4–manifolds and their (equivariant) intersection forms

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4–manifolds

We distinguish between *smooth 4–manifolds* and *topological 4–manifolds*. Recall that the former has charts to open sets of $\mathbb{R}^4$ such that the transition maps are diffeomorphisms. For topological 4–manifolds we only demand that the transition maps are homeomorphisms. Clearly every smooth 4–manifold is also a topological 4–manifold.

Intersection forms

Given a closed 4–manifold W we can represent any $a, b \in H_2(W)$ by embedded transverse surfaces A, B and we define the intersection number $a \cdot b$ to be the sum over the signs of all intersection points of A and B. This gives rise to the symmetric intersection form

$$H_2(W;\mathbb{Z}) \times H_2(W;\mathbb{Z}) \to \mathbb{Z},$$

which gives rise to a non–singular intersection

$$Q_W : H_2(W;\mathbb{Z})/\text{tor} \times H_2(W;\mathbb{Z})/\text{tor} \to \mathbb{Z}.$$

Alternative we can define Q_W via

$$H_2(W;\mathbb{Z}) \to H^2(W;\mathbb{Z}) \to \text{Hom}(H_2(W;\mathbb{Z}),\mathbb{Z}).$$

Question. Which non–singular symmetric forms can appear as intersection forms of top/smooth 4–manifolds?

Examples

The intersection form of $S^2 \times S^2$ is presented by

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

the intersection form of $\mathbb{CP}^2$ is given by (1) and intersection form of $\overline{\mathbb{CP}}^2$ is given by (-1). Finally for

$$W = \{(x_0 : x_1 : x_2 : x_3) \in \mathbb{CP}^3 | x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0\}$$

(the K3–surface) the intersection form is represented by $2(-E_8) \oplus 3H$ where

$$H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, E_8 = \begin{pmatrix} 2 & 1 & & & & & & \\ 1 & 2 & 1 & & & & & \\ & 1 & 2 & 1 & & & & \\ & & 1 & 2 & 1 & & & \\ & & & 1 & 2 & 1 & & 1 \\ & & & & 1 & 2 & 1 & \\ & & & & & 1 & 2 & \\ & & & & 1 & & & 2 \end{pmatrix}.$$

Question. Can E_8 appear as intersection form of a smooth/top. 4–manifold?

Intersection forms of topological 4–manifolds

Question. Which non–singular symmetric forms can appear as intersection forms of topological 4–manifolds?

Freedman proved 1982 the following for which he got the Fields medal 1986.

Answer.
(a) Any non–singular symmetric form Q is the intersection form of a topological 4–manifolds.
(b) If Q is even, then there is a unique such top. 4–manifold
(c) If Q is odd, then there exist exactly two such top. 4–manifolds.

Here Q is called *odd* if $Q(a,a)$ is odd for all a, otherwise Q is *even*. Note that (b) implies the topological Poincaré conjecture in dimension 4.

Intersection forms of smooth 4–manifolds I

There are very many positive definite intersection forms, and there is no good way to classify them. The picture for indefinite forms is very different.

It follows from algebra that any indefinite odd intersection form is of the form $n_+(1) \oplus n_-(-1)$, in particular it is realized by $n_+\mathbb{CP}^2 \# n_-\overline{\mathbb{CP}}^2$.

Also, if Q is indefinite and even, then $Q = nE_8 \oplus kH$ for $n \in \mathbb{Z}, k > 0$.

Rochlin in 1954 showed that if Q is the intersection form of a smooth 4–manifold and if Q is even, then the signature is divisible by 16. For example if $Q = nE_8 \oplus kH$, then n is even.

This implies that E_8 can not be the intersection form of a smooth 4–manifold.

Intersection forms of smooth 4–manifolds II

A form Q is called *positive definite* if $Q(a,a) > 0$ for all a. For example E_8 is positive definite.

Theorem (Donaldson 1982). If Q is the intersection form of a smooth 4–manifold and if Q is positive definite, then $Q = n(1)$.

In particular there is no smooth 4–manifold with $Q = E_8 \# E_8$. But what about $Q = -2E_8 \oplus 2H$?

Theorem (Furuta). If $Q = n2E_8 \oplus kH, k > 0$ is the intersection form of a smooth 4–manifold, then $k \geq 2|n| + 1$.

11/8–Conjecture. If $Q = n2E_8 \oplus kH, k > 0$ is the intersection form of a smooth 4–manifold, then $k \geq 3|n|$.

Note that all $Q = n2E_8 \oplus kH$ with $k \geq 3|n|$ can be realized by smooth 4–manifolds.

Equivariant intersection forms

Let $\pi = \pi_1(W)$ and let $\tilde{W}$ be the universal cover of W. Note that $H_*(\tilde{W})$ is a $\mathbb{Z}[\pi]$–module because of the Deck transformations. We write $H_2(W;\mathbb{Z}[\pi]) = H_2(\tilde{W})$.

We define the equivariant intersection form

$$\tilde{Q}_W : H_2(W;\mathbb{Z}[\pi]) \times H_2(W;\mathbb{Z}[\pi]) \to \mathbb{Z}[\pi]$$

as follows. Given $a, b \in H_2(W;\mathbb{Z}[\pi]) = H_2(\tilde{W})$ we represent them by embedded transverse surfaces A, B and we define

$$\tilde{Q}_W(a,b) = \sum_{g \in \pi} (A \cdot gB) g^{-1} \in \mathbb{Z}[\pi].$$

Alternative we can define $\tilde{Q}_W$ via Poincaré duality and the evaluation homomorphism:

$$H_2(W;\mathbb{Z}[\pi]) \cong H^2(W;\mathbb{Z}[\pi]) \to \mathrm{Hom}(H_2(W;\mathbb{Z}[\pi]), \mathbb{Z}[\pi]).$$

The equivariant intersection form is hermitian.

Equivariant intersection forms with $\pi = \mathbb{Z}$

We now restrict to $\pi_1(W) = \mathbb{Z}$. We write

$$H_*(W; \mathbb{Z}[\pi]) = H_*(W; \mathbb{Z}[t^{\pm 1}]).$$

If $A(t)$ is a matrix representing $\tilde{Q}_W$ on $H_2(W; \mathbb{Z}[t^{\pm 1}])/\text{tor}$, then $A(t) = A(t^{-1})$.

Example. If X is a simply connected 4–manifold, then for $W = S^1 \times S^3 \# X$ we have $\pi_1(W) = \mathbb{Z}$ and

$$H_2(W; \mathbb{Z}[t^{\pm 1}]) = H_2(X; \mathbb{Z}) \otimes \mathbb{Z}[t^{\pm 1}].$$

Furthermore a matrix representing Q_X also represents $\tilde{Q}_W$.

Question. Which hermitian non–singular forms over $\mathbb{Z}[t^{\pm 1}]$ are equivariant intersection forms of top/smooth 4–manifolds?

We will use the following throughout. If the matrix $A(t)$ represents $\tilde{Q}_W$ on $H_2(W; \mathbb{Z}[t^{\pm 1}])/\text{tor}$, then $A(1) = A(t = 1)$ represents Q_W on $H_2(W; \mathbb{Z})/\text{tor}$.

Topological 4–manifolds with $\pi_1 = \mathbb{Z}$

Question. Which hermitian non–singular forms over $\mathbb{Z}[t^{\pm 1}]$ are equivariant intersection forms of topological 4–manifolds?

This question was answered by Freedman and Quinn:

Theorem. Any hermitian non–singular form $A(t)$ over $\mathbb{Z}[t^{\pm 1}]$ is the equivariant intersection form of a topological 4–manifold W. Furthermore W is unique if $A(1)$ is even, and there exist two such W is $A(1)$ is odd.

Smooth 4–manifolds with $\pi_1 = \mathbb{Z}$ and $A(1)$ indefinite

Note every hermitian non–singular form $A(t)$ over $\mathbb{Z}[t^{\pm 1}]$ is the equivariant intersection form of a topological 4–manifold W since we have restrictions on $A(1)$.

On the other hand, if $A(t)$ is congruent to $A(1)$ over $\mathbb{Z}[t^{\pm 1}]$, i.e. if there exists a matrix $P(t)$ such that $P(t)A(t)P(t)^t = A(1)$, and if $A(1)$ is the intersection form of a smooth 4–manifold X, then $A(t)$ is the equivariant intersection form of $S^1 \times S^3 \# X$.

Somewhat surprisingly, $A(t)$ is often congruent to $A(1)$. More precisely:

Theorem (Hambleton–Teichner). Let $A(t)$ be a hermitian matrix over $\mathbb{Z}[t^{\pm 1}]$ of rank r. Write $s = \text{sign}(A(1))$. If $r - |s| \geq 6$, then $A(t)$ is congruent to $A(1)$.

Question. Does the conclusion hold for $r - |s| > 0$, i.e. for $A(1)$ indefinite?

Smooth 4–manifolds with $\pi_1 = \mathbb{Z}$ and $A(1)$ positive definite

Now consider

$$A(t) = \begin{pmatrix} 1+x+x^2 & x+x^2 & 1+x & x \\ x+x^2 & 1+x+x^2 & x & 1+x \\ 1+x & x & 2 & 0 \\ x & 1+x & 0 & 2 \end{pmatrix}$$

with $x = t + t^{-1}$. This matrix is hermitian and non–singular. Furthermore $A(1)$ is congruent to the identity matrix.

By Freedman $A(t)$ is the equivariant intersection form of a unique topological 4–manifold W. On the other hand Hambleton–Teichner showed that $A(t)$ is not congruent to $A(1)$. So W is not of the form $S^1 \times S^3 \# X$.

Question. Is W smoothable?

In joint work with Hambleton–Melvin–Teichner we proved the following.

Theorem. W is not smoothable.

Proof of theorem. We have W with

$$\tilde{Q}_W = A(t) = \begin{pmatrix} 1+x+x^2 & x+x^2 & 1+x & x \\ x+x^2 & 1+x+x^2 & x & 1+x \\ 1+x & x & 2 & 0 \\ x & 1+x & 0 & 2 \end{pmatrix}$$

where $x = t + t^{-1}$. Furthermore $A(1)$ is congruent to id, i.e. positive definite.

Let W_n be the n–fold cover of W corresponding to $\pi_1(W) = \mathbb{Z} \to \mathbb{Z}/n$. Note that the signature s and the Euler characteristic χ are multiplicative under finite covers. We get

$$\begin{aligned} H_2(W_n) &= H_2(W_n) - H_1(W_n) + H_0(W_n) \\ &\quad -H_3(W_n) + H_4(W_n) \\ &= \chi(W_n) = n\chi(W) \\ &= nH_2(W) = ns(W) \\ &= s(W_n). \end{aligned}$$

Therefore W_n is again positive definite and we can apply Donaldson's theorem.

Indeed, Q_{W_n} is not congruent to id for $n \geq 3$, hence W_n is not smoothable, hence W is not smoothable.

Question.

Let $A(t)$ be a matrix such that $A(1)$ is congruent to id. We saw that if $A(t)$ is the equivariant intersection form of a 4–manifold W, then Q_{W_n} has to be congruent to id for all n. Does this imply that $A(t)$ is already congruent to $A(1)$?

If the answer is yes, then this would be a big step towards classifying all equivariant intersection forms of smooth 4–manifolds with $\pi_1 = \mathbb{Z}$.