THE MEMBERSHIP PROBLEM FOR 3-MANIFOLD GROUPS IS SOLVABLE

STEFAN FRIEDL AND HENRY WILTON

Abstract. We show that the Membership Problem for finitely generated subgroups of 3-manifold groups is solvable.

1. Introduction

The classical group-theoretic decision problems were formulated by Dehn [De11] in work on the topology of surfaces. He considered in particular the following questions about finite presentations $\langle A|R\rangle$ for a group π:

(1) the Word Problem, which asks for an algorithm to determine whether or not a word in the generators A represents the trivial element of π;
(2) the Conjugacy Problem, which asks for an algorithm to determine whether or not two words in the generators A represent conjugate elements of π.

In this context another question arises naturally:

(3) the Membership Problem, where the goal is to determine whether a given element of a group lies in a specified subgroup.

Note that a solution to the Conjugacy Problem and also a solution to the Membership Problem each give a solution to the Word Problem. The initial hope that these problems might always be solvable was dashed by Novikov [No55] and Boone [Bo58] who showed that there exist finitely presented groups with unsolvable Word Problem.

It is therefore natural to ask for which classes of groups the above problems can be solved. In this paper we will discuss the case of 3-manifold groups, i.e. fundamental groups of compact 3-manifolds. We now give a short summary of results on decision problems for 3-manifold groups. We refer to the survey paper [AFW14] for a much more detailed discussion.

Our understanding of 3-manifold groups has expanded rapidly over the recent years. In particular it is a consequence of the Geometrization Theorem due to Perelman [Pe02, Pe03a, Pe03b, MT07, MT14, BBBMP10] and work of Hempel [He87] that 3-manifold groups are residually finite, which gives rise to a solution to the Word Problem. The first solution to the Conjugacy Problem was given by Préaux [Pr06, Pr12] (extending earlier work of Sela [Sel93] on knot groups). A conceptually easy solution for fundamental groups of orientable 3-manifolds is given by the recent result [HWZ13] that these groups are 'conjugacy separable'.

The Word Problem and the Conjugacy Problem for fundamental groups of orientable 3-manifold groups can thus be solved by translating the problem to dealing with the corresponding problems for finite groups, which in turn can be solved trivially. The same approach cannot work for the Membership Problem since 3-manifold groups are not 'subgroup separable', see e.g. [BKS87, NW01]. The main goal of this paper is to show that the Membership Problem has nonetheless a uniform solution for 3-manifold groups. More precisely, we have the following theorem.

Theorem 1. *There exists an algorithm which takes as input a finite presentation $\pi = \langle A|R\rangle$ of a 3-manifold group, a finite set of words $w_1, \dots, w_k$ in A and a word z in A and which determines whether or not the element $z \in \pi$ lies in the subgroup of π generated by $w_1, \dots, w_k$.*

The four main ingredients in the proof are the following:

(1) The resolution of the Tameness Conjecture by Agol [Ag04] and Calegari–Gabai [CG06].
(2) An algorithm of Kapovich–Weidmann–Miasnikov [KWM05] which deals with the Membership Problem of the fundamental group of a graph of groups.
(3) Algorithms of Jaco–Letscher–Rubinstein [JLR02], Jaco–Rubinstein [JR03] and Jaco–Tollefson [JT95] which determine the prime decomposition and the JSJ decomposition of a given triangulated 3-manifold.
(4) The Virtually Compact Special Theorem of Agol [Ag13] and Wise [Wi12a, Wi12b].

The paper is organized as follows. In Section 2 we recall the solutions to the Word Problem and the Conjugacy Problem for 3-manifold groups. In Section 3 we discuss our main theorem in more detail. In Section 4 we recall several basic results and algorithms, and we give the proof of our main theorem in Section 5. Finally, in Section 6 we will quickly discuss a slightly different approach to the proof of our main theorem and we will raise a few questions.

Conventions. All 3-manifolds and surfaces are assumed to be compact and connected. We furthermore assume that all graphs are connected. Finally we assume that all classes of groups are closed under isomorphism.

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2. The Word Problem and the Conjugacy Problem for 3-manifold groups

We start out with introducing several definitions which we will need throughout this paper.

(1) Given a set A we denote by $F(A)$ the free group generated by A. As usual we will freely go back and forth between words in A and elements represented by these words in $F(A)$.

(2) A *finite presentation* $\langle A|R\rangle$ is a finite set A together with a finite set R of elements in $F(A)$. We follow the usual convention that by $\langle A|R\rangle$ we indicate at the same time the finite data and also the group

$$F(A)/\langle\langle R\rangle\rangle,$$

i.e. the quotient of $F(A)$ by the normal closure $\langle\langle R\rangle\rangle$ of R in $F(A)$. In the notation we will for the most part not distinguish between elements in $F(A)$ and the elements they represent in $\langle A|R\rangle$.

(3) A *finite presentation for a group* π is a finite presentation $\langle A|R\rangle$ such that $\pi \cong \langle A|R\rangle$. We say that a group π is *finitely presentable* if it admits a finite presentation.

Before we start with the solution to the Membership Problem for 3-manifold groups it is worth looking at the solution to the Word Problem and to the Conjugacy Problem.

Recall that a group π is called *residually finite* if given any non-trivial $g \in \pi$ there exists a homomorphism $f\colon \pi \to G$ to a finite group G such that $f(g)$ is non-trivial. It is consequence of the Geometrization Theorem [Th82, Pe02, Pe03a, Pe03b] and of work of Hempel [He87] that 3-manifold groups are residually finite.

The following well-known lemma thus gives a solution to the Word Problem for 3-manifold groups.

Lemma 2. *There exists an algorithm which takes as input a finite presentation $\pi = \langle A|R\rangle$ and an element $w \in F(A)$, and which, if π is residually finite, determines whether or not w represents the trivial element.*

Proof. Let $\pi = \langle A|R\rangle$ be a finitely presented group and let $w \in F(A)$. We suppose that π is residually finite. If w represents the trivial element, then we will see this eventually by writing down systematically all words which are products of conjugates of elements in $R \cup R^{-1}$. On the other hand, if w does not represent the trivial word, then by residual finiteness there exists a homomorphism $f\colon \pi \to G$ to a finite group with $f(g) \neq e$. Since $\langle A|R\rangle$ is a finite presentation we can recursively enumerate all homomorphisms from π to finite groups. After finitely many steps we will thus detect that g is indeed non-trivial. □

As we mentioned in the introduction, Préaux, extending Sela's work on knot groups [Sel93], proved that the Conjugacy Problem is solvable for the fundamental groups of orientable [Pr06] and non-orientable [Pr12] 3-manifolds. (Note that, in contrast to many other group properties, solvability of the Conjugacy Problem does not automatically pass to finite extensions [CM77].)

It is natural to ask whether there also exists a solution to the Conjugacy Problem along the lines of Lemma 2. In the following we say that a group π is *conjugacy separable* if, given any non-conjugate $g, h \in \pi$, there exists a homomorphism $f\colon \pi \to G$ to a finite group G such that $f(g)$ and $f(h)$ are non-conjugate. A slight variation on the proof of Lemma 2 also shows that the Conjugacy Problem is solvable if the given group is conjugacy separable.

Hamilton, the second author and Zalesskii [HWZ13], building on the recent work of Agol [Ag13] and Wise [Wi12a, Wi12b] and work of Minasyan [Min12] showed that fundamental groups of orientable 3-manifold N are conjugacy separable. This result gives another solution to the Conjugacy Problem for fundamental groups of *orientable* 3-manifolds.

3. The statement of the main theorem

Let $\mathcal{C}$ be a class of finitely presentable groups. We say that the *Membership Problem is solvable in* $\mathcal{C}$ if there exists an algorithm which takes as input a finite presentation $\langle A|R\rangle$, a finite set of words $w_1, \dots, w_k$ in A and a word z in A and which, if $\langle A|R\rangle$ is a presentation for a group π in $\mathcal{C}$, determines whether or not the element $z \in \pi$ lies in the subgroup generated by $w_1, \dots, w_k \in \pi$.

The following theorem is now a reformulation of our main theorem.

Theorem 3. *The Membership Problem is solvable for the class of 3-manifold groups.*

In Section 2 we saw that separability properties of fundamental groups can be used to solve the Word Problem and the Conjugacy Problem for fundamental groups of (orientable) 3-manifold.

The 'right' notion of separability in the context of the Membership Problem is the separability of subgroups. More precisely, in the following we say that a subset Γ of a group π is *separable* if given any $g \notin \Gamma$ there exists a homomorphism $f\colon \pi \to G$ to a finite group such that $f(g) \notin f(\Gamma)$. We say that a group π is *subgroup separable* if every finitely generated subgroup is separable. The proof of Lemma 2 can easily be modified to show that the Membership Problem is solvable for the class of finitely presentable groups which are subgroup separable.

Scott [Sc78] showed that the fundamental groups of Seifert fibred 3-manifolds are subgroup separable. Furthermore, it follows from work of Agol [Ag13] and Wise [Wi12a, Wi12b], together with the proof of the Tameness Conjecture by Agol [Ag04] and Calegari–Gabai [CG06] that fundamental groups of hyperbolic 3-manifolds are subgroup separable. (The precise references for this statement can be found in [AFW15].)

On the other hand, there are many examples of fundamental groups of prime 3-manifolds which are not subgroup separable, see e.g. [BKS87, NW01]. We thus see that we cannot hope to prove Theorem 3 in the general case by appealing to separability properties only.

The key idea in the proof of Theorem 3 is to apply a theorem of Kapovich–Weidmann–Miasnikov [KWM05] which provides a solution to the Membership Problem for fundamental groups of graphs of groups if various conditions are satisfied. We will apply this theorem twice, once to reduce the problem to the case of prime 3-manifolds, and then later on to deal with the case of prime 3-manifolds with non-trivial JSJ decomposition.

The proof of Theorem 3 is organized as follows. In Section 4 we will first make some preliminary observations. In Section 5.1 we formulate the aforementioned main theorem of [KWM05] which we will use in Section 5.2 to argue that it suffices to prove our main theorem for closed, orientable, prime 3-manifolds. In Sections 5.3, 5.4 and 5.5 we will show that one can use the main theorem of [KWM05] to deal with the fundamental groups of closed, orientable, prime 3-manifolds with non-trivial JSJ decomposition.

4. Preliminary results

4.1. **Basic algorithms.** We start out with several basic lemmas which we will need time and again during the paper. The statements of the lemmas are well-known to experts, but we include proofs for the reader's convenience. At a first reading of the paper it might nonetheless be better to skip this section.

Lemma 4. *There exists an algorithm which takes as input two finite presentations $\langle A|R\rangle$ and $\langle A'|R'\rangle$ and which finds an isomorphism $\langle A|R\rangle \to \langle A'|R'\rangle$, if such an isomorphism exists.*

Here, by 'finds an isomorphism $\langle A|R\rangle \to \langle A'|R'\rangle$' we mean that the algorithm finds a map $A \to F(A')$ which descends to an isomorphism $\langle A|R\rangle \to \langle A'|R'\rangle$.

Proof. Let $\pi = \langle A|R\rangle$ and $\pi' = \langle A'|R'\rangle$ be finite presentations. We denote by T the subgroup of $F(A)$ normally generated by R. Similarly we define T'.

Note that a homomorphism $\varphi\colon \langle A\rangle \to \langle A'\rangle$ descends to an isomorphism $\varphi\colon \pi \to \pi'$ if and only if the following two conditions hold:

(1) for all $r \in R$ we have $\varphi(r) \in T'$, and
(2) there exists a homomorphism $\psi\colon \langle A'\rangle \to \langle A\rangle$ such that $\psi(r') \in T$ for all $r' \in R'$, such that $\psi(\phi(g))g^{-1} \in T$ for all $g \in A$ and such that $\phi(\psi(g'))(g')^{-1} \in T'$ for all $g' \in A'$.

We run the following Turing machines simultaneously:

(1) A Turing machine which produces a list of all words in A of π which represent the trivial word, i.e. a Turing machine which produces a list of all elements of T.
(2) A Turing machine which produces a list of all homomorphisms $\varphi\colon \langle A\rangle \to \langle A'\rangle$, i.e. a Turing machine which outputs all $|A|$-tuples of elements in $\langle A'\rangle$.
(3) A Turing machine which produces a list of all words in A' of π' which represent the trivial word.
(4) A Turing machine which produces a list of all homomorphisms $\varphi\colon \langle A'\rangle \to \langle A\rangle$.

Now suppose there exists an isomorphism $\pi \to \pi'$. It is clear that after finitely many steps we will find a pair $\varphi\colon \langle A\rangle \to \langle A'\rangle$ and $\psi\colon \langle A'\rangle \to \langle A\rangle$ such that the following conditions hold:

(1) for all $r \in R$ we have $\varphi(r) \in T'$,
(2) for all $r' \in R'$ we have $\psi(r') \in T$
(3) for all $g \in A$ we have $\psi(\phi(g))g^{-1} \in T$, and
(4) for all $g' \in A'$ we have $\phi(\psi(g'))(g')^{-1} \in T'$.

□

In the following we say that a subgroup Γ of a group π is a *retract* if there exists a retraction $r\colon \pi \to \Gamma$, i.e. a homomorphism with $r(g) = g$ for all $g \in \Gamma$. Almost the same argument as in Lemma 4 also proves the following lemma.

Lemma 5. *There exists an algorithm which takes as input two finite presentations $\pi = \langle A|R\rangle$ and $\pi' = \langle A'|R'\rangle$ and which finds a map $f\colon \pi \to \pi'$ and a left-inverse $g\colon \pi' \to \pi$ to f, if π is isomorphic to a retract of π'.*

We also have the following lemma.

Lemma 6. *There exists an algorithm which takes as input a finite presentation $\pi = \langle A|R\rangle$ and two finite set of words X and Y in A, and which certifies that the two sets X and Y generate the same subgroup of π, if this is indeed the case.*

Proof. The algorithm enumerates all elements in $\langle X, \langle\langle R\rangle\rangle\,\rangle \subset F(A)$ and it enumerates all elements in $\langle Y, \langle\langle R\rangle\rangle\,\rangle \subset F(A)$. Note that X and Y generate the same subgroup of π if and only if $X \subset \langle Y, \langle\langle R\rangle\rangle\,\rangle$ and $Y \subset \langle X, \langle\langle R\rangle\rangle\,\rangle$. If this is indeed the case, then this will be verified after finitely many steps. □

Lemma 7. *There exists an algorithm which takes as input two finite presentations $\Gamma = \langle A|R\rangle$ and $\pi = \langle B|S\rangle$, a homomorphism $f\colon \Gamma \to \pi$ and finite a set of elements in π which generate a finite-index subgroup $\pi_0 \subset \pi$, and which gives as output a set of coset representatives for $\Gamma_0 := f^{-1}(\pi_0)$ in Γ and a finite presentation $\langle A_0|R_0\rangle$ together with an isomorphism $\langle A_0|R_0\rangle \to \Gamma_0$.*

Proof. We start out with the following claim.

Claim. There exists an algorithm which takes as input a finite presentation $\pi = \langle B|S\rangle$ and a finite set X of elements in π such that $\pi_0 = \langle X\rangle$ is a finite-index subgroup of π, and which finds a homomorphism $\varphi\colon \pi \to G$ to a finite group G and a subgroup $G_0 \subset G$ such that $\pi_0 = \varphi^{-1}(G_0)$.

First note that there exists indeed a homomorphism $\varphi\colon \pi \to G$ onto a finite group and a subgroup G_0 of G such that $\pi_0 = \varphi^{-1}(G_0)$. For example, we could take G to be the quotient of π by the core of $\langle X\rangle$ and G_0 the image of π_0 in this quotient. (Here recall that given a subgroup H of π the core of H is given by the normal subgroup $\cap_{g\in\pi} gHg^{-1}$; if H is a finite index subgroup of π then the core is also a finite index subgroup of π.)

The algorithm now goes through all epimorphisms from π to finite groups G. For each epimorphism $\varphi\colon \pi \to G$ onto a finite group we also consider all finite-index subgroups $G_0 \subset G$. We then calculate a set of coset representatives for $G_0 \subset G$. By enumerating the elements in π_0 and determining their images under φ we can then find preimages of the coset representatives, i.e. we can find a set of coset representatives for $\varphi^{-1}(G_0)$. Using the Reidemeister-Schreier process we can then determine a finite set of generators Y for $\varphi^{-1}(G_0)$. If X and Y generate the same group, then this will be certified by the algorithm of Lemma 6. By the above discussion this algorithm will terminate after finitely many steps. This concludes the proof of the claim.

Now suppose we are given two finite presentations $\Gamma = \langle A|R\rangle$ and $\pi = \langle B|S\rangle$, a homomorphism $f\colon \Gamma \to \pi$ and a set of elements in π which generate a finite-index subgroup $\pi_0 \subset \pi$. We apply the above claim to $\pi_0 \subset \pi$. We write $H := (\varphi\circ f)(\Gamma)$. Note that $H_0 := (\varphi \circ f)(\Gamma_0) = (\varphi \circ f)(\Gamma) \cap \varphi(\pi_0)$. We can evidently find coset representatives for $H_0 \subset H$. As in the proof of the claim we can furthermore find preimages of the coset representatives under the map $\varphi \circ f$, which are then coset representatives for $\Gamma_0 \subset \Gamma$. Using the Reidemeister–Schreier process we can now find a finite presentation $\langle A_0 | R_0 \rangle$ together with an isomorphism $\langle A_0 | R_0 \rangle \to \Gamma_0$. $\square$

Lemma 8. *There exists an algorithm which, given a finite presentation $\pi = \langle A | R \rangle$, determines a list of all finite-index subgroups; more precisely, it provides a list of finite presentations $\pi_i = \langle A_i | R_i \rangle$ and monomorphisms $f_i \colon \pi_i \to \pi$ such that any finite-index subgroup of π agrees with $f_i(\pi_i)$ for some i.*

Proof. Given a finite presentation $\pi = \langle A | R \rangle$ we can list all homomorphisms to finite groups. Furthermore, given a homomorphism $\pi \to G$ to a finite group and a subgroup $G_0 \subset G$ we saw in the previous proof that we can determine coset representatives for $f^{-1}(G_0) \subset \pi$, and the Reidemeister–Schreier procedure gives a finite presentation for $f^{-1}(G_0)$ together with a map to π. As we saw in the proof of the previous lemma, any finite index subgroup of π arises that way. $\square$

Lemma 9. *There exists an algorithm which given a finite presentation $\Gamma = \langle A | R \rangle$ and a finite set of elements $X \subset F(A)$ certifies that the subgroup $\langle X \rangle \subset \Gamma$ is normal, if this is the case.*

Proof. Let $\Gamma = \langle a_1, \dots, a_l | r_1, \dots, r_m \rangle$ be a finite presentation and let $X = \{g_1, \dots, g_k\} \subset F(A)$. Note that $g_1, \dots, g_k$ generate a normal subgroup in Γ if and only if $a_i g_j a_i^{-1}$ lie in $\langle X, \langle\langle R \rangle\rangle \rangle$ for any i and j.

But if this is the case, then this can be certified by enumerating all elements in $\langle X, \langle\langle R \rangle\rangle \rangle$ and after finitely many steps we will have verified that indeed all the elements $a_i g_j a_i^{-1}$ lie in $\langle X, \langle\langle R \rangle\rangle \rangle$. $\square$

Lemma 10. *There exists an algorithm which given two finitely presented groups $\Gamma = \langle A | R \rangle$ and $\pi = \langle B | S \rangle$ and a finite set of elements $X \subset \Gamma$ certifies that the subgroup $\langle X \rangle \subset \Gamma$ is normal, and that the quotient of Γ by the normal subgroup $\langle X \rangle$ is isomorphic to π.*

Proof. Let $X = \{g_1, \dots, g_k\}$ be a finite set of elements in a finite presentation $\Gamma = \langle a_1, \dots, a_l | r_1, \dots, r_m \rangle$. We first apply the algorithm from the proof of Lemma 9 which allows us to certify that $\langle g_1, \dots, g_k \rangle \subset \Gamma$ is normal. If $\Gamma / \langle g_1, \dots, g_k \rangle$ is isomorphic to π, then the group given by the finite presentation

$$\langle a_1, \dots, a_l | r_1, \dots, r_m, g_1, \dots, g_k \rangle$$

is isomorphic to π. But by Lemma 4 the existence of such an isomorphism can be certified after finitely many steps. $\square$

4.2. Preliminary observations on the Membership Problem.

In this section we will prove two elementary lemmas dealing with the Membership Problem.

Lemma 11. *There exists an algorithm which takes as input a finite presentation $\pi = \langle A | R \rangle$, a finite set $X \subset \pi$ such that $\langle X \rangle$ is a finite-index subgroup of π and an element $g \in \pi$ and decides whether or not $g \in \pi$ lies in $\langle X \rangle$.*

Proof. By the proof of Lemma 7 we can find a homomorphism $f \colon \pi \to G$ to a finite group and a subgroup $G_0 \subset G$ such that $\pi_0 = f^{-1}(G_0)$. Given an element

$g \in \pi$ we now only have to determine whether or not $f(g)$ lies in G_0, which can be done trivially. □

Before we state our next lemma we need to give two more definitions:

(1) We say that a group π is *virtually isomorphic to a retract* of a group Γ if there exists a finite-index subgroup π_0 of π which is isomorphic to a retract of Γ.
(2) A class $\mathcal{C}$ of groups is *recursively enumerable* if there exists a Turing machine that outputs a list of finite presentations, all presenting groups in the class $\mathcal{C}$, such that any group in $\mathcal{C}$ is isomorphic to at least one of the groups defined by those finite presentations.

We can now formulate the following lemma.

Lemma 12. *Let $\mathcal{C}, \mathcal{D}$ be classes of finitely presentable groups, and suppose that every group in $\mathcal{C}$ is virtually isomorphic to a retract of a group in $\mathcal{D}$. If*

(1) *the Membership Problem is solvable in $\mathcal{D}$, and*
(2) *$\mathcal{D}$ is recursively enumerable,*

then the Membership Problem is also solvable in $\mathcal{C}$.

Proof. Let $\pi = \langle A|R\rangle$ be a finite presentation of a group in $\mathcal{C}$. Using Lemma 8 we may enumerate all subgroups of finite index in π. Because $\mathcal{D}$ is recursively enumerable, we will by Lemma 5 eventually find a finite-index subgroup π_0 of π, a finite presentation $\langle B|S\rangle$ for a group Γ in $\mathcal{D}$, a homomorphism $f\colon \pi_0 \to \Gamma$ and a left-inverse $g\colon \Gamma \to \pi_0$ to f. Note that f is in particular injective.

Let H be a finitely generated subgroup of $\pi = \langle A|R\rangle$, specified by a finite set of elements. By Lemma 7, we may compute a generating set for $H_0 = H \cap \pi_0$ and a set of coset representatives $h_1, \dots, h_k$ for H_0 in H. Now, if $g \in \pi$ then, for each i, we may by Lemma 11 determine whether or not $h_i^{-1}g \in \pi_0$. If there is no such i then evidently $g \notin H$. Otherwise, if $h_i^{-1}g \in \pi_0$ then, using the solution to the membership problem in $f(\pi_0) \subset \Gamma$, we may determine whether or not $h_i^{-1}g \in H_0$. The lemma now follows from the observation that $g \in H$ if and only if $h_i^{-1}g \in H_0$ for some i. □

5. The Membership Problem for 3-manifold groups

5.1. **The theorem of Kapovich–Weidmann–Miasnikov.** We first recall that a graph of groups $\mathcal{G}$ is a finite graph with vertex set $V = V(\mathcal{G})$ and edge set $E = E(\mathcal{G})$ together with vertex groups $G_v, v \in V$ and edge groups $H_e, e \in E$ and monomorphisms $i_e\colon H_e \to G_{i(e)}$ and $t_e\colon H_e \to G_{t(e)}$. Also recall that Serre [Ser80] associated to a graph $\mathcal{G}$ of finitely presented groups a finitely presented group, which is referred to as the *fundamental group of* $\mathcal{G}$.

We also need the following definitions:

(1) A *decorated group* is a pair $(G, \{H_i\}_{i\in I})$ where G is a group and $\{H_i\}_{i\in I}$ is a finite indexed set of subgroups of G indexed by I. We refer to G as the *vertex group* and to each $H_i, i \in I$ as an *edge group*.
(2) We say that two decorated groups $(G, \{H_i\}_{i\in I})$ and $(K, \{L_j\}_{j\in J})$ are *isomorphic*, written as
$$(G, \{H_i\}_{i\in I}) \cong (K, \{L_j\}_{j\in J}),$$

if there exists an isomorphism $\varphi\colon G \to K$, a bijection $\psi\colon I \to J$ and elements $k_j \in K, j \in J$ such that $\varphi(H_i) = k_{\psi(i)} L_{\psi(i)} k_{\psi(i)}^{-1}$ for any $i \in I$.

(3) A *subdecoration* of a decorated group $(G, \{H_i\}_{i\in I})$ is a pair $(G, \{H_j\}_{j\in J})$, where $J \subset I$ is a subset.

(4) A *finite decorated group presentation* is a pair

$$(\langle A|R\rangle, \{\langle X_i|S_i\rangle, f_i\}_{i\in I})$$

where $\langle A|R\rangle$ is a finite presentation, where I is a finite set, and each $\langle X_i|S_i\rangle$ is a finite presentation and each $f_i\colon \langle X_i|S_i\rangle \to \langle A|R\rangle$ is a monomorphism. Note that a finite decorated group presentation $(\langle A|R\rangle, \{\langle X_i|S_i\rangle, f_i\}_{i\in I})$ defines a decorated group $(\langle A|R\rangle, \{f_i(\langle X_i|S_i\rangle)\}_{i\in I})$.

(5) A *finite presentation for a decorated group* $(G, \{H_i\}_{i\in I})$ is a finite decorated group presentation $(\langle A|R\rangle, \{\langle X_i|S_i\rangle, f_i\}_{i\in I})$ such that

$$(G, \{H_i\}_{i\in I}) \cong (\langle A|R\rangle, \{f_i(\langle X_i|S_i\rangle)\}_{i\in I}).$$

We say that a decorated group is *finitely presentable* if it admits a finite decorated group presentation.

We say that a class $\mathcal{P}$ of decorated groups is *recursively enumerable* if there exists a Turing machine that outputs a list of finite decorated group presentations, each presenting a group in $\mathcal{P}$, such that any decorated group in $\mathcal{P}$ is isomorphic to a decorated group defined by one of those finite decorated group presentations.

Finally, given a class $\mathcal{P}$ of decorated groups we say that a graph of groups with vertex groups $\{G_v\}_{v\in V}$ and edge groups $\{H_e\}_{e\in E}$ is *based on* $\mathcal{P}$ if given any vertex v the pair $(G_v, \{i_e(H_e)\}_{e\in E, i(e)=v})$ is equivalent to a subdecoration of a decorated group in $\mathcal{P}$.

The following theorem is basically due to Kapovich–Weidmann–Miasnikov.

Theorem 13. (Kapovich–Weidmann–Miasnikov) *Let $\mathcal{P}$ be a class of decorated groups such that the following hold:*

(I) *There is an algorithm which for each finite decorated group presentation $(\langle A|R\rangle, \{\langle X_i|S_i\rangle, f_i\}_{i\in I})$ of a decorated group in $\mathcal{P}$, each $i \in I$, and each finite set $Y \subset F(A)$ determines whether or not a given element in $\langle A|R\rangle$ lies in the double coset $\langle f_i(X_i)\rangle\langle Y\rangle \subset \langle A|R\rangle$.*

(II) *The Membership Problem is solvable for the class of all vertex groups appearing in $\mathcal{P}$.*

(III) *Every edge group is* slender*, meaning that every subgroup of an edge group is finitely generated.*

(IV) *There is an algorithm which for each finite decorated group presentation $(\langle A|R\rangle, \{\langle X_i|S_i\rangle, f_i\}_{i\in I})$ of a decorated group in $\mathcal{P}$, each $i \in I$ and each finite set $Y \subset F(A)$ computes a finite generating set for the intersection $\langle f_i(X_i)\rangle \cap \langle Y\rangle \subset \langle A|R\rangle$.*

(V) *The class $\mathcal{P}$ is recursively enumerable.*

Then the Membership Problem is solvable for the class of fundamental groups of all graphs of groups based on $\mathcal{P}$.

Proof. In this proof we assume some familiarity with [KWM05]. We start out with the following claim.

Claim. *If a class of decorated groups $\mathcal{P}$ satisfies Condition I, then it also satisfies*

(I') *There is an algorithm which for each finite decorated group presentation $(\langle A|R\rangle, \{\langle X_i|S_i\rangle, f_i\}_{i\in I})$ of a decorated group in $\mathcal{P}$, each $i \in I$, and each finite set $Y \subset F(A)$ determines whether or not for a given element a we have $J := a\langle f_i(X_i)\rangle \cap \langle Y\rangle \neq \emptyset$, and if $J \neq \emptyset$, gives as output an element in J.*

First note that if G and H are subgroups of a group π and if $b \in \pi$, then it follows trivially that $bG \cap H \neq \emptyset$ if and only if $b^{-1} \in GH := \{gh \,|\, g \in G, h \in H\}$. So if $\mathcal{P}$ satisfies Condition I, then we can determine whether or not $J := a\langle f_i(X_i)\rangle \cap \langle Y\rangle \neq \emptyset$. Now suppose that $J \neq \emptyset$. We now have to find an element in J. This means that we have to find an element in

$$a\langle f_i(X_i), \langle\langle R\rangle\rangle\,\rangle \,\cap\, \langle Y, \langle\langle R\rangle\rangle\,\rangle \subset F(A).$$

But we can just enumerate all elements in $a\langle f_i(X_i), \langle\langle R\rangle\rangle\,\rangle$ and we can enumerate all elements in $\langle Y, \langle\langle R\rangle\rangle\,\rangle$, and since the intersection is non-trivial we will eventually find an element which lies in both sets. This concludes the proof of the claim.

It now follows from Conditions I', II and IV that any graph of finitely presented groups based on $\mathcal{P}$ is 'benign' in the sense of [KWM05, Definition 5.6]. The solvability of the Membership Problem follows from [KWM05, Theorem 5.13] combined with the following claim.

Claim. *There exists an algorithm which takes as input a finite presentation $\langle A|R\rangle$ for the fundamental group of a graph of groups in $\mathcal{C}$ and which gives as output a finite presentation of a graph of groups based on $\mathcal{P}$ together with an isomorphism from $\langle A|R\rangle$ to the fundamental group of the graph of groups.*

Let $\langle A|R\rangle$ be a finite presentation for the fundamental group of a graph of groups in $\mathcal{C}$. By assumption the class $\mathcal{P}$ of decorated groups is recursively enumerable. It is clear that one can then also recursively enumerate the class of all decorated groups which are subdecorations of decorated groups in $\mathcal{P}$. It furthermore follows from Lemma 4 that the class of isomorphisms between edge groups in $\mathcal{P}$ is also recursively enumerable. It is now straightforward to see that the class of graphs of groups based on $\mathcal{P}$ is also recursively enumerable. By Lemma 4 we will eventually find an isomorphism from $\langle A|R\rangle$ to the finite presentation of the fundamental group of a graph of groups based on $\mathcal{P}$.

□

We then have the following immediate corollary to Theorem 13. See also [KWM05, Corollary 5.16].

Corollary 14. *Let $\mathcal{C}$ be a recursively enumerable class of groups for which the Membership Problem is solvable. Then the Membership Problem is solvable for the class of groups which are isomorphic to finite free products of groups in $\mathcal{C}$.*

Proof. We denote by $\mathcal{P}$ the class of decorated groups $(G, \{e\})$ with $G \in \mathcal{C}$. It follows easily from our assumptions that Conditions I to V of Theorem 13 are satisfied for $\mathcal{P}$. By Theorem 13 the Membership Problem is solvable for the class of groups which are isomorphic to fundamental groups of graphs of groups based on $\mathcal{P}$. The corollary now follows from the observation that if the underlying graph is a tree, then the corresponding fundamental group is in this case just the free product of the vertex groups. □

5.2. **The reduction to the case of closed, orientable, prime 3-manifolds.** The goal of this section is to prove the following proposition.

Proposition 15. *If the Membership Problem is solvable for the class of fundamental groups of all closed, orientable, prime* 3*-manifolds, then it is also solvable for the class of fundamental groups of all* 3*-manifolds.*

In the following we say that a class $\mathcal{M}$ of 3-manifolds is *recursively enumerable* if there exists a Turing machine that outputs a list of finite simplicial spaces, each representing a manifold in $\mathcal{M}$, such that any manifold in $\mathcal{M}$ is homeomorphic to one of those simplicial spaces. In the proof of Proposition 15 we will need the following theorem which is basically a combination of work of Moise [Mo52, Mo77] and Jaco–Rubinstein [JR03] or alternatively Jaco–Tollefson [JT95].

Theorem 16. *The class of closed, orientable, prime 3-manifolds is recursively enumerable.*

Proof. By Moise's Theorem [Mo52, Mo77] every 3-manifold N admits a finite triangulation, i.e. N can be written as a finite simplicial complex.

Note that there exists an algorithm which checks whether or not a given finite simplicial complex represents a closed orientable 3-manifold. Indeed, one only needs to check whether the link of each vertex is a 2-sphere, and whether the third homology group is non-zero. Furthermore, by work of Jaco–Rubinstein [JR03] and also by Jaco–Tollefson [JT95, Algorithm 7.1] there exists an algorithm which given a triangulated closed, orientable 3-manifold N determines whether or not the manifold is prime.

We thus go through all finite simplicial complexes and we keep the ones which represent closed, prime 3-manifolds. By Moise's Theorem any 3-manifold will eventually appear in this list of finite simplicial complexes. □

Proof of Proposition 15. Let N be a 3-manifold. Note that N admits a finite cover M which is orientable. We denote by $W = M \cup_{\partial M = \partial M} M$ the double of M which is now a closed, orientable 3-manifold. Note that the 'folding map' $W \to M$ is a retraction onto M. This implies in particular that the folding induces a homomorphism $\pi_1 W \to \pi_1 M$ which is a left inverse to the inclusion induced map $\pi_1 M \to \pi_1 W$. Furthermore, it is a consequence of the Kneser–Milnor Prime Decomposition theorem that W is the connected sum of finitely many prime 3-manifolds, in particular $\pi_1 W$ is the free product of fundamental groups of closed, orientable, prime 3-manifolds.

We now write

$$\begin{aligned}
\mathcal{M} &:= \text{class of all 3-manifolds,}\\
\mathcal{M}^{\text{or}} &:= \text{class of all orientable 3-manifolds,}\\
\overline{\mathcal{M}}^{\text{or}} &:= \text{class of all closed, orientable 3-manifolds,}\\
\overline{\mathcal{M}}^{\text{or}}_{\text{pr}} &:= \text{class of all closed, orientable, prime 3-manifolds.}
\end{aligned}$$

We furthermore denote by $\mathcal{G}, \mathcal{G}^{\rm or}, \overline{\mathcal{G}}^{\rm or}$ and $\overline{\mathcal{G}}^{\rm or}_{\rm pr}$ the corresponding classes of fundamental groups. By assumption the Membership Problem is solvable in $\overline{\mathcal{G}}^{\rm or}_{\rm pr}$.

It follows from Theorem 16 that $\overline{\mathcal{M}}^{\rm or}_{\rm pr}$ is recursively enumerable. Using the 2-skeleton of a triangulation one can write down a presentation for the fundamental group of any finite connected simplicial complex. It follows that $\overline{\mathcal{G}}^{\rm or}_{\rm pr}$ is also recursively enumerable. It is now a consequence of the aforementioned Kneser–Milnor Prime Decomposition theorem and from Corollary 14 that the Membership Problem is solvable for $\overline{\mathcal{G}}^{\rm or}$.

It follows from the above discussion that any group in $\mathcal{G}$ is virtually isomorphic to a retract of a group in $\overline{\mathcal{G}}^{\rm or}$. Since any group in $\overline{\mathcal{G}}^{\rm or}$ is the free product of finitely many groups in $\overline{\mathcal{G}}^{\rm or}_{\rm pr}$, and since $\overline{\mathcal{G}}^{\rm or}_{\rm pr}$ is recursively enumerable it now follows that $\overline{\mathcal{G}}^{\rm or}$ is also recursively enumerable. It now follows from Lemma 12 that the Membership Problem is solvable in $\mathcal{G}$. □

5.3. **The JSJ decomposition of 3-manifolds.** In this paper, by a hyperbolic 3-manifold we mean a compact, orientable 3-manifold such that the boundary is a possibly empty union of tori, such that the interior admits a complete hyperbolic structure. By a Seifert fibred manifold we always mean an orientable Seifert fibred manifold.

If N is a closed, orientable, prime 3-manifold then the Geometrization Theorem of Thurston [Th82] and Perelman [Pe02, Pe03a, Pe03b] says that there exists a collection of incompressible tori such that N cut along the tori consists of components which are either Seifert fibred or hyperbolic. A minimal collection of such tori is furthermore unique up to isotopy. The elements of a minimal collection of such tori are called the *JSJ tori* of N and the components of N cut along the JSJ tori are the *JSJ components of* N. The Geometrization Theorem thus in particular shows that $\pi_1(N)$ is the fundamental group of a graph of groups where the vertex groups are fundamental groups of hyperbolic 3-manifolds and Seifert fibred manifolds and where the edge groups correspond to the boundary tori of the JSJ components.

We now consider the following classes of decorated groups:

$$\begin{array}{rcc} \mathcal{P}_{\rm hyp} & = & \begin{array}{c}\text{all decorated groups isomorphic to } (\pi_1(N), \{\pi_1(T_i)\}_{i\in I}), \text{ where} \\ N \text{ is a hyperbolic 3-manifold and } \{T_i\}_{i\in I} \text{ are the boundary tori of } N,\end{array} \\ \mathcal{P}_{\rm sfs} & = & \begin{array}{c}\text{all decorated groups isomorphic to } (\pi_1(N), \{\pi_1(T_i)\}_{i\in I}), \text{ where} \\ N \text{ is a Seifert fibered 3-manifold and } \{T_i\}_{i\in I} \text{ are the boundary tori of } N,\end{array} \\ \mathcal{P}_{\rm cl} & = & \begin{array}{c}\text{all decorated groups isomorphic to } (\pi_1(N), \emptyset), \text{ where} \\ N \text{ is a closed 3-manifold which is hyperbolic or Seifert fibered.}\end{array} \end{array}$$

(Note that the definitions of these classes do not depend on the choice of base points and path connecting the base points. In the first two examples of decorated groups, $I = \pi_0(\partial N)$ is the set of boundary components of N.) We also write $\mathcal{P} = \mathcal{P}_{\rm hyp} \cup \mathcal{P}_{\rm sfs} \cup \mathcal{P}_{\rm cl}$.

We recall that in order to prove Theorem 3 it suffices by Proposition 15 to show that the Membership Problem is solvable for the class of fundamental groups of closed, orientable, prime 3-manifolds. By the Geometrization Theorem it therefore suffices to prove the Membership Problem for the class of fundamental groups of graphs of groups which are based on $\mathcal{P}$.

We now argue that $\mathcal{P}$ satisfies Properties I to V from Theorem 13.

(I) In Theorems 20 and 23 we will see that fundamental groups of Seifert fibred manifolds and hyperbolic 3-manifolds are double coset separable, i.e. any product GH of finitely generated groups G and H is separable. An argument as in the proof of Lemma 2 now shows that $\mathcal{P}$ satisfies Condition I.

(II) The proof of Condition I also shows that Condition II holds.

(III) It is obvious that Condition III holds.

(IV) In Proposition 27 we will show that $\mathcal{P}$ satisfies Condition IV.

It suffices, therefore, to verify Condition V, which is a consequence of the following theorem of Jaco–Tollefson [JT95], also proved in Jaco–Letscher–Rubinstein [JLR02].

Theorem 17. *The classes $\mathcal{P}_{\text{hyp}}, \mathcal{P}_{\text{sfs}}$ and $\mathcal{P}_{\text{cl}}$ are recursively enumerable. In particular $\mathcal{P} = \mathcal{P}_{\text{hyp}} \cup \mathcal{P}_{\text{sfs}} \cup \mathcal{P}_{\text{cl}}$ is recursively enumerable.*

Proof. First note that any hyperbolic 3-manifold and any Seifert fibred manifold appears as the JSJ component of a closed, orientable, prime 3-manifold. By Theorem 16 the class of closed, orientable, prime 3-manifolds is recursively enumerable. For each such triangulated 3-manifold we can determine the JSJ components using the algorithm of Jaco–Tollefson [JT95] and also Jaco–Letscher–Rubinstein [JLR02]. We can thus recursively enumerate the class of all JSJ components of closed, orientable, prime 3-manifolds, in particular, by the above, we can recursively enumerate the class of 3-manifolds which are either hyperbolic or Seifert fibred.

For the manifolds with non-trivial boundary we can furthermore by [JT95, Algorithm 8.1] decide whether or not the 3-manifold is Seifert fibred.

The theorem is now an straightforward consequence of the above algorithms. □

5.4. **Subgroups of fundamental groups of Seifert fibred manifolds and of hyperbolic 3-manifolds.** In this section we will recall well-known results about subgroups of fundamental groups of Seifert fibred manifolds and of hyperbolic 3-manifolds. We will in particular see that the class of decorated groups $\mathcal{P}$ from Section 5.3 satisfies Conditions I and II. We will also need some of the results from this section in the next section when we deal with Condition IV.

In the following, given a subgroup Γ of a group π we say that Γ *is a virtual retract of* π if there exists a finite-index subgroup π_0 which contains Γ and such that Γ is a retract of π_0.

The following theorem is proved implicitly by Scott [Sc78].

Theorem 18. *Any finitely generated subgroup of a surface group is a virtual retract.*

Note that finitely generated subgroups which are virtual retracts are in particular separable, see e.g. [AFW15, (G.10)]. The above theorem thus implies that surface groups are subgroup separable. For the record we note that this furthermore implies the following theorem; see [Sc78, Sc85] for details.

Theorem 19. *Fundamental groups of Seifert fibred manifolds are subgroup separable.*

In fact a somewhat stronger statement holds true. In order to state the result we recall that a group π is called *double-coset separable* if given any finitely generated subgroups $G, H \subset \pi$ the product $GH \subset \pi$ is separable. The following theorem was proved by Niblo [Ni92] building on the aforementioned work of Scott [Sc78].

Theorem 20. *The fundamental group of any Seifert fibred manifold is double-coset separable.*

We now turn to the study of fundamental groups of hyperbolic 3-manifolds. The first key result is the Tameness Theorem of Agol [Ag04] and Calegari–Gabai [CG06]:

Theorem 21. *Let N be a hyperbolic 3-manifold and $\Gamma \subset \pi := \pi_1(N)$ a finitely generated subgroup. Then precisely one of the following holds:*

1. *either Γ is a relatively quasiconvex subgroup of π, or*
2. *there exists a finite-index subgroup π_0 of π which contains Γ as a normal subgroup with $\pi_0/\Gamma \cong \mathbb{Z}$.*

We will not be concerned with the precise definition of 'relatively quasiconvex'. We will use Theorem 21 only in conjunction with the following theorem, which is a consequence of work of Haglund [Hag08] and the Virtually Compact Special Theorem which was proved by Wise [Wi12a, Wi12b] for hyperbolic 3-manifolds with boundary and by Agol [Ag13] for closed hyperbolic 3-manifolds. (See also [CDW12] or [SaW15]. We refer to [AFW15] for details and precise references.)

Theorem 22. *Let N be a hyperbolic 3-manifold and $\Gamma \subset \pi := \pi_1(N)$ a finitely generated subgroup. If Γ is a relatively quasiconvex subgroup of π, then Γ is a virtual retract of π.*

The combination of Theorems 21 and 22 implies that the fundamental group of a hyperbolic 3-manifold is subgroup separable; we refer to [AFW15, (G.10) and (G.11)] for details. In fact, by work of Wise and Hruska, the following stronger result holds.

Theorem 23. *The fundamental group of any hyperbolic 3-manifold is double-coset separable.*

Proof. Let N be a hyperbolic 3-manifold and let G, H be finitely generated subgroups of $\pi = \pi_1 N$. We need to prove that the double-coset GH is separable. Suppose, therefore, that $g \in \pi \setminus GH$.

Suppose first that G is a virtual fibre in π. Replacing π by a subgroup of finite index, we may assume that we have an epimorphism

$$\eta \colon \pi \to \mathbb{Z}$$

with kernel G. Then $\eta(g) \notin \eta(H)$. For any n such that $\eta(g) \notin n\mathbb{Z}$ and $\eta(H) \subseteq n\mathbb{Z}$, the concatenation

$$\pi \xrightarrow{\eta} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$$

separates g from GH, as required.

We may therefore assume that both G and H are relatively quasiconvex in π. By a result of Hruska [Hr10, Corollary 1.6], G and H are both quasi-isometrically embedded, and therefore the double coset GH is separable by a theorem of Wise [Wi12a, Theorem 16.23]. □

5.5. **Computing generating sets for intersections.** In order to prove Theorem 3 it now suffices to show that the class of decorated groups $\mathcal{P}$ which we introduced in Section 5.3 satisfies Condition V. We will deal with this issue in this section.

Lemma 24. *There exists an algorithm which takes as input a finite decorated group presentation $\Pi = (\langle A|R\rangle, \{\langle X_i|S_i\rangle, f_i\}_{i\in I})$ and a finite subset Y of $F(A)$ and which, if Π represents a decorated group in $\mathcal{P}_{\text{hyp}}$, gives for each $i \in I$ as output a finite generating set for $\langle f_i(X_i)\rangle \cap \langle Y\rangle$ as a subgroup of $\langle A|R\rangle$.*

Proof. Let $\Pi = (\langle A|R\rangle, \{\langle X_i|S_i\rangle, f_i\}_{i\in I})$ be a finite decorated group presentation and let Y be a finite subset of $F(A)$. We suppose that $(\langle A|R\rangle, \{\langle f_i(X_i)\rangle\}_{i\in I})$ is isomorphic to $(\pi_1(N), \{\pi_1(T_i)\}_{i\in I})$ where N is a hyperbolic 3-manifold and $T_i, i \in I$ are the boundary components of N. Let $i \in I$. We write $P = \langle f_i(X_i)\rangle$ and $\Gamma = \langle Y\rangle \subset \pi$. By Theorems 21 and 22 precisely one of the following happens:

(a) either there exists either a finite-index subgroup π_0 of π and a retraction $r\colon \pi_0 \to \Gamma$, or
(b) there exists a finite-index subgroup π_0 and a homomorphism $p\colon \pi_0 \to \mathbb{Z}$ such that $\Gamma = \operatorname{Ker} p$.

In the former case the algorithms of Lemmas 8 and 5 will find such π_0 and r. In the latter case again a naïve search using the Reidemeister–Schreier algorithm together with Lemma 10 will find such π_0 and p. In either case, by Lemma 7 we can compute generators for $P_0 = \pi_0 \cap P$.

In case (b) we have $\Gamma \cap P = \operatorname{Ker} p|_{P_0}$ which can be computed by standard linear algebra. In case (a) we note that $\Gamma \cap P = r(P_0) \cap P_0$. Using the solution to the word problem in π (see Lemma 2), we can determine whether or not all generators of $r(P_0)$ and P_0 commute, i.e. we can determine whether or not $[r(P_0), P_0] = 1$.

First suppose that $[r(P_0), P_0] = 1$. Recall that P_0 is the fundamental group of a boundary torus of a hyperbolic 3-manifold. It is well-known, see e.g. [AFW15, Theorem 3.1], that this implies that P_0 is a maximal abelian subgroup of π_0. It now follows that $r(P_0) \subseteq P_0$, which implies that $\Gamma \cap P = r(P_0)$.

Now suppose that $[r(P_0), P_0] \neq 1$. The fact that N is hyperbolic implies by [AFW15, Corollary 3.11] that the centralizer of any non-identity element in π_0 is abelian. It now follows that $r(P_0) \cap P_0 = 1$ and so $\Gamma \cap P = 1$. □

We now also consider the following class of decorated groups:

$\mathcal{P}_{\text{product}} =$ all decorated groups isomorphic to $(\pi_1(S^1 \times \Sigma), \{\pi_1(T_i)\}_{i\in I})$, where Σ is a surface and $T_i, i \in I = \pi_0(\partial N)$ are the components of $\partial(S^1 \times \Sigma)$.

We then have the following lemma.

Lemma 25. *There exists an algorithm which takes as input a finite decorated group presentation $\Pi = (\langle A|R\rangle, \{\langle X_i|S_i\rangle, f_i\}_{i\in I})$ and a finite subset Y of $F(A)$ and which, if Π represents a subdecoration for a decorated group in $\mathcal{P}_{\text{product}}$, gives for each $i \in I$ as output a finite generating set for $\langle f_i(X_i)\rangle \cap \langle Y\rangle$ as a subgroup of $\langle A|R\rangle$.*

Proof. Let $(\langle A|R\rangle, \{\langle X_i|S_i\rangle, f_i\}_{i\in I})$ be a finite decorated group presentation and Y a finite subset of $F(A)$. We suppose that $(\langle A|R\rangle, \{\langle f_i(X_i)\rangle\}_{i\in I})$ is isomorphic to $(\pi_1(S^1 \times \Sigma), \{\pi_1(T_i)\}_{i\in I})$ where Σ is a surface and $T_i, i \in I$ are some boundary components of $S^1 \times \Sigma$. Let $i \in I$. We write $\pi = \langle A|R\rangle$, $P = \langle f_i(X_i)\rangle$ and $\Gamma = \langle Y\rangle$.

By Theorem 18 every finitely generated subgroup of the surface group $\pi_1(\Sigma)$ is a virtual retract. It follows easily that every finitely generated subgroup of $\pi_1(S^1 \times \Sigma) = \mathbb{Z} \times \pi_1(\Sigma)$ is a virtual retract. Therefore, the algorithms of Lemmas 8 and 5 will find a finite-index subgroup π_0 of π and a retraction $r : \pi_0 \to \Gamma$. As in the proof of Lemma 24 we can compute generators for $P_0 = \pi_0 \cap P$.

Again, we note that $\Gamma \cap P = r(P_0) \cap P_0$. An explicit computation again determines whether or not $[r(P_0), P_0] = 1$. If so then, just as before, because P_0 is maximal abelian we have $r(P_0) \subseteq P_0$ and so $r(P_0) = \Gamma \cap P$. If not then, by the commutative transitivity of $\pi_1(\Sigma)$, we deduce that $\Gamma \cap P = r(P_0) \cap P_0$ is contained in the centre Z_0 of π_0 and so it suffices to compute $r(P_0) \cap Z_0$. But now $r(P_0) \cap Z_0$ can be seen in the abelianization of π_0, and so can be computed by elementary linear algebra. $\square$

Lemma 26. *There exists an algorithm which takes as input a finite decorated group presentation $\Pi = (\langle A|R\rangle, \{\langle X_i|S_i\rangle, f_i\}_{i\in I})$ and a finite subset Y of $F(A)$ and which, if Π represents a decorated group in $\mathcal{P}_{\text{sfs}}$, gives for each $i \in I$ as output a finite generating set for $\langle f_i(X_i)\rangle \cap \langle Y\rangle$ as a subgroup of $\langle A|R\rangle$.*

Proof. Let $(\langle A|R\rangle, \{\langle X_i|S_i\rangle, f_i\}_{i\in I})$ be a finite decorated group presentation and Y a finite subset of $F(A)$. We suppose that $(\langle A|R\rangle, \{\langle f_i(X_i)\rangle\}_{i\in I})$ is isomorphic to $(\pi_1(N), \{\pi_1(T_i)\}_{i\in I})$ where N is a Seifert fibered space and $\{T_i\}_{i\in I}$ are the boundary components of N. Let $i \in I$. We write $\pi = \langle A|R\rangle$, $P = \langle f_i(X_i)\rangle$ and $\Gamma = \langle Y\rangle$.

By [He76, Theorem 11.10] there exists a finite cover of N which is a product $S^1 \times \Sigma$. Using Lemma 8 we now enumerate all finite-index subgroups of π. We can furthermore enumerate all fundamental groups of products $S^1 \times \Sigma$, Σ a surface, and using an obvious generalization of Lemma 4 we will eventually find a finite-index subgroup π_0 of π, a presentation $\langle A|R\rangle$ of $\pi_1(S^1 \times \Sigma)$, Σ a surface, and an isomorphism $g \colon \pi_0 \to \Gamma := \langle A|R\rangle$ such that $g(P \cap \pi_0)$ is the fundamental group of a boundary component of $S^1 \times \Sigma$.

Using Lemma 7 we can find a generating set Y_0 for $\Gamma \cap \pi_0$, a finite presentation $\langle X_0|S_0\rangle$, and an isomorphism $f_0 \colon \langle X_0|S_0\rangle \to P \cap \pi_0$. We now apply the algorithm of Lemma 25 to the finite decorated group presentation $(\langle A|R\rangle, \{\langle X_0|S_0\rangle, f_0\})$ and the finite set Y_0. The algorithm then gives us a generating set for $(\Gamma \cap \pi_0) \cap (P \cap \pi_0) = (\Gamma \cap \pi_0) \cap P$.

Note that $(\Gamma \cap \pi_0) \cap P$ is a finite index subgroup of $\Gamma \cap P$, which in turn is a subgroup of $P \cong \mathbb{Z}^2$. It is now straightforward to list the (finitely many) subgroups of $P \cong \mathbb{Z}^2$ which contain $(\Gamma \cap \pi_0) \cap P$ as a finite index subgroup. For each of these subgroups we pick a finite number of generators and using the fact that π is subgroup separable, see Theorem 19, we can check whether the generators lie in P and in Γ. $\square$

We are now ready to prove that the class $\mathcal{P}$ of decorated groups satisfies Condition IV. More precisely, we have the following proposition.

Proposition 27. *There exists an algorithm which takes as input a finite decorated group presentation* $\Pi = (\langle A|R\rangle, \{\langle X_i|S_i\rangle, f_i\}_{i\in I})$ *and a finite subset* Y *of* $F(A)$ *and which, if* Π *represents a decorated group in* $\mathcal{P}$*, gives for each* $i \in I$ *as output a finite generating set for* $\langle f_i(X_i)\rangle \cap \langle Y\rangle$ *as a subgroup of* $\langle A|R\rangle$*.*

Proof. Let $(\langle A|R\rangle, \{\langle X_i|S_i\rangle, f_i\}_{i\in I})$ be a finite decorated group presentation which represents a decorated group in $\mathcal{P}$ and let $Y \subset F(A)$ be a finite set. Let $i \in I$. By the solution to the word problem for $\langle A|R\rangle$ we can determine whether or not $f_i(X_i)$ generates the trivial group. If it does, then there is nothing to show.

Now suppose that $f_i(X_i)$ does not generate the trivial group. By definition of $\mathcal{P}$ it follows that Π represents an element in $\mathcal{P}_{\text{hyp}}$ or it represents an element in $\mathcal{P}_{\text{sfs}}$.

Using Theorem 17 and an obvious generalization of Lemma 4 we can now certify that $(\langle A|R\rangle, \{\langle f_i(X_i)\rangle\}_{i\in I})$ is isomorphic to a subdecoration of a decorated group in $\mathcal{P}_{\text{hyp}}$, or we can certify that $(\langle A|R\rangle, X)$ is isomorphic to a subdecoration of a decorated group in $\mathcal{P}_{\text{sfs}}$. (It follows from basic facts in 3-manifold topology that only one of the two cases can occur, but this fact is irrelevant for the proof of this proposition.) In the former case we now apply the algorithm from Lemma 24, while in the latter case we apply the algorithm from Lemma 26 $\square$

6. Alternative approaches and open questions

In the proof of our main theorem we used two big theorems on fundamental groups of hyperbolic 3-manifolds: the Tameness Theorem of Agol [Ag04] and Calegari–Gabai [CG06] and Theorem 22 which is a consequence of the Virtually Compact Special Theorem of Agol [Ag13] and Wise [Wi12a, Wi12b]. The Tameness Theorem is indispensable: it is needed to control geometrically infinite subgroups. However, it is quite possible that one could also prove Theorem 1 without appealing to the Virtually Compact Special Theorem. For example, Gitik [Gi96] and Kapovich [Ka96] showed that the Membership Problem is solvable for quasiconvex subgroups of word-hyperbolic groups. It is now straightforward to see that one can prove our main theorem by appealing to this result and to appealing to Theorem 22 only for hyperbolic 3-manifolds with non-empty boundary. It is now an interesting question whether one can also replace Theorem 22 by more general methods from geometric group theory.

In the following we consider the class of decorated presentations of *toral relatively hyperbolic groups.* That is, we consider

$$\mathcal{H} = \begin{array}{l}\text{all decorated groups isomorphic to } (\pi, \{\Gamma_i\}_{i\in I}), \text{ where } \pi \text{ is a group which}\\ \text{is hyperbolic relative to the finite collection of f.g. abelian subgroups } \{\Gamma_i\}_{i\in I}.\end{array}$$

It is now fairly straightforward to see that one can prove our main theorem without referring to Theorem 22 if one can give affirmative answers to the following three questions.

The first is a generalization of the above-mentioned work of Gitik and Kapovich to the toral relatively hyperbolic setting.

Question 28. *Does there exist an algorithm which takes as input a finite decorated group presentation $\Pi = (\langle A|R\rangle, \{\langle X_i|S_i\rangle, f_i\}_{i\in I})$ and a finite subset Y of $F(A)$ and which, if Π represents a decorated group in $\mathcal{H}$ and if Y generates a relatively quasi-convex subgroup of $\langle A|R\rangle$, determines whether or not a given element in $\langle A|R\rangle$ lies in $\langle Y\rangle \subset \langle A|R\rangle$?*

The second question generalizes the first question to double cosets.

Question 29. *Does there exist an algorithm which takes as input a finite decorated group presentation $\Pi = (\langle A|R\rangle, \{\langle X_i|S_i\rangle, f_i\}_{i\in I})$, a finite subset Y of $F(A)$ and an index $i \in I$ and which, if Π represents a decorated group in $\mathcal{H}$ and if Y generates a relatively quasi-convex subgroup of $\langle A|R\rangle$, determines whether or not a given element in $\langle A|R\rangle$ lies in the double coset $\langle f_i(X_i)\rangle\langle Y\rangle \subset \langle A|R\rangle$?*

The final question asks for an algorithm to compute the intersection of a relatively quasiconvex subgroup and a maximal parabolic subgroup.

Question 30. *Does there exist an algorithm which takes as input a finite decorated group presentation $\Pi = (\langle A|R\rangle, \{\langle X_i|S_i\rangle, f_i\}_{i\in I})$ and a finite subset Y of $F(A)$ and which, if Π represents a decorated group in $\mathcal{H}$, gives for each $i \in I$ as output a finite generating set for $\langle f_i(X_i)\rangle \cap \langle Y\rangle$ as a subgroup of $\langle A|R\rangle$?*

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Fakultät für Mathematik, Universität Regensburg, Germany
E-mail address: `sfriedl@gmail.com`

Department of Mathematics, University College London, UK
E-mail address: `hwilton@math.ucl.ac.uk`