Symplectic 4–manifolds and fibered 3–manifolds

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Symplectic 4–manifolds.

Definition. A 4–manifold M is called symplectic if there exists a closed 2–form ω such that $\omega \wedge \omega \neq 0$ everywhere.

Thurston proved the following in 1976.

Theorem. If N is a closed fibered 3–manifold, then $S^1 \times N$ is symplectic.

We can now show that the converse holds:

Theorem (F–Vidussi 2008). Let N be a closed 3–manifold. If $S^1 \times N$ is symplectic, then N is fibered.

In fact using constructions of symplectic forms in an earlier paper (generalizing Thurston and Fernandez–Gray–Morgan) we can completely determine the symplectic cone, i.e. which $a \in H^2(S^1 \times N; \mathbb{R})$ can be represented by a symplectic form.

Twisted Alexander polynomials.

We use twisted Alexander polynomials as our main tool.

Definition. Let N a 3–manifold, $\phi \in H^1(N;\mathbb{Z}) = \mathrm{Hom}(\pi_1(N),\mathbb{Z})$ and $\tilde{\pi} \subset \pi = \pi_1(N)$ a finite index subgroup. Consider the twisted $\mathbb{Z}[t^{\pm 1}]$–module

$$H_1(N;\mathbb{Z}[\pi/\tilde{\pi}][t^{\pm 1}]).$$

We denote by $\Delta^{\pi/\tilde{\pi}}_{N,\phi}$ its order, called twisted Alexander polynomial.

E.g. if $H_1(N;\mathbb{Z}[\pi/\tilde{\pi}][t^{\pm 1}]) = \bigoplus \mathbb{Z}[t^{\pm 1}]/p_i(t)$, then $\Delta^{\pi/\tilde{\pi}}_{N,\phi} = \prod p_i(t)$.

Example. If $N = S^3 \setminus K$, $\phi \in H^1(S^3 \setminus K;\mathbb{Z}) \cong \mathbb{Z}$ a generator and $\tilde{\pi} = \pi$, then $\Delta^{\pi/\tilde{\pi}}_{N,\phi} = \Delta_K$ is the ordinary Alexander polynomial of the knot K.

Fibered manifolds and twisted Alexander polynomials

Defn. Given N and $\phi \in H^1(N;\mathbb{Z}) = \mathrm{Hom}(\pi_1(N),\mathbb{Z})$ we say that (N,ϕ) fibers if there exists a fibration $p : N \to S^1$ with $p_* = \phi : \mathrm{Hom}(\pi_1(N),\pi_1(S^1))$.

The following is due to Cha, Goda–Kitano–Morifuji, F–Kim:

Theorem. If (N,ϕ) fibers, then for any finite index $\tilde{\pi} \subset \pi$ we have that
(1) $\Delta_{N,\phi}^{\pi/\tilde{\pi}}$ is monic,
(2) the degree of $\Delta_{N,\phi}^{\pi/\tilde{\pi}}$ is determined by $||\phi||_T$.

(Monic means that the top coefficient equals ± 1)

The theorem generalizes the fact that for a fibered knot K the Alexander polynomial Δ_K is monic and $\deg \Delta_K = 2\mathrm{genus} K$.

Symplectic manifolds and twisted Alexander polynomials

The following generalizes a result of Kronheimer.

Theorem (F–Vidussi). If $S^1 \times N$ is symplectic, then there exists $\phi \in H^1(N;\mathbb{Z})$ such that for any finite index subgroup $\tilde{\pi} \subset \pi$ we have that
(1) $\Delta_{N,\phi}^{\pi/\tilde{\pi}}$ is monic,
(2) the degree of $\Delta_{N,\phi}^{\pi/\tilde{\pi}}$ is determined by $||\phi||_T$.

Proof. (1) Use Taubes' results on Seiberg–Witten invariants of all finite (and symplectic!) covers of $S^1 \times N$.
(2) Apply Meng–Taubes to get information on Alexander polynomials.

So we have to show that twisted Alexander polynomials detect fibered 3–manifolds.

The main theorem.

Theorem (F–Vidussi). Let N be a 3–manifold with empty or toroidal boundary. Let $\phi \in H^1(N;\mathbb{Z})$ such that for any finite index subgroup $\tilde{\pi} \subset \pi$ we have that
(A) $\Delta_{N,\phi}^{\pi/\tilde{\pi}}$ is monic,
(B) the degree of $\Delta_{N,\phi}^{\pi/\tilde{\pi}}$ is determined by $||\phi||_T$,
then (N, ϕ) fibers over S^1.

Corollary. The collection of Seiberg–Witten invariants of all finite covers of $S^1 \times N$ 'knows' whether $S^1 \times N$ is symplectic or not.

The first ingredient of the proof. Throughout assume we have a closed 3–manifold N and $\phi \in H^1(N;\mathbb{Z})$ primitive. Let $\Sigma \subset N$ be a connected Thurston norm minimizing surface dual to ϕ.

Theorem A. If (N,ϕ) satisfies (A) and (B) for any finite index $\tilde{\pi} \subset \pi$, then for either inclusion $\iota_\pm$

$$\pi_1(\Sigma) \to \pi_1(N \setminus \nu\Sigma)$$

induces an isomorphism of prosolvable completions.

Note that $\varphi : A \to B$ induces an isomorphism of prosolvable completions if and only if for any finite solvable group S we have a bijection

$$\varphi^* : \mathrm{Hom}(B,S) \to \mathrm{Hom}(A,S),$$

and if for any $\beta : B \to S$ we have

$$\mathrm{Im}(A \to B \to S) = \mathrm{Im}(B \to S).$$

Note that Theorem A can not be enough to conclude that $\pi_1(\Sigma) \to \pi_1(N \setminus \nu\Sigma)$ is an isomorphism, e.g. given an Alexander polynomial one knot K the map $\mathbb{Z} \to \pi_1(S^3 \setminus K)$ induces an isomorphism of prosolvable completions.

The three ingredients. Throughout assume have N and $\phi \in H^1(N;\mathbb{Z})$ primitive, $\Sigma \subset N$ Thurston norm minimizing dual to ϕ.

Theorem A. If (N,ϕ) satisfies (A) and (B) for any $\tilde{\pi} \subset \pi$, then for either inclusion $\iota_\pm$

$$\pi_1(\Sigma) \to \pi_1(N \setminus \nu\Sigma)$$

induces an isomorphism of prosolvable completions.

The following is well–known for hyperbolic mfds.

Theorem B (F–Aschenbrenner). Let W any 3–manifold, then $\pi_1(W)$ is virtually residually p.

Building on a result of Agol we show:

Theorem C. If $\pi_1(N \setminus \nu\Sigma)$ is residually finite solvable and if

$$\pi_1(\Sigma) \to \pi_1(N \setminus \nu\Sigma)$$

induces an isomorphism of prosolvable completions, then $\pi_1(\Sigma) \to \pi_1(N \setminus \nu\Sigma)$ is an isomorphism.

Proof of main theorem using Theorems A, B and C

Theorem (F–Vidussi). Let N be a 3–manifold with empty or toroidal boundary. Let $\phi \in H^1(N;\mathbb{Z})$ such that for any finite index subgroup $\tilde{\pi} \subset \pi$ we have that
(A) $\Delta^{\pi/\tilde{\pi}}_{N,\phi}$ is monic,
(B) the degree of $\Delta^{\pi/\tilde{\pi}}_{N,\phi}$ is determined by $||\phi||_T$,
then (N,ϕ) fibers over S^1.

Proof. Given (N,ϕ) let $p : \tilde{N} \to N$ a finite cover. Write $\tilde{\phi} = p^{-1}(\phi)$. Then
(1) (N,ϕ) fibers if and only if $(\tilde{N},\tilde{\phi})$ fibers.
(2) (N,ϕ) satisfies (A) and (B) if and only if $(\tilde{N},\tilde{\phi})$ does.
So by Theorem B we only have to prove the theorem for N with $\pi_1(N)$ residually p, in particular we can assume that $\pi_1(N)$ (and hence $\pi_1(N \setminus \nu\Sigma)$) is residually finite solvable. We can also assume that ϕ is primitive. Theorems A and C now give the main theorem.

Proof of Theorem A (1).
Assume we have (N, ϕ) such that for any finite index subgroup $\tilde{\pi} \subset \pi$ we have that
(A) $\Delta_{N,\phi}^{\pi/\tilde{\pi}}$ is monic,
(B) the degree of $\Delta_{N,\phi}^{\pi/\tilde{\pi}}$ determined by $||\phi||_T$.
We claim that $\iota_\pm : \pi_1(\Sigma) \to \pi_1(N \setminus \nu\Sigma)$ induces an isomorphism of prosolvable completions, i.e. for any finite solvable group S the map

$$\iota_\pm^* : \mathrm{Hom}(\pi_1(N \setminus \nu\Sigma), S) \to \mathrm{Hom}(\pi_1(\Sigma), S)$$

is a bijection and for any $\beta : \pi_1(N \setminus \nu\Sigma) \to S$ we have

$$\mathrm{Im}(\pi_1(\Sigma) \to \pi_1(N \setminus \Sigma) \to S) = \mathrm{Im}(\pi_1(N \setminus \Sigma) \to S).$$

A M–V argument shows that (A) and (B) imply that for any $\alpha : \pi_1(N) \to G$ with G finite, we have

$$\begin{array}{rcl} \mathrm{Im}\{\pi_1(\Sigma) \to \pi_1(N) \to G\} & = & \mathrm{Im}\{\pi_1(N \setminus \nu\Sigma) \to G\} \\ H_1(\Sigma; \mathbb{Z}[G]) & \xrightarrow{\cong} & H_1(N \setminus \nu\Sigma; \mathbb{Z}[G]) \end{array}$$

hence

$$\pi_1(\Sigma)/[\mathrm{Ker}(\alpha), \mathrm{Ker}(\alpha)] \xrightarrow{\cong} \pi_1(N \setminus \nu\Sigma)/[\mathrm{Ker}(\alpha), \mathrm{Ker}(\alpha)].$$

Note that this is only information for homomorphisms from $\pi_1(N)$ to a finite group.

Proof of Theorem A (2). Note that with G trivial we get $H_1(\Sigma;\mathbb{Z}) \xrightarrow{H}_1 (N \setminus \nu\Sigma;\mathbb{Z})$, i.e. the conditions above hold for S any finite abelian group.

Now we have to show that $\iota_\pm : \pi_1(\Sigma) \to \pi_1(N \setminus \nu\Sigma)$ looks like an isomorphism 'on the level of finite metabelian groups'. For example let $\beta : \pi_1(N \setminus \nu\Sigma) \to S$ be a homomorphism to a finite metabelian group. We need that

$$\text{Im}\{\pi_1(\Sigma) \to \pi_1(N\setminus\nu\Sigma) \to S\} = \text{Im}\{\pi_1(N\setminus\nu\Sigma) \to S\}.$$

The problem is that we can a priori not extend $\beta : \pi_1(N \setminus \nu\Sigma) \to S$ to a homomorphism from $\pi_1(N)$.

Now let P be the abelian group $P = S/[S,S]$ and write $n = |P|$ and $H := H_1(N \setminus \Sigma;\mathbb{Z})$. Since $\iota_\pm : H_1(\Sigma) \xrightarrow{\cong} H_1(N \setminus \nu\Sigma)$ we can extend $\pi : \pi_1(N \setminus \nu\Sigma) \to H \to H/nH$ (a characteristic quotient) to

$$\pi_1(N) \to \mathbb{Z} \ltimes H/nH \to \mathbb{Z}/k \ltimes H/nH.$$

for some k. (So we reduced the solvability length of S to extend the homomorphism from $\pi_1(N\setminus\nu\Sigma)$ over $\pi_1(N)$).

Proof of Theorem A (3). Recall that we started with a map $\beta : \pi_1(N\backslash\nu\Sigma) \to S$ to a finite metabelian group. We write $P = S/[S,S]$, $n = |P|$ and $H := H_1(N \setminus \Sigma)$. We can extend

$$\begin{array}{c}\pi : \pi_1(N \setminus \nu\Sigma) \to H \to H/nH \text{ to} \\ \alpha : \pi_1(N) \to \mathbb{Z} \ltimes H/nH \to \mathbb{Z}/k \ltimes H/nH.\end{array}$$

(reduced solvability length of S to extend over $\pi_1(N)$).

On the other hand we saw that for any $\alpha : \pi_1(N) \to G$, we have

$$\pi_1(\Sigma)/[\text{Ker}(\alpha), \text{Ker}(\alpha)] \xrightarrow{\cong} \pi_1(N\backslash\nu\Sigma)/[\text{Ker}(\alpha), \text{Ker}(\alpha)].$$

With $G = \mathbb{Z}/k \ltimes H/nH$ we immediately get that

$$\pi_1(\Sigma)/[\text{Ker}(\pi), \text{Ker}(\pi)] \xrightarrow{\cong} \pi_1(N\backslash\nu\Sigma)/[\text{Ker}(\pi), \text{Ker}(\pi)].$$

Put differently, with $\pi : \pi_1(N \setminus \nu\Sigma) \to H \to H/nH$ a homomorphism to an abelian group we now get metabelian information again, i.e. we recuperated the 'solvability length' we gave up in order to extend a homomorphism to $\pi_1(N)$. But

$$\begin{array}{ccl}\pi_1(N \setminus \nu\Sigma) & \to & \pi_1(N \setminus \nu\Sigma)/[\text{Ker}(\pi), \text{Ker}(\pi)] \\ & \searrow & \downarrow \\ & & S.\end{array}$$

So we get the result for finite metabelian S.
We now induct on solvability length of S.

Proof of Theorem B.

Theorem B (F–Aschenbrenner). $\pi_1(N)$ is virtually residually p.

It is well–known that finitely generated linear groups (subgroups of $\mathrm{GL}(n, \mathbb{C})$) are virtually residually p for almost all primes p. In particular hyperbolic 3–manifold groups are virtually residually p.

An argument similar to Hempel's proof that 3–manifold groups are residually finite now shows that all 3–manifold groups are virtually residually p.

Proof of Theorem C (1).

Let $\Sigma \subset N$. We have two inclusions $\iota_\pm : \Sigma \to M$.

Theorem C. If $\pi_1(N \setminus \nu\Sigma)$ is residually finite solvable and if

$$\iota_\pm : \pi_1(\Sigma) \to \pi_1(N \setminus \nu\Sigma)$$

induce an isomorphism of prosolvable completions, then $\pi_1(\Sigma) \to \pi_1(N \setminus \nu\Sigma)$ is an isomorphism.

The main tool is a theorem of Ian Agol. We need:

Definition. A group π is called RFRS if π is residually finite solvable and 'the rank of finite index subgroups grows quickly with the index'.

Remark.
(1) Free groups and surface groups are RFRS.
(2) Most 3–manifold groups are not RFRS.
(3) Are hyperbolic 3–manifold groups virtually RFRS?
(4) The $\pi_1(M)$ above is RFRS since 'solvably' $\pi_1(M)$ looks the same as a surface group.

Proof of Theorem C (2).

Agol's amazing theorem:

Theorem (Agol). Let W 3–manifold, $\pi_1(W)$ RFRS and $\phi \in H^1(W)$. Then there exists a finite solvable cover $p: \hat{W} \to W$ such that $p^*(\phi)$ lies on the closure of a fibered cone.

(In particular W is virtually fibered.)

We need a slightly different version.

Theorem (Agol). Let $M = N \setminus \nu\Sigma$ and W the double of $M = N \setminus \nu\Sigma$. If $\pi_1(M)$ is RFRS then there exists a homomorphism $\pi_1(W) \to \pi_1(M) \to S$ to a finite solvable group S such that for the corresponding cover $p: \hat{W} \to W$ of W the element $\hat{\Sigma} = p^*(\Sigma)$ lies on the closure of a fibered cone.

Proof of Theorem C (3).

Recall we have $\Sigma \subset N$ and write $M = N \setminus \nu\Sigma$. We assume that $\pi_1(M)$ is residually finite solvable and that

$$\iota_\pm : \pi_1(\Sigma) \to \pi_1(M)$$

induce an isomorphism of prosolvable completions. Let W the double of $M = N \setminus \nu\Sigma$. We want to show that M is a product, i.e. Σ lies in the *interior* of a fibered cone of W.

Since $\pi_1(\Sigma)$ is RFRS, $\pi_1(M)$ is also RFRS. By Agol there exists a solvable cover $p : \hat{W} \to W$ of W such that $\hat{\Sigma} = p^*(\Sigma)$ lies on the *closure* of a fibered cone.

Without loss of generality we can assume that already Σ lies in the closure of a fibered cone (Since $\hat{\Sigma}$ is a fiber iff Σ is a fiber).

But how do we get from Σ in the closure of a fibered cone to Σ in the interior of a fibered cone?

Proof of Theorem C (4).

Recall we have $\Sigma \subset N$ and write $M = N \setminus \nu\Sigma$. We assume that

$$\iota_\pm : \pi_1(\Sigma) \to \pi_1(M)$$

induce an isomorphism of prosolvable completions. Let W be the double of $M = N \setminus \nu\Sigma$. We also assume that Σ lies in the closure of a fibered cone. We have to show that Σ lies in the interior of a fibered cone.

If Σ lies in the cone on the *boundary* of a fibered face, then using the natural involution on the double W one can see that it lies on the boundary of at least *two* fibered faces.

But algebraically W looks like $\Sigma \times S^1$, i.e. Σ lies in the interior of a face of the Alexander norm (the algebraic version of the Thurston norm). But for fibered classes the Alexander norm agrees with the Thurston norm, which gives a contradiction.