LINKING FORMS REVISITED

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Abstract. We show that the $\mathbb{Q}/\mathbb{Z}$-valued linking forms on rational homology spheres are (anti-) symmetric and we compute the linking form of a 3-dimensional rational homology sphere in terms of a Heegaard splitting. Both results have been known to a larger or lesser degree, but it is difficult to find rigorous down-to-earth proofs in the literature.

1. Introduction

Let M be a $(2n+1)$-dimensional rational homology sphere. In Section 2.2 we recall the definition of the *linking form*

$$\lambda_M \colon H_n(M;\mathbb{Z}) \times H_n(M;\mathbb{Z}) \ \to \ \mathbb{Q}/\mathbb{Z}.$$

It follows easily from the definition that it is bilinear and non-singular. This form, whose definition goes back to Seifert [12, 13], has since then appeared frequently both in the study of high-dimensional manifolds [6, 14, 15] and in low dimensional topology [7, 1, 3].

The following proposition states a key property of linking forms.

Proposition 1.1. *Let M be a $(2n+1)$-dimensional rational homology sphere. If n is odd, then the linking form λ_M on $H_n(M;\mathbb{Z})$ is symmetric otherwise it is anti-symmetric.*

The statement of the proposition is well-known. But to the best of our knowledge there are not many proofs in the literature. A somewhat informal proof can be found in the original paper by Seifert [12, p. 814]. Linking forms have been generalized by Blanchfield and many others to more general coefficients, where the corresponding linking forms are also well-known to be hermitian. But there are again very few rigorous proofs for these statements, in fact we are only aware of the recent paper by Powell [9]. We give a rigorous quick proof of Proposition 2.6 and we expect that the same approach can be used to reprove the hermitianness statement of Powell [9]. To keep the paper short and readable we will not attempt to carry out this generalization.

The following proposition recalls the arguably most frequently used calculation of linking forms on 3-manifolds.

Proposition 1.2. *The linking form of the 3-dimensional lens space $L(p,q)$ is isometric to the form*

$$\begin{array}{rcl} \mathbb{Z}_p \times \mathbb{Z}_p & \to & \mathbb{Q}/\mathbb{Z} \\ (a,b) & \mapsto & -\frac{q}{p} \cdot a \cdot b. \end{array}$$

This proposition is essential in the classification of lens spaces up to homotopy equivalence, in fact Whitehead [16] showed that two lens spaces are homotopy equivalent if and only if their linking forms are isometric.

In the literature, except for the precise sign in the formula, many proofs of Proposition 1.2 or of equivalent statements can be found. In fact many textbooks in algebraic topology contain a proof, see e.g. [5, p. 306], [8, Chapter 69] and [2, p. 364], except that none of these proofs address the precise sign in the calculation. All these proofs work very explicitly with lens spaces and it is not evident how they generalize say to other 3-manifolds.

We will now explain how to calculate the linking form of any rational homology sphere in terms of a Heegaard splitting. We will then see that this calculation gives in particular a proof of Proposition 1.2.

Throughout this paper, given $g \in \mathbb{N}$ we adopt the following notation:

(1) We denote by X_g a handlebody of genus g and we equip it with an orientation. We denote by Z_g a copy of X_g.
(2) We write $F_g = \partial X_g = \partial Z_g$. We equip F_g with the orientation coming from the boundary orientation of X_g.
(3) We denote by $a_1, \dots, a_g, b_1, \dots, b_g \in H_1(F_g; \mathbb{Z})$ a symplectic basis for $H_1(F_g; \mathbb{Z})$ such that $a_1, \dots, a_g$ form a basis for $H_1(X_g; \mathbb{Z})$. Recall that "symplectic basis" means that the intersection form of F_g with respect to this basis is given by the matrix
$$\begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}$$
where we denote by I_g the $g \times g$-identity matrix.
(4) Given a self-diffeomorphism φ of the genus g surface F_g we write $M(\varphi) := X_g \cup_\varphi Z_g$ where we identify $x \in F_g = \partial X_g$ with $\varphi(x) \in \partial Z_g$. We give $M(\varphi)$ the orientation which turns the inclusion $X_g \to M(\varphi)$ into an orientation-preserving embedding. Furthermore we denote by
$$\begin{pmatrix} A_\varphi & B_\varphi \\ C_\varphi & D_\varphi \end{pmatrix}$$
the matrix that represents $\varphi_*\colon H_1(F_g; \mathbb{Z}) \to H_1(F_g; \mathbb{Z})$ with respect to the ordered basis $a_1, \dots, a_g, b_1, \dots, b_g$.

One of the first theorems in 3-manifold topology states that every closed 3-manifold can be written as $M(\varphi)$ for some g and some diffeomorphism $\varphi\colon F_g \to F_g$. (Here and throughout this paper all manifolds are understood to be compact, oriented and path-connected.) The following theorem thus gives a calculation of the linking form for any 3-dimensional rational homology sphere.

Theorem 1.3. *Let $g \in \mathbb{N}$ and let $\varphi\colon F_g \to F_g$ be an orientation-preserving diffeomorphism. If $M(\varphi)$ is a rational homology sphere, then $B_\varphi \in M(g \times g, \mathbb{Z})$ is invertible and the linking form of $M(\varphi)$ is isometric to the form*
$$\begin{aligned} \mathbb{Z}^g/B_\varphi^T\mathbb{Z}^g \times \mathbb{Z}^g/B_\varphi^T\mathbb{Z}^g \to\ & \mathbb{Q}/\mathbb{Z} \\ (v, w) \mapsto\ & v^T(B_\varphi^{-1}A_\varphi)w. \end{aligned}$$

Remark.

(1) In Theorem 3.5 we will state precisely what isomorphism $\mathbb{Z}/B_\varphi^T\mathbb{Z}^g \to H_1(M(\varphi); \mathbb{Z})$ we use.
(2) As we mentioned above, the previous calculations of linking forms that we are aware of do not address the sign question of the formula, i.e. they only determine the linking form up to a fixed sign. We tried exceedingly hard to determine the sign correctly. Nonetheless, one should take our sign with a grain of salt. After we first thought that we had definitely determined the correct sign, we found three more sign errors.

(3) One could make the case that the statement of Theorem 1.3 is at least implicit in [10] as explained by Seifert [12, p. 827]. But the calculation provided in that paper is not very rigorous by today's standards and it is also very hard to decypher for a modern reader, even if the reader is able to understand arcane German. To the best of our knowledge we provide the first proof of Theorem 1.3 that is rigorous and that only uses singular homology and cohomology. Also, similar to our proof of the symmetry of linking forms, we think that our approach to calculating linking forms can be generalized quite easily to compute twisted linking forms of a closed 3-manifold in terms of a Heegaard splitting.

We now return to lens spaces. We denote by $X = Z = S^1 \times D^2$ the solid torus and we write $F = \partial X = \partial Z$. We equip S^1, $S^1 \times D^2$ and $F = \partial X = S^1 \times S^1$ with the standard orientation. Note that with these conventions $a = [S^1 \times 1]$ and $b = [1 \times S^1]$ form a symplectic basis for the torus ∂X. Let $p, q \in \mathbb{N}$ be coprime. We pick $r, s \in \mathbb{N}$ such that $r(-q) - ps = 1$. We write

$$A = \begin{pmatrix} r & p \\ s & -q \end{pmatrix}$$

and we denote by $\varphi\colon F \to F$ the diffeomorphism such that φ_* with respect to the basis given by $a = [S^1 \times 1]$ and $b = [1 \times S^1]$ is represented by the matrix A. By [11, Chapter 1.5], the resulting 3-manifold is precisely the lens space $L(p, q)$. (To avoid confusion, note that a respectively b in [11, Chapter 1.5] are called *longitude* and *meridian*, in particular our ordered basis $\{a, b\}$ is the same basis as in [11, Chapter 1.5], except with the opposite order. Thus our matrix A is the matrix given [11, p. 24] with rows and columns swapped.) Theorem 1.3 says that the linking form of $L(p, q)$ is isometric to the form

$$\begin{array}{rcl} \mathbb{Z}/p \times \mathbb{Z}/p & \mapsto & \mathbb{Q}/\mathbb{Z} \\ (v, w) & \mapsto & v \cdot \frac{r}{p} \cdot w. \end{array}$$

Note that multiplication by q is an isomorphism of $\mathbb{Z}/p$ and that $r(-q) \equiv 1 \mod p$. This shows that the above form is isometric to the form

$$\begin{array}{rcl} \mathbb{Z}/p \times \mathbb{Z}/p & \mapsto & \mathbb{Q}/\mathbb{Z} \\ (v, w) & \mapsto & v \cdot \frac{q^2 r}{p} \cdot w \;=\; -v \cdot \frac{q}{p} \cdot w. \end{array}$$

which implies Proposition 1.2.

Remark. One of the ideas of the proof is to reduce the calculation of Poincaré duality of a 3-manifold to the well-known calculation of Poincaré duality of the Heegaard surface F of $M(\varphi) = X \cup_F Z$. A similar approach has been used in [4] to reduce the calculation of the Blanchfield form of a knot to the Poincaré duality of a Seifert surface.

Remark. Given an $(2n + 1)$-dimensional manifold M one can also define a linking form on the torsion submodule of $H_n(M; \mathbb{Z})$. The same argument as in the proof of Proposition 2.6 shows that it is symmetric. In the 3-dimensional context it should not be very hard to generalize Theorem 1.3 to the case of 3-manifolds that are not rational homology spheres.

This paper is organized as follows. In Section 2.1 we recall basic facts on the cup product and the cap product with coefficients. In Section 2.2 we recall the definition of the linking form and in Section 2.3 we provide the proof that linking forms are (anti-) symmetric. Finally in Section 3 we provide the proof of Theorem 1.3.

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2. Preliminaries

This section recalls the definition of the linking form as well as some standard facts of algebraic topology. References include [2, 8, 5, 14, 6, 13].

Before we start out the discussion of the properties of the cup product and the cap product we want to point out that Bredon [2] defines the coboundary map as $\delta_n = (-1)^{n+1}\partial^*_{n+1}$ whereas most other books, e.g. Munkres [8] and Hatcher [5] define the coboundary map as $\delta_n = \partial^*_{n+1}$. We choose to follow the latter convention. These sign conventions influence some of the formulas, e.g. the diagram in Lemma 2.3 commutes only up to the sign $(-1)^{k+1}$, whereas following the approach of Bredon the diagram in Lemma 2.3 would commute.

2.1. **The cup product and the cap product.** Let X be a topological space and let G, H be abelian groups. The usual definition of the cup product as provided in [2, Chapter VI.4] generalizes to a cup product

$$\cup_\otimes \colon C^k(X;G) \times C^l(X;H) \;\to\; C^{k+l}(X;G\otimes H).$$

A slightly lengthy but uneventful calculation shows that for $f \in C^k(X;G)$ and $g \in C^l(X;H)$ we have

$$\delta(f \cup_\otimes g) \;=\; \delta(f) \cup_\otimes g + (-1)^k \cdot \varphi \cup_\otimes \delta(g) \;\in\; C^{k+l}(X;G\otimes H). \tag{2.1}$$

This implies that the above cup product on cochains descends to a cup product

$$\cup_\otimes \colon H^k(X;G) \times H^l(X;H) \;\to\; H^{k+l}(X;G\otimes H).$$

We denote by $\Theta\colon G\otimes H \to H \otimes G$ the obvious isomorphism. Then for $\varphi \in H^k(X;G)$ and $\psi \in H^l(X;H)$ the usual proof of the (anti-) symmetry of the cup product can be used to show that

$$\Theta_*(\varphi \cup_\otimes \psi) \;=\; (-1)^{kl} \cdot \psi \cup_\otimes \varphi \;\in\; H^{k+l}(X;H\otimes G). \tag{2.2}$$

If $H = \mathbb{Z}$, then using the obvious isomorphism $\nu\colon G\otimes\mathbb{Z} \to G$ we obtain the cup product

$$\cup\colon H^k(X;G) \times H^l(X;\mathbb{Z}) \;\to\; H^{k+l}(X;G\otimes\mathbb{Z}) \xrightarrow{\nu_*} H^{k+l}(X;G).$$

The same holds if $G = \mathbb{Z}$ and H is some arbitrary abelian group.

Now let G be an abelian group and let (X, U) be a pair of topological spaces. The usual definition of the cap product as provided in [2, Chapter VI.5] generalizes to a cap product

$$\cap\colon H^k(X;G) \times H_l(X,U;\mathbb{Z}) \;\to\; H_{l-k}(X,U;G).$$

If X is path-connected, then we make the identification $H_0(X;G) = G$ via the augmentation map. In this case we refer to

$$\begin{array}{rcl} \langle\;,\;\rangle\colon H^k(X;G) \times H_k(X;\mathbb{Z}) & \to & H_0(X;G) = G \\ (\varphi,\sigma) & \mapsto & \langle\varphi,\sigma\rangle := \varphi\cap\sigma \end{array}$$

as the *Kronecker pairing.* The following lemma is a slight generalization of the properties of the more common cap product as provided in [2, Chapter VI].

Lemma 2.1. *Let G be an abelian group and let (X, U) be a pair of topological spaces.*

(1) *Let $f\colon (X,U) \to (Z,V)$ be a map of pairs. If $\xi \in H^k(Z;G)$ and $\sigma \in H_l(X,U;\mathbb{Z})$, then*
$$f_*(f^*(\xi) \cap \sigma) \;=\; \xi \cap f_*(\sigma) \;\in\; H_{l-k}(Z,V;G).$$

(2) *If $\varphi \in H^k(X;G)$, $\psi \in H^l(X;\mathbb{Z})$ and $\sigma \in H_m(X,U;\mathbb{Z})$, then*
$$(\varphi \cup \psi) \cap \sigma \;=\; \varphi \cap (\psi \cap \sigma) \;\in\; H_{m-k-l}(X,U;G).$$

Now let M be an n-dimensional manifold. (Recall that all manifolds are assumed to be compact, oriented and path-connected.) As usual we denote by $[M] \in H_n(M, \partial M;\mathbb{Z})$ the fundamental class. Let G be an abelian group. The Poincaré duality theorem says that the map
$$\begin{array}{rcl} \cap[M]\colon H^k(M;G) & \to & H_{n-k}(M,\partial M;G) \\ \varphi & \mapsto & \varphi \cap [M] \end{array}$$
is an isomorphism. We denote by $\mathrm{PD}_M^G\colon H_{n-k}(M,\partial M;G) \to H^k(M;G)$ the inverse.

The following proposition is [2, Theorem VI.9.2]. Note that in this instance the different sign convention of Bredon does not affect the outcome.

Proposition 2.2. *Let M be an n-dimensional manifold and let G be an abelian group. We denote by $k\colon \partial M \to M$ the inclusion map. Then for any $p \in \mathbb{N}_0$ the following diagram commutes up to the sign $(-1)^p$:*
$$\begin{array}{ccc} H_{n-p}(M,\partial M;G) & \xleftarrow{\;\cap[M]\;} & H^p(M;G) \\ \downarrow \partial & & \downarrow k^* \\ H_{n-p-1}(\partial M;G) & \xleftarrow{\;\cap[\partial M]\;} & H^p(\partial M;G). \end{array}$$

2.2. **The definition of linking form on rational homology spheres.** Let X be a topological space. We denote by $\beta\colon H^k(X;\mathbb{Q}/\mathbb{Z}) \to H^{k+1}(X;\mathbb{Z})$ the *Bockstein homomorphism* which arises from the short exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ of coefficients. We define similarly the Bockstein homomorphism $\beta\colon H_k(X;\mathbb{Q}/\mathbb{Z}) \to H_{k-1}(X;\mathbb{Z})$.

Lemma 2.3. *Let Z be an m-dimensional compact manifold. For any $k \in \{0,\dots,m-1\}$ the diagram*
$$\begin{array}{ccc} H_{m-k-1}(Z;\mathbb{Z}) & \xleftarrow{\;\cap[Z]\;} & H^{k+1}(Z;\mathbb{Z}) \\ \beta\uparrow & & \beta\uparrow \\ H_{m-k}(Z;\mathbb{Q}/\mathbb{Z}) & \xleftarrow{\;\cap[Z]\;} & H^k(Z;\mathbb{Q}/\mathbb{Z}) \end{array}$$
commutes up to the sign $(-1)^{k+1}$.

Proof. The lemma is basically [8, Lemma 69.2], except that in the reference the sign is not specified. The sign comes from the following general fact: Let X be a topological space and let G be an abelian group. Furthermore let $\varphi \in C^k(X;G)$ and let $\sigma\colon \Delta^l \to X$ be a singular l-simplex. If $k \le l$, then a straightforward calculation shows that
$$\partial(\varphi \cap \sigma) \;=\; (-1)^{k+1} \cdot \delta\varphi \cap \sigma \;+\; (-1)^k \cdot (\varphi \cap \partial\sigma).$$
(If one takes the different sign conventions into account, this equality is exactly [2, Proposition VI.5.1].) In our case σ is a cycle that represents the fundamental class of Z. It is now clear that in our diagram the sign $(-1)^{k+1}$ appears. We leave the details of the precise argument to the reader. □

Now let M be an $(2n+1)$-dimensional rational homology sphere with $n \geq 1$. In this case the Bockstein homomorphisms in homology and cohomology in dimension n are in fact isomorphisms. We denote by Ω the composition

$$\begin{array}{ccccccc} H_n(M;\mathbb{Z}) & \xrightarrow{\mathrm{PD}_M^{\mathbb{Z}}} & H^{n+1}(M;\mathbb{Z}) & \xrightarrow{\beta^{-1}} & H^n(M;\mathbb{Q}/\mathbb{Z}) & \xrightarrow{\mathrm{ev}} & \mathrm{Hom}_{\mathbb{Z}}(H_n(M;\mathbb{Z}),\mathbb{Q}/\mathbb{Z}) \\ & & & & \varphi & \mapsto & (\sigma \mapsto \langle \varphi, \sigma\rangle). \end{array}$$

of Poincaré duality, the inverse Bockstein and the Kronecker evaluation map.

Definition. The *linking form* of a $(2n+1)$-dimensional rational homology sphere M is the form

$$\lambda_M \colon H_n(M;\mathbb{Z}) \times H_n(M;\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$$

defined by $\lambda_M(a,b) = \Omega(a)(b)$.

We summarize some key properties of the linking form in the following lemma.

Lemma 2.4. *Let M be a $(2n+1)$-dimensional rational homology sphere. Then the following statements hold:*

(1) *λ_M is bilinear and non-singular (i.e. Ω is an isomorphism),*
(2) *given a and b in $H_n(M;\mathbb{Z})$, we have*

$$\lambda_M(a,b) \;=\; \big\langle (\beta^{-1} \circ \mathrm{PD}_M^{\mathbb{Z}})(a) \cup \mathrm{PD}_M^{\mathbb{Z}}(b), [M] \big\rangle,$$

(3) *if n is odd, then the linking form λ_M is symmetric, otherwise it is anti-symmetric.*

Proof. It is clear that λ_M is bilinear. To show that λ_M is non-singular we need to show that all three homomorphisms in the definition of Ω are isomorphisms. Clearly we only have to argue that the last homomorphism is an isomorphism, but this in turn is an immediate consequence of the universal coefficient theorem and the fact that $\mathbb{Q}/\mathbb{Z}$ is injective.

We turn to the proof of (2). By the definition of the Kronecker pairing we have

$$\big\langle (\beta^{-1} \circ \mathrm{PD}_M^{\mathbb{Z}})(a) \cup \mathrm{PD}_M^{\mathbb{Z}}(b), [M] \big\rangle \;=\; \big((\beta^{-1} \circ \mathrm{PD}_M^{\mathbb{Z}})(a) \cup \mathrm{PD}_M^{\mathbb{Z}}(b) \big) \cap [M].$$

Next, using the second statement of Lemma 2.1 and the fact that by definition we have $\mathrm{PD}_M^{\mathbb{Z}}(b) \cap [M] = b$, we deduce that this expression reduces to $(\beta^{-1} \circ \mathrm{PD}_M^{\mathbb{Z}})(a) \cap b$. Looking back at Definition 2.2, this is nothing but the linking form applied to a and b, as claimed.

We postpone the proof of (3) to the next section. □

Lemma 2.4 might remind the reader of the intersection form of even-dimensional manifolds. In fact, since the proof of Theorem 1.3 will relate the linking form of $M(\varphi)$ to the intersection form of the Heegaard surface F, we briefly recall the definition of this latter form. Namely, given a closed oriented surface F, the *intersection form* of F with rational coefficients

$$Q_F \colon H_1(F;\mathbb{Q}) \times H_1(F;\mathbb{Q}) \to \mathbb{Q}$$

is defined as

$$Q_F(x,y) \;:=\; \big\langle \mathrm{PD}_F^{\mathbb{Q}}(x) \cup \mathrm{PD}_F^{\mathbb{Q}}(y), [F] \big\rangle \;=\; \big(\mathrm{PD}_F^{\mathbb{Q}}(x) \cup \mathrm{PD}_F^{\mathbb{Q}}(y) \big) \cap [F].$$

It follows immediately from Lemma 2.1 (2) that for $x, y \in H_1(F;\mathbb{Q})$ we have

$$\begin{aligned} Q_F(x,y) \;&=\; \big(\mathrm{PD}_F^{\mathbb{Q}}(x) \cup \mathrm{PD}_F^{\mathbb{Q}}(y) \big) \cap [F] \\ &=\; \mathrm{PD}_F^{\mathbb{Q}}(x) \cap \big(\mathrm{PD}_F^{\mathbb{Q}}(y) \cap [F] \big) \;=\; \mathrm{PD}_F^{\mathbb{Q}}(x) \cap y \;=\; \big\langle \mathrm{PD}_F^{\mathbb{Q}}(x), y \big\rangle. \end{aligned} \tag{2.3}$$

2.3. **Symmetry of the linking form.** In this section, we shall give a short algebraic proof that the linking form is (anti-) symmetric. The idea is to use the definition of the linking form in terms of the cup product, see Lemma 2.4 (2).

Throughout this section we denote by $\nu\colon \mathbb{Q}/\mathbb{Z}\otimes\mathbb{Z}\to\mathbb{Q}/\mathbb{Z}$ and $\nu\colon \mathbb{Z}\otimes\mathbb{Q}/\mathbb{Z}\to\mathbb{Q}/\mathbb{Z}$ the obvious isomorphisms. Now recall that by definition we can decompose the cup product $\cup$ as

$$H^k(M;\mathbb{Z})\times H^l(M;\mathbb{Q}/\mathbb{Z}) \xrightarrow{\cup_\otimes} H^{k+l}(M;\mathbb{Z}\otimes\mathbb{Q}/\mathbb{Z}) \xrightarrow{\nu_*} H^{k+l}(M;\mathbb{Q}/\mathbb{Z}). \tag{2.4}$$

Lemma 2.5. *Let X be a topological space. For any $x\in H^k(X;\mathbb{Q}/\mathbb{Z})$ and $y\in H^l(X;\mathbb{Q}/\mathbb{Z})$, we have*

$$\nu_*(\beta(x)\cup_\otimes y) \;=\; (-1)^{k+1}\cdot\nu_*(x\cup_\otimes\beta(y)) \;\in\; H^{k+l+1}(X;\mathbb{Q}/\mathbb{Z}).$$

Proof. We denote by ρ the canonical projection from $\mathbb{Q}$ to $\mathbb{Q}/\mathbb{Z}$. Pick f in $C^k(X;\mathbb{Q})$ and g in $C^l(X;\mathbb{Q})$ so that $[\rho_*(f)]=x$ and $[\rho_*(g)]=y$. The usual mild diagram chase in the definition of the Bockstein homomorphism shows that there exist unique cocycles $\beta(f)$ in $C^{k+1}(X;\mathbb{Z})$ and $\beta(g)$ in $C^{l+1}(X;\mathbb{Z})$ which satisfy $\iota_*(\beta(f))=\delta(f)$ and $\iota_*(\beta(g))=\delta(g)$; here δ denotes the coboundary map and ι denotes the inclusion map $\mathbb{Z}\to\mathbb{Q}$. Using (2.1) together with the definition of $\beta(f)$ and $\beta(g)$, we have the following equality in $H^{k+l+1}(X;\mathbb{Q}\otimes\mathbb{Q})$:

$$0=[\delta(f\cup_\otimes g)]=[\delta(f)\cup_\otimes g+(-1)^k\cdot f\cup_\otimes\delta(g)]=[\iota_*(\beta(f))\cup_\otimes g+(-1)^k\cdot f\cup_\otimes\iota_*(\beta(g))]. \tag{2.5}$$

In order to relate the right hand side of (2.5) to the expressions which appear in the statement of the lemma, we consider the following commutative diagram of group homomorphisms

$$\begin{array}{ccccc}
\mathbb{Z}\otimes\mathbb{Q} & \xrightarrow{\iota\otimes\mathrm{id}} & \mathbb{Q}\otimes\mathbb{Q} & \xleftarrow{\mathrm{id}\otimes\iota} & \mathbb{Q}\otimes\mathbb{Z}\\
\downarrow{\scriptstyle \mathrm{id}\otimes\rho} & & \downarrow{\scriptstyle\mu} & & \downarrow{\scriptstyle\rho\otimes\mathrm{id}}\\
\mathbb{Z}\otimes\mathbb{Q}/\mathbb{Z} & & \mathbb{Q} & & \mathbb{Q}/\mathbb{Z}\otimes\mathbb{Z}\\
& \searrow{\scriptstyle\nu} & \downarrow{\scriptstyle\rho} & \swarrow{\scriptstyle\nu} & \\
& & \mathbb{Q}/\mathbb{Z}, & &
\end{array} \tag{2.6}$$

where ν and μ stand for the obvious multiplication maps. Using (2.5) and the commutativity of (2.6), we get the following equality in $H^{k+l+1}(X;\mathbb{Q}/\mathbb{Z})$:

$$\begin{aligned}
0 &= (p\circ\mu)_*\big([i_*(\beta(f))\cup_\otimes g \;+\; (-1)^k\cdot f\cup_\otimes i_*(\beta(g))]\big)\\
&= \big[\nu_*\big(\beta(f)\cup_\otimes\rho_*(g)\big) \;+\; (-1)^k\cdot\nu_*\big(\rho_*(f)\cup_\otimes\beta(g)\big)\big]\\
&= \nu_*\big([\beta(f)]\cup_\otimes[\rho_*(g)]\big) \;+\; (-1)^k\cdot\nu_*\big([\rho_*(f)]\cup_\otimes[\beta(g)]\big)\\
&= \nu_*(\beta(x)\cup_\otimes y) \;+\; (-1)^k\cdot\nu_*(x\cup_\otimes\beta(y)).
\end{aligned}$$

Note that the third equality follows from the fact that $\beta(f)\in C^{k+1}(X;\mathbb{Z})$, $\rho_*(g)\in C^k(X;\mathbb{Q}/\mathbb{Z})$, $\rho_*(f)\in C^l(X;\mathbb{Q}/\mathbb{Z})$ and $\beta(g)\in C^{l+1}(X;\mathbb{Z})$ are cocycles. The lemma now follows immediately. □

We can now finally provide the proof of Lemma 2.4. More precisely, we have the following proposition.

Proposition 2.6. *Let M be a $(2n+1)$-dimensional rational homology sphere. If n is odd, then the linking form λ_M is symmetric, otherwise it is anti-symmetric.*

Proof. Given a and b in $H_n(M;\mathbb{Z})$, we set $x := \beta^{-1}(\mathrm{PD}_M^{\mathbb{Z}}(a))$ and $y := \beta^{-1}(\mathrm{PD}_M^{\mathbb{Z}}(b))$. Using Lemma 2.4 (2), the factorization described in (2.4) and Lemma 2.5, we obtain

$$\begin{aligned}\lambda_M(a,b) &= \left\langle(\beta^{-1}\circ \mathrm{PD}_M^{\mathbb{Z}})(a)\cup \mathrm{PD}_M^{\mathbb{Z}}(b),[M]\right\rangle \\ &= \langle \nu_*(x\cup_\otimes \beta(y)),[M]\rangle \;=\; (-1)^{n+1}\cdot\langle \nu_*(\beta(x)\cup_\otimes y),[M]\rangle.\end{aligned} \tag{2.7}$$

Since $n(n+1)$ is even it follows from (2.2) that

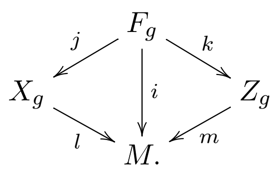
$$\lambda_M(a,b) = (-1)^{n+1}\cdot \langle \nu_*(y\cup_\otimes \beta(x)),[M]\rangle.$$

Proceeding as in (2.7), this is nothing but $\lambda_M(b,a)$, which concludes the proof of the proposition. □

3. Proof of Theorem 1.3

Our proof of Theorem 1.3 decomposes into two main steps. First, we provide a convenient presentation of $H_1(M;\mathbb{Z})$, then we compute the linking form. We recall some of the notation from the introduction and we add a few more definitions which shall be used throughout this chapter.

(1) We denote by X_g a fixed handlebody of genus g and we equip it with an orientation. We denote by Z_g a copy of X_g.
(2) We write $F_g = \partial X_g = \partial Z_g$.
(3) We denote by $a_1,\dots,a_g,b_1,\dots,b_g \in H_1(F_g;\mathbb{Z})$ a symplectic basis for $H_1(F_g;\mathbb{Z})$ such that $a_1,\dots,a_g$ form a basis for $H_1(X_g;\mathbb{Z})$. In particular the intersection numbers are given by $a_i\cdot b_j = \delta_{ij}, b_i\cdot a_j = -\delta_{ij}, a_i\cdot a_j = 0$ and $b_i\cdot b_j = 0$ for $i = 1,\dots,g$. Note that this implies that $b_1,\dots,b_g$ represent the zero element in $H_1(X_g;\mathbb{Z})$. By a slight abuse of notation we also denote by $a_i \in H_1(X_g;\mathbb{Z})$ the image of a_i under the inclusion induced map $H_1(F_g;\mathbb{Z}) \to H_1(X_g;\mathbb{Z})$.
(4) Sometimes we will use the bases of (3) to make the identifications $H_1(F_g;\mathbb{Z}) = \mathbb{Z}^{2g}$ and $H_1(X_g;\mathbb{Z}) = \mathbb{Z}^g$. Furthermore, since Z_g is a copy of X_g we can use the same basis as for $H_1(X_g;\mathbb{Z})$ to make the identification $H_1(Z_g;\mathbb{Z}) = \mathbb{Z}^g$.
(5) Given a self-diffeomorphism φ of the genus g surface F_g we write $M(\varphi) := X_g \cup_\varphi Z_g$ where we identify $x \in F_g = \partial X_g$ with $\varphi(x) \in \partial Z_g$. Furthermore we denote by
$$\begin{pmatrix} A_\varphi & B_\varphi \\ C_\varphi & D_\varphi\end{pmatrix}$$
the matrix that represents $\varphi_*\colon H_1(F_g;\mathbb{Z}) \to H_1(F_g;\mathbb{Z})$ with respect to the ordered basis $a_1,\dots,a_g,b_1,\dots,b_g$. If φ is understood, then we drop it from the notation.
(6) The following diagram summarizes the various inclusion maps arising in the subsequent discussion:



(7) If g is understood, then we drop it from the notation.

3.1. **A presentation for $H_1(M;\mathbb{Z})$.** We start with an elementary lemma.

Lemma 3.1. *Let φ be a self-diffeomorphism of $F = F_g$. Then the following two statements hold:*

(1) *The matrix $(A\ B)$ defines an epimorphism,*
(2) $AB^T = BA^T$.

Proof. Since φ_* is a symplectic automorphism of $H_1(F;\mathbb{Z})$, it follows that the matrix $R := \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ preserves the symplectic matrix $J := \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}$. In other words $R^T JR = J$. From this latter equation, we deduce that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} D^T & -B^T \\ -C^T & A^T \end{pmatrix} = \begin{pmatrix} I_g & 0 \\ 0 & I_g \end{pmatrix}. \tag{3.1}$$

Consequently, we obtain $(A\ B)\begin{pmatrix} D^T \\ -C^T \end{pmatrix} = I_g$ and thus the matrix $(A\ B)$ defines an epimorphism, proving the first assertion; the second assertion also follows from (3.1) by considering the top-right term of (3.1). □

Using this lemma, we can now provide a presentation matrix for $H_1(M(\varphi);\mathbb{Z})$.

Proposition 3.2. *Let φ be a self-diffeomorphism of $F = F_g$. Then the following statements hold:*

(1) *The abelian group $H_1(M;\mathbb{Z})$ is generated by $i_*(a_1), \ldots, i_*(a_g)$ and with respect to this generating set, $-B^T$ is a presentation matrix. More precisely, the homomorphism $\mathbb{Z}^g \to H_1(M;\mathbb{Z})$ given by $e_r \mapsto i_*(a_r)$ is an epimorphism and its kernel is given by $-B^T\mathbb{Z}^g$.*
(2) *If $M = M(\varphi)$ is a 3-dimensional rational homology sphere, then $\det(B) \neq 0$, i.e. B is invertible over the rationals.*

Proof. Recall that we denote by $j\colon F \to X$ and $k\colon F \to Z$ the inclusion maps. Since all the spaces involved are connected, the Mayer-Vietoris sequence of $M = X \cup_F Z$ yields the exact sequence

$$H_1(F;\mathbb{Z}) \xrightarrow{\begin{pmatrix} j_* \\ -k_* \end{pmatrix}} \begin{matrix} H_1(X;\mathbb{Z}) \\ \oplus \\ H_1(Z;\mathbb{Z}) \end{matrix} \xrightarrow{l_* \oplus m_*} H_1(M;\mathbb{Z}) \to 0. \tag{3.2}$$

Recalling our choice of bases, we observe that j_* is represented by $(I_g\ 0)$ and that k_* is represented by $(A\ B)$. Using the first statement of Lemma 3.1, k_* is surjective and thus we see that the exact sequence displayed in (3.2) reduces to

$$\ker(k_*) \xrightarrow{j_*} H_1(X;\mathbb{Z}) \to H_1(M;\mathbb{Z}) \to 0.$$

Elementary linear algebra together with the second statement of Lemma 3.1 imply that $\ker(k_*) = \ker(A\ B)$ is equal to $\operatorname{im}(-B^T\ A)$. Since j_* is represented by the matrix $(I_g\ 0)$, we deduce that the restriction of j_* to $\ker(A\ B)$ is represented by $-B^T$, as desired. This concludes the proof of the first statement.

The second statement of the proposition is an immediate consequence of the first statement. □

3.2. **The computation of the linking form.** Recall that we denote by $i\colon F \to M$ the inclusion. The proof of Theorem 1.3 is based on the following observation. If we manage to find a map

$\theta\colon H_1(M;\mathbb{Z}) \to H_1(F;\mathbb{Q}/\mathbb{Z})$ which makes the diagram

$$\begin{array}{ccccc} H_1(M;\mathbb{Z}) & \xrightarrow{\mathrm{PD}^{\mathbb{Z}}_M} & H^2(M;\mathbb{Z}) & \xrightarrow{\beta^{-1}} & H^1(M;\mathbb{Q}/\mathbb{Z}) \\ \downarrow \theta & & & & \downarrow i^* \\ H_1(F;\mathbb{Q}/\mathbb{Z}) & & \xrightarrow{\mathrm{PD}^{\mathbb{Q}/\mathbb{Z}}_F} & & H^1(F;\mathbb{Q}/\mathbb{Z}) \end{array} \tag{3.3}$$

commute, then we can reduce the calculation of the Poincaré duality in the 3-manifold M to the much-better understood Poincaré duality of the surface F and it will be fairly easy to compute the linking form.

Indeed, assuming such a map θ exists, we claim that the computation of

$$\lambda_M \circ (i_* \times i_*)\colon H_1(F;\mathbb{Z}) \times H_1(F;\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$$

boils down to the computation of $\theta \circ i_*$. More precisely, for $v, w \in H_1(F;\mathbb{Z})$ we apply successively the definition of the linking form, the naturality of the evaluation map (which is a consequence of Lemma 2.1) and the commutativity of (3.3) to obtain that:

$$\begin{aligned} \lambda_M(i_*(v), i_*(w)) &= \big\langle (\beta^{-1} \circ \mathrm{PD}^{\mathbb{Z}}_M \circ i_*)(v), i_*(w) \big\rangle_M = \big\langle (i^* \circ \beta^{-1} \circ \mathrm{PD}^{\mathbb{Z}}_M \circ i_*)(v), w \big\rangle_F \\ &= \big\langle (\mathrm{PD}^{\mathbb{Q}/\mathbb{Z}}_F \circ \theta \circ i_*)(v), w \big\rangle_F \in \mathbb{Q}/\mathbb{Z}. \end{aligned} \tag{3.4}$$

Summarizing, the proof of Theorem 1.3 now decomposes into two steps: firstly, we define the map $\theta\colon H_1(M;\mathbb{Z}) \to H_1(F;\mathbb{Q}/\mathbb{Z})$ (and check that it makes (3.3) commute) and secondly, we compute $\theta \circ i_*$. To carry out the first step, define θ as the composition

$$H_1(M;\mathbb{Z}) \xrightarrow{\beta^{-1}} H_2(M;\mathbb{Q}/\mathbb{Z}) \xrightarrow{p_*} H_2(M,X;\mathbb{Q}/\mathbb{Z}) \xleftarrow{\cong} H_2(Z,F;\mathbb{Q}/\mathbb{Z}) \xrightarrow{\partial} H_1(F;\mathbb{Q}/\mathbb{Z}) \tag{3.5}$$

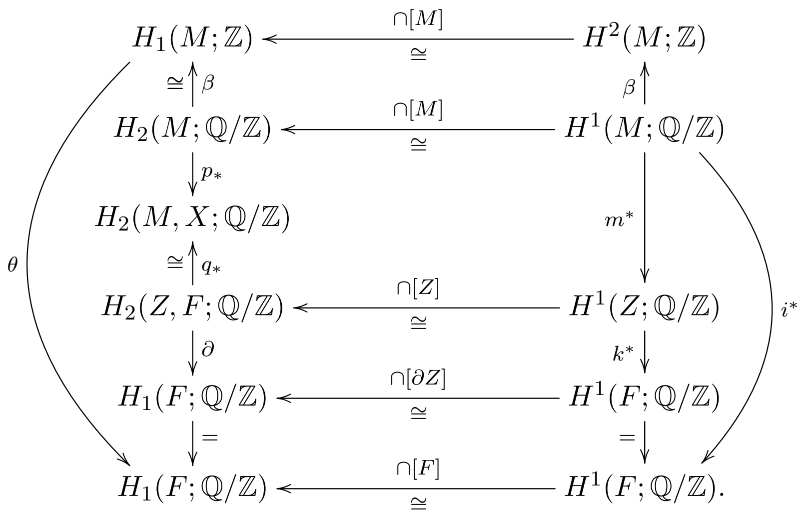
of the following maps: the inverse Bockstein homomorphism, the map induced by the obvious map $p\colon (M,\emptyset) \to (M,X)$, the inverse of the excision isomorphism and the connecting homomorphism of the long exact sequence of the pair (Z,F) with $\mathbb{Q}/\mathbb{Z}$-coefficients.

Lemma 3.3. *The homomorphism θ defined in (3.5) makes (3.3) commute. More precisely, we have*

$$\mathrm{PD}^{\mathbb{Q}/\mathbb{Z}}_F \circ \theta = i^* \circ \beta^{-1} \circ \mathrm{PD}^{\mathbb{Z}}_M\colon H_1(M;\mathbb{Z}) \to H^1(F;\mathbb{Q}/\mathbb{Z}).$$

Proof. We consider the maps of pairs $p\colon (M,\emptyset) \to (M,X)$ and $q\colon (Z,F) \to (M,X)$. We equip $Z \subset M$ with the orientation that turns the inclusion $Z \to M$ into an orientation-preserving map. Note that with this choice of orientation we have $p_*([M]) = q_*([Z])$ and that $[\partial Z] = -[\partial X] = -[F]$.

Recall that by definition, capping with the fundamental class is the inverse of the Poincaré duality isomorphism. Keeping this in mind, the lemma will be proved if we manage to show that the following diagram commutes:



Indeed, starting from the upper right corner and traveling to the lower left corner, the leftmost path produces the map $\theta \circ (\mathrm{PD}_M^{\mathbb{Z}})^{-1}$, while the rightmost path produces the map $(\mathrm{PD}_F^{\mathbb{Q}/\mathbb{Z}})^{-1} \circ i^* \circ \beta^{-1}$.

The top square commutes by Lemma 2.3, to be precise, it commutes since in our case we have $(-1)^2 = 1$. The third square anti-commutes by Proposition 2.2. The bottom square anti-commutes since the boundary orientation of Z is the opposite of the orientation of F (recall that we had equipped F with the orientation coming from X), in particular we have $[\partial Z] = -[F]$. Finally the second square (or first pentagon, depending on your point of view), commutes by applying the second statement of Lemma 2.1. More precisely, applying the first statement of Lemma 2.1 to the two maps $p\colon (M,\emptyset) \to (M,X)$ and $q\colon (Z,F) \to (M,X)$ and using that $p_*([M]) = q_*([Z])$ we obtain that for every φ in $H^1(M;\mathbb{Q}/\mathbb{Z})$, we have

$$p_*(\varphi\cap[M]) \;=\; p_*(p^*(\varphi)\cap[M]) \;=\; \varphi\cap p_*([M]) \;=\; \varphi\cap q_*([Z]) \;=\; q_*(q^*(\varphi)\cap[Z]) \;=\; q_*(m^*(\varphi)\cap[Z]).$$

□

In the remainder of this paper we use the following notation:

(1) We denote by i the inclusion map $F \to M$.
(2) We denote by $\rho\colon \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ the canonical projection.
(3) We denote by $\Phi_{\mathbb{Z}}\colon \mathbb{Z}^g \to H_1(F;\mathbb{Z})$ the map that is given by $\Phi_{\mathbb{Z}}(e_r) = a_r$, similarly we define $\Phi_{\mathbb{Q}}\colon \mathbb{Q}^g \to H_1(F;\mathbb{Q})$ and $\Phi_{\mathbb{Q}/\mathbb{Z}}\colon (\mathbb{Q}/\mathbb{Z})^g \to H_1(F;\mathbb{Q}/\mathbb{Z})$. We will use on several occasions that for $\mathbb{Z}^g \subset \mathbb{Q}^g$ the maps $\Phi_{\mathbb{Z}}$ and $\Phi_{\mathbb{Q}}$ agree.
(4) If in (3) we replace the a_r by b_r we obtain maps that we denote by $\Psi_{\mathbb{Z}}$, $\Psi_{\mathbb{Q}}$ and $\Psi_{\mathbb{Q}/\mathbb{Z}}$.

The next proposition deals with the computation of $\theta \circ i_*\colon H_1(F;\mathbb{Z}) \to H_1(F;\mathbb{Q}/\mathbb{Z})$ on the span of $a_1,\dots,a_g \in H_1(F;\mathbb{Z})$.

Proposition 3.4. *For any $v \in \mathbb{Z}^g$ the following equality holds:*

$$(\theta \circ i_*)(\Phi_{\mathbb{Z}}(v)) \;=\; -\Psi_{\mathbb{Q}/\mathbb{Z}}(B^{-1}Av) \;\in\; H_1(F;\mathbb{Q}/\mathbb{Z}).$$

Proof. In this proof we will mostly drop all inclusion maps from the notation, especially if we work on the chain level. We denote by $\widetilde{a}_1,\dots,\widetilde{a}_g$, $\widetilde{b}_1,\dots,\widetilde{b}_g$ singular chains in F that represent $a_1,\dots,a_g,b_1,\dots,b_g$. Let $v = (v_1,\dots,v_g) \in \mathbb{Z}^g$. We denote by $\widetilde{\Phi}_{\mathbb{Z}}\colon \mathbb{Z}^g \to C_1(F;\mathbb{Z})$ the map that is given by $\widetilde{\Phi}_{\mathbb{Z}}(e_r) = \widetilde{a}_r$ and we denote by $\widetilde{\Psi}_{\mathbb{Z}}\colon \mathbb{Z}^g \to C_1(F;\mathbb{Z})$ the map that is given by $\widetilde{\Psi}_{\mathbb{Z}}(e_r) = \widetilde{b}_r$. We make the obvious adjustments in the notation when we use other coefficients.

Claim.

(1) There exists $x \in C_2(X;\mathbb{Q})$ with $\partial_{\mathbb{Q}}(x) = \widetilde{\Psi}_{\mathbb{Q}}(B^{-1}Av) \in C_1(F;\mathbb{Q}) \subset C_1(X;\mathbb{Q})$.
(2) There exists $z \in C_2(Z;\mathbb{Q})$ with $\partial_{\mathbb{Q}}(z) = \widetilde{\Phi}_{\mathbb{Z}}(v) - \widetilde{\Psi}_{\mathbb{Q}}(B^{-1}Av) \in C_1(F;\mathbb{Q}) \subset C_1(Z;\mathbb{Q})$.

Note that $\widetilde{\Psi}_{\mathbb{Q}}(B^{-1}Av)$ is a rational linear combination of $\widetilde{b}_1,\dots,\widetilde{b}_g$. Since each $\widetilde{b}_r$ is null-homologous in X we see that $\widetilde{\Psi}_{\mathbb{Q}}(B^{-1}Av)$ is null-homologous in $C_*(X;\mathbb{Q})$. This shows that there exists a singular 2-chain $x \in C_2(X;\mathbb{Q})$ with $\partial_{\mathbb{Q}}(x) = \widetilde{\Psi}_{\mathbb{Q}}(B^{-1}Av)$.

We make the usual identification $H_1(Z;\mathbb{Z}) = \mathbb{Z}^g$ and $H_1(Z;\mathbb{Q}) = \mathbb{Q}^g$ coming from the fact that Z is a copy of X. Under this identification the map $k_* \circ \Phi_{\mathbb{Q}}\colon \mathbb{Q}^g \to H_1(Z;\mathbb{Q}) = \mathbb{Q}^g$ is by definition given by the matrix A and the map $k_* \circ \Psi_{\mathbb{Q}}\colon \mathbb{Q}^g \to H_1(Z;\mathbb{Q}) = \mathbb{Q}^g$ is by definition given by the matrix B. Putting these two observations together we see that in $H_1(Z;\mathbb{Q}) = \mathbb{Q}^g$ we have the equality:

$$k_*\big(\Phi_{\mathbb{Q}}(v) - \Psi_{\mathbb{Q}}(B^{-1}Av)\big) \;=\; (k_* \circ \Phi_{\mathbb{Q}})(v) - (k_* \circ \Psi_{\mathbb{Q}})(B^{-1}Av) \;=\; Av - BB^{-1}Av \;=\; 0.$$

Put differently, the singular 1-chain $\widetilde{\Phi}_{\mathbb{Q}}(v) - \widetilde{\Psi}_{\mathbb{Q}}(B^{-1}Av) = \widetilde{\Phi}_{\mathbb{Z}}(v) - \widetilde{\Psi}_{\mathbb{Q}}(B^{-1}Av)$ is null-homologous in $C_*(Z;\mathbb{Q})$, i.e. there exists a singular 2-chain $z \in C_2(Z;\mathbb{Q})$ with $\partial_{\mathbb{Q}}(z) = \widetilde{\Psi}_{\mathbb{Z}}(v) - \widetilde{\Psi}_{\mathbb{Q}}(B^{-1}Av)$. This concludes the proof of the claim.

From the definition of the Bockstein homomorphism $\beta\colon H_2(M;\mathbb{Q}/\mathbb{Z}) \to H_1(M;\mathbb{Z})$ as a connecting homomorphism and the above properties of x and z, it follows immediately that

$$\beta([\rho_*(z+x)]) \;=\; i_*(\Phi_{\mathbb{Z}}(v)) \;\in\; H_1(M;\mathbb{Q}/\mathbb{Z}).$$

To conclude the proof of the lemma, recall that the map θ is defined as the composition

$$H_1(M;\mathbb{Z}) \xrightarrow{\beta^{-1}} H_2(M;\mathbb{Q}/\mathbb{Z}) \xrightarrow{p_*} H_2(M,X;\mathbb{Q}/\mathbb{Z}) \xleftarrow{\cong} H_2(Z,F;\mathbb{Q}/\mathbb{Z}) \xrightarrow{\partial_{\mathbb{Q}/\mathbb{Z}}} H_1(F;\mathbb{Q}/\mathbb{Z}).$$

Using the definition of the relative homology group $H_2(M,X;\mathbb{Q}/\mathbb{Z})$ and the previous computation, it follows that

$$(p_* \circ \beta^{-1} \circ i_*(\Phi_{\mathbb{Z}}(v))) \;=\; p_*([\rho_*(z+x)]) \;=\; \rho_*([z]).$$

Since z is already a singular chain in $C_2(Z,F;\mathbb{Q}/Z)$ it suffices to prove the following claim.

Claim. We have $\partial_{\mathbb{Q}/\mathbb{Z}}(\rho_*([z])) = -\Psi_{\mathbb{Q}/\mathbb{Z}}(B^{-1}Av) \in H_1(F;\mathbb{Q}/\mathbb{Z})$.

By the choice of z we have $\partial_{\mathbb{Q}}(z) = \widetilde{\Phi}_{\mathbb{Z}}(v) - \widetilde{\Psi}_{\mathbb{Q}}(B^{-1}Av) \in C_1(F;\mathbb{Q})$. This implies that $\partial_{\mathbb{Q}/\mathbb{Z}}(\rho_*([z])) = [\rho_*(\widetilde{\Phi}_{\mathbb{Z}}(v)) - \rho_*(\widetilde{\Psi}_{\mathbb{Q}}(B^{-1}Av))] \in H_1(F;\mathbb{Q}/\mathbb{Z})$. But $\widetilde{\Phi}_{\mathbb{Z}}(v)$ is an integral class, so we have $\partial_{\mathbb{Q}/\mathbb{Z}}(\rho_*([z])) = -\Psi_{\mathbb{Q}/\mathbb{Z}}(B^{-1}Av) \in H_1(F;\mathbb{Q}/\mathbb{Z})$. This concludes the proof of the proposition. □

We can now provide the proof of Theorem 1.3. In fact we will prove the following slightly more precise statement:

Theorem 3.5. *Let $g \in \mathbb{N}$ and let $\varphi\colon F_g \to F_g$ be an orientation-preserving diffeomorphism. Suppose that $M(\varphi)$ is a rational homology sphere. Then the following statements hold:*

(1) *The above homomorphism $\Phi_{\mathbb{Z}}\colon \mathbb{Z}^g \to H_1(M(\varphi);\mathbb{Z})$ descends to an isomorphism*

$$\Phi\colon \mathbb{Z}^g/B_\varphi^T \xrightarrow{\cong} H_1(M(\varphi);\mathbb{Z}).$$

(2) *The matrix $B_\varphi \in M(g \times g,\mathbb{Z})$ has non-zero determinant.*

(3) *The isomorphism* Φ *from* (1) *defines an isometry from the form*

$$\begin{aligned} \mathbb{Z}^g/B_\varphi^T\mathbb{Z}^g \times \mathbb{Z}^g/B_\varphi^T\mathbb{Z}^g &\to \mathbb{Q}/\mathbb{Z} \\ (v,w) &\mapsto v^T(B_\varphi^{-1}A_\varphi)w \end{aligned}$$

to the linking form of $M(\varphi)$.

Proof. Note that statements (1) and (2) have already been proved in Proposition 3.2. Therefore it is enough to show that for all $v, w \in \mathbb{Z}^g$ we have

$$\lambda_M(i_*(\Phi_\mathbb{Z}(v)), i_*(\Phi_\mathbb{Z}(w))) \ = \ v^T(B^{-1}A)w \in \mathbb{Q}/\mathbb{Z}.$$

Combining (3.4) with Proposition 3.4, we obtain the equality

$$\begin{aligned} \lambda_M(i_*(\Phi_\mathbb{Z}(v)), i_*(\Phi_\mathbb{Z}(w))) &= \big\langle(\mathrm{PD}_F^{\mathbb{Q}/\mathbb{Z}} \circ \theta \circ i_*)(\Phi_\mathbb{Z}(v)), \Phi_\mathbb{Z}(w)\big\rangle_F \\ &= -\big\langle \mathrm{PD}_F^{\mathbb{Q}/\mathbb{Z}}(\Psi_{\mathbb{Q}/\mathbb{Z}}(B^{-1}Av)), \Phi_\mathbb{Z}(w)\big\rangle_F. \end{aligned} \tag{3.6}$$

The commutativity of the diagram

$$\begin{array}{ccccc} H_1(F;\mathbb{Q}) & \xrightarrow{\mathrm{PD}_F^\mathbb{Q}} & H^1(F;\mathbb{Q}) & \xrightarrow{\mathrm{ev}} & \mathrm{Hom}(H_1(F;\mathbb{Q}),\mathbb{Q}) \\ \downarrow \rho_* & & \downarrow \rho_* & & \downarrow \rho_* \\ H_1(F;\mathbb{Q}/\mathbb{Z}) & \xrightarrow{\mathrm{PD}_F^{\mathbb{Q}/\mathbb{Z}}} & H^1(F;\mathbb{Q}/\mathbb{Z}) & \xrightarrow{\mathrm{ev}} & \mathrm{Hom}(H_1(F;\mathbb{Q}),\mathbb{Q}/\mathbb{Z}) \end{array}$$

now implies that

$$\big\langle \mathrm{PD}_F^{\mathbb{Q}/\mathbb{Z}}(\Psi_{\mathbb{Q}/\mathbb{Z}}(B^{-1}Av)), \Phi_\mathbb{Z}(w)\big\rangle_F \ = \ \rho_*\big(\big\langle \mathrm{PD}_F^\mathbb{Q}(\Psi_\mathbb{Q}(B^{-1}Av)), \Phi_\mathbb{Q}(w)\big\rangle_F\big). \tag{3.7}$$

By the calculation of the intersection form of the surface F given in (2.3) we have

$$\rho_*\big(\big\langle \mathrm{PD}_F^\mathbb{Q}(\Psi_\mathbb{Q}(B^{-1}Av)), \Phi_\mathbb{Q}(w)\big\rangle_F\big) \ = \ Q_F\big(\Psi_\mathbb{Q}(B^{-1}Av), \Phi_\mathbb{Q}(w)\big). \tag{3.8}$$

Finally we recall that the a_r and b_r form a symplectic basis for $H_1(F;\mathbb{Z})$, i.e. with respect to this basis the intersection form Q_F is represented by the matrix $\left(\begin{smallmatrix} 0 & I_g \\ -I_g & 0 \end{smallmatrix}\right)$. In our context this implies that

$$Q_F\big(\Psi_\mathbb{Q}(B^{-1}Av), \Phi_\mathbb{Q}(w)\big) \ = \ v^TB^{-1}Aw. \tag{3.9}$$

The desired statement now follows from the combination of (3.6), (3.7), (3.8) and (3.9). □

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