Splitting numbers of links

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Abstract. The splitting number of a link is the minimal number of crossing changes between different components required, on any diagram, to convert it to a split link. We introduce new techniques to compute the splitting number, involving covering links and Alexander invariants. As an application, we completely determine the splitting numbers of links with 9 or fewer crossings. Also, with these techniques, we either reprove or improve upon the lower bounds for splitting numbers of links computed by J. Batson and C. Seed using Khovanov homology.

1. Introduction

Any link in S^3 can be converted to the split union of its component knots by a sequence of crossing changes between different components. Following J. Batson and C. Seed [BS13], we define the *splitting number* of a link L, denoted by $\operatorname{sp}(L)$, as the minimal number of crossing changes in such a sequence.

We present two new techniques for obtaining lower bounds for the splitting number. The first approach uses covering links, and the second method arises from the multivariable Alexander polynomial of a link.

Our general covering link theorem is stated as Theorem 3.2. Theorem 1.1 below gives a special case which applies to 2-component links L with unknotted components and odd linking number. Note that the splitting number is equal to the linking number modulo two. If we take the 2-fold branched cover of S^3 with branching set a component of L, then the preimage of the other component is a knot in S^3, which we call a *2-fold covering knot* of L. Also recall that the *slice genus* of a knot K in S^3 is defined to be the minimal genus of a surface F smoothly embedded in D^4 such that $\partial(D^4, F) = (S^3, K)$.

Theorem 1.1. *Suppose L is a 2-component link with unknotted components. If $\operatorname{sp}(L) = 2k+1$, then any 2-fold covering knot of L has slice genus at most k.*

Theorem 3.2 also has other useful consequences, given in Corollaries 3.5 and 3.6, dealing with the case of even linking numbers, for example. Three covering link arguments which use these corollaries are given in Section 7.

Our Alexander polynomial method is efficacious for two component links when the linking number is one and at least one component is knotted. By looking at the effect of a crossing change on the Alexander module we obtain the following result:

Theorem 1.2. *Suppose L is 2-component link with Alexander polynomial $\Delta_L(s,t)$. If $\mathrm{sp}(L) = 1$, then $\Delta_L(s,1) \cdot \Delta_L(1,t)$ divides $\Delta_L(s,t)$.*

We will use elementary methods explained in Lemma 2.1 and our techniques from covering links and Alexander polynomials to obtain lower bounds on the splitting number for links with 9 or fewer crossings. Together with enough patience with link diagrams, this is sufficient to determine the splitting number for all these links. Our results for links up to 9 crossings are summarised by Table 3 in Section 6.

In [BS13], Batson and Seed defined a spectral sequence from the Khovanov homology of a link L which converges to the Khovanov homology of the split link with the same components as L. They showed that this spectral sequence gives rise to a lower bound on $\mathrm{sp}(L)$, and by computing it for links up to 12 crossings, they gave many examples for which this lower bound is strictly stronger than the lower bound coming from linking numbers. They determined the splitting number of some of these examples, while some were left undetermined.

We revisit the examples of Batson and Seed and show that our methods are strong enough to recover their lower bounds. Furthermore we show that for several cases our methods give more information. In particular, we completely determine the splitting numbers of all the examples of Batson and Seed. We refer the reader to Section 5 for more details.

Organisation of the paper

We start out, in Section 2, with some basic observations on the splitting number of a link. In Section 3.1 we prove Theorem 3.2, which is a general result on the effect of crossing changes on covering links, and then we provide an example in Section 3.2. We give a proof of Theorem 1.2 in Sections 4.1 and 4.2 and we illustrate its use with an example in Section 4.3. The examples of Batson and Seed are discussed in Section 5, with Section 5.1 focussing on examples which use Theorem 1.2, and Section 5.2 on examples which require Theorem 1.1. A 3-component example of Batson and Seed is discussed in Section 5.3. Next, our results on the splitting numbers of links with 9 crossings or fewer are given in Section 6, with some particular arguments used to obtain these results described in Section 7.

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2. Basic observations

A link is *split* if it is a split union of knots. We recall from the introduction that the *splitting number* $\mathrm{sp}(L)$ of a link L is defined to be the minimal number of crossing changes which one needs to make on L, each crossing change between different components, in order to obtain a split link.

We note that this differs from the definition of 'splitting number' which occurs in [Ada96, Shi12]; in these papers crossing changes of a component with itself are permitted.

Given a link L we say that a non-split sublink with all of the linking numbers zero is *obstructive*. (All obstructive sublinks which occur in the applications of this paper will be Whitehead links.) We then define $c(L)$ to be the maximal size of a collection of distinct obstructive sublinks of L, such that any two sublinks in the collection have at most one component in common. Note that c is zero for trivial links.

As another example consider the link $L9a54$ shown in Figure 1. The sublink $L_1 \sqcup L_3$ is an unlink, while both $L_1 \sqcup L_2$ and $L_2 \sqcup L_3$ are Whitehead links, hence are obstructive. Thus $c(L) = 2$.

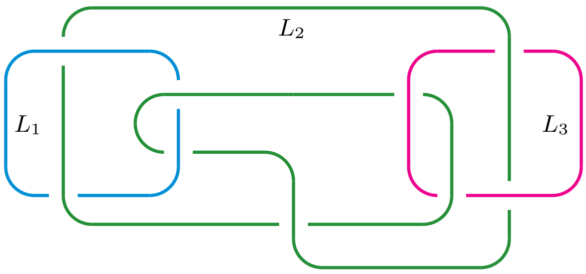

FIGURE 1. The link $L9a54$.

Finally we discuss the link J in Figure 17. It has four components J_1, J_2, J_3, J_4 and $J_1 \cup J_3$ and $J_2 \cup J_4$ each form a Whitehead link. It follows that $c(J) = 2$.

In practice it is straightforward to obtain lower bounds for $c(L)$. In most cases it is also not too hard to determine $c(L)$ precisely.

Now we have the following elementary lemma.

Lemma 2.1. *Let $L = L_1 \sqcup \cdots \sqcup L_m$ be a link. Then*

$$\mathrm{sp}(L) \equiv \sum_{i>j} \mathrm{lk}(L_i, L_j) \mod 2$$

and

$$\mathrm{sp}(L) \geq \sum_{i>j} |\,\mathrm{lk}(L_i, L_j)| + 2c(L).$$

Proof. Given a link L we write

$$a(L) = \sum_{i>j} |\,\mathrm{lk}(L_i, L_j)|.$$

Note that a crossing change between two different components always changes the value of a by precisely one. Since a of the unlink is zero we immediately obtain the first statement.

If we do a crossing change between two components with non-zero linking number, then a goes down by at most one, whereas c stays the same or increases by one. On the other hand, if we do a crossing change between two components with zero linking number, then a goes up by one and c decreases by at most one, since the two components belong to at most one obstructive sublink in any maximal collection whose cardinality realises $c(L)$. It now follows that $a(L) + 2c(L)$ decreases with each crossing change between different components by at most one. $\square$

The right hand side of the second inequality is greater than or equal to the lower bound $b_{\mathrm{lk}}(L)$ of [BS13, Section 5]. In some cases the lower bound coming from Lemma 2.1 is stronger. For example, let L be two split copies of the Borromean rings. For this L we have $c(L) = 2$, giving a sharp lower bound on the splitting number of 4, whereas $b_{\mathrm{lk}}(L) = 2$.

3. Covering link calculus

In this section, first we prove our main covering link result, Theorem 3.2, showing that covering links can be used to give lower bounds on the splitting number. Then we show how to extract Theorem 1.1 and three other useful corollaries from Theorem 3.2. In Section 3.2 we present an example of this approach.

3.1. Crossing changes and covering links

The following definition is a special case of the notion of a covering link occurring in [Koh93, Method 5] and [CK08], for example.

Definition 3.1. Let $L = L_1 \sqcup \cdots \sqcup L_m$ be an m-component link with L_i unknotted. We denote the double branched cover of S^3 with branching set the unknot L_i by $p\colon S^3 \to S^3$. We refer to $p^{-1}(L \smallsetminus L_i)$ as the *2-fold covering link of L with respect to L_i.*

We note that a choice of orientation of a link induces an orientation of its covering links.

In the theorem below we use the term *internal band sum* to refer to the operation on an oriented link L described as follows. The data for the move is an embedding $f\colon D^1 \times D^1 \subset S^3$ such that $f(D^1 \times D^1) \cap L = f(\{-1, 1\} \times D^1)$, the orientation of $f(\{-1\} \times D^1)$ agrees with that of L and the orientation of $f(\{1\} \times D^1)$ is opposite to that of L. The output is a new oriented link given by $(L \smallsetminus f(\{-1, 1\} \times D^1)) \cup f(D^1 \times \{-1, 1\})$, after rounding corners. The new link has the orientation induced from L.

Theorem 3.2. *Let $L = L_1 \sqcup \cdots \sqcup L_m$ be an m-component link and suppose that L_i is unknotted for some fixed i. Fix an orientation of L. Suppose L can be transformed to a split link by $\alpha + \beta$ crossing changes involving distinct components, where α of these involve L_i and β of these do not involve L_i. Then the 2-fold covering link J of L with respect to L_i can be altered by performing α internal band sums and 2β crossing changes between different components to the split union of $2(m-1)$ knots comprising two copies of L_j, for each $j \neq i$.*

Proof. We may assume $i = 1$. We begin by investigating the effect of crossing changes on the 2-fold covering link with respect to the first component L_1 of a link L.

Type A. First we consider crossing changes between the branching component L_1 and another component, say L_2. Such a crossing change lifts to a rotation of the preimage J of L_2 around the lift $\widetilde{L_1}$ of L_1, as shown in Figure 2. The top left and middle left diagrams show a link before and after a crossing change, in a cylindrical neighbourhood which contains an interval from each of L_1 and L_2. To branch over L_1, which is the component running down the centre of the cylinders, cut along the surface which is shown in the diagrams. The results of taking the top left and middle left diagrams, cutting, and glueing two copies together, are shown in the top right and middle right diagrams respectively.

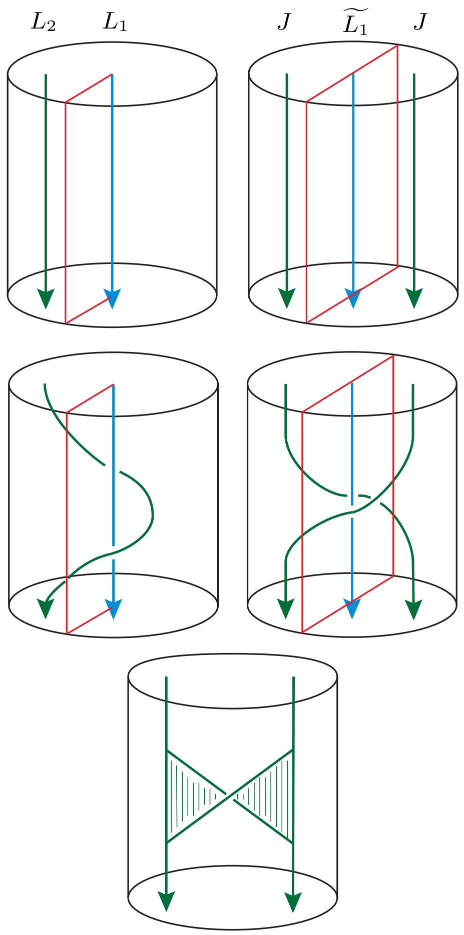

FIGURE 2. The effect of a crossing change on a 2-fold covering link where one component is the branching set.

After forgetting the branching set, the same effect on the lift of L_2 can be achieved by adding a band to J; see the bottom diagram of Figure 2. By ignoring the band, we obtain the top right diagram with the branching component removed. If we instead use the band to make an internal band sum, we obtain the middle right diagram with the branching set removed. Note that this band is attached to J in such a way that orientations are preserved. This holds no matter what choice of orientations were made for L. Thus we see that a crossing change between L_1 and L_2 corresponds to an internal band sum on the covering link.

Type B. Consider a crossing change which does not involve L_1, say between L_2 and L_3. Such a crossing change can be realised by ± 1 Dehn surgery on a circle which has zero linking number with L and which bounds an embedded disc, say D, in S^3 that intersects L in two points of opposite signs, one point of L_2 and one point of L_3. By performing the Dehn surgeries, and then taking the branched cover over L_1, we produce the covering link of the link obtained by the crossing change.

Note that the preimage of the disc D in the double branched cover consists of 2 disjoint discs, each of which intersects the covering link transversally in two points with opposite signs, one point of the preimage of L_2 and one point of the preimage of L_3. As an alternative construction, we can take the branched cover and then perform ± 1 Dehn surgeries along the boundary circles of the preimage discs. This gives the same the covering link. From this it follows that a single crossing change between L_2 and L_3 corresponds to two crossing changes on the covering link.

Note that when there is more than one crossing change, of either type, the corresponding surgery discs and bands associated to the covering link are disjoint.

Recall that the link L can be altered to become the split union of m knots $L_1, \dots, L_m$ by α crossing changes of Type A and β crossing changes of Type B. By the above arguments, the 2-fold covering link of L with respect to the first component L_1 can be altered to become the corresponding covering link of the split link, which is the split union $L_2 \sqcup L_2 \sqcup \cdots \sqcup L_m \sqcup L_m$, by α internal band sums and 2β crossing changes. □

In the following result, $g_4(K)$ denotes the slice genus of a knot K in S^3, namely the minimal genus of a smoothly embedded connected oriented surface in D^4 whose boundary is K.

Corollary 3.3. *Under the same hypotheses as Theorem 3.2, the 2-fold covering link of L with respect to L_i bounds a smoothly embedded oriented surface F in D^4 which has no closed components and has Euler characteristic*

$$\chi(F) = 2(m-1) - \alpha - 4\beta - 4\sum_{k \neq i} g_4(L_k).$$

In addition, if there is some $j \neq i$ such that each L_k with $k \neq j$ is involved in some crossing change with L_j, then F is connected.

Proof. Once again we may assume that $i = 1$. Let J be the 2-fold covering link of L with respect to L_1.

An internal band sum can be inverted by performing another band sum, while the inverse of a crossing change is also a crossing change. Hence by Theorem 3.2 we can also obtain the covering link J from the split union $L_2 \sqcup L_2 \sqcup \cdots \sqcup L_m \sqcup L_m$ by performing α internal band sums and 2β crossing changes.

Choose surfaces V_j embedded in D^4 with $\partial V_j = L_j$ and genus $g_4(L_j)$. Take a split union $V_2 \sqcup V_2 \sqcup \cdots \sqcup V_m \sqcup V_m$ in D^4. The boundary of these surfaces is the split union $L_2 \sqcup L_2 \sqcup \cdots \sqcup L_m \sqcup L_m$. The covering link J can be realised as the boundary of a surface obtained from the split union of the surfaces by attaching α bands and 2β clasps in S^3. As pointed out in the proof of Theorem 3.2, the surgery discs and bands associated to crossing changes are disjoint. Pushing slightly into D^4, we obtain an immersed surface in D^4 bounded by J; each clasp gives a transverse intersection. As usual, we remove the intersections by cutting out a disc neighbourhood of the intersection point from each sheet and glueing a twisted annulus which is a Seifert surface for the Hopf link. This gives a smoothly embedded oriented surface F in D^4 bounded by the covering link J. Note that each band attached changes the Euler characteristic of the surface by -1, while each twisted annulus used to remove an intersection point changes the Euler characteristic by -2. Therefore the resulting surface F has Euler characteristic

$$\chi(F) = \sum_{k=2}^{m} 2 \cdot (1 - 2g_4(V_k)) - \alpha - 4\beta$$

which is equal to the claimed value.

The final conclusion of the corollary states (when $i = 1$) that F is connected if there is some $j \neq 1$ such that each L_k with $k \neq j$ is involved in some crossing change with L_j. To see this, observe that a crossing change involving L_j and L_1 joins the two copies of V_j; a crossing change involving L_j and L_k with $j, k \geq 2$ joins one of the two copies of V_j to one of the two copies of V_k and joins the other copy of V_j to the other copy of V_k. Under the hypothesis, it follows that F is connected. □

Corollary 3.3 has some useful consequences of its own. Considering the case of $m = 2$, $\alpha = 2k + 1$, $\beta = 0$, and $g_4(L_k) = 0$, we obtain Theorem 1.1 stated in the introduction.

Theorem 1.1. *Suppose L is a 2-component link with unknotted components. If* $\operatorname{sp}(L) = 2k + 1$*, then any 2-fold covering knot of L has slice genus at most k.*

Remark 3.4.

(1) In the proof of Corollary 3.3, when $\beta = 0$, we construct an embedded surface F without local maxima. Therefore in order to show, using Theorem 1.1, that a link of linking number one with unknotted components has splitting number at least three, it suffices to show that the covering link is not a ribbon knot.
(2) Different choices of orientation on a link J can change the minimal genus of a connected surface which J bounds in D^4. Since the splitting number is independent of orientations, in applications we will choose the orientation which gives the strongest lower bound. This remark will be relevant in Section 5.3.
(3) If L is a non-split 2-component link, then the surface F of Corollary 3.3 is automatically connected, by the last sentence of that corollary.

The following is another useful consequence of Corollary 3.3.

Corollary 3.5. *Suppose L is a 2-component link with unknotted components and* $\operatorname{sp}(L) = 2$*. Then any 2-fold covering link of L is weakly slice; that is, bounds an annulus smoothly embedded in D^4.*

Proof. First note that a 2-fold covering link has two components, since the linking number is even by Lemma 2.1. Applying Corollary 3.3 with $m = 2$, $\alpha = 2$, $\beta = 0$, and $g_4(L_k) = 0$, the conclusion follows. □

We state one more corollary to Theorem 3.2. Let $\operatorname{sp}_i(L)$ be the minimal number of crossing changes between distinct components not involving L_i required to transforms L to a split link. By convention, $\operatorname{sp}_i(L)$ is infinite if we must make a crossing change involving L_i in order to split L.

Corollary 3.6 (c.f. [Koh93, Method 5])**.** *For a link $L = L_1 \sqcup \cdots \sqcup L_m$ and its 2-fold covering link J with respect to L_i, we have* $\operatorname{sp}_i(L) \geq \frac{1}{2}\operatorname{sp}(J)$*.*

Proof. This follows from Theorem 3.2 with $\alpha = 0$. □

We remark that the above results generalise to n-fold covering links in a reasonably straightforward manner. One can also draw analogous conclusions when the branching component is knotted. We do not address these generalisations here, since the results stated above are sufficient for the applications considered in this paper.

3.2. An example of the covering link technique

To illustrate the use of the method developed in Section 3.1, we now apply it to prove that the splitting number of the 2-component link $L9a30$ is three. More applications of

Theorem 1.1 and Corollaries 3.5 and 3.6 will be discussed later; see Sections 5.2, 5.3, 6, and 7 for instance. In this paper, we use the link names employed in the LinkInfo database [CLb]. The link $L9a30$ is shown in Figure 3. It is a two component link of linking number one with unknotted components. Recall that the splitting number is determined modulo 2 by the linking number by Lemma 2.1. It is easy to see from Figure 3 that 3 crossing changes suffice, so the splitting number is either one or three.

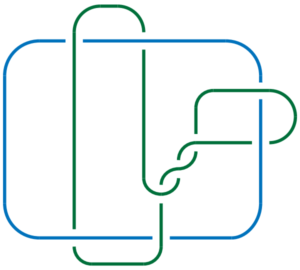

FIGURE 3. The link $L9a30$.

To see that $\mathrm{sp}(L9a30) \neq 1$, we take a 2-fold cover branched over one of the components, and check that the resulting knot is not slice. Figure 4 shows the result of an isotopy which was made in preparation for taking a branched cover on the left, and the knot obtained as the preimage of $L9a30$ after deleting the preimage of the branching component on the right.

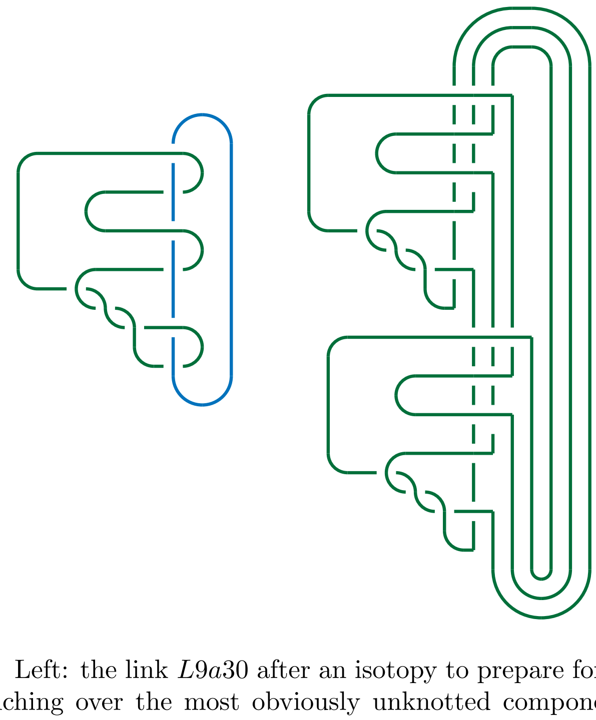

FIGURE 4. Left: the link $L9a30$ after an isotopy to prepare for taking the cover branching over the most obviously unknotted component. Right: the knot which arises as the covering link after taking two-fold branched cover and deleting the branching set.

The knot on the right of Figure 4 after a simplifying isotopy is shown in Figure 5; it is a twist knot with a negative clasp and 7 positive half twists. This knot is well known not to be a slice knot, a fact which was first proved by A. Casson and C. Gordon [CG78, CG86]. Therefore, by Theorem 1.1, the splitting number of $L9a30$ is at least three, as claimed.

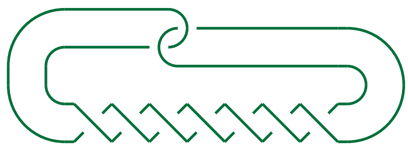

FIGURE 5. The covering knot on the right of Figure 4 after an isotopy.

4. Alexander invariants

In this section we will recall the definition of Alexander modules and polynomials of oriented links. We then show how Alexander modules are affected by a crossing change which then allows us to prove Theorem 1.2.

4.1. Crossing changes and the Alexander module

Throughout this section, given an oriented m-component link L, the oriented meridians are denoted by $\mu_1, \dots, \mu_m$. Note that $\mu_1, \dots, \mu_m$ give rise to a basis for $H_1(S^3 \smallsetminus \nu L; \mathbb{Z})$. We will henceforth use this basis to identify $H_1(S^3 \smallsetminus \nu L; \mathbb{Z})$ with $\mathbb{Z}^m$. Let R be a subring of $\mathbb{C}$ and let $\psi \colon \mathbb{Z}^m \to F$ be a homomorphism to a free abelian group. We denote the induced map

$$\pi_1(S^3 \smallsetminus \nu L) \longrightarrow H_1(S^3 \smallsetminus \nu L; \mathbb{Z}) = \mathbb{Z}^m \xrightarrow{\psi} F$$

by ψ as well. We can then consider the corresponding Alexander module

$$H_1^{\psi}(S^3 \smallsetminus \nu L; R[F])$$

and the order of the Alexander module is denoted by

$$\Delta_L^{\psi} \in R[F] = \operatorname{ord}_{R[F]}(H_1^{\psi}(S^3 \smallsetminus \nu L; R[F])).$$

(We refer to [Hil02] for the definition of the order of a $R[F]$-module.) If ψ is the identity, then we drop ψ from the notation and we obtain the usual multivariable Alexander polynomial Δ_L.

Note that what we term the Alexander module has also been called the "link module" in the literature e.g. [Kaw96b]. The following proposition relates the Alexander modules of two oriented links which differ by a crossing change.

Proposition 4.1. *Let L and L' be two oriented m-component links which differ by a single crossing change. Let R be a subring of $\mathbb{C}$ and let $\psi \colon \mathbb{Z}^m \to F$ be a homomorphism to a free abelian group. Then there exists a diagram*

$$\begin{tikzcd}
 & R[F] \arrow[dr] & & R[F] \arrow[dl] & \\
 & & M \arrow[dl] \arrow[dr] & & \\
 & H_1^{\psi}(S^3 \smallsetminus \nu L; R[F]) \arrow[dl] & & H_1(S^3 \smallsetminus \nu L'; R[F]) \arrow[dr] & \\
0 & & & & 0
\end{tikzcd}$$

where M is some $R[F]$-module and where the diagonal sequences are exact.

The formulation of this proposition is somewhat more general than what is strictly needed in the proof of Theorem 1.2. We hope that this more general formulation will be applicable, in future work, to the computation of unlinking numbers; see the beginning of Section 6 for the definition of the unlinking number of a link.

Proof. We write $X = S^3 \smallsetminus \nu L$ and $X' = S^3 \smallsetminus \nu L'$. We pick two open disjoint discs D_1 and D_2 in the interior of D^2 and we write

$$\begin{aligned} B &= (D^2 \smallsetminus (D_1 \cup D_2)) \times [0,1], \\ S &= (D^2 \smallsetminus (D_1 \cup D_2)) \times \{0,1\} \cup_{\partial D^2 \times \{0,1\}} S^1 \times [0,1]. \end{aligned}$$

Put differently, S is the 'top and bottom boundary' of B together with the outer cylinder $S^1 \times [0,1]$.

Since L and L' are related by a single crossing change there exists a subset Y of X and continuous injective maps $f\colon B \to X$ and $f'\colon B' \to X'$ with the following properties:

(1) $X = Y \cup f(B)$ and $Y \cap f(B) = f(S)$,
(2) $X' = Y \cup f'(B)$ and $Y \cap f'(B) = f'(S)$.

We can now state the following claim.

Claim. *There exists a short exact sequence*

$$R[F] \longrightarrow H_1^\psi(Y; R[F]) \longrightarrow H_1^\psi(X; R[F]) \longrightarrow 0.$$

By a slight abuse of notation we now write $B = f(B)$ and $S = f(S)$. We then consider the following Mayer-Vietoris sequence:

$$\begin{aligned} \cdots \longrightarrow H_1^\psi(S; R[F]) \xrightarrow{i_* \oplus j_*} H_1^\psi(B; R[F]) \oplus H_1^\psi(Y; R[F]) \longrightarrow H_1^\psi(X; R[F]) \\ \longrightarrow H_0^\psi(S; R[F]) \xrightarrow{i_* \oplus j_*} H_0^\psi(B; R[F]) \oplus H_0^\psi(Y; R[F]) \end{aligned}$$

where $i\colon S \to B$ and $j\colon S \to Y$ are the two inclusion maps. We need to study the relationships between the homology groups of S and B. We make the following observations: By [HS97, Section VI.3] we have the following commutative diagram

$$\begin{tikzcd} H_0^\psi(S; R[F]) \arrow[r] \arrow[d, "\cong"'] & H_0^\psi(B; R[F]) \arrow[d, "\cong"'] \\ R[F]/\{\psi(g)v - v\}_{v \in R[F],\, g \in \pi_1(S)} \arrow[r] & R[F]/\{\psi(g)v - v\}_{v \in R[F],\, g \in \pi_1(B)}. \end{tikzcd}$$

Here the horizontal maps are induced by the inclusion $S \to B$ and the vertical maps are isomorphisms. The map $i_*\colon \pi_1(S) \to \pi_1(B)$ is surjective; it follows that the bottom horizontal map is an isomorphism. Hence the top horizontal map is also an isomorphism. The above Mayer-Vietoris sequence thus simplifies to the following sequence

$$H_1^\psi(S; R[F]) \xrightarrow{i_* \oplus j_*} H_1^\psi(B; R[F]) \oplus H_1^\psi(Y; R[F]) \longrightarrow H_1^\psi(X; R[F]) \longrightarrow 0.$$

We note that the space B is homotopy equivalent to a wedge of two circles m and n. Furthermore, S is homotopy equivalent to the wedge of m, n and another curve m' which is homotopic to m in B. By another slight abuse of notation we now replace B and S by these wedges of circles and we view B and S as CW-complexes with precisely one 0-cell in the obvious way. We denote by $p\colon \tilde{S} \to S$ and $p\colon \tilde{B} \to B$ the coverings corresponding to the homomorphisms $\pi_1(S) \to \pi_1(B) \to \pi_1(X) \xrightarrow{\psi} F$. Note that we can and will view $\tilde{B}$

as a subset of $\tilde{S}$. We now pick pre-images $\tilde{m}, \tilde{n}$ and $\tilde{m}'$ of m, n and m' under the covering map $p\colon \tilde{S} \to S$. Note that $\{\tilde{m}, \tilde{n}\}$ is a basis for $C_1(B; R[F]) = C_1(\tilde{B})$ and $\{\tilde{m}, \tilde{n}, \tilde{m}'\}$ is a basis for $C_1(S; R[F]) = C_1(\tilde{S})$. The kernel of the map $C_1(S; R[F]) \to C_1(B; R[F])$ is given by $R[F] \cdot (m - m')$. We thus obtain the following commutative diagram of chain complexes with exact rows

$$\begin{tikzcd}
0 \arrow[r] & R[F]\cdot(m-m') \arrow[r] \arrow[d] & C_1(S;R[F]) \arrow[r, "i_*"] \arrow[d] & C_1(B;R[F]) \arrow[r] \arrow[d] & 0 \\
& 0 \arrow[r] & C_0(S;R[F]) \arrow[r, "i_*"] & C_0(B;R[F]) \arrow[r] & 0.
\end{tikzcd}$$

It now follows easily from the diagram, or more formally from the snake lemma, that

(1) $$\ker(i_*\colon H_1^\psi(S;R[F]) \longrightarrow H_1^\psi(B;R[F])) \cong R[F]\cdot(m-m')$$

and that

(2) $$\operatorname{coker}(i_*\colon H_1^\psi(S;R[F]) \longrightarrow H_1^\psi(B;R[F])) = 0.$$

Finally we consider the following commutative diagram

$$\begin{tikzcd}
R[F]\cdot(m-m') \arrow[r] \arrow[d] & H_1^\psi(Y;R[F]) \arrow[r] \arrow[d, "\begin{pmatrix}0\\ \mathrm{Id}\end{pmatrix}"'] & H_1^\psi(X;R[F]) \arrow[r] \arrow[d, "="] & 0 \\
H_1^\psi(S;R[F]) \arrow[r] & H_1^\psi(B;R[F]) \oplus H_1^\psi(Y;R[F]) \arrow[r] & H_1^\psi(X;R[F]) \arrow[r] & 0.
\end{tikzcd}$$

We had already seen above that the bottom horizontal sequence is exact. It now follows from (1), (2) and some modest diagram chasing that the top horizontal sequence is also exact. This concludes the proof of the claim.

Precisely the same proof shows that there exists a short exact sequence

$$R[F] \longrightarrow H_1^\psi(Y;R[F]) \longrightarrow H_1^\psi(X';R[F]) \longrightarrow 0.$$

(Use $B = f'(B), S = f'(S)$ instead of $B = f(B), S = f(S)$). Combining these two short exact sequences now gives the desired result, by taking $M := H_1^\psi(Y;R[F])$. $\square$

4.2. The Alexander polynomial obstruction

Using Proposition 4.1 we can prove the following obstruction to the splitting number being equal to 1.

Theorem 4.2. *Let L be a 2-component oriented link. We denote the Alexander polynomial of L by $\Delta_L(s,t)$. If the splitting number of L equals one, then $\Delta_L(s,1)\cdot\Delta_L(1,t)$ divides $\Delta_L(s,t)$.*

Let $L = J \cup K$ be an oriented link with splitting number equal to one. We denote the Alexander polynomials of J and K by Δ_J and Δ_K respectively. It follows from Lemma 2.1 that the linking number satisfies $|\operatorname{lk}(J,K)| = 1$. Therefore by the Torres condition $|\Delta_L(1,1)| = 1$ and we have that

$$\Delta_L(s,1) = \Delta_J(s) \text{ and } \Delta_L(1,t) = \Delta_K(t).$$

We can thus reformulate the statement of the theorem as saying that if $L = J \cup K$ is an oriented link with splitting number equal to one, then $\Delta_J(s)$ and $\Delta_K(t)$ both divide $\Delta_L(s,t)$.

Proof. Let $L = J \cup K$ be an oriented link with splitting number equal to one. We denote by $\psi\colon H_1(S^3 \smallsetminus L;\mathbb{Z}) \to \langle s,t \,|\, [s,t]=1\rangle$ the map which is given by sending the meridian of J to s and by sending the meridian of K to t. We write $\Lambda := \mathbb{Z}[s^{\pm1},t^{\pm1}]$.

In the following we also denote by ψ the map $H_1(S^3 \smallsetminus J;\mathbb{Z}) \to \langle s,t \,|\, [s,t]=1\rangle$, which is given by sending the meridian of J to s. Note that with this convention we have an isomorphism

$$H_1^{\psi}(S^3 \smallsetminus J;\Lambda) = H_1(S^3 \smallsetminus J;\mathbb{Z}[s^{\pm1}]) \otimes_{\mathbb{Z}[s^{\pm1}]} \Lambda$$

and we obtain that

$$\operatorname{ord}_{\Lambda}(H_1^{\psi}(S^3 \smallsetminus J;\Lambda)) = \Delta_J(s). \tag{3}$$

Similarly we define a map $H_1(S^3 \smallsetminus K;\mathbb{Z}) \to \langle s,t \,|\, [s,t]=1\rangle$ by sending the meridian of K to t. We see that

$$\operatorname{ord}_{\Lambda}(H_1^{\psi}(S^3 \smallsetminus K;\Lambda)) = \Delta_K(t). \tag{4}$$

We denote the split link with components J and K by $J \sqcup K$. The Mayer-Vietoris sequence for $S^3 \smallsetminus (J \sqcup K)$ which comes from splitting along the separating 2-sphere S gives rise to an exact sequence

$$\begin{aligned} 0 \longrightarrow H_1^{\psi}(S^3 \smallsetminus J;\Lambda) \oplus H_1^{\psi}(S^3 \smallsetminus K;\Lambda) &\longrightarrow H_1^{\psi}(S^3 \smallsetminus (J \sqcup K);\Lambda) \\ &\xrightarrow{h} H_0(S;\Lambda) \longrightarrow H_0^{\psi}(S^3 \smallsetminus J;\Lambda) \oplus H_0^{\psi}(S^3 \smallsetminus K;\Lambda). \end{aligned}$$

We recall that by [HS97, Section VI] we have, for any connected space X with a ring homomorphism $\psi\colon \pi_1(X) \to \Lambda$ we have

$$H_0^{\psi}(X;\Lambda) = \Lambda / \{\psi(g)v - v \,|\, v \in \Lambda, g \in \pi_1(X)\}.$$

It follows easily that $H_0^{\psi}(S;\Lambda) \cong \Lambda$ and that $H_0^{\psi}(S^3 \smallsetminus J;\Lambda)$ and $H_0^{\psi}(S^3 \smallsetminus K;\Lambda)$ are Λ-torsion. In particular we see that the last map in the above long exact sequence has a nontrivial kernel. By the exactness of the Mayer-Vietoris sequence above it follows that the map h has nontrivial image.

Since L has splitting number one we can do one crossing change involving both J and K to turn L into $J \sqcup K$. The conclusion of Proposition 4.1 together with the above Mayer-Vietoris sequence gives rise to a diagram of maps as follows:

$$\begin{tikzcd} & \Lambda \arrow[r, "f"] & M \arrow[r, two heads, "p"] \arrow[d, "g"] & H_1^{\psi}(S^3 \smallsetminus L;\Lambda) \\ H_1^{\psi}(S^3 \smallsetminus J;\Lambda) \oplus H_1^{\psi}(S^3 \smallsetminus K;\Lambda) \arrow[rr, hook] & & H_1^{\psi}(S^3 \smallsetminus (J \sqcup K);\Lambda) \arrow[r, "h"] & \Lambda, \end{tikzcd}$$

where the top and bottom horizontal sequences are exact, and where the map h is nontrivial. In particular note that p gives rise to an isomorphism $M/f(\Lambda) \cong H_1^{\psi}(S^3 \smallsetminus L;\Lambda)$, and that g gives rise to an epimorphism

$$H_1^{\psi}(S^3 \smallsetminus L;\Lambda) \cong M/f(\Lambda) \longrightarrow H_1^{\psi}(S^3 \smallsetminus (J \sqcup K);\Lambda) \,/\, (g \circ f)(\Lambda). \tag{5}$$

Next we will prove the following claim.

Claim. *The map*

$$H_1^{\psi}(S^3 \smallsetminus J;\Lambda) \oplus H_1^{\psi}(S^3 \smallsetminus K;\Lambda) \longrightarrow H_1^{\psi}(S^3 \smallsetminus J \sqcup K;\Lambda) \,/\, (g \circ f)(\Lambda)$$

is a monomorphism.

We consider the following commutative diagram

$$\begin{tikzcd}
 & & \Lambda \arrow[d, "g\circ f"] & \\
0 \arrow[r] & H_1^\psi(S^3\setminus J;\Lambda)\oplus H_1^\psi(S^3\setminus K;\Lambda) \arrow[r, hook] & H_1^\psi(S^3\setminus(J\sqcup K);\Lambda) \arrow[r, "h"] \arrow[d] & \Lambda \arrow[d] \\
 & & \frac{H_1^\psi(S^3\setminus(J\sqcup K);\Lambda)}{(g\circ f)(\Lambda)} \arrow[r, "h"] & \Lambda/(h\circ g\circ f)(\Lambda),
\end{tikzcd}$$

where the bottom vertical maps are the obvious projection maps. Furthermore, as above the map h in the middle sequence is nontrivial.

We first note that the bottom left group is Λ-torsion. Indeed, in the discussion preceding the proof we saw that $\Delta_L(s,t)\neq 0$. This implies that the homology group $H_1^\psi(S^3\setminus L;\Lambda)$ is Λ-torsion. But by (5) this also implies that the bottom left group of the diagram is Λ-torsion.

It follows that in the square the composition of maps given by going down and then right factors through a Λ-torsion group. On the other hand we have seen that the map $h\colon H_1^\psi(S^3\setminus(J\sqcup K);\Lambda)\to\Lambda$ is nontrivial. By the commutativity of the square and by the fact that the down-right composition of maps factors through a Λ-torsion group it now follows that the projection map $\Lambda\to\Lambda/(h\circ g\circ f)(\Lambda)$ cannot be an isomorphism. But this just means that the composition

$$\Lambda \xrightarrow{g\circ f} H_1^\psi(S^3\setminus(J\sqcup K);\Lambda)\xrightarrow{h}\Lambda$$

is nontrivial, and in particular injective. Put differently, we have

$$\operatorname{im}\Big(g\circ f\colon\Lambda\longrightarrow H_1^\psi(S^3\setminus(J\sqcup K);\Lambda)\Big)\cap\ker(h)=0.$$

By the exactness of the middle horizontal sequence we thus see that the intersection of the images of $(g\circ f)(\Lambda)$ and of $H_1^\psi(S^3\setminus J;\Lambda)\oplus H_1^\psi(S^3\setminus K;\Lambda)$ in $H_1^\psi(S^3\setminus(J\sqcup K);\Lambda))$ is trivial. It follows that the map

$$H_1^\psi(S^3\setminus J;\Lambda)\oplus H_1^\psi(S^3\setminus K;\Lambda)\longrightarrow H_1^\psi(S^3\setminus J\sqcup K;\Lambda)/(g\circ f)(\Lambda)$$

is indeed a monomorphism. This concludes the proof of the claim.

Before we continue with the proof we recall that if

$$0\longrightarrow A\longrightarrow B\longrightarrow C\longrightarrow 0$$

is a short exact sequence of Λ-modules, then by [Hil02, Part 1.3.3] the orders of the modules are related by the following equality

$$\operatorname{ord}_\Lambda(B)=\operatorname{ord}_\Lambda(A)\cdot\operatorname{ord}_\Lambda(C). \tag{6}$$

We thus see, from the claim and from (6), (3) and (4) that

$$\Delta_J(s)\cdot\Delta_K(t)\mid\operatorname{ord}_\Lambda\Big(H_1^\psi(S^3\setminus J\sqcup K;\Lambda)/(g\circ f)(\Lambda)\Big).$$

On the other hand it follows from (5) and again from (6) that

$$\operatorname{ord}_\Lambda\Big(H_1^\psi(S^3\setminus(J\sqcup K);\Lambda)/(g\circ f)(\Lambda)\Big)\,\Big|\,\operatorname{ord}_\Lambda\Big(H_1^\psi(S^3\setminus L;\Lambda)\Big).$$ □

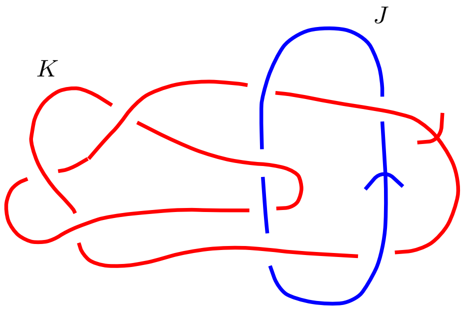

FIGURE 6. The link $L9a29$, with splitting number 3.

4.3. An example of the Alexander polynomial technique

We consider the oriented link $L = K \sqcup J = L9a29$ from Figure 6. It has linking number one and it is not hard to see that one can turn it into a split link using three crossing changes between the two components. The multivariable Alexander polynomial of L is

$$\Delta_L(s,t) \quad = \quad s - s^2 + t - st + s^2t - t^2 + st^2 - s^2t^2 + t^3 - st^3 + s^2t^3 - t^4 + st^4.$$

It is straightforward to see that $\Delta_J(s) \cdot \Delta_K(t) = 1 - t + t^2$ does not divide $\Delta_L(s,t)$. It thus follows from Theorem 4.2 that the splitting number of L is three.

This is one of the instances of the use of the Alexander polynomial which is cited in Section 6, in Table 3 (method 4). The other computations listed in that table as using this method are performed in a similar fashion; see the LinkInfo tables [CLb] for the multivariable Alexander polynomials of the other 9 crossing links, which are $L9a24$, $L9n13$, $L9n14$ and $L9n17$. Since these are two component links of linking number 1, the Alexander polynomials of the components can be obtained by substituting either $t = 1$ or $s = 1$ into the multivariable Alexander polynomial in $\mathbb{Z}[s^{\pm 1}, t^{\pm 1}]$.

5. The examples of Batson and Seed

In [BS13], Batson and Seed constructed a spectral sequence from the Khovanov homology of a link L to the Khovanov homology of the split link with the same components as L. This spectral sequence gives rise to a lower bound on the splitting number, given by the lowest page on which their spectral sequence collapses.

Batson and Seed computed the lower bound for all links up to 12 crossings and they showed that it provides more information than basic linking number observations (see our Lemma 2.1) for 17 links. The lower bound they computed will be denoted by $b(L)$. One of the 17 links is a 3-component link with 12 crossings, for which $b(L) = 3$ while the sum of the absolute values of the linking numbers is one. The remaining 16 links have 2-components and satisfy $\mathrm{lk}(L) = \pm 1$ and $b(L) = 3$. One of these has 11 crossings, and 15 of these have 12 crossings. Batson and Seed determined the splitting numbers for 7 links among these 17 links, while for the other 10 cases the splitting numbers are listed as being either 3 or 5. This information is given in [BS13, Table 3].

In this section we revisit these links to reprove or improve the results in [BS13]. In particular we completely determine the splitting numbers by using our methods.

5.1. Using the Alexander polynomial

We first apply our Alexander polynomial method to the examples of [BS13] with at least one knotted component. This will reprove their splitting number results for these links. Before we turn to the links of [BS13, Table 3], we will discuss a link with 13-crossings in detail, which is also discussed in [BS13].

A 13-crossing example. Consider the 2-component link L shown in Figure 7. It is the link denoted by ${}^2n^{13}_{8862}$ in [BS13].

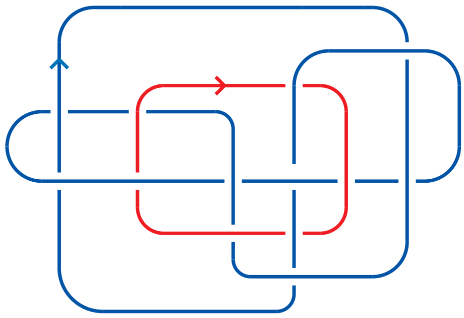

FIGURE 7. A 2-component link of linking number 1 whose splitting number equals 3.

Note that one component is an unknot and the other is a trefoil. We refer to the unknotted component as J and to the knotted component as K. It is not hard to see that L can be turned into a split link using three crossing changes. On the other hand the linking number is 1, so it follows from Lemma 2.1 that the splitting number is either one or three. The invariant $b(L)$ shows that the splitting number of L is in fact three.

We will now use Theorem 4.2 to give another proof that the splitting number of L equals three. We used Kodama's program *knotGTK* to show that

$$\begin{aligned}\Delta_L(s,t) = &-s^8t^4 + s^7t^5 + 4s^8t^3 - 5s^6t^5 - 6s^8t^2 - 9s^7t^3 + 13s^6t^4 + 11s^5t^5 + 4s^8t \\ &+ 17s^7t^2 - 6s^6t^3 - 37s^5t^4 - 14s^4t^5 - s^8 - 12s^7t - 10s^6t^2 + 45s^5t^3 - 4s^6 \\ &+ 52s^4t^4 + 11s^3t^5 + 3s^7 + 12s^6t - 24s^5t^2 - 74s^4t^3 - 44s^3t^4 - 5s^2t^5 \\ &+ 2s^5t + 51s^4t^2 + 67s^3t^3 + 23s^2t^4 + st^5 + 3s^5 - 13s^4t - 46s^3t^2 - 39s^2t^3 \\ &- 7st^4 - s^4 + 11s^3t + 25s^2t^2 + 15st^3 + t^4 - 3s^2t - 9st^2 - 3t^3 + 2t^2.\end{aligned}$$

It is straightforward to see (we used Maple) that

$$\Delta_L(s,1)\cdot\Delta_L(1,t) = \Delta_J(s)\cdot\Delta_K(t) = 1 - t + t^2$$

does not divide $\Delta_L(s,t)$. Thus it follows from Theorem 4.2 that the splitting number of L is not one. By the above observations we therefore see that the splitting number of L is equal to three.

Seven 12-crossing examples. In [BS13, Table 3], Batson and Seed give seven examples of 2-component 12 crossing links which have linking number equal to one and for which $b(L)$ detects that the splitting number is three.

In Table 1, we list the links together with their Dowker-Thistlethwaite (DT) codes and multivariable Alexander polynomials. The translation between the names we use (following

Link	DT code	Alexander polynomial
$L12n1342$	$(14,\overline{6},\overline{10},16,\overline{4},\overline{18}),$ $(\overline{20},22,8,\overline{24},\overline{2},12)$	$s^2t^4 - st^4 - s^2t^2 + s^2t + st^2 + t^3 - t^2 - s + 1$
$L12n1350$	$(14,\overline{6},\overline{10},\overline{16},\overline{4},20),$ $(12,22,\overline{8},24,2,18)$	$-s^4t^3 + s^3t^4 + s^4t^2 + s^3t^3 - 2s^2t^4 - 2s^3t^2 + s^2t^3$ $+ st^4 + s^2t^2 + s^3 + s^2t - 2st^2 - 2s^2 + st + t^2 + s - t$
$L12n1357$	$(14,\overline{6},\overline{10},16,\overline{4},22),$ $(20,2,8,24,12,18)$	$-s^4t^4 + s^4t^3 + s^3t^4 - s^4t^2 + s^4t - s^2t^3$ $+ s^2t^2 - s^2t + t^3 - t^2 + s + t - 1$
$L12n1363$	$(14,\overline{6},\overline{10},\overline{18},\overline{4},22),$ $(2,20,\overline{8},24,12,16)$	$2s^2t^2 - 3s^2t - 3st^2 + 2s^2 + 5st + 2t^2 - 3s - 3t + 2$
$L12n1367$	$(14,\overline{6},\overline{10},\overline{18},\overline{4},24),$ $(2,12,22,\overline{8},16,20)$	$s^4t^2 + s^3t^3 - s^2t^4 - s^4t - 2s^3t^2 + st^4 + s^3t$ $+ s^2t^2 + st^3 + s^3 - 2st^2 - t^3 - s^2 + st + t^2$
$L12n1374$	$(14,\overline{6},\overline{10},20,\overline{4},16),$ $(2,12,22,24,8,18)$	$s^4t^3 + s^3t^4 - s^4t^2 - s^3t^3 - s^2t^4 + s^4t + st^4$ $- s^4 + s^2t^2 - t^4 + s^3 + t^3 - s^2 - st - t^2 + s + t$
$L12n1404$	$(14,\overline{8},\overline{18},\overline{12},\overline{22},\overline{4}),$ $(20,2,24,\overline{6},16,\overline{10})$	$2s^2t^3 - st^4 - 2s^2t^2 - st^3 + t^4$ $+ 3st^2 + s^2 - st - 2t^2 - s + 2t$

TABLE 1. Seven 12-crossing links and their Alexander polynomials

LinkInfo [CLb]) and the convention used in [BS13] is given by $L12nX = {}^2a^{12}_{X+4196}$. All these Alexander polynomials are irreducible. For each link, both components are trefoils, so $\Delta_L(s,1) = 1-s+s^2$ and $\Delta_L(1,t) = 1-t+t^2$ do not divide the multivariable Alexander polynomial. Thus it follows from Theorem 4.2 that the splitting number of each of these links is at least three, which recovers the results of Batson and Seed. Inspection of the diagrams shows that the splitting numbers are indeed equal to 3.

5.2. Using the covering link technique

Batson and Seed, in [BS13, Table 3], give nine further examples of links which have two unknotted components and linking number ± 1. They list these links as having splitting number either three or five. Translating notation again, we have: $L11a372 = {}^2a^{11}_{739}$, $L12aX = {}^2a^{12}_{X+1288}$ and $L12nY = {}^2n^{12}_{Y+4196}$.

Table 2 lists the results of our computations, giving the slice genus of the knot obtained by taking a 2-fold branched cover of S^3, branched over one of the components, the method which we use to compute the slice genus, and the resulting splitting number obtained by the methods of Section 3.1.

The methods we use to compute the slice genus of the covering knot are as follows. First, the slice genus of a knot is bounded below by half the absolute value of its signature $\sigma(K) = \operatorname{sign}(A + A^T)$, where A is a Seifert matrix of K, by [Mur67, Theorem 9.1]. We used a Python software package of the first author to compute $\sigma(K)$. The Rasmussen s-invariant [Ras10] also gives a lower bound by $|s(K)| \leq 2g_4(K)$. We used `JavaKh` of J. Green and S. Morrison to compute $s(K)$.

We can also prove that a knot is not slice using the twisted Alexander polynomial [KL99], denoted $\Delta^\chi_K(s) \in \mathbb{Q}(\zeta_q)[s^{\pm 1}]$, where ζ_q is a primitive q-th root of unity, associated to the p-fold cyclic cover X_p of the knot exterior X and a character $\chi \colon TH_1(X_p;\mathbb{Z}) \to \mathbb{Z}_q$. For slice knots, there exists a metaboliser of the $\mathbb{Q}/\mathbb{Z}$ valued linking form on $H_1(X_p;\mathbb{Z})$, such that for characters χ which vanish on the metaboliser, the twisted Alexander polynomial factorises (up to a unit) as $g(s)\overline{g(s)}$, for some $g \in \mathbb{Q}(\zeta_q)[s^{\pm 1}]$. By checking that this condition does not hold for all metabolisers, we can prove that a knot is not slice. (For each covering

Link	DT code	Covering knot slice genus	Method for slice genus	Splitting number
$L11a372$	$(12, 14, 16, 20, 18),$ $(10, 2, 4, 22, 8, 6)$	2	$\sigma = -4$	5
$L12a1233$	$(12, 14, 16, 18, 20),$ $(2, 24, 4, 6, 22, 8, 10)$	2	$\sigma = -4$	5
$L12a1264$	$(12, 14, 16, 20, 18),$ $(2, 24, 4, 22, 8, 6, 10)$	2	$\sigma = -4$	5
$L12a1384$	$(14, 8, 16, 24, 18, 20),$ $(2, 22, 4, 10, 12, 6)$	2	$\sigma = -4$	5
$L12n1319$	$(12, 14, 16, 24, \overline{18}),$ $(2, 10, 22, \overline{20}, \overline{8}, 4, 6)$	1	$\sigma = -2$	3
$L12n1320$	$(12, \overline{14}, \overline{16}, 24, \overline{18}),$ $(6, 22, \overline{20}, \overline{8}, \overline{4}, \overline{2}, 10)$	1	$p = 3,\ q = 7,$ $\Delta^{\chi}(s) = 7s^2 + 15s + 7$	3
$L12n1321$	$(12, 14, 16, 24, \overline{18}),$ $(10, 2, 22, \overline{20}, \overline{8}, 4, 6)$	2	$s = 4$	5
$L12n1323$	$(12, 14, 18, 16, \overline{20}),$ $(10, 2, 24, 6, \overline{22}, \overline{8}, 4)$	2	$\sigma = -4$	5
$L12n1326$	$(12, 14, \overline{18}, 24, \overline{20}),$ $(8, 2, 4, \overline{22}, \overline{10}, \overline{6}, \overline{16})$	1	$p = 3,\ q = 7,$ $\Delta^{\chi}(s) = 7s^2 - 71s + 7$	3

TABLE 2. Nine links and their covering knot slice genus

knot K to which we apply this, all metabolisers give the same polynomial Δ_K^{χ}.) Our computations of twisted Alexander polynomials were performed using a Maple program written by C. Herald, P. Kirk and C. Livingston [HKL10].

The invariants discussed above give us lower bounds on the slice genera of the covering knots. We do not need to know the precise slice genera in order to obtain lower bounds. Nevertheless we point out that we are able to determine them. In each case we found the requisite crossing changes to split the link, so an application of Theorem 1.1 gives us an upper bound on the slice genus of the covering knot, which implies that the above lower bounds are sharp.

For the links $L11a372$, $L12a1233$, $L12a1264$, $L12a1384$, $L12n1321$ and $L12n1323$, we are able to show that the splitting numbers of these links is 5. Amusingly we use Khovanov homology, in the guise of the s-invariant, to compute that the slice genus of the covering knot of $L12n1321$ is 2. We remark that this knot has $\sigma = -2$, which is only sufficient to show that the splitting number is at least 3.

For the other links, as in Section 5.1, our obstruction gives the same information as the Batson-Seed lower bound, namely that the splitting number is at least three. For the links $L12n1319$, $L12n1320$ and $L12n1326$ we looked at the diagrams and found the crossing changes to verify that the splitting number is indeed 3.

We present one example in detail, the link $L11a372$, which is shown as given by LinkInfo on the left of Figure 8, while on the right the link $L11a372$ is shown after an isotopy, to prepare for making a diagram of a covering link. It is easy to see from the diagram that the splitting number is at most 5. The link $L11a372$ has $b(L11a372) = 3$.

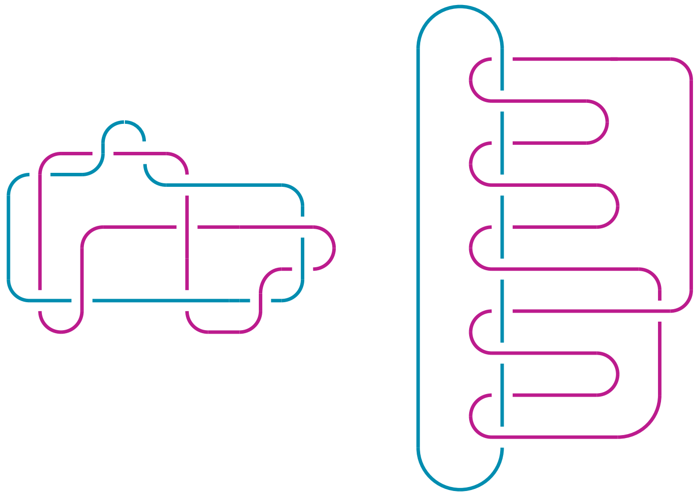

Figure 8. Left: the link $L11a372$. Right: after an isotopy.

The 2-fold covering link obtained by branching over the left hand component is shown in Figure 9. This turns out to be the knot 7_5, which according to KnotInfo [CLa] has $|\sigma| = 4$ and slice genus 2. Therefore by Theorem 1.1, the splitting number is 5.

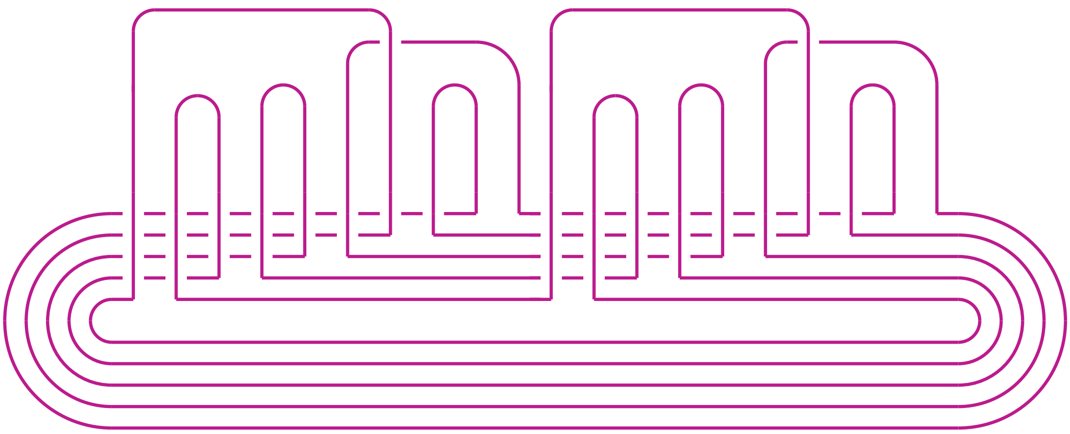

Figure 9. The covering link obtained by taking a 2-fold branched covering over the left hand component of the link in Figure 8.

5.3. A 3-component example

There is one final link listed in [BS13, Table 3] as having splitting number either three or five, namely the 3-component link $L := L12a1622$, which is shown in Figure 10. In the notation of [BS13], L is the link ${}^3a^{12}_{2910}$.

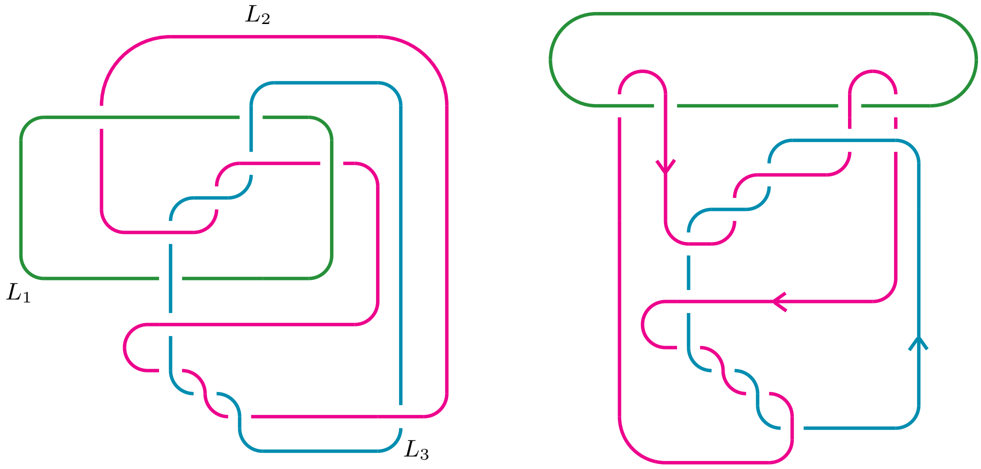

FIGURE 10. Left: the link $L12a1622$. Right: the same link, after an isotopy to prepare for taking a covering link by branching over the top component.

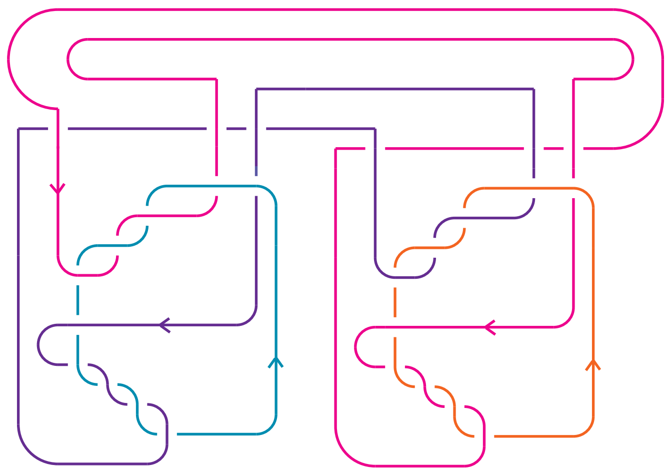

FIGURE 11. The 2-fold covering link of the link in Figure 10, branched over L_1.

We show that the splitting number of L is in fact 5. Note that the components are unknotted, and the only nonzero linking number is between L_2 and L_3, which have $|\operatorname{lk}(L_2, L_3)| = 1$. Thus the splitting number is odd by Lemma 2.1. It is easy to find 5 crossing changes which suffice.

We begin by showing that three crossing changes involving just L_2 and L_3 do not suffice to split the link. We take the 2-fold covering link J with respect to L_1. The result of an isotopy to prepare for taking such a covering is shown on the right of Figure 10. The resulting covering link J is shown in Figure 11. The link J has splitting number 10 by Lemma 2.1, with a sharp lower bound given by the sum of the absolute values of the linking numbers between the components. By Corollary 3.6, we have that $\operatorname{sp}_1(L) \geq 5$.

Combining $\mathrm{sp}_1(L) \geq 5$ with the linking number, it follows that if $\mathrm{sp}(L) \leq 3$, then exactly one crossing change involving (L_2, L_3) is required to split the link, and there can be either two additional (L_1, L_2) crossing changes, or two (L_1, L_3) crossing changes. We will give the argument to show that the first possibility cannot happen; the argument discounting the second possibility is analogous.

Suppose that two (L_1, L_2) crossing changes and one (L_2, L_3) crossing change yields the unlink. Applying Corollary 3.3 (with $m = 3$, $\alpha = 2$, $\beta = 1$, $g_4(L_k) = 0$), it follows that the covering link $J \subset S^3$ bounds an oriented surface F of Euler characteristic $2(3-1)-2-4 = -2$ which is smoothly embedded in D^4 and has no closed component. Also, F is connected by the last part of Corollary 3.3 since both L_1 and L_3 are involved in some crossing change with L_2. Since J has 4 components, F is a 3-punctured disc. That is, J is weakly slice.

To show that this cannot be the case for J, we use the link signature invariant, which is defined similarly to the knot signature: for a link J, choose a surface V in S^3 bounded by J (V may be disconnected), define the Seifert pairing on $H_1(V)$ and an associated Seifert matrix A as usual. Then the *link signature* of J is defined by $\sigma(J) = \mathrm{sign}(A + A^T)$. Due to K. Murasugi [Mur67], if an m-component link J bounds a smoothly embedded oriented surface F in D^4, we have $|\sigma(J)| \leq 2g(F) + m - b_0(F)$ where $g(F)$ is the genus and $b_0(F)$ is the 0th Betti number of F. For our covering link J, since it bounds a 3-punctured disc in D^4, we have $|\sigma(J)| \leq 3$. Here we orient J as in Figure 11; this orientation is obtained using the orientations of L_2 and L_3 shown on the right of Figure 10. On the other hand, a computation aided by a Python software package of the first author shows that $\sigma(J) = -7$. From this contradiction it follows that one (L_2, L_3) crossing change and two (L_1, L_2) crossing changes never split L.

6. Links with 9 or fewer crossings

In Table 3 we give the splitting numbers for the links of 9 crossings or fewer, together with the method which is used to give a sharp lower bound for the splitting number. The entry in the method column of the table refers to the list below.

In the case of 2-component links with unknotted components and linking number one, knowing that the *unlinking number* is greater than one implies that the splitting number is at least three. Recall that by definition the unlinking number of an m-component link L is the minimal number of crossing changes required to convert L to the m-component unlink. Note that for this link invariant, crossing changes of a component with itself are in general permitted.

In Method 3 below, we will make use of computations of unlinking numbers made by P. Kohn in [Koh93], where making considerable use of his earlier work in [Koh91], he computed the unlinking numbers of 2-component links with 9 or fewer crossings, in all but 5 cases.

(1) Using Lemma 2.1, the linking numbers determine the lower bound for the splitting number, by providing a lower bound or by fixing the splitting number modulo 2.
(2) A combination of linking numbers and either one or two Whitehead links as a sublink determine a lower bound for the splitting number. That is, Lemma 2.1 provides a sharp lower bound, with $c(L) \neq 0$.
(3) This is a link where the sum of the linking numbers is one and the components are unknotted, but which does not have unlinking number one, and so cannot have splitting number one. Therefore the splitting number is at least three.

Link L	sp(L)	Method	Link L	sp(L)	Method	Link L	sp(L)	Method
$L2a1$	1	(1)	$L8n6$	4	(1)	$L9a42$	2	(1)
$L4a1$	2	(1)	$L8n7$	4	(1)	$L9a43$	3	(1)
$L5a1$	2	(1)	$L8n8$	4	(1)	$L9a44$	3	(1)
$L6a1$	2	(1)	$L9a1$	2	(1)	$L9a45$	3	(1)
$L6a2$	3	(1)	$L9a2$	2	(1)	$L9a46$	3	(3)
$L6a3$	3	(1)	$L9a3$	2	(1)	$L9a47$	4	(2)
$L6a4$	2	(1)	$L9a4$	2	(1)	$L9a48$	4	(1)
$L6a5$	3	(1)	$L9a5$	2	(1)	$L9a49$	4	(1)
$L6n1$	3	(1)	$L9a6$	2	(1)	$L9a50$	4	(2)
$L7a1$	2	(1)	$L9a7$	2	(1)	$L9a51$	4	(1)
$L7a2$	2	(1)	$L9a8$	2	(1)	$L9a52$	4	(2)
$L7a3$	2	(1)	$L9a9$	2	(1)	$L9a53$	2	(1)
$L7a4$	2	(1)	$L9a10$	2	(1)	$L9a54$	4	(2)
$L7a5$	1	(1)	$L9a11$	2	(1)	$L9a55$	4	(1)
$L7a6$	3	(3)	$L9a12$	2	(1)	$L9n1$	2	(1)
$L7a7$	3	(1)	$L9a13$	2	(1)	$L9n2$	2	(1)
$L7n1$	2	(1)	$L9a14$	2	(1)	$L9n3$	2	(1)
$L7n2$	2	(1)	$L9a15$	2	(1)	$L9n4$	2	(1)
$L8a1$	2	(1)	$L9a16$	2	(1)	$L9n5$	2	(1)
$L8a2$	2	(1)	$L9a17$	2	(1)	$L9n6$	2	(1)
$L8a3$	2	(1)	$L9a18$	2	(1)	$L9n7$	2	(1)
$L8a4$	2	(1)	$L9a19$	2	(1)	$L9n8$	2	(1)
$L8a5$	2	(1)	$L9a20$	3	(3)	$L9n9$	2	(1)
$L8a6$	2	(1)	$L9a21$	1	(1)	$L9n10$	2	(1)
$L8a7$	2	(1)	$L9a22$	3	(3)	$L9n11$	2	(1)
$L8a8$	3	(3)	$L9a23$	3	(1)	$L9n12$	2	(1)
$L8a9$	3	(3)	$L9a24$	3	(4)	$L9n13$	3	(4)
$L8a10$	3	(1)	$L9a25$	3	(1)	$L9n14$	3	(4)
$L8a11$	3	(1)	$L9a26$	3	(3)	$L9n15$	3	(1)
$L8a12$	4	(1)	$L9a27$	1	(1)	$L9n16$	3	(1)
$L8a13$	4	(1)	$L9a28$	3	(1)	$L9n17$	3	(4)
$L8a14$	4	(1)	$L9a29$	3	(4)	$L9n18$	4	(1)
$L8a15$	3	(1)	$L9a30$	3	(3)	$L9n19$	4	(1)
$L8a16$	3	(3)	$L9a31$	1	(1)	$L9n20$	3	(1)
$L8a17$	4	(1)	$L9a32$	3	(1)	$L9n21$	3	(1)
$L8a18$	4	(1)	$L9a33$	3	(1)	$L9n22$	3	(1)
$L8a19$	2	(1)	$L9a34$	2	(1)	$L9n23$	4	(2)
$L8a20$	4	(1)	$L9a35$	2	(1)	$L9n24$	4	(2)
$L8a21$	4	(1)	$L9a36$	4	(5)	$L9n25$	2	(2)
$L8n1$	2	(1)	$L9a37$	2	(1)	$L9n26$	4	(2)
$L8n2$	2	(1)	$L9a38$	2	(1)	$L9n27$	4	(2)
$L8n3$	4	(1)	$L9a39$	2	(1)	$L9n28$	4	(2)
$L8n4$	4	(1)	$L9a40$	4	(5)	·	·	·
$L8n5$	2	(1)	$L9a41$	2	(1)	·	·	·

TABLE 3. Splitting numbers of links with 9 or fewer crossings.

For the 2-component case (all which use this method have two components apart from $L9a46$ and $L8a16$), we know that this link does not have unlinking number one by [Koh93]. Kohn did not explicitly give an argument that the unlinking number of $L9a30$ is at least 2, but we computed the splitting number of $L9a30$ in Section 3.2.

For the 3-component links $L8a16$ and $L9a46$, we show that the splitting number (and unlinking number) is not one in Sections 7.2 and 7.3 respectively.

(4) A 2-component link of linking number one, with at least one component knotted. The Alexander polynomials of the components do not divide the multivariable Alexander polynomial of the link, so by Theorem 4.2 the splitting number must be at least 3. See Section 4.3 for an example of this argument in action, for the link $L9a29$.

(5) A 2-component link with unknotted components and linking number zero modulo two. For the link $L9a36$, we note that the unlinking number is not two by [Koh93]. Thus the splitting number must be at least 4. For the link $L9a40$, we show that the splitting number is not two in Section 7.1.

We remark that some of the splitting numbers in the table are also given in [BS13].

7. Arguments for the splitting number of particular links

7.1. The link $L9a40$

The link $L9a40$ is shown on the left of Figure 12. We claim that the splitting number of $L9a40$ is four. Note that the linking number is zero, so the splitting number is either two or four, since it is easy to see from the diagram that four crossing changes suffice to split the link.

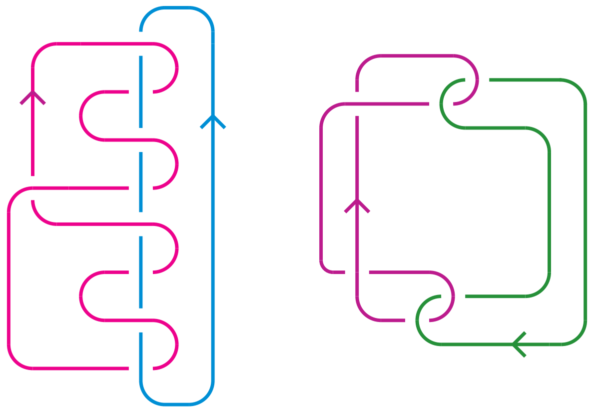

FIGURE 12. Left: the link $L9a40$. Right: the 2-fold branched cover with respect to the right hand component.

To show that the splitting number cannot be two, we consider the 2-component link obtained by taking the 2-fold covering link with respect to the right hand component, which is shown on the right of Figure 12. This is the link $L6a1$. By Corollary 3.5, if $\mathrm{sp}(L9a40) = 2$, then $L6a1$ would bound an annulus smoothly embedded in D^4. Thus, any internal band sum of $L6a1$, which is a knot, would have slice genus at most one. But the band sum of $L6a1$ shown in Figure 13 is the knot 7_5, which has signature 4 and smooth slice genus 2. It follows that the splitting number of $L9a40$ is four as claimed.

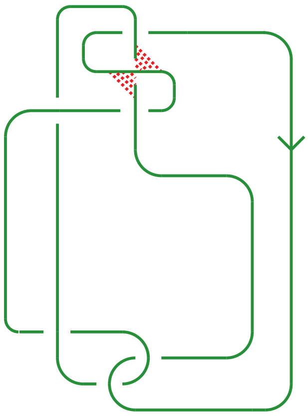

FIGURE 13. A band sum of $L6a1$ to produce the knot 7_5. The band which was added is coloured in with dotted lines; it has a half twist in it.

7.2. The link $L8a16$

The link $L8a16$ is shown in Figure 14. The components are labelled L_1, L_2 and L_3. The linking number $|\operatorname{lk}(L_1, L_2)| = 1$, and the other linking numbers are trivial. We claim that $\operatorname{sp}(L8a16) = 3$. It is not hard to find three crossing changes which work; for example change all three of the crossings where L_2 passes over L_1 in Figure 14. By this observation and Lemma 2.1 the splitting number is either one or three. We therefore need to show that it is not possible to split the link with a single crossing change. (We remark that this is the same as showing that the unlinking number is greater than one, since the components are unknotted.)

By linking number considerations a single crossing change would have to involve L_1 and L_2. To discard this eventuality, we will take a 2-fold covering link branched over L_3.

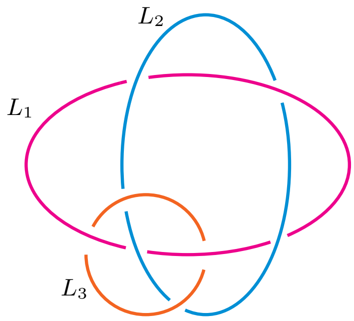

FIGURE 14. The link $L8a16$.

The left of Figure 15 shows the link $L8a16$ after an isotopy; the right hand picture shows the 2-fold cover branched over L_3. Call this link J. The sum of linking numbers $\sum_{i<j} |\operatorname{lk}(J_i, J_j)| = 6$, so $\operatorname{sp}(J) \geq 6$ by Lemma 2.1 (in fact $\operatorname{sp}(J) = u(J) = 6$). Therefore, by Corollary 3.6, we see that $\operatorname{sp}_3(L8a16) \geq 3$. (Recall that $\operatorname{sp}_i(L)$ denotes the splitting number of L where the component L_i is not involved in any crossing changes.) Thus, as claimed, it is not possible to split the link in a single crossing.

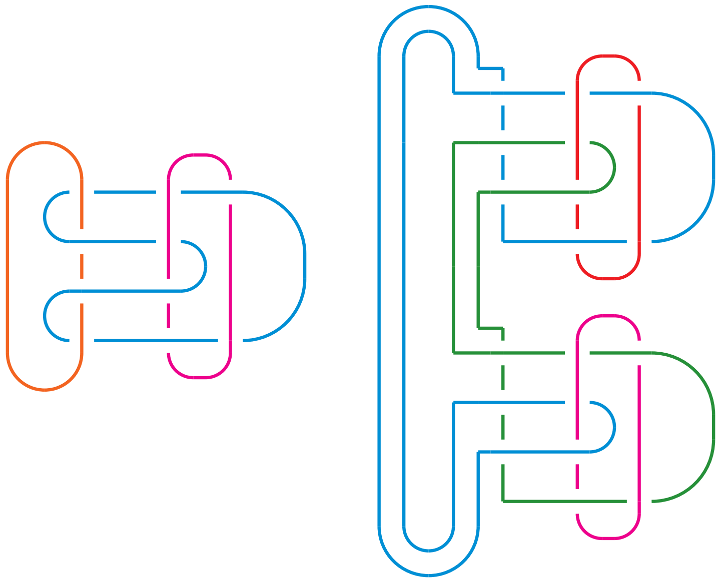

FIGURE 15. The link $L8a16$ after an isotopy (left), and the 2-fold cover branched over L_3 (right).

7.3. The link $L9a46$

The link $L9a46$ is shown on the left of Figure 16. We claim that $\mathrm{sp}(L9a46) = 3$. It is not hard to find three changes which suffice. For example, in Figure 16, change the crossings where L_1 passes under L_2.

Note that $|\operatorname{lk}(L_1, L_2)| = 1$. Therefore if one crossing change suffices, it must be between L_1 and L_2. We need to show that this is not possible. For this purpose we apply Corollary 3.6 again. We will take a 2-fold covering link branched over L_3. In preparation for this, the link from the left of Figure 16 is shown, after an isotopy, on the right of Figure 16.

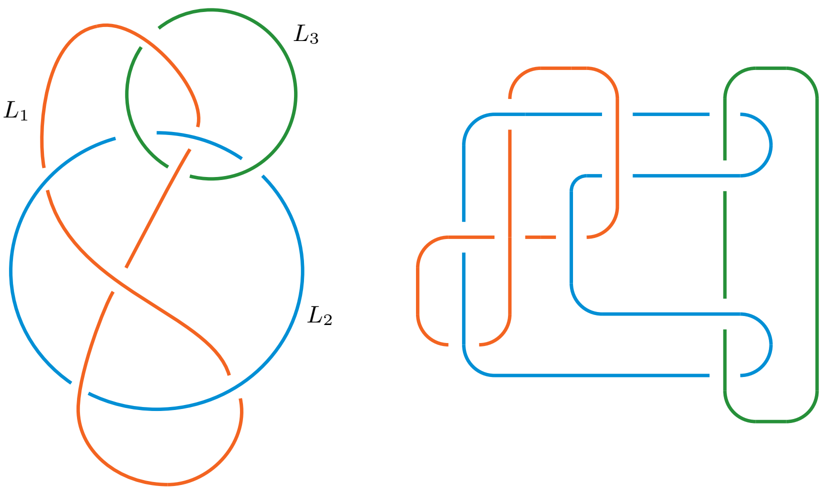

FIGURE 16. Left: the link $L9a46$. Right: the link $L9a46$ after an isotopy.

Taking the cover branched over the right hand component of the link on the right of Figure 16, we obtain the 2-fold covering link J shown in Figure 17.

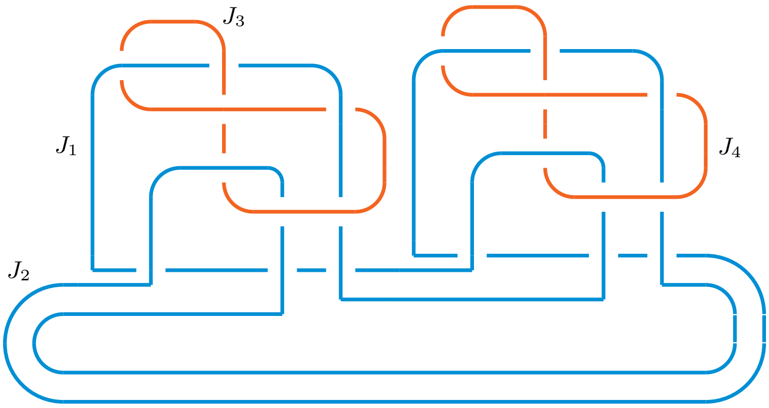

FIGURE 17. The 2-fold covering link of the link $L9a46$ from the right of Figure 16.

We need to see that the link J of Figure 17 has splitting number at least 6. Observe that $|\operatorname{lk}(J_1, J_4)| = |\operatorname{lk}(J_2, J_3)| = 1$. Moreover, the sublinks $J_1 \sqcup J_3$ and $J_2 \sqcup J_4$ are Whitehead links. It now follows from Lemma 2.1 that $\operatorname{sp}(J) \geq 6$. By Corollary 3.6, we obtain that $\operatorname{sp}_3(L9a46) \geq 3$. It thus follows from the above discussion that $\operatorname{sp}(L9a46) = 3$.

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