The Thurston norm via Fox calculus

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(joint work with Kevin Schreve and Stephan Tillmann)

Let N be a compact orientable 3-manifold. The *Thurston seminorm* of a class $\phi \in H^1(N;\mathbb{Z}) = H_2(N, \partial N;\mathbb{Z})$ is defined as

$$x(\phi) := \min\big\{\chi_-(\Sigma)\,|\,\Sigma \subseteq N \text{ properly embedded surface dual to } \phi\big\}.$$

Here, given a surface Σ with connected components $\Sigma_1 \cup \cdots \cup \Sigma_k$, we define its complexity to be $\chi_-(\Sigma) = \sum_{i=1}^k \max\{-\chi(\Sigma_i), 0\}$. A class $\phi \in H^1(N;\mathbb{R})$ is called *fibered* if it can be represented by a non-degenerate closed 1-form. By [19] an integral class $\phi \in H^1(N;\mathbb{Z}) = \mathrm{Hom}(\pi_1(N),\mathbb{Z})$ is fibered if and only if there exists a fibration $p\colon N \to S^1$ such that $p_* = \phi : \pi_1(N) \to \pi_1(S^1) = \mathbb{Z}$.

Thurston [18] showed that x is a seminorm on $H^1(N;\mathbb{Z})$ which extends to a seminorm on $H^1(N;\mathbb{R})$. He also considered the norm ball

$$\mathcal{N}_N := \{\phi \in H_1(N;\mathbb{R})\,:\,x(\phi) \leq 1\}$$

and the corresponding dual norm ball

$$\mathcal{P}_N := \{v \in H_1(N;\mathbb{R})\,:\,\phi(v) \leq 1 \text{ for all } \phi \in \mathcal{N}_N\}.$$

He showed that $\mathcal{P}_N$ is a polytope with integral vertices, i.e. with vertices in $\mathrm{Im}\{H_1(N;\mathbb{Z})/\mathrm{torsion} \to H_1(N;\mathbb{R})\}$. Furthermore, Thurston showed that we can turn $\mathcal{P}_N$ into a marked polytope $\mathcal{M}_N$ which has the property that a cohomology class $\phi \in H^1(N;\mathbb{R})$ is fibered if and only if it pairs maximally with a marked vertex, i.e. if and only if there exists a marked vertex v of $\mathcal{M}_N$ such that

$$\phi(v) > \phi(w) \text{ for all } v \neq w \in \mathcal{P}_N.$$

Now let $\pi = \langle x, y\,|\,r\rangle$ be a presentation with two generators and one relator, such that r is cyclically reduced and such that $b_1(\pi) = 2$. In [9] we associated to such a presentation π a marked polytope $\mathcal{M}_\pi$ in $H_1(\pi;\mathbb{R})$ as follows:

(1) We start at the origin and walk across $H_1(\pi;\mathbb{Z}) = \mathbb{Z}^2$ as dictated by the word r which we start reading from the left.
(2) We take the convex hull of all the points reached in (1). We furthermore mark all vertices which get hit only once by the path in (1).
(3) We take the midpoints of all the squares in the convex hull that touch a vertex of the polytope defined in (2). We mark a midpoint if all the corresponding vertices in (2) are marked.
(4) We take the marked polytope corresponding to the set of points in (3) and denote it by $\mathcal{M}_\pi$.

An alternative, more formal definition of $\mathcal{M}_\pi$ is given in [9] in terms of the Fox derivatives of r. The main result of [10] says the following.

Theorem 1. *Let N be an irreducible, compact, orientable 3-manifold that admits a presentation $\pi = \langle x, y\,|\,r\rangle$ as above. Then*

$$\mathcal{M}_N = \mathcal{M}_\pi.$$

The following corollary gives the statement of the theorem in a slightly more informal fashion. The method for reading off the fibered classes in $H^1(N;\mathbb{R})$ from r is closely related to Brown's algorithm [5].

Corollary 2. *If N is an irreducible, compact, orientable 3-manifold that admits a presentation $\pi = \langle x, y \,|\, r\rangle$ as above, then the Thurston norm and the set of fibered classes can be read off from the relator r.*

In the proof of Theorem 1 we use the definition of $\mathcal{M}_\pi$ in terms of Fox derivatives. This makes it possible to relate the polytope $\mathcal{M}_\pi$ to the chain complex of the universal cover of the 2-complex corresponding X to the presentation π. This makes it possible to study the 'size' of $\mathcal{M}_\pi$ using twisted Reidemeister torsions of X. These twisted Reidemeister torsions agree with twisted Reidemeister torsions of N since X is simple homotopy equivalent to N.

In the following we denote by $\mathcal{P}_N$ and $\mathcal{P}_\pi$ the polytopes $\mathcal{M}_N$ and $\mathcal{M}_\pi$ without the markings. At this point the proof of Theorem 1 breaks up into three parts:

(1) We first show that $\mathcal{P}_N \subset \mathcal{P}_\pi$. Put differently, we need to show that $\mathcal{P}_\pi$ is 'big enough' to contain $\mathcal{P}_N$. We show this using the main theorem of [11] which says that twisted Reidemeister torsions detect the Thurston norm of N. This result in turn relies on the work of Agol [1], Liu [14], Przytycki-Wise [16, 15] and Wise [21] which says in this context that Agol's virtual fibering theorem [1] applies.
(2) Next we need to show the reverse inclusion $\mathcal{P}_\pi \subset \mathcal{P}_N$. This means that we need to show that $\mathcal{P}_\pi$ is 'not bigger than necessary'. At this stage it is crucial that r is cyclically reduced. By [20] this implies that all summands in the Fox derivative $\frac{\partial r}{\partial x}$ are distinct elements in the group ring $\mathbb{Z}[\pi]$. Using the fact that $\pi_1(N)$ is residually torsion-free elementary-amenable (which is a consequence of the aforementioned papers [1, 2, 14, 16, 15, 21] and a result of Linnell–Schick [13]) and using the non-commutative Reidemeister torsions of [6, 8, 12] we show that indeed $\mathcal{P}_\pi \subset \mathcal{P}_N$.
(3) Finally we need to show that the markings of $\mathcal{M}_N$ and $\mathcal{M}_\pi$ agree. We prove this using Novikov-Sikorav homology [4, 17].

References

[1] I. Agol, *Criteria for virtual fibering*, J. Topol. **1** (2008), no. 2, 269–284.

[2] I. Agol, *The virtual Haken conjecture*, with an appendix by I. Agol, D. Groves and J. Manning, Documenta Math. **18** (2013), 1045–1087.

[3] M. Aschenbrenner, S. Friedl and H. Wilton, *3-manifold groups*, Preprint (2013)

[4] R. Bieri, *Deficiency and the geometric invariants of a group*, With an appendix by Pascal Schweitzer, J. Pure Appl. Algebra **208** (2007), 951–959.

[5] K. S. Brown, *Trees, valuations, and the Bieri-Neumann-Strebel invariant*, Invent. Math. **90** (1987), 479-504.

[6] T. Cochran, *Noncommutative knot theory*, Algebr. Geom. Topol. **4** (2004), 347–398.

[7] N. Dunfield, *Alexander and Thurston norms of fibered 3–manifolds*, Pacific J. Math. **200** (2001), no. 1, 43–58.

[8] S. Friedl, *Reidemeister torsion, the Thurston norm and Harvey's invariants*, Pac. J. Math. **230** (2007), 271–296.

[9] S. Friedl and S. Tillmann, *Two-generator one-relator groups and marked polytopes*, preprint, 2015.
[10] S. Friedl, K. Schreve, S. Tillmann, *Thurston norm via Fox calculus*, preprint, 2015.
[11] S. Friedl and S. Vidussi, *The Thurston norm and twisted Alexander polynomials*, Preprint (2012), to be published by J. Reine Ang. Math.
[12] S. Harvey, *Higher–order polynomial invariants of 3–manifolds giving lower bounds for the Thurston norm*, Topology **44** (2005), 895–945.
[13] P. Linnell and P. Schick, *Finite group extensions and the Atiyah conjecture*, J. Amer. Math. Soc. **20** (2007), no. 4, 1003–1051.
[14] Y. Liu, *Virtual cubulation of nonpositively curved graph manifolds*, J. Topol. **6** (2013), no. 4, 793–822.
[15] P. Przytycki and D. Wise, *Graph manifolds with boundary are virtually special*, J. Topology **7** (2014), 419–435.
[16] P. Przytycki and D. Wise, *Mixed* 3*-manifolds are virtually special*, preprint (2012).
[17] J.–C. Sikorav, *Homologie de Novikov associée à une classe de cohomologie réelle de degré un*, thèse, Orsay, 1987.
[18] W. P. Thurston, *A norm for the homology of 3–manifolds*, Mem. Amer. Math. Soc. **59**, no. 339 (1986), 99–130.
[19] D. Tischler, *On fibering certain foliated manifolds over* S^1, Topology **9** (1970), 153–154.
[20] C. Weinbaum, *On relators and diagrams for groups with one defining relation*, Illinois J. Math. **16** (1972), 308–322.
[21] D. Wise, *From riches to RAAGs:* 3*-manifolds, right–angled Artin groups, and cubical geometry*, CBMS Regional Conference Series in Mathematics, 2012.