THE L^2–ALEXANDER TORSION OF 3-MANIFOLDS

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Abstract. We introduce L^2-Alexander torsions for 3-manifolds, which can be viewed as a generalization of the L^2-Alexander polynomial of Li–Zhang. We state the L^2-Alexander torsions for graph manifolds and we partially compute them for fibered manifolds. We furthermore show that given any irreducible 3-manifold there exists a coefficient system such that the corresponding L^2-torsion detects the Thurston norm.

1. Introduction

1.1. **The L^2-Alexander torsion.** An *admissible triple* (N, ϕ, γ) consists of a prime orientable compact 3–dimensional manifold N with empty or toroidal boundary, a class $\phi \in H^1(N; \mathbb{R}) = \mathrm{Hom}(\pi_1(N), \mathbb{R})$ and a homomorphism $\gamma\colon \pi_1(N) \to G$ such that $\phi\colon \pi_1(N) \to \mathbb{R}$ factors through γ. We say that an admissible triple (N, ϕ, γ) is *rational* if ϕ is a rational cohomology class.

Given an admissible triple (N, ϕ, γ) we use the L^2–torsion, see e.g. [Lü02] for details, to introduce in Section 4.2 the *L^2–Alexander torsion* $\tau^{(2)}(N, \phi, \gamma)$ which is a function

$$\tau^{(2)}(N, \phi, \gamma) : \mathbb{R}^+ \to [0, \infty).$$

We say that two functions $f, g\colon \mathbb{R}^+ \to [0, \infty)$ are *equivalent*, written as $f \doteq g$, if there exists an $r \in \mathbb{R}$, such that $f(t) = t^r g(t)$ for all $t \in \mathbb{R}^+$. The equivalence class of $\tau^{(2)}(N, \phi, \gamma)$ is a well–defined invariant of (N, ϕ, γ). If γ is the identity homomorphism, then we will drop it from the notation, i.e., we just write $\tau^{(2)}(N, \phi)$.

As we explain in more detail in [DFL14a], the *L^2–Alexander torsion* $\tau^{(2)}(N, \phi, \gamma)$ can be viewed as a 'twisted' invariant of the pair (N, ϕ), and in particular it can be viewed as a cousin to the twisted Alexander polynomial [Li01, FV10] and the higher-order Alexander polynomials [Co04, Ha05] of 3-manifolds.

Given any $\phi \in H^1(N; \mathbb{R})$, if we take $t = 1$ and $\gamma = \mathrm{id}$ we obtain the usual L^2-torsion of a 3-manifold. The third author and Schick [LS99] showed that it is completely determined by the sum of the volumes of the hyperbolic pieces in the JSJ-decomposition of the underlying 3-manifold. We refer to Theorem 8.1 for details.

1.2. **The degree of the L^2-Alexander torsion.** We are interested in the behavior of the L^2-Alexander torsion for the limits $t \to 0$ and $t \to \infty$. We say that a function

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f is *monomial in the limit* if there exist $d, D \in \mathbb{R}$ and non-zero real numbers c, C such that

$$\lim_{t\to 0} \frac{f(t)}{t^d} = c \text{ and } \lim_{t\to\infty} \frac{f(t)}{t^D} = C.$$

We refer to $\deg f(t) := D - d$ as the *degree* of f. Furthermore we say f is *monic* if $c = C = 1$.

Note that the notion of being monomial in the limit, being monic and the degree only depend on the equivalence class of the function.

1.3. **Calculations of the L^2-Alexander torsion.** In order to state our results on L^2-Alexander torsions for certain classes of 3-manifolds we need one more definition. Let N be a 3–manifold and let $\phi \in H^1(N;\mathbb{Z}) = \operatorname{Hom}(\pi_1(N), \mathbb{Z})$. The *Thurston norm* of ϕ is defined as

$$x_N(\phi) = \min\{\chi_-(\Sigma) \,|\, \Sigma \subset N \text{ properly embedded surface dual to } \phi\}.$$

Here, given a surface Σ with connected components $\Sigma_1 \cup \cdots \cup \Sigma_k$, we define $\chi_-(\Sigma) = \sum_{i=1}^k \max\{-\chi(\Sigma_i), 0\}$. Thurston [Th86] showed that x_N defines a (possibly degenerate) norm on $H^1(N;\mathbb{Z})$. It can be extended to a norm on $H^1(N;\mathbb{R})$ which we also denote by x_N.

In Section 8.3 we will prove the following theorem.

Theorem 1.1. *Let $N \neq S^1 \times D^2, S^1 \times S^2$ be a graph manifold. Then for any non-trivial $\phi \in H^1(N;\mathbb{Q})$ and any representative τ of $\tau^{(2)}(N,\phi)$ we have*

$$\tau(t) = \begin{cases} 1, & \text{if } t \leq 1 \\ t^{x_N(\phi)}, & \text{if } t \geq 1. \end{cases}$$

In particular $\tau^{(2)}(N,\phi)$ is monomial in the limit and it is monic of degree $x_N(\phi)$.

Let N be a 3–manifold and let $\phi \in H^1(N;\mathbb{Q}) = \operatorname{Hom}(\pi_1(N), \mathbb{Q})$ be non–trivial. We say that ϕ *is fibered* if there exists a fibration $p\colon N \to S^1$ and an $r \in \mathbb{Q}$ such that the induced map $p_*\colon \pi_1(N) \to \pi_1(S^1) = \mathbb{Z}$ coincides with $r \cdot \phi$. With this definition we can now formulate the following theorem which we will prove in Section 8.2.

Theorem 1.2. *Let (N, ϕ, γ) be a rational admissible triple with $N \neq S^1 \times D^2, S^1 \times S^2$ such that $\phi \in H^1(N;\mathbb{Q})$ is fibered. We denote by $T \geq 1$ the entropy of the monodromy (see Section 8.2). Then there exists a representative τ of $\tau^{(2)}(N,\phi,\gamma)$ such that*

$$\tau(t) = \begin{cases} 1, & \text{if } t \leq \frac{1}{T}, \\ t^{x_N(\phi)}, & \text{if } t \geq T. \end{cases}$$

In particular $\tau^{(2)}(N,\phi,\gamma)$ is monomial in the limit and it is monic of degree $x_N(\phi)$.

1.4. **The symmetry of L^2-Alexander torsions.** For completeness we recall the main result of [DFL14b]. In that paper we showed that if (N,ϕ,γ) is an admissible triple, then for any representative τ of $\tau^{(2)}(N,\phi,\gamma)$ there exists a $k\in\mathbb{R}$ such that

$$\tau^{(2)}(t^{-1}) = t^k\cdot\tau^{(2)}(t). \tag{1}$$

This result thus says that the L^2-Alexander torsion is symmetric. This result can in particular be viewed as an analogue of the fact that the ordinary and the twisted Alexander polynomials of 3-manifolds are symmetric. (See [HSW10, FKK12]).

Now suppose that $\phi\in H^1(N;\mathbb{Z})$. Then there is an interesting relation to the Thurston norm $x_N(\phi)$. Namely, we can use the stronger equivalence relation that two functions $f,g\colon\mathbb{R}^+\to[0,\infty)$ are *equivalent*, if there exists an $m\in\mathbb{Z}$, such that $f(t)=t^m g(t)$ for all $t\in\mathbb{R}^+$, and still $\tau^{(2)}(N,\phi,\gamma)$ is a well-defined invariant. In [DFL14b] we prove that we have $k\in\mathbb{Z}$ in the equation (1), and the reduction of k mod 2 is an invariant of (N,ϕ,γ), which turns out to be $x_N(\phi)$ mod 2.

1.5. **The L^2-Alexander torsion of knot complements.** An important special case is given by knot exteriors. Let K be an oriented knot in S^3. We denote by νK an open tubular neighborhood of K and we refer to $X(K):=S^3\setminus\nu K$ as the *exterior of* K. Furthermore we denote by $\phi_K\in H^1(X(K);\mathbb{Z})=\mathrm{Hom}(\pi_1(X(K)),\mathbb{Z})$ the epimorphism which sends the oriented meridian to 1. If $\gamma\colon\pi_1(X(K))\to G$ is a homomorphism, such that $(X(K),\phi_K,\gamma)$ forms an admissible triple, then we write

$$\tau^{(2)}(K,\gamma):=\tau^{(2)}(X(K),\phi_K,\gamma)\colon\mathbb{R}^+\to[0,\infty).$$

It follows from (1) that $\tau^{(2)}(K,\gamma)$ does not depend on the orientation of K. Of particular interest is the invariant $\tau^{(2)}(K):=\tau^{(2)}(K,\mathrm{id})$. We will see in Section 7.2 that the resulting L^2-Alexander torsion is basically the same as the L^2–Alexander invariant introduced by Li and Zhang [LZ06a, LZ06b].

The calculations from the previous section also specialize to the case of knots. For example, Theorem 1.1 implies that for any (iterated) torus knot K we have

$$\tau^{(2)}(K)=(t\mapsto\max\{1,t^{2\ \mathrm{genus}(K)-1}\}),$$

where genus(K) denotes the minimal genus of a Seifert surface of K. This equality was first proved by Ben Aribi [BA13a, BA13b] and generalizes an earlier result of the first author and Wegner [DW10, DW13]. The combination of Theorem 1.1 together with the aforementioned work of the third author and Schick [LS99] gives us the following theorem which states that the L^2-Alexander torsion detects the unknot. This result was first proved by Ben Aribi [BA13a, BA13b].

Theorem 1.3. *A knot $K\subset S^3$ is trivial if and only if $\tau^{(2)}(K)\doteq(t\mapsto\max\{1,t\}^{-1})$.*

For knots it is enlightening to consider the coefficient system given by $\gamma = \phi_K \in H^1(X(K);\mathbb{Z}) = \operatorname{Hom}(\pi_1(X(K),\mathbb{Z})$. In order to state the result we factor the Alexander polynomial $\Delta_K(z) \in \mathbb{Z}[z^{\pm 1}]$ as

$$\Delta_K(z) = C \cdot z^m \cdot \prod_{i=1}^{k} (z - a_i),$$

with some $C \in \mathbb{Z} \setminus \{0\}, m \in \mathbb{Z}$ and $a_1, \dots, a_k \in \mathbb{C} \setminus \{0\}$. In Section 7.3 we then prove that

$$\tau^{(2)}(K,\gamma)(t) \doteq C \cdot \prod_{i=1}^{k} \max\{|a_i|, t\} \cdot \max\{t, 1\}^{-1}.$$

In particular $\tau^{(2)}(K,\gamma)$ is a piecewise monomial function that is determined by the ordinary Alexander polynomial.

1.6. **The L^2-Alexander torsion and the Thurston norm.** In this final section we want to relate the L^2-Alexander torsion to the Thurston norm for more general types of 3-manifolds. In this context we can not show that L^2-Alexander torsions are monomial in the limit. In Section 6 we will therefore generalize the notion of degree from functions that are monomial in the limit to more general types of functions.

With that definition of a degree we can show that L^2-Alexander torsions corresponding to certain epimorphisms γ give lower bounds on the Thurston norm.

Theorem 1.4. *Let $(N, \phi, \gamma\colon \pi_1(N) \to G)$ be an admissible triple. If G is virtually abelian, i.e., if G admits a finite index subgroup that is abelian, then*

$$\deg \tau^{(2)}(N, \phi, \gamma) \leq x_N(\phi).$$

Using the virtual fibering theorem of Agol [Ag08, Ag13], Liu [Liu13], Przytycki–Wise [PW12, PW14] and Wise [Wi12a, Wi12b] we will prove that there exists a homomorphism γ onto a virtually abelian group such that the L^2-Alexander torsion in fact determines the Thurston norm. More precisely, we have the following theorem.

Theorem 1.5. *Let N be a prime 3-manifold with empty or toroidal boundary that is not a closed graph manifold. Then there exists an epimorphism $\gamma\colon \pi_1(N) \to G$ onto a virtually abelian group such that the projection map $\pi_1(N) \to H_1(N;\mathbb{Z})/torsion$ factors through γ and such that for any $\phi \in H^1(N;\mathbb{R})$ the function $\tau^{(2)}(N,\phi,\gamma)$ is monomial in the limit with*

$$\deg \tau^{(2)}(N, \phi, \gamma) = x_N(\phi).$$

Conventions. We assume that all groups are finitely generated and that all 3–manifolds are orientable, compact and connected, unless we explicitly say otherwise.

Given a ring R we will view all modules as left R-modules, unless we say explicitly otherwise. Furthermore, given a matrix $A \in M_{m,n}(R)$, by a slight abuse of notation, we denote by $A\colon R^m \to R^n$ the R-homomorphism of left R-modules obtained by right multiplication with A and thinking of elements in R^m as the only row in a $(1,m)$-matrix.

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2. Hilbert $\mathcal{N}(G)$-modules and the Fuglede–Kadison determinant

In this section we will recall the definition and some basic properties of Hilbert $\mathcal{N}(G)$-modules and the Fuglede–Kadison determinant.

At a first reading of the paper it is enough to know that given any group G and any matrix A over $\mathbb{R}[G]$ (which is not necessarily a square matrix) one can, under slight technical assumptions, associate to A its Fuglede–Kadison determinant $\det_{\mathcal{N}}(G)(A) \in \mathbb{R}^+$. Some of the key properties of the Fuglede–Kadison determinant are summarized in Proposition 2.1.

2.1. **The dimension of Hilbert $\mathcal{N}(G)$–modules.** Let G be a group. We denote by $\mathcal{N}(G)$ the algebra of G–equivariant bounded linear operators from $l^2(G)$ to $l^2(G)$. Following [Lü02, Definition 1.5] we define a *Hilbert $\mathcal{N}(G)$-module* to be a Hilbert space V together with a linear isometric left G-action such that there exists a Hilbert space H and an isometric linear G-embedding of V into the tensor product of Hilbert spaces $H \otimes l^2(G)$ with the G-action given by the G-action on the second factor. A *map of Hilbert $\mathcal{N}(G)$-modules* $f\colon V \to W$ is a bounded G-equivariant operator.

For example, the Hilbert space $l^2(G)^m$ with the obvious left G–action is a Hilbert $\mathcal{N}(G)$-module with $H = \mathbb{R}^m$. In the following we will view elements of $l^2(G)^m$ as row vectors with entries in $l^2(G)$. In particular, if A is an $m \times n$-matrix over $\mathbb{R}[G]$, then A acts by right multiplication on $l^2(G)^m$. Here, as indicated already in the conventions, we view elements in $l^2(G)^m$ as row vectors. The matrix A thus defines a map $l^2(G)^m \to l^2(G)^n$. This map is in fact a map of Hilbert $\mathcal{N}(G)$-modules.

Let V be a Hilbert $\mathcal{N}(G)$-module. One can associate to V the *von Neumann dimension* $\dim_{\mathcal{N}(G)}(V) \in [0,\infty]$. We will not recall the definition, instead we refer to [Lü02, Definition 1.10] for details. We only note that the von Neumann dimension has many of the usual properties of dimensions. For example, if V is a Hilbert $\mathcal{N}(G)$-module, then $\dim_{\mathcal{N}(G)}(V) = 0$ if and only if $V = 0$. We refer to [Lü02, Theorem 1.12] and [Lü02, Theorem 6.29] for many more properties.

2.2. **Definition of the Fuglede–Kadison determinant.** Let G be a group and let A be an $m \times n$-matrix over the group ring $\mathbb{R}[G]$. In this section we recall the definition of the Fuglede–Kadison determinant of A.

As we mentioned above A defines a map of Hilbert $\mathcal{N}(G)$-modules $l^2(G)^m \to l^2(G)^n$. We consider the *spectral density function of* A which is defined as

$$\begin{aligned} F_A : \mathbb{R} &\to [0,\infty) \\ \lambda &\mapsto \sup\left\{ \dim_{\mathcal{N}(G)}(L) \;\middle|\; \begin{array}{l} L \subset l^2(G)^m \text{ a Hilbert } \mathcal{N}(G)\text{-submodule of } l^2(G)^m \\ \text{such that } \|Ax\| \leq \lambda \cdot \|x\| \text{ for all } x \in L \end{array} \right\}. \end{aligned}$$

By [Lü02, Section 2] the function F is a monotone non–decreasing right–continuous function. Clearly $F_A(\lambda) = 0$ for $\lambda < 0$.

In the following let $F\colon \mathbb{R} \to [0,\infty)$ be a monotone non–decreasing, right–continuous bounded function. We then denote by dF the unique measure on the Borel σ–algebra on $\mathbb{R}$ which has the property that for a half open interval $(a,b]$ with $a < b$ we have

$$dF((a,b]) = F(b) - F(a).$$

Now we return to the $m \times n$-matrix A over the group ring $\mathbb{R}[G]$. (We could consider more generally matrices over the von Neumann algebra $\mathcal{N}(G)$, but we restrict ourselves to matrices over $\mathbb{R}[G]$.)

The *Fuglede–Kadison determinant* of A is defined as

$$\det_{\mathcal{N}(G)}(A) := \begin{cases} \exp\left(\int_{(0,\infty)} \ln(\lambda) dF_A\right), & \text{if } \int_{(0,\infty)} \ln(\lambda) dF_A > -\infty, \\ 0, & \text{if } \int_{(0,\infty)} \ln(\lambda) dF_A = -\infty. \end{cases}$$

We say A is of *determinant class* if $\int_{(0,\infty)} \ln(\lambda) dF > -\infty$. It follows immediately from [Lü02, Section 3.7] that this definition agrees with the definition given, in a more general setup, in [Lü02, Section 3.2].

2.3. **Properties of the Fuglede–Kadison determinant.** For future reference we recall in the following two propositions some of the main properties of the Fuglede–Kadison determinant. Both propositions follow easily from the definitions and from [Lü02, Theorem 3.14].

Proposition 2.1. *Let G be a group and let A be a matrix over $\mathbb{R}[G]$. Then the following hold.*

(1) *Swapping two columns or two rows of A does not change the Fuglede–Kadison determinant.*
(2) *Adding a column of zeros or a row of zeros does not change the Fuglede–Kadison determinant.*
(3) *Right multiplication of a column by $\pm g$ with $g \in G$ does not change the Fuglede–Kadison determinant.*
(4) *If G is a subgroup of a group H, then we can also view A as a matrix over $\mathbb{R}[H]$ and*
$$\det_{\mathcal{N}(H)}(A) = \det_{\mathcal{N}(G)}(A).$$
(5) *If A is a matrix over $\mathbb{R}$ such that the usual determinant* $\det(A)$ *is non-zero, then* $\det_{\mathcal{N}(G)}(A) = |\det(A)|$.

(6) *We denote by* $\overline{A}$ *the matrix which is obtained by applying the involution of* G, $g \mapsto g^{-1}$, *to each entry of* A. *Then*
$$\det_{\mathcal{N}(G)}\left(\overline{A}^t\right) = \det_{\mathcal{N}(G)}(A).$$

Let $\widehat{G} \subset G$ be a subgroup of index d and let $f\colon V \to W$ be a homomorphism between two based free left-$\mathbb{R}[G]$ modules. We pick representatives for $G/\widehat{G}$. Multiplying the basis elements with all the representatives turns V and W into based free left-$\mathbb{R}[\widehat{G}]$-modules. In particular, if A is a $k \times l$-matrix over $\mathbb{R}[G]$ then the above procedure turns A into $dk \times dl$-matrix which we denote by $\iota_G^{\widehat{G}}(A)$. The fact that there is some slight indeterminacy in the definition of $\iota_G^{\widehat{G}}(A)$ will not play a role.

Proposition 2.2. *Let* $\widehat{G} \subset G$ *be a subgroup of finite index and let* $f\colon V \to W$ *be a homomorphism between two based free left-*$\mathbb{R}[G]$ *modules. Then*
$$\det_{\mathcal{N}(\widehat{G})}(f) = \det_{\mathcal{N}(G)}(f)^{[G:\widehat{G}]}.$$
In particular f *is of determinant class viewed as a map of Hilbert-*$\mathcal{N}(\widehat{G})$*-modules if and only if it is of determinant class viewed as a map of Hilbert-*$\mathcal{N}(G)$*-modules. Equivalently, if* A *is a matrix over* $\mathbb{R}[G]$, *then* $\det_{\mathcal{N}(\widehat{G})}\left(\iota_G^{\widehat{G}}(A)\right) = \det_{\mathcal{N}(G)}(A)^{[G:\widehat{G}]}$.

2.4. **The rank of a square matrix.** Let G be a group and let A be a $k \times k$-matrix over $\mathbb{R}[G]$. We define the *rank of* A as follows:
$$\operatorname{rank}_G(A) := k - \dim_{\mathcal{N}(G)}\left(l^2(G)^k/\overline{l^2(G)^k A}\right).$$
Note that by [Lü02, Lemma 2.11(11)] we have $\operatorname{rank}_G(\overline{A}^t A) = \operatorname{rank}_G(A)$. We say that A has *full rank* if $\operatorname{rank}(A) = k$.

We have the following characterization of matrices of full rank which is an immediate consequence of [Lü02, Theorem 1.12 (1) and (2)].

Lemma 2.3. *Let* G *be a group and let* A *be a* $k \times k$*-matrix over* $\mathbb{R}[G]$. *Then* A *has full rank if and only if the map* $l^2(G)^k \to l^2(G)^k$ *given by right multiplication by* A *is injective.*

2.5. **Properties of the regular Fuglede–Kadison determinant.** Given a square matrix A over $\mathbb{R}[G]$ we define the *regular Fuglede–Kadison determinant* as
$$\det^{\mathrm{r}}_{\mathcal{N}(G)}(A) := \begin{cases} \det_{\mathcal{N}(G)}(A), & \text{if } G \text{ has full rank,} \\ 0, & \text{otherwise.} \end{cases}$$
The following proposition collects several key properties of the regular Fuglede–Kadison determinant. The proposition is again a straightforward consequence of the definitions and of [Lü02, Theorems 1.12 and 3.14].

Proposition 2.4. *Let* G *be a group and let* A *be a* $k \times k$*-matrix over* $\mathbb{R}[G]$. *Then the following hold.*

(1) *We have* $\det^{\mathrm{r}}_{\mathcal{N}(G)}(A) \neq 0$ *if and only if* A *is of determinant class and it has full rank.*
(2) *If we swap two columns or two rows of* A*, then the regular Fuglede–Kadison determinant stays unchanged.*
(3) *If we multiply a row or a column of* A *by* $\lambda \in \mathbb{R} \setminus \{0\}$*, then the regular Fuglede–Kadison determinant is multiplied by* $|\lambda|$.
(4) *Right multiplication of a column or left multiplication of a row by some* $g \in G$ *does not change the regular Fuglede–Kadison determinant.*
(5) *If* G *is a subgroup of a group* H*, then we can also view* A *also as a matrix over* $\mathbb{R}[H]$ *and*
$$\det^{\mathrm{r}}_{\mathcal{N}(H)}(A) = \det^{\mathrm{r}}_{\mathcal{N}(G)}(A).$$
(6) *If* A *is a matrix over* $\mathbb{R}$*, then* $\det^{\mathrm{r}}_{\mathcal{N}(G)}(A) = |\det(A)|$*, where* $\det(A) \in \mathbb{R}$ *denotes the usual determinant. In particular,* $\det^{\mathrm{r}}_{\mathcal{N}(G)}(\mathrm{id}) = 1$.
(7) *If* $\widehat{G}$ *is a finite index subgroup of* G*, then*
$$\det^{\mathrm{r}}_{\mathcal{N}(\widehat{G})}\left(\iota_G^{\widehat{G}}(A)\right) = \det^{\mathrm{r}}_{\mathcal{N}(G)}(A)^{[G:\widehat{G}]}.$$
(8) *If* A *is a square matrix over* $\mathbb{R}[G]$ *of the same size as* A*, then*
$$\det^{\mathrm{r}}_{\mathcal{N}(G)}(A \cdot B) = \det^{\mathrm{r}}_{\mathcal{N}(G)}(A) \cdot \det^{\mathrm{r}}_{\mathcal{N}(G)}(B).$$
(9) *If* $l \times l$*-matrix and* C *a* $l \times k$*-matrix, then*
$$\det^{\mathrm{r}}_{\mathcal{N}(G)}\begin{pmatrix} A & 0 \\ C & B \end{pmatrix} = \det^{\mathrm{r}}_{\mathcal{N}(G)}(A) \cdot \det^{\mathrm{r}}_{\mathcal{N}(G)}(B).$$

2.6. **The class $\mathcal{G}$.** In order to state the next theorem we need the notion of a 'sofic' group. In fact, we will not give the definition, but we note that by [ES06, Theorem 1] the following hold:

(1) the class of sofic groups contains the class of residually amenable groups,
(2) any subgroup and any finite index extension of a sofic group is again sofic.

It follows from (1) that the following classes of groups are sofic:

(1) residually finite groups,
(2) 3-manifold groups, since they are residually finite, see [Hem87], and
(3) virtually solvable groups.

Here recall that if $\mathcal{P}$ is a property of groups, then a group is said to be *virtually* $\mathcal{P}$ if the group admits a finite index normal subgroup that satisfies $\mathcal{P}$.

The following theorem was proved by Elek and Szabó [ES05]. (See also [Lü94, Sc01, Cl99] for special cases.)

Theorem 2.5. *Let* G *be a group that is sofic. Then the following hold.*

(1) *Any square matrix over* $\mathbb{Q}[G]$ *is of determinant class.*
(2) *If* A *is a square matrix over* $\mathbb{Z}[G]$*, then* $\det_{\mathcal{N}(G)}(A) \geq 1$.
(3) *If* A *is an invertible matrix over* $\mathbb{Z}[G]$*, then* $\det_{\mathcal{N}(G)}(A) = 1$.

Proof. Let G be a group that is sofic. By the main result of [ES05] any square matrix A over $\mathbb{Z}[G]$ is of determinant class with $\det_{\mathcal{N}(G)}(A) \geq 1$.

If A admits in fact an inverse matrix B over $\mathbb{Z}[G]$, then it follows from Proposition 2.4 that

$$\det_{\mathcal{N}(G)}(A)\cdot\det_{\mathcal{N}(G)}(B) = \det^r_{\mathcal{N}(G)}(A)\cdot\det^r_{\mathcal{N}(G)}(B) = \det^r_{\mathcal{N}(G)}(AB) = \det^r_{\mathcal{N}(G)}(\mathrm{id}) = 1.$$

By the above both $\det_{\mathcal{N}(G)}(A)$ and $\det_{\mathcal{N}(G)}(B)$ are at least one, it follows that $\det_{\mathcal{N}(G)}(A) = 1$.

Finally, if A is a square matrix over $\mathbb{Q}[G]$, then we can write $A = r\cdot B$ with $r \in \mathbb{Q}$ and B a matrix over $\mathbb{Z}[G]$. It follows immediately from the aforementioned result of [ES05] and from the definitions that A is also of determinant class. $\square$

Now we denote by $\mathcal{G}$ the class of all sofic groups G. To the best of our knowledge it is not known whether there exist finitely presented groups that are not sofic. Moreover, we do not know whether any matrix A over any real group ring is of determinant class.

2.7. **Fuglede–Kadison determinants and the Mahler measures.** In general it is very difficult to calculate the Fuglede–Kadison determinant of a matrix over a group ring $\mathbb{R}[G]$. In this section we will recall the well-known fact that if G is abelian, then the Fuglede–Kadison determinant can be expressed in terms of a Mahler measure, which makes the Fuglede-Kadison determinant much more accessible.

First, let $p \in \mathbb{R}[z_1^{\pm1}, \dots, z_k^{\pm1}]$ be a multivariable Laurent polynomial. If $p = 0$, then its Mahler measure is defined as $m(p) = 0$. Otherwise the *Mahler measure* of p is defined as

$$m(p) := \exp\left(\int_0^{2\pi} \dots \int_0^{2\pi} \ln\left|p\left(e^{it_1}, \dots, e^{it_k}\right)\right|\, dt_1 \dots dt_k\right).$$

Note that the Mahler measure is multiplicative, i.e., for any multivariable Laurent polynomials p, q we have $m(pq) = m(p) + m(q)$. If $p \in \mathbb{R}[z^{\pm1}]$ is a one-variable polynomial then we can write $p(z) = D\cdot z^n \cdot \prod_{i=1}^{l}(z - b_i)$, where $D \in \mathbb{R}, n \in \mathbb{Z}$ and $b_1, \dots, b_l \in \mathbb{C}$. It follows from Jensen's formula that

$$m(p) = |D|\cdot\prod_{i=1}^{l}\max\{1, |b_i|\}. \tag{2}$$

If H is a free abelian group of rank k and $p \in \mathbb{R}[H]$ is non-zero, then we pick an isomorphism $f\colon \mathbb{Z}^k \xrightarrow{\cong} H$ which induces an isomorphism $\mathbb{R}[\mathbb{Z}^k] = \mathbb{R}[z_1^{\pm1}, \dots, z_k^{\pm1}] \cong \mathbb{R}[H]$ and we define the *Mahler measure* of p as $m(p) := m(f_*^{-1}(p))$. Note that this is independent of the choice of f. Also note that if H is the trivial group and $p \in \mathbb{R}\setminus\{0\}$, then $m(p) = |p|$.

The following lemma relates the regular Fuglede-Kadison determinant to the Mahler measure.

Lemma 2.6. *Let H be a free abelian group and let A be a square matrix over $\mathbb{R}[H]$. Then*
$$\det^{\mathrm{r}}_{\mathcal{N}(H)}(A) = m(\det_{\mathbb{R}[H]}(A))$$
where $\det_{\mathbb{R}[H]}(A) \in \mathbb{R}[H]$ is the usual determinant of the matrix A.

Proof. Let H be a free abelian group and let A be a $k \times k$-matrix over $\mathbb{R}[H]$. It follows from Lemma 2.3 that A has full rank if and only if multiplication by A is an injective map on $\mathbb{R}[H]^k$. But the latter is of course equivalent to $\det(A) \in \mathbb{R}[H]$ being non-zero.

Now we suppose that $\det(A) \neq 0$. By the above we have $\det^{\mathrm{r}}_{\mathcal{N}(H)}(A) = \det_{\mathcal{N}(H)}(A)$, and the desired equality $\det_{\mathcal{N}(H)}(A) = m(\det(A))$ is proved in [Ra12, Section 1.2], building on [Lü02, Exercise 3.8]. □

Given a finite set S we denote by $\mathbb{R}[S]$ the $\mathbb{R}$-vector space spanned freely by the elements of S. Given $n \in \mathbb{N}$ we denote by $M(n \times n, \mathbb{R}[S])$ the set of all $n \times n$-matrices with entries in $\mathbb{R}[S]$. Note that $M(n \times n, \mathbb{R}[S])$ is a finite dimensional real vector space and we endow it with the usual topology. Now we have the following useful corollary to Lemma 2.6.

Corollary 2.7. *Let G be a group that is virtually abelian. Then for any finite subset S of G the function*
$$\det^{\mathrm{r}}_{\mathcal{N}(G)} \colon M(n \times n, \mathbb{R}[S]) \to [0, \infty)$$
is continuous.

Proof. Since G is virtually abelian there exists a finite index subgroup of G that is abelian. Since G is finitely generated, see the conventions we apply throughout, it follows that any finite index subgroup is also finitely generated. Since any finitely generated abelian subgroup admits a finite index torsion-free abelian subgroup there exists a finite index subgroup $\widehat{G}$ of G that is torsion-free abelian.

We pick representatives $g_1, \ldots, g_d$ for $G/\widehat{G}$. Given a matrix A over $\mathbb{R}[G]$ we define the matrix $\iota_G^{\widehat{G}}(A)$ over $\mathbb{R}[\widehat{G}]$ using this ordered set of representatives. It is straightforward to verify that there exists a finite subset $\widehat{S}$ of $\widehat{G}$ such that the map
$$\iota_G^{\widehat{G}} \colon M(n \times n, \mathbb{R}[G] \to M(dn \times dn, \mathbb{R}[\widehat{G}])$$
restricts to a map
$$\iota_G^{\widehat{G}} \colon M(n \times n, \mathbb{R}[S] \to M(dn \times dn, \mathbb{R}[\widehat{S}])$$
and that this map is continuous. By Proposition 2.4 it thus suffices to show that
$$\det^{\mathrm{r}}_{\mathcal{N}(\widehat{G})} \colon M(n \times n, \mathbb{R}[\widehat{S}]) \to [0, \infty)$$
is continuous. But the continuity of this function is a consequence of Lemma 2.6 and the continuity of the Mahler measure of multivariable polynomials of bounded degree, see [Bo98, p. 127]. □

Finally we conclude with the following lemma.

Lemma 2.8. *Let G be a group, $g \in G$ an element of infinite order and let $t \in \mathbb{R}^+$. Then*

$$\det^{\mathrm{r}}_{\mathcal{N}(G)}(1 - tg) = \max\{1, t\}.$$

Proof. By Proposition 2.4 (2.4) we have $\det^{\mathrm{r}}_{\mathcal{N}(G)}(1 - tg) = \det^{\mathrm{r}}_{\mathcal{N}(\langle g\rangle)}(1 - tg)$. By Lemma 2.6 we know that $\det^{\mathrm{r}}_{\mathcal{N}(\langle g\rangle)}(1 - tg)$ equals the Mahler measure of $1 - tg$, viewed as a polynomial in g. By (2) we have $m(1 - tg) = m((-t)(g - t^{-1})) = |-t| \cdot \max\{1, t^{-1}\} = \max\{1, t\}$. □

3. The L^2–torsion of complexes over group rings

We first recall some definitions (see [Lü02, Definitions 1.16 and 3.29]). Let G be a group and let

$$0 \longrightarrow C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \dots \longrightarrow C_1 \xrightarrow{\partial_1} C_0 \longrightarrow 0$$

be a complex of length n of finitely generated free based left $\mathbb{R}[G]$-modules. Here by 'based' we mean that all the C_i's are equipped with a basis as free left $\mathbb{R}[G]$-modules. Note that the basing turns each $\mathcal{N}(G) \otimes_{\mathbb{R}[G]} C_i$ naturally into an $\mathcal{N}(G)$-module and the resulting boundary maps $\mathrm{id} \otimes \partial_i \colon \mathcal{N}(G) \otimes_{\mathbb{R}[G]} C_i \to \mathcal{N}(G) \otimes_{\mathbb{R}[G]} C_{i-1}$ are maps of $\mathcal{N}(G)$-modules. Given $i \in \{0, \dots, n\}$ we write

$$\begin{array}{rcl} Z_i(C_*) & := & \mathrm{Ker}\{\mathcal{N}(G) \otimes_{\mathbb{R}[G]} C_i \xrightarrow{\mathrm{id} \otimes \partial_i} \mathcal{N}(G) \otimes_{\mathbb{R}[G]} C_{i-1}\}, \\ B_i(C_*) & := & \mathrm{Im}\{\mathcal{N}(G) \otimes_{\mathbb{R}[G]} C_{i+1} \xrightarrow{\mathrm{id} \otimes \partial_{i+1}} \mathcal{N}(G) \otimes_{\mathbb{R}[G]} C_i\}, \\ H_i(C_*) & := & Z_i(C_*)/\overline{B_i(C_*)}, \end{array}$$

where $\overline{B_i(C_*)}$ denotes the closure of $B_i(C_*)$ in the Hilbert space $\mathcal{N}(G) \otimes_{\mathbb{R}[G]} C_i$. Furthermore we denote by

$$b_i^{(2)}(C_*) := \dim_{\mathcal{N}(G)} H_i(C_*)$$

the i-th L^2-Betti number of C_*. We say that the complex C_* is *weakly acyclic* if all its L^2-Betti numbers vanish.

If the complex is not weakly acyclic, or if at least one of the boundary maps is not of determinant class, then we define $\tau^{(2)}(C_*) := 0$. (Note that this convention differs from the one used in [Lü02].) If the complex is weakly acyclic and if the boundary maps are of determinant class then its L^2–torsion is defined as follows:

$$\tau^{(2)}(C_*) := \prod_{i=1}^{n} \det_{\mathcal{N}(G)}(\partial_i)^{(-1)^i} \in (0, \infty).$$

Note that we take the *multiplicative inverse of the exponential* of the L^2–torsion defined in the monograph [Lü02]. Our convention of using the multiplicative inverse follows the convention established in [Tu86, Tu01, Tu02a].

The following two lemmas are analogues to [Tu01, Theorem 2.2].

Lemma 3.1. *Let G be a group. Let*

$$0 \longrightarrow \mathbb{R}[G]^k \xrightarrow{B} \mathbb{R}[G]^{k+l} \xrightarrow{A} \mathbb{R}[G]^l \longrightarrow 0$$

be a complex. Let $L \subset \{1, \ldots, k+l\}$ be a subset of size l. We write

$$\begin{array}{rcl} A(L) & := & \textit{rows in } A \textit{ corresponding to } L, \\ B(L) & := & \textit{result of deleting the columns of } B \textit{ corresponding to } L. \end{array}$$

If $\det^{\mathrm{r}}_{\mathcal{N}(G)}(A(L)) \neq 0$, then

$$\tau^{(2)}(\textit{based complex}) = \det{}^{\mathrm{r}}_{\mathcal{N}(G)}(B(L)) \cdot \det{}^{\mathrm{r}}_{\mathcal{N}(G)}(A(L))^{-1}.$$

Proof. We obtain the following short exact sequence of $\mathbb{R}G$-chain complexes (written as columns) where i and p are the canonical inclusions and projections corresponding to L.

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{R}[G]^k \arrow[r, "\mathrm{id}"] \arrow[d, "B[L]"] & \mathbb{R}[G]^k \arrow[r] \arrow[d, "B"] & 0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{R}[G]^k \arrow[r, "i"] \arrow[d] & \mathbb{R}[G]^{k+l} \arrow[r, "p"] \arrow[d, "A"] & \mathbb{R}[G]^l \arrow[r] \arrow[d, "A[L|"] & 0 \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & \mathbb{R}[G]^l \arrow[r, "\mathrm{id}"] \arrow[d] & \mathbb{R}[G]^l \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

If we apply $L^2(G) \otimes_{\mathbb{R}[G]} -$, we obtain a short exact sequence of Hilbert $\mathcal{N}(G)$-chain complexes. Now the claim follows by a direct application of the weakly exact long L^2-homology sequences and the sum formula for L^2-torsion to it, see [Lü02, Theorem 1.21 on page 27 and Theorem 3.35 on page 142]. □

The next lemma is proved analogously, we leave the details to the reader.

Lemma 3.2. *Let G be a group. Let*

$$0 \longrightarrow \mathcal{N}(G)^j \xrightarrow{C} \mathcal{N}(G)^k \xrightarrow{B} \mathcal{N}(G)^{k+l-j} \xrightarrow{A} \mathcal{N}(G)^l \longrightarrow 0$$

be a complex. Let $L \subset \{1,\dots,k+l-j\}$ be a subset of size l and $J \subset \{1,\dots,k\}$ a subset of size j. We write

$$\begin{array}{rcl} A(J) & := & \textit{rows in } A \textit{ corresponding to } J,\\ B(J,L) & := & \textit{result of deleting the columns of } B \textit{ corresponding to } J\\ & & \textit{and deleting the rows corresponding to } L\\ C(L) & := & \textit{columns of } C \textit{ corresponding to } L. \end{array}$$

If $\det^{\mathrm{r}}_{\mathcal{N}(G)}(A(J)) \neq 0$ and $\det^{\mathrm{r}}_{\mathcal{N}(G)}(C(L)) \neq 0$, then

$$\tau^{(2)}(\textit{based complex}) = \det\nolimits^{\mathrm{r}}_{\mathcal{N}(G)}(B(J,L)) \cdot \det\nolimits^{\mathrm{r}}_{\mathcal{N}(G)}(A(J))^{-1} \cdot \det\nolimits^{\mathrm{r}}_{\mathcal{N}(G)}(C(L))^{-1}.$$

3.1. **The L^2-torsion and the Mahler measures.** Given a free abelian group H we now denote by $\mathbb{R}(H)$ the quotient field of $\mathbb{R}[H]$. For $f = pq^{-1} \in \mathbb{R}(H)$ we define $m(f) := m(p)m(q)^{-1}$. Given a chain complex of based $\mathbb{R}[H]$-modules C_* we denote by $\tau(C_*) \in \mathbb{R}(H)$ the Reidemeister torsion of $\mathbb{R}(H) \otimes_{\mathbb{R}[H]} C_*$ as defined in [Tu01, Theorem 2.6 on page 11]. Note that by definition $\tau(C_*) = 0$ if and only if C_* is not acyclic.

We can now formulate the following useful proposition.

Proposition 3.3. *Let H be a free abelian group and let C_* be a chain complex of based $\mathbb{R}[H]$-modules. Then*

$$\tau^{(2)}(C_*) = m\big(\tau(\mathbb{R}(H) \otimes_{\mathbb{R}[H]} C_*)\big).$$

Proof. We observe from [Lü02, Lemma 1.34 on page 35] that C_* is L^2-acyclic if and only if $C_*^{(0)} := \mathbb{R}(H) \otimes_{\mathbb{R}[H]} C_*$ is acyclic. Put differently, $\tau^{(2)}(C_*) = 0$ if and only if $m(\tau(C_*^{(0)})) = 0$. Hence we can assume without loss of generality that both torsions are non-zero. This implies that $C_*^{(0)}$ is contractible as $\mathbb{R}(H)$-chain complex and we can choose a $\mathbb{R}(H)$-chain contraction. Then

$$\tau(C_*^{(0)}) = \det\nolimits_{\mathbb{R}(H)}((c+\gamma)_{\mathrm{ev}} \colon C_{\mathrm{ev}}^{(0)} \to C_{\mathrm{odd}}^{(0)}) \quad \in \mathbb{R}(H)^{\times}.$$

Clearing denominators we can find an element $x \in \mathbb{R}[H]$ and $\mathbb{R}[H]$-homomorphisms $\gamma_n' \colon C_n \to C_{n+1}$ such that over the quotient field $\mathbb{R}(H)$ the composite of $l_x \circ \gamma_n$ is γ_n', where l_x is left multiplication with x. We get

$$\tau(C_*^{(0)}) = \det\nolimits_{\mathbb{R}[H]}((l_x \circ c + \gamma')_{\mathrm{ev}} \colon C_{\mathrm{ev}} \to C_{\mathrm{odd}}) \cdot \det\nolimits_{\mathbb{R}[H]}(l_x \colon C_{\mathrm{odd}} \to C_{\mathrm{odd}})^{-1} \quad \in \mathbb{R}(H)^{\times}.$$

On the other hand, we conclude from [Lü02, Lemma 3.41 on page 146] applied to the weak chain contraction given by (γ', l_x) that

$$\tau^{(2)}(C_*) = \det\nolimits_{\mathcal{N}(H)}((l_x \circ c + \gamma')_{\mathrm{ev}} \colon C_{\mathrm{ev}} \to C_{\mathrm{odd}}) \cdot \det\nolimits_{\mathcal{N}(H)}(l_x \colon C_{\mathrm{odd}} \to C_{\mathrm{odd}})^{-1} \quad \in (0,\infty).$$

Now the claim follows from Lemma 2.6. ∎

4. Admissible triples and the L^2–Alexander torsion

4.1. Admissible triples. Let π be a group, $\phi \in \operatorname{Hom}(\pi, \mathbb{R})$ a non-trivial homomorphism and $\gamma\colon \pi \to G$ a homomorphism. We say that (π, ϕ, γ) form an *admissible triple* if $\phi\colon \pi \to \mathbb{R}$ factors through γ, i.e., if there exists a homomorphism $G \to \mathbb{R}$ such that the following diagram commutes:

$$\begin{tikzcd} \pi \arrow[r, "\gamma"] \arrow[dr, "\phi"'] & G \arrow[d] \\ & \mathbb{R}. \end{tikzcd}$$

Note that if $\gamma\colon \pi \to G$ is a homomorphism such that the projection map $\pi \to H_1(\pi;\mathbb{Z})/\text{torsion}$ factors through γ, then (π, ϕ, γ) is an admissible triple for any $\phi \in \operatorname{Hom}(\pi, \mathbb{R})$.

If N is a prime 3–manifold with empty or toroidal boundary, $\phi \in H^1(N;\mathbb{R}) = \operatorname{Hom}(\pi_1(N), \mathbb{R})$ and $\gamma\colon \pi_1(N) \to G$, then we say that (N, ϕ, γ) form an *admissible triple* if $(\pi_1(N), \phi, \gamma)$ form an admissible triple. Note that this is consistent with the definition given in the introduction.

Let $(\pi, \phi, \gamma\colon \pi \to G)$ be an admissible triple and let $t \in \mathbb{R}^+$. We consider the ring homomorphism

$$\begin{array}{rccc} \kappa(\phi, \gamma, t)\colon & \mathbb{Z}[\pi] & \to & \mathbb{R}[G] \\ & \sum\limits_{i=1}^{n} a_i w_i & \mapsto & \sum\limits_{i=1}^{n} a_i t^{\phi(w_i)} \gamma(w_i). \end{array}$$

Note that this ring homomorphism allows us to view $\mathbb{R}[G]$ and $\mathcal{N}(G)$ as $\mathbb{Z}[\pi]$-right modules via right multiplication. Given a matrix $A = (a_{ij})_{ij}$ over $\mathbb{Z}[\pi]$ we furthermore write

$$\kappa(\phi, \gamma, t)(A) := \big(\kappa(\phi, \gamma, t)(a_{ij})\big)_{ij}.$$

4.2. Definition of the L^2–Alexander torsion of CW–complexes and manifolds. Let X be a finite CW–complex. We write $\pi = \pi_1(X)$. Let $\phi \in H^1(X;\mathbb{R}) = \operatorname{Hom}(\pi, \mathbb{R})$ and let $\gamma\colon \pi \to G$ be a homomorphism to a group. Finally let $t \in \mathbb{R}^+$. Recall that $\kappa(\phi, \gamma, t) : \mathbb{Z}[\pi] \to \mathbb{R}[G]$ defines a right $\mathbb{Z}[\pi]$-module structure on $\mathbb{R}[G]$ and $\mathcal{N}(G)$. Now we denote by $\widetilde{X}$ the universal cover of X. The deck transformation induces a natural left $\mathbb{Z}[\pi]$–action on $C_*(\widetilde{X})$. We consider the chain complex $\mathbb{R}[G] \otimes_{\mathbb{Z}[\pi]} C_*(\widetilde{X})$ of left $\mathbb{R}[G]$-modules, where the $\mathbb{R}[G]$-action is given by left multiplication on $\mathbb{R}[G]$. Now we pick an ordering and and an orientation of the cells of X and we pick a lift of the cells of X to $\widetilde{X}$. Note that the chosen lifts, orderings and orientations of the cells endow each $\mathbb{R}[G] \otimes_{\mathbb{Z}[\pi]} C_i(\widetilde{X})$ with a basis as a free left $\mathbb{R}[G]$-module. We then denote by

$$\tau^{(2)}(X, \phi, \gamma, t) \in [0, \infty)$$

the corresponding torsion, as defined in Section 3. We thus obtain a function

$$\begin{array}{rccl} \tau^{(2)}(X,\phi,\gamma)\colon & \mathbb{R}^+ & \to & [0,\infty) \\ & t & \mapsto & \tau^{(2)}(X,\phi,\gamma,t) \end{array}$$

that we call the L^2*–Alexander torsion of* (X,ϕ,γ). It follows immediately from the definitions and Proposition 2.1 (2.1) and (2.1) that the function $\tau^{(2)}(X,\phi,\gamma)$ does not depend on the orderings and the orientations of the cells. On the other hand, using [Lü02, Theorem 3.35 (5)] one can easily show that a change of lifts changes the L^2-Alexander torsion function by multiplication by $t \mapsto t^r$ for some $r \in \mathbb{R}$. Put differently, the equivalence class of $\tau^{(2)}(X,\phi,\gamma)\colon \mathbb{R}^+ \to [0,\infty)$ is a well-defined invariant of (X,ϕ,γ).

Let (N,ϕ,γ) be an admissible triple where N is a compact smooth manifold. We pick a CW–structure X for N. We define

$$\tau^{(2)}(N,\phi,\gamma) := \tau^{(2)}(X,\phi,\gamma)\colon \mathbb{R}^+ \to [0,\infty).$$

Note that the equivalence class of $\tau^{(2)}(N,\phi,\gamma)$ is a well-defined invariant of (N,ϕ,γ). This is an immediate consequence of the above discussion, of [Ch74, Theorem 1], of a slight generalization of [Lü02, Theorem 3.96 (1)], and the observation that for a matrix A which represents the trivial element in $\mathrm{Wh}(G)$ we have $\det^{\mathrm{r}}_{\mathcal{N}(G)}(\kappa(\phi,g,t)) \doteq 1$. The last statement is a straightforward consequence of the definitions and of Proposition 2.4.

If $\gamma = \mathrm{id}\colon \pi_1(N) \to \pi_1(N)$ is the identity map, then we drop γ from the notation, i.e., we write $\tau^{(2)}(N,\phi) := \tau^{(2)}(N,\phi,\mathrm{id})$ and we refer to it as the *full* L^2*-Alexander torsion of* (N,ϕ).

Remark. In the above discussion we restricted ourselves to $t \in \mathbb{R}^+$. Verbatim the same discussion shows that we could also take $t \in \mathbb{C}\setminus\{0\}$, which then gives rise to a function $\tau^{(2)}(N,\phi,\gamma)\colon \mathbb{C}\setminus\{0\} \to [0,\infty)$. But it follows from the argument of [LZ06a, Theorem 7.1] that for any $t \in \mathbb{C}\setminus\{0\}$ we have $\tau^{(2)}(N,\phi,\gamma)(t) = \tau^{(2)}(N,\phi,\gamma)(|t|)$. We therefore do not loose any information by restricting ourselves to viewing $\tau^{(2)}(N,\phi,\gamma)$ as a function on $\mathbb{R}^+$.

5. Basic properties of the L^2–Alexander torsion

We recall that we say that two functions $f,g\colon \mathbb{R}^+ \to [0,\infty)$ are equivalent, written as $f \doteq g$, if there exists an $r \in \mathbb{R}$, such that $f(t) = t^r g(t)$ for all $t \in \mathbb{R}^+$. Note that if two functions are equivalent, then the evaluations at $t = 1$ agree.

The following lemma is an immediate consequence of the definitions and of Proposition 2.1 (2.1).

Lemma 5.1. *Let* $(N,\phi,\gamma\colon \pi \to G)$ *be an admissible triple and let* $\varphi\colon G \to H$ *be a monomorphism. Then*

$$\tau^{(2)}(N,\phi,\varphi\circ\gamma) \doteq \tau^{(2)}(N,\phi,\gamma).$$

The lemma in particular shows that for L^2-Alexander torsion we can restrict ourselves to γ being an epimorphism. The following lemma follows immediately from the definitions:

Lemma 5.2. *Let (N, ϕ, γ) be an admissible triple, and let $r \in \mathbb{R}$, then*
$$\tau^{(2)}(N, r\phi, \gamma)(t) \doteq \tau^{(2)}(N, \phi, \gamma)(t^r).$$

Lemma 5.3. *Let $(N, \phi, \gamma \colon \pi = \pi_1(N) \to G)$ be an admissible triple. Let $p \colon \widehat{N} \to N$ be a finite regular cover such that $\operatorname{Ker}(\gamma) \subset \widehat{\pi} := \pi_1(\widehat{N})$. We write $\widehat{\phi} := p^*\phi$ and we denote by $\widehat{\gamma}$ the restriction of γ to $\widehat{\pi}$. Then*
$$\tau^{(2)}(\widehat{N}, \widehat{\phi}, \widehat{\gamma})(t) \doteq \left(\tau^{(2)}(N, \phi, \gamma)(t)\right)^{[\widehat{N}:N]}.$$

Proof. We write $\pi = \pi_1(N)$ and $\widehat{\pi} := \pi_1(\widehat{N})$. We first note that by Lemma 5.1 we can and will assume that γ is surjective. Now we write $\widehat{G} := \operatorname{Im}(\widehat{\gamma})$.

We pick a CW–structure X for N and we denote by $\widehat{X}$ the cover of X corresponding to the finite cover $\widehat{N}$ of N. We furthermore denote by $\widetilde{X}$ the universal cover of X, which is of course also the universal cover of $\widehat{X}$.

We pick lifts of the cells of X to $\widetilde{X}$. These turn $\mathbb{R}[G] \otimes_{\mathbb{Z}[\pi]} C_*(\widetilde{X})$ into a chain complex of based free left $\mathbb{R}[G]$-modules. We also pick representatives for $\pi/\widehat{\pi}$. By taking all the translates of the above lifts of the cells by all the representatives we can view $\mathbb{R}[\widehat{G}] \otimes_{\mathbb{Z}[\widehat{\pi}]} C_i(\widetilde{X})$ as a chain complex of based left $\mathbb{R}[\widehat{G}]$-modules. For the remainder of this proof we view $\tau^{(2)}(N, \phi, \gamma)$ and $\tau^{(2)}(\widehat{N}, \widehat{\phi}, \widehat{\gamma})$ as defined using these bases.

Now we fix a $t \in \mathbb{R}^+$. Henceforth we view $\mathbb{R}[G]$ as a right $\mathbb{Z}[\pi]$-module via $\kappa(\phi, \gamma, t)$, and we view $\mathbb{R}[\widehat{G}]$ as a right $\mathbb{Z}[\widehat{\pi}]$-module via $\kappa(\widehat{\phi}, \widehat{\gamma}, t)$. For each i we then consider the map
$$\begin{array}{rcl} \mathbb{R}[\widehat{G}] \otimes_{\mathbb{Z}[\widehat{\pi}]} C_i(\widetilde{X}) & \to & \mathbb{R}[G] \otimes_{\mathbb{Z}[\pi]} C_i(\widetilde{X}) \\ \sum_i p_i \otimes \sigma_i & \mapsto & \sum_i p_i \otimes \sigma_i. \end{array}$$
It is straightforward to see that these maps are well-defined maps of left $\mathbb{R}[\widehat{G}]$-modules. Furthermore, it follows easily from the assumption $\operatorname{Ker}(\gamma) \subset \widehat{\pi} := \pi_1(\widehat{N})$ that these maps are in fact isomorphisms of left $\mathbb{R}[\widehat{G}]$-modules. Finally note that the maps are obviously chain maps.

It now follows from [Lü02, Theorem 1.35 (9)] that $\mathbb{R}[\widehat{G}] \otimes_{\mathbb{Z}[\widehat{\pi}]} C_i(\widetilde{X})$ is weakly acyclic if and only if $\mathbb{R}[G] \otimes_{\mathbb{Z}[\pi]} C_i(\widetilde{X})$ is weakly acyclic. We can thus restrict ourselves to the case that both are weakly acyclic. It then follows from Proposition 2.2 that
$$\begin{array}{rcl} \tau^{(2)}(\widehat{N}, \widehat{\phi}, \widehat{\gamma})(t) = \tau\left(\mathbb{R}[\widehat{G}] \otimes_{\mathbb{Z}[\widehat{\pi}]} C_i(\widetilde{X})\right)^{[\widehat{N}:N]} & = & \left(\tau\left(\mathbb{R}[G] \otimes_{\mathbb{Z}[\pi]} C_i(\widetilde{X})\right)\right)^{[\widehat{N}:N]} \\ & = & \left(\tau^{(2)}(N, \phi, \gamma)(t)\right)^{[\widehat{N}:N]}. \end{array}$$
□

It follows immediately from the definitions that if (N, ϕ, γ) is an admissible triple, then so is $(N, -\phi, \gamma)$ and $\tau^{(2)}(N, -\phi, \gamma)(t) \doteq \tau^{(2)}(N, \phi, \gamma)(t^{-1})$. In [DFL14b] we will prove the following theorem, which together with the above discussion implies that $\tau^{(2)}(N, -\phi, \gamma) \doteq \tau^{(2)}(N, \phi, \gamma)$.

Theorem 5.4. *Let (N, ϕ, γ) be an admissible triple and let τ be a representative of $\tau^{(2)}(N, \phi, \gamma)$. Then there exists an $r \in \mathbb{R}$ such that*

$$\tau(t^{-1}) = t^r \cdot \tau(t) \text{ for any } t \in \mathbb{R}_{>0}.$$

Furthermore, if $\phi \in H^1(N; \mathbb{Z})$, then there exists a representative τ of $\tau^{(2)}(N, \phi, \gamma)$ and an $n \in \mathbb{Z}$ with $n \equiv x_N(\phi)$ mod 2 such that

$$\tau(t^{-1}) = t^n \cdot \tau(t) \text{ for any } t \in \mathbb{R}_{>0}.$$

We conclude this section with a discussion of the L^2-Alexander torsions of 3-manifolds with a non-trivial JSJ decomposition.

Theorem 5.5. *Let N be an irreducible 3–manifold and $\phi \in H^1(N; \mathbb{R})$. We denote by $T_1, \dots, T_k$ the collection of JSJ tori and we denote by $N_1, \dots, N_l$ the JSJ pieces. Let $\gamma\colon \pi_1(N) \to G$ be a homomorphism such that the restriction to each JSJ torus has infinite image. For $i = 1, \dots, l$ we denote by $\phi_i \in H^1(N_i; \mathbb{R})$ and $\gamma_i\colon \pi_1(N_i) \to G$ the restriction of ϕ and γ to N_i. Then*

$$\tau^{(2)}(N, \phi, \gamma) \doteq \prod_{i=1}^{l} \tau^{(2)}(N_i, \phi_i, \gamma_i).$$

In the proof of Theorem 5.5 we will need the following lemma.

Lemma 5.6. *Let T be a torus and let $\phi \in H^1(T; \mathbb{R})$. Let $\gamma\colon \pi_1(T) \to G$ be a homomorphism with infinite image. Then*

$$\tau^{(2)}(T, \phi, \gamma) \doteq 1.$$

Proof. We first note that by Lemma 5.1 we can assume that γ is surjective. In particular this implies G is an infinite, finitely generated abelian group. Note that G contains a finite index subgroup which is free abelian. Since a finite cover of a torus is once again a torus we can by Lemma 5.3 assume, without loss of generality, that G is already free abelian.

Now we pick a CW-structure for T with one 0–cell p, two 1–cells x, y and one 2–cell i. We write $\pi = \pi_1(T)$, we denote $\widetilde{T}$ the universal cover of T and we denote by x and y the elements in π defined by the eponymous cells. Note that for appropriate lifts of the cells the based chain complex $C_*(\widetilde{T}; \mathbb{Z})$ is isomorphic to

$$0 \to \mathbb{Z}[\pi] \xrightarrow{\begin{pmatrix} 1-y & x-1 \end{pmatrix}} \mathbb{Z}[\pi]^2 \xrightarrow{\begin{pmatrix} 1-x \\ 1-y \end{pmatrix}} \mathbb{Z}[\pi] \to 0.$$

Let $t \in \mathbb{R}^+$. Now we tensor the above chain complex with $\mathbb{R}[G]$, viewed as a $\mathbb{Z}[\pi]$-module via the representation $\kappa(\phi, \gamma, t)$. We then obtain the chain complex

$$C_* \;:=\; 0 \to \mathbb{R}[G] \xrightarrow{\begin{pmatrix} 1 - t^{\phi(y)}\gamma(y) & t^{\phi(x)}\gamma(x) - 1 \end{pmatrix}} \mathbb{R}[G]^2 \xrightarrow{\begin{pmatrix} 1 - t^{\phi(x)}\gamma(x) \\ 1 - t^{\phi(y)}\gamma(y) \end{pmatrix}} \mathbb{R}[G] \to 0.$$

It follows from Proposition 3.3 that

$$\tau^{(2)}(T, \phi, \gamma) = m\big(\tau(\mathbb{R}(G) \otimes_{\mathbb{R}[H]} C_*)\big).$$

Note that $\gamma(x)$ or $\gamma(y)$ is a non-trivial element in the free-abelian group H. It is now straightforward to see that $\mathbb{R}(G) \otimes_{\mathbb{R}[H]} C_*$ is acyclic. Furthermore an elementary calculation, see e.g. [Tu01, Theorem 2.2], shows that $\tau(\mathbb{R}(G) \otimes_{\mathbb{R}[H]} C_*) = \pm 1$. It thus follows that $\tau^{(2)}(T, \phi, \gamma) \doteq 1$.

□

Now we can give the proof of Theorem 5.5.

Proof of Theorem 5.5. We can and will pick a CW-structure for N such that each JSJ torus and thus also each JSJ component corresponds to a subcomplex.

Let $t \in \mathbb{R}^+$. We view $\mathcal{N}(G)$ as a left $\mathbb{Z}[\pi_1(N)]$-module via $\kappa(\phi, \gamma, t)$ and as a module over each $\mathbb{Z}[\pi_1(T_i)]$ and each $\mathbb{Z}[\pi_1(N_i)]$ via restriction. Since the restriction of γ to each JSJ torus has infinite image it follows from Lemma 5.6 that each $\mathcal{N}(G) \otimes_{\mathbb{Z}[\pi_1(T_i)]} C_*(\widetilde{T_i})$ is weakly acyclic. It then follows from a Mayer–Vietoris argument that the chain complex $\mathcal{N}(G) \otimes_{\mathbb{Z}[\pi_1(N]} C_*(\widetilde{N})$ is weakly acyclic if and only if all of the $\mathcal{N}(G) \otimes_{\mathbb{Z}[\pi_1(N_i)]} C_*(\widetilde{N_i})$ are weakly acyclic. In particular the theorem holds if one of the chain complexes is not weakly acyclic.

Thus we can now assume that all of the above chain complexes are weakly acyclic. The theorem then follows from Lemma 5.6 and the multiplicativity of L^2-torsions for short exact sequences, see [Lü02, Theorem 3.35 (1)]. (Recall once again that the L^2-torsion in [Lü02] is minus the logarithm of our L^2-torsion.) □

6. The degree of functions

Let $f\colon \mathbb{R}^+ \to [0, \infty)$ be a function. If $f(t) = 0$ for arbitrarily small t, then we define the *degree of f at* 0 to be $\deg_0(f) := \infty$. Otherwise we define the *degree of f at* 0 to be

$$\deg_0(f) := \liminf_{t \to 0} \frac{\ln(f(t))}{\ln(t)} \in \mathbb{R} \cup \{-\infty\}.$$

Similarly, if $f(t) = 0$ for arbitrarily large t, then we define the *degree of f at* ∞ to be $-\infty$. Otherwise we define the *degree of f at* ∞ as

$$\deg_\infty(f) := \limsup_{t \to \infty} \frac{\ln(f(t))}{\ln(t)} \in \mathbb{R} \cup \{\infty\}.$$

We follow the usual convention of extending addition on $\mathbb{R}$ partly to $\mathbb{R}\cup\{-\infty\}\cup\{\infty\}$, i.e.

(1) for $a \in \mathbb{R}$ we define $a + \infty := a$ and $a + (-\infty) := -\infty$, and
(2) we define $\infty + \infty := \infty$ and $-\infty + (-\infty) := -\infty$.

As usual we also define $a - b := a + (-b)$. If $\deg_\infty(f) - \deg_0(f)$ is defined, then we define the *degree of f* as

$$\deg(f) := \deg_\infty(f) - \deg_0(f).$$

If $\deg_\infty(f) - \deg_0(f)$ is undefined, then we set $\deg(f) := -\infty$.

In the following we say that a function $f\colon \mathbb{R}^+ \to [0,\infty)$ is *piecewise monomial* if we can find $0 = t_0 < t_1 < t_2 \cdots < t_k < t_{k+1} := \infty$, $d_0, \dots, d_k \in \mathbb{Z}$ and furthermore non-zero real numbers $C_0, \dots, C_k$ such that

$$f(t) = C_i t^{d_i} \text{ for all } t \in [t_i, t_{i+1}) \cap \mathbb{R}^+.$$

We say that a function $f\colon \mathbb{R}^+ \to [0,\infty)$ is *eventually monomial* if there exist $0 = s < S < \infty$, $d, D \in \mathbb{R}$ and non-zero real numbers c, C such that

$$f(t) = ct^d \text{ for } t \in (0,s) \text{ and } f(t) = Ct^D \text{ for } t \in (S,\infty).$$

Finally we recall that a function $f\colon \mathbb{R}^+ \to [0,\infty)$ is *monomial in the limit* if there exist $d, D \in \mathbb{R}$ and non-zero real numbers c, C such that

$$\lim_{t\to 0} \frac{f(t)}{t^d} = c \text{ and } \lim_{t\to\infty} \frac{f(t)}{t^D} = C.$$

We summarize some properties of the degree function in the following lemma. We leave the elementary proof to the reader.

Lemma 6.1. *Let $f, g\colon \mathbb{R}^+ \to [0,\infty)$ be functions.*

(1) *If $f = 0$ is the zero function, then we have* $\deg_\infty(f) = -\infty$ *and* $\deg_0(f) = \infty$ *and thus* $\deg(f) = -\infty - \infty = -\infty$.
(2) *If f is monomial in the limit with d and D as in the definition, then* $\deg(f) = D - d$.
(3) *If $f = a_r t^r + a_{r+1} t^{r+1} + \cdots + a_s t^s$ is a polynomial with $a_r \neq 0$ and $a_s \neq 0$, then* $\deg(f) = s - r$.
(4) *If one of f or g is monomial in the limit, then* $\deg(f \cdot g) = \deg(f) + \deg(g)$.
(5) *If* $\deg(f) \in \mathbb{R}$, *then* $\deg(\frac{1}{f}) = -\deg(f)$.
(6) *If $f \doteq g$, then* $\deg(f) = \deg(g)$.
(7) *If $s \in [0,\infty)$ and if* $\deg(f) \in \mathbb{R}$, *then* $\deg(f^s) = s\deg(f)$.

7. The L^2–Alexander torsion for knots

Recall that given an oriented knot K we denote by νK an open tubular neighborhood of K and we write $X(K) = S^3 \setminus \nu K$ the exterior of K. Observe that $X(K)$ is a compact 3-manifold whose boundary consists in a single torus $\partial\nu K$. Furthermore we denote by $\phi_K \in H^1(X(K);\mathbb{Z}) = \operatorname{Hom}(\pi_1(X(K)),\mathbb{Z})$ the usual *abelianization* which is the epimorphism which sends the oriented meridian to 1. An *admissible homomorphism* is a homomorphism $\gamma\colon \pi_1(X(K)) \to G$ such that ϕ_K factors through γ. Note that if γ is admissible, then $(X(K), \phi_K, \gamma)$ is an admissible triple and we define

$$\tau^{(2)}(K,\gamma) := \tau^{(2)}(X(K), \phi_K, \gamma)\colon \mathbb{R}^+ \to [0,\infty).$$

If γ is the identity homomorphism, then we write $\tau^{(2)}(K) := \tau^{(2)}(K,\gamma)$ and we refer to $\tau^{(2)}(K)$ as the *full L^2-Alexander torsion of K*.

It follows from the symmetry of the L^2-Alexander torsion, see the discussion preceding Theorem 5.4, that these definitions do not depend on the orientation of K. We will henceforth only work with unoriented knots, and given a knot K we mean by ϕ_K either one of the two elements in $H^1(X(K);\mathbb{Z})$ that correspond to an epimorphism under the canonical isomorphism $H^1(X(K);\mathbb{Z}) = \operatorname{Hom}(\pi_1(X(K)),\mathbb{Z})$. Note that either ϕ_K sends the meridian of K to ± 1. It will not matter which of the two possible choices for ϕ_K we take.

In Section 7.1 we show how one can use Fox derivatives in the calculation of $\tau^{(2)}(K,\gamma)$. In Section 7.2 we will use this calculation to show that the full L^2-Alexander torsion of K is basically the same as the L^2-Alexander invariant of Li–Zhang. In Section 7.3 we will use $\gamma = \phi_K$ as the coefficient system and we will see that the resulting L^2-Alexander torsion is determined by the ordinary Alexander polynomial $\Delta_K(z) \in \mathbb{Z}[z^{\pm 1}]$ of K.

7.1. **Fox derivatives.** In the following we denote by F the free group with generators $g_1,\dots,g_k$. We then denote by $\frac{\partial}{\partial g_i}\colon \mathbb{Z}[F] \to \mathbb{Z}[F]$ the *Fox derivative with respect to g_i*, i.e., the unique $\mathbb{Z}$-linear map such that

$$\frac{\partial g_i}{\partial g_i} = 1, \quad \frac{\partial g_j}{\partial g_i} = 0 \text{ for } i \neq j \text{ and } \frac{\partial uv}{\partial g_i} = \frac{\partial u}{\partial g_i} + u\frac{\partial v}{\partial g_i} \text{ for all } u, v \in F.$$

We refer to [Fo53] for the basic properties of the Fox derivatives.

We can now formulate the following lemma which can be viewed as a slight generalization of [DW10, Theorem 3.2] and [DW13, Theorem 3.5].

Lemma 7.1. *Let K be a knot and let $\pi = \pi_1(X(K))$ denote its group. Consider an admissible homomorphism $\gamma\colon \pi \to G$ and let $\langle g_1,\dots,g_k \,|\, r_1,\dots,r_{k-1}\rangle$ be a deficiency one presentation for π. (We could for example take a Wirtinger presentation for π.) We denote by $B = (\frac{\partial r_j}{\partial g_i})$ the $(k-1)\times k$–matrix over $\mathbb{Z}[\pi]$ that is given by taking all Fox derivatives of all relations. We pick any $i \in \{1,\dots,k\}$ such that $\gamma(g_i)$ is an element of infinite order. We denote by B_i the result of deleting the i-th column of B. Then we have*

$$\tau^{(2)}(K,\gamma) \doteq \operatorname{det}^{\mathrm{r}}_{\mathcal{N}(G)}(\kappa(\phi_K,\gamma,t)(B_i)) \cdot \max\{1, t^{\phi_K(g_i)}\}^{-1}.$$

Proof. We write $\phi = \phi_K$. We denote by Y the 2–complex with one 0–cell, k 1–cells and $k-1$ 2–cells that corresponds to the given presentation. It is well-known, see e.g. [GKM05, Section 3] or [FJR11, p. 458], that Y is simple homotopy equivalent to

$X(K)$. The argument at the end of Section 4.2 shows that we can use Y to calculate $\tau^{(2)}(K)$.

It follows basically from the definition of the Fox derivatives that we can lift the cells of Y to the universal cover $\widetilde{Y}$ such that the chain complex $C_*(\widetilde{Y};\mathbb{Z})$ is isomorphic to

$$0 \longrightarrow \mathbb{Z}[\pi]^{k-1} \xrightarrow{B} \mathbb{Z}[\pi]^k \xrightarrow{\begin{pmatrix}1-g_1\\ \vdots \\ 1-g_k\end{pmatrix}} \mathbb{Z}[\pi] \longrightarrow 0.$$

Again we refer to [Fo53] for details. This implies that for any $t \in \mathbb{R}^+$ the chain complex $\mathbb{R}[G] \otimes_{\mathbb{Z}[\pi]} C_*(\widetilde{Y})$ is isomorphic to

$$0 \longrightarrow \mathbb{R}[G]^{k-1} \xrightarrow{\kappa(\phi,\gamma,t)(B)} \mathbb{R}[G]^k \xrightarrow{\begin{pmatrix}1-t^{\phi(g_1)}\gamma(g_1)\\ \vdots \\ 1-t^{\phi(g_k)}\gamma(g_k)\end{pmatrix}} \mathbb{R}[G] \longrightarrow 0.$$

Since $\gamma(g_i)$ it follows from Lemma 2.8 that $\det^{\mathrm{r}}_{\mathcal{N}(G)}\big(1-t^{\phi(g_i)}\gamma(g_i)\big) = \max\{1, t^{\phi(g_i)}\}$. The lemma now follows immediately from Lemma 3.1. □

7.2. **The L^2-Alexander invariant of Li–Zhang.** Let K be a knot. We pick a Wirtinger presentation $\langle g_1,\dots,g_k \,|\, r_1,\dots,r_{k-1}\rangle$ for $\pi = \pi_1(X(K))$. We denote by $B = (\frac{\partial r_j}{\partial g_i})$ the $(k-1)\times k$–matrix over $\mathbb{Z}[\pi]$ that is given by taking all Fox derivatives of all relations. We pick any $i \in \{1,\dots,k\}$ and we denote by B_i the result of deleting the i-th column of B. The *L^2–Alexander invariant $\Delta_K^{(2)}$ of K* is then defined as the function

$$\begin{array}{rcl}\Delta_K^{(2)}\colon \mathbb{C}\setminus\{0\} & \to & [0,\infty)\\ t & \mapsto & \det^{\mathrm{r}}_{\mathcal{N}(G)}\big(\kappa(\phi,\gamma,t)(B_i)\big).\end{array}$$

This invariant was first introduced by Li–Zhang [LZ06a, Section 7] and [LZ06b, Section 3], using slightly different conventions. In these papers it is also implicitly proved that the function $\Delta_K^{(2)}$, as an invariant of K, is well-defined up to multiplication by a function of the form $t \mapsto |t|^n$ with $n \in \mathbb{Z}$. Furthermore, Li–Zhang [LZ06a, LZ06b] implicitly, and Dubois–Wegner [DW13, Proposition 3.2] explicitly showed that for any $t \in \mathbb{C}\setminus\{0\}$ we have $\Delta_K^{(2)}(t) = \Delta_K^{(2)}(|t|)$. To be consistent with our other conventions we henceforth view $\Delta_K^{(2)}$ as a function defined on $\mathbb{R}^+$.

It now follows from Lemma 7.1 and the fact that every generator of a Wirtinger presentation is a meridian that

$$\tau^{(2)}(K) \doteq \Delta_K^{(2)} \cdot \max\{1,t\}^{-1}.$$

This shows that the full L^2-Alexander torsion and the L^2-Alexander invariant are essentially the same invariant.

7.3. **The L^2–Alexander torsion and the one-variable Alexander polynomial.** We will now see that given a knot K the ordinary Alexander polynomial $\Delta_K(z) \in \mathbb{Z}[z^{\pm 1}]$ determines the L^2-Alexander torsion corresponding to the abelianization. More precisely, we have the following proposition.

Proposition 7.2. *Let K be a knot and let $\Delta_K(z) \in \mathbb{Z}[z^{\pm 1}]$ be a representative of the Alexander polynomial of K. We write*

$$\Delta_K(z) = C \cdot z^m \cdot \prod_{i=1}^{k} (z - a_i),$$

where $C \in \mathbb{Z} \setminus \{0\}, m \in \mathbb{Z}$ and $a_1, \ldots, a_k \in \mathbb{C} \setminus \{0\}$. Then

$$\tau^{(2)}(K, \phi_K) \doteq C \cdot \prod_{i=1}^{k} \max\{|a_i|, t\} \cdot \max\{1, t\}^{-1}.$$

Note that the proposition can also be proved using Proposition 3.3 and the fact that the Reidemeister torsion of a knot corresponding to the abelianization equals $\Delta_K(z) \cdot (z-1)^{-1}$, see [Tu01] for details.

Proof. Let K be a knot. We write $\phi = \phi_K$. Let $\langle g_1, \ldots, g_k \,|\, r_1, \ldots, r_{k-1} \rangle$ be a Wirtinger presentation for $\pi = \pi_1(X(K))$. We again denote by $B = (\frac{\partial r_j}{\partial g_i})$ the $(k-1) \times k$–matrix over $\mathbb{Z}[\pi]$ that is given by taking all Fox derivatives of all relations. We pick any $i \in \{1, \ldots, k\}$ and we denote by B_i the result of deleting the i-th column of B.

We now apply the ring homomorphism $\phi \colon \mathbb{Z}[\pi] \to \mathbb{Z}[\langle z \rangle] = \mathbb{Z}[z^{\pm 1}]$ to all entries of B_i and we denote the resulting matrix by $A_i(z)$. Note that by [CF63, Chapter VIII.3] we have $\det(A_i(z)) = \Delta_K(z)$.

Given $t \in \mathbb{R}^+$ we denote by $A_i(tz)$ the matrix over $\mathbb{R}[z^{\pm 1}]$ that is given by substituting z by tz. Similarly we denote by $\Delta_K(tz)$ the polynomial over $\mathbb{R}[z^{\pm 1}]$ which is given by substituting z by tz. It follows immediately from the definitions that $\kappa(\phi, \phi, t)(B_i) = A_i(tz)$. Also note that $\det(A_i(tz)) = \Delta_K(tz)$.

By Lemmas 7.1 and by the discussion in Section 3.1 we have

$$\begin{array}{rcl}
\tau^{(2)}(K, \phi_K)(t) & \doteq & \det{}^{\mathrm{r}}_{\mathcal{N}(\langle z \rangle)}(A_i(tz)) \cdot \max\{1, t\}^{-1} \\
& = & m(\det(A_i(tz)) \cdot \max\{1, t\}^{-1} \\
& = & m(\Delta_K(tz)) \cdot \max\{1, t\}^{-1} \\
& = & m\left(C \cdot (zt)^m \prod_{i=1}^{k} (tz - a_i)\right) \cdot \max\{1, t\}^{-1} \\
& \doteq & m\left(C \cdot (zt)^m \prod_{i=1}^{k} (z - a_i t^{-1})\right) \cdot \max\{1, t\}^{-1} \\
& = & C \cdot \prod_{i=1}^{k} \max\{1, t^{-1} \cdot |a_i|\} \cdot \max\{1, t\}^{-1} \\
& \doteq & C \cdot \prod_{i=1}^{k} \max\{t, |a_i|\} \cdot \max\{1, t\}^{-1}.
\end{array}$$

□

We obtain the following corollary:

Corollary 7.3. *Given any knot K the L^2-Alexander torsion $\tau^{(2)}(K,\phi_K)$ is a piecewise monomial function with*

$$\deg \tau^{(2)}(K,\phi_K) = \deg \Delta_K(t).$$

Furthermore $\tau^{(2)}(K,\phi_K)$ is monic if and only if $\Delta_K(t)$ is monic.

Proof. Let $\Delta(z) = \Delta_K(z) \in \mathbb{Z}[z^{\pm 1}]$ be a representative of the Alexander polynomial of K. We write $\Delta(z) = C \cdot z^m \cdot \prod_{i=1}^k (z - a_i)$, where $C \in \mathbb{Z} \setminus \{0\}, m \in \mathbb{Z}$ and $a_1, \dots, a_k \in \mathbb{C}$ such that $|a_1| \le |a_2| \le \dots \le |a_k|$. By Proposition 7.2 we have

$$\tau^{(2)}(K,\phi_K) \doteq \theta(t) := C \cdot \prod_{i=1}^k \max\{|a_i|, t\} \cdot \max\{1,t\}^{-1}.$$

It follows immediately that $\tau^{(2)}(K,\phi_K)$ is a piecewise polynomial. Note that

$$\begin{array}{rcll} \theta(t) & = & C \cdot t^k, & \text{for } t \ge \max\{1, |a_1|, \dots, |a_k|\}, \text{ and} \\ \theta(t) & = & C \cdot \prod_{i=1}^k |a_i|, & \text{for } t \le \min\{1, |a_1|, \dots, |a_k|\}. \end{array}$$

We thus see that

$$\deg \tau^{(2)}(K,\phi_K) = \deg \theta = k.$$

It is well–known that the Alexander polynomial is symmetric, i.e., $\Delta_K(z) = z^l \Delta_K(z^{-1})$ for some $l \in \mathbb{Z}$. It follows in particular that the set of zeros is closed under inversion, i.e. $\{a_1, \dots, a_k\} = \{a_1^{-1}, \dots, a_k^{-1}\}$ as a set with multiplicities. This implies that $\left|\prod_{i=1}^k a_i\right| = 1$. It now follows that $\tau^{(2)}(K,\phi_K)$ is monic if and only if $\Delta_K(z)$ is monic. $\square$

Proposition 7.2 shows that the L^2-Alexander torsion $\tau^{(2)}(K,\phi_K)$ contains a lot of the essential information of the ordinary Alexander polynomial $\Delta_K(t)$. Nonetheless, some information gets lost. For example, let $K = T_{p,q}$ be the (p,q)-torus knot. It is well-known that

$$\Delta_{T_{p,q}}(t) = \frac{(t^{pq}-1)(t-1)}{(t^p-1)(t^q-1)}.$$

This is a polynomial of degree $(p-1)(q-1)$ and all the zeros are roots of unity. It thus follows from Proposition 7.2 that

$$\tau^{(2)}(T_{p,q},\phi_K) = \max\{1,t\}^{(p-1)(q-1)-1}.$$

(In fact we will see in Theorem 8.8 that this equality holds for any admissible epimorphism γ.) In particular, if we consider the torus knots $T_{3,7}$ and $T_{4,5}$, then it is now straightforward to see that all L^2-Alexander torsions agree, but that the ordinary Alexander polynomials are different.

8. Topological and geometric information contained in the L^2–Alexander torsions

8.1. **The volume of a 3-manifold.** We define the volume $\operatorname{Vol}(N)$ of a prime orientable 3–manifold N with empty or toroidal boundary to be the sum of the volumes of the hyperbolic pieces in the JSJ decomposition. We can now state the following theorem, which is a slight reformulation of [LS99, Theorem 0.7].

Theorem 8.1. *If N is a prime 3–manifold with empty or toroidal boundary, then for any $\phi \in H^1(N;\mathbb{R})$ we have*

$$\tau^{(2)}(N,\phi)(t=1) \; = \; \exp\left(\frac{1}{6\pi}\operatorname{Vol}(N)\right). \tag{3}$$

We make two remarks on the differences between the above formulation and the formulation of [LS99, Theorem 0.7].

(1) If N is a prime 3–manifold N with empty or toroidal boundary, then either $N \cong S^1 \times D^2$ or the boundary of N is incompressible. (See e.g., [Ne99, p. 221]).
(2) In this paper we also use a slightly different convention for L^2-torsions compared to [LS99]. Tracing through the differences one notices, that the L^2-torsions differ by a sign, a factor of $\frac{1}{2}$ and by taking the logarithm.

8.2. **Fibered classes and the L^2–Alexander torsion.** Let G be a group with finite generating set S. Given $g \in G$ we denote by $\ell_S(g)$ the minimal length of a word in S representing g. The *entropy* of a homomorphism $f\colon G \to G$ is defined as

$$h(f) := \max\left\{ \limsup_{n\to\infty} \left(\ell_S(f^n(g))\right)^{\frac{1}{n}} \;\middle|\; g \in S\right\}.$$

Note that the entropy is independent of the choice of S. We refer to [FLP79, p. 185] for details. (Note though that we take the exponential of the entropy as defined in [FLP79].)

Now let Σ be a surface and let $f\colon \Sigma \to \Sigma$ be a self-diffeomorphism. Choose $x \in \Sigma$ and a path w in X from $f(x)$ to x. Define the entropy of f to be the entropy of the group automorphism $\alpha(f,x,w)\colon \pi_1(\Sigma,x) \xrightarrow{\pi_1(f,x)} \pi_1(\Sigma,f(x)) \xrightarrow{t_w} \pi_1(\Sigma,x)$, where t_w is given by conjugation with the path w. One easily checks that this definition is independent of the choice of x and w and the entropy depends only on the homotopy class of f.

Note that if Σ is a closed surface with $\chi(\Sigma) < 0$ and if f is pseudo-Anosov, then by [FLP79, p. 195] the entropy $h(f)$ equals the dilatation of f.

Now we have the following theorem.

Theorem 8.2. *Let (N,ϕ,γ) be a an admissible triple such that $\phi \in H^1(N;\mathbb{Q})$ is fibered and such that $G \in \mathcal{G}$. Then there exists a representative τ of $\tau^{(2)}(N,\phi,\gamma)$ such* *that for $T = h(f)$ we have*

$$\tau(t) = \begin{cases} 1, & \text{if } t \in (0, 1/T), \\ t^{x_N(\phi)}, & \text{if } t > T. \end{cases}$$

In particular $\tau^{(2)}(N, \phi, \gamma)$ is monomial in the limit with degree $x_N(\phi)$ and furthermore $\tau^{(2)}(N, \phi, \gamma)$ is monic.

The proof of Theorem 8.2 will require the remainder of Section 8.2. We first need to recall a theorem of Caray–Farber–Mathai [CFM97]. In the following, given a group G we denote by $\mathrm{GL}(n, \mathcal{N}(G))$ the group of invertible $n \times n$–matrices with entries in $\mathcal{N}(G)$. Secondly, if $f \in \mathcal{N}(G)$ then we write $\mathrm{tr}_G(f) := f(1)$, where $1 \in l^2(G)$ denotes the unit element. Furthermore, if $A = (a_{ij})$ is an $n \times n$–matrix over $\mathcal{N}(G)$, then we define

$$\mathrm{tr}_G(A) := \sum_{i=1}^{n} \mathrm{tr}_G(a_{ii}).$$

We can now formulate the following theorem of Caray–Farber–Mathai [CFM97, Theorem 1.10 (e)].

Theorem 8.3. *Let $t \in \mathbb{R}^+$ and let*

$$\begin{array}{rcl} A\colon [0,t] & \to & \mathrm{GL}(n, \mathcal{N}(G)) \\ s & \mapsto & A(s) \end{array}$$

be a continuous piecewise smooth map, then

$$\det{}^{\mathrm{r}}_{\mathcal{N}(G)}(A(t)) = \det{}^{\mathrm{r}}_{\mathcal{N}(G)}(A(0)) \cdot \exp\left(\int_0^t \mathrm{Re}\ \mathrm{tr}_G \left(A(s)^{-1} \cdot \frac{d}{ds} A|_s \right) ds \right).$$

Before we continue we need to introduce the entropy of a matrix over $\mathbb{R}[G]$. Given $p = \sum_{g \in G} a_g g \in \mathbb{R}[G]$ we write

$$|p|_1 := \sum_{g \in G} |a_g|$$

and given a $k \times l$-matrix $A = (a_{ij})$ over $\mathbb{R}[G]$ we write

$$\|A\|_1 := k \cdot l \cdot \max\{|a_{ij}|_1 \mid i = 1, \ldots, k, j = 1, \ldots, l\}.$$

We then define

$$h(A) := \lim_{k \to \infty} \left(\|A^k\|_1 \right)^{\frac{1}{k}}.$$

Recall that we denote by $\mathcal{G}$ the class of all sofic groups, the class $\mathcal{G}$ contains in particular the class of all groups that are residually amenable.

Proposition 8.4. *Let G be a group, let $\phi\colon G \to \mathbb{Z}$ be an epimorphism, and let $y \in G$ be an element $y \in G$ with $\phi(y) = 1$. We write $H = \mathrm{Ker}(\phi)$ [1]. Let P, Q be two square*

[1] Jerome: K denotes a knot in the paper. I change the notation for $\mathrm{Ker}(\phi)$.

matrices over $\mathbb{Z}[H]$ of the same size which are invertible over $\mathbb{Z}[H]$. If $G \in \mathcal{G}$, then for $T = h(QP^{-1})$ we have

$$\det^{\mathrm{r}}_{\mathcal{N}(G)}(P - tyQ) = \begin{cases} 1, & \text{if } t \in (0, 1/T), \\ t^n, & \text{if } t > T. \end{cases}$$

Proof. We write $A = QP^{-1}$ and $T = h(A)$. Since $G \in \mathcal{G}$ it follows from Proposition 2.4 and Theorem 2.5 that

$$\det^{\mathrm{r}}_{\mathcal{N}(G)}(P - tyQ) = \det^{\mathrm{r}}_{\mathcal{N}(G)}(\mathrm{id} - tyQP^{-1}) \cdot \det^{\mathrm{r}}_{\mathcal{N}(G)}(P) = \det^{\mathrm{r}}_{\mathcal{N}(G)}(\mathrm{id} - tyA).$$

Now let $s \in (0, \frac{1}{T})$. It follows easily from the definition of $h(A)$ that the power series

$$\sum_{i=0}^{\infty} s^i (yA)^i$$

converges in the operator norm and that it is an inverse to $P(s) = \mathrm{id} - syA$.

For any $t \in (0, \frac{1}{T})$ we can thus apply Theorem 8.3 and we obtain that

$$\begin{aligned} \det^{\mathrm{r}}_{\mathcal{N}(G)}(\mathrm{id} - tyA) &= \det^{\mathrm{r}}_{\mathcal{N}(G)}(P(t)) \\ &= \det^{\mathrm{r}}_{\mathcal{N}(G)}(P(0)) \cdot \exp\left(\int_0^1 \mathrm{Re}\ \mathrm{tr}_G\left(P(s)^{-1} \tfrac{d}{ds} P|_s\right) ds\right) \\ &= \exp\left(\int_0^1 \mathrm{Re}\ \mathrm{tr}_G\left(\left(\textstyle\sum_{i=0}^{\infty} (tyA)^i\right)(-yA)\right) dt\right) \\ &= \exp\left(\int_0^1 \mathrm{Re}\ \mathrm{tr}_G\left(\textstyle\sum_{i=0}^{\infty} -t^i (yA)^{i+1}\right) dt\right). \end{aligned}$$

Note that

$$\mathrm{tr}_G\left(\sum_{i=0}^{\infty} -t^i (yA)^{i+1}\right) = \sum_{i=0}^{\infty} \mathrm{tr}_G(-t^i (yA)^{i+1}).$$

Also note that any entry of $(yA)^{i+1}$ is of the form

$$\sum_{j=1}^{l} a_j g_j \text{ with } a_1, \dots, a_l \in \mathbb{Z} \text{ and } g_1, \dots, g_l,$$

where $\phi(g_1) = \cdots = \phi(g_l) = i + 1$. It follows immediately that $\mathrm{tr}_G(-t^i(yA)^{i+1}) = 0$ for all $i \geq 0$. We thus see that

$$\det^{\mathrm{r}}_{\mathcal{N}(G)}(\mathrm{id} - tyA) = 1 \text{ for all } t \in (0, \frac{1}{T}).$$

Now suppose that $t > T$. It follows from Theorem 2.5 and from the above that

$$\begin{aligned} \det^{\mathrm{r}}_{\mathcal{N}(G)}(\mathrm{id} - tyA) &= \det^{\mathrm{r}}_{\mathcal{N}(G)}(tAy(t^{-1}y^{-1}A^{-1} - \mathrm{id})) \\ &= t^n \ \det^{\mathrm{r}}_{\mathcal{N}(G)}(Ay) \ \det^{\mathrm{r}}_{\mathcal{N}(G)}(t^{-1}y^{-1}A^{-1} - \mathrm{id}) = t^n. \end{aligned}$$

□

We are now finally in a position to prove Theorem 8.2.

Proof of Theorem 8.2. Given a space X and a map $f\colon X \to X$ we denote by

$$M(X,f) := X \times [-1,1] \;/\; (x,-1) \sim (f(x),1)$$

the corresponding mapping torus. Note that $M(X,f)$ has a canonical projection map $M(X,f) \to S^1 = [-1,1]/(-1 \sim 1)$ and we refer to the epimorphism $\pi_1(M(X,f)) \to \pi_1(S^1) = \mathbb{Z}$ as the canonical epimorphism to $\mathbb{Z}$. It is well-known that two homotopic maps $f_0, f_1\colon X \to X$ give rise to simple homotopy equivalent mapping tori.

Now let (N,ϕ,γ) be an admissible triple such that $\phi \in H^1(N;\mathbb{Q})$ is fibered. By Lemma 5.2 we only need to consider the case that ϕ is a primitive element in $H^1(N;\mathbb{Z})$. Note that we can identify N with $M(\Sigma,f)$ for some connected surface Σ and self-diffeomorphism $f\colon \Sigma \to \Sigma$ in such a way that $\phi \in H^1(N;\mathbb{Z}) = \operatorname{Hom}(\pi_1(N),\mathbb{Z})$ agrees with the canonical epimorphism $\pi_1(M(\Sigma,f)) \to \mathbb{Z}$. We consider the case that $\chi(\Sigma) \neq 0$. The proof in the case that $\chi(\Sigma) = 0$ is basically the same and is left to the reader.

We pick a CW-structure for Σ with one 0-cell p, n 1-cells $g_1,\dots,g_n$ and one 2-cell which by a slight abuse of notation we denote again by Σ. By another slight abuse of notation we denote the elements in $\pi_1(\Sigma,p)$ represented by $g_1,\dots,g_n$ by the same symbols. By the Cellular Approximation Theorem the diffeomorphism f is homotopic to a cellular map f'. In fact one can see 'by hand' that f' can be chosen such that $f(p) = p$ and such that $f'(g_i)$ is represented by the path traced out by the word $f_*(g_i)$ in the generators $g_1,\dots,g_n$.

We write $I = [-1,1]$. Given a cell c of Σ we denote by $c \times I$ the product cell. Furthermore we denote by μ the element in $\pi = \pi_1(N) = \pi_1(M(\Sigma,f'),p)$ that is represented by the loop $p \times I$. Note that the product CW-structure on $\Sigma \times I$ descends to a CW-structure on $M(\Sigma,f')$. By the above we know that $N = M(\Sigma,f)$ is homotopy equivalent to $M(\Sigma,f')$. Since the Whitehead group of fibered 3-manifolds is trivial, see [Wa78], these two spaces are in fact simple homotopy equivalent. We thus have $\tau^{(2)}(N,\phi,\gamma) \doteq \tau^{(2)}(M(\Sigma,f'),\phi,\gamma)$.

Now we collect the cells of $M(\Sigma,f')$ according to their dimensions and we order them as follows:

$$\{\Sigma \times I\}\ \{\Sigma, g_1 \times I,\dots,g_n \times I\}\ \{g_1,\dots,g_n,p \times I\}\ \{p\}.$$

It is straightforward to see that for an appropriate lift of the above ordered sets of cells of $M(\Sigma,f')$ to the universal cover, the resulting chain complex of the universal cover is isomorphic to

$$0 \longrightarrow \mathbb{Z}[\pi] \xrightarrow{B_3} \mathbb{Z}[\pi] \oplus \mathbb{Z}[\pi]^n \xrightarrow{B_2} \mathbb{Z}[\pi]^n \oplus \mathbb{Z}[\pi] \xrightarrow{B_1} \mathbb{Z}[\pi] \longrightarrow 0$$

where

$$B_3 = \begin{pmatrix} 1-\mu & * \dots & * \end{pmatrix},\ B_2 = \begin{pmatrix} * & * \\ \mathrm{id}_n - \mu A & * \end{pmatrix},\ B_1 = \begin{pmatrix} * \\ 1-\mu \end{pmatrix},$$

and where in turn the (i,j)-entry of the $n \times n$-matrix A is given by $\frac{\partial f_*(g_i)}{\partial g_j}$ and where the $*$'s indicate matrices of an appropriate size. Note that each entry of the i-th row of A is a sum of at most $\ell_S(f_*(g_i))$ elements in π, possibly equipped with a minus sign. It thus follows immediately from the definitions that

$$h(A) \le h\big(f_*\colon \pi_1(\Sigma,p) \to \pi_1(\Sigma,p)\big).$$

By Lemma 2.8 we have $\det^{\mathrm{r}}_{\mathcal{N}(G)}(1-t\gamma(\mu))^{-1} = \max\{1,t\}$. By the definitions and by Lemma 3.2 we therefore have

$$\begin{array}{rl} & \tau(N,\phi,\gamma) \\ \doteq & \tau^{(2)}\left(\mathbb{R}[\pi] \xrightarrow{\kappa(\phi,\gamma,t)(B_3)} \mathbb{R}[\pi]\oplus\mathbb{R}[\pi]^n \xrightarrow{\kappa(\phi,\gamma,t)(B_2)} \mathbb{R}[\pi]^n\oplus\mathbb{R}[\pi] \xrightarrow{\kappa(\phi,\gamma,t)(B_1)} \mathbb{R}[\pi]\right) \\ = & \det^{\mathrm{r}}_{\mathcal{N}(G)}(1-t\gamma(\mu))^{-1}\ \det^{\mathrm{r}}_{\mathcal{N}(G)}(\mathrm{id}-t\gamma(\mu)\gamma(A))\ \det^{\mathrm{r}}_{\mathcal{N}(G)}(1-t\gamma(\mu))^{-1} \\ = & \max\{1,t\}^{-2}\ \det_{\mathcal{N}(G)}(\mathrm{id}-t\gamma(\mu)\gamma(A)). \end{array}$$

Now the theorem follows from Proposition 8.4, the observation that $h(\gamma(A)) \le h(A)$ and the well-known fact, see e.g. [Th86, Theorem 3], that a fiber surface is Thurston norm minimizing, i.e., $n-2 = x_N(\phi)$. □

8.3. **L^2-Alexander torsions of graph manifolds.** First we recall that a graph manifold is a 3-manifold for which all its JSJ components are Seifert fibered spaces. The following theorem gives the computation of the L^2-Alexander torsions of Seifert fibered spaces. The proof of the theorem is a variation of the proof of [Lü02, Theorem 3.105].

Theorem 8.5. *Let (N,ϕ,γ) be an admissible triple with $N \ne S^1\times D^2$ and $N \ne S^1\times S^2$. Suppose that N is a Seifert fibered 3-manifold such that the image of a regular fiber under γ is an element of infinite order, then*

$$\tau^{(2)}(N,\phi,\gamma) \doteq \max\{1,t^{x_N(\phi)}\}.$$

Now we obtain the following result.

Theorem 8.6. *Let (N,ϕ,γ) be an admissible triple with $N \ne S^1\times D^2$ and $N \ne S^1\times S^2$. Suppose that N is a graph manifold and that given any JSJ component of N the image of a regular fiber under γ is an element of infinite order, then*

$$\tau^{(2)}(N,\phi,\gamma) \doteq \max\{1,t^{x_N(\phi)}\}.$$

In particular $\tau^{(2)}(N,\phi,\gamma)$ is monomial in the limit with degree $x_N(\phi)$ and furthermore $\tau^{(2)}(N,\phi,\gamma)$ is monic.

Proof. We denote by $N_i, i=1,\dots,k$ the JSJ components of N, which by assumption are Seifert fibered spaces. Note that for each i we have $N_i \ne S^1\times D^2$ and $N_i \ne S^1\times S^2$. For $i=1,\dots,k$ we write $\phi_i = \phi|_{N_i}$ and we write $\gamma_i = \gamma|_{\pi_1(N_i)}$. By our assumption on γ and by Theorem 8.5 we have $\tau(N_i,\phi_i,\gamma_i) \doteq \max\{1,t^{x_{N_i}(\phi_i)}\}$. Furthermore, note that the Seifert fibered structure of a Seifert fibered 3-manifold restricts to a fibration of any boundary torus. It now follows from our assumption on γ that the restriction of γ to any JSJ torus has infinite image. Thus it follows from Theorem 5.5 that

$$\tau^{(2)}(N,\phi,\gamma) \doteq \prod_{i=1}^{k} \tau^{(2)}(N_i,\phi,\gamma_i) \doteq \prod_{i=1}^{k} \max\{1, t^{x_{N_i}(\phi_i)}\} = \max\left\{1, t^{\sum_{i=1}^{k} x_{N_i}(\phi_i)}\right\}.$$

The theorem follows from [EN85, Proposition 3.5] which shows that $\sum_{i=1}^{k} x_{N_i}(\phi_i) = x_N(\phi)$. □

8.4. **Applications to knot theory.** We denote by $\mathcal{K}$ the minimal set of oriented knots that contains the unknot and that is closed under the connect sum operation and under cabling. Note that $\mathcal{K}$ contains torus knots, and more generally iterated torus knots. We recall the following well-known lemma.

Lemma 8.7. *Let K be a knot. Then the following are equivalent:*

(1) K lies in $\mathcal{K}$.
(2) $X(K)$ is a graph manifold with the property that the regular fiber of any Seifert fibered piece is non-zero in $H_1(X(K);\mathbb{Z})$.
(3) $X(K)$ is a graph manifold.

Here the implication (1) ⇒ (2) is not hard to verify. The implication (2) ⇒ (3) is trivial and the implication (3) ⇒ (1) is [Go83, Corollary 4.2].

We can now state and prove the following theorem.

Theorem 8.8. *Let K be a knot in $\mathcal{K}$. Then for any admissible epimorphism $\gamma\colon \pi_1(X(K)) \to G$ we have*

$$\tau^{(2)}(K,\gamma) \doteq \max\left\{1, t^{2\,\mathrm{genus}(K)-1}\right\}.$$

In particular, if $K = T_{p,q}$ is the (p,q)-torus knot, then

$$\tau^{(2)}(T_{p,q},\gamma) \doteq \max\left\{1, t^{(p-1)(q-1)-1}\right\}.$$

Proof. First note that if K is the trivial knot, then $X(K) = S^1 \times D^2$. In this case we identify $\pi_1(X(K))$ with the infinite cyclic group generated by μ. Since X is simple homotopy equivalent to a circle it follows from the definitions and from Lemma 2.8 that

$$\tau^{(2)}(K,\gamma) \doteq \tau^{(2)}\left(0 \longrightarrow \mathbb{R}[\langle\mu\rangle] \xrightarrow{1-t\mu} \mathbb{R}[\langle\mu\rangle] \longrightarrow 0\right) \doteq \max\{1,t\}^{-1}.$$

Now let K be a non-trivial knot. It is well-known and straightforward to show that $x_{X(K)}(\phi_K) = 2\,\mathrm{genus}(K) - 1$. Now we suppose that K lies in $\mathcal{K}$. By Lemma 8.7 the knot exterior $X(K)$ is a graph manifold with the property that the regular fiber of any Seifert fibered piece represents is non-zero in $H_1(X(K);\mathbb{Z})$. Since γ is admissible we can appeal to Theorem 8.6 to obtain the desired result. The statement for torus knots follows from the well-known fact that the genus of the (p,q)-torus knot is $\frac{1}{2}(p-1)(q-1)$. □

The combination of Theorems 8.1 and 8.8 and Lemma 8.7 immediately implies Theorem 1.3.

9. The degree of the L^2–Alexander torsion and the Thurston norm I

The goal of this section is to prove the following theorem.

Theorem 9.1. *Let (N,ϕ,γ) be an admissible triple where γ is an epimorphism onto a virtually abelian group. Then*

$$\deg \tau^{(2)}(N,\phi,\gamma) \leq x_N(\phi).$$

Theorem 9.1 is an immediate consequence of the following two propositions. Here note that the first proposition holds without the assumption that the image of γ is virtually abelian. We also expect the second statement to hold without any restrictions, but as of now we can not provide a proof.

Proposition 9.2. *Let $(N,\phi,\gamma\colon \pi_1(N)\to G)$ be an admissible triple. We write $H = \operatorname{Ker}(\phi\colon G\to\mathbb{Z})$ and we pick $\mu\in G$ with $\phi(\mu)=1$. Then there exist $k,l\in\mathbb{N}$ with $k-l = x_N(\phi)$ and a square matrix A over $\mathbb{Z}[H]$ such that*

$$t \mapsto \max\{1,t\}^{-l}\cdot \det\nolimits^{\mathrm{r}}_{\mathcal{N}(G)}\left(A + t\mu\begin{pmatrix}\mathrm{id}_k & 0\\ 0 & 0\end{pmatrix}\right)$$

is a representative of $\tau^{(2)}(N,\phi,\gamma)$.

Proposition 9.3. *Let G be a virtually abelian group and let $\phi\colon G\to\mathbb{Z}$ be an epimorphism. We write $H=\operatorname{Ker}(\phi\colon G\to\mathbb{Z})$. Let $\mu\in G$ with $\phi(G)=1$ and let A be a square matrix over $\mathbb{Z}[H]$. Then*

$$\deg\left(t \mapsto \max\{1,t\}^{-l}\cdot \det\nolimits^{\mathrm{r}}_{\mathcal{N}(G)}\left(A + t\mu\begin{pmatrix}\mathrm{id}_k & 0\\ 0 & 0\end{pmatrix}\right)\right) \leq k-l.$$

9.1. **Proof of Proposition 9.2.** In the proof of Proposition 9.2 we will need the following lemma.

Lemma 9.4. *Let G be a group and let $\phi\colon G\to\mathbb{Z}$ be an epimorphism. We write $H=\operatorname{Ker}(\phi\colon G\to\mathbb{Z})$ and we pick $\mu\in G$ with $\phi(\mu)=1$. Given $n\times n$-matrices Y_1, Y_2 over $\mathbb{Z}[H]$ and $\nu\in G$ with $\phi(\nu)=r$ there exists an $(r+1)n\times(r+1)n$-matrix A over $\mathbb{Z}[H]$ such such that*

$$\det\nolimits^{\mathrm{r}}_{\mathcal{N}(G)}\begin{pmatrix}\mathrm{id}_n & Y_1\\ -t^r\nu\,\mathrm{id}_n & Y_2\end{pmatrix} = \det\nolimits^{\mathrm{r}}_{\mathcal{N}(G)}\left(A + t\mu\begin{pmatrix}\mathrm{id}_{rn} & 0\\ 0 & 0\end{pmatrix}\right)$$

for any $t\in\mathbb{R}^+$.

Proof. By Proposition 2.4 and elementary matrix transformations we have

$$\begin{aligned}
&\det{}^{\mathrm{r}}_{\mathcal{N}(G)}\begin{pmatrix}\mathrm{id}_n & Y_1\\ -t^r\nu\,\mathrm{id}_n & Y_2\end{pmatrix}\\
=\;&\det{}^{\mathrm{r}}_{\mathcal{N}(G)}\begin{pmatrix}\mathrm{id}_n & Y_1\\ -t^r\mu^r\,\mathrm{id}_n & \mu^r\nu^{-1}Y_2\end{pmatrix}\\
=\;&\det{}^{\mathrm{r}}_{\mathcal{N}(G)}\begin{pmatrix}\mathrm{id}_n & 0 & \dots & 0 & Y_1\\ 0 & \mathrm{id}_n & & & 0\\ 0 & \ddots & \ddots & & \vdots\\ 0 & & 0 & \mathrm{id}_n & 0\\ -t^r\mu^r\,\mathrm{id}_n & \dots & -t^2\mu^2\,\mathrm{id}_n & -t\mu\,\mathrm{id}_n & \mu^r\nu^{-1}Y_2\end{pmatrix}\\
=\;&\det{}^{\mathrm{r}}_{\mathcal{N}(G)}\begin{pmatrix}\mathrm{id}_n & 0 & \dots & 0 & Y_1\\ -t\mu\,\mathrm{id}_n & \mathrm{id}_n & & & 0\\ 0 & \ddots & \ddots & & \vdots\\ 0 & & -t\mu\,\mathrm{id}_n & \mathrm{id}_n & 0\\ 0 & \dots & 0 & -t\mu\,\mathrm{id}_n & \mu^r\nu^{-1}Y_2\end{pmatrix}.
\end{aligned}$$

By swapping the rows appropriately and multiplying them by -1 we get the matrix of the desired form. By Proposition 2.4 these procedures do not change the regular Fuglede–Kadison determinant. □

Lemma 9.5. *Let G be a group and let $\phi\colon G \to \mathbb{Z}$ be an epimorphism. Let $\mu \in G$ with $\phi(\mu) \neq 0$ and let $w \in H = \operatorname{Ker}(\phi)$. Then for any $t \in \mathbb{R}^+$ we have*

$$\det{}^{\mathrm{r}}_{\mathcal{N}(G)}\begin{pmatrix}1 & 1\\ -t\mu & -w\end{pmatrix} = \max\{1, t\}.$$

Proof. We first note that by subtracting the first column from the second column and by multiplying the second column by $-w^{-1}$ on the right we turn the given matrix into the matrix

$$\begin{pmatrix}1 & 0\\ -t\mu & 1-t\mu w^{-1}\end{pmatrix}.$$

Note that $\phi(\mu w^{-1}) \neq 0$, in particular μw^{-1} is an element of G of infinite order. The lemma now follows immediately from Proposition 2.4 and Lemma 2.8. □

We are now ready to give the proof of Proposition 9.2.

Proof of Proposition 9.2. Let $(N, \phi, \gamma\colon \pi_1(N) \to G)$ be an admissible triple. We write $H = \operatorname{Ker}(\phi\colon G \to \mathbb{Z})$ and we pick $\mu \in G$ with $\phi(\mu) = 1$. It follows easily from [Tu02b, Section 1] that we can find an oriented surface $\Sigma \subset N$ with components $\Sigma_1, \dots, \Sigma_l$ and non-zero $r_1, \dots, r_l \in \mathbb{N}$ with the following properties:

(1) $r_1[\Sigma_1] + \dots + r_l[\Sigma_l]$ is dual to ϕ,
(2) $\sum_{i=1}^{l} -r_i\chi(\Sigma_i) \leq x_N(\phi)$,
(3) $N \setminus \Sigma$ is connected.

For $i = 1, \dots, l$ we pick disjoint oriented tubular neighborhoods $\Sigma_i \times [0,1]$ and we identify Σ_i with $\Sigma_i \times \{0\}$. We write $M := N \setminus \cup_{i=1}^{l} \Sigma_i \times [0,1]$. We pick once and for all a base point p in M and we denote by $\widetilde{N}$ the universal cover of N. We write $\pi = \pi_1(N,p)$. For $i = 1, \dots, l$ we also pick a curve ν_i based at p which intersects Σ_i precisely once in a positive direction and does not intersect any other component of Σ. Note that $\phi(\nu_i) = r_i$. By a slight abuse of notation we denote $\gamma(\nu_i)$ also by ν_i. Finally for $i = 1, \dots, l$ we write $n_i = \chi(\Sigma_i) + 2$.

Following [Fr14, Section 4] we now pick an appropriate CW–structure for N and we pick appropriate lifts of the cells to the universal cover. The resulting boundary maps are described in detail [Fr14, Section 4]. In order to keep the notation manageable we now restrict to the case $l = 2$.

It then follows from the discussion in [Fr14, Section 4] and the definitions that

$$\tau(t) := \tau^{(2)} \left(0 \longrightarrow \mathbb{R}[G]^4 \xrightarrow{B_3} \mathbb{R}[G]^{5+n_1+n_2} \xrightarrow{B_2} \mathbb{R}[G]^{5+n_1+n_2} \xrightarrow{B_1} \mathbb{R}[G]^4 \longrightarrow 0 \right)$$

is a representative for $\tau^{(2)}(N,\phi,\gamma)$, where B_3, B_2, B_1 are matrices of the form

$$\begin{aligned}
B_3 &= \begin{pmatrix} * & 0 & 0 & 0 \\ 0 & 0 & * & 0 \\ 1 & 1 & 0 & 0 \\ -t^{r_1}\nu_1 & -z_1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -t^{r_2}\nu_2 & -z_2 \\ 0 & * & 0 & * \end{pmatrix} \\
B_2 &= \begin{pmatrix} * & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 0 & 0 & 0 \\ \mathrm{id}_{n_1} & 0 & * & 0 & 0 & 0 & * \\ -t^{r_1}\nu_1 \,\mathrm{id}_{n_1} & 0 & 0 & * & 0 & 0 & * \\ 0 & \mathrm{id}_{n_2} & 0 & 0 & * & 0 & * \\ 0 & -t^{r_2}\nu_2 \,\mathrm{id}_{n_2} & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * \end{pmatrix}, \\
B_1 &= \begin{pmatrix} 1 & 0 & * & 0 & 0 & 0 & 1 & 0 & * \\ -t^{r_1}\nu_1 & 0 & 0 & * & 0 & 0 & -x_1 & 0 & * \\ 0 & 1 & 0 & 0 & * & 0 & 0 & 1 & * \\ 0 & -t^{r_2}\nu_2 & 0 & 0 & 0 & * & 0 & -x_2 & * \end{pmatrix}
\end{aligned}$$

with $x_1, x_2, z_1, z_2 \in H$ and where all the entries of the matrices marked by $*$ lie in $\mathbb{Z}[H]$.

It follows from Lemma 9.5 and Proposition 2.4 that

$$\det^{\mathrm{r}}_{\mathcal{N}(G)}\begin{pmatrix} 1 & 1 & 0 & 0 \\ -t^{r_1}\nu_1 & -z_1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -t^{r_2}\nu_2 & -z_2 \end{pmatrix} = \det^{\mathrm{r}}_{\mathcal{N}(G)}\begin{pmatrix} 1 & 0 & 1 & 0 \\ -t^{r_1}\nu_1 & 0 & -x_1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -t^{r_2}\nu_2 & 0 & -x_2 \end{pmatrix} = \max\{1, t^{r_1+r_2}\}.$$

Thus it follows from Lemma 3.2 that

$$\tau(t) = \max\{1, t^{r_1+r_2}\}^{-1} \cdot \det^{\mathrm{r}}_{\mathcal{N}(G)}\begin{pmatrix} \mathrm{id}_{n_1} & 0 & * \\ -t^{r_1}\nu_1\,\mathrm{id}_{n_1} & 0 & * \\ 0 & \mathrm{id}_{n_2} & * \\ 0 & -t^{r_2}\nu_2\,\mathrm{id}_{n_2} & * \end{pmatrix} \cdot \max\{1, t^{r_1+r_2}\}^{-1}.$$

Now we set $l = 2r_1 + 2r_2$ and $k = r_1n_1 + r_2n_2$. It is then straightforward to see that if we apply the proof of Lemma 9.4 twice then we can turn the above matrix into a matrix of the desired form. We leave the elementary details to the reader. □

9.2. **Proof of Proposition 9.3.** It is clear that the following proposition, together with elementary properties of the degree function, implies Proposition 9.3.

Proposition 9.6. *Let G be a virtually abelian group. Let $m \geq k$ be natural numbers. Let A be an $m \times m$-matrix over $\mathbb{Z}[G]$ and let B be a $k \times k$-matrix over $\mathbb{Z}[G]$. Then*

$$\deg\left(t \mapsto \det^{\mathrm{r}}_{\mathcal{N}(G)}\left(A + t\begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}\right)\right) \leq k.$$

Proof of Proposition 9.6. For $t \in \mathbb{R}^+$ we define

$$f(t) := \det^{\mathrm{r}}_{\mathcal{N}(G)}\left(A + t\begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}\right).$$

It suffices to prove the following claim.

Claim.

$$\deg_0(f(t)) \geq 0 \text{ and } \deg_\infty(f(t)) \leq k.$$

We start out with $\deg_0(f(t))$. If $f(t) = 0$ for arbitrarily small t, then there is nothing to prove. Now we suppose that this is not the case. It follows from Corollary 2.7 that

$$\begin{aligned} \lim_{t\to 0} f(t) &= \lim_{t\to 0} \det^{\mathrm{r}}_{\mathcal{N}(G)}\left(A + t\begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}\right) \\ &= \det^{\mathrm{r}}_{\mathcal{N}(G)} \lim_{t\to 0}\left(A + t\begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}\right) = \det^{\mathrm{r}}_{\mathcal{N}(G)}(A) \in [0, \infty). \end{aligned}$$

In particular we see that $\ln(f(t))$ is bounded from the above for sufficiently small t. It now follows that

$$\deg_0(f(t)) = \lim_{t\to 0} \frac{\ln f(t)}{\ln t} \geq 0.$$

Now we turn to $\deg_\infty(f(t))$. We write

$$A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$$

where A_1 is a $k \times k$-matrix. It then follows from Proposition 2.4 and Corollary 2.7 that

$$\begin{array}{rcl} \lim_{t\to\infty} \frac{1}{t^k} f(t) & = & \lim_{t\to\infty} \frac{1}{t^k} \det^{\mathrm{r}}_{\mathcal{N}(G)} \begin{pmatrix} A_1 + tB & A_2 \\ A_3 & A_4 \end{pmatrix} \\ & = & \lim_{t\to\infty} \det^{\mathrm{r}}_{\mathcal{N}(G)} \begin{pmatrix} t^{-1}A_1 + B & t^{-1}A_2 \\ A_3 & A_4 \end{pmatrix} \\ & = & \det^{\mathrm{r}}_{\mathcal{N}(G)} \lim_{t\to\infty} \begin{pmatrix} t^{-1}A_1 + B & t^{-1}A_2 \\ A_3 & A_4 \end{pmatrix} \\ & = & \det^{\mathrm{r}}_{\mathcal{N}(G)} \begin{pmatrix} B & 0 \\ A_3 & A_4 \end{pmatrix} \in [0,\infty). \end{array}$$

It thus follows that $\ln\left(\frac{1}{t^k} f(t)\right)$ is bounded from the above for sufficiently large t. Now we see that

$$\deg_\infty(f(t)) = \lim_{t\to\infty} \frac{\ln f(t)}{\ln t} = \lim_{t\to\infty} \frac{\ln\left(t^k \frac{1}{t^k} f(t)\right)}{\ln t} = k + \lim_{t\to\infty} \frac{\ln\left(\frac{1}{t^k} f(t)\right)}{\ln t} \le k.$$

This concludes the proof of the claim and thus of the proposition. □

10. The degree of the L^2–Alexander torsion and the Thurston norm II

In Section 8.6 we had already seen that 'most' L^2-Alexander torsions detect the Thurston norm of a graph manifold. In this section we will show that also for all other prime 3-manifolds there exists an L^2-Alexander torsion which detects the Thurston norm. More precisely, the goal of this section is to prove the following theorem.

Theorem 10.1. *Let N be a prime 3-manifold with empty or toroidal boundary that is not a closed graph manifold. Then there exists an epimorphism $\gamma\colon \pi_1(N) \to G$ onto a virtually abelian group such that the projection map $\pi_1(N) \to H_1(N;\mathbb{Z})/torsion$ factors through γ and such that for any $\phi \in H^1(N;\mathbb{R})$ the function $\tau^{(2)}(N,\phi,\gamma)$ is monomial in the limit with*

$$\deg \tau^{(2)}(N,\phi,\gamma) = x_N(\phi).$$

10.1. **The virtual fibering theorem.** Before we recall the virtual fibering theorem of Agol [Ag08] we need to recall a few definitions. First of all, given a 3-manifold N we say that a class $\phi \in H^1(N;\mathbb{R})$ is *quasi-fibered* if ϕ is the limit of fibered classes in $H^1(N;\mathbb{R})$. We will also use the notion of a group π being *residually finite rationally solvable* (*RFRS*). In fact one can treat this notion as a black box. We thus provide the definition only for completeness' sake. A group is RFRS if there exists a filtration of π by subgroups $\pi = \pi_0 \supseteq \pi_1 \supseteq \pi_2 \cdots$ such that

(1) $\bigcap_i \pi_i = \{1\}$,
(2) for any i the group π_i is a normal, finite-index subgroup of π,
(3) for any i the map $\pi_i \to \pi_i/\pi_{i+1}$ factors through $\pi_i \to H_1(\pi_i;\mathbb{Z})/\text{torsion}$.

The following is a straightforward consequence of the virtual fibering theorem of Agol [Ag08, Theorem 5.1] (see also [FK14, Theorem 5.1] and [FV12, Corollary 5.2]).

Theorem 10.2. *Let N be a prime 3-manifold with empty or toroidal boundary. Suppose that $\pi_1(N)$ is virtually RFRS. Then there exists a finite regular cover $p\colon \widehat{N} \to N$ such that for every class $\phi \in H^1(N;\mathbb{R})$ the class $p^*\phi \in H^1(\widehat{N};\mathbb{R})$ is quasi-fibered.*

The following theorem was proved by Agol [Ag13] and Wise [Wi12a, Wi12b] in the hyperbolic case. It was proved by Liu [Liu13] and Przytycki–Wise [PW14] for graph manifolds with boundary and it was proved by Przytycki–Wise [PW12] for manifolds with a non-trivial JSJ decomposition and at least one hyperbolic piece in the JSJ decomposition.

Theorem 10.3. *If N is an irreducible 3-manifold that is not a closed graph manifold, then $\pi_1(N)$ is virtually RFRS.*

10.2. **Continuity of degrees.** Given a group G, a homomorphism $\phi\colon G \to \mathbb{R}$ and $t \in \mathbb{R}^+$ we consider the ring homomorphism

$$\begin{array}{rccc} \kappa(\phi,t)\colon & \mathbb{R}[G] & \to & \mathbb{R}[G] \\ & \sum\limits_{i=1}^{n} a_i g_i & \mapsto & \sum\limits_{i=1}^{n} a_i t^{\phi(g_i)} g_i. \end{array}$$

As usual, given a matrix A over $\mathbb{R}[G]$ we define $\kappa(\phi,t)(A)$ by applying $\kappa(\phi,t)$ to each entry of A.

Recall that in Section 6 we associated to many functions $f\colon \mathbb{R}^+ \to [0,\infty)$ a degree $\deg(f)$ with values in $\mathbb{R}\cup\{\pm\infty\}$. Now we endow $\mathbb{R}\cup\{\pm\infty\}$ with the usual topology, i.e., the topology on $\mathbb{R}$ with a 'point at infinity on the left' and a 'point at infinity on the right'.

We have the following proposition.

Proposition 10.4. *Let G be virtually abelian group and let A be a square matrix over $\mathbb{Z}[G]$ such that $\det^{\mathrm{r}}_{\mathcal{N}(G)}(A) \neq 0$. Then the map*

$$\begin{array}{rcl} \operatorname{Hom}(G,\mathbb{R}) & \to & \mathbb{R}\cup\{\pm\infty\} \\ \phi & \mapsto & \deg\left(\begin{array}{rcl} \mathbb{R}^+ & \to & [0,\infty) \\ t & \mapsto & \det^{\mathrm{r}}_{\mathcal{N}(G)}\big(\kappa(\phi,t)(A)\big) \end{array}\right) \end{array}$$

takes values in $[0,\infty)$ and it is a (possibly degenerate) norm.

Before we continue we recall that given a free abelian group F and $p \in \mathbb{R}[F]$ we denote by $m(p)$ the Mahler measure. In the proof of Proposition 10.4 we will need the following lemma.

Lemma 10.5. *Let F be a free abelian group and let $p \in \mathbb{R}[F]$ be non-zero. We write $p = \sum_{f \in F} p_f \cdot f$. Then for any $\phi \in \operatorname{Hom}(F, \mathbb{R})$ we have*

$$\deg\big(t \mapsto m(\kappa(\phi,t)(p))\big) = \max\{\phi(f) - \phi(g) \mid f, g \in F \text{ with } p_f \neq 0 \text{ and } p_g \neq 0\}.$$

Proof. Let $\phi \in \operatorname{Hom}(F, \mathbb{R})$. We write

$$\begin{array}{rcl} d & = & \min\{\phi(f) \mid f \in F \text{ with } a_f \neq 0\}, \text{ and} \\ D & = & \max\{\phi(f) \mid f \in F \text{ with } a_f \neq 0\}. \end{array}$$

We write $K = \operatorname{Ker}(\phi)$. Then there exist $p_1, \dots, p_r \in \mathbb{R}[K]$ and $g_1, \dots, g_r \in F$ with

$$d = \phi(g_1) < \phi(g_2) < \dots < \phi(g_r) = D$$

such that $p = p_1 g_1 + \dots + p_r g_r$. Note that $p_1 \neq 0$ and $p_r \neq 0$. By the continuity of the Mahler measure, see Corollary 2.7 and [Bo98, p. 127], we have

$$\lim_{t \to \infty} \frac{m(\kappa(\phi,t)(p))}{t^D} = \lim_{t \to \infty} m\left(p_1 g_1 \frac{t^{\phi(g_1)}}{t^D} + \dots + p_r g_r \frac{t^{\phi(g_r)}}{t^D}\right) = m(p_r g_r) \neq 0.$$

It thus follows that $\deg_\infty(t \mapsto m(\kappa(\phi,t)(p))) = D$. Basically the same argument also shows that $\deg_0(t \mapsto m(\kappa(\phi,t)(p))) = d$. Putting these two equalities together gives the desired result. $\square$

We can now give the proof of Proposition 10.4.

Proof of Proposition 10.4. In the proof of Corollary 2.7 we already saw that G admits a finite index subgroup F that is torsion-free abelian. We pick representatives $g_1, \dots, g_d$ for G/F. Given a matrix B over $\mathbb{R}[G]$ we define the matrix $\iota_G^F(B)$ over $\mathbb{R}[F]$ as in Section 2.3, using this ordered set of representatives. It follows easily from the definitions that for any $\phi \in \operatorname{Hom}(G, \mathbb{R})$ and any $t \in \mathbb{R}^+$ we have

$$\iota_G^F(\kappa(\phi,t)(A)) = \kappa(\phi|_F, t)\left(\iota_G^F(A)\right). \tag{4}$$

Now we denote by $p \in \mathbb{Z}[F]$ the determinant of $\iota_G^F(A)$. It follows from (4), Proposition 2.4 and Lemma 2.6 that

$$\operatorname{det}^{\mathrm{r}}_{\mathcal{N}(G)}\big(\kappa(\phi,t)(A)\big) = m\big(\kappa(\phi,t)(p)\big)^{\frac{1}{[G:F]}} \text{ for any } \phi \in \operatorname{Hom}(G, \mathbb{R}) \text{ and } t \in \mathbb{R}^+. \tag{5}$$

If we apply (5) to $t = 1$, then we see that our assumption that $\operatorname{det}^{\mathrm{r}}_{\mathcal{N}(G)}(A) \neq 0$ implies in particular that $p \neq 0$. Furthermore, by the combination of (5) and Lemma 6.1 (7) it suffices to show that the map

$$\begin{array}{rcl} \operatorname{Hom}(F, \mathbb{R}) & \to & \mathbb{R} \cup \{\pm\infty\} \\ \psi & \mapsto & \deg\big(t \mapsto m(\kappa(\psi,t)(p))\big) \end{array}$$

takes values in $[0, \infty)$ and that it is a (possibly degenerate) norm. But since $p \neq 0$ this is an immediate consequence of Lemma 10.5. $\square$

10.3. **The proof of Theorem 10.1.**

Proof. Let N be a prime 3-manifold which is not a closed graph manifold. It follows from Theorems 10.2 and 10.3 that there exists a finite regular cover $p\colon \widehat{N} \to N$ such that given any $\phi \in H^1(N;\mathbb{R})$ the pull-back $p^*\phi \in H^1(\widehat{N};\mathbb{R})$ is quasi-fibered.

Now we denote by $\widehat{\gamma}\colon \pi_1(\widehat{N}) \to H := H_1(\widehat{N};\mathbb{Z})/\text{torsion}$ the canonical epimorphism. By Theorem 8.2 we have

$$\deg \tau^{(2)}(\widehat{N},\widehat{\gamma},\psi) = x_{\widehat{N}}(\psi) \text{ for any fibered } \psi \in H^1(\widehat{N};\mathbb{Q}). \tag{6}$$

It follows from Proposition 10.4 and from the fact that $x_{\widehat{N}}$ is a norm that both sides of (6) are continuous in ψ. It thus follows that we also have

$$\deg \tau^{(2)}(\widehat{N},\widehat{\gamma},\psi) = x_{\widehat{N}}(\psi) \text{ for any quasi-fibered } \psi \in H^1(\widehat{N};\mathbb{R}).$$

In particular the equality holds for any $p^*\phi$ with $\phi \in H^1(N;\mathbb{R})$.

Now we consider the projection homomorphism

$$\gamma\colon \pi_1(N) \to G := \pi_1(N)/\operatorname{Ker}\{\widehat{\phi}\colon \pi_1(\widehat{N}) \to H\}.$$

(Note that $\operatorname{Ker}\{\widehat{\phi}\colon \pi_1(\widehat{N}) \to H\}$ is indeed normal in $\pi_1(N)$.) It follows from the above, from Lemma 5.3 and the multiplicativity of the Thurston norm under finite covers, see Gabai [Ga83, Corollary 6.13], that for any $\phi \in H^1(N;\mathbb{R})$ we have

$$\deg \tau^{(2)}(N,\gamma,\phi) = \frac{1}{[N:\widehat{N}]} \deg \tau^{(2)}(\widehat{N},\widehat{\gamma},p^*\phi) = \frac{1}{[N:\widehat{N}]} x_{\widehat{N}}(p^*\phi) = x_N(\phi).$$

□

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