

SPATIAL GRAPHS

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ABSTRACT. We give a definition of spatial graphs and we introduce the notion of a graphical neighborhood that generalizes the notion of a tubular neighborhood of a knot.

Before we talk about spatial graphs we need to settle what we mean by a graph.

Definition. An *abstract graph* G is a triple (V, E, φ) where V is a finite non-empty set, E is a finite set and φ is a map

$$\varphi: E \rightarrow \{\text{subsets of } V \text{ with one or two elements}\}.$$

The elements of V are called *vertices of G* and the elements of E are called the *edges of G* . Furthermore, given $e \in E$ the points in $\varphi(e)$ are called the *endpoints of e* .

Our goal is to introduce the notion of a “spatial graph”. The idea is that a spatial graph should be a graph in S^3 , the same way that a link is a union of circles in S^3 . The problem is to make precise what “in S^3 ” is supposed to mean.

In the following we first recall the definition of a link and the notion of a tubular neighborhood of a link. We then proceed with the definition of a spatial graph and a discussion of graphical neighborhoods which are a generalization of tubular neighborhoods.

To us it seems like the most natural definition of a link is to define it as a closed 1-dimensional submanifold of S^3 . (It is worth pointing out that the definitions of a link in the various textbooks [BZH14, p. 1], [Li97, p. 1], [Ro90, p. 2 and 100] and [Ka96, p. 4] are all somewhat different, but they lead to equivalent theories.) One advantage of the above definition is that it allows us to dip into the well-developed theory of smooth manifolds.

One of the key technical tools in studying links are tubular neighborhoods. In this context there are again different definitions that are being used. Therefore we fix our conventions. Let M be a 3-manifold. (Here and throughout by a 3-manifold we mean an orientable, connected 3-dimensional manifold, possibly with boundary). Furthermore let C be a compact proper 1-dimensional submanifold of M . (Here proper means that $\partial C = C \cap \partial M$ and that C intersects ∂M transversally.) A *tubular embedding of C* is an embedding $F: C \times \overline{B^2} \rightarrow M$ with $F(P, 0) = P$ for all $P \in C$ such that $F(\partial C \times \overline{B^2}) = \partial M \cap F(C \times \overline{B^2})$ and such that $F(C \times \overline{B^2})$ is a submanifold with corners of M . A *tubular neighborhood of C* is the image of a tubular embedding.

Remark. Note that if C has non-empty boundary, then N is submanifold with non-empty corners, thus strictly speaking it is not a smooth submanifold. Fortunately in practice this is not a problem. For example we are mostly interested in considering the exterior

$E_C := M \setminus \overset{\circ}{N}$ where $\overset{\circ}{N}$ is the interior of N . The exterior E_C is a smooth manifold with corners, but by “straightening of corners”, see [Wa16, Proposition 2.6.2] we can view $E_C = M \setminus \overset{\circ}{N}$ as a smooth manifold in a canonical way.

The following theorem shows that tubular neighborhoods always exist.

Theorem 0.1. [Wa16, Theorem 2.3.3] *Every compact proper 1-dimensional manifold of every compact 3-manifold admits a tubular neighborhood.*

The following theorem, see [Wa16, Chapter 2.5], shows that tubular neighborhoods are unique in an appropriate sense.

Theorem 0.2. *Let M be a 3-manifold and let C be a proper 1-dimensional submanifold of M . If N and N' are two tubular neighborhoods of C , then there exists a map $\Phi: M \times [0, 1] \rightarrow M$ with the following properties:*

- (1) *if given $t \in [0, 1]$ we denote by $\Phi_t: M \rightarrow M$ the map defined by $\Phi_t(P) := \Phi(P, t)$, then each $\Phi_t: M \rightarrow M$ is a diffeomorphism,*
- (2) *each Φ_t restricts to the identity on C ,*
- (3) $\Phi_0 = \text{id}_M$,
- (4) *the restriction of Φ_1 to N defines a diffeomorphism from N to N' .*

Given a closed 1-dimensional manifold C of a 3-manifold M we pick a tubular neighborhood N . We denote by $\overset{\circ}{N} = N \setminus \partial N$ the interior of N . We refer to $M_C := M \setminus \overset{\circ}{N}$ as the *exterior of C* . The tubular neighborhood N and the exterior of C have the following four useful properties:

- (a) by Theorem 0.2 the diffeomorphism type of the exterior is in fact well-defined, i.e. it is independent of the choice of the tubular embedding,
- (b) M_C is a compact 3-manifold,
- (c) M_C is a deformation retract of $M \setminus C$,
- (d) C is a deformation retract of the tubular neighborhood N .

After this long discussion of links and 1-dimensional submanifolds of 3-manifolds we turn to the notion of a spatial graph. Our goal is to give a definition of spatial graphs that has the following two features:

- (1) the definition is flexible enough to incorporate all “reasonable examples”, for example we want the pictures shown in Figure 1 to be examples of spatial graphs (here we follow the usual topological convention of viewing S^3 as $\mathbb{R}^3 \cup \{\infty\}$),
- (2) the definition ensures that spatial graphs have good technical properties, which we interpret as asking for an analogue of tubular neighborhoods, in the sense that we want an analogue of the above statements (a), (b), (c) and (d).

In preparation for our definition of a spatial graph we need to introduce the notion of an arc.

Definition. An *arc* in S^3 is a subset E of S^3 for which there exists a map $\varphi: [0, 1] \rightarrow S^3$ with the following properties:

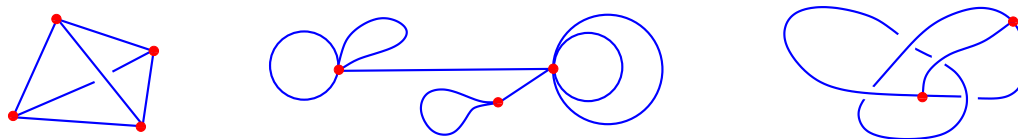


FIGURE 1.

- (1) the map φ is smooth, i.e. all derivatives are defined on the open interval $(0, 1)$ and they extend to continuous maps on the closed interval $[0, 1]$ that we also call derivatives,
- (2) the first derivative $\varphi'(t)$ is non-zero for all $t \in [0, 1]$,
- (3) the restriction of φ to $(0, 1)$ is injective,
- (4) $\varphi((0, 1)) \cap \varphi(\{0, 1\}) = \emptyset$ and
- (5) $\varphi((0, 1)) = E$.

Given an arc E as above we refer to φ and $\varphi(1)$ as the *endpoints of E* . (Note that the endpoints of E are *not* points on E .)



FIGURE 2. Illustration of arcs with one or two endpoints.

Remark. If an arc has two different endpoints, then an arc is precisely the same as the image of an embedding of the interval $[0, 1]$ in S^3 . The more technical features of the definition are necessary to get the right degree of control in the case that an arc has only one endpoint.

Now we can give our definition of a spatial graph.

Definition. A *spatial graph* G is a pair (V, E) with the following properties:

- (1) V is a finite non-empty subset of S^3 .
- (2) E is a subset of S^3 with the following properties:
 - (a) E is disjoint from V ,
 - (b) E has finitely many components,
 - (c) each component of E is an arc and the endpoints of each arc lie on V .

We refer to the points in V as the *vertices of G* and we refer to the components of E as the *edges of G* . Furthermore, given a spatial graph $G = (V, E)$ we write $|G| = V \cup E \subset S^3$. A spatial graph defines an abstract graph in obvious way.

It should be clear that Figure 1 shows three examples of spatial graphs. In the figure the red dots correspond to the vertices and the blue segments correspond to the edges.

Next we want to introduce a generalization of the tubular neighborhood for a link. We start out with the following definition.

Definition. Let $G = (V, E)$ be a spatial graph with vertex set $V = \{v_1, \dots, v_m\}$. A *small neighborhood* of V is a compact 3-dimensional manifold X of S^3 with components X_1, \dots, X_m such that for each $i \in \{1, \dots, m\}$ there exists a diffeomorphism $\Phi_i: \overline{B^3} \rightarrow X_i$ with $\Phi_i = v_i$ and such that for each $r \in (0, 1]$ the image $\Phi_i(S_r^2)$ is a submanifold of $S^3 \setminus V$ that is transverse to the submanifold E of $S^3 \setminus V$. (Here S_r^2 is the sphere of radius r in \mathbb{R}^3 around the origin.)

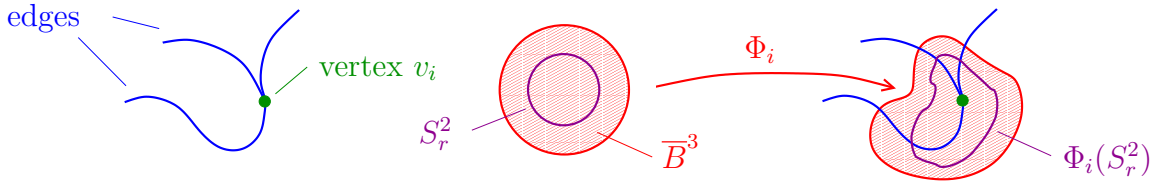


FIGURE 3. Illustration of a small neighborhood.

We point out that $E \cap (S^3 \setminus \overset{\circ}{X})$ is a proper submanifold of $S^3 \setminus \overset{\circ}{X}$. Now we can define a graphical neighborhood of a spatial graph.

Definition. Let $G = (V, E)$ be a spatial graph. A *graphical neighborhood* of (V, E) is a subset N of S^3 that can be written as a union $N = X \cup Y$ where X is a small neighborhood of V and Y is a tubular neighborhood of $E \cap (S^3 \setminus \overset{\circ}{X})$ in $S^3 \setminus \overset{\circ}{X}$.

Remark. As remarked above, after “straightening of corners”, we can view $S^3 \setminus \overset{\circ}{N}$ as a smooth manifold in a canonical way.

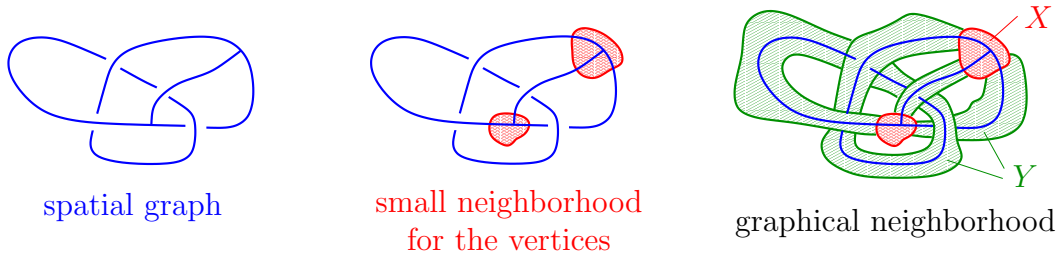


FIGURE 4.

In the following we will see that every spatial graph admits a graphical neighborhood and that graphical neighborhoods have properties that are very similar to the properties of tubular neighborhoods.

We start out with the following theorem that plays the role of Theorem 0.1.

Theorem 0.3. *Every spatial graph admits a graphical neighborhood.*

Sketch of proof. Let $G = (V, E)$ be a spatial graph. It is fairly elementary to see that V admits a small neighborhood X (Here one needs to use properties (1) and (2) of arcs). By the Tubular Neighborhood Theorem 0.1 there exists a tubular neighborhood Y of the proper submanifold $E \cap (S^3 \setminus \overset{\circ}{X})$ of $S^3 \setminus \overset{\circ}{X}$. Then $X \cup Y$ is a graphical neighborhood of G . \square

The most difficult aspect is to show that graphical neighborhoods are unique in an appropriate sense. The following theorem plays the role of Theorem 0.2.

Theorem 0.4. *Let G be a spatial graph. If N and N' are two graphical neighborhoods of G , then there exists a map $\Phi: S^3 \times [0, 1] \rightarrow S^3$ with the following properties:*

- (1) *if given $t \in [0, 1]$ we denote by $\Phi_t: S^3 \rightarrow S^3$ the map defined by $\Phi_t(P) := \Phi(P, t)$, then each $\Phi_t: S^3 \rightarrow S^3$ is a diffeomorphism,*
- (2) *each Φ_t is the identity on the vertex set and it preserves each edge setwise,*
- (3) $\Phi_0 = \text{id}$,
- (4) *we have $\Phi_1(N) = N'$.*

The proof of Theorem 0.4 is somewhat involved. We refer to [?] for more details. The next proposition summarizes some of the key properties of graphical neighborhoods.

Proposition 0.5. *Let G be a spatial graph and let N be a graphical neighborhood of G .*

- (1) *N contains $|G|$ in the interior $\overset{\circ}{N} = N \setminus \partial N$ of N ,*
- (2) *$|G|$ is a deformation retract of N ,*
- (3) *∂N is a deformation retract of $N \setminus |G|$,*
- (4) *the exterior E_G is a compact 3-manifold that is a deformation retract of $S^3 \setminus |G|$, and*
- (5) *the diffeomorphism type of the exterior $E_G := S^3 \setminus \overset{\circ}{N}$ does not depend on the choice of N .*

Sketch of proof. Let $N = X \cup Y$ be a graphical neighborhood of G . Statement (1) is immediate. Statement (4) is a consequence of of statement (3). Statements (2) and (3) are a fairly immediate consequence of the fact that Y is a product and the observation that for every component X' of X the pair $(X', E \cap X')$ is homeomorphic to a pair (\overline{B}^3, R) where R consist of straight segments emanating from the origin. Finally (5) is an immediate consequence of Theorem 0.4 and the above discussion of smoothening corners. More details will be provided in [?]. \square

Remark. (1) One can easily generalize the definition of a spatial graph in S^3 to a spatial graph in any closed 3-manifold. The above results can also easily be generalized to this more general context. We will consider this more general case in [?].

- (2) In the literature sometimes a spatial graph is defined as a PL-embedding of a graph in S^3 and as a neighborhood one uses a regular neighborhood as defined in [RS72, p. 33]. By [RS72, Theorem 3.8] these are unique in an appropriate sense and by [RS72, Corollary 3.30] the PL-spatial graph is a deformation retract of each regular neighborhood. It is not so clear though whether there are analogues of statements (3) and (4) of Proposition 0.5.

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