SYMPLECTIC $\mathbf{S^1 \times N^3}$ AND SUBGROUP SEPARABILITY

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Let N be a closed 3–manifold. Thurston [Th76] showed that if N admits a fibration over S^1, then $S^1 \times N$ is symplectic, i.e. it can be endowed with a closed, non–degenerate 2–form ω.

It is natural to ask whether the converse of this statement holds true. We can state this problem in the following form:

Conjecture 1. *Let N be a closed 3–manifold. If $S^1 \times N$ is symplectic, then there exists $\phi \in H^1(N;\mathbb{Z})$ such that (N,ϕ) fibers over S^1.*

Here we say that (N,ϕ) fibers over S^1 if the homotopy class of maps $N \to S^1$ determined by $\phi \in H^1(N;\mathbb{Z}) = [N,S^1]$ contains a representative that is a fiber bundle over S^1.

Assuming the Geometrization Conjecture, it is possible to prove that the problem is reduced to the study of irreducible 3–manifolds, and we will henceforth make that assumption for N.

In [FV06a] we related this problem to the study of twisted Alexander polynomials of N, and in particular we proved the following, that is a weaker version of the main result of [FV06a]:

Theorem 1. *Let N be an irreducible 3–manifold such that $S^1 \times N$ admits a symplectic structure. Then there exists a primitive $\phi \in H^1(N;\mathbb{Z})$ such that for any epimorphism $\alpha : \pi_1(N) \to G$ onto a finite group G the associated 1–variable twisted Alexander polynomial $\Delta^{\alpha}_{N,\phi} \in \mathbb{Z}[t^{\pm 1}]$ is non–zero.*

Recall that the 1–variable twisted Alexander polynomial $\Delta^{\alpha}_{N,\phi}$ associated to the pair (N,ϕ) is defined as the $\mathbb{Z}[t^{\pm 1}]$–order of the twisted Alexander module $H_1(N;\mathbb{Z}[G][t^{\pm 1}])$. Note that $\Delta^{\alpha}_{N,\phi} \neq 0$ if and only if $H_1(N;\mathbb{Z}[G][t^{\pm 1}])$ is $\mathbb{Z}[t^{\pm 1}]$–torsion.

Given $\alpha : \pi_1(N) \to G$, denote the corresponding regular cover of N cover by N_G. Note that, if $S^1 \times N$ is symplectic, so is $S^1 \times N_G$. The ingredients of the proof of Theorem 1 are now the following: Taubes' results on the Seiberg-Witten invariants of symplectic 4–manifolds, the relation, proved by Meng and Taubes, between the Seiberg–Witten invariants of N_G and the ordinary ordinary Alexander polynomial Δ_{N_G}, and finally a relation obtained in [FV06a] between Δ_{N_G} and $\Delta^{\alpha}_{N,\phi}$.

Theorem 1 says in particular that the following conjecture implies Conjecture 1 for irreducible manifolds.

Conjecture 2. *Let $\phi \in H^1(N;\mathbb{Z})$ be a primitive class such that $\Delta^{\alpha}_{N,\phi} \neq 0$ for all $\alpha : \pi_1(N) \to G$, then (N,ϕ) fibers over S^1.*

To state our main theorem we need the following definition.

Definition. A subgroup $A \subset \pi$ is *separable* if for all $g \in \pi \setminus A$, there exists an epimorphism $\alpha : \pi \to G$ to a finite group G such that $\alpha(g) \notin \alpha(A)$.

We have the following result, proven in [FV06b]:

Theorem 2. *Let N be an irreducible 3–manifold. Let $\phi \in H^1(N;\mathbb{Z})$ be a primitive class such that $\Delta^{\alpha}_{N,\phi} \neq 0$ for all epimorphisms $\alpha : \pi_1(N) \to G$ to a finite group. Let $\Sigma \subset N$ be an embedded surface dual to ϕ having minimal genus. If $\pi_1(\Sigma) \subset \pi_1(N)$ is separable, then (N, ϕ) fibers.*

The question of which subgroups of the fundamental group of a Haken manifold are separable has been studied extensively. In particular, the fact that abelian subgroups are separable (cf. [LN91] and [Ha01]) and that incompressible surfaces in Seifert fibered spaces are classified leads to the following corollary.

Corollary 1. *Conjecture 1 holds for irreducible manifolds with vanishing Thurston norm and for graph manifolds.*

This corollary in particular implies that if N_K is the 0–surgery on a knot K of genus $g(K) = 1$, and $S^1 \times N_K$ is symplectic, then K is a trefoil or the figure–8 knot. This answers a question of Kronheimer [Kr98].

Scott [Sc78] showed that any subgroup of a hyperbolic 2–manifolds is separable. It has been conjectured by Thurston [Th82] that all (surface) subgroups of hyperbolic 3–manifolds are separable. Clearly a positive solution to Thurston's conjecture would imply Conjecture 1 for hyperbolic manifolds. Furthermore suitable subgroup separability properties of the hyperbolic pieces in the geometric decomposition can be shown to imply Conjecture 1 for all irreducible manifolds.

We conclude with a short outline of the proof of Theorem 2. Let $M = N \setminus \nu\Sigma$. We have two embeddings $i_\pm : \Sigma \to \partial M$. By Stallings' theorem, (N, ϕ) fibers if the inclusion induced maps $i_\pm : \pi_1(\Sigma) \to \pi_1(M)$ are isomorphisms. Since Σ has minimal genus we know that the maps $i_\pm : \pi_1(\Sigma) \to \pi_1(M)$ are injective and that $\pi_1(M) \to \pi_1(N)$ is injective.

Assume, by contradiction, that one of the $i_\pm$ is not an isomorphism. We use the corresponding inclusion to view $\pi_1(\Sigma)$ and $\pi_1(M)$ as subgroups of $\pi_1(N)$.

Given an epimorphism $\alpha : \pi_1(N) \to G$ to any finite group G we have a long exact Mayer–Vietoris sequence

$$H_1(N;\mathbb{Z}[G][t^{\pm 1}]) \to H_0(\Sigma;\mathbb{Z}[G])\otimes\mathbb{Z}[t^{\pm 1}] \to H_0(M;\mathbb{Z}[G])\otimes\mathbb{Z}[t^{\pm 1}] \to H_0(N;\mathbb{Z}[G][t^{\pm 1}]).$$

Now consider the ranks of the modules over $\mathbb{Z}[t^{\pm 1}]$: we have

$$\begin{aligned}
\text{rank}_{\mathbb{Z}[t^{\pm 1}]}(H_1(N;\mathbb{Z}[G][t^{\pm 1}])) &= 0 \text{ since } \Delta^{\alpha}_{N,\phi} \neq 0,\\
\text{rank}_{\mathbb{Z}[t^{\pm 1}]}(H_0(N;\mathbb{Z}[G][t^{\pm 1}])) &= 0 \text{ since } \phi \neq 0,\\
\text{rank}_{\mathbb{Z}[t^{\pm 1}]}(H_0(\Sigma;\mathbb{Z}[G]) \otimes \mathbb{Z}[t^{\pm 1}]) &= \text{rank}_{\mathbb{Z}}(H_0(\Sigma;\mathbb{Z}[G])),\\
\text{rank}_{\mathbb{Z}[t^{\pm 1}]}(H_0(M;\mathbb{Z}[G]) \otimes \mathbb{Z}[t^{\pm 1}]) &= \text{rank}_{\mathbb{Z}}(H_0(M;\mathbb{Z}[G])).
\end{aligned}$$

Therefore

$$\frac{|G|}{|\alpha(\pi_1(\Sigma))|} = \text{rank}_{\mathbb{Z}}(H_0(\Sigma;\mathbb{Z}[G])) = \text{rank}_{\mathbb{Z}}(H_0(M;\mathbb{Z}[G])) = \frac{|G|}{|\alpha(\pi_1(M))|}.$$

In particular we get that $\alpha(\pi_1(\Sigma)) = \alpha(\pi_1(M)) \subset G$. On the other hand it follows immediately from the assumption that $\pi_1(\Sigma) \subset \pi_1(N)$ is separable, and from the assumption that $\pi_1(\Sigma) \to \pi_1(M)$ is not an epimorphism, that there exists an epimorphism $\alpha : \pi_1(N) \to G$ to a finite group with $\alpha(\pi_1(\Sigma)) \neq \alpha(\pi_1(M))$. This contradiction concludes the proof of Theorem 2.

References

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