

# THE $L^2$ -ALEXANDER TORSION IS SYMMETRIC

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ABSTRACT. We show that the  $L^2$ -Alexander torsion of a 3-manifold is a symmetric function. This can be viewed as a generalization of the symmetry of the Alexander polynomial of a knot.

## 1. INTRODUCTION

An *admissible triple*  $(N, \phi, \gamma)$  consists of an irreducible, orientable, compact 3-manifold  $N \neq S^1 \times D^2$  with empty or toroidal boundary, a class  $\phi \neq 0 \in H^1(N; \mathbb{Z}) = \text{Hom}(\pi_1(N), \mathbb{Z})$  and a homomorphism  $\gamma: \pi_1(N) \rightarrow G$  such that  $\phi$  factors through  $\gamma$ . In [DFL14a, DFL14b] we used the  $L^2$ -torsion (see e.g. [Lü02]) to associate to an admissible triple  $(N, \phi, \gamma)$  the  $L^2$ -Alexander torsion  $\tau^{(2)}(N, \phi, \gamma)$  which is a function

$$\tau^{(2)}(N, \phi, \gamma): \mathbb{R}_{>0} \rightarrow \mathbb{R}_{\geq 0}$$

that is well-defined up to multiplication by a function of the type  $t \mapsto t^m$  for some  $m \in \mathbb{Z}$ . We recall the definition in Section 6.

The goal of this paper is to show that the  $L^2$ -Alexander torsion is symmetric. In order to state the precise symmetry result we need to recall that given a 3-manifold  $N$  the *Thurston norm* [Th86] of some  $\phi \in H^1(N; \mathbb{Z}) = \text{Hom}(\pi_1(N), \mathbb{Z})$  is defined as

$$x_N(\phi) := \min\{\chi_-(S) \mid S \subset N \text{ properly embedded surface dual to } \phi\}.$$

Here, given a surface  $S$  we define its complexity as  $\chi_-(S) := -\chi(S')$ , where  $S'$  is the result of deleting all components from  $S$  that are disks or spheres. Thurston [Th86] showed that  $x_N$  is a, possibly degenerate, norm on  $H^1(N; \mathbb{Z})$ . Now we can formulate the main result of this paper.

**Theorem 1.1.** *Let  $(N, \phi, \gamma)$  be an admissible triple. Then for any representative  $\tau$  of  $\tau^{(2)}(N, \phi, \gamma)$  there exists an  $n \in \mathbb{Z}$  with  $n \equiv x_N(\phi) \pmod{2}$  such that*

$$\tau(t^{-1}) = t^n \cdot \tau(t) \text{ for any } t \in \mathbb{R}_{>0}.$$

It is worth looking at the case that  $N = S^3 \setminus \nu K$  is the complement of a tubular neighborhood  $\nu K$  of an oriented knot  $K \subset S^3$ . We denote by  $\phi_K: \pi_1(N) \rightarrow \mathbb{Z}$  the

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epimorphism sending the oriented meridian to 1. Let  $\gamma: \pi_1(N) \rightarrow G$  be a homomorphism such that  $\phi_K$  factors through  $\gamma$ . We define

$$\tau^{(2)}(K, \gamma) := \tau^{(2)}(S^3 \setminus \nu K, \phi_K, \gamma).$$

If we take  $\gamma = \text{id}$  to be the identity, then we showed in [DFL14b] that

$$\tau^{(2)}(K, \text{id}) = \Delta_K^{(2)}(t) \cdot \max\{1, t\},$$

where  $\Delta_K^{(2)}(t): \mathbb{R}_{>0} \rightarrow \mathbb{R}_{\geq 0}$  denotes the  $L^2$ -Alexander invariant of Li-Zhang [LZ06a, LZ06b] which was also studied in [DW10, DW13, BA13a, BA13b].

If we take  $\gamma = \phi_K$ , then we showed in [DFL14b] that the  $L^2$ -Alexander torsion  $\tau^{(2)}(K, \phi_K)$  is fully determined by the Alexander polynomial  $\Delta_K(t)$  of  $K$  and that in turn  $\tau^{(2)}(K, \phi_K)$  almost determines the Alexander polynomial  $\Delta_K(t)$ . In this sense the  $L^2$ -Alexander torsion can be viewed as a ‘twisted’ version of the Alexander polynomial, and at least morally it is related to the twisted Alexander polynomial of Wada [Wa94] and to the higher-order Alexander polynomials of Cochran [Co04] and Harvey [Ha05]. We refer to [DFL14a] for more on the relationship and similarities between the various twisted invariants.

If  $K$  is a knot, then any Seifert surface is dual to  $\phi_K$  and it immediately follows that  $x(\phi_K) \leq \max\{2 \cdot \text{genus}(K) - 1, 0\}$ . In fact an elementary argument shows that for any *non-trivial* knot we have the equality  $x(\phi_K) = 2 \cdot \text{genus}(K) - 1$ . In particular the Thurston norm of  $\phi_K$  is odd. We thus obtain the following corollary to Theorem 1.1.

**Theorem 1.2.** *Let  $K \subset S^3$  be an oriented non-trivial knot and let  $\gamma: \pi_1(N) \rightarrow G$  be a homomorphism such that  $\phi_K$  factors through  $\gamma$ . Then there exists an odd  $n$  with*

$$\tau^{(2)}(K, \gamma)(t^{-1}) = t^n \cdot \tau^{(2)}(K, \gamma)(t) \text{ for any } t \in \mathbb{R}_{>0}.$$

The proof of Theorem 1.1 has many similarities with the proof of the main theorem in [FKK12] which in turn builds on the ideas of Turaev [Tu90, Tu01, Tu02]. In an attempt to keep the proof as short as possible we will on several occasions refer to [FKK12] and [Tu90] for definitions and results.

**Conventions.** All manifolds are assumed to be connected, orientable and compact. All CW-complexes are assumed to be finite and connected. If  $G$  is a group then we equip  $\mathbb{C}[G]$  with the involution given by complex conjugation and by  $\bar{g} := g^{-1}$  for  $g \in G$ . We extend this involution to matrices over  $\mathbb{C}[G]$  by applying the involution to each entry. Given a ring  $R$  we will view all modules as left  $R$ -modules, unless we say explicitly otherwise. Furthermore, given a matrix  $A \in M_{m,n}(R)$  we denote by  $A: R^m \rightarrow R^n$  the  $R$ -homomorphism of left  $R$ -modules obtained by right multiplication with  $A$  and thinking of elements in  $R^m$  as the only row in a  $(1, m)$ -matrix.

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## 2. EULER STRUCTURES

In this section we recall the notion of an Euler structure of a pair of CW-complexes and manifolds which is due to Turaev. We refer to [Tu90, Tu01, FKK12] for full details. Throughout this paper, given a space  $X$ , we denote by  $\mathcal{H}_1(X)$  the first integral homology group viewed as a multiplicative group.

**2.1. Euler structures on CW-complexes.** Let  $X$  be a CW-complex of dimension  $m$  and let  $Y$  be a proper subcomplex. We denote by  $p: \tilde{X} \rightarrow X$  the universal covering of  $X$  and we write  $\tilde{Y} := p^{-1}(Y)$ . An *Euler lift* is a set of cells in  $\tilde{X}$  such that each cell of  $X \setminus Y$  is covered by precisely one of the cells in the Euler lift.

Using the canonical left action of  $\pi = \pi_1(X)$  on  $\tilde{X}$  we obtain a free and transitive action of  $\pi$  on the set of cells of  $\tilde{X} \setminus \tilde{Y}$  lying over a fixed cell in  $X \setminus Y$ . If  $c$  and  $c'$  are two Euler lifts, then we can order the cells such that  $c = \{c_{ij}\}$  and  $c' = \{c'_{ij}\}$  and such that for each  $i$  and  $j$  the cells  $c_{ij}$  and  $c'_{ij}$  lie over the same  $i$ -cell in  $X \setminus Y$ . In particular there exist unique  $g_{ij} \in \pi$  such that  $c'_{ij} = g_{ij} \cdot c_{ij}$ . We denote the projection map  $\pi \rightarrow \mathcal{H}_1(X)$  by  $\Psi$ . We define

$$c'/c := \prod_{i=0}^m \prod_j \Psi(g_{ij})^{(-1)^i} \in \mathcal{H}_1(X).$$

We say that  $c$  and  $c'$  are *equivalent* if  $c'/c \in \mathcal{H}_1(X)$  is trivial. An equivalence class of Euler lifts will be referred to as an *Euler structure*. We denote by  $\text{Eul}(X, Y)$  the set of Euler structures. If  $Y = \emptyset$  then we will also write  $\text{Eul}(X) = \text{Eul}(X, Y)$ .

Given  $g \in \mathcal{H}_1(X)$  and  $e \in \text{Eul}(X, Y)$  we define  $g \cdot e \in \text{Eul}(X, Y)$  as follows: pick representatives  $c$  for  $e$  and  $\tilde{g} \in \pi_1(X)$  for  $g$ , then act on one  $i$ -cell of  $c$  by  $g^{(-1)^i}$ . The resulting Euler lift represents an element in  $\text{Eul}(X, Y)$  which is independent of the choice of the cell. We denote by  $g \cdot e$  the Euler structure represented by this new Euler lift. This defines a free and transitive  $\mathcal{H}_1(X)$ -action on  $\text{Eul}(X, Y)$ , with  $(g \cdot e)/e = g$ .

If  $(X', Y')$  is a cellular subdivision of  $(X, Y)$ , then there exists a canonical  $\mathcal{H}_1(X)$ -equivariant bijection  $\sigma: \text{Eul}(X, Y) \rightarrow \text{Eul}(X', Y')$  which is defined as follows: Let  $e \in \text{Eul}(X, Y)$  and pick an Euler lift for  $(X, Y)$  which represents  $e$ . There exists a unique Euler lift for  $(X', Y')$  such that the cells in the Euler lift of  $(X', Y')$  are contained in the cells of the Euler lift of  $(X, Y)$ . We denote by  $\sigma(e)$  the Euler structure represented by this Euler lift. This map equals the map of Turaev [Tu90, Section 1.2].

**2.2. Euler structures of smooth manifolds.** Let  $N$  be a manifold and let  $\partial_0 N \subset \partial N$  be a union of components of  $\partial N$  such that  $\chi(N, \partial_0 N) = 0$ . A *triangulation* of  $N$  is a pair  $(X, t)$  where  $X$  is a simplicial complex and  $t: |X| \rightarrow N$  is a homeomorphism.

Throughout this section we write  $Y := t^{-1}(\partial_0 N)$ . For the most part we will suppress  $t$  from the notation. Following [Tu90, Section I.4.1] we consider the projective system of sets  $\{\text{Eul}(X, Y)\}_{(X, t)}$  where  $(X, t)$  runs over all  $C^1$ -triangulations of  $N$  and where the maps are the  $\mathcal{H}_1(N)$ -equivariant bijections between these sets induced either by  $C^1$ -subdivisions or by smooth isotopies in  $N$ . We define  $\text{Eul}(N, \partial_0 N)$  by identifying the sets  $\{\text{Eul}(X, Y)\}_{(X, t)}$  via these bijections. We refer to  $\text{Eul}(N, \partial_0 N)$  as the set of *Euler structures on  $(N, \partial_0 N)$* . For a  $C^1$ -triangulation  $X$  of  $N$  we get a canonical  $\mathcal{H}_1(N)$ -equivariant bijection  $\text{Eul}(X, Y) \rightarrow \text{Eul}(N, \partial_0 N)$ .

### 3. THE $L^2$ -TORSION OF A MANIFOLD

**3.1. The  $L^2$ -torsion of a chain complex.** First we recall some key properties of the Fuglede-Kadison determinant and the definition of the  $L^2$ -torsion of a chain complex of free based left  $\mathbb{C}[G]$ -modules. Throughout the section we refer to [Lü02] and to [DFL14b] for details and proofs.

We fix a group  $G$ . Let  $A$  be a  $k \times l$ -matrix over  $\mathbb{C}[G]$ . There exists the notion of  $A$  being of ‘determinant class’. (To be more precise, we view the  $k \times l$ -matrix  $A$  as a map  $\mathcal{N}(G)^l \rightarrow \mathcal{N}(G)^k$ , where  $\mathcal{N}(G)$  is the von Neumann algebra of  $G$ , and then there is the notion of being of ‘determinant class’.) We treat this entirely as a black box, but we note that if  $G$  is residually amenable, e.g., a 3-manifold group [He87] or solvable, then by [ES05] any matrix over  $\mathbb{Q}[G]$  is of determinant class. If the matrix  $A$  is not of determinant class then for the purpose of this paper we define  $\det_{\mathcal{N}(G)}(A) = 0$ . On the other hand, if  $A$  is of determinant class, then we define

$$\det_{\mathcal{N}(G)}(A) := \text{Fuglede-Kadison determinant of } A \in \mathbb{R}_{>0}.$$

Here we do not assume that  $A$  is a square matrix. In an attempt to keep the paper as short as possible we will not provide the, somewhat lengthy, definition of the Fuglede-Kadison determinant. Instead we summarize a few key properties in the following theorem which is a consequence of [Lü02, Example 3.12] and [Lü02, Theorem 3.14].

**Theorem 3.1.** (1) *If  $A$  is a square matrix with complex entries such that the usual determinant  $\det(A) \in \mathbb{C}$  is non-zero, then  $\det_{\mathcal{N}(G)}(A) = |\det(A)|$ .*  
 (2) *The determinant does not change if we swap two rows or two columns.*  
 (3) *Right multiplication of a column by  $\pm g$ ,  $g \in G$  does not change the determinant.*  
 (4) *For any matrix  $A$  over  $\mathbb{C}[G]$  we have  $\det_{\mathcal{N}(G)}(A) = \det_{\mathcal{N}(G)}(\overline{A}^t)$ .*

Note that (2) implies that when we study Fuglede-Kadison determinants of homomorphisms we can work with unordered bases. Now let

$$C_* = \left( 0 \rightarrow C_l \xrightarrow{\partial_l} C_{l-1} \xrightarrow{\partial_{l-1}} \dots C_1 \xrightarrow{\partial_1} C_0 \rightarrow 0 \right)$$

be a chain complex of free left  $\mathbb{C}[G]$ -modules. We refer to [Lü02] for the definition of the  $L^2$ -Betti numbers  $b_i^{(2)}(C_*) \in \mathbb{R}_{\geq 0}$ . Now suppose that the chain complex is

equipped with bases  $B_i \subset C_i$ ,  $i = 0, \dots, l$ . If one of the  $L^2$ -Betti numbers  $b_i^{(2)}(C_*)$  is non-zero or if one of the boundary maps is not of determinant class, then we define the  $L^2$ -torsion  $\tau^{(2)}(C_*, B_*) := 0$ . Otherwise we define the  $L^2$ -torsion to be

$$\tau^{(2)}(C_*, B_*) := \prod_{i=1}^l \det_{\mathcal{N}(G)}(A_i)^{(-1)^i} \in \mathbb{R}_{>0}$$

where the  $A_i$  denote the boundary matrices corresponding to the given bases. This definition is the multiplicative inverse of the exponential of the  $L^2$ -torsion as defined in [Lü02, Definition 3.29].

**3.2. The twisted  $L^2$ -torsion of CW-complexes and manifolds.** Let  $(X, Y)$  be a pair of CW-complexes and let  $e \in \text{Eul}(X, Y)$ . We denote by  $p: \tilde{X} \rightarrow X$  the universal covering of  $X$  and we write  $\tilde{Y} := p^{-1}(Y)$ . The deck transformation turns  $C_*(\tilde{X}, \tilde{Y})$  naturally into a chain complex of left  $\mathbb{Z}[\pi_1(X)]$ -modules.

Now let  $G$  be a group and let  $\varphi: \pi_1(X) \rightarrow \text{GL}(d, \mathbb{C}[G])$  be a representation. We view elements of  $\mathbb{C}[G]^d$  as row vectors. Right multiplication via  $\varphi(g)$  thus turns  $\mathbb{C}[G]^d$  into a right  $\mathbb{Z}[\pi_1(X)]$ -module. We consider the chain complex

$$C_*^\varphi(X, Y; \mathbb{C}[G]^d) := \mathbb{C}[G]^d \otimes_{\mathbb{Z}[\pi_1(X)]} C_*(\tilde{X}, \tilde{Y})$$

of left  $\mathbb{C}[G]$ -modules. Let  $e \in \text{Eul}(X, Y)$ . We pick an Euler lift  $\{c_{ij}\}$  that represents  $e$ . Throughout this paper we denote by  $v_1, \dots, v_d$  the standard basis for  $\mathbb{C}[G]^d$ . We equip the chain complex  $C_*^\varphi(X, Y; \mathbb{C}[G]^d)$  with the basis provided by the  $v_k \otimes c_{ij}$ . Therefore we can define

$$\tau^{(2)}(X, Y, \varphi, e) := \tau^{(2)}(C_*^\varphi(X, Y; \mathbb{C}[G]^d), \{v_k \otimes c_{ij}\}) \in \mathbb{R}_{\geq 0}.$$

**Lemma 3.2.** (1) *The number  $\tau^{(2)}(X, Y, \varphi, e)$  is well-defined.*

(2) *If  $g \in \mathcal{H}_1(X)$ , then*

$$\tau^{(2)}(X, Y, \varphi, ge) = \det_{\mathcal{N}(G)}(\varphi(g^{-1})) \cdot \tau^{(2)}(X, Y, \varphi, e).$$

(3) *If  $(X', Y')$  is a cellular subdivision of  $(X, Y)$  and if  $e' \in \text{Eul}(X', Y')$  is the Euler structure corresponding to  $e$ , then*

$$\tau^{(2)}(X', Y', \varphi, e') = \tau^{(2)}(X, Y, \varphi, e).$$

The proofs are completely analogous to the proofs for ordinary Reidemeister torsion as given in [Tu01, FKK12]. In the interest of space we will not provide the proofs.

Finally let  $N$  be a manifold and let  $\partial_0 N \subset \partial N$  be a union of components of  $\partial N$  with  $\chi(N, \partial_0 N) = 0$ . Let  $G$  be a group and let  $\varphi: \pi_1(N) \rightarrow \text{GL}(d, \mathbb{C}[G])$  be a representation. Let  $e \in \text{Eul}(N, \partial_0 N)$ . Recall that for any  $C^1$ -triangulation  $f: X \rightarrow N$  we get a bijection  $\text{Eul}(X, Y) \xrightarrow{f_*} \text{Eul}(N, \partial_0 N)$ . We define

$$\tau^{(2)}(N, \partial_0 N, \varphi, e) := \tau^{(2)}(X, Y, \varphi \circ f_*, f_*^{-1}(e)).$$

By Lemma 3.2 (3) and the discussion in [Tu90] the invariant  $\tau^{(2)}(N, \partial_0 N, \varphi, e) \in \mathbb{R}_{\geq 0}$  is well-defined, i.e., independent of the choice of the triangulation.

#### 4. DUALITY FOR TORSION OF MANIFOLDS EQUIPPED WITH EULER STRUCTURES

**4.1. The algebraic duality theorem for  $L^2$ -torsion.** Let  $G$  be a group and let  $V$  be a right  $\mathbb{C}[G]$ -module. We denote by  $\overline{V}$  the left  $\mathbb{C}[G]$ -module with the same underlying abelian group but with the module structure given by  $p \cdot \overline{v} := v \cdot \overline{p}$  for  $p \in \mathbb{C}[G]$  and  $v \in V$ . If  $V$  is a left  $\mathbb{C}[G]$ -module then we can consider  $\text{Hom}_{\mathbb{C}[G]}(V, \mathbb{C}[G])$ , the set of all left  $\mathbb{C}[G]$ -module homomorphisms. Since the range  $\mathbb{C}[G]$  is a  $\mathbb{C}[G]$ -bimodule we can naturally view  $\text{Hom}_{\mathbb{C}[G]}(V, \mathbb{C}[G])$  as a right  $\mathbb{C}[G]$ -module.

In the following let  $C_*$  be a chain complex of length  $m$  of left  $\mathbb{C}[G]$ -modules with boundary operators  $\partial_i$ . Suppose that  $C_*$  is equipped with a basis  $B_i$  for each  $C_i$ . We denote by  $C^\#$  the *dual chain complex* whose chain groups are the  $\mathbb{C}[G]$ -left modules  $C_i^\# := \overline{\text{Hom}_{\mathbb{C}[G]}(C_{m-i}, \mathbb{C}[G])}$  and where the boundary map  $\partial_i^\# : C_{i+1}^\# \rightarrow C_i^\#$  is given by  $(-1)^{m-i} \partial_{m-i}^*$ . This means that for any  $c \in C_{m-i}$  and  $d \in C_{i+1}^\#$  we have  $\partial_i^\#(d)(c) = (-1)^{m-i} d(\partial_{m-i}(c))$ . We denote by  $B_*^\#$  the bases of  $C^\#$  dual to the bases  $B_*$ .

**Lemma 4.1.** *If  $\tau^{(2)}(C_*, B_*) = 0$ , then  $\tau^{(2)}(C_*^\#, B_*^\#) = 0$ , otherwise we have*

$$\tau^{(2)}(C_*, B_*) = \tau^{(2)}(C_*^\#, B_*^\#)^{(-1)^{m+1}}.$$

*Proof.* By the proof of [Lü02, Theorem 1.35 (3)] the  $L^2$ -Betti numbers of  $C_*$  vanish if and only if the  $L^2$ -Betti numbers of  $C_*^\#$  vanish. In particular, if either  $L^2$ -Betti number does not vanish, then both torsions are zero.

Now we suppose that the  $L^2$ -Betti numbers of  $C_*$  vanish. We denote by  $A_i$  the corresponding matrices of the boundary maps of  $C_*$ . The boundary matrices of the chain complex  $C_*^\#$  with respect to the basis  $B_*^\#$  are given by  $(-1)^{m-i} \overline{A_i}^t$ . Now the lemma is an immediate consequence of the definitions and of Theorem 3.1 (4).  $\square$

**4.2. The duality theorem for manifolds.** Before we state our main technical duality theorem we need to introduce two more definitions.

- (1) Let  $G$  be a group and let  $\varphi : \pi \rightarrow \text{GL}(d, \mathbb{C}[G])$  be a representation. We denote by  $\varphi^\dagger$  the representation which is given by  $g \mapsto \overline{\varphi(g^{-1})}^t$ .
- (2) Let  $N$  be an  $m$ -manifold and let  $e \in \text{Eul}(N, \partial N)$ . Pick a triangulation  $X$  for  $N$ . We denote by  $Y$  the subcomplex corresponding to  $\partial N$ . Let  $X^\dagger$  be the CW-complex that is given by the cellular decomposition of  $N$  dual to  $X$ . Pick an Euler lift  $\{c_{ij}\}$  that represents  $e \in \text{Eul}(X, Y) = \text{Eul}(N, \partial N)$ . For any  $i$ -cell  $c$  in  $\widetilde{X} \setminus \widetilde{Y}$  we denote by  $c^\dagger$  the unique oriented  $(m-i)$ -cell in  $\widetilde{X}^\dagger$  which has intersection number  $+1$  with  $c$ . The Euler lift  $\{c_{ij}^\dagger\}$  defines an element in  $\text{Eul}(X^\dagger) = \text{Eul}(N)$  that we denote by  $e^\dagger$ . This map is an  $\mathcal{H}_1(N)$ -equivariant bijection and we denote the inverse map  $\text{Eul}(N, \partial N) \rightarrow \text{Eul}(N)$  again by

$e \mapsto e^\dagger$ . We refer to [Mu84, Chapter 70], [Tu01, Section 14] and [FKK12, Section 4] for details.

**Theorem 4.2.** *Let  $N$  be an  $m$ -manifold. Let  $G$  be a group and let  $\varphi: \pi(N) \rightarrow GL(d, \mathbb{C}[G])$  be a representation. Let  $e \in \text{Eul}(N, \partial N)$ . Then either both  $\tau^{(2)}(N, \partial N, \varphi, e)$  and  $\tau^{(2)}(N, \varphi^\dagger, e^\dagger)$  are zero, or the following equality holds:*

$$\tau^{(2)}(N, \partial N, \varphi, e) = \tau^{(2)}(N, \varphi^\dagger, e^\dagger)^{(-1)^{m+1}}.$$

*Proof.* Pick a triangulation  $X$  for  $N$  and denote by  $Y$  the subcomplex corresponding to  $\partial N$ . Let  $X^\dagger$  be the CW-complex which is given by the cellular decomposition of  $N$  dual to  $X$ . We identify  $\pi = \pi_1(X) = \pi_1(N) = \pi_1(X^\dagger)$ . We pick an Euler lift  $\{c_{ij}\}$  which represents  $e \in \text{Eul}(N, \partial N) = \text{Eul}(X, Y)$ . We denote by  $c_{ij}^\dagger$  the corresponding dual cells. The theorem follows from the definitions and the following claim.

*Claim.* Either both  $\tau^{(2)}(C_*^\varphi(X, Y; \mathbb{C}[G]^d), \{v_k \otimes c_{ij}\})$  and  $\tau^{(2)}(C_*^{\varphi^\dagger}(X^\dagger; \mathbb{C}[G]^d), \{v_k \otimes c_{ij}^\dagger\})$  are zero, or the following equality holds:

$$\tau^{(2)}(C_*^\varphi(X, Y; \mathbb{C}[G]^d), \{v_k \otimes c_{ij}\}) = \tau^{(2)}(C_*^{\varphi^\dagger}(X^\dagger; \mathbb{C}[G]^d), \{v_k \otimes c_{ij}^\dagger\})^{(-1)^{m+1}}.$$

In order to prove the claim we first note that there is a unique, sesquilinear paring

$$\begin{aligned} C_{m-i}(\tilde{X}, \tilde{Y}) \times C_i(\tilde{X}^\dagger) &\rightarrow \mathbb{Z}[\pi] \\ (a, b) &\mapsto \langle a, b \rangle := \sum_{g \in \pi} (a \cdot gb)g^{-1}. \end{aligned}$$

such that  $a \cdot b^\dagger = \delta_{ab}$  for any two cells  $a$  and  $b$  of  $\tilde{X} \setminus \tilde{Y}$ . Here *sesquilinear* means that for any  $a \in C_{m-i}(\tilde{X}, \tilde{Y})$ ,  $b \in C_i(\tilde{X}^\dagger)$  and  $p, q \in \mathbb{Z}[\pi]$  we have  $\langle pa, qb \rangle = q \langle a, b \rangle \bar{p}$ . It is straightforward to see that the pairing is non-singular. It follows immediately from [Tu01, Claim 14.4]) that these maps give rise to well-defined maps

$$\begin{aligned} C_i(\tilde{X}, \tilde{Y}) &\rightarrow \overline{\text{Hom}_{\mathbb{Z}[\pi]}(C_{m-i}(\tilde{X}^\dagger), \mathbb{Z}[\pi])} \\ a &\mapsto (b \mapsto \langle a, b \rangle) \end{aligned}$$

that define an isomorphism of based chain complexes of right  $\mathbb{Z}[\pi]$ -modules. In fact it follows easily from the definitions that the maps define an isomorphism

$$(C_*(\tilde{X}, \tilde{Y}), \{c_{ij}\}) \rightarrow \overline{(\text{Hom}_{\mathbb{Z}[\pi]}(C_{m-*}(\tilde{X}^\dagger), \mathbb{Z}[\pi]), \{(c_{ij}^\dagger)^*\})}$$

of based chain complexes of left  $\mathbb{Z}[\pi]$ -modules. Tensoring these chain complexes with  $\mathbb{C}[G]^d$  we obtain an isomorphism

$$\begin{aligned} & \left( \mathbb{C}[G]^d \otimes_{\mathbb{Z}[\pi]} C_*(\tilde{X}, \tilde{Y}), \{v_k \otimes c_{ij}\} \right) \\ & \rightarrow \left( \mathbb{C}[G]^d \otimes_{\mathbb{Z}[\pi]} \overline{\text{Hom}_{\mathbb{Z}[\pi]}(C_{m-*}(\tilde{X}^\dagger), \mathbb{Z}[\pi]), \{v_k \otimes (c_{ij}^\dagger)^*\}} \right) \end{aligned}$$

of based chain complexes of  $\mathbb{C}[G]$ -modules. Furthermore the maps

$$\begin{aligned} \mathbb{C}[G]^d \otimes_{\mathbb{Z}[\pi]} \overline{\text{Hom}_{\mathbb{Z}[\pi]}(C_i(\widetilde{X}^\dagger), \mathbb{Z}[\pi])} &\rightarrow \overline{\text{Hom}_{\mathbb{C}[G]}(C_i^{\varphi^\dagger}(X^\dagger; \mathbb{C}[G]^d), \mathbb{C}[G])} \\ v \otimes f &\mapsto \left( \begin{array}{ccc} C_i^{\varphi^\dagger}(X^\dagger; \mathbb{C}[G]^d) & \rightarrow & \mathbb{C}[G] \\ w \otimes \sigma & \mapsto & v\varphi(\overline{f(\sigma)})\overline{w}^t \end{array} \right) \end{aligned}$$

induce an isomorphism

$$(C_*^\varphi(X, Y; \mathbb{C}[G]^d), \{v_k \otimes c_{ij}\}) \rightarrow (C_*^{\varphi^\dagger}(X^\dagger; \mathbb{C}[G]^d)^\#, \{(v_k \otimes c_{ij}^\dagger)^\#\})$$

of based chain complexes of  $\mathbb{C}[G]$ -modules. The claim follows from Lemma 4.1.  $\square$

## 5. TWISTED $L^2$ -TORSION OF 3-MANIFOLDS

**5.1. Canonical structures on tori.** Let  $T$  be a torus. We equip  $T$  with a CW-structure with one 0-cell  $p$ , two 1-cells  $x$  and  $y$  and one 2-cell  $s$ . We write  $\pi = \pi_1(T, p)$  and by a slight abuse of notation we denote by  $x$  and  $y$  the elements in  $\pi$  represented by  $x$  and  $y$ . We denote by  $\widetilde{T}$  the universal cover of  $T$ . There exist lifts of the cells such that the chain complex  $C_*(\widetilde{T})$  of left  $\mathbb{Z}[\pi]$ -modules with respect to the bases given by these lifts is of the form

$$(1) \quad 0 \rightarrow \mathbb{Z}[\pi] \xrightarrow{\begin{pmatrix} y-1 & 1-x \end{pmatrix}} \mathbb{Z}[\pi]^2 \xrightarrow{\begin{pmatrix} 1-x \\ 1-y \end{pmatrix}} \mathbb{Z}[\pi] \rightarrow 0.$$

We refer to the corresponding Euler structure of  $T$  as the *canonical Euler structure on  $T$* . Given a group  $G$  we say that a representation  $\varphi: \pi \rightarrow \text{GL}(1, \mathbb{C}[G])$  is *monomial* if for any  $x \in \pi$  we have  $\varphi(x) = zg$  for some  $z \in \mathbb{C}$  and  $g \in G$ . The following is [DFL14b, Lemma 5.6].

**Lemma 5.1.** *Let  $\varphi: \pi_1(T) \rightarrow \text{GL}(1, \mathbb{C}[G])$  be a monomial representation such that  $b_*^{(2)}(T; \mathbb{C}[G]) = 0$  and  $e$  be the canonical Euler structure on  $T$ . Then  $\tau^{(2)}(T, \varphi, e) = 1$ .*

**5.2. Chern classes on 3-manifolds with toroidal boundary.** Let  $N$  be a 3-manifold with toroidal incompressible boundary and let  $e \in \text{Eul}(N, \partial N)$ . Let  $X$  be a triangulation for  $N$ . We denote the subcomplexes corresponding to the boundary components of  $N$  by  $S_1 \cup \dots \cup S_b$ . We denote by  $p: \widetilde{X} \rightarrow X$  and  $p_i: \widetilde{S}_i \rightarrow S_i, i = 1, \dots, b$  the universal covering maps of  $X$  and  $S_i, i = 1, \dots, b$ . For each  $i$  we identify a component of  $p^{-1}(S_i)$  with  $\widetilde{S}_i$

Pick an Euler lift  $c$  that represents  $e$ . For each boundary torus  $S_i$  pick an Euler lift  $\tilde{s}_i$  to  $\widetilde{S}_i \subset p^{-1}(S_i) \subset \widetilde{X}$  that represents the canonical Euler structure. The set of cells  $\{\tilde{s}_1, \dots, \tilde{s}_b, c\}$  defines an Euler structure  $K(e)$  for  $N$ , which only depends on  $e$ . Put differently, we defined a map  $K: \text{Eul}(N, \partial N) \rightarrow \text{Eul}(N)$  which is easily seen to be  $\mathcal{H}_1(N)$ -equivariant. Given  $e \in \text{Eul}(N)$  there exists a unique element  $g \in \mathcal{H}_1(N)$  such that  $e = g \cdot K(e^\dagger)$ . Following Turaev [Tu02, p. 11] and [FKK12, Section 6.3] we define  $c_1(e) := g \in H_1(N; \mathbb{Z})$  and we refer to  $c_1(e)$  as the *Chern class of  $e$* .



**5.3. Torsions of 3-manifolds.** Let  $\pi$  and  $G$  be groups and let  $\varphi: \pi \rightarrow \mathrm{GL}(1, \mathbb{C}[G])$  be a monomial representation. By the multiplicativity of the Fuglede-Kadison determinant, see [Lü02, Theorem 3.14], given  $g \in \pi$  the invariant  $\det_{\mathcal{N}(G)}(\varphi(g))$  only depends on the homology class of  $g$ . Put differently,  $\det_{\mathcal{N}(G)} \circ \varphi: \pi \rightarrow \mathbb{R}_{\geq 0}$  descends to a map  $\det_{\mathcal{N}(G)} \circ \varphi: H_1(\pi; \mathbb{Z}) \rightarrow \mathbb{R}_{\geq 0}$ .

**Theorem 5.2.** *Let  $N$  be a 3-manifold which is either closed or which has toroidal, incompressible boundary. Let  $G$  be a group and let  $\varphi: \pi(N) \rightarrow \mathrm{GL}(1, \mathbb{C}[G])$  be a monomial representation such that  $b_*^{(2)}(\partial N; \mathbb{C}[G]) = 0$ . For any  $e \in \mathrm{Eul}(N)$  we have*

$$\tau^{(2)}(N, \partial N, \varphi, e^\dagger) = \det_{\mathcal{N}(G)}(\varphi(c_1(e))) \cdot \tau^{(2)}(N, \varphi, e).$$

*Proof.* The assumption that  $b_*^{(2)}(\partial N; \mathbb{C}[G]) = 0$  together with the proof of [Lü02, Theorem 1.35 (2)] implies that  $b_*^{(2)}(N; \mathbb{C}[G]) = 0$  if and only if  $b_*^{(2)}(N, \partial N; \mathbb{C}[G]) = 0$ . If both are non-zero, then both torsions  $\tau^{(2)}(N, \partial N, \varphi, e^\dagger)$  and  $\tau^{(2)}(N, \varphi, e)$  are zero. For the remainder of this proof we assume that  $b_*^{(2)}(N; \mathbb{C}[G]) = 0$ .

Pick a triangulation  $X$  for  $N$ . As usual denote by  $Y$  the subcomplex corresponding to  $\partial N$ . Let  $e \in \mathrm{Eul}(N)$ . Pick an Euler lift  $c_*$  which represents  $e^\dagger \in \mathrm{Eul}(N, \partial N) = \mathrm{Eul}(X, Y)$ . Denote the components of  $Y$  by  $Y_1 \cup \cdots \cup Y_b$  and pick  $\tilde{s}_*^1, \dots, \tilde{s}_*^b$  as in the previous section. We write  $\tilde{s}_* = \tilde{s}_*^1 \cup \cdots \cup \tilde{s}_*^b$ . Denote by  $\{\tilde{s}_* \cup c_*\}$  the resulting Euler lift for  $X$ . Recall that this Euler lift represents  $K(e^\dagger)$ .

*Claim.*

$$\tau^{(2)}(C_*^\varphi(X, Y; \mathbb{C}[G]), \{c_*\}) = \tau^{(2)}(C_*^\varphi(X; \mathbb{C}[G]), \{\tilde{s}_* \cup c_*\}).$$

We consider the following short exact sequence of chain complexes

$$0 \rightarrow \bigoplus_{i=1}^b C_*^\varphi(Y_i; \mathbb{C}[G]) \rightarrow C_*^\varphi(X; \mathbb{C}[G]) \rightarrow C_*^\varphi(X, Y; \mathbb{C}[G]) \rightarrow 0,$$

with the bases  $\{s_*^i\}_{i=1, \dots, b}$ ,  $\{\tilde{s}_* \cup c_*\}$  and  $\{c_*\}$ . These bases are in fact compatible, in the sense that the middle basis is the image of the left basis together with a lift of the right basis. By Lemma 5.1 we have  $\tau^{(2)}(C_*^\varphi(Y_i; \mathbb{C}[G]), \{\tilde{s}_*^i\}) = 1$  for  $i = 1, \dots, b$ . Now it follows from the multiplicativity of torsion, see [Lü02, Theorem 3.35], that

$$\tau^{(2)}(C_*^\varphi(X, Y; \mathbb{C}[G]), \{c_*\}) = \tau^{(2)}(C_*^\varphi(X; \mathbb{C}[G]), \{c_* \cup \tilde{s}_*\}).$$

Here we used that the complexes are acyclic. This concludes the proof of the claim.

Finally it follows from this claim, the definitions and Lemma 3.2 that

$$\begin{aligned} \tau^{(2)}(N, \partial N, \varphi, e^\dagger) &= \tau^{(2)}(C_*^\varphi(X, Y; \mathbb{C}[G]), \{c_*\}) = \tau^{(2)}(C_*^\varphi(X; \mathbb{C}[G]), \{\tilde{s}_* \cup c_*\}) \\ &= \tau^{(2)}(N, \varphi, K(e^\dagger)) = \tau^{(2)}(N, \varphi, c_1(e)^{-1}e) \\ &= \det_{\mathcal{N}(G)}(\varphi(c_1(e))) \cdot \tau^{(2)}(N, \varphi, e). \end{aligned}$$

□

6. THE SYMMETRY OF THE  $L^2$ -ALEXANDER TORSION

Let  $(N, \phi, \gamma: \pi_1(N) \rightarrow G)$  be an admissible triple and let  $e \in \text{Eul}(N)$ . Given  $t \in \mathbb{R}_{>0}$  we consider the representation  $\gamma_t: \pi_1(N) \rightarrow \text{GL}(1, \mathbb{C}[G])$  that is given by  $\gamma_t(g) := (t^{\phi(g)}\gamma(g))$ . We denote by  $\tau^{(2)}(N, \phi, \gamma, e)$  the function

$$\begin{aligned} \tau^{(2)}(N, \phi, \gamma, e): \mathbb{R}_{>0} &\rightarrow \mathbb{R}_{>0} \\ t &\mapsto \tau^{(2)}(N, \gamma_t, e). \end{aligned}$$

For another  $e' \in \text{Eul}(N)$  we have  $e' = ge$  for some  $g \in \mathcal{H}_1(N)$ . By Lemma 3.2

$$\tau^{(2)}(N, \phi, \gamma, ge)(t) = t^{-\phi(g)}\tau^{(2)}(N, \phi, g, e)(t) \text{ for all } t \in \mathbb{R}_{>0}.$$

Put differently, the functions  $\tau^{(2)}(N, \phi, \gamma, e)$  and  $\tau^{(2)}(N, \phi, \gamma, ge)$  are equivalent. We denote by  $\tau^{(2)}(N, \phi, \gamma)$  the equivalence class of the functions  $\tau^{(2)}(N, \phi, \gamma, e)$  and we refer to  $\tau^{(2)}(N, \phi, \gamma)$  as the  $L^2$ -Alexander torsion of  $(N, \phi, \gamma)$ .

*Proof of Theorem 1.1.* Let  $e \in \text{Eul}(N)$  and  $t \in \mathbb{R}_{>0}$ . We write  $\tau = \tau^{(2)}(N, \gamma, \phi, e)$ . Note that  $(\gamma_t)^\dagger = \gamma_{t^{-1}}$ . It follows from Theorems 4.2 and 5.2 that

$$\begin{aligned} \tau(t) &= \tau^{(2)}(N, \gamma, \phi, e) = \tau^{(2)}(N, \gamma_t, e) = \tau^{(2)}(N, \partial N, (\gamma_t)^\dagger, e^\dagger) = \tau^{(2)}(N, \partial N, \gamma_{t^{-1}}, e^\dagger) \\ &= \det_{\mathcal{N}(G)}(\gamma_{t^{-1}}(c_1(e))) \cdot \tau^{(2)}(N, \gamma_{t^{-1}}, e) = \det_{\mathcal{N}(G)}(t^{-\phi(c_1(e))}c_1(e)) \cdot \tau^{(2)}(N, \gamma_{t^{-1}}, e) \\ &= t^{-\phi(c_1(e))} \cdot \tau^{(2)}(N, \gamma_{t^{-1}}, e) = t^{-\phi(c_1(e))} \cdot \tau(t^{-1}). \end{aligned}$$

Now it suffices to show that for any  $\phi \in H^1(N; \mathbb{Z})$  we have  $\phi(c_1(e)) = x_N(\phi) \pmod{2}$ .

So let  $S$  be a Thurston norm minimizing surface which is dual to some  $\phi \in H^1(N; \mathbb{Z})$ . Since  $N$  is irreducible and since  $N \neq S^1 \times D^2$  we can arrange that  $S$  has no disk components. Therefore we have

$$x_N(\phi) \equiv \chi_-(S) \equiv b_0(\partial S) \pmod{2\mathbb{Z}}.$$

On the other hand, by [Tu02, Lemma VI.1.2] and [Tu02, Section XI.1] we have that  $b_0(\partial S) \equiv c_1(e) \cdot S \pmod{2\mathbb{Z}}$  where  $c_1(e) \cdot S$  is the intersection number of  $c_1(e) \in H_1(N) = \mathcal{H}_1(N)$  with  $S$ . Since  $S$  is dual to  $\phi$ , we obtain the desired equality

$$\phi(c_1(e)) \equiv c_1(e) \cdot S \equiv b_0(\partial S) \equiv \chi_-(S) \equiv x_N(\phi) \pmod{2\mathbb{Z}}.$$

□

Finally, a *real admissible triple*  $(N, \phi, \gamma)$  is defined like an admissible triple, except that now we also allow  $\phi$  to lie in  $H^1(N; \mathbb{R}) = \text{Hom}(\pi_1(N), \mathbb{R})$ . The same definition as in Section 6 associates to  $(N, \phi, e)$  a function  $\tau^{(2)}(N, \phi, e): \mathbb{R}_{>0} \rightarrow \mathbb{R}_{\geq 0}$  that is well-defined up to multiplication by a function of the form  $t \mapsto t^r$  for some  $r \in \mathbb{R}$ . The same argument as in the proof of Theorem 1.1 gives us the following result.

**Theorem 6.1.** *Let  $(N, \phi, \gamma)$  be a real admissible triple. Then for any representative  $\tau$  of  $\tau^{(2)}(N, \phi, \gamma)$  there exists an  $r \in \mathbb{R}$  such that  $\tau(t^{-1}) = t^r \cdot \tau(t)$  for any  $t \in \mathbb{R}_{>0}$ .*

The only difference to Theorem 1.1 is that for real cohomology classes  $\phi \in H^1(N; \mathbb{R})$  we can not relate the exponent  $r$  to the Thurston norm of  $\phi$ .

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