NONTRIVIAL ALEXANDER POLYNOMIALS OF KNOTS AND LINKS

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ABSTRACT. In this paper we present a sequence of link invariants, defined from twisted Alexander polynomials, and discuss their effectiveness in distinguishing knots. In particular, we recast and extend by geometric means a recent result of Silver and Williams on the nontriviality of twisted Alexander polynomials for nontrivial knots. Furthermore building on results in [7] we prove that these invariants decide if a genus one knot is fibered. Finally we also show that these invariants distinguish all mutants with up to 12 crossings.

1. Definition of the invariant and main results

Let $L \subset S^3$ be an oriented m–component link, and denote by $X(L) = S^3 \setminus \nu L$ its exterior. Let $R = \mathbb{Z}$ or $R = \mathbb{F}_p$: given a representation $\alpha : \pi_1(X(L)) \to \mathrm{GL}(R,k)$ we can consider the associated multivariable twisted Alexander polynomial $\Delta_L^\alpha \in R[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$ (where $t_1, \dots, t_m$ correspond to a basis of $H_1(X(L))$ determined by the meridians to each link component), well–defined up to units. In Section 2.1 we recall the details of the definition.

Let now $\alpha : \pi_1(X) \to S_k$ be a homomorphism into the symmetric group. Using the action of S_k on R^k by permutation of the coordinates, we get a representation $\pi_1(X) \to S_k \to \mathrm{GL}(R,k)$ that we will denote by α as well. Consider now the set of representations of $\pi_1(X(L))$ in the symmetric group, modulo conjugation:

$$\mathcal{R}_k(L) = \{\alpha : \pi_1(X(L)) \to S_k\}/\sim$$

where two representations are equivalent if they are the same up to conjugation by an element in S_k. Given $\alpha : \pi_1(X(L)) \to S_k$ the polynomial Δ_L^α depends only on the equivalence class $[\alpha]$ of α in $\mathcal{R}_k(L)$. We now define the invariant

$$\Delta_L^k = \prod_{[\alpha] \in \mathcal{R}_k(L)} \Delta_L^\alpha \in R[t_1^{\pm 1}, \dots, t_m^{\pm 1}].$$

We will illustrate the effectiveness of this invariant by discussing some of the topological information that it carries, and by using explicit calculations we show its ability to distinguish many examples of inequivalent mutant knots.

Our first result relates the link invariants Δ_L^k with epimorphisms of the link group onto finite groups, which will lead to a useful topological interpretation (cf. Lemma 2.3). Precisely, consider an epimorphism $\gamma : \pi_1(X(L)) \to G$, where G is a finite group of order $k = |G|$. Using the left action of G on its group ring we can define a representation, denoted with the same symbol, $\gamma : \pi_1(X(L)) \to \mathrm{Aut}_R(R[G]) \cong \mathrm{GL}(R,k)$. We have the following:

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Proposition 1.1. *Let $\gamma : \pi_1(X(L)) \to G$ be a homomorphism to a finite group G of order k. Then Δ_L^γ divides Δ_L^k.*

This relation is crucial in proving the following theorem, which shows that the sequence Δ_L^k detects the unknot and the Hopf link.

Theorem 1.2. *Let $L \subset S^3$ be an oriented link which is neither the unknot nor the Hopf link (with either orientation). Then there exists a k such that $\Delta_L^k \neq \pm 1 \in \mathbb{Z}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$.*

In fact we will show that if L is neither the unknot nor the Hopf link, then there exists an epimorphism $\gamma : \pi_1(X(L)) \to G$ to a finite group such that $\Delta_L^\gamma \neq 1$. (This result is nontrivial when $m = 1$ or 2.) For the case of knots this provides a different approach to a recent result by Silver and Williams [22].

The proof is based on the relation between twisted Alexander polynomials and covers of the link exterior, using ideas from previous papers by the authors [6, 7], combined with information on the topology of those covers arising from the work in [19, 16, 3].

If K is a fibered knot, its ordinary Alexander polynomial is monic. The following result, combining results from [5] and [7], generalizes that assertion to Δ_K^k and shows that, at least in some cases, the converse holds true.

Theorem 1.3. *Let $K \subset S^3$ be a fibered knot, then $\Delta_K^k \in \mathbb{Z}[t^{\pm 1}]$ is monic for any k. Conversely, if Δ_K^k is monic for all k and if K is a genus one knot, then K is fibered.*

Note that the converse also holds for knots whose exteriors has fundamental group that satisfies suitable subgroup separability properties. We refer the interested reader to [7] for details (the results in [7] are only stated for closed 3–manifolds, but they also hold for 3–manifolds with toroidal boundary).

For a knot K the calculation of Δ_K^k can be done using the program *KnotTwister* [8]. Our computations in Section 4 confirm that Δ_K^k are very strong knot invariants. For example computing $\Delta_K^5 \in \mathbb{F}_{13}[t^{\pm 1}]$ distinguishes all pairs and triples of mutants with up to 12 crossings (cf. Section 4 for the definition of mutants). In Section 4 we also show that Δ_K^4 is not determined by either HOMFLY polynomial, Khovanov homology or Knot Floer homology.

The paper is organized as follows. In Section 2 we give a precise definition of twisted Alexander polynomials and discuss some basic properties. In particular we give a proof of Proposition 1.1. In Section 3 we give the proofs of Theorems 1.2 and 1.3. We conclude the paper in Section 4 with several examples and questions.

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2. Twisted Alexander polynomials and finite covers

2.1. **Twisted Alexander modules and their polynomials.** In this section we give the precise definition of the (twisted) Alexander polynomials. Twisted Alexander polynomials were introduced, for the case of knots, in 1990 by Lin [18], and further generalized to links by Wada [25]. We follow the approach taken by Cha [2] and [6].

For the remainder of this section let N be a 3–manifold (by which we always mean a compact, connected and oriented 3–manifold) and denote by $H := H_1(N)/\text{Tor}H_1(N)$ the maximal free abelian quotient of $\pi_1(N)$. Furthermore let F be a free abelian group and let R be $\mathbb{Z}$ or the field $\mathbb{F}_p := \mathbb{Z}/p\mathbb{Z}$ where p is a prime number.

Now let $\phi \in \text{Hom}(H, F)$ be a non–trivial homomorphism. Through the homomorphism ϕ, $\pi_1(N)$ acts on F by translations. Furthermore let $\alpha : \pi_1(N) \to \text{GL}(R, k)$ be a representation. We write $R^k[F] = R^k \otimes_R R[F]$. We get a representation

$$\begin{array}{rccl} \alpha \otimes \phi : & \pi_1(N) & \to & \text{Aut}(R^k[F]) \\ & g & \mapsto & \left(\sum_i a_i \otimes f_i \mapsto \sum_i \alpha(g)(a_i) \otimes (f_i + \phi(g)\right). \end{array}$$

We can therefore view $R^k[F]$ as a left $\mathbb{Z}[\pi_1(N)]$–module. Note that this module structure commutes with the natural $R[F]$–multiplication on $R^k[F]$.

Let $\tilde{N}$ be the universal cover of N. Note that $\pi_1(N)$ acts on the left on $\tilde{N}$ as the group of deck transformation. The chain groups $C_*(\tilde{N})$ are in a natural way right $\mathbb{Z}[\pi_1(N)]$–modules, with the right action on $C_*(\tilde{N})$ defined via $\sigma \cdot g := g^{-1}\sigma$, for $\sigma \in C_*(\tilde{N})$. We can form by tensor product the chain complex $C_*(\tilde{N}) \otimes_{\mathbb{Z}[\pi_1(N)]} R^k[F]$. Now define $H_1(N; R^k[F]) := H_1(C_*(\tilde{N}) \otimes_{\mathbb{Z}[\pi_1(N)]} R^k[F])$, which inherits the structure of $R[F]$–module.

The $R[F]$–module $H_1(N; R^k[F])$ is a finitely presented and finitely related $R[F]$–module since $R[F]$ is Noetherian. Therefore $H_1(N; R^k[F])$ has a free $R[F]$–resolution

$$R[F]^r \xrightarrow{S} R[F]^s \to H_1(N; R^k[F]) \to 0$$

of finite $R[F]$–modules, where we can always assume that $r \geq s$.

Definition 2.1. The *twisted Alexander polynomial* of (N, α, ϕ) is defined to be the order of the $R[F]$–module $H_1(N; R^k[F])$, i.e. the greatest common divisor of the $s \times s$ minors of the $s \times r$–matrix S. It is denoted by $\Delta^\alpha_{N,\phi} \in R[F]$.

Note that this definition only makes sense since $R[F]$ is a UFD. It is well–known that $\Delta^\alpha_{N,\phi}$ is well–defined only up to multiplication by a unit in $R[F]$ and its definition is independent of the choice of the resolution.

When ϕ is the identity map on H, we will simply write Δ^α_N. Also, we will write $\Delta_{N,\phi}$ in the case that $\alpha : \pi_1(N) \to \text{GL}(\mathbb{Z}, 1)$ is the trivial representation.

If $N = X(L)$ is the exterior of an oriented ordered link $L = L_1 \cup \cdots \cup L_m$, then we write Δ^α_L for the twisted Alexander polynomial of $X(L)$. Also, we can identify H with the free abelian group generated by $t_1, \ldots, t_m$ and we can view the corresponding twisted Alexander polynomial Δ^α_L as an element in $R[t_1^{\pm 1}, \ldots, t_m^{\pm 1}]$.

2.2. **Twisted Alexander polynomials and homomorphisms to finite groups.** Let N be a 3–manifold and let $\gamma : \pi_1(N) \to G$ be an epimorphism onto a finite group G of order $k = |G|$. We get the induced regular representation $\gamma : \pi_1(N) \to G \to \text{Aut}(R[G])$ where $g \in G$ acts on $R[G]$ by left multiplication. Since $R[G] \cong R^{|G|}$ we can identify $\text{Aut}_R(R[G]) = \text{GL}(R, k)$. It is easy to see that the isomorphism type of the $R[H]$–module $H_1(N; R^k[H])$ does not depend on the identification $\text{Aut}_R(R[G]) = \text{GL}(R, k)$.

The following lemma clearly implies Proposition 1.1.

Lemma 2.2. *Let $\gamma : \pi_1(N) \to G$ be an epimorphism onto a finite group G of order k. Then there exists a homomorphism $\alpha : \pi_1(N) \to S_k$ such that the corresponding representation*

$$\alpha : \pi_1(N) \to S_k \to GL(R,k)$$

is given by the regular representation $\gamma : \pi_1(N) \to G \to GL(R,k)$.

Proof. Denote the elements of G by $g_1,\dots,g_k$. Since γ defines an action on the set $G = \{g_1,\dots,g_k\}$ via left multiplication we get an induced map $\alpha : \pi_1(N) \to S_k$. Clearly the corresponding representation

$$\alpha : \pi_1(N) \to S_k \to \mathrm{GL}(R,k)$$

is isomorphic to the regular representation $\gamma : \pi_1(N) \to G \to \mathrm{GL}(R,k)$. □

2.3. **Twisted Alexander polynomials and finite covers.** For the remainder of this section let $\gamma : \pi_1(N) \to G$ be an epimorphism onto a finite group G of order k, and take $R = \mathbb{Z}$. Denote the induced G–cover of N by $\pi : N_G \to N$. Also, denote by H_G the maximal free abelian quotient of $\pi_1(N_G)$: the map $\pi_* : H_G \to H$ is easily seen to have maximal rank, hence in particular $b_1(N_G) \geq b_1(N)$. Given any homomorphism $\phi : H \to F$ to a free abelian group F we can consider the induced homomorphism $\phi_G := \pi^*\phi : H_G \to F$. In particular, when ϕ is the identity map on H, we have $\phi_G = \pi_* : H_G \to H$.

We can now formulate the relationship between the twisted Alexander polynomials of N and the untwisted Alexander polynomial of N_G.

Lemma 2.3. [6] *Let $\gamma : \pi_1(N) \to G$ be an epimorphism onto a finite group G and $\pi : N_G \to N$ the induced G-cover. Then*

$$\Delta_N^\gamma = \Delta_{N_G,\pi_*} \in \mathbb{Z}[H].$$

Finally, we need to rewrite the Alexander polynomial Δ_{N_G,π_*} in terms of the full Alexander polynomial of N_G; their relation is the following.

Proposition 2.4. [6][23] *Let N be a 3–manifold with non–empty toroidal boundary, and let N_G be the 3–manifold defined as above. Furthermore let $\Delta_{N_G} \in \mathbb{Z}[H_G]$ be the (ordinary multivariable) Alexander polynomial. Then we have the following equality in $\mathbb{Z}[H]$:*
If $b_1(N_G) > 1$, then

$$(1)\quad \Delta_{N_G,\pi_*} = \begin{cases} \pi_*(\Delta_{N_G}) & \text{if } b_1(N) > 1, \\ (t^{div\pi_*} - 1)\pi_*(\Delta_{N_G}) & \text{if } b_1(N) = 1,\ Im\,\pi_* = \langle t^{div\,\pi_*}\rangle,\ t \in H \text{ indivisible.} \end{cases}$$

If $b_1(N_G) = 1$, then $b_1(N) = 1$ and

$$(2)\qquad \Delta_{N_G,\pi_*} = \pi_*(\Delta_{N_G}).$$

3. Proof of Theorems 1.2 and 1.3

3.1. **Proof of Theorem 1.2.** The topological ingredient of the proof is a result on the virtual Betti number of link exteriors. This result can be deduced quite directly from [3, Theorem 1.3], but it is perhaps appropriate, in order to illustrate its nature, to break down the proof to emphasize the role of the JSJ decomposition of a link exterior.

We start with the following results.

Theorem 3.1. [19, 16] *Let N be an irreducible 3–manifold containing an essential torus or annulus S; up to a lift to a finite cover, we can assume that S is non–separating. Then S is either the fiber of a fibration over S^1, or the virtual Betti number $vb_1(N)$ of N is infinite.*

Remark. Recall that having virtual Betti number $vb_1(N)$ infinite means that N admits finite covers of arbitrarily large Betti number. *A priori*, the covers do not have to be regular: however, to any finite cover $\hat{N}$ with fundamental group $\hat{\pi}$ we can canonically associate a finite regular cover $\bar{N}$ determined by the subgroup $\bar{\pi} := \cap_{p\in\pi_1(N)} p\hat{\pi}p^{-1}$. This subgroup is clearly a normal subgroup of both $\hat{\pi}$ and $\pi_1(N)$. Also, since $\hat{\pi} \subset \pi_1(N)$ is of finite index we see easily that $\bar{\pi}$ is in fact the intersection of finitely many subgroups of $\pi_1(N)$ of finite index. Therefore $\bar{\pi} \subset \hat{\pi} \subset \pi_1(N)$ is of finite index as well, and $\bar{N}$ is a finite cover. From standard arguments, we have $b_1(\bar{N}) \geq b_1(\hat{N}) \geq b_1(N)$, so we can assume that N admits finite *regular* covers of arbitrarily large Betti number. The set of left cosets $\pi_1(N)/\bar{\pi}$ is a finite group, that we denote by G, hence $\bar{\pi}$ is the kernel of an epimorphism $\gamma : \pi_1(N) \to G$, so that $\bar{N} = N_G$.

Theorem 3.2. [3, Theorem 2.7] *Let N be an irreducible 3–manifold with non–empty incompressible boundary all of whose components are tori. Suppose that the interior of N has a complete hyperbolic structure of finite volume. Then $vb_1(N) = \infty$.*

The topological ingredient in the proof of Theorem 1.2 is then the following observation.

Lemma 3.3. *Let $L = L_1 \cup \cdots \cup L_m \subset S^3$ be an oriented link which is neither the unknot nor the Hopf link. Then $vb_1(X(L)) = \infty$.*

Proof. First note that if L is a split link, i.e. if $X(L) = S^3 \setminus \nu L$ is reducible, then $\pi_1(X(L))$ maps onto a free group with two generators, which implies that $vb_1(X(L)) = \infty$ (cf. e.g. [16]).

We can therefore now assume that L is not a split link. In particular no component of L bounds a disk in the complement of the components. By Dehn's Lemma this implies that the boundary of $X(L)$ is incompressible. As $X(L)$ is irreducible and has boundary, $X(L)$ is Haken, hence it admits a geometric decomposition along a (possibly empty) family of essential tori $\mathcal{T}$. We will break the argument in subcases.

First assume that $\mathcal{T}$ is non-empty. Clearly $X(L)$ cannot be covered by a torus bundle over S^1 since $X(L)$ has boundary. It therefore follows from Theorem 3.1 that $vb_1(X(L)) = \infty$.

Now assume that $\mathcal{T}$ is empty. By Thurston's geometrization of Haken manifolds we deduce that either $X(L)$ is Seifert-fibered or the interior of $X(L)$ has a complete hyperbolic structure of finite volume.

In the hyperbolic case, Theorem 3.2 asserts that $vb_1(X(L)) = \infty$.

We are left with the Seifert–fibered case. The classification of Seifert links (see [4, Chapter II]) shows that L is the link obtained by removing m fibers, regular or singular, from the (p, q)–Seifert fibration of S^3, where (p, q) are coprime integers or $(0, \pm 1)$. Depending on the type of the orbifold quotient (see Jaco [13, Chapter VIII]), $X(L)$ either contains essential tori or is *special*. In the former case, Theorem 3.1 implies $vb_1(X(L)) = \infty$ right away. If $X(L)$ is special, checking case–by–case, L is either: a (nontrivial) (p, q)–torus knot, obtained by removing a regular fiber; the union of the unknot and its (p, q)–cable, obtained by removing a regular fiber and the fiber with multiplicity p (whose exterior is the *p/q–cable space*); one of a family of 3–component links obtained by removing a regular fiber and the two singular fibers. In the last two cases, we can identify an essential, non–separating *cabling* annulus joining a regular and a singular fiber of the Seifert fibration. With the exception of the Hopf link with either orientations (corresponding to $q = \pm 1$) these annuli do not fiber $X(L)$ by [4, Theorem 11.2]. For a (p, q)–torus knot traced on a torus T, the annulus $X(L) \cap T$ is the only essential annulus, and it is separating, so we pass to some finite cover. However, this cover cannot be an annulus bundle over S^1 ($T^2 \times I$ or the twisted I-bundle over a Klein bottle), as by [11, Theorems 10.5, 10.6] the only manifolds covered by $T^2 \times I$ are $T^2 \times I$ itself and the twisted I-bundle over a Klein bottle, which does not embed in S^3. It follows that, with the exception of the Hopf link with either orientation (for whom $X(L) = T^2 \times I$), all these links have $vb_1(X(L)) = \infty$ by Theorem 3.1. □

The following theorem, together with Proposition 1.1, immediately implies Theorem 1.2.

Theorem 3.4. *Let $L = L_1 \cup \cdots \cup L_m \subset S^3$ be an oriented link which is neither the unknot nor the Hopf link. Then there exists an epimorphism $\gamma : \pi_1(X(L)) \to G$ onto a finite group G such that $\Delta_L^\gamma \neq \pm 1$.*

Proof. Since L is neither the unknot nor the Hopf link, Lemma 3.3 implies that there exists a cover $X(L)_G$ with $b_1(X(L)_G) > 2$. As $X(L)_G$ has non–empty boundary all of whose components are tori, Corollary II.4.4 of [24] implies that the sum of the coefficients of $\Delta_{X(L)_G}$ is zero. Hence, by Proposition 2.4 the sum of the coefficients of $\Delta_{X(L)_G,\pi_*}$ is zero as well, hence, by Lemma 2.3, $\Delta_{X(L)}^\gamma$ cannot be ± 1. □

When L is the unknot or the Hopf link, $X(L)$ is homeomorphic to $S^1 \times D^2$ and $T^2 \times I$ respectively. In particular, the maximal (free) abelian cover $\widehat{X}(L)$ is contractible. Given any representation $\pi_1(X(L)) \to \mathrm{GL}(R, k)$, we have

$$H_1(X(L); R^k[H_1(X(L))]) \cong H_1(\widehat{X}(L); R^k) = 0,$$

where the first isomorphism follows from the Eckmann–Shapiro lemma. As the corresponding twisted Alexander module is trivial, $\Delta_L^k = 1$ for all k. This implies that the sequence Δ_L^k detects the unknot and the Hopf link.

3.2. **Proof of Theorem 1.3.** Let $K \subset S^3$ be a fibered knot; it is shown in [5] that Δ_K^α is monic for any representation $\alpha : \pi_1(X(K)) \to \mathrm{GL}(\mathbb{Z}, k)$ (cf. also [2] and [10]). This clearly implies that Δ_K^k is monic for all k.

Now let K be a genus one knot such that Δ_K^k is monic for all k. We denote by N_K the zero framed surgery along K. Gabai [9] showed that K is fibered if and only if N_K is fibered. Clearly, N_K has vanishing Thurston norm. Under this hypothesis we show, in [7] that if N_K is not fibered, then there exists an epimorphism $\beta : \pi_1(N_K) \to G$ onto a finite group G such that $\Delta_{N_K}^\beta = 0$.

Now consider the homomorphism

$$\gamma : \pi_1(X(K)) \to \pi_1(N_K) \to G.$$

Since $\pi_1(X(K)) \to \pi_1(N_K)$ is an epimorphism, it follows from the 5–term exact sequence (cf. [1, Chapter VII, Corollary 6.4]) that $H_1(N_K; R^k[t^{\pm 1}])$ is a quotient of $H_1(X(K); R^k[t^{\pm 1}])$, hence there exists a polynomial $p(t) \in \mathbb{Z}[t^{\pm 1}]$ such that $\Delta_K^\gamma = p(t)\Delta^\beta_{N_K}$ (cf. also [15]). In particular $\Delta^\gamma_{X(K)} = 0$. But then Theorem 1.3 follows from Proposition 1.1.

4. Calculations

A natural test for invariants is their ability to detect mutation. Recall that two knots K_1 and K_2 are called mutants if there exists a ball in S^3 whose boundary meets the knots in 4 points, such that removing the ball, rotating it by π around an axis (in a way which preserves the 4 points), and gluing it back turns K_1 into K_2. Figure 1 shows perhaps the

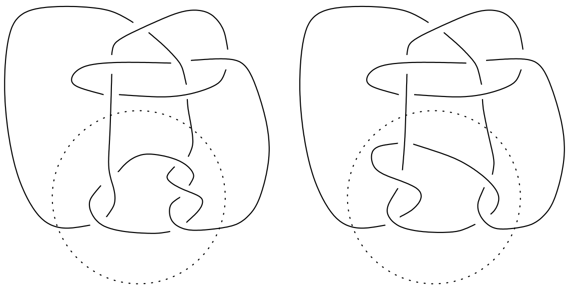

Figure 1. The Conway knot and the Kinoshita–Terasaka knot.

most famous pair of mutants, namely the Conway knot 11_{401} and the Kinoshita–Terasaka knot 11_{409}. (Here we use Knotscape notation for knots with more than 10 crossings, it is organized so that non-alternating knots are appended to alternating ones instead of using 'a' and 'n' superscripts.) In both cases there exist seven equivalence classes of abelian homomorphisms $\pi_1(X(K)) \to S_5$ and one non–abelian equivalence class of homomorphisms $\pi_1(X(K)) \to S_5$. Using *KnotTwister* we can compute their invariants:

$$\Delta^5_{11_{401}} = 1 + 6t + 9t^2 + 12t^3 + t^5 + 3t^6 + t^7 + 3t^8 + t^9 + 12t^{11} + 9t^{12} + 6t^{13} + t^{14}$$
$$\Delta^5_{11_{409}} = 1 + 11t + 12t^2 + 10t^3 + 5t^4 + 11t^5 + 4t^6 + 11t^7 + 5t^8 + 10t^9 + 12t^{10} + 11t^{11} + t^{12}$$

where both polynomials are considered in $\mathbb{F}_{13}[t^{\pm 1}]$. Note that Wada [25] used parabolic representations to $\mathrm{SL}(\mathbb{F}_{17}, 2)$ to show that these two knots can be distinguished using twisted Alexander polynomials (cf. also [12]).

We have computed $\Delta_K^k \in \mathbb{F}_{13}[t^{\pm 1}]$ for all groups of mutant 11–crossing knots for the smallest value of k that distinguishes the mutants. The results are tabled in the Appendix. We also computed Δ^5_K for all groups of mutant 12–crossing knots, and again we verified that Δ^5_K distinguishes the mutant knots.

We can summarize these computations in the following lemma.

Lemma 4.1. *Let K_1, K_2 be a mutant pair with 12 crossings or less. Then*

$$\Delta^5_{K_1} \neq \Delta^5_{K_2} \in \mathbb{F}_{13}[t^{\pm 1}].$$

Note that the results of Section 3 can be interpreted as stating that the sequence Δ^k_K detects the unknot, the trefoil knot and the figure–8 knot (which are the only fibered genus one knots). This raises the question about how effectively the sequence Δ^k_K at distinguishes knots in general.

In fact, we can use Δ_K^k to examine pairs of knots for whom other invariants are inconclusive. For example, the knots 10_{40} and 10_{103} are alternating knots with the same HOMFLY polynomial (hence same Jones and Alexander polynomial) and the same signature. As Ng [20, p. 292] points out this implies by [17] and [21] that 10_{40} and 10_{103} also have the same Khovanov homology and the same knot Floer homology. One can verify that $\Delta^3_{10_{40}} = \Delta^3_{10_{103}} \in \mathbb{F}_{13}[t^{\pm 1}]$ and that $\mathcal{R}_4(10_{40})$ and $\mathcal{R}_4(10_{103})$ have eight elements each. Furthermore in $\mathbb{F}_{13}[t^{\pm 1}]$ we have

$$\begin{array}{rcl} \Delta^4_{10_{40}} &=& 1+8t^2+t^3+12t^4+8t^5+\cdots+8t^{176}+t^{178} \\ \Delta^4_{10_{103}} &=& 1+11t+12t^2+4t^3+2t^4+3t^5+\cdots+12t^{170}+11t^{171}+t^{172}. \end{array}$$

So the invariant Δ_K^4 is neither determined by Khovanov homology nor by Knot Floer homology.

Appendix

The following table lists all mutant pairs of knots with 11 crossings, together with the degrees of $\Delta_K^k \in \mathbb{F}_{13}[t^{\pm 1}]$ (for the smallest k which distinguishes the mutants) and the first 5 terms of Δ_K^k. All computations take place in the ring $\mathbb{F}_{13}[t^{\pm 1}]$. Note that the first five pairs of mutants are also distinguished by Δ_K^5.

Knot	k	$\#\mathcal{R}_k(K)$	$\deg(\Delta_K^k)$	Lowest and highest terms of $\Delta_K^k \in \mathbb{F}_{13}[t^{\pm 1}]$
11_{44}	3	7	160	$1+12t+2t^2+7t^3+8t^4+12t^5+\cdots+12t^{159}+t^{160}$
11_{47}	3	7	160	$1+12t+11t^2+11t^3+7t^4+3t^5+\cdots+12t^{159}+t^{160}$
11_{57}	3	7	160	$1+12t+5t^2+3t^3+2t^4+10t^5+\cdots+12t^{159}+t^{160}$
11_{231}	3	7	160	$1+12t+t^2+7t^3+12t^4+7t^5+\cdots+12t^{159}+t^{160}$
11_{438}	3	7	118	$1+11t+9t^2+4t^3+t^5+\cdots+11t^{117}+t^{118}$
11_{442}	3	7	118	$1+11t+3t^2+3t^3+t^4+10t^5+\cdots+11t^{117}+t^{118}$
11_{440}	3	7	88	$1+10t+t^2+12t^3+9t^4+5t^5+\cdots+10t^{87}+t^{88}$
11_{441}	3	7	76	$1+10t+10t^2+11t^3+3t^4+2t^5+\cdots+10t^{75}+t^{76}$
11_{443}	3	7	160	$1+2t+2t^2+8t^3+t^4+9t^5+\cdots+2t^{159}+t^{160}$
11_{445}	3	7	160	$1+2t+6t^2+3t^3+6t^4+9t^5+\cdots+2t^{159}+t^{160}$
11_{19}	5	13	496	$1+9t+2t^2+12t^3+6t^4+6t^5+\cdots+4t^{495}+12t^{496}$
11_{25}	5	12	460	$1+2t+7t^2+9t^3+t^4+2t^5+\cdots+2t^{459}+t^{460}$
11_{24}	5	10	388	$1+9t+11t^2+t^3+10t^4+t^5+\cdots+9t^{387}+t^{388}$
11_{26}	5	9	352	$1+2t+7t^2+8t^3+8t^4+8t^5+\cdots+11t^{351}+12t^{352}$
11_{251}	5	11	424	$1+8t+7t^2+5t^3+12t^4+4t^5+\cdots+5t^{423}+12t^{424}$
11_{253}	5	11	424	$1+8t+3t^2+5t^3+t^4+9t^5+\cdots+5t^{423}+12t^{424}$
11_{252}	5	9	352	$1+2t+5t^3+6t^4+11t^5+\cdots+11t^{351}+12t^{352}$
11_{254}	5	10	388	$1+9t+3t^2+2t^3+t^4+6t^5+\cdots+9t^{387}+t^{388}$
11_{402}	5	17	466	$1+4t+3t^2+7t^3+8t^4+6t^5+\cdots+9t^{465}+12t^{466}$
11_{410}	5	15	418	$1+10t+10t^2+4t^3+2t^4+5t^5+\cdots+3t^{417}+12t^{418}$
11_{403}	5	9	352	$1+12t+4t^2+7t^3+7t^4+7t^5+\cdots+t^{351}+12t^{352}$
11_{411}	5	9	352	$1+12t+3t^2+9t^3+2t^4+t^5+\cdots+t^{351}+12t^{352}$
11_{406}	5	17	288	$1+8t+2t^2+7t^3+2t^4+5t^5+\cdots+8t^{287}+t^{288}$
11_{412}	5	19	408	$1+2t+t^2+12t^3+3t^4+5t^5+\cdots+2t^{407}+t^{408}$
11_{407}	5	12	336	$1+7t+4t^2+5t^3+5t^4+6t^5+\cdots+7t^{335}+t^{336}$
11_{413}	5	12	340	$1+t+2t^2+t^4+3t^5+\cdots+t^{339}+t^{340}$
11_{408}	5	15	568	$1+4t+3t^3+6t^4+5t^5+\cdots+4t^{567}+t^{568}$
11_{414}	5	16	604	$1+11t^2+10t^3+7t^4+5t^5+\cdots+2t^{602}+12t^{604}$
11_{518}	5	12	220	$1+t+12t^2+12t^3+3t^4+11t^5+\cdots+t^{219}+t^{220}$
11_{519}	5	11	228	$1+3t+7t^2+11t^3+3t^4+5t^5+\cdots+10t^{227}+12t^{228}$

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