SYMPLECTIC $\mathbf{S^1 \times N^3}$, SUBGROUP SEPARABILITY, AND VANISHING THURSTON NORM

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Abstract. Let N be a closed, oriented 3–manifold. A folklore conjecture states that $S^1 \times N$ admits a symplectic structure if and only if N admits a fibration over the circle. We will prove this conjecture in the case when N is irreducible and its fundamental group satisfies appropriate subgroup separability conditions. This statement includes 3–manifolds with vanishing Thurston norm, graph manifolds and 3–manifolds with surface subgroup separability (a condition satisfied conjecturally by all hyperbolic 3–manifolds). Our result covers, in particular, the case of 0–framed surgeries along knots of genus one. The statement follows from the proof that twisted Alexander polynomials decide fiberability for all the 3–manifolds listed above. As a corollary, it follows that twisted Alexander polynomials decide if a knot of genus one is fibered.

Dedicated to the memory of Xiao-Song Lin

1. Introduction

Let N be a 3–manifold. Throughout the paper, unless otherwise stated, we will assume that all 3–manifolds are closed, oriented and connected, all surfaces are oriented, and all homology and cohomology groups have integer coefficients.

Thurston [Th76] showed that if N admits a fibration over S^1, then $S^1 \times N$ is symplectic, i.e. it can be endowed with a closed, non–degenerate 2–form ω.

It is natural to ask whether the converse of this statement holds true. Interest on this question was motivated by Taubes' results in the study of symplectic 4-manifolds (see [Ta94, Ta95]), that gave some initial evidence to an affirmative answer to this question. We can state this problem in the following form:

Conjecture 1. *Let N be a 3–manifold. If $S^1 \times N$ is symplectic, then there exists $\phi \in H^1(N)$ such that (N, ϕ) fibers over S^1.*

Here we say that (N, ϕ) fibers over S^1 if the homotopy class of maps $N \to S^1$ determined by $\phi \in H^1(N) = [N, S^1]$ contains a representative that is a fiber bundle over S^1; in that case, we will also say that ϕ is a *fibered class.*

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In this paper, we will restrict ourselves to study Conjecture 1 for irreducible 3–manifolds. We remark that this requirement, assuming the Geometrization Conjecture, is not restrictive, as discussed in [McC01] or [Vi99]. For sake of clarity, we will remark such assumption in our statements, but it is quite possibly redundant.

In [FV06a], we suggested an approach to Conjecture 1 based on the study of twisted Alexander polynomials $\Delta^{\alpha}_{N,\phi}$ of N associated to some $\phi \in H^1(N)$ and an epimorphism α of the fundamental group of N onto a finite group G. This approach, while relying on results from Seiberg-Witten theory and symplectic topology, embeds Conjecture 1 in questions related with group theory for 3–manifold groups and the theory of covering spaces.

Precisely, in [FV06a] we showed that Conjecture 1 is implied by the following (perhaps stronger) conjecture:

Conjecture 2. *Let N be a 3–manifold and let $\phi \in H^1(N)$ be a primitive class such that for any epimorphism onto a finite group $\alpha : \pi_1(N) \to G$ the twisted Alexander polynomial $\Delta^{\alpha}_{N,\phi} \in \mathbb{Z}[t^{\pm 1}]$ is monic and* $\deg \Delta^{\alpha}_{N,\phi} = |G| \, \|\phi\|_T + 2 div \, \phi_G$. *Then (N, ϕ) fibers over S^1.*

(Notation and definitions relevant to this conjecture are presented in Sections 2 and 3.)

Specifically, Theorem 4.3 of [FV06a] asserts that the conditions on the twisted Alexander polynomial $\Delta^{\alpha}_{N,\phi}$ required in Conjecture 2 are satisfied by the Künneth component in $H^1(N)$ of the class $[\omega] \in H^2(S^1 \times N)$ of an integral symplectic form on $S^1 \times N$.

In this paper we will collect some dividends from this approach. Precisely, we will show in Theorem 4.2 that Conjecture 2 holds true for a class of 3–manifolds whose fundamental group satisfies appropriate subgroup separability conditions. For sake of exposition, we will quote here a slightly weaker version of Theorem 4.2. In order to state it, recall that a subgroup $A \subset \pi_1(N)$ of the fundamental group of a 3-manifold is *separable* if for any $g \in \pi_1(N) \setminus A$ there exists an epimorphism $\alpha : \pi_1(N) \to G$, where G is a finite group, such that $\alpha(g) \notin \alpha(A)$.

We have the following:

Theorem 1. *Let N be an irreducible 3–manifold and let $\phi \in H^1(N)$ be a primitive class such that for any epimorphism onto a finite group $\alpha : \pi_1(N) \to G$ the twisted Alexander polynomial $\Delta^{\alpha}_{N,\phi}$ is non–zero. If the subgroup carried by a connected minimal genus representative of the class Poincaré dual to ϕ is separable, then (N, ϕ) fibers over S^1.*

Manifolds with surface subgroup separability include Seifert manifolds and, perhaps more importantly, it is conjectured that all hyperbolic 3–manifolds satisfy surface subgroup separability.

We point out the somewhat surprising fact that Theorem 1 states that, in the cases under consideration, Conjecture 2 holds under the apparently much weaker assumption that the twisted Alexander polynomial is non–zero for all epimorphisms onto a finite group. Furthermore, combined with the results of [FV06a], this result amounts to the assertion that in the situation above the set of Seiberg–Witten invariants of all finite covers of $S^1 \times N$ decide the existence of a symplectic structure.

Because of their relevance, we quote some corollaries of this result.

Corollary 1. *Let N be an irreducible 3–manifold with vanishing Thurston norm. If $S^1 \times N$ is symplectic, then (N, ϕ) fibers over S^1 for all $\phi \in H^1(N) \setminus \{0\}$.*

Recall that 0–surgeries along a knot are irreducible (cf. [Ga87]). We can therefore apply Corollary 1 to the case where the 3–manifold is obtained as 0–surgery $N(K)$ of S^3 along a knot K of genus 1. Combined with [BZ67] and [Ga83] it implies that if $S^1 \times N(K)$ is symplectic, then K is a trefoil or the figure–8 knot. This answers in the affirmative, for the genus 1 case, Question 7.11 of Kronheimer in [Kr99] and in particular gives a new proof (see [FV06a] for the original proof) of the fact that if K is the genus–1 pretzel knot $(5, -3, 5)$, $S^1 \times N(K)$ is not symplectic, a question raised in [Kr98]. Note also that this corollary completely characterizes product symplectic manifolds with trivial canonical class.

Corollary 1 follows from Theorem 1 together with a result of Long and Niblo [LN91]. In Section 2 we will also provide a direct and largely self–contained proof based on, and phrased in terms of, Seiberg-Witten theory for symplectic 4-manifolds.

Theorem 1 asserts the completeness of twisted Alexander polynomials for deciding if a pair (N, ϕ) satisfying the hypothesis of this theorem is fibered. In particular, as the 0–surgery of S^3 along K is fibered if and only if K is, we deduce the following (cf. also [FV06b]).

Corollary 2. *Twisted Alexander polynomials decide if a knot of genus 1 is fibered.*

Building upon Theorem 1 and the geometric decomposition of Haken manifolds we reduce, in Theorem 5.5, the proof of Conjecture 2 to an appropriate condition of subgroup separability for the hyperbolic components. A corollary of Theorem 5.5 is worth mentioning:

Corollary 3. *Let N be an irreducible 3–manifold and let $\phi \in H^1(N)$ be a primitive class such that for any epimorphism onto a finite group $\alpha : \pi_1(N) \to G$ the twisted Alexander polynomial $\Delta^{\alpha}_{N,\phi}$ is non–zero. Denote by N' the union of its Seifert components. Then $(N', \phi|_{N'})$ fibers over S^1. In particular, if N is a graph manifold, Conjecture 2 holds true.*

The kind of techniques discussed here are applied in [FV07] to study the more general class of symplectic 4–manifolds admitting a free circle action, obtaining results similar to the ones presented in this paper.

2. A Seiberg-Witten Proof of Corollary 1

Our first goal is to give a proof of Corollary 1 that is as much as possible self–contained, and that is based on quite standard results of Seiberg-Witten theory for symplectic 4-manifolds and their 3–dimensional counterpart. In particular, this proof avoids to deal directly with the somewhat convoluted definition of Seiberg-Witten invariants for 3–manifolds with $b_1(N) = 1$.

Remember that, for all $\phi \in H^1(N)$, the *Thurston (semi)norm* of ϕ is defined by minimizing the complexity of the representatives of the class Poincaré dual to ϕ, namely

$$||\phi||_T = \min\{\chi_-(S) \,|\, S \subset N \text{ embedded surface dual to } \phi\}.$$

Here, given a surface S with connected components $S_1 \cup \cdots \cup S_k$, we define $\chi_-(S) = \sum_{i=1}^k \max\{-\chi(S_i), 0\}$. Requiring linearity on rays, this extends to a (semi)norm on $H^1(N, \mathbb{R})$ (cf. [Th86]).

We have the following straightforward observation.

Lemma 2.1. *Let N be an irreducible 3–manifold, and let $\varphi \in H^1(N, \mathbb{R})$ be a non–trivial element with $\|\varphi\|_T = 0$. Then N contains a non–separating essential torus T.*

Proof. It is well–known (see e.g. [Th86]) that if the Thurston norm vanishes for some non–trivial $\varphi \in H^1(N, \mathbb{R})$ then there exists also a non–trivial $\phi \in H^1(N)$ with $||\phi||_T = 0$. Let S be a (possibly disconnected) embedded surface dual to ϕ with $\chi_-(S) = 0$. Since cutting S along compressing disks would increase χ_- we can assume that each component of S is incompressible. The hypothesis of irreducibility excludes the case of spheres. Since S is non–trivial in homology there exists a connected component T of S that is non–separating. Clearly T satisfies the conditions of the statement. $\square$

We are ready to prove the following theorem, that obviously implies Corollary 1.

Theorem 2.2. *Let N be an irreducible 3-manifold such that $S^1 \times N$ admits a symplectic form ω whose cohomology class admits Künneth decomposition $[\omega] = [dt] \wedge \varphi + \eta$, where $\varphi \in H^1(N, \mathbb{R})$. If $\|\varphi\|_T = 0$ then any $\phi \in H^1(N, \mathbb{R}) \setminus \{0\}$ can be represented by a closed, non–degenerate 1-form; in particular, if ϕ is an integral class, it can be represented by a fibration.*

Proof. By Lemma 2.1, we can assume the existence of a non–separating essential torus T in N. In the case when T is a fiber of a fibration over S^1, N is a torus bundle. This means that N is the mapping torus of a self–diffeomorphism ψ of T^2 classified, up to isotopy, by an element of $SL(2, \mathbb{Z})$. The first cohomology group of N is identified with $\mathbb{Z} \oplus H^1(T^2)^\psi$, where $H^1(T^2)^\psi$ is the invariant part of the fiber cohomology (so that $1 \leq b_1(N) \leq 3$), and the Thurston norm vanishes on all of $H^1(N)$. Also, the entire $H^1(N) \setminus \{0\}$ is composed of fibered classes, and any nonzero element of the DeRahm cohomology is represented by a (unique up to isotopy) closed, non–degenerate 1–form (see [Th86]).

We will show now that the case where T is a fiber is the only possible one. Let us assume, by contradiction, that T is not a fiber. As N is irreducible and contains a non–separating essential torus that is not a fiber, it follows from [Ko87, Lemma 1 and Proposition 7] (cf. also [Lu88]) that the virtual Betti number of N is infinite, and in particular there exists a finite cover $p : \hat{N} \to N$ with first Betti number $b_1(\hat{N}) > 3$. (Note that this is excluded in the fibered case: any cover of a torus bundle is itself a torus bundle, hence the first Betti number, as observed before, is at most 3.) The 4–dimensional Seiberg-Witten polynomial of $S^1 \times \hat{N}$ (that has $b_+(S^1 \times \hat{N}) = b_1(\hat{N}) > 1$) coincides, with suitable identification of the orientations, with the Seiberg-Witten polynomial $SW_{\hat{N}}$ of $\hat{N}$, an element of $\mathbb{Z}[H^2(\hat{N})]$ (see e.g. [Kr99]). In particular all basic classes K_i are pull-backs, and we will identify them with elements of $H^2(\hat{N})$.

As there exists a finite covering map $p : S^1 \times \hat{N} \to S^1 \times N$, the manifold $S^1 \times \hat{N}$ is naturally endowed with the symplectic form $\hat{\omega} := p^*\omega$, which has Künneth component $\hat{\varphi} := p^*\varphi \in H^2(\hat{N}, \mathbb{R})$. In particular, the canonical class is a basic class of $S^1 \times \hat{N}$, hence is (identified with) an element of $H^2(\hat{N})$ that we denote by $\hat{K}$. We will exploit now the two main results of Seiberg-Witten theory for symplectic 4–manifolds with $b_+ > 1$, contained in [Ta94] and [Ta95]. First, the Seiberg-Witten invariant of $\hat{K}$ is equal to 1. Second, "more constraints" on the basic classes K_i implies (as $K_i \cdot \hat{\omega} = K_i \cdot \hat{\varphi}$, the products respectively in $S^1 \times \hat{N}$ and $\hat{N}$) that

$$0 \leq |K_i \cdot \hat{\varphi}| \leq \hat{K} \cdot \hat{\varphi},$$

and if the latter vanishes, $K_i = \hat{K} = 0$.

The 3–dimensional adjunction inequality for $\hat{N}$ (or, if preferred, McMullen's inequality relating the Alexander and the Thurston norm), asserts now that

$$|\hat{K} \cdot \hat{\varphi}| \leq \|\hat{\varphi}\|_T = |\deg p| \|\varphi\|_T = 0,$$

where the penultimate equality follows from [Ga87]. This, together with Taubes' "more constraints", implies that $\hat{K}$ is the only basic class and is trivial, which implies in turn that $SW(\hat{N}) = 1 \in \mathbb{Z}[H^2(\hat{N})]$. But it is well-known (see [Tu01]) that, as $b_1(\hat{N}) > 3$, the sum of the coefficients of $SW(\hat{N})$ (that equals, by [MeT96], the sum of coefficients of the Alexander polynomial of N) must vanish. We get therefore a contradiction, which completes the proof. □

Note that Theorem 2.2 covers the case of symplectic 4–manifolds of the form $S^1 \times N$ having a trivial canonical class. In fact, for these manifolds, the Thurston norm of N must vanish, as discussed in [Vi03].

3. Twisted Alexander polynomials

In this section we are going to define (twisted) Alexander polynomial associated to a representation of the fundamental group of a closed 3–manifold onto a finite group, first introduced for the case of knots in [Li01] (for a broader definition see e.g. [FK06]).

Let N be a closed 3–manifold and let $\phi : H_1(N) \to \mathbb{Z} = \langle t \rangle$ be a non–trivial homomorphism. We will think of ϕ, when useful, as an element of either $Hom(H_1(N), \mathbb{Z})$ or $H^1(N)$. Through the homomorphism ϕ, $\pi_1(N)$ acts on $\mathbb{Z}$ by translations. Furthermore let $\alpha : \pi_1(N) \to G$ be an epimorphism onto a finite group G. The composition of (α, ϕ) with the diagonal on $\pi_1(N)$ gives an action of $\pi_1(N)$ on $G \times \mathbb{Z}$, which extends to a ring homomorphism from $\mathbb{Z}[\pi_1(N)]$ to the $\mathbb{Z}[t^{\pm 1}]$–linear endomorphisms of $\mathbb{Z}[G \times \mathbb{Z}] = \mathbb{Z}[G][t^{\pm 1}]$. This induces a left $\mathbb{Z}[\pi_1(N)]$–structure on $\mathbb{Z}[G][t^{\pm 1}]$.

Now let $\tilde{N}$ be the universal cover of N. Note that $\pi_1(N)$ acts on the left on $\tilde{N}$ as group of deck transformation. The chain groups $C_*(\tilde{N})$ are in a natural way right $\mathbb{Z}[\pi_1(N)]$–modules, with the right action on $C_*(\tilde{N})$ defined via $\sigma \cdot g := g^{-1}\sigma$, for $\sigma \in C_*(\tilde{N})$. We can form by tensor product the chain complex $C_*(\tilde{N}) \otimes_{\mathbb{Z}[\pi_1(N)]} \mathbb{Z}[G][t^{\pm 1}]$. Now define $H_i(N; \mathbb{Z}[G][t^{\pm 1}]) := H_i(C_*(\tilde{N}) \otimes_{\mathbb{Z}[\pi_1(N)]} \mathbb{Z}[G][t^{\pm 1}])$, which inherit the structure of $\mathbb{Z}[t^{\pm 1}]$–modules. These module take the name of twisted Alexander modules.

Our goal is to define an invariant out of $H_1(N; \mathbb{Z}[G][t^{\pm 1}])$. First note that endowing N with a finite cell structure we can view $C_*(\tilde{N}) \otimes_{\mathbb{Z}[\pi_1(N)]} \mathbb{Z}[G][t^{\pm 1}]$ as finitely generated $\mathbb{Z}[t^{\pm 1}]$–modules. The $\mathbb{Z}[t^{\pm 1}]$–module $H_1(N; \mathbb{Z}[G][t^{\pm 1}])$ is now a finitely presented and finitely related $\mathbb{Z}[t^{\pm 1}]$–module since $\mathbb{Z}[t^{\pm 1}]$ is Noetherian. Therefore $H_1(N; \mathbb{Z}[G][t^{\pm 1}])$ has a free $\mathbb{Z}[t^{\pm 1}]$–resolution

$$\mathbb{Z}[t^{\pm 1}]^r \xrightarrow{S} \mathbb{Z}[t^{\pm 1}]^s \to H_1(N; \mathbb{Z}[G][t^{\pm 1}]) \to 0$$

of finite $\mathbb{Z}[t^{\pm 1}]$–modules. Without loss of generality we can assume that $r \geq s$.

Definition 3.1. The *twisted Alexander polynomial* of (N, α, ϕ) is defined to be the order of the $\mathbb{Z}[t^{\pm 1}]$–module $H_1(N; \mathbb{Z}[G][t^{\pm 1}])$, i.e. the greatest common divisor of the $s \times s$ minors of the $s \times r$–matrix S. It is denoted by $\Delta^{\alpha}_{N,\phi} \in \mathbb{Z}[t^{\pm 1}]$, and it is well–defined up to units of $\mathbb{Z}[t^{\pm 1}]$.

Note that this definition only makes sense since $\mathbb{Z}[t^{\pm 1}]$ is a UFD. It is well–known (see e.g. [FV06a]) that, up to sign, there is a unique choice of $\Delta^{\alpha}_{N,\phi} \in \mathbb{Z}[t^{\pm 1}]$ symmetric under the natural involution of $\mathbb{Z}[t^{\pm 1}]$.

If G is the trivial group we will drop α from the notation. With these conventions, $\Delta_{N,\phi} \in \mathbb{Z}[t^{\pm 1}]$ is the ordinary 1–variable Alexander polynomial associated to ϕ.

Remark. The 1-variable twisted Alexander polynomial defined above can also be described in as the specialization of a multivariable twisted Alexander polynomial taking values in $\mathbb{Z}[H]$, where H is the free part of $H_1(N)$. This polynomial, in turn, is related with the ordinary Alexander polynomial of the G–cover N_G of N and then, thanks to [MeT96], to the Seiberg-Witten invariants of $S^1 \times N_G$. These observations constitute the starting point of the connection between Conjecture 2 and Conjecture 1. See [FV06a] for details.

4. Proof of the Main Theorem

We will now discuss our main result. Before turning to the statement, we recall the definition of subgroup separability.

Definition 4.1. Let π be a group and $A \subset \pi$ a subgroup. We say that A is *separable* if for any $g \in \pi \setminus A$ there exists a finite group G and an epimorphism $\alpha : \pi \to G$ such that $\alpha(g) \notin \alpha(A)$. A group π is called *subgroup separable* (respectively *surface subgroup separable*) if any finitely generated subgroup $A \subset \pi$ (respectively any surface group $A \subset \pi$) is separable in π.

Subgroup separable groups are often also called locally extended residually finite (LERF).

We are in position now to present our main result.

Theorem 4.2. *Let N be an irreducible 3–manifold and let $\phi \in H^1(N)$ a primitive class such that for any epimorphism onto a finite group $\alpha : \pi_1(N) \to G$ the twisted Alexander polynomial $\Delta^\alpha_{N,\phi}$ is non–zero. If ϕ is dual to a connected incompressible embedded surface S such that $\pi_1(S)$ is separable in $\pi_1(N)$, then (N, ϕ) fibers over S^1.*

We point out that the condition that S is connected is not restrictive. Indeed, McMullen [McM02] showed that if $\phi \in H^1(N)$ is primitive and $\Delta_{N,\phi} \neq 0$, then ϕ is dual to a connected incompressible surface.

For the proof of Theorem 4.2 we will make use of the following standard result:

Lemma 4.3. *Let X be a connected space, $\alpha : \pi_1(X) \to G$ a group homomorphism such that $G/Im(\alpha)$ is finite. Then*

$$H_0(X; \mathbb{Z}[G]) \cong \mathbb{Z}^{|G/Im(\alpha)|}.$$

In fact, the set of components of the (possibly disconnected) finite cover of X defined by α gives a basis for the $\mathbb{Z}$–module $H_0(X; \mathbb{Z}[G])$ via the Eckmann–Shapiro lemma.

Also, we will need two well–known properties of twisted Alexander modules.

Lemma 4.4. *Let N be a 3–manifold, $\phi \in H^1(N)$ primitive and $\alpha : \pi_1(N) \to G$ an epimorphism to a finite group. Then*

1. *$\Delta^\alpha_{N,\phi} \neq 0$ if and only if $H_1(N; \mathbb{Z}[G][t^{\pm 1}])$ is $\mathbb{Z}[t^{\pm 1}]$–torsion.*
2. *If $X \subset N$ is a subset, then $rank_{\mathbb{Z}[t^{\pm 1}]}(H_0(X; \mathbb{Z}[G][t^{\pm 1}])) = 0$ if and only if ϕ is non–trivial on $H_1(X)$. Furthermore, if ϕ vanishes on $H_1(X)$, then*

$$rank_{\mathbb{Z}[t^{\pm 1}]}(H_0(X; \mathbb{Z}[G][t^{\pm 1}])) = rank_{\mathbb{Z}}(H_0(X; \mathbb{Z}[G])) = |G|/|\alpha(\pi_1(X))|.$$

Proof. The first part is a well–known property of orders. For the second part note that if ϕ is non–trivial on $H_1(X)$, then it follows from Lemma 4.3 applied to $\alpha \times \phi : \pi_1(X) \to \mathbb{Z} \times G$ that $H_0(X; \mathbb{Z}[G][t^{\pm 1}])$ has finite rank over $\mathbb{Z}$. In particular $H_0(X; \mathbb{Z}[G][t^{\pm 1}])$ is $\mathbb{Z}[t^{\pm 1}]$–torsion. On the other hand, if ϕ is trivial on $H_1(X)$, then $H_0(X; \mathbb{Z}[G][t^{\pm 1}]) = H_0(X; \mathbb{Z}[G]) \otimes \mathbb{Z}[t^{\pm 1}]$, and the lemma follows from Lemma 4.3. □

We are now ready to prove Theorem 4.2.

Proof of Theorem 4.2. Let $S \subset N$ be a connected incompressible embedded surface dual to ϕ such that $\pi_1(S)$ is separable in $\pi_1(N)$. Let $M := N \setminus \nu S$, and denote by $\iota_\pm$ the positive and negative inclusions of S into M. Since S is incompressible, $\iota_\pm : \pi_1(S) \to \pi_1(M)$ is injective, by Dehn's Lemma. Furthermore it is well–known that $\pi_1(M) \to \pi_1(N)$ is injective as well. Denote $A := \pi_1(S)$, $B := \pi_1(M)$: the previous observations allow us to consider A (with either inclusion) and B as subgroups of $\pi_1(N)$. Given an epimorphism $\alpha : \pi_1(N) \to G$, this determines, by restriction, an homomorphism from these subgroups to G, that we will denote by α as well.

We will prove that the hypothesis on ϕ imply that either inclusion $A \subset B$ is in fact an isomorphism. Assume, by contradiction, that there exist an element $g \in B \setminus A$. Since by assumption $\pi_1(S)$ is separable, there exist a finite group G and an epimorphism $\alpha : \pi_1(N) \to G$ such that $\alpha(g) \notin \alpha(A)$; in particular this implies $|\alpha(A)| < |\alpha(B)|$.

We claim that this contradicts the hypothesis that $\Delta^{\alpha}_{N,\phi}$ is non–zero. In fact, using the homomorphisms $\alpha : A \to G$, $\alpha : B \to G$ we define twisted homology modules for S and M. These are related with the twisted Alexander modules of N by the exact sequence

$$\begin{array}{ccccccccc} & & & & \dots & \to & H_1(N; \mathbb{Z}[G][t^{\pm 1}]) & & \\ \to & H_0(S; \mathbb{Z}[G]) \otimes \mathbb{Z}[t^{\pm 1}] & \xrightarrow{t\iota_+ - \iota_-} & H_0(M; \mathbb{Z}[G]) \otimes \mathbb{Z}[t^{\pm 1}] & & \to & H_0(N; \mathbb{Z}[G][t^{\pm 1}]) & \to & 0. \end{array}$$

(We refer to [FK06] for details.) Concerning the terms of the previous sequence, Lemma 4.4 implies that $H_i(N; \mathbb{Z}[G][t^{\pm 1}]) \otimes \mathbb{Q}(t) = 0$ for $i = 0, 1$.

We are in position now to reach the contradiction. Tensoring the above exact sequence with $\mathbb{Q}(t)$ we now see that

$$\begin{array}{rcl} \operatorname{rank}_{\mathbb{Z}}(H_0(S; \mathbb{Z}[G])) & = & \operatorname{rank}_{\mathbb{Q}(t)}(H_0(S; \mathbb{Z}[G]) \otimes \mathbb{Q}(t)) \\ & = & \operatorname{rank}_{\mathbb{Q}(t)}(H_0(M; \mathbb{Z}[G]) \otimes \mathbb{Q}(t)) = \operatorname{rank}_{\mathbb{Z}}(H_0(M; \mathbb{Z}[G])). \end{array}$$

It then follows, applying Lemma 4.3, that

$$\frac{|G|}{|\alpha(A)|} = \operatorname{rank}_{\mathbb{Z}}(H_0(S; \mathbb{Z}[G])) = \operatorname{rank}_{\mathbb{Z}}(H_0(M; \mathbb{Z}[G])) = \frac{|G|}{|\alpha(B)|},$$

which contradicts $|\alpha(A)| < |\alpha(B)|$, hence (reverting to the standard notation) the maps $\iota_\pm : \pi_1(S) \to \pi_1(M)$ are isomorphisms.

Completing the proof is now a standard exercise: note that $\mathrm{Ker}\{\pi_1(N) \to \mathbb{Z}\}$ is an infinite amalgamated product

$$\mathrm{Ker}\{\pi_1(N) \to \mathbb{Z}\} = \dots \pi_1(M) *_{\pi_1(S)} *\pi_1(M) *_{\pi_1(S)} *\pi_1(M) \dots,$$

where the maps $\pi_1(S) \to \pi_1(M)$ are given by $\iota_\pm$. Since $\iota_\pm$ are isomorphisms it follows immediately that $\mathrm{Ker}\{\pi_1(N) \to \mathbb{Z}\} \cong \pi_1(M)$, in particular $\mathrm{Ker}\{\pi_1(N) \to \mathbb{Z}\}$ is finitely generated. Since N is irreducible it now follows from Stallings' theorem [St62] that (N, ϕ) fibers over S^1. $\square$

Remark. (1) Scott [Sc78] showed that the fundamental groups of surfaces and Seifert manifolds are subgroup separable. On the other hand it is known (cf. [BKS87] and [NW01]) that fundamental groups of graph manifolds and knot complements are in general not subgroup separable. It is not known whether they are surface subgroup separable or not. Thurston [Th82, p. 380] asked whether fundamental groups of hyperbolic 3–manifolds are subgroup separable, and various results in this direction are known (see e.g. [LR05, Gi99]). We refer to [LR05] for more information on 3–manifolds and subgroup separability.
(2) A connected Thurston norm minimizing embedded surface S is incompressible, but the converse is in general not true. Since there exist separable incompressible Seifert surfaces for hyperbolic knots which are not of minimal genus (cf. [AS04]) it might be useful to include, as in the statement above, non–Thurston norm minimizing surfaces. Note that in the case that (N, ϕ) fibers over S^1 an incompressible surface dual to ϕ is Thurston norm minimizing and unique up to isotopy.
(3) The proof of Theorem 4.2 carries over to the case that N has toroidal boundary.

Whereas subgroup separability is in the general case not completely understood, the following result of Long and Niblo has particular relevance for us (see [LN91]).

Theorem 4.5. *(Long-Niblo) Let N be a Haken manifold, and $T \subset N$ an embedded incompressible torus. Then $\pi_1(T)$ is separable in $\pi_1(N)$.*

This result has been further generalized by Hamilton, who proved in [Ha01] that any abelian subgroup is separable in the fundamental group of Haken manifolds.

The following proposition, therefore, completes the claim that Corollary 1 follows from Theorem 4.2:

Proposition 4.6. *Let N be an irreducible 3–manifold and $\phi \in H^1(N)$ a primitive class with $||\phi||_T = 0$, such that for any epimorphism onto a finite group $\alpha : \pi_1(N) \to G$ the twisted Alexander polynomial $\Delta^\alpha_{N,\phi}$ is non–zero. Then (N, ϕ) fibers over S^1.*

Proof. As observed above, the assumption $\Delta_{N,\phi} \neq 0$ implies that we can find a connected Thurston norm minimizing embedded surface $S \subset N$ dual to ϕ. Clearly S is a torus since N is closed, irreducible and $||\phi||_T = 0$. The subgroup of $\pi_1(N)$ carried by S is separable, hence the statement follows from Theorem 4.2. $\square$

5. The JSJ decomposition

As pointed out in the previous section, 3–manifolds do not satisfy, in general, subgroup separability, and it is not clear whether the weaker condition of surface subgroup separability required in the hypothesis of Theorem 4.2 holds or not. Instead, there is more expectation that some condition of subgroup separability is satisfied by hyperbolic 3–manifolds. The goal of this section is to use the Geometrization Theorem for Haken manifolds and the results of the previous sections to reduce the proof of Conjecture 2 to a suitable condition of surface subgroup separability for hyperbolic manifolds. For manifolds not already geometric, this is a more direct, and perhaps more realistic requirement, than the hypothesis of Theorem 4.2.

We will start by recalling some standard definitions and results. (For notation and general results on 3–manifold topology we refer to [Bo02] and [He76].)

Let $T_1, \dots, T_s \subset N$ be a family of incompressible embedded tori. We call $\{T_1, \dots, T_s\}$ a *torus decomposition* if the (closures of the) components of N cut along $\cup_{j=1}^s T_j$ are either Seifert manifolds or they are simple. (Here simple means that any incompressible properly embedded torus or annulus is boundary parallel.)

We call $\{T_1, \dots, T_s\}$ a *JSJ decomposition* if any proper subfamily fails to satisfy the conditions above. By the work of Jaco–Shalen and Johannson a JSJ decomposition is unique up to isotopy. The Geometrization Theorem for Haken manifolds asserts that the interior of the simple factors of the decomposition admits a hyperbolic metric of finite volume.

We are interested in the JSJ decomposition because of the following theorem.

Theorem 5.1. [EN85, Theorem 4.2] *Let N be a 3–manifold, $\phi \in H^1(N)$ and $\{T_1, \dots, T_s\}$ a JSJ decomposition. Then (N, ϕ) fibers over S^1 if and only if $(N_i, \phi|_{N_i})$ fibers over S^1 for every component N_i of N cut along $\cup_{j=1}^s T_i$.*

This result reduces the problem of fiberability of a 3–manifold to the study of its JSJ components. It is natural, within our approach, to assume that a conjecture similar to Conjecture 2 holds for manifolds with toroidal boundary. However note that even if for some nonfibered factor $N_i \subset N$ an epimorphism $\alpha : \pi_1(N_i) \to G$ detects nonfiberedness, there is no reason why that epimorphism should extend to $\pi_1(N)$. This issue will not cause particular difficulty for the Seifert components but it is more delicate for the hyperbolic components. This is analogous to the problem faced in proving residual finiteness for a Haken manifolds starting from the residual finiteness of its JSJ components and, in fact, our strategy employs the pattern of [He87].

Before we state the next theorem we recall that given a torus T and $\phi \in H^1(T)$, (T, ϕ) fibers over S^1 if and only if $\phi \neq 0$.

Theorem 5.2. *Let N be an irreducible 3–manifold and $\phi \in H^1(N)$ a primitive class. Assume that for any epimorphism $\alpha : \pi_1(N) \to G$ onto a finite group G the twisted Alexander polynomial $\Delta^{\alpha}_{N,\phi} \in \mathbb{Z}[t^{\pm 1}]$ is non–zero. Let $T \subset N$ be an incompressible* *embedded torus. Then either $\phi|_T \in H^1(T)$ is non–zero, or (N, ϕ) fibers over S^1 with fiber T.*

Note that this gives in particular another proof that the examples in the proof of [FV06a, Theorem 5.1] are not symplectic.

Proof. We start by considering the case where T is non–separating. Assume that $\phi|_T = 0$. Denote the result of cutting N along T by M. Consider the Mayer–Vietoris sequence

$$\begin{array}{ccccccc} & & & & & & H_1(N; \mathbb{Z}[t^{\pm 1}]) \\ \to & H_0(T; \mathbb{Z}[t^{\pm 1}]) & \xrightarrow{\iota_- - \iota_+} & H_0(M; \mathbb{Z}[t^{\pm 1}]) & \to & & H_0(N; \mathbb{Z}[t^{\pm 1}]). \end{array}$$

As $\phi|_T = 0$ and $\Delta_{N,\phi} \neq 0$, Lemma 4.4 implies respectively that $H_0(T; \mathbb{Z}[t^{\pm 1}])$ is a free $\mathbb{Z}[t^{\pm 1}]$–module and that $H_i(N; \mathbb{Z}[t^{\pm 1}])$ are $\mathbb{Z}[t^{\pm 1}]$–torsion modules. Lemma 4.4 requires then that $\phi|_M = 0$. This implies that T is dual to ϕ and it follows from Proposition 4.6 that (N, ϕ) fibers over S^1.

Now assume that T is separating. We will show that $\phi|_T$ cannot be zero. Denote the two components of N cut along T by M_1 and M_2. Since ϕ is non–zero and the map $H_1(M_1) \oplus H_1(M_2) \to H_1(N)$ is an epimorphism it follows that $\phi|_{M_i}$ is non–zero for at least one i. Furthermore an almost identical argument as above shows that if $\phi|_{M_i}, i = 1, 2$ were both non–zero, then $\phi|_T$ would be non–zero as well. So we can now assume that $\phi|_{M_i}$ is non–zero for $i = 1$ and zero for $i = 2$.

Since the kernel of $H_1(T) \to H_1(M_2)$ is nontrivial by Lefschetz duality, and since $\pi_1(T) = H_1(T)$ it follows that the injective map $\pi_1(T) \to \pi_1(M_2)$ is not an isomorphism. We can therefore find $g \in \pi_1(M_2) \setminus \pi_1(T)$. Since T is incompressible we can view $\pi_1(T)$ and $\pi_1(M_2)$ as subgroups of $\pi_1(N)$. By Theorem 4.5 we can now find an epimorphism $\alpha : \pi_1(N) \to G$ onto a finite group G such that $|\alpha(\pi_1(T))| < |\alpha(\pi_1(M_2))|$. In particular $\mathrm{rank}_{\mathbb{Z}}(H_0(T; \mathbb{Z}[G])) > \mathrm{rank}_{\mathbb{Z}}(H_0(M_2; \mathbb{Z}[G]))$. Now consider the following Mayer–Vietoris sequence

$$\begin{array}{ccccccc} & & & & & & H_1(N; \mathbb{Z}[G][t^{\pm 1}]) \\ \to & H_0(T; \mathbb{Z}[G][t^{\pm 1}]) & \xrightarrow{\iota_- - \iota_+} & \oplus_{i=1}^2 H_0(M_i; \mathbb{Z}[G][t^{\pm 1}]) & \to & & H_0(N; \mathbb{Z}[G][t^{\pm 1}]). \end{array}$$

It follows from Lemma 4.4 that, if $\phi|_T = 0$, $H_0(T; \mathbb{Z}[G][t^{\pm 1}])$ and $H_0(M_2; \mathbb{Z}[G][t^{\pm 1}])$ are free $\mathbb{Z}[t^{\pm 1}]$–modules of ranks $\mathrm{rank}_{\mathbb{Z}}(H_0(T; \mathbb{Z}[G]))$ and $\mathrm{rank}_{\mathbb{Z}}(H_0(M_2; \mathbb{Z}[G]))$. Furthermore, as $\phi|_{M_1} \neq 0$ and $\Delta^{\alpha}_{N,\phi} \neq 0$, all other modules are $\mathbb{Z}[t^{\pm 1}]$–torsion modules. However this condition cannot hold since $\mathrm{rank}_{\mathbb{Z}}(H_0(T; \mathbb{Z}[G])) > \mathrm{rank}_{\mathbb{Z}}(H_0(M_2; \mathbb{Z}[G]))$, hence $\phi|_T \neq 0$. □

In view of Proposition 4.6, we will restrict our interest to the classes $\phi \in H^1(N)$ with strictly positive Thurston norm. We can apply Theorem 5.2 to the tori of the JSJ decomposition to prove the following result.

Proposition 5.3. *Let N be an irreducible 3–manifold and $\phi \in H^1(N)$ primitive with strictly positive Thurston norm. Let $T_1, \ldots, T_s \subset N$ be the JSJ decomposition. Denote* *the components of N cut along $\cup_{j=1}^{s} T_i$ by $N_1, \dots, N_r$, and let $\phi_i = \phi|_{N_i}$. Assume that for any epimorphism $\alpha : \pi_1(N) \to G$ onto a finite group G the twisted Alexander polynomial $\Delta_{N,\phi}^{\alpha} \in \mathbb{Z}[t^{\pm 1}]$ is non–zero. Then for for any epimorphism $\alpha : \pi_1(N) \to G$ onto a finite group G and for any $i \in \{1, \dots, r\}$ the twisted Alexander polynomial $\Delta_{N_i,\phi_i}^{\alpha} \in \mathbb{Z}[t^{\pm 1}]$ is non–zero.*

Proof. We can apply Theorem 5.2 to conclude that ϕ is non–trivial when restricted to $T_i, i = 1, \dots, s$. Therefore, for any epimorphism $\alpha : \pi_1(N) \to G$ onto a finite group G the twisted Alexander module $H_1(T_i; \mathbb{Z}[G][t^{\pm 1}])$ is $\mathbb{Z}[t^{\pm 1}]$–torsion for all $i = 1, \dots, s$.

Now consider the Mayer–Vietoris exact sequence

$$\to \bigoplus_{j=1}^{s} H_1(T_j; \mathbb{Z}[G][t^{\pm 1}]) \to \bigoplus_{i=1}^{r} H_1(N_i; \mathbb{Z}[G][t^{\pm 1}]) \to H_1(N; \mathbb{Z}[G][t^{\pm 1}]) \to \dots .$$

Since $H_1(T_j; \mathbb{Z}[G][t^{\pm 1}]), j = 1, \dots, s$ and $H_1(N; \mathbb{Z}[G][t^{\pm 1}])$ are $\mathbb{Z}[t^{\pm 1}]$–torsion, it follows that $H_1(N_i; \mathbb{Z}[G][t^{\pm 1}]), i = 1, \dots, r$ are $\mathbb{Z}[t^{\pm 1}]$–torsion.

This concludes the proof of the proposition. □

Remark. Note that, along the previous lines, it is possible to prove a statement analogous to Proposition 5.3 asserting that, if $\Delta_{N,\phi}^{\alpha}$ is monic, so are the $\Delta_{N_i,\phi_i}^{\alpha}$.

Theorem 5.2 will allow us to control completely the Seifert components of the JSJ decomposition of a 3–manifold N that satisfies the hypothesis of the Theorem and, under suitable assumption of separability, the hyperbolic components.

Before formulating this assumption, we need to recall first some results and definitions.

First, we will use the classification of incompressible surfaces in Seifert manifolds. We recall the following theorem ([Ja80, Theorem VI.34] and [Hat, Proposition 1.11]).

Theorem 5.4. *Let N be a (compact, orientable) Seifert manifold. If Σ is a connected (orientable) incompressible surface in N, then one of the following holds:*

1. *Σ is a vertical annulus or torus.*
2. *Σ is a horizontal non–separating surface fibering N as a surface bundle over S^1.*
3. *Σ is a boundary–parallel annulus.*
4. *Σ is a horizontal surface separating N in two twisted I–bundles over a compact surface.*

(Here, a surface in N is called *vertical* (resp. *horizontal*) if it is the union of fibers (resp. transverse to all fibers) of some Seifert fibration of N.)

Second, observe that given a number n the group $\mathbb{Z} \oplus \mathbb{Z}$ has precisely one characteristic subgroup of index n^2, namely $n(\mathbb{Z} \oplus \mathbb{Z})$. Now let N be a 3–manifold with empty or toroidal boundary. Given a prime p we say that $K \subset \pi_1(N)$ is *p–boundary characteristic* if for any component T of ∂M the group $K \cap \pi_1(T)$ is the characteristic subgroup of $\pi_1(T)$ of order p^2. We denote by $C_p(N)$ the set of all finite index subgroups of $\pi_1(N)$ which are p–boundary characteristic. (If N has empty boundary, this is just the set of finite index subgroups.)

We have the following.

Theorem 5.5. *Let N be an irreducible 3–manifold and let $\phi \in H^1(N)$ be a primitive class such that for any epimorphism onto a finite group $\alpha : \pi_1(N) \to G$ the twisted Alexander polynomial $\Delta^{\alpha}_{N,\phi}$ is non–zero. Then the following hold:*

1. *$(N', \phi|_{N'})$ fibers over S^1, where N' is the union of the Seifert components.*
2. *Assume that any hyperbolic component N_i satisfies the condition that, for an incompressible surface $S_i \subset N_i$ Poincaré dual to $\phi|_{N_i}$ and any $g \in \pi_1(N_i) \setminus \pi_1(S_i)$, there are infinitely many primes p such that there exist an epimorphism $\pi_1(N_i) \to G$ onto a finite group G with $\alpha(g) \notin \alpha(\pi_1(S_i))$ and $Ker(\alpha) \in C_p(N_i)$. Then (N, ϕ) fibers over S^1.*

Proof. If N has a trivial JSJ decomposition, then N is either a Seifert fibered manifold, or is hyperbolic. In this case the result follows from Theorem 4.2 because Seifert manifolds satisfy subgroup separability, respectively because of the separability assumption of the hypothesis.

We can assume therefore that N has a nontrivial JSJ decomposition and ϕ has strictly positive Thurston norm. In light of Theorem 5.1, we want to show that for each component N_i, the pair (N_i, ϕ_i) is fibered. First note that it follows from Proposition 5.3 that ϕ_i is nontrivial. However, ϕ_i is not necessarily primitive, even if ϕ is. Denote by φ_i a primitive class in $H^1(N_i)$ with the property that $\phi_i = n\varphi_i$. Clearly (N_i, ϕ_i) fibers over S^1 if and only if (N_i, φ_i) fibers over S^1. Since $\Delta_{N_i,n\varphi_i}(t) = \Delta_{N_i,\varphi_i}(t^n)$, it follows that $\Delta_{N_i,\varphi_i} \neq 0$ so that we can find in N_i a connected minimal genus representative Σ_i of the class Poincaré dual to φ_i.

At this point, we will treat separately Seifert and hyperbolic components.

Let N_i be a Seifert component. For any component T of ∂N_i, the intersection $\Sigma_i \cap T$ is homologically essential, as $\phi|_T$ is a multiple of its Poincaré dual, and the former is nonzero by Theorem 5.2. The knowledge of incompressible surfaces in Seifert manifolds contained in Theorem 5.4 will allow us quite easily to show that Σ_i is a fiber.

As the intersection of Σ_i with the boundary components is homologically essential, Σ_i can satisfy only Case (1) and Case (2) of Theorem 5.4, and we claim that also in Case (1) Σ_i fibers N_i over S^1. This follows by applying *verbatim* the proof of Theorem 4.2 to the surface Σ_i in N_i, using the condition that $\Delta^{\alpha}_{N_i,\varphi_i} \neq 0$ and the fact that the isomorphic image of the abelian group $\pi_1(\Sigma_i) = \mathbb{Z} \subset \pi_1(N_i) \subset \pi_1(N)$ is *separable in* $\pi_1(N)$, by the aforementioned result of Hamilton ([Ha01]). Together with Theorem 5.1 this concludes the proof of (1).

Now let N_i be a hyperbolic component. We write $N_i^c = \cup_{j \neq i} N_j$. It follows from [He87, Lemma 4.1] that for all but finitely many prime numbers p there exist $K_j \in C_p(N_j)$ for all $j \neq i$. It then follows from [He87, Theorem 2.2] that in fact for all but finitely many prime numbers p there exists $K' \in C_p(N_i^c)$.

Assume, by contradiction, that Σ_i is not a fiber of N_i. By the above remark and the separability hypothesis, we can find a prime number p such that there exists $K' \in C_p(N_i^c)$ and such that there exists an epimorphism $\pi_1(N_i) \to G$ onto a finite group G with $|\alpha(\pi_1(\Sigma_i)| < |\alpha(\pi_1(N_i))|$ and $\mathrm{Ker}(\alpha) \in C_p(N_i)$. Applying again [He87, Theorem 2.2] we conclude that there exists a finite index subgroup $K \subset \pi_1(N)$ such that $K \cap \pi_1(N_i) \subset \mathrm{Ker}(\alpha)$. We can and will assume that K is normal. Now consider the epimorphism $\beta : \pi_1(N) \to H = \pi_1(N)/K$. Its restriction to $\pi_1(N_i)$ fits into the following commutative diagram

$$\begin{tikzcd} \pi_1(N_i) \arrow[r] \arrow[dr] & \pi_1(N_i)/(K \cap \pi_1(N_i)) \subset \pi_1(N)/K = H \arrow[d] \\ & \pi_1(N_i)/\mathrm{Ker}(\alpha) = G. \end{tikzcd}$$

Since $|\alpha(\pi_1(\Sigma_i))| < |\alpha(\pi_1(N_i))|$ it follows that $|\beta(\pi_1(\Sigma_i))| < |\beta(\pi_1(N_i))|$. Following again the argument in the proof of Theorem 4.2 we deduce that $\Delta^{\beta}_{N_i,\varphi_i} = 0$. But this is a contradiction to Proposition 5.3.

This shows that (N_i, ϕ_i) fibers over S^1. Together with (1) and Theorem 5.1 this concludes the proof of (2). $\square$

If N has no hyperbolic components, we have the following:

Corollary 5.6. *If N is a graph manifold (i.e. all components in the JSJ decomposition are Seifert manifolds), then Conjecture 1 holds for N.*

This corollary is particularly significant in light of [NW01], that asserts that a graph manifold satisfies subgroup separability if and only if it has trivial JSJ decomposition.

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