ON HIGH-DIMENSIONAL REPRESENTATIONS OF KNOT GROUPS

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Abstract. Given a hyperbolic knot K and any $n \geq 2$ the abelian representations and the holonomy representation each give rise to an $(n-1)$-dimensional component in the $\mathrm{SL}(n,\mathbb{C})$-character variety. A component of the $\mathrm{SL}(n,\mathbb{C})$-character variety of dimension $\geq n$ is called high-dimensional.

It was proved by Cooper and Long that there exist hyperbolic knots with high-dimensional components in the $\mathrm{SL}(2,\mathbb{C})$-character variety. We show that given any non-trivial knot K and sufficiently large n the $\mathrm{SL}(n,\mathbb{C})$-character variety of K admits high-dimensional components.

1. Introduction

Given a knot $K \subset S^3$ we denote by $E_K = S^3 \setminus \nu K$ the knot exterior and we write $\pi_K = \pi_1(E_K)$. Furthermore, given a group G and $n \in \mathbb{N}$ we denote by $X(G, \mathrm{SL}(n,\mathbb{C}))$ the $\mathrm{SL}(n,\mathbb{C})$-character variety. We recall the precise definition in Section 2. It is straightforward to see that the abelian representations of a knot group π_K give rise to an $(n-1)$-dimensional subvariety of $X(\pi_K, \mathrm{SL}(n,\mathbb{C}))$ consisting solely of characters of abelian representations (see [HMnP15, Sec. 2]).

If K is hyperbolic, then we denote by $\widetilde{\mathrm{Hol}}\colon \pi_K \to \mathrm{SL}(2,\mathbb{C})$ a lift of the holonomy representation. For $n \geq 2$ we denote by $\zeta_n\colon \mathrm{SL}(2,\mathbb{C}) \to \mathrm{SL}(n,\mathbb{C})$ the, up to conjugation, unique rational irreducible representation of $\mathrm{SL}(2,\mathbb{C})$. Menal-Ferrer and Porti [MFP12b, MFP12a] showed that for any n the representation $\rho_n := \zeta_n \circ \widetilde{\mathrm{Hol}}$ is a smooth point of the $\mathrm{SL}(n,\mathbb{C})$-representation variety $R(\pi_k, \mathrm{SL}(n,\mathbb{C}))$. Moreover, Menal-Ferrer and Porti proved that the corresponding character χ_{ρ_n} is a smooth point on the character variety $X(\pi_K, \mathrm{SL}(n,\mathbb{C}))$, it is contained in a unique component of dimension $n-1$ [MFP12b, Theorem 0.4]. Also, the deformations of reducible representations studied in [HM14] and [BAH15] give rise to $(n-1)$-dimensional components in the character variety $X(\pi_K, \mathrm{SL}(n,\mathbb{C}))$.

The above discussion shows that given any knot K the character variety $X(\pi_K, \mathrm{SL}(n,\mathbb{C}))$ contains an $(n-1)$-dimensional subvariety consisting of abelian representations and if K is hyperbolic $X(\pi_K, \mathrm{SL}(n,\mathbb{C}))$ also contains an $(n-1)$-dimensional subvariety that contains characters of irreducible representations.

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This motivates the following definition. Given a knot K we say that a component of $X(\pi_K, \mathrm{SL}(n,\mathbb{C}))$ is *high-dimensional* if its dimension is greater than $n-1$. We summarize some known facts about the existence and non-existence of high-dimensional components of character varieties of knot groups.

- For $n = 3$ and K a non-alternating torus knot the variety $X(\pi_K, \mathrm{SL}(3,\mathbb{C}))$ has 3-dimensional components, whereas for alternating torus knots $X(\pi_K, \mathrm{SL}(3,\mathbb{C}))$ has only 2-dimensional components. In particular for alternating torus knots $X(\pi_K, \mathrm{SL}(3,\mathbb{C}))$ does not contain any high-dimensional components. For more details see [MP16, Thm. 1.1].
- For $n = 3$ and $K = 4_1$ the variety $X(\pi_K, \mathrm{SL}(3,\mathbb{C}))$ has five 2-dimensional components. Three of the five components contain characters of irreducible representations. There are no higher-dimensional components. (See [HMnP15, Thm. 1.2].)
- It was proved by Cooper and Long [CL96, Sec. 8] that for a given n there exists an alternating hyperbolic knot K_n in S^3 such that the $\mathrm{SL}(2,\mathbb{C})$-character variety admits a component of dimension at least n.

The main result of this note is to prove that the $\mathrm{SL}(n,\mathbb{C})$-character variety of every non-trivial knot admits high-dimensional components for n sufficiently large.

Theorem 1.1. *Let $K \subset S^3$ be a non-trivial knot. Then for all $N \in \mathbb{N}$ there exists an $n \geq N$ such that the character variety $X(\pi_K, \mathrm{SL}(n,\mathbb{C}))$ contains a high-dimensional component.*

Given a group G we now denote by $X^{irr}(G, \mathrm{SL}(n,\mathbb{C}))$ the character variety corresponding to irreducible representations. We refer to Section 3 for the precise definition. The following is now a more refined version of Theorem 1.1.

Theorem 1.2. *Let $K \subset S^3$ be a non-trivial knot. Then given any $N \in \mathbb{N}$ there exists a $p \geq N$, such that $X^{irr}(\pi_K, \mathrm{SL}(p,\mathbb{C}))$ contains a high-dimensional component.*

In the special case of the figure-eight knot we obtain a refined quantitative result:

Corollary 1.3. *Let $K \subset S^3$ be the figure-eight knot. Then for all $n \in \mathbb{N}$ the representation variety $X(\pi_K, \mathrm{SL}(10n,\mathbb{C}))$ has a component C of dimension at least $4n^2 - 1$. Moreover, C contains characters of irreducible representations.*

Remark. For a free group F_r we have: $\dim X(F_r, \mathrm{SL}(n,\mathbb{C}))/(n^2-1) = (r-1)$, and hence $\limsup_{n\to\infty} \dim X(G, \mathrm{SL}(n,\mathbb{C}))/(n^2-1) \leq (r-1)$ if G is generated by r elements. It follows from Corollary 1.3 that for the figure-eight knot $K = 4_1$

$$1/25 \leq \limsup_{n\to\infty} \Big(\dim X(\pi_K, \mathrm{SL}(n,\mathbb{C}))/(n^2-1) \Big) \leq 1$$

holds.

2. Representation and character varieties

Before we provide the proof of Theorem 1.1 we recall some definitions and facts. The general reference for representation and character varieties is Lubotzky's and Magid's book [LM85].

Given two representations $\rho_1\colon G\to \mathrm{GL}(n_1,\mathbb{C})$ and $\rho_2\colon G\to \mathrm{GL}(n_2,\mathbb{C})$ we define the *direct sum* $\rho_1\oplus\rho_2\colon G\to \mathrm{GL}(n_1+n_2,\mathbb{C})$ by

$$\big(\rho_1\oplus\rho_2\big)(\gamma)=\left(\begin{array}{c|c}\rho_1(\gamma) & 0\\ \hline 0 & \rho_2(\gamma)\end{array}\right).$$

Definition. We call a representation $\rho\colon G\to \mathrm{GL}(n,\mathbb{C})$ *reducible* if there exists a non-trivial subspace $V\subset\mathbb{C}^n$, $0\neq V\neq\mathbb{C}^n$, such that V is $\rho(G)$-stable. The representation ρ is called *irreducible* or *simple* if it is not reducible. A *semisimple* representation is a direct sum of simple representations.

Let $G=\langle g_1,\dots,g_r\rangle$ be a finitely generated group. A $\mathrm{SL}(n,\mathbb{C})$-representation is a homomorphism $\rho\colon G\to\mathrm{SL}(n,\mathbb{C})$. The $\mathrm{SL}(n,\mathbb{C})$-representation variety is

$$R(G,\mathrm{SL}(n,\mathbb{C}))\ =\ \mathrm{Hom}(G,\mathrm{SL}(n,\mathbb{C}))\ \subset\ \mathrm{SL}(n,\mathbb{C})^r\ \subset\ M_n(\mathbb{C})^r\ \cong\ \mathbb{C}^{n^2r}.$$

The representation variety $R(G,\mathrm{SL}(n,\mathbb{C}))$ is an affine algebraic set. It is contained in $\mathrm{SL}(n,\mathbb{C})^r$ via the inclusion $\rho\mapsto\big(\rho(g_1),\dots,\rho(g_r)\big)$, and it is the set of solutions of a system of polynomial equations in the matrix coefficients.

The group $\mathrm{SL}(n,\mathbb{C})$ acts by conjugation on $R(G,\mathrm{SL}(n,\mathbb{C}))$. More precisely, for $A\in\mathrm{SL}(n,\mathbb{C})$ and $\rho\in R(G,\mathrm{SL}(n,\mathbb{C}))$ we define $(A.\rho)(g)=A\rho(g)A^{-1}$ for all $g\in G$. In what follows we will write $\rho\sim\rho'$ if there exists an $A\in\mathrm{SL}(n,\mathbb{C})$ such that $\rho'=A.\rho$, and we will call ρ and ρ' *equivalent*. For $\rho\in R(G,\mathrm{SL}(n,\mathbb{C}))$ we define its *character* $\chi_\rho\colon G\to\mathbb{C}$ by $\chi_\rho(\gamma)=\mathrm{tr}(\rho(\gamma))$. We have $\rho\sim\rho'\Rightarrow\chi_\rho=\chi_{\rho'}$. Moreover, if ρ and ρ' are semisimple, then $\rho\sim\rho'$ if and only if $\chi_\rho=\chi_{\rho'}$. (See Theorems 1.27 and 1.28 in Lubotzky's and Magid's book [LM85].)

The *algebraic quotient* or *GIT quotient* for the action of $\mathrm{SL}(n,\mathbb{C})$ on $R(G,\mathrm{SL}(n,\mathbb{C}))$ is called the *character variety*. This quotient will be denoted by $X(G,\mathrm{SL}(n,\mathbb{C}))=R(G,\mathrm{SL}(n,\mathbb{C}))\mathbin{/\!\!/}\mathrm{SL}(n,\mathbb{C})$. The character variety is not necessarily an irreducible affine algebraic set. Work of C. Procesi [Pro76] implies that there exists a finite number of group elements $\{\gamma_i\mid 1\le i\le M\}\subset G$ such that the image of $t\colon R(G,\mathrm{SL}(n,\mathbb{C}))\to\mathbb{C}^M$ given by

$$t(\rho)=\big(\chi_\rho(\gamma_1),\dots,\chi_\rho(\gamma_M)\big)$$

can be identified with the affine algebraic set $X(G,\mathrm{SL}(n,\mathbb{C}))\cong t(R(G,\mathrm{SL}(n,\mathbb{C})))$, see also [LM85, p. 27]. This justifies the name *character variety*. For an introduction to algebraic invariant theory see Dolgachev's book [Dol03]. For a brief introduction to $\mathrm{SL}(n,\mathbb{C})$-representation and character varieties of groups see [Heu16].

Example 2.1. For a free group F_r of rank r we have $R(F_r, \mathrm{SL}(n,\mathbb{C})) \cong \mathrm{SL}(n,\mathbb{C})^r$ is an irreducible algebraic variety of dimension $r(n^2-1)$, and the dimension of the character variety $X(F_k, \mathrm{SL}(n,\mathbb{C}))$ is $(r-1)(n^2-1)$.

The first homology group of the knot exterior is isomorphic to $\mathbb{Z}$. A canonical surjection $\varphi\colon \pi_K \to \mathbb{Z}$ is given by $\varphi(\gamma) = \mathrm{lk}(\gamma, K)$ where lk denotes the linking number in S^3 (see [BZH13, 3.B]). Hence, every *abelian* representation of a knot group π_K factors through $\varphi\colon \pi_K \to \mathbb{Z}$. Here, we call ρ *abelian* if its image is abelian. Therefore, we obtain for each non-zero complex number $\eta \in \mathbb{C}^*$ an abelian representation $\eta^\varphi\colon \pi_K \to \mathrm{GL}(1,\mathbb{C}) = \mathbb{C}^*$ given by $\gamma \mapsto \eta^{\varphi(\gamma)}$. Notice that a 1-dimensional representation is always irreducible.

Let W be a finite dimensional $\mathbb{C}$-vector space. For every representation $\rho\colon G \to \mathrm{GL}(W)$ the vector space W turns into a $\mathbb{C}[G]$-left module via ρ. This $\mathbb{C}[G]$-module will be denoted by W_ρ or simply W if no confusion can arise. Notice that every finite dimensional $\mathbb{C}$-vector space W which is a $\mathbb{C}[G]$-left module gives a representation $\rho\colon G \to \mathrm{GL}(W)$, and by fixing a basis of W we obtain a matrix representation.

The following lemma follows from Proposition 1.7 in [LM85] and the discussion therein.

Lemma 2.2. *Any group epimorphism $\alpha\colon G \twoheadrightarrow F$ between finitely generated groups induces a closed embedding $R(F, \mathrm{SL}(n,\mathbb{C})) \hookrightarrow R(G, \mathrm{SL}(n,\mathbb{C}))$ of algebraic varieties, and an injection*

$$X(F, \mathrm{SL}(n,\mathbb{C})) \hookrightarrow X(G, \mathrm{SL}(n,\mathbb{C}))\,.$$

Let $H \le G$ be a subgroup of finite index. Then the restriction of a representation $\rho\colon G \to \mathrm{SL}(n,\mathbb{C})$ to H will be denoted by $\mathrm{res}^G_H\,\rho$ or simply by $\rho|_H$ if no confusion can arise. This restriction is compatible with the action by conjugation and it induces a regular map $\nu\colon X(G, \mathrm{SL}(n,\mathbb{C})) \to X(H, \mathrm{SL}(n,\mathbb{C}))$. In what follows we will make use of the following result of A.S. Rapinchuk which follows directly from [Rap98, Lemma 1]

Lemma 2.3. *If $H \le G$ is a subgroup of finite index k, then*

$$\nu\colon X(G, \mathrm{SL}(n,\mathbb{C})) \to X(H, \mathrm{SL}(n,\mathbb{C}))$$

has finite fibers.

2.1. **The induced representation.** Let G be a group and let $H \le G$ be a subgroup of finite index k. Given a representation $\alpha\colon H \to \mathrm{GL}(m,\mathbb{C})$ we refer to the representation of G that is given by left multiplication by G on

$$\mathbb{C}[G] \otimes_{\mathbb{C}[H]} \mathbb{C}^m$$

as the induced representation. We denote by $e_1, \dots, e_m$ the standard basis of $\mathbb{C}^m$ and we pick representatives $g_1, \dots, g_k$ of G/H. It is straightforward to see that $g_i \otimes e_j$ with

$i \in \{1,\dots,k\}$ and $j \in \{1,\dots,m\}$ form a basis for $\mathbb{C}[\pi] \otimes_{\mathbb{C}[\tilde{\pi}]} \mathbb{C}^m$ as a complex vector space. Using the ordered basis

$$g_1 \otimes e_1, \dots, g_1 \otimes e_m, \dots, g_k \otimes e_1, \dots, g_k \otimes e_m$$

the induced representation can be viewed as a representation $\operatorname{ind}_H^G \alpha\colon G \to \mathrm{GL}(mk,\mathbb{C})$. If $\alpha\colon H \to \mathrm{SL}(m,\mathbb{C})$ is a representation into the special linear group, then for $g \in G$ a priori the determinant of $\operatorname{ind}_H^G \alpha(g)$ is in $\{\pm 1\}$. But it is straightforward to see that if m is even, then $\operatorname{ind}_H^G \alpha$ defines in fact a representation $G \to \mathrm{SL}(mk,\mathbb{C})$.

Lemma 2.4. *Let m be even, and let $H \le G$ be a subgroup of finite index k. Then the map*

$$\iota\colon R(H,\mathrm{SL}(m,\mathbb{C})) \to R(G,\mathrm{SL}(mk,\mathbb{C}))$$

given by $\iota(\alpha) = \operatorname{ind}_H^G \alpha$ is an injective algebraic map. It depends on the choice of a system of representatives, and it is compatible with the action of $\mathrm{SL}(m,\mathbb{C})$ *and* $\mathrm{SL}(mk,\mathbb{C})$ *respectively.*

Moreover, the corresponding regular map (which does not depend on the choice of a system of representatives)

$$\bar{\iota}\colon X(H,\mathrm{SL}(m,\mathbb{C})) \to X(G,\mathrm{SL}(mk,\mathbb{C}))$$

has finite fibers.

Proof. A very detailed proof of the first statement can be found in [CR90, §10.A] (see also [LM85, pp. 9–10] and [Rap98]). The second part is Lemma 3 from [Rap98] □

We are now in the position to prove our main result:

Proof of Theorem 1.1. Let K be a non-trivial knot. We write $\pi_K = \pi_1(E_K)$. Cooper, Long, and Reid [CLR97, Theorem 1.3] (see also [But04, Corollary 6]) showed that G admits a finite-index subgroup H that admits an epimorphism $\alpha\colon H \to F_2$ onto a free group on two generators. It is clear, see Example 2.1, that $R(F_2,\mathrm{SL}(m,\mathbb{C})) \cong \mathrm{SL}(m,\mathbb{C})^2$, and

$$\dim X(F_2,\mathrm{SL}(m,\mathbb{C})) \;=\; m^2-1.$$

It follows from Lemma 2.2 that the variety $X(H,\mathrm{SL}(m,\mathbb{C}))$ has a component of dimension at least m^2-1. We denote by k the index of H in G, and we will suppose that m is even. Then it follows from Lemma 2.4 that $X(G,\mathrm{SL}(mk,\mathbb{C}))$ contains an irreducible component of dimension at least m^2-1.

Now for all $m > k$ we have $m^2-1 > mk-1$. Therefore, for a given $N \in \mathbb{N}$ we choose an even $m \in \mathbb{N}$, $m > k$, such that $n := mk \ge N$. The character variety $X(\pi_K,\mathrm{SL}(n,\mathbb{C}))$ contains an irreducible component whose dimension is bigger than $m^2-1 > mk-1 = n-1$. □

3. Proof of Theorem 1.2

We let $R^{irr}(G,\mathrm{SL}(n,\mathbb{C}))\subset R(G,\mathrm{SL}(n,\mathbb{C}))$ denote the Zariski-open subset of irreducible representations. The set $R^{irr}(G,\mathrm{SL}(n,\mathbb{C}))$ is invariant by the $\mathrm{SL}(n,\mathbb{C})$-action, and we will denote by $X^{irr}(G,\mathrm{SL}(n,\mathbb{C}))\subset X(G,\mathrm{SL}(n,\mathbb{C}))$ its image in the character variety. Notice that $X^{irr}(G,\mathrm{SL}(n,\mathbb{C}))$ is an orbit space for the action of $\mathrm{SL}(n,\mathbb{C})$ on $R^{irr}(G,\mathrm{SL}(n,\mathbb{C}))$ (see [New78, Chap. 3, §3]).

Before we can give the proof of Theorem 1.2 we need to introduce several further definitions. These notations are classic (see [Ser77, Bro94] for more details).

Let H and K be two subgroups of finite index of G, and let $\alpha\colon H\to \mathrm{GL}(W)$ be a linear representation. Then for all $g\in G$ we obtained the *twisted* representation $\alpha^g\colon gHg^{-1}\to\mathrm{GL}(W)$ given by

$$\alpha^g(x)=\alpha(g^{-1}xg),\ \text{for } x\in gHg^{-1}.$$

Notice that the twisted representation α^g is irreducible or semisimple if and only if α is irreducible or semisimple respectively.

Now, we choose a set of representatives S of the (K,H) double cosets of G. For $s\in S$, we let $H_s=sHs^{-1}\cap K\leq K$. We obtain a homomorphism $\mathrm{res}^{sHs^{-1}}_{H_s}\alpha^s\colon H_s\to\mathrm{GL}(W)$ by restriction of α^s to H_s. The representation $\mathrm{res}^G_K\mathrm{ind}^G_H\alpha$ is equivalent to the direct sum of twisted representations:

$$\mathrm{res}^G_K\,\mathrm{ind}^G_H\,\alpha\ \cong\ \bigoplus_{s\in S}\mathrm{ind}^K_{H_s}\,\mathrm{res}^{sHs^{-1}}_{H_s}\,\alpha^s\,. \tag{1}$$

Equation (1) takes a simple form if $H=N=K$ is a normal subgroup of finite index of G. We obtain:

$$\mathrm{res}^G_N\,\mathrm{ind}^G_N\,\alpha\cong\bigoplus_{s\in S}\alpha^s \tag{2}$$

where S is a set of representatives of the N cosets of G.

In what follows we will make use of the following lemmas:

Lemma 3.1. *Let G be a group, $H\leq G$ a subgroup of finite index, and $\rho\colon G\to\mathrm{GL}(V)$ be a representation. If $\mathrm{res}^G_H\rho\colon H\to\mathrm{GL}(V)$ is a semisimple, then $\rho\colon G\to\mathrm{GL}(V)$ is semisimple.*

Proof. This is Theorem 1.5 in [Weh73]. □

Lemma 3.2. *Let G be a group, $H\leq G$ a subgroup of finite index, and $\alpha\colon H\to\mathrm{GL}(W)$ be a representation. If α is irreducible, then $\mathrm{ind}^G_H\alpha$ is semisimple.*

Proof. We can choose a normal subgroup $N\trianglelefteq G$ of finite index such that $N\leq H$. More precisely, we can take

$$N=\bigcap_g gHg^{-1}$$

to be the *normal core* of H in G. We choose a set of representatives S of the (N,H) double cosets of G. In this case we obtain that $H_s=sHs^{-1}\cap N=s(H\cap N)s^{-1}=N$, and the double coset NsH is equal to sH since $N \subset H$ is normal. Therefore, equation (1) gives:

$$\operatorname{res}_N^G \operatorname{ind}_H^G \alpha \cong \bigoplus_{s \in S} \operatorname{res}_N^{sHs^{-1}} \alpha \,.$$

Now, $\operatorname{res}_N^{sHs^{-1}} \alpha\colon N \to \mathrm{GL}(W)$ is a *twist* of $\alpha|_N$ i.e. for all $g \in N$ we have

$$\operatorname{res}_N^{sHs^{-1}} \alpha(g) = \alpha(s^{-1}gs) = (\alpha|_N)^s(g)\,.$$

By Clifford's theorem [Weh73, Theorem 1.7], we obtain that $\alpha|_N$ is semisimple. We have that $\alpha|_N = \alpha_1 \oplus \cdots \oplus \alpha_k$ is a direct sum of simple representations. Therefore,

$$\left(\operatorname{ind}_H^G \alpha\right)\big|_N \cong \bigoplus_{s \in S} \alpha_1^s \oplus \cdots \oplus \alpha_k^s$$

is the direct sum of irreducible representations. This proves that $\left(\operatorname{ind}_H^G \alpha\right)\big|_N$ is semisimple, and it follows from Lemma 3.1 that $\operatorname{ind}_H^G \alpha$ is semisimple. □

Corollary 3.3. *Let G be a group,and let $N \triangleleft G$ a normal subgroup of finite index. If $\alpha\colon N \to \mathrm{SL}(V)$ is irreducible, then $\operatorname{ind}_N^G(\alpha)$ is semisimple.*

Moreover, if $\operatorname{ind}_N^G(\alpha) \cong \rho_1 \oplus \cdots \oplus \rho_l$ is a decomposition of $\operatorname{ind}_N^G(\alpha)$ into irreducible representations $\rho_j\colon G \to \mathrm{SL}(V_j)$, then $\dim V$ divides $\dim V_j$ and hence

$$\dim V \;\le\; \dim V_j \;\le\; \dim(V) \cdot [G : N]\,.$$

Proof. The first part follows directly from equation (2) and Lemma 3.1 since α^s is irreducible for all $s \in G$. Notice that S is now a set of representatives of the cosets G/N. Moreover, we obtain

$$\bigoplus_{s \in S} \alpha^s \;\cong\; \operatorname{res}_N^G \operatorname{ind}_N^G \alpha \;\cong\; \rho_1|_N \oplus \cdots \oplus \rho_l|_N\,.$$

If $\rho_j|_N$ is irreducible, then it must be isomorphic to one of the twisted representations α^s. Otherwise $\rho_j|_N$ is isomorphic to a direct sum of twisted representations α^s, $s \in S$, and hence $\dim V_j$ is a multiple of $\dim V$. □

Lemma 3.4. *Let G be a group and let $H \le G$ be a finite index subgroup.*

If there exists a surjective homomorphism $\varphi\colon H \twoheadrightarrow F_2$ onto a free group of rank two, then there exists a normal subgroup $N \trianglelefteq G$ of finite index such that $N \le H$, and $\varphi(N) \le F_2$ is a free group of finite rank $r \ge 2$.

Proof. Let N be the normal core of H i.e. $N = \bigcap_{g \in G} gHg^{-1} \trianglelefteq G$. The normal subgroup is a finite index subgroup of G, and $N \le H$. Now, $H \ge \varphi^{-1}(\varphi(N)) \ge N$, and therefore $\varphi^{-1}(\varphi(N))$ is also of finite index in H (and hence in G). Hence, $\varphi(N) \trianglelefteq F_2$ is of finite index, and $\varphi(N)$ is a free group of rank $r = [F_2 : \varphi(N)] + 1 \ge 2$. □

The following theorem implies in particular Theorem 1.2 from the introduction.

Theorem 3.5. *Let $K \subset S^3$ be a non-trivial knot. Then there exists $k \in \mathbb{N}$ such that for all even $m \in \mathbb{N}$ there exists $p \in \mathbb{N}$ such that $m \le p \le mk$, and*

$$\dim X^{irr}(\pi_K, \mathrm{SL}(p,\mathbb{C})) \ge \frac{m^2-k}{k}\,.$$

In particular, for m even with $m > k^2$ there exists $p \in \mathbb{N}$ such that $m \le p < m\sqrt{m}$, and

$$\dim X^{irr}(\pi_K, \mathrm{SL}(p,\mathbb{C})) \ge \frac{m^2-k}{k} > km - 1 \ge p-1\,.$$

Proof. By Lemma 3.4 there exists a finite index normal subgroup $N \trianglelefteq \pi_K$ of the knot group π_K, and an epimorphism $\psi\colon N \twoheadrightarrow F_2$. We put $k = [\pi_K : N]$.

For all even $m \in \mathbb{N}$, we obtain a regular map $\psi^*\colon X^{irr}(F_2, \mathrm{SL}(m,\mathbb{C})) \to X(N, \mathrm{SL}(m,\mathbb{C}))$ and we let denote $C \subset X(N, \mathrm{SL}(m,\mathbb{C}))$ the image of ψ^*. By Chevalley's theorem, the set $C \subset X(N, \mathrm{SL}(m,\mathbb{C}))$ is constructible (see [Bor91, 10.2]). Again, by Chevalley's theorem the image $D := \bar\iota(C)$, $\bar\iota\colon X(N, \mathrm{SL}(m,\mathbb{C})) \to X(\pi_K, \mathrm{SL}(km,\mathbb{C}))$, is also a constructible set. Notice that $\dim C = \dim D = m^2 - 1$ since ϕ^* is an embedding and $\bar\iota$ has finite fibers.

If D contains a character of an irreducible representation, then $D \cap X^{irr}(\pi_K, \mathrm{SL}(km,\mathbb{C}))$ contains a Zariski-open subset of $\bar D$ which is of dimension $m^2 - 1 \ge (m^2-k)/k$. Hence the conclusion of the theorem is satisfied for $p = km$.

If D does not contain irreducible representations, then $D \subset X^{red}(\pi_K, \mathrm{SL}(km,\mathbb{C}))$. In this case we can choose for a given $\chi \in D$ a semisimple representation ρ such that $\chi_\rho = \chi$. Now, we follow the argument in the proof of Corollary 3.3 and we obtain $\rho \sim \rho_1 \oplus \cdots \oplus \rho_l$ where $\rho_j\colon \pi_K \to \mathrm{GL}(p_j,\mathbb{C})$ and $m \le p_j < km$. For the l-tuple $(p_1,\dots,p_l)$ we consider the regular map

$$\Phi_{(p_1,\dots,p_l)}\colon R(\pi_K, \mathrm{SL}(p_1,\mathbb{C})) \times \cdots \times R(\pi_K, \mathrm{SL}(p_l,\mathbb{C})) \times (\mathbb{C}^*)^{l-1} \to R(\pi_K, \mathrm{SL}(km,\mathbb{C}))$$

given by

$$\Phi_{(p_1,\dots,p_l)}(\rho_1,\dots,\rho_l,\lambda_1,\dots,\lambda_{l-1}) = \bigoplus_{i=1}^{l-1} (\rho_i \otimes \lambda_i^{p_l\varphi}) \oplus \big(\rho_l \otimes (\lambda_1^{-p_1}\cdots\lambda_{l-1}^{-p_{l-1}})^\varphi\big)\,.$$

The map $\Phi_{(p_1,\dots,p_l)}$ induces a map between the character varieties

$$\bar\Phi_{(p_1,\dots,p_l)}\colon X(\pi_K, \mathrm{SL}(p_1,\mathbb{C})) \times \cdots \times X(\pi_K, \mathrm{SL}(p_l,\mathbb{C})) \times (\mathbb{C}^*)^{l-1} \to X^{red}(\pi_K, \mathrm{SL}(km,\mathbb{C}))\,.$$

The restriction of $\bar\Phi_{(p_1,\dots,p_l)}$ to

$$X^{irr}(\pi_K, \mathrm{SL}(p_1,\mathbb{C})) \times \cdots \times X^{irr}(\pi_K, \mathrm{SL}(p_l,\mathbb{C})) \times (\mathbb{C}^*)^{l-1}$$

has finite fibers and we denote the image par $D_{(p_1,\dots,p_l)}$. Again, by Chevalley's theorem, the image $D_{(p_1,\dots,p_l)} \subset X^{red}(\pi_K, \mathrm{SL}(km,\mathbb{C}))$ is a constructible set, and

$$\dim D_{(p_1,\dots,p_l)} = \sum_{j=1}^{l} \dim X(\pi_K, \mathrm{SL}(p_j,\mathbb{C})) + l - 1\,. \tag{3}$$

By Corollary 3.3 D is covered by finitely many sets of the form $D_{(p_1,\dots,p_l)}$. Since $\dim D = m^2-1$ there must be at least one set $D_{(p_1,\dots,p_l)}$ of dimension at least m^2-1. If we apply (3) to this choice we obtain that

$$\sum_{j=1}^{l} \dim X(\pi_K, \mathrm{SL}(p_j,\mathbb{C})) \ge m^2 - l\,.$$

In particular there exists a j such that the corresponding summand is greater or equal than $\frac{m^2-l}{l}$. Note that from $m \le p_j \le mk$ for $j=1,\dots,l$ and $p_1+\dots+p_l = mk$ it follows that $l \le k$ which in turn implies that $\frac{m^2-l}{l} \ge \frac{m^2-k}{k}$. Summarizing we see that

$$\dim X^{irr}(\pi_K, \mathrm{SL}(p_j,\mathbb{C})) \ge \frac{m^2-k}{k}\,.$$

This concludes the proof of the first statement of the theorem.

The second statement follows from the first statement using some elementary algebraic inequalities. □

4. The character variety of the figure-eight knot

The aim of this section is to prove Corollary 1.3. In order to study the character variety of the figure-eight knot we have to address the question under which conditions the induced representation is irreducible. This is a quite classical subject and we follow Serre's exposition in [Ser77].

Lemma 4.1 (Schur's lemma and its converse)**.** *Let $\rho\colon G \to \mathrm{GL}(W)$ be a representation. If ρ is simple, then $\mathrm{Hom}_{\mathbb{C}[G]}(W,W) \cong \mathbb{C}$. Conversely, if $\rho\colon G \to \mathrm{GL}(W)$ is semisimple and $\mathrm{Hom}_{\mathbb{C}[G]}(W,W) \cong \mathbb{C}$, then ρ is simple.*

Proof. A proof of Schur's lemma can be found in [CR06, §27], and its converse is an easy exercise. Notice that the hypothesis *semisimple* is essential for the converse of Schur's lemma (see [CR06, p. 189]). □

4.1. **The adjoint isomorphism.** Let Q, R be rings and let ${}_QS$, ${}_RT_Q$, ${}_RU$ be modules (here the location of the indices indicates whether these are left or right module structures). Then $\mathrm{Hom}_R(T,U)$ is a left Q-module via $qf(v) = f(vq)$ for all $v \in V$, and

$$\mathrm{Hom}_Q\big(S, \mathrm{Hom}_R(T,U)\big) \cong \mathrm{Hom}_R\big(T \otimes_Q S, U\big) \tag{4}$$

(see [Rot09, Theorem 2.76]).

Lemma 4.2. *Let G be a group and $H \le G$ a subgroup of finite index. For each $\mathbb{C}[H]$-left module W and a $\mathbb{C}[G]$-left module V we obtain*

$$\mathrm{Hom}_{\mathbb{C}[G]}\big(V, \mathrm{ind}_H^G W\big) \cong \mathrm{Hom}_{\mathbb{C}[H]}\big(\mathrm{res}_H^G V, W\big)\,, \tag{5}$$

and

$$\mathrm{Hom}_{\mathbb{C}[H]}\big(W, \operatorname{res}_H^G V\big) \cong \mathrm{Hom}_{\mathbb{C}[G]}\big(\operatorname{ind}_H^G W, V\big). \tag{6}$$

Proof. For proving (5) we apply (4) with $Q = \mathbb{C}[G]$, $R = \mathbb{C}[H]$, $S = V$, $T = \mathbb{C}[G]$, and $U = W$:

$$\mathrm{Hom}_{\mathbb{C}[G]}\big(V, \mathrm{Hom}_{\mathbb{C}[H]}(\mathbb{C}[G], W)\big) \cong \mathrm{Hom}_{\mathbb{C}[H]}\big(\mathbb{C}[G] \otimes_{\mathbb{C}[G]} V, W\big).$$

Since $H \le G$ is of finite index we obtain that the coiduced module $\operatorname{coind}_H^G(W) := \mathrm{Hom}_{\mathbb{C}[H]}(\mathbb{C}[G], W)$ and the induced module $\operatorname{ind}_H^G(W)$ are isomorphic as $\mathbb{C}[G]$-left modules (see [Bro94, III (5.9)]). Moreover, $\operatorname{res}_H^G(U)$ and $\mathbb{C}[G] \otimes_{\mathbb{C}[G]} U$ are isomorphic as left $\mathbb{C}[H]$-modules, and (5) follows.

In oder to prove (6) we apply (4) with $Q = \mathbb{C}[H]$, $R = \mathbb{C}[G]$, $S = W$, $T = \mathbb{C}[G]$ and $U = V$:

$$\mathrm{Hom}_{\mathbb{C}[H]}\big(W, \mathrm{Hom}_{\mathbb{C}[G]}(\mathbb{C}[G], V)\big) \cong \mathrm{Hom}_{\mathbb{C}[G]}\big(\mathbb{C}[G] \otimes_{\mathbb{C}[H]} W, V\big).$$

The left $\mathbb{C}[H]$-module $\mathrm{Hom}_{\mathbb{C}[G]}(\mathbb{C}[G], V)$ is isomorphic to $\operatorname{res}_H^G(V)$, and hence (6) follows. ☐

4.2. **Mackey's irreducibility criterion.** Let $H \le G$ be a subgroup of finite index, and let $\alpha\colon H \to \mathrm{GL}(W)$ be a representation. For $s \in G$ we obtain the conjugate representation $\alpha^s\colon sHs^{-1} \to \mathrm{GL}(W)$. We define $H_s := sHs^{-1} \cap H$, and we set $W_s := \operatorname{res}_{H_s}^{sHs^{-1}} W_{\alpha^s}$. In what follows we call two semisimple representations V and V' of G *disjoint* if $\mathrm{Hom}_{\mathbb{C}[G]}(V, V') = 0$

Lemma 4.3 (Mackey's criterion)**.** *Let $H \le G$ be a subgroup of finite index. We suppose that $\alpha\colon H \to \mathrm{GL}(W)$ is an irreducible representation. Then $\operatorname{ind}_H^G \alpha$ is irreducible if and only if for all $s \in G - H$ the $\mathbb{C}[H_s]$-modules W_s and $\operatorname{res}_{H_s}^H W$ are disjoint.*

Proof. It follows from Lemma 3.2 that $\operatorname{ind}_H^G \alpha$ is semisimple. Therefore, by Lemma 4.1 it follows that $\operatorname{ind}_H^G \alpha$ is irreducible if and only if $\mathrm{Hom}_{\mathbb{C}[G]}(\operatorname{ind}_H^G W, \operatorname{ind}_H^G W) \cong \mathbb{C}$. We choose a system S of the (H, H) double cosets of G. Then we obtain that

$$\begin{aligned}
\mathrm{Hom}_{\mathbb{C}[G]}(\operatorname{ind}_H^G W, \operatorname{ind}_H^G W) &\cong \mathrm{Hom}_{\mathbb{C}[H]}(W, \operatorname{res}_H^G \operatorname{ind}_H^G W), && \text{(by (6))}\\
&\cong \bigoplus_{s \in S} \mathrm{Hom}_{\mathbb{C}[H]}(W, \operatorname{ind}_{H_s}^H W_s), && \text{(by (1) for } K = H)\\
&\cong \bigoplus_{s \in S} \mathrm{Hom}_{\mathbb{C}[H_s]}(\operatorname{res}_{H_s}^H W, W_s), && \text{(by (5))}.
\end{aligned}$$

Now, if $s \in H$, then $H_s = H$ and $W_s \cong W$, and $\mathrm{Hom}_H(W, W) \cong \mathbb{C}$ since W is an irreducible $\mathbb{C}[H]$-module (see Lemma 4.1). Therefore, $\mathrm{Hom}_{\mathbb{C}[G]}(\operatorname{ind}_H^G W, \operatorname{ind}_H^G W) \cong \mathbb{C}$ if and only if $\mathrm{Hom}_{\mathbb{C}[H_s]}(\operatorname{res}_{H_s}^H W, W_s) = 0$ for all $s \in G - H$. ☐

Corollary 4.4. *G be a group,and let $N \trianglelefteq G$ be a normal subgroup of finite index. If $\alpha\colon N \to \mathrm{SL}(V)$ is irreducible, then $\operatorname{ind}_N^G(\alpha)$ is irreducible if and only if α and α^s are non-equivalent for all $s \in G - N$.*

Proof. We apply Lemma 4.3 in the case $H = N$ is a normal subgroup. It follows that $N_s = N$ and α^s is the twisted representation. Notice that two irreducible representations are disjoint if and only if they are not equivalent. □

4.3. **Example.** Let $K = \mathfrak{b}(\alpha,\beta) \subset S^3$ be a two-bridge knot. A presentation of the knot group $\pi_{\alpha,\beta}$ is given by

$$\pi_{\alpha,\beta} = \langle s,t \mid l_s s = t l_s\rangle \quad \text{where } l_s = s^{\epsilon_1}t^{\epsilon_2}\cdots t^{\epsilon_{\alpha-1}}, \text{ and } \epsilon_k = (-1)^{[k\beta/\alpha]}.$$

We consider the following representation of $\pi_{\alpha,\beta}$ into the symmetric group: $\delta\colon \pi_{\alpha,\beta} \to \mathcal{S}_\alpha$ given by:

$$\delta(s) = (1)(2,2n+1)(3,2n)\cdots(n+1,n+2), \text{ and } \delta(a) = (1,2,\ldots,\alpha)\,, \tag{7}$$

where $\alpha = 2n+1$ and $a = ts^{-1}$. The image of δ is a dihedral group. We adopt the convention that permutations act on the right on $\{1,\ldots,\alpha\}$, and hence $\pi_{\alpha,\beta}$ acts on the right. We put $N = \mathrm{Ker}(\delta)$ and $H = \mathrm{Stab}(1) = \{g \in G \mid 1^{\delta(g)} = 1\}$. We have $N \le H$, $N \trianglelefteq \pi_{\alpha,\beta}$, $[\pi_{\alpha,\beta} : H] = \alpha$, and $[H : N] = 2$.

The irregular covering of E_K corresponding to H has been studied since the beginning of knot theory. K. Reidemeister calculated a presentation of H. Moreover, he showed that the total space of the corresponding irregular branched covering $(\widehat{S}^3, \widehat{K}) \to (S^3, \mathfrak{b}(\alpha,\beta))$ is simply connected. He proved also that the branching set $\widehat{K}$ consists of $(n+1)$ unknotted components (see [Rei29, Rei74]). G. Burde proved in [Bur71] that $\widehat{S}^3$ is in fact the 3-sphere and he determined the nature on the branching set explicitly in [Bur88]. More recently, G. Walsh studied the regular branched covering corresponding to N [Wal05]. She proved that the corresponding branching set is a great circle link in S^3.

Let us consider the figure-eight knot $\mathfrak{b}(5,3)$ and its group $\pi_{5,3}$:

$$\pi_{5,3} \cong \langle s,t \mid st^{-1}s^{-1}ts = tst^{-1}s^{-1}t\rangle \cong \langle s,a \mid a^{-1}s^{-1}asa^{-1}sas^{-1}a^{-1}\rangle \tag{8}$$

where $a = ts^{-1}$. In this case we have that $\delta\colon \pi_{5,3} \to \mathcal{S}_5$ is given by

$$\delta(s) = (1)(2,5)(3,4) \quad \text{and} \quad \delta(a) = (1,2,3,4,5)\,. \tag{9}$$

For the figure-eight knot, the link $\widehat{K}_{4_1}$ has a particular simple form (see Figure 1 and [BZH13, Example 14.22]). If we fill in the component $\hat{k}_0$, then $\widehat{K}_{4_1}$ transforms into

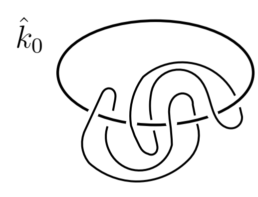

FIGURE 1. The link $\widehat{K}_{4_1} \subset \widehat{S}^3$.

the trivial link of two components. Therefore $H/\langle\langle y_0\rangle\rangle \cong F_2$ where $\langle\langle y_0\rangle\rangle$ denotes the normal subgroup of H generated by the meridian y_0 of $\hat{k}_0$.

More precisely, we can use the Reidemeister–Schreier method [ZVC80] for finding a presentation for H: $\{1, a, a^2, a^3, a^4\}$ is a Schreier representative system for the right cosets modulo H. Hence generators of H are

$$y_0 = s,\ y_i = a^i s a^{i-5},\ i = 1,2,3,4, \quad \text{and } y_5 = a^5\,. \tag{10}$$

We obtain defining relations r_i, $i = 0,1,2,3,4$, for H by expressing

$$r_i = a^i(a^{-1}s^{-1}asa^{-1}sas^{-1}a^{-1})a^{-i}$$

as a word in the y_j:

$$r_0 = y_5^{-1}y_1^{-1}y_2^2y_1^{-1},\ r_1 = y_0^{-1}y_1y_3y_2^{-1},\ r_2 = y_4^{-1}y_5y_0y_5^{-1}y_4y_3^{-1},$$
$$r_3 = y_3^{-1}y_4y_0y_4^{-1},\ r_4 = y_2^{-1}y_3y_1y_5y_0^{-1}y_5^{-1}\,.$$

It follows that $H/\langle\langle y_0\rangle\rangle \cong \langle y_1, y_2, y_3, y_4, y_5 \mid y_3, y_1 = y_2, y_5\rangle \cong F(y_1, y_4)$. Therefore, a surjection $\psi\colon H \twoheadrightarrow F_2 = F(x,y) \cong$ is given by

$$\psi(y_0) = \psi(y_3) = \psi(y_5) = 1, \quad \psi(y_1) = \psi(y_2) = x \quad \text{and} \quad \psi(y_4) = y\,. \tag{11}$$

We have $y_0 \in H - N$, and $y_0 \in \operatorname{Ker}(\psi)$.

We need also generators of N: we have $y_0 \notin N$ hence Reidemeister-Schreier gives that N is generated by

$$y_iy_0^{-1}, \quad y_5, \quad y_0^2, \quad y_0y_i, \quad y_0y_5y_0^{-1} \quad \text{where } i = 1,2,3,4. \tag{12}$$

Lemma 4.5. *Let $\beta\colon F_2 \to \mathrm{SL}(2m,\mathbb{C})$ be irreducible. Then $\alpha = \beta\circ\psi\colon H \to F_2 \to \mathrm{SL}(2m,\mathbb{C})$ and $\alpha|_N\colon H \to F_2 \to \mathrm{SL}(2m,\mathbb{C})$ are also irreducible.*

Proof. If $\beta\colon F_2 \to \mathrm{GL}(m,\mathbb{C})$ is irreducible, then $\alpha = \beta\circ\psi$ is also irreducible since $\psi\colon H \to F_2$ is surjective (the representations α and β have the same image). Similarly, (12) and (11) give that $\psi|_N\colon N \to F_2$ is are also surjective (β and $\alpha|_N$ have the same image). □

We obtain a component of representations $X_0 \subset X(H, \mathrm{SL}(2n,\mathbb{C}))$ with $\dim X_0 \geq 4n^2 - 1$, and X_0 contains irreducible representations. In order to apply Lemma 4.3 we notice that $\{1, a, a^2\}$ is a representative system for the (H,H)-double cosets

$$\pi_{5,3} = H \sqcup H\,a\,H \sqcup H\,a^2\,H\,.$$

More precisely, we have $HaH = Ha \sqcup Ha^4$, and $Ha^2H = Ha^2 \sqcup Ha^3$. We have also

$$H_a = H \cap aHa^{-1} = N = H_{a^2} = H \cap a^2Ha^{-2}$$

since an element in the image of the dihedral representation $\delta\colon \pi_{5,3} \to \mathcal{S}_5$, given by (9), which fixes two numbers is the identity.

For the rest of this section we let $G := \pi_{5,3}$ denote the group of the figure-eight knot.

Lemma 4.6. *Let $\beta\colon F(x,y) \to \mathrm{GL}(m,\mathbb{C})$ be given given by $\beta(x) = A$ and $\beta(y) = B$. If β is irreducible, then $\rho = \operatorname{ind}_H^G(\beta\circ\psi)$ is irreducible.*

Proof. We let $\alpha = \beta \circ \psi$ denote the corresponding representation of H. By Lemma 4.5 we obtain that $\alpha|_N$, $(\alpha|_N)^a$ and $(\alpha|_N)^{a^2}$ are irreducible. Hence by Lemma 4.3 we obtain that $\rho = \operatorname{ind}_H^G \alpha$ is irreducible if and only if

$$\alpha|_N \not\cong (\alpha|_N)^a \quad \text{and} \quad \alpha|_N \not\cong (\alpha|_N)^{a^2}.$$

The element $y_0^2 \in N$ and $\psi(y_0) = 1$ implies $\alpha|_N(y_0^2) = I_m$. Let $\beta(x) = A$ and $\beta(y) = B$ where $A, B \in \mathrm{GL}(m, \mathbb{C})$. We have

$$a^{-1} y_0^2 a = a^{-1} s^2 a = a^{-5} \cdot a^4 s a^{-1} \cdot a s a^{-4} \cdot a^5 = y_5^{-1} y_4 y_1 y_5$$

and

$$a^{-2} y_0^2 a^2 = a^{-2} s^2 a^2 = a^{-5} \cdot a^3 s a^{-2} \cdot a^2 s a^{-3} \cdot a^5 = y_5^{-1} y_3 y_2 y_5.$$

Now, α is given by

$$\alpha(y_0) = \alpha(y_3) = \alpha(y_5) = I_m, \ \alpha(y_1) = \alpha(y_2) = A, \quad \text{and } \alpha(y_4) = B.$$

Therefore, $(\alpha|_N)^a(y_0^2) = BA$ and $(\alpha|_N)^{a^2}(y_0^2) = A$. Now, if β is irreducible, then $A \neq I_n$, and $AB \neq I_n$. Hence $\alpha|_N \not\cong (\alpha|_N)^a$ and $\alpha|_N \not\cong (\alpha|_N)^{a^2}$. ☐

Proof of Corollary 1.3. The subgroup $H \leq \pi_{5,3}$ is of index 5, and $\psi \colon H \to F_2$ is a surjective homomorphism onto a free group of rank 2. Now, by Lemma 4.5 and Lemma 4.6, and the same argument as in the proof of Theorem 1.2, we obtain that for all $m \in \mathbb{N}$ the character variety $X(\pi_{5,3}, \mathrm{SL}(10m, \mathbb{C}))$ has a component C of dimension at least $4m^2 - 1$. Finitely, Lemma 4.6 implies that C contains characters of irreducible representations. ☐

More explicitly, if $\beta \colon F(x, y) \to \mathrm{GL}(m, \mathbb{C})$ is a representation given by $\beta(x) = A$ and $\beta(y) = B$ then, following the construction given in Section 2.1, the induced representation $\rho = \operatorname{ind}_H^G(\beta \circ \psi) \colon \pi_{5,3} \to \mathrm{GL}(5m, \mathbb{C})$ is given by

$$\rho(s) = \begin{pmatrix} I_m & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & B \\ 0 & 0 & 0 & I_m & 0 \\ 0 & 0 & A & 0 & 0 \\ 0 & A & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \rho(t) = \begin{pmatrix} 0 & A & 0 & 0 & 0 \\ I_m & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & B \\ 0 & 0 & 0 & I_m & 0 \\ 0 & 0 & A & 0 & 0 \end{pmatrix}.$$

Here, s and t are the generators of $\pi_{5,3}$ from (8), and $I_m \in \mathrm{GL}(m, \mathbb{C})$ is the identity matrix.

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