [FK13, page 296] we can assume that M cut along S is the union of two disjoint (not necessarily connected) manifolds $M_\pm$ such that $R_\pm \subset \partial M_\pm$ and such that each component of $M_\pm$ contains a component of $R_\pm$. Furthermore we can assume that S has no disk or spherical components, i.e. $\chi_-(S) = -\chi(S)$.

Now we make the following observations:

(1) From $H_*^{\alpha}(M, R_\pm; \mathbb{F}^k) = 0$ it follows that the maps $H_*^{\alpha}(R_\pm; \mathbb{F}^k) \to H_*(M; \mathbb{F}^k)$ are isomorphisms, hence $b_*^{\alpha}(R_\pm; \mathbb{F}^k) = b_*^{\alpha}(M; \mathbb{F}^k)$.
(2) From $H_*^{\alpha}(M, R_\pm; \mathbb{F}^k) = 0$ it follows that the maps $H_*^{\alpha}(R_\pm; \mathbb{F}^k) \to H_*(M; \mathbb{F}^k)$ are injective. Since the inclusion $R_\pm \to M$ factors through $R_\pm \to M_\pm$ we see that the maps $H_*^{\alpha}(R_\pm; \mathbb{F}^k) \to H_*^{\alpha}(M_\pm; \mathbb{F}^k)$ are also injective.
(3) From $H_*^{\alpha}(M, R_\pm; \mathbb{F}^k) = 0$ it also follows that the maps $H_*^{\alpha}(R_\pm; \mathbb{F}^k) \to H_*(M; \mathbb{F}^k)$ are surjective. Since the inclusion $R_\pm \to M$ factors through $M_\pm \to M$ we see that the maps $H_*^{\alpha}(M_\pm; \mathbb{F}^k) \to H_*^{\alpha}(M; \mathbb{F}^k)$ are also surjective.

(4) Since each component of $M_\pm$ contains a component of $R_\pm$ we obtain from Lemma 2.2 that the maps $H_0^\alpha(R_\pm;\mathbb{F}^k)\to H_0^\alpha(M_\pm;\mathbb{F}^k)$ are surjective. By (2) the maps are in fact isomorphisms. In particular $b_0^\alpha(M_\pm;\mathbb{F}^k)=b_0^\alpha(M;\mathbb{F}^k)$ and $H_0^\alpha(M_\pm,R_\pm;\mathbb{F}^k)=0$.

(5) It follows from the long exact sequence of the triple $(M,M_\pm,R_\pm)$, excision, Proposition 2.1 and (4) that

$$H_2^\alpha(M_\pm,R_\pm;\mathbb{F}^k)\cong H_3^\alpha(M,M_\pm;\mathbb{F}^k)\cong H_3^\alpha(M_\mp,S;\mathbb{F}^k)\cong H_0^\alpha(M_\mp,R_\mp;\mathbb{F}^k)=0.$$

Thus the maps $H_2^\alpha(R_\pm;\mathbb{F}^k)\to H_2^\alpha(M_\pm;\mathbb{F}^k)$ are surjective, but we already know from (1) that they are furthemore injective. Therefore we have isomorphisms $H_2^\alpha(R_\pm;\mathbb{F}^k)\cong H_2^\alpha(M_\pm;\mathbb{F}^k)$.

Now we consider the Mayer–Vietoris sequence with twisted coefficients (see e.g. [FK06, Section 3] for details) for the decomposition of M along S:

$$\begin{array}{l} 0\longrightarrow H_2^\alpha(S;\mathbb{F}^k)\longrightarrow H_2^\alpha(M_-;\mathbb{F}^k)\oplus H_2^\alpha(M_+;\mathbb{F}^k)\longrightarrow H_2^\alpha(M;\mathbb{F}^k)\longrightarrow \\ \longrightarrow H_1^\alpha(S;\mathbb{F}^k)\longrightarrow H_1^\alpha(M_-;\mathbb{F}^k)\oplus H_1^\alpha(M_+;\mathbb{F}^k)\longrightarrow H_1^\alpha(M;\mathbb{F}^k)\longrightarrow \\ \longrightarrow H_0^\alpha(S;\mathbb{F}^k)\longrightarrow H_0^\alpha(M_-;\mathbb{F}^k)\oplus H_0^\alpha(M_+;\mathbb{F}^k)\longrightarrow H_0^\alpha(M;\mathbb{F}^k)\longrightarrow 0.\end{array}$$

It follows from (3) that the long exact sequence splits into three short exact sequences. It follows from the bottom short exact sequence and from (1) and (4) that

$$b_0^\alpha(S;\mathbb{F}^k)=b_0^\alpha(M;\mathbb{F}^k)=b_0^\alpha(R_\pm;\mathbb{F}^k).$$

Similarly it follows from the top short exact sequence, from (1) and (5) that

$$b_2^\alpha(S;\mathbb{F}^k)=b_2^\alpha(M;\mathbb{F}^k)=b_2^\alpha(M_\pm;\mathbb{F}^k)=b_2^\alpha(R_\pm;\mathbb{F}^k).$$

Furthermore it follows from the second short exact sequence, from (1) and (2) that

$$b_1^\alpha(S;\mathbb{F}^k)\geq b_1^\alpha(M;\mathbb{F}^k)=b_1^\alpha(R_\pm;\mathbb{F}^k).$$

Putting everything together we see that

$$\begin{array}{rcl} k\chi_-(S)\;=\;-k\chi(S)\;=\;-\chi^\alpha(S) &=& b_1^\alpha(S;\mathbb{F}^k)-b_0^\alpha(S;\mathbb{F}^k)-b_2^\alpha(S;\mathbb{F}^k)\\ &\geq& b_1^\alpha(R_\pm;\mathbb{F}^k)-b_0^\alpha(R_\pm;\mathbb{F}^k)-b_2^\alpha(R_\pm;\mathbb{F}^k)\\ &=& -\chi^\alpha(R_\pm)\;=\;-k\chi(R_\pm)\;=\;k\chi_-(R_\pm).\end{array}$$

□

References

[Ag08] I. Agol, *Criteria for virtual fibering*, J. Topol. **1** (2008), no. 2, 269–284.

[AD15] I. Agol and N. Dunfield, *Certifying the Thurston norm via* $\mathrm{SL}(2,\mathbb{C})$*-twisted homology*, preprint arXiv:1501.02136, to appear in the Thurston memorial conference proceedings, Princeton University Press.

[CF13] J. Cha and S. Friedl, *Twisted torsion invariants and link concordance*, Forum Math. 25 (2013), no. 3, 471–504.

[FK06] S. Friedl and T. Kim, *Thurston norm, fibered manifolds and twisted Alexander polynomials*, Topology **45** (2006), no. 6, 929–953.

[FK13] S. Friedl and T. Kim, *Taut sutured manifolds and twisted homology*, Math. Res. Letter 20 (2013), no. 02, 1-15.

[Ga83] D. Gabai, *Foliations and the topology of 3-manifolds*, J. Differential Geom. **18** (1983), no. 3, 445–503.

[HSW10] J. Hillman, D. Silver and S. Williams, *On reciprocality of twisted Alexander invariants*, Algebr. Geom. Topol. 10, No. 2, 1017–1026 (2010).

[Liu13] Y. Liu, *Virtual cubulation of nonpositively curved graph manifolds*, J. Topol. 6 (2013), no. 4, 793–822.

[PW12] P. Przytycki and D. Wise, *Mixed 3-manifolds are virtually special*, preprint arXiv:1205.6742.

[Wi12] D. Wise, *The structure of groups with a quasi-convex hierarchy*, 181 pages, preprint (2012), downloaded on April 15, 2012 from the conference webpage for the NSF-CBMS conference '3-Manifolds, Artin Groups and Cubical Geometry', from August 1–5, 2011 held at the CUNY Graduate Center.

Fakultät für Mathematik, Universität Regensburg, Germany
E-mail address: sfriedl@gmail.com

Department of Mathematics, Konkuk University, Seoul 05029, Republic of Korea
E-mail address: tkim@konkuk.ac.kr