# PICARD'S THEOREM

STEFAN FRIEDL

Abstract. We give a summary for the proof of Picard's Theorem. The proof is for the most part an excerpt of [F].

## 1. Introduction

*Definition.* Let $U \subset \mathbb{C}$ be an open subset. A function $f: U \to \mathbb{C}$ is *holomorphic* if for any $z_0 \in U$ the limit

$$\frac{d}{dz} f(z_0) \;:=\; f'(z_0) \;:=\; \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \;\in \mathbb{C}$$

exists.

*Example.* We had seen in Analysis III that polynomials, the exponential function, and more generally functions defined by power series are holomorphic. Moreover products, fractions, sums and compositions of holomorphic functions are again holomorphic.

In Analysis III we had shown that holomorphic functions have many surprising properties. One of the best known results is Liouville's theorem:

**Theorem 1.1. (Liouville's Theorem)** *Every bounded holomorphic map $f: \mathbb{C} \to \mathbb{C}$ is constant.*

The following is a straightforward corollary to Liouvill'es Theorem.

**Corollary 1.2.** *If $f: \mathbb{C} \to \mathbb{C}$ is a holomorphic function, then the image of $f$ is dense, i.e. given any $z \in \mathbb{C}$ and any $r > 0$ there exists a $w \in \mathbb{C}$ with $f(w) \in B_r(z)$.*

The following theorem is a significant strengthening of this corollary.

**Theorem 1.3. (Picard's Theorem)** *Let $f: \mathbb{C} \to \mathbb{C}$ be a non-constant holomorphic function. Then there exists at most one $z \in \mathbb{C}$ which does not lie in the image of $f$.*

*Example.* Picard's Theorem is optimal as is shown by the holomorphic function $f(z) = \exp(z)$, whose image equals $\mathbb{C} \setminus \{0\}$.

## 2. Covering Theory

### 2.1. Homotopic maps.

*Definition.* Let $X$ be a topological space (e.g. a metric space, e.g. a subset of $\mathbb{R}^n$)

(1) Two paths $\gamma_0, \gamma_1 \colon [0,1] \to X$ with the same starting point $P := \gamma_0(0) = \gamma_1(0)$ and the same endpoint $Q := \gamma_0(1) = \gamma_1(1)$ are called *homotopic in* $X$, if there exists a map

$$\begin{array}{rcl} \Gamma\colon [0,1]\times[0,1] & \to & X \\ (t,s) & \mapsto & \Gamma(t,s), \end{array}$$

with the following properties

(a) for every $t \in [0,1]$ we have $\Gamma(t,0) = \gamma_0(t)$ and $\Gamma(t,1) = \gamma_1(t)$,

(b) for every $s \in [0,1]$ we have $\Gamma(0,s) = P$ and $\Gamma(0,s) = Q$.

Put differently, a homotopy between two paths consists of a "continuous" family of paths $\{\Gamma(-,s)\}_{s\in[0,1]}$ from $P$ to $Q$ which interpolates between the paths $\gamma_0$ and $\gamma_1$.

(2) We say $X$ is *simply connected* if each loop $\gamma$ is homotopic to the constant path given by $\delta(t) := \gamma(0)$ for all $t \in [0,1]$.

*Examples.*

(1) Every convex subset of $\mathbb{R}^n$ is simply connected,
(2) the circle $S^1$ and $\mathbb{R}^2 \setminus \{0\}$ are not simply connected.

*Definition.*

(1) We write

$$\mathbb{D} \;=\; \{z \in \mathbb{C} \,|\, |z|^2 < 1\}.$$

(2) We write $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ which we equip with the topology where every set of the form $(\mathbb{C} \setminus \overline{B_r(0)}) \cup \{\infty\}$ is open.
(3) Given $U \subset \overline{\mathbb{C}}$ we denote by $\overline{U}$ the closure of $U$ in $\overline{\mathbb{C}}$.
(4) A *Jordan curve* is the image of an injective map $S^1 \to \overline{\mathbb{C}}$.

We can now formulate one of the key results in complex analysis. A proof is for example provided in [GM, Chapter I.3] and [La, Chapter X].

**Theorem 2.1. (Riemann Mapping Theorem + Carathéodory Theorem)** *Let $U$ be a simply connected open subset of $\mathbb{C}$ with $U \neq \mathbb{C}$. Then the following statements hold:*

(1) *There exists a biholomorphism $\phi\colon U \to \mathbb{D}$.*
(2) *If $\partial\overline{U}$ is a Jordan curve, then $\phi$ extends to a homeomorphism $\overline{U} \to \mathbb{C}$.*

*Example.* We define

$$\mathbb{H} \;=\; \{(x,y) \in \mathbb{R}^2 \,|\, y > 0\} \;=\; \{z = x + iy \in \mathbb{C} \,|\, y > 0\}.$$

It is straightforward to verify that the maps

$$\begin{array}{rcl} \Phi\colon \mathbb{H} & \to & \mathbb{D} \\ z & \mapsto & \frac{z-i}{z+i} \end{array} \quad\text{and}\quad \begin{array}{rcl} \Psi\colon \mathbb{D} & \to & \mathbb{H} \\ w & \mapsto & \frac{i+iw}{1-w} \end{array}$$

are biholomorphisms that are inverse to one-another and that extend to homeomorphisms between $\overline{\mathbb{H}}$ and $\overline{\mathbb{D}}$.

## 2.2. **Covering maps.**

*Definition.* Let $p\colon X \to B$ be a map between topological spaces.

(1) We say an open subset $U \subset B$ is *uniformly covered*, if $p^{-1}(U)$ is the union of disjoint open subsets $\{V_i\}_{i\in I}$ with the property, that the restriction of $p$ to each subset $V_i$ is a homeomorphism.

(2) We say the map $p\colon X \to B$ is a *covering*, if it is surjective and if for every $b \in B$ there exists an open neighborhood $U$ of $b$ which is uniformly covered.

*Example.* The map

$$\begin{array}{rcl} p\colon \mathbb{R} & \to & S^1 \\ \varphi & \mapsto & e^{i\varphi} \end{array}$$

is a covering. Indeed, let $P = e^{i\alpha}$ be a point in $S^1$. We pick the open neighborhood

$$U \; := \; \left\{ e^{i\varphi} \,\middle|\, \varphi \in \left(\alpha - \tfrac{\pi}{4}, \alpha + \tfrac{\pi}{4}\right) \right\}.$$

Then

$$p^{-1}(U) \; := \; \bigsqcup_{j\in\mathbb{Z}} \underbrace{\left(\alpha - \tfrac{\pi}{4} + 2\pi j, \alpha + \tfrac{\pi}{4} + 2\pi j\right)}_{=:V_j},$$

and for each $j \in \mathbb{Z}$ the restriction of $p$ to $V_j \to U$ is a homeomorphism. This example is illustrated in Figure 1.

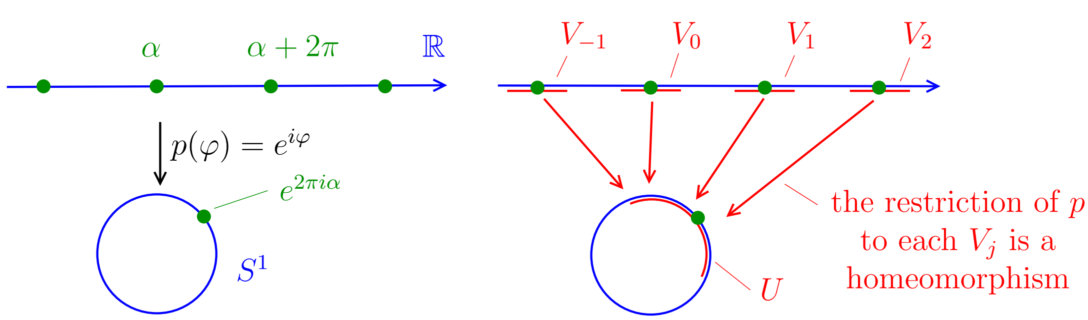


FIGURE 1.

The following theorem is a standard result in topology.

**Theorem 2.2.** *Let $X$ be a (reasonable) path-connected topological space. Then there exists a unique covering $p\colon \widetilde{X} \to X$ where $\widetilde{X}$ is simply-connected. This covering is called the* universal covering of $X$.

*Example.* The universal covering of the torus $S^1 \times S^1 = \mathbb{R}^2/\mathbb{Z}^2$ is $\mathbb{R}^2$.

**Proposition 2.3.** *Let $p\colon (X, x_0) \to (B, b_0)$ be a covering map of two pointed topological spaces. Furthermore let $Z$ be a (reasonable) path-connected topological space and let*

*$f\colon (Z,z_0) \to (B,b_0)$ be a map. If $Z$ is simply-connected, then there exists a lift, i.e. a commutative diagram*

$$\begin{array}{ccc} & \text{there exists a lift } \widetilde{f} \nearrow & (X,x_0) \\ & & \downarrow p \\ (Z,z_0) & \xrightarrow[f]{} & (B,b_0) \end{array}$$

*Proof.* Let $z$ be a point in $Z$. We pick a path $\gamma$ from $z_0$ to $z$. Then $f\circ\gamma$ is a path from $b_0$ to $f(z)$. Since $p$ is a covering we can lift the path to a path $\widetilde{f\circ\gamma}\colon [0,1] \to X$ with starting point $\big(\widetilde{f\circ\gamma}\big)(0) = x_0$. We define $\widetilde{f}(z) = \big(\widetilde{f\circ\gamma}\big)(1)$. The hypothesis that $Z$ is simply-connected implies that $\widetilde{f}(z)$ does not depend on the choice of the path $\gamma$. □

*Example.* We can lift any path $[0,1] \to B$ to a path $[0,1] \to X$. But the lift of a loop is not necessarily again a loop.

## 3. Hyperbolic geometry

### 3.1. **Möbius transformations.**

*Definition.*

(1) The *hyperbolic disk* is the manifold

$$\mathbb{D} \;=\; \{z \in \mathbb{C} \,|\, |z|^2 < 1\}$$

together with the Riemannian structure which assigns to the tangent space $T_z\mathbb{D} = \mathbb{R}^2$ at a point $z \in \mathbb{D}$ the positive-definite symmetric form

$$\begin{array}{rcl} g_z\colon \mathbb{R}^2 \times \mathbb{R}^2 & \to & \mathbb{R} \\ (v,w) & \mapsto & \dfrac{2}{(1-|z|^2)^2}\langle v,w\rangle \end{array}$$

where, as above, $\langle v,w\rangle$ denotes the usual scalar product on $\mathbb{R}^2$.

*Definition.* Let $(M,g)$ be a connected Riemannian manifold.

(1) Given $v \in T_PM$ we write $\|v\| = \sqrt{g_P(v,v,)}$.
(2) We define the *length of a smooth path* $\gamma\colon [a,b] \to M$ as

$$\ell_M(\gamma) \;:=\; \int_{t=a}^{t=b} \|\gamma'(t)\|\,dt.$$

(3) Let $P$ and $Q$ be two points on $M$. We define

$$d_M(P,Q) \;:=\; \inf\{\ell(\gamma)\,|\,\gamma \text{ is a smooth path in } M \text{ from } P \text{ to } Q\}.$$

This defines a metric on $M$.

*Example.* We consider the Riemannian manifold $(M, g) = \mathbb{D}$. For $r \in (0,1)$ we consider the smooth path $\gamma_r\colon [0,r] \to \mathbb{D}$ given by $t \mapsto t$. Then

$$\lim_{r\to 1} \ell(\gamma) \;=\; \lim_{r\to 1} \int_{t=0}^{r} \frac{2}{1-r^2}\, dt \;=\; \infty.$$

In fact this curve defines the shortest path from 0 to $r$. Thus we see that $(M, g) = \mathbb{D}$ has infinite diameter.

*Examples.* In Figure 2 we see three different decompositions of $\mathbb{D}$ into subsets. In each of the three cases these subsets have the same hyperbolic size, even though they differ in the usual euclidean sense.

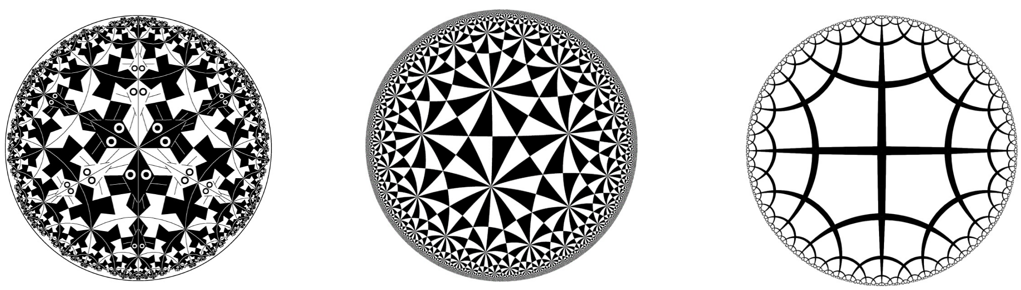

FIGURE 2.

*Definition.* A *hyperbolic line* is a subset of $\mathbb{D}$ of one of the following two types:

(1) it is the intersection of $\mathbb{D}$ with a euclidean line through the origin,
(2) it is the intersection of $\mathbb{D}$ with a euclidean circle that intersects the circle $S^1 = \partial\overline{\mathbb{D}}$ orthogonally.

In each of the two cases we refer to the intersection of the euclidean object with $S^1 = \partial\overline{\mathbb{D}}$ as the *endpoints* of the hyperbolic line. We refer to Figure 3 for an illustration.

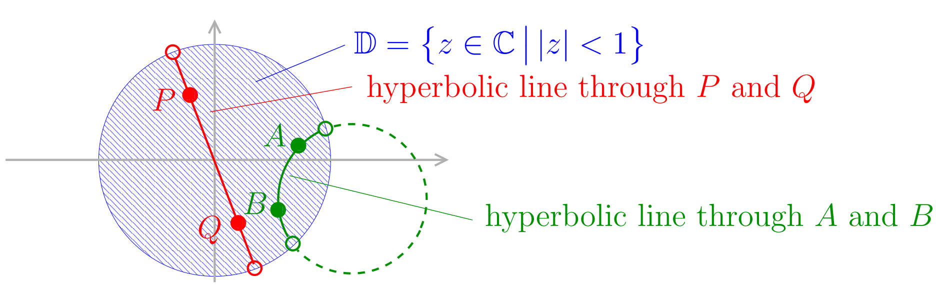


FIGURE 3.

*Definition.* A *Möbius transformation* of $\mathbb{D}$ is a map of the form

$$\begin{array}{rcl} \mathbb{D} & \to & \mathbb{D} \\ z & \mapsto & e^{i\theta}\cdot\frac{z-a}{1-\overline{a}z} \end{array}$$

for some $\theta\in\mathbb{R}$ and $a\in\mathbb{D}$.

*Example.* Rotations around the origin are precisely the Möbius transformations fixing 0.

The following proposition summarizes key properties of Möbius transformations. All statements can be proved in an elementary fashion.

**Proposition 3.1.**

1. *Compositions of Möbius transformations are again Möbius transformations and the inverse of a Möbius transformation is again a Möbius transformation.*
2. *Möbius transformations are biholomorphisms, they are isometries, they are orientation-preserving and they send hyperbolic lines to hyperbolic lines.*
3. *Given any two hyperbolic lines there exists a Möbius transformation that sends one to the other.*
4. *Given $P,Q\in\mathbb{D}$ and non-zero vectors $v\in T_P\mathbb{D}$ and $w\in T_Q\mathbb{D}$ with $\|v\|_g=\|w\|_g$ there exists a Möbius transformation $\phi$ with $\phi(P)=Q$ and $d\phi_P(v)=w$.*
5. *Let $\phi$ and $\psi$ be two Möbius transformations. Suppose there exists a $P\in\mathbb{D}$ such that $\phi(P)=\psi(P)$ and such that $d\phi_P=d\psi_P$. Then $\phi=\psi$.*

### 3.2. Möbius structures.

*Definition.* Let $M$ be a 2-dimensional manifold without boundary. A *Möbius structure* for $M$ is a family of homeomorphisms $\{\Phi_i\colon U_i\to V_i\}_{i\in I}$ from open subsets of $M$ to open subsets of $\mathbb{D}$ such that $\bigcup\limits_{i\in I}U_i=M$ and such that for any $i,j\in I$ the transition map

$$\underbrace{\Phi_i(U_i\cap U_j)}_{\subset\mathbb{D}}\xrightarrow{\left(\Phi_i|_{U_i\cap U_j}\right)^{-1}}U_i\cap U_j\xrightarrow{\Phi_j|_{U_i\cap U_j}}\underbrace{\Phi_j(U_i\cap U_j)}_{\subset\mathbb{D}}$$

is given by a Möbius transformation. Sometimes we refer to a manifold together with a Möbius structure as a *Möbius manifold.*

The following lemma says that Möbius structures are useful for solving two problems at once: we can use them to show that a manifold is a complex manifold and we can use them to show that a manifold has a hyperbolic structure.

**Lemma 3.2.** *Let $M$ be a surface and let $\{\Phi_i\colon U_i\to V_i\}_{i\in I}$ be a Möbius structure for $M$. Then the following hold:*

1. *The charts form a holomorphic atlas for $M$, in particular $M$ is a complex 1-dimensional manifold.*
2. *The manifold $M$ admits a unique Riemannian structure $g$ such that all the charts in the atlas $\{\Phi_i\colon U_i\to V_i\}_{i\in I}$ are isometries.*

*Definition.* A *Möbius map* between two Möbius manifolds is a map that, with respect to the charts of the Möbius structures, is given by Möbius transformations.

The following lemma is a fairly straightforward consequence of Proposition 3.1 (5).

**Lemma 3.3.** *Let $\Phi, \Psi\colon M \to N$ be two Möbius maps between two Möbius manifolds. Suppose that $M$ is path-connected. If $\Phi$ and $\Psi$ agree on an non-empty open connected subset, then they agree everywhere.*

### 3.3. **Complete Riemannian manifolds.**

*Definition.*

(1) We say that a metric space $(X, d)$ is *complete* if every Cauchy sequence in $(X, d)$ converges.
(2) We say that a connected Riemannian manifold $(M, g)$ is *complete* if the corresponding metric space $(M, d_M)$ is complete.

*Examples.*

(1) The metric space given by $\mathbb{R}^n$ and the euclidean metric is complete.
(2) The metric space given by the open ball $B^n$ and the euclidean metric is not complete. Indeed the sequence $a_n = 1 - \frac{1}{n}$ is a Cauchy sequence, but it does not converge in the open ball $B^n$. Similarly we see that for example the metric space $\mathbb{C} \setminus \{0, 1\}$ with the usual euclidean metric is not complete.

The following proposition gives a useful criterion for a metric space to be complete.

**Proposition 3.4.** *Every compact metric space is complete.*

3.4. **The complex manifold $H_4/\sim$.** In the following let $Q_1 = 1, Q_2 = i, Q_3 = -1$ and $Q_0 = Q_4 = -i$. We denote by $H_4$ the closed, non-compact subset of $\mathbb{D}$ that is bounded by the four hyperbolic lines with endpoints $Q_k, Q_{k+1}$ where $k = 0, 1, 2, 3$.

We can and will pick a Möbius transformation $\Phi_1$ that restricts to the reflection in the $x$-axis on the hyperbolic line with endpoints $Q_2$ and $Q_3$. Similarly we can and will pick a Möbius transformation $\Phi_2$ that restricts to the reflection in the $x$-axis on the hyperbolic line with endpoints $Q_2$ and $Q_1$. We refer to Figure 4 for an illustration.

Given $k \in \{1, 2\}$ we declare any point $P$ on the line with the endpoints $Q_k$ and $Q_{k+1}$ to be equivalent to $\Phi_{3-k}(P)$. We denote by $\sim$ the equivalence relation that is generated by these equivalences.

**Proposition 3.5.**

(1) *The topological space $H_4/\sim$ admits a Möbius structure. In particular it is a 1-dimensional complex manifold.*
(2) *The Riemannian structure on $H_4/\sim$ coming from* (1) *and Lemma 3.2 is complete.*

*Proof.* We start out with the proof of (1). We have to show that $H_4/\sim$ admits a Möbius structure. We denote by $p\colon H_4 \to H_4/\sim$ the projection map. In the following let $P \in H_4/\sim$.

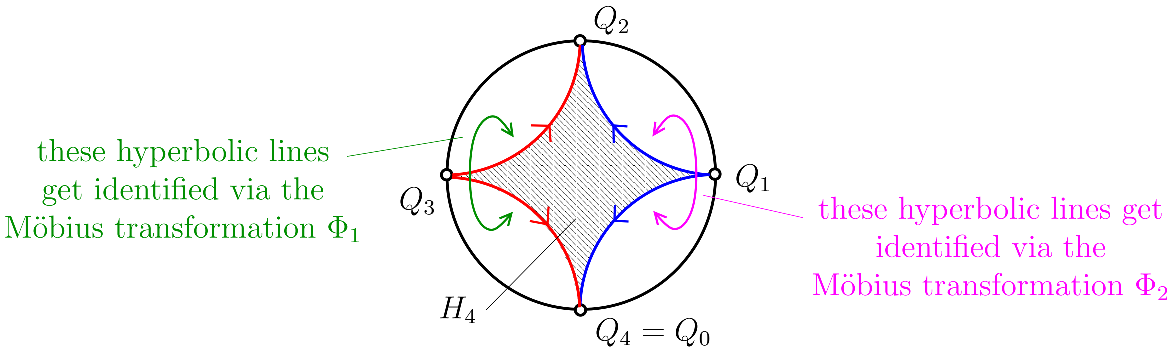


FIGURE 4.

(1) If $P = p(z)$ for some $z \in \mathring{H}_4$, then

$$\begin{array}{rcl} \psi_P\colon p\big(\mathring{H}_4\big) & \to & \mathring{H}_4 \\ p(w) & \mapsto & w \end{array}$$

is a chart around $P = p(z)$.

(2) Now suppose that $P = p(z)$ where $z$ lies on $\partial H_4$. In the following we deal with the case that $z$ lies on the edge from $Q_1$ to $Q_2$. All other cases are dealt with almost the same way. We pick an $r > 0$ such that $B_{2r}(z)$ intersects no other component of $\partial H_4$. We consider the map

$$\begin{array}{rcl} \psi_P\colon p(B_r(z) \cap H_4) \cup p(\Phi_2(B_r(z)) \cap H_4) & \to & B_r(z) \\ p(w) & \mapsto & \left\{ \begin{array}{ll} w, & \text{if } w \in B_r(z) \cap H_4 \\ \Phi_2^{-1}(w), & \text{if } w \in \Phi_1^{-1}(B_r(z)) \cap H_4. \end{array} \right. \end{array}$$

This map is a chart around $P = p(z)$.

The charts that we just constructed form an atlas for $H_4/\sim$. It follows immediately from the definitions that all transition maps are given by Möbius transformations. It follows that the maps $\psi_P$ form a Möbius structure for $H_4/\sim$. It follows that $H_4/\sim$ is a 1-dimensional complex manifold and that these charts satisfy the conditions of Lemma 3.2. This concludes the proof of (1).

We continue with the proof of (2). We have to show that the Riemannian structure on $H_4/\sim$ coming from (1) and Lemma 3.2 is complete. By the argument of page 5 any bounded sequence in $H_4/\sim$ stays within a compact subset. Hence it converges by Proposition 3.4. □

**Proposition 3.6.** *There exists a biholomorphism $H_4/\sim \to \mathbb{C} \setminus \{0, 1\}$.*

In the proof of Proposition 3.6 we need the following standard result from complex analysis.

**Proposition 3.7. (Schwarz Reflection Principle)** *Let $U$ be an open subset of the upper half-plane $\{z \in \mathbb{C} \,|\, \mathrm{Im}(z) \geq 0\}$ and let $f\colon U \to \mathbb{C}$ be a continuous function with the following properties:*

(1) *$f$ is holomorphic on $\mathring{U} = \{z \in U \,|\, \mathrm{Im}(z) > 0\}$,*
(2) *$f$ only assumes real values on $U \cap \mathbb{R}$.*

*We set $U' := \{\overline{z} \,|\, z \in U\}$, i.e. $U'$ is the reflection of $U$ in the $x$-axis. Then the function*

$$\begin{aligned} \tilde{f}\colon U \cup U' &\to \mathbb{C} \\ z &\mapsto \begin{cases} f(z), & \text{if } z \in U, \\ \overline{f(\overline{z})}, & \text{if } z \in U', \end{cases} \end{aligned}$$

*is holomorphic.*

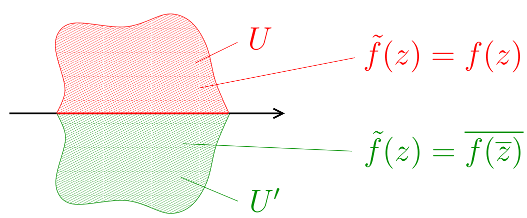


FIGURE 5. Illustration of the Schwarz reflection principle.

*Proof of Proposition 3.6.* We denote by $\Delta$ the open subset of $\mathbb{D}$ bounded by the three hyperbolic lines with endpoints $-1, 1$ and $i$. (We refer to Figure 6 for an illustration of $\Delta$.) It follows from applying the Riemann Mapping Theorem 2.1 that there exists a biholomorphism $\Psi\colon \Delta \to \mathbb{D}$ that extends to a homeomorphism $\Psi\colon \overline{\Delta} \to \overline{\mathbb{D}}$. We combine $\Psi$ with the above biholomorphism from $\mathbb{D}$ to $\mathbb{H}$. This way we obtain a biholomorphism $\Phi\colon \Delta \to \mathbb{H}$ that extends to a homeomorphism $\Phi\colon \overline{\Delta} \to \overline{\mathbb{H}}$.

After possibly applying a Möbius transformation of $\mathbb{H}$ we can assume that $\Phi(i) = \infty$ and that $\Phi(\{-1, 1\}) = \{0, 1\}$.

Note that $\Phi$ restricts to a homeomorphism $\Phi\colon \partial\overline{\Delta} \to \partial\overline{\mathbb{H}} = \mathbb{R} \cup \{\infty\}$. By the above we have $\Phi(\partial\overline{\Delta}) = \mathbb{R} \setminus \{0, 1\}$. Now we consider the map

$$\begin{aligned} \Psi\colon H_4 &\to \mathbb{C} \setminus \{0, 1\} \\ z &\mapsto \begin{cases} \Phi(z), & \text{if } z \in \overline{\Delta}, \\ \overline{\Phi(\overline{z})}, & \text{if } \overline{z} \in \overline{\Delta}, \end{cases} \end{aligned}$$

Note that by the above we have $f((-1, 1)) = (0, 1)$. Thus it follows from the Schwarz Reflection Principle, see Proposition 3.7, that the restriction of $\Psi$ to the interior of $H_4$ is holomorphic. By construction of $\sim$ we know that if $P$ and $Q$ are two equivalent points on $\partial H_4$, then $\Psi(P) = \Psi(Q)$. Therefore the map $\Psi$ factors through a map $H_4/\sim \to \mathbb{C} \setminus \{0, 1\}$.

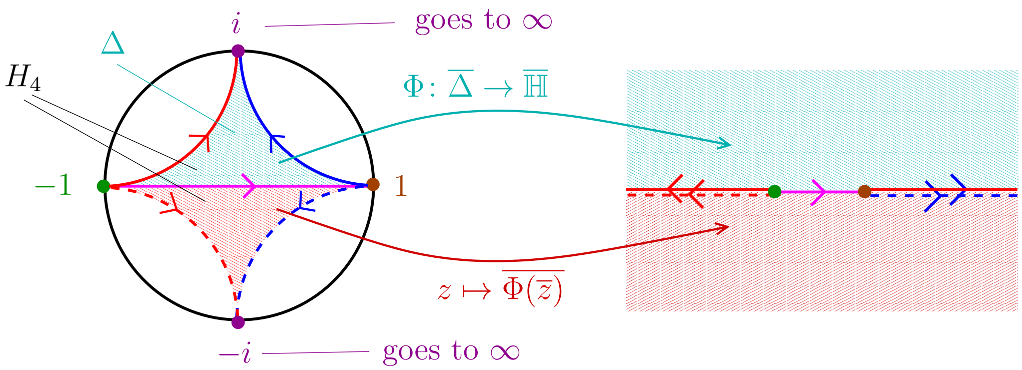


FIGURE 6. Illustration of the proof of Proposition 3.6

Using once again the Schwarz Reflection Principle one can show that this map is in fact a biholomorphism. We leave the verification of this step as an exercise to the reader. □

### 3.5. **The universal cover of $H_4/\sim$.**

**Proposition 3.8.** *Let $H_4/\sim$ be the three-punctured sphere with the Möbius structure constructed in Proposition 3.5. Then there exists a covering map $p\colon \mathbb{D} \to H_4/\sim$ which is a local biholomorphism.*

*Remark.*

(1) We consider again the covering $p\colon \mathbb{D} \to H_4/\sim$ that we had just constructed. The interior of $H_4$ is uniformly covered. In particular we see that

$$\mathbb{D} \setminus \underbrace{p^{-1}(\partial H_4)}_{\text{“1-dimensional”}} = \bigsqcup \text{copies of } \mathring{H}_4.$$

Put differently, up to the “one-dimensional” subset $p^{-1}(\partial H_4)$ we can cover $\mathbb{D}$ by infinitely many disjoint copies of the open hyperbolic octagon. Such a decomposition is often called a *tessellation*. This tessellation is shown in Figure 7.

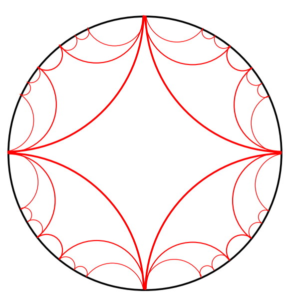

FIGURE 7.

(2) Since $\mathbb{D}$ is simply-connected we see that $p\colon \mathbb{D} \to H_4/\sim$ is the universal covering of $H_4/\sim$.

*Proof of Proposition 3.8.* In the following we will show that there exists a Möbius map $p\colon \mathbb{D} \to M := H_4/\sim$ that is in fact a covering map.

For the remainder of the proof we adopt the following notation and conventions.

(1) We view $\mathring{H}_4$ as a subset of $\mathbb{D}$ and also of $H_4/\sim$.
(2) Given $Z \in \mathbb{D}$ we denote by

$$\begin{array}{rcl} \gamma_Z\colon [0,1] & \to & \mathbb{D} \\ t & \mapsto & t\cdot Z \end{array}$$

the radial path from the origin $0 \in \mathbb{D}$ to $Z$.

(3) We say that $U \subset \mathbb{D}$ is *convex* if it is convex in the usual euclidean sense, i.e. if given any $P, Q \in U$ the points $t\cdot P + (1-t)\cdot Q$, $t \in [0,1]$ also lie in $U$.

Our first goal is to construct a suitable map $p\colon \mathbb{D} \to M = H_4/\sim$. So let $Z \in \mathbb{D}$. We write $\gamma = \gamma_Z$. We say $t \in [0,1]$ is *good* if there exists a convex open neighborhood $U$ of $\gamma([0,t])$ and a map $\Psi\colon U \to H_4/\sim$ which has the following properties:

(i) $\Psi$ is a Möbius map,
(ii) it restricts to the identity on $\mathring{H}_4 \cap U \subset H_4/\sim$.

We start out with the following lemma.

**Lemma 3.9.** *The point $t = 1$ is good.*

*Proof.* We set

$$T := \{t \in [0,1] \mid t \text{ is good}\}.$$

We start out with the following observations:

(1) We have that $0 \in T$ since the identity on the open neighborhood $\mathring{H}_4$ of $0$ has the required properties.[1]
(2) Being good is clearly an open condition, hence $T \subset [0,1]$ is open.

We want to show that $T = [0,1]$. By the above, and since $[0,1]$ is connected, it suffices to show that $s := \sup(T)$ is good.

Since $0 \in S$ and since $T$ is open we see that $s > 0$. Therefore there exists an increasing sequence $\{t_n\}_{n\in\mathbb{N}}$ of good numbers $t_n \in [0,t)$ that converges to $s$. Since each $t_n$ is good we can pick for each $n$ a convex open neighborhood $U_n$ of $\gamma([0,t_n])$ and a map $\Psi_n\colon U_n \to H_4/\sim$ that has the desired properties (i) and (ii).

Let $m,n \in \mathbb{N}$. The Möbius maps $\Psi_m$ and $\Psi_n$ agree on the non-empty open connected subset $\mathring{H}_4 \cap U_n \cap U_m$. It follows from Lemma 3.3 that the maps $\Psi_m$ and $\Psi_n$ agree on $U_n \cap U_m$.

By the continuity of $\gamma$ the sequence $\{\gamma(t_n)\}_{n\in\mathbb{N}}$ converges to $\gamma(t)$, in particular it is a Cauchy sequence with respect to $d_{\mathbb{D}}$. Since the maps $\Psi_n$ are Möbius maps they are local

[1]Hereby we use that for each point $Z \in \mathring{H}_4$ the identity map $\mathrm{id}\colon \mathring{H}_4 \to \mathring{H}_4$ is part of the Möbius structure of $H_4/\sim$.

isometries, hence it follows that $\{\Psi_n(\gamma(t_n))\}_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to $d_M$. By Proposition 3.5 the metric space $(M = H_4/\sim, d_M)$ is complete. Therefore the Cauchy sequence $\{\Psi_n(\gamma(t_n))\}_{n\in\mathbb{N}}$ converges to a point $S \in H_4/\sim$.

It follows easily from Proposition 3.1 (3) that there exists an $r > 0$ and a Möbius isomorphism $\Omega\colon B_r^{\mathbb{D}}(0) \to B_r^M(S)$ such that $\Omega(0) = S$. We pick an $n$ such that $\Psi_n(\gamma(t_n)) \in B_r^M(S)$ and such that $d_{\mathbb{D}}(\gamma(t), \gamma(s)) < r$. We write $t = t_n, \Psi = \Psi_n, U = U_n$ and we write $X = \gamma(t)$. Let $v \in T_M\mathbb{D}$ be a non-zero vector. We let $w := d(\Omega^{-1} \circ \Psi)_M(v)$. The Möbius maps $\Omega$ and $\Psi$ are in particular local isometries, therefore we have $\|w\| = \|v\|$. It follows from Proposition 3.1 (4) that there exists a Möbius transformation $\Theta$ with $\Theta(X) = \Omega^{-1}(\Psi(X))$ and such that $d\Theta_X(v) = w$.

The situation is illustrated in Figure 8. The Möbius maps $\Omega \circ \Theta\colon B_r^{\mathbb{D}}(X) \to H_4/\sim$ and

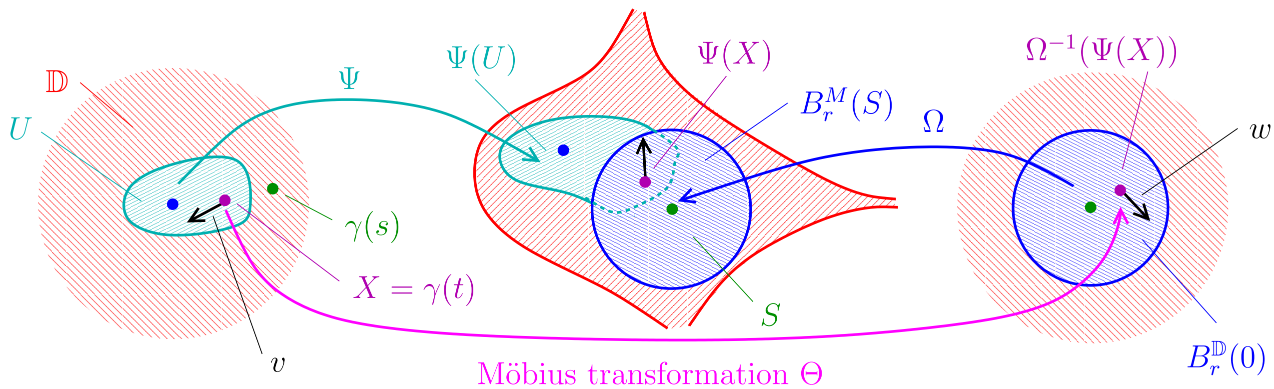


FIGURE 8.

$\Psi\colon U \to H_4/\sim$ agree at $X = \gamma(t)$ and we have $d(\Omega \circ \Theta)_X(v) = d\Psi_X(v)$. It follows from Lemma 3.3 that $\Omega \circ \Theta$ and $\Psi$ agree on the path-connected set $B_r^{\mathbb{D}}(X) \cap U$.

Now we consider the map

$$\begin{array}{rcl} \Phi\colon U \cup B_r^{\mathbb{D}}(\gamma(t)) & \to & H_4/\sim \\ Q & \mapsto & \left\{ \begin{array}{ll} \Psi(Q), & \text{if } Q \in U, \\ \Omega^{-1}(\Theta(Q)), & \text{if } Q \in B_r^{\mathbb{D}}(\gamma(t)). \end{array} \right. \end{array}$$

By the above discussion this map is well-defined and it is locally a Möbius map, hence it is a Möbius map. It follows from $d_{\mathbb{D}}(\gamma(t), \gamma(s)) < r$ and the fact that $\gamma$ is a geodesic in $\mathbb{D}$ that $\gamma([t, s]) \subset B_r^{\mathbb{D}}(\gamma(t))$, in particular $U \cup B_r^{\mathbb{D}}(\gamma(t))$ is an open subset of $\mathbb{D}$ that contains $\gamma([0, t]) \cup \gamma([t, s]) = \gamma([0, s])$. It is now straightforward to see that we can find a *convex* open subset $W$ of $U \cup B_r^{\mathbb{D}}(\gamma(t))$ that contains $\gamma([0, s])$. The Möbius map $\Phi\colon W \to H_4/\sim$ now certifies that $s$ is also good. □

Thus we have now shown that there exists a convex open neighborhood $U$ of $\gamma([0, 1])$ and a local isometry $\Psi\colon U \to H_4/\sim$ which agrees with the identity map on $U \cap \mathring{H}_4$. We define $p(Z) := \Psi(Z)$. By Lemma 3.3 this definition does not depend on the choice of $U$ and $\Psi$.

The following lemma now concludes the proof of Proposition 3.8

**Lemma 3.10.**

(1) *The map* $p\colon \mathbb{D} \to H_4/\sim$ *is a Möbius map.*
(2) *The map* $p\colon \mathbb{D} \to H_4/\sim$ *is a covering map.*

*Proof.*

(1) It follows basically from the construction of $p$ that it is a Möbius map.
(2) First we show that the map $p\colon \mathbb{D} \to M = H_4/\sim$ is surjective. Recall that by construction $p$ is the identity on $\mathring{H}_4$. Since $p$ is continuous it follows that the restriction of $p$ to $H_4 \to M = H_4/\sim$ is surjective.

    It remains to show that every $Q \in H_4/\sim$ admits a uniformly covered neighborhood. Let $Q \in H_4/\sim$. We pick an $r > 0$ such that an $r$-neighborhood around $Q$ is isometric to $B_r^{\mathbb{D}}(0)$. Then one can show fairly easily that $U = B_{r/2}^{\mathbb{D}}(Q)$ is uniformly covered. □

□

## 4. The proof of Picard's Theorem

**Theorem 4.1. (Picard's Theorem)** *Let* $f\colon \mathbb{C} \to \mathbb{C}$ *be a non-constant holomorphic function. Then there exists at most one* $z \in \mathbb{C}$ *which does not lie in the image of* $f$*.*

*Proof.* Let $f\colon \mathbb{C} \to \mathbb{C}$ be a holomorphic function such that there exist two different complex numbers $a, b$ that do not lie in the image of $f$. We need to show that $f$ is constant. We consider the biholomorphism

$$\begin{array}{rcl} \alpha\colon \mathbb{C} \setminus \{a,b\} & \to & \mathbb{C} \setminus \{0,1\} \\ z & \mapsto & \dfrac{z-a}{b-a}. \end{array}$$

Furthermore we denote by $\beta\colon \mathbb{C}\setminus\{0,1\} \to H_4/\sim$ the biholomorphism from Proposition 3.6. Now we consider the following diagram of maps

$$\begin{array}{ccccccc} & & & & & & \mathbb{D} \\ & & & & & & \downarrow p \\ \mathbb{C} & \xrightarrow{f} & \mathbb{C} \setminus \{a,b\} & \xrightarrow[\cong]{\alpha} & \mathbb{C} \setminus \{0,1\} & \xrightarrow[\cong]{\beta} & H_4/\sim \end{array}$$

where $p\colon \mathbb{D} \to H_4/\sim$ is the covering map from Proposition 3.8.

Since $\mathbb{C}$ is simply connected we can appeal to Proposition 2.3 to obtain a lift of the map $\beta \circ \alpha \circ f\colon \mathbb{C} \to H_4/\sim$ to the universal cover. More precisely, there exists a map

$\widetilde{\beta \circ \alpha \circ f}\colon \mathbb{C} \to \mathbb{D}$ such that the following diagram commutes

$$\begin{array}{ccccccc} & & & & & & \mathbb{D} \\ & & & \nearrow^{\widetilde{\beta\circ\alpha\circ f}} & & & \downarrow \Phi \\ \mathbb{C} & \xrightarrow[f]{} & \mathbb{C} \setminus \{a,b\} & \xrightarrow[\alpha]{\cong} & \mathbb{C} \setminus \{0,1\} & \xrightarrow[\beta]{\cong} & H_4/\sim . \end{array}$$

It follows from the fact that $\beta \circ \alpha \circ f$ is holomorphic and the fact that $p$ is a local biholomorphism, that the map $\widetilde{\beta \circ \alpha \circ f}\colon \mathbb{C} \to \mathbb{D}$ is also holomorphic. But $\mathbb{D}$ is of course bounded. Therefore it follows from Liouville's Theorem 1.1 that $\widetilde{\beta \circ \alpha \circ f}$ is constant. But then $\beta \circ \alpha \circ f$ is also constant. Since $\alpha$ and $\beta$ are biholomorphisms this implies that $f$ itself is already constant. □

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