

ALGEBRAIC TOPOLOGY I + II

STEFAN FRIEDL

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1. TOPOLOGICAL SPACES

1.1. The definition of a topological space. We recall the definition of a topological space from Analysis IV.

Definition. A *topological space* is a pair $(X, \mathcal{T})$, where X is a set and $\mathcal{T}$ is a topology on X , i.e. $\mathcal{T}$ is a set of subsets of X with the following properties:

- (1) the empty set and the entire set X are contained in $\mathcal{T}$,
- (2) the intersection of *finitely many* sets in $\mathcal{T}$ is again a set in $\mathcal{T}$,
- (3) the union of *arbitrarily many* sets in $\mathcal{T}$ is again a set in $\mathcal{T}$.

The sets in $\mathcal{T}$ are called *open*.

Example.

- (1) Let (X, d) be a metric space. A subset U of X is called *open* if for every $x \in U$ there exists an $\epsilon > 0$ such that $B_\epsilon(x) := \{y \in X \mid d(x, y) < \epsilon\}$ is contained in U . We had already seen in Analysis II that

$$\mathcal{T} := \text{all open subsets of } (X, d)$$

is a topology on X . In the following we consider $\mathbb{R}^n$ as a metric space with the euclidean metric and we always view $\mathbb{R}^n$ with the resulting topology, unless we say explicitly otherwise.

- (2) Let X be a set, then $\mathcal{T} = \{\emptyset, X\}$ is a topology on X . This topology is sometimes called the *trivial topology on X* .
- (3) Let X be a set and let $\mathcal{T}$ be the power set of X , i.e. the set of all subsets of X . Then $\mathcal{T}$ is also a topology on X . Put differently, $\mathcal{T}$ is the topology such that all subsets are open. This topology is usually referred to as the *discrete topology on X* .
- (4) Let $X = \mathbb{R}$ and let $\mathcal{T}$ be defined as follows:

$$U \in \mathcal{T} : \Longleftrightarrow \text{either } U = \emptyset \text{ or } U \text{ is the complement of finitely many points in } \mathbb{R}.$$

For example the sets $\emptyset, \mathbb{R} \setminus \{\pi\}, \mathbb{R} \setminus \{-1, \sqrt{2}\}$ and also $\mathbb{R}$ (since it is the complement of zero points) lie in $\mathcal{T}$. It follows easily from elementary set theory that $\mathcal{T}$ is a topology on $X = \mathbb{R}$.

- (5) We consider the set

$$X := \mathbb{R}^n \cup \{\infty\},$$

i.e. X consists of $\mathbb{R}^n$ and an extra point ∞ . We say $U \subset X$ is open¹, if both of the following two conditions are satisfied:

- (a) for each point $x \in U \cap \mathbb{R}^n$ there exists an $\epsilon > 0$ such that $B_\epsilon(x) \subset U$,
- (b) if $\infty \in U$, then there exists a $C > 0$ such that $\{x \in \mathbb{R}^n \mid \|x\| > C\} \subset U$.

It is straightforward to see that this defines indeed a topology on X . For $n = 1$ we had introduced this topological space in Analysis IV and we had referred to it as the “line with a point at infinity”. We now refer to $\mathbb{R}^n \cup \{\infty\}$ as “ $\mathbb{R}^n$ with a point at infinity”.

¹If we want to specify a topology, it suffices to specify which subsets are called “open”.

(6) We consider the set

$$X := \mathbb{R} \cup \{*\},$$

i.e. X consists of $\mathbb{R}$ and an extra point $*$. We say $U \subset X$ is open, if the following two conditions are satisfied:

- (a) for each point $x \in U \cap \mathbb{R}$ there exists an $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subset U$,
- (b) if $*$ $\in U$, then there exists an $\epsilon > 0$ such that $(-\epsilon, 0) \cup (0, \epsilon) \subset U$.

We had seen in Analysis IV that this is indeed a topology on X . We refer to this topological space as the “line with two zeros”.

(7) If $(X, \mathcal{T})$ is a topological space and if $Y \subset X$ is a subset, then

$$\mathcal{S} := \{Y \cap U \mid U \in \mathcal{T}\}$$

is a topology on Y . We refer to $\mathcal{S}$ as the *subspace topology* on Y . Unless we say something else we consider each subset Y of $\mathbb{R}^n$ always as a topological space with respect to the subspace topology.

Now we recall several definitions from Analysis IV.

Definition. Let X be a topological space.²

- (1) Let $A \subset X$ be a subset. We say $U \subset X$ is a *neighborhood* of A if there exists an open set V such that $A \subset V \subset U$. We say U is an *open neighborhood* of A , if U is furthermore open.
- (2) We say X is *Hausdorff*, if given any two points $x \neq y$ there exist disjoint open neighborhoods U of x and V of y .

Example.

- (1) If $X = \mathbb{R}$ and $A = [0, 2)$, then $U = (-1, 3]$ and $V = (-2, \infty)$ are neighborhoods of A in X .
- (2) We had already seen in Analysis II Proposition 1.8 that metric spaces are Hausdorff. Furthermore we had seen in Analysis IV that the line with a point at infinity is also Hausdorff and the same argument shows that $\mathbb{R}^n$ with a point at infinity is Hausdorff. On the other hand we had seen in Analysis IV that the line with two zeros is not Hausdorff.
- (3) A straightforward exercise shows that a topological space X is Hausdorff if and only if the diagonal $D = \{(x, x) \mid x \in X\}$ is a closed subset of $X \times X$.

Definition. Let X be a topological space and let A be a subset of X .

- (1) The *interior* $\overset{\circ}{A}$ is defined as the union of all open sets of X that are contained in A .
- (2) We say A is *closed*, if $X \setminus A$ is open.
- (3) The *closure* $\overline{A}$ of A is defined as the intersection of all closed sets in X that contain A .
- (4) The *boundary of A in X* is defined as $\partial A := \overline{A} \setminus \overset{\circ}{A}$.

²As usual we suppress the topology from the notation, i.e. we write “let X be a topological space” instead of the more precise “let $(X, \mathcal{T})$ be a topological space”.

Example. We consider $X = \mathbb{R}$ and A is the half-open interval $[-1, 2)$. Then the interior of A is the open interval $(-1, 2)$ and the closure of A is the closed interval $[-1, 2]$. Furthermore $\partial A = \{-1, 2\}$.

It follows immediately from the axioms of a topology that the interior of a set is an open set. Furthermore it is straightforward to see that the union of finitely many closed sets is again closed and that the intersection of arbitrarily many closed sets is again closed. It follows easily that the closure of a subset is closed.

Definition. Let X be a topological space. An *open covering* of X is a family $\{U_i\}_{i \in I}$ of open subsets of X with

$$X = \bigcup_{i \in I} U_i.$$

We say a topological space X is *compact* if for each open covering $\{U_i\}_{i \in I}$ of X there exist finitely many indices $i_1, \dots, i_k \in I$ such that

$$X = U_{i_1} \cup \dots \cup U_{i_k}.$$

Example. The Heine–Borel Theorem says that a subset A of $\mathbb{R}^n$ is compact if and only if it is bounded and closed.

We recall the following well-known lemma.

Lemma 1.1. *Let X be a topological space and let $A \subset X$ be a compact subset. If X is Hausdorff, then A is a closed subset of X .*

For completeness' sake we provide the proof.

Proof. Let X be a Hausdorff space and let $A \subset X$ be a compact subset. We want to show that $X \setminus A$ is open. It suffices to prove the following claim.

Claim. Let $x \in X \setminus A$. Then there exists an open neighborhood V of x that is contained in $X \setminus A$.

We apply the Hausdorff-property to x and every $y \in A$. For every $y \in A$ we obtain disjoint open neighborhoods U_y of y and V_y of x . Evidently we have

$$A = \bigcup_{y \in A} \{y\} \subset \bigcup_{y \in A} (U_y \cap A) \subset A.$$

Thus we see that $\{U_y \cap A\}_{y \in A}$ is an open covering of A . Since A is compact there exist $y_1, \dots, y_k$ such that

$$A = \bigcup_{i=1}^k (U_{y_i} \cap A).$$

Now we consider

$$V := \bigcap_{i=1}^k V_{y_i}.$$

Since V is the intersection of finitely many open sets, it is open itself. Furthermore V does not intersect any of the U_{y_i} , $i = 1, \dots, k$. Hence it V is disjoint from von $A \subset U_{y_1} \cup \dots \cup U_{y_k}$. This concludes the proof of the claim. $\square$

Definition. We say a map $f: X \rightarrow Y$ between two topological spaces X and Y is *continuous*, if for each open set U in Y the preimage $f^{-1}(U)$ is open in X .

Example. We consider the set $X = \{A, B, C, D\}$ where the topology is given by the set $\mathcal{T} := \{\emptyset, \{A\}, \{C\}, \{D, A, B\}, \{B, C, D\}, X\}$. Then it follows easily from the definitions that the map

$$\begin{aligned} S^1 &\rightarrow X \\ e^{it} &\mapsto \begin{cases} A, & \text{if } t \in (-\frac{\pi}{4}, \frac{\pi}{4}), \\ B, & \text{if } t = \frac{\pi}{4}, \\ C & \text{if } t \in (\frac{\pi}{4}, \frac{3\pi}{4}), \\ D & \text{if } t = \frac{3\pi}{4}. \end{cases} \end{aligned}$$

is continuous. This map is illustrated in Figure 1.

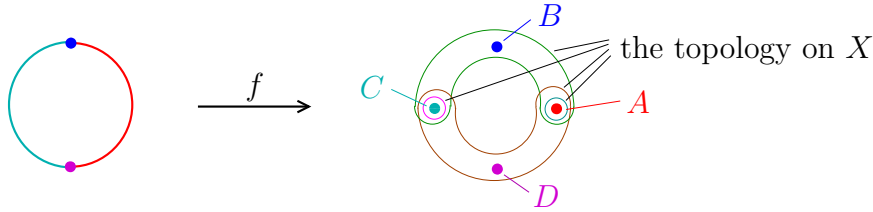


FIGURE 1.

It is straightforward to see that the composition of two continuous maps is again continuous. For maps between metric spaces we obtain the same notion of continuity as in Analysis II.

The following lemma states perhaps the most important feature of compact sets.

Lemma 1.2.

- (1) Let $f: X \rightarrow Y$ be a continuous map. If X is compact, then $f(X)$ is also compact.
- (2) Let $f: X \rightarrow \mathbb{R}$ be a continuous map. If X is compact, then f assumes its maximum and its minimum.
- (3) Let $f: X \rightarrow Y$ be a continuous map. If X is compact and if Y is Hausdorff, then $f(X)$ is a closed subset of Y .

Proof. In Analysis II we had proved the first two statements for metric spaces, the proof for topological spaces is verbatim the same. The third statement is an immediate consequence of Lemma 1.1 and the first statement. $\square$

Definition. We say a map $f: X \rightarrow Y$ between two topological spaces X and Y is a *homeomorphism* if the following three properties are satisfied:

- (1) f is continuous,
- (2) f is bijective,
- (3) $f^{-1}: Y \rightarrow X$ is also continuous.

If there exists a homeomorphism between X and Y we say that X and Y are *homeomorphic* and sometimes we write $X \cong Y$.

The following proposition, that we had proved in Analysis IV, gives an often useful criterion for showing that a map is a homeomorphism.

Proposition 1.3. *Let $f: X \rightarrow Y$ be a bijective continuous map between topological spaces. If X is compact and if Y is Hausdorff, then f is a homeomorphism.*

Example. We consider the map

$$\begin{aligned} \Phi: S^n &\rightarrow \mathbb{R}^n \cup \{\infty\} \\ (x_1, \dots, x_{n+1}) &\mapsto \begin{cases} \left(\frac{x_1}{1-x_{n+1}}, \dots, \frac{x_n}{1-x_{n+1}} \right), & \text{if } x_{n+1} < 1, \\ \infty, & \text{if } x_{n+1} = 1. \end{cases} \end{aligned}$$

where we equip $\mathbb{R}^n \cup \{\infty\}$ with the topology that we had introduced on page 13. Outside of the “North pole” $(0, \dots, 0, 1)$ this map is just the stereographic projection that is illustrated in Figure 2. This map is easily seen to be continuous³ and a bijection. Furthermore S^n is compact by Heine-Borel and $\mathbb{R}^n \cup \{\infty\}$ is Hausdorff, as we had just pointed out above. Hence it follows from Proposition 1.3 that Φ is a homeomorphism.

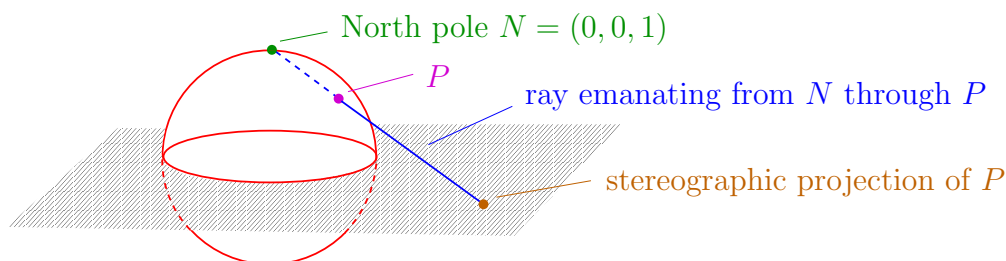


FIGURE 2. Stereographic projection from $S^2 \setminus \{(0, 0, 1)\}$ onto $\mathbb{R}^2$.

Remark. If two topological spaces are homeomorphic, then they have the same topological properties, i.e. they share all properties that are defined purely in terms of the topology. For example, if X and Y are homeomorphic, then X is Hausdorff if and only if Y is Hausdorff, X is compact if and only if Y is compact and so on.

Convention. Henceforth any map between two topological spaces is assumed to be continuous, unless we say explicitly otherwise.

Definition.

³Is that really so easy?

- (1) We say that a subset $A \subset \mathbb{R}^n$ is *convex*, if for any two distinct points P and Q in A the segment $\overline{PQ} := \{tP + (1-t)Q \mid t \in [0, 1]\}$ also lies in A .
- (2) Given a subset S of $\mathbb{R}^k$ the *convex hull* of S is defined as the intersection of all convex subsets of $\mathbb{R}^k$ that contain S . Since the intersection of convex sets is again convex we see that the convex hull of S is a convex subset of $\mathbb{R}^k$.

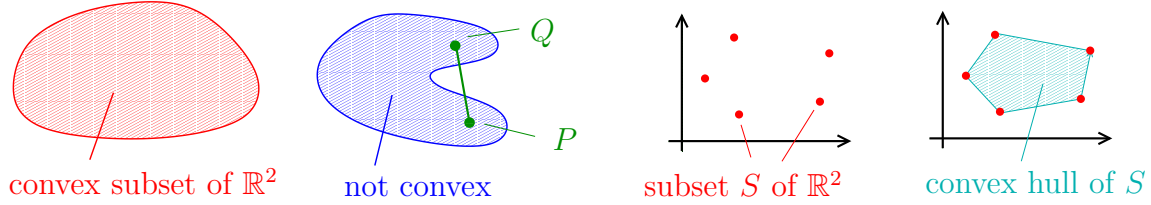


FIGURE 3.

Example. The convex hull of points $P_1, \dots, P_n \in \mathbb{R}^k$ is easily seen to be given by the set

$$\left\{ \sum_{i=1}^n t_i P_i \mid t_1, \dots, t_n \in \mathbb{R}_{\geq 0} \text{ and } \sum_{i=1}^n t_i = 1 \right\}.$$

The following lemma gives a useful criterion for showing that subsets of $\mathbb{R}^n$ are homeomorphic to an open or to a closed ball.

Lemma 1.4.

- (1) Let A be a bounded open convex subset of $\mathbb{R}^n$, then A is homeomorphic to the open n -dimensional ball $B^n := \{x \in \mathbb{R}^n \mid \|x\| < 1\}$.
- (2) Let A be a bounded closed convex subset of $\mathbb{R}^n$ such that the interior of A is non-empty. Then it follows that A is homeomorphic to the closed n -dimensional ball $\overline{B}^n = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$. More precisely there exists a homeomorphism $f: A \rightarrow \overline{B}^n$ with $\Phi(\partial A) = S^{n-1}$.

Examples.

- (1) It follows from Lemma 1.4 that the open cube $(0, 1)^n$ is homeomorphic to B^n . More generally, it follows from Lemma 1.4 that for any $r, s \in \mathbb{N}_0$ the product of balls $B^r \times B^s \subset \mathbb{R}^r \times \mathbb{R}^s = \mathbb{R}^{r+s}$ is homeomorphic to B^{r+s} .
- (2) It follows from Lemma 1.4 that any triangle, i.e. any subset of $\mathbb{R}^2$ of the form $A = \{P + sv + tw \mid s, t \in [0, 1] \text{ and } s + t \leq 1\}$ where $P \in \mathbb{R}^2$ and v, w are two linearly independent vectors, is homeomorphic to $\overline{B}^2$.

Proof.

- (2) Let A be a bounded closed convex subset of $\mathbb{R}^n$ such that the interior of A is non-empty. After a translation we can assume that 0 lies in the interior of A . Given

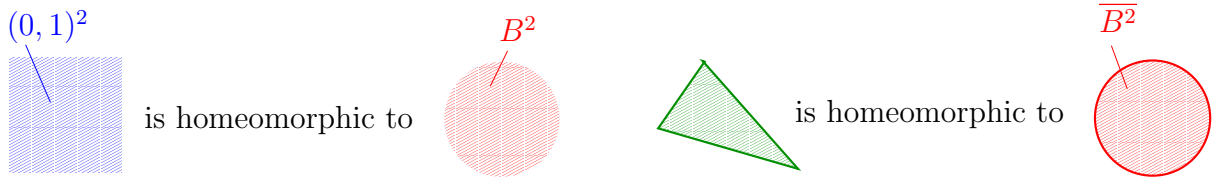


FIGURE 4.

$x \in A \setminus \{0\}$ we define

$$\rho(x) := \sup \{ \|rx\| \mid r \in \mathbb{R}_{>0} \text{ and } rx \in A \}$$

and

$$f(x) := x \cdot \frac{\rho(x)}{\|x\|}.$$

Since A is closed we have $f(x) \in A$.

Claim. The map $\rho: A \rightarrow \mathbb{R}$ is positive, bounded and continuous.

Since A is bounded it follows that ρ is bounded. It follows from the definition of ρ that $\rho(x)$ is always positive. Therefore it remains to show that ρ is continuous. Now let $x \in A$ and let $\epsilon > 0$. It suffices to show the following two statements:

- (a) there exists an open neighborhood U of x such that $\rho(y) > \rho(x) - \epsilon$ for all $y \in U$,
- (b) there exists an open neighborhood V of x such that $\rho(y) < \rho(x) + \epsilon$ for all $y \in V$.

We first show the existence of U . After possibly replacing ϵ by $\min\{\frac{1}{2}\rho(x), \epsilon\}$ we can suppose that $\epsilon \in (0, \rho(x))$.

Since 0 lies in the interior of A there exists an $\eta > 0$ such that $B_\eta(0) \subset A$. Since A is convex, the convex hull C of $B_\eta(0) \cup \{f(x)\}$ is still contained⁴ in A . Furthermore, since $B_\eta(0)$ is open it is straightforward to see that $C' := C \setminus \{f(x)\}$ is an open subset of $\mathbb{R}^n$. We denote by $S_{\rho(x)-\epsilon}^{n-1}$ the sphere of radius $\rho(x) - \epsilon$ around 0. The point $x \cdot \frac{\rho(x)-\epsilon}{\|x\|}$ lies on $S_{\rho(x)-\epsilon}^{n-1}$ and it lies in C' . Since C' is open there exists an open neighborhood U' on $S_{\rho(x)-\epsilon}^{n-1}$ that is contained in C' . We set

$$U := \{rz \mid z \in U' \text{ and } r \in (0, 1)\}.$$

This is an open neighborhood of x and for any $y \in U$ we have $(\rho(x) - \epsilon) \cdot y \in A$, i.e. for any $y \in A$ we have $\rho(y) \geq \rho(x) - \epsilon$. Thus we have shown that the desired neighborhood U exists. The existence of V is proved in a very similar way. We refer to [Be, Chapter 11.3] for full details. This concludes the proof of the claim.

⁴Why is that the case?

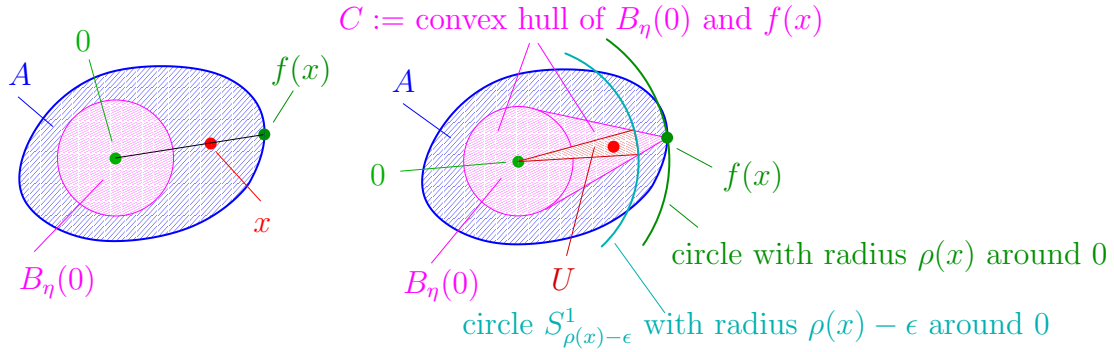


FIGURE 5. Illustration for the proof of Lemma 1.4.

We now consider the map

$$\begin{aligned} \Phi: A &\rightarrow \overline{B}^n \\ x &\mapsto \begin{cases} x \cdot \frac{1}{\rho(x)}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases} \end{aligned}$$

It is straightforward to verify that this map is bijective⁵ and it follows from the above claim⁶ that Φ is continuous. It now follows from Proposition 1.3 that Φ is a homeomorphism. We leave it as an exercise to verify that $\Phi(\partial A) = S^{n-1}$.

- (1) The first claim follows from (2) by applying (2) to the closure $\overline{A}$ ⁷ and by restricting the resulting homeomorphism $\Phi: \overline{A} \rightarrow \overline{B}^n$ to the interior. $\square$

We recall two definitions about metric spaces.

- (1) Given a subset A of a metric space X the diameter is defined as

$$\text{diameter}(A) := \sup\{d(a, b) \mid a, b \in A\} \in \mathbb{R}_{\geq 0} \cup \{\infty\}.$$

- (2) For any metric space X , any $x \in X$ and any non-empty subset A of X we refer to

$$d(x, A) := \inf\{d(x, a) \mid a \in A\} \in \mathbb{R}_{\geq 0}$$

as the *distance from x to A* .

For example the diameter of the cube $[0, 1]^n \subset \mathbb{R}^n$ is easily seen to be the n -th root of 2.

The following Lemma of Lebesgue was already stated and proved in Analysis II. As a warm-up exercise we prove it again.

⁵It follows easily from the convexity of A , the hypothesis that $\overset{\circ}{A} \neq \emptyset$ and the definition of Φ that for each $v \in S^{n-1}$ the map Φ restricts to a bijection on the “ray defined by v ”, i.e. it restricts to a bijection

$$A \cap \{rv \mid r \in \mathbb{R}_{>0}\} \rightarrow \overline{B}^n \cap \{rv \mid r \in \mathbb{R}_{>0}\}.$$

⁶From the continuity of δ it follows that Φ is continuous on $A \setminus \{0\}$. From the fact that δ is bounded it follows easily that Φ is continuous in 0.

⁷Why is the closure of a convex set again a convex set?

Lemma 1.5. (Lebesgue's Lemma) *Let K be a compact metric space and let $\{U_i\}_{i \in I}$ be an open covering of K . Then there exists a $\delta > 0$ such that for every subset A with $\text{diameter}(A) < \delta$ there exists an $i \in I$ with $A \subset U_i$.*

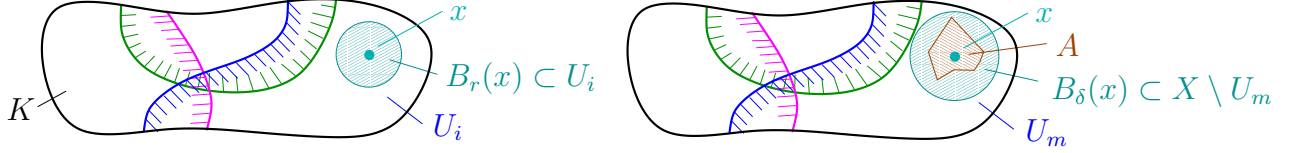


FIGURE 6. Illustration of the proof of Lebesgue's Lemma.

Proof. Since K is compact we can cover K with finitely many of the U_i 's. Put differently, without loss of generality we can assume that $I = \{1, \dots, n\}$ is a finite set.

If there exists an $i \in \{1, \dots, n\}$ with $K = U_i$, then any $\delta > 0$ has the desired property. Now suppose that this is not the case, i.e. we suppose that for any i we have $U_i \subsetneq K$.⁸ We first prove the following claim.

Claim. The function

$$\begin{aligned} f: K &\rightarrow \mathbb{R} \\ x &\mapsto f(x) := \frac{1}{n} \sum_{i=1}^n d(x, K \setminus U_i) \end{aligned}$$

is continuous and positive.

It follows easily from the definitions and the triangle inequality that f is continuous. Now let $x \in K$. We want to show that $f(x) > 0$. Since $K = U_1 \cup \dots \cup U_n$ there exists an i with $x \in U_i$. Since U_i is open there exists an $r > 0$ with $B_r(x) \subset U_i$. It follows that $d(x, K \setminus U_i) \geq r$, hence $f(x) \geq \frac{r}{n}$. This concludes the proof of the claim.

Since f is continuous and since K is compact it follows from Lemma 1.2 that the function f has a global minimum δ on K . It follows from the claim that this minimum δ is greater than 0. Now we want to show that this δ has the desired property. More precisely, we want to prove the following claim.

Claim. Let A be a subset of K with $\text{diameter}(A) < \delta$. Then there exists an $i \in \{1, \dots, n\}$ with $A \subset U_i$.

Let $x \in A$. We choose an $m \in \{1, \dots, n\}$ so that $d(x, K \setminus U_m)$ is maximal. We want to show that $A \subset U_m$. Since the diameter of A is less than δ we have $A \subset B_\delta(x)$. Thus it suffices to show that $B_\delta(x) \subset U_m$. Put differently, it we need to show that $d(x, K \setminus U_m) \geq \delta$. Indeed we have

$$\begin{aligned} d(x, K \setminus U_m) &\geq \frac{1}{n} \sum_{i=1}^n d(x, K \setminus U_i) = f(x) \geq \delta. \\ &\quad \uparrow \qquad \qquad \qquad \uparrow \\ &\text{since } d(x, K \setminus U_m) \geq d(x, K \setminus U_i) \quad \text{by the choice of } \delta \end{aligned}$$

⁸Where do we use in the subsequent argument that $U_i \subsetneq K$?

This concludes the proof of the claim. $\square$

In many cases the following corollary is even more useful.

Corollary 1.6. *Let $f: [0, 1]^n \rightarrow X$ be a map from the cube $[0, 1]^n$ to a topological space X and let $\{V_i\}_{i \in I}$ be an open covering of X . Then there exists an $N > 0$ such that for any $a_1, \dots, a_n \in \{0, \dots, N-1\}$ there exists an $i \in I$ such that*

$$f\left(\left[\frac{a_1}{N}, \frac{a_1+1}{N}\right] \times \dots \times \left[\frac{a_n}{N}, \frac{a_n+1}{N}\right]\right) \subset V_i.$$

The corollary thus says that if $f: [0, 1]^n \rightarrow X$ is a map, and if we are given an open covering of X , then we can always find a small grid on the cube, such that each cube of the grid gets mapped into one of the open sets covering X .

Proof. Let $f: [0, 1]^n \rightarrow X$ be a map from the cube $[0, 1]^n$ to a topological space and let $\{V_i\}_{i \in I}$ be an open covering of X . We apply Lemma 1.5 to the open covering $U_i := f^{-1}(V_i)$, $i \in I$ of K and we obtain a $\delta > 0$ such that for each subset A with $\text{diameter}(A) < \delta$ there exists an $i \in I$ with $A \subset U_i = f^{-1}(V_i)$, which means that $f(A) \subset V_i$. The corollary follows from the observation that for sufficiently large N any cube of side length $\frac{1}{N}$ has diameter less than δ . $\square$

1.2. Constructions of more topological spaces. In this section we will recall several ways to build new examples of topological spaces out of existing topological spaces.

Definition. Let $X_1, \dots, X_k$ be topological spaces. We say $U \subset X_1 \times \dots \times X_k$ is *open* if for each $(x_1, \dots, x_k) \in U$ there exist open neighborhoods U_i of x_i in X_i with $U_1 \times \dots \times U_k \subset U$.

In Analysis IV we had seen that the above definition does indeed define a topology on $X_1 \times \dots \times X_k$. We refer to this topology as the *product topology* on $X_1 \times \dots \times X_k$. We had seen in Analysis IV that the product $X_1 \times \dots \times X_k$ is Hausdorff if and only if each X_i is Hausdorff and that the product $X_1 \times \dots \times X_k$ is compact if and only if each X_i is compact.

Example. The topological space $\mathbb{R}^k \times \mathbb{R}^l$ is easily seen to be homeomorphic to $\mathbb{R}^{k+l}$. More precisely, the obvious map

$$\begin{aligned} \mathbb{R}^k \times \mathbb{R}^l &\rightarrow \mathbb{R}^{k+l} \\ ((x_1, \dots, x_k), (y_1, \dots, y_l)) &\mapsto (x_1, \dots, x_k, y_1, \dots, y_l) \end{aligned}$$

is a homeomorphism.

In the following we refer to $S^1 \times S^1$ as the *torus* and we refer to $(S^1)^n$ as the *n -dimensional torus*. We had seen in Analysis IV that the torus is indeed homeomorphic to the “standard torus” in $\mathbb{R}^3$.

Definition. Let $\sim$ be an equivalence relation on a topological space X . We denote by $p: X \rightarrow X/\sim$ the canonical projection map from X onto the set of equivalence classes $X/\sim$. We say $U \subset X/\sim$ is open if $p^{-1}(U) \subset X$ is open.

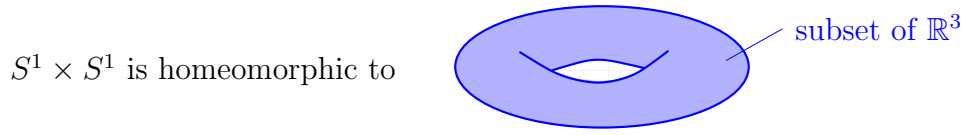


FIGURE 7.

It is straightforward to verify that this does indeed define a topology on $X/\sim$ and that the projection map $p: X \rightarrow X/\sim$ is continuous. We refer to this topology on $X/\sim$ as the *quotient topology* and we refer to $X/\sim$ as a quotient space of X . If X is compact, then it follows from Lemma 1.2 that $X/\sim$ is also compact.

Example. Let X be a topological space and let $A \subset X$ be a subset. For $P, Q \in X$ we define

$$P \sim Q \iff P = Q \text{ or } P, Q \text{ both lie in } A.$$

This is easily seen to be an equivalence relation on X . We write $X/A := X/\sim$. In Analysis IV we had for example seen that $\overline{B^2}/S^1$ is homeomorphic to S^2 . More generally it follows easily from Proposition 1.3 that the map

$$\begin{aligned} \Omega: \overline{B^n}/S^{n-1} &\rightarrow S^n \\ [r \cdot v] &\mapsto \cos(\pi r) \cdot (0, \dots, 0, 1) + \sin(\pi r) \cdot v, \text{ with } r \in [0, 1], v \in S^{n-1} \end{aligned}$$

is a homeomorphism. For $n = 1$ this homeomorphism is illustrated in Figure 8.

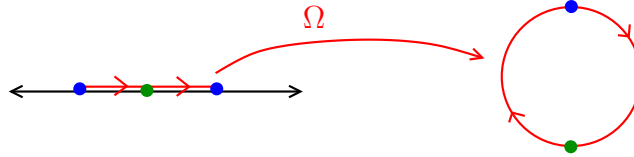


FIGURE 8.

We also recall the following definition from Analysis IV.

Definition. Let X be a set and let G be a group with trivial element e .

- (1) An *action* of G on X is a map

$$\begin{aligned} G \times X &\rightarrow X \\ (g, x) &\mapsto g \cdot x \end{aligned}$$

with the following properties

$$\begin{aligned} e \cdot x &= x, & \text{for all } x \in X, \\ g \cdot (h \cdot x) &= (gh) \cdot x, & \text{for all } x \in X \text{ and } g, h \in G. \end{aligned}$$

- (2) The action is called *free*, if $g \cdot x = x$ for some $x \in X$ implies that $g = e$.
 (3) We say G acts *transitively*, if for any x and y in X there exists a $g \in G$ with $g \cdot x = y$.

- (4) If X is a topological space, then we say that the action is *continuous*, if for every $g \in G$ the map

$$\begin{aligned} X &\rightarrow X \\ x &\mapsto g \cdot x \end{aligned}$$

is continuous.⁹

It is straightforward to verify that if a group G acts on a set X , then

$$x \sim y \iff \text{there exists a } g \in G \text{ such that } g \cdot x = y$$

is an equivalence relation on X . We write

$$X/G := X/\sim.$$

If X is a topological space, then we view X/G as a topological space equipped with the quotient topology. On page 23 we had pointed out that the projection map $X \rightarrow X/G$ is continuous. In Lemma 18.9 of Analysis IV we had seen the projection map is also open.

Examples.

- (1) The group $G = \mathbb{Z}^n$ acts on $X = \mathbb{R}^n$ by addition. This action is evidently free and continuous. In Analysis IV we had seen that

$$\begin{aligned} \mathbb{R}^n/\mathbb{Z}^n &\rightarrow (S^1)^n \\ [(t_1, \dots, t_n)] &\mapsto (e^{2\pi i t_1}, \dots, e^{2\pi i t_n}) \end{aligned}$$

is a homeomorphism.

- (2) Let $X = \mathbb{R} \times (-1, 1)$ and $G = \mathbb{Z}$. The map

$$\begin{aligned} \mathbb{Z} \times (\mathbb{R} \times [-1, 1]) &\rightarrow \mathbb{R} \times [-1, 1] \\ (n, (x, y)) &\mapsto (x + n, (-1)^n y) \end{aligned}$$

defines an action that is free and continuous. The quotient is homeomorphic to the Möbius band.¹⁰

- (3) Let $X = S^n$ and $G = \{\pm 1\}$. The map

$$\begin{aligned} \{\pm 1\} \times S^n &\rightarrow S^n \\ (\epsilon, P) &\mapsto \epsilon \cdot P \end{aligned}$$

defines an action that is free and continuous. We refer to the quotient space $S^n/\{\pm 1\}$ as the *n-dimensional real projective space* $\mathbb{RP}^n$.¹¹ In exercise sheet 1 we will show that $\mathbb{RP}^1$ is homeomorphic to S^1 .

⁹If the action is continuous, then the map $x \mapsto g \cdot x$ is in fact a homeomorphism with inverse map given by $x \mapsto g^{-1} \cdot x$.

¹⁰Depending on the point of view, this can of course also be taken as the definition of the Möbius band.

¹¹For $n = 2$ we sometimes refer to $\mathbb{RP}^2$ as the real projective plane.

- (4) Let $X = \mathbb{R}^{n+1} \setminus \{0\}$ and let $G = \mathbb{R} \setminus \{0\}$, where we view G as a group via multiplication. The map

$$\begin{aligned} (\mathbb{R} \setminus \{0\}) \times (\mathbb{R}^{n+1} \setminus \{0\}) &\rightarrow \mathbb{R}^{n+1} \setminus \{0\} \\ (r, P) &\mapsto r \cdot P \end{aligned}$$

defines an action that is free and continuous. It is straightforward to see that the map

$$\begin{aligned} \mathbb{RP}^n = S^n / \{\pm 1\} &\rightarrow (\mathbb{R}^{n+1} \setminus \{0\}) / (\mathbb{R} \setminus \{0\}) \\ [P] &\mapsto [P] \end{aligned}$$

defines a homeomorphism. Our new point of view on real projective spaces has the advantage that we can create a different class of examples of topological spaces by replacing $\mathbb{R}$ by $\mathbb{C}$. More precisely, let $X = \mathbb{C}^{n+1} \setminus \{0\}$ and let $G = \mathbb{C} \setminus \{0\}$. The map

$$\begin{aligned} (\mathbb{C} \setminus \{0\}) \times (\mathbb{C}^{n+1} \setminus \{0\}) &\rightarrow \mathbb{C}^{n+1} \setminus \{0\} \\ (z, P) &\mapsto z \cdot P \end{aligned}$$

defines an action that is free and continuous. The quotient space $(\mathbb{C}^{n+1} \setminus \{0\}) / (\mathbb{C} \setminus \{0\})$ is called the *n-dimensional complex projective space* $\mathbb{CP}^n$. In exercise sheet 1 we will show that $\mathbb{CP}^1$ is homeomorphic to S^2 .^{12 13}

- (5) Let $X = \mathbb{R}$ and $G = \{\pm 1\}$. The map

$$\begin{aligned} \{\pm 1\} \times \mathbb{R} &\rightarrow \mathbb{R} \\ (\epsilon, x) &\mapsto \epsilon \cdot x \end{aligned}$$

defines an action that is continuous. But this action is not free, since $(-1) \cdot 0 = 0$, but -1 is not the trivial element in the group $G = \{\pm 1\}$. In Analysis IV we had shown that $\mathbb{R} / \{\pm 1\}$ is homeomorphic to the half-open interval $[0, \infty)$.

Finally we recall Lemma 17.4 from Analysis IV which is used so frequently, that later on we will no longer cite it explicitly.

Lemma 1.7. *Let $\sim$ be an equivalence relation on a topological space X and let $f: X \rightarrow Y$ be a continuous map with the property that $f(x) = f(y)$ whenever $x \sim y$. Then there exists a unique continuous map $g: X / \sim \rightarrow Y$, such that $f = g \circ p$, i.e. such that the following*

¹²Here is a fun fact: if we consider $\mathbb{CP}^4 / \sim$ modulo the equivalence relation defined by complex conjugation, i.e. where $[(z, w)] \sim [(\bar{z}, \bar{w})]$, then the quotient space is homeomorphic to S^4 . This statement is not entirely trivial to prove, see e.g. [Hn, Mas4].

¹³The definition of projective space makes sense for any field, even for fields of non-zero characteristic. If $\mathbb{F}$ is a field of characteristic p we can still define $\mathbb{FP}^n = (\mathbb{F}^{n+1} \setminus \{0\}) / (\mathbb{F} \setminus \{0\})$. A priori this is just a set, but together with the Zariski topology it actually becomes an unusual, but interesting topological space. For any field the map

$$\begin{aligned} \mathbb{FP}^n = (\mathbb{F}^{n+1} \setminus \{0\}) / (\mathbb{F} \setminus \{0\}) &\rightarrow \{\text{all one-dimensional subspaces of } \mathbb{F}^{n+1}\} \\ [v] &\mapsto \mathbb{F} \cdot v \end{aligned}$$

is a bijection. Thus we can view $\mathbb{FP}^n$ as the set of all lines through the origin.

diagram of maps commutes:

$$\begin{array}{ccc} X & \xrightarrow{p} & X/\sim \\ & \searrow f & \downarrow g \\ & & Y. \end{array}$$

1.3. Further examples of topological spaces. In this section we will recall several other examples of topological spaces that we had already introduced in Analysis IV. Before we do so, we recall the following definition: Suppose we are given a relation $\sim$ on a set X .¹⁴ We say $x, y \in X$ are equivalent if there exists a sequence $x = x_1, \dots, x_k = y$ of elements in X such that for all $i = 1, \dots, k-1$ the following holds: either $x_i \sim x_{i+1}$ or $x_{i+1} \sim x_i$. We then write again $x \sim y$ if x and y are equivalent. It is straightforward to see that this is now indeed an equivalence relation. We say it is *generated* by the initial relation.

Example. The relations $x \sim (x+1)$ with $x \in \mathbb{R}$ on $\mathbb{R}$ generate the familiar equivalence relation $x \sim y \Leftrightarrow x - y \in \mathbb{Z}$.

We will now see that many familiar spaces can be described as quotient spaces.

(1) We consider $X = [0, 1] \times [0, 1] \subset \mathbb{R}^2$ and the equivalence relation which is generated by

$$(x, 0) \sim (x, 1) \quad \text{for all } x \in [0, 1].$$

The quotient topological space $X/\sim$ is obtained from the square $X = [0, 1] \times [0, 1]$ by identifying each point on the upper edge with the corresponding point on the lower edge. Put differently, “we glue the upper edge to the lower edge”. Using Proposition 1.3 one can easily show that the map

$$\begin{aligned} X/\sim &\rightarrow S^1 \times [0, 1] \\ (s, t) &\mapsto (e^{2\pi i t}, s) \end{aligned}$$

is a homeomorphism. We refer to $X/\sim$ and also to $S^1 \times [0, 1]$ as a cylinder or an annulus.

the points $(x, 1)$ and $(x, 0)$ are equivalent

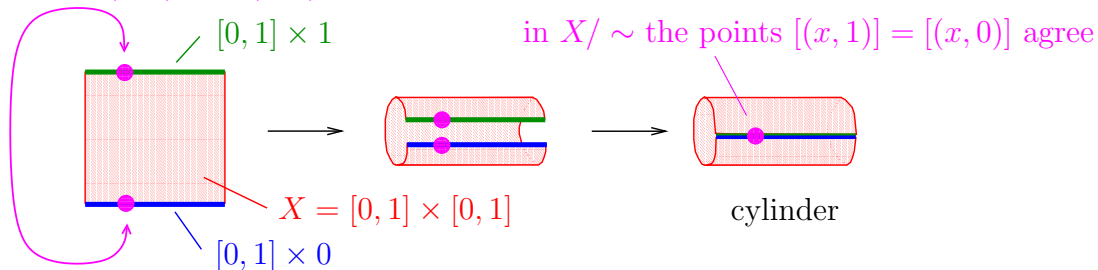


FIGURE 9. Construction of a cylinder by gluing two edges.

¹⁴A relation on a set X is nothing but a subset A of $X \times X$, for $x, y \in X$ we write $x \sim y$ if $(x, y) \in A$.

- (2) Now we consider $X = [0, 1] \times [0, 1] \subset \mathbb{R}^2$, this time with the equivalence relation which is generated by

$$(0, y) \sim (1, 1 - y) \quad \text{for all } y \in [0, 1].$$

The quotient space $X/\sim$ is obtained from the square X by gluing the edge on the left to the edge on right, but this time “we glue the edges with a twist”. This space is homeomorphic to the Möbius band as defined on page 27.¹⁵

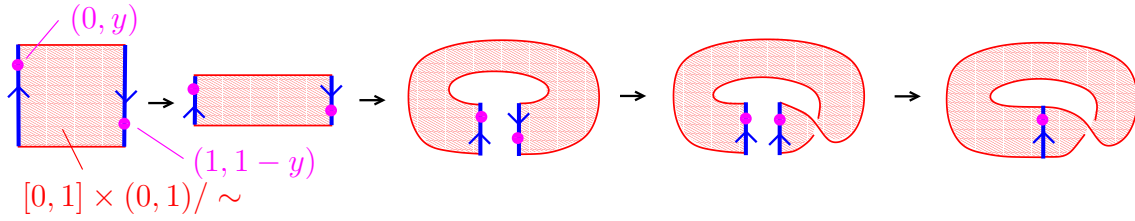


FIGURE 10. The Möbius band.

- (3) Now we consider again $X = [0, 1] \times [0, 1] \subset \mathbb{R}^2$, but this time with the equivalence relation which is generated by

$$(x, 0) \sim (x, 1) \quad \text{for all } x \in [0, 1]$$

and by

$$(0, y) \sim (1, y) \quad \text{for all } y \in [0, 1].$$

Put differently, we obtain the quotient space $X/\sim$ by gluing the top edge to the bottom edge and the left-hand edge to the right-hand edge, each time without adding a twist. Figure 11 illustrates that the quotient space $X/\sim$ is indeed a torus. The inclusion

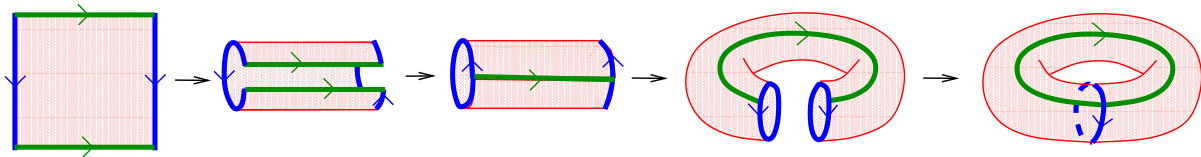


FIGURE 11. By gluing of the opposite edges of a square we obtain a torus.

$X \rightarrow \mathbb{R}^2$ descends to a map $X/\sim \rightarrow \mathbb{R}^2/\mathbb{Z}^2$. Using Proposition 1.3 one can show that this is a homeomorphism.¹⁶

¹⁵In fact the map $[0, 1] \times [0, 1] \rightarrow \mathbb{R} \times [-1, 1]$ given by $(x, y) \mapsto (x, 2y - 1)$ induces a homeomorphism $[0, 1] \times [0, 1]/\sim \rightarrow (\mathbb{R} \times [-1, 1])/\mathbb{Z}$.

¹⁶Why is $\mathbb{R}^2/\mathbb{Z}^2$ Hausdorff?

- (4) Now we consider $X = [0, 1] \times [0, 1] \subset \mathbb{R}^2$, this time with the equivalence relation which is generated by

$$(x, 0) \sim (x, 1) \quad \text{for all } x \in [0, 1]$$

and by

$$(0, y) \sim (1, 1 - y) \quad \text{for all } y \in [0, 1].$$

Thus we obtain the quotient space $X/\sim$ by gluing the upper edge to the lower edge and by gluing the edge on the left to the edge on the right with a twist. The resulting topological space is called the *Klein bottle*.

every map from the Klein bottle to $\mathbb{R}^3$ has a self-intersection

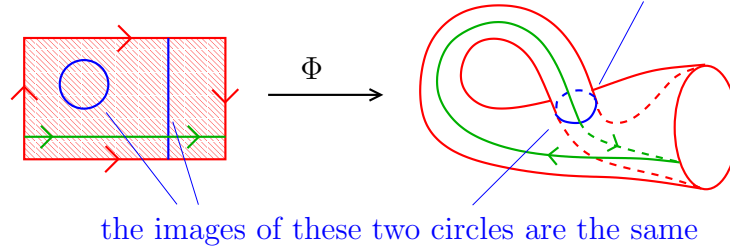


FIGURE 12. The definition of the Klein bottle.

- (5) We denote by E_8 the regular octagon in $\mathbb{C}$ with the vertices $Q_k = e^{2\pi i k/16}$ where $k = 1, 3, \dots, 15$. Given two points $A, B \in \mathbb{R}^2 = \mathbb{C}$ we denote by $\overline{AB}$ the euclidean segment from A to B . Furthermore, given $\varphi \in \mathbb{R}$ we denote by $r_\varphi: \mathbb{C} \rightarrow \mathbb{C}$ the reflection in the euclidean line $\{te^{i\varphi} \mid t \in \mathbb{R}\}$. We denote by $\sim$ the equivalence relation on E_8 which is generated by

$$P \in \overline{Q_{2k-1}Q_{2k+1}} \sim r_{2\pi(2k+2)/16}(P) \in \overline{Q_{2k+3}Q_{2k+5}}$$

for $k = 0, 1, 4, 5$. These four relations are sketched in Figure 13. We refer to the topological space $E_8/\sim$ as the *surface of genus 2*. The equivalence relation is sketched in Figure 13. Furthermore, in Figure 14 we indicate how $E_8/\sim$ is related to the actual “physical” surface of genus 2 in $\mathbb{R}^3$. The same way we can define the *surface of genus g* for any $g \geq 3$. More precisely, we take a regular $4g$ -gon in $\mathbb{C}$ and for $j = 1, \dots, g$ we identify the $(4j + 1)$ -st edge with the $(4j + 3)$ -rd edge and the $(4j + 2)$ -nd edge with the $(4j + 4)$ -th edge. Hereby each identification is given by a reflection. For $g = 3$ this construction is sketched in Figure 15.

Finally we refer to the sphere S^2 as the *surface of genus 0* and we refer to the torus $S^1 \times S^1$ as the *surface of genus 1*.

We have now introduced many examples of topological spaces, some of them, for example the surface of genus 2, can easily be pictured as subsets of “our universe”, i.e. they can be viewed as subsets of $\mathbb{R}^3$. This raises the following question.

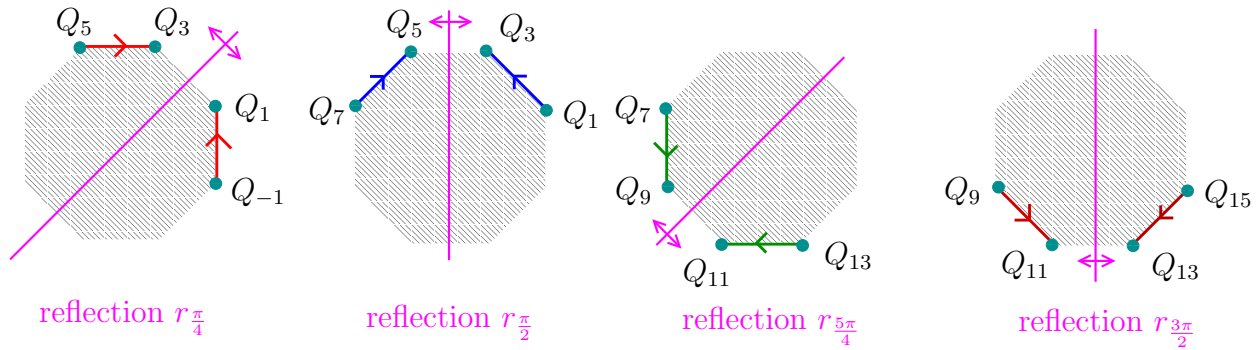


FIGURE 13.

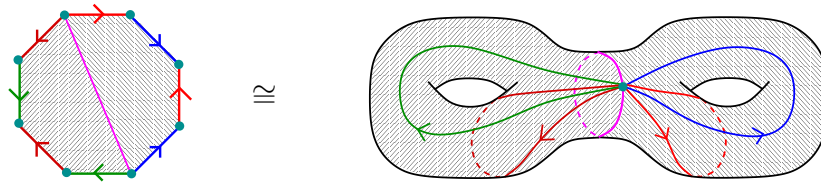


FIGURE 14.

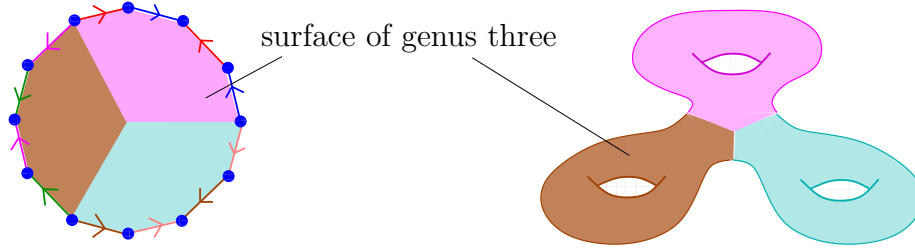


FIGURE 15. The surface of genus three.

Question 1.8. Can the real projective space $\mathbb{RP}^2$ and the Klein bottle be viewed¹⁷ as subsets of $\mathbb{R}^3$?

1.4. The two notions of connected topological spaces. We had already seen in Analysis II that there are two notions of connectedness of a topological space. We recall the definitions and we recall some basic facts.

Definition.

- (1) We say that a topological space X is *path-connected* if given any two points x and y there exists a path from x to y , i.e. there exists a map $\gamma: [0, 1] \rightarrow X$ with $\gamma(0) = x$ and $\gamma(1) = y$.

¹⁷The word “view” is intentionally vague.

(2) We say that a topological space X is *connected* if the following holds¹⁸

$$\text{if } X = U \sqcup V \text{ is the disjoint union of open sets } U, V \implies X = U \text{ or } X = V.$$

Both notions of connectedness have their advantages and disadvantages. For example, if a space is path-connected, it is often easy to verify so, one just needs to find enough paths. In particular it follows almost immediately from the definitions that any convex subset of $\mathbb{R}^n$ is path-connected. On the other hand it is often easier to verify that a given space X is not connected, one just needs to find disjoint open non-empty sets that cover X .

For most spaces the two notions of connectedness agree. For example we have the following lemma which we had proved as Lemma 6.2 of Analysis III.

Lemma 1.9. *Every path-connected topological space is also connected.*

The converse to Lemma 1.9 is unfortunately not always correct. For example consider the space

$$X := \{(0, y) \mid y \in [-1, 1]\} \cup \{(x, \sin(\frac{1}{x})) \mid x \in (0, \pi]\} \subset \mathbb{R}^2.$$

The space X is the union of the graph $x \mapsto \sin(\frac{1}{x})$ for $x \in (0, \pi]$ and an interval on the y -axis. We sketch the space X in Figure 16. In Analysis III we had seen that X is connected but not path-connected. The fact that an example of a space that is connected but not

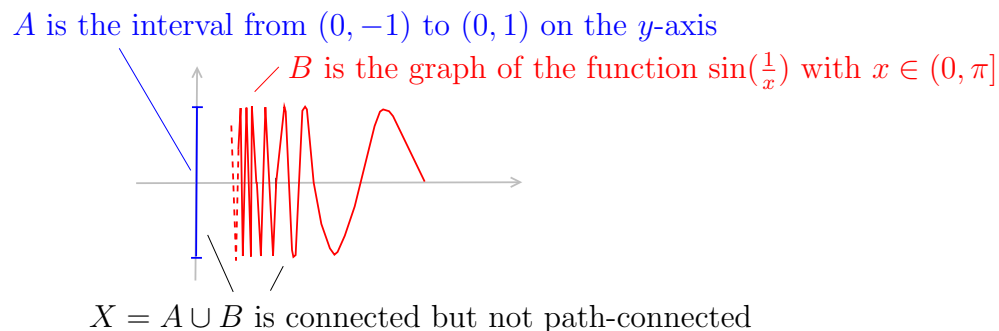


FIGURE 16.

path-connected is so “wild” indicates that perhaps for most “reasonable” spaces the two notions of connectedness and path-connectedness agree.

The next lemma gives a criterion for the union of (path-) connected subsets to be (path-) connected.

Lemma 1.10. *Let X be a topological space and let $\{B_i\}_{i \in I}$ be a family of subsets such that the intersection $\bigcap_{i \in I} B_i$ is non-empty.*

(1) *If each B_i , $i \in I$ is path-connected, then $\bigcup_{i \in I} B_i$ is path-connected.*

¹⁸Recall that we write $X = U \sqcup V$ if X is the disjoint union of U and V , i.e. if $X = U \cup V$ and if $U \cap V = \emptyset$.

- (2) If each B_i , $i \in I$ is connected, then $\bigcup_{i \in I} B_i$ is connected.

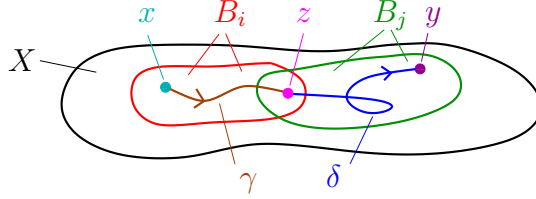


FIGURE 17. Illustration for the proof of Lemma 1.10 (1).

Proof. Let X be a topological space and let $\{B_i\}_{i \in I}$ be a family of subsets such that the intersection $\bigcap_{i \in I} B_i$ contains a point z . We write $Z := \bigcup_{i \in I} B_i$.

- (1) We suppose that each B_i , $i \in I$ is path-connected. So let $x, y \in Z$. There exists an $i \in I$ with $x \in B_i$ and there exists a $j \in I$ with $y \in B_j$. Since B_i and B_j are path-connected we can pick a path $\gamma: [0, 1] \rightarrow B_i$ from x to z and we can pick a path $\delta: [0, 1] \rightarrow B_j$ from z to y . It is clear that

$$\begin{aligned} [0, 1] &\rightarrow Z \\ t &\mapsto \begin{cases} \gamma(2t), & \text{if } t \in [0, \frac{1}{2}] \\ \delta(2t - 1), & \text{if } t \in [\frac{1}{2}, 1] \end{cases} \end{aligned}$$

is a path¹⁹ from x to y .

- (2) We suppose that each B_i , $i \in I$ is connected. We write $Z = U \sqcup V$ where U and V are open subsets. Without loss of generality we can assume that $z \in U$. For each $i \in I$ we have a decomposition $B_i = (U \cap B_i) \sqcup (V \cap B_i)$ into disjoint open subsets (here we use the definition of the subspace topology). Since B_i is connected and since $U \cap B_i \neq \emptyset$ we see that $B_i = U \cap B_i$. Thus we have shown that

$$U = \bigcup_{i \in I} (B_i \cap U) = \bigcup_{i \in I} B_i = Z. \quad \square$$

In exercise sheet 1 we will prove the following lemma which is sometimes useful.

Lemma 1.11.

- (1) Let X and Y be topological spaces. Then X and Y are (path)-connected if and only if $X \times Y$ is (path)-connected.
- (2) Let X be a topological space and let A and B be connected subsets of X . If $A \cap B \neq \emptyset$, then $A \cup B$ is also connected.

By a clever use of the notion of connectedness one can sometimes prove statements which at first glance have nothing to do with connectedness. For example we can now prove the following lemma.

¹⁹It is straightforward to verify that this map is indeed continuous.

Lemma 1.12. *The topological spaces $\mathbb{R}$ and $\mathbb{R}^2$ are not homeomorphic.*

Example. The proof of Lemma 1.12 also shows that the interval $[0, 1]$ is not homeomorphic to the square $[0, 1]^2$. On the other hand there exist surjective maps $[0, 1] \rightarrow [0, 1]^2$, these are usually called *space filling curve*. We refer to [Sa] for a long list of constructions of such maps. We now see that a space filling curve cannot be injective, since otherwise it would, by Proposition 1.3, be a homeomorphism.

Later on we will study the question whether for $k \neq l$ the topological spaces $\mathbb{R}^k$ and $\mathbb{R}^l$ can be homeomorphic.

Proof. Let us suppose that there exists a homeomorphism $\varphi: \mathbb{R} \rightarrow \mathbb{R}^2$. Let $P \in \mathbb{R}$ be a point. Then φ restricts to a homeomorphism from $\mathbb{R} \setminus \{P\}$ to $\mathbb{R}^2 \setminus \{\varphi(P)\}$. The topological space $\mathbb{R} \setminus \{P\} = (-\infty, P) \cup (P, \infty)$ is not path-connected²⁰, while one can easily verify that $\mathbb{R}^2 \setminus \{\varphi(P)\}$ is path-connected. But as we had remarked on page 17, homeomorphic spaces have the same topological properties. In particular two homeomorphic spaces are either both path-connected or they are both not path-connected. Thus we have obtained a contradiction. $\square$

We recall that a topological space X is called *discrete*, if it is equipped with the discrete topology. This is equivalent to saying that each point in X is an open set in X . For example, if we consider $X = \mathbb{Z}$ with the subspace topology coming from $\mathbb{R}$, then $X = \mathbb{Z}$ is discrete.

Lemma 1.13.

- (1) *If X is connected and if Y consists of components $Y_i, i \in I$, then given any map $f: X \rightarrow Y$ there exists an $i \in I$ with $f(X) \subset Y_i$.*
- (2) *Every map $f: X \rightarrow Y$ from a connected topological space X to a discrete topological space is constant.*

The second statement was proved in Analysis IV. Almost the same elementary proof also implies the first statement.

1.5. The (path-) components of a topological spaces.

Definition. Let x and y be two points in a topological space X . We say that the points x and y are *path-equivalent*, if there exists a map $\gamma: [0, 1] \rightarrow X$ with $\gamma(0) = x$ and $\gamma(1) = y$.

Lemma 1.14. *For every topological space X the notion of path-equivalence is indeed an equivalence relation.*

Proof. All properties of an equivalence relation are trivial except possibly for transitivity. The fact that path-equivalence is transitive follows immediately from the argument in Lemma 1.10 (2). $\square$

Definition. Let X be a topological space.

²⁰Why is this space not path-connected? How can one show this?

- (1) We call the path-equivalence classes of points the *path-components* of X .
- (2) We denote by $\pi_0(X)$ the set of path-components of X .²¹

Example.

- (1) In [Bk, Chapter 9.2] it is shown that the general linear group

$$\mathrm{GL}(n, \mathbb{R}) = \{A \in M(n \times n; \mathbb{R}) \mid \det(A) \neq 0\} \subset M(n \times n; \mathbb{R}) = \mathbb{R}^{n^2}$$

has precisely two path-components, namely

$$\mathrm{SL}(n, \mathbb{R}) = \{A \in \mathrm{GL}(n, \mathbb{R}) \mid \det(A) > 0\} \quad \text{and} \quad \{A \in \mathrm{GL}(n, \mathbb{R}) \mid \det(A) < 0\}.$$

Put differently, two invertible matrices lie in the same path-component if and only if the signs of their determinants agree. The same statement also holds for $O(n)$, more precisely $O(n)$ has precisely two path-components, namely

$$\mathrm{SO}(n, \mathbb{R}) = \{A \in O(n, \mathbb{R}) \mid \det(A) > 0\} \quad \text{and} \quad \{A \in O(n, \mathbb{R}) \mid \det(A) < 0\}.$$

- (2) It is straightforward to verify that each point in $\mathbb{Q}$ is in fact already a path-component of $\mathbb{Q}$.

The following lemma follows immediately from the definitions.

Lemma 1.15. *Let X be a topological space. Then the following statements hold:*

- (1) X is the disjoint union of its path-components,
- (2) each path-component of X is path-connected,
- (3) each path-component C of X is a maximal path-connected subset, i.e. if $C \subsetneq D$, then D is not path-connected.

Our goal now is to define the “components” of a topological space that should play the same role as the path-components, i.e. we want to introduce the notion of “components” such that the statement of Lemma 1.15 with “path-connected” replaced by “connected” and “path-component” replaced by “component”.

We start out with the following definition.

Definition. Let x and y be two points in a topological space X . We say that the points x and y are *equivalent*, if there exists a connected subset of X that contains both x and y .

Lemma 1.16. *For every topological space X the above notion of equivalence is indeed an equivalence relation.*

Proof. All properties of an equivalence relation are trivial except for transitivity. So suppose that x and y are equivalent and that y and z are equivalent. By definition there exists a connected subset A that contains x and y and there exists a connected subset B that contains y and z . By Lemma 1.10 (2) the union $A \cup B$ is connected and it thus follows that x and z are equivalent. $\square$

²¹The notation “ $\pi_0(X)$ ” might look rather odd at the moment, but it will become clearer later on why this is a perfectly sensible notation.

Definition. Let X be a topological space. We call the equivalence classes of points on X the *components* of X .

Example.

- (1) The topological space illustrated in Figure 16 has two path-components but only a single component.
- (2) It is straightforward to verify that each point in $\mathbb{Q}$ is in fact already a component of $\mathbb{Q}$.

The following is now the promised analogue of Lemma 1.17.

Lemma 1.17. *Let X be a topological space. Then the following statements hold:*

- (1) *X is the disjoint union of its components,*
- (2) *each component of X is connected,*
- (3) *each component C of X is a maximal connected subset, i.e. if $C \subsetneq D$, then D is not connected.*

Proof. Let X be a topological space. The first statement is clear. Now let C be a component of X . We pick $c \in C$. For each $d \in C$ there exists by definition a connected subset Y_d of X that contains c and d . We have

$$C = \bigcup_{c \in C} \{c\} \subset \bigcup_{c \in C} Y_c \subset C \quad \text{which implies that} \quad C = \bigcup_{c \in C} Y_c.$$

$\uparrow$

by definition all points in Y_d are equivalent to c , thus we have $Y_d \subset C$

It follows from Lemma 1.10 that C is connected. Finally the last statement of the lemma follows immediately from the definitions. $\square$

We conclude the discussion of the components of a topological space with the following lemma.

Lemma 1.18. *Let X be a topological space.*

- (1) *Every component of X is closed.*
- (2) *If X has only finitely many components, then each component is also open.*

Example. We consider $X = \mathbb{Q}$. There every point is already a component. We see that every component is closed, but in this case no component is an open subset. Therefore we cannot drop the condition on the number of components in the second statement of the lemma.

Proof. Let X be a topological space.

- (1) Let C be a component of X . We need to show that $X \setminus C$ is open. So let $P \in X \setminus C$. By Lemma 1.17 we know that $C \cup \{P\}$ is not connected. Therefore there exists a decomposition $C \cup \{P\} = U \sqcup V$ with U and V open subsets such that both U and V are non-empty. Without loss of generality $C \cap U \neq \emptyset$. Since C is connected and since $C = (U \cap C) \sqcup (V \cap C)$ we see that $C = U \cap C$. Since V is also non-empty we see that $V = \{P\}$. By the definition of the subspace topology that means that there exists an open subset W of X such $W \cap (C \cup \{P\}) = V = \{P\}$. But this

shows that W is an open neighborhood of P that does not intersect C . We have thus shown that the complement of C is open.

- (2) Now suppose that X has only finitely many components. Let C be a component of X . We already know from (1) that C is closed. By definition the complement C is the union of all other components. Since there are only finitely many and since they are closed by (1) it follows that the complement of C is closed, i.e. C is open. $\square$

Remark. Let X be a topological space. We say that the points x and y are *quasi-equivalent*, if the following holds

$$\text{if } X = U \sqcup V^{22} \text{ with } U, V \text{ open} \implies \text{either } x, y \in U \text{ or } x, y \in V.$$

It is straightforward to see that this indeed an equivalence on X . The equivalence classes of this equivalence relation are called quasi-components. It follows immediately from the definitions that components are contained in quasi-components. But the notions are different, for example it is shown in [SS, p. 137] that there exist topological spaces such that quasi-components are not connected.

1.6. Local properties.

Definition. Let P be a property of topological spaces. We say a topological space X is *locally P* if given any $Q \in X$ and any neighborhood U of Q there exists an open neighborhood V of Q that is contained in U that has the property P .

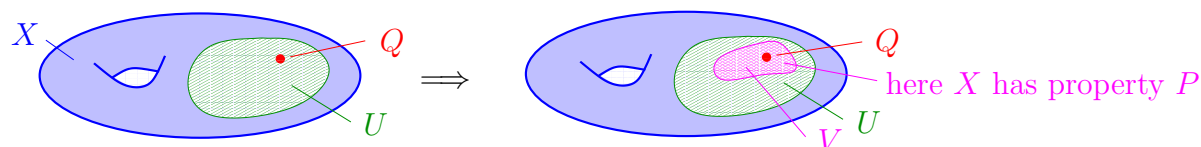


FIGURE 18.

Examples.

- (1) Every open subset of $\mathbb{R}^n$ is locally path-connected. Indeed, suppose A is an open subset of $\mathbb{R}^n$. Let $Q \in A$ and let U be a neighborhood of Q in A . Then there exists an $\epsilon > 0$ such that $B_\epsilon(x) \subset U$, but the ϵ -ball $B_\epsilon(x)$ is of course path-connected.
- (2) The space $X = (-2, 1) \cup [3, 6] \subset \mathbb{R}$ with the usual topology is not path-connected but it is locally path-connected.²³ On the other hand $X = \mathbb{Q}$ is not locally path-connected.
- (3) The line with two zeros, i.e. the topological space $\mathbb{R} \cup \{*\}$ with the topology from page 14, is not Hausdorff, but one can easily show that $\mathbb{R} \cup \{*\}$ is locally Hausdorff.

²³Why is that?

Lemma 1.19. *Let X be a topological space that is locally path-connected topological space. Then*

$$X \text{ is connected} \iff X \text{ is path-connected.}$$

Proof. The “ $\Leftarrow$ ”-direction is an immediate consequence of Lemma 1.9. We will see in exercise sheet 1 that if X is locally path-connected, then the converse also holds. $\square$

Remark. The term “locally P ” is often used in the literature with subtly different definitions. Here are three definitions that are often used in the literature:

- (A) A topological space X is called *locally P* , if given any $Q \in X$ and any neighborhood U of Q there exists an open neighborhood V of Q that is contained in U that has the property P .
- (B) A topological space X is called *locally P* , if given any $Q \in X$ there exists an open neighborhood V of Q that has the property P .
- (C) A topological space X is called *locally P* , if given any $Q \in X$ there exists a neighborhood²⁴ V of Q that has the property P .

Here definition (A) is the one we are using throughout the paper. Definitions (B) and (C) are less strict, for example any path-connected topological space is also locally path-connected according to definitions (B) and (C). Furthermore, one can easily see that according to the definition (C) the space $\mathbb{R}$ is locally compact, but it is not locally compact according to definitions (A) and (B). So if one sees the notion “locally P ” in the literature one needs to look up the definition to make sure what convention is used.

Definition. Let P be a property of maps between topological spaces. We say that a map $f: X \rightarrow Y$ is *locally P* if given any $Q \in X$ there exists an open neighborhood U of Q and an open neighborhood V of $f(Q)$ such that $f: U \rightarrow V$ has the property P .

Example. A map $f: X \rightarrow Y$ between two topological spaces is a *local homeomorphism* if given any $Q \in X$ there exists an open neighborhood U of Q and an open neighborhood V of $f(Q)$ such that $f: U \rightarrow V$ is a homeomorphism. For example the projection map $f: \mathbb{R} \rightarrow \mathbb{R}/\mathbb{Z}$ is a local homeomorphism.²⁵

We will often use the statement of the following lemma without making reference to the lemma.

Lemma 1.20. *Let $f: X \rightarrow Y$ be a map between two topological spaces (for once we do not assume that f is continuous). If f is locally continuous, then f itself is continuous.*

Proof. Let $V \subset Y$ be open. We need to show that $f^{-1}(V)$ is open. It suffices to show that given any $Q \in f^{-1}(V)$ there exists an open neighborhood U of P with $U \subset f^{-1}(V)$. So let $Q \in f^{-1}(V)$. Since f is locally continuous there exists an open neighborhood A of Q and an open neighborhood B of $f(Q)$ such that $f: A \rightarrow B$ is continuous. We put

²⁴Recall that a neighborhood does not have to be open.

²⁵Why is it a local homeomorphism?

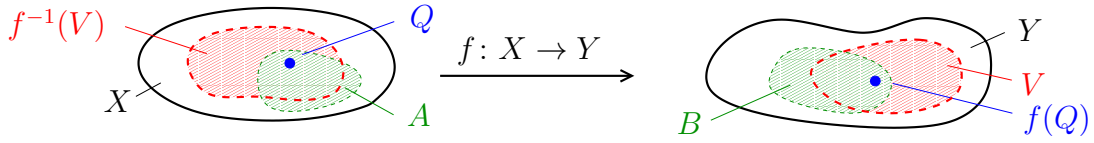


FIGURE 19. Illustration for the proof of Lemma 1.20

$U = f^{-1}(B \cap V)$. Since $f: A \rightarrow B$ is continuous the set U is indeed an open neighborhood of Q . $\square$

1.7. Graphs and topological realizations of graphs. We start out with the following rather abstract definition.

Definition.

- (1) An *abstract graph* G is a triple (V, E, φ) where V is a non-empty set, E is a set and φ is a map

$$\varphi: E \rightarrow \{\text{subsets of } V \text{ with one or two elements}\}.$$

The elements of V are called *vertices of G* and the elements of E are called the *edges of G* . Furthermore, given $e \in E$ the points in $\varphi(e)$ are called the *endpoints of e* .

- (2) We say that two abstract graphs (V, E, φ) and (V', E', φ') are *isomorphic*, if there exist bijections $\Phi: V \rightarrow V'$ and $\Psi: E \rightarrow E'$ such that for any $e \in E$ we have $\varphi'(\Psi(e)) = \Phi(\varphi(e))$.

Examples.

- (A) We take $V = \{A, B, C\}$, $E = \{e, f, g, h\}$ and φ to be the map that is defined by $\varphi(e) = \{A, B\}$, $\varphi(f) = \{B, C\}$, $\varphi(g) = \{B, C\}$ and $\varphi(h) = \{A\}$. Then (V, E, φ) is an abstract graph.
- (B) Let $a, b \in \mathbb{Z}$. We denote by $G_{a,b}$ the abstract graph with vertex set $V = \mathbb{Z}$ and edge set $\{\{x, y\} \mid x = y \pm a \text{ or } x = y \pm b\}$. The map φ is given by sending each edge $e = \{x, y\}$ to $\{x, y\}$.
- (C) We take V to be the set of all subway stations in München and we take E to be the set of all segments of the subway system. The map φ then assigns to each $e \in E$ the two stations that are connected by the segment e .

Definition. We say an abstract graph (V, E, φ) is *connected* if for any two vertices $v, w \in V$ there exist vertices $v = v_0, v_1, \dots, v_n = w$ and edges $e_0, \dots, e_{n-1}$ with $\varphi(e_i) = \{v_i, v_{i+1}\}$ for $i = 0, \dots, n-1$.

Examples. We now want to determine whether or not the above examples (A), (B) and (C) of graphs are connected.

- (A) This graph is connected, for example if we take $v = A$ and $w = C$, then $v_0 = A$, $v_1 = B$ and $v_2 = C$ and $e_0 = e$ and $e_1 = f$ have the desired property.

- (B) Let $a, b \in \mathbb{Z}$. The graph $G_{a,b}$ is connected if and only if for any two integers $x, y \in \mathbb{Z}$ there exist $n, m \in \mathbb{Z}$ such that $x - y = ma + nb$. This is the case if and only if a and b are coprime.
- (C) This graph is connected since any two subway stations are connected by subway (one might have to change lines but that is of no concern to us).

So far this definition has little to do with topology. We will change this by introducing topological graphs. Before we can do so we need the following definition.

Definition. Given a bounded 1-dimensional submanifold M of $\mathbb{R}^n$ that is homeomorphic to $(0, 1)$ we refer to the points in $\partial M \setminus M$ as the *endpoints* of M .²⁶

For example, given any $n \in \mathbb{N}$, for $M = (0, 1) \times 0 \in \mathbb{R} \times \mathbb{R}^{n-1} = \mathbb{R}^n$ the endpoints are given by $(0, 0)$ and $(1, 0)$.²⁷ More examples are given in Figure 20.

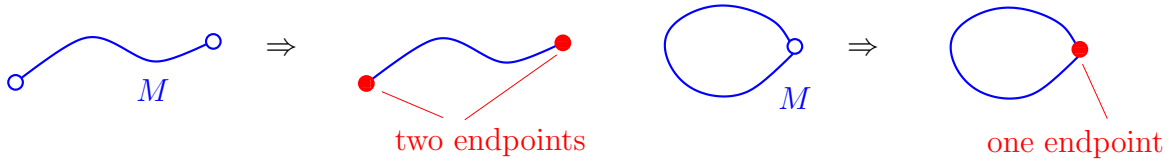


FIGURE 20.

Definition. A *topological graph* G is a pair (V, E) with the following properties:

- (1) V is a non-empty discrete subset of $\mathbb{R}^n$.
- (2) E is a subset of $\mathbb{R}^n$ with the following properties:
 - (a) E is disjoint from V ,
 - (b) each component of E is a bounded 1-dimensional submanifold of $\mathbb{R}^n$ that is homeomorphic to the open interval $(0, 1)$,
 - (c) for each component e of E the endpoints of e lie in V ,
 - (d) every bounded subset of $\mathbb{R}^n$ intersects only finitely many components of E .²⁸

We refer to the points in V as the *vertices* of G and we refer to the components of E as the *edges* of G . Furthermore, given a topological graph $G = (V, E)$ we write $|G| = V \cup E \subset \mathbb{R}^n$.

We sketch several examples of topological graphs in Figure 21. The red dots correspond to the vertices and the blue segments correspond to the edges.²⁹

²⁶Alternatively we could define the endpoints as the points in $\overline{M} \setminus M$.

²⁷This sounds obvious, but if one goes through the proof carefully one sees that the argument for $n = 1$ is slightly different from the argument for $n \geq 2$.

²⁸What is an example of (V, E) that satisfies all axioms except for 2 (d)?

²⁹The starting point here are of course the pictures, in the sense that the definition is an attempt to turn the pictures of graphs into a precise mathematical object.

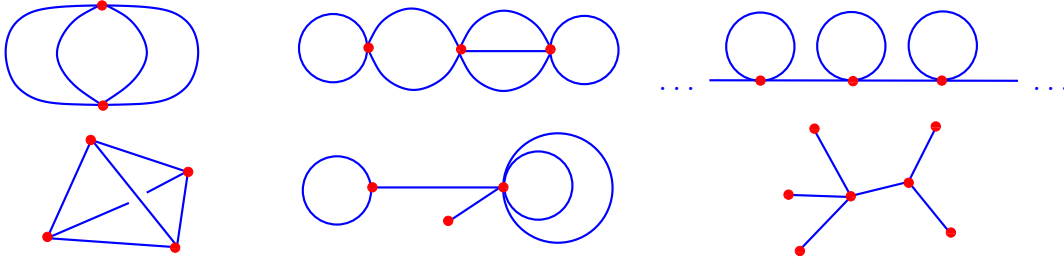


FIGURE 21.

Definition. Given a topological graph (V, E) the *underlying abstract graph* is defined as the abstract graph (V', E', φ') given by setting

$$V := V', \quad E' := \text{the set of components of } E$$

and for $e \in E'$ we define $\varphi(e)$ to be the endpoints³⁰ of e . We say a topological graph G is a *topological realization* of an abstract graph H , if the underlying abstract graph of G is isomorphic to H .

Examples. In Figure 22 we give two topological realizations of the graph from Example (A). In Figure 23 we show topological realizations of the graphs $G_{2,3}$ and $G_{0,2}$. Finally in Figure 24 we show a topological realization of the subway system of München.

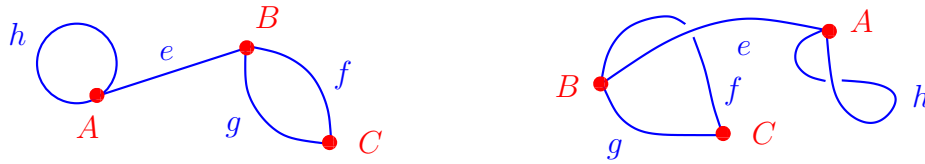


FIGURE 22.

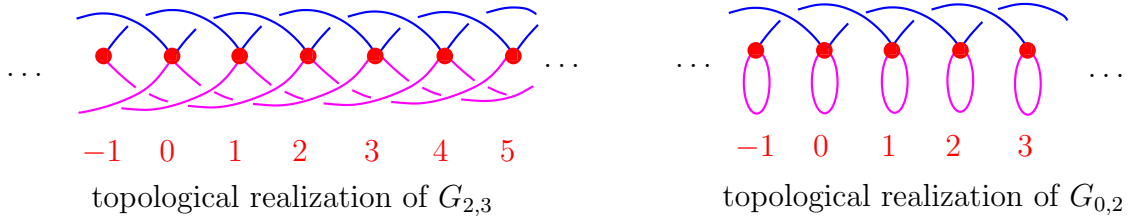


FIGURE 23.

³⁰Here we use implicitly the fact that the set of endpoints of a bounded 1-dimensional submanifold of $\mathbb{R}^n$ that is homeomorphic to the open interval $(0, 1)$ consists of either one or two points. Why is that the case?



FIGURE 24. The subway system of München.

In the following we will often go back and forth between topological graphs and abstract graphs without distinguishing these two different types of graphs in the notation. Also, very often, given a topological graph $G = (V, E)$ we will not distinguish between G and the topological space $|G|$. Sometimes we will just talk about “graphs” and it will be clear from the context what type of graph we mean.

For the most part the two points of view are quite similar, for example given a topological graph $G = G(V, e)$ the topological space $|G|$ is connected if and only if the underlying abstract graph is connected.³¹

Remark. The origins of graph theory go back to Euler and the problem on the seven bridges of Königsberg. The details of this story can be found on

https://en.wikipedia.org/wiki/Seven_Bridges_of_Koenigsberg

Euler’s original article can be found here

<http://eulerarchive.maa.org//docs/originals/E053.pdf>

1.8. The basis of a topology. In this section we recall a few more technical definitions from Analysis IV. Let X be a set and let $\mathcal{B} = \{B_i\}_{i \in I}$ be a family of subsets of X . We say $\mathcal{B}$ has the *basis property* if the following two conditions are satisfied:

- (B1) For any $x \in X$ there exists an $i \in I$ such that $x \in B_i$.
- (B2) Let $i, j \in I$ and let $x \in B_i \cap B_j$. Then there exists a $k \in I$ such that $x \in B_k$ and $B_k \subset B_i \cap B_j$.

In Analysis IV we had proved the following lemma.

Lemma 1.21. *Let X be a set and let $\mathcal{B} = \{B_i\}_{i \in I}$ be a family of subsets of X . We define*

$$\mathcal{T}(\mathcal{B}) := \{V \subset X \mid \text{for every } x \in V \text{ there exists an } i \in I \text{ such that } x \in B_i \subset V\}.$$

³¹We will not prove this statement and we will therefore not make use of it.

If $\mathcal{B}$ has the basis property, then $\mathcal{T}$ is a topology on X .

We refer to $\mathcal{T}(\mathcal{B})$ as the *topology on X generated by $\mathcal{B}$* . Conversely, if $\mathcal{T}$ is a given topology on a set X and if $\mathcal{B}$ is a family of subsets of X which has the basis property and which satisfies $\mathcal{T} = \mathcal{T}(\mathcal{B})$, then we say that $\mathcal{B}$ is a *basis of the topology $\mathcal{T}$ of X* .

Examples.

- (1) If (X, d) is a metric space, then the set of all balls $B_r(x)$ with $r > 0$ and $x \in X$ satisfy (B1) and (B2).³² The topology generated by this family of subsets is precisely the topology we already introduced on page 13.
- (2) For $X = \mathbb{R}^2$ the set $\mathcal{B} = \{[a, b] \times [c, d] \mid a, b, c, d \in \mathbb{Q}\}$ of rectangles with rational vertices is also a basis for the standard topology on $\mathbb{R}^2$. In particular we see that $\mathbb{R}^2$, and similarly also any $\mathbb{R}^n$, admit a basis for the topology that consists of only countably many sets.

Remark. In Lemma 16.2 of Analysis IV we showed that a family of subsets $\mathcal{B}$ that has the basis property is the basis for a given topology $\mathcal{T}$ if all sets in $\mathcal{B}$ are open in $\mathcal{T}$ and if any open set in $\mathcal{T}$ is a union of sets in $\mathcal{B}$.

Sometimes the following lemma can be useful.

Lemma 1.22. *Let $\mathcal{B}$ be a basis for the topology of a topological space X . Furthermore let $\{U_j\}_{j \in J}$ be an open covering of X , i.e. $\{U_j\}_{j \in J}$ is a family of open sets with $\bigcup_{j \in J} U_j = X$. Then*

$$\{B \in \mathcal{B} \mid \text{there exists some } j \in J \text{ with } B \subset U_j\}$$

is also a basis for the topology of X .

Proof. We write

$$\mathcal{C} := \{B \in \mathcal{B} \mid \text{there exists some } j \in J \text{ with } B \subset U_j\}.$$

Since $\mathcal{C} \subset \mathcal{B}$ we have $\mathcal{T}(\mathcal{C}) \subset \mathcal{T}(\mathcal{B})$. It remains to show the reverse inclusion $\mathcal{T}(\mathcal{B}) \subset \mathcal{T}(\mathcal{C})$. It suffices to prove the following claim: given any $x \in X$ and any $B \in \mathcal{B}$ with $x \in B$ there exists a $C \in \mathcal{C}$ such that $x \in C \subset B$. So let $x \in X$ and let $B \in \mathcal{B}$ with $x \in B$. By hypothesis there exists a j such that $x \in U_j$. Since U_j is open and since $\mathcal{B}$ is a basis for the topology of X there exists a $B' \in \mathcal{B}$ with $x \in B' \subset U_j$. By (B2) there exists a $B'' \in \mathcal{B}$ with $x \in B''$ and $B'' \subset B \cap B' \subset U_j$. By definition we have $B'' \in \mathcal{C}$. $\square$

Example. As we had pointed out above, $\mathcal{B} = \{B_r(x) \mid r > 0 \text{ and } x \in \mathbb{R}^n\}$ is a basis for the usual topology of $\mathbb{R}^n$. We denote by $\mathcal{U}$ the family of all open sets that are contained in an open ball of radius 1. Then the open sets in $\mathcal{U}$ cover $\mathbb{R}^n$. It follows from Lemma 1.22 that

$$\{B_r(x) \mid r \in (0, 1) \text{ and } x \in \mathbb{R}^n\}$$

is also a basis for $\mathbb{R}^n$.

³²Why is (B2) satisfied?

In Analysis IV we had also proved the following lemma.

Lemma 1.23. *Let $f: X \rightarrow Y$ be a map between topological spaces. Let $\mathcal{B}$ be a basis for the topology of Y . Then the following holds*

$$f \text{ is continuous} \iff \text{for each } B \in \mathcal{B} \text{ the preimage } f^{-1}(B) \text{ is open in } X.$$

Finally recall that a map $f: X \rightarrow Y$ between topological spaces is called *open* if the image of every open subset of X is open in Y . We have the following analogue to Lemma 1.23.

Lemma 1.24. *Let $f: X \rightarrow Y$ be a map between topological spaces. (Here for once we do not assume that f is continuous.) Let $\mathcal{B} = \{B_i\}_{i \in I}$ be a basis for the topology of X . Then the following holds*

$$f \text{ is open} \iff \text{for each } i \in I \text{ the image } f(B_i) \text{ is open in } Y.$$

We leave the elementary proof as an exercise to the reader.

1.9. Manifolds. We now recall the definitions of a topological manifold and of a manifold. Both were already given in Analysis IV.

Definition. Let X be a topological space.

- (1) We say X is *second-countable* if there exists a countable basis for the topology.
- (2) An *n -dimensional chart for X at a point $x \in X$* is a homeomorphism $\Phi: U \rightarrow V$ where U is an open neighborhood of x and V is one of the following:
 - (i) V either an open subset of $\mathbb{R}^n$ or
 - (ii) V is an open subset of the half-space $H_n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$ and $\Phi(x)$ lies on $E_{n-1} = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n = 0\}$.
- In the former case we say that Φ is a chart of type (i) in the latter case we say that Φ is a chart of type (ii).
- (3) We say X is an *n -dimensional topological manifold*, if X is second-countable and Hausdorff, and if for every $x \in X$ there exists an n -dimensional chart $\Phi: U \rightarrow V$ at x .
- (4) We say that a point x on a topological manifold is a *boundary point* if x does not admit a chart of type (i).³³ We denote by ∂X the set of all boundary points of X .
- (5) We say a topological manifold X is *closed* if X is compact with $\partial X = \emptyset$.³⁴
- (6) An *n -dimensional atlas for X* is a family of n -dimensional charts $\{\Phi_i: U_i \rightarrow V_i\}_{i \in I}$ such that $\bigcup_{i \in I} U_i = X$.
- (7) An atlas $\{\Phi_i: U_i \rightarrow V_i\}_{i \in I}$ is called *smooth* if for all $i, j \in I$ the transition map

$$\Phi_j \circ \Phi_i^{-1}: \Phi_i(U_i \cap U_j) \rightarrow \Phi_j(U_i \cap U_j)$$

is smooth.

³³It might sound more natural to define a boundary point to be a point that admits a chart of type (ii). In fact we will see later that a point on a topological manifold admits *either* a chart of type (i) or a chart of type (ii). But the proof of this statement is not trivial and as of now we can not prove this statement.

³⁴The latter condition just means that all points in X admit a chart of type (i).

- (8) An n -dimensional manifold is a pair $(X, \mathcal{A})$ consisting of an n -dimensional topological manifold X together with a smooth atlas $\mathcal{A}$ for X .

Remark. In the literature “topological manifolds” are sometimes referred to as “manifolds”, and what we now call “manifolds” are referred to as “smooth manifolds”.

In Analysis IV we had already seen that many familiar topological spaces are manifolds. For example we had seen that the following are manifolds:

- (1) open subsets of $\mathbb{R}^n$,
- (2) submanifolds of $\mathbb{R}^k$, in particular all spheres $S^n \subset \mathbb{R}^{n+1}$,
- (3) the surfaces of genus g as defined on page 28.

Furthermore we had seen that products of manifolds are again manifolds. For example the n -dimensional torus $(S^1)^n$ is an n -dimensional manifold.

Topological manifolds are particularly important and also particularly nice topological spaces. For example it is straightforward to see that every topological manifold is locally path-connected.³⁵

Lemma 1.25.

- (1) Every topological manifold is locally path-connected.
- (2) A topological manifold is connected if and only if it is path-connected.³⁶

Proof. Let X be an n -dimensional topological manifold.

- (1) Since X is locally homeomorphic to an open subset of $\mathbb{R}^n$ it follows easily from the fact, proved on page 35, that open subsets of $\mathbb{R}^n$ are locally path-connected, that X itself is locally path-connected.
- (2) The second statement is an immediate consequence of (1) and of Lemma 1.19. $\square$

On several occasions we will need the following, slightly technical lemma.

Lemma 1.26. *Let M be an n -dimensional topological manifold and let $\Phi: U \rightarrow V$ be a chart. Let $A \subset V$ be a subset that is compact. Then the corresponding subset $\Phi^{-1}(A)$ is a closed subset of M .³⁷*

Proof. Let M be an n -dimensional topological manifold and let $\Phi: U \rightarrow V$ be a chart. Let $\Psi = \Phi^{-1}: V \rightarrow U$ be the inverse map. Let $A \subset V$ be a subset that is compact. It follows from Lemma 1.2 that $\Psi(A)$ is a compact subset of M . But M is Hausdorff by definition of a topological manifold. Hence $\Psi(A) = \Phi^{-1}(A)$ is closed in M . $\square$

Lemma 1.27. *Let M be a topological manifold. If M is compact, then ∂M is also compact.*

³⁵This follows from the fact that every point on a topological manifold has an open neighborhood that is homeomorphic to an open ball, but an open ball is of course locally path-connected.

³⁶We had already seen this statement as Lemma 5.2 in Analysis IV.

³⁷It does not suffice that A is a closed subset of the topological space V . Indeed, $A = V$ is a closed subset of V , but $U = \Phi^{-1}(V)$ is in general not a closed subset of M .

Proof. Let M be a compact topological manifold. We want to show that ∂M is also compact. We first claim that ∂M is closed, or equivalently that $M \setminus \partial M$ is open. But the openness statement follows immediately from the definition of ∂M and from the observation that having a chart of type (i) is an open condition. We now see that ∂M is a closed subset of the compact topological space M . It follows from Lemma 1.1 that M itself is compact. $\square$

We recall a few more definitions from Analysis IV.

Definition.

- (1) A map $f: M \rightarrow N$ between manifolds is called *smooth* if for any chart Φ of M and any chart Ψ of N , in the respective atlases, the concatenation $\Psi \circ f \circ \Phi^{-1}$ is smooth.³⁸
- (2) We say a map $f: M \rightarrow N$ between manifolds is a *diffeomorphism* if f is smooth, if f is a bijection and if $f^{-1}: N \rightarrow M$ is also smooth.

Now we return to group actions since these are one of the best ways to construct new topological spaces and manifolds.

Definition. Let X be a topological space and let G be a group that acts on X .

- (1) We say that G acts *properly* if for every two points x and y in X there exist open neighborhoods U of x and V of y such that the set $\{g \in G \mid gU \cap V \neq \emptyset\}$ is finite.³⁹
- (2) If X is a manifold, then we say that the action is *smooth* if for any $g \in G$ the map

$$\begin{aligned} X &\rightarrow X \\ x &\mapsto g \cdot x \end{aligned}$$

is smooth.

Definition. The group actions (1), (2) and (3) on page 24 are proper and smooth. Whereas group action (4) is smooth, but not proper.

The following proposition is the combination of Satz 18.11 and Satz 19.2 from Analysis IV.

Proposition 1.28. *Let G be a group that acts freely, properly and continuously on a (closed) n -dimensional topological manifold M . Then the following hold:*

- (1) *the quotient space M/G is also a (closed) n -dimensional topological manifold,*
- (2) *furthermore, if M is a manifold and if G acts in fact smoothly, then M/G admits a smooth atlas such that the projection map $p: M \rightarrow M/G$ is a local diffeomorphism.*

Example. The action

$$\begin{aligned} \mathbb{Z}^n \times \mathbb{R}^n &\rightarrow \mathbb{R}^n \\ (z, v) &\mapsto z + v \end{aligned}$$

³⁸If M is m -dimensional and if N is n -dimensional, then the map $\Psi \circ f \circ \Phi^{-1}$ is a map from an open subset of $\mathbb{R}^m$ or from an open subset of $\{(x_1, \dots, x_m) \mid x_m \geq 0\}$ to $\mathbb{R}^n$, hence the notion of smooth makes sense for such a map.

³⁹In particular, any action by a finite group is proper.

is free, proper and smooth. It follows from Proposition 1.28 that $\mathbb{R}^n/\mathbb{Z}^n$ is an n -dimensional manifold. Similarly one can now show that the Möbius band $(\mathbb{R} \times [-1, 1])/\mathbb{Z}$ is a 2-dimensional manifold such that the boundary is diffeomorphic to S^1 . Furthermore one can easily show that the real projective space $\mathbb{RP}^n = S^n/\{\pm 1\}$ is an n -dimensional manifold.

1.10. The classification of 1-dimensional manifolds. One of the goals of topology is to completely classify suitable classes of topological spaces. For example one could try to classify all connected topological manifolds of a given dimension. The following proposition gives a complete classification in the 1-dimensional case.

Proposition 1.29. *Let M be a connected 1-dimensional topological manifold. Then the homeomorphism type of M is given by the following table⁴⁰*

	without boundary	with boundary
compact	the circle S^1	the closed interval $[0, 1]$
non-compact	the open interval $(0, 1)$	the half-open interval $[0, 1)$.

Note that the adjectives “Hausdorff” and “second-countable” in the definition of a topological manifold are both necessary:

- (1) The line with two zeros satisfies all axioms of a manifold except for the Hausdorff axiom.⁴¹
- (2) If one drops the condition that a topological manifold has to be second-countable, then there is one more example of a connected 1-dimensional topological manifold, namely the “long line”, see [Ga2] and

[http://en.wikipedia.org/wiki/Long_line_\(topology\)](http://en.wikipedia.org/wiki/Long_line_(topology))

Proposition 1.29 is proved in [Ga2]. In the following we only deal with the case that M is closed. Put differently, we prove the following proposition.

Proposition 1.30. *Let M be a compact connected closed 1-dimensional topological manifold. Then M is homeomorphic to S^1 .⁴²*

In the proof of Proposition 1.30 we will need the following lemma.

Lemma 1.31. *Let M be a 1-dimensional topological manifold and let $\Phi: U \rightarrow (-2, 2)$ be a chart. Let $[a, b] \subset (-2, 2)$ be a compact interval. Then $\Phi^{-1}([a, b])$ is a closed subset of M .*

Proof. Since Φ is a homeomorphism the map $\Phi^{-1}: (-2, 2) \rightarrow U$ is also continuous. Since $[a, b]$ is compact it follows from Lemma 1.2 that $\Phi^{-1}([a, b])$ is also compact. But any compact subspace of a Hausdorff space is closed.⁴³ But M is a topological manifold, in particular M is Hausdorff. Therefore $\Phi^{-1}([a, b])$ is a closed subset of M . $\square$

⁴⁰Why is the half-open interval $[0, 1)$ not homeomorphic to the open interval $(0, 1)$?

⁴¹Does there exist a compact topological space that also satisfies all axioms of a closed manifold except for the Hausdorff axiom?

⁴²The proof of Proposition 1.30 is not an official part of the course.

⁴³This statement is Satz 1.12 from Analysis IV.

Definition. Let M be a 1-dimensional topological manifold.

- (1) A *c-parametrization* for M is a homeomorphism $\Psi: [-1, 1] \rightarrow C$ from the compact interval $[-1, 1]$ to a closed subset C of M . We say the c-parametrization is *open* if $\Psi((-1, 1))$ is an open subset of M .
- (2) We say that two c-parametrizations Ψ and Ψ' are *almost disjoint* if

$$\Psi((-1, 1)) \cap \Psi'((-1, 1)) = \emptyset.$$

- (3) A *c-atlas* is a family of c-parametrizations $\{\Psi_i: [-1, 1] \rightarrow C_i\}_{i \in I}$ such that the union of the C_i 's equals M . We call the c-atlas open if all c-parametrizations in the c-atlas are open. We say the c-atlas *almost disjoint* if all c-parametrizations in the c-atlas are almost-disjoint.

Using Lemma 1.31 one can prove the following lemma that is illustrated in Figure 25. We leave the proof as a voluntary exercise.

Lemma 1.32. *Let M be a 1-dimensional topological manifold and let $\Psi_1: [-1, 1] \rightarrow C_1$ and $\Psi_2: [-1, 1] \rightarrow C_2$ be two open c-parametrizations such that $C_1 \cap C_2$ is non-empty but such that C_2 is not contained in C_1 . Then*

$$[-1, 1] \setminus \Psi_1^{-1}(\Psi_2((-1, 1))) = [a, b]$$

for some $-1 \leq a < b \leq 1$.

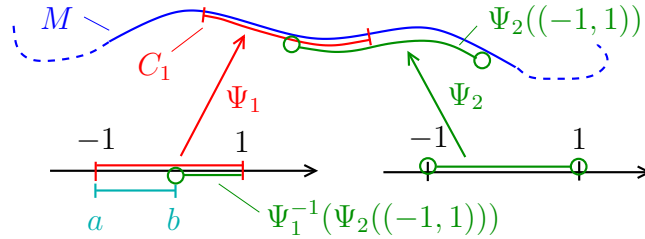


FIGURE 25. Illustration of Lemma 1.32.

Now we can provide the proof of Proposition 1.30.

Proof of Proposition 1.30. Let M be a compact connected closed 1-dimensional topological manifold. We start out with the following claim.

Claim. There exists a finite open almost-disjoint c-atlas $\Psi_i: [-1, 1] \rightarrow C_i$ with $i = 1, \dots, k$.

Since M is assumed to be closed there exists for each $P \in M$ a chart $\Phi_P: U_P \rightarrow (-2, 2)$ of type (i) such that $\Phi_P(P) = 0$. The open sets $\Phi_P^{-1}((-1, 1))$ are an open covering of M .⁴⁴ Since M is compact there exist $P_1, \dots, P_k$ such that the open sets $\Phi_{P_i}^{-1}((-1, 1))$, $i = 1, \dots, k$ are an open covering of M . It follows that the union of the sets $C_i := \Phi_{P_i}^{-1}([-1, 1])$,

⁴⁴Here we use implicitly the following fact: if U is open in M , and if V is open in U , then V is also open in M . Why is that?

$i = 1, \dots, k$ is also all of M . It follows from Lemma 1.31 that each C_i is a closed subset of M .

For $i = 1, \dots, k$ we now consider the homeomorphism Ψ_i that is given by

$$\Psi_i := (\Phi_{P_i}|_{C_i})^{-1}: [-1, 1] \xrightarrow{\cong} C_i.$$

By the above discussion these are open c-parametrizations. After possibly removing some c-parametrizations we can assume that no C_i is contained in any other C_j .

We now make the following modifications to our c-atlas. If for some $i < j$ we have $\Psi_i((-1, 1)) \cap \Psi_j((-1, 1)) \neq \emptyset$, then it follows from Lemma 1.32 that

$$[-1, 1] \setminus \Psi_i^{-1}(\Psi_j((-1, 1))) = [a, b]$$

for some $-1 \leq a < b \leq 1$. We now replace Ψ_i by the map

$$[-1, 1] \xrightarrow{\text{linear homeomorphism}} [a, b] \xrightarrow{\Psi_i} M.$$

Repeating this process we obtain strongly open c-parametrizations that are furthermore almost-disjoint. This concludes the proof of the claim.

We say that a c-atlas is *simple* if given any $P \in M$ there exist at most two c-parametrizations that contain P .

Claim. Any almost-disjoint open c-atlas is simple.

This follows from the fact that any point P on a 1-dimensional topological manifold has the property, that given any open neighborhood U there exists an open neighborhood V of x , contained in U , such that $V \setminus \{P\}$ contains precisely two components. If a point $P \in M$ is contained in k c-parametrizations of an open almost-disjoint c-atlas, then one can easily show that P admits an open neighborhood U such that for any open neighborhood V of P contained in U the set $V \setminus \{P\}$ has at least k components. This concludes the proof of the claim.

Now let $\Psi_i: C_i \rightarrow [-1, 1]$ and $\Psi_j: C_j \rightarrow [-1, 1]$ be two c-parametrizations of our c-atlas such that $C_i \cap C_j$ consists of a single point. Since the c-parametrizations are almost disjoint we have $C_i \cap C_j \subset \Psi_i(\{\pm 1\})$ and $C_i \cap C_j \subset \Psi_j(\{\pm 1\})$. After possibly replacing $\Psi_i(t)$ by $\Psi_i(-t)$ and/or $\Psi_j(t)$ by $\Psi_j(-t)$ we can assume that $\Psi_i(1) = \Psi_j(-1)$. We consider the map

$$\begin{aligned} \Psi: [-1, 1] &\rightarrow C_1 \cup C_2 \\ x &\mapsto \begin{cases} \Psi_i(2x + 1), & \text{if } x \in [-1, 0], \\ \Psi_j(2x - 1), & \text{if } x \in [0, 1]. \end{cases} \end{aligned}$$

It is straightforward to verify that Ψ is a bijection and continuous. It follows from Proposition 1.3 that Ψ is in fact a homeomorphism.⁴⁵

By iteratively applying the procedure from the previous claim to our c-atlas we obtain a finite almost-disjoint simple c-atlas⁴⁶ $\Psi_i: C_i \rightarrow [-1, 1]$, $i = 1, \dots, k$ such that for any $i \neq j$ either $C_i \cap C_j$ is empty or it consists of two points.

⁴⁵Here we use that M is Hausdorff, hence the subset $C_i \cup C_j$ is also Hausdorff.

⁴⁶Note that we no longer claim that the resulting c-atlas is open.

Now pick a c-parametrization $\Psi_i: [-1, 1] \rightarrow C_i$ from the c-atlas. Since M is closed and since $[-1, 1]$ is not closed we see that Ψ_i is not a homeomorphism,⁴⁷ in particular $C_i \neq M$. Since M is connected⁴⁸ there exists another c-parametrization $\Psi_j: [-1, 1] \rightarrow C_j$ such that $C_i \cap C_j \neq \emptyset$. By our choice of c-atlas the set $C_i \cap C_j$ consists of two points.

After possibly replacing $\Psi_i(t)$ by $\Psi_i(-t)$ we can assume that $\Psi_i(1) = \Psi_j(-1)$ and $\Psi_i(-1) = \Psi_j(1)$. We consider the map

$$\begin{aligned} \Psi: [-1, 1] / -1 \sim 1 &\rightarrow C_i \cup C_j \\ x &\mapsto \begin{cases} \Psi_i(2x + 1), & \text{if } x \in [-1, 0], \\ \Psi_j(2x - 1), & \text{if } x \in [0, 1]. \end{cases} \end{aligned}$$

It is straightforward to verify that ψ is a bijection⁴⁹ and continuous.⁵⁰ It follows from Proposition 1.3 that Ψ is in fact a homeomorphism.

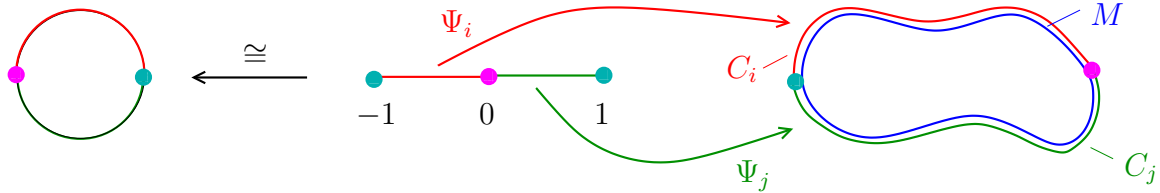


FIGURE 26.

Thus we have found a homeomorphism from $[-1, 1] / -1 \sim 1 = S^1$ to $C_1 \cup C_2$. It remains to prove the following claim.

Claim. We have $C_i \cup C_j = M$.

The c-atlas we work with is simple and almost-disjoint. Since $\Psi_i(\pm 1) = \Psi_j(\mp 1)$ it follows that C_i and C_j did not intersect any other C_k . This shows that $C_i \cup C_j$ and $\bigcup_{k \neq i, j} C_k$ are disjoint. Since both are closed and since M is connected and since $C_i \cup C_j$ is non-empty it follows that $M = C_i \cup C_j$. $\square$

If we consider manifolds instead of topological manifolds and diffeomorphisms instead of homeomorphism we get the same answer. More precisely the following proposition holds.

⁴⁷Careful reading shows that here we make use of the fact that a half-open interval is not homeomorphic to an open interval.

⁴⁸Why does this follow from connectedness?

⁴⁹Here we use that we identify -1 and 1 on the left hand side.

⁵⁰It is continuous at $[-1] = [1]$ since $\Psi_i(-1) = \Psi_j(1)$.

Proposition 1.33. *Let M be a connected 1-dimensional manifold. Then the diffeomorphism type of M is given by the following table*

	without boundary	with boundary
compact	the circle S^1	the closed interval $[0, 1]$
non-compact	the open interval $(0, 1)$	the half-open interval $[0, 1)$.

If M has no boundary, then this proposition is proved in the appendix of [Mi2]. The case of manifolds with boundary can be reduced to the case of manifolds without boundary by first considering the interior of the manifold.

1.11. Orientations of manifolds. In this section we recall the definition of an orientation on a manifold that we had already given in Analysis IV.

Definition. Let V be a real vector space of dimension $k \geq 1$

- (1) We say that two bases of V are *equivalent* if the determinant of the base change matrix is positive.
- (2) An equivalence class of bases is called an *orientation* for V . An *oriented vector space* is a vector space together with an orientation.
- (3) Let $(V, \mathcal{O})$ be an oriented vector space. We say $\{v_1, \dots, v_k\}$ is a *positive basis*, if $\{v_1, \dots, v_k\}$ lies in $\mathcal{O}$, otherwise we say that it is a negative basis.

Remark.

- (1) It is straightforward to see that any real vector space of dimension ≥ 1 admits precisely two orientations. Given an orientation $\mathcal{O}$ for some vector space V we denote by $-\mathcal{O}$ the other orientation.
- (2) We always view $\mathbb{R}^n$ as equipped with the orientation such that the canonical basis $\{e_1, \dots, e_n\}$ is a positive basis.

Definition. Let V and W be oriented vector spaces. We say an isomorphism $\Phi: V \rightarrow W$ is *orientation-preserving* if the image of a positive basis of V is a positive basis of W . Otherwise we say that Φ is *orientation-reversing*.

Definition. Let M be a k -dimensional manifold with $k \geq 1$.

- (1) A *pre-orientation* for M is a map which assigns to each tangent space⁵¹ $T_P M$ an orientation.
- (2) We say a *pre-orientation is continuous with respect to a chart* $\Phi: U \rightarrow V$ ⁵² if for any connected component U' of U the map $\Phi_*: T_P M = T_P U \rightarrow T_{\Phi(P)} V = \mathbb{R}^k$ is either orientation-preserving for all $P \in U'$ or it is orientation-reversing for all $P \in U'$.

⁵¹We do not recall the definition of the tangent space of a manifold. This definition was for example given in Analysis IV. If M is a submanifold of $\mathbb{R}^n$, then $T_P M$ is essentially the same as the usual tangent space of a submanifold.

⁵²As usual we mean any chart in the given atlas of the manifold M .

- (3) We say a pre-orientation is an *orientation* if it is continuous with respect to every chart.⁵³
- (4) A manifold together with an orientation is called an *oriented manifold*.
- (5) A manifold that admits an orientation is called *orientable*, otherwise we say that the manifold is *non-orientable*.

It is relatively straightforward to show that the spheres S^n , the n -dimensional torus $(S^1)^n$ and all surfaces of genus $g \geq 0$ are orientable. On the other hand, in Lemma 9.16 of Analysis IV we had shown that the Möbius band is not orientable. In Figures 27 and 28 we see that the Klein bottle and the real projective plane $\mathbb{RP}^2$ contain the Möbius band as a submanifold of codimension 0. It follows that these two manifolds are not orientable either.⁵⁴

Convention. In the following we will always view S^n as an oriented manifold with the convention that a basis $v_1, \dots, v_n \in T_P(S^n)$ is positive if $\det(P, v_1, \dots, v_n) > 0$. For S^1 we obtain the “counterclockwise orientation”.

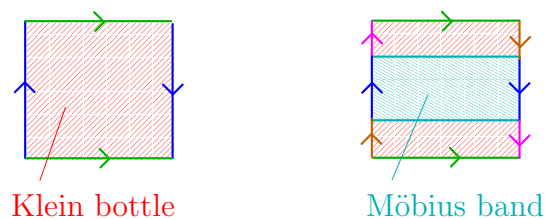


FIGURE 27. The Möbius band is a submanifold of the Klein bottle.

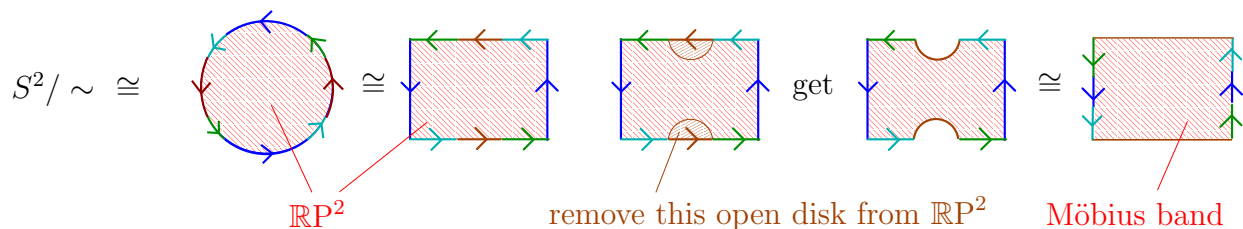


FIGURE 28. The Möbius band is a submanifold of $\mathbb{RP}^2$.

The following proposition is now a refinement of Proposition 1.28. We leave the proof as an exercise to the reader.

⁵³As we had shown in Analysis IV, it suffices to show continuity for some family of charts that cover M . We do not need to verify it for all charts in the given atlas.

⁵⁴In fact in Figure 28 we show the slightly more precise statement, that if we remove an open disk from $\mathbb{RP}^2$, then we obtain a manifold that is diffeomorphic to the Möbius band. We will make use of this fact later on.

Proposition 1.34. *Let G be a group that acts freely, properly and smoothly on an oriented n -dimensional manifold M . If for each $g \in G$ the map*

$$\begin{array}{ccc} M & \rightarrow & M \\ P & \mapsto & g \cdot P \end{array}$$

is orientation-preserving, then M/G admits an orientation such that the projection map $p: M \rightarrow M/G$ is orientation-preserving.

Example. The action of $G = \{\pm 1\}$ on S^n is orientation-preserving if and only if n is odd. It follows from Proposition 1.34 that for n odd the manifold $\mathbb{RP}^n = S^n / \pm 1$ is orientable.

We had just seen that $\mathbb{RP}^2$ is not orientable. We conclude this section with the following question.

Question 1.35. *Is $\mathbb{RP}^{2n}$ orientable for $n \geq 2$?*

2. DIFFERENTIAL TOPOLOGY

Differential topology is the study of manifolds and their submanifolds viewed as topological spaces. In this section we will state, without proof but with precise references, some technical results which later on will make our life much easier.

2.1. The Tubular Neighborhood Theorem.

Definition. A map $\varphi: N \rightarrow M$ between two manifolds is called an *embedding* if the following conditions are satisfied:

- (1) $\varphi: N \rightarrow \varphi(N)$ is a homeomorphism,⁵⁵⁵⁶
- (2) φ is smooth,
- (3) for each $P \in N$ the induced map $\varphi_*: T_P N \rightarrow T_{\varphi(P)} M$ is a monomorphism.

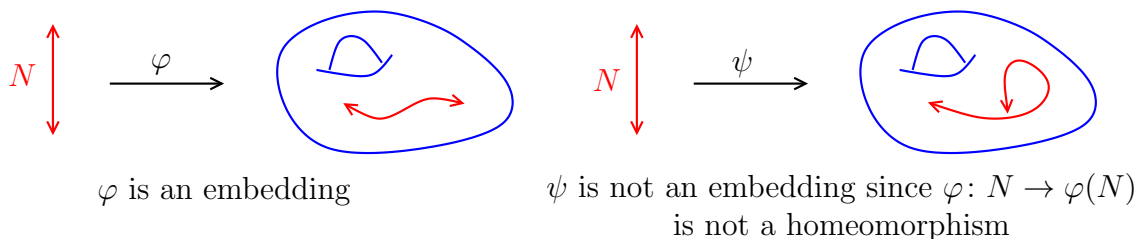


FIGURE 29.

The following proposition is proved in [Le2, Proposition 5.2]. We had proved a very similar statement as Satz 11.1 in Analysis II.

Proposition 2.1. *Let $\varphi: N \rightarrow M$ be an embedding of an n -dimensional manifold N into a manifold M . Then $\varphi(N)$ is an n -dimensional submanifold of M .*

The following theorem is a key technical tool in the study of manifolds.

Theorem 2.2. (Tubular Neighborhood Theorem) *Let M be an orientable m -dimensional manifold and let N be a 1-dimensional submanifold. Then there exists an embedding*

$$F: N \times \overline{B}^{m-1} \rightarrow M$$

*such that $F(P, 0) = P$ for all $P \in N$. Furthermore, if M and N are oriented, then we can find such F that is furthermore orientation-preserving.*⁵⁷⁵⁸

⁵⁵Hereby $\varphi(N) \subset M$ is endowed with the subspace topology.

⁵⁶If N is compact it follows from Proposition 1.3 that it suffices to verify that φ is injective and continuous.

⁵⁷Hereby we view $\overline{B}^{m-1} \subset \mathbb{R}^{m-1}$ as an oriented manifold in the usual way and we endow $N \times \overline{B}^{m-1}$ with the product orientation.

⁵⁸Does the conclusion of the theorem also hold if M is not assumed to be oriented?

Example. The map

$$\begin{aligned} S^1 &\rightarrow S^1 \times S^1 \\ z &\mapsto (z, 1) \end{aligned}$$

is an embedding. The map

$$\begin{aligned} S^1 \times [-1, 1] &\rightarrow S^1 \times S^1 \\ (z, t) &\mapsto (z, e^{it}) \end{aligned}$$

has all the desired properties of the Tubular Neighborhood Theorem. This map is also illustrated in Figure 30.

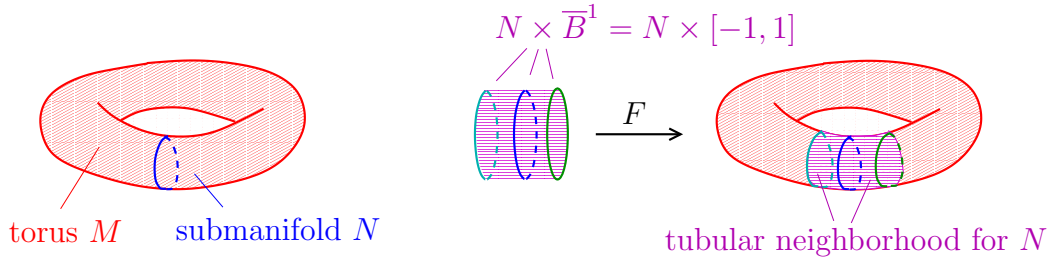


FIGURE 30. Illustration of the Tubular Neighborhood Theorem.

We will not prove the theorem, we refer to [Ko, Section III.2] or alternatively to [Br, Theorem 11.14] and [Le2, Theorem 6.24] for a proof. In the following we often refer to $F(N \times \overline{B}^{m-1})$ as a *tubular neighborhood* of N .

Before we can discuss to what degree the tubular neighborhood is unique we need to introduce one more definition.

Definition. Let M be a manifold and let W be another manifold. We say that two embeddings $F, G: W \rightarrow M$ are *smoothly isotopic*, if there exists a *smooth isotopy* from F to G , i.e. a smooth map

$$\begin{aligned} H: W \times [0, 1] &\rightarrow M \\ (z, t) &\mapsto H(z, t) \end{aligned}$$

such that the following hold:

- (1) We have $H(P, 0) = F(P)$ and $H(P, 1) = G(P)$ for all $P \in W$,
- (2) for each $t \in [0, 1]$ the map $W \rightarrow M$ given by $P \mapsto H(P, t)$ is an embedding.

Now we can formulate the following proposition that is proved in [Ko, Corollary III.3.2].

Proposition 2.3. *Let M be an m -dimensional manifold and let N be a 1-dimensional submanifold of M . Then any two embeddings $F, G: N \times \overline{B}^{m-1} \rightarrow M \setminus \partial M$ such that $F(P, 0) = G(P, 0) = P$ for all $P \in N$ are isotopic.*

The following proposition can also be proved using the methods of [Ko, Section III.2].

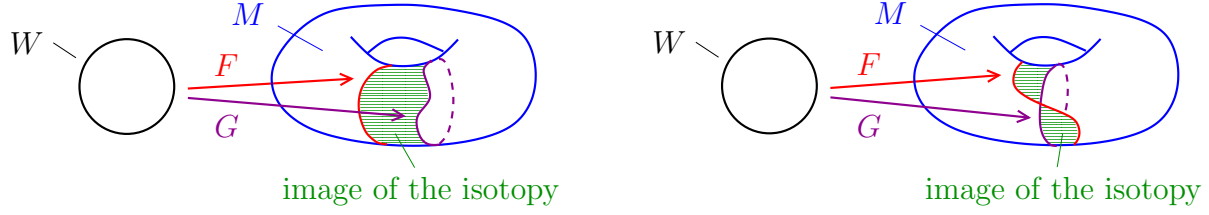


FIGURE 31. Illustration of isotopic maps.

Theorem 2.4. (Extension Theorem) *Let $f: N \rightarrow M$ be an embedding of a manifold with boundary into a manifold M such that $f(N) \subset M \setminus \partial M$. Then there exists an embedding⁵⁹*

$$f: (N \sqcup (\partial N \times [0, 1])) / \partial N = \partial N \times 0 \rightarrow M$$

which restricts to the original f on N and such that the image lies in $M \setminus \partial M$.

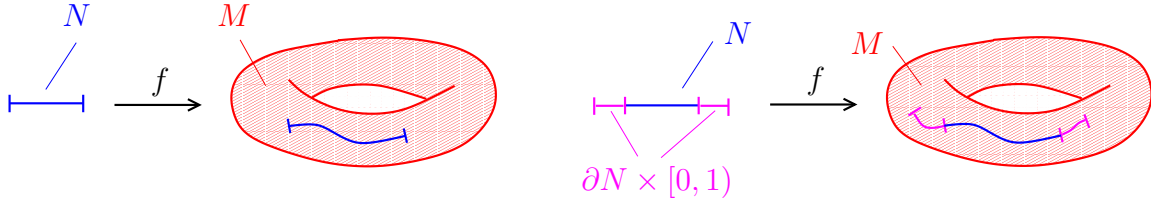


FIGURE 32. Illustration of the Extension Theorem.

We also have the following proposition which is at times quite useful.

Proposition 2.5. *Let M be a connected manifold. Then given any two points P and Q in $M \setminus \partial M$ there exists a diffeomorphism $\Phi: M \rightarrow M$ with $\Phi(P) = Q$.*

Sketch of the proof. This proposition is a typical example of a statement that is easy to visualize, but where it is quite tricky to write down a rigorous proof. Therefore we just give a sketch of the proof. A complete proof is given on page 22 of [Mi2].

So let P and Q be two different points on an n -dimensional manifold M . Since M is connected it follows from Lemma 1.25 that there exists a path $\gamma: [0, 1] \rightarrow M$ with $\gamma(0) = P$ and $\gamma(1) = Q$. With some more effort one can show that $\gamma: [0, 1] \rightarrow M$ can be chosen to be an embedding.

It follows from the Extension Theorem 2.4 that there exists an embedding⁶⁰

$$f: [-1, 2] \rightarrow M$$

⁵⁹The space $(N \sqcup (\partial N \times [0, 1])) / \partial N = \partial N \times 0$ is hereby meant to be the space $(N \sqcup (\partial N \times [0, 1])) / \sim$ where $P \in \partial N$ is equivalent to $P \times \{0\}$ on $\partial N \times [0, 1]$.

⁶⁰Here $[-1, 2]$ is the result of attaching the interval $[0, 1]$ to the boundary points 0 and 1 of $N = [0, 1]$, i.e. we extend $N = [0, 1]$ by adding the interval $[0, 1]$ to “the left and the right” and we obtain $[-1, 2]$.

such that $f(t) = \gamma(t)$ for $t \in [0, 1]$ and it follows from the Tubular Neighborhood Theorem 2.2 that there exists in fact an embedding

$$f: [-1, 2] \times \overline{B}^{n-1} \rightarrow M$$

such that $f(t, 0) = \gamma(t)$ for $t \in [0, 1]$. It is relatively straightforward to show that there exists a diffeomorphism

$$\Psi: [-1, 2] \times \overline{B}^{n-1} \rightarrow [-1, 2] \times \overline{B}^{n-1}$$

such that $\Psi = \text{id}$ on a neighborhood of the boundary of $[-1, 2] \times \overline{B}^{n-1}$ and such that furthermore $\Psi(0, 0) = (1, 0)$. Then the map

$$\begin{aligned} \Phi: M &\rightarrow M \\ P &\mapsto \begin{cases} P, & \text{if } P \notin f([-1, 2] \times \overline{B}^{n-1}), \\ f(\Psi(f^{-1}(P))), & \text{if } P \in f([-1, 2] \times \overline{B}^{n-1}) \end{cases} \end{aligned}$$

has the desired properties. $\square$

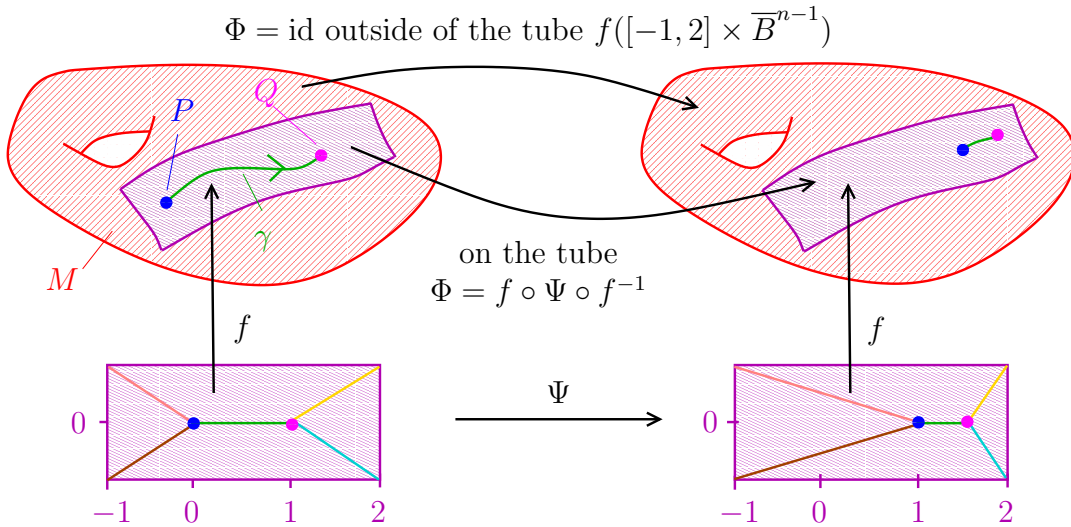


FIGURE 33. Sketch for the proof of Proposition 2.5

We conclude this section with the formulation of the Collar Neighborhood Theorem. The statement is similar in flavor to the Tubular Neighborhood Theorem 2.2 and the Extension Theorem 2.4.

Definition. Let M be a manifold. A *collar neighborhood* of ∂M is defined as the image of an embedding $f: \partial M \times [0, 1) \rightarrow M$ that is the identity map on $\partial M \times \{0\} = \partial M$.

Now we can state the Collar Neighborhood Theorem [Le2, Theorem 9.25].

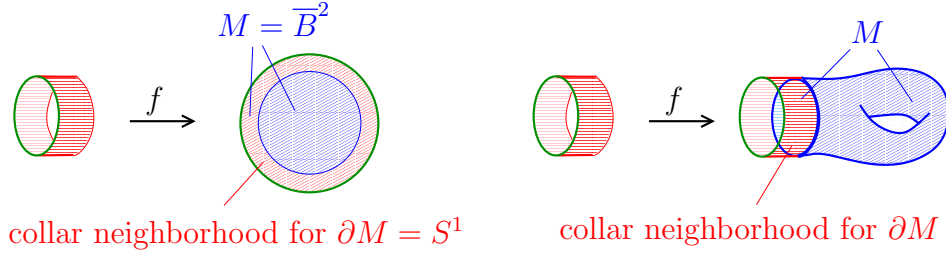


FIGURE 34.

Theorem 2.6. (Collar Neighborhood Theorem) *The boundary of any manifold M has a collar neighborhood.*

2.2. The connected sum operation. We introduce one last way to produce more examples of manifolds and topological spaces, out of given manifolds.

Definition. Let M and M' be two oriented n -dimensional manifolds and furthermore let $\varphi: \overline{B}^n \rightarrow M \setminus \partial M$ and $\varphi': \overline{B}^n \rightarrow M' \setminus \partial M'$ be embeddings where φ is orientation-preserving and φ' is orientation-reversing. We define the *connected sum* M and M' as

$$M \# M' := (M \setminus \varphi(B^n)) \sqcup (M' \setminus \varphi'(B^n)) / \{\varphi(P) = \varphi'(P) \mid P \in S^{n-1}\}.$$

Remark. The connected sum is defined for two *oriented* n -dimensional manifolds. In general the definition does depend on the choice of orientation. More precisely, there exist oriented n -dimensional manifolds M and M' such that $M \# M'$ is not diffeomorphic (in fact not even homeomorphic) to $M \# (-M')$, where $-M'$ denotes as usual the manifold M' with the opposite orientation. We will see such examples later on in Algebraic Topology II.

Proposition 2.7. *Let M and M' be two oriented n -dimensional manifolds.*

- (1) *For any choice of embeddings⁶¹ $\varphi: \overline{B}^n \rightarrow M \setminus \partial M$ and $\varphi': \overline{B}^n \rightarrow M' \setminus \partial M'$ the resulting connected sum $M \# M'$ is an n -dimensional manifold.*
- (2) *If M and M' are connected, then the diffeomorphism type of $M \# M'$ does not depend on the choice of φ and φ' .*

In the proof of Proposition 2.7 we will need the following theorem of Richard Palais [Pa1].

Theorem 2.8. *Let M be an n -dimensional manifold. Furthermore let $\varphi: \overline{B}^n \rightarrow M \setminus \partial M$ and $\psi: \overline{B}^n \rightarrow M \setminus \partial M$ be two orientation-preserving embeddings. Then there exists a map*

$$F: M \times [0, 1] \rightarrow M$$

with the following properties:

- (1) *we have $F(-, 0) = \text{id}$,*
- (2) *for each $P \in \partial M$ and for any $t \in [0, 1]$ we have $F(P, t) = P$,*
- (3) *for each $P \in \overline{B}^n$ we have $F(\varphi(P), 0) = \psi(P)$,*

⁶¹Of course we demand that φ is orientation-preserving and that φ' is orientation-reversing.

- (4) for each $t \in [0, 1]$ the map $x \mapsto F(x, t)$ is an orientation-preserving diffeomorphism of M .

In particular the map $f: M \rightarrow M$ given by $f(P) = F(P, 1)$ is an orientation-preserving diffeomorphism that satisfies $f \circ \varphi = \psi$.

We can now provide the proof of Proposition 2.7.

Proof.

- (1) By the Extension Theorem 2.4 we can extend the embeddings φ and φ' to embeddings $\Phi: 2\overline{B}^n \rightarrow M \setminus \partial M$ and $\Phi': 2\overline{B}^n \rightarrow M' \setminus \partial M'$.⁶² We need to find an atlas for $M \# M'$. For points in $M \setminus \varphi(\overline{B}^n)$ and $M' \setminus \varphi'(\overline{B}^n)$ we take the restrictions of the charts of M and M' to these open subsets. Now let $x \in \varphi(S^{n-1}) = \varphi'(S^{n-1})$. We pick a chart $\Gamma: U \rightarrow V \subset \mathbb{R}^{n-1}$ for $\varphi^{-1}(x) \in S^{n-1}$ on the $(n-1)$ -dimensional manifold S^{n-1} . We consider the map

$$\begin{aligned} \Phi(U \times [1, 2)) \cup \Phi'(U \times [1, 2)) &\rightarrow D \times (-1, 1) \subset \mathbb{R}^{n-1} \times \mathbb{R} = \mathbb{R}^n \\ P &\mapsto \begin{cases} (\Gamma(Q), 1-t), & \text{if } P = \Phi(Q, t), \text{ with } Q \in U, t \in [1, 2), \\ (\Gamma(Q), t-1), & \text{if } P = \Phi'(Q, t), \text{ with } Q \in U, t \in [1, 2). \end{cases} \end{aligned}$$

It is straightforward to verify that this map is well-defined and a homeomorphism and that these charts, together with the other charts form a smooth atlas for $M \# M'$.

- (2) The second statement is harder to prove. The key ingredient is Theorem 2.8. We leave the details to the reader. $\square$

Example.

- (1) The connected sum of an n -dimensional manifold with S^n is diffeomorphic to M .
(2) Slightly informally the connected sum of two oriented manifolds M and M' can be described as follows: we remove an open ball from each manifold and then glue M and M' together, along the boundary spheres that we had just created, in such a way that the orientations of the boundaries are opposite. In Figure 35 we sketch that the connected sum of two tori is the surface of genus two.⁶³

2.3. Knots and their complements. Before we introduce knots we recall several different ways how we can describe the 3-sphere S^3 . More precisely, by definition and the discussion on page 17 we have

$$S^3 = \{(z, w) \in \mathbb{C} \mid |w|^2 + |z|^2 = 1\} = \{(w, x, y, z) \mid w^2 + x^2 + y^2 + z^2 = 1\} \cong \mathbb{R}^3 \cup \{\infty\}.$$

$\uparrow$
via stereographic projection

⁶²Here we use that adding $\partial\overline{B}^n \times [0, 1]$ to $\overline{B}^n$ along the boundary gives the closed ball of radius 2.

⁶³The picture should be pretty convincing, but it would be fairly painful to try to prove the statement rigorously using the official definition of the surface of genus 2 as the quotient of an octagon. This is a typical example where one is quite content to work with a convincing picture instead of providing a rigorous, but much less instructive argument.

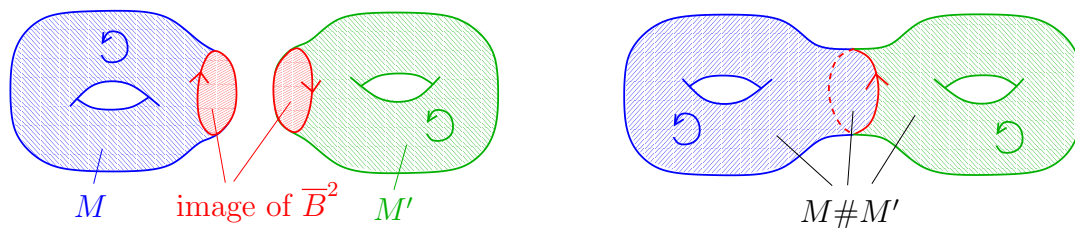


FIGURE 35.

We will go back and forth between these models without mentioning these maps. In particular we will always view $\mathbb{R}^3$ as a subset of $S^3 = \mathbb{R}^3 \cup \{\infty\}$.

Definition. A *knot* is a 1-dimensional submanifold of $S^3 = \mathbb{R}^3 \cup \{\infty\}$ that is diffeomorphic to S^1 . We say that the knot is *oriented* if it is oriented as a 1-dimensional manifold.

Remark.

- (1) It follows from Proposition 2.1 that a subset of S^3 is a knot if and only if it is the image of an embedding $S^1 \rightarrow S^3$.⁶⁴
- (2) The definition of a knot is supposed to model the “physical objects” that we have in mind and that are sketched in Figure 36. It is therefore perhaps at first not clear why we consider knots in $S^3 = \mathbb{R}^3 \cup \{\infty\}$ instead of knots in $\mathbb{R}^3$. The reason is that topologists prefer, if possible, to work with compact spaces. In particular the compact space $S^3 = \mathbb{R}^3 \cup \{\infty\}$ is often strongly preferable to the non-compact space $\mathbb{R}^3$.

In Figure 36 we give three examples of knots. More precisely, we show 1-dimensional submanifolds of $\mathbb{R}^3$ that we view as 1-dimensional submanifolds of $\mathbb{R}^3 \cup \{\infty\} = S^3$.⁶⁵ We now want to say that two knots are equivalent if one can be “deformed” into the other. The following definition makes this idea precise.

Definition. We say that *two knots K and J are equivalent* if there exists a smooth isotopy from K to J , i.e. a smooth map

$$\begin{aligned} F: S^1 \times [0, 1] &\rightarrow S^3 \\ (z, t) &\mapsto F(z, t) \end{aligned}$$

⁶⁴To be more precise, the “if”-direction is a consequence of Proposition 2.1. On the other hand, if $K \subset S^3$ is a knot, then by definition there exists a diffeomorphism $\varphi: S^1 \rightarrow K \subset S^3$ and it follows easily from the definitions that the map φ viewed as a map from $S^1 \rightarrow S^3$ is an embedding whose image is precisely K .

⁶⁵In principle it is possible to give a precise description of these three knots. For example the trivial knot is defined as $\{(x, y, 0) \mid x^2 + y^2 = 1\} \subset S^3 = \mathbb{R}^3 \cup \{\infty\}$. Later on we will give a precise definition of the trefoil. It is also clear that one can give a precise description of the figure-8 knot, but this description would be painful to write down and it would not add to our understanding. We therefore stick with the picture, with the understanding, that if somebody was challenging us, we could write down a precise descriptions in coordinates. But it is considered very impolite to challenge a topologist to give a rigorous description.

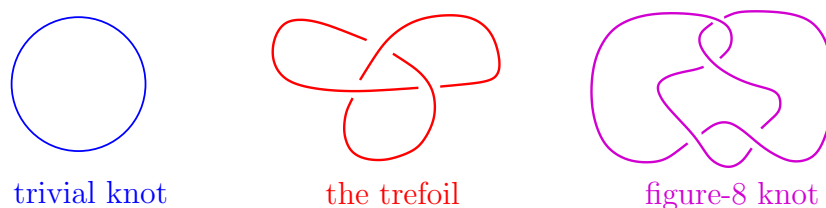


FIGURE 36. The trivial knot, the trefoil knot and the figure-8 knot.

such that the following hold:

- (1) We have $F(S^1 \times \{0\}) = K$ and $F(S^1 \times \{1\}) = J$,
- (2) for each $t \in [0, 1]$ the map $S^1 \rightarrow S^3$ given by $z \mapsto F(z, t)$ is an embedding.

If K and J are in fact oriented knots, then we also want the following condition to be satisfied:

- (3) the maps

$$\begin{array}{ccc} S^1 & \rightarrow & K \\ z & \mapsto & F(z, 0) \end{array} \quad \text{and} \quad \begin{array}{ccc} S^1 & \rightarrow & J \\ z & \mapsto & F(z, 1) \end{array}$$

are both orientation preserving.⁶⁶

Remark. A smooth isotopy F from a knot K to a knot J consists of a “smooth” family $F(S^1, s)$ of knots with parameter $s \in [0, 1]$.

For example the three knots shown in Figure 37 are equivalent. Usually we do not distinguish two knots if they are equivalent. For example, any knot that is equivalent to a trivial knot is called *trivial knot*.

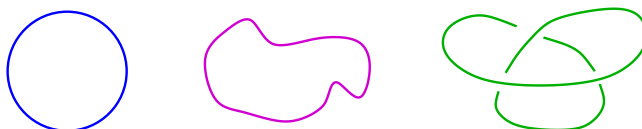


FIGURE 37. The three knots are equivalent.

Playing around with pictures for some time shows that it might be quite difficult to show that the trefoil is equivalent to the trivial knot. This arouses the suspicion, that the trefoil is in fact not equivalent to the trivial knot. This raises the following question.

Question 2.9. *How can we show that the trefoil is not the trivial knot?*

The following proposition allows us to turn this question into the question, whether two topological spaces are homeomorphic.

⁶⁶Recall that on page 50 we said that we always view S^1 as an oriented manifold, where the orientation is the “counterclockwise orientation”.

Proposition 2.10. *If K and J are two equivalent knots, then the knot complements $S^3 \setminus K$ and $S^3 \setminus J$ are diffeomorphic, in particular homeomorphic.*

Remark. In fact the converse to Proposition 2.10 also holds. If K and J are two knots such that the knot complements $S^3 \setminus K$ and $S^3 \setminus J$ are diffeomorphic, then K and J are equivalent. One might think that this should be quite straightforward to show, but it turns out to be surprisingly difficult. A proof was given only in 1989 by Cameron Gordon and John Luecke [GL]. The proof builds on the work of William Thurston for which he got the fields medals in 1982.

Proof. Let

$$\begin{aligned} F: S^1 \times [0, 1] &\rightarrow S^3 \\ (z, t) &\mapsto F(z, t) \end{aligned}$$

be a smooth isotopy from a knot K to a knot J . By the Isotopy Extension Theorem, see [Ko, Theorem II.5.2] for the precise statement and the proof thereof, we can extend the smooth isotopy F to a smooth isotopy of S^3 . More precisely, there exists a smooth map

$$\begin{aligned} G: S^3 \times [0, 1] &\rightarrow S^3 \\ (z, t) &\mapsto G(z, t) \end{aligned}$$

such that the following hold:

- (1) We have $G(z, 0) = z$ for all $z \in S^3$, in particular $G(K \times \{0\}) = K$,
- (2) $G(K \times \{1\}) = J$, and
- (3) for each $t \in [0, 1]$ the map $S^3 \rightarrow S^3$ given by $z \mapsto G(z, t)$ is a diffeomorphism.

Then the map

$$\begin{aligned} \Phi: S^3 &\rightarrow S^3 \\ z = G(z, 0) &\mapsto G(z, 1) \end{aligned}$$

is a diffeomorphism with $\Phi(K) = J$. In particular the map Φ restricts to a diffeomorphism $\Phi: S^3 \setminus K \rightarrow S^3 \setminus J$. $\square$

We conclude this discussion of knots with a definition.

Definition. Let K be an oriented knot in S^3 . By the Tubular Neighborhood Theorem 2.2 there exists an orientation-preserving embedding

$$F: K \times \overline{B}^2 \rightarrow S^3$$

such that $F(P, 0) = P$ for all $P \in K$. For any $P \in K$ we refer to the oriented submanifold⁶⁷ $F(\{P\} \times S^1)$ as a *meridian* of K .

It follows easily from Proposition 2.3 that any two meridians of a given knot K are smoothly isotopic in the knot complement $S^3 \setminus K$. More casually speaking, a meridian is a curve which circles once around a knot where the orientation is given by the right-hand-rule, i.e. if the thumb points into the direction of the knot, then the fingers point into the direction of the meridian.

⁶⁷It follows easily from Proposition 2.1 that $F(P \times S^1)$ is indeed a submanifold.

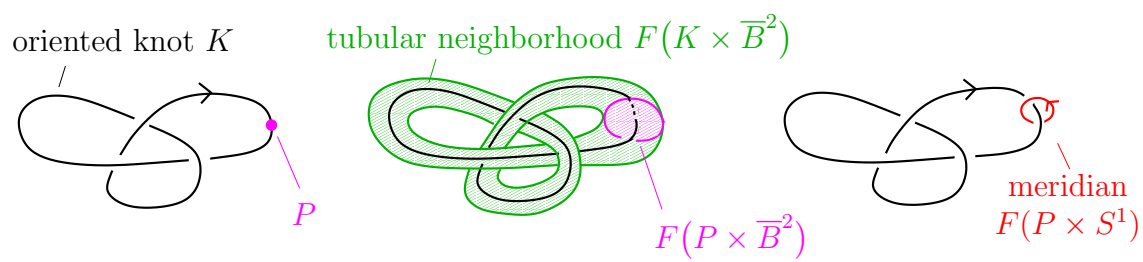


FIGURE 38.

3. HOW CAN WE SHOW THAT TWO TOPOLOGICAL SPACES ARE NOT HOMEOMORPHIC?

In the previous two sections we collected many examples of topological spaces. In many cases it was relatively straightforward to see that two topological spaces are homeomorphic. For example we wrote down an explicit homeomorphism from $\mathbb{R}^n/\mathbb{Z}^n$ to $(S^1)^n$. Sometimes it is rather painful to explicitly write down a homeomorphism, for example it was not particularly easy to find a homeomorphism from the open cube $(0,1)^n$ to the open ball $B^n = \{x \in \mathbb{R}^n \mid \|x\| < 1\}$. Nonetheless, if two topological spaces are homeomorphic, then usually one can write down an explicit homeomorphism.

On the other hand, how can we tell that two topological spaces are not homeomorphic? There are a couple of trivial criteria: being compact, connected, Hausdorff are properties of topological spaces that are shared by homeomorphic topological spaces. This implies for example that the closed cube $[0,1]^n$ is not homeomorphic to the open cube $(0,1)^n$, since the former topological space is compact and the latter is not.

But these criteria are of no use when it comes to studying more subtle examples. For example they do not suffice to address the following questions:

- (1) Can surfaces with different genera be homeomorphic?
- (2) Given $k \in \mathbb{Z}_{\geq 0}$ we denote by U the complement of k points in $\mathbb{R}^2$. If $k \neq l$, is it possible that U_k and U_l are homeomorphic?
- (3) Is the complement of the trefoil homeomorphic to the complement of the trivial knot?

In each case we suspect that the answer is no. Certainly we do not succeed in finding a homeomorphism, but how can we show that this is not due to lack of imagination, but that there is indeed no homeomorphism?

In Analysis III and IV we had seen two tools which appear to be useful. We first recall the relevant result from Analysis IV. In Analysis IV we associated to each manifold M and each $k \in \mathbb{Z}_{\geq 0}$ the de Rham cohomology group $H_{dR}^k(M)$. We showed that two diffeomorphic manifolds have isomorphic de Rham cohomology groups. We used de Rham cohomology groups to show that the 2-dimensional sphere S^2 is not diffeomorphic to the torus $S^1 \times S^1$.⁶⁸ This approach has several disadvantages:

- (1) the de Rham cohomology groups are only defined for manifolds,
- (2) the de Rham cohomology groups can only be used to show that two manifolds are not *diffeomorphic*, a priori they cannot be used to show that two manifolds are not *homeomorphic*,
- (3) as of right now we have few tools to compute de Rham cohomology groups.

In Algebraic Topology III we will fix all these problems. In Algebraic Topology I we will instead try to revive some of the ideas of Analysis III.

- (1) In Analysis III, given a subset U of $\mathbb{C}$, we had defined a *path* in U to be a map $\gamma: [a,b] \rightarrow U$. If $\gamma(a) = \gamma(b)$, then we say that γ is a *loop*.

⁶⁸In fact we used a little trick: we showed really that $S^2 \setminus \{P\}$ cannot be diffeomorphic to $S^1 \times S^1 \setminus \{Q\}$, since for the former we have $H_{dR}^1(S^2 \setminus \{P\}) = 0$ whereas for the latter we have $H_{dR}^1(S^1 \times S^1 \setminus \{Q\}) \neq 0$.

- (2) Two paths $\gamma_0, \gamma_1: [a, b] \rightarrow U$ with the same starting point $P := \gamma_0(a) = \gamma_1(a)$ and the same endpoint $Q := \gamma_0(b) = \gamma_1(b)$ are called *homotopic in U* , if there exists a map

$$\begin{aligned} \Gamma: [a, b] \times [0, 1] &\rightarrow U \\ (t, s) &\mapsto \Gamma(t, s), \end{aligned}$$

with the following properties

- (a) for every $t \in [a, b]$ we have

$$\Gamma(t, 0) = \gamma_0(t) \text{ and } \Gamma(t, 1) = \gamma_1(t),$$

- (b) for every $s \in [0, 1]$ we have

$$\Gamma(a, s) = P \quad \text{and} \quad \Gamma(b, s) = Q.$$

Put differently, a homotopy between two paths consists of a “continuous” family of paths $\{\Gamma(-, s)\}_{s \in [0, 1]}$ from P to Q which interpolates between the paths γ_0 and γ_1 . In Figure 39 we illustrate a homotopy between two paths.

- (3) A loop $\gamma: [a, b] \rightarrow U$ is called *null-homotopic* if it is homotopic to the constant path given by $\delta(t) := \gamma(a)$ for all $t \in [a, b]$.
 (4) We say U is *simply connected* if each loop is null-homotopic.

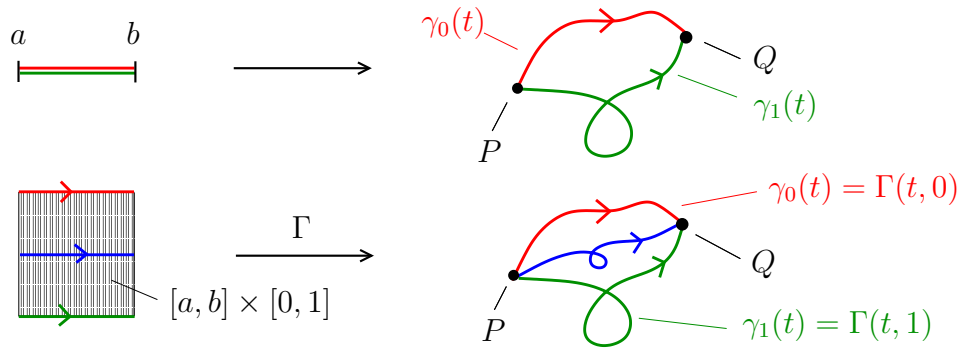


FIGURE 39. Schematic image of a homotopy Γ between two paths γ_0 and γ_1 from P to Q .

Example. In Analysis III we had already seen that any loop in $\mathbb{R}^2 = \mathbb{C}$ is null-homotopic. Indeed, we let $\gamma: [0, 1] \rightarrow \mathbb{C}$ be a loop with starting and endpoint P . Then

$$\begin{aligned} \Gamma: [a, b] \times [0, 1] &\rightarrow \mathbb{R}^n \\ (t, s) &\mapsto \gamma(t) \cdot (1 - s) + P \cdot s \end{aligned}$$

is a homotopy between the loop γ and the constant path at P . The fact that every loop in $\mathbb{C}$ is null-homotopic is illustrated in Figure 40.

We now recall the following definition from Analysis III.

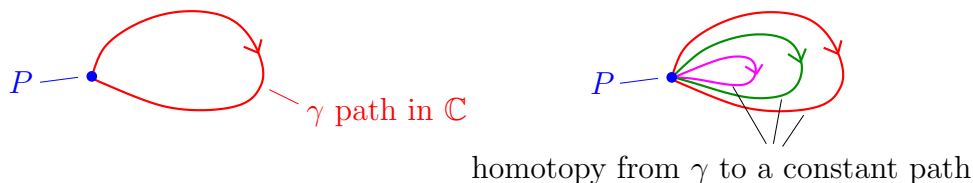


FIGURE 40.

Definition. Let $U \subset \mathbb{C}$ be an open subset, let $f: U \rightarrow \mathbb{C}$ be a holomorphic function and let $\gamma: [a, b] \rightarrow U$ be a continuously differentiable path.⁶⁹ The *path-integral of f along γ* is defined as

$$\int_{\gamma} f(z) dz := \int_{t=a}^{t=b} f(\gamma(t)) \cdot \gamma'(t) dt \in \mathbb{C}.$$

Examples.

- (1) Let $U = \mathbb{C} \setminus \{0\}$, let $f(z) = \frac{1}{z}$ and let $\gamma: [0, 1] \rightarrow \mathbb{C} \setminus \{0\}$ be the loop given by $\gamma(t) = e^{2\pi it}$, i.e. γ is a loop that “goes once around the origin”. Then

$$\int_{\gamma} f(z) dz := \int_{t=0}^{t=1} f(\gamma(t)) \cdot \gamma'(t) dt = \int_{t=0}^{t=1} \frac{1}{e^{2\pi it}} \cdot e^{2\pi it} \cdot 2\pi i dt = 2\pi i.$$

- (2) If γ is a constant path, then $\gamma'(t) \equiv 0$. This shows that the path-integral along a constant path is zero.

The following proposition is Satz 10.2 from Analysis III.

Proposition 3.1. *Let $U \subset \mathbb{C}$ be an open subset, let $f: U \rightarrow \mathbb{C}$ be a holomorphic function and let $\gamma, \delta: [a, b] \rightarrow U$ be two continuously differentiable paths. If γ and δ are homotopic, then*

$$\int_{\gamma} f(z) dz = \int_{\delta} f(z) dz.$$

The following corollary is a consequence of Proposition 3.1 and of the above calculations.

Corollary 3.2. *The loop $\gamma: [0, 1] \rightarrow \mathbb{C} \setminus \{0\}$ given by $\gamma(t) = e^{2\pi it}$ is not homotopic to a constant path, i.e. γ is not null-homotopic.*

If two subsets of $\mathbb{C}$ are homeomorphic, then evidently either both are simply connected or none is. From Corollary 3.2 it now follows that $\mathbb{R}^2 = \mathbb{C}$ and $\mathbb{R}^2 \setminus \{(0, 0)\} = \mathbb{C} \setminus \{0\}$ are not homeomorphic.

This second approach to showing that two topological spaces are different also has several disadvantages:

- (1) A priori these notions are defined only for subsets of $\mathbb{C}$,

⁶⁹A path $\gamma: [a, b] \rightarrow U$ is called continuously differentiable if it is differentiable, if its derivative γ' is continuous on (a, b) and if the derivative extends to a continuous map on the interval $[a, b]$.

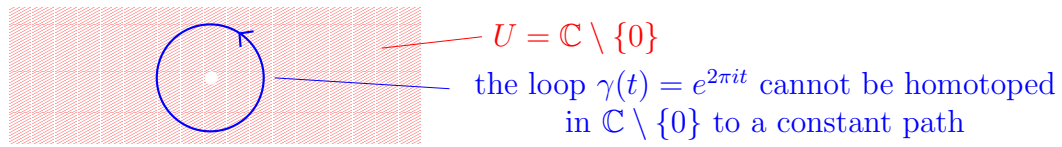


FIGURE 41.

- (2) this approach does not allow us to show that the two subsets $U = \mathbb{C} \setminus \{0\}$ and $V = \mathbb{C} \setminus \{0, 1\}$ are not homeomorphic, since neither is simply connected.

The first objection is of course easy to rectify, since the definitions make sense for any topological space. But if we work with more general situations a new problem arises: how can we show, without using complex analysis, that a topological space is not simply connected? In the following chapters we will address all these issues.

In particular, when it comes to distinguishing $U = \mathbb{C} \setminus \{0\}$ and $V = \mathbb{C} \setminus \{0, 1\}$, then naively we would say that V has “more” loops that are not null-homotopic than U . In the subsequent chapter we will in particular make precise what “more” means.

4. THE FUNDAMENTAL GROUP

4.1. Homotopy classes of paths. In this section we extend some of the ideas and notions about paths that we had already introduced in Analysis III to the more general setting of paths in topological spaces.

Definition. Let X be a topological space.

- (1) A *path* in X is a⁷⁰ map $\gamma: [a, b] \rightarrow X$. We call $\gamma(a)$ the *starting point* of γ and we call $\gamma(b)$ the *endpoint* of γ . Often we say that γ is a *path from $\gamma(a)$ to $\gamma(b)$* .
- (2) A *loop* is a path for which the starting point and the endpoint coincide.

Convention. In the following, if we do not specify the domain of a path or a loop, then it is understood to be $[0, 1]$.

Definition. Let X be a topological space.

- (1) Let $\gamma_0, \gamma_1: [a, b] \rightarrow X$ be two paths with the same starting point P and the same endpoint Q . A *homotopy between the paths γ_0 and γ_1* is a map

$$\begin{aligned} \Gamma: [a, b] \times [0, 1] &\rightarrow U \\ (t, s) &\mapsto \Gamma(t, s), \end{aligned}$$

with the following properties

- (a) for every $t \in [a, b]$ we have

$$\Gamma(t, 0) = \gamma_0(t) \text{ and } \Gamma(t, 1) = \gamma_1(t),$$

- (b) for every $s \in [0, 1]$ we have

$$\Gamma(a, s) = P \quad \text{and} \quad \Gamma(b, s) = Q.$$

- (2) If there exists a homotopy between two paths γ_0 and γ_1 , then we say that γ_0 and γ_1 are *homotopic*, and we write $\gamma_0 \simeq \gamma_1$.
- (3) We say a loop $\gamma: [a, b] \rightarrow X$ is *null-homotopic*, if it is homotopic to the constant path given by $\gamma(t) = \gamma(a)$ for all $t \in [a, b]$.

Examples.

- (A) Let $\gamma_0, \gamma_1: [a, b] \rightarrow \mathbb{R}^n$ be two paths in $\mathbb{R}^n$ with same starting point and the same endpoint. Then

$$\begin{aligned} \Gamma: [a, b] \times [0, 1] &\rightarrow \mathbb{R}^n \\ (t, s) &\mapsto \gamma_0(t) \cdot (1 - s) + \gamma_1(t) \cdot s \end{aligned}$$

is a homotopy between γ_0 and γ_1 . We had already illustrated this fact in Figure 40.

- (B) Let $X = \mathbb{C} \setminus \{0\} = \mathbb{R}^2 \setminus \{(0, 0)\}$. As we had just seen in Corollary 3.2, the path

$$\begin{aligned} \gamma: [0, 1] &\rightarrow X \\ t &\mapsto e^{2\pi i t} = (\cos(2\pi t), \sin(2\pi t)) \end{aligned}$$

is not null-homotopic in X .

⁷⁰Recall that all maps are understood to be continuous, unless we say something else.

(C) Let $Y = S^1 \times (0, 2)$ and consider the path

$$\begin{aligned} \tilde{\gamma}: [0, 1] &\rightarrow Y = S^1 \times (0, 2) \\ t &\mapsto (e^{2\pi i t}, 1). \end{aligned}$$

This path is not null-homotopic. This can be seen as follows: we consider the embedding

$$\begin{aligned} \Phi: Y = S^1 \times (0, 2) &\rightarrow X = \mathbb{C} \setminus \{0\} \\ (z, t) &\mapsto z \cdot t. \end{aligned}$$

If $\tilde{\gamma}$ was null-homotopic in Y , then there would exist a homotopy

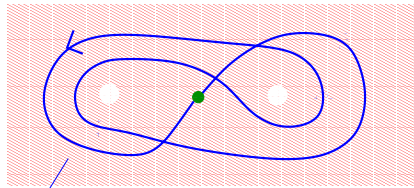
$$\tilde{F}: [0, 1] \times [0, 1] \rightarrow Y$$

from $\tilde{\gamma}$ to the constant path $\tilde{\delta}(t) := (1, 1)$, $t \in [0, 1]$. But then

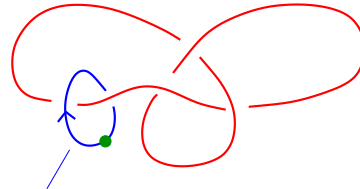
$$F := \Phi \circ \tilde{F}: [0, 1] \times [0, 1] \rightarrow X$$

would be a homotopy in $X = \mathbb{C} \setminus \{0\}$ from the path $\gamma := \Phi \circ \tilde{\gamma}$ to the constant path $\delta := \Phi \circ \tilde{\delta}$. But $\gamma = \Phi \circ \tilde{\gamma}$ is precisely the path γ from example (B) which we had shown not to be null-homotopic in $X = \mathbb{C} \setminus \{0\}$.

(D) We consider the two loops in Figure 42. In both cases it is hard to see how these loops could possibly be homotopic to a constant path, but it is also difficult to find an argument why that should not be possible.⁷¹



loop in $\mathbb{C} \setminus \{\pm 1\}$



loop in the complement of a knot

FIGURE 42.

(E) We consider the real projective space $\mathbb{RP}^2$. Here and throughout the lecture notes we make the identification⁷²

$$\begin{aligned} \mathbb{RP}^2 = S^2 / \sim_P &\xrightarrow{\cong} \overline{B}^2 / \sim \\ [(x, y, z)] &\mapsto [(x, y)] \end{aligned}$$

⁷¹For example the given loop in $\mathbb{C} \setminus \{0, 1\}$ is actually null-homotopic in $\mathbb{C} \setminus \{0\}$ and it is null-homotopic in $\mathbb{C} \setminus \{1\}$. In particular our trick of using our results from Analysis III does not work in this case.

⁷²More precisely, the given map is easily seen to be a homeomorphism and we use this homeomorphism to go back and forth between $\mathbb{RP}^2 = S^2 / \sim$ and $\overline{B}^2 / \sim$.

where $\sim$ is the equivalence relation on $\overline{B}^2$ generated by $P \sim -P$ for $P \in S^1 = \partial\overline{B}^2$. Then

$$\begin{aligned} \gamma: [0, 1] &\rightarrow \overline{B}^2 / \sim \\ t &\mapsto (\cos(\pi t), \sin(\pi t)) \end{aligned}$$

is a loop, see Figure 43.⁷³ But is it null-homotopic?

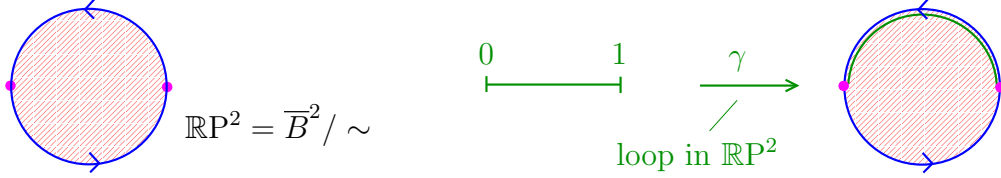


FIGURE 43.

(F) On page 16 we considered the set $X = \{A, B, C, D\}$ with a slightly unusual topology and gave an example of a continuous non-constant map $f: S^1 \rightarrow X$. The question arises, is this map null-homotopic?

Remark. Let X be a topological space and let $x_0 \in X$. The maps

$$\begin{aligned} \{\text{loops } \gamma: [0, 1] \rightarrow X \text{ with } \gamma(0) = \gamma(1) = x_0\} &\Leftrightarrow \{\text{maps } f: S^1 \rightarrow X \text{ with } f(1) = x_0\} \\ (\gamma: [0, 1] \rightarrow X) &\mapsto (z = e^{2\pi it} \mapsto \gamma(t)) \\ (t \mapsto f(e^{2\pi it})) &\leftarrow (f: S^1 \rightarrow X) \end{aligned}$$

are evidently bijections. Thus we can think of loops in (X, x_0) as maps from S^1 to X that send 1 to x_0 .

The following lemma will be proved in exercise sheet 3. It gives a useful criterion for a loop to be null-homotopic.

Lemma 4.1. *Let X be a topological space and let $x_0 \in X$. Let $\gamma: [0, 1] \rightarrow X$ be a loop with $\gamma(0) = \gamma(1) = x_0$. We denote by*

$$\begin{aligned} \varphi: S^1 &\rightarrow X \\ z = e^{2\pi it} &\mapsto \gamma(t) \end{aligned}$$

the corresponding map from S^1 to X . Then γ is null-homotopic if and only if there exists a map $\Phi: \overline{B}^2 \rightarrow X$ so that $\Phi|_{S^1} = \varphi$.

Before we continue with the examples we need to gather a few more elementary facts about homotopies of paths.

Lemma 4.2. *Let X be a topological space and let $x, y \in X$. Then “homotopy” is an equivalence relation on the set of paths $\gamma: [a, b] \rightarrow X$ from x to y .*

⁷³Why is it a loop?

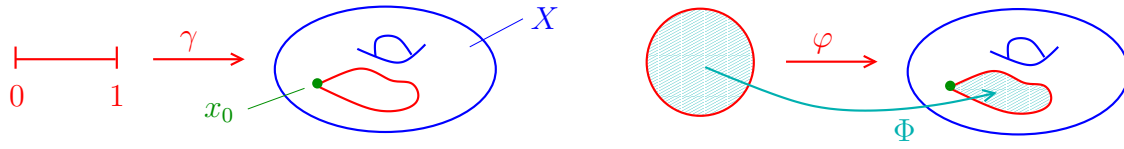


FIGURE 44. Illustration of Lemma 4.1.

Proof. In order to simplify the notation we only work with paths defined on the interval $[0, 1]$. We have to show that “homotopy” satisfies the three properties of an equivalence relation. Let $\alpha, \beta, \gamma: [0, 1] \rightarrow X$ be three paths from x to y .

- (1) The “constant homotopy” given by $\Gamma(t, s) := \gamma(t)$ defines a homotopy between γ and γ .
- (2) Let Γ be a homotopy from α to β . Then the map Γ' defined by

$$\Gamma'(t, s) := \Gamma(t, 1 - s)$$

is a homotopy from β to α .

- (3) Let Γ be a homotopy from α to β and let Δ be a homotopy from β to γ . Then the map

$$\begin{aligned} [0, 1] \times [0, 1] &\rightarrow X \\ (t, s) &\mapsto \begin{cases} \Gamma(t, 2s), & \text{if } s \in [0, \frac{1}{2}], \\ \Delta(t, 2s - 1), & \text{if } s \in (\frac{1}{2}, 1] \end{cases} \end{aligned}$$

is a homotopy from α to γ .⁷⁴ This part of the proof is illustrated in Figure 45. $\square$

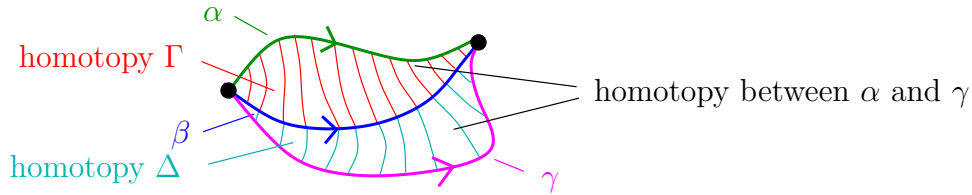


FIGURE 45.

Definition. Given a path $f: [0, 1] \rightarrow X$ in a topological space X we denote by $[f]$ the equivalence class of f with respect to the equivalence relation which is given by homotopies. We call $[f]$ the *homotopy class* of f .

Definition. Let $\gamma: [a, b] \rightarrow X$ be a path in a topological space and let $\varphi: [a, b] \rightarrow [a, b]$ be a map⁷⁵ with $\varphi(a) = a$ and $\varphi(b) = b$. We refer to $\gamma \circ \varphi: [a, b] \rightarrow X$ as a *reparametrization* of γ .

⁷⁴It is straightforward to verify that the map is indeed continuous.

⁷⁵Note that φ is not required to be a homeomorphism.

The following lemma says that reparametrizing does not change the homotopy class of a path.

Lemma 4.3. *Let $\gamma: [a, b] \rightarrow X$ be a path in a topological space. If a path $\delta: [a, b] \rightarrow X$ is obtained from γ by a reparametrization, then γ and δ are homotopic.*

Proof. Let $\gamma: [a, b] \rightarrow X$ be a path in a topological space and let $\varphi: [a, b] \rightarrow [a, b]$ be a map with $\varphi(a) = a$, $\varphi(b) = b$. Then a homotopy between γ and $\gamma \circ \varphi$ is given by⁷⁶

$$\begin{aligned} P: [a, b] \times [0, 1] &\rightarrow X \\ (t, s) &\mapsto \gamma(t \cdot s + \varphi(t) \cdot (1 - s)). \end{aligned} \quad \square$$

In Analysis III we had already seen that we can concatenate two paths if the endpoint of the first path agrees with the starting point of the second path. We use this observation to define the product of two paths.

Definition. Let X be a topological space and let $\alpha: [a, b] \rightarrow X$ and $\beta: [c, d] \rightarrow X$ be two paths with $\alpha(b) = \beta(c)$. We define the *product of α and β* as the path $\alpha * \beta$ which is given by⁷⁷

$$\begin{aligned} \alpha * \beta: [0, 1] &\rightarrow X \\ t &\mapsto \begin{cases} \alpha(a + 2t(b - a)), & \text{if } t \in [0, \frac{1}{2}] \\ \beta(c + (2t - 1)(d - c)), & \text{if } t \in (\frac{1}{2}, 1]. \end{cases} \end{aligned}$$

The product of two paths is thus given by “first running along α ” and then “running along β ”. Hereby we reparametrized the path so that the domain of $\alpha * \beta$ is $[0, 1]$. This convention might look slightly odd, but later on we will mostly work with paths that are defined on the interval $[0, 1]$.

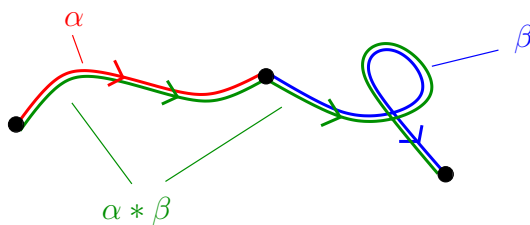


FIGURE 46. Definition of the product of two paths.

The proof of the following lemma is elementary and is left as an exercise. The statement of the lemma is illustrated in Figure 47.

⁷⁶Where do we actually use the hypothesis that $\varphi(a) = a$ and $\varphi(b) = b$?

⁷⁷Again it is straightforward to see that our hypothesis $\alpha(b) = \beta(c)$ implies that $\alpha * \beta: [0, 1] \rightarrow X$ is a continuous map.

Lemma 4.4. Let $\alpha, \alpha': [a, b] \rightarrow X$ and $\beta, \beta': [c, d] \rightarrow X$ be two sets of paths in a topological space X such that $\alpha(b) = \alpha'(b) = \beta(c) = \beta'(c)$. Then

$$\alpha \simeq \alpha' \text{ and } \beta \simeq \beta' \implies \alpha * \beta \simeq \alpha' * \beta'.$$

Put differently, if $[\alpha] = [\alpha']$ and $[\beta] = [\beta']$, then $[\alpha * \beta] = [\alpha' * \beta']$.

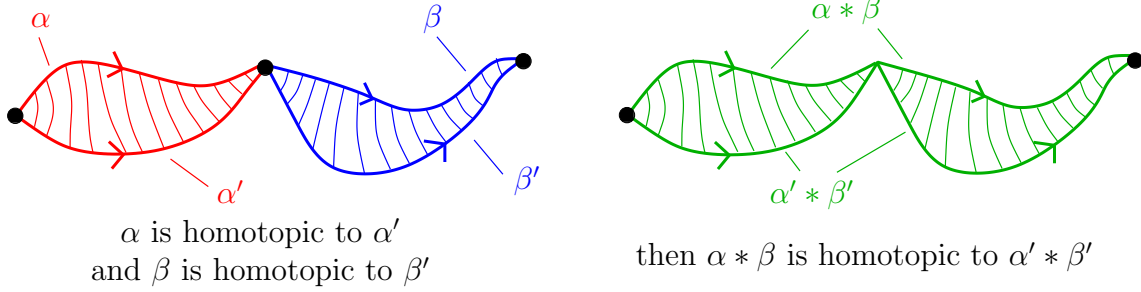


FIGURE 47. Illustration of Lemma 4.4.

Definition. Let $[\alpha]$ and $[\beta]$ be two equivalence classes of paths in a topological space X with $\alpha(1) = \beta(0)$. We define the⁷⁸ *product of the equivalence classes* $[\alpha]$ and $[\beta]$ as

$$[\alpha] \cdot [\beta] = [\alpha * \beta].$$

The following proposition says that the product of equivalence classes of paths shares many properties of the product structure of a group.

Proposition 4.5. Let $\alpha, \beta, \gamma: [0, 1] \rightarrow X$ be three paths in a topological space X . Then the following hold:

- (1) If $\alpha(1) = \beta(0)$ and $\beta(1) = \gamma(0)$, then⁷⁹

$$[\alpha] \cdot ([\beta] \cdot [\gamma]) = ([\alpha] \cdot [\beta]) \cdot [\gamma].$$

- (2) For $x \in X$ we denote by e_x the constant path that is given by $e_x(t) := x$, $t \in [0, 1]$. Then the following equality holds

$$[e_{\alpha(0)}] \cdot [\alpha] = [\alpha] = [\alpha] \cdot [e_{\alpha(1)}].$$

- (3) We denote by $\bar{\alpha}$ the path that is defined by $\bar{\alpha}(t) := \alpha(1 - t)$, $t \in [0, 1]$. Then the following equalities hold

$$[\alpha] \cdot [\bar{\alpha}] = [e_{\alpha(0)}] \quad \text{and} \quad [\bar{\alpha}] \cdot [\alpha] = [e_{\alpha(1)}].$$

Proof. Let $\alpha, \beta, \gamma: [0, 1] \rightarrow X$ be three paths in a topological space X .

⁷⁸According to Lemma 4.4 this definition does not depend on the choice of the representatives α and β of the equivalence classes.

⁷⁹Put differently, the paths $\alpha * (\beta * \gamma)$ and $(\alpha * \beta) * \gamma$ are homotopic. Note that these paths are similar in the sense that both paths “run through” the points on α , β and γ , but at different speeds. In general the paths are not equal, for example we have $(\alpha * (\beta * \gamma))(\frac{1}{2}) = \alpha(1)$ and $((\alpha * \beta) * \gamma)(\frac{1}{2}) = \beta(1)$.

(1) Suppose that $\alpha(1) = \beta(0)$ and $\beta(1) = \gamma(0)$. We need to show that

$$\alpha * (\beta * \gamma) \simeq (\alpha * \beta) * \gamma.$$

We consider the map

$$\begin{aligned} \Phi: [0, 1] &\rightarrow X \\ t &\mapsto \begin{cases} \alpha(3t), & \text{if } t \in [0, \frac{1}{3}], \\ \beta(3t - 1), & \text{if } t \in (\frac{1}{3}, \frac{2}{3}], \\ \gamma(3t - 2), & \text{if } t \in (\frac{2}{3}, 1]. \end{cases} \end{aligned}$$

According to our hypothesis this map is indeed continuous. We consider the maps

$$\begin{aligned} p: [0, 1] &\rightarrow [0, 1] & q: [0, 1] &\rightarrow [0, 1] \\ t &\mapsto \begin{cases} \frac{2}{3}t, & \text{if } t \in [0, \frac{1}{2}], \\ \frac{4}{3}t - \frac{1}{3}, & \text{if } t \in (\frac{1}{2}, 1] \end{cases} & \text{and} & t \mapsto \begin{cases} \frac{4}{3}t, & \text{if } t \in [0, \frac{1}{2}], \\ \frac{2}{3}t + \frac{1}{3}, & \text{if } t \in (\frac{1}{2}, 1] \end{cases} \end{aligned}$$

Then

$$\begin{array}{ccccccc} \alpha * (\beta * \gamma) & = & \Phi \circ p & \simeq & \Phi & \simeq & \Phi \circ q = (\alpha * \beta) * \gamma \\ & & \uparrow & & \uparrow & & \uparrow \\ & & \text{by definition} & & \text{by Lemma 4.3} & & \text{by definition} \end{array}$$

(2) We consider the map

$$\begin{aligned} q: [0, 1] &\rightarrow [0, 1] \\ t &\mapsto \begin{cases} 0, & \text{if } t \in [0, \frac{1}{2}], \\ 2t - 1, & \text{if } t \in (\frac{1}{2}, 1]. \end{cases} \end{aligned}$$

Then

$$\begin{array}{ccc} e_{\alpha(0)} * \alpha & = & \alpha \circ q \simeq \alpha. \\ & \uparrow & \uparrow \\ & \text{by definition} & \text{by Lemma 4.3} \end{array}$$

This shows that $[e_{\alpha(0)}] \cdot [\alpha] = [e_{\alpha(0)} * \alpha] = [\alpha]$. Almost the same argument also implies that $[\alpha] = [\alpha] \cdot [e_{\alpha(1)}]$.

(3) We write $P = \alpha(0)$. Note that $\alpha * \bar{\alpha}$ is a loop in P . We consider the map

$$\begin{aligned} F: [0, 1] \times [0, 1] &\rightarrow X \\ (t, s) &\mapsto \begin{cases} \alpha(2t \cdot s), & \text{if } t \in [0, \frac{1}{2}], \\ \alpha(2s - 2t \cdot s), & \text{if } t \in (\frac{1}{2}, 1]. \end{cases} \end{aligned}$$

It is straightforward to verify that this map is indeed continuous. For $s = 0$ we obtain the constant path at P and for $s = 1$ we obtain the path $\alpha * \bar{\alpha}$. For any $s \in [0, 1]$ we furthermore have $F(0, s) = F(1, s) = P$. Thus F is a homotopy from the constant path to $\alpha * \bar{\alpha}$ which implies that $[e_{\alpha(0)}] = [\alpha * \bar{\alpha}] = [\alpha] \cdot [\bar{\alpha}]$. Almost the same argument shows that $[\bar{\alpha} * \alpha] = [e_{\alpha(1)}]$. $\square$

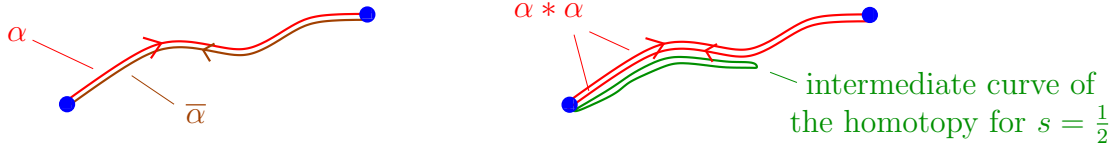


FIGURE 48. Illustration of the proof of Proposition 4.5 (3).

4.2. The fundamental group of a pointed topological space. In Proposition 4.5 we had seen that the product of homotopy classes of paths shares many properties of the multiplication in a group. Nonetheless the homotopy classes of paths do not form a group, since the product of two paths is not defined, unless the endpoint of the first path agrees with the starting point of the second path.

We will resolve this problem by restricting our attention to loops.

Definition. Let X be a topological space and let $x_0 \in X$ be a point. A *loop in (X, x_0)* is a path $f: [0, 1] \rightarrow X$ ⁸⁰ with $f(0) = f(1) = x_0$, i.e. x_0 is the starting as well as the endpoint of f .

Proposition 4.6. *Let X be a topological space and $x_0 \in X$. The set*

$$\pi_1(X, x_0) := \{\text{homotopy classes of loops in } (X, x_0)\}$$

together with the product map $[\alpha] \cdot [\beta] := [\alpha * \beta]$ forms a group where the neutral element is represented by the constant path at x_0 and where the inverse of $[\gamma]$ is given by $[\bar{\gamma}]$.

Proof. It is clear that the product of any two loops in (X, x_0) is defined. According to Lemma 4.4 the product of paths descends to a well-defined map

$$\begin{aligned} \pi_1(X, x_0) \times \pi_1(X, x_0) &\rightarrow \pi_1(X, x_0) \\ ([\alpha], [\beta]) &\mapsto [\alpha] \cdot [\beta] := [\alpha * \beta]. \end{aligned}$$

According to Proposition 4.5 (1) this product map satisfies the associativity axiom. The trivial element is according to Proposition 4.5 (2) given by the homotopy class of the constant loop $[e_{x_0}]$. Furthermore according to Proposition 4.5 (3) the inverse of the homotopy class of a loop $\gamma: [0, 1] \rightarrow X$ in x_0 is given by the homotopy class of the loop defined by $\bar{\gamma}(t) := \gamma(1 - t)$, $t \in [0, 1]$. $\square$

Definition. We call $\pi_1(X, x_0)$ the *fundamental group of X with respect to the base point x_0* .

One of our main goals of Algebraic Topology I will be to determine the fundamental group of the topological spaces that we had introduced in the earlier sections. This goal will keep us busy for many weeks.

Examples.

⁸⁰Note that part of the definition of “loop in (X, x_0) ” is that the domain is the interval $[0, 1]$.

- (1) We consider $X = \mathbb{R}^n$ and let $x_0 \in \mathbb{R}^n$ be an arbitrary point. As we had pointed out on page 66, all loops in (X, x_0) are null-homotopic. More precisely, if $\gamma: [0, 1] \rightarrow \mathbb{R}^n$ is a loop in $(\mathbb{R}^n, x_0)$, then

$$\begin{aligned} F: [0, 1] \times [0, 1] &\rightarrow \mathbb{R}^n \\ (t, s) &\mapsto \gamma(t) \cdot (1 - s) + x_0 \cdot s \end{aligned}$$

is a homotopy between the loop γ and the constant loop e_{x_0} . Thus it follows that $\pi_1(X, x_0) = 0$.⁸¹

- (2) We say that a subset $X \subset \mathbb{R}^n$ is *star-shaped* if there exists an $x \in X$ such that for all $y \in X$ the segment $xt + y(1 - t)$, $t \in [0, 1]$ lies in X . For example open and closed balls and open and closed rectangles are star-shaped. Furthermore all convex subsets are star-shaped. Precisely the same proof as in (1) shows that the $\pi_1(X, x) = 0$.

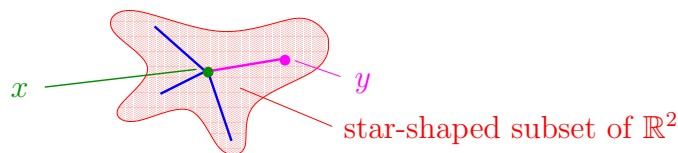


FIGURE 49.

Let X be a proper subset⁸² of $\mathbb{C}$, let $x_0 \in X$ be a point and let $w \in \mathbb{C} \setminus X$ be a point in the complement of X . Then it follows from Satz 10.2 and Satz 11.3 of Analysis III⁸³ that the map⁸⁴

$$\begin{aligned} \pi_1(X, x_0) &\rightarrow \mathbb{Z} \\ [\gamma] &\mapsto \frac{1}{2\pi i} \int_{\gamma} \frac{1}{z-w} dz \end{aligned}$$

is well-defined. Furthermore it is straightforward to see⁸⁵ that this map is in fact a group homomorphism. It follows from the calculation on page 64 that in many cases, e.g. if $X = S^1$, $x_0 = 1$ and $w = 0$, this group homomorphism is an epimorphism. We have thus proved the following lemma.

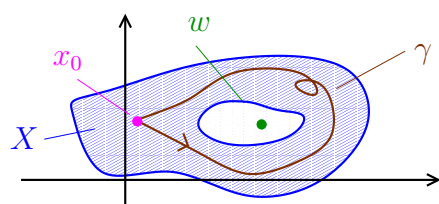
⁸¹If $\pi_1(X, x_0)$ is the trivial group, then we usually write $\pi_1(X, x_0) = 0$. This notation is commonly used, even though it is not entirely logical: we use the multiplicative notation for the product structure on the group $\pi_1(X, x_0)$, and it would therefore make more sense to write $\pi_1(X, x_0) = 1$ if the group $\pi_1(X, x_0)$ is trivial.

⁸²We say A is a proper subset of B if $A \subset B$ and if $A \neq B$.

⁸³Here Satz 10.2 says that the integral depends only on the homotopy class of γ . Satz 11.3 says that the integral, divided by $2\pi i$, is indeed an integer.

⁸⁴Here we make use of the fact that in Kapitel 9 of Analysis III we also defined the path-integral of a holomorphic function $f: U \rightarrow \mathbb{C}$ along a path $\gamma: [a, b] \rightarrow U$ that is only continuous and not necessarily continuously differentiable. This more general notion of a path-integral agrees for continuously differentiable paths with the definition we gave on page 64.

⁸⁵This follows immediately from the additivity of path-integrals, see e.g. Lemma 9.6 of Analysis III.



the map

$$\begin{aligned} \pi_1(X, x_0) &\rightarrow \mathbb{Z} \\ [\gamma] &\mapsto \frac{1}{2\pi i} \int_{\gamma} \frac{1}{z-w} dz \end{aligned}$$

is a homomorphism

FIGURE 50.

Lemma 4.7. *There exists an epimorphism $\pi_1(S^1, 1) \rightarrow \mathbb{Z}$.*

This raises the following question.

Question 4.8. *Is the above epimorphism $\pi_1(S^1, 1) \rightarrow \mathbb{Z}$ in fact an isomorphism?*

The definition of the fundamental group of a topological space X relies on the choice of a base point x_0 . The following proposition shows that, at least for path-connected topological spaces, the choice of the base point does not affect the isomorphism type of the fundamental group.

Proposition 4.9. *Let X be a topological space and let x_0 and x_1 be two points in X . If $p: [0, 1] \rightarrow X$ is a path from x_0 to x_1 , then*

$$\begin{aligned} p_*: \pi_1(X, x_1) &\rightarrow \pi_1(X, x_0) \\ [\gamma] &\mapsto [p * \gamma * \bar{p}] \end{aligned}$$

is a group isomorphism.

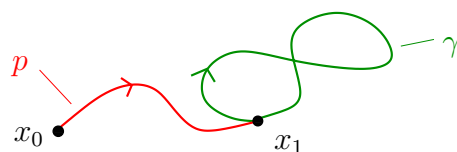


FIGURE 51. Schematic sketch of Proposition 4.9.

Proof. Let X be a topological space and let x_0 and x_1 be two points in X that are connected via a path $p: [0, 1] \rightarrow X$ from x_0 to x_1 . Let γ be a loop in (X, x_1) . Then $p * \gamma * \bar{p}$ is evidently a loop in (X, x_0) . From Lemma 4.4 and Proposition 4.5 it follows that the homotopy class of $p * \gamma * \bar{p}$ depends only on the choice of the homotopy class of γ , i.e. the map

$$\begin{aligned} \Phi: \pi_1(X, x_1) &\rightarrow \pi_1(X, x_0) \\ [\gamma] &\mapsto [p * \gamma * \bar{p}] \end{aligned}$$

is well-defined. Furthermore for two loops γ, δ in (X, x_1) we have

$$\begin{aligned} \Phi([\gamma]) * \Phi([\delta]) &= [p * \gamma * \bar{p}] * [p * \delta * \bar{p}] = [p * \gamma * \bar{p} * p * \delta * \bar{p}] \\ &= [p * \gamma * e_{x_1} * \delta * \bar{p}] = [p * (\gamma * \delta) * \bar{p}] = \Phi([\gamma * \delta]). \end{aligned}$$

$\uparrow$ $\uparrow$
 Proposition 4.5 (3) and Lemma 4.4 Proposition 4.5 (2) and Lemma 4.4

Thus we have shown that Φ is a group homomorphism. Furthermore Φ is even a group isomorphism, since an inverse map is given by⁸⁶

$$\begin{aligned} \Psi: \pi_1(X, x_0) &\rightarrow \pi_1(X, x_1) \\ [\delta] &\mapsto [\bar{p} * \delta * p]. \end{aligned} \quad \square$$

We obtain the following corollary:

Corollary 4.10. *Let X be a path-connected topological space and let $x_0, x_1 \in X$. Then the groups $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ are isomorphic.*

Example. Let X be a star-shaped subset of $\mathbb{R}^n$. Above we had seen that there exists an $x \in X$ such that $\pi_1(X, x) = 0$. It now follows from Corollary 4.10 that $\pi_1(X, y) = 0$ for any point $y \in X$. Put differently, the fundamental group of every star-shaped subset of $\mathbb{R}^n$ with respect to any base point is trivial.

If X is a path-connected topological space, then Corollary 4.10 states that the isomorphism type of the fundamental group does not depend on the choice of the base point. Sometimes we suppress the base point from the notation and we denote by $\pi_1(X)$ the isomorphism type of the fundamental group.

Definition. Let X be a path-connected topological space. If $\pi_1(X) = 0$ ⁸⁷, then we say that X is simply connected.⁸⁸

The following lemma we will be proved in exercise sheet 4.

Lemma 4.11. *Let X be a topological space and let P, Q be two points in X . If X is simply connected, then any two paths from P to Q are homotopic.*

We had already seen on page 74 that $\mathbb{R}^n$ and more generally, all star-shaped subsets of $\mathbb{R}^n$ are simply connected. The following statement is much more interesting:

Proposition 4.12. *For any $n \geq 2$ the sphere S^n is simply connected.*

In the proof of Proposition 4.12 we will need the following lemma.

Lemma 4.13. *For any $P \in S^n$ the space $S^n \setminus \{P\}$ is simply connected.*

⁸⁶Why is $\Phi \circ \Psi$ the identity on $\pi_1(X, x_0)$ and why is $\Psi \circ \Phi$ the identity on $\pi_1(X, x_1)$?

⁸⁷Since X is a path-connected topological space it is irrelevant which base point we consider.

⁸⁸It is straightforward to see that for subsets of $\mathbb{C}$ we obtain the same notion of “simply connectedness” as on page 63.

Proof. On page 17 we had shown that there exists a homeomorphism $f: S^n \rightarrow \mathbb{R}^n \cup \{\infty\}$. We set $Q := f^{-1}(\infty)$. Thus f restricts to homeomorphism $f: S^n \setminus \{Q\} \rightarrow \mathbb{R}^n$. Now let P be any other point on S^n . There exists a homeomorphism⁸⁹ $g: S^n \rightarrow S^n$ with $g(P) = Q$. Then $f \circ g: S^n \rightarrow \mathbb{R}^n \cup \{\infty\}$ restricts to a homeomorphism $S^n \setminus \{P\} \rightarrow \mathbb{R}^n$. But $\mathbb{R}^n$ is simply connected, hence $S^n \setminus \{P\}$ is also simply connected. $\square$

Now we can turn to the proof of Proposition 4.12.

Proof of Proposition 4.12.

Here is a very short “proof”: Let $x_0 \in S^n$ and let $\gamma: [0, 1] \rightarrow S^n$ be a loop in x_0 . We have to show that γ is null-homotopic. We pick a point P that does not lie in the image of γ . By Lemma 4.13 the space $S^n \setminus \{P\}$ is simply connected, hence γ is null-homotopic in $S^n \setminus \{P\}$, in particular it is null-homotopic in S^n .

This sounds convincing, except for one crucial gap: we did not justify why there exists a point P that does not lie in the image of γ . In fact such a point does not need to exist since there are surjective (and as always continuous) maps from an interval to any S^n .

We write

$$N := (0, \dots, 0, 1) \quad \text{“North pole”} \quad \text{and} \quad U := S^n \setminus \{N\},$$

and also

$$S := (0, \dots, 0, -1) \quad \text{“South pole”} \quad \text{and} \quad V := S^n \setminus \{S\}.$$

We fix a base point x_0 on S^n that is neither N nor S . By Lemma 4.13 it suffices to show that any loop in (S^n, x_0) is homotopic to a loop in $U = S^n \setminus \{N\}$.

So let γ be a loop in (S^n, x_0) .

Claim. There exists a subdivision

$$0 = t_0 < t_1 < \dots < t_{k-1} < t_k = 1,$$

such that the following hold:

- (1) for each $i \in \{0, \dots, k-1\}$ we have $\gamma([t_i, t_{i+1}]) \subset U$ or $\gamma([t_i, t_{i+1}]) \subset V$, and
- (2) for each $i \in \{1, \dots, k-1\}$ we have $\gamma(t_i) \neq N$ and $\gamma(t_i) \neq S$.

By Corollary 1.6 applied to $\gamma: [0, 1] \rightarrow S^n$ and $S^n = U \cup V$ there exists a k such that for any $i \in \{0, \dots, k-1\}$ the image $\gamma([\frac{i}{k}, \frac{i+1}{k}])$ lies in U or in V . In particular the subdivision $t_i = \frac{i}{k}$, $i = 0, \dots, k$ satisfies condition (1).

Now suppose that there exists an $i \in \{1, \dots, k-1\}$ with $\gamma(t_i) = N$. Then neither $\gamma([t_{i-1}, t_i])$ nor $\gamma([t_i, t_{i+1}])$ can be contained in $U = S^n \setminus \{N\}$. So by (1) both have to be contained in V , in particular we have $\gamma([t_{i-1}, t_{i+1}]) \subset V$. But then we can remove t_i from the subdivision. Iterating this process we can assume that for each every i we have

⁸⁹The existence of the homeomorphism follows from Proposition 2.5. But one can also easily give an elementary proof of that fact. More precisely, using basic facts from linear algebra one can show that there exists an orthogonal matrix A with $A \cdot P = Q$. Then the map $g: S^n \rightarrow S^n$ that is given by multiplication by A has the desired property.

$\gamma(t_i) \neq N$. Similarly we can arrange for every i we have $\gamma(t_i) \neq S$. This concludes the proof of the claim.

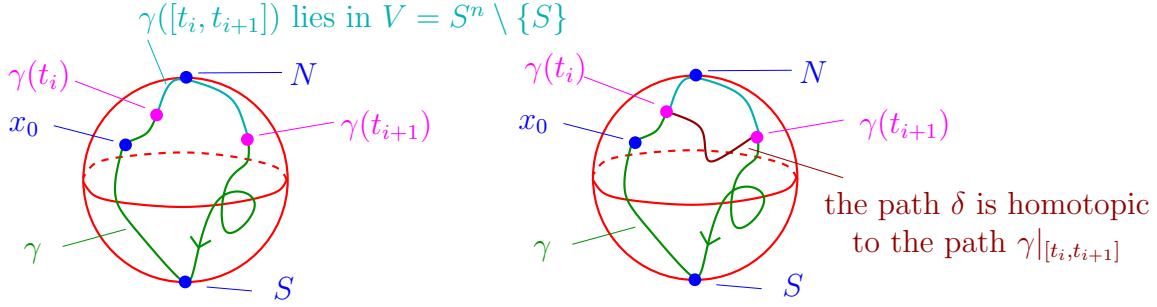


FIGURE 52. Illustration of Proposition 4.12.

As mentioned before, by Lemma 4.13 it now suffices to prove the following claim.

Claim. The loop $\gamma: [0, 1] \rightarrow S^n$ in (S^n, x_0) is homotopic to a path that does not hit N .

We pick a subdivision as in the previous claim. Suppose there exists an $i \in \{0, \dots, k-1\}$ such that $\gamma([t_i, t_{i+1}])$ hits N . By the choice of our subdivision we know that $\gamma([t_i, t_{i+1}])$ lies entirely in $V = S^n \setminus \{S\}$.⁹⁰ Since $S^n \setminus \{N, S\}$ is path-connected and since $\gamma(t_i) \neq N, S$, $\gamma(t_{i+1}) \neq N, S$ we can pick a path $\delta: [t_i, t_{i+1}] \rightarrow U \cap V = S^n \setminus \{N, S\}$ that connects $\gamma(t_i)$ and $\gamma(t_{i+1})$. Since the paths $\gamma|_{[t_i, t_{i+1}]}$ and δ both lie in the simply connected set $V = S^n \setminus \{S\}$ they are homotopic by Lemma 4.11. In the loop γ we now replace $\gamma([t_i, t_{i+1}])$ by δ . We obtain a new loop that is homotopic to the original loop and that does not hit N on the interval $[t_i, t_{i+1}]$. Iterating this procedure gives us the desired homotopy to a loop that does not hit N . This concludes the proof of the claim.⁹¹ $\square$

⁹⁰We know it lies in U or V , but since it hits N it cannot lie in U .

⁹¹Where did we use in the proof that $n \geq 2$?

5. CATEGORIES AND FUNCTORS

In this section we recall the notion of a category and of a functor that we already encountered in Analysis IV. These notions might initially appear to rather abstract, but they play an essential role in algebraic topology and they are regularly used in all branches of pure mathematics.

5.1. Definition and examples of categories.

Definition. A category $\mathcal{C}$ consists of the following data:

- (1) A class⁹² $\text{Ob}(\mathcal{C})$ of mathematical objects which are called the *objects of the category*,
- (2) for each pair (X, Y) of objects there exists a set⁹³ of *morphisms* $\text{Mor}(X, Y)$,
- (3) for any three objects X, Y and Z there exists a map

$$\begin{aligned} \text{Mor}(X, Y) \times \text{Mor}(Y, Z) &\rightarrow \text{Mor}(X, Z) \\ (f, g) &\mapsto g \circ f, \end{aligned}$$

that satisfies the following axioms:

- (A) (associativity): For every $f \in \text{Mor}(W, X)$, $g \in \text{Mor}(X, Y)$ and $h \in \text{Mor}(Y, Z)$ we have

$$(h \circ g) \circ f = h \circ (g \circ f) \in \text{Mor}(W, Z).$$

- (B) (identity): For every object X there exists a morphism $\text{id}_X \in \text{Mor}(X, X)$ with the property that

$$\begin{aligned} \text{id}_X \circ f &= f \quad \text{for all } f \in \text{Mor}(Z, X), \quad \text{and} \\ f \circ \text{id}_X &= f \quad \text{for all } f \in \text{Mor}(X, Y). \end{aligned}$$

Example.

- (a) We call the category $\mathcal{C}$ with

$$\begin{aligned} \text{Ob}(\mathcal{C}) &:= \text{all sets,} \\ \text{Mor}(X, Y) &:= \text{all maps from } X \text{ to } Y, \end{aligned}$$

together with the usual composition of maps the *category of all sets*.

⁹²As in Analysis IV we will not give a definition of what “class” means. Informally speaking a “class” is a generalization of the notion of a “set”. For example it does not make sense to talk of the “set of all groups”, the same way as it does not make sense to talk of the “set of all sets”. But it does make sense to the talk of “class of all groups”. We refer to

[https://en.wikipedia.org/wiki/Class_\(set_theory\)](https://en.wikipedia.org/wiki/Class_(set_theory))

for details. Truth be told, most mathematicians learn from logicians at some point when to say “class” instead of “set”, but many mathematicians do not know exactly what they are doing at that point.

⁹³One could ask why $\text{Mor}(X, Y)$ has to be a set, why not just a class? To the best of my knowledge the axioms on morphisms that we write down also make sense if $\text{Mor}(X, Y)$ is only required to be a class. Here’s a long discussion which gives a satisfactory answer to people in category theory:

<http://mathoverflow.net/questions/48810/why-need-the-morphisms-to-form-a-set>

I myself do not understand the answer.

(b) We consider the category $\mathcal{C}$ with

$$\begin{aligned}\text{Ob}(\mathcal{C}) &:= \text{all sets,} \\ \text{Mor}(X, Y) &:= \begin{cases} \{\text{id}_X\}, & \text{if } X = Y \\ \emptyset, & \text{if } X \neq Y. \end{cases}\end{aligned}$$

The composition map only needs to be defined if $X = Y = Z$ ⁹⁴, and in this case we define $\text{id}_X \circ \text{id}_X := \text{id}_X$. One can easily convince oneself that this defines a category.

(c) Let $\mathbb{K}$ be a field. We refer to the category $\mathcal{C}$ with

$$\begin{aligned}\text{Ob}(\mathcal{C}) &:= \text{all } \mathbb{K}\text{-vector spaces,} \\ \text{Mor}(X, Y) &:= \text{Hom}_{\mathbb{K}}(X, Y) = \text{all } \mathbb{K}\text{-homomorphisms from } X \text{ to } Y,\end{aligned}$$

with the usual composition of homomorphisms as the *category of $\mathbb{K}$ -vector spaces*.

(d) We refer to the category $\mathcal{G}$ with

$$\begin{aligned}\text{Ob}(\mathcal{G}) &:= \text{all groups,} \\ \text{Mor}(X, Y) &:= \text{Hom}(X, Y) = \text{all group homomorphisms from } X \text{ to } Y,\end{aligned}$$

with the usual composition of group homomorphisms as the *category of groups*.

(d) We refer to the category $\mathcal{A}$ with

$$\begin{aligned}\text{Ob}(\mathcal{A}) &:= \text{all abelian groups,} \\ \text{Mor}(X, Y) &:= \text{Hom}(X, Y) = \text{all group homomorphisms from } X \text{ to } Y,\end{aligned}$$

with the usual composition of group homomorphisms as the *category of abelian groups*.

(f) We refer to the category $\mathcal{T}$ with

$$\begin{aligned}\text{Ob}(\mathcal{T}) &:= \text{all topological spaces,} \\ \text{Mor}(X, Y) &:= \mathcal{C}(X, Y) := \text{all continuous maps from } X \text{ to } Y,\end{aligned}$$

with the usual composition of maps as the *category of topological spaces*. A *pointed topological space* is a pair (X, x_0) , where X is a topological space and x_0 is a point in X . We refer to the category $\mathcal{P}$ with

$$\begin{aligned}\text{Ob}(\mathcal{P}) &:= \text{all pointed topological spaces,} \\ \text{Mor}((X, x_0), (Y, y_0)) &:= \text{all continuous maps } f \text{ from } X \text{ to } Y \text{ with } f(x_0) = y_0\end{aligned}$$

with the usual composition of maps as the *category of pointed topological spaces*.

(g) We refer to the category $\mathcal{M}$ with

$$\begin{aligned}\text{Ob}(\mathcal{M}) &:= \text{all manifolds,} \\ \text{Mor}(X, Y) &:= \mathcal{C}(X, Y) := \text{all smooth maps from } X \text{ to } Y,\end{aligned}$$

with the usual composition of maps as the *category of manifolds*. We define the category of *pointed manifolds* the same way as we defined the category of *pointed topological spaces*.

⁹⁴If $X \neq Y$ then $\text{Mor}(X, Y) = \emptyset$, but then $\text{Mor}(X, Y) \times \text{Mor}(Y, Z)$ is also the empty set for any Z .

- (h) A *pair of topological spaces* is a pair (X, A) , where X is a topological space and A is a subset of X . We refer to the category $\mathcal{Q}$ with

$$\begin{aligned}\mathrm{Ob}(\mathcal{Q}) &:= \text{all pairs of topological spaces,} \\ \mathrm{Mor}((X, A), (Y, B)) &:= \text{all continuous maps } f \text{ from } X \text{ to } Y \text{ with } f(A) \subset B\end{aligned}$$

with the usual composition of maps as the *category of pairs of topological spaces*.

There exist also more “exotic” examples of categories.

- (i) Let G be an arbitrary group. We consider

$$\begin{aligned}\mathrm{Ob}(\mathcal{C}) &:= \{\text{a set with a unique element } *\}, \\ \mathrm{Mor}(*, *) &:= G.\end{aligned}$$

We define

$$\begin{aligned}\mathrm{Mor}(*, *) \times \mathrm{Mor}(*, *) &\rightarrow \mathrm{Mor}(*, *) \\ (f, g) &\mapsto g \circ f := gf\end{aligned}$$

via the group structure on $\mathrm{Mor}(*, *) = G$. The axioms of a category can be easily deduced from the group axioms of G .

The name “morphisms” suggests that morphisms are necessarily maps. But as we will later see, this is not necessarily always the case.

5.2. Functors.

Definition. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. A *covariant functor* $F: \mathcal{C} \rightarrow \mathcal{D}$ consists of a map

$$F: \mathrm{Ob}(\mathcal{C}) \rightarrow \mathrm{Ob}(\mathcal{D})$$

and for all $X, Y \in \mathcal{C}$ there exists furthermore a map

$$F: \mathrm{Mor}(X, Y) \rightarrow \mathrm{Mor}(F(X), F(Y)),$$

such that following axioms are satisfied

- (F1) $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ for all $X \in \mathrm{Ob}(\mathcal{C})$,
(F2) for $\phi \in \mathrm{Mor}(X, Y)$ and $\psi \in \mathrm{Mor}(Y, Z)$ the following equality holds

$$F(\psi \circ \phi) = F(\psi) \circ F(\phi) \in \mathrm{Mor}(F(X), F(Z)).$$

Convention. For a covariant functor F and $\phi \in \mathrm{Mor}(X, Y)$ we often write ϕ_* instead of $F(\phi)$. The second axiom then becomes $(\psi \circ \phi)_* = \psi_* \circ \phi_*$.

We now give some examples of functors.

Examples.

- (1) Let $\mathcal{V}$ be the category of real vector spaces and let W be a real vector space. Then the map

$$\begin{aligned} F: \text{Ob}(\mathcal{V}) &\rightarrow \text{Ob}(\mathcal{V}) \\ V &\mapsto \text{Hom}_{\mathbb{R}}(W, V), \end{aligned}$$

together with the maps

$$\begin{aligned} \text{Mor}(U, V) &\mapsto \text{Mor}(F(U), F(V)) \\ (\phi: U \rightarrow V) &\mapsto \left(\begin{array}{ccc} \text{Hom}(W, U) & \rightarrow & \text{Hom}(W, V) \\ & f \mapsto & \phi \circ f \end{array} \right) \end{aligned}$$

defines a covariant functor.

- (2) Let $\mathcal{P}$ be the category of pointed manifolds and let $\mathcal{V}$ be the category of real vector spaces. In Analysis IV we had seen that

$$\begin{aligned} F: \text{Ob}(\mathcal{P}) &\rightarrow \text{Ob}(\mathcal{V}) \\ (M, P) &\mapsto T_P M, \end{aligned}$$

together with the maps

$$\begin{aligned} \text{Mor}((M, P), (N, Q)) &\mapsto \text{Mor}(T_P M, T_Q N) \\ f &\mapsto f_* = Df(P) \end{aligned}$$

is a covariant functor from the category of pointed manifolds to the category of real vector spaces.

We also have the following definition which looks almost the same as the previous definition.

Definition. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. A *contravariant functor* $F: \mathcal{C} \rightarrow \mathcal{D}$ consists of a map

$$F: \text{Ob}(\mathcal{C}) \rightarrow \text{Ob}(\mathcal{D})$$

and for all $X, Y \in \mathcal{C}$ there exists furthermore a map

$$\text{Mor}(X, Y) \rightarrow \text{Mor}(F(Y), F(X))$$

such that following axioms are satisfied:

- (F1) $F(\text{id}_X) = \text{id}_{F(X)}$ for all $X \in \text{Ob}(\mathcal{C})$,
- (F2) for any $\phi \in \text{Mor}(X, Y)$ and $\psi \in \text{Mor}(Y, Z)$ we have

$$F(\psi \circ \phi) = F(\phi) \circ F(\psi) \in \text{Mor}(F(Z), F(X)).$$

The definition of a contravariant functor looks very much the same as the definition of a covariant functor. The difference is that a contravariant functor “reverses the direction of maps” and the orders of $F(\psi)$ and $F(\phi)$ in the second axiom are swapped.

Convention. For a contravariant functor F and $\phi \in \text{Mor}(X, Y)$ we often write ϕ^* instead of $F(\phi)$. The second axiom then becomes $(\psi \circ \phi)^* = \phi^* \circ \psi^*$.

Examples.

- (1) Let $\mathcal{V}$ be the category of real vector spaces. Then the map

$$\begin{aligned} F: \text{Ob}(\mathcal{V}) &\rightarrow \text{Ob}(\mathcal{V}) \\ V &\mapsto V^* := \text{Hom}_{\mathbb{R}}(V, \mathbb{R}) =: \text{dual vector space of } V, \end{aligned}$$

together with the maps

$$\begin{aligned} \text{Mor}(U, V) &\mapsto \text{Mor}(V^*, U^*) \\ (\phi: U \rightarrow V) &\mapsto \left(\begin{array}{ccc} \text{Hom}(V, \mathbb{R}) & \rightarrow & \text{Hom}(U, \mathbb{R}) \\ f & \mapsto & f \circ \phi \end{array} \right) \end{aligned}$$

is a contravariant functor.

- (2) Let $\mathcal{T}$ be the category of topological spaces and let $\mathcal{V}$ be the category of real vector spaces. Then the map

$$\begin{aligned} F: \text{Ob}(\mathcal{T}) &\rightarrow \text{Ob}(\mathcal{V}) \\ X &\mapsto C(X, \mathbb{R}) := \text{set of continuous functions } X \rightarrow \mathbb{R}, \end{aligned}$$

together with the maps

$$\begin{aligned} \text{Mor}(X, Y) &\mapsto \text{Mor}(C(Y), C(X)) \\ (\phi: X \rightarrow Y) &\mapsto \left(\begin{array}{ccc} C(Y) & \rightarrow & C(X) \\ f & \mapsto & f \circ \phi \end{array} \right) \end{aligned}$$

is a contravariant functor.

- (3) Let $\mathcal{M}$ be the category of manifolds, let $\mathcal{V}$ be the category of real vector spaces and let $k \in \mathbb{Z}_{\geq 0}$. In Analysis IV we had seen that the map

$$\begin{aligned} F: \text{Ob}(\mathcal{M}) &\rightarrow \text{Ob}(\mathcal{V}) \\ X &\mapsto H^k(M) := k\text{-th de Rham cohomology group of } M, \end{aligned}$$

together with the maps

$$\begin{aligned} \text{Mor}(M, N) &\mapsto \text{Mor}(H^k(N), H^k(M)) \\ (f: M \rightarrow N) &\mapsto \left(\begin{array}{ccc} f^*: H^k(N) & \rightarrow & H^k(M) \\ \omega & \mapsto & f^*\omega \end{array} \right) \end{aligned}$$

is a contravariant functor.

We will not see any other examples of contravariant functors in Algebraic Topology I and II. We will come back to them when we study cohomology groups in Algebraic Topology III.

5.3. The fundamental group as functor. Now, after recalling the definition of categories and functors, we turn back to the study of fundamental groups.

Let $f: X \rightarrow Y$ be a map between topological spaces and let $x_0 \in X$ be a base point. If $\gamma: [0, 1] \rightarrow X$ is a loop in the point x_0 , then $f \circ \gamma: [0, 1] \rightarrow Y$ is a loop in the point $f(x_0)$. It is straightforward to see that

$$\begin{aligned} f_*: \pi_1(X, x_0) &\rightarrow \pi_1(Y, f(x_0)) \\ [\gamma] &\mapsto [f \circ \gamma] \end{aligned}$$

is a well-defined map, i.e. $f_*([\gamma])$ is independent of the choice of the representative of the homotopy class.⁹⁵ Furthermore it follows easily from the definitions⁹⁶ that f_* is a group homomorphism. We call f_* the *induced map*. It also follows easily from the definitions that

$$(\text{id}_X)_* = \text{id}_{\pi_1(X, x_0)}, \quad \text{for all pointed pairs } (X, x_0),$$

$$(g \circ f)_* = g_* \circ f_*, \quad \text{for all maps } f: (X, x_0) \rightarrow (Y, y_0) \text{ and } g: (Y, y_0) \rightarrow (Z, z_0).$$

Summarizing we just convinced ourselves of the validity of the following proposition.

Proposition 5.1. *Let $\mathcal{P}$ be the category of pointed topological spaces and let $\mathcal{G}$ be the category of groups. Then the map*

$$\begin{aligned} \text{Ob}(\mathcal{P}) &\rightarrow \text{Ob}(\mathcal{G}) \\ (X, x_0) &\mapsto \pi_1(X, x_0) \end{aligned}$$

together with the maps

$$\begin{aligned} \text{Mor}((X, x_0), (Y, y_0)) &\rightarrow \text{Mor}(\pi_1(X, x_0), \pi_1(Y, y_0)) \\ (f: (X, x_0) \rightarrow (Y, y_0)) &\mapsto \left(\begin{array}{ccc} f_*: \pi_1(X, x_0) &\rightarrow & \pi_1(Y, y_0) \\ [\gamma] &\mapsto & [f \circ \gamma] \end{array} \right) \end{aligned}$$

is a covariant functor.

In Analysis IV we had already seen on several occasions that it was useful to know that the de Rham cohomology groups are functorial. We will now see that the functoriality of the fundamental groups can also be helpful for studying certain questions.

Definition. Let X be a topological space and let $A \subset X$ be a subspace. We say A is a *retract* of X if there exists a retraction $r: X \rightarrow A$, i.e. a map⁹⁷ with $r(a) = a$ for all $a \in A$.

Examples.

- (1) Let X be a topological space. Then any point $x_0 \in X$ is a retract of X , indeed, the map $r(x) = x_0$ for $x \in X$ is a retraction.
- (2) The circle $S^1 \times \{1\}$ is a retract of the torus $S^1 \times S^1$, in fact a retraction is given by $r(z, w) = (z, 1)$.
- (3) The circle S^1 is a retract of $\overline{B}^2 \setminus \{(0, 0)\}$, indeed, the map

$$\begin{aligned} \overline{B}^2 \setminus \{(0, 0)\} &\rightarrow S^1 \\ z &\mapsto \frac{z}{|z|} \end{aligned}$$

is a retraction. (Hereby we consider $\overline{B}^2$ as subset of $\mathbb{C}$.)

⁹⁵Indeed, suppose that γ and δ are two loops in (X, x_0) that are homotopic. Let F be a homotopy between γ and δ . Then $f \circ F$ is a homotopy between $f \circ \gamma$ and $f \circ \delta$. We had used almost the same argument on page 67.

⁹⁶If γ and δ are loops in x_0 , then it follows immediately from the definitions that $(f \circ \gamma) * (f \circ \delta) = f \circ (\gamma * \delta)$.

⁹⁷As always, a map is understood to be continuous. If we did not demand continuity, then any non-empty subset of X would be a retract of X .

- (4) The set $S^0 = \{-1, 1\}$ is not a retract of $\overline{B}^1 = [-1, 1]$. Indeed, since $[-1, 1]$ is connected and since S^0 is discrete it follows from Lemma 1.13 that any map from $[-1, 1]$ to S^0 has to be constant. Thus there does not exist a map $f: [-1, 1] \rightarrow S^0$ with $f(-1) = -1$ and $f(1) = 1$.

We can now push the last example up by one dimension.

Lemma 5.2. *The circle S^1 is not a retract of $\overline{B}^2$.*

Proof. We denote by $i: S^1 \rightarrow \overline{B}^2$ the inclusion map. Suppose there exists a retraction $r: \overline{B}^2 \rightarrow S^1$. By definition this means that $r \circ i = \text{id}_{S^1}$. We obtain the following commutative diagram of maps between topological spaces

$$\begin{array}{ccc}
 & \overline{B}^2 & \\
 i \nearrow & & \searrow r \\
 S^1 & \xrightarrow{r \circ i = \text{id}_{S^1}} & S^1
 \end{array}
 \quad \text{which induces} \quad
 \begin{array}{ccc}
 & \pi_1(\overline{B}^2, 1) & \\
 i_* \nearrow & & \searrow r_* \\
 \pi_1(S^1, 1) & \xrightarrow{(r \circ i)_* = (\text{id}_{S^1})_*} & \pi_1(S^1, 1).
 \end{array}$$

It follows from the functoriality of fundamental groups that $r_* \circ i_* = (r \circ i)_*$, i.e. the diagram on the right hand side commutes. By the functoriality we know that $(\text{id}_{S^1})_*$ is the identity of $\pi_1(S^1, 1)$.

We had seen on page 76 that $\pi_1(\overline{B}^2, 1)$ is the trivial group whereas we had seen on page 74 that $\pi_1(S^1, 1)$ is not the trivial group. The lower horizontal map in the second diagram is thus an isomorphism of a non-trivial group, but the upper map factors through the trivial group, i.e. the composition of i_* and r_* cannot be an isomorphism of a non-trivial group.⁹⁸ Thus we have obtained a contradiction. $\square$

The last example and the last lemma now raise the following question.

Question 5.3. *Let $n \geq 2$. Is S^n a retract of $\overline{B}^{n+1}$?*

At the moment we cannot answer this question. We will first need to develop new invariants before we can return to this question. We will not be able to answer this question for many months.

Almost exactly the same argument as in the proof of Lemma 5.2 gives a proof of the following lemma:

Lemma 5.4. *Let X be a topological space and let $A \subset X$ be a retract of X . Let $a \in A$. We denote by $i: A \rightarrow X$ the inclusion map.*

- (1) *There exists an epimorphism $\varphi: \pi_1(X, a) \rightarrow \pi_1(A, a)$ such that $\varphi \circ i_*$ is the identity on $\pi_1(A, a)$.*
- (2) *The inclusion induced map $\pi_1(A, a) \rightarrow \pi_1(X, a)$ is a monomorphism.*

In exercise sheet 4 we will write down the formal proof of the lemma.

⁹⁸Why not?

Example. We had just seen on page 84 that the circle $S^1 \times \{1\}$ is a retract of the torus $S^1 \times S^1$. Furthermore we had seen on page 74 that $\pi_1(S^1, 1)$ is not the trivial group. It follows from Lemma 5.4 that the fundamental group of the torus is also non-trivial. In Proposition 4.12 we had seen that $\pi_1(S^2) = 0$. Thus we see that the torus $S^1 \times S^1$ and the sphere S^2 are not homeomorphic. In Analysis IV we had only shown that they are not diffeomorphic.

Lemma 5.5. *Let E_8 be the regular octagon from the definition of the surface of genus 2, see page 28 for details, and let $\Sigma = E_8 / \sim$ be the surface of genus 2. Let P and Q be two adjacent vertices of E_8 . Then the following hold:*

(1) *The map*

$$\begin{aligned} \gamma: S^1 &\rightarrow E_8 / \sim \\ e^{2\pi it} &\mapsto [t \cdot P + (1 - t) \cdot Q] \end{aligned}$$

is a homeomorphism onto its image.

(2) *There exists a map $\Phi: E_8 / \sim \rightarrow S^1 \times S^1$ such that the following diagram commutes*

$$\begin{array}{ccc} & S^1 & \\ \gamma \swarrow & & \searrow z \mapsto (z, 1) \\ E_8 / \sim & \xrightarrow{\Phi} & S^1 \times S^1. \end{array}$$

(3) *The subset $\gamma(S^1)$ is a retract of $\Sigma = E_8 / \sim$.*

(4) *There exists an epimorphism $\pi_1(\Sigma, P) \rightarrow \mathbb{Z}$ that sends $[\gamma]$ onto 1.*⁹⁹

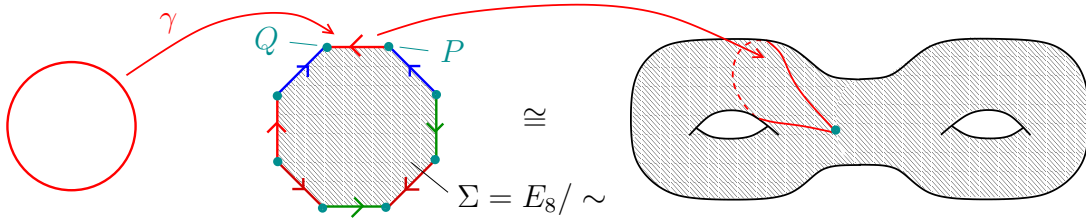


FIGURE 53. Illustration of Lemma 5.5.

Remark. The lemma shows in particular that the four loops that we drew in Figure 14 on the surface of genus 2 represent non-trivial elements in the fundamental group. Almost the same way one can show that in the surface of genus g , $g \geq 2$, defined as a quotient of a regular $4g$ -gon, each edge gives rise to a non-trivial element of the fundamental group.

Proof.

(1) It is clear that γ is continuous and injective. We had shown in Analysis IV that $\Sigma = E_8 / \sim$ is Hausdorff. It follows from Proposition 1.3 that γ is a homeomorphism onto its image.

⁹⁹Recall that by the bijection on page 68 we view maps $f: S^1 \rightarrow \Sigma$ with $f(1) = P$ as elements in $\pi_1(\Sigma, P)$.

- (2) The map Φ is sketched in Figure 54 on the top. It is given by sending all points in the lower pentagon to a single point, such a map can be extended continuously to the upper pentagon in such a way that the remaining four sides of the upper pentagon get sent homeomorphically to the four sides of the square.^{100 101} This map Φ has the desired property.
- (3) The retraction is now given by the concatenation of the maps

$$\Sigma \xrightarrow{\Phi} S^1 \times S^1 \xrightarrow{(z,w) \mapsto z} S^1 \xrightarrow{\gamma} \Sigma.$$

- (4) This statement is a consequence of (3), Lemma 5.4 and Lemma 4.7. $\square$

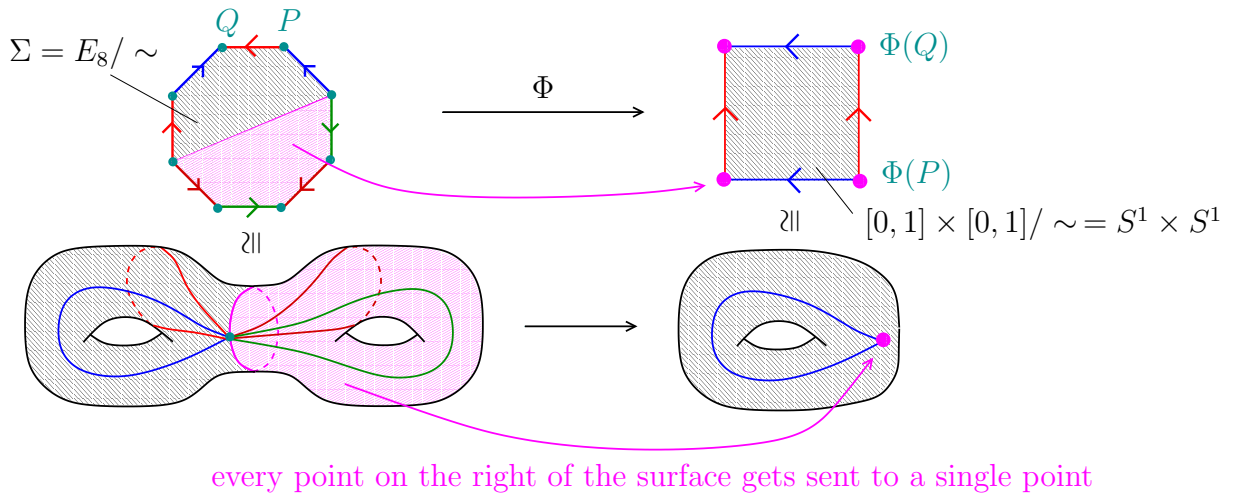


FIGURE 54.

This lemma concludes our discussion of some of the basic properties of fundamental groups. So far we could either show that the fundamental group of a given topological space is trivial, or we could show it is non-trivial by invoking Lemma 4.7. In the next section we will start with a much more systematic discussion of fundamental groups which will allow us to determine the fundamental groups of many topological spaces.

¹⁰⁰It is clearly slightly painful to write down this map explicitly. It is easier to write down a similar map, which maps a square X onto a triangle Y in such a way that one side of the square gets sent to a vertex of the triangle and such that the remaining three sides of the square get sent to the three sides of the triangle. For example the map

$$\begin{aligned} X := [0, 1] \times [0, 1] &\rightarrow Y := \{(x, y) \in \mathbb{R}^2 \mid x \in [0, 1], y \in [0, x]\} \\ (x, y) &\mapsto (x, xy) \end{aligned}$$

is of that form.

¹⁰¹In the lower part of Figure 54 we illustrate the map for the surfaces viewed as subsets of $\mathbb{R}^3$. There the map Φ is given by crushing the right-hand half of the surface of genus 2 to a single point.

6. FUNDAMENTAL GROUPS AND COVERINGS

We have now introduced the fundamental group of a (pointed) topological space and we have proved several basic facts. But the only proof that fundamental groups are non-trivial relied on a fact from complex analysis. In particular, so far we have not yet managed to compute a single fundamental group that is not the trivial group. For example, we know that $\pi_1(S^1)$ is non-trivial, but we do not know what group it is.

In this chapter we will compute many non-trivial fundamental groups using the theory of covering spaces. Before we delve into the theory of covering spaces we insert a short section on the cardinality of sets.

6.1. The cardinality of sets. In this short section we introduce the notion of the cardinality of a set. This notion plays a role in all branches of mathematics.

Definition.

- (1) We say that two given sets S and T are *equinumerous* if there exists a bijection $\varphi: S \rightarrow T$.¹⁰²
- (2) Given a set S we define the *cardinality* $\#S$ of S as the equinumerosity equivalence class of S in the class of all sets. Put differently we have

$$\#S := \text{class of all sets } T \text{ that are equinumerous with } S.$$

- (3) If S is a finite set, then we write $\#S = n$ for the unique $n \in \mathbb{N}_0$ which satisfies $\#S = \#\{1, \dots, n\}$. If S infinite, then sometimes we write $\#S = \infty$.¹⁰³
- (4) Let S be a set. If $\#S = n$ for some $n \in \mathbb{N}_0$ or $\#S = \#\mathbb{N}$, then we say that S is *countable*, otherwise we say that S is *uncountable*.

Example. We have $\#(\mathbb{R} \setminus \{0\}) = \#\mathbb{R}$. Indeed a bijection is given by

$$\begin{aligned} \mathbb{R} \setminus \{0\} &\rightarrow \mathbb{R} \\ x &\mapsto \begin{cases} x, & \text{if } x \in \mathbb{R} \setminus \mathbb{N}_0, \\ x - 1, & \text{if } x \in \mathbb{N}. \end{cases} \end{aligned}$$

Remark.

- (1) Another way of putting the definition of cardinality is that given two sets S and T we have $\#S = \#T$ if and only if there exists a bijection from S to T .
- (2) In Analysis I we had seen that $\mathbb{Q}$ is countable and that $\mathbb{R}$ is not countable, in particular $\#\mathbb{Q} \neq \#\mathbb{R}$.
- (3) If S is countable, then any subset T is also countable.¹⁰⁴

¹⁰²It follows immediately from the definitions that being equinumerous defines an equivalence relation on the class of all sets.

¹⁰³The latter notation is somewhat dangerous, since $\#\mathbb{Q} = \infty$ and $\#\mathbb{R} = \infty$ but we had seen in Analysis I that $\#\mathbb{Q} \neq \#\mathbb{R}$ since the former set is countable whereas the latter set is not.

¹⁰⁴After excluding the case that S is finite and the case that T is finite one sees that one has to prove the following: given any infinite subset T of $\mathbb{N}$ there exists a bijection $\varphi: \mathbb{N} \rightarrow T$. Such a bijection is given by defining $\varphi(i)$ as the i -th smallest element of T .

Definition. Let S and T be two sets.

- (1) We write $\#S \leq \#T$ if there exists a monomorphism $\varphi: S \rightarrow T$.
- (2) We write $S < T$ if $\#S \leq \#T$ but $\#S \neq \#T$.

Examples.

- (1) We have $\#\mathbb{Q} < \#\mathbb{R}$ but we had just pointed out that $\#\mathbb{Q} \neq \#\mathbb{R}$ since $\#\mathbb{Q} = \#\mathbb{N}$ but $\#\mathbb{R} \neq \#\mathbb{N}$. Therefore $\#\mathbb{Q} < \#\mathbb{R}$.
- (2) It follows from the previous remark that a set S is countable if and only if $S \leq \#\mathbb{N}$.
- (3) Given a set X we denote by $P(X)$ its power set, i.e. the set of all subsets of X . For every X we have $\#X < \#P(X)$.¹⁰⁵ Indeed, the map

$$\begin{aligned} X &\rightarrow P(X) \\ x &\mapsto \{x\} \end{aligned}$$

is an injection, hence $\#X \leq \#P(X)$. Now let $f: X \rightarrow P(X)$ be a map. We have to show that it cannot be surjective. Indeed, the set

$$Y := \{x \in X \mid x \notin f(x)\}$$

is not in the range of f : if there did exist a $z \in X$ with $f(z) = Y$, then $z \in Y$ if and only if $z \notin Y$, which is a contradiction.

Theorem 6.1. (Bernstein-Schröder)¹⁰⁶ *Let S and T be two sets. Then*

$$\#S \leq \#T \text{ and } \#T \leq \#S \implies \#S = \#T.$$

Example. Let U be an open subset of $\mathbb{R}$.¹⁰⁷ Then $\#U = \#\mathbb{R}$. Indeed, the inclusion $U \rightarrow \mathbb{R}$ is of course injective. Furthermore, since U is open there exists an open interval (a, b) that is a subset of U . But any open interval is homeomorphic to $\mathbb{R}$, so there exists an injective map $\mathbb{R} \xrightarrow{\cong} (a, b) \subset U$. It follows from the Bernstein-Schröder-Theorem that there exists a bijection from U to $\mathbb{R}$.

Sketch of proof. Let $f: A \rightarrow B$ and $g: B \rightarrow A$ be two injective maps between two sets A and B . Without loss of generality we can assume that A and B are disjoint.¹⁰⁸ We denote by $\sim$ the equivalence relation on $A \sqcup B$ that is generated by $a \sim f(a)$ for $a \in A$ and by $g \sim g(b)$ for $b \in B$. Note that if $a \in A$ lies in the image of g , then by the injectivity of f there exists a unique element $g^{-1}(a) \in B$ and we have $a \sim g^{-1}(a)$. The same remark works with the roles of f and g reversed.

¹⁰⁵This statement is known as Cantor's Theorem.

¹⁰⁶The theorem was initially stated by Cantor without a proof and it was first proved by Felix Bernstein (1878-1956) and Ernst Schröder (1841-1902) in 1896. Sometimes it is also called the Cantor-Bernstein-Schröder theorem.

¹⁰⁷Every open subset of $\mathbb{R}$ is the disjoint union of open intervals (a_i, b_i) , $i \in I$. The number of intervals can be finite, but it can also be countably infinite and in fact it can also be uncountably infinite. For example the complement of the Cantor set consists of uncountably many open intervals.

¹⁰⁸If A and B are not disjoint, then we replace A by $A' = A \times \{0\}$ and B by $B' = B \times \{1\}$, then A' and B' are disjoint, since any two elements in A' and B' now have different second coordinate.

Since $A \sqcup B$ is the disjoint union of all equivalence classes it suffices to prove the following claim.¹⁰⁹

Claim. For each equivalence class I there exists a bijection $A \cap I \rightarrow B \cap I$.

So let I be an equivalence class¹¹⁰ and let $a \in A \cap I$.¹¹¹ The subset $I \subset A \sqcup B$ consists of all elements equivalent to a . The equivalence class I is then of one of the following three types:¹¹²

			A		B		A		B		A		B	
(1)	a'	...	$f^{-1}(g^{-1}(a))$,		$g^{-1}(a)$,		a ,		$f(a)$,		$g(f(a))$,		$f(g(f(a)))$,	...
(2)	b'	...	$f^{-1}(g^{-1}(a))$,		$g^{-1}(a)$,		a ,		$f(a)$,		$g(f(a))$,		$f(g(f(a)))$,	...
(3)	...	...	$f^{-1}(g^{-1}(a))$,		$g^{-1}(a)$,		a ,		$f(a)$,		$g(f(a))$,		$f(g(f(a)))$,	...

In case (1) the map f defines a bijection $A \cap I \rightarrow B \cap I$, in case (2) the map g^{-1} defines a bijection $A \cap I \rightarrow B \cap I$ and in case (3) both f and g^{-1} define a bijection $A \cap I \rightarrow B \cap I$. This concludes the proof of the claim. $\square$

Remark. The *continuum hypothesis*, first formulated by Cantor in 1878, states that there exists no set S with $\#\mathbb{Q} < \#S < \#\mathbb{R}$. Any attempt at proving this statement relies on first of all making precise what “sets” are supposed to be or what axioms they are supposed to satisfy. As wikipedia writes “the answer to this problem is independent of ZFC set theory (that is, Zermelo–Fraenkel set theory with the axiom of choice included), so that either the continuum hypothesis or its negation can be added as an axiom to ZFC set theory, with the resulting theory being consistent if and only if ZFC is consistent.” This independence was proved in 1963 by Paul Cohen for which he obtained a Fields medal in 1966. More useful information can as always be found on wikipedia:

https://en.wikipedia.org/wiki/Continuum_hypothesis

6.2. Covering spaces. Now we return to the study of topological spaces.

Definition. Let $p: X \rightarrow B$ be a map between topological spaces.

¹⁰⁹Indeed, recall that equivalence classes always give a disjoint decomposition of a set. In our case, if $I_j, j \in J$ denote the set of equivalence classes of $\sim$ on $A \sqcup B$, then we have

$$A \sqcup B = \bigsqcup_{j \in J} I_j \quad \text{and therefore also} \quad A = \bigsqcup_{j \in J} (A \cap I_j) \quad \text{and} \quad B = \bigsqcup_{j \in J} (B \cap I_j).$$

Therefore it suffices to give a bijection $A \cap I_j \rightarrow B \cap I_j$ for every $j \in J$.

¹¹⁰We gave the definition of an equivalence class in Analysis IV. Unfortunately in Analysis IV we forgot to say that an equivalence class has to be non-empty.

¹¹¹There always exists an $a \in A \cap I$. Indeed, I is by definition non-empty. If it contains an element in A we are done. Otherwise it contains an element in B , but then it also contains $g(b) \in A$, since $b \sim g(b)$.

¹¹²Note that we do not claim that the elements in the given sequences are distinct, for example $f: A \rightarrow B$ could be a bijection and $g = f^{-1}$ its inverse, then the equivalence class consists only of the two elements a and $g(a)$.

- (1) We say an open subset $U \subset B$ is *uniformly covered*, if $p^{-1}(U)$ is the union of disjoint open subsets $\{V_i\}_{i \in I}$ with the property, that the restriction of p to each subset V_i is a homeomorphism.
- (2) We say the map $p: X \rightarrow B$ is a *covering*, if it is surjective and if for every $b \in B$ there exists an open neighborhood U of b which is uniformly covered.

Remark. If U is a uniformly covered subset of B , then clearly any open subset of U is also uniformly covered. For example it follows that if $p: X \rightarrow B$ is a covering and if B is a manifold, then any point $b \in B$ admits also a connected uniformly covered neighborhood.¹¹³

Examples.

- (a) The map

$$\begin{aligned} p: \mathbb{R} &\rightarrow S^1 \\ \varphi &\mapsto e^{i\varphi} \end{aligned}$$

is a covering. Indeed, let $P = e^{i\alpha}$ be a point in S^1 . We pick the open neighborhood

$$U := \{e^{i\varphi} \mid \varphi \in (\alpha - \frac{\pi}{4}, \alpha + \frac{\pi}{4})\}.$$

Then

$$p^{-1}(U) := \bigsqcup_{j \in \mathbb{Z}} \underbrace{(\alpha - \frac{\pi}{4} + 2\pi j, \alpha + \frac{\pi}{4} + 2\pi j)}_{=: V_j},$$

and for each $j \in \mathbb{Z}$ the restriction of p to $V_j \rightarrow U$ is a homeomorphism. This example is illustrated in Figure 55.

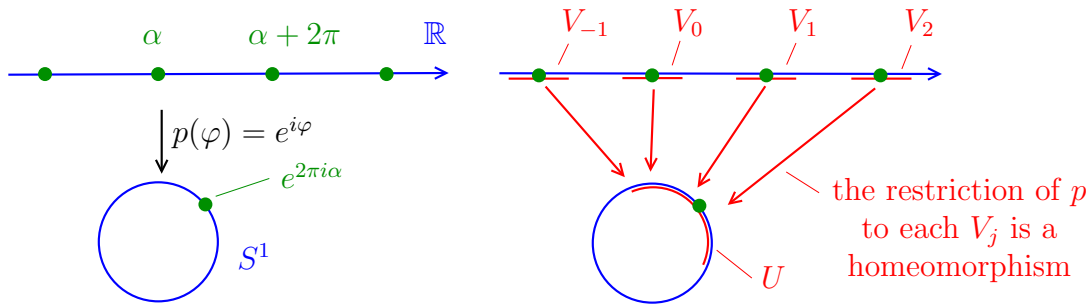


FIGURE 55.

- (b) We consider the “infinite helix”

$$H := \{(re^{it}, t) \mid r \in (1, 2) \text{ and } t \in \mathbb{R}\}$$

together with the “annulus”

$$A := \{re^{it} \mid r \in (1, 2) \text{ and } t \in \mathbb{R}\}.$$

¹¹³Why does this follow for manifolds? Why does this not hold for all topological spaces B ?

The map

$$\begin{aligned} p: H &\rightarrow A \\ (re^{it}, t) &\mapsto re^{it} \end{aligned}$$

is a covering. This statement is illustrated in Figure 56. The proof of the statement is similar to the proof in the previous example.

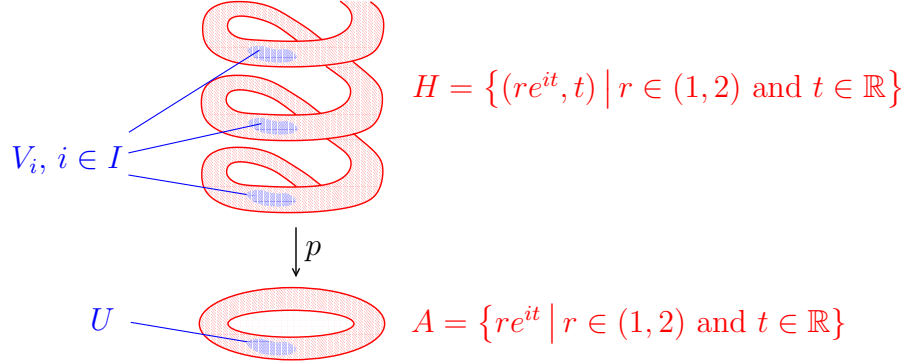


FIGURE 56.

(c) The map

$$\begin{aligned} p: \mathbb{R}^n &\rightarrow \mathbb{R}^n / \mathbb{Z}^n \\ x &\mapsto [x] = x + \mathbb{Z}^n \end{aligned}$$

is a covering. Indeed, let $[x] = [(x_1, \dots, x_n)]$ be a point in $\mathbb{R}^n / \mathbb{Z}^n$. We consider

$$W := \{(y_1, \dots, y_n) \mid |y_i - x_i| < \frac{1}{2}\} \subset \mathbb{R}^n$$

and we put $U := p(W)$. Then U is an open¹¹⁴ neighborhood of $[x]$. Furthermore

$$p^{-1}(U) = \bigcup_{i \in \mathbb{Z}^n} (W + i),$$

and for each i the restriction of p to $W + i \subset \mathbb{R}^n$ is a homeomorphism.¹¹⁵

(d) For every n the map

$$\begin{aligned} p: S^n &\rightarrow \mathbb{RP}^n = S^n / x \sim -x \\ x &\mapsto [x] = \{x, -x\} \end{aligned}$$

is a covering. Indeed, for $[x] \in \mathbb{RP}^n$ we set

$$W = \{y \in S^n \mid \underbrace{x \cdot y}_{\text{scalar product}} > 0\}.$$

¹¹⁴The set U is open since, as we had mentioned on page 24, the projection map $p: \mathbb{R}^n \rightarrow \mathbb{R}^n / \mathbb{Z}^n$ is open.

¹¹⁵It is straightforward to verify that the map $p: V_i := W + i \rightarrow U$ is a bijection. It follows from the remark on page 23 that the projection map p is continuous and as we had just mentioned in the previous footnote, the projection map is also open. It follows that the map $p: V_i = W + i \rightarrow U$ is a homeomorphism.

Then $p(W)$ is an open neighborhood of $[x]$ and it is uniformly covered by the two open sets W and $-W$.

(e) The projection map

$$\begin{aligned} p: S^1 &\rightarrow [-1, 1] \\ (x, y) &\mapsto x \end{aligned}$$

is not a covering. Indeed, the point $b = 1 \in [-1, 1]$ has no open neighborhood that is uniformly covered. This example is illustrated in Figure 57.

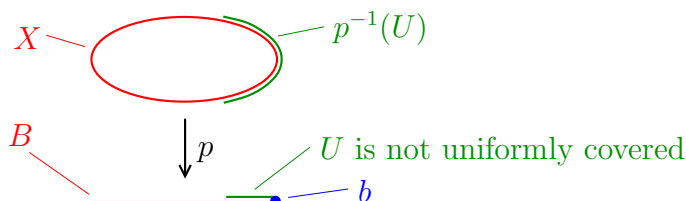


FIGURE 57.

Now we can formulate the following lemma that will be proved in exercise sheet 5.

Lemma 6.2. *Let $p: X \rightarrow B$ be a covering of topological spaces. If B is a connected topological space, then for any two points P and Q in B we have*

$$\#p^{-1}(P) = \#p^{-1}(Q).$$

Example. For a general map $p: X \rightarrow B$ between topological spaces the number of preimages of a point depends of course on the point chosen in B . For instance in the above example (e) we have $\#p^{-1}(0) = 2$ but $\#p^{-1}(1) = 1$.

Definition. Let $p: X \rightarrow B$ be a covering of a connected topological space.

- (1) We define the *degree* $[X : B]$ of p as the cardinality of $\#p^{-1}(b)$ for some $b \in B$.¹¹⁶
- (2) If the degree is a finite number n , then we say that p is an *n -fold covering* of B . Otherwise we say that p is an *infinite covering*. If the cardinality is countable, then sometimes we call it a *countable covering*.

Examples.

- (1) We had just seen that the map $p: S^n \rightarrow S^n / \sim$ is a 2-fold covering of the real projective space $S^n / \sim = \mathbb{RP}^n$.
- (2) As usual we consider $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ as a subset of $\mathbb{C}$. Then the map $p: S^1 \rightarrow S^1$ given by $p(z) := z^n$ is a covering. Indeed, let $z = e^{2\pi i \varphi}$ be a point in

¹¹⁶It follows from Lemma 6.2 that this definition does not depend on the choice of b .

S^1 . Then

$$U := \{e^{2\pi i\psi} \mid \psi \in (\varphi - \frac{1}{4}, \varphi + \frac{1}{4})\}$$

has the desired property. Indeed, we have

$$p^{-1}(U) = \bigsqcup_{j=1}^n \underbrace{\{e^{2\pi i\psi} \mid \psi \in (\frac{1}{n}(\varphi - \frac{1}{4}) + \frac{j}{n}, \frac{1}{n}(\varphi + \frac{1}{4}) + \frac{j}{n})\}}_{=: V_j},$$

and the restriction of p to each set V_j is easily seen to be a homeomorphism. This shows that p is a covering and it is clear that p is an n -fold covering.

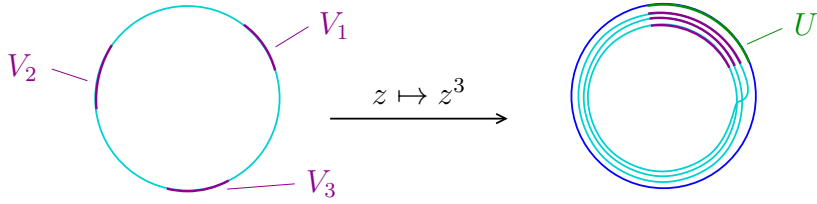


FIGURE 58.

(3) We consider the map

$$p: \overbrace{\mathbb{R} \times [-1, 1]/(x, y) \sim (x+1, y)}^{\text{annulus}} \rightarrow \overbrace{\mathbb{R} \times [-1, 1]/(x, y) \sim (x + \frac{1}{2}, -y)}^{\text{Möbius band}} \\ [(x, y)] \rightarrow [(x, y)].$$

It is straightforward to show that p is a 2-fold covering. Thus we can view the annulus as a 2-fold covering of the Möbius band. This covering is illustrated in Figure 59.

- (4) If in the previous example we replace the interval $[-1, 1]$ by the circle $[-1, 1]/-1 \sim 1$, then on the left hand side we obtain the torus and on the right hand side we obtain the Klein bottle. This shows that there exists a degree 2 covering map from the torus to the Klein bottle.
- (5) In Figure 60 we sketch three coverings of topological graphs. The coverings are hereby given by sending edges of a certain color, with the indicated direction, to the edge of the same color with the indicated direction.¹¹⁷

Question 6.3. *We had just seen that there exists a covering map $S^2 \rightarrow \mathbb{RP}^2$ of degree two and that there exists a covering map from the torus to the Klein bottle of degree two. This raises the question: between which surfaces do there exist covering maps? More precisely we can ask the following:*

- (1) *Does there exist a covering map from the torus $S^1 \times S^1$ to the sphere S^2 ?*
- (2) *Does there exist a covering map from the sphere S^2 to the torus $S^1 \times S^1$?*

¹¹⁷It is a good exercise to convince oneself, that the indicated maps are in fact coverings.

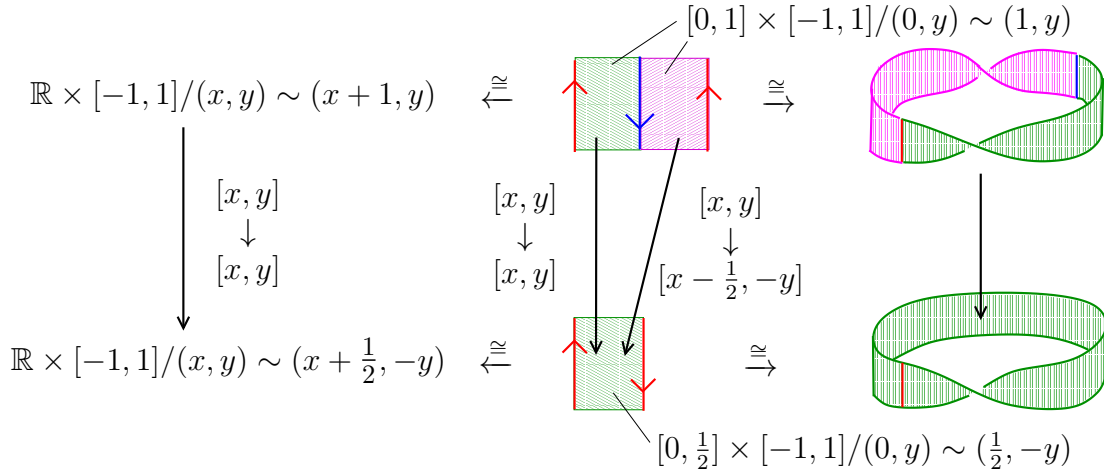


FIGURE 59.

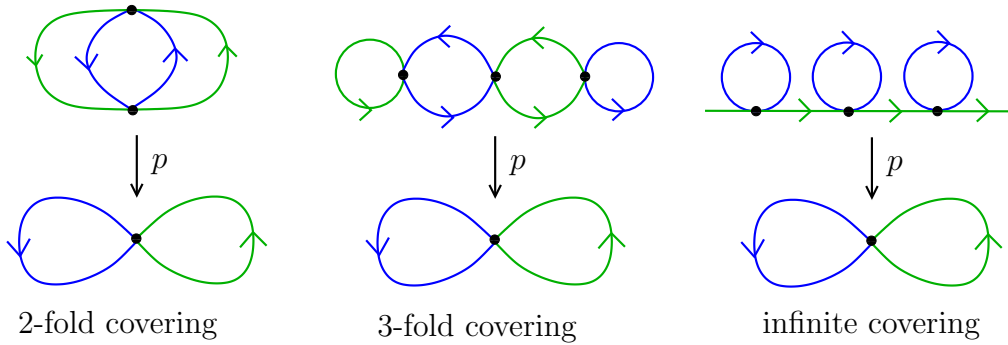


FIGURE 60.

The following lemma summarizes some basic facts about coverings.

Lemma 6.4. *Let $p: X \rightarrow B$ be a covering of topological spaces. Then the following hold:*

- (1) *The map $p: X \rightarrow B$ is open.*¹¹⁸
- (2) *For any $b \in B$ the preimage $p^{-1}(b)$ is a discrete subset of X .*
- (3) *If B is Hausdorff, then X is also Hausdorff.*
- (4) *If p is a finite covering and if B is compact, then X is also compact.*
- (5) *If p is an infinite covering, then X is non-compact.*

Proof. Let $p: X \rightarrow B$ be a covering of topological spaces.

- (1) Let U be an open subset of X . We need to show that $p(U)$ is open. It suffices to show that for each $P \in p(U)$ there exists an open neighborhood V such that $P \in V \subset p(U)$. Since p is a covering there exists an open neighborhood W of P such that $p^{-1}(W)$ is

¹¹⁸Recall that this means that the image of each open set in X is open in B .

the union of disjoint open subsets $\{W_i\}_{i \in I}$ with the property that the restriction of p to each subset W_i is a homeomorphism.

Since $P \in p(U)$ there exists a $Q \in U$ with $p(Q) = P$. There exists an $i \in I$ with $Q \in W_i$. Since W_i is open and since U is open the set $U \cap W_i$ is also open in X . Since $p: W_i \rightarrow W$ is a homeomorphism and since $U \cap W_i$ is open in W_i the set $p(U \cap W_i)$ is also open in $p(W_i) = W$. But since W is open the set $V := p(U \cap W_i)$ is also an open subset of B . It is the desired open neighborhood of P with $P \in V \subset U$.

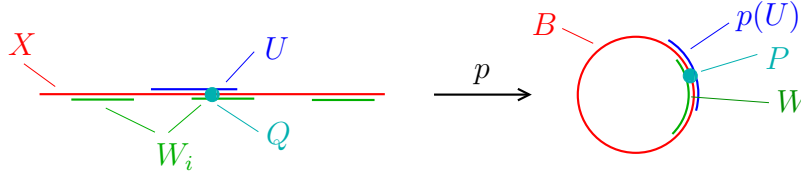


FIGURE 61. Illustration for the proof of Lemma 6.4 (1).

- (2) Let $b \in B$. Since $p: X \rightarrow B$ is a covering there exists an open neighborhood U of b such that $p^{-1}(U)$ is the union of disjoint open subsets $\{V_i\}_{i \in I}$ with the property that the restriction of p to each subset V_i is a homeomorphism. For each $i \in I$ there exists a unique point $x_i \in V_i$ with $p(x_i) = b$ and the preimage $p^{-1}(b)$ consists of precisely the x_i , $i \in I$. Since each V_i is open in X it follows from the definition of the subspace topology that each $x_i \in p^{-1}(b)$ is an open subset of $p^{-1}(b)$. Put differently, the subspace topology on $p^{-1}(b)$ is the discrete topology. But this is equivalent to saying that $p^{-1}(b)$ is a discrete subset of X .
- (3) This statement is relatively easy to prove and it is an exercise in exercise sheet 5.
- (4) Let $p: X \rightarrow B$ be an n -fold covering of a compact space B . Let $\{U_i\}_{i \in I}$ be an open covering of X . Let $b \in B$. We pick an open connected neighborhood M of b that is uniformly covered. For each $x \in p^{-1}(b)$ we denote by N_x the component of $p^{-1}(M)$ that contains x . For each point $x \in p^{-1}(b)$ we pick an $i_x \in I$ with $x \in U_{i_x}$. We write

$$W_b := \bigcap_{x \in p^{-1}(b)} p(U_{i_x} \cap N_x).$$

Claim. The preimage $p^{-1}(W_b)$ is contained in the union of $n = \#p^{-1}(b)$ subsets of our open covering $\{U_i\}_{i \in I}$.

Since M is uniformly covered the map $p: N_x \rightarrow M$ is a homeomorphism for any $x \in p^{-1}(b)$. We denote by $p_x: N_x \rightarrow M$ this homeomorphism. We write $F := p^{-1}(b)$. We then have

$$p^{-1}(W_b) = \bigcup_{y \in F} p_y^{-1} \left(\bigcap_{x \in F} p_x(U_{i_x} \cap N_x) \right) = \bigcup_{y \in F} \bigcap_{x \in F} \overbrace{p_y^{-1}(p_x(U_{i_x})) \cap p_y^{-1}(p_x(N_x))}^{\subset U_{i_y} \cap N_y} \subset \bigcup_{y \in F} U_{i_y}.$$

$\underbrace{\hspace{10em}}_{= U_{i_y} \text{ if } y = x} \quad \underbrace{\hspace{10em}}_{= N_y}$

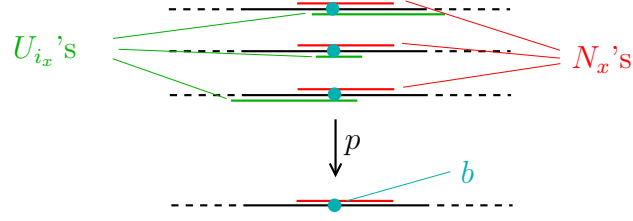


FIGURE 62. Illustration for the proof of Lemma 6.4 (4).

This concludes the proof of the claim.

By (1) each $p(U_{i_x} \cap N_x)$ is an open subset of B and since $p^{-1}(b)$ is a finite set we see that W_b is an open subset of B . Furthermore b is contained in each $p(U_{i_x} \cap N_x)$, so W_b is in fact an open neighborhood of $b \in B$. Since B is compact we can cover B by finitely many of these open sets, i.e. there exist $b_1, \dots, b_k \in B$ with $B = W_{b_1} \cup \dots \cup W_{b_k}$. The preimages $p^{-1}(W_{b_1}), \dots, p^{-1}(W_{b_k})$ cover X . By the above claim each preimage $p^{-1}(W_{b_i})$ is contained in the union of $n = \#p^{-1}(b_i)$ subsets of our open covering $\{U_i\}_{i \in I}$. Summarizing we have shown that we can cover X by $n \cdot k$ sets out of the collection $\{U_i\}_{i \in I}$.

- (5) Let $p: X \rightarrow B$ be an infinite covering. If B is non-compact, then it follows from Lemma 1.2 that X is also non-compact. So now suppose that B is compact. By definition we can cover B by open sets that are uniformly covered. Since B is compact we can cover B by finitely many open sets $U_1, \dots, U_k$ that are uniformly covered. Without loss of generality we can assume that U_k is not contained in $U_1 \cup \dots \cup U_{k-1}$.¹¹⁹ Now let P be a point in $U_k \setminus (U_1 \cup \dots \cup U_{k-1})$. Furthermore let V_i , $i \in I$ be the disjoint open subsets of X such that $p^{-1}(U_k)$ is the union of the V_i 's and such that the restriction of p to each V_i is a homeomorphism. For $i \in I$ we denote by $Q_i \in V_i$ the unique point with $p(Q_i) = P$. Now we consider the open covering

$$X = p^{-1}(U_1) \cup \dots \cup p^{-1}(U_{k-1}) \cup \bigcup_{i \in I} V_i.$$

Note that for $i \in I$ the point Q_i is contained only in V_i . This means that we cannot remove any of the open sets V_i , $i \in I$ from the open covering. Since I is an infinite it follows that X does not satisfy the definition of a compact set, i.e. X is not compact. $\square$

We recall two definitions about actions from page 44 and we introduce one new definition.

Definition. Let G be a group which acts continuously on a topological space X .

- (1) We say G acts *freely* if $g \cdot x = x$ for $g \in G$ and $x \in X$ implies that $g = e$.
- (2) We say G acts *properly* if for any two points x and y in X there exist open neighborhoods U of x and V of y such that the set $\{g \in G \mid gU \cap V \neq \emptyset\}$ is finite.

¹¹⁹Indeed, if U_k was contained in $U_1 \cup \dots \cup U_{k-1}$, then we could remove U_k from the collection of open sets and we continue with the collection $U_1, \dots, U_{k-1}$.

- (3) We say G *acts discretely*, if for each $x \in X$ there exists an open neighborhood U so that

$$U \cap gU = \emptyset \text{ for all } g \neq e.$$

Lemma 6.5. *Let G be a group which acts continuously on a topological space X .*

- (1) *If G is finite, then the action is proper.*
- (2) *If G acts discretely, then it also acts freely.*
- (3) *If X is a Hausdorff space and if the action by G is free and proper, then the action is also discrete.*

Here the first two statements follow immediately from the definitions, and the third statement was proved in Analysis IV Lemma 18.12

Example.

- (A) The group $G = \mathbb{Z}^n$ acts on $X = \mathbb{R}^n$ by addition. This action is discrete. Indeed, for each $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ the open neighborhood

$$U = \{y = (y_1, \dots, y_n) \in \mathbb{R}^n \mid \max\{|x_1 - y_1|, \dots, |x_n - y_n|\} < \frac{1}{2}\}$$

has the desired property.

- (B) Let $X = \mathbb{R} \times [-1, 1]$ and $G = \mathbb{Z}$. The action

$$\begin{aligned} \mathbb{Z} \times (\mathbb{R} \times [-1, 1]) &\rightarrow \mathbb{R} \times [-1, 1] \\ (n, (x, y)) &\mapsto (x + n, (-1)^n y) \end{aligned}$$

is discrete. Indeed, for any $(x, y) \in X$ the open neighborhood

$$U = \{(a, b) \mid |a - x| < \frac{1}{2} \text{ and } b \in [-1, 1]\}$$

has the desired property.

- (C) Let $X = S^n$ and $G = \{\pm 1\}$. The map

$$\begin{aligned} \{\pm 1\} \times S^n &\rightarrow S^n \\ (\epsilon, P) &\mapsto \epsilon \cdot P \end{aligned}$$

defines an action that is discrete. Indeed, for any $P \in S^n$ the open hemisphere

$$U = \{Q \in S^n \mid \underbrace{P \cdot Q}_{\text{scalar product}} > 0\}$$

has the desired property.¹²⁰

- (D) Every discrete action is free, but as we will see in this example, the converse does not hold. Let $X = \mathbb{R}^{n+1} \setminus \{0\}$ and $G = \mathbb{R} \setminus \{0\}$. The map

$$\begin{aligned} (\mathbb{R} \setminus \{0\}) \times (\mathbb{R}^{n+1} \setminus \{0\}) &\rightarrow \mathbb{R}^{n+1} \setminus \{0\} \\ (r, P) &\mapsto r \cdot P \end{aligned}$$

¹²⁰Alternatively one can argue, that the action is obviously free and S^n is Hausdorff, hence the action is discrete by Lemma 6.5 (1) and (3).

defines an action. It is straightforward to see that the action is free. But the action is not discrete. Indeed, it is easy to see that for each open non-empty set U there exist uncountably many real numbers $r \neq 0$ with $U \cap rU \neq \emptyset$.

(E) We consider the following two self-homeomorphisms of $\mathbb{R}^2$:

$$\begin{aligned} A: \mathbb{R}^2 &\rightarrow \mathbb{R}^2 & B: \mathbb{R}^2 &\rightarrow \mathbb{R}^2 \\ (x, y) &\mapsto (x+1, 1-y) & \text{and} & & (x, y) &\mapsto (x, y+1). \end{aligned}$$

We denote by G the subgroup of all homeomorphisms of $\mathbb{R}^2$ that is generated by A and B . This means¹²¹

$$G = \begin{array}{l} \text{all self-homeomorphisms of } \mathbb{R}^2 \text{ that can be written} \\ \text{as a finite concatenation of the maps } A, B, A^{-1} \text{ and } B^{-1}. \end{array}$$

The group G acts by definition on $\mathbb{R}^2$. In exercise sheet 5 we will see that this action of G on $\mathbb{R}^2$ is discrete.

We consider again the square $X = [0, 1] \times [0, 1]$ with the equivalence relation that is generated by $(0, y) \sim (1, 1 - y)$ and $(x, 0) \sim (x, 1)$. It is straightforward to see that the map

$$\begin{aligned} X/\sim &\rightarrow \mathbb{R}^2/G \\ [(x, y)] &\mapsto [(x, y)] \end{aligned}$$

is a homeomorphism. Put differently, $\mathbb{R}^2/G$ is homeomorphic to the Klein bottle.

(F) Let

$$X = S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}$$

and let $G = \mathbb{Z}_p$ for some $p \in \mathbb{N}$. Furthermore let $q \in \mathbb{Z}$ with $\gcd(p, q) = 1$. Then the map

$$\begin{aligned} \mathbb{Z}_p \times S^3 &\rightarrow S^3 \\ (k + p\mathbb{Z}, (z_1, z_2)) &\mapsto (e^{2\pi i k/p} z_1, e^{2\pi i k q/p} z_2) \end{aligned}$$

defines a smooth, orientation-preserving and discrete action of $\mathbb{Z}_p$ on S^3 .¹²² We denote by $L(p, q) = S^3/\mathbb{Z}_p$ the quotient space. It follows from Propositions 1.28 and 1.34 that $L(p, q)$ is again a closed, compact, orientable 3-dimensional-manifold. We refer to the manifolds $L(p, q)$ as *lens spaces*.¹²³ We consider two special cases:

- (a) For $p = 1$ we only have the action of the trivial group on S^3 and the quotient space $L(1, q)$ is therefore just the original sphere S^3 .
- (b) For $p = 2$ we have q is odd, so for $k = 0$ we have $e^{2\pi i k/2} = e^{2\pi i k q/2} = 1$ and for $k = 1$ we have $e^{2\pi i k/2} = e^{2\pi i k q/2} = -1$. Put differently, the action of $\mathbb{Z}_2$ on S^3 is the same as the action of $\mathbb{Z}_2 = \{\pm 1\}$ that we had used in the definition of the real projective space $\mathbb{RP}^n$. This shows that $L(2, q) = \mathbb{RP}^n$.

¹²¹For example the map $B \circ A \circ A \circ B^{-1} \circ A$ lies in G .

¹²²It is straightforward to verify that the map does indeed define an action. It is clear that the action is smooth and a direct calculation shows that the action is orientation-preserving. One can also easily verify that the action is free. It follows from Lemma 6.5 (1) and (3) that the action is discrete.

¹²³The name “lens space” comes from an alternative description of these manifolds.

We discuss the last type of examples in slightly more detail. For $p \in \mathbb{N}$ and $q \in \mathbb{N}$ with $\gcd(p, q) = 1$ we just introduced a new topological space, namely the lens space $L(p, q)$. It is natural to ask, for which parameters (p, q) are these spaces homeomorphic?

Lemma 6.6. *Let $p \in \mathbb{N}$ and furthermore let $q, r \in \mathbb{Z}$ with $\gcd(p, q) = \gcd(p, r) = 1$. If $q \equiv \pm r^{\pm 1} \pmod{p}$,¹²⁴ then $L(p, q)$ and $L(p, r)$ are homeomorphic.¹²⁵*

Proof. Let $p \in \mathbb{N}$ and let $q, r \in \mathbb{Z}$ with $\gcd(p, q) = \gcd(p, r) = 1$. We suppose that $q \equiv \pm r^{\pm 1} \pmod{p}$. We define a map

$$L(p, q) \rightarrow L(p, r)$$

as follows:

$$\begin{array}{lll} \text{Case 1: } q \equiv r \pmod{p}. & [(z_1, z_2)] & \mapsto [(z_1, z_2)] \\ \text{Case 2: } q \equiv -r \pmod{p}. & [(z_1, z_2)] & \mapsto [(z_1, \overline{z_2})] \\ \text{Case 3: } q \equiv r^{-1} \pmod{p}. & [(z_1, z_2)] & \mapsto [(z_2, z_1)] \\ \text{Case 4: } q \equiv -r^{-1} \pmod{p}. & [(z_1, z_2)] & \mapsto [(z_2, \overline{z_1})] \end{array}$$

In all four cases it is straightforward to verify that the given map is well-defined and that it is in fact a homeomorphism. $\square$

Proof. Let $p \in \mathbb{N}$ and let $q, r \in \mathbb{Z}$ with $\gcd(p, q) = \gcd(p, r) = 1$. First suppose that $q \equiv \pm r \pmod{p}$. In that case (p, q) and (p, r) define exactly the same equivalence relation on S^3 , so the quotient spaces are also the same.¹²⁶ Now suppose that $q \equiv \pm r^{-1} \pmod{p}$. It is straightforward to verify that the map

$$\begin{array}{ll} L(p, q) & \rightarrow L(p, r) \\ [(z_1, z_2)] & \mapsto [(z_1, \overline{z_2})] \end{array}$$

is a well-defined map that is in fact a homeomorphism. $\square$

In the proof of Lemma 6.6 it was quite straightforward to write down the homeomorphism. This raises the question whether there are homeomorphisms between more lens spaces that are harder to find, or whether all other pairs of lens spaces are non-homeomorphic. More precisely, we have the following question.

Question 6.7. *Let $p, \tilde{p} \in \mathbb{N}$ and let $q, \tilde{q} \in \mathbb{Z}$ with $\gcd(p, q) = \gcd(\tilde{p}, \tilde{q}) = 1$. If $L(p, q)$ and $L(\tilde{p}, \tilde{q})$ are homeomorphic, does it follow that $p = \tilde{p}$ and $q \equiv \pm \tilde{q}^{\pm 1} \pmod{p}$?*

This question will keep us busy for a very long time.

We had just seen on page 98 that the group $\{\pm 1\}$ acts freely and continuously on any sphere. We had also seen that any finite cyclic group $\mathbb{Z}_p$ acts freely and continuously on the 3-sphere. This raises the question, whether such an action by $\mathbb{Z}_p$ also exists on the 2-sphere. More generally we can ask the following question.

¹²⁴By our hypothesis we have $\gcd(p, r) = 1$. This implies that there exists a multiplicative inverse of r modulo p , i.e. there exists an s with $r \cdot s \equiv 1 \pmod{p}$. We write $r^{-1} := s$.

¹²⁵In particular any lens space is of the form $L(p, q)$ with $q \in \{0, \dots, p-1\}$.

¹²⁶They are not only homeomorphic, they are really the same.

Question 6.8. Given $n \in \mathbb{N}$, which finite cyclic groups can act freely and continuously on the n -dimensional sphere S^n ?¹²⁷

We will come back to this question in Algebraic Topology II.

The following proposition connects the notion of discrete actions to the theory of covering spaces.

Proposition 6.9. *Let X be a topological space together with a discrete and continuous action by a group G . Then the canonical projection $p: X \rightarrow X/G$ is a covering.*

Example. This proposition, combined with examples (A) and (C), gives in particular new proofs that the maps $\mathbb{R}^n \rightarrow \mathbb{R}^n/\mathbb{Z}^n$ and $S^n \rightarrow S^n/\sim = \mathbb{RP}^n$ are covering maps.

Proof. Let G be a group which acts discretely and continuously on a topological space X . We want to show that the projection map $p: X \rightarrow X/G$ is a covering. Let $[x] \in X/G$. Since the group acts discretely there exists an open neighborhood V of $x \in X$ such that

$$V \cap gV = \emptyset \text{ for all } g \neq e.$$

We put $U := p(V)$. The set U is an open neighborhood of $[x] = p(x)$.¹²⁸ The preimage $p^{-1}(U)$ is the disjoint union of the open subsets¹²⁹ $gV, g \in G$. The restriction of the map p to any gV is a continuous, bijective¹³⁰ map $gV \rightarrow U$ and according to Footnote 128 this map is furthermore open. It follows that the restriction of p to any gV is in fact a homeomorphism. Thus we see that U is uniformly covered. Since this holds for any point in X/G we have shown that the projection map $p: X \rightarrow X/G$ is a covering. $\square$

The following proposition is a slight generalization of the previous proposition.

Proposition 6.10. *Let X be a topological space together with a discrete and continuous action by a group G . Furthermore let H be a subgroup of G . Then the canonical projection*

$$\begin{aligned} p: X/H &\rightarrow X/G \\ [x] &\mapsto [x] \end{aligned}$$

is a covering.

For H the trivial group we recover the statement of Proposition 6.9. In fact the proof of Proposition 6.10 is almost identical to the proof of Proposition 6.9. We leave the details to the reader.

Examples.

¹²⁷Any cyclic group $\mathbb{Z}_k, k \neq 0$ admits a non-trivial action on S^n where $j \in \mathbb{Z}_k$ acts by rotation around the x -axis by the angle $2\pi \frac{j}{k}$. But this action is not free, since the points $(-1, 0, \dots, 0)$ and $(0, \dots, 0, 1)$ are fixed.

¹²⁸Here we use the following fact from Analysis IV: If G is a group which acts continuously on a topological space X , then the projection map $X \rightarrow X/G$ is open.

¹²⁹Why is $p^{-1}(U) = \bigcup_{g \in G} gV$ and why are the sets gV disjoint?

¹³⁰Why is the map bijective?

- (1) Let $X = \mathbb{R} \times [-1, 1]$ and $G = \mathbb{Z}$. Once again we consider the discrete action

$$\begin{aligned} \mathbb{Z} \times (\mathbb{R} \times [-1, 1]) &\rightarrow \mathbb{R} \times [-1, 1] \\ (n, (x, y)) &\mapsto (x + \tfrac{1}{2}n, (-1)^n y). \end{aligned}$$

Let $H = 2\mathbb{Z}$. According to Proposition 6.10 the projection map $p: X/H \rightarrow X/G$ is a covering. Put differently, all the horizontal maps in the following commutative diagram are coverings:

$$\begin{array}{ccc} (\mathbb{R} \times [-1, 1])/2\mathbb{Z} & \xrightarrow{\quad\quad\quad} & (\mathbb{R} \times [-1, 1])/\mathbb{Z} \\ \downarrow = & & \downarrow = \\ (\mathbb{R} \times [-1, 1])/(x, y) \sim (x + \tfrac{1}{2}2k, (-1)^{2k}y) & \longrightarrow & (\mathbb{R} \times [-1, 1])/(x, y) \sim (x + \tfrac{1}{2}k, (-1)^k y) \\ \downarrow = & & \downarrow = \\ \underbrace{(\mathbb{R} \times [-1, 1])/(x, y) \sim (x + k, y)}_{=\text{the annulus}} & \longrightarrow & \underbrace{(\mathbb{R} \times [-1, 1])/(x, y) \sim (x + \tfrac{1}{2}k, (-1)^k y)}_{=\text{Möbius band}} \end{array}$$

Each point on X/G has precisely two preimages, so the covering is in fact a 2-fold covering. Thus we recover the result from page 94 that the annulus is a 2-fold covering of the Möbius band.

- (2) We consider again the group G of self-homeomorphisms of $\mathbb{R}^2$ that is generated by

$$\begin{aligned} A: \mathbb{R}^2 &\rightarrow \mathbb{R}^2 & \text{and} & & B: \mathbb{R}^2 &\rightarrow \mathbb{R}^2 \\ (x, y) &\mapsto (x + 1, 1 - y) & & & (x, y) &\mapsto (x, y + 1). \end{aligned}$$

We let H be the subgroup of G generated by A^2 and B . Note that A^2 is the map given by

$$\begin{aligned} A^2: \mathbb{R}^2 &\xrightarrow{A} \mathbb{R}^2 & \xrightarrow{A} & \mathbb{R}^2 \\ (x, y) &\mapsto (x + 1, 1 - y) & \mapsto & \underbrace{((x + 1) + 1, (1 - (1 - y)))}_{=(x+2, y)}. \end{aligned}$$

Note that A^2 and B are two homeomorphisms that commute, so any element in H is in fact of the form

$$\begin{aligned} A^{2k} \circ B^l: \mathbb{R}^2 &\rightarrow \mathbb{R}^2 \\ (x, y) &\mapsto (x + 2k, y + l) \end{aligned}$$

for some $k, l \in \mathbb{Z}$. By Proposition 6.10 the projection map $p: \mathbb{R}^2/H \rightarrow \mathbb{R}^2/G$ is a covering map. We already know that $\mathbb{R}^2/G$ is the Klein bottle. It follows immediately from the above description of H , $\mathbb{R}^2/H = \mathbb{R}^2/(2\mathbb{Z} \oplus \mathbb{Z})$, so $\mathbb{R}^2/H$ is just the 2-dimensional torus. As on page 94 we see that we can view the 2-dimensional torus as a 2-fold covering of the Klein bottle.

6.3. The lifting of paths.

Definition. Let $p: X \rightarrow B$ be a map between topological spaces and let $f: [a, b] \rightarrow B$ be a path. Let $x \in X$ be a point with $p(x) = f(a)$. A path $g: [a, b] \rightarrow X$ is called a *lift of f to the starting point x* , if $g(a) = x$ and if $p \circ g = f$, i.e. if the following diagram of maps commutes:

$$\begin{array}{ccc} & & X \\ & \nearrow g & \downarrow p \\ [a, b] & \xrightarrow{f} & B. \end{array}$$

Examples.

(1) We consider the map

$$\begin{array}{ccc} p: \mathbb{R} & \rightarrow & S^1 \\ t & \mapsto & e^{it} \end{array}$$

and the path

$$\begin{array}{ccc} f: [0, b] & \rightarrow & S^1 \\ t & \mapsto & e^{2it} \end{array}$$

with starting point $f(0) = 1$. Let $x = 4\pi$. Then $p(x) = p(4\pi) = 1$ and

$$\begin{array}{ccc} g: [0, b] & \rightarrow & \mathbb{R} \\ t & \mapsto & 4\pi + 2t \end{array}$$

is a lift of f to the starting point 4π . Note that if $b = \pi$, then f is a loop in $(S^1, 1)$, but the lift is not a loop in $(\mathbb{R}, 4\pi)$. Put differently, the lift of a loop is not necessarily again a loop. This example is illustrated in Figure 63.

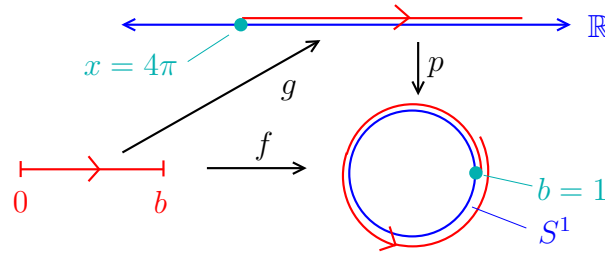


FIGURE 63.

(2) In Figure 64 we illustrate the lifting of a path in the covering $p: H \rightarrow A$ of an infinite helix over an annulus that we had considered on page 92.

Proposition 6.11. *Let $p: X \rightarrow B$ be a covering and let $f: [0, 1] \rightarrow B$ be a path. Then given any $x \in X$ with $p(x) = f(0)$ there exists a unique lift $g: [0, 1] \rightarrow X$ of f to the starting point x .*

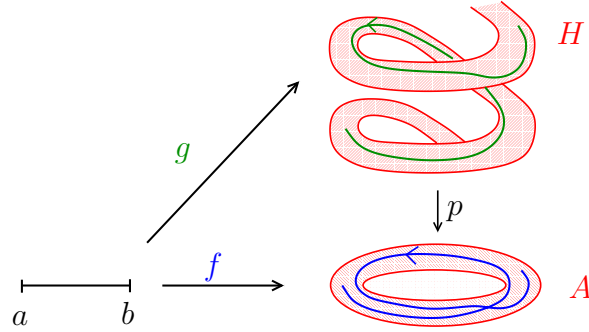


FIGURE 64.

Proof. Let $p: X \rightarrow B$ be a covering, let $f: [0, 1] \rightarrow B$ be a path and let x be a point in X with $p(x) := f(0)$. We first show the existence of a lift and then we prove the uniqueness of the lift.

(a) We consider

$$S := \{s \in [0, 1] \mid \text{there exists a lift of } f|_{[0, s]} \text{ to the starting point } x\}.$$

Note that S is non-empty since 0 has the desired property. We put $t := \sup(S)$. We want to show that $t = 1$ and that $t \in S$. It suffices to show that there exists an $\epsilon > 0$ such that $[t, t + \epsilon) \cap [0, 1]$ still lies in S . We set $b := f(t)$. Since p is a covering there exists an open neighborhood U of $b \in B$, which is uniformly covered. As a reminder, this means that $p^{-1}(U)$ is the union of disjoint open subsets $\{V_i\}_{i \in I}$ with the property that the restriction of p to each subset V_i is a homeomorphism. Since U is open and since f is continuous there exists an $\epsilon > 0$ such that $f((t - \epsilon, t + \epsilon) \cap [0, 1]) \subset U$.

By the definition of t there exists an $s \in (t - \epsilon, t] \cap S$. We denote by g the lift of $f|_{[0, s]}$ the starting point x . There exists a unique $i \in I$ such that $g(s) \in V_i$. We denote by $p_i: V_i \rightarrow U$ the homeomorphism which is given by the restriction of p to V_i .

We consider the map

$$\begin{aligned} h: [0, t + \epsilon) \cap [0, 1] &\rightarrow X \\ z &\mapsto \begin{cases} g(z), & \text{if } z \in [0, s], \\ p_i^{-1}(f(z)), & \text{if } z \in (s, t + \epsilon) \cap [0, 1]. \end{cases} \end{aligned}$$

Since $g(s) = p_i^{-1}(f(s))$ we see that this map is indeed continuous. It is clear that this defines a lift of f to the starting point x .

(b) Now we show the uniqueness of the lift. The proof of that statement is similar to the proof of the existence of a lift. So suppose that $g, h: [0, 1] \rightarrow X$ are two lifts of f to the starting point x . We want to show that $g = h$. We consider

$$S := \{s \in [0, 1] \mid g(t) = h(t) \text{ for all } t \in [0, s]\}.$$

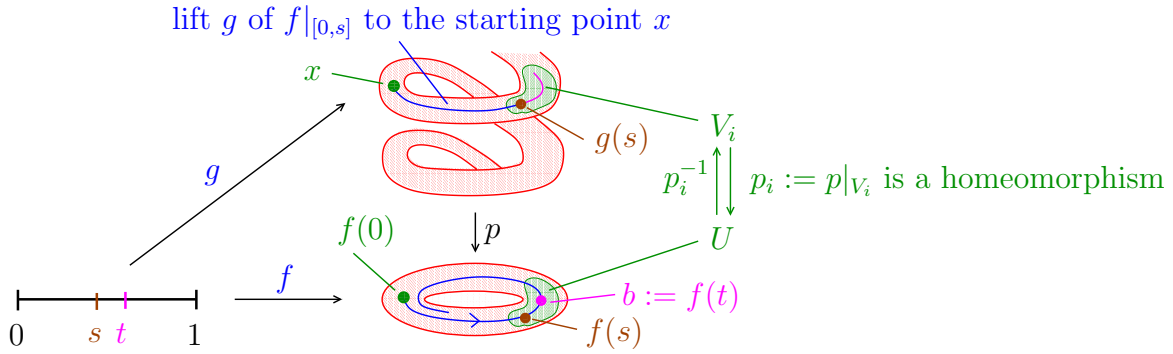


FIGURE 65.

Note that $0 \in S$, i.e. S is non-empty. We put $t := \sup(S)$. We want to show that $t = 1$ and that $t \in S$. Again it suffices to show that there exists an $\epsilon > 0$ such that $[t, t + \epsilon) \cap [0, 1]$ still lies in S .

We put $b := f(t)$. Since p is a covering there exists an open neighborhood U of $b \in B$ which is uniformly covered. Since U is open and since f is continuous there exists an $\epsilon > 0$ such that $f((t - \epsilon, t + \epsilon) \cap [0, 1]) \subset U$. We pick an $s \in (t - \epsilon, t] \cap [0, 1]$. There exists a unique $i \in I$ such that $g(s) \in V_i$. We denote by $p_i: V_i \rightarrow U$ the homeomorphism which is given by the restriction of p to V_i .

Since $g(s) = h(s)$ it follows that $h(s) \in V_i$. Then it follows that $g(z) \in V_i$ and $h(z) \in V_i$ for all $z \in (t - \epsilon, t + \epsilon) \cap [0, 1]$. It follows from the definitions that

$$g(z) = p_i^{-1}(f(z)) \quad \text{and} \quad h(z) = p_i^{-1}(f(z))$$

for all $z \in (t - \epsilon, t + \epsilon) \cap [0, 1]$. In particular $[t, t + \epsilon) \cap [0, 1]$ lies in S . $\square$

6.4. The lifting of homotopies. In the last chapter we had seen that given a covering map $p: X \rightarrow B$ any path $f: [0, 1] \rightarrow B$ can be lifted to a path $g: [0, 1] \rightarrow X$. In this chapter we want to study the question whether the lifts of homotopic paths are still homotopic.

Before we start this discussion we first generalize the notion of a lift.

Definition. Let $p: X \rightarrow B$ be a covering and let $f: Y \rightarrow B$ be a map between topological spaces. A *lift of f to X* is a map¹³¹ $\tilde{f}: Y \rightarrow X$ such that $p \circ \tilde{f} = f$, i.e. such that the following diagram of maps commutes:

$$\begin{array}{ccc} & & X \\ & \nearrow \tilde{f} & \downarrow p \\ Y & \xrightarrow{f} & B. \end{array}$$

¹³¹Once again, all maps are understood to be continuous.

Proposition 6.12. *Let $p: X \rightarrow B$ be a covering, let B be a topological space, let Y be a topological manifold¹³² and let $f: Y \times [0, 1] \rightarrow B$ be a map. Furthermore let $\tilde{f}: Y \times 0 \rightarrow X$ be a lift of $f|_{Y \times 0}$. Let*

$$\tilde{f}: Y \times [0, 1] \rightarrow X$$

be the map that is uniquely determined¹³³ by the property that for each $y \in Y$ the path $t \mapsto \tilde{f}(y, t)$ is the lift of the path $t \mapsto f(y, t)$ to the starting point $\tilde{f}(y, 0)$. Then the map $\tilde{f}: Y \times [0, 1] \rightarrow X$ is continuous.

The statement of Proposition 6.12 is sketched in Figure 66. One way of formulating Proposition 6.12 is that a “continuous” family of paths in B lifts again to a “continuous” family of paths in X .

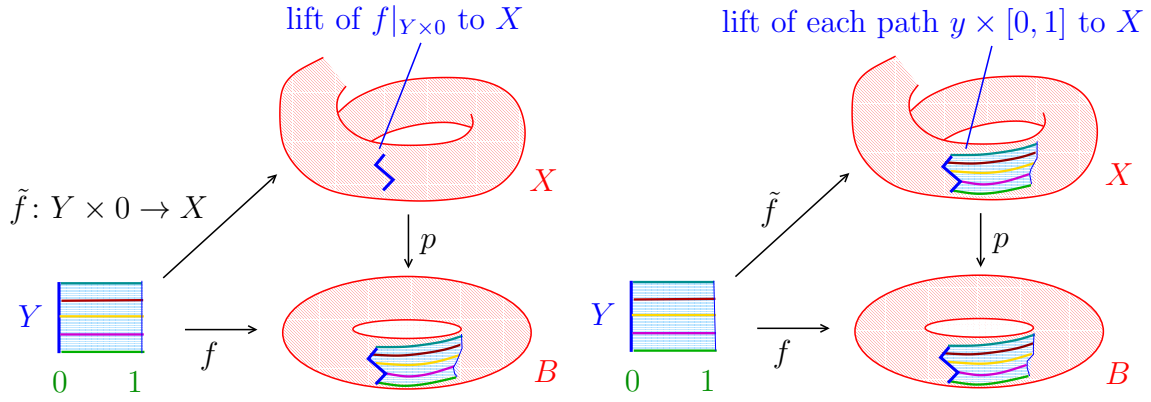


FIGURE 66. Schematic image for the lift of f .

Proof. Let $p: X \rightarrow B$ be a covering, let $f: Y \times [0, 1] \rightarrow B$ be a continuous map and let $\tilde{f}: Y \times 0 \rightarrow X$ be a continuous lift of $f|_{Y \times 0}$. (Here and throughout this proof we do not suppose that all maps are continuous.) We begin with the following definitions:

- (1) A *good set* $U \times I$ consists of a connected subset $U \subset Y$ and an open interval¹³⁴ $I \subset [0, 1]$ such that $f(U \times I)$ lies in a uniformly covered subset of B .
- (2) We say a good set $U \times I$ is *very good*, if there exists a $t \in I$ such that the restriction of $\tilde{f}$ to $U \times \{t\}$ is continuous.
- (3) Let $y \in Y$ and $t \in [0, 1]$. A *(very) good neighborhood* $U \times I$ is a (very) good set $U \times I$ with $y \in U$ and $t \in I$.

¹³²In our only application we will have $Y = [0, 1]$, but it is good to keep the roles of Y and $[0, 1]$ separate.

¹³³The existence and the uniqueness of this map is an immediate consequence of Proposition 6.11.

¹³⁴We say $A \subset [0, 1]$ is an open interval, if A is an interval and if A is open with respect to the subspace topology of $[0, 1]$. Put differently, $A = (a, b)$, or $A = [0, a)$ or $A = (b, 1]$ or $A = [0, 1]$ for $0 \leq a \leq b \leq 1$.

It follows easily¹³⁵ from the continuity of f , the hypothesis that Y is a manifold and the fact that $p: X \rightarrow B$ is a covering that each point (y, t) admits a good neighborhood.¹³⁶

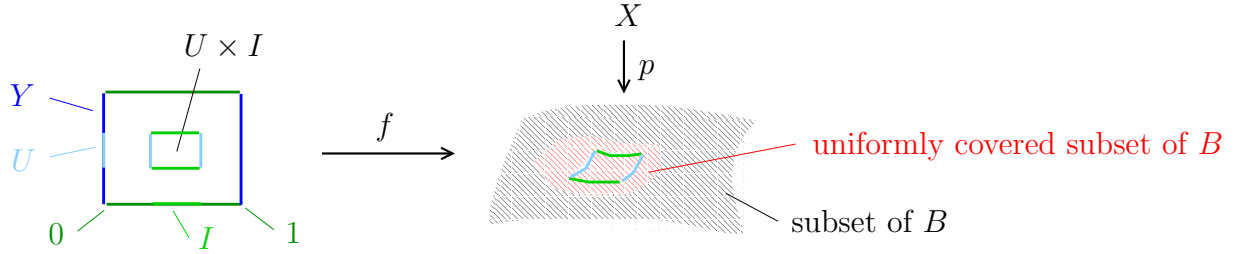


FIGURE 67. Schematic image of a good neighborhood.

Claim. Let $U \subset Y$ open and let $I \subset [0, 1]$ be an open interval such that $U \times I$ is very good. Then $\tilde{f}$ is continuous on $U \times I$.

Let $U \subset Y$ be an open connected subset and let $I \subset [0, 1]$ be an interval such that $U \times I$ is very good. This means in particular that there exists a $t \in I$ such that

$$\begin{aligned} U &\rightarrow X \\ y &\mapsto \tilde{f}(y, t) \end{aligned}$$

is continuous.

According to our hypothesis there exists a uniformly covered subset $W \subset B$ such that $f(U \times I) \subset W$. Since $U \times I$ is connected we can without loss of generality suppose that W is connected. Since W is uniformly covered the preimage $p^{-1}(W)$ is the union of disjoint open subsets $\{V_j\}_{j \in J}$, with the property, that the restriction of p to each subset V_j is a homeomorphism.

We now want to show that there exists a $j \in J$ with $\tilde{f}(U \times I) \subset V_j$. We choose a $y \in U$. Then there exists a unique $j \in J$ such that $\tilde{f}(y, t) \in V_j$. Since $\tilde{f}$ is continuous $U \times \{t\}$ and since U is connected it follows from Lemma 1.13¹³⁷ that $\tilde{f}(U \times \{t\}) \subset V_j$. Now let $z \in U$. We just showed that $\tilde{f}(z, t) \in V_j$. By the construction of $\tilde{f}$ the map $\tilde{f}$ is continuous on $\{z\} \times I$. Since I is connected it follows again from Lemma 1.13 that $\tilde{f}(\{z\} \times I) \subset V_j$. We have thus shown that $\tilde{f}(U \times I) \subset V_j$. This part of the argument is sketched in Figure 68.

We denote by $p_j: V_j \rightarrow W$ the homeomorphism which is given by the restriction of p to V_j . Since $\tilde{f}(U \times I) \subset V_j$ it follows that on $U \times I$ the map $\tilde{f}$ is given by $\tilde{f} = p_j^{-1} \circ f$. In

¹³⁵Why does this follow?

¹³⁶Here we use the fact, established in Lemmas 1.9 and 1.25, that the topological manifold Y is locally connected.

¹³⁷We apply Lemma 1.13 to the continuous map $U \times \{t\} \rightarrow p^{-1}(W)$ from the connected set U to the set $p^{-1}(W)$ with components V_j , $j \in J$.

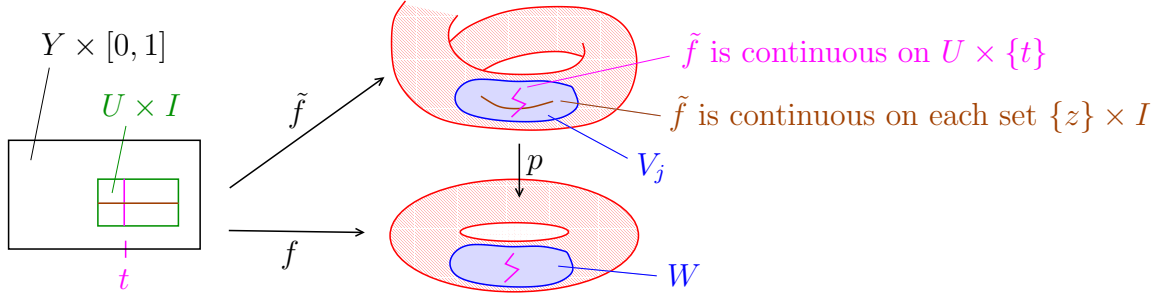


FIGURE 68.

particular the map $\tilde{f}$ is a composition of continuous maps, so it is continuous on $U \times I$. Thus we have proved the claim.

Claim. Let $y \in Y$. We consider

$$T := \{t \in [0, 1] \mid \text{there exists a very good neighborhood of } (y, t)\}.$$

We claim that $T = [0, 1]$.

Thus let $y \in Y$. Since $[0, 1]$ is connected it suffices to show the following three statements:

- (1) $T \neq \emptyset$,
- (2) T is open, and
- (3) T is closed.

We will now prove these three statements:

- (1) As we had pointed out above, every point in $Y \times I$, in particular the point $(y, 0)$, admits a good neighborhood. By our hypothesis $\tilde{f}$ is continuous on $Y \times \{0\}$. It follows that the good neighborhood is in fact very good.
- (2) It is easy to see that T is open. Indeed, suppose that we are given a very good neighborhood $U \times I$ of (y, t) . Then $U \times I$ is also a very good neighborhood for any $(y, s) \in U \times I$. Therefore the open neighborhood I of t also lies in T .
- (3) It remains to show that T is closed. Put differently, we need to show that T equals its closure $\overline{T}$. Thus let $t \in \overline{T}$. As we already pointed out, there exists a good neighborhood $U \times I$ of (y, t) . Since $t \in \overline{T}$ there exists an $s \in T \cap I$. Since $s \in T$ it follows from the above claim that there exists a very good open neighborhood $V \times J$ of (y, s) . By the previous claim the restriction of $\tilde{f}$ to $V \times J$ is continuous, in particular it is continuous on $(U \cap V) \times \{s\}$. We denote by W the component of $U \cap V$ that contains y . Then $W \times I$ is a very good neighborhood of (y, t) . This step is illustrated in Figure 69.

This concludes the proof of the claim.

We have just shown that given any $y \in Y$ and any $t \in [0, 1]$ there exists a very good neighborhood $U_{y,t}$ of (y, t) . By the first claim the map $\tilde{f}$ is continuous on each $U_{y,t}$. But

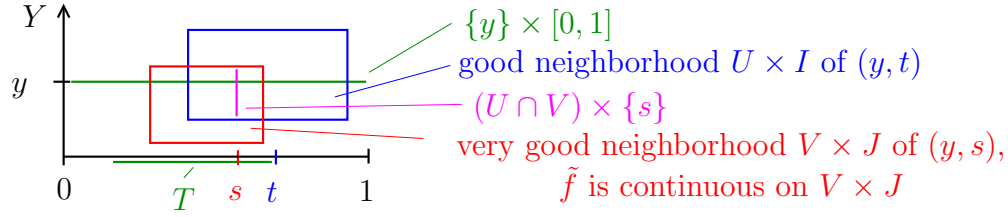


FIGURE 69.

the open subsets $U_{y,t}$ cover all of $Y \times [0, 1]$. It follows from Lemma 1.20 that $\tilde{f}$ is continuous on $Y \times [0, 1]$. $\square$

Corollary 6.13. *Let $p: X \rightarrow B$ be a covering and let $f, g: [0, 1] \rightarrow B$ be two paths. Let $\tilde{f}, \tilde{g}: [0, 1] \rightarrow X$ be two lifts with the same starting point. If f and g are homotopic, then the endpoints of $\tilde{f}$ and $\tilde{g}$ agree and the paths $\tilde{f}$ and $\tilde{g}$ are homotopic.*

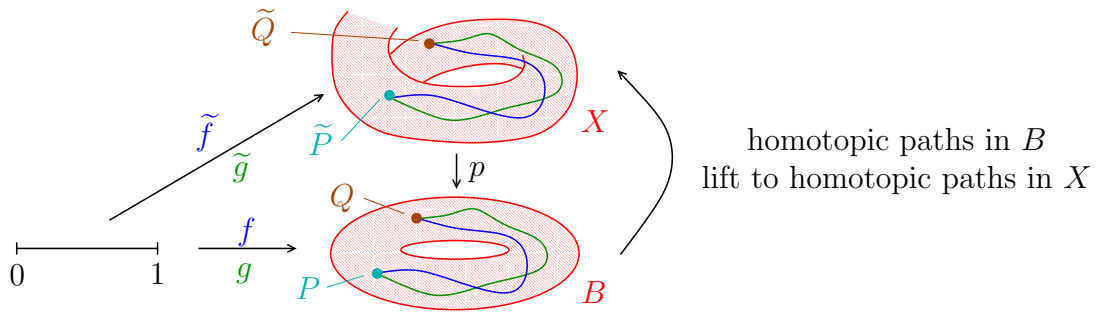


FIGURE 70. Illustration of Corollary 6.13.

Proof. We write $P := f(0) = g(0)$ and $Q := f(1) = g(1)$. We denote by $\tilde{P} := \tilde{f}(0) = \tilde{g}(0)$ the common starting point of the lifts and we write $\tilde{Q} := \tilde{f}(1)$. Let

$$\begin{aligned} H: [0, 1] \times [0, 1] &\rightarrow B \\ (t, s) &\mapsto H(t, s) \end{aligned}$$

be a homotopy between the paths f and g . As a reminder, H is a map such that

$$H(t, 0) = f(t) \quad \text{and} \quad H(t, 1) = g(t) \quad \text{for all } t \in [0, 1],$$

and such that

$$H(0, s) = P \quad \text{and} \quad H(1, s) = Q \quad \text{for all } s \in [0, 1].$$

On $\{0\} \times [0, 1]$ we consider the lift $\tilde{H}$ of H which is given by $\tilde{H}(0, s) = \tilde{P}$. Furthermore we denote by $\tilde{H}: [0, 1] \times [0, 1] \rightarrow X$ the map which is defined as follows: for each $s \in [0, 1]$

the path $t \mapsto \tilde{H}(t, s)$ is the unique lift of the path $t \mapsto H(t, s)$ to the starting point $\tilde{P}$ lift. According to Proposition 6.12 the map $\tilde{H}$ is in fact continuous. By definition of $\tilde{H}$ we have

$$\tilde{H}(t, 0) = \tilde{f}(t) \quad \text{and} \quad \tilde{H}(t, 1) = \tilde{g}(t) \quad \text{for all } t \in [0, 1],$$

For each $s \in [0, 1]$ we have $p(\tilde{H}(1, s)) = Q$. Since $\tilde{H}$ is continuous we obtain a map

$$\begin{aligned} [0, 1] &\rightarrow p^{-1}(Q) \subset X \\ s &\mapsto \tilde{H}(1, s). \end{aligned}$$

Since $[0, 1]$ is connected it follows from Lemma 1.13 (2) and Lemma 6.4 (2) that $\tilde{H}(1, s) = \tilde{Q}$ for all $s \in [0, 1]$. In particular we obtain that

$$\tilde{f}(1) = \tilde{H}(1, 0) = \tilde{Q} = \tilde{H}(1, 1) = \tilde{g}(1).$$

Thus it follows that $\tilde{H}$ is indeed a homotopy between the paths $\tilde{f}$ and $\tilde{g}$. $\square$

Definition. Let $p: X \rightarrow B$ be a map between topological spaces and let $x_0 \in X$ and $b_0 \in B$. We say a map $p: X \rightarrow B$ is a *covering of pointed topological spaces*, if $p: X \rightarrow B$ is a covering and if $p(x_0) = b_0$.

Corollary 6.14. *For any covering $p: (X, x_0) \rightarrow (B, b_0)$ of pointed topological spaces the induced map $p_*: \pi_1(X, x_0) \rightarrow \pi_1(B, b_0)$ is a monomorphism.*

Proof. We have¹³⁸

$$\begin{aligned} [f] \in \ker(\pi_1(X, x_0) \rightarrow \pi_1(B, b_0)) &\implies [p \circ f] = e \in \pi_1(B, b_0) \implies p \circ f \simeq e_{b_0} \\ &\implies f \simeq e_{x_0} \implies [f] = e \in \pi_1(X, x_0). \\ &\quad \uparrow \end{aligned}$$

by Corollary 6.13, since f is the lift of $p \circ f$ and e_{x_0} is the lift of e_{b_0} to the starting point x_0 $\square$

Example. In Question 6.3 (1) we had asked whether there exists a covering map from the torus $S^1 \times S^1$ to the sphere S^2 . By Proposition 4.12 we know that $\pi_1(S^2) = 0$ whereas we had seen on page 86 that $\pi_1(S^1 \times S^1)$ is non-trivial. Using Corollary 6.14 we can now answer Question 6.3 (1) in the negative: there is no covering map from $S^1 \times S^1$ to S^2 .

We conclude the section with the following lemma.

Lemma 6.15. *Let $p: (X, x_0) \rightarrow (B, b_0)$ be a covering of pointed topological spaces.*

- (1) *Let f be a loop in (B, b_0) . We denote by $\tilde{f}$ the lift of f to the starting point x_0 . Then the following holds:*

$$\tilde{f} \text{ is a loop in } (X, x_0) \iff [f] \text{ lies in } p_*(\pi_1(X, x_0)) \subset \pi_1(B, b_0).$$

- (2) *Now we suppose that X is simply connected. Let f and g loops in (B, b_0) and let $\tilde{f}$ and $\tilde{g}$ be the lifts of f and g to the starting point x_0 . Then the following holds:*

$$f \text{ and } g \text{ are homotopic loops} \iff \text{the endpoints of } \tilde{f} \text{ and } \tilde{g} \text{ agree.}$$

¹³⁸We use the notation from page 71: for $y \in Y$ we denote by e_y the constant path $e_y(t) := y, t \in [0, 1]$.

(3) If X is path-connected, then $[X : B] = [\pi_1(B, b_0) : p_*(\pi_1(X, x_0))]$.

Remark. In exercise sheet 13 we use Lemma 6.15 to answer Question 6.3 (2) in the negative.

Proof.

- (1) We prove the “ $\Rightarrow$ ”-direction. So let f be a loop in (B, b_0) . We denote by $\tilde{f}$ the lift of f to the starting point x_0 . If $\tilde{f}$ is a loop in (X, x_0) , then it defines an element in $\pi_1(X, x_0)$ and we have $[f] = [p \circ \tilde{f}] = p_*([\tilde{f}]) \in p_*(\pi_1(X, x_0))$. The proof of the “ $\Leftarrow$ ”-direction is an exercise in exercise sheet 6.¹³⁹
- (2) The “ $\Rightarrow$ ”-direction is an immediate consequence of Corollary 6.13. The proof of the “ $\Leftarrow$ ”-direction is once again an exercise in exercise sheet 6. For that direction we need that X is simply connected.
- (3) This statement is proved in exercise sheet 13. □

6.5. Group actions and fundamental groups. The following proposition finally allows us to determine many fundamental groups. This proposition is the reward for the many rather theoretical results of the previous sections.

Theorem 6.16. *Let X be a simply connected topological space. Let G be a group which acts continuously and discretely on X . We choose an $x \in X$. We denote by $p: X \rightarrow X/G$ the canonical projection map. Then the map*

$$\begin{aligned} G &\mapsto \pi_1(X/G, [x]) \\ g &\mapsto [p \circ (\text{path in } X \text{ from } x \text{ to } g \cdot x)] \end{aligned}$$

is a well-defined isomorphism of groups. In particular we have $\pi_1(X/G) \cong G$.

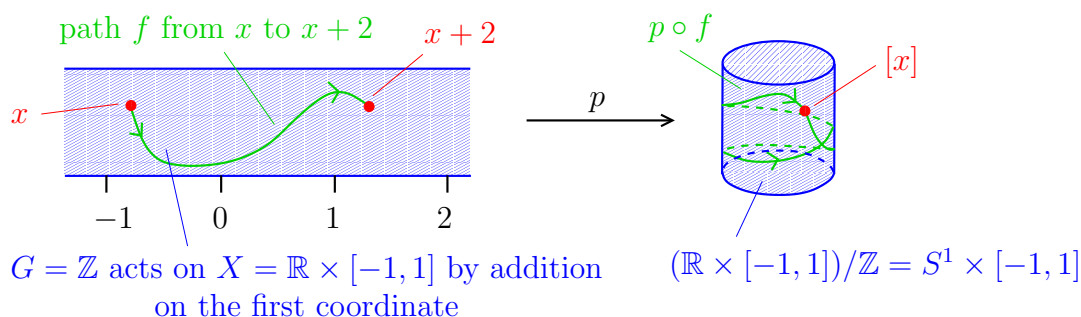


FIGURE 71. Illustration of Theorem 6.16.

Examples.

¹³⁹Note that $[f]$ lies in $p_*(\pi_1(X, x_0)) \subset \pi_1(B, b_0)$ means that there exists a loop g in (X, x_0) such that $p \circ g$ is homotopic to f . A priori it does not mean that there exists a loop g with $p \circ g = f$.

- (1) We consider the action of $G = \mathbb{Z}$ on the simply connected topological space $X = \mathbb{R}$. We denote by $p: \mathbb{R} \rightarrow \mathbb{R}/\mathbb{Z}$ the canonical projection map. It follows immediately from Theorem 6.16 that the map

$$\begin{aligned} \mathbb{Z} &\rightarrow \pi_1(\mathbb{R}/\mathbb{Z}, 0) \\ n &\mapsto \underbrace{\left[\begin{array}{ll} [0, 1] & \rightarrow \mathbb{R}/\mathbb{Z} \\ t & \mapsto p(nt) = [nt] \end{array} \right]}_{\substack{\text{the map } t \mapsto nt \text{ is a} \\ \text{path from } 0 \text{ to } 0 + n}} \end{aligned}$$

is an isomorphism of groups. In particular $\pi_1(\mathbb{R}/\mathbb{Z}, 0) \cong \mathbb{Z}$. We denote by $\phi: \mathbb{R}/\mathbb{Z} \rightarrow S^1$ the homeomorphism given by $\phi([t]) = e^{2\pi it}$. If we compose the above isomorphism with the isomorphism $\phi_*: \pi_1(\mathbb{R}/\mathbb{Z}, 0) \rightarrow \pi_1(S^1, 1)$, then we see that

$$\begin{aligned} \Phi: \mathbb{Z} &\rightarrow \pi_1(S^1, 1) \\ n &\mapsto \left[\begin{array}{ll} [0, 1] & \rightarrow S^1 \\ t & \mapsto e^{2\pi int} \end{array} \right] \end{aligned}$$

is an isomorphism of groups. In particular we finally see that $\pi_1(S^1) \cong \pi_1(\mathbb{R}/\mathbb{Z}) \cong \mathbb{Z}$. We had suspected this result for a while, but now we finally have a proof. Another way of reformulating that Φ is an isomorphism is to say that every loop in $(S^1, 1)$ is homotopic to a loop of the form $t \mapsto e^{2\pi int}$ for a uniquely determined $n \in \mathbb{Z}$. Furthermore we obtain an affirmative answer to Question 4.8: the epimorphism

$$\begin{aligned} \Psi: \pi_1(S^1, 1) &\rightarrow \mathbb{Z} \\ [\gamma] &\mapsto \frac{1}{2\pi i} \int_{\gamma} \frac{1}{z} dz \end{aligned}$$

is an isomorphism.¹⁴⁰ We will provide many more examples for Theorem 6.16 immediately after the proof of Theorem 6.16.

- (2) In the exercises we will use Theorem 6.16 to show that the fundamental group of the topological space $X = \{A, B, C, D\}$ from page 16 is isomorphic to $\mathbb{Z}$. In particular we show that the loop given on page 16 is not null-homotopic. This answers the question we had posed on page 68.

We now provide the proof of Theorem 6.16.

Proof of Theorem 6.16. ¹⁴¹ Let X be a simply connected topological space. Let G be a group which acts continuously and discretely on X . We choose an $x \in X$. We denote by $p: X \rightarrow X/G$ the canonical projection map. We consider the map

$$\begin{aligned} \Phi: G &\mapsto \pi_1(X/G, [x]) \\ g &\mapsto [p \circ (\text{path in } X \text{ from } x \text{ to } gx)]. \end{aligned}$$

¹⁴⁰Indeed, it is straightforward to see that $\Psi \circ \Phi: \mathbb{Z} \rightarrow \pi_1(S^1, 1) \rightarrow \mathbb{Z}$ is the identity. The first map Φ is an isomorphism, so $\Psi = \Phi^{-1}$ is also an isomorphism.

¹⁴¹It is perhaps helpful to read the proof with the example of the action of $\mathbb{Z}^n$ on $\mathbb{R}^n$ in mind.

- (3) Now we prove that Φ is surjective. We choose an arbitrary element $c \in \pi_1(X/G, [x])$ and represent it by a loop f in $(X/G, [x])$. According to Proposition 6.11 there exists a lift $\tilde{f}: [0, 1] \rightarrow X$ of the path f to the starting point x . By definition we have $p \circ \tilde{f} = f$. In particular we obtain for the endpoint of $\tilde{f}$ that $p(\tilde{f}(1)) = [x]$, therefore it follows that $\tilde{f}(1) = gx$ for some $g \in G$. Thus we have proved that

$$\Phi(g) = [p \circ (\text{path from } x \text{ to } gx)] = [p \circ \tilde{f}] = [f] = c.$$

- (4) We conclude the proof of the proposition by showing that Φ is injective. Let $g \in G$. Then the following holds:

$$\begin{aligned} \Phi(g) = e &\implies [p \circ (\text{path } f \text{ from } x \text{ to } gx)] = e \in \pi_1(X/G, [x]) \\ &\implies p \circ f \simeq e_{[x]} \text{ in } X/G \\ &\implies f(1) = e_x(1) \implies gx = x \implies g = e. \\ &\quad \uparrow \qquad \qquad \qquad \uparrow \\ &\text{by Corollary 6.13, since } f \text{ is a lift of } p \circ f \text{ and } e_x \quad \text{since the action is discrete,} \\ &\text{is a lift of } e_{[x]} \text{ to the starting point } x \quad \text{hence free} \end{aligned}$$

This shows that Φ is indeed injective.

We have thus shown that Φ is a well-defined isomorphism. $\square$

We can now determine the fundamental groups of many of the topological spaces that we have encountered so far.

Corollary 6.17. *We have the following isomorphisms of fundamental groups*

$$\begin{aligned} \text{(A)} \quad & \pi_1(n\text{-dimensional torus}) = \pi_1(\mathbb{R}^n/\mathbb{Z}^n) \cong \mathbb{Z}^n \\ & \text{in particular} \\ & \pi_1(S^1) = \pi_1(\mathbb{R}/\mathbb{Z}) \cong \mathbb{Z}, \\ \text{(B)} \quad & \text{for any } n \neq 0, 1 \\ & \pi_1(\text{real projective space } \mathbb{RP}^n) = \pi_1(S^n/\{\pm 1\}) \cong \mathbb{Z}_2, \\ \text{(C)} \quad & \pi_1(\text{Möbius band}) = \pi_1((\mathbb{R} \times [-1, 1])/\mathbb{Z}) \cong \mathbb{Z}, \\ \text{(D)} \quad & \pi_1(\text{lens space } L(p, q)) = \pi_1(S^3/\mathbb{Z}_p) \cong \mathbb{Z}_p. \end{aligned}$$

Proof. By Proposition 4.12 we know that for any $n \geq 2$ the sphere S^n is simply connected. Furthermore we know by the discussion on page 74 that $\mathbb{R}^n$, $n \geq 1$, and that the strip $\mathbb{R} \times [-1, 1]$ are simply connected. We had seen on page 98 that all the actions are discrete. The actions are of course also continuous. The corollary is now an immediate consequence of Theorem 6.16. $\square$

Thus we have now finally determined the fundamental groups of many spaces. In Proposition 4.12 we had seen that $\pi_1(S^2) = 0$ whereas we have now seen that $\pi_1(S^1 \times S^1) \cong \mathbb{Z}^2$. The fundamental groups are not isomorphic, hence the sphere S^2 and the torus $S^1 \times S^1$ are not homeomorphic.¹⁴³

¹⁴³In Analysis IV we had shown a weaker result: we had shown, using de Rham cohomology, that S^2 is not diffeomorphic to the torus. In general the two notions of “homeomorphic” and “diffeomorphic” are not

We also obtained a partial answer to Question 6.7: if two lens spaces $L(p, q)$ and $L(\tilde{p}, \tilde{q})$ are homeomorphic, then $p = \tilde{p}$. But as of right now we cannot say anything about the parameters q and $\tilde{q}$.

Examples. Theorem 6.16 not only gives us the fact that $\pi_1(X/G)$ is isomorphic to the group G , but it also gives us an explicit isomorphism. We will now study this isomorphism in two examples to explicitly determine non-trivial elements in the fundamental group of the torus and the real projective space.

(1) As a reminder, the torus is defined as¹⁴⁴

$$T = \mathbb{R}^2/\mathbb{Z}^2 = [0, 1] \times [0, 1]/(x, 0) \sim (x, 1) \text{ and } (0, y) \sim (1, y).$$

The explicit isomorphism of Theorem 6.16

$$\Phi: \mathbb{Z}^2 \rightarrow \pi_1(T, (0, 0))$$

assumes in particular the following values

$$\Phi((1, 0)) = x := \underbrace{\begin{bmatrix} [0, 1] & \rightarrow & [0, 1] \times [0, 1]/\sim \\ t & \mapsto & [(t, 0)] \end{bmatrix}}_{t \mapsto (t, 0) \text{ is a path from } (0, 0) \text{ to } (1, 0)}$$

and

$$\Phi((0, 1)) = y := \underbrace{\begin{bmatrix} [0, 1] & \rightarrow & [0, 1] \times [0, 1]/\sim \\ t & \mapsto & [(0, t)] \end{bmatrix}}_{t \mapsto (0, t) \text{ is a path from } (0, 0) \text{ to } (0, 1)}.$$

In the description of the torus as a subset of $\mathbb{R}^3$, see Figure 73, these two loops correspond to the “longitude” and the “meridian”.

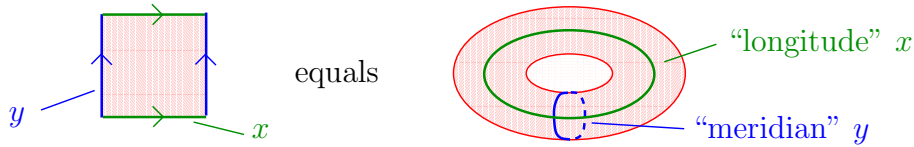


FIGURE 73.

the same. There exist manifolds that are homeomorphic but not diffeomorphic. In fact John Milnor [Mil] showed in 1956 that there exist manifolds that are homeomorphic to S^7 but that are not diffeomorphic to S^7 .

¹⁴⁴Here we are slightly generous in our usage of the equality sign. More precisely, strictly speaking $\mathbb{R}^2/\mathbb{Z}^2$ is not the same as $[0, 1] \times [0, 1]/\sim$. But there exists a canonical homeomorphism between these two spaces that we use to identify these spaces.

- (2) Now we consider the real projective space $\mathbb{RP}^2 = S^2/\{\pm 1\} = S^2/x \sim -x$ and as usual we denote by $p: S^2 \rightarrow \mathbb{RP}^2$ the projection. We put $Q := (1, 0, 0)$. We consider the following path in S^2 :

$$\begin{aligned} f: [0, 1] &\rightarrow S^2 \\ t &\mapsto (\cos(\pi t), \sin(\pi t), 0). \end{aligned}$$

This is a path from Q to $-Q = (-1) \cdot Q$. Since -1 is the one non-trivial element of the group $G = \{\pm 1\}$ it follows from the explicit isomorphism of Theorem 6.16 that the equivalence class of $p \circ f$ is precisely the one non-trivial element of the fundamental group $\pi_1(\mathbb{RP}^2, p(Q)) \cong \{\pm 1\}$.¹⁴⁵ Since $\pi_1(\mathbb{RP}^2, p(Q))$ is isomorphic to $\mathbb{Z}_2$ we see that $p \circ f$ represents an element of order 2, i.e. $(p \circ f) * (p \circ f)$ represents the trivial element in the fundamental group. Put differently, $(p \circ f) * (p \circ f)$ is null-homotopic. This one can see quite explicitly. More precisely, we have

$$\begin{aligned} [p \circ f] \cdot [p \circ f] &= [(p \circ f) * (p \circ f)] = [(p \circ (-f)) * (p \circ f)] = [p \circ (f * (-f))] \\ &= \left[p \circ \underbrace{\left(\begin{array}{ccc} f: [0, 1] & \rightarrow & S^2 \\ t & \mapsto & (\cos(2\pi t), \sin(2\pi t), 0) \end{array} \right)}_{\text{loop } g \text{ in } S^2} \right]. \end{aligned}$$

The loop g in (S^2, Q) is clearly null-homotopic.¹⁴⁶ But then the loop $p \circ g$ is also null-homotopic. This example is illustrated in Figure 74.

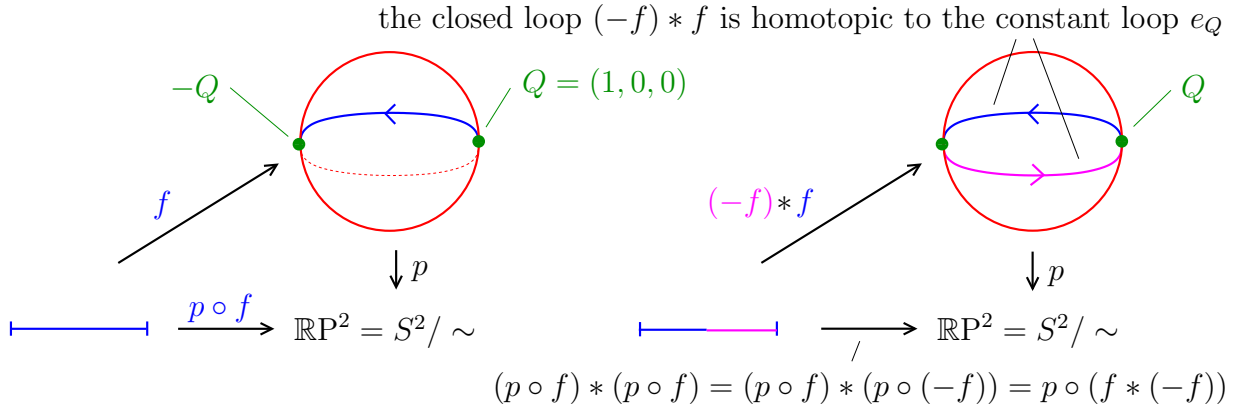


FIGURE 74.

¹⁴⁵In particular we have now shown that the loop in $\mathbb{RP}^2$ given on page 68 is not null-homotopic.

¹⁴⁶More precisely, it is straightforward to write down an explicit homotopy from g to the constant path e_Q . Alternatively one can also deduce that g is null-homotopic from our calculation $\pi_1(S^2, Q) = 0$.

Example. We consider the Klein bottle. Recall that on page 99 we had introduced the group G that is given by all self-homeomorphisms of $\mathbb{R}^2$ that can be written as concatenations of the two self-homeomorphisms

$$\begin{aligned} A: \mathbb{R}^2 &\rightarrow \mathbb{R}^2 & B: \mathbb{R}^2 &\rightarrow \mathbb{R}^2 \\ (x, y) &\mapsto (x+1, 1-y) & \text{and} & & (x, y) &\mapsto (x, y+1) \end{aligned}$$

and their inverses. In exercise sheet 5 we had seen that G acts discretely on $\mathbb{R}^2$ and we had seen on page 99 that $\mathbb{R}^2/G$ is homeomorphic to the Klein bottle. It follows from Theorem 6.16 that

$$\pi_1(\text{Klein bottle}) \cong G = \text{group that is generated by the homeomorphisms } A \text{ and } B.$$

The maps A and B do not commute. Indeed, the map $A \circ B: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is given by

$$(x, y) \mapsto (x+1, (1-(y+1))) = (x+1, -y)$$

whereas the map $B \circ A: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is given by

$$(x, y) \mapsto (x+1, (1-y)+1) = (x+1, 2-y).$$

Thus we see that the fundamental group of the Klein bottle is a non-abelian group.

In Figure 75 we show the two elements b and c of the fundamental group of the Klein bottle $K = [0, 1] \times [0, 1] / \sim$ which correspond to B and $C = B^{-1} \circ A$ under the isomorphism of Theorem 6.16 and the homeomorphism from page 99. Since the two homeomorphisms B and $C(x, y) = (x+1, -y)$ do not commute, the corresponding elements $[b]$ and $[c]$ of $\pi_1(K)$ do not commute either.

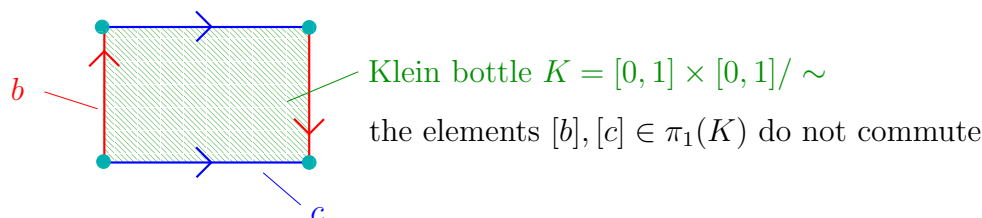


FIGURE 75.

The previous example showed that fundamental groups can be non-abelian groups. In fact, as we will see later, “most” fundamental groups are non-abelian. This will pose several major problems. For example, what is a good way to describe a non-abelian group? How can we decide whether or not two non-abelian groups are isomorphic?

6.6. The fundamental group of the product of two topological spaces. The next proposition says that the fundamental groups of the product $A \times B$ of two topological spaces A and B is the direct product of the fundamental groups of A and B . Before we state the proposition explicitly we recall the definition of the direct product of two groups.

Let G and H two groups. We can form the cartesian product

$$G \times H := \{(g, h) \mid g \in G \text{ and } h \in H\}.$$

This is again a group with the group structure given by

$$(g_1, h_1) \cdot (g_2, h_2) := (g_1 g_2, h_1 h_2)$$

for $g_1, g_2 \in G$ and $h_1, h_2 \in H$. We call the group $G \times H$ the *direct product of the groups G and H* .

We can now formulate the promised proposition.

Proposition 6.18. *Let A and B be two topological spaces and let $a_0 \in A$ and $b_0 \in B$. We consider the maps*

$$\begin{array}{ccc} i: A & \rightarrow & A \times B \\ a & \mapsto & (a, b_0) \end{array} \quad \text{and} \quad \begin{array}{ccc} j: B & \rightarrow & A \times B \\ b & \mapsto & (a_0, b). \end{array}$$

Then the map

$$\begin{array}{ccc} \Phi: \pi_1(A, a_0) \times \pi_1(B, b_0) & \rightarrow & \pi_1(A \times B, (a_0, b_0)) \\ (x, y) & \rightarrow & i_*(x) \cdot j_*(y) \end{array}$$

is an isomorphism.

Example. It follows from Proposition 6.18 and the above calculations of fundamental groups that

$$\pi_1(S^1 \times L(p, q)) \cong \pi_1(S^1) \times \pi_1(L(p, q)) \cong \mathbb{Z} \times \mathbb{Z}_p.$$

Of course we can also apply Proposition 6.18 several times and we see once again that

$$\pi_1(\underbrace{S^1 \times \cdots \times S^1}_{n \text{ times}}) \cong \underbrace{\pi_1(S^1) \times \cdots \times \pi_1(S^1)}_{n \text{ times}} \cong \underbrace{\mathbb{Z} \times \cdots \times \mathbb{Z}}_{n \text{ times}} \cong \mathbb{Z}^n.$$

Finally it follows from Propositions 4.12 and 6.18 that for any $k, l \geq 2$ we have

$$\pi_1(S^k \times S^l) \cong \pi_1(S^k) \times \pi_1(S^l) \cong 0 \times 0 = 0.$$

In particular we see that the fundamental group cannot distinguish the products of spheres of dimension ≥ 2 . For example at the moment we cannot show that the two 6-dimensional manifolds $S^3 \times S^3$ and $S^2 \times S^2 \times S^2$ are not homeomorphic.

In the proof of Proposition 6.18 we will need the following purely group-theoretic lemma.

Lemma 6.19. *Let $\varphi_1: G_1 \rightarrow \pi$ and $\varphi_2: G_2 \rightarrow \pi$ be two group homomorphisms. If for every $g_1 \in G_1$ and $g_2 \in G_2$ the images $\varphi_1(g_1) \in \pi$ and $\varphi_2(g_2) \in \pi$ commute, i.e. if we have $\varphi_1(g_1) \cdot \varphi_2(g_2) = \varphi_2(g_2) \cdot \varphi_1(g_1)$, then the map*

$$\begin{array}{ccc} \varphi_1 \times \varphi_2: G_1 \times G_2 & \rightarrow & \pi \\ (g_1, g_2) & \mapsto & \varphi_1(g_1) \cdot \varphi_2(g_2) \end{array}$$

is a homomorphism.¹⁴⁷

¹⁴⁷In fact the converse also holds, if $\varphi_1 \times \varphi_2$ is a homomorphism, then for every $g_1 \in G_1$ and $g_2 \in G_2$ the images $\varphi_1(g_1) \in \pi$ and $\varphi_2(g_2) \in \pi$ commute. Why is that?

Proof. We write $\Phi = \varphi_1 \times \varphi_2$. Let $g_1, g'_1 \in G_1$ and let $g_2, g'_2 \in G_2$. We have

$$\begin{aligned} \Phi(g_1, g_2) \cdot \Phi(g'_1, g'_2) &= \varphi_1(g_1) \cdot \varphi_2(g_2) \cdot \varphi_1(g'_1) \cdot \varphi_2(g'_2) \\ &= \varphi_1(g_1) \cdot \varphi_1(g'_1) \cdot \varphi_2(g_2) \cdot \varphi_2(g'_2) = \varphi_1(g_1 g'_1) \cdot \varphi_2(g_2 g'_2) \\ &\quad \uparrow \\ &\quad \text{by our hypothesis} \end{aligned} = \Phi(g_1 g'_1, g_2 g'_2).$$

□

Proof of Proposition 6.18. We need to prove the following three statements.

- (1) The map Φ is a group homomorphism.
- (2) The homomorphism Φ is a monomorphism.
- (3) The homomorphism Φ is an epimorphism.

We start with the proof of the first statement. By Lemma 6.19 and by definition it suffices to show that for any loop $\alpha: [0, 1] \rightarrow A$ in a_0 and any loop $\beta: [0, 1] \rightarrow B$ in b_0 we have

$$i_*([\alpha]) \cdot j_*([\beta]) = j_*([\beta]) \cdot i_*([\alpha]) \in \pi_1(A \times B, (a_0, b_0))$$

or equivalently, that

$$(i \circ \alpha) * (j \circ \beta) \simeq (j \circ \beta) * (i \circ \alpha).$$

We consider the map

$$\begin{aligned} \Psi = \alpha \times \beta: [0, 1] \times [0, 1] &\rightarrow A \times B \\ (x, y) &\mapsto (\alpha(x), \beta(y)). \end{aligned}$$

For $s = \{0, 1\}$ we also consider the maps

$$\begin{aligned} x_s: [0, 1] &\rightarrow [0, 1] \times [0, 1] & \text{and} & & y_s: [0, 1] &\rightarrow [0, 1] \times [0, 1] \\ t &\mapsto (t, s) & & & t &\mapsto (s, t). \end{aligned}$$

We then have

$$\begin{array}{ccccc} \text{since endpoint of } x_0 \text{ equals starting point of } y_1 & & \text{same argument backwards} & & \\ & \downarrow & & \downarrow & \\ (i \circ \alpha) * (j \circ \beta) = (\Psi \circ x_0) * (\Psi \circ y_1) & = & \Psi \circ (x_0 * y_1) & \simeq & \Psi \circ (y_0 * x_1) = (j \circ \beta) * (i \circ \alpha). \\ & \uparrow & & \uparrow & \\ \text{since } i \circ \alpha = \Psi \circ x_0 & & x_0 * y_1 \text{ and } y_0 * x_1 \text{ are both paths in } [0, 1] \times [0, 1] \text{ with} & & \\ \text{and } j \circ \beta = \Psi \circ y_1 & & \text{same starting point } (0, 0) \text{ and same endpoint } (1, 1), & & \\ & & \text{they are homotopic since } [0, 1] \times [0, 1] \text{ is convex} & & \end{array}$$

All the maps in the above argument are illustrated in Figure 76.

The other two statements needed for the proof of the proposition are an exercise in exercise sheet 6. □

Summarizing we have now finally managed to calculate many non-trivial fundamental groups. Nonetheless, the list of spaces for which we cannot yet calculate the fundamental group is still pretty long, it includes

- (1) graphs,
- (2) knot complements,
- (3) the complement of more than one point in $\mathbb{C}$, e.g. $\mathbb{C} \setminus \{0, 1\}$, and

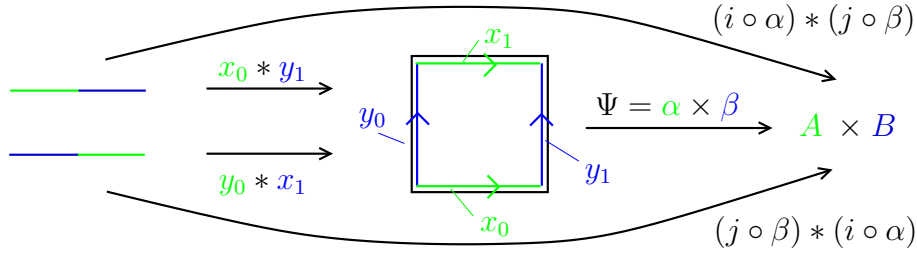


FIGURE 76.

- (4) surfaces of genus ≥ 2 ,
- (5) connected sums of manifolds.

In the following chapters we will introduce more methods for determining the fundamental group of a given topological space. But first we give in the next section two applications of our results.

6.7. Applications: the Fundamental Theorem of Algebra and the Borsuk-Ulam Theorem. We first give an alternative proof of the Fundamental Theorem of Algebra that we had first proved in Analysis III. Somewhat surprisingly this proof does not make any use of complex analysis.

Theorem 6.20. (Fundamental Theorem of Algebra) *Every nonconstant polynomial with coefficients in $\mathbb{C}$ has a zero in $\mathbb{C}$.*

Proof. Let $q(z)$ be a polynomial with complex coefficients with $q(z) \neq 0$ for all $z \in \mathbb{C}$ with $|z| = r$. We consider the path

$$\begin{aligned} f_r^{q(z)} : [0, 1] &\rightarrow S^1 \\ s &\mapsto \frac{q(re^{2\pi is})/q(r)}{|q(re^{2\pi is})/q(r)|} \end{aligned}$$

Note that $f_r^{q(z)}(s=0) = f_r^{q(z)}(s=1) = 1$, i.e. $f_r^{q(z)}$ is a loop in $(S^1, 1)$. Now let

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

be a polynomial of degree $n \geq 1$. We assume that $p(z)$ has no zeros in $\mathbb{C}$. For $t \in [0, 1]$ we also consider the polynomial

$$p_t(z) := a_n z^n + t(a_{n-1} z^{n-1} + \cdots + a_1 z + a_0).$$

Note that $p_1(z) = p(z)$ and $p_0(z) = a_n z^n$. Since $a_n \neq 0$ there exists an $s > 0$ such that¹⁴⁸

$$|a_n z^n| > |a_{n-1} z^{n-1} + \cdots + a_1 z + a_0|$$

¹⁴⁸In fact we can write it down explicitly: it follows easily from the triangle inequality that any $s \in \mathbb{R}$ such that $s \cdot |a_n| > |a_{n-1} + \cdots + a_0|$ will work.

for any z with $|z| = s$. This inequality implies that for any $t \in [0, 1]$ and any z with $|z| = s$ we have $p_t(z) \neq 0$, i.e. $p_t(z)$ has no zeros on the circle $|z| = s$.

We obtain the following sequence of homotopy equivalences of loops:

$$\begin{array}{ccc}
 \text{follows from the definition} & & \text{follows from } p_0(z) = a_n z^n \\
 \downarrow & & \downarrow \\
 \text{constant loop at } 1 & = & f_0^{p(z)} \simeq f_s^{p(z)} = f_s^{p_1(z)} \simeq f_s^{p_0(z)} = \text{the loop } \begin{array}{c} [0, 1] \rightarrow S^1 \\ t \mapsto e^{2\pi i n t} \end{array} \\
 \uparrow & & \uparrow \\
 \text{the maps } f_r^{p(z)} \text{ with } r \in [0, s] \text{ are defined,} & & \text{the maps } f_s^{p_t(z)} \text{ with } t \in [0, 1] \text{ are defined,} \\
 \text{since the polynomials } p(z) \text{ has no zeros} & & \text{since the polynomials } p_t(z) \text{ have no zeros on } |z| = s, \\
 \text{and they form a homotopy from } f_0^{p(z)} \text{ to } f_s^{p(z)} & & \text{and they form a homotopy from } f_s^{p_1(z)} \text{ to } f_s^{p_0(z)}
 \end{array}$$

On page 112 we had seen that the loop on the right defines a non-trivial element in the fundamental group $\pi_1(S^1, 1)$, but we had just shown that it is homotopic to the constant path. Thus we have obtained a contradiction. $\square$

The following theorem goes back to the Polish mathematicians Karol Borsuk (1905-1982) and Stanislaw Ulam (1909-1984).¹⁴⁹ It was first proved in 1932.

Theorem 6.21. (Borsuk-Ulam Theorem) *For every map $f: S^2 \rightarrow \mathbb{R}^2$ there exists a pair of antipodal points x and $-x$ in S^2 with $f(x) = f(-x)$.*

Example.

- (1) It follows from the Borsuk-Ulam Theorem that there exist two antipodal points on earth with the same temperature and the same barometric pressure.
- (2) The Borsuk-Ulam Theorem implies in particular that there is no injective map from $S^2 \rightarrow \mathbb{R}^2$.
- (3) The statement of the Borsuk-Ulam Theorem also makes sense for 2 replaced by any $n \in \mathbb{N}$. For $n = 1$ one can prove the statement using elementary methods.¹⁵⁰ We will address the problem for $n \geq 3$ in Algebraic Topology III.

Proof. We prove the theorem by contradiction. So suppose there exists a map $f: S^2 \rightarrow \mathbb{R}^2$ such that $f(x) \neq f(-x)$ for all $x \in S^2$. This allows to define the map

$$\begin{aligned}
 g: S^2 &\rightarrow S^1 \\
 x &\mapsto \frac{f(x) - f(-x)}{\|f(x) - f(-x)\|}.
 \end{aligned}$$

Note that $g(-x) = -g(x)$ for all $x \in S^2$. Now let η be the loop that is defined by

$$\begin{aligned}
 \eta: [0, 1] &\rightarrow S^2 \\
 s &\mapsto \eta(s) = (\cos(2\pi s), \sin(2\pi s), 0)
 \end{aligned}$$

and let $h = g \circ \eta: [0, 1] \rightarrow S^1$ be the composition of η with g . We pick a $c \in \mathbb{R}$ such that $h(0) = e^{2\pi i c}$. We denote by $p: \mathbb{R} \rightarrow S^1$, $t \mapsto e^{2\pi i t}$, the usual covering map. We note that

¹⁴⁹Ulam became much more famous through his work on the hydrogen bomb.

¹⁵⁰How would you prove the statement?

for $z, w \in \mathbb{R}$ we have

$$p(z) = p(w) \iff z - w \in \mathbb{Z} \quad \text{and} \quad p(z) = -p(w) \iff z - w \in \frac{1}{2} + \mathbb{Z}.$$

By Proposition 6.11 we can lift the path $h: [0, 1] \rightarrow S^1$ to a path $\tilde{h}: [0, 1] \rightarrow \mathbb{R}$ with $\tilde{h}(0) = c$. We refer to Figure 77 for an illustration of all the definitions.

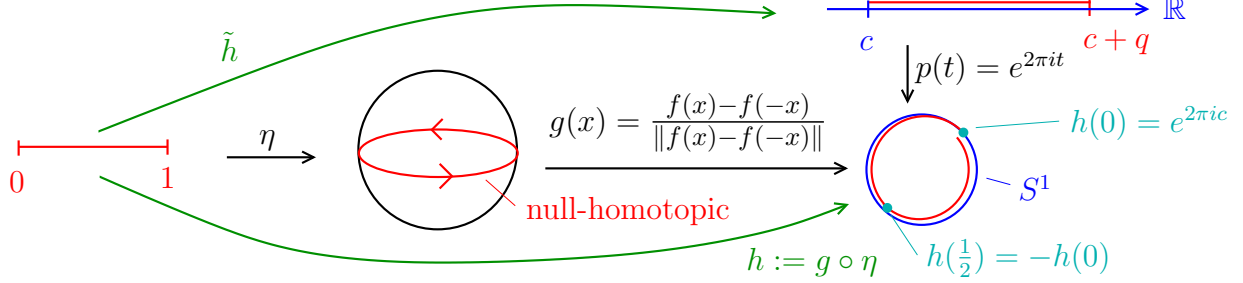


FIGURE 77. Illustration of the Borsuk-Ulam Theorem 6.21

Given any $s \in [0, \frac{1}{2}]$ we have

$$p(\tilde{h}(s + \frac{1}{2})) = h(s + \frac{1}{2}) = g(\eta(s + \frac{1}{2})) = g(-\eta(s)) = -g(\eta(s)) = -h(s) = -p(\tilde{h}(s)).$$

$\uparrow$ by the definition of η $\uparrow$ since $g(-x) = -g(x)$ for all $x \in S^2$

By the above discussion of p this implies that

$$\tilde{h}(s + \frac{1}{2}) - \tilde{h}(s) =: \frac{q(s)}{2}$$

with $q(s)$ an odd integer. The map $q: [0, \frac{1}{2}] \rightarrow \mathbb{Z}$ is the difference of two continuous maps, hence continuous. It follows from Lemma 1.13 that it is constant. We denote this constant value by q . Since q is odd it is in particular non-zero.

It follows from the above that

$$\tilde{h}(1) = \tilde{h}(\frac{1}{2}) + \frac{q}{2} = \tilde{h}(0) + \frac{q}{2} + \frac{q}{2} = \tilde{h}(0) + q.$$

Since q is non-zero we obtain from Lemma 6.15 that $h = g \circ \eta$ is not null-homotopic. But h was the composition $h = g \circ \eta: [0, 1] \rightarrow S^2 \rightarrow S^1$. By Proposition 4.12 the loop η is null-homotopic in S^2 which implies by Proposition 5.1 that $h = g \circ \eta$ is null-homotopic in S^1 . Thus we have obtained a contradiction. $\square$

7. HOMOTOPY EQUIVALENT TOPOLOGICAL SPACES

7.1. Homotopic maps. In this section we will introduce an important category where the morphisms are no longer maps. Before we can do so we first need to introduce the following definition.

Definition. Let $f_0, f_1: X \rightarrow Y$ be two maps between two topological spaces. A *homotopy* between the maps f_0 and f_1 is a map

$$\begin{aligned} F: X \times [0, 1] &\rightarrow Y \\ (x, t) &\mapsto F(x, t) \end{aligned}$$

such that for all $x \in X$ we have

$$F(x, 0) = f_0(x) \quad \text{and} \quad F(x, 1) = f_1(x).$$

If there exists a homotopy between the maps f_0 and f_1 , then we say that f_0 and f_1 are *homotopic* and we write $f_0 \simeq f_1$.

Examples.

- (1) Let $X \subset \mathbb{R}^n$ be a star-shaped subset. Recall that this means that there exists an $x_0 \in X$ such that for all $x \in X$ the segment $x_0 \cdot (1 - t) + x \cdot t, t \in [0, 1]$ lies in X . For example all balls and also all of $\mathbb{R}^n$ are star-shaped. We consider the maps

$$\begin{aligned} f_0: X &\rightarrow X & \text{and} & & f_1 = \text{id}: X &\rightarrow X \\ x &\mapsto x_0 & & & x &\mapsto x. \end{aligned}$$

Put differently, f_0 is the constant map which sends every point to the point x_0 and f_1 is the identity. Then

$$\begin{aligned} F: X \times [0, 1] &\rightarrow X \\ (x, t) &\mapsto \underbrace{x_0 \cdot (1 - t) + x \cdot t}_{\in X \text{ since } X \text{ is star-shaped}} \end{aligned}$$

is a homotopy between f_0 and f_1 .

- (2) In Figure 78 we show four maps α, β, γ and δ from the circle S^1 to the annulus $X = B_2(0) \setminus B_1(0)$. The maps α and β are homotopic, whereas the maps γ and δ look they are not homotopic.



the maps α and β are homotopic

the maps γ and δ appear not to be homotopic

FIGURE 78.

Remark. Let $\gamma_0, \gamma_1: [a, b] \rightarrow X$ be two maps from an interval to a topological space. We just said that the two maps are homotopic if there exists a map $\Gamma: [a, b] \times [0, 1] \rightarrow Y$ such that

$$(i) \quad \Gamma(t, 0) = \gamma_0(t) \quad \text{and} \quad \Gamma(t, 1) = \gamma_1(t) \quad \text{for all } t \in [a, b].$$

We can also view these two maps as paths. If they have the same starting point $\gamma_0(a) = \gamma_1(a) =: P$ and the same endpoint $\gamma_0(b) = \gamma_1(b) =: Q$, then we had said on page 66 that the paths are homotopic if there exists a map $\Gamma: [a, b] \times [0, 1] \rightarrow Y$ that satisfies (i) and that satisfies also

$$(ii) \quad \Gamma(a, s) = P \quad \text{and} \quad \Gamma(b, s) = Q \quad \text{for all } s \in [0, 1].$$

It is somewhat unfortunate that there are two different notions of homotopies. But it should be clear from the context which notion we mean, in particular when we say that “two paths are homotopic” we always mean a homotopy of the latter type.

Sometimes, if we want to avoid confusion, we say

γ_0 and γ_1 are *path-homotopic* if the homotopy fixes the starting point and the endpoint and we say

γ_0 and γ_1 are *freely-homotopic* if the homotopy is not required to fix the starting point and the endpoint.

Clearly two path-homotopic paths are also freely homotopic. But in general the converse does not hold. For example, consider the two paths

$$\begin{array}{ccc} \alpha: [0, 1] & \rightarrow & Y = \mathbb{C} \setminus \{0\} \\ x & \mapsto & e^{2\pi i x} \end{array} \quad \text{and} \quad \begin{array}{ccc} \beta: [0, 1] & \rightarrow & Y = \mathbb{C} \setminus \{0\} \\ x & \mapsto & 1. \end{array}$$

Here α goes counterclockwise once around the origin and β is the constant path. We had seen in Corollary 3.2 that α and β are not path-homotopic. On the other hand they are freely homotopic. Indeed a free homotopy is given by

$$\begin{array}{ccc} F: [0, 1] \times [0, 1] & \rightarrow & Y = \mathbb{C} \setminus \{0\} \\ (x, t) & \mapsto & e^{2\pi i x(1-t)}. \end{array}$$

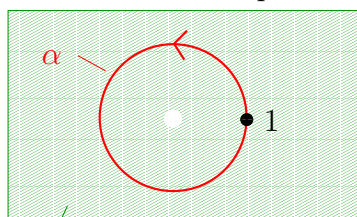
This free homotopy is illustrated in Figure 79.

In Lemma 4.2 we had shown that “homotopy of paths” is an equivalence relation on the set of paths between two points. The same way, with verbatim the same proof, one can prove the following lemma:

Lemma 7.1. *Let X be and Y topological spaces. Then “homotopy of maps” forms an equivalence relation on the set of maps from X to Y .*

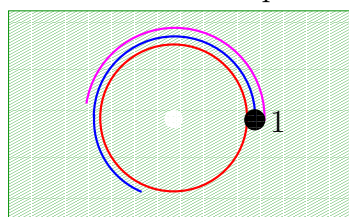
Definition. Given two topological spaces X and Y and a map $f: X \rightarrow Y$ we denote by $[f]$ the equivalence class of f . Furthermore we denote by $[X, Y]$ the set of homotopy equivalence classes of maps from X to Y .

the path α is not path-homotopic
to a constant path



$X = \mathbb{C} \setminus \{0\}$

the path α is freely homotopic
to a constant path



if we do not have to fix the “second” endpoint,
then we can continuously turn α into a constant path

FIGURE 79.

Example. We write $I = [0, 1]$. A slight generalization of the argument in the previous example shows that for any path-connected topological space X the set $[I, X]$ consists of a single element, i.e. all maps $I = [0, 1] \rightarrow X$ are freely homotopic. In exercise sheet 7 we generalize this statement to show that if A is a convex subset of $\mathbb{R}^n$ and if X is a path-connected topological space, then any two maps $f, g: A \rightarrow X$ are homotopic.

We will prove the following lemma in exercise sheet 7.

Lemma 7.2. *Let X, Y and Z topological spaces. If $f, f': X \rightarrow Y$ and $g, g': Y \rightarrow Z$ are homotopic maps, then the map $g \circ f$ is homotopic to $g' \circ f'$.*

We now define the *homotopy category* $\mathcal{H}$ of topological spaces by

$$\begin{aligned} \text{Ob}(\mathcal{H}) &:= \text{all topological spaces,} \\ \text{Mor}(X, Y) &:= [X, Y] \end{aligned}$$

with the composition¹⁵¹

$$\begin{aligned} \text{Mor}(X, Y) \times \text{Mor}(Y, Z) &\rightarrow \text{Mor}(X, Z) \\ ([f], [g]) &\mapsto [g \circ f]. \end{aligned}$$

In this category the morphisms are not maps but they are equivalence classes of maps.

7.2. The fundamental groups of homotopy equivalent topological spaces.

Definition. Let X be and Y topological spaces.

- (1) A map $f: X \rightarrow Y$ is a *homotopy equivalence between X and Y* , if f admits a *homotopy inverse*, i.e. a map $g: Y \rightarrow X$ with $g \circ f \simeq \text{id}_X$ and $f \circ g \simeq \text{id}_Y$.
- (2) If there exists a homotopy equivalence between X and Y , then we say that X and Y are *homotopy equivalent* and we write $X \simeq Y$.

¹⁵¹The fact that this map is well-defined is a consequence of Lemma 7.2.

- (3) We say X is *contractible*, if X is homotopy equivalent to a topological space which consists of precisely one point.

The name “homotopy equivalence” and the notation $\simeq$ already suggests that this is an equivalence relation on the class of topological spaces. This is indeed the case.

Lemma 7.3. *The relation “homotopy equivalence” is an equivalence relation on the class of topological spaces.*

Proof. It is clear that the only property of an equivalence relation that requires any thought is transitivity. If $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ are maps between topological spaces that admit homotopy inverses $f': Y \rightarrow X$ and $g': Z \rightarrow Y$, then it is straightforward to verify that $f' \circ g'$ is a homotopy inverse to $g \circ f$. $\square$

We will now give a long list of examples of homotopy equivalent spaces.

Examples.

- (1) Let X be a star-shaped subset of $\mathbb{R}^n$ and let $x_0 \in X$ be a point such that for any $x \in X$ the segment from x_0 to x lies in X . We consider the maps

$$\begin{array}{ccc} f: X & \rightarrow & \{x_0\} \\ x & \mapsto & x_0 \end{array} \quad \text{and} \quad \begin{array}{ccc} g: \{x_0\} & \rightarrow & X \\ x_0 & \mapsto & x_0. \end{array}$$

Then $f \circ g = \text{id}_{\{x_0\}}$ and we had just seen on page 123 that $g \circ f \simeq \text{id}_X$. Thus we showed that any star-shaped subspace of $\mathbb{R}^n$, e.g. any convex subset of $\mathbb{R}^n$ or $\mathbb{R}^n$ itself, is homotopy equivalent to a point, i.e. it is contractible.

- (2) It follows from the previous example together with Lemma 7.3 that for any $n \in \mathbb{N}$ and any $k \in \mathbb{N}$ the topological spaces $\mathbb{R}^n$ and $\mathbb{R}^k$ are homotopy equivalent.
- (3) We consider the Möbius band

$$X = [0, 1] \times [-1, 1] / (0, y) \sim (1, -y)$$

and the circle

$$Y = [0, 1] / 0 \sim 1.$$

We consider the maps

$$\begin{array}{ccc} f: X & \rightarrow & Y \\ [(x, y)] & \mapsto & [x] \end{array} \quad \text{and} \quad \begin{array}{ccc} g: Y & \rightarrow & X \\ [x] & \mapsto & [(x, 0)]. \end{array}$$

Then $f \circ g = \text{id}_Y$ and furthermore we have $g \circ f \simeq \text{id}_X$ as can be seen using the following homotopy

$$\begin{array}{ccc} F: X \times [0, 1] & \rightarrow & X \\ ([x, y], t) & \mapsto & [(x, y \cdot (1 - t))]. \end{array}$$

The Möbius band is thus homotopy equivalent to the circle.

Remark. If X and Y are homotopy equivalent topological spaces, then it is straightforward to verify that X is path-connected if and only if Y is path-connected. We leave the verification of this statement as an exercise to the reader.

Definition. We say that a subset A of a topological space X is a *deformation retract* of X , if there exists a *deformation retraction* from X to A , i.e. a map $F: X \times [0, 1] \rightarrow X$ with the following properties:

- (1) for every $x \in X$ we have $F(x, 0) = x$ and $F(x, 1) \in A$,
- (2) for every $a \in A$ we have $F(a, t) = a$ for all $t \in [0, 1]$.

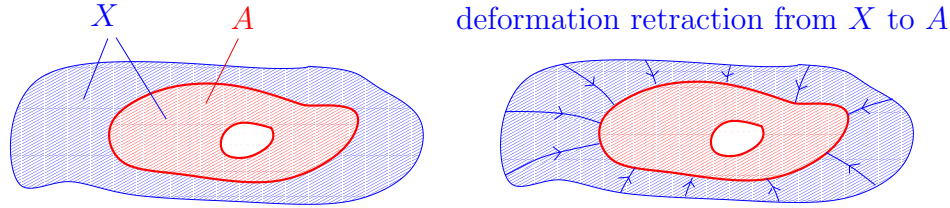


FIGURE 80. Schematic picture of a deformation retraction.

Examples.

- (1) For any topological space X and any interval $I \subset \mathbb{R}$ containing 0 the subspace $X \times \{0\}$ is a deformation retract of $X \times I$. In fact the deformation retraction is given by

$$\begin{aligned} (X \times I) \times [0, 1] &\rightarrow X \times I \\ ((x, s), t) &\mapsto (x, s \cdot (1 - t)). \end{aligned}$$

- (2) In exercise sheet 7 we will see that for any $n \in \mathbb{N}$ the sphere S^{n-1} is a deformation retract of $\mathbb{R}^n \setminus \{0\}$.
- (3) It follows easily from the previous examples that the central circle $[0, 1] \times \{0\}$ of the Möbius band is a deformation retract of the Möbius band $[0, 1] \times [-1, 1] / \sim$.

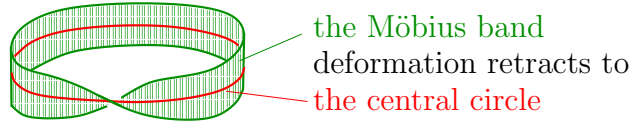


FIGURE 81.

In exercise sheet 7 we will prove the following elementary lemma.

Lemma 7.4. *Let A be a subset of a topological space X which admits a deformation retraction F from X to A . Then the map*

$$\begin{aligned} X &\rightarrow A \\ x &\mapsto F(x, 1) \end{aligned}$$

is a homotopy equivalence. In particular the deformation retract A is homotopy equivalent to the total space X .

The lemma often gives a convenient way for showing that two spaces are homotopy equivalent. For example the lemma, together with the previous example implies that the sphere S^{n-1} and $\mathbb{R}^n \setminus \{0\}$ are homotopy equivalent.

Proposition 7.5. *Let $f: X \rightarrow Y$ be a map between topological spaces which is a homotopy equivalence. Let $x_0 \in X$ be a point. We put $y_0 := f(x_0)$. Then the induced map*

$$f_*: \pi_1(X, x_0) \rightarrow \pi_1(Y, y_0)$$

is an isomorphism.

Example. It follows from Lemma 7.4, Proposition 7.5, Corollary 6.17, Proposition 4.12 and the previous example that for any $n \geq 2$ and any $Q \in \mathbb{R}^n$ we have

$$\pi_1(\mathbb{R}^n \setminus \{Q\}) \cong \pi_1(\mathbb{R}^n \setminus \{0\}) \cong \begin{cases} \mathbb{Z}, & \text{if } n = 2, \\ 0, & \text{if } n \geq 3. \end{cases}$$

Proof. Let $f: X \rightarrow Y$ be a map between topological spaces which is a homotopy equivalence. Let $x_0 \in X$ be a point. We put $y_0 = f(x_0)$. We pick a homotopy inverse $g: Y \rightarrow X$ of f . We put $x_1 = g(y_0)$. According to our hypothesis we have $\text{id}_X \simeq g \circ f$, i.e. there exists a homotopy H between id_X and $g \circ f$.¹⁵²

We denote by

$$\begin{aligned} \alpha: [0, 1] &\rightarrow X \\ t &\mapsto H(x_0, t) \end{aligned}$$

the path with which the homotopy H connects the points

$$H(x_0, 0) = (\text{id}_X)(x_0) = x_0 \quad \text{and} \quad H(x_0, 1) = (g \circ f)(x_0) = x_1.$$

Claim. The maps

$$\begin{aligned} (g \circ f)_*: \pi_1(X, x_0) &\rightarrow \pi_1(X, x_1) \\ [s] &\mapsto [(g \circ f) \circ s] \end{aligned} \quad \text{and} \quad \begin{aligned} \pi_1(X, x_0) &\rightarrow \pi_1(X, x_1) \\ [s] &\mapsto [\bar{\alpha} * s * \alpha] \end{aligned}$$

are identical.

Let $s: [0, 1] \rightarrow X$ be a loop in (X, x_0) . We want to show that the loops $(g \circ f) \circ s$ and $\bar{\alpha} * s * \alpha$ are path-homotopic. We consider the map¹⁵³

$$G: [0, 1] \times [0, 1] \rightarrow X$$

$$(r, t) \mapsto \begin{cases} \bar{\alpha}(4r), & \text{if } r \in [0, \frac{1}{4}t], \\ H\left(s\left(\frac{r - \frac{1}{4}t}{1 - \frac{1}{2}t}\right), 1 - t\right), & \text{if } r \in (\frac{1}{4}t, 1 - \frac{1}{4}t), \\ \alpha(4r - 3), & \text{if } r \in [1 - \frac{1}{4}t, 1]. \end{cases}$$

¹⁵²Can we already conclude from this fact that $g \circ f$ induces the identity map on fundamental groups? Unfortunately not, for example $g \circ f$ does not even induce a map on $\pi_1(X, x_0)$, since $g \circ f$ sends the base point x_0 to the point x_1 , which in general is different from x_0 .

¹⁵³The map G is defined as follows: for $t \in [0, 1]$ the loop $r \mapsto G(r, t)$ is given by traveling along $\bar{\alpha}$ from x_1 to $\bar{\alpha}(t) = \alpha(1 - t)$, then one follows the loop given by $H(s(-), t)$, and then one returns along α back to x_1 .

It is straightforward to verify that G is indeed a continuous map with

$$G(0, t) = G(1, t) = x_1 \quad \text{for all } t \in [0, 1]$$

and which satisfies

$$G(r, 0) = ((g \circ f) \circ s)(r) \quad \text{and} \quad G(r, 1) = (\bar{\alpha} * s * \alpha)(r) \quad \text{for all } r \in [0, 1].$$

This shows that G is in fact a path-homotopy between the loops $(g \circ f) \circ s$ and $\bar{\alpha} * s * \alpha$ ¹⁵⁴

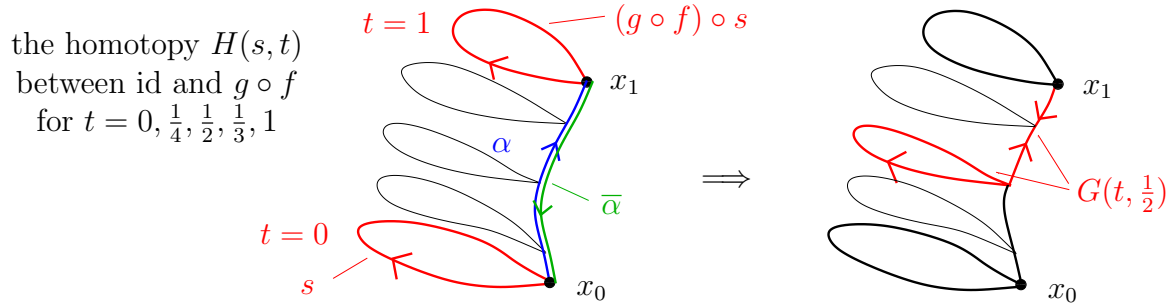


FIGURE 82. Schematic image of H and G .

in (X, x_1) . Thus we have proved the claim.

We consider the following diagram:

$$\begin{array}{ccc} \pi_1(X, x_0) & \xrightarrow{[s] \mapsto [\bar{\alpha} * s * \alpha]} & \pi_1(X, x_1) \\ \downarrow f_* & \searrow g_* & \downarrow f_* \\ \pi_1(Y, y_0) & \xrightarrow{[t] \mapsto [(f \circ \bar{\alpha}) * t * (f \circ \alpha)]} & \pi_1(Y, y_1). \end{array}$$

We make the following observations:

- (1) By the claim and Proposition 4.9 the map $g_* \circ f_*: \pi_1(X, x_0) \rightarrow \pi_1(X, x_1)$ is an isomorphism.
- (2) This implies that the map $f_*: \pi_1(X, x_0) \rightarrow \pi_1(Y, y_0)$ is a monomorphism.
- (3) If we replace the roles of f and g , then the argument of the claim and Proposition 4.9 shows that the map $f_* \circ g_*: \pi_1(Y, y_0) \rightarrow \pi_1(Y, y_1)$ is also an isomorphism.
- (4) This implies that the map $f_*: \pi_1(X, x_1) \rightarrow \pi_1(Y, y_1)$ is an epimorphism.
- (5) We consider the diagram

$$\begin{array}{ccc} \pi_1(X, x_0) & \xrightarrow{[s] \mapsto [\bar{\alpha} * s * \alpha]} & \pi_1(X, x_1) \\ \downarrow f_* & \cong & \downarrow f_* \\ \pi_1(Y, y_0) & \xrightarrow{[t] \mapsto [(f \circ \bar{\alpha}) * t * (f \circ \alpha)]} & \pi_1(Y, y_1). \end{array}$$

¹⁵⁴To be precise, G is a homotopy between $(g \circ f) \circ s$ and a suitable parametrization of $\bar{\alpha} * s * \alpha$.

The horizontal maps are isomorphisms by Proposition 4.9. It follows immediately from the definitions of the maps that the diagram commutes.

- (6) Since the right vertical map is an epimorphism and since the horizontal maps are isomorphisms we now see that the left vertical map is also an epimorphism,
- (7) The combination of (2) and (6) shows that the left vertical map is an isomorphism.

□

In Lemma 1.12 we had already seen that the topological spaces $\mathbb{R}$ and $\mathbb{R}^2$ are not homeomorphic. Now we can prove a similar statement in one dimension higher.

Lemma 7.6. *The topological spaces $\mathbb{R}^2$ and $\mathbb{R}^3$ are not homeomorphic.*

Unfortunately we are still not able to settle the question whether for $k \neq l \geq 3$ the topological spaces $\mathbb{R}^k$ and $\mathbb{R}^l$ can be homeomorphic. Before we can settle this question we will have to develop a completely new set of tools.

Proof. Let us suppose that there exists a homeomorphism $f: \mathbb{R}^2 \rightarrow \mathbb{R}^3$. Let $P \in \mathbb{R}^2$ be a point. We put $Q := f(P)$. Then f restricts to a homeomorphism from $\mathbb{R}^2 \setminus \{P\}$ to $\mathbb{R}^3 \setminus \{Q\}$. But then we obtain that

$$\begin{array}{ccccc} \mathbb{Z} & \cong & \pi_1(\mathbb{R}^2 \setminus \{P\}) & \cong & \pi_1(\mathbb{R}^3 \setminus \{Q\}) & = & 0. \\ \uparrow & & \uparrow & & \uparrow & & \\ \text{page 128} & & \text{since } f \text{ is a homeomorphism} & & \text{page 128} & & \end{array}$$

Thus we obtained a contradiction.

□

7.3. The wedge of two topological spaces.

Definition. Let A and B be two topological spaces and let $a \in A$ and $b \in B$ be two points. We then write¹⁵⁵

$$A \vee B := A \sqcup B / a = b.$$

The topological space $A \vee B$ is called the *wedge of A and B* .

Remark. The notation in the definition of the wedge of two topological spaces does not mention that the definition depends on the choice of $a \in A$ and $b \in B$. If A or B is disconnected, then it is easy to see in examples that in general the definition of the wedge $A \vee B$ depends on the choice of component from which we take a or b .

But fortunately in many situations the homeomorphism type of $A \vee B$ does not depend on the choices of a and b . For example, if A and B are connected manifolds, then it follows from Proposition 2.5 that the homeomorphism type of $A \vee B$ does not depend on the choices of a and b .

Examples.

- (1) The space $A \vee B$ should be viewed as the result of gluing A and B along the points a and b . The wedge of two circles is shown in Figure 83.

¹⁵⁵More precisely, $A \vee B = A \sqcup B / \sim$ where $\sim$ is the equivalence relation generated by $a = b$.

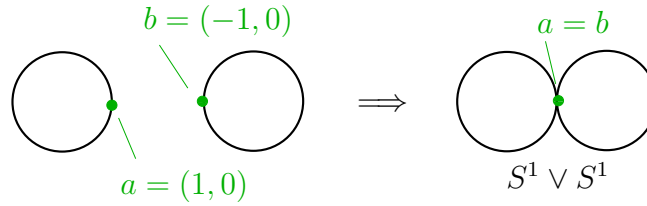


FIGURE 83. The wedge of two circles.

- (2) If G is a connected topological graph with one vertex and n edges, then it is straightforward to see that G is homeomorphic to the wedge of n circles.
- (3) In exercise sheet 7 we will see that the wedge of two circles is homotopy equivalent to $\mathbb{C} \setminus \{\pm 1\}$. More precisely, we will show that

$$S^1 \vee S^1 = \underbrace{\{z \in \mathbb{C} \mid |z + 1| = 1\} \cup \{z \in \mathbb{C} \mid |z - 1| = 1\}}_{\text{two circles that meet at the origin}}$$

is a deformation retract of $\mathbb{C} \setminus \{\pm 1\}$.

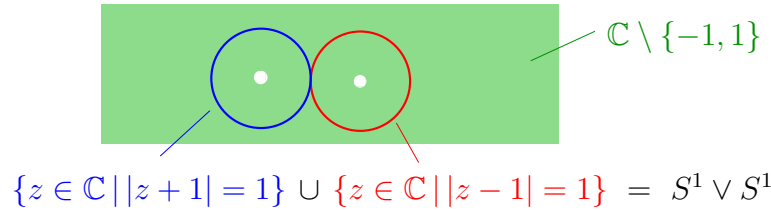


FIGURE 84.

We conclude the discussion of homotopy equivalences with another look at graphs.

Definition. Let $G = (V, E)$ be a finite graph, i.e. a graph such that V and E are finite.¹⁵⁶

- (1) We refer to

$$\chi(G) = \#V - \#E = \text{number of vertices} - \text{number of edges}$$

as the *Euler characteristic* of G .

- (2) A *tree* is a finite connected graph with Euler characteristic 1.
- (3) Given a vertex v and an edge e we write

$$f(v, e) := \begin{cases} 0, & \text{if no endpoint of } e \text{ equals } v, \\ 1, & \text{if } e \text{ has two endpoints, one of which equals } v, \\ 2, & \text{if } e \text{ has one endpoint and this one endpoint equals } v. \end{cases}$$

¹⁵⁶Note that the subsequent definition makes sense for abstract graphs and for topological graphs.

(4) The *valence* of a vertex v is defined as

$$\text{valence}(v) := \sum_{\text{edge } e} f(v, e).$$

In a slightly less formal way, the valence of a vertex v is the number of edges at v , but hereby we have to count any loop at v twice, since it “goes twice into v ”.

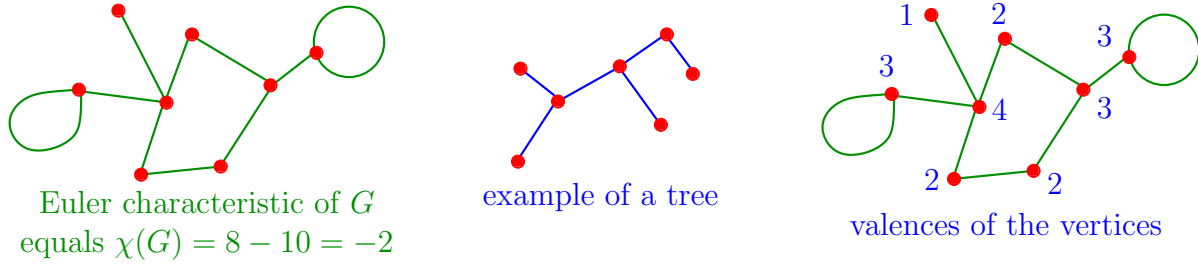


FIGURE 85. Examples of trees.

Proposition 7.7. *Let T be a topological graph that is a tree. Then the following hold:*

- (1) *If T has at least one edge, then it admits a vertex of valence one.*
- (2) *The tree T is contractible.*
- (3) *The fundamental group $\pi_1(T)$ is trivial.*

Proof. Let T be a tree. Then the following hold:

- (1) We suppose that T has at least one edge. Note that a vertex with valence 0 is an isolated point. Since T has at least one edge and since it is by definition connected, we see that the tree T does not have a vertex of valence 0. Furthermore we have

$$\sum_{\text{vertex } v} \text{valence}(v) = \sum_{\text{vertex } v} \sum_{\text{edge } e} f(v, e) = \sum_{\text{edge } e} \underbrace{2}_{\text{edge each contributes 2 to the sum}} = \sum_{\text{edge } e} 2 = 2 \cdot \# \text{ edges} = 2 \cdot \# \text{ vertices} - 2.$$

since $\chi(T) = 1$

This equality can only hold if there is at least one vertex with valence less than 2. By the above the valence has to be at least one, so there exists a vertex of valence one.

- (2) We prove that any tree is contractible by induction on the number of edges. If a tree has zero edges, then it consists of precisely one point, in particular it is even homeomorphic to a point. Now suppose that we know that all trees with n edges are contractible. Let $T = (V, E)$ be a tree with $n + 1$ edges. By (1) it admits a vertex v of valence one. We denote by e the unique edge with v as an endpoint. We write $T' = (V \setminus \{v\}, E \setminus e)$. We reduced the number of vertices and of edges by one, so the Euler characteristic is unchanged, i.e. $\chi(T') = \chi(T) = 1$, so T' is again a tree.

We denote by u the other endpoint of e . We consider the maps

$$f: T \rightarrow T' \quad \text{and} \quad g: T' \rightarrow T$$

$$x \mapsto \begin{cases} u, & \text{if } x \in \bar{e}, \\ x, & \text{otherwise.} \end{cases} \quad \text{and} \quad x \mapsto x.$$

We claim that g is a homotopy inverse to f . Evidently $f \circ g = \text{id}_{T'}$. We need to show that $g \circ f \simeq \text{id}_T$. We pick a homeomorphism $\varphi: [0, 1] \rightarrow \bar{e}$ with $\varphi(0) = u$ and $\varphi(1) = v$. Then

$$H: T \times [0, 1] \rightarrow T$$

$$(x, t) \mapsto \begin{cases} \varphi(\varphi^{-1}(x) \cdot t), & \text{if } x \in \bar{e}, \\ x, & \text{otherwise.} \end{cases}$$

is a homotopy from $g \circ f$ to id_T .

We have thus shown that T is homotopy equivalent to T' . By our induction hypothesis T' is homotopy equivalent to a point. So it follows from Lemma 7.3 that T itself is also homotopy equivalent to a point, i.e. T is contractible.

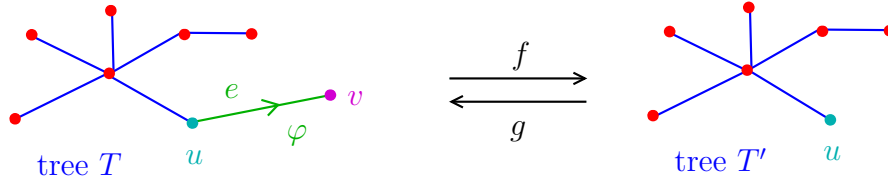


FIGURE 86. Illustration for the proof that every tree is contractible.

(3) It follows from (2) and Proposition 7.5 that the fundamental group of T is trivial. $\square$

We now provide a few more definitions on graphs.

Definition. Let $G = (V, E)$ be a graph.

- (1) A *subgraph* of $G = (V, E)$ is a graph $G' = (V', E')$ such that $V' \subset V$ and such that E' is a union of components of E .
- (2) We say an edge e is *adjacent to a subgraph* G' if e has two endpoints where one of the endpoints lies on G' and the other does not.
- (3) A *maximal tree* for G is a subgraph that is a tree and which does not admit an adjacent edge.

Remark.

- (1) If an edge e is adjacent to a subgraph G' , then it is not part of the subgraph. Indeed, by definition of a subgraph, for any edge that is part of the subgraph also its endpoints have to lie on the subgraph.

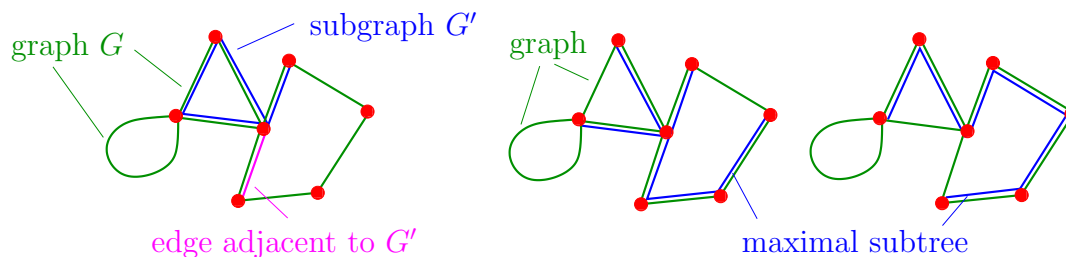


FIGURE 87.

- (2) The example of Figure 87 shows that a graph usually has many different maximal subtrees.
- (3) A maximal tree is often also called a *spanning tree*.

We conclude this discussion on graphs with the following proposition.

Proposition 7.8. *Let $G = (V, E)$ be a finite connected topological graph.*

- (1) *The graph G admits a maximal tree.*
- (2) *Every maximal tree of G contains all vertices of G .*
- (3) *The graph G is homotopy equivalent to the wedge of $1 - \chi(G)$ circles.*
- (4) *Two finite connected topological graphs with the same Euler characteristic are homotopy equivalent.*

We will study later the question, whether the converse to statement (4) holds: if two finite connected graphs are homotopy equivalent, does it follow that the Euler characteristics are the same?

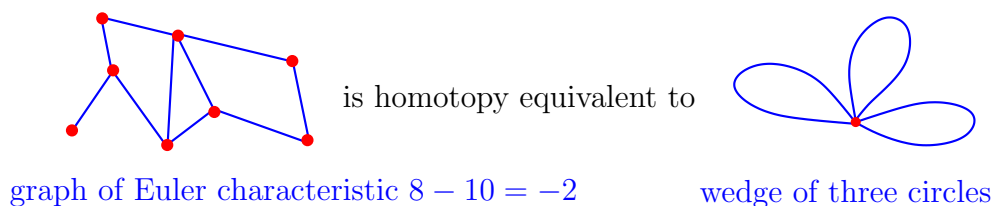


FIGURE 88.

We only provide a sketch of the proof.

Sketch of the proof. Let $G = (V, E)$ be a finite connected topological graph.

- (1) We pick a vertex v . We define $T_0 = \{v\}$ and view it as a subtree of G . We now define a sequence of trees $T_0, \dots, T_k$ by iteratively attaching an adjacent edge to the previous tree. This process stops after finitely many steps since G is finite. We always add an edge and a vertex, thus we see that $\chi(T_k) = \dots = \chi(T_0) = 1$, i.e. T_k is a tree. By definition it is in fact a maximal tree. This argument is illustrated in Figure 89 on the left.

- (2) Let T be a subgraph of G that is a tree. Suppose that there exists a vertex w that does not lie in T . We will show that T is not maximal. Let v be a vertex in T . Since G is connected there exists a sequence of edges $e_0, \dots, e_n$ with the following properties:
- (a) v is a vertex of e_0 ,
 - (b) for each $i \in \{0, \dots, n-1\}$ a vertex of e_i is also a vertex of e_{i+1} , and
 - (c) w is a vertex of e_n .

Then there exists an $i \in \{0, \dots, n-1\}$ such that a vertex of e_i lies in T and the other one does not. But then e_i is an edge adjacent to T . This shows that T was not maximal. This argument is illustrated in Figure 89 on the right.

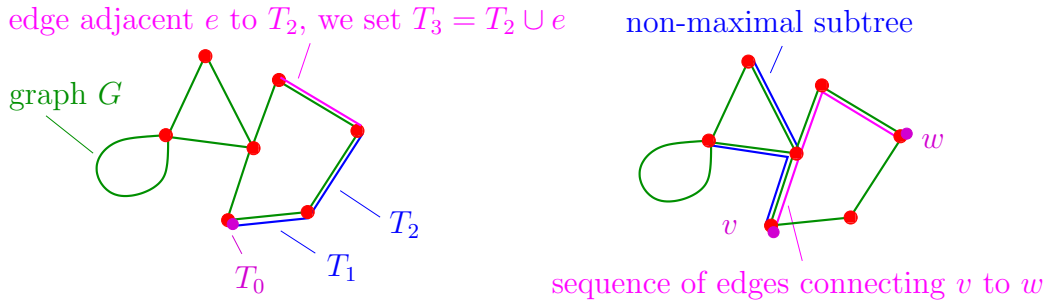


FIGURE 89. Illustration for the proofs of the first two statements of Proposition 7.8.

- (3) Let G be a finite topological graph. We write $n = 1 - \chi(G)$. We prove the statement by induction on the number of edges in a maximal tree. If a maximal tree has zero edges, then it consists of a single vertex. By (2) this means that the original graph has only one vertex and it has n edges. It follows from the discussion on page 131 that the graph is in fact homeomorphic to the wedge of n circles.

So suppose that we know the desired statement whenever the maximal tree has k edges. Now suppose that G has a maximal tree T with $k+1$ edges.

By Proposition 7.7 the tree $T = (W, F)$ has a vertex v of valence one. We denote by e the one edge in T such that v is an endpoint of e . We denote by u the other endpoint of v . By the definition of an edge there exists a homeomorphism $\varphi: [0, 1] \rightarrow \bar{e}$ such that $\varphi(0) = u$ and $\varphi(1) = v$. We denote by $f_1, \dots, f_k$ the edges in $G \setminus T$ that have v as an endpoint.

We first consider the case that for $j = 1, \dots, k$ the edge f_j has another endpoint w_j . We pick homeomorphisms $\psi_j: [0, 1] \rightarrow \bar{f}_j$, $j = 1, \dots, k$ such that $\psi_j(0) = v$ and $\psi_j(1) = w_j$. We now denote by G' a graph that has the vertices $V \setminus \{v\}$, the edges $E \setminus \{e, f_1, \dots, f_k\}$ and which has new edges f'_j , $j = 1, \dots, k$ with endpoints u and w_j .¹⁵⁷ We pick homeomorphisms $\psi'_j: [0, 1] \rightarrow \bar{f}'_j$, $j = 1, \dots, k$ such that $\psi'_j(0) = u$

¹⁵⁷At that point one has to argue why such a graph G' exists. One way of seeing this is to take the graph $G \subset \mathbb{R}^n$ and view it as a graph of $G \subset \mathbb{R}^{n+1}$. In $\mathbb{R}^{n+1}$ we can connect the points u and w_j by disjoint edges, without intersecting the previous graph.

and $\psi'_j(1) = w_j$. We consider the maps¹⁵⁸

$$g: G \rightarrow G' \quad \quad \quad g': G' \rightarrow G$$

$$x \mapsto \begin{cases} \psi'_j(\psi_j^{-1}(x)), & \text{if } x \in \bar{f}_j, \\ u, & \text{if } x \in \bar{e}, \\ x, & \text{otherwise.} \end{cases} \quad \text{and} \quad x \mapsto \begin{cases} \varphi(2t), & \text{if } x = \psi'_j(t), t \in [0, \frac{1}{2}], \\ \psi_j(2t - 1), & \text{if } x = \psi'_j(t), t \in [\frac{1}{2}, 1], \\ x, & \text{otherwise.} \end{cases}$$

We leave it as an exercise to show that $g \circ g': G' \rightarrow G'$ and $g' \circ g: G \rightarrow G$ are homotopic to the identity map.¹⁵⁹ This shows that G is homotopy equivalent to G' for which $T' = (W \setminus \{v\}, F \setminus e)$ is a maximal tree. Since T' has one edge less than T it follows from our induction hypothesis that G' is homotopy equivalent to the wedge of $1 - \chi(G') = 1 - \chi(G) = n$ circles. It follows from Lemma 7.3 that G itself is homotopy equivalent to the wedge of n circles.

The case that there are edges in $G \setminus E$ for which v is the only endpoint is dealt with in a very similar way. We leave this case as an exercise to the reader. A related statement will be proved in exercise sheet 7.

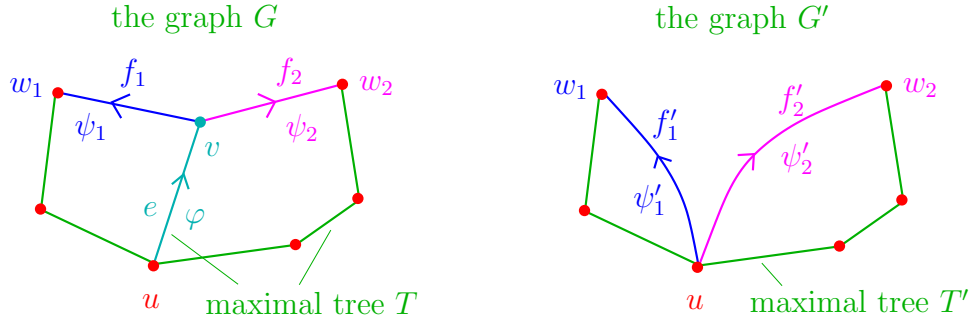


FIGURE 90. Illustration for the proof of the third two statement of Proposition 7.8.

- (4) The last statement is an immediate consequence of (3) and Lemma 7.3. □

¹⁵⁸One can easily verify that these maps are in fact continuous.

¹⁵⁹Note that if $n = 0$, then this is precisely the construction from the proof of Proposition 7.7.

8. BASICS OF GROUP THEORY

Before we continue with developing the theory of fundamental groups of topological spaces we want to introduce several basic constructions in group theory and we want state several facts which we will use throughout the remainder of this course.

8.1. Free abelian groups and finitely generated abelian groups. Let S be a non-empty set. We write

$$\mathbb{Z}^S := \text{all maps from } S \text{ to } \mathbb{Z}.$$

Furthermore, for S the empty set we write $\mathbb{Z}^S := \{0\}$. We consider $\mathbb{Z}^S$ with the obvious group structure, i.e. the group structure that is given by

$$\begin{aligned} \mathbb{Z}^S \times \mathbb{Z}^S &\rightarrow \mathbb{Z}^S \\ (f, g) &\mapsto \left(\begin{array}{c} S \rightarrow \mathbb{Z} \\ s \mapsto f(s) + g(s) \end{array} \right). \end{aligned}$$

In the following, given any set S , we are mostly interested in the subgroup

$$\mathbb{Z}^{(S)} := \text{all maps from } S \text{ to } \mathbb{Z} \text{ which are non-zero for only finitely many } s \in S.$$

We refer to $\mathbb{Z}^{(S)}$ as the *free abelian group generated by S* or as the *free abelian group on the generating set S* . We refer to the cardinality of S as the *rank of $\mathbb{Z}^{(S)}$* . We will use the following convention. Given $s \in S$ we denote by s also the map

$$\begin{aligned} S &\rightarrow \mathbb{Z} \\ t &\mapsto \begin{cases} 1, & \text{if } t = s, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

This map lies of course in $\mathbb{Z}^{(S)}$. Thus we can view S as a subset of $\mathbb{Z}^{(S)}$. Furthermore given $s_1, \dots, s_k \in S$ and $n_1, \dots, n_k \in \mathbb{Z}$ we obtain the corresponding element $n_1 s_1 + \dots + n_k s_k$ of $\mathbb{Z}^{(S)}$. Sometimes $n_1 s_1 + \dots + n_k s_k$ is called a *formal linear combination* of elements in S . Every element of $\mathbb{Z}^{(S)}$ can be written as such a formal linear combination of finitely many elements of S . Indeed, for $f \in \mathbb{Z}^{(S)}$ we can write¹⁶⁰

$$f := \sum_{s \in S \text{ with } f(s) \neq 0} f(s) \cdot s.$$

The free abelian group generated by S has the following “universal property”:

Lemma 8.1. *Let S be a set and let $\phi: S \rightarrow G$ be a map to an abelian group G . Then there exists a unique homomorphism $\psi: \mathbb{Z}^{(S)} \rightarrow G$ that makes the following diagram commute*

$$\begin{array}{ccc} S & \hookrightarrow & \mathbb{Z}^{(S)} \\ & \searrow \phi & \downarrow \psi \\ & & G. \end{array}$$

¹⁶⁰It is straightforward to see that both sides define the same map $S \rightarrow \mathbb{Z}$, so they are the same element of $\mathbb{Z}^{(S)}$.

Proof. We consider the map¹⁶¹

$$\begin{aligned} \psi: \mathbb{Z}^{(S)} &\rightarrow G \\ f = \sum_{i=1}^n n_i s_i &\mapsto \psi(f) := \sum_{i=1}^n n_i \phi(s_i). \end{aligned}$$

It is straightforward to verify that ψ has the desired property and that it is the unique such homomorphism. $\square$

Remark. Let S be a set. Sometimes in the literature a *free abelian group on the generating set S* is defined as a group π together with a map $\varphi: S \rightarrow \pi$ which has the following universal property: given any map $\phi: S \rightarrow G$ to an abelian group G there exists a unique homomorphism $\psi: \pi \rightarrow G$ that makes the following diagram commute

$$\begin{array}{ccc} S & \xrightarrow{\quad} & \pi \\ & \searrow \phi & \downarrow \psi \\ & & G. \end{array}$$

In this language Lemma 8.1 can be reinterpreted as saying that a free abelian group on the generating set S exists. The “usual universal property” argument shows that any two free abelian groups on the generating set S are isomorphic. More precisely, let $\varphi': S \rightarrow \pi'$ be another free abelian group on the generating set S . By the universal property of φ and φ' there exist homomorphisms $\psi: \pi \rightarrow \pi'$ and $\psi': \pi' \rightarrow \pi$ that make the following diagrams commute:

$$\begin{array}{ccc} S & \xrightarrow{\varphi} & \pi \\ & \searrow \varphi' & \downarrow \psi \\ & & \pi' \end{array} \quad \text{and} \quad \begin{array}{ccc} S & \xrightarrow{\varphi'} & \pi' \\ & \searrow \varphi & \downarrow \psi' \\ & & \pi. \end{array}$$

In particular we get the commutative diagrams

$$\begin{array}{ccc} S & \xrightarrow{\varphi} & \pi \\ & \searrow \varphi' & \downarrow \psi \\ & & \pi' \\ & \searrow \varphi & \downarrow \psi' \\ & & \pi \end{array} \quad \text{which simplifies to} \quad \begin{array}{ccc} S & \xrightarrow{\varphi} & \pi \\ & \searrow \varphi & \downarrow \psi' \circ \psi \\ & & \pi \end{array} \quad \text{and of course we also have} \quad \begin{array}{ccc} S & \xrightarrow{\varphi} & \pi \\ & \searrow \varphi & \downarrow \text{id}_\pi \\ & & \pi. \end{array}$$

It follows from the uniqueness part of the universal property applied to the two diagrams on the right that $\psi' \circ \psi = \text{id}_\pi$. The same way one shows that $\psi \circ \psi' = \text{id}_{\pi'}$. In particular ψ and ψ' are isomorphisms.

Definition. We say that a group G is a *free abelian group*, if it is isomorphic to a free abelian group $\mathbb{Z}^{(S)}$ for some set S .

¹⁶¹Hereby we assume that G is written as an additive group. If G is a multiplicative group we would need to replace $n_i \cdot \phi(s_i)$ by $\phi(s_i)^{n_i}$.

Lemma 8.2.

- (1) Any subgroup G of a free abelian group F is again a free abelian group.
- (2) If $\varphi: \mathbb{Z}^{(S)} \rightarrow \mathbb{Z}^{(T)}$ is a monomorphism, then¹⁶² $\#S \leq \#T$.
- (3) If $\varphi: \mathbb{Z}^{(S)} \rightarrow \mathbb{Z}^{(T)}$ is an epimorphism, then $\#S \geq \#T$.

Proof. The first two statements are proved in [Rt, Theorem 9.3]. Now let $\varphi: \mathbb{Z}^{(S)} \rightarrow \mathbb{Z}^{(T)}$ be an epimorphism. For each $t \in T$ there exists therefore some $\phi(t) \in S$ with $\varphi(\phi(t)) = t$. By Lemma 8.1 there exists a unique homomorphism $\psi: \mathbb{Z}^{(T)} \rightarrow \mathbb{Z}^{(S)}$ with $\psi(t) = \phi(t)$ for all $t \in T$. Thus we have $\varphi(\psi(t)) = \varphi(\phi(t)) = t$ for all t , so $\varphi \circ \psi = \text{id}$, in particular ψ is a monomorphism. It follows from (2) that $\#T \leq \#S$. $\square$

Definition.

- (1) If F is a free abelian group, then we define the *rank* of F as the cardinality of any set S with $F \cong \mathbb{Z}^{(S)}$.¹⁶³
- (2) Given an abelian group G we say that $S \subset G$ is a *generating set* if the canonical homomorphism $\mathbb{Z}^{(S)} \rightarrow G$ is an epimorphism.¹⁶⁴
- (3) Given an abelian group G we say that $S \subset G$ is a *basis* if the canonical homomorphism $\mathbb{Z}^{(S)} \rightarrow G$ is an isomorphism.
- (4) We say that an abelian group G is *finitely generated* if it admits a finite generating set.

Example. Let S be a finite set with m elements $s_1, \dots, s_m$. Let $\phi: S \rightarrow \mathbb{Z}^m$ be the map given by $\phi(s_i) = e_i$. Then the resulting homomorphism $\Psi: \mathbb{Z}^{(S)} \rightarrow \mathbb{Z}^m$ is an isomorphism. Put differently, $\mathbb{Z}^m$ is a free abelian group of rank m .

We use this opportunity to recall the classification of finitely generated abelian groups which had been proved in the algebra course. A proof is for example also provided in [Hu, Chapter II.2].

Theorem 8.3. (Classification of finitely generated abelian groups) *Let A be a finitely generated abelian group. Then there exist non-zero natural numbers $a_1, \dots, a_k$ with $a_i | a_{i+1}$ for $i = 1, \dots, k-1$ and an $r \in \mathbb{N}_0$ such that*

$$A \cong \mathbb{Z}^r \oplus \bigoplus_{i=1}^k \mathbb{Z}_{a_i}.$$

¹⁶²Hereby we denote by $\#S$ and $\#T$ the cardinalities of the sets S and T , as defined on page 88.

¹⁶³It follows from Lemma 8.2 and the Bernstein-Schröder Theorem 6.1 that the rank of a free abelian group is well-defined.

¹⁶⁴Hereby we mean by “canonical map $\mathbb{Z}^{(S)} \rightarrow G$ ” the unique homomorphism $\psi: \mathbb{Z}^{(S)} \rightarrow G$ from Lemma 8.1 that makes the following diagram commute

$$\begin{array}{ccc} S & \xrightarrow{\quad} & \mathbb{Z}^{(S)} \\ & \searrow \phi & \downarrow \psi \\ & & G \end{array}$$

where $S \rightarrow G$ is the inclusion map.

Furthermore the numbers $a_1, \dots, a_k$ and r are uniquely determined by A .

Definition. Given a finitely generated abelian group A we denote the direct sum of the cyclic groups on the right hand side of Theorem 8.3 as the *canonical form* of A .

Example. The Chinese Remainder Theorem says that for any coprime m and n we have an isomorphism

$$\begin{aligned} \mathbb{Z}_{mn} &\xrightarrow{\cong} \mathbb{Z}_m \times \mathbb{Z}_n = \mathbb{Z}_m \oplus \mathbb{Z}_n \\ k + mn\mathbb{Z} &\mapsto (k + m\mathbb{Z}, k + n\mathbb{Z}). \end{aligned}$$

For example consider $A = \mathbb{Z}_3 \oplus \mathbb{Z}_5$. It is not in the form of Theorem 8.3, but we have the isomorphism $\mathbb{Z}_3 \oplus \mathbb{Z}_5 \cong \mathbb{Z}_{15}$, which is of the desired form. Thus $\mathbb{Z}_{15}$ is the canonical form of $\mathbb{Z}_3 \oplus \mathbb{Z}_5$.

We will not provide a proof for Theorem 8.3, but we will sketch a proof for the following lemma. The third statement of the lemma is a somewhat weaker version of Theorem 8.3.

Lemma 8.4.

- (1) Let V be an $m \times n$ -matrix over $\mathbb{Z}$. For any two matrices $A \in \text{GL}(m, \mathbb{Z})$ and $B \in \text{GL}(n, \mathbb{Z})$ we have an isomorphism

$$\mathbb{Z}^m / AVB\mathbb{Z}^n \cong \mathbb{Z}^m / V\mathbb{Z}^n.$$

- (2) Let V be an $m \times n$ -matrix over $\mathbb{Z}$. Then there exist two matrices $A \in \text{GL}(m, \mathbb{Z})$ and $B \in \text{GL}(n, \mathbb{Z})$ ¹⁶⁵ such that¹⁶⁶

$$A \cdot V \cdot B = \begin{pmatrix} p_1 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & 0 & p_t & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

for some $p_1, \dots, p_t \in \mathbb{N}$.

- (3) Let A be a finitely generated abelian group. Then there exists an $r \geq 0$ and $p_1, \dots, p_t \in \mathbb{N}$ such that

$$A \cong \mathbb{Z}^r \oplus \bigoplus_{i=1}^t \mathbb{Z}_{p_i}.$$

As mentioned before, we will only provide sketch for the proof of the lemma.

Sketch of proof.

- (1) The proof of the first statement is elementary. It is left as an exercise in exercise sheet 8.

¹⁶⁵The matrices A and B are invertible over $\mathbb{Z}$, i.e. their determinants are ± 1 .

¹⁶⁶Hereby the last row and column of zeros can be arbitrarily large.

- (2) First note that if an $m \times n$ -matrix P is obtained from an $m \times n$ -matrix Q by swapping two rows, then P is obtained by left-multiplying Q by an $m \times m$ -matrix of the form¹⁶⁷

$$\begin{pmatrix} \text{id} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \text{id} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \text{id} \end{pmatrix}.$$

Similarly if P is obtained from an $m \times n$ -matrix Q by adding the k -th row to the l -th row, then P is obtained by left-multiplying Q by an $m \times m$ -matrix of the form

$$\begin{pmatrix} \text{id} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & \text{id} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \text{id} \end{pmatrix}$$

where the extra 1 sits in the (k, l) -entry. Both these $m \times m$ -matrices lie in $\text{GL}(m, \mathbb{Z})$. If we do the same operations for columns, then this corresponds to right multiplication by the same type of matrices in $\text{GL}(n, \mathbb{Z})$ as above.

Now let V be an $m \times n$ -matrix over $\mathbb{Z}$. If V is the zero matrix, then there is nothing to show. So suppose that V is not the zero matrix. We perform the following steps:

- (a) After swapping rows and columns we can arrange that the absolute value of the $(1, 1)$ -entry is less or equal than the absolute value of all other entries. After possibly multiplying the first column by -1 we can assume that the $(1, 1)$ -entry v_{11} is positive.
- (b) By adding and subtracting the first column from the other columns we can arrange that all entries in the first row, except for the $(1, 1)$ -entry, lie between 0 and $v_{11} - 1$.
- (c) The same way as in (2) we can arrange that all entries in the first column, except for the $(1, 1)$ -entry, lie between 0 and $v_{11} - 1$.

Now we have to distinguish two cases:

- (a) If there exists outside of the $(1, 1)$ -entry another entry in the first row or column that is non-zero, then we start again with (1). Note that the new entry $(1, 1)$ -entry will be less than before. So this process will come to a halt after finitely many steps.

¹⁶⁷Here is the reality check whether that is really correct: we have

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & d \\ a & b \end{pmatrix}.$$

- (b) If all entries in the first row and column, except for the $(1, 1)$ -entry are zero, then

$$V = \begin{pmatrix} v_{11} & 0 \\ 0 & V' \end{pmatrix}$$

for some $(m-1) \times (n-1)$ -matrix V' and we perform the same operations on the smaller matrix V' .

- (3) Let A be a finitely generated abelian group. By the definition of a finitely generated abelian group there exists an epimorphism $\psi: \mathbb{Z}^m \rightarrow A$. Since $\mathbb{Z}$ is a Noetherian ring we know that $\ker(\psi)$ is again a finitely generated free abelian group of rank $n \leq m$. In particular there exists an isomorphism $v: \mathbb{Z}^n \rightarrow \ker(\psi)$. We denote by V the $m \times n$ -matrix that represents the resulting map $\mathbb{Z}^n \rightarrow \ker(\psi) \subset \mathbb{Z}^m$ with respect to the standard bases. Then $A \cong \mathbb{Z}^m / V\mathbb{Z}^n$. We pick matrices A and B as in (2). From (1) it follows that $\mathbb{Z}^m / V\mathbb{Z}^n$ is isomorphic to $\mathbb{Z}^m / AVB\mathbb{Z}^n$. But the latter is isomorphic to $\mathbb{Z}_{p_1} \oplus \cdots \oplus \mathbb{Z}_{p_t} \oplus \mathbb{Z}^{m-n}$. $\square$

Definition. Let A be a finitely generated abelian group.

- (1) We refer to the r in the statement of Theorem 8.3 as the *rank* of the finitely generated abelian group A . We denote it by $\text{rank}(A)$.¹⁶⁸
 (2) We refer to

$$\text{tor}(A) = \{a \in A \mid \text{there exists an } n \in \mathbb{N} \text{ with } na = 0\}$$

as the *torsion subgroup* of A .¹⁶⁹ If $\text{tor}(A)$ is the trivial subgroup, then we say that A is *torsion-free*.

In the following two lemmas we collect several basic facts about finitely generated abelian groups. Most of the statements follow more or less directly from Theorem 8.3. We will not prove the lemmas and we leave it as an exercise to deduce the statements from Theorem 8.3 and the algebra course you took.

Lemma 8.5. *Let*

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

be a short exact sequence of abelian groups¹⁷⁰ such that at least two of these groups are finitely generated. Then the following holds:

- (1) *all three groups are finitely generated abelian groups,*
 (2) $\text{rank}(B) = \text{rank}(A) + \text{rank}(C)$,
 (3) *if A and C are torsion-free, then B is also torsion-free.*

We also have the following lemma.

¹⁶⁸If A is a finitely generated free abelian group, then this definition of rank agrees with the definition given for free abelian groups on page 137.

¹⁶⁹Why is it a subgroup?

¹⁷⁰Recall that this means that the map $A \rightarrow B$ is injective, that $B \rightarrow C$ is surjective and that the image of $A \rightarrow B$ agrees with the kernel of $B \rightarrow C$.

Lemma 8.6.

(1) If A, B and C are finitely generated abelian groups, then

$$A \oplus C \cong B \oplus C \implies A \cong B.$$

(2) If A is a subgroup of a finitely generated abelian group B , then $\text{tor}(A)$ is a subgroup of $\text{tor}(B)$ and $\text{rank}(A) \leq \text{rank}(B)$.

(3) If A is a subgroup of $\mathbb{Z}^r$, then A is a free abelian group with rank less or equal than r .¹⁷¹

(4) If A is a subgroup of $\mathbb{Z}^r$ with $\text{rank}(A) < r$, then A is a proper subgroup, i.e. $A \neq \mathbb{Z}^r$.

Abelian groups that are not finitely generated are not classified and they tend to be significantly harder. For example the following lemma shows that not every torsion-free abelian group is a free abelian group.

Lemma 8.7. *The abelian group $(\mathbb{Q}, +)$ is torsion-free but not free abelian.*

Proof. Clearly $(\mathbb{Q}, +)$ is torsion-free. If it was free abelian, then there would exist an epimorphism to $\mathbb{Z}$. In exercise sheet 8 we show that such an epimorphism cannot exist. $\square$

8.2. The free product of groups. We first recall the definition of the direct product of two groups G and H that we had already given on page 118. The direct product of two groups G and H is defined as

$$G \times H := \{(g, h) \mid g \in G \text{ and } h \in H\}$$

together with the group structure that is given by

$$(g_1, h_1) \cdot (g_2, h_2) := (g_1 g_2, h_1 h_2)$$

for $g_1, g_2 \in G$ and $h_1, h_2 \in H$. We call the group $G \times H$ the *direct product of the groups G and H* . Via the monomorphisms

$$\begin{array}{ccc} G & \rightarrow & G \times H \\ g & \mapsto & (g, e) \end{array} \quad \text{and} \quad \begin{array}{ccc} H & \rightarrow & G \times H \\ h & \mapsto & (e, h) \end{array}$$

we can view G and H as subgroups of $G \times H$. Note that these subgroups commute, i.e. for every $g \in G$ and $h \in H$ we have the equality $(g, e)(e, h) = (g, h) = (e, h)(g, e)$.

We will now introduce the “free product” $G * H$ of two groups G and H . The definition is perhaps initially harder than the definition of the “direct product” $G \times H$. But for many, perhaps even most applications, the free product is more useful.

So let G be and H be two groups. We consider the set $G * H$ of all finite sequences $(x_1, \dots, x_m)$ such that the following conditions are satisfied:

- (1) each x_i lies in one of the groups G or H ,
- (2) no x_j is the neutral element of G or of H ,
- (3) any two consecutive x_j ’s lie in two different groups.

¹⁷¹In fact this statement is already contained in Lemma 8.2.

Hereby we also allow the “empty sequence” $()$.

Let $(x_1, \dots, x_m)$ and $(y_1, \dots, y_n)$ be two elements in $G * H$. We define the product of these two elements as follows:

- (1) If x_m and y_1 lie in different groups, then we define

$$(x_1, \dots, x_m) \cdot (y_1, \dots, y_n) := (x_1, \dots, x_m, y_1, \dots, y_n).$$

- (2) If x_m and y_1 lie in the same group, then there exists a j such that $x_{m+1-i} = y_i^{-1}$ for $i = 1, \dots, j$, and such that $x_{m-j} \neq y_{j+1}^{-1}$. We define¹⁷²

$$(x_1, \dots, x_m) \cdot (y_1, \dots, y_n) := (x_1, \dots, \underbrace{x_{m-j} \cdot y_{j+1}}_{\neq e}, \dots, y_n).$$

Another way of stating the product formula is as follows: given two sequences $(x_1, \dots, x_m)$ and $(y_1, \dots, y_n)$ we stack them together $(x_1, \dots, x_m, y_1, \dots, y_n)$ and then we delete any occurrence of a subsequence of the form a, a^{-1} for $a \in G$ or $a \in H$ and if a subsequence is of the form a, b with $a, b \in G$ or $a, b \in H$, then we replace it by ab .

We henceforth refer to $G * H$ together with this product structure as the *free product* of G and H . We have the following important lemma.

Lemma 8.8. *The free product of two groups G and H is again a group. The neutral element of $G * H$ is hereby given by the empty sequence $()$ and the inverse of an element $(x_1, \dots, x_m) \in G * H$ is given by*

$$(x_1, \dots, x_m)^{-1} = (x_m^{-1}, \dots, x_1^{-1}) \in G * H.$$

Proof. It is clear that the empty sequence is a neutral element. It follows immediately from the definition of the product that the inverse of an element $(x_1, \dots, x_m) \in G * H$ is given by $(x_m^{-1}, \dots, x_1^{-1})$. It remains to show that the multiplication satisfies associativity.

In principle one can show associativity naively “by hand”, but there are surprisingly many cases one needs to distinguish, so that the proof becomes rather painful.

We will therefore follow a different approach. We write $W = G * H$ and we denote by $(P(W), \circ)$ the group of all permutations of W , i.e. $P(W)$ is the set of all bijections from W to W and the group structure is given by composition of maps. For $g \in G$ we consider the map¹⁷³

$$\begin{aligned} \lambda_g: W &\rightarrow W \\ (x_1, \dots, x_m) &\mapsto \begin{cases} (g, x_1, x_2, \dots, x_m), & \text{if } x_1 \in H, \\ (gx_1, x_2, \dots, x_m), & \text{if } x_1 \in G \text{ and } g \neq x_1^{-1}, \\ (x_2, \dots, x_m), & \text{if } g = x_1^{-1}. \end{cases} \end{aligned}$$

It is now straightforward to see that for any $g, g' \in G$ and any $(x_1, \dots, x_m) \in W$ we have $g(g' \cdot (x_1, \dots, x_m)) = (gg') \cdot (x_1, \dots, x_m)$. This means that $\lambda_g \circ \lambda_{g'} = \lambda_{gg'}$. It is now

¹⁷²Since x_m and y_1 lie in the same group, the elements x_{m-j} and y_{j+1} also lie in the same group, so it makes sense to consider $x_{m-j} \cdot y_{j+1}$.

¹⁷³It is straightforward to see that the sequences on the right do indeed lie in $G * H$.

straightforward to see that the map

$$\begin{aligned}\lambda: G &\rightarrow P(W) \\ g &\mapsto \lambda_g\end{aligned}$$

is a group homomorphism. Similarly we define $\lambda: H \rightarrow P(W)$. Now we consider the map

$$\begin{aligned}\lambda: W &\rightarrow P(W) \\ (x_1, \dots, x_n) &\mapsto \lambda_{x_1} \circ \dots \circ \lambda_{x_n}.\end{aligned}$$

This map is injective since $\lambda(x_1, \dots, x_n)$ applied to the empty word $()$ gives the word $(x_1, \dots, x_n)$. Hence $\lambda(x_1, \dots, x_n)$ is not the identity map from $W \rightarrow W$, i.e. $\lambda(x_1, \dots, x_n)$ is non-trivial in $P(W)$.

Furthermore the map $\lambda: W \rightarrow P(W)$ satisfies $\lambda(x \cdot y) = \lambda_x \circ \lambda_y$. Put differently, λ defines an injective map $\lambda: (W, \cdot) \rightarrow (P(W), \circ)$ that preserves the product structure. But since $(P(W), \circ)$ satisfies the associativity law we now see that $(W, \cdot)$ also satisfies the associativity law. $\square$

The following lemma summarizes two main properties of the free product of two groups.

Lemma 8.9. *Let G and H groups. Then the following holds:*

(1) *The maps*

$$\begin{aligned}i: G &\rightarrow G * H \\ g &\mapsto \begin{cases} (g), & \text{if } g \neq e, \\ (), & \text{if } g = e \end{cases} \quad \text{and} \quad \begin{aligned}j: H &\rightarrow G * H \\ h &\mapsto \begin{cases} (h), & \text{if } h \neq e, \\ (), & \text{if } h = e \end{cases}\end{aligned}\end{aligned}$$

are monomorphisms.

(2) *For any two group homomorphisms $\alpha: G \rightarrow A$ and $\beta: H \rightarrow A$ there exists a unique group homomorphism $\gamma: G * H \rightarrow A$ such that the following diagram commutes*

$$\begin{array}{ccc} G * H & \xleftarrow{i} & G \\ j \uparrow & \searrow \gamma & \downarrow \alpha \\ H & \xrightarrow{\beta} & A. \end{array}$$

Example. Let G and H be two groups. We denote by $\alpha: G \rightarrow G \times H$ and $\beta: H \rightarrow G \times H$ the obvious inclusion maps. It follows from Lemma 8.9 there exists a unique homomorphism $\gamma: G * H \rightarrow G \times H$ which makes the following diagram commute:

$$\begin{array}{ccc} G * H & \xleftarrow{i} & G \\ j \uparrow & \searrow \gamma & \downarrow \alpha \\ H & \xrightarrow{\beta} & G \times H. \end{array}$$

Remark. In the literature the direct product of two groups often gets introduced in a slightly different way, namely by a “universal property”. More precisely, let G be and H be two groups. The *direct product of G and H* is then defined as a group K together with two

homomorphisms $i: G \rightarrow K$ and $j: H \rightarrow K$ which satisfy the following universal property: for every two group homomorphisms $\alpha: G \rightarrow A$ and $\beta: H \rightarrow A$ there exists a *unique* group homomorphism $\gamma: K \rightarrow A$ such that the following diagram commutes

$$\begin{array}{ccc} K & \xleftarrow{i} & G \\ j \uparrow & \searrow \gamma & \downarrow \alpha \\ H & \xrightarrow{\beta} & A. \end{array}$$

Lemma 8.9 can now be reinterpreted as saying that the direct product of two groups always exists. An argument that is very similar to the one we had given on page 138 shows that the direct product is unique in the following sense: If $(K, i: G \rightarrow K, j: H \rightarrow K)$ and $(K', i': G \rightarrow K', j': H \rightarrow K')$ both satisfy the above universal property, then there exists a unique isomorphism $\Phi: K \rightarrow K'$ such that $i' = \Phi \circ i$ and $j' = \Phi \circ j$.¹⁷⁴

Proof. It is clear that the maps i and j are injective, furthermore it follows immediately from the definition of the group structure on $G * H$ that the maps are group homomorphisms.

Now let $\alpha: G \rightarrow A$ and $\beta: H \rightarrow A$ be two group homomorphisms. For an element $(g_1, h_1, \dots, g_k, h_k) \in G * H$ with $g_i \in G, h_i \in H, i = 1, \dots, k$ we define

$$\gamma(g_1, h_1, \dots, g_k, h_k) := \alpha(g_1) \cdot \beta(h_1) \cdot \dots \cdot \alpha(g_k) \cdot \beta(h_k).$$

The same way we define γ for all other elements in $G * H$. Now one can show easily that γ is a group homomorphism.

It remains to show the uniqueness of γ . Any element in $G * H$ is a product of elements of the form $i(g)$, $g \in G$ and $j(h)$, $h \in H$. It follows that any homomorphism $\delta: G * H \rightarrow A$ is already uniquely determined by the values of δ on all elements of the form $i(g)$, $g \in G$ and $j(h)$, $h \in H$. Put differently, there is at most one homomorphism $\delta: G * H \rightarrow A$ with $\delta(g) = \alpha(i(g))$ for all $g \in G$ and $\delta(h) = \beta(j(h))$ for all $h \in H$. $\square$

Remark.

- (1) We will use the monomorphisms from Lemma 8.9 to view G and H as subgroups of $G * H$.
- (2) If there is no danger of confusion, then we often shorten $(x_1, \dots, x_m) \in G * H$ to the notation $x_1 \dots x_m$. This notation is coherent with the convention (1). Note that if $g \in G$ and $h \in H$ are non-trivial elements, then the sequences (g, h) and (h, g) both lie in $G * H$ and they are different. With the convention that we just introduced that means that $g \cdot h \neq h \cdot g$. Put differently, a non-trivial element of G never commutes with a non-trivial element of H .
- (3) The construction of the free product of two groups generalizes in an obvious way to the free group of finitely many, in fact even to infinitely many groups.

¹⁷⁴Why does $K = G \times H$ with the obvious maps $i: G \rightarrow K$ and $j: H \rightarrow K$ not satisfy this universal property? Or does it?

Let t be a symbol. We define

$$\langle t \rangle := \{ \dots, t^{-2}, t^{-1}, e, t, t^2, \dots \}.$$

The set $\langle t \rangle$ admits an obvious group structure that is given by $t^i \cdot t^j := t^{i+j}$. This group is called *the infinite cyclic group generated by t* . The map $\mathbb{Z} \rightarrow \langle t \rangle$, $n \mapsto t^n$ is evidently an isomorphism.

For a set S we refer to

$$\langle S \rangle := \text{free product of the groups } \langle s \rangle, s \in S$$

as the *free group on the set S* . Via the map $s \mapsto (s)$ we can and will view S as a subset of $\langle S \rangle$. For example, if $S = \{t_1, \dots, t_k\}$, then we obtain the group

$$\langle t_1, \dots, t_k \rangle := \langle t_1 \rangle * \dots * \langle t_k \rangle$$

which we also refer to as the *free group on the generators $t_1, \dots, t_k$* . Each element of the free group $F(t_1, \dots, t_k)$ is thus of the form

$$t_{r_1}^{s_1} t_{r_2}^{s_2} \dots t_{r_l}^{s_l}$$

with $r_i \neq r_{i+1}$ for $i = 1, \dots, l-1$ and $s_i \neq 0$ for $i = 1, \dots, l$. Sometimes we denote the free group on k generators just by F_k .

Example. We consider the free group $\langle x, y \rangle$ on generators x and y . Elements in this group are of the form $xy^3x^{-2}y$ or $y^{-2}xy$. The multiplication is the “obvious one”, for example

$$(xy^3x^{-1}y^2) \cdot (y^{-2}x^3yxy^{-1}) = xy^3x^2yxy^{-1}.$$

The group is not abelian since $xy \neq yx$.¹⁷⁵

Lemma 8.10. *Let S be a set, let G be an arbitrary group and let $g: S \rightarrow G$ be a map. Then there exists a unique homomorphism $\varphi: \langle S \rangle \rightarrow G$ with $\varphi(s) = g(s)$ for $s \in S$.*

Proof. For $s \in S$ the map

$$\begin{array}{ccc} \langle s \rangle & \rightarrow & G \\ s^n & \mapsto & g(s)^n \end{array}$$

is evidently the unique homomorphism $\varphi_s: \langle s \rangle \rightarrow G$ with $\varphi_s(s) = g(s)$. It now follows from the obvious generalization of Lemma 8.9 to the case of the free product of arbitrarily many groups that there exists a unique homomorphism $\varphi: \langle S \rangle \rightarrow G$ with $\varphi(s) = \varphi_s(s) = g(s)$ for $s \in S$. $\square$

Remark. Let S be a set. A *free group on S* is a group K together with a map $f: S \rightarrow K$ which satisfies the following universal property: for any map $g: S \rightarrow G$ from the set S to a

¹⁷⁵Why is $xy \neq yx$?

group G ¹⁷⁶ there exists a unique group homomorphism $\varphi: K \rightarrow G$ such that the following diagram commutes

$$\begin{array}{ccc} S & \xrightarrow{f} & K \\ & \searrow g & \downarrow \varphi \\ & & G. \end{array}$$

Lemma 8.10 says that the obvious map $S \rightarrow \langle S \rangle$ is a free group on S in the above sense. As before, a slight variation on the argument on page 138 shows that the free group on S is unique in an appropriate sense.

Definition.

- (1) A group G is called *free*, if it is isomorphic to the free group $\langle S \rangle$ on some set S .
- (2) Given a group G we say that $S \subset G$ is a *generating set* if the canonical homomorphism $\langle S \rangle \rightarrow G$ is an epimorphism.¹⁷⁷ Sometimes we also say that the elements in S are *generators of G* .
- (3) We say that a group G is *finitely generated* if it admits a finite generating set.

Example.

- (1) As on page 99 we consider the following two self-homeomorphisms of $\mathbb{R}^2$:

$$\begin{array}{ccc} A: \mathbb{R}^2 & \rightarrow & \mathbb{R}^2 \\ (x, y) & \mapsto & (x+1, 1-y) \end{array} \quad \text{and} \quad \begin{array}{ccc} B: \mathbb{R}^2 & \rightarrow & \mathbb{R}^2 \\ (x, y) & \mapsto & (x, y+1) \end{array}$$

and we denote by G the subgroup of all homeomorphisms of $\mathbb{R}^2$ that is generated by A and B . This means that

$$G = \begin{array}{l} \text{all self-homeomorphisms of } \mathbb{R}^2 \text{ that can be written} \\ \text{as a finite concatenation of the maps } A, B, A^{-1} \text{ and } B^{-1}. \end{array}$$

We had seen on page 99 that $\mathbb{R}^2/G$ equals the Klein bottle K and using Theorem 6.16 we had seen on page 117 that $\pi_1(\text{Klein bottle}) \cong G$. The group G is finitely generated in the above sense. More precisely, it follows immediately from the definition of G that $S = \{A, B\} \subset G$ is a generating set for G .

- (2) For any set S there exists by Lemma 8.10 a unique homomorphism $\langle S \rangle \rightarrow \mathbb{Z}^{(S)}$ that sends $s \in S$ to s viewed as an element in $\mathbb{Z}^{(S)}$. Since the set $S \subset \mathbb{Z}^{(S)}$ generates $\mathbb{Z}^{(S)}$ we see that this homomorphism is an epimorphism. It follows that any finitely generated abelian group in the sense of page 139 is also a finitely generated group in the above sense.

¹⁷⁶Note that in contrast to the Lemma 8.1 group G is now not necessarily abelian.

¹⁷⁷Hereby we mean by “canonical map $\langle S \rangle \rightarrow G$ ” the unique homomorphism $\psi: \langle S \rangle \rightarrow G$ from Lemma 8.10 that makes the following diagram commute

$$\begin{array}{ccc} S & \hookrightarrow & \langle S \rangle \\ & \searrow \phi & \downarrow \psi \\ & & G \end{array}$$

where $S \rightarrow G$ is the inclusion map.

The subgroup of a free abelian group is again a free abelian group. Does the same hold for free groups? We record this as a question for future reference.

Question 8.11. *Is every subgroup of a free group again a free group?*

The next lemma says in particular that free groups on two sets are isomorphic if and only if the cardinalities of the sets are the same.

Lemma 8.12. *Let S and T be two sets.*

- (1) *If there exists an epimorphism $\langle S \rangle \rightarrow \langle T \rangle$, then $\#S \geq \#T$.*
- (2) *If $\langle S \rangle$ is isomorphic to $\langle T \rangle$, then $\#S = \#T$.*
- (3) *If given $n \in \mathbb{N}$ we denote by F_n the free group on n generators, then for $k \neq l$ the groups F_k and F_l are not isomorphic.*

Proof.

- (1) We first recall that we had just seen in the previous example that for any set U there exists a unique epimorphism $\psi_U: \langle U \rangle \rightarrow \mathbb{Z}^{(U)}$ that sends $u \in U$ to u viewed as an element in $\mathbb{Z}^{(U)}$.

Now let S and T be two sets and let $\varphi: \langle S \rangle \rightarrow \langle T \rangle$ be an epimorphism. By Lemma 8.1 there exists a unique homomorphism $\tilde{\varphi}: \mathbb{Z}^{(S)} \rightarrow \mathbb{Z}^{(T)}$ that sends $s \in S$ to $\psi_T(\varphi(s)) \in \mathbb{Z}^{(T)}$. We obtain the following diagram of homomorphisms

$$\begin{array}{ccc} \langle S \rangle & \xrightarrow{\varphi} & \langle T \rangle \\ \psi_S \downarrow & & \downarrow \psi_T \\ \mathbb{Z}^{(S)} & \xrightarrow{\tilde{\varphi}} & \mathbb{Z}^{(T)}. \end{array}$$

It follows immediately from the definition of $\tilde{\varphi}$ that this diagram commutes.¹⁷⁸ Since the top horizontal and since the vertical maps are epimorphisms we see that the bottom horizontal map is also an epimorphism. It follows from Lemma 8.2 that $\#S \geq \#T$.

- (2) Let $f: \langle S \rangle \rightarrow \langle T \rangle$ be an isomorphism. We apply (1) to f and to f^{-1} and we obtain that $\#S \geq \#T$ and $\#T \geq \#S$. It follows from the Bernstein-Schröder Theorem 6.1 that $\#S = \#T$.

- (3) This statement is an immediate consequence of (2).¹⁷⁹ □

We had already seen that some groups can be written as a direct product of two groups in different ways. For example it follows from the Chinese Remainder Theorem that the groups $\mathbb{Z}_3 \times \mathbb{Z}_{35}$ and $\mathbb{Z}_5 \times \mathbb{Z}_{21}$ are isomorphic. The following theorem says that the situation is much more rigid for free products. Before we can state the theorem we have to introduce the following definition.

¹⁷⁸Here we use that a homomorphism $\gamma: \langle S \rangle \rightarrow G$ is uniquely determined by the values of γ on S . Put differently, if $\alpha: \langle S \rangle \rightarrow G$ and $\beta: \langle S \rangle \rightarrow G$ are two homomorphisms with $\alpha(s) = \beta(s)$ for all $s \in S$, then it follows immediately from the uniqueness statement of Lemma 8.10 that $\alpha = \beta$.

¹⁷⁹Note that if we just want to prove statement (3), then in the proof of (2) we could replace the slightly esoteric Lemma 8.2 by the more conventional Lemma 8.5 that was proved in the Algebra course.

Definition. A group G is *indecomposable* if it is not isomorphic to the free product of two non-trivial groups.

The following lemma will be proved in exercise sheet 8.

Lemma 8.13. *Abelian groups and finite groups are indecomposable.*

Now we can state the Grushko¹⁸⁰ Decomposition Theorem.

Theorem 8.14. *Let G be a finitely generated group.*

- (1) *The group G is isomorphic to the free product $G = A_1 * \cdots * A_k$ of non-trivial indecomposable groups.*
- (2) *The isomorphism types of $A_1, \dots, A_k$ are unique up to permutation. More precisely, if $A_1, \dots, A_k$ and $B_1, \dots, B_l$ are indecomposable finitely generated groups such that*

$$A_1 * \cdots * A_k \cong B_1 * \cdots * B_l,$$

then $k = l$ and furthermore there exists a permutation $\sigma \in S_k$ such that $A_i \cong B_{\sigma(i)}$ for $i = 1, \dots, k$.

Proof. A proof for the existence of the decomposition of any finitely generated group G into indecomposable groups was first given by Igor Grushko [Gr]. See also [LS, Theorem IV.1.8] for a proof.¹⁸¹ The uniqueness statement follows from a refinement of Grushko's proof due to John Stallings [St2, p. 168f]. See also [Ne, Theorem 4.1] for a more direct reference. $\square$

Example. It follows from Lemma 8.13 and from the Grushko Decomposition Theorem 8.14 that the free product $\mathbb{Z}_3 * \mathbb{Z}_{35}$ is not isomorphic to the free product $\mathbb{Z}_5 * \mathbb{Z}_{21}$.

8.3. An alternative definition of the free product of groups. For some purposes it is sometimes useful to work with a different model for the free product of two groups.

Let A and B be two groups. We set¹⁸²

$$S(A, B) := \{\text{all finite sequences } (g_1, \dots, g_k) \text{ with each } g_i \text{ in } A \text{ or } B\}$$

and we denote by $\sim$ the equivalence relation on $S(A, B)$ that is generated by the relations

$$(g_1, \dots, g_l, e, g_{l+2}, \dots, g_k) \sim (g_1, \dots, g_l, g_{l+2}, \dots, g_k)$$

and by the relations

$$(g_1, \dots, g_l, g_{l+1}, \dots, g_k) \sim (g_1, \dots, g_l g_{l+1}, \dots, g_k)$$

¹⁸⁰Igor Grushko was a Russian mathematician. The theorem was proved in 1940.

¹⁸¹Why is this not trivial? If G is indecomposable, we are done. If G is not indecomposable, then we can write G as the product of two non-trivial groups; if they are indecomposable, we are done. If not, we decompose and so on. But why does this process stop after finitely many steps? This is non-trivial and that is the content of Grushko's original paper.

¹⁸²This definition is similar to the definition of $A * B$, but now we allow that elements of the sequence are trivial and we allow that consecutive elements lie in the same group.

if both g_l and g_{l+1} lie in A or both lie in B . It is now straightforward to verify that the map

$$\begin{aligned} S(A, B)/\sim \times S(A, B)/\sim &\rightarrow S(A, B)/\sim \\ [(g_1, \dots, g_k)], [h_1, \dots, h_l] &\mapsto [(g_1, \dots, g_k, h_1, \dots, h_l)] \end{aligned}$$

is well-defined and that it defines a group structure on $S(A, B)$.¹⁸³

Proposition 8.15. *Let A and B be two groups. Then the map¹⁸⁴*

$$\begin{aligned} \Phi: S(A, B)/\sim &\rightarrow A * B \\ [(g_1, \dots, g_k)] &\mapsto g_1 \cdots g_k \end{aligned}$$

is well-defined and it is an isomorphism. The inverse map is given by

$$\begin{aligned} \Psi: A * B &\rightarrow S(A, B)/\sim \\ (g_1, \dots, g_k) &\mapsto [(g_1, \dots, g_k)]. \end{aligned}$$

In the proof of Proposition 8.15 we will need the following little lemma.

Lemma 8.16. *Let $\Phi: G \rightarrow H$ and $\Psi: H \rightarrow G$ be two group homomorphisms such that $\Phi \circ \Psi = \text{id}_H$ and such that Ψ is an epimorphism. Then Φ is an isomorphism.*

The lemma can be summarized as follows, if we are given a commutative diagram of the form

$$\begin{array}{ccc} & G & \\ \Psi \nearrow & & \searrow \Phi \\ H & \xrightarrow{\text{id}} & H, \end{array}$$

then Φ is an isomorphism.¹⁸⁵

Proof. It follows from $\Phi \circ \Psi = \text{id}_H$ that Φ is an epimorphism. It remains to show that Φ is a monomorphism. So let $g \in \ker(\Phi)$. Since Ψ is an epimorphism there exists an $h \in H$ with $\Psi(h) = g$. Since $\Phi \circ \Psi = \text{id}_H$ we see that $h = \Phi(\Psi(h)) = \Phi(g) = e$. Thus $h = e$ and thus $g = \Psi(h) = \Psi(e) = e$. $\square$

Proof. It is straightforward to verify that the map

$$\begin{aligned} \Phi: S(A, B)/\sim &\rightarrow A * B \\ [(g_1, \dots, g_k)] &\mapsto g_1 \cdots g_k \end{aligned}$$

¹⁸³The neutral element is again represented by the empty sequence $()$ and the inverse of $[(g_1, \dots, g_k)]$ is given by $[(g_k^{-1}, \dots, g_1^{-1})]$. This time associativity is not a problem. Why not?

¹⁸⁴Recall that we use the canonical monomorphisms from Lemma 8.9 to view A and B as subgroups of $A * B$. If we want to be more precise, then we could say that Φ is given by $[(g_1, \dots, g_k)] \mapsto (g_1) \cdots (g_k)$.

¹⁸⁵Is Ψ also necessarily an isomorphism?

is well-defined and that it is a homomorphism. This homomorphism is surjective since $A * B$ is a subset of $S(A, B)$. Next we consider the map¹⁸⁶

$$\begin{aligned}\Psi: A * B &\rightarrow S(A, B)/\sim \\ (g_1, \dots, g_k) &\mapsto [(g_1, \dots, g_k)].\end{aligned}$$

It is straightforward to see that Ψ is a homomorphism. It is clear that $\Phi \circ \Psi = \text{id}_{A*B}$. It is not so obvious that $\Psi \circ \Phi = \text{id}_{S(A, B)/\sim}$. In fact we will proceed in a slightly different fashion: to show that Φ is an isomorphism it suffices, according to Lemma 8.16, to show that Ψ is surjective. This in turn follows from the following claim.

Claim. Any element in $S(A, B)$ is equivalent to an element in $A * B$.

We prove the claim by induction on the length of a sequence in $S(A, B)$. Clearly the claim holds for all sequences of length 0. Suppose we already know the claim for all sequences of length $\leq k - 1$. Let $(g_1, \dots, g_k)$ be a sequence of length k . If $(g_1, \dots, g_k) \in A * B$, then we are done. Now suppose that this is not the case. Then one of the following occurs:

- (1) either there exists an $i \in \{1, \dots, k\}$ with $g_i = e$, or
- (2) there exists an $i \in \{1, \dots, k - 1\}$ with $g_i, g_{i+1} \in A$, or
- (3) there exists an $i \in \{1, \dots, k - 1\}$ with $g_i, g_{i+1} \in B$.

In the first case the sequence is equivalent to $(g_1, \dots, g_{i-1}, g_{i+1}, \dots, g_k)$. In the second and third case the sequence is equivalent to $(g_1, \dots, g_{i-1} \cdot g_{i+1}, \dots, g_k)$. In all three cases we have thus shown that our original sequence is equivalent to a sequence of length $k - 1$. So we are done by our induction hypothesis. This concludes the proof of the claim. $\square$

Let A and B be two groups. In the following we will use the isomorphism of Proposition 8.15 to identify the group $S(A, B)/\sim$ with the free product $A * B$. Both groups have their advantages:

- (1) In the group $S(A, B)/\sim$ it is easier to write down the product of two elements, we just need to juxtapose sequences of elements in A and B .
- (2) On the other hand in the group $A * B$ it is trivial to check whether two elements are the same. For example for two non-trivial elements $a \in A$ and $b \in B$ the elements (a, b) and (b, a) of $A * B$ are by definition different. But from the definition of $S(A, B)/\sim$ it is not immediately clear that $[(a, b)]$ and $[(b, a)]$ are indeed different.

¹⁸⁶Put differently, $A * B \rightarrow S(A, B)/\sim$ is the homomorphism that is given by the universal property of $A * B$.

9. THE SEIFERT-VAN KAMPEN THEOREM I

We are still lacking the computation of fundamental groups of even fairly simple topological spaces. We will now take a new approach to calculating fundamental groups. Namely in the following two chapters we will formulate and prove results which allow us to calculate the fundamental group of a given topological space from the fundamental groups of suitable subspaces.

9.1. The Seifert–van Kampen theorem I. The idea of the Seifert-van Kampen theorem is to reduce the calculation of the fundamental group of a topological spaces X to the determination of fundamental groups of “simpler subsets” of X .

Let X be a topological space and let $U, V \subset X$ be two subsets with $X = U \cup V$ and with $U \cap V \neq \emptyset$. We choose a base point $x_0 \in U \cap V$ and we consider the following inclusion maps and the corresponding induced homomorphisms of the fundamental groups:

$$\begin{array}{ccc} U \cap V & \xrightarrow{\quad} & U \\ \downarrow & & \downarrow i \\ V & \xrightarrow{j} & X = U \cup V \end{array} \quad \text{and} \quad \begin{array}{ccc} \pi_1(U \cap V, x_0) & \xrightarrow{\quad} & \pi_1(U, x_0) \\ \downarrow & & \downarrow i_* \\ \pi_1(V, x_0) & \xrightarrow{j_*} & \pi_1(X, x_0) = \pi_1(U \cup V, x_0). \end{array}$$

Our goal is to determine the fundamental group of the total space $X = U \cup V$ in terms of the fundamental groups of U , V , $U \cap V$ and the induced maps $\pi_1(U \cap V, x_0) \rightarrow \pi_1(U, x_0)$ and $\pi_1(U \cap V, x_0) \rightarrow \pi_1(V, x_0)$.

We start out with the following lemma:

Lemma 9.1. *Let X be a topological space and let $U, V \subset X$ be two open subsets with $X = U \cup V$ and such that $U \cap V \neq \emptyset$. We choose a base point $x_0 \in U \cap V$. If $U \cap V$ is path-connected, then the homomorphism*

$$\pi_1(U, x_0) * \pi_1(V, x_0) \rightarrow \pi_1(X, x_0),$$

which is induced by the homomorphisms $\pi_1(U, x_0) \rightarrow \pi_1(X, x_0)$ and $\pi_1(V, x_0) \rightarrow \pi_1(X, x_0)$ and by Lemma 8.9 (2), is surjective.

Example.

(1) We consider the sphere

$$X = S^2 = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\},$$

with the covering $X = U \cup V$ given by the two open subsets

$$\begin{aligned} U &= \{(x, y, z) \mid x^2 + y^2 + z^2 = 1 \text{ and } z > -\tfrac{1}{2}\}, \\ V &= \{(x, y, z) \mid x^2 + y^2 + z^2 = 1 \text{ and } z < \tfrac{1}{2}\}. \end{aligned}$$

We pick the base point $x_0 = (1, 0, 0) \in U \cap V$. Here the intersection $U \cap V$ is path-connected, therefore we can apply Lemma 9.1. It says that we have an epimorphism

$$\pi_1(U, x_0) * \pi_1(V, x_0) \rightarrow \pi_1(S^2, x_0).$$

The subspaces U and V are homeomorphic to an open disk,¹⁸⁷ thus the fundamental groups are trivial. It follows that the trivial group¹⁸⁸ surjects onto $\pi_1(S^2, x_0)$, thus $\pi_1(S^2, x_0)$ itself is trivial. Precisely the same argument shows that $\pi_1(S^n) = 0$ for $n \geq 3$.

We had already obtained this result in Proposition 4.12. In fact, as we will see shortly, the proof of Lemma 9.1 has clear parallels to the proof of Proposition 4.12.

(2) We consider the sphere

$$X = S^1 = \{e^{it} \mid t \in \mathbb{R}\},$$

with the covering $X = U \cup V$ given by the two open subsets

$$U := \{e^{it} \mid t \in (-\frac{\pi}{4}, \frac{5\pi}{4})\}$$

$$V := \{e^{it} \mid t \in (\frac{3\pi}{4}, \frac{9\pi}{4})\}.$$

We pick the base point $x_0 = 1$. The subsets U and V are both homeomorphic to an open interval, hence $\pi_1(U, x_0) = \pi_1(V, x_0) = 1$, but of course we know from Corollary 6.17 that $\pi_1(X, x_0) \cong \mathbb{Z}$. Thus we see that $\pi_1(U, x_0) * \pi_1(V, x_0) \rightarrow \pi_1(X, x_0)$ is not an epimorphism. This is not a contradiction to Lemma 9.1, since in this example the intersection $U \cap V$ is not path-connected.

We now turn to the proof of Lemma 9.1.

Proof of Lemma 9.1. Let X be a topological space and let $U, V \subset X$ be two open subsets with $X = U \cup V$ and such that $U \cap V \neq \emptyset$. We choose a base point $x_0 \in U \cap V$. It suffices to prove the following claim.¹⁸⁹

Claim. Every loop $s: [0, 1] \rightarrow X = U \cup V$ in x_0 is homotopic¹⁹⁰ to a loop which is the product of finitely many loops in (U, x_0) or in (V, x_0) .

So let $s: [0, 1] \rightarrow X = U \cup V$ in x_0 be a loop. Since U and V are open we can use Corollary 1.6, to find a subdivision

$$0 = t_0 < t_1 < t_2 < \dots < t_k < t_{k+1} = 1$$

of the interval $[0, 1]$ such that the image of each $s_i := s|_{[t_i, t_{i+1}]}$ lies in U or in V . The same argument as in the proof of Proposition 4.12 shows that we can assume that each $s(t_i)$ lies in $U \cap V$.¹⁹¹ The problem is that even though the paths s_i lie in U respectively in V , they are not necessarily loops in x_0 . But since $U \cap V$ is path-connected we can find for each

¹⁸⁷A homeomorphism is for example given by stereographic projection.

¹⁸⁸Here we use the obvious fact that the free product of two times the trivial group is again the trivial group.

¹⁸⁹Why does it suffice to prove this claim?

¹⁹⁰Here we work of course with the notion of path-homotopy.

¹⁹¹Indeed, if $s(t_i)$ lies in $U \setminus V$, then $s([t_{i-1}, t_i])$ and $s([t_i, t_{i+1}])$ cannot lie in V , hence they both lie in U . This implies that $s([t_{i-1}, t_{i+1}])$ lies in U , and therefore we can remove t_i from the subdivision.

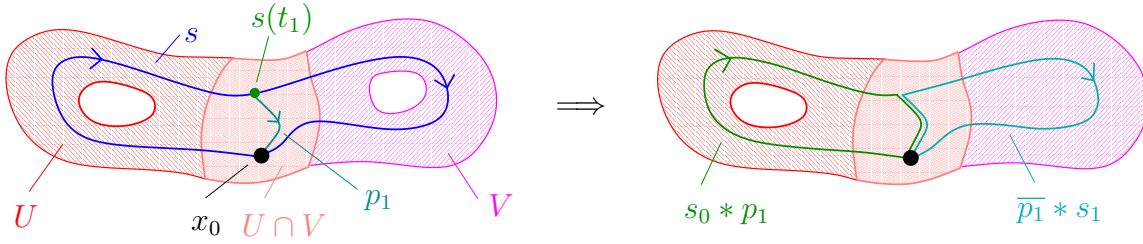


FIGURE 91. Schematic image for the proof of Lemma 9.1.

$i \in \{1, \dots, k\}$ a path p_i in $U \cap V$ which connects $s(t_i) \in U \cap V$ to the base point $x_0 \in U \cap V$. Now we have the following homotopies:

$$\begin{array}{ll}
 \text{by Lemma 4.3, since } s & \text{since each } p_i * \bar{p}_i \\
 \text{is a reparametrization} & \text{is null-homotopic} \\
 \text{of } s_0 * \dots * s_k & \\
 \downarrow & \downarrow \\
 s \simeq s_0 * \dots * s_k & \simeq s_0 * p_1 * \bar{p}_1 * s_1 * p_2 * \bar{p}_2 * s_2 * p_3 * \bar{p}_3 * \dots * \bar{p}_k * s_k \\
 & \simeq \underbrace{(s_0 * p_1)}_{\text{loop in } (U, x_0) \text{ or in } (V, x_0)} * \underbrace{(\bar{p}_1 * s_1 * p_2)}_{\text{loop in } (U, x_0) \text{ or in } (V, x_0)} * \underbrace{(\bar{p}_2 * s_2 * p_3)}_{\text{loop in } (U, x_0) \text{ or in } (V, x_0)} * (\bar{p}_3 * \dots * (\bar{p}_k * s_k)).
 \end{array}$$

Thus we have shown that s is homotopic to a product of loops in (U, x_0) and (V, x_0) .¹⁹² $\square$

Theorem 9.2. (Seifert-van Kampen) *Let X be a topological space and let $U, V \subset X$ be two open subsets with $X = U \cup V$ and with $U \cap V \neq \emptyset$. We choose a base point $x_0 \in U \cap V$. If $U \cap V$ is simply connected, then the map*

$$\pi_1(U, x_0) * \pi_1(V, x_0) \rightarrow \pi_1(X, x_0)$$

that is induced by $\pi_1(U, x_0) \rightarrow \pi_1(X, x_0)$ and $\pi_1(V, x_0) \rightarrow \pi_1(X, x_0)$ and Lemma 8.9 (2), is an isomorphism.

Remark. One might wonder whether the hypothesis that $U \cap V$ is simply connected is really necessary. But this is indeed the case. For example we could take $X = S^1 \times [-1, 1]$ with $U = S^1 \times [-1, \frac{1}{2})$ and $V = S^1 \times (-\frac{1}{2}, 1]$. In this case we have $\pi_1(X) \cong \mathbb{Z}$ which is not isomorphic to the non-abelian group $\pi_1(U) * \pi_1(V) \cong \mathbb{Z} * \mathbb{Z}$.

We postpone the proof of the Seifert-van Kampen Theorem 9.2 to the next section and we first discuss several examples and applications of the theorem.

¹⁹²The proof of Lemma 9.1 is arguably easier to follow than the proof of Proposition 4.12. The reason is that in the special case Proposition 4.12 we introduced more notation, e.g. North pole and South pole, than was perhaps strictly necessary. All we really needed was that we could cover S^2 by two open simply connected subsets U and V such that $U \cap V$ is path-connected. The precise choice of U and V is irrelevant.

Let A and B be two topological spaces and let $a \in A$ and $b \in B$ be two points. Recall that on page 130 we had defined the wedge $A \vee B$ of A and B as

$$A \vee B := A \sqcup B / a = b.$$

We illustrate this definition in Figure 92. In the following we will often identify A and B with their respective images in $A \vee B$. Another way of thinking about the wedge of A and B is to say that the wedge $X = A \vee B$ can be decomposed into two subsets A and B such that $A \cap B$ is a point. Since $A \cap B$ is a point it is in particular simply connected. Thus it sounds like one should be able to apply the Seifert-van Kampen Theorem 9.2. This is almost correct, except that in the Seifert-van Kampen Theorem we have to deal with decompositions of a space X into *open* subsets. We therefore need to introduce a somewhat technical definition.



FIGURE 92. The wedge of a circle with a torus.

Definition. We say a point x in a topological space is *simple*, if there exists an open neighborhood U of x such that x is a deformation retract of U .

Examples.

- (1) Every point on a topological manifold admits an open neighborhood that is homeomorphic to a ball and every open ball deformation retracts to any of its points. This shows that every point on a topological manifold is simple. In particular every point on a sphere is simple.
- (2) We consider $X = \mathbb{C} \setminus \{\frac{1}{n} \mid n \in \mathbb{N}\}$. The point $x = 0$ is not simple, “since any open neighborhood of 0 has non-trivial fundamental group”. In exercise sheet 8 we will provide the full details for the proof that $x = 0$ is not a simple point of X .

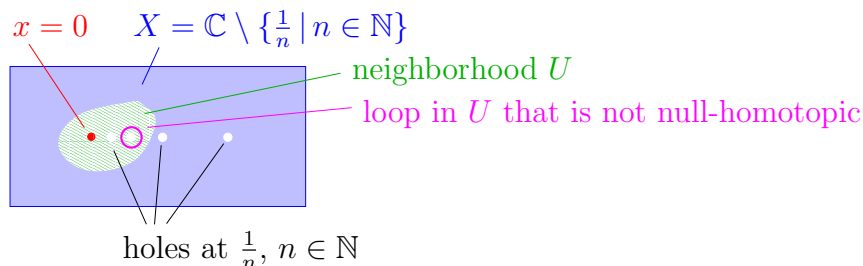


FIGURE 93. Example of a point that is not simple.

Proposition 9.3. *Let A and B be two path-connected topological spaces and let $a \in A$ and $b \in B$ be simple points. Then the inclusion maps induce an isomorphism¹⁹³*

$$\pi_1(A, a) * \pi_1(B, b) \xrightarrow{\cong} \pi_1(A \vee B, a = b).$$

Proof. We write $X = A \vee B$ and $x_0 = \{a, b\} \in A \vee B$. We pick an open neighborhood C in A of a that deformation retracts to a . Furthermore we pick an open neighborhood D of b in B that deformation retracts to b . We consider $U := A \cup D$ and $V := B \cup C$, each viewed as subsets of X .¹⁹⁴

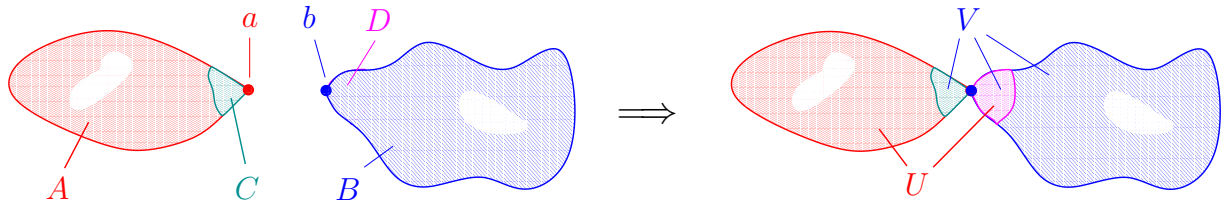


FIGURE 94. Schematic image for the proof of Proposition 9.3.

Now we consider the following diagram

$$\begin{array}{ccc} \pi_1(A, a) * \pi_1(B, b) & \longrightarrow & \pi_1(X, x_0) \\ \downarrow & & \downarrow = \\ \pi_1(U, x_0) * \pi_1(V, x_0) & \longrightarrow & \pi_1(X, x_0). \end{array}$$

We make the following observations:

- (1) The vertical and horizontal maps are all induced by the various inclusions of spaces. In particular the diagram commutes.
- (2) It follows from the definition of the quotient topology on $A \vee B = (A \sqcup B) / \sim$ that U and V are open subsets of X .
- (3) By the choice of C and D the subset $U \cap V = C \cup D$ of $X = A \vee B$ deformation retracts to x_0 .¹⁹⁵ This implies by Lemma 7.4 and Proposition 7.5 that $U \cap V$ is simply connected.
- (4) By (2) and (3) we can apply Seifert-van Kampen Theorem 9.2 to the decomposition $A \vee B = U \cup V$ and we obtain that the bottom horizontal map of the above diagram is an isomorphism.

¹⁹³More precisely, the two obvious inclusions $A \rightarrow A \vee B$ and $B \rightarrow A \vee B$ induce homomorphisms $\pi_1(A, a) \rightarrow \pi_1(A \vee B, a = b)$ and $\pi_1(B, b) \rightarrow \pi_1(A \vee B, a = b)$. Combined with Lemma 8.9 (2) we obtain the homomorphism $\pi_1(A, a) * \pi_1(B, b) \rightarrow \pi_1(A \vee B, a = b)$.

¹⁹⁴So to be precise, U is really the image of $A \cup D$ under the projection map $A \sqcup B \rightarrow A \sqcup B / a = b$.

¹⁹⁵Why is that the case?

- (5) By the choice of C it follows easily that A is a deformation retract of $U = A \cup C$. In particular it follows from Lemma 7.4 that the inclusion $A \rightarrow U = A \cup C$ is a homotopy equivalence. The same way we see that the inclusion $B \rightarrow V = B \cup D$ is a homotopy equivalence.
- (6) It follows from (5) together with Proposition 7.5 that the two inclusion induced maps $\pi_1(A, a) \rightarrow \pi_1(U, a)$ and $\pi_1(B, b) \rightarrow \pi_1(V, b)$ are isomorphisms, which implies that the left vertical map in the above commutative diagram is an isomorphism.
- (7) We deduce that the top horizontal map is, as desired, also an isomorphism. $\square$

Examples.

- (1) We consider again the wedge $S^1 \vee S^1$ with $a = (1, 0)$ and $b = (-1, 0)$. These points are simple.¹⁹⁶ By the discussion on page 112 we know that $\pi_1(S^1, a) = \langle x \rangle$ ¹⁹⁷ and

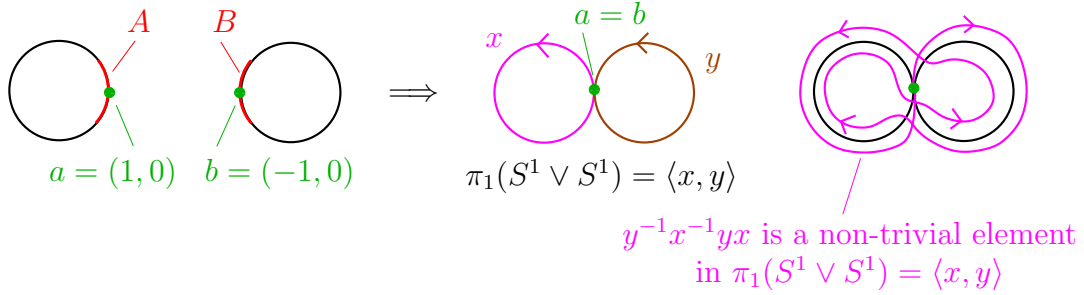


FIGURE 95. The wedge of two circles.

$\pi_1(S^1, b) = \langle y \rangle$ where x and y are represented by loops that go “once around the circle”, as illustrated in Figure 95. Proposition 9.3 says that the inclusion maps induce an isomorphism

$$\langle x, y \rangle = \langle x \rangle * \langle y \rangle = \pi_1(S^1, a) * \pi_1(S^1, b) \xrightarrow{\cong} \pi_1(S^1 \vee S^1, a = b).$$

- (2) We consider the loop γ from page 67 in $\mathbb{C} \setminus \{\pm 1\}$ that is also illustrated in Figure 96 on the left. We had already pointed out on page 131 that there exists a deformation retraction f from $\mathbb{C} \setminus \{\pm 1\}$ to the wedge of two circles that is given by

$$S^1 \vee S^1 = \underbrace{\{z \in \mathbb{C} \mid |z + 1| = 1\} \cup \{z \in \mathbb{C} \mid |z - 1| = 1\}}_{\text{two circles that meet at the origin}}.$$

¹⁹⁶This follows either from the general statement that any point on a topological manifold is simple, or more simply, by taking A and B to be small “intervals” around a and b .

¹⁹⁷Recall that $\langle x \rangle$ denotes the free group generated by x , i.e. $\langle x \rangle = \{\dots, x^{-1}, e = x^0, x, x^2, \dots\}$.

We then have $f_*([\gamma]) = [f \circ \gamma] = \langle y^{-1}x^{-1}yx \rangle \in \langle x, y \rangle = \pi_1(S^1 \vee S^1)$, which is non-trivial. Thus we see that $f \circ \gamma$ is not null-homotopic in $S^1 \vee S^1$ which implies that γ is not null-homotopic in $\mathbb{C} \setminus \{\pm 1\}$.¹⁹⁸

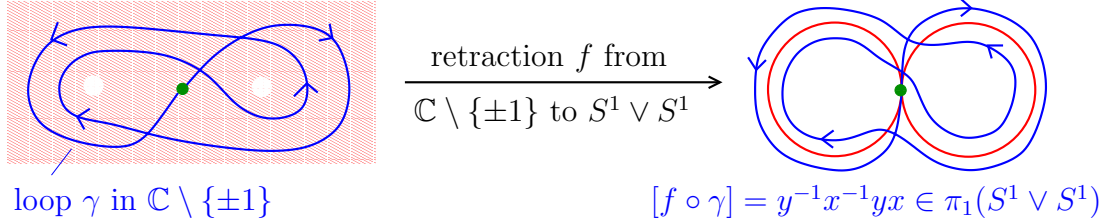


FIGURE 96.

- (3) The same argument as in (1) shows of course that the fundamental group of a wedge of k circles is isomorphic to the free group on k generators, where each generator corresponds precisely to a loop around one of the k circles.

Lemma 9.4.

- (1) Let G be a finite connected topological graph with v vertices and e edges. Then

$$\pi_1(G) \cong \text{free group on } 1 - \chi(G) = 1 + e - v \text{ generators.}$$

- (2) Two finite connected graphs are homotopy equivalent if and only if they have the same Euler characteristic.

Proof.

- (1) Let G be a finite connected topological graph with v vertices and e edges. In Proposition 7.8 we had shown that the graph G is homotopy equivalent to the wedge of $1 - \chi(G) = 1 + e - v$ circles. Thus it follows from Propositions 7.5 and 9.3 that $\pi_1(G)$ is isomorphic to the free group on $1 - \chi(G) = 1 + e - v$ generators.
- (2) Let G and G' be two finite connected graphs. If $\chi(G) = \chi(G')$, then we had already seen in Proposition 7.8 that G and G' are homotopy equivalent. On the other hand, if $\chi(G) \neq \chi(G')$, then it follows from (1) together with Lemma 8.12 that G and G' are not homotopy equivalent. $\square$

As a last example in this section we would like to determine the fundamental group of

$$X = (\mathbb{R} \sqcup (\mathbb{Z} \times S^2)) / n \sim (n, (0, 0, -1)).$$

The space X consists of an infinite line where to each $n \in \mathbb{Z}$ we attach a 2-dimensional sphere. The space X is illustrated in Figure 97. Given $k \in \mathbb{N}$ we denote by X_k the space

¹⁹⁸We could also have applied Seifert-van Kampen Theorem 9.2 to the covering $\mathbb{C} \setminus \{\pm 1\} = U \cup V$ where $U = \{z \in \mathbb{C} \setminus \{-1\} \mid \operatorname{Re}(z) < \frac{1}{2}\}$ and $V = \{z \in \mathbb{C} \setminus \{1\} \mid \operatorname{Re}(z) > -\frac{1}{2}\}$. Both sets U and V are open and they are homotopy equivalent to a circle.

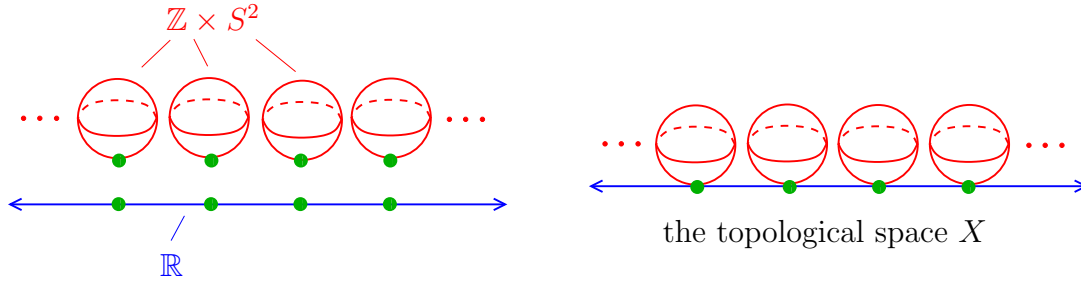


FIGURE 97.

that is obtained by X by restricting oneself to the interval $(-k - \frac{1}{2}, k + \frac{1}{2})$, i.e.

$$X_k = \left(\left(-k - \frac{1}{2}, k + \frac{1}{2} \right) \sqcup (\{-k, \dots, k-1, k\} \times S^2) \right) / n \sim (n, (0, 0, -1)).$$

Note that X_k can be viewed as the wedge of an open interval with $2k + 1$ spheres. It follows immediately from Proposition 4.12, Proposition 9.3 and a straightforward induction argument that $\pi_1(X_k) = 0$ for all k . Furthermore one can easily convince oneself that the sets X_k are open. It now follows from the following lemma that $\pi_1(X)$ is also trivial.

Lemma 9.5. *Let X be a path-connected topological space and let $x_0 \in X$. Furthermore let $X_k, k \in \mathbb{N}$ be a sequence of subsets of X such that the following hold:*

- (1) *each X_k is open,*
- (2) *each X_k is simply connected,*
- (3) *the sequence X_k is nested, i.e. for each k we have $X_k \subset X_{k+1}$,*
- (4) *we have $\bigcup_{k \in \mathbb{N}} X_k = X$.*

Then $\pi_1(X, x_0) = 0$.

The proof of the lemma is an exercise in exercise sheet 9.

9.2. Proof of the Seifert-van Kampen Theorem 9.2. After this long list of applications we finally provide the proof of the Seifert-van Kampen Theorem 9.2.

Proof. Let X be a topological space and let $U, V \subset X$ be two open subsets with $X = U \cup V$ and with $U \cap V \neq \emptyset$ such that $U \cap V$ is simply connected. We choose a base point $x_0 \in U \cap V$. We write $A := \pi_1(U, x_0)$ and $B := \pi_1(V, x_0)$. We need to show that the map

$$\Phi: A * B = \pi_1(U, x_0) * \pi_1(V, x_0) \rightarrow \pi_1(X, x_0)$$

that is induced by the inclusions $U \rightarrow X$ and $V \rightarrow X$ is an isomorphism. Taking into account Lemma 9.1 it suffices to show that the map is injective. To prove injectivity it is enough to show that there exists a map $\Psi: \pi_1(X, x_0) \rightarrow A * B$ such that $\Psi \circ \Phi = \text{id}_{A*B}$. We will prove the existence of such a map Ψ in three steps:

- (1) we first construct a map $\bar{\Psi}: \{\text{loops in } (X, x_0)\} \rightarrow A * B$,

(2) then we show that $\overline{\Psi}$ descends to a well-defined map

$$\begin{aligned} \Psi: \pi_1(X, x_0) &\rightarrow A * B \\ [g] &\mapsto \overline{\Psi}(g) \end{aligned}$$

(3) and finally we show that $\Psi \circ \Phi = \text{id}_{A*B}$.

For the remainder of the proof we will identify $A * B$ with $S(A, B)/\sim$ using the isomorphism given by Proposition 8.15.

We start out with step (1). Let $g: [0, 1] \rightarrow X$ be a loop in (X, x_0) . We apply Corollary 1.6 to the map $g: [0, 1] \rightarrow X$ and the open cover $X = U \cup V$. We obtain an N such that for every $i \in \{0, \dots, N-1\}$ the image of the interval $[\frac{i}{N}, \frac{i+1}{N}]$ under the map g lies in U or in V . We introduce a few definitions.

- (a) We say a subinterval of $[0, 1]$ can be *colored green* if the image of the subinterval under the map g lies in U and we say a interval can be *colored blue* if the image of the subinterval lies in V . Note that the image of a subinterval lies in $U \cap V$ if and only if the subinterval can be colored green as well as blue.
- (b) A *coloring* is a map $\alpha: \{0, \frac{1}{N}, \dots, 1 - \frac{1}{N}\} \rightarrow \{\text{blue}, \text{green}\}$ such that each subinterval $[i, i + \frac{1}{N}]$ can be colored by $\alpha(i)$. This definition is illustrated in Figure 98.

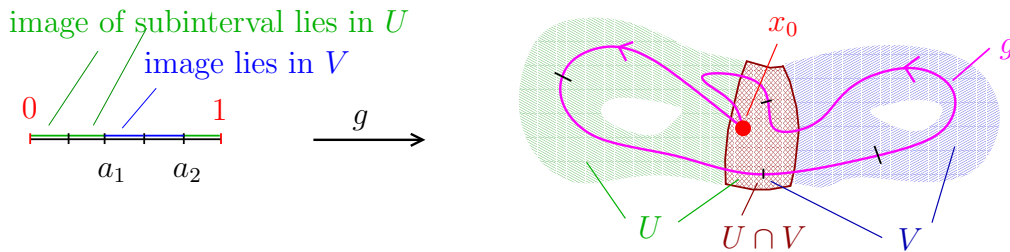


FIGURE 98.

- (c) Given a coloring α we denote by $0 < a_1 < \dots < a_{k-1} < 1$ the numbers where the coloring of the subintervals changes from one color to another. This definition is also illustrated in Figure 98. Note that for each a_i the intervals $[a_i - \frac{1}{N}, a_i]$ and $[a_i, a_i + \frac{1}{N}]$ can be colored in two different colors, which implies that $g(a_i) \in U \cap V$.
- (d) A *basing* for a coloring α is a collection of paths $p_1, \dots, p_{k-1}$ in $U \cap V$ such that each path p_i goes from x_0 to $g(a_i)$.¹⁹⁹

¹⁹⁹Such basing exists since each $g(a_i)$ lies in $U \cap V$ and since $U \cap V$ is path-connected.

Now given a coloring α and a basing $p_1, \dots, p_{k-1}$ we define^{200 201}

$$\bar{\Psi}(g) := \left[\underbrace{[g|_{[0,a_1]} * \bar{p}_1]}_{\text{loop in } (U, x_0) \text{ or } \text{loop in } (V, x_0)}, \underbrace{[p_1 * g|_{[a_1,a_2]} * \bar{p}_2]}_{\text{loop in } (U, x_0) \text{ or } \text{loop in } (V, x_0)}, \dots, \underbrace{[p_{k-1} * g|_{[a_{k-1},1]}]}_{\text{loop in } (U, x_0) \text{ or } \text{loop in } (V, x_0)} \right] \in S(A, B) / \sim = A * B.$$

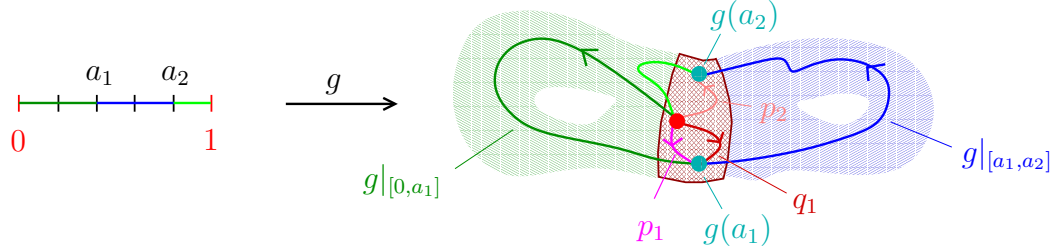


FIGURE 99.

Our next goal is to prove the following claim.

Claim. The definition of $\bar{\Psi}(g)$ does not depend on any of the choices made.

We need to prove the following three statements:

- (a) given N and a coloring the definition does not depend on the choice of the basings,
- (b) given N the definition does not depend on the choice of the coloring,
- (c) the definition does not depend on the choice of N .

We first fix a coloring α and we show that the definition of $\bar{\Psi}(g)$ does not depend on the choice of the basing. First we pick a different path q_1 from x_0 to $g(a_1)$ instead of the original path p_1 . (See Figure 99 for an illustration.) In this case we have

$$\begin{aligned} & \text{definition of } \bar{\Psi}(g) \text{ using the basing } p_1, p_2, \dots, p_{k-1} = \\ &= [g|_{[0,a_1]} * \bar{p}_1, [p_1 * g|_{[a_1,a_2]} * \bar{p}_2], \dots, [p_{k-1} * g|_{[a_{k-1},1]}]] \\ &= [g|_{[0,a_1]} * \bar{p}_1 * q_1 * \bar{q}_1, [q_1 * \bar{q}_1 * p_1 * g|_{[a_1,a_2]} * \bar{p}_2], \dots, [p_{k-1} * g|_{[a_{k-1},1]}]] \\ & \quad \uparrow \\ & \text{since } q_1 * \bar{q}_1 \text{ is null-homotopic} \\ &= [g|_{[0,a_1]} * q_1, [\bar{q}_1 * g|_{[a_1,a_2]} * \bar{p}_2], \dots, [p_{k-1} * g|_{[a_{k-1},1]}]] \\ & \quad \uparrow \\ & \bar{p}_1 * q_1 \text{ and } \bar{q}_1 * p_1 \text{ are null-homotopic since } U \cap V \text{ is simply connected}^{202} \\ &= \text{definition of } \bar{\Psi}(g) \text{ using the basing } q_1, p_2, \dots, p_{k-1}. \end{aligned}$$

²⁰⁰Each path $p_m * g|_{[a_m, a_{m+1}]} * \bar{p}_{m+1}$ is a loop in (U, x_0) or (V, x_0) , so each path represents an element in $\pi_1(U, x_0)$ or $\pi_1(V, x_0)$. Thus we obtain a sequence of elements in $\pi_1(U, x_0)$ and $\pi_1(V, x_0)$. The equivalence class of this sequence defines an element in $S(A, B) / \sim = A * B$.

²⁰¹This definition is almost the same as in the proof of Lemma 9.1.

Precisely the same argument shows that the choices of the other paths p_i are also irrelevant. This concludes the proof of (a).

Now we show that the definition does not depend on the choice of the coloring. So suppose we have two colorings α and β . We can turn one coloring into the other by a finite sequence of color swaps of a single subinterval. Therefore it suffices to consider the case that there exists a single i where the values for $\alpha(i)$ and $\beta(i)$ differ.

We consider the case, sketched in Figure 100, with $\alpha(i) = \text{blue}$ and $\beta(i) = \text{green}$ such that the common coloring to the left is green and the common coloring to the right is blue. All other cases are dealt with in a similar way.²⁰³

Let $0 = a_0 < a_1 < \dots < a_k = 1$ be the decomposition for α . We pick a basing $p_1, \dots, p_{k-1}$ for α . There exists a unique m with $a_m = i$. The corresponding decomposition for the coloring β is given by $0 = a_0 < a_1 < \dots < a_m + \frac{1}{N} < a_{m+1} < a_k = 1$. We write $q = g|_{[a_m, a_m + \frac{1}{N}]}$. Since $[a_m, a_m + \frac{1}{N}]$ can be colored green and blue we see that q lies actually in $U \cap V$. We have

$$\begin{aligned} \text{definition of } \bar{\Psi}(g) \text{ using the coloring } \alpha \text{ and the basing } p_1, \dots, p_{k-1} &= \\ = [g|_{[0, a_1]} * \bar{p}_1, \dots, [p_{m-1} * g|_{[a_{m-1}, a_m]} * \bar{p}_m], [p_m * g|_{[a_m, a_{m+1}]} * \bar{p}_{m+1}], \dots] \\ = [g|_{[0, a_1]} * \bar{p}_1, \dots, [p_{m-1} * g|_{[a_{m-1}, a_m + \frac{1}{N}]} * \bar{q} * \bar{p}_m], [p_m * q * g|_{[a_m + \frac{1}{N}, a_{m+1}]} * \bar{p}_{m+1}], \dots] \\ \uparrow \end{aligned}$$

since $g|_{[a_{m-1}, a_m]} \simeq g|_{[a_{m-1}, a_m + \frac{1}{N}]} * \bar{q}$ and $g|_{[a_m, a_m + \frac{1}{N}]} \simeq q * g|_{[a_m + \frac{1}{N}, a_{m+1}]}$

= definition of $\bar{\Psi}(g)$ using the coloring β and basing $p_1, \dots, p_{m-1}, p_m * q, p_{m+1}, \dots, p_{k-1}$.

(We refer to Figure 100 for an illustration of the proof.) This concludes the proof of (b).

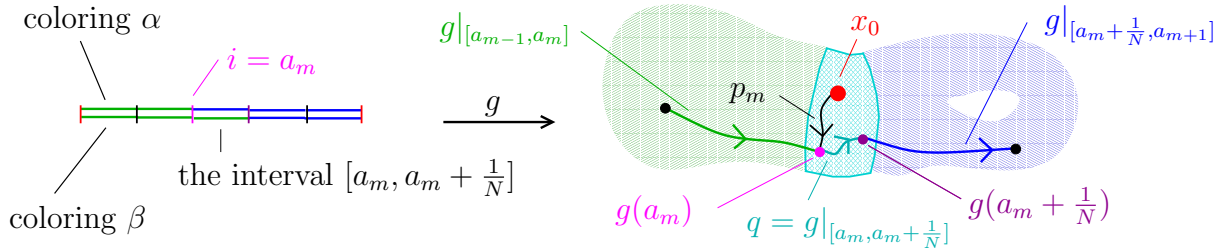


FIGURE 100.

It remains to prove (c). First suppose that instead of dividing $[0, 1]$ into N intervals we divide it into $m \cdot N$ intervals for some $m \in \mathbb{N}$. A coloring for the subdivision into N subintervals then also gives rise to a coloring into $m \cdot N$ subintervals, and it follows

²⁰²This is the only time that in the proof we use the hypothesis that $U \cap V$ is simply connected. In fact if one studies the argument carefully one sees that we only used that the maps $\pi_1(U \cap V, x_0) \rightarrow \pi_1(U, x_0)$ and $\pi_1(U \cap V, x_0) \rightarrow \pi_1(V, x_0)$ are trivial.

²⁰³In fact the only case which requires a slightly modified argument is that the coloring to the left and right are the same, i.e. either both are blue or both are green. We leave it as an exercise to the reader to perform that argument.

immediately from the definitions that we end up with the same element of $A * B$. Now if had chosen a different N' instead, then we can replace N and N' by $N \cdot N'$ and by the argument we just made we obtain the same element of $A * B$. This concludes the proof of (c). Summarizing, we are now done with the proof of the claim.

We turn to step (2) of the above program. We need to prove the following claim.

Claim. If $g, g': [0, 1] \rightarrow X$ are two homotopic loops in (X, x_0) , then $\overline{\Psi}(g) = \overline{\Psi}(g')$.

Let $g, g': [0, 1] \rightarrow X$ be two homotopic loops in (X, x_0) . We pick a homotopy

$$\begin{aligned} H: [0, 1] \times [0, 1] &\rightarrow X \\ (t, s) &\mapsto H^s(t). \end{aligned}$$

from the loop g to the loop g' . We apply Corollary 1.6 to the map $H: [0, 1] \times [0, 1] \rightarrow X$ and the open cover $X = U \cup V$. We obtain an N such that for each choice of $i, j \in \{0, \frac{1}{N}, \dots, 1 - \frac{1}{N}\}$ the image of the square $[i, i + \frac{1}{N}] \times [j, j + \frac{1}{N}]$ under the map H lies in U or in V . Given $j \in \{0, \frac{1}{N}, \dots, 1\}$ we now define

$$\begin{aligned} g(j): [0, 1] &\rightarrow X \\ t &\mapsto H^j(t) \end{aligned}$$

Note that each $g(j)$ is a loop in x_0 and that $g(0) = g$ and $g(1) = g'$. Therefore it suffices to show that for any $j \in \{0, \frac{1}{N}, \dots, 1 - \frac{1}{N}\}$ we have $\overline{\Psi}(g(j)) = \overline{\Psi}(g(j + \frac{1}{N}))$. So let $j \in \{0, \frac{1}{N}, \dots, 1 - \frac{1}{N}\}$.

As above we say that a subset of $[0, 1] \times [0, 1]$ can be *colored green* if the image of the subset under the map H lies in U and we say the subset can be *colored blue* if the image lies in V . We pick a map $\alpha: \{0, \frac{1}{N}, \dots, 1 - \frac{1}{N}\} \rightarrow \{\text{blue}, \text{green}\}$ such that for each i the value $\alpha(i)$ is a color with which the square $[i, i + \frac{1}{N}] \times [\frac{j}{N}, \frac{j+1}{N}]$ can be colored. Note that α also defines a coloring for the loops $g(j)$ and $g(j + \frac{1}{N})$.

We define $0 = a_0 < a_1 < \dots < a_k = 1$ as above for the coloring α . Furthermore we pick a basing $p_i, i = 1, \dots, k-1$ for the loop $g(j)$. For $i = 0, \dots, k$ we write $q_i = H^{[j, j + \frac{1}{N}]}|_{a_i}$. Put differently, q_j connects a point on the loop $g(j)$ to the corresponding point on the loop $g(j + \frac{1}{N})$. We now have

$$\begin{aligned} \overline{\Psi}(g(j)) &= \overline{\Psi}(g(j)) \text{ defined using the coloring } \alpha \text{ and the basing } p_1, \dots, p_{k-1} \\ &= [\dots * [p_m * \underbrace{H^j|_{[a_m, a_{m+1}]}}_{=g(j)|_{[a_m, a_{m+1}]}} * \bar{p}_{m+1}] * \dots] \\ &= [\dots * [p_m * q_m * \bar{q}_m * H^j|_{[a_m, a_{m+1}]} * q_{m+1} * \bar{q}_{m+1} * \bar{p}_{m+1}] * \dots] \\ &\quad \uparrow \\ &\text{since each } q_m * \bar{q}_m \text{ is null-homotopic} \\ &= [\dots * [p_m * q_m * \underbrace{H^{j + \frac{1}{N}}|_{[a_m, a_{m+1}]}}_{=g(j + \frac{1}{N})|_{[a_m, a_{m+1}]}} * \bar{q}_{m+1} * \bar{p}_{m+1}] * \dots] \\ &\quad \uparrow \\ &\text{since } H \text{ restricts to homotopies from } \bar{q}_m * H^j|_{[a_m, a_{m+1}]} * q_{m+1} \text{ to } H^{j + \frac{1}{N}}|_{[a_m, a_{m+1}]} \text{ for } m = 0, \dots, k-1 \\ &= \overline{\Psi}(g(j + \frac{1}{N})) \text{ defined using } \alpha \text{ and the basing } p_1 * q_1, \dots, p_{k-1} * q_{k-1}. \end{aligned}$$

This last part of the proof is illustrated in Figure 101. This concludes the proof of the claim and thus of part (2) of our three step program.

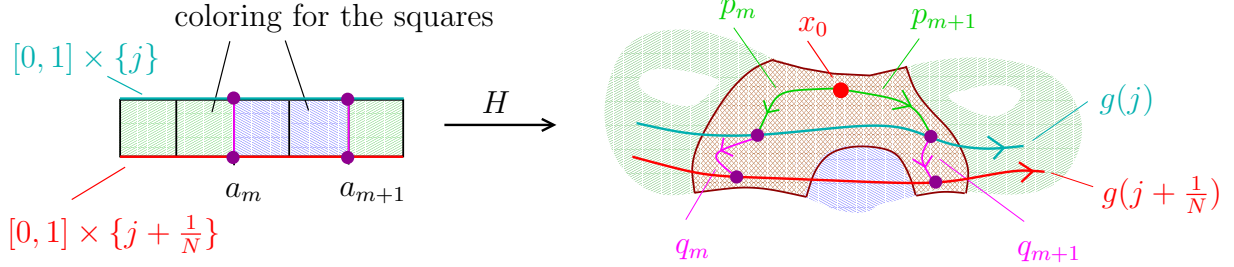


FIGURE 101.

Summarizing we now showed that the map $\bar{\Psi}$ descends to a well-defined map

$$\begin{aligned} \Psi: \pi_1(X, x_0) &\rightarrow A * B \\ [g] &\mapsto [\bar{\Psi}(g)]. \end{aligned}$$

It remains to prove the following claim.

Claim. We have $\Psi \circ \Phi = \text{id}_{A*B}$.

So let $[y_1] \cdots [y_k] \in A * B$ where $y_1, \dots, y_k$ are loops in (U, x_0) and (V, x_0) . Then

$$\begin{aligned} \text{definition of } \Phi: A * B = \pi_1(U, x_0) * \pi_1(V, x_0) &\rightarrow \pi_1(X, x_0) \\ \downarrow & \\ (\Psi \circ \Phi)([y_1] \cdots [y_k]) &= \Psi([y_1 * \cdots * y_k]) = [\bar{\Psi}(y_1 * \cdots * y_k)] = [y_1 * \cdots * y_k] \\ &= [y_1] \cdots [y_k]. \end{aligned}$$

$\uparrow$
 by definition of $\bar{\Psi}$, where we use $N = k$ and where
 we can use the constant paths as basings since
 $y_1, \dots, y_k$ are already loops in (U, x_0) and (V, x_0)

This concludes the proof of the claim and thus also the proof of the Seifert-van Kampen Theorem 9.2 $\square$

9.3. More examples: surfaces and the connected sum of manifolds. Given $g \in \mathbb{N}_0$ we denote by Σ_g the surface of genus g . In this section we first want to prove the following proposition.

Proposition 9.6. *For $g \neq h$ the surface Σ_g of genus g is not homeomorphic to the surface Σ_h of genus h .*

As on several earlier occasions we want to prove this result by considering what happens if we remove a point from the topological spaces.

Lemma 9.7. *For any point P on the surface Σ_g the complement $\Sigma_g \setminus \{P\}$ is homotopy equivalent to the wedge of $2g$ circles.*

Proof. We prove the lemma for the surface Σ of genus 2. The proof for $g \geq 3$ is almost identical. So let $\Sigma = E_8 / \sim$ as sketched in Figure 102. We denote by O the origin in $E_8 \subset \mathbb{C}$. We first show that $\Sigma \setminus \{O\}$ is homotopy equivalent to the wedge of 4 circles.

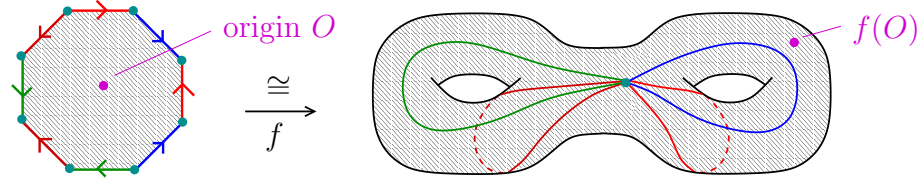


FIGURE 102.

Claim. There exists a deformation retraction from $\Sigma \setminus \{O\} = (E_8 \setminus \{O\}) / \sim$ to $\partial E_8 / \sim$.

First we consider the map²⁰⁴²⁰⁵

$$\begin{aligned} r: E_8 \setminus \{O\} &\rightarrow \partial E_8 \\ re^{i\varphi} &\mapsto \text{unique point } se^{i\varphi} \text{ on } \partial E_8. \end{aligned}$$

This map “projects points radially outward towards the boundary”. The deformation retraction is now given by “pushing outward radially”, i.e. it is given by the map

$$\begin{aligned} ((E_8 \setminus \{O\}) / \sim) \times [0, 1] &\mapsto (E_8 \setminus \{O\}) / \sim \\ ([P], t) &\mapsto [P \cdot (1 - t) + r(P) \cdot t]. \end{aligned}$$

This concludes the proof of the claim.

Claim. The space $\partial E_8 / \sim$ is homeomorphic to the wedge of 4 circles.

A homeomorphism is given by sending the point on $\partial E_8 / \sim$ that corresponds to the vertices to the common point of the four circles, and by sending the edges that get identified to the same circle. We refer to Figure 103 for an illustration. We leave it as an exercise to write down the proof in detail.

The two claims put together show that $\Sigma \setminus \{O\}$ is homotopy equivalent to the wedge of 4 circles.

Now let P be some other point on Σ . By Proposition 2.5 there exists a homeomorphism $\Phi: \Sigma \rightarrow \Sigma$ with $\Phi(P) = O$. This shows that $\Sigma \setminus \{P\}$ is homeomorphic to $\Sigma \setminus \{O\}$. Since the latter topological space is homotopy equivalent to the wedge of 4 circles it follows that the former is also homotopy equivalent to the wedge of 4 circles. $\square$

Now we can turn to the proof of Proposition 9.6.

²⁰⁴Note that this map cannot be extended continuously to O .

²⁰⁵Here we use polar coordinates, i.e. r, s are positive real numbers.

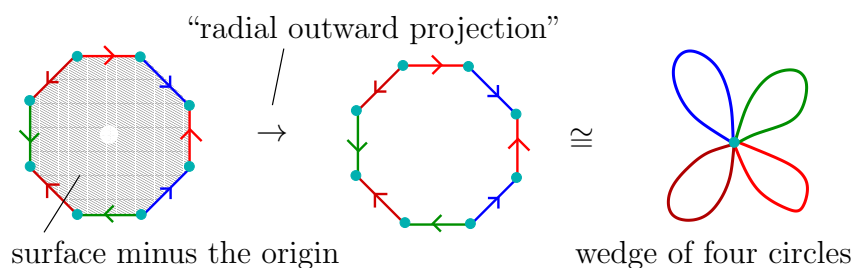


FIGURE 103.

Proof of Proposition 9.6. Let S and T be two surfaces of genus g and h . Suppose that there exists a homeomorphism $f: S \rightarrow T$. Pick a point $P \in S$. We write $Q = f(P)$. The homeomorphism f restricts to a homeomorphism $S \setminus \{P\} \rightarrow T \setminus \{Q\}$. It now follows that

$$\langle x_1, \dots, x_{2g} \rangle \cong \pi_1(\text{wedge of } 2g \text{ circles}) \xrightarrow{\cong} \pi_1(S \setminus \{P\}) \xrightarrow{f_*} \pi_1(T \setminus \{Q\}) \cong \langle y_1, \dots, y_{2h} \rangle.$$

$\uparrow$ see the example on page 159 by Lemma 9.7 and Proposition 7.5 same argument as before $\uparrow$

It follows from Lemma 8.12 that $g = h$. □

For many applications the following theorem, which is a variation on the Seifert-van Kampen Theorem 9.8, is very useful.

Theorem 9.8. (Seifert-van Kampen for manifolds) *Let M be an m -dimensional manifold and let $R, S \subset M$ be two m -dimensional submanifolds such that the following hold:*

- (1) $M = R \cup S$,
- (2) $R \cap S$ is a component of ∂R and it is a component of ∂S ,
- (3) $R \cap S$ is simply connected.

We pick a base point $x_0 \in R \cap S$. Then the inclusion induced map

$$\pi_1(R, x_0) * \pi_1(S, x_0) \rightarrow \pi_1(M, x_0)$$

is an isomorphism.

Example. Let $n \geq 3$ and let $H_+ := \{(x_1, \dots, x_{n+1}) \in S^n \mid x_{n+1} \geq 0\}$ be the closed upper hemisphere and define similarly the closed lower hemisphere H_- . Then H_+, H_- are submanifolds of S^n that satisfy the hypothesis of Theorem 9.8 and we obtain once again that $\pi_1(S^n) \cong \pi_1(H_+) * \pi_1(H_-) = \{e\}$.

We will of course prove Theorem 9.8 by reducing it to Theorem 9.2. As we will see, the proof is very similar to the proof of Proposition 9.3 where we determined the fundamental group of a wedge of two spaces.

Proof of Theorem 9.8. Let M be an m -dimensional manifold and let $R, S \subset M$ be two m -dimensional submanifolds such that the following hold:

- (1) $M = R \cup S$,

- (2) $R \cap S$ is a union of components of ∂R and it is a union of components of ∂S ,
- (3) $R \cap S$ is non-empty and simply connected.

We pick a base point $x_0 \in R \cap S$. Furthermore we pick maps $f: \partial R \times [0, 1) \rightarrow R$ and $g: \partial S \times [0, 1) \rightarrow S$ that are provided by the Collar Neighborhood Theorem 2.6. We set

$$U := R \cup \underbrace{g((R \cap S) \times [0, 1))}_{\subset S \times [0, 1)} \quad \text{and} \quad V := S \cup \underbrace{f((R \cap S) \times [0, 1))}_{\subset R \times [0, 1)}.$$

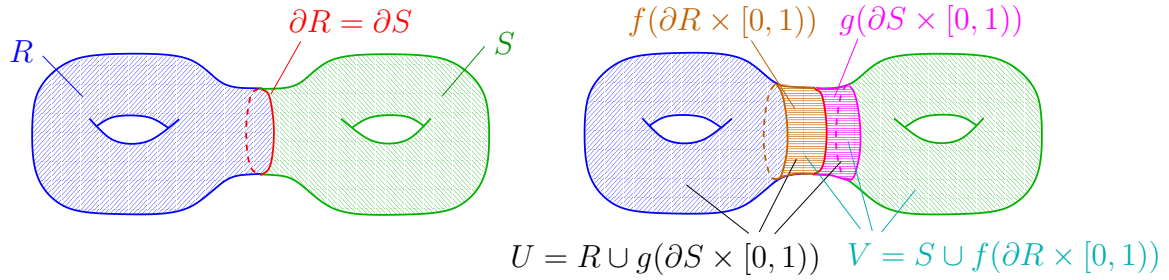


FIGURE 104. Illustration of the proof of Theorem 9.8.

We consider the following diagram

$$\begin{array}{ccc} \pi_1(R, x_0) * \pi_1(S, x_0) & \longrightarrow & \pi_1(M, x_0) \\ \downarrow & & \downarrow = \\ \pi_1(U, x_0) * \pi_1(V, x_0) & \longrightarrow & \pi_1(M, x_0). \end{array}$$

Similar to the proof of Proposition 9.3 one can now show that the following holds:

- (1) the diagram commutes,
- (2) U is an open subset of M and R is a deformation retract of U , similar for V and S , in particular the vertical map on the left is an isomorphism,
- (3) the intersection

$$U \cap V = \underbrace{g((R \cap S) \times [0, 1))}_{\subset S \times [0, 1)} \cup \underbrace{f((R \cap S) \times [0, 1))}_{\subset R \times [0, 1)} \cong (R \cap S) \times (-1, 1)$$

is homotopy equivalent to $R \cap S$, in particular it is simply connected by our hypothesis,

- (4) by the Seifert-van Kampen Theorem 9.2 the bottom horizontal map is an isomorphism,
- (5) hence the top horizontal map is also an isomorphism. □

We conclude this section by considering the connected sum of two manifolds of dimension ≥ 3 . More precisely, we have the following proposition:

Proposition 9.9. *Let $n \geq 3$ and let M and M' be two connected oriented n -dimensional manifolds. Then*

$$\pi_1(M \# M') \cong \pi_1(M) * \pi_1(M').$$

Examples.

- (1) Let p, q be two coprime integers and let r, s be another two coprime integers. We can then consider the lens spaces $L(p, q)$ and $L(r, s)$. It follows from Proposition 9.9 and Corollary 6.17 that

$$\pi_1(L(p, q) \# L(r, s)) \cong \mathbb{Z}_p * \mathbb{Z}_r.$$

It follows for example from this calculation, from Lemma 8.13 and from the Grushko Decomposition Theorem 8.14 that the two topological spaces $L(3, 1) \# L(35, 1)$ and $L(5, 1) \# L(21, 1)$ are not homeomorphic.

- (2) Let M and N be two connected oriented n -dimensional manifolds. We denote by $-N$ the same manifold as N but with the opposite orientation. It follows from Proposition 9.9 that $\pi_1(M \# N) \cong \pi_1(M \# (-N))$.

Proof of Proposition 9.9. Let M and M' be two connected oriented n -dimensional manifolds and let $\varphi: \overline{B}^n \rightarrow M$ and $\varphi': \overline{B}^n \rightarrow M'$ be two embeddings where φ is orientation-preserving and φ' is orientation-reversing. Recall that the connected sum of M and M' is defined as

$$M \# M' := \left(\underbrace{(M \setminus \varphi(B^n))}_{=: X} \sqcup \underbrace{(M' \setminus \varphi'(B^n))}_{=: X'} \right) / \varphi(P) = \varphi'(P) \text{ for } P \in S^{n-1}.$$

We pick a base point $x_0 \in X \cap X' = \varphi(S^{n-1}) = \varphi'(S^{n-1})$.

Claim. The inclusion induced maps $\pi_1(X, x_0) \rightarrow \pi_1(M, x_0)$ and $\pi_1(X', x_0) \rightarrow \pi_1(M', x_0)$ are isomorphisms.

We show that the inclusion induced map $\pi_1(X, x_0) \rightarrow \pi_1(M, x_0)$ is an isomorphism. We write $Y = \varphi(\overline{B}^n)$. It follows from Proposition 2.1 that X and Y are n -dimensional submanifolds of M . Note that $X \cup Y = M$ and that $X \cap Y = \partial Y \cong S^{n-1}$. Since $n \geq 3$ we obtain from Proposition 4.12 that $X \cap Y$ is simply connected. We obtain that

$$\begin{array}{ccc} \pi_1(X, x_0) & \xrightarrow{\cong} & \pi_1(X, x_0) * \pi_1(\overline{B}^n, x_0) & \xrightarrow{\cong} & \pi_1(M). \\ \uparrow & & \uparrow & & \uparrow \\ & & \text{since } \pi_1(\overline{B}^n, x_0) = \{e\} & & \text{Theorem 9.8} \end{array}$$

The same way we can show that the inclusion induced map $\pi_1(X', x_0) \rightarrow \pi_1(M', x_0)$ is an isomorphism. This concludes the proof the claim.

We now see that²⁰⁶

$$\begin{array}{ccc} \pi_1(M \# M, x_0) & \xleftarrow{\cong} & \pi_1(X, x_0) * \pi_1(X', x_0) & \xrightarrow{\cong} & \pi_1(M, x_0) * \pi_1(M', x_0). \\ \uparrow & & \uparrow & & \uparrow \\ & \text{by Theorem 9.8, since} & & \text{above claim} & \\ & X \cap X' \cong S^{n-1} \text{ is simply-connected} & & & \end{array}$$

□

Remark. We cannot apply the Seifert-van Kampen Theorem 9.2 to the connected sum of two surfaces, since in this case the fundamental group of the overlap $R \cap R'$ in the proof of Proposition 9.9 is not simply connected. In particular we cannot use this approach to determine the fundamental group of the surface of genus 2, which by Figure 35 is the connected sum of two tori. Our next goal is to formulate and prove a generalization of the Seifert-van Kampen Theorem 9.2 which will allow us to determine the fundamental group of the surface of genus g for $g \geq 2$.

²⁰⁶Here the directions of the arrow indicate in which direction the obvious inclusion induced maps point.

10. PRESENTATIONS OF GROUPS AND AMALGAMATED PRODUCTS

We have now made some progress, for example we have shown that surfaces of different genus are not homeomorphic. But this proof is perhaps slightly unsatisfactory in so far as we used the trick of removing a point, instead of dealing directly with the fundamental groups of surfaces.

In the next section we will cover the generalized Seifert-van Kampen Theorem. It will allow us to compute many more fundamental groups. In particular we will determine the fundamental groups of surfaces and knot complements.

Before we can state the generalized Seifert-van Kampen Theorem we need to introduce more concepts from group theory. We will do so in this chapter.

10.1. Basic definitions in group theory. Let G be a group. Recall that a subgroup H is called *normal* if for any $g \in G$ we have $g^{-1}Hg = H$. If $H \subset G$ is normal, then we often write $H \triangleleft G$. If $H_i, i \in I$ is a family of normal subgroups of G , then it is straightforward to see that the intersection $\bigcap_{i \in I} H_i$ is again a normal subgroup of G .

Let H be a subgroup of G . The subgroup H defines an equivalence relation on G by defining $g \sim \tilde{g}$ if there exists an $h \in H$ with $gh = \tilde{g}$. Put differently, $g \sim \tilde{g}$ if $\tilde{g}^{-1}g \in H$. The equivalence classes are precisely the sets of the form $gH = \{gh \mid h \in H\}$ for some $g \in G$. We denote the set of equivalence classes by G/H .

Remark. Let H be a normal subgroup of G .

(1) For any $h \in H$ and $a, b \in G$ we have

$$ahbH = ab \cdot \underbrace{b^{-1}hb}_{\text{in } H \text{ since } H \text{ is normal}} H = abH.$$

(2) If $aH = bH$, then for any $c, d \in G$ we have

$$cadH = cb \underbrace{(b^{-1}a)}_{\in H} d = \underbrace{cbd}_{\substack{\uparrow \\ \text{above remark (1)}}} H.$$

We will now see that the group structure on G descends to a group structure on G/H . More precisely, given aH and bH we set

$$aH \cdot bH := abH.$$

It follows immediately from the above remark (1) that this definition does not depend on the representatives a and b . It is now straightforward to see that this defines indeed a group structure on G/H . The following lemma plays the same role as Lemma 1.7 does for topological spaces.

Lemma 10.1. *Let G be a group and let H be a normal subgroup. Let $\varphi: G \rightarrow \Gamma$ be a group homomorphism with the property that $H \subset \ker(\varphi)$. Then there exists a unique*

homomorphism $\psi: G/H \rightarrow \Gamma$ such that the following diagram commutes

$$\begin{array}{ccc} G & \longrightarrow & G/H \\ & \searrow \varphi & \downarrow \psi \\ & & \Gamma. \end{array}$$

Proof. For $gH \in G/H$ we put $\psi(gH) := \varphi(g)$. Since $H \subset \ker(\varphi)$ this definition is independent of the choice of the representatives g .²⁰⁷ It is straightforward to verify that ψ is a homomorphism and that it is uniquely determined by the desired property. $\square$

10.2. Presentation of groups.

Definition. Let G be a group and let $A \subset G$ be a subset. We define

$$\text{subgroup of } G \text{ generated by } A := \langle A \rangle := \bigcap_{\substack{A \subset H \subset G \\ H \text{ subgroup}}} H,$$

i.e. $\langle A \rangle$ is the intersection of all subgroups H of G which contain A . This is again a subgroup of G , namely the smallest²⁰⁸ subgroup of G which contains A . Similarly we define

$$\text{subgroup of } G \text{ normally generated by } A := \langle\langle A \rangle\rangle := \bigcap_{A \subset H \triangleleft G} H,$$

i.e. $\langle\langle A \rangle\rangle$ is the intersection of all *normal* subgroups H of G which contain A . This is again a normal subgroup of G , namely the smallest normal subgroup of G which contains A .

The following lemma follows immediately from the definitions and from the fact that in an abelian group every subgroup is normal.

Lemma 10.2. *Let G be a group and let $A \subset G$ be a subset.*

- (1) *If A is a subgroup, then $\langle A \rangle = A$.*
- (2) *If A is a normal subgroup, then $\langle\langle A \rangle\rangle = A$.*
- (3) *If G is abelian, then $\langle\langle A \rangle\rangle = \langle A \rangle$.*

The following lemma gives a more explicit description of $\langle A \rangle$ and $\langle\langle A \rangle\rangle$.

Lemma 10.3. *Let G be a group and let $A \subset G$ be a subset.*

- (1) *Then $\langle A \rangle$ equals the set of all elements of G which can be written as products of elements in A and their inverses, i.e.*

$$\langle A \rangle = \{a_1^{\epsilon_1} \cdots a_k^{\epsilon_k} \mid a_1, \dots, a_k \in A \text{ and } \epsilon_1, \dots, \epsilon_k \in \{-1, 1\}\}.$$

- (2) *The subset $\langle\langle A \rangle\rangle$ equals the set of all elements of G which can be written as products of conjugates of elements in A and their inverses, i.e.*

$$\langle\langle A \rangle\rangle = \{g_1 a_1^{\epsilon_1} g_1^{-1} \cdots g_k a_k^{\epsilon_k} g_k^{-1} \mid a_1, \dots, a_k \in A, g_1, \dots, g_k \in G \text{ and } \epsilon_1, \dots, \epsilon_k \in \{-1, 1\}\}.$$

²⁰⁷Indeed, if $gH = \tilde{g}H$, then $g = \tilde{g}h$ for some $h \in H$, therefore $\varphi(g) = \varphi(\tilde{g}h) = \varphi(\tilde{g}) \cdot \varphi(h) = \varphi(\tilde{g})$.

²⁰⁸Here “smallest” means that if B is another subgroup that contains A , then $A \subset B$. This statement is obvious, since B appears in the definition of $\langle A \rangle$ as one of the H ’s.

Example. On page 99 we had introduced a subgroup of the group H of all self-homeomorphisms of $\mathbb{R}^2$ that was generated by two given self-homeomorphisms A and B . The lemma shows that the definition of “being generated” by a subset used on page 99 agrees with the above more formal definition.

Proof. We only prove the second statement. The proof of the first statement is very similar. So let G be a group and let $A \subset G$ be a subset. The subgroup $\langle\langle A \rangle\rangle$ contains all elements of A and their inverses. The normal subgroup $\langle\langle A \rangle\rangle$ then contains also all elements of the form $ga^{\pm 1}g^{-1}$, $g \in G, a \in A$. Since $\langle\langle A \rangle\rangle$ is a subgroup it is also closed under multiplication, i.e. it contains products of such elements. It follows that $\langle\langle A \rangle\rangle$ contains the right-hand side of (2). Conversely it is straightforward to see that the right-hand side is indeed a normal²⁰⁹ subgroup of G that contains A , hence it is contained in the left-hand side of (2). $\square$

Lemma 10.4. *Let G be a group and let $A \subset G$ be a subset. Let $\alpha: G \rightarrow \Gamma$ be a group homomorphism such that $\alpha(a) = e \in \Gamma$ for all $a \in A$. Then there exists a unique group homomorphism $\beta: G/\langle\langle A \rangle\rangle \rightarrow \Gamma$ such that the following diagram commutes:*

$$\begin{array}{ccc} G & \longrightarrow & G/\langle\langle A \rangle\rangle \\ & \searrow \alpha & \downarrow \beta \\ & & \Gamma. \end{array}$$

Proof. Let G be a group and let $A \subset G$ be a subset. Furthermore let $\alpha: G \rightarrow \Gamma$ be a group homomorphism such that $\alpha(a) = e \in \Gamma$ for all $a \in A$. This means that $A \subset \ker(\alpha)$. Since $\ker(\alpha)$ is a normal subgroup it follows from the definition of $\langle\langle A \rangle\rangle$ that $\langle\langle A \rangle\rangle \subset \ker(\alpha)$. The proposition now follows immediately from Lemma 10.1. $\square$

The following definition will subsequently play an important role:

Definition. Let X be a set and let R be a subset of the free group $\langle X \rangle$ generated by the set X . We define

$$\langle X | R \rangle := \langle X \rangle / \langle\langle R \rangle\rangle.$$

We refer to the elements of X as *generators of $\langle X | R \rangle$* and we refer to the elements of R as the *relators of $\langle X | R \rangle$* .

Examples.

- (1) For any $k \in \mathbb{N}_0$ we have

$$\langle g | g^k \rangle = \langle g \rangle / \langle\langle g^k \rangle\rangle = \langle g \rangle / \langle g^k \rangle \cong \mathbb{Z}_k.$$

$\uparrow$
since $\langle g \rangle$ is abelian it follows from Lemma 10.2 that $\langle\langle g^k \rangle\rangle = \langle g^k \rangle$

- (2) The group $\langle x_1, \dots, x_k | \rangle$ which has no relators is just the free group on the generators $x_1, \dots, x_k$.

²⁰⁹Why is the right-hand side normal?

(3) We consider the group

$$\pi = \langle x, y \mid [x, y] \rangle$$

where $[x, y] := xyx^{-1}y^{-1}$ is the commutator of x and y . What can we say about this group? Modulo $\langle\langle [x, y] \rangle\rangle$ we have²¹⁰

$$\begin{array}{ccc} yx = xyx^{-1}y^{-1} \cdot yx = xy & & xy^{-1} = y^{-1} \cdot yx \cdot y^{-1} = y^{-1} \cdot xy \cdot y^{-1} = y^{-1}x. \\ \uparrow & \text{and} & \uparrow \\ \text{Remark (1) on page 171} & & \text{calculation on the left and Remark (2) on page 171} \end{array}$$

Similarly one can show that the equalities $yx^{-1} = x^{-1}y$ and $x^{-1}y^{-1} = y^{-1}x^{-1}$ hold in $\pi = \langle x, y \mid [x, y] \rangle$. This shows that in π the elements $x^{\pm 1}$ and $y^{\pm 1}$ commute. This means that in π an arbitrary product of powers of x and y can be reordered to equal a product of the $x^m y^n$. For example in π we have the equality

$$\begin{array}{ccccccc} x^3 y^2 x^{-1} y & = & x^3 y \cdot yx^{-1} \cdot y & = & x^3 y \cdot x^{-1} y \cdot y & = & x^3 \cdot yx^{-1} \cdot y^2 = x^3 \cdot x^{-1} y \cdot y^2 = x^2 y^3. \\ & & \uparrow & & \uparrow & & \uparrow \\ & & \text{since } yx^{-1} = x^{-1}y & & \text{since } yx^{-1} = x^{-1}y & & \end{array}$$

By Lemma 8.10 there exists a unique homomorphism $\alpha: \langle x, y \rangle \rightarrow \mathbb{Z}^2$ which satisfies $\alpha(x) = (1, 0)$ and $\alpha(y) = (0, 1)$. Note that

$$\alpha([x, y]) = \alpha(xyx^{-1}y^{-1}) = \alpha(x) + \alpha(y) - \alpha(x) - \alpha(y) = 0.$$

Thus by Lemma 10.4 there exists a unique group homomorphism

$$\beta: \pi = \langle x, y \rangle / \langle\langle [x, y] \rangle\rangle = \langle x, y \mid [x, y] \rangle \rightarrow \mathbb{Z}^2$$

with $\beta(x) = (1, 0)$ and $\beta(y) = (0, 1)$. This homomorphism is evidently an epimorphism. It is also a monomorphism, indeed:

- (a) we had just seen in the previous example that any element in π is equivalent to an element of the form $x^m y^n$,
- (b) furthermore $\beta(x^m y^n) = (m, n)$ is zero if and only if $x^m y^n = x^0 y^0$ is the trivial element in π .

Frequently we will write relators as equations, for example we often write

$$\langle x, y \mid yx^{-1}y^{-1} \rangle = \langle x, y \mid yx^{-1}y^{-1} = e \rangle = \langle x, y \mid xy = yx \rangle.$$

Because of this notation one often refers to a relator also as a relation.

(4) The same way as in the above example one can show that there exists an isomorphism

$$\beta: \langle x_1, \dots, x_m \mid [x_i, x_j] = e \text{ for all } i, j \in \{1, \dots, m\} \rangle \xrightarrow{\cong} \mathbb{Z}^m$$

which sends each x_i to the i -th standard basis vector e_i of $\mathbb{Z}^m$.

²¹⁰Here we follow the standard convention that in the notation we do not distinguish between an element in $\langle X \rangle$ and an element in $\langle X \mid R \rangle$, the same way as usually we do not distinguish in the notation between an element in $\mathbb{Z}$ and an element in $\mathbb{Z}_5 = \mathbb{Z}/5\mathbb{Z}$.

(5) We consider the following three self-homeomorphisms of $\mathbb{R}^2$:

$$\begin{aligned} A: \mathbb{R}^2 &\rightarrow \mathbb{R}^2 & B: \mathbb{R}^2 &\rightarrow \mathbb{R}^2 & \tilde{A}: \mathbb{R}^2 &\rightarrow \mathbb{R}^2 \\ (s, t) &\mapsto (s+1, 1-t) & \text{and} & (s, t) \mapsto (s, t+1) & \text{and} & (s, t) \mapsto (s+1, -t). \end{aligned}$$

We already know the first two maps from page 99. We denote by G the subgroup of all homeomorphisms of $\mathbb{R}^2$ that is generated by A and B . We are interested in the group G since $\mathbb{R}^2/G$ is the Klein bottle K and by Theorem 6.16 we have $\pi_1(K) \cong G$.

Note that $\tilde{A} = B^{-1}A$. Let $X = \{x, y\}$ be the set that consists of two elements. By Lemma 8.10 there exists a unique homomorphism $\alpha: \langle x, y \rangle \rightarrow G$ with $\alpha(x) = \tilde{A}$ and $\alpha(y) = B$. Now let R be the subset of $\langle X \rangle = \langle x, y \rangle$ that consists of the single element xyx^{-1} . We want to consider $\alpha(yxyx^{-1}) \in G$, which is by definition a map from $\mathbb{R}^2$ to $\mathbb{R}^2$. For any point $(s, t) \in \mathbb{R}^2$ this map is given by

$$\begin{aligned} \alpha(yxyx^{-1})(s, t) &= B\tilde{A}B\tilde{A}^{-1}(s, t) = B\tilde{A}B(s-1, -t) \\ &= B\tilde{A}(s-1, -t+1) = B(s, t-1) = (s, t). \end{aligned}$$

This shows that $\alpha(yxyx^{-1})$ is the identity map of $\mathbb{R}^2$, i.e. it is the trivial element in G . It follows from Lemma 10.4 that α descends to a homomorphism

$$\beta: \langle x, y \rangle / \langle\langle yxyx^{-1} \rangle\rangle = \langle x, y \mid yxyx^{-1} \rangle \rightarrow G.$$

This homomorphism is an epimorphism, since the image of β is the group generated by $\tilde{A} = B^{-1}A$ and B , but this group is the same as the group generated by A and B , i.e. the image of β is G . Is the homomorphism β also a monomorphism? We will come back to this question later on.

We continue with the following definitions.

Definition.

- (1) A *presentation* is a set X together with a subset $R \subset \langle X \rangle$. We call the elements of X the *generators* and we call the elements of R the *relators*²¹¹ of the presentation.
- (2) Let π be a group. A *presentation for π* is a presentation (X, R) together with an isomorphism

$$\pi \xrightarrow{\cong} \langle X \mid R \rangle.$$

Very often one just says $\langle X \mid R \rangle$ is a presentation for π and the explicit isomorphism is suppressed from the notation.

- (3) We say that a group π is *finitely generated*, if π admits a presentation with finitely many generators.²¹²
- (4) We say a group π is *finitely presented*, if π admits a presentation with finitely many generators and finitely many relators.

We now give some examples of finitely generated and finitely presented groups.

²¹¹The relators are often also called relations.

²¹²This definition of a finitely generated group is equivalent to the one we gave on page 148. Why is that?

Examples.

- (1) Every group admits a presentation. Indeed, let π be a group. Then we set $X := \pi$, i.e. X is the set of elements of π . By Lemma 8.10 there exists a unique homomorphism $\alpha: \langle X \rangle \rightarrow \pi$ with $\alpha(g) = g$ for all $g \in X = \pi$. We set $R := \ker(\alpha)$. Then

$$\langle X | R \rangle = \langle X \rangle / \langle\langle R \rangle\rangle = \langle X \rangle / \langle\langle \ker(\alpha) \rangle\rangle = \langle X \rangle / \ker(\alpha) \xrightarrow[\alpha]{\cong} G.$$

↑
by Lemma 10.2 since $R = \ker(\alpha)$ is a normal subgroup

- (2) We had just seen on page 174 that

$$\mathbb{Z}^n \cong \langle x_1, \dots, x_n \mid [x_i, x_j] = e \text{ with } i, j \in \{1, \dots, n\} \rangle$$

which shows that $\mathbb{Z}^n$ is a finitely presented group.

- (3) In fact every finitely generated abelian group is also finitely presented. Indeed, let A be a finitely generated abelian group. Then

$$\begin{array}{ccc} A \cong \bigoplus_{i=1}^n \mathbb{Z}_{a_i} & = & \langle x_1, \dots, x_n \mid [x_i, x_j] = e \text{ for } i, j = 1, \dots, n, x_i^{a_i} = e \text{ for } i = 1, \dots, n \rangle. \\ \uparrow & & \uparrow \\ \text{Theorem 8.3} & & \text{as in (2)} \\ \text{for some } a_i \in \mathbb{N}_0 & & \end{array}$$

- (4) There exist groups which are finitely generated but not finitely presented. In fact there are countably many isomorphism types of finitely presented groups,²¹³ whereas there are uncountably many isomorphism types of finitely generated groups.²¹⁴ A proof for both statements is given in Chapter 14 of [Rb].
- (5) We had seen on page 148 that the fundamental group of the Klein bottle is finitely generated. But is it also finitely presented?

Remark. Every finitely presented group admits infinitely many finite presentations. Indeed, let $X = \{x_1, \dots, x_k\}$ be a finite set and let $r = \{r_1, \dots, r_l\} \subset \langle X \rangle = \langle x_1, \dots, x_k \rangle$ also be a finite set. Then for any word s in $r_1, \dots, r_l$, i.e. any element s of $\langle r_1, \dots, r_l \rangle$, we have the equality²¹⁵

$$(1) \quad \langle x_1, \dots, x_k \mid r_1, \dots, r_l \rangle = \langle x_1, \dots, x_k \mid r_1, \dots, r_l, s \rangle$$

and for any symbol x and any word s in $x_1, \dots, x_k$ we have an isomorphism²¹⁶

$$(2) \quad \begin{array}{ccc} \langle x_1, \dots, x_k \mid r_1, \dots, r_l \rangle & \xrightarrow{\cong} & \langle x_1, \dots, x_k, x \mid r_1, \dots, r_l, x \cdot s \rangle. \\ x_i & \mapsto & x_i \end{array}$$

²¹³That statement is pretty obvious.

²¹⁴This statement is much less obvious. The idea is that in a free group $\langle x_1, \dots, x_k \rangle$ there are uncountably many infinite sequences of elements $\{r_1, r_2, \dots\}$. The difficulty is to show that there exist uncountably many sequences that give raise to non-isomorphic groups.

²¹⁵It is straightforward to see that $\langle\langle r_1, \dots, r_l \rangle\rangle = \langle\langle r_1, \dots, r_l, s \rangle\rangle$ as subsets of $\langle\langle x_1, \dots, x_k \rangle\rangle$. This shows that the two groups in (1) are identical. On the other hand the presentations are different, since $\{r_1, \dots, r_l\}$ and $\{r_1, \dots, r_l, s\}$ are two different subsets of $\langle x_1, \dots, x_k \rangle$.

The above two methods, and their inverses, of obtaining new presentations out of a given one are called *Tietze transformations*.²¹⁷ Any two finite presentations for some given group are in fact related by a finite sequence of these two Tietze transformations. This fact is for example proved in Proposition II.2.2 of [LS].

10.3. The abelianization of a group. The following definition should already be familiar from the algebra course.

Definition. Let π be a group. We refer to

$$[\pi, \pi] := \langle \underbrace{\{ [x, y] \mid x, y \in \pi \}}_{=xyx^{-1}y^{-1}} \rangle$$

as the *commutator subgroup* of π .

We start out with the following elementary lemma.

Lemma 10.5. *Let π be a group. Then the following hold:*

- (1) *The group $\pi/[\pi, \pi]$ is abelian.*
- (2) *We have*

$$[\pi, \pi] := \left\{ \prod_{i=1}^n [x_i, y_i] \mid x_1, y_1, \dots, x_n, y_n \in \pi \right\}.$$

Remark. Note that in general not every element of the commutator subgroup is a commutator. For example, if $\pi = \langle a, b, c, d \rangle$ is the free group on four generators a, b, c, d , then we have $[a, b] \cdot [c, d] \in [\pi, \pi]$, but it is not a commutator.²¹⁸

Proof. Let π be a group.

- (1) For any two elements $x, y \in \pi$ we have

$$xy[\pi, \pi] = yx \cdot x^{-1}y^{-1}xy[\pi, \pi] = yx \cdot \underbrace{[x^{-1}, y^{-1}]}_{\in [\pi, \pi]}[\pi, \pi] = yx[\pi, \pi].$$

This shows that $\pi/[\pi, \pi]$ is indeed abelian.

- (2) Note that the inverse of a commutator is again a commutator, i.e. given $x, y \in \pi$ we have

$$[x, y]^{-1} = (xyx^{-1}y^{-1})^{-1} = yxy^{-1}x^{-1} = [y, x].$$

Furthermore a conjugate of a commutator is again a commutator. Indeed, given $x, y, g \in \pi$ we have

$$\begin{aligned} g[x, y]g^{-1} &= gxyx^{-1}y^{-1}g^{-1} = gxx^{-1}gyg^{-1}gx^{-1}g^{-1}gy^{-1}g^{-1} \\ &= (gxx^{-1})(gyg^{-1})(gxg^{-1})^{-1}(gyg^{-1})^{-1} = [gxg^{-1}, gyg^{-1}]. \end{aligned}$$

²¹⁶The fact that the map given by $x_i \mapsto x_i$, $i = 1, \dots, k$ induces an isomorphism is an exercise in exercise sheet 9.

²¹⁷Heinrich Tietze (1880-1964) was an Austrian mathematician.

²¹⁸Why not?

It follows from this discussion and from Lemma 10.3 that each element of the commutator subgroup can be written as a product of commutators, i.e.

$$[\pi, \pi] := \left\{ \prod_{i=1}^n [x_i, y_i] \mid x_1, y_1, \dots, x_n, y_n \in \pi \right\}. \quad \square$$

Definition. Let π be a group. We call

$$\pi_{\text{ab}} := \pi / [\pi, \pi]$$

the *abelianization* of π .²¹⁹

The following proposition summarizes some of the key properties of the abelianization of groups.

Proposition 10.6.

- (1) If π is abelian, then $\pi_{\text{ab}} = \pi$.
- (2) Let π be a group and let $\alpha: \pi \rightarrow H$ be a homomorphism to an abelian group. Then there exists a unique homomorphism $\beta: \pi_{\text{ab}} = \pi / [\pi, \pi] \rightarrow H$ such that the following diagram commutes:

$$\begin{array}{ccc} \pi & \xrightarrow{\quad} & \pi_{\text{ab}} = \pi / [\pi, \pi] \\ & \searrow \alpha & \downarrow \beta \\ & & H. \end{array}$$

In particular for any group π and any abelian group G the map

$$\begin{array}{ccc} \text{Hom}(\pi_{\text{ab}}, G) & \rightarrow & \text{Hom}(\pi, G) \\ (\alpha: \pi_{\text{ab}} \rightarrow G) & \mapsto & (\pi \rightarrow \pi_{\text{ab}} \xrightarrow{\alpha} G) \end{array}$$

is a bijection.

- (3) Let G and H be two groups. There exists a unique isomorphism

$$(G * H)_{\text{ab}} \xrightarrow{\cong} G_{\text{ab}} \times H_{\text{ab}}$$

which makes the following diagram commute:

$$\begin{array}{ccc} G * H & \longrightarrow & G \times H \\ \downarrow & & \downarrow \\ (G * H)_{\text{ab}} & \xrightarrow{\cong} & G_{\text{ab}} \times H_{\text{ab}}, \end{array}$$

where the other three maps are the obvious projection maps.

²¹⁹The abelianization of a group π is sometimes also written as $H_1(\pi)$. The reason for that somewhat strange looking notation will become clear in Algebraic Topology II.

- (4) Let $F = \langle x_1, \dots, x_m \rangle$ be the free group on m generators $x_1, \dots, x_m$. Then the map²²⁰

$$\begin{aligned} \Phi: F_{\text{ab}} = F/[F, F] &\rightarrow \mathbb{Z}^m \\ [x_i] &\mapsto e_i := \text{the } i\text{-th standard basis vector} \end{aligned}$$

is an isomorphism.

- (5) Let $\pi = \langle x_1, \dots, x_m \mid r_1, \dots, r_n \rangle$ be a finitely presented group. We denote by

$$\Phi: \langle x_1, \dots, x_m \rangle \rightarrow \mathbb{Z}^m$$

the homomorphism that is given by $\Phi(x_i) = e_i$. There exists a unique isomorphism

$$\Psi: \pi/[\pi, \pi] \xrightarrow{\cong} \mathbb{Z}^m / \langle \Phi(r_1), \dots, \Phi(r_n) \rangle$$

such that the following diagram commutes

$$\begin{array}{ccc} \langle x_1, \dots, x_m \rangle & \xrightarrow[\Phi]{x_i \mapsto e_i} & \mathbb{Z}^m \\ \downarrow & & \downarrow \\ \pi = \langle x_1, \dots, x_m \rangle / \langle\langle r_1, \dots, r_n \rangle\rangle & & \\ \downarrow & & \downarrow \\ \pi_{\text{ab}} = \pi/[\pi, \pi] & \xrightarrow{\Psi} & \mathbb{Z}^m / \langle \Phi(r_1), \dots, \Phi(r_n) \rangle. \end{array}$$

- (6) We denote by $\mathcal{G}$ the category of groups and we denote by $\mathcal{A}$ the category of abelian groups. Then the maps

$$G \mapsto G_{\text{ab}} = G/[G, G]$$

and

$$(\varphi: G \rightarrow H) \mapsto \left(\begin{array}{ccc} \varphi_*: G_{\text{ab}} = G/[G, G] & \rightarrow & H_{\text{ab}} = H/[H, H] \\ [g] & \mapsto & [\varphi(g)] \end{array} \right)$$

define a covariant functor from $\mathcal{G}$ to $\mathcal{A}$.

- (7) Isomorphic groups have isomorphic abelianizations.

Examples.

- (1) By Proposition 10.6 (1) and (3) we have

$$(\mathbb{Z}_3 * \mathbb{Z}_{35})_{\text{ab}} \cong \mathbb{Z}_3 \times \mathbb{Z}_{35} \quad \text{and} \quad (\mathbb{Z}_5 * \mathbb{Z}_{21})_{\text{ab}} \cong \mathbb{Z}_5 \times \mathbb{Z}_{21}.$$

It follows from the Chinese Remainder Theorem that the abelianizations are isomorphic. On the other hand on page 150 we had already observed that the Grushko Decomposition Theorem 8.14 implies that the groups $\mathbb{Z}_3 * \mathbb{Z}_{35}$ and $\mathbb{Z}_5 * \mathbb{Z}_{21}$ are not isomorphic. Thus we have seen that non-isomorphic groups can have isomorphic abelianizations.

²²⁰By Lemma 8.10 there exists a unique homomorphism $F \rightarrow \mathbb{Z}^m$ with $x_i \mapsto e_i$. By (1) this map descends to a homomorphism $F/[F, F] \rightarrow \mathbb{Z}^m$. The way we write down the map is a little bit sloppy, since we only specify the map on the generators of $F/[F, F]$.

- (2) We give an explicit example for statement (5). Let

$$\pi = \langle x, y \mid x^2 y^{-2} x, y^{-1} x^3 y x \rangle.$$

Let $\Phi: \langle x, y \rangle \rightarrow \mathbb{Z}^2$ be the epimorphism given by $\Phi(x) = (1, 0)$ and $\Phi(y) = (0, 1)$. Then the abelianization of π is given by²²¹

$$\pi_{\text{ab}} = \mathbb{Z}^2 / \langle \underbrace{\Phi(x^3 y^{-2})}_{=\begin{pmatrix} 3 \\ -2 \end{pmatrix}}, \underbrace{\Phi(y^{-1} x^3 y x)}_{=\begin{pmatrix} 4 \\ 0 \end{pmatrix}} \rangle = \mathbb{Z}^2 / \begin{pmatrix} 3 & 4 \\ -2 & 0 \end{pmatrix} \mathbb{Z}^2 \cong \mathbb{Z}_8.$$

- (3) Let S be a set. It is straightforward to show that the abelianization of the free group $\langle S \rangle$ generated by S is isomorphic to the free abelian group $\mathbb{Z}^{(S)}$ generated by S .
- (4) Given $k \in \mathbb{N}$ we denote by S_k the permutation group on k elements and we denote by $A_k \subset S_k$ the alternating group, i.e. the group of all permutations with positive sign. In the algebra course it was shown that for $k \geq 5$ the group A_k is simple, i.e. the only normal subgroups of A_k are the trivial group and the group A_k itself.²²² The commutator subgroup $[A_k, A_k]$ is normal. Since A_k is not abelian we cannot have $[A_k, A_k] = \{e\}$. Thus we see that $[A_k, A_k] = A_k$, which implies that the abelianization of A_k is the trivial group.

Now we provide the proof of Proposition 10.6.

Proof of Proposition 10.6.

- (1) The first statement is obvious.²²³
- (2) Let π be a group and let $\alpha: \pi \rightarrow H$ be a homomorphism to an abelian group. Then $[\pi, \pi]$ lies in the kernel of α . The desired statement now follows immediately from Lemma 10.1.
- (3) The maps

$$\begin{aligned} \Phi: (G * H)_{\text{ab}} &\rightarrow G_{\text{ab}} \times H_{\text{ab}} \\ [(x_1, \dots, x_k)] &\mapsto \left(\left[\prod_{x_i \in G} x_i \right], \left[\prod_{x_i \in H} x_i \right] \right) \quad \text{and} \quad \Psi: G_{\text{ab}} \times H_{\text{ab}} \rightarrow (G * H)_{\text{ab}} \\ &([g], [h]) \mapsto [g \cdot h] \end{aligned}$$

are easily seen to be well-defined, to be homomorphisms and to be inverses of one another.

- (4) This statement follows immediately from the statements (1) and (3) together with the observation that $\langle x_1, \dots, x_m \rangle = \langle x_1 \rangle * \dots * \langle x_m \rangle$, that $\langle x_i \rangle_{\text{ab}} \cong \mathbb{Z}$ and that $\mathbb{Z}^m = \mathbb{Z} \times \dots \times \mathbb{Z}$.
- (5) It is straightforward to see that there exists a unique homomorphism

$$\Psi: \pi / [\pi, \pi] \rightarrow \mathbb{Z}^m / \langle \Phi(r_1), \dots, \Phi(r_n) \rangle$$

²²¹Why is the group isomorphic to $\mathbb{Z}_8$?

²²²A proof of this statement is also given in any self-respecting algebra book, see e.g. [Rb, p. 69].

²²³Why?

that makes the diagram of statement (5) commute. It remains to show that Ψ is an isomorphism. We write $F = \langle x_1, \dots, x_m \rangle$ and we write $G = \langle y_1, \dots, y_n \rangle$. We consider the following diagram

$$\begin{array}{ccc}
 G_{\text{ab}} = G/[G, G] & \xrightarrow[\cong]{[y_i] \mapsto e_i} & \mathbb{Z}^n \\
 \downarrow [y_i] \mapsto [r_i] & & \downarrow e_i \mapsto \Phi(r_i) \\
 F_{\text{ab}} = F/[F, F] & \xrightarrow[\cong]{\Phi([x_i]) = e_i} & \mathbb{Z}^m \\
 \downarrow & & \downarrow \\
 \pi_{\text{ab}} = \pi/[\pi, \pi] & \xrightarrow{\Psi} & \mathbb{Z}^m \\
 & & \downarrow \\
 & & \langle \Phi(r_1), \dots, \Phi(r_n) \rangle
 \end{array}$$

Note that the diagram commutes. By (4) the top two horizontal maps are isomorphisms. The bottom vertical maps are clearly epimorphisms and it is clear that the images of the top vertical maps lie in the kernel of the bottom vertical maps. It now follows from an elementary “diagram chase” that the bottom horizontal map is also an isomorphism.²²⁴ We leave the details to the reader.

- (6) This statement follows easily from the definitions and the observation that the image of a commutator under a group homomorphism is once again a commutator.
- (7) The last statement is an immediate consequence of (6). $\square$

10.4. The amalgamated product of groups. In the formulation of the generalized version of the Seifert-van Kampen theorem we will need to replace the free product of two groups by the “amalgamated product” of two groups.

In Section 8.2 we had first defined the free product via an explicit construction. We had then explained that one can also characterize the free product via a universal property. In this section we will proceed the other way round, we will first state the desired universal property and then we will explicitly construct a group which satisfies the universal property.

Proposition 10.7. *Let $\alpha: G \rightarrow A$ and $\beta: G \rightarrow B$ be two group homomorphisms. Then there exists a triple*

$$(A *_G B, \varphi: A \rightarrow A *_G B, \psi: B \rightarrow A *_G B)$$

²²⁴More precisely, we use the following fact: suppose we are given a commutative diagram

$$\begin{array}{ccc}
 A & \xrightarrow{\cong} & A' \\
 \downarrow & & \downarrow \\
 B & \xrightarrow{\cong} & B' \\
 \downarrow & & \downarrow \\
 C & \longrightarrow & C'
 \end{array}$$

of group homomorphisms where the bottom vertical maps are epimorphisms, where the top two horizontal maps are isomorphisms and where the kernels of the bottom vertical maps are the images of the top vertical maps. Then the bottom horizontal map is also an isomorphism. This general statement can be proved by a “diagram chase”. This statement can also be viewed as a special case of the “five lemma”.

where $A *_G B$ is a group and φ and ψ are homomorphisms with the following properties:

(a) the following diagram commutes:

$$\begin{array}{ccc} G & \xrightarrow{\alpha} & A \\ \beta \downarrow & & \downarrow \varphi \\ B & \xrightarrow[\psi]{} & A *_G B, \end{array}$$

(b) if $(H', \varphi': A \rightarrow H', \psi': B \rightarrow H')$ is another triple which satisfies property (a), then there exists a unique homomorphism $\Theta: A *_G B \rightarrow H'$ such that $\varphi' = \Theta \circ \varphi$ and $\psi' = \Theta \circ \psi$, i.e. such that the following diagram commutes

$$\begin{array}{ccc} G & \xrightarrow{\alpha} & A \\ \beta \downarrow & & \downarrow \varphi \\ B & \xrightarrow[\psi]{} & A *_G B \end{array} \begin{array}{c} \nearrow \varphi' \\ \searrow \Theta \\ \xrightarrow[\psi']{} \end{array} H'.$$

Proof. Let $\alpha: G \rightarrow A$ and $\beta: G \rightarrow B$ be two group homomorphisms. As usual we view A and B as subgroups of $A * B$. We put

$$A *_G B := A * B / \langle\langle \{\alpha(g)\beta(g)^{-1}\}_{g \in G} \rangle\rangle$$

and we define $\varphi: A \rightarrow A *_G B$ as the composition of the obvious homomorphisms

$$A \hookrightarrow A * B \twoheadrightarrow A * B / \langle\langle \{\alpha(g)\beta(g)^{-1}\}_{g \in G} \rangle\rangle,$$

and precisely the same way we define the homomorphism $\psi: B \rightarrow A *_G B$. It follows from the construction of $A *_G B$ that $\varphi(\alpha(g)) = \psi(\beta(g))$ for all $g \in G$. Therefore the triple $(A *_G B, \varphi: A \rightarrow A *_G B, \psi: B \rightarrow A *_G B)$ satisfies property (a).

Now let $(H', \varphi': A \rightarrow H', \psi': B \rightarrow H')$ be another triple which satisfies property (a). We consider the following diagram

$$\begin{array}{ccccc} G & \xrightarrow{\alpha} & A & & \\ \beta \downarrow & & \downarrow \varphi & \nearrow \varphi' & \\ B & \xrightarrow[\psi]{} & A *_G B = \frac{A * B}{\langle\langle \alpha(g)\beta(g)^{-1} \rangle\rangle} & \xrightarrow[\psi']{} & H' \\ & \searrow \Phi & & & \end{array}$$

Φ

where $\Phi: A * B \rightarrow H'$ is the unique homomorphism determined by ψ' and φ' . Since $\varphi' \circ \alpha = \psi' \circ \beta$ it follows that for any $g \in G$ we have

$$\Phi(\alpha(g) \cdot \beta(g)^{-1}) = \Phi(\alpha(g)) \cdot \Phi(\beta(g))^{-1} = \varphi'(\alpha(g)) \cdot \psi'(\beta(g))^{-1} = e.$$

Put differently, all elements of the form $\alpha(g)\beta(g)^{-1}$ in $A * B$ lie in the kernel of Φ . By Lemma 10.4 there exists a unique homomorphism $\Theta: A * B / \langle\langle \{\alpha(g)\beta(g)^{-1}\}_{g \in G} \rangle\rangle \rightarrow H'$ which makes the following diagram commute

$$\begin{array}{ccc} A * B & \xrightarrow{\quad} & A *_G B = A * B / \langle\langle \{\alpha(g)\beta(g)^{-1}\}_{g \in G} \rangle\rangle \\ & \searrow \Phi & \downarrow \Theta \\ & & H'. \end{array}$$

This proves the existence of Θ . The uniqueness follows from the observation that another choice of Θ' would give another choice for Φ , but this homomorphism is unique. $\square$

Definition. Let $\alpha: G \rightarrow A$ and $\beta: G \rightarrow B$ be two group homomorphisms. We write

$$A *_G B := A * B / \langle\langle \{\alpha(g)\beta(g)^{-1}\}_{g \in G} \rangle\rangle$$

and we refer to $A *_G B$ as the *amalgamated product of A and B with amalgam G* .²²⁵

Remark. A variation on the argument on page 138 shows that if

$$(H, \varphi: A \rightarrow H, \psi: B \rightarrow H)$$

is a triple that has both the properties (a) and (b) of Proposition 10.7, then there exists a unique isomorphism $\Theta: A *_G B \xrightarrow{\cong} H$ such that $\varphi(a) = \Theta(a)$ for all $a \in A$ and $\psi(b) = \Theta(b)$ for all $b \in B$. Often we will use this uniquely defined isomorphism to identify H with the amalgamated product $A *_G B$.

Example. Let $G = \mathbb{Z}_2$, $A = \mathbb{Z}_4$ and $B = \mathbb{Z}_6$ and consider the homomorphisms

$$\begin{array}{ccc} \alpha: G & \rightarrow & A \\ [k] & \mapsto & [2k] \end{array} \quad \text{and} \quad \begin{array}{ccc} \beta: G & \rightarrow & B \\ [k] & \mapsto & [3k]. \end{array}$$

Furthermore we consider the homomorphisms²²⁶

$$\begin{array}{ccc} \varphi: A = \mathbb{Z}_4 & \rightarrow & \text{SL}(2, \mathbb{Z}) \\ k & \mapsto & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^k \end{array} \quad \text{and} \quad \begin{array}{ccc} \varphi: B = \mathbb{Z}_6 & \rightarrow & \text{SL}(2, \mathbb{Z}) \\ k & \mapsto & \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}^k. \end{array}$$

It is straightforward to verify that²²⁷ $\varphi \circ \alpha = \psi \circ \beta$, therefore there exists by Proposition 10.7 a homomorphism

$$A *_G B = \mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6 \rightarrow \text{SL}(2, \mathbb{Z}).$$

More surprisingly, this homomorphism is in fact an isomorphism. We refer to [Rs, p. 218] and [Se, p. 35] for a proof.

²²⁵The notation is a little misleading since the homomorphisms α and β do not get mentioned in the notation, even though they play of course an important role. In particular different choices of α and β can lead to non-isomorphic amalgamated products.

²²⁶Are these really homomorphisms?

²²⁷Indeed, the one non-trivial element of $G = \mathbb{Z}_2$ gets sent under both maps to $-\text{id}$.

The following lemma summarizes two important special cases of the amalgamated product of groups.

Lemma 10.8.

- (1) For two groups A and B and $G = \{e\}$ the trivial group, the corresponding amalgamated product is the usual free product, i.e.

$$A *_{\{e\}} B = A * B.$$

- (2) If $\alpha: G \rightarrow A$ is a group homomorphism and if $\beta: G \rightarrow \{e\}$ is the trivial homomorphism, then

$$A *_G \{e\} = A / \langle\langle \alpha(G) \rangle\rangle.$$

Proof. Both statements follow immediately from the definition

$$A *_G B := A * B / \langle\langle \{\alpha(g)\beta(g)^{-1}\}_{g \in G} \rangle\rangle. \quad \square$$

The following lemma gives us a convenient description of the amalgamated product of two finitely presented groups.

Lemma 10.9. *Let*

$$A = \langle x_1, \dots, x_k \mid r_1, \dots, r_p \rangle$$

and

$$B = \langle y_1, \dots, y_l \mid s_1, \dots, s_q \rangle$$

be two finitely presented groups. Furthermore let $\alpha: G \rightarrow A$ and $\beta: G \rightarrow B$ be two group homomorphisms and let $g_1, \dots, g_m$ be generators of G . Then

$$A *_G B = \langle x_1, \dots, x_k, y_1, \dots, y_l \mid r_1, \dots, r_p, s_1, \dots, s_q \text{ and } \alpha(g_i) = \beta(g_i), i = 1, \dots, m \rangle.$$

Examples.

- (1) Let $A_1 = \langle x_1, y_1 \rangle$ and $A_2 = \langle x_2, y_2 \rangle$ be two free groups and for $i = 1, 2$ let $\varphi_i: \langle t \rangle \rightarrow A_i$ be the homomorphism that is given by $\varphi_i(t^n) := [x_i, y_i]^n$. Then we obtain from Lemma 10.9 that

$$A_1 *_{\langle t \rangle} A_2 = \langle x_1, y_1, x_2, y_2 \mid [x_1, y_1] = [x_2, y_2] \rangle.$$

- (2) Let $A = \langle a \rangle$ and let $B = \langle b \rangle$ be two infinite cyclic groups and let $G = \langle x, y \mid [x, y] \rangle$ be a group that is isomorphic to $\mathbb{Z}^2$. We consider the homomorphism $\alpha: G \rightarrow A$ that is given by $\alpha(x) = a$ and $\alpha(y) = e$ and we consider the homomorphism $\beta: G \rightarrow B$ that is given by $\beta(x) = e$ and $\beta(y) = b$. Then

$$A *_G B = \langle a, b \mid \alpha(x) = \beta(x), \alpha(y) = \beta(y) \rangle = \langle a, b \mid a = e, e = b \rangle = \langle \mid \rangle = \{e\}.$$

$\uparrow$

Lemma 10.9

$\uparrow$

Tietze transformation (2), see page 177

This shows in particular that the amalgamated product of two non-trivial groups can in fact be trivial.

Proof of Lemma 10.9. We need to show that

$$\underbrace{\langle\langle \{\alpha(g)\beta(g)^{-1}\}_{g \in G} \rangle\rangle}_{\text{definition of } A *_G B} = \underbrace{\langle\langle \{\alpha(g_i)\beta(g_i)^{-1}\}_{i=1, \dots, m} \rangle\rangle}_{\text{right-hand side of the proposition}}.$$

Put differently, we need to prove the following claim.

Claim. In the free product $A * B$ we have

$$\langle\langle \{\alpha(g)\beta(g)^{-1}\}_{g \in G} \rangle\rangle = \langle\langle \{\alpha(g_i)\beta(g_i)^{-1}\}_{i=1, \dots, m} \rangle\rangle.$$

It is clear that the right-hand side is contained in the left-hand side. To show the reverse inclusion we need to show that for each $g \in G$ we have $\alpha(g)\beta(g)^{-1} \in \langle\langle \{\alpha(g_i)\beta(g_i)^{-1}\}_{i=1, \dots, m} \rangle\rangle$. We start out with two observations:

(1) We first consider the case $g = g_i^{-1}$ for some i . Then

$$\alpha(g)\beta(g)^{-1} = \alpha(g_i^{-1})\beta(g_i^{-1})^{-1} = \alpha(g_i)^{-1}(\alpha(g_i)\beta(g_i^{-1}))^{-1}\alpha(g_i),$$

thus $\alpha(g)\beta(g)^{-1}$ is a conjugate of the inverse of $\alpha(g_i)\beta(g_i^{-1})$, in particular it lies in $\langle\langle \{\alpha(g_i)\beta(g_i)^{-1}\}_{i=1, \dots, m} \rangle\rangle$.

(2) Note that if a, b and c are elements in a group π such that ac and b lie in a normal subgroup Γ , then $abc = ac \cdot c^{-1}bc$ also lies in Γ .

We now turn to the proof of the claim. We define the *length* of $g \in G$ as

$$\ell(g) = \min\{n \mid \text{we can write } g = g_{i_1}^{\epsilon_1} \cdots g_{i_n}^{\epsilon_n} \text{ for } \epsilon_j \in \{-1, 1\} \text{ and } i_j \in \{1, \dots, m\}\}.$$

Since $g_1, \dots, g_m$ are generators of G any $g \in G$ has a length.

We prove the claim by induction on the length of $g \in G$. For $\ell(g) = 0$ there is nothing to prove. Now suppose we have shown the claim for all elements of length $n - 1$. Now let $g \in G$ be an element of length n . Thus we can write $g = g_{i_1}^{\epsilon_1} \cdots g_{i_n}^{\epsilon_n}$ for $\epsilon_1, \dots, \epsilon_n \in \{-1, 1\}$ and $i_1, \dots, i_n \in \{1, \dots, m\}$. We then have

$$\begin{aligned} \alpha(g)\beta(g)^{-1} &= \alpha(g)\beta(g^{-1}) = \alpha(g_{i_1}^{\epsilon_1} \cdots g_{i_n}^{\epsilon_n}) \cdot \beta(g_{i_n}^{-\epsilon_n} \cdots g_{i_1}^{-\epsilon_1}) \\ &= \underbrace{\alpha(g_{i_1}^{\epsilon_1}) \cdots \alpha(g_{i_{n-1}}^{\epsilon_{n-1}})}_{=:a} \cdot \underbrace{\alpha(g_{i_n}^{\epsilon_n}) \cdot \beta(g_{i_n}^{-\epsilon_n})}_{=:b} \cdot \underbrace{\beta(g_{i_{n-1}}^{-\epsilon_{n-1}}) \cdots \beta(g_{i_1}^{-\epsilon_1})}_{=:c}. \end{aligned}$$

Note that b lies in the normal subgroup $\langle\langle \{\alpha(g_i)\beta(g_i)^{-1}\}_{i=1, \dots, m} \rangle\rangle$ by (1) and ac lies in the normal subgroup $\langle\langle \{\alpha(g_i)\beta(g_i)^{-1}\}_{i=1, \dots, m} \rangle\rangle$ by induction. Thus $abc = \alpha(g)\beta(g)^{-1}$ lies in $\langle\langle \{\alpha(g_i)\beta(g_i)^{-1}\}_{i=1, \dots, m} \rangle\rangle$ by (2). $\square$

In the previous example we had seen that the amalgamated product of two non-trivial groups can be trivial. We conclude this introduction to amalgamated products with the following proposition that is proved in [Se, p. 3]. It gives a criterion for the amalgamated product to be non-trivial.

Proposition 10.10. *Let $\alpha: G \rightarrow A$ and $\beta: G \rightarrow B$ be two group homomorphisms. If α and β are monomorphisms, then the canonical homomorphisms $A \rightarrow A *_G B$ and $B \rightarrow A *_G B$ are also monomorphisms.*

11. THE GENERAL SEIFERT-VAN KAMPEN THEOREM

11.1. The formulation of the general Seifert-van Kampen Theorem. With the group theoretic preparations from the previous chapter we can now formulate and prove the general Seifert-van Kampen Theorem.

Theorem 11.1. (Seifert–van Kampen) *Let X be a topological space and let $X = U \cup V$ be a decomposition of X in two open subsets U and V such that $U \cap V$ is non-empty and path-connected. Let $x_0 \in U \cap V$. Then there exists an isomorphism*

$$\Phi: \pi_1(U, x_0) *_{\pi_1(U \cap V, x_0)} \pi_1(V, x_0) \xrightarrow{\cong} \pi_1(X, x_0)$$

such that the following diagram commutes:

$$\begin{array}{ccccc} \pi_1(U \cap V, x_0) & \xrightarrow{\quad} & \pi_1(U, x_0) & & \\ \downarrow & & \downarrow & \searrow & \\ \pi_1(V, x_0) & \xrightarrow{\quad} & \pi_1(U, x_0) *_{\pi_1(U \cap V, x_0)} \pi_1(V, x_0) & \xrightarrow[\cong]{\Phi} & \pi_1(X, x_0) \\ & \searrow & & & \\ & & & & \end{array}$$

Here all other maps are the obvious inclusion induced homomorphisms.

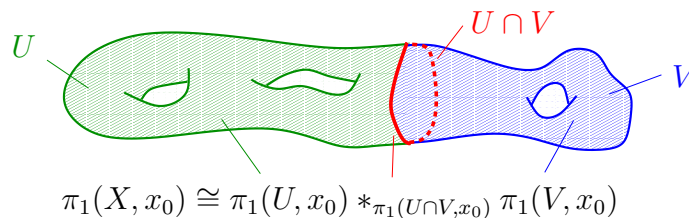


FIGURE 105. Illustration of Theorem 11.1.

The proof of the general Seifert-van Kampen Theorem is a modification of the proof of the more “basic” Seifert-van Kampen Theorem 9.2 where we had assumed that $U \cap V$ is in fact simply connected. We only sketch the proof of the general Seifert-van Kampen Theorem 11.1. The full details are for example given in Section 1.2 of [Ha].

Sketch of the proof. The various inclusions of topological spaces give rise to the following commutative diagram:

$$\begin{array}{ccc} \pi_1(U \cap V, x_0) & \xrightarrow{i} & \pi_1(U, x_0) \\ j \downarrow & & \searrow \\ \pi_1(V, x_0) & \xrightarrow{\quad} & \pi_1(X, x_0) \end{array}$$

By Proposition 10.7 there exists a unique homomorphism

$$\Phi: \pi_1(U, x_0) *_{\pi_1(U \cap V, x_0)} \pi_1(V, x_0) \rightarrow \pi_1(X, x_0)$$

such that the following diagram commutes:

$$\begin{array}{ccccc} \pi_1(U \cap V, x_0) & \xrightarrow{i} & \pi_1(U, x_0) & & \\ j \downarrow & & \downarrow k & & \\ \pi_1(V, x_0) & \xrightarrow{l} & \pi_1(U, x_0) *_{\pi_1(U \cap V, x_0)} \pi_1(V, x_0) & \xrightarrow{\Phi} & \pi_1(X, x_0) \\ & \searrow & & \nearrow & \\ & & & & \pi_1(X, x_0) \end{array}$$

It remains to show that Φ is surjective and that Φ is injective.

We first show surjectivity. We consider the following commutative diagram²²⁸ of homomorphisms

$$\begin{array}{ccc} & \pi_1(U, x_0) *_{\pi_1(U \cap V, x_0)} \pi_1(V, x_0) & \\ & \nearrow \quad \searrow \Phi & \\ \pi_1(U, x_0) * \pi_1(V, x_0) & \xrightarrow{\quad} & \pi_1(X, x_0) \end{array}$$

By Lemma 9.1 the horizontal map is surjective. It follows that the diagonal map Φ on the right is also surjective.

Now we turn to the proof of the injectivity of Φ . We write $A = \pi_1(U, x_0)$, $B = \pi_1(V, x_0)$ and $G = \pi_1(U \cap V, x_0)$. As on page 150 we consider the set

$$S(A, B) := \{\text{all finite sequences } (x_1, \dots, x_k) \text{ with each } x_i \text{ in } A \text{ or } B\}.$$

Furthermore we denote by $\sim_G$ the equivalence relation on $S(A, B)$ generated by the relation

$$(x_1, \dots, x_l, e, x_{l+2}, \dots, x_k) \sim (x_1, \dots, x_l, x_{l+2}, \dots, x_k)$$

where e is trivial in A or B , by the relation

$$(x_1, \dots, x_l, x_{l+1}, \dots, x_k) \sim (x_1, \dots, x_l x_{l+1}, \dots, x_k)$$

if both x_l and x_{l+1} lie in A or both lie in B , and also by the relation

$$(x_1, \dots, x_l, i(g), x_{l+2}, \dots, x_k) \sim (x_1, \dots, x_l, j(g), x_{l+2}, \dots, x_k)$$

for $g \in G$. Similar to Proposition 8.15 one can now show that the map

$$\begin{array}{ccc} S(A, B)/\sim_G & \rightarrow & A *_G B \\ [(x_1, \dots, x_k)] & \mapsto & x_1 \cdots x_k \end{array}$$

is a well-defined bijection. We will use this isomorphism to identify $S(A, B)/\sim_G$ with $A *_G B$.

The proof of the present theorem is now almost identical to the proof of Theorem 9.2, one just needs to replace the discussion involving $S(A, B)/\sim$ by the corresponding objects in $S(A, B)/\sim_G$. More precisely, in Footnote 202 we had pointed out where in the proof of

²²⁸What are the maps, and why does the diagram commute?

Theorem 9.2 we used the hypothesis that $U \cap V$ is simply-connected. At that point in the proof of Theorem 9.2 we need to replace $A * B = S(A, B) / \sim$ with $A *_G B = S(A, B) / \sim_G$. Full details for the proof can be found for example in Section 1.2 of [Ha]. $\square$

The same way that we deduced Theorem 9.8 from Theorem 9.2 we can now obtain the following theorem from Theorem 11.1.

Theorem 11.2. (Seifert-van Kampen for manifolds) *Let M be an m -dimensional manifold and let $R, S \subset M$ be two m -dimensional submanifolds such that the following hold:*

- (1) $M = R \cup S$,
- (2) $R \cap S$ is a component of ∂R and it is a component of ∂S ,

We pick a base point $x_0 \in R \cap S$. Then the inclusion induced map

$$\pi_1(R, x_0) *_{\pi_1(R \cap S, x_0)} \pi_1(S, x_0) \rightarrow \pi_1(M, x_0)$$

is an isomorphism.

Example. On page 167 we had seen that the earlier Seifert-van Kampen Theorem 9.8 can be applied to show that $\pi_1(S^n) = 0$ for $n \geq 3$. If we replace Theorem 9.8 by the more general Theorem 11.2, then we can also determine the fundamental group of $\pi_1(S^2)$ using the “Seifert-van Kampen method”. More precisely, let $H_+ := \{(x, y, z) \in S^2 \mid z \geq 0\}$ be the closed upper hemisphere and define similarly the closed lower hemisphere H_- . Then H_+, H_- are submanifolds of S^2 that satisfy the hypothesis of Theorem 11.2 and we obtain once again that $\pi_1(S^2) \cong \pi_1(H_+) *_{\mathbb{Z}} \pi_1(H_-) = \{e\} *_{\mathbb{Z}} \{e\} = \{e\}$.

11.2. The fundamental groups of surfaces. We now want to use the Seifert-van Kampen Theorem 11.1 to determine the fundamental groups of surfaces of genus $g \geq 2$. Before we do so we consider the torus, the Klein bottle and the real projective plane to gain some confidence in our newly acquired methods.²²⁹

- (1) We first consider the torus $T = [0, 1] \times [0, 1] / \sim$. In the following we will use the following notation, see also Figure 106 for an illustration:
 - (a) We denote by Q the point on T corresponding to the four vertices of the square.
 - (b) We write $T = U \cup V$ where

$$U = T \setminus \overline{B_{\frac{1}{8}}(\frac{1}{2}, \frac{1}{2})} \subset T \quad \text{and} \quad V = B_{\frac{1}{4}}(\frac{1}{2}, \frac{1}{2}) \subset T,$$

i.e. U is the complement of a closed disk and V is an open disk and $U \cap V$ is an open annulus.

- (c) We let $P = (r, r) \in U \cap V$ and we denote by p the direct path from Q to P .
- (d) We pick a loop γ in $(U \cap V, P)$ that “goes clockwise around the annulus” and that represents a generator of the infinite cyclic group $\pi_1(U \cap V, P) \cong \mathbb{Z}$.²³⁰

²²⁹The following examples can also be handled using Theorem 11.2 instead of Theorem 11.1. The advantage of using Theorem 11.1 at this point is that in the illustrations one can draw $U \cap V$ better if U and V are open subsets.

²³⁰Since $U \cap V$ is an open annulus its fundamental group is isomorphic to $\mathbb{Z}$.

- (e) We denote by x and y the two loops in (T, Q) corresponding to the horizontal and the vertical edge of the square. By a slight abuse of notation we denote the corresponding elements in $\pi_1(T, Q)$ by x and y as well.

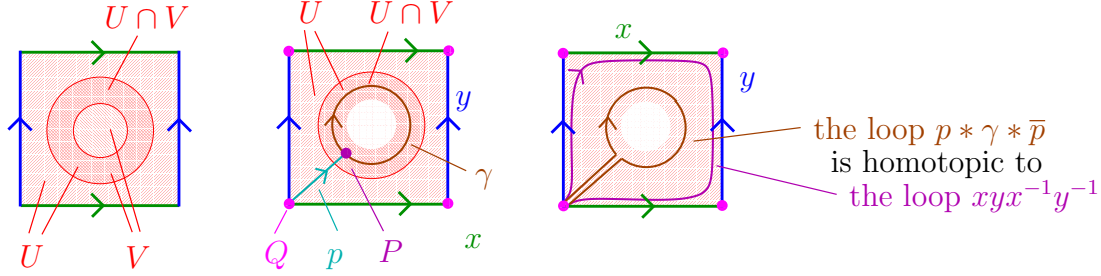


FIGURE 106.

We now have

$$\begin{array}{ll}
 \pi_1(T, P) \cong \pi_1(U, P) *_{\pi_1(U \cap V, P)} \pi_1(V, P) & \begin{array}{l} \uparrow \\ \text{Seifert-van Kampen Theorem 11.1} \end{array} \\
 \uparrow & \uparrow \\
 \pi_1(U, P) / \langle\langle i_*(\pi_1(U \cap V, P)) \rangle\rangle & \begin{array}{l} \uparrow \\ \text{by Lemma 10.8 we have } A *_G \{e\} = A / \langle\langle G \rangle\rangle \end{array} \\
 \uparrow & \uparrow \\
 \pi_1(U, Q) / \langle\langle p * \gamma * \bar{p} \rangle\rangle & \begin{array}{l} \uparrow \\ \text{by Lemma 4.9 the path } p \text{ gives} \\ \text{rise to an isomorphism } p_* \end{array} \\
 \uparrow & \uparrow \\
 \pi_1(U, Q) / \langle\langle xyx^{-1}y^{-1} \rangle\rangle & \begin{array}{l} \uparrow \\ \text{since } p * \gamma * \bar{p} \text{ is homotopic to } xyx^{-1}y^{-1} \end{array}
 \end{array}$$

As in the proof of Lemma 9.7 we can show that $W := \partial([0, 1] \times [0, 1]) / \sim$ is a deformation retract of U and that W is homeomorphic to the wedge of two circles. By the discussion on page 159 we can therefore make the identification $\pi_1(W, P) = \langle x, y \rangle$, see also Figure 107. We now continue with the above calculation of $\pi_1(T, P)$. We have

$$\begin{array}{ll}
 (*) = \pi_1(U, Q) / \langle\langle xyx^{-1}y^{-1} \rangle\rangle & \begin{array}{l} \xrightarrow{\cong} \\ \uparrow \end{array} \\
 \uparrow & \begin{array}{l} \text{by Lemma 7.4 and Proposition 7.5} \end{array} \\
 \langle x, y \rangle / \langle\langle xyx^{-1}y^{-1} \rangle\rangle & = \langle x, y \mid [x, y] \rangle \cong \mathbb{Z}^2. \\
 \uparrow & \uparrow \\
 \text{since } W \text{ is the wedge of the circles } x \text{ and } y & \text{see page 174}
 \end{array}$$

Thus we have reproved that the fundamental group of the torus is isomorphic to the free abelian group $\mathbb{Z}^2$.

- (2) We turn to the Klein bottle K . We perform almost the same argument as for the torus, we just replace the situation illustrated in Figure 107 by the situation illustrated in

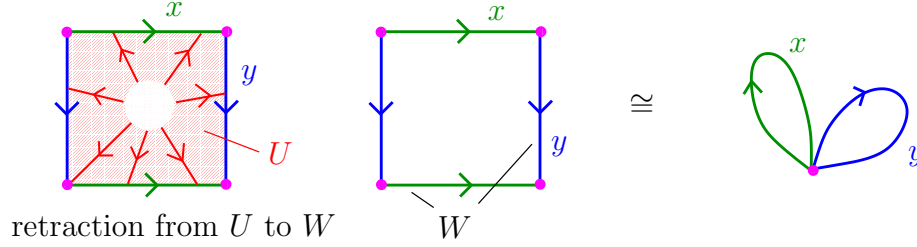


FIGURE 107.

Figure 108. The same argument as above shows that²³¹

$$\pi_1(K) \cong \langle x, y \mid yxyx^{-1} \rangle.$$

Using this presentation and using Proposition 10.6 (5) we obtain in particular that the abelianization of $\pi_1(K)$ is given by

$$\pi_1(K)_{\text{ab}} \cong \langle x, y \mid yxyx^{-1} \rangle_{\text{ab}} \cong \mathbb{Z}^2 / \begin{pmatrix} 0 \\ 2 \end{pmatrix} \mathbb{Z} \cong \mathbb{Z} \oplus \mathbb{Z}_2.$$

On page 117 we had already given a very different calculation of $\pi_1(K)$. More precisely, we had introduced the group G that is given by all self-homeomorphisms of $\mathbb{R}^2$ that can be written as concatenations of the two self-homeomorphisms

$$\begin{aligned} A: \mathbb{R}^2 &\rightarrow \mathbb{R}^2 & B: \mathbb{R}^2 &\rightarrow \mathbb{R}^2 \\ (s, t) &\mapsto (s+1, 1-t) & \text{and} & & (s, t) &\mapsto (s, t+1) \end{aligned}$$

and their inverses and we had shown that $\pi_1(K) \cong G$. On page 175 we had already shown that there exists a homomorphism $\Phi: \langle x, y \mid yxyx^{-1} \rangle \rightarrow G$ with $\Phi(x) = B^{-1}A$ and $\Phi(y) = B$. This homomorphism is in fact an isomorphism.²³²

- (3) Next we consider again the real projective space $\mathbb{RP}^2$. Hereby we use the description of $\mathbb{RP}^2$ as

$$\mathbb{RP}^2 = \overline{B}^2 / \sim \quad \text{where for } z \in S^1 \text{ we have } z \sim -z.$$

²³¹In particular this shows that $\pi_1(K)$ is finitely presented. This answers the question we had raised on page 176.

²³²This can be seen as follows. Denote by H the group of all selfhomeomorphisms of $\mathbb{R}^2$. We consider the diagram

$$\begin{array}{ccc} \langle x, y \mid yxyx^{-1} \rangle & \xrightarrow{\cong} & \pi_1(K) \xleftarrow{\cong} \text{subgroup of } H \\ & \searrow \Phi & \text{generated by } A, B. \\ & & x \mapsto B^{-1}A, y \mapsto B \end{array}$$

Here the isomorphisms to $\pi_1(K)$ are the ones given by the two calculations of the fundamental group. If one goes carefully through the definitions of the two horizontal maps, then one can see that the diagram commutes. Since two of the three maps are isomorphisms and since the diagram commutes it follows that Φ is also an isomorphism.

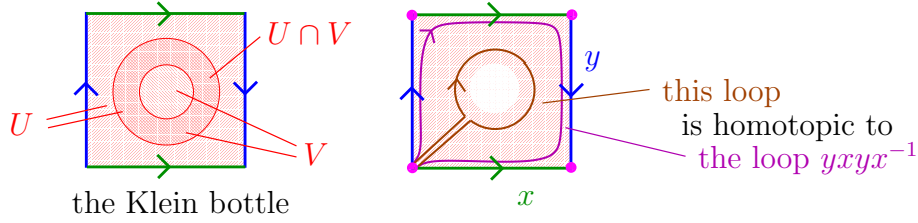


FIGURE 108.

As in the previous two examples we decompose $\mathbb{RP}^2$ into an open disk V and the complement of a closed disk U such that $U \cap V$ is an annulus. We consider the base point $Q = [1] = [-1] \in \mathbb{RP}^2$ and we denote by α the loop

$$\begin{aligned} \alpha: [0, 1] &\rightarrow \mathbb{RP}^2 \\ t &\mapsto [e^{i\pi t}]. \end{aligned}$$

By a slight abuse of notation we denote by α also the corresponding element in $\pi_1(\mathbb{RP}^2, Q)$. Furthermore we denote by W the image of α in $\mathbb{RP}^2$. Evidently α is a generator of $\pi_1(W, Q)$. Then, with similar arguments as in the torus case, we see that

$$\begin{array}{ccccc} \pi_1(\mathbb{RP}^2) & \cong & \pi_1(U, Q) / \langle\langle p * \gamma * \bar{p} \rangle\rangle & \cong & \pi_1(W, Q) / \langle\langle p * \gamma * \bar{p} \rangle\rangle & \cong & \langle\alpha\rangle / \langle\alpha^2\rangle & \cong & \mathbb{Z}_2. \\ \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\ \text{Seifert-van Kampen Theorem 11.1} & & \text{since } W \text{ is a deformation} & & \text{since } \pi_1(W, Q) = \langle\alpha\rangle & & & & \\ \text{and Lemma 10.8 (2)} & & \text{retract of } U & & \text{and } [p * \gamma * \bar{p}] = \alpha^2 & & & & \end{array}$$

In particular we obtain the same result, as we should, as in Corollary 6.17.

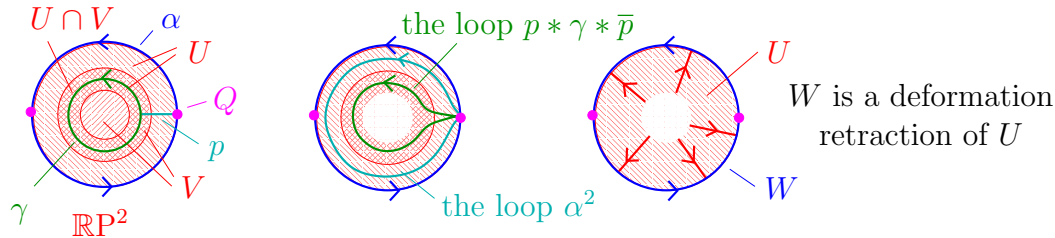


FIGURE 109.

- (4) Now that we have gained some confidence that the approach of determining fundamental groups using Seifert-van Kampen Theorem 11.1 does indeed give the right fundamental groups we turn to the surfaces of higher genus that we could not handle with our previous methods.

So now we consider the surface $F = E_8 / \sim$ of genus 2. As in the torus case we write $F = U \cup V$ where V is an open disk and where U is the complement of a closed disk such that $U \cap V$ is an annulus. In the following we use basically the same notation as in the torus case or alternatively we refer to Figures 110 and 111 for the

notation. We write $W = \partial E_8 / \sim$. By the proof of Proposition 9.6 we know that W is a deformation retraction of U and that W is homeomorphic to the wedge of four circles corresponding to x_1, y_1, x_2, y_2 . See also Figure 111 for an illustration. Basically

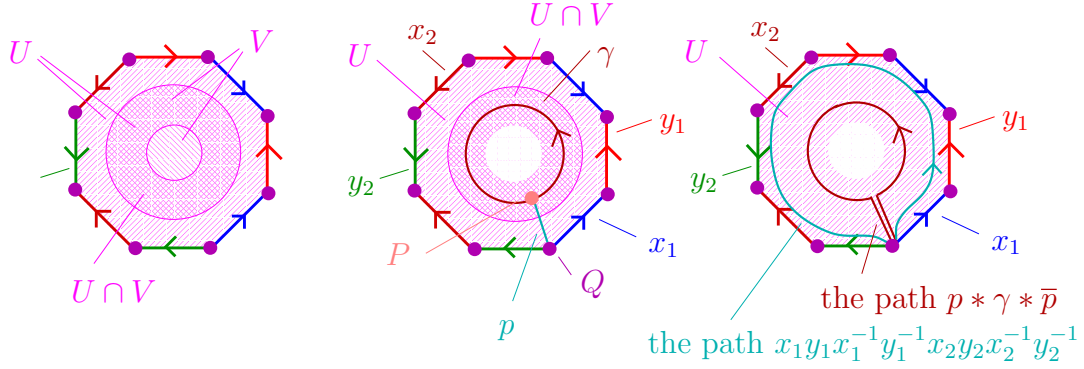


FIGURE 110.

exactly the same argument as in the torus case shows that

$$\begin{aligned}
 \pi_1(E_8 / \sim, P) &\cong \pi_1(U, Q) / \langle \langle \overbrace{x_1 y_1 x_1^{-1} y_1^{-1} x_2 y_2 x_2^{-1} y_2^{-1}}^{\text{homotopic to } p * \gamma * \bar{p}} \rangle \rangle \\
 &\cong \pi_1(W, Q) / \langle \langle [x_1, y_1][x_2, y_2] \rangle \rangle \\
 &\cong \langle x_1, y_1, x_2, y_2 \rangle / \langle \langle [x_1, y_1][x_2, y_2] \rangle \rangle = \langle x_1, y_1, x_2, y_2 \mid [x_1, y_1][x_2, y_2] \rangle.
 \end{aligned}$$

Thus we have finally succeeded in calculating the fundamental group of the surface of genus 2.

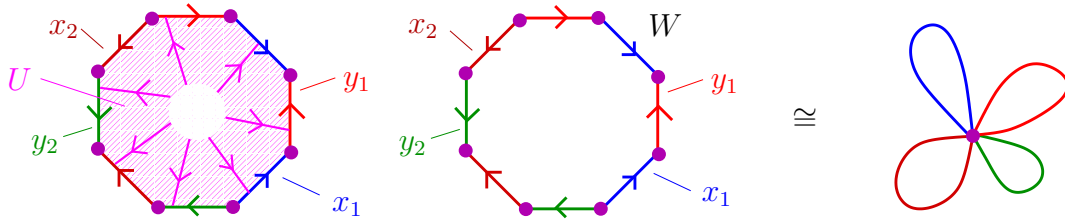


FIGURE 111.

The following gives in particular a new proof for Proposition 9.6.

Proposition 11.3.

(1) For any $g \in \mathbb{N}_0$ we have

$$\pi_1(\text{surface of genus } g) \cong \langle x_1, y_1, \dots, x_g, y_g \mid [x_1, y_1] \cdots [x_g, y_g] \rangle$$

and we have

$$\pi_1(\text{surface of genus } g)_{\text{ab}} \cong \mathbb{Z}^{2g}.$$

- (2) For $g \neq h$ the surfaces of genus g and h are not homeomorphic.

Proof. Let $g \in \mathbb{N}_0$. For $g = 0$ we obtain the fundamental group from Proposition 4.12 and for $g = 1$ we obtain the presentation from the discussion on page 189. For $g \geq 2$ the presentation follows from a straightforward generalization of the calculation preceding the proposition.

The stated abelianization of the fundamental group is an immediate consequence of Proposition 10.6 (5).

Finally for $g \neq h$ we obtain from the above calculation and from the classification of finitely generated abelian groups that we gave in Theorem 8.3 that the abelianizations of the fundamental groups of the surfaces of genus g and h are not isomorphic. Hence the surfaces are not homeomorphic. $\square$

Remark.

- (1) In Figure 112 we show loops that represent the generators x_1, y_1, x_2, y_2 of the fundamental group for the surface of genus 2.

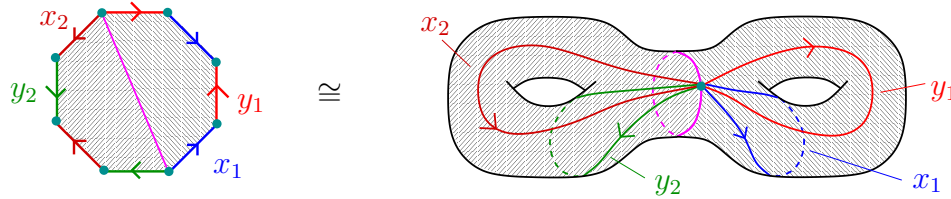


FIGURE 112.

- (2) As explained in Figure 35 we can view the surface Σ of genus 2 as the connected sum of two copies of the torus T . We can also use this point of view to determine the fundamental group of the surface Σ of genus 2. More precisely, we have $\Sigma = F_1 \cup F_2$ with $F_1 = T \setminus B^2$ and $F_2 = T \setminus B^2$ where $F_1 \cap F_2 = \partial F_1 = \partial F_2 \cong S^1$ and we obtain, modulo details, that^{233 234}

$$\begin{array}{ccc}
 \text{Seifert-van Kampen Theorem 11.2} & & \\
 \downarrow & & \\
 \pi_1(\Sigma) & \cong & \pi_1(F_1) *_{\pi_1(F_1 \cap F_2)} \pi_1(F_2) \\
 & \cong & \langle x_1, y_1 \rangle *_{\langle t \rangle} \langle x_2, y_2 \rangle \cong \langle x_1, x_2, y_1, y_2 \mid [x_1, y_1] = [x_2, y_2] \rangle. \\
 \uparrow & & \uparrow \\
 \text{with } t \mapsto [x_1, y_1], t \mapsto [x_2, y_2] & & \text{see page 184}
 \end{array}$$

²³³It is a good exercise to fill in the details.

²³⁴Why is this group isomorphic to the above group $\langle x_1, y_1, x_2, y_2 \mid [x_1, y_1][x_2, y_2] \rangle$?

Let π be the fundamental group of the surface of genus g . We have just shown that $\pi_{\text{ab}} \cong \mathbb{Z}^{2g}$. This raises the question, whether perhaps π is isomorphic to $\mathbb{Z}^{2g}$ or to the free group F_{2g} on $2g$ generators. In exercise sheet 10 we will see that π is non-abelian, in particular π is not isomorphic to $\mathbb{Z}^{2g}$. It is much harder to determine whether π is a free group. We record this as a question for later.

Question 11.4. *Let π be the fundamental group of the surface of genus g . Is π isomorphic to the free group F_{2g} on $2g$ generators?*

11.3. Non-orientable surfaces. In Section 2.2 we had introduced the connected sum of two connected *oriented* manifolds of the same dimension. Given a connected 2-dimensional manifold F we will now define the connected sum of F with the real projective plane $\mathbb{RP}^2$ and we will see that it is well-defined.²³⁵ In the following we always use the description of $\mathbb{RP}^2$ as

$$\mathbb{RP}^2 = \overline{B}^2 / \sim \quad \text{where for } z \in S^1 \text{ we have } z \sim -z$$

and we view $\frac{1}{2}\overline{B}^2 = \{z \in \mathbb{C} \mid |z| \leq \frac{1}{2}\}$ as a subset of $\mathbb{RP}^2$. Now let F be a connected 2-dimensional manifold. We pick an embedding $\varphi: \overline{B}^2 \rightarrow F$ and we define

$$F \# \mathbb{RP}^2 := (F \setminus \varphi(B^2)) \sqcup (\mathbb{RP}^2 \setminus \frac{1}{2}B^2) / \{\varphi(z) = \frac{1}{2}z \mid z \in S^1\}.$$

The following proposition says in particular that in this context the connected sum is well-defined, even though $\mathbb{RP}^2$ is non-orientable.

Proposition 11.5. *Let F be a connected 2-dimensional manifold.*

- (1) *The diffeomorphism type of $F \# \mathbb{RP}^2$ is well-defined.*
- (2) *The connected sum $F \# \mathbb{RP}^2$ is a connected, non-orientable 2-dimensional manifold.*

From Proposition 11.5 we now obtain many new examples of closed non-orientable 2-dimensional manifolds beyond the Klein bottle. For example the connected sum of the surface of genus g with $\mathbb{RP}^2$ or the connected sum $\mathbb{RP}^2 \# \mathbb{RP}^2$ are closed non-orientable 2-dimensional manifolds.

Sketch of the proof. The proof of Proposition 2.7 (1) shows that $F \# \mathbb{RP}^2$ is a connected 2-dimensional manifold. The connected sum $F \# \mathbb{RP}^2$ contains $\mathbb{RP}^2 \setminus \frac{1}{2}B^2$, i.e. it contains the real projective plane minus an open disk. We had already seen in Figure 28 that $\mathbb{RP}^2$ minus an open disk is a Möbius band. For convenience we recall the argument in Figure 11.3. Thus we see that $F \# \mathbb{RP}^2$ contains a Möbius band, i.e. it contains a non-orientable submanifold of codimension 0. Therefore $F \# \mathbb{RP}^2$ cannot be orientable.

²³⁵It turns out that the connected sum of two connected n -dimensional manifolds is always well-defined if one of the manifolds is non-orientable. The reason is that given any non-orientable n -dimensional manifold M and any open ball $B^n \subset M$ there exists a diffeomorphism of M that restricts to an *orientation-reversing* diffeomorphism of B^n . Unfortunately this is not written down very well in the literature. See e.g. [FM, p. 160] and

<https://math.stackexchange.com/questions/620165>

for more information. We will not make use of this fact.

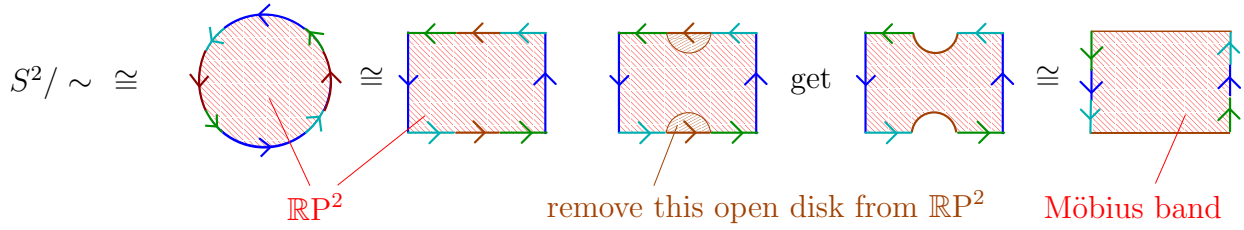


FIGURE 113.

We now turn to the proof that the connected sum is well-defined. Let $\varphi: \overline{B}^2 \rightarrow F$ and $\psi: \overline{B}^2 \rightarrow F$ be two embeddings. Again, using the same proof as in Proposition 2.7 and using [Pa1, Theorem B] we see that it suffices to consider the case that φ and ψ differ by an orientation reversing self-diffeomorphism of $\overline{B}^2$, in fact it suffices to consider the case that $\psi(z) = \varphi(\bar{z})$ for all $z \in \overline{B}^2 \subset \mathbb{R}^2 = \mathbb{C}$. So we need to show that there exists a diffeomorphism

$$(F \setminus \varphi(B^2)) \sqcup (\mathbb{RP}^2 \setminus \tfrac{1}{2}B^2) / \varphi(z) \sim \tfrac{1}{2}z \cong (F \setminus \varphi(B^2)) \sqcup (\mathbb{RP}^2 \setminus \tfrac{1}{2}B^2) / \varphi(\bar{z}) \sim \tfrac{1}{2}\bar{z}.$$

The equivalence relation $\varphi(\bar{z}) \sim \tfrac{1}{2}\bar{z}$ for $z \in S^1$ is the same as the equivalence relation $\varphi(z) \sim \tfrac{1}{2}z$ for $z \in S^1$. So we want to prove the following claim.

Claim. We denote by $\theta: \tfrac{1}{2}S^1 \rightarrow \tfrac{1}{2}S^1$ the map that is given by complex conjugation. Then there exists a diffeomorphism

$$(F \setminus \varphi(B^2)) \sqcup (\mathbb{RP}^2 \setminus \tfrac{1}{2}B^2) / \varphi(z) \sim \tfrac{1}{2}z \cong (F \setminus \varphi(B^2)) \sqcup (\mathbb{RP}^2 \setminus \tfrac{1}{2}B^2) / \varphi(z) \sim \theta(\tfrac{1}{2}z).$$

The map θ extends to a diffeomorphism

$$\begin{aligned} \Theta: \mathbb{RP}^2 \setminus \tfrac{1}{2}B^2 &\rightarrow \mathbb{RP}^2 \setminus \tfrac{1}{2}B^2 \\ [z] &\mapsto [\bar{z}]. \end{aligned}$$

The desired diffeomorphism is then given by

$$\begin{aligned} (F \setminus \varphi(B^2)) \sqcup (\mathbb{RP}^2 \setminus \tfrac{1}{2}B^2) / \varphi(z) \sim \tfrac{1}{2}z &\rightarrow (F \setminus \varphi(B^2)) \sqcup (\mathbb{RP}^2 \setminus \tfrac{1}{2}B^2) / \varphi(z) \sim \theta(\tfrac{1}{2}z) \\ Q &\mapsto \begin{cases} Q & \text{if } Q \in F \setminus \varphi(B^2), \\ \Theta(Q) & \text{if } Q \in \mathbb{RP}^2 \setminus \tfrac{1}{2}B^2. \end{cases} \quad \square \end{aligned}$$

We want to do one more calculation of a fundamental group, namely we want to determine the fundamental group of $\mathbb{RP}^2 \# \mathbb{RP}^2$. We record the result in the following lemma.

Lemma 11.6. *We have*

$$\pi_1(\mathbb{RP}^2 \# \mathbb{RP}^2) \cong \langle x, y \mid x^2 = y^2 \rangle.$$

Proof. In order to distinguish the two copies of $\mathbb{RP}^2$ we write $P = \mathbb{RP}^2$ and $Q = \mathbb{RP}^2$. The proof is of course a variation on the proof of Proposition 9.9. We have

$$\pi_1(\mathbb{RP}^2 \# \mathbb{RP}^2) = \pi_1(P \# Q) \cong \pi_1(P \setminus \tfrac{1}{2}B^2) *_{\pi_1(\frac{1}{2}S^1)} \pi_1(Q \setminus \tfrac{1}{2}B^2).$$

Seifert-van Kampen Theorem 11.2

We pick a generator g for $\pi_1(\frac{1}{2}S^1)$. In the calculation of the fundamental group of the real projective plane on page 190 we had seen that $\pi_1(P \setminus \frac{1}{2}B^2) \cong \langle x \rangle$ and that the inclusion induced map $\pi_1(\frac{1}{2}S^1) \rightarrow \pi_1(P \setminus \frac{1}{2}B^2)$ is given by $g \mapsto x^2$. The same holds for Q with $\pi_1(Q \setminus \frac{1}{2}B^2) = \langle y \rangle$. Summarizing we see that

$$\begin{aligned} \pi_1(\mathbb{RP}^2 \# \mathbb{RP}^2) &\cong \pi_1(P \setminus \tfrac{1}{2}B^2) *_{\pi_1(\frac{1}{2}S^1)} \pi_1(Q \setminus \tfrac{1}{2}B^2) \\ &\cong \text{amalgamated product of } \left(\begin{array}{ccc} \langle g \rangle & \rightarrow & \langle x \rangle \\ g & \mapsto & x^2 \end{array} \text{ and } \begin{array}{ccc} \langle g \rangle & \rightarrow & \langle y \rangle \\ g & \mapsto & y^2 \end{array} \right) \\ &\cong \langle x, y \mid x^2 = y^2 \rangle = \langle x, y \mid x^2 y^{-2} \rangle. \\ &\quad \uparrow \\ &\text{Lemma 10.9} \end{aligned}$$

□

Now the question is whether $\mathbb{RP}^2 \# \mathbb{RP}^2$ is a “new” 2-dimensional manifold or whether we have already seen it before. A first step towards understanding the fundamental group $\pi_1(\mathbb{RP}^2 \# \mathbb{RP}^2) = \langle x, y \mid x^2 = y^2 \rangle$ is to determine its abelianization. Using Proposition 10.6 (5) we see that

$$\pi_1(\mathbb{RP}^2 \# \mathbb{RP}^2)_{\text{ab}} \cong \langle x, y \mid x^2 y^{-2} \rangle_{\text{ab}} \cong \mathbb{Z}^2 / \begin{pmatrix} 2 \\ -2 \end{pmatrix} \mathbb{Z} \cong \mathbb{Z} \oplus \mathbb{Z}_2.$$

Interestingly we obtain the same abelianization as we got on page 190 for the abelianization of the fundamental group of the Klein bottle. This raises the question whether $\mathbb{RP}^2 \# \mathbb{RP}^2$ is homeomorphic to the Klein bottle K .

At first glance it is not clear what the answer should be. We had just seen that

$$\pi_1(\mathbb{RP}^2 \# \mathbb{RP}^2) \cong \langle x, y \mid x^2 = y^2 \rangle$$

whereas we had seen on page 190 that

$$\pi_1(K) \cong \langle x, y \mid yxyx^{-1} \rangle.$$

So at the very least there is no obvious isomorphism between these two groups. We therefore decide to have a closer look at the two topological spaces involved.

- (1) We had just seen in Figure 113 that removing an open disk from $\mathbb{RP}^2$ we obtain the Möbius band. This shows that $\mathbb{RP}^2 \# \mathbb{RP}^2$ is given by gluing two Möbius bands along their common boundary.
- (2) In Figure 114 we see a closed curve in the Klein bottle that decomposes the Klein bottle into two Möbius bands.

Put differently, we see that both $\mathbb{RP}^2 \# \mathbb{RP}^2$ and the Klein bottle are the result of gluing two Möbius bands along their boundary. It is now straightforward to see that the Klein bottle is homeomorphic to $\mathbb{RP}^2 \# \mathbb{RP}^2$.²³⁶

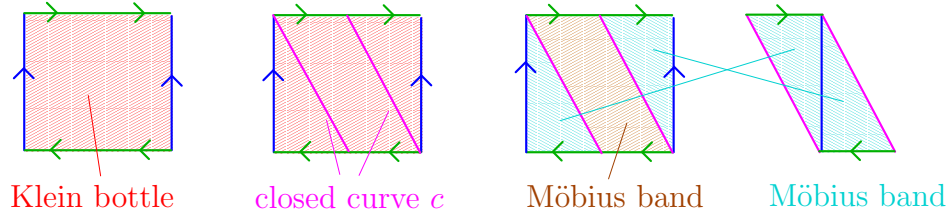


FIGURE 114.

Given $k \in \mathbb{N}$ we now denote by $k \cdot \mathbb{RP}^2$ the connected sum of k copies of the real projective plane.²³⁷ By Proposition 11.5 the topological space $k \cdot \mathbb{RP}^2$ is a non-orientable 2-dimensional manifold.

We conclude this section with the following lemma.

Lemma 11.7. *Let $k \in \mathbb{N}$. Then*

$$\pi_1(k \cdot \mathbb{RP}^2) \cong \langle x_1, \dots, x_k \mid x_1^2 \cdots x_k^2 \rangle \quad \text{and} \quad \pi_1(k \cdot \mathbb{RP}^2)_{\text{ab}} \cong \mathbb{Z}^{k-1} \oplus \mathbb{Z}_2.$$

Proof. A straightforward induction argument and a generalization of the argument of Lemma 11.6 gives the desired presentation for $\pi_1(k \cdot \mathbb{RP}^2)$.²³⁸ The abelianization can be determined easily from the presentation using Proposition 10.6 (5). We leave the details to the reader. $\square$

11.4. The classification of closed 2-dimensional (topological) manifolds. The following proposition now gives the complete classification of connected closed topological 2-dimensional manifolds up to homeomorphism.

Theorem 11.8. *Every connected, closed 2-dimensional topological manifold is homeomorphic to either*

- (1) *a surface of genus g for some unique $g \in \mathbb{N}_0$ or*
- (2) *to $k \cdot \mathbb{RP}^2$ for some unique $k \in \mathbb{N}$.*

Proof. By Radó's Theorem [Rd] every connected, closed 2-dimensional topological manifold can be “triangulated”. A proof that every connected, closed, triangulated 2-dimensional topological manifold is homeomorphic to a surface of genus g or to $k \cdot \mathbb{RP}^2$ for some k is for example given in [ST] or alternatively in [FW, Gl] or in [GX, Theorem 6.2].

²³⁶We now know that the groups $\pi_1(\mathbb{RP}^2 \# \mathbb{RP}^2) \cong \langle x, y \mid x^2 = y^2 \rangle$ and $\pi_1(K) \cong \langle x, y \mid yxyx^{-1} \rangle$ are isomorphic. But what is an explicit isomorphism between these two groups?

²³⁷To be more precise, we define $0 \cdot \mathbb{RP}^2 = S^2$ and iteratively $k \cdot \mathbb{RP}^2 := (k-1) \cdot \mathbb{RP}^2 \# \mathbb{RP}^2$ for $k \in \mathbb{N}$.

²³⁸Why is for $k = 2$ this group isomorphic to the group given in Lemma 11.6?

It remains to prove the uniqueness statement. This means that we have to show that the topological spaces listed in (1) and (2) are pairwise non-homeomorphic. It follows from Proposition 11.3 and Lemma 11.7 together with the classification of finitely generated groups, see Theorem 8.3, that the abelianizations of the fundamental groups of the topological manifolds listed in (1) and (2) are pairwise non-isomorphic. But then it follows from Proposition 10.6 (6) that also the fundamental groups are pairwise non-isomorphic, which in turn implies that the given topological manifolds are pairwise not homeomorphic. $\square$

The following theorem gives the classification of connected, closed 2-dimensional manifolds up to diffeomorphism.

Theorem 11.9. *Let Σ be a connected, closed 2-dimensional manifold.*

- (1) *If Σ is orientable, then it is diffeomorphic to a surface of genus g for some unique $g \in \mathbb{N}_0$.*
- (2) *If Σ is non-orientable, then it is diffeomorphic to $k \cdot \mathbb{RP}^2$ for some unique $k \in \mathbb{N}$.*

Proof. The fact that any connected, closed 2-dimensional manifold is diffeomorphic to one of the given manifolds is [Hi, Theorems 9.3.5 and 9.3.10]. The uniqueness statement for g and k is an immediate consequence of the uniqueness statement of Theorem 11.8. $\square$

Example. Let $g \in \mathbb{N}$ and denote by Σ_g the surface of genus g . The connected sum $\Sigma_g \# \mathbb{RP}^2$ contains a Möbius band as a submanifold of codimension 0,²³⁹ hence $\Sigma_g \# \mathbb{RP}^2$ is non-orientable. It follows from Theorem 11.9 that $\Sigma_g \# \mathbb{RP}^2$ is diffeomorphic to $k \cdot \mathbb{RP}^2$ for some unique $k \in \mathbb{N}$. In exercise sheet 11 we will determine k as a function of g .

11.5. The classification of 2-dimensional (topological) manifolds with boundary.

We now turn to the classification of compact 2-dimensional (topological) manifolds with boundary. We first prove the following proposition.

Proposition 11.10. *Let F be a 2-dimensional topological manifold with $\partial F \neq \emptyset$. Then the boundary $Y := \partial F$ is a 1-dimensional topological manifold which has no boundary, i.e. such that $\partial Y = \emptyset$.*

Remark. It might come as a surprise that we do not yet know the statement of the proposition. In fact in Satz 4.3 of Analysis IV we had shown that the boundary of any n -dimensional manifold is an $(n - 1)$ -dimensional manifold. But we did not show the corresponding statement for topological manifolds. In fact at the moment we can prove the statement only for topological manifolds of dimension *two*. We will deal with the case of higher-dimensional topological manifolds in Algebraic Topology II.

The key ingredient in the proof of Proposition 11.10 is the following lemma.

Lemma 11.11. *Let $H = \{(x, y) \in \mathbb{R}^2 \mid y \geq 0\}$ be the closed upper half-space in $\mathbb{R}^2$. If V is an open subset of H with $V \cap \partial H \neq \emptyset$, then V is not homeomorphic to an open subset of $\mathbb{R}^2$.*

²³⁹This follows from the observation that $\Sigma_g \# \mathbb{RP}^2$ contains $\mathbb{RP}^2$ minus a disk, which by Figure 28 is a Möbius band.

For example it follows from the lemma that the half-open square $(0, 1) \times [0, 1)$ is not homeomorphic to any open subset of $\mathbb{R}^2$.

Proof. Let V be an open subset of the closed upper half-space H which contains a point $Q \in \partial H = \{(x, 0) \mid x \in \mathbb{R}\}$. Furthermore let U be an open subset of $\mathbb{R}^2$. Suppose there exists a homeomorphism $\Phi: U \rightarrow V$. We write $\Psi := \Phi^{-1}$ and we write $P = \Psi(Q)$.

Since U is open in $\mathbb{R}^2$ we can pick a $\sigma > 0$ with $B_\sigma(P) \subset U$. Since V is open in H and since Ψ is continuous there exists an $\epsilon > 0$ with $B_\epsilon(Q) \cap H \subset V$ and with $\Psi(B_\epsilon(Q)) \subset B_\sigma(P)$. Furthermore, by the continuity of Φ there exists an $\eta > 0$ with $\Phi(B_\eta(P)) \subset B_\epsilon(Q)$. We refer to Figure 115 for an illustration. We consider the following commutative diagram of maps

$$\begin{array}{ccc} & (B_\epsilon(Q) \setminus \{Q\}) \cap H & \\ \Phi \nearrow & & \searrow \Psi = \Phi^{-1} \\ B_\eta(P) \setminus \{P\} & \xrightarrow{i} & B_\sigma(P) \setminus \{P\}, \end{array}$$

where $i: B_\eta(P) \setminus \{P\} \rightarrow B_\sigma(P) \setminus \{P\}$ denotes the inclusion map. Now we pick a base point $x \in B_\eta(P) \setminus \{P\}$. By the functoriality of fundamental groups, the above commutative diagram of maps gives rise to the following commutative diagram

$$\begin{array}{ccc} & \pi_1((B_\epsilon(Q) \setminus \{Q\}) \cap H, \Phi(x)) & \\ \Phi_* \nearrow & & \searrow \Psi_* = \Phi_*^{-1} \\ \pi_1(B_\eta(P) \setminus \{P\}, x) & \xrightarrow{i_*} & \pi_1(B_\sigma(P) \setminus \{P\}, x). \end{array}$$

We make the following observations:

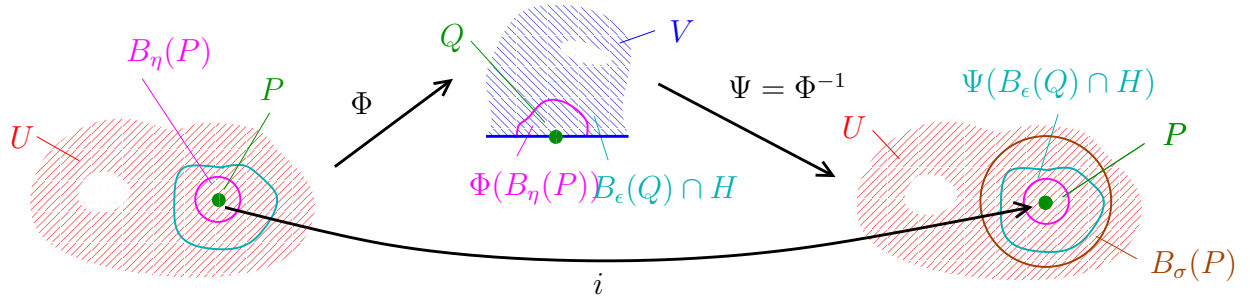


FIGURE 115.

- (1) the topological spaces on the left and on the right are homotopy equivalent to S^1 , in particular the fundamental groups are non-trivial,

- (2) the inclusion map $B_\eta(P) \setminus \{P\} \rightarrow B_\sigma(P) \setminus \{P\}$ is a homotopy equivalence,²⁴⁰ it follows from Proposition 7.5 that $i_*: \pi_1(B_\eta(P) \setminus \{P\}, x) \rightarrow \pi_1(B_\sigma(P) \setminus \{P\}, x)$ is an isomorphism,
- (3) the topological space $(B_\epsilon(Q) \setminus \{Q\}) \cap H$ is star-shaped²⁴¹, which implies by the discussion on page 76 that $\pi_1((B_\epsilon(Q) \setminus \{Q\}) \cap H, \Phi(x)) = 0$.

Summarizing we just showed that an isomorphism of non-trivial groups factors through the trivial group. This is not possible and hence we have obtained the desired contradiction. $\square$

We can now turn to the proof of Proposition 11.10.

Proof of Proposition 11.10. Let F be a 2-dimensional topological manifold with $\partial F \neq \emptyset$. We first recall some definitions.

- (a) A *2-dimensional chart for F at a point $x \in X$* is a homeomorphism $\Phi: U \rightarrow V$ where U is an open neighborhood of x and V is one of the following:
 - (i) V is an open subset of $\mathbb{R}^2$ or
 - (ii) V is an open subset of the half-space $H := \{(x, y) \in \mathbb{R}^2 \mid y \geq 0\}$ and $\Phi(x)$ lies on $\partial H = \{(x, y) \in \mathbb{R}^2 \mid y = 0\}$.
 In the former case we say that Φ is a chart of type (i) in the latter case we say that Φ is a chart of type (ii).
- (b) We say that a point x on a topological manifold is a *boundary point* if x does not admit a chart of type (i). We denote by ∂X the set of all boundary points of X .

Now we turn to the proof that ∂F is a 1-dimensional topological manifold with no boundary.²⁴² We start out with the following claim.

Claim. Let x be a point on F . Then x admits either a chart of type (i) or it admits a chart of type (ii).

If a point did admit a chart $\Phi: U \rightarrow V$ of type (i) and a chart $\Psi: X \rightarrow Y$ of type (ii), then $\Phi \circ \Psi^{-1}: \Psi(U \cap X) \rightarrow \Phi(U \cap X)$ would give rise to a homeomorphism from an open set of H containing a point on $\partial H = \{(x, 0) \mid x \in \mathbb{R}\}$ to an open set in $\mathbb{R}^2$. But by Lemma 11.11 such a homeomorphism cannot exist. This concludes the proof of the claim.

Now let $P \in F$ be a point on the boundary. Let $\Phi: U \rightarrow V$ be a chart of type (ii) for P . It follows from the claim and the definitions that $\partial F \cap U = \Phi^{-1}(\partial H \cap V)$. Thus we see that Φ restricts to a chart $\Phi: \partial F \cap U \rightarrow \partial H \cap V$ of type (i), here we view $\partial H \cap V$ as an open subset of $\partial H = \mathbb{R}$. This shows that ∂F admits an atlas where all charts are charts of type (i). Since subspaces of Hausdorff-spaces are Hausdorff and since subspaces of second-countable spaces are second-countable it now follows that ∂F is indeed a 1-dimensional topological manifold with no boundary points. $\square$

²⁴⁰Why is it a homotopy equivalence? Does this follow from Lemma 7.4?

²⁴¹Why is that?

²⁴²The proof of this statement is basically the same as in the case of manifolds that we gave in Satz 4.1 and 4.3 of Analysis IV. There the key ingredient was that an open subset of $\mathbb{R}^n$ cannot be diffeomorphic to an open subset of $H = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$ that contains a boundary point. For $n = 2$ we just proved the same statement with “diffeomorphic” replaced by “homeomorphic”.

Given a closed connected 2-dimensional manifold Σ and $n \in \mathbb{N}_0$ we now define Σ *minus n open disks* as the result of removing the interiors of n disjoint closed disks.²⁴³ The result is a compact, connected 2-dimensional manifold with n boundary components. Using theorem 2.8 one can show that the diffeomorphism type of the resulting manifold does not depend on the choice of the closed disks. Given $g, n \in \mathbb{N}_0$ we write

$$\Sigma_{g,n} := \text{the surface of genus } g \text{ minus } n \text{ open disks}$$

and for $k \in \mathbb{N}$ and $n \in \mathbb{N}_0$ we write

$$R_{k,n} := \text{the manifold } k \cdot \mathbb{RP}^2 \text{ minus } n \text{ open disks.}$$

Examples.

- (1) The surface $\Sigma_{0,2}$ is the result of removing two open disjoint disks from the 2-sphere. The resulting topological space is homeomorphic to a cylinder. We refer to Figure 116 for an illustration of this statement.

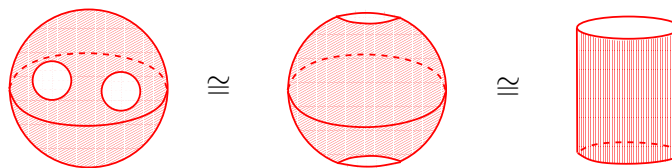


FIGURE 116.

- (2) We had seen in Figure 113 that the result of removing an open disk from $\mathbb{RP}^2$ is the Möbius band, i.e. $R_{1,1}$ is the Möbius band.

Theorem 11.12. *Every connected, compact 2-dimensional topological manifold is homeomorphic to either*

- (1) *the surface $\Sigma_{g,n}$ for unique $g, n \in \mathbb{N}_0$ or*
- (2) *to the surface $R_{k,n}$ for unique $k \in \mathbb{N}$ and $n \in \mathbb{N}_0$.*

Sketch of the proof. Let F be a connected, compact 2-dimensional topological manifold. By Lemma 1.27 and Proposition 11.10 the boundary ∂F is a compact 1-dimensional topological manifold with no boundary. Since ∂F is compact it has finitely many components that we denote by $C_1, \dots, C_n$. Each component C_i is compact and has no boundary. Thus it follows from Propositions 1.29 that for each $i = 1, \dots, n$ there exists a homeomorphism $f_i: S^1 \rightarrow C_i$.

We denote by $\hat{F}$ the result of gluing a disk to each boundary component. More precisely, we consider

$$\hat{F} := \left(F \sqcup \bigsqcup_{i=1}^n \overline{B}^2 \times \{i\} \right) / \sim \text{ where } f_i(P) \sim P \text{ for } P \in S^1 \text{ and } i \in \{1, \dots, n\}.$$

²⁴³Why did we not make the definition as “the result of removing n disjoint open disks”? Wouldn’t that be much easier?

The resulting topological space $\widehat{F}$ is a connected, closed, 2-dimensional topological manifold²⁴⁴. Furthermore the homeomorphism type of $\widehat{F}$ does not depend on the choices of the homeomorphisms $f_1, \dots, f_n$.²⁴⁵

Now we apply Theorem 11.8. There are two possible conclusions:

- (1) Either there exists a homeomorphism $\Phi: \Sigma_g \rightarrow \widehat{F}$ from the surface of genus g to $\widehat{F}$ for a unique $g \in \mathbb{N}_0$. Then Φ restricts to a homeomorphism

$$\Phi: \Sigma_{g,n} = \Sigma_g \setminus \bigcup_{i=1}^n \Phi^{-1}(B^2 \times \{i\}) \rightarrow F.$$

- (2) Or there exists a homeomorphism $\Phi: k \cdot \mathbb{RP}^2 \rightarrow \widehat{F}$ for a unique $k \in \mathbb{N}$. Then Φ restricts to a homeomorphism

$$\Phi: R_{k,n} = k \cdot \mathbb{RP}^2 \setminus \bigcup_{i=1}^n \Phi^{-1}(B^2 \times \{i\}) \rightarrow F.$$

In both cases n is uniquely determined as the number of boundary components of F . Thus we have shown that F is homeomorphic to either $\Sigma_{g,n}$ or $R_{k,n}$ for some $g \in \mathbb{N}_0$ or $k \in \mathbb{N}$ and $n \in \mathbb{N}_0$ which are furthermore unique. $\square$

The following theorem gives us the analogous classification statement for manifolds with boundary up to *diffeomorphism*.

Theorem 11.13. *Let Σ be a connected, compact 2-dimensional manifold.*

- (1) *If Σ is orientable, then Σ is diffeomorphic to the surface $\Sigma_{g,n}$ for some unique $g, n \in \mathbb{N}_0$.*
- (2) *If Σ is non-orientable, then Σ is diffeomorphic to the surface $R_{k,n}$ for some unique $k \in \mathbb{N}$ and $n \in \mathbb{N}_0$.*

Proof. The theorem is proved in a very similar fashion as Theorem 11.12. One needs to replace Proposition 1.29 by Proposition 1.33. Furthermore we have to use Theorem 11.9 instead of Theorem 11.8. Full details are given in [Hi, Theorem 9.3.7]. $\square$

We conclude this chapter with a short discussion of the properties of fundamental groups of 2-dimensional manifolds. We recall that the fundamental group of the real projective space $\mathbb{RP}^2$ is isomorphic to $\mathbb{Z}_2$, therefore it is torsion and in particular it is not torsion-free. It is less clear whether or not the fundamental groups of the other 2-dimensional manifolds are torsion-free. More precisely, we have the following question.

Question 11.14.

²⁴⁴This statement requires more justification, we skip the details, hence we refer to it only as a “sketch of proof”.

²⁴⁵This statement also needs a justification. This is the second reason why it is called “sketch of a proof”.

(1) Let $g \in \mathbb{N}$. Is the fundamental group

$$\pi_1(\text{surface of genus } g) \cong \langle x_1, y_1, \dots, x_g, y_g \mid [x_1, y_1] \cdots [x_g, y_g] \rangle$$

torsion-free?

(2) Let $k \geq 2$. Is the fundamental group

$$\pi_1(k \cdot \mathbb{RP}^2) \cong \langle x_1, \dots, x_k \mid x_1^2 \cdots x_k^2 \rangle$$

torsion-free?

We will come back to this question at a much later stage.

11.6. Retractions onto boundary components of 2-dimensional manifolds. Let X be a topological space and let $A \subset X$ be a subset. Recall that we say that A is a *retract* of X if there exists a retraction $r: X \rightarrow A$, i.e. a map with $r(a) = a$ for all $a \in A$. We also recall the statement of Lemma 5.2.

Lemma 5.2. *The circle S^1 is not a retract of $\overline{B}^2$.*

We quickly recall the proof of Lemma 5.2.

Proof. We denote by $i: S^1 \rightarrow \overline{B}^2$ the inclusion map. Suppose there exists a retraction $r: \overline{B}^2 \rightarrow S^1$, i.e. a map with $r \circ i = \text{id}_{S^1}$. We obtain the commutative diagram

$$\begin{array}{ccc} & \overline{B}^2 & \\ i \nearrow & & \searrow r \\ S^1 & \xrightarrow{r \circ i = \text{id}_{S^1}} & S^1 \end{array} \quad \text{which induces} \quad \begin{array}{ccc} & \underbrace{\pi_1(\overline{B}^2, 1)}_{=0} & \\ i_* \nearrow & & \searrow r_* \\ \underbrace{\pi_1(S^1, 1)}_{\cong \mathbb{Z}} & \xrightarrow{(r \circ i)_* = (\text{id}_{S^1})_*} & \underbrace{\pi_1(S^1, 1)}_{\cong \mathbb{Z}} \end{array}$$

But a commutative diagram as given on the right hand side cannot exist since an isomorphism of a non-trivial group cannot factor through the trivial group. $\square$

Given $g \in \mathbb{N}_0$ and $n \in \mathbb{N}$ we define, as on page 201

$$\Sigma_{g,n} := \text{the surface of genus } g \text{ minus } n \text{ open disks.}$$

For example $\Sigma_{0,1}$ is the result of removing an open disk from S^2 , i.e. $\Sigma_{0,1}$ is a disk. In light of Lemma 5.2 the following question arises:

Question 11.15. *Let $g \in \mathbb{N}_0$ and let $n \in \mathbb{N}$. Let C be a boundary component of $\Sigma_{g,n}$. Is C a retract of $\Sigma_{g,n}$?*

Lemma 5.2 says that the answer to Question 11.15 is no if $g = 0$ and $n = 1$. The proof of Lemma 5.2 suggests that if we want to answer Question 11.15, then given a boundary component C of $\Sigma_{g,n}$ we should try to understand the inclusion induced map $\pi_1(C) \rightarrow \pi_1(\Sigma_{g,n})$.

Lemma 11.16. *Let $g \in \mathbb{N}_0$ and let $n \in \mathbb{N}$. We denote the boundary components of $\Sigma_{g,n}$ by $C_1, \dots, C_n$. Then the following statements hold:*

- (1) *There exists a presentation*

$$\pi_1(\Sigma_{g,n}) = \langle x_1, y_1, \dots, x_g, y_g, z_1, \dots, z_n \mid [x_1, y_1] \cdots [x_g, y_g] \cdot z_1 \cdots z_n \rangle$$

such that for any $i \in \{1, \dots, n\}$ the element z_i is represented by a curve that goes once around C_i . We refer to Figure 117 for an illustration.

- (2) *If $g \geq 1$, then for any $i \in \{1, \dots, n\}$ the inclusion induced map $\pi_1(C_i) \rightarrow \pi_1(\Sigma_{g,n})$ is a monomorphism.*

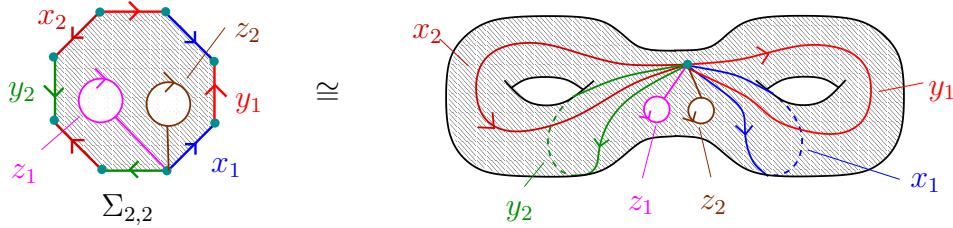


FIGURE 117. Illustration of the curves in Lemma 11.16.

Sketch of the proof. Let $g \in \mathbb{N}_0$ and let $n \in \mathbb{N}$.

- (1) The following argument is very similar to the argument on the pages 189 and 192 and it relies heavily on the notation from Figure 118. We have

$$\begin{aligned} \pi_1(\Sigma_{g,n}) &\stackrel{\text{Seifert-van Kampen Theorem 11.1}}{\downarrow} \pi_1(U) *_{\pi_1(U \cap V)} \underbrace{\pi_1(V)}_{=\{e\}} \stackrel{\text{Lemma 10.8}}{\downarrow} \pi_1(U) / \langle\langle \pi_1(U \cap V) \rangle\rangle \\ &= \langle x_1, y_1, \dots, x_g, y_g, z_1, \dots, z_n \rangle / \langle\langle [x_1, y_1] \cdots [x_g, y_g] \cdot z_1 \cdots z_n \rangle\rangle. \\ &\uparrow \\ &\text{see Figure 118} \end{aligned}$$

- (2) Now suppose that $g \geq 1$. Let $i \in \{1, \dots, n\}$. The image of $\pi_1(C_i)$ in $\pi := \pi_1(\Sigma_{g,n})$ is the subgroup generated by z_i . It suffices to show that z_i is an element of infinite order in π . It follows immediately from Lemmas 8.10 and 10.4 that there exists a homomorphism

$$\langle x_1, y_1, \dots, x_g, y_g, z_1, \dots, z_n \mid [x_1, y_1] \cdots [x_g, y_g] \cdot z_1 \cdots z_n \rangle \rightarrow \langle x_1, y_1, \dots, x_g, y_g \rangle$$

with

$$\begin{aligned} x_j &\mapsto x_j & \text{for } j = 1, \dots, g \\ y_j &\mapsto y_j & \text{for } j = 1, \dots, g \end{aligned} \quad \text{and} \quad \begin{aligned} z_i &\mapsto ([x_1, y_1] \cdots [x_g, y_g])^{-1} \\ z_j &\mapsto e & \text{for } j \neq i. \end{aligned}$$

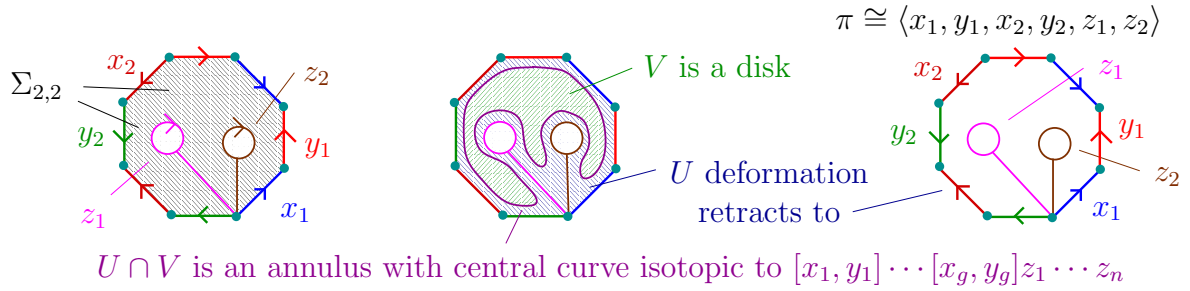


FIGURE 118. Illustration for the proof of Lemma 11.16.

The image of z_i has infinite order in the free group $\langle x_1, y_1, \dots, x_g, y_g \rangle$,²⁴⁶ hence z_i is already an element of infinite order in π . $\square$

We can now answer Question 11.15.

Lemma 11.17. *Let $g \in \mathbb{N}_0$.*

- (1) *The unique boundary component of $\Sigma_{g,1}$ is not a retract of $\Sigma_{g,1}$.*
- (2) *For $k \geq 2$ every boundary component of $\Sigma_{g,k}$ is a retract of $\Sigma_{g,k}$.*

Proof. Let $g \in \mathbb{N}_0$.

- (1) The case $g = 0$ was dealt with in Lemma 5.2. Now we assume that $g \geq 1$. We denote by C the unique boundary component of $\Sigma := \Sigma_{g,1}$. We had just seen in Lemma 11.16 that $\pi_1(C) \rightarrow \pi_1(\Sigma)$ is a monomorphism if $g \geq 1$. So we cannot apply the same argument as in Lemma 5.2 where we studied the induced map on the level of fundamental groups. The idea now is to study instead the induced map on the *abelianizations* of the fundamental groups.

We denote by $i: C \rightarrow \Sigma$ the inclusion map and we suppose that there exists a retraction $r: \Sigma \rightarrow C$, i.e. a map with $r \circ i = \text{id}_C$. We write $\pi_1(C) = \langle t \rangle$ and we use the isomorphism from Lemma 11.16 (1) to make the identification

$$\pi_1(\Sigma) = \langle x_1, y_1, \dots, x_g, y_g, z \mid [x_1, y_1] \cdots [x_g, y_g] \cdot z \rangle$$

where z corresponds to t under the inclusion $C = \partial\Sigma \rightarrow \Sigma$. We obtain the commutative diagram

$$\begin{array}{ccc}
 & \Sigma & \\
 i \nearrow & & \searrow r \\
 C & & C \\
 \text{roi}=\text{id}_C \nearrow & & \searrow
 \end{array}
 \quad \text{which induces}
 \quad
 \begin{array}{ccc}
 & \pi_1(\Sigma)_{\text{ab}} & \\
 i_* \nearrow & & \searrow r_* \\
 \underbrace{\pi_1(C)_{\text{ab}}}_{=\langle t \rangle} & \xrightarrow{(r \circ i)_* = \text{id}} & \underbrace{\pi_1(C)_{\text{ab}}}_{=\langle t \rangle}
 \end{array}$$

by Proposition 10.6 (6)
the commutative diagram

In $\pi_1(\Sigma)$ we have

$$z = ([x_1, y_1] \cdots [x_g, y_g])^{-1} \in \pi_1(\Sigma).$$

²⁴⁶Why does it have infinite order?

This shows that z lies in the commutator subgroup of $\pi_1(\Sigma)$, i.e. z represents the trivial element in $\pi_1(\Sigma)_{\text{ab}}$. Thus we have shown that the left diagonal map is the zero map. But the bottom horizontal map is an isomorphism of a non-trivial group and the diagram commutes. We have thus obtained a contradiction.

- (2) The retraction from $\Sigma_{2,n}$ to a boundary component is sketched in Figure 119. The general case is treated the same way.²⁴⁷ $\square$

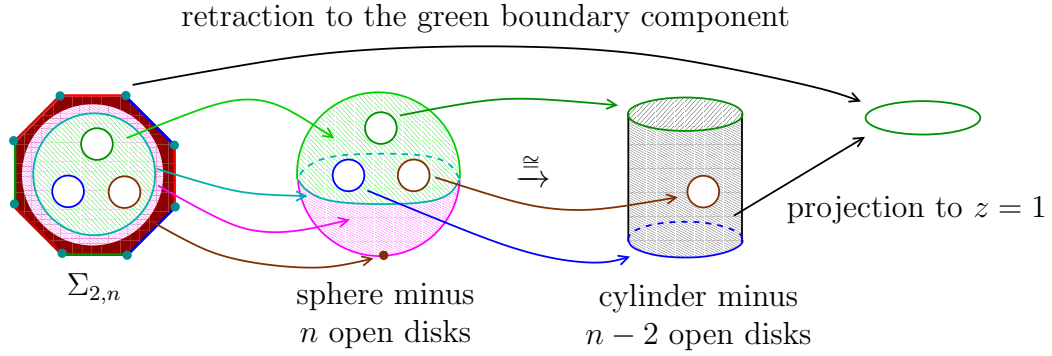


FIGURE 119.

²⁴⁷The proof is somewhat similar to the proof of Lemma 5.5.

12. EXAMPLES: KNOTS AND MAPPING TORI

In this section we will study two the fundamental groups of two more types of topological spaces. First we study the fundamental groups of the complements of knots, and then we introduce mapping tori and we study their fundamental groups.

12.1. An excursion into knot theory (*).²⁴⁸ We first recall that we have different descriptions of the 3-dimensional sphere S^3 , i.e. we have

$$S^3 = \{(w, x, y, z) \mid w^2 + x^2 + y^2 + z^2 = 1\} = \{(z, w) \in \mathbb{C} \mid |w|^2 + |z|^2 = 1\} = \mathbb{R}^3 \cup \{\infty\}.$$

$\uparrow$
via stereographic projection

We now add another point of view to the above.

Lemma 12.1. *We consider the following subspaces of S^3 :*

$$A := \{(z, w) \in \mathbb{C} \mid |w|^2 + |z|^2 = 1 \text{ and } |z|^2 \geq \tfrac{1}{2}\}$$

and

$$B := \{(z, w) \in \mathbb{C} \mid |w|^2 + |z|^2 = 1 \text{ and } |w|^2 \geq \tfrac{1}{2}\}.$$

Then the following hold:

- (1) A and B are solid tori, i.e. diffeomorphic to $S^1 \times \overline{B}^2$,
- (2) $S^3 = A \cup B$,
- (3) $A \cap B = \partial A = \partial B$ is a torus, i.e. diffeomorphic to $S^1 \times S^1$.

This lemma thus says that we can decompose S^3 as the union of two solid tori that overlap only on their common boundary. This says in particular that we can decompose $\mathbb{R}^3 \cup \{\infty\} = S^3$ into two solid tori A and B . This is not particularly easy to visualize. One of the solid tori is the standard solid torus in $\mathbb{R}^3$. The other solid torus $B = c \times \overline{B}^2$ is given by the circle $c = z\text{-axis} \cup \{\infty\}$, and for each point on c we have a disk through c that touches ∂V . For example for $c = 0$ the corresponding disk is the disk containing the origin in the xy -plane. See also Figure 120 for an illustration.

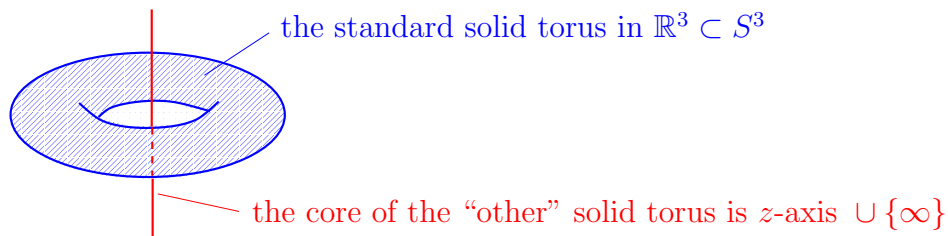


FIGURE 120.

²⁴⁸This section is not part of the course Algebraic Topology I.

Proof.

- (1) We consider the map

$$\begin{aligned} \Phi: S^1 \times \overline{B}^2 &\rightarrow B = \{(z, w) \in \mathbb{C} \mid |w|^2 + |z|^2 = 1 \text{ and } |w|^2 \geq \tfrac{1}{2}\} \\ (a, b) &\mapsto \left(\frac{a}{\sqrt{|a|^2 + |b|^2}}, \frac{b}{\sqrt{|a|^2 + |b|^2}} \right) \end{aligned}$$

together with the map Ψ in the reverse direction given by

$$\left(\frac{z}{|z|}, \frac{w}{|z|} \right) \leftarrow (z, w).$$

It is now straightforward to see that Φ and Ψ are inverse maps. In particular Φ is a diffeomorphism. This shows that B is a solid torus. Exactly the same argument shows that A is a solid torus.

- (2) Now let $(z, w) \in S^3$. From $|z|^2 + |w|^2 = 1$ it follows that $|z|^2 \geq \frac{1}{2}$ or $|w|^2 \geq \frac{1}{2}$, i.e. (z, w) lies in A or it lies in B .
- (3) The only points in S^3 which lie in both A and B are the points (z, w) with $|z|^2 = \frac{1}{2}$ and $|w|^2 = \frac{1}{2}$. It is straightforward to see that these are precisely the points on ∂A and ∂B and that they describe a torus. $\square$

Now we want to finally turn to the study of knots. In particular we would like to study the knots that are illustrated in Figure 121. In Question 2.9 we had raised the question, how can one show that the trefoil is not trivial?

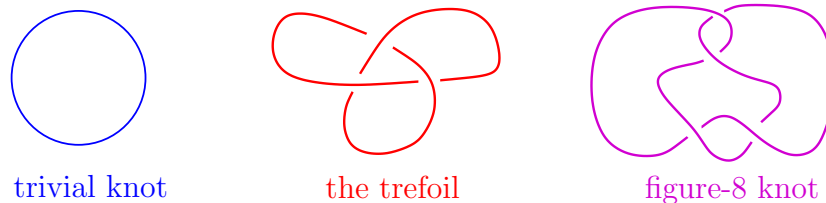


FIGURE 121. The trivial knot, the trefoil knot and the figure-8 knot.

We now have some tools to attack this question. We had seen in Proposition 2.10 that if K and J are equivalent knots, then the knot complements $S^3 \setminus K$ and $S^3 \setminus J$ are diffeomorphic. In particular if $\pi_1(S^3 \setminus K)$ and $\pi_1(S^3 \setminus J)$ are not isomorphic, then K and J are not equivalent. Thus we arrive at the following variation on Question 2.9.

Question 12.2. *Can the fundamental group be used to distinguish the trefoil from the trivial knot?*

Before we continue we want to give a precise description of the trivial knot and the trefoil knot. We define the *trivial knot* as the knot²⁴⁹

$$U := \{(e^{it}, 0) \mid t \in \mathbb{R}\} \subset \{(z, w) \in \mathbb{C} \mid |w|^2 + |z|^2 = 1\}$$

or equivalently

$$U := \{(\cos t, \sin t, 0) \mid t \in \mathbb{R}\} \subset \mathbb{R}^3 \cup \{\infty\}.$$

Now we want to give a rigorous definition of the trefoil knot. We consider the curve $C = \{(2t, 3t) \mid t \in \mathbb{R}\}$ on the torus $T = [0, 2\pi] \times [0, 2\pi] / \sim$. We define the trefoil K as the image of C under the map

$$\begin{aligned} \Phi: T = [0, 2\pi] \times [0, 2\pi] / \sim &\rightarrow S^3 \\ (s, t) &\mapsto \left(\frac{1}{\sqrt{2}} e^{is}, \frac{1}{\sqrt{2}} e^{it} \right). \end{aligned}$$

This definition of the trefoil is illustrated in Figure 122.

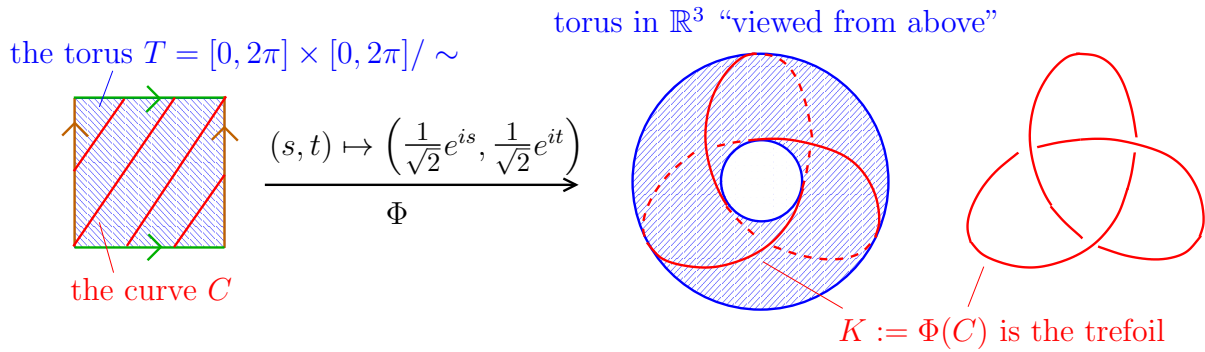


FIGURE 122.

Now we turn to the calculation of the fundamental groups of the complements of the trivial knot and the trefoil knot.

Proposition 12.3. *If U is the trivial knot, then $S^3 \setminus U$ is diffeomorphic to $S^1 \times \mathbb{C}^2$, in particular $\pi_1(S^3 \setminus U) \cong \mathbb{Z}$.*

Proof. Note that

$$\begin{aligned} S^3 \setminus U &= \{(z, w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1\} \setminus \{(z, 0) \mid |z| = 1\} \\ &= \{(z, w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1 \text{ and } |w| > 0\}. \end{aligned}$$

Now basically the same argument as in the proof of Lemma 12.1 shows that the map

$$\begin{aligned} \Phi: S^1 \times \mathbb{C} &\rightarrow S^3 \setminus U = \{(z, w) \in \mathbb{C}^2 \mid |w|^2 + |z|^2 = 1 \text{ and } |w| > 0\} \\ (a, b) &\mapsto \left(\frac{a}{\sqrt{|a|^2 + |b|^2}}, \frac{b}{\sqrt{|a|^2 + |b|^2}} \right) \end{aligned}$$

²⁴⁹The name U comes from the fact that the trivial knot is often also called the *unknot*.

is a diffeomorphism. Since $S^1 \times \overline{B}^2$ is homotopy equivalent to S^1 we obtain from the above and from Proposition 7.5 that $\pi_1(S^3 \setminus U) = \pi_1(S^1 \times \mathbb{C}) = \pi_1(S^1) \cong \mathbb{Z}$. $\square$

Remark. Let U be the trivial knot. A careful reading of the proof of Proposition 12.3 shows that a generator of $\pi_1(S^3 \setminus U)$ is given by a meridian that we had defined on page 60. We refer to Figure 123 for an illustration of the trivial knot and a meridian.

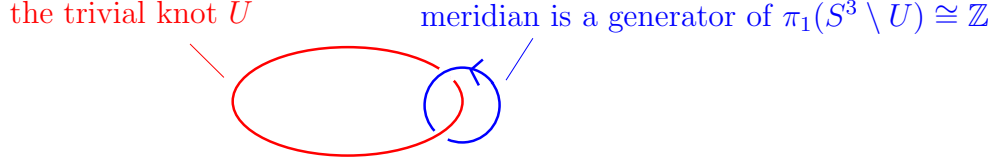


FIGURE 123.

Proposition 12.4. *If K is the trefoil, then*

$$\pi_1(S^3 \setminus K) \cong \langle x, y \mid x^2 = y^3 \rangle.$$

Proof. By Lemma 12.1 there exists a diffeomorphism

$$\Phi: S^3 \xrightarrow{\cong} \underbrace{(S^1 \times \overline{B}^2)}_{=:A} \cup_{S^1 \times S^1 = S^1 \times S^1} \underbrace{(\overline{B}^2 \times S^1)}_{=:B}$$

which is, up to scaling by a factor of $\sqrt{2}$, the identity on $T := S^1 \times S^1$. We identify the trefoil with its image under Φ , i.e. we have

$$K = \{(e^{2si}, e^{3si}) \mid s \in \mathbb{R}\} \subset T.$$

We write $A_K = A \setminus K$, $B_K = B \setminus K$ and $T_K = T \setminus K$. Note that A_K and B_K are submanifolds of $X_K := S^3 \setminus K$ and with²⁵⁰ $A_K \cap B_K = \partial A_K = \partial B_K = T_K$ and with $A_K \cup B_K = X_K$. Note that $x := S^1 \times \{0\}$ is a deformation retract of A_K and that $y := \{0\} \times S^1$ is a deformation retract of B_K . Finally note that T_K retracts onto a parallel copy of K . More precisely, for $\epsilon > 0$ sufficiently small, the subset T_K retracts onto the loop

$$z = \{(e^{2si+\epsilon}, e^{3si}) \mid s \in \mathbb{R}\} \subset T_K.$$

By an abuse of notation we denote by x, y and z also the loops corresponding to the circles x, y and z with the obvious orientation. Then the inclusion induced maps

$$\begin{array}{ccccc} \pi_1(A_K) & \leftarrow & \pi_1(T_K) & \rightarrow & \pi_1(B_K) \\ i_*(g) & \leftarrow & g & \mapsto & j_*(g) \end{array} \quad \text{become} \quad \begin{array}{ccccc} \langle x \rangle & \leftarrow & \langle z \rangle & \rightarrow & \langle y \rangle \\ x^2 & \leftarrow & z & \mapsto & y^3. \end{array}$$

Thus we see that

$$\pi_1(X_K) = \pi_1(A_K) *_{\pi_1(T_K)} \pi_1(B_K) \cong \langle x \rangle *_{x^2=y^3} \langle y \rangle = \langle x, y \mid x^2 = y^3 \rangle.$$

Seifert-van Kampen Theorem 11.2

above discussion

Lemma 10.9

$\square$

²⁵⁰Here we mean by ∂A_K and ∂B_K the boundary of A_K and B_K viewed as submanifolds of S^3 .

Now the question is whether or not $\pi_1(S^3 \setminus K) = \langle x, y \mid x^2 = y^3 \rangle$ is isomorphic to $\pi_1(S^3 \setminus U) = \mathbb{Z}$. As usual we first consider the abelianization of $\pi_1(S^3 \setminus K)$, but it is straightforward to see, using Proposition 10.6 that the abelianization is isomorphic to $\mathbb{Z}$.²⁵¹²⁵²

The following lemma says that the group $\pi_1(S^3 \setminus K) = \langle x, y \mid x^2 = y^3 \rangle$ is not isomorphic to $\pi_1(S^3 \setminus U) = \mathbb{Z}$ which by the above discussion implies in particular that the trefoil is non-trivial.

Lemma 12.5. *The group $\langle x, y \mid x^2 = y^3 \rangle$ is not isomorphic to $\mathbb{Z}$.*

Proof. It follows from Lemma 10.4 and a straightforward calculation that there exists a unique homomorphism

$$\Phi: \langle x, y \mid x^2 = y^3 \rangle \rightarrow S_3 = \text{permutation group on three elements}$$

with²⁵³

$$\Phi(x) = \sigma := (1\ 2) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

and

$$\Phi(y) = \tau := (1\ 2\ 3) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}.$$

The permutations σ and τ do not commute, hence the image of Φ is a non-abelian subgroup of S_3 .²⁵⁴ On the other hand the abelian group $\mathbb{Z}$ cannot admit an epimorphism onto a non-abelian group²⁵⁵. This concludes the proof of the lemma. $\square$

There exists a relatively straightforward algorithm for computing a presentation of the fundamental group of any knot complement. We refer to [Li2, Chapter VI.3] for details. In particular one can use fundamental groups to show that the figure-8 knot is also non-trivial and also that it is not equivalent to the trefoil.

²⁵¹In fact that is not a coincidence, we will see in Algebraic Topology II that for any knot J the abelianization of $\pi_1(S^3 \setminus J)$ is isomorphic to $\mathbb{Z}$.

²⁵²If one goes very carefully through the definitions and calculations then one can see that a meridian once again becomes a generator of the abelianization.

²⁵³Recall that for pairwise disjoint $a_1, \dots, a_k \in \{1, \dots, n\}$ we denote by $(a_1\ a_2\ \dots\ a_k) \in S_n$ the permutation that is given by

$$\begin{aligned} \{1, \dots, n\} &\rightarrow \{1, \dots, n\} \\ m &\mapsto \begin{cases} a_{i+1}, & \text{if } m = a_i \text{ for some } i \in \{1, \dots, k-1\}, \\ a_1 & \text{if } m = a_k, \\ m, & \text{otherwise.} \end{cases} \end{aligned}$$

Put differently, the numbers $(a_1, \dots, a_k)$ get cyclically permuted and all other numbers stay the same.

²⁵⁴In fact it is easy to see that the only non-abelian subgroup of S_3 is S_3 itself, i.e. Φ is an epimorphism onto S_3 .

²⁵⁵Why not?

12.2. Mapping tori. In this section we first introduce a new way to construct new topological spaces out of given topological spaces.

Definition. Let X be a topological space and let $f: X \rightarrow X$ be a homeomorphism. We refer to

$$M(X, f) := X \times [0, 1] / (x, 0) \sim (f(x), 1)$$

as the *mapping torus* of (X, f) . We refer to the map

$$\begin{aligned} M(X, f) = X \times [0, 1] / (x, 0) \sim (f(x), 1) &\rightarrow S^1 \\ [(x, t)] &\mapsto e^{2\pi i t} \end{aligned}$$

as the *canonical projection onto S^1* . Sometimes X is called the *fiber* of the mapping torus and f is called the *monodromy* of the mapping torus.

Examples.

- (1) We consider the topological space $[0, 1]$ and let $f: [0, 1] \rightarrow [0, 1]$ be the self-homeomorphism of $[0, 1]$ that is given by $f(w) = 1 - w$. It follows immediately from the definitions that the corresponding mapping torus $M([0, 1], f)$ is precisely the Möbius band as defined on page 27. We refer to Figure 124 for an illustration.

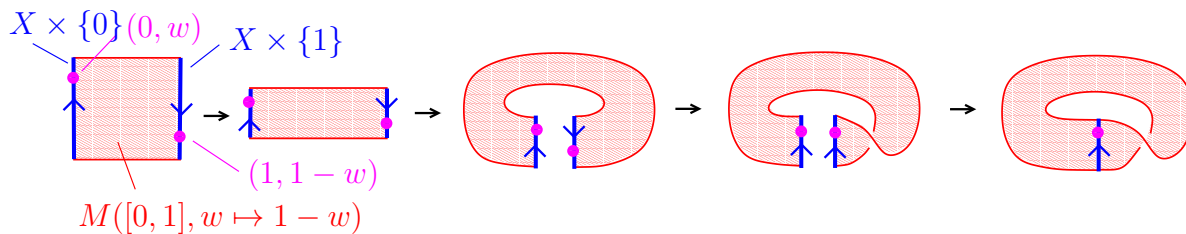


FIGURE 124. The Möbius band as a mapping torus.

- (2) Let $X = S^1$ and let $f: X \rightarrow X$ be the homeomorphism that is given by $f(z) = \bar{z}$. Then the map

$$\begin{aligned} \overbrace{([0, 1] \times [0, 1]) / (x, 0) \sim (x, 1), (0, y) \sim (1, 1 - y)}^{\text{Klein bottle}} &\mapsto M(S^1, f) \\ [(x, y)] &\mapsto [(e^{2\pi i y}, x)] \end{aligned}$$

is easily seen to give a homeomorphism from the Klein bottle to the mapping torus (S^1, f) .²⁵⁶ This homeomorphism is also illustrated in Figure 125.

- (3) Given any topological space X we have a canonical homeomorphism

$$\begin{aligned} M(X, \text{id}) &\cong X \times S^1 \\ [(x, t)] &\mapsto (x, e^{2\pi i t}). \end{aligned}$$

²⁵⁶What does the mapping torus $M(S^1, z \mapsto -z)$ look like?

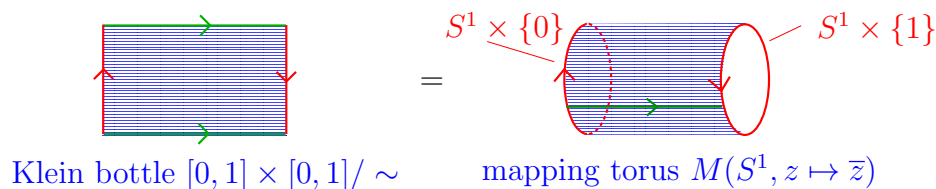


FIGURE 125.

- (4) In [Ro, Chapter I.10] and [BZH, Section 5.C] it is shown that the complement of the trefoil knot and the complement of the figure-8 knot are homeomorphic to a mapping torus, where the fiber is given by removing a point from the torus.
- (5) If one wants to construct new interesting topological spaces out of a given topological space we need to find “non-trivial” self-homeomorphisms. For example let Σ be an oriented surface and let $C \subset \Sigma$ be an oriented curve on Σ , i.e. C is an oriented submanifold diffeomorphic to S^1 . By the Tubular Neighborhood Theorem 2.2 there exists an orientation preserving embedding $\Phi: S^1 \times [-1, 1] \rightarrow \Sigma$ such that $\Phi: S^1 \times \{0\} \rightarrow C$ is an orientation preserving diffeomorphism. We refer to the homeomorphism²⁵⁷

$$\begin{aligned} \Sigma &\rightarrow \Sigma \\ P &\mapsto \begin{cases} P, & \text{if } P \notin \Phi(S^1 \times [-1, 1]), \\ \Phi(z e^{2\pi \frac{t+1}{2} i}, t) & \text{if } P = \Phi(z, t) \text{ for some } z \in S^1, t \in [-1, 1] \end{cases} \end{aligned}$$

as the *Dehn twist along the curve C* . The definition of a Dehn twist is illustrated in Figure 126.

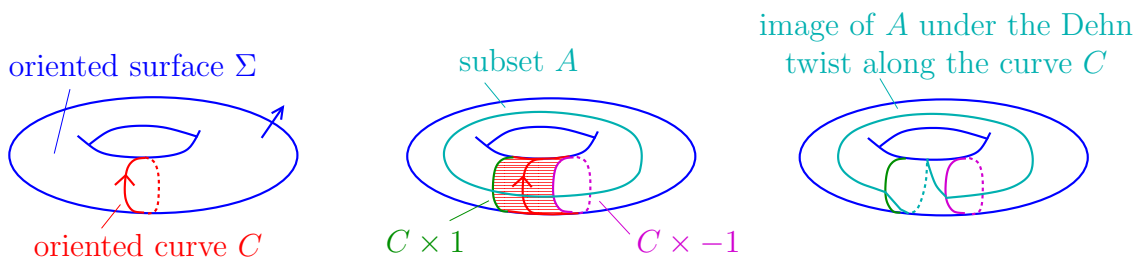


FIGURE 126.

Definition. Let X and Y be topological spaces. Two homeomorphisms $f, g: X \rightarrow Y$ are called *isotopic*, if there exists an *isotopy* from f to g , i.e. a map

$$\begin{aligned} H: X \times [0, 1] &\rightarrow Y \\ (x, t) &\mapsto H(x, t) \end{aligned}$$

such that the following hold:

²⁵⁷It is straightforward to see that the map is indeed continuous and a bijection. What is the inverse map?

- (1) We have $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for all $x \in X$,
- (2) for each $t \in [0, 1]$ the map $X \rightarrow Y$ given by $x \mapsto H(x, t)$ is a homeomorphism.

Examples.

- (1) We consider $X = [0, 1]$ and the two self-homeomorphisms given by $f(x) = x$ and by $g(x) = x^2$. Then²⁵⁸

$$\begin{aligned} H: X \times [0, 1] &\rightarrow [0, 1] \\ (x, t) &\mapsto f(x) \cdot (1 - t) + g(x) \cdot t \end{aligned}$$

is an isotopy from f to g .

- (2) Lickorish [Li] showed that any self-homeomorphism of a compact orientable 2-dimensional manifold is isotopic to the concatenation of finitely many Dehn twists.

The following lemma will be proved in exercise sheet 11.

Lemma 12.6.

- (1) Any self-homeomorphism of $[0, 1]$ is isotopic to either $f(x) = x$ or to $g(x) = 1 - x$.
- (2) Any self-homeomorphism of S^1 is isotopic to either $f(z) = z$ or to $g(z) = \bar{z}$.

The following lemma will also be proved in exercise sheet 11.

Lemma 12.7. *Let X be a topological space. If $f, g: X \rightarrow X$ are two homeomorphisms that are isotopic, then the mapping tori $M(X, f)$ and $M(X, g)$ are homeomorphic.*

Example. We consider $X = S^1$ and the self-homeomorphism that is given by $g(z) = -z$. The map $H(z, t) = e^{\pi i t} z$ defines an isotopy from $f = \text{id}$ to g . It follows from Lemma 12.7 that the mapping torus $M(S^1, g(z) = -z)$ is homeomorphic to the torus $M(S^1, \text{id}) = S^1 \times S^1$.

We can now determine which surfaces are mapping tori.

Proposition 12.8. *The annulus, the Möbius band, the torus and the Klein bottle are, up to homeomorphism, the only connected, compact 2-dimensional topological manifolds that can be written as a mapping torus $M(X, f)$ of a connected 1-dimensional topological manifold X .*

Proof. We had just shown in the examples (1), (2) and (3) on page 212 that

$$\begin{array}{ll} \text{annulus } S^1 \times [0, 1] &\cong M([0, 1], \text{id}) \\ \text{Möbius band} &\cong M([0, 1], x \mapsto 1 - x) \end{array} \quad \text{and} \quad \begin{array}{ll} \text{torus } S^1 \times S^1 &\cong M(S^1, \text{id}) \\ \text{Klein bottle} &\cong M(S^1, z \mapsto \bar{z}). \end{array}$$

In particular all these four manifolds are homeomorphic to mapping tori. It remains to show that these are the only connected, compact 2-dimensional topological manifolds that can be written as a mapping torus.

It follows from Proposition 1.29 that any connected compact topological 1-dimensional manifold is homeomorphic to $[0, 1]$ or to S^1 . We know from Lemma 12.6 that any self-homeomorphism is isotopic to one of the self-homeomorphisms that we had written down

²⁵⁸Let $t \in [0, 1]$. Why is the map $[0, 1] \rightarrow [0, 1]$ given by $x \mapsto H(x, t)$ a homeomorphism?

above. But then it follows from Lemma 12.7 that any mapping torus is homeomorphic to one from the above list. $\square$

For proving statements about mapping tori the following lemma gives a useful alternative point of view. We leave the elementary proof as an exercise to the reader.

Lemma 12.9. *Let X be a topological space and let $f: X \rightarrow X$ be a homeomorphism. Then the following hold:*

(1) *The map*²⁵⁹

$$\begin{aligned} \mathbb{Z} \times (X \times \mathbb{R}) &\rightarrow X \times \mathbb{R} \\ (n, (x, t)) &\mapsto (f^n(x), t + n) \end{aligned}$$

*defines a continuous, proper and discrete action*²⁶⁰ *of the group $\mathbb{Z}$ on $X \times \mathbb{R}$.*

(2) *The map*

$$\begin{aligned} M(X, f) = X \times [0, 1] / \sim &\rightarrow (X \times \mathbb{R}) / \mathbb{Z} \\ [(x, t)] &\mapsto [(x, t)] \end{aligned}$$

is a homeomorphism.

In the following, given a topological space X and a homeomorphism $f: X \rightarrow X$ we denote the corresponding quotient space by $(X \times \mathbb{R}) / \mathbb{Z}_f$ and we will identify the mapping torus $M(X, f)$ with $(X \times \mathbb{R}) / \mathbb{Z}_f$ using the homeomorphism of Lemma 12.9.

Remark. It follows from Lemma 12.9 together with Proposition 1.28 that if X is a topological n -dimensional manifold and f is a homeomorphism, then the corresponding mapping torus $M(X, f) = (X \times \mathbb{R}) / \mathbb{Z}_f$ is a topological $(n + 1)$ -dimensional manifold. Furthermore, if X is in fact an n -dimensional manifold and $f: X \rightarrow X$ is a diffeomorphism, then almost the same argument shows that the mapping torus $M(X, f) = (X \times \mathbb{R}) / \mathbb{Z}_f$ is an $(n + 1)$ -dimensional manifold.

Lemma 12.10. *Let X be a topological space and let $f: X \rightarrow X$ be a homeomorphism. Then the map*

$$\begin{aligned} X \times \mathbb{R} &\rightarrow (X \times \mathbb{R}) / \mathbb{Z}_f = M(X, f) \\ (x, t) &\mapsto [(x, t)] \end{aligned}$$

is a covering of infinite degree. Furthermore for $k \in \mathbb{N}$ the map

$$\begin{aligned} M(X, f^k) = (X \times \mathbb{R}) / \mathbb{Z}_{f^k} &\rightarrow (X \times \mathbb{R}) / \mathbb{Z}_f = M(X, f) \\ [(x, t)] &\mapsto [(x, k \cdot t)] \end{aligned}$$

is a k -fold covering.

Proof. Let X be a topological space and let $f: X \rightarrow X$ be a homeomorphism. The first statement is an immediate consequence of Lemma 12.9 and Proposition 6.10.

²⁵⁹Here $f^n(x)$ means $\underbrace{(f \circ \dots \circ f)}_{n\text{-times}}(x)$.

²⁶⁰Why is it an action?

Now let $k \in \mathbb{N}$. We consider the following commutative diagram of maps:

$$\begin{array}{ccc}
 (X \times \mathbb{R})/\mathbb{Z}_{f^k} & \xrightarrow{[(x,t)] \mapsto [(x,k \cdot t)]} & (X \times \mathbb{R})/\mathbb{Z}_f \\
 \searrow [(x,t)] \mapsto [(x,k \cdot t)] & & \nearrow [(x,t)] \mapsto [(x,t)] \\
 & (X \times \mathbb{R})/(k \cdot \mathbb{Z})_f &
 \end{array}$$

The left diagonal map is easily seen to be well-defined and to be a homeomorphism. By Proposition 6.10 the right diagonal map is a k -fold covering. It follows that the top horizontal map is also a k -fold covering. $\square$

Example.

- (1) We apply Lemma 12.10 to $X = [0, 1]$ and the homeomorphism that is given by $f(x) := 1 - x$. Note that $f^2(x) = f(f(x)) = f(1 - x) = 1 - (1 - x) = x$ for all x , i.e. $f^2 = \text{id}$. Thus we recover the statement from page 94 that the annulus $M([0, 1], \text{id})$ is a 2-fold covering of the Möbius band $M([0, 1], f(x) = 1 - x)$.
- (2) We apply Lemma 12.10 to $X = S^1$ and the self-homeomorphism of $X = S^1$ that is given by $f(z) := \bar{z}$. As in the previous example we see that $f^2 = \text{id}$. Thus we get, after the discussion on page 102, a new proof that the torus $M(S^1, \text{id})$ is a 2-fold covering of the Klein bottle $M(S^1, f(z) = \bar{z})$.

Our next goal is to determine the fundamental groups of mapping tori. Before we can do so we need to introduce a new concept from group theory.

Definition. Let N be a group, let $k \in \mathbb{N}_0$ and let $\varphi: N \rightarrow N$ be an isomorphism such that $\varphi^k = \text{id}$.²⁶¹ We define the *semidirect product of N and $\mathbb{Z}_k$ with respect to φ* as the group $N \rtimes_{\varphi} \mathbb{Z}_k$ where the underlying set is given by the direct product $N \times \mathbb{Z}_k$,²⁶² but where the group multiplication²⁶³ is given by

$$(h, m) \cdot (h', m') := (h \cdot \varphi^m(h'), m + m').$$

If φ is understood from the context, then we drop it from the notation, i.e. we write $N \rtimes \mathbb{Z}_k$ instead of $N \rtimes_{\varphi} \mathbb{Z}_k$.

Examples.

- (1) Let N be a group and $\varphi = \text{id}_N$, then $N \rtimes_{\text{id}} \mathbb{Z} = N \times \mathbb{Z}$ is the direct product of N and $\mathbb{Z}$.
- (2) Let $G = \mathbb{Z}_2$ and let $N = \mathbb{Z}_n$ for some $n \in \mathbb{N}_0$. We denote by $\varphi: \mathbb{Z}_n \rightarrow \mathbb{Z}_n$ the isomorphism that is given by multiplication by -1 . Clearly this isomorphism satisfies $\varphi^2 = \text{id}$, thus we can form the semidirect product $\mathbb{Z}_n \rtimes_{\varphi} \mathbb{Z}_2$, which is called

²⁶¹Note that this condition is automatically satisfied if $k = 0$, i.e. for any isomorphism φ of any group we always have $\varphi^0 = \text{id}$.

²⁶²In particular, if N is finite and $k \in \mathbb{N}$, then the cardinality of $N \rtimes \mathbb{Z}_k$ is given by $|N| \cdot |k|$.

²⁶³It is an elementary argument to show that $N \rtimes_{\varphi} \mathbb{Z}$ with the given multiplication is indeed a group.

the n -th dihedral group D_n . It has $2n$ elements. As an example, the third dihedral group D_3 is isomorphic to the permutation group S_3 via the isomorphism²⁶⁴

$$\begin{aligned}\Psi: \mathbb{Z}_3 \rtimes \mathbb{Z}_2 &\rightarrow S_3 \\ (h, m) &\mapsto (1\ 2\ 3)^h \cdot (1\ 2)^m.\end{aligned}$$

Remark. In exercise sheet 11 we will see that $N = \{(h, 0) \mid h \in N\} \subset N \rtimes \mathbb{Z}_k$ is a normal subgroup of $N \rtimes \mathbb{Z}_k$. Put differently, we have $N \triangleleft N \rtimes \mathbb{Z}_k$. This is the reason for the symbol “ $\rtimes$ ” in the notation of the semidirect product.

We have the following elementary lemma.

Lemma 12.11. *Let G be a group and let $\gamma: G \rightarrow \mathbb{Z}$ be an epimorphism. We pick an element $t \in G$ with $\gamma(t) = 1$ and we write $N := \ker(\gamma)$. We denote by φ the isomorphism*

$$\begin{aligned}\varphi: N &\rightarrow N \\ h &\mapsto tht^{-1}.\end{aligned}$$

*Then the map*²⁶⁵

$$\begin{aligned}\Psi: G &\rightarrow N \rtimes_{\varphi} \mathbb{Z} \\ g &\mapsto (gt^{-\gamma(g)}, \gamma(g))\end{aligned}$$

is an isomorphism of groups.

Proof. It is straightforward to verify that Ψ is injective and bijective. It remains to show that Ψ is a homomorphism: given $g, h \in G$ we have indeed

$$\begin{aligned}\Psi(g \cdot h) &= (ght^{-\gamma(gh)}, \gamma(gh)) = (gt^{-\gamma(g)} \cdot \underbrace{t^{\gamma(g)} \cdot ht^{-\gamma(h)} \cdot t^{-\gamma(g)}}_{=\varphi^{\gamma(g)}(ht^{-\gamma(h)})}, \gamma(g) + \gamma(h)) \\ &= (gt^{-\gamma(g)}, \gamma(g)) \cdot (ht^{-\gamma(h)}, \gamma(h)) = \Psi(g) \cdot \Psi(h). \quad \square\end{aligned}$$

The following proposition shows that the fundamental group of a mapping torus $M(X, f)$ is isomorphic to semidirect product of $\pi_1(X)$ with $\mathbb{Z}$.

Proposition 12.12. *Let X be a path-connected topological space and let $f: X \rightarrow X$ be a homeomorphism. Then there exists an isomorphism $\psi: \pi_1(M(X, f)) \cong \pi_1(X) \rtimes \mathbb{Z}$ such that the following diagram commutes*

$$\begin{array}{ccccc}\pi_1(X) & \longrightarrow & \pi_1(M(X, f)) & \longrightarrow & \pi_1(S^1) \\ \downarrow & & \cong \downarrow \psi & & \downarrow \cong \\ \pi_1(X) & \longrightarrow & \pi_1(X) \rtimes \mathbb{Z} & \longrightarrow & \mathbb{Z}.\end{array}$$

²⁶⁴First note that a straightforward calculation shows that $(1\ 2)^{-1} \cdot (1\ 2\ 3) \cdot (1\ 2) = (1\ 3\ 2) = (1\ 2\ 3)^{-1}$. This implies that also for any $n \in \mathbb{Z}_2$ and $s \in \mathbb{Z}$ we have $(1\ 2)^{-n} \cdot (1\ 2\ 3)^s \cdot (1\ 2)^n = (1\ 2\ 3)^{s(-1)^n}$. Now for any $(h, m), (h', m') \in \mathbb{Z}_n \rtimes_{\varphi} \mathbb{Z}_2$ we have

$$\begin{aligned}\Psi(h, m)\Psi(h', m') &= (1\ 2\ 3)^h \cdot (1\ 2)^m \cdot (1\ 2\ 3)^{h'} \cdot (1\ 2)^{m'} = (1\ 2\ 3)^h \cdot (1\ 2)^m (1\ 2)^{-m} (1\ 2\ 3)^{(-1)^m h'} (1\ 2)^m \cdot (1\ 2)^{m'} \\ &= (1\ 2\ 3)^{h+(-1)^m h'} \cdot (1\ 2)^{m+m'} = \Psi(h + (-1)^m h', m + m') = \Psi((h, m) \cdot (h', m')).\end{aligned}$$

We have thus shown that Ψ is a homomorphism.

²⁶⁵Note that $\gamma(gt^{-\gamma(g)}) = \gamma(g) + \gamma(t)(-\gamma(g)) = 0$, i.e. $gt^{-\gamma(g)}$ does indeed lie in $N = \ker(\gamma)$.

Hereby the top-left horizontal map is induced by the inclusion $X = X \times \{0\} \rightarrow M(X, f)$ and the top-right horizontal map is induced by the canonical projection map $M(X, f) \rightarrow S^1$. The bottom-left horizontal map is given by the natural inclusion into the first term of the semidirect product and the bottom-right horizontal map is given by the projection onto the second term of the semidirect product.²⁶⁶

In the proof of Proposition 12.12 we will need the following lemma which is a straightforward generalization of Theorem 6.16 and the proof thereof. We leave the details of the proof to the reader.

Lemma 12.13. *Let X be a path-connected²⁶⁷ topological space. Let G be a group which acts continuously and discretely on X . We choose an $x \in X$. We denote by $p: X \rightarrow X/G$ the corresponding covering map.²⁶⁸ Then the map*

$$\begin{array}{ccc} \pi_1(X/G, [x]) & \rightarrow & G \\ [\gamma: [0, 1] \rightarrow X/G] & \mapsto & \begin{array}{l} \text{unique } g \in G \text{ such that for the unique lift}^{269} \\ \tilde{\gamma}: [0, 1] \rightarrow X \text{ of the path } \gamma \text{ with } \tilde{\gamma}(0) = x \\ \text{we have } \tilde{\gamma}(1) = g \cdot x. \end{array} \end{array}$$

is an epimorphism with kernel isomorphic to $\pi_1(X, x)$.

We can now give the proof of Proposition 12.12.

Proof of Proposition 12.12. We have

$$\begin{array}{ccccccc} \pi_1(M(X, f)) & = & \pi_1((X \times \mathbb{R})/\mathbb{Z}_f) & \cong & \pi_1(X \times \mathbb{R}) \rtimes \mathbb{Z} & \cong & \pi_1(X) \rtimes \mathbb{Z}. \\ \uparrow & & \uparrow & & \uparrow & & \\ \text{Lemma 12.9} & & \text{Lemmas 12.13 and 12.11} & & & & \end{array}$$

Thus we have obtained an isomorphism $\psi: \pi_1(M(X, f)) \xrightarrow{\cong} \pi_1(X) \rtimes \mathbb{Z}$. We leave the verification that the given diagram commutes to the reader. $\square$

Remark. A careful reading of the proof of the proposition shows that we can identify the automorphism in the semidirect product. More precisely, let X be a path-connected topological space and let $f: X \rightarrow X$ be a homeomorphism. Let x be a base point and let γ be path from $f(x)$ to x . We denote by φ the automorphism of $\pi_1(X, x)$ that is given by

$$\begin{array}{ccccc} \pi_1(X, x) & \xrightarrow{f_*} & \pi_1(X, f(x)) & \xrightarrow[\text{Prop. 4.9}]{\cong} & \pi_1(X, x). \\ & & \searrow & \nearrow & \\ & & & \text{=: } \varphi & \end{array}$$

Then $\pi_1(M(X, f)) \cong \pi_1(X, x) \rtimes_{\varphi} \mathbb{Z}$.

We conclude this section with a discussion of several examples.

²⁶⁶Note that both maps on the bottom are group homomorphisms.

²⁶⁷In Theorem 6.16 we had assumed that X is simply connected.

²⁶⁸Recall that by Proposition 6.9 the map $p: X \rightarrow X/G$ is a covering.

²⁶⁹Recall that by Lemma 6.11 there exists a unique lift $\tilde{\gamma}$ of γ to the starting point $\tilde{\gamma}(0) = x$.

Examples.

- (1) Let $K = M(S^1, f(z) := \bar{z})$ be the Klein bottle. Then the automorphism $f_* =: \varphi$ of $\pi_1(S^1) = \mathbb{Z}$ is multiplication by -1 . It follows from the previous remark²⁷⁰ that

$$\pi_1(\text{Klein bottle}) \cong \mathbb{Z} \rtimes_{\varphi} \mathbb{Z}.$$

- (2) Let $A \in \text{SL}(n, \mathbb{Z})$ be a matrix. In exercise sheet 4 we proved the following two statements:

- (a) the map

$$\begin{aligned} f(A): \mathbb{R}^n/\mathbb{Z}^n &\rightarrow \mathbb{R}^n/\mathbb{Z}^n \\ v &\mapsto Av \end{aligned}$$

is a homeomorphism,

- (b) if we identify $\pi_1(\mathbb{R}^n/\mathbb{Z}^n, 0)$ with $\mathbb{Z}^n$ as in Theorem 6.16, then we obtain the following commutative diagram

$$\begin{array}{ccc} \pi_1(\mathbb{R}^n/\mathbb{Z}^n, 0) & \xrightarrow{f(A)_*} & \pi_1(\mathbb{R}^n/\mathbb{Z}^n, 0) \\ \downarrow = & & \downarrow = \\ \mathbb{Z}^n & \xrightarrow{v \mapsto Av} & \mathbb{Z}^n. \end{array}$$

It follows from the above remark that $\pi_1(M(\mathbb{R}^n/\mathbb{Z}^n, f(A))) \cong \mathbb{Z}^n \rtimes_A \mathbb{Z}$. Note that this implies that $\pi_1(M(\mathbb{R}^n/\mathbb{Z}^n, f(A)))$ is solvable.²⁷¹

- (3) Let X be a connected topological space. Then the mapping torus $M(X, \text{id}_X)$ is homeomorphic to the product $X \times S^1$. By the above remark we have

$$\pi_1(X \times S^1) = \pi_1(M(X, \text{id}_X)) \cong \pi_1(X) \rtimes_{\text{id}} \mathbb{Z} = \pi_1(X) \times \mathbb{Z} = \pi_1(X) \times \pi_1(S^1).$$

So in this space case we obtain the same result as in Proposition 6.18.

²⁷⁰On page 190 we had seen that $\pi_1(K) \cong \langle x, y | yxyx^{-1} \rangle$. What is an isomorphism between $\langle x, y | yxyx^{-1} \rangle$ and $\mathbb{Z} \rtimes_{\varphi} \mathbb{Z}$?

²⁷¹Recall that a group G is solvable if there exist subgroups $G_0 = \{1\} \subset G_1 \subset G_2 \subset \cdots \subset G_k = G$ such that for $i = 0, \dots, k-1$ the group G_i is normal in G_{i+1} and such that G_{i+1}/G_i is abelian. In our case the filtration is given by $\{0\} \subset \mathbb{Z}^n \subset \mathbb{Z}^n \rtimes_A \mathbb{Z}$.

13. LIMITS

The goal of this section is to determine the fundamental group of “unbounded spaces”, e.g. we would like to determine the fundamental group of the surface of “infinite genus” illustrated in Figure 127.²⁷² One idea might be to view the surface of “infinite genus” as the limit of a sequence given by compact surfaces. But then we need to make precise what we mean by a “limit of topological spaces” and “limit of the corresponding fundamental groups”. We will do so in the following sections.

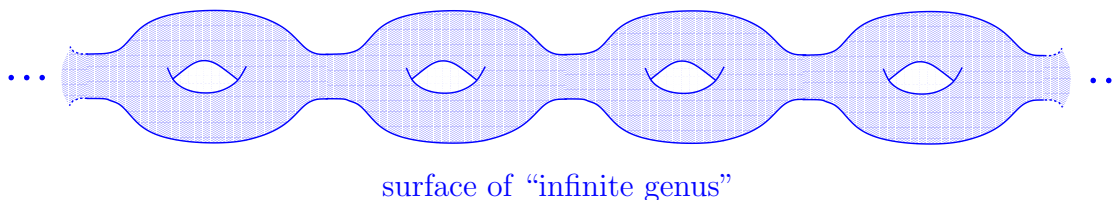


FIGURE 127.

13.1. Preordered and directed sets.

Definition. Let X be a set and let $\leq$ be a relation on X .

- (1) The relation is called *reflexive* if for any $x \in X$ we have $x \leq x$.
- (2) The relation is called *transitive* if for any $x, y, z \in X$ we have

$$x \leq y \text{ and } y \leq z \implies x \leq z.$$

- (3) A reflexive and transitive relation is called a *preorder*.
- (4) A set together with a preorder is called a *preordered set*.
- (5) A *directed set* is a preordered set such that for any $x, y \in X$ there exists a $z \in X$ with $x \leq z$ and $y \leq z$.

Examples.

- (1) Let X be a subset of $\mathbb{R}$ and denote by $\leq$ the usual “less or equal” relation on the real numbers. Then $(X, \leq)$ is a directed set. In particular $(\mathbb{N}, \leq)$ is a directed set.
- (2) Given any set X the relation “ $\leq$ ” where $x \leq y$ only if $x = y$ is a preorder, which we call the *trivial preorder* on X .
- (3) Let M be a set. We denote by $\mathcal{P}(M)$ the power set of M , i.e. the set of all subsets of M . Then “being a subset”, i.e. the relation given by “ $\subseteq$ ”, defines a preorder on M . It is straightforward to see that $(\mathcal{P}(M), \subseteq)$ is a preordered set, in fact it is even a directed set.
- (4) The set $X = \{0, x, y\}$ with the trivial relations $0 \leq 0, x \leq x, y \leq y$ and the non-trivial relations $0 \leq x$ and $0 \leq y$ is a preordered set. We refer to $(\{0, x, y\}, \leq)$ as the *push-pull set*. Note that this is not a directed set, since there is no $z \in X$ with $x \leq z$ and $y \leq z$.

²⁷²Actually we also need to make precise what we mean by the surface of “infinite genus”.

- (5) We consider the set $\mathbb{N}$. The divisibility relation $k|l$ for $k, l \in \mathbb{N}$ is a preorder on $\mathbb{N}$ and with this preorder $\mathbb{N}$ is in fact a directed set.²⁷³

13.2. The direct limit of a direct system.

Definition. Let $(I, \leq)$ be a preordered set²⁷⁴ and let $\mathcal{C}$ be a category. A *direct system in the category $\mathcal{C}$ over I* is a family of objects $\{X_i\}_{i \in I}$ in $\mathcal{C}$ indexed by I , together with a family of morphisms $\{f_{ij}: X_i \rightarrow X_j\}$ for all $i, j \in I$ with $i \leq j$ such that the following two conditions are satisfied:

- (1) $f_{ii} = \text{id}_{X_i}$ for all $i \in I$,
- (2) $f_{ik} = f_{jk} \circ f_{ij}$ for all $i, j, k \in I$ with $i \leq j \leq k$.

Examples.

- (A) We consider the directed set $I = (\mathbb{N}, \leq)$ and the category of groups. For $i \in \mathbb{N}$ we consider the group $\mathbb{Z}^i$ and for $i \leq j$ we denote by

$$\begin{aligned} f_{ij}: \mathbb{Z}^i &\rightarrow \mathbb{Z}^j \\ (x_1, \dots, x_i) &\mapsto (x_1, \dots, x_i, 0, \dots, 0) \end{aligned}$$

the inclusion map. Then $(\{\mathbb{Z}^i\}_{i \in \mathbb{N}}, \{f_{ij}\}_{i \leq j})$ forms clearly a direct system.

- (B) Again we consider $I = (\mathbb{N}, \leq)$ and the category of groups. Given $i \in \mathbb{N}$ we define

$$F_i := \langle x_1, \dots, x_i \rangle = \text{the free group on } i \text{ generators}$$

and given $i \leq j$ we set

$$\begin{aligned} f_{ij}: F_i &\rightarrow F_j \\ x_k &\mapsto x_k, \text{ for } k = 1, \dots, i. \end{aligned}$$

This is clearly a direct system of groups.

- (C) Given $r \in \mathbb{N} \cup \{\infty\}$ we set

$$X_r := \left(0, r + \frac{1}{2}\right) \cup \underbrace{\bigcup_{j=1}^r \left\{z \in \mathbb{C} \mid \left|z - \left(j + \frac{1}{4}i\right)\right| = \frac{1}{4}\right\}}_{\text{circle of radius } \frac{1}{4} \text{ around } j + \frac{1}{4}i \in \mathbb{C}} \subset \mathbb{C}.$$

For example, for $r \in \mathbb{N}$ the set X_r is the subset of $\mathbb{C} = \mathbb{R}^2$ that is given by the interval $(0, r + \frac{1}{2})$ with r circles attached. We refer to Figure 128 for an illustration.

Now we consider the directed set $(\mathbb{N}, \leq)$ and for $r \leq s$ we denote by $f_{rs}: X_r \rightarrow X_s$ the obvious inclusion map. This is clearly a direct system of topological spaces.

- (D) We can generalize example (C) as follows. Let $(I, \leq)$ be a preordered set and let $\{X_i\}_{i \in I}$ be a family of topological spaces such that for any $i \leq j$ we have $X_i \subset X_j$. Given $i \leq j$ we denote by $\iota_{ij}: X_i \rightarrow X_j$ the inclusion map. Then $(\{X_i\}_{i \in I}, \{\iota_{ij}\}_{i \leq j})$ is a preordered system in the category of topological spaces.

²⁷³Indeed, given $k, l \in \mathbb{N}$ we have $k|kl$ and $l|kl$.

²⁷⁴In the literature one often demands that I is a directed set, see e.g. [Ha, p. 243], but this is not necessary.

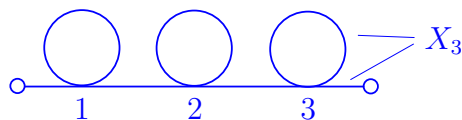


FIGURE 128.

- (E) Let $(\{0, x, y\}, \leq)$ be the push-pull set and let $\mathcal{C}$ be a category. A direct system in the category $\mathcal{C}$ over the preordered set $(\{0, x, y\}, \leq)$ consists of three objects A_0, A_x, A_y , the identity morphisms $\text{id}_{A_0}, \text{id}_{A_x}, \text{id}_{A_y}$ and morphisms $f_x: A_0 \rightarrow A_x$ and $f_y: A_0 \rightarrow A_y$. We refer to such a system as a *pushout system* and often we arrange the morphisms in a diagram as follows

$$\begin{array}{ccc} A_0 & \xrightarrow{f_x} & A_x \\ f_y \downarrow & & \\ & & A_y. \end{array}$$

Definition. Let $(I, \leq)$ be a preordered set and let $\mathcal{C}$ be a category. Suppose we are given a direct system $(\{X_i\}_{i \in I}, \{f_{ij}\}_{i \leq j})$ in the category $\mathcal{C}$. A *direct limit*²⁷⁵ of the direct system is an object Y in $\mathcal{C}$ together with morphisms $g_i: X_i \rightarrow Y, i \in I$ in the category $\mathcal{C}$ such that the following two conditions are satisfied:

- (1) For all $i \leq j$ we have $g_j \circ f_{ij} = g_i: X_i \rightarrow Y$, i.e. for all $i \leq j$ the following diagram commutes

$$\begin{array}{ccc} X_j & \xrightarrow{g_j} & Y \\ f_{ij} \uparrow & & \nearrow g_i \\ X_i & \xrightarrow{g_i} & Y. \end{array}$$

- (2) If we are given another object Y' and morphisms $g'_i: X_i \rightarrow Y', i \in I$ that satisfy (1), then there exists a unique morphism $F: Y \rightarrow Y'$ such that for all $i \leq j$ the following diagram commutes²⁷⁶

$$\begin{array}{ccccc} X_j & \xrightarrow{g'_j} & & & Y' \\ & \searrow g_j & & \nearrow F & \\ f_{ij} \uparrow & & Y & \xrightarrow[\exists!]{F} & Y' \\ & \nearrow g_i & & \nwarrow g'_i & \\ X_i & \xrightarrow{g'_i} & & & Y' \end{array}$$

²⁷⁵Somewhat confusingly a “direct limit” is often also called a “projective limit” or “colimit of a direct system”.

²⁷⁶The notation $\exists!$ means that there exists a uniquely defined morphism that makes the diagram commute.

The “usual argument” for objects that satisfy a universal property²⁷⁷ shows, that if the direct limit exists, then it is unique in an appropriate sense. We denote the direct limit by $\lim_{\rightarrow} X_i$ or sometimes by $\varinjlim X_i$.²⁷⁸

Example. Now we discuss the direct limits of the direct systems we had discussed above.

- (A) We consider the above direct system in the category of groups that is given by the free abelian groups $\mathbb{Z}^i$, $i \in \mathbb{N}$, with the obvious inclusion homomorphisms $\mathbb{Z}^i \rightarrow \mathbb{Z}^j$ for $i \leq j$. We claim that²⁷⁹

$$\varinjlim \mathbb{Z}^i = \mathbb{Z}^{(\mathbb{N})} = \{(x_1, x_2, \dots) \mid x_i \in \mathbb{Z} \text{ but only finitely many } x_i \text{'s are non-zero}\},$$

where the maps $\mathbb{Z}^j \rightarrow \varinjlim \mathbb{Z}^i = \mathbb{Z}^{(\mathbb{N})}$, $j \in \mathbb{N}$ are the obvious inclusion maps. Indeed, given a group Y' and homomorphisms $g'_j: \mathbb{Z}^j \rightarrow Y'$ that satisfy condition (1) we define

$$\begin{aligned} F: \varinjlim \mathbb{Z}^i = \mathbb{Z}^{(\mathbb{N})} &\rightarrow Y' \\ (x_1, \dots, x_j, 0, \dots) &\mapsto g'_j(x_1, \dots, x_j). \end{aligned}$$

It is straightforward to see that this definition is well-defined, i.e. independent of the choice of x_k .²⁸⁰ It is clear that the diagram (2) commutes. Furthermore, since $(x_1, \dots, x_j, 0, \dots) = g_j(x_1, \dots, x_j)$ it is clear that F is the unique homomorphism that makes the diagram (2) commute.

Thus we have now shown that the direct limit is the free abelian group on an infinite countable generating set.

- (B) We consider the above direct system in the category of groups that is given by the free groups $F_i := \langle x_1, \dots, x_i \rangle$, $i \in \mathbb{N}$, with the obvious inclusion homomorphisms $F_i \rightarrow F_j$ for $i \leq j$. Similar to the proof of (A) one can show that

$$\varinjlim F_i = \langle x_1, x_2, x_3, \dots \rangle = \text{free group on the generators } x_1, x_2, x_3, \dots$$

where the maps $F_j \rightarrow \varinjlim F_i = \langle x_1, x_2, \dots \rangle$, $j \in \mathbb{N}$ are the obvious maps. Put differently, the direct limit is the free group on an infinite countable generating set.

- (C) We continue with the direct system $(X_r, r \in \mathbb{N})$ in the category of topological spaces from above. Perhaps not surprisingly the direct limit is the interval $(0, \infty)$ with infinitely many circles attached, i.e.

$$\varinjlim X_r = X_\infty = (0, \infty) \cup \bigcup_{j=1}^{\infty} \left\{ z \in \mathbb{C} \mid \left| z - \left(j + \frac{1}{4}i \right) \right| = \frac{1}{4} \right\} \subset \mathbb{C},$$

where the maps $X_s \rightarrow \varinjlim X_r = X_\infty$, $s \in \mathbb{N}$ are the obvious inclusion maps. We leave the verification of this statement as an exercise.

²⁷⁷See for instance the discussion on page 138 for an example of such an argument.

²⁷⁸Hereby we ignore the maps f_{ij} and g_i in the notation. Usually these maps are clear from the context.

²⁷⁹Why is the direct limit not given by $\mathbb{Z}^{\mathbb{N}}$, i.e. the group of all infinite sequences?

²⁸⁰Here we did not demand that $x_k = 0$.

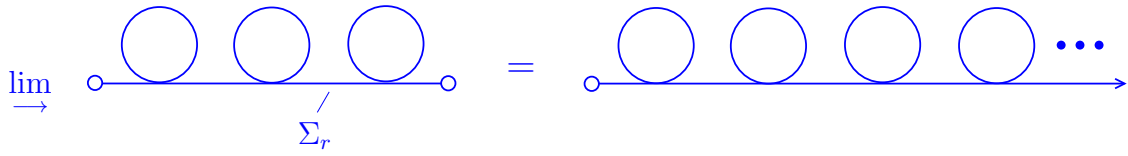


FIGURE 129.

- (D) Let $(I, \leq)$ be a preordered set and let $\{X_i\}_{i \in I}$ be a family of topological spaces such that for any $i \leq j$ we have $X_i \subset X_j$. In the category of topological spaces we have

$$\lim_{\rightarrow} X_i = X := \bigcup_{i \in I} X_i,$$

where the topology on $X = \bigcup_{i \in I} X_i$ is given by the rule that $U \subset X$ is open if and only if $U \cap X_i$ is open for all $i \in I$. Furthermore the maps $g_j: X_j \rightarrow X = \bigcup_{i \in I} X_i, j \in \mathbb{N}$ are the obvious inclusion maps.²⁸¹ The verification of this statement is an exercise in exercise sheet 12.

Definition. If the direct limit of a pushout system

$$\begin{array}{ccc} A_0 & \xrightarrow{f_x} & A_x \\ f_y \downarrow & & \\ A_y & & \end{array}$$

exists, then we refer to it as the *pushout* of the *pushout system*. The defining property of the pushout Z can be summarized in the following commutative diagram²⁸²

$$\begin{array}{ccccc} A_0 & \xrightarrow{f_x} & A_x & & \\ f_y \downarrow & & \downarrow g_x & \searrow g'_x & \\ A_y & \xrightarrow{g_y} & \text{pushout } Z & \xrightarrow{\exists!} & Z' \\ & \searrow g'_y & & & \end{array}$$

Examples.

- (1) We first consider the pushout in the category of groups. By Proposition 10.7 the pushout of a pushout system in the category of groups

$$\begin{array}{ccc} G & \xrightarrow{\alpha} & A \\ \beta \downarrow & & \downarrow \\ B & & \end{array} \quad \text{is given by} \quad \begin{array}{ccc} G & \xrightarrow{\alpha} & A \\ \beta \downarrow & & \downarrow \\ B & \longrightarrow & A *_G B = \frac{A * B}{\langle\langle \{\alpha(g) \cdot \beta(g)^{-1}\}_{g \in G} \rangle\rangle}. \end{array}$$

²⁸¹Why are the maps g_j continuous, i.e. why are they morphisms in the category of topological spaces.

²⁸²Here we suppress the morphism $g_0: A_0 \rightarrow Z$ from the diagram since it is given by $g_x \circ f_x = g_y \circ f_y$.

- (2) Now we consider the pushout in the category of topological spaces. The pushout of a pushout system of topological spaces

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ g \downarrow & & \downarrow \\ Z & & \end{array} \quad \text{is given by} \quad \begin{array}{ccc} X & \xrightarrow{f} & Y \\ g \downarrow & & \downarrow \\ Z & \longrightarrow & Y \sqcup_X Z := (Y \sqcup Z)/f(x) \sim g(x). \end{array}$$

Hereby we equip $Y \sqcup_X Z$ with the quotient topology induced from the obvious topology on $Y \sqcup Z$. We leave it as an exercise to show that this is indeed the pushout.

- (3) The direct limit of a direct system does not always exist. For example, denote by $\mathcal{C}$ the category with two objects, namely the trivial group and the group $\mathbb{Z}_2$, and where the morphisms are the usual group homomorphisms. Then the pushout system

$$\begin{array}{ccc} 0 & \longrightarrow & \mathbb{Z}_2 \\ \downarrow & & \\ \mathbb{Z}_2 & & \end{array} \quad \text{does not have a direct limit} \quad \begin{array}{ccc} 0 & \longrightarrow & \mathbb{Z}_2 \\ \downarrow & & \downarrow \beta \\ \mathbb{Z}_2 & \xrightarrow{\alpha} & C \end{array}$$

since there is no commutative diagram of the form

with $C = 0$ or $C = \mathbb{Z}_2$ such that in the following two diagrams there exists the desired diagonal maps from C to $Y' = \mathbb{Z}_2$:²⁸³

$$\begin{array}{ccc} 0 & \longrightarrow & \mathbb{Z}_2 \\ \downarrow & & \downarrow \beta \\ \mathbb{Z}_2 & \xrightarrow{\alpha} & C \end{array} \quad \begin{array}{c} \nearrow x \mapsto x \\ \searrow x \mapsto 0 \end{array} \quad \begin{array}{ccc} 0 & \longrightarrow & \mathbb{Z}_2 \\ \downarrow & & \downarrow \beta \\ \mathbb{Z}_2 & \xrightarrow{\alpha} & C \end{array} \quad \begin{array}{c} \nearrow x \mapsto 0 \\ \searrow x \mapsto x \end{array}$$

and

Proposition 13.1. *The direct limit of any direct system exists in the following categories:*

- (0) the category of sets,
- (1) the category of topological spaces,
- (2) the category of groups,
- (3) the category of abelian groups,
- (4) the category of rings,
- (5) the category of R -modules, where R is a commutative ring.

Proof. Let $(I, \leq)$ be a preordered set and let $\mathcal{C}$ be one of the six given categories. Furthermore let $(\{X_i\}_{i \in I}, \{f_{ij}: X_i \rightarrow X_j\}_{i \leq j})$ be a direct system in $\mathcal{C}$.

- (0),(1) If we work in the category of sets, then we define the direct limit as the set

$$\lim_{\longrightarrow} X_i := \left(\bigsqcup_{i \in I} X_i \right) / \sim \quad \text{where } x \sim f_{ij}(x) \text{ for all } i \leq j, x \in X_i.$$

²⁸³Why is there no such diagram with $C = 0$ or $C = \mathbb{Z}_2$ and homomorphisms α and β ?

If we work in the category of topological spaces, then $\bigsqcup_{i \in I} X_i$ is equipped with the topology where a subset U is open if and only if $U \cap X_i$ is open for all i .²⁸⁴

- (2) If we work in the category of groups, then we cannot use the definition of (0) and (1), since the union of groups is not a group. Therefore we define the direct limit via the “smallest group that contains all X_i ”, i.e. we define it as the group

$$\lim_{\rightarrow} X_i := \left(\bigast_{i \in I} X_i \right) / \langle \langle \{f_{ij}(x) \cdot x^{-1} \mid i \leq j \text{ and } x \in X_i\} \rangle \rangle.$$

- (3),(4),(5) If we work in the category of abelian groups, or in the category of rings or in the category of R -modules, then we define²⁸⁵

$$\lim_{\rightarrow} X_i := \left(\bigoplus_{i \in I} X_i \right) / \{f_{ij}(x) - x \mid i \leq j \text{ and } x \in X_i\}.$$

Furthermore in all six cases we denote by $g_j: X_j \rightarrow \lim_{\rightarrow} X_i$, $j \in I$ the obvious maps. It is now straightforward to verify that in all five categories we provided a direct limit. We leave the details to the reader. $\square$

Remark. Suppose we are given a direct system $(\{X_i\}_{i \in I}, \{f_{ij}: X_i \rightarrow X_j\}_{i \leq j})$ in any of the four algebraic categories mentioned in the previous proposition. We now assume that $(I, \leq)$ is not only a preordered set, but that it is in fact a directed set. Then we can also define

$$\lim_{\rightarrow} X_i := \left(\bigsqcup_{i \in I} X_i \right) / \sim \text{ where } x \sim f_{ij}(x) \text{ for all } i \leq j, x \in X_i$$

and we denote by $g_j: X_j \rightarrow \lim_{\rightarrow} X_i$, $j \in I$, the obvious maps. If we work in the category of groups, then $\lim_{\rightarrow} X_i$ is naturally a group as follows: if we are given $a \in X_i$ and $b \in X_j$, by definition of a directed set there exists $k \in I$ with $i \leq k$ and $j \leq k$, and we define

$$[a] \cdot [b] := \underbrace{f_{ik}(a)}_{\in X_k} \cdot \underbrace{f_{jk}(b)}_{\in X_k} \in \lim_{\rightarrow} X_i.$$

It follows easily from the definition of a direct system that this definition of the product structure is independent of the choice of k and that this defines indeed a group structure on $\lim_{\rightarrow} X_i$. Similarly we deal with the case that we work in the other three algebraic categories.

Question. Let $(\{X_i\}_{i \in I}, \{f_{ij}\}_{i \leq j})$ be a direct system of topological spaces. We can now form the direct limit $\lim_{\rightarrow} X_i$ of the topological spaces. The maps f_{ij} also induce homomorphisms on fundamental groups. The question is, when does “taking limits commute with taking

²⁸⁴Note that for the above example of the pushout we do *not* get the same topological space (why not?) but we get a topological space that is homeomorphic (why?) to the above pushout. Since we only care about the direct limit up to homeomorphism this makes no difference.

²⁸⁵Here we really mod out by the subgroup generated by the differences $f_{ij}(x) - x$.

fundamental groups”, i.e. under what circumstances do we have²⁸⁶

$$\pi_1\left(\varinjlim X_i\right) = \varinjlim \pi_1(X_i) ?$$

Example. We have already seen two instances where the equality of the question holds.

- (1) Suppose X is a topological space that is the union of two open subsets U and V such that U, V and $U \cap V$ are path-connected. We fix a base point in $U \cap V$. Then

$$\pi_1\left(\varinjlim \begin{array}{c} U \cap V \rightarrow U \\ \downarrow \\ V \end{array}\right) \underset{\text{see page 225}}{=} \pi_1(X) \underset{\substack{\text{Seifert-van Kampen} \\ \text{Theorem 11.1}}}{=} \pi_1(U) *_{\pi_1(U \cap V)} \pi_1(V) \underset{\text{see page 224}}{=} \varinjlim \begin{array}{c} \pi_1(U \cap V) \rightarrow \pi_1(U) \\ \downarrow \\ \pi_1(V) \end{array}$$

- (2) Let X be a path-connected topological space and let $X_k, k \in \mathbb{N}$ be a sequence of subsets of X such that the following hold:

- (a) each X_k is open,
- (b) each X_k is simply connected,
- (c) the sequence X_k is nested, i.e. for each k we have $X_k \subset X_{k+1}$,
- (d) we have $\bigcup_{k \in \mathbb{N}} X_k = X$.

Then for any $w \in X_1$ we have²⁸⁷

$$\pi_1\left(\varinjlim X_i, w\right) \underset{\text{see page 225}}{=} \pi_1\left(\bigcup_{i \in \mathbb{N}} X_i, x_0\right) \underset{\text{Lemma 9.5}}{=} \{e\} = \varinjlim \{e\} = \varinjlim \pi_1(X_i, w).$$

The following proposition generalizes the second example.

Proposition 13.2. *Let $X_i, i \in \mathbb{N}$ be a sequence of topological spaces such that the following holds:*

- (1) *the sequence is nested, i.e. for each $i \in \mathbb{N}$ we have $X_i \subset X_{i+1}$, and*
- (2) *each X_i is open in X_{i+1} .*

Then for any $w \in X_1$ we have

$$\pi_1\left(\varinjlim X_i, w\right) = \varinjlim \pi_1(X_i, w).$$

We will now see that the proof of Proposition 13.2 is a generalization of the ideas behind the proof of Lemma 9.5.

Proof of Proposition 13.2. Let $X_i, i \in \mathbb{N}$ be a nested sequence of topological spaces such that each X_i is open in X_{i+1} . Let $w \in X_1$. We write $X = \bigcup_{i \in \mathbb{N}} X_i = \varinjlim X_i$.

²⁸⁶As it is this question is not well-phrased, since fundamental groups depend on base points and we ignored them in the above discussion.

²⁸⁷Recall that by the discussion on page 221 the topological spaces $\{X_i\}_{i \in \mathbb{N}}$ together with the inclusion maps form a direct system in the category of topological spaces. The direct limit is given by the union of all X_i 's, where the union is given a suitable topology. Furthermore note that the inclusion induced maps $\pi_1(X_i, w) \rightarrow \pi_1(X_j, w)$ turn the groups $\{\pi_1(X_i, w)\}_{i \in \mathbb{N}}$ into a direct system in the category of groups.

Claim. For every $i \in \mathbb{N}$ the subset X_i is open in X .

By the definition of the topology on X we have to show that for each $j \in \mathbb{N}$ the intersection $X_i \cap X_j$ is open in X_j . Since the sequence is nested it follows that for $j \leq i$ we have $X_i \cap X_j = X_j$, in particular $X_i \cap X_j$ is open in X_j . Now we need to show that X_i is open in any X_{i+k} with $k \in \mathbb{N}_0$. We do so by induction on k . For $k = 0$ the statement is clear. Suppose we already know that X_i is open in X_{i+k} . Then X_i is an open subset in X_{i+k} and X_{i+k} is in turn an open subset of X_{i+k+1} by our hypothesis. It follows that X_i is open in X_{i+k+1} .²⁸⁸ This concludes the proof of the claim.

Given $k \leq l$ we denote by φ_{kl} the inclusion induced map $\pi_1(X_k, w) \rightarrow \pi_1(X_l, w)$. We denote by $\Phi: \varinjlim \pi_1(X_i, w) \rightarrow \pi_1(X, w)$ the unique homomorphism that makes the following diagram commute for any $k \leq l$:

$$\begin{array}{ccc}
 \pi_1(X_l, w) & & \\
 \uparrow \varphi_{kl} & \searrow \psi_l & \\
 \pi_1(X_k, w) & \xrightarrow{\psi_k} & \varinjlim \pi_1(X_i, w) \xrightarrow{\Phi} \pi_1(X, w).
 \end{array}$$

We need to show that Φ is an isomorphism. First we show that the map Φ is surjective. Let $g \in \pi_1(X, w)$. We represent g by a loop $\gamma: [0, 1] \rightarrow X$. Since each X_k is open in $X = \varinjlim X_i$ we obtain immediately from Corollary 1.6 and the hypothesis that the X_k 's are nested that there exists a $k \in \mathbb{N}$ such that the image of γ lies in X_k . We consider the following commutative diagram

$$\begin{array}{ccc}
 & \varinjlim \pi_1(X_i, w) & \xrightarrow{\Phi} \pi_1(X, w). \\
 \nearrow & & \\
 \pi_1(X_k, w) & \xrightarrow{[\gamma] \mapsto g} &
 \end{array}$$

By our choice of k we see that g lies in the image of the curved map, hence it also lies in the image of Φ . This concludes the proof that Φ is surjective.

Now we show that the map Φ is injective. So let $g \in \varinjlim \pi_1(X_i, w)$. Since $(\mathbb{N}, \leq)$ is a directed set we can use the description of the direct limit from page 226, i.e. we can write

$$\varinjlim \pi_1(X_i, w) = \left(\bigsqcup_{i \in \mathbb{N}} \pi_1(X_i, w) \right) / \sim \quad \text{where } g \sim \varphi_{kl}(g) \text{ for all } k \leq l, g \in \pi_1(X_k, w).$$

In particular we see that there exists a k such that g lies in the image of the map $\psi_k: \pi_1(X_k, w) \rightarrow \varinjlim \pi_1(X_i, w)$. We pick a loop $\gamma: [0, 1] \rightarrow X_k$ such that $\psi_k([\gamma]) = g$. Now suppose that $\Phi(g)$ is trivial. This means that there exists a homotopy H in X from γ to the constant path. As above we can apply Corollary 1.6 to deduce that there exists an

²⁸⁸Here we use the following general fact: if U is an open subset in V and V is an open subset of W , then U is an open subset of W . Why is that true?

$l \geq k$ such that the image of H lies in X_l . This implies that $\varphi_{kl}([\gamma])$ is trivial in $\pi_1(X_l, w)$. We obtain the following diagram

$$\begin{array}{ccc}
 \pi_1(X_l, w) & & \\
 \uparrow [\gamma] \mapsto e & \searrow & \\
 \pi_1(X_k, w) & \xrightarrow{[\gamma] \mapsto g} & \varinjlim \pi_1(X_i, w) \xrightarrow{g \mapsto e} \pi_1(X, w).
 \end{array}$$

It follows immediately that g itself is trivial. This concludes the proof that Φ is injective. $\square$

Examples.

- (1) As on page 221 we denote by X_∞ the interval $(0, \infty)$ with infinitely many circles attached. We pick the base point $w = \frac{1}{4}$. We then have ²⁸⁹

$$\begin{array}{ccccccc}
 \pi_1(X_\infty, w) & = & \pi_1(\varinjlim X_r, w) & = & \varinjlim \pi_1(X_r, w) & = & \varinjlim \langle x_1, \dots, x_r \rangle = \langle x_1, x_2, \dots \rangle. \\
 \uparrow & & \uparrow & & \uparrow & & \uparrow \\
 \text{see page 223} & & \text{Proposition 13.2} & & \text{Lemma 9.4} & & \text{see page 223}
 \end{array}$$

Furthermore the proof of Lemma 9.4 and the above argument show that the generators $x_1, x_2, \dots$ of $\pi_1(X_\infty)$ can be chosen such that x_n , $n \in \mathbb{N}$, corresponds to a loop which goes “once around the n -th circle”.

- (2) We now consider

$$\tilde{X} := \mathbb{R} \cup \bigcup_{j \in \mathbb{Z}} \underbrace{\left\{ z \in \mathbb{C} \mid \left| z - \left(j + \frac{1}{4}i \right) \right| = \frac{1}{4} \right\}}_{\text{circle of radius } \frac{1}{4} \text{ around } j + \frac{1}{4}i \in \mathbb{C}} \subset \mathbb{C}.$$

Put differently, $\tilde{X}$ is the real axis with infinitely many circles attached. The same argument as in (1) shows that $\pi_1(\tilde{X})$ is the free group on infinitely many generators x_n , $n \in \mathbb{Z}$. In Figure 130 we illustrate a covering map $p: \tilde{X} \rightarrow X = S^1 \vee S^1$. By Corollary 6.14 the map $\pi_1(\tilde{X}) \rightarrow \pi_1(X)$ is a monomorphism. We had just seen that $\pi_1(\tilde{X})$ is a free group on infinitely generators and by the discussion on page 158 we know that $\pi_1(X)$ is a free group on two generators. In fact, with appropriate

²⁸⁹The third equality follows from Lemma 9.4, to be more precise, it follows from the fact that for any $r \leq s$ the inclusion map $\iota_{rs}: X_r \rightarrow X_s$ and the isomorphisms of Lemma 9.4 give rise to a commutative diagram

$$\begin{array}{ccc}
 \pi_1(X_r, w) & \xrightarrow{\cong} & \langle x_1, \dots, x_r \rangle \\
 (\iota_{rs})_* \downarrow & & \downarrow x_i \mapsto x_i \\
 \pi_1(X_s, w) & \xrightarrow{\cong} & \langle x_1, \dots, x_s \rangle.
 \end{array}$$

The fact that this diagram commutes follows easily from the proof of Lemma 9.4. We leave the details to the reader.

orientations and conventions this shows that the homomorphism

$$\begin{aligned} \pi_1(\tilde{X}) = \langle \dots, x_{-2}, x_{-1}, x_0, x_1, x_2, \dots \rangle &\rightarrow \langle s, t \rangle = \pi_1(X) \\ x_n &\mapsto s^n t s^{-n} \end{aligned}$$

is a monomorphism. In particular we see that the free group $\langle s, t \rangle$ contains a subgroup that is “much larger” than the original free group itself.

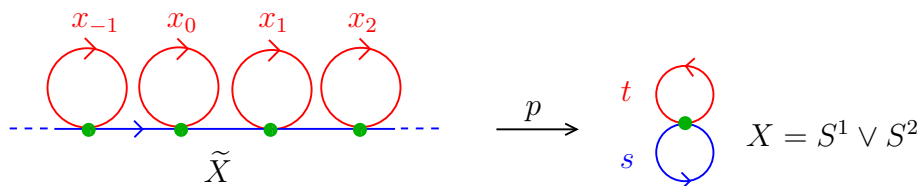


FIGURE 130.

- (3) An argument similar to (1) shows that $\pi_1(\mathbb{C} \setminus \mathbb{Z})$ is a free group generated by $x_n, n \in \mathbb{Z}$ where x_n is a loop which goes “once around the n -th hole”. We leave the details as an exercise to the reader. We refer to Figure 131 for an illustration.

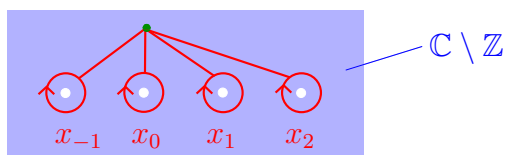


FIGURE 131.

Finally we return to the initial goal of this section, namely to determine the fundamental group of the surface of “infinite genus” that we illustrate in Figure 132. Before we can calculate its fundamental group we have to give a precise description of the surface of infinite genus. We write

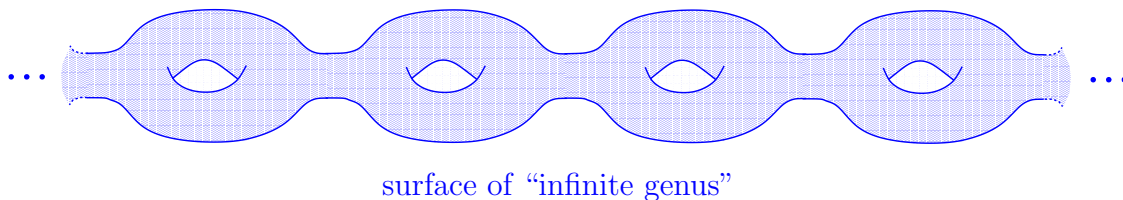


FIGURE 132.

$F :=$ the torus minus two open disks.

We define $X_0 = S^1 \times [-1, 1]$ and for $i \geq 0$ we define iteratively

$X_{i+1} := X_i$ with a copy of F attached to each of the two boundary components of X_i .

Note that each X_i is a surface of genus $2i$ with two open disks removed. We define

$$\text{surface of infinite genus} := \lim_{\rightarrow} X_i = \bigcup_{i \in \mathbb{N}_0} X_i.$$

It should be clear that this definition gives a rigorous description of the surface that we had sketched in Figure 132.

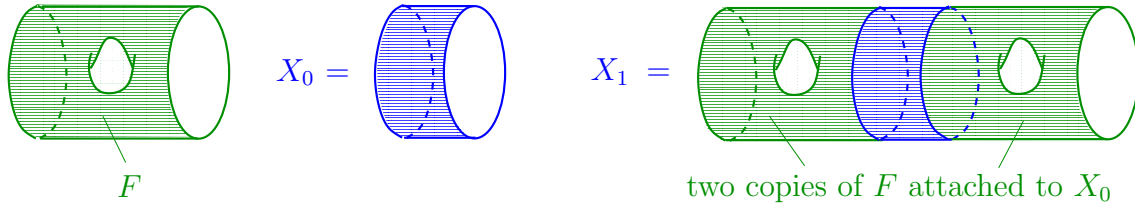


FIGURE 133.

Lemma 13.3. *The fundamental group of the surface of infinite genus is the free group on infinitely many generators.*

Remark. More generally, the fundamental group of any non-compact 2-dimensional manifold is a free group, this is for example proved in [Sti, p. 142].

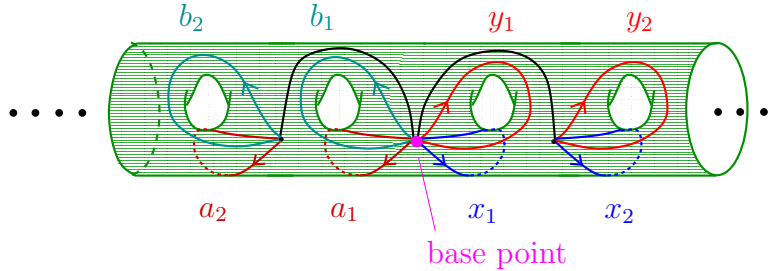


FIGURE 134. A set of generators for the fundamental group of a surface of infinite genus.

Proof. By Lemma 11.16 we have²⁹⁰

$$\pi_1(F) = \langle a, b, c, c' \mid c = [a, b] \cdot c' \rangle = \langle x, y, z, z' \mid z = [x, y] \cdot z' \rangle$$

where the generators z, z' and c, c' correspond to the two boundary components. We refer to Figure 135 for an illustration. An induction argument, using the Seifert-van Kampen

²⁹⁰This is indeed a consequence of Lemma 11.16, the only change is that we reversed the orientation of one of the boundary curves.

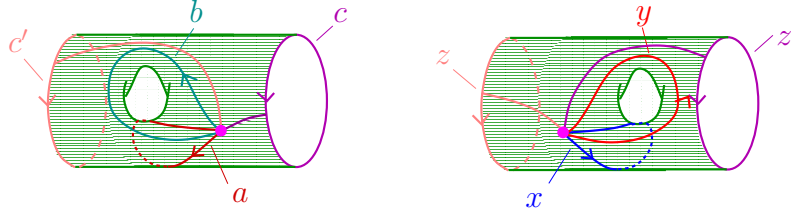


FIGURE 135.

Theorem 11.2 for manifolds together with Lemma 10.9 and the above presentations of $\pi_1(F)$ shows that for any $l \geq k$ we have a commutative diagram

$$\begin{array}{ccc}
 \pi_1(X_k) & \xrightarrow{\cong} & \left\langle \begin{array}{c} x_1, y_1, \dots, x_k, y_k \\ a_1, b_1, \dots, a_k, b_k \end{array} \middle| \begin{array}{c} z_k, c_k \\ \prod_{j=1}^k [a_j, b_j] c_k = \prod_{j=1}^k [x_j, y_j] z_k \end{array} \right\rangle \\
 \downarrow & & \downarrow \begin{array}{c} c_k \mapsto \prod_{j=k+1}^l [a_j, b_j] c_l \\ z_k \mapsto \prod_{j=k+1}^l [x_j, y_j] z_l \end{array} \\
 \pi_1(X_l) & \xrightarrow{\cong} & \left\langle \begin{array}{c} x_1, y_1, \dots, x_l, y_l \\ a_1, b_1, \dots, a_l, b_l \end{array} \middle| \begin{array}{c} z_l, c_l \\ \prod_{j=1}^l [a_j, b_j] c_l = \prod_{j=1}^l [x_j, y_j] z_l \end{array} \right\rangle
 \end{array}$$

where the left vertical map is induced by the inclusion $X_k \rightarrow X_l$. We leave the details to the energetic reader. For any $l \geq k$ we also have the following commutative diagram of

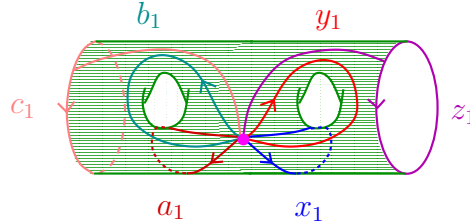


FIGURE 136.

homomorphisms:

$$\begin{array}{ccc}
 \left\langle \begin{array}{c} x_1, y_1, \dots, x_k, y_k \\ a_1, b_1, \dots, a_k, b_k \end{array} \middle| p \right\rangle & \xrightarrow[p \mapsto \prod_{j=1}^k [a_j, b_j] c_k]{\cong} & \left\langle \begin{array}{c} x_1, y_1, \dots, x_k, y_k \\ a_1, b_1, \dots, a_k, b_k \end{array} \middle| \begin{array}{c} z_k, c_k \\ \prod_{j=1}^k [a_j, b_j] c_k = \prod_{j=1}^k [x_j, y_j] z_k \end{array} \right\rangle \\
 \downarrow & & \downarrow \begin{array}{c} c_k \mapsto \prod_{j=k+1}^l [a_j, b_j] c_l \\ z_k \mapsto \prod_{j=k+1}^l [x_j, y_j] z_l \end{array} \\
 \left\langle \begin{array}{c} x_1, y_1, \dots, x_l, y_l \\ a_1, b_1, \dots, a_l, b_l \end{array} \middle| p \right\rangle & \xrightarrow[p \mapsto \prod_{j=1}^l [a_j, b_j] c_l]{\cong} & \left\langle \begin{array}{c} x_1, y_1, \dots, x_l, y_l \\ a_1, b_1, \dots, a_l, b_l \end{array} \middle| \begin{array}{c} z_l, c_l \\ \prod_{j=1}^l [a_j, b_j] c_l = \prod_{j=1}^l [x_j, y_j] z_l \end{array} \right\rangle
 \end{array}$$

The horizontal homomorphisms are in fact isomorphisms, this is for example a consequence of the Tietze transformations from page 177.²⁹¹

In the following we write $U_i = X_i \setminus \partial X_i$, $i \in \mathbb{N}_0$. Note that each U_i is in fact an open subset of Σ_∞ . We denote by Σ_∞ the surface of infinite genus. We have

$$\begin{array}{ccccccc} & & \text{Proposition 13.2} & & \text{by the above two} & & \\ & & \downarrow & & \text{commutative diagrams} & & \\ \pi_1(\Sigma_\infty) & = & \pi_1(\varinjlim U_i) & = & \varinjlim \pi_1(U_i) & = & \varinjlim \pi_1(X_i) & \xrightarrow{\quad} & \varinjlim \left\langle \begin{matrix} x_1, y_1, \dots, x_i, y_i \\ a_1, b_1, \dots, a_i, b_i \end{matrix} p \right\rangle & = & \left\langle \begin{matrix} x_1, y_1, \dots \\ a_1, b_1, \dots \end{matrix} p \right\rangle. \\ \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\ \text{since } \Sigma_\infty = \bigcup_{i \in \mathbb{N}_0} U_i & & \text{since } \pi_1(U_i) \xrightarrow{\cong} \pi_1(X_i) & & & & & & \text{see example on} & & \text{page 223} \end{array}$$

This concludes the proof of the lemma. $\square$

13.3. Gluing formula for fundamental groups and HNN-extensions (*).²⁹²

Definition. Let X be a topological space, let A and B be two subsets of X and let $f: A \rightarrow B$ be a homeomorphism. We write

$$X(f) := X(f: A \rightarrow B) := X / \sim \quad \text{where } f(a) \sim b \text{ for } a \in A.$$

Examples.

- (1) Let X be the torus with two open disks removed. We denote by A and B the two boundary components and we endow them with the orientation coming from a fixed orientation of the torus. Let $f: A \rightarrow B$ be an orientation-reversing diffeomorphism. Then $X(f: A \rightarrow B)$ is diffeomorphic to the surface of genus 2. This statement is illustrated in Figure 137.²⁹³
- (2) Let Y be a topological space and let $f: Y \rightarrow Y$ be a homeomorphism. We consider $X = Y \times [0, 1]$ and by a slight abuse of notation we denote by $f: Y \times \{0\} \rightarrow Y \times \{1\}$ also the map defined by $f(y, 0) = (f(y), 1)$. It follows immediately from the definitions that

$$X(f) = (Y \times [0, 1])(f) = M(X, f),$$

where $M(X, f)$ denotes the mapping torus that we had introduced on page 212.

Our goal now is to determine $\pi_1(X(f))$ in terms of the fundamental groups of X , A and the induced map $f_*: \pi_1(A) \rightarrow \pi_1(B)$. Perhaps not surprisingly, we again need a new construction from group theory.

²⁹¹Indeed, we write $X = \{x_1, y_1, \dots, x_m, y_m, a_1, b_1, \dots, a_m, b_m\}$, $u = \prod_{j=1}^m [a_j, b_j]$, $v = \prod_{j=1}^m [x_j, y_j]$, $c = c_m$ and $z = z_m$, then

$$\langle X, z, c \mid uc = vz \rangle = \langle X, z, c \mid z = v^{-1}uc \rangle \xleftarrow{\cong} \langle X, c \rangle \xrightarrow{\cong} \langle X, c, p \mid p = uc \rangle = \langle X, c, p \mid X, c = u^{-1}p \rangle \xleftarrow{\cong} \langle X, p \rangle$$

where all three isomorphisms are the isomorphisms of the Tietze transformation (2). It is straightforward to see that the concatenation equals for $m = k, l$ precisely the horizontal maps in the diagram.

²⁹²This section is not part of the Algebraic Topology I course.

²⁹³What topological space do we obtain if f is chosen to be orientation-preserving?

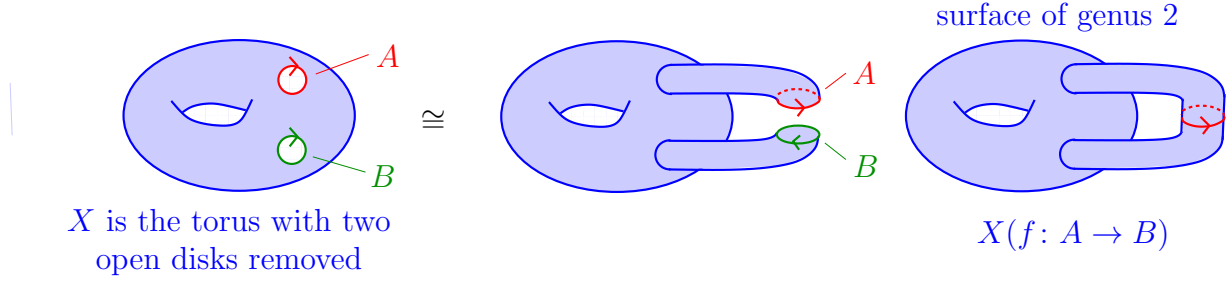


FIGURE 137.

Definition. Let π and Γ be two groups and let $\alpha, \beta: \Gamma \rightarrow \pi$ be two homomorphisms. We refer to

$$\langle \pi, t \mid \alpha(\Gamma) = t\beta(\Gamma)t^{-1} \rangle := (\pi * \langle t \rangle) / \langle\langle \{\alpha(g)t\beta(g)^{-1}t^{-1}\}_{g \in \Gamma} \rangle\rangle$$

as the *HNN-extension* of (π, α, β) .

Remark.

- (1) If $\pi = \langle x_1, \dots, x_k \mid r_1, \dots, r_l \rangle$ is a finitely presented group and if $g_1, \dots, g_m$ is a generating set for Γ , then it follows easily from the definitions that the HNN-extension of (π, α, β) is given by

$$\langle x_1, \dots, x_k, t \mid r_1, \dots, r_l, \alpha(g_i) = t\beta(g_i)t^{-1} \text{ for } i = 1, \dots, m \rangle.$$

- (2) If $\alpha, \beta: \Gamma \rightarrow \pi$ are monomorphisms, then it follows from [Se, p. 45] that the obvious map $\pi \rightarrow \langle \pi, t \mid \alpha(\Gamma) = t\beta(\Gamma)t^{-1} \rangle$ is also a monomorphism.

Examples.

- (1) Let $\pi = \Gamma$, let $\alpha = \text{id}$ and let $\beta: \pi \rightarrow \pi$ be an isomorphism. Then it is straightforward to see that the map

$$\begin{aligned} \Phi: \pi \rtimes_{\beta} \mathbb{Z} &\rightarrow \langle \pi, t \mid \alpha(\Gamma) = t\beta(\Gamma)t^{-1} \rangle \\ (g, n) &\mapsto g \cdot t^{-n} \end{aligned}$$

is an isomorphism.²⁹⁴ Put differently, we have now shown that a semidirect product is a special case of an HNN-extension.

- (2) Let π be a group and let Γ be the trivial group. It follows immediately from the definitions that the corresponding HNN-extension is the free product $\pi * \langle t \rangle$.

²⁹⁴As a reality check, here is the calculation that the given map is indeed a homomorphism: let (g, m) and (h, n) be two elements in $\pi \rtimes_{\beta} \mathbb{Z}$, then

$$\begin{aligned} \Phi(g, m)\Phi(h, n) &= gt^{-m} \cdot ht^{-n} = gt^{-m} \cdot t^m \beta^m(h) t^{-m} \cdot t^{-n} = g\beta^m(h)t^{-m-n} = \Phi(g\beta^m(h), m+n) \\ &\quad \uparrow \\ &\text{since } h = \alpha(h) = t\beta(h)t^{-1} = t \cdot t\beta(\beta(h))t^{-1} \cdot t^{-1} = \dots &= \Phi((g, m)(h, n)). \end{aligned}$$

This shows that Φ is indeed a homomorphism.

- (3) Let π and Γ be two groups and let $\alpha, \beta: \Gamma \rightarrow \pi$ be two homomorphisms. Then there exists an epimorphism

$$\langle \pi, t \mid \alpha(\Gamma) = t\beta(\Gamma)t^{-1} \rangle \rightarrow \mathbb{Z}$$

that is uniquely determined by $g \mapsto 0$ if $g \in \pi$ and $t^n \mapsto n$.

Theorem 13.4. *Let X be a path-connected topological space, let A and B be two open path-connected subsets of X and let $f: A \rightarrow B$ be a homeomorphism. We pick a base point $x \in A$ and we pick a path γ in X from x to $f(x)$. We write $\Gamma = \pi_1(A, x)$. Furthermore we denote by*

$$\alpha: \Gamma = \pi_1(A, x) \rightarrow \pi_1(X, x)$$

the inclusion induced map and we denote by

$$\beta: \Gamma = \pi_1(A, x) \xrightarrow{f_*} \pi_1(B, f(x)) \rightarrow \pi_1(X, f(x)) \xrightarrow{\gamma_*} \pi_1(X, x)$$

the concatenation of the maps induced by f , the inclusion and the base point change isomorphism from Proposition 4.9, using the path γ . Then there exists an isomorphism

$$\Phi: \pi_1(X(f), x) \xrightarrow{\cong} \langle \pi_1(X, x), t \mid \alpha(\Gamma) = t\beta(\Gamma)t^{-1} \rangle.$$

which has the following two properties:

- (1) *The diagram*

$$\begin{array}{ccc} & \pi_1(X, x) & \\ \swarrow & & \searrow \\ \pi_1(X(f), x) & \xrightarrow{\Phi} & \langle \pi_1(X, x), t \mid \alpha(\Gamma) = t\beta(\Gamma)t^{-1} \rangle \end{array}$$

with the obvious diagonal maps commutes.

- (2) *The closed loop in $X(f)$ defined by γ corresponds under the isomorphism Φ to t .*

Example. As on page 233 we consider the surface X that is given by removing two open disks from a torus. We refer to the two boundary components as A and B . We equip A and B with the orientations given in Figure 138. We pick a diffeomorphism $f: A \rightarrow B$ that preserves the orientation. Then $X(f)$ is the surface of genus two.

We pick a base point $x \in A$. We pick a loop a in (A, x) we pick loops c, d in (X, x) , a path γ in X from x to $f(x)$ and a loop b in $(B, f(x))$ as indicated in Figure 138. Finally we write $\tilde{b} = \gamma * b * \bar{\gamma}$. By Lemma 11.16 and as in the proof of Lemma 13.3 we have an isomorphism

$$\pi_1(X) = \langle c, d, a, \tilde{b} \mid a = [c, d] \cdot \tilde{b} \rangle.$$

We write $\Gamma = \pi_1(A, x)$. Note that $\Gamma = \langle a \rangle$ and $\alpha(a) = a$ and $\beta(a) = \tilde{b}$. It follows from Theorem 13.4 and the remark on page 234 that

$$\begin{array}{ccccc}
 \pi_1(X(f), x) & \cong & \langle \pi_1(X), t \mid \alpha(\Gamma) = t\beta(\Gamma)t^{-1} \rangle & = & \langle c, d, a, \tilde{b}, t \mid a = [c, d] \cdot \tilde{b}, a = t\tilde{b}t^{-1} \rangle \\
 \uparrow & & \uparrow & & \uparrow \\
 & \text{Theorem 13.4} & & \text{remark on page 234} & \\
 & = \langle c, d, \tilde{b}, t \mid [c, d] \cdot \tilde{b} = t\tilde{b}t^{-1} \rangle & = & \langle c, d, \tilde{b}, t \mid \underbrace{[c, d] \cdot \tilde{b}t\tilde{b}^{-1}t^{-1}}_{=[c, d][\tilde{b}, t]} \rangle \\
 \uparrow & & & & \\
 & \text{Tietze transformations} & & &
 \end{array}$$

Thus we have now obtained the same presentation that we had already obtained in Proposition 11.3.

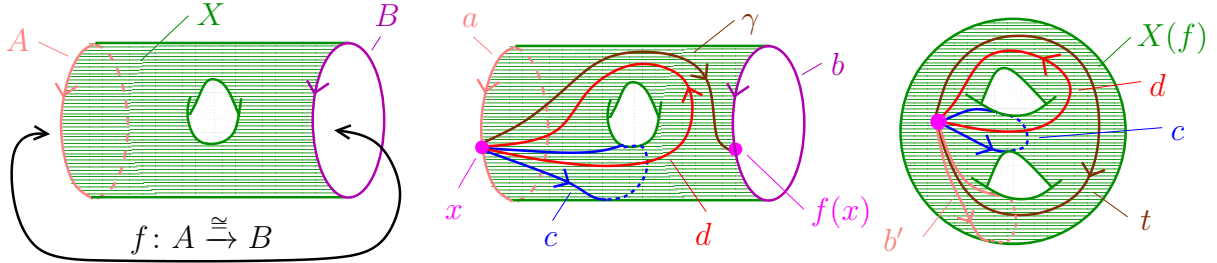


FIGURE 138.

*Sketch of the proof of Theorem 13.4.*²⁹⁵ Let X be a path-connected topological space, let A and B be two open path-connected subsets of X and let $f: A \rightarrow B$ be a homeomorphism. We pick a base point $x \in A$ and we pick a path γ in X from x to $f(x)$. For $i \in \mathbb{Z}$ we now define $X_i := X \times \{i\}$, $A_i := A \times \{i\}$, $B_i := B \times \{i\}$ and $x_i := \{x\} \times \{i\}$. Furthermore we define

$$\tilde{X} := \left(\bigcup_{i \in \mathbb{Z}} X_i \right) / (a, i) \sim (f(a), i - 1) \quad \text{for } a \in A \text{ and } i \in \mathbb{Z}.$$

There is an obvious action of $\mathbb{Z}$ on $\tilde{X}$ given by $n \cdot [(x, i)] := [(x, i + n)]$ for $(x, i) \in X_i$. It is straightforward to see that this action is discrete and continuous. It follows easily from the definitions that the map

$$\begin{array}{ccc}
 \Phi: X(f) & \rightarrow & \tilde{X}/\mathbb{Z} \\
 [x] & \mapsto & [(x, 0)]
 \end{array}$$

is a homeomorphism. We refer to Figure 139 for an illustration.

Given $i \in \mathbb{Z}$ we write $\pi_i := \pi_1(X_i, x_i)$ and $\Gamma_i = \pi_1(A_i, x_i)$.

²⁹⁵As we had seen on pages 233 and 234, we can view Theorem 13.4 as a generalization of Proposition 12.12. In fact the subsequent proof is partly modelled on the proof of Proposition 12.12.

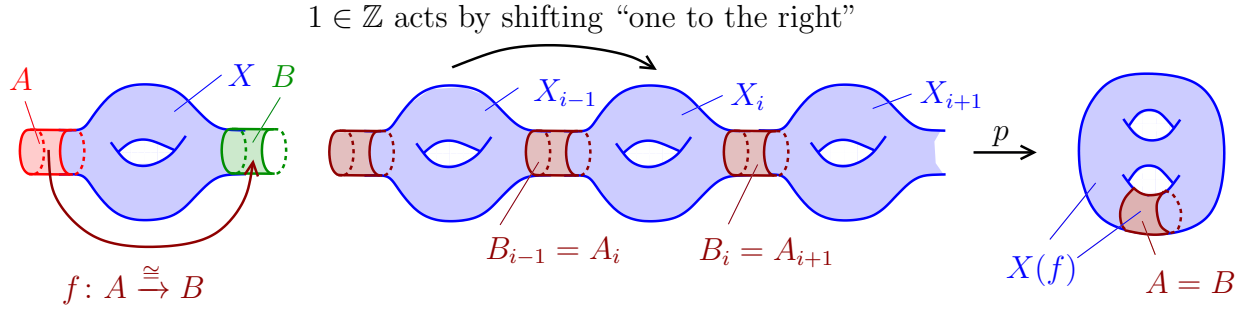


FIGURE 139.

Claim. There exists an isomorphism²⁹⁶

$$\pi_1(\tilde{X}) \cong \dots \pi_{-2} *_{\Gamma_{-1}} \pi_{-1} *_{\Gamma_0} \pi_0 *_{\Gamma_1} \pi_1 *_{\Gamma_2} \pi_2 \dots$$

Given any $n \in \mathbb{N}$ we write

$$\tilde{X}_n := \left(\bigcup_{i=-n}^n X_i \right) / (a, i) \sim (f(a), i-1) \quad \text{for } a \in A, i \in \{-n+1, \dots, n\}.$$

It follows from the Seifert-van Kampen Theorem 11.1 applied iteratively $2n$ times that

$$\pi_1(\tilde{X}_n) \cong \pi_{-n} *_{\Gamma_{-n+1}} \pi_{-n+1} \dots \pi_{n-1} *_{\Gamma_n} \pi_n.$$

We now see that

$$\begin{array}{ccccccc} \pi_1(\tilde{X}) & = & \pi_1(\varinjlim \tilde{X}_n) & = & \varinjlim \pi_1(\tilde{X}_n) & = & \dots \pi_{-2} *_{\Gamma_{-1}} \pi_{-1} *_{\Gamma_0} \pi_0 *_{\Gamma_1} \pi_1 *_{\Gamma_2} \pi_2 \dots \\ & & \uparrow & & \uparrow & & \\ & & \text{Proposition 13.2} & & \text{above calculation} & & \end{array}$$

This concludes the proof of the claim.

We now denote by φ the automorphism of the infinite amalgamated product of the claim that “shifts every term by one to the right”. Then we obtain an isomorphism²⁹⁷

$$\begin{array}{ccc} \text{above homeomorphism} & & \text{Lemma 12.11 and the above claim} \\ \downarrow & & \downarrow \\ \pi_1(X) & \cong & \pi_1(\tilde{X}/\mathbb{Z}) \cong \left(\dots \pi_{-1} *_{\Gamma_0} \pi_0 *_{\Gamma_1} \pi_1 \dots \right) \rtimes_{\varphi} \mathbb{Z} \xrightarrow{\cong} \langle \pi_1(X, x), t \mid \alpha(\Gamma) = t\beta(\Gamma)t^{-1} \rangle \\ & & (g_i, n) \mapsto gt^{-i} \cdot t^n, \text{ where } g_i \in \pi_i \end{array} \quad \square$$

²⁹⁶This claim is somewhat imprecise, we do not explain what exactly we mean by “...” and we are not careful with the base points. This is one reason why this argument is referred to as a “sketch” of a proof.

²⁹⁷Here we also skip many details.

Remark. One of the key inputs in the proof of Theorem 13.4 was the Seifert-van Kampen Theorem 11.1. If we use instead the Seifert-van Kampen Theorem for manifolds, see Theorem 11.2, then we obtain a “manifold version” of Theorem 13.4. We leave it as an exercise to formulate that version and to make the necessary changes to the proof.

13.4. The inverse limit of an inverse system. For completeness’ sake we also introduce the inverse limit of an inverse system, even though it will not play a role in the intermediate future.

Definition. Let $(I, \leq)$ be a preordered set and let $\mathcal{C}$ be a category. An *inverse system in the category $\mathcal{C}$ over I* is a family of objects $\{X_i\}_{i \in I}$ in $\mathcal{C}$ indexed by I , together with a family of morphisms $\{f_{ji}: X_j \rightarrow X_i\}$ for all $i, j \in I$ with $i \leq j$ ²⁹⁸ such that the following two conditions are satisfied:

- (1) $f_{ii} = \text{id}_{X_i}$ for all $i \in I$,
- (2) $f_{ki} = f_{ji} \circ f_{kj}$ for all $i, j, k \in I$ with $i \leq j \leq k$.

Examples.

- (A) For each $i \in \mathbb{N}$ we define $X_i := \mathbb{Z}^i$ and for $i \leq j$ we denote by

$$\begin{aligned} f_{ji}: \mathbb{Z}^j &\rightarrow \mathbb{Z}^i \\ (x_1, \dots, x_j) &\mapsto (x_1, \dots, x_i) \end{aligned}$$

the projection onto the first i coordinates. Then $(\{\mathbb{Z}^i\}_{i \in \mathbb{N}}, \{f_{ji}\}_{i \leq j})$ forms an inverse system in the category of groups.

- (B) Let p be a prime. For each $i \in \mathbb{N}$ we define $X_i := \mathbb{Z}_{p^i}$ and for $i \leq j$ we denote by

$$\begin{aligned} f_{ji}: \mathbb{Z}_{p^j} &\rightarrow \mathbb{Z}_{p^i} \\ n + p^j \mathbb{Z} &\mapsto n + p^i \mathbb{Z} \end{aligned}$$

the obvious projection map. Then $(\{\mathbb{Z}_{p^i}\}_{i \in \mathbb{N}}, \{f_{ji}\}_{i \leq j})$ forms an inverse system in the category of rings.

- (C) We let $I = \mathbb{N}$, but this time we use the preorder given by divisibility, i.e. we write $k \leq l$ if and only if $k|l$. For each $k \in \mathbb{N}$ we define $X_k := \mathbb{Z}_k$ and for $k|l$ we denote by

$$\begin{aligned} f_{lk}: \mathbb{Z}_l &\rightarrow \mathbb{Z}_k \\ n + l\mathbb{Z} &\mapsto n + k\mathbb{Z} \end{aligned}$$

the obvious projection map. Then $(\{\mathbb{Z}_k\}_{k \in \mathbb{N}}, \{f_{lk}\}_{k|l})$ forms an inverse system in the category of rings.

The following definition is the “mirror” image of the definition of the direct limit of a direct system.

Definition. Let $(I, \leq)$ be a preordered set and let $\mathcal{C}$ be a category. Furthermore suppose we are given an inverse system $(\{X_i\}_{i \in I}, \{f_{ji}\}_{i \leq j})$ in the category $\mathcal{C}$. An *inverse limit*²⁹⁹ of

²⁹⁸So here for $i \leq j$ we have a morphism $X_j \rightarrow X_i$ instead of $X_i \rightarrow X_j$ in a direct system.

²⁹⁹Somewhat confusingly an “inverse limit” is often also called an “inductive limit” or “limit of an inverse system”.

the inverse system is an object Y in $\mathcal{C}$ together with morphisms $g_i: Y \rightarrow X_i$, $i \in I$, in the category $\mathcal{C}$ such that the following two conditions are satisfied:

- (1) For all $i \leq j$ we have $f_{ji} \circ g_j = g_i: Y \rightarrow X_i$, i.e. for all $i \leq j$ the following diagram commutes

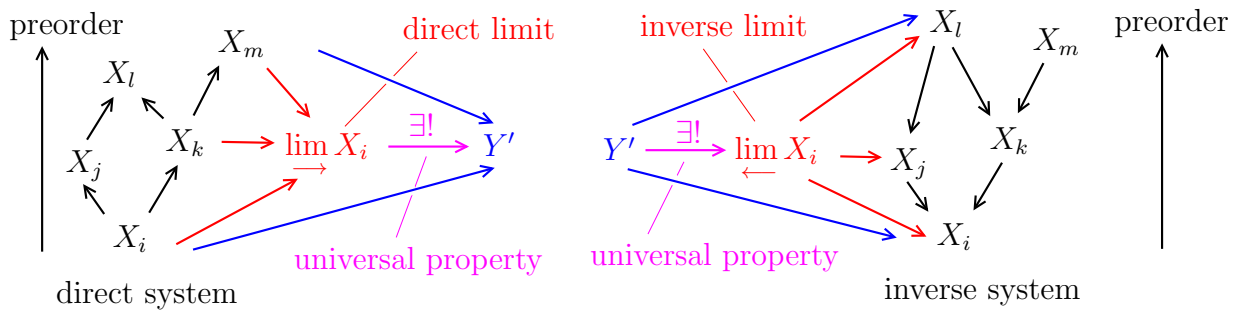
$$\begin{array}{ccc} Y & \xrightarrow{g_j} & X_j \\ & \searrow g_i & \downarrow f_{ji} \\ & & X_i \end{array}$$

- (2) If we are given another object Y' and morphisms $g'_i: Y' \rightarrow X_i$, $i \in I$, that satisfy (1), then there exists a unique morphism $F: Y' \rightarrow Y$ such that for all $i \leq j$ the following diagram commutes

$$\begin{array}{ccccc} & & g'_j & \xrightarrow{\quad} & X_j \\ & \nearrow & & \nearrow g_j & \downarrow f_{ji} \\ Y' & \xrightarrow[\exists!]{F} & Y & \searrow g_i & X_i \\ & \searrow & & \searrow g'_i & \end{array}$$

The “usual argument” for objects that satisfy a universal property shows, that if the inverse limit exists, then it is unique in an appropriate sense. We denote the inverse limit by $\varprojlim X_i$ or sometimes by $\varprojlim_I X_i$.

In the following diagram we compare schematically the definition of the direct limit of a direct system to the inverse limit of an inverse system:³⁰⁰



³⁰⁰The difference between a direct limit and an inverse limit is not the direction of the arrows between the X_i . In fact given a preorder “ $\leq$ ” one could define a preorder “ $\widetilde{\leq}$ ” via $a \widetilde{\leq} b$ if and only if $b \leq a$, hence a direct system over $(I, \leq)$ is an inverse system over $(I, \widetilde{\leq})$. The real difference is whether the X_i map to the limit, or whether the limit maps to the X_i .

Examples. In order to get an idea of how to construct inverse limits we first consider the special case that the preorder on I is trivial, in the sense that we have $i \leq j$ if and only if $i = j$.

- (D) Let I be a set with the trivial preorder. Suppose that for any $i \in I$ we are given a set X_i . Together with the identity maps of the X_i this defines an inverse system over I in the category of sets. We claim that the inverse limit is given by³⁰¹

$$\lim_{\leftarrow} X_i = \prod_{i \in I} X_i$$

together with the maps g_i that are given by the projection onto the i -component. We now verify this statement. Since we only have $i \leq j$ if $i = j$ there is nothing to verify in part (1) of the definition of the inverse limit. Now suppose we are given a set Y' and maps $g'_i: Y' \rightarrow X_i$. Then we define

$$\begin{aligned} F: Y' &\rightarrow \lim_{\leftarrow} X_i \\ y &\mapsto \left(\begin{array}{ccc} I & \rightarrow & \prod_{i \in I} X_i \\ i & \mapsto & g'_i(y) \end{array} \right). \end{aligned}$$

It is straightforward to verify that F is uniquely determined and that it has the desired property. We have thus determined the inverse limit $\lim_{\leftarrow} X_i$.

- (E) Let I be a set with the trivial preorder and suppose that for any $i \in I$ we are given a topological space X_i . Together with the identity maps of the X_i this defines an inverse system over I in the category of topological spaces. As in (D) we define

$$\lim_{\leftarrow} X_i = \prod_{i \in I} X_i$$

and each map g_i is again given by the projection onto the i -component.

To conclude that we have found an inverse limit in the category of topological spaces we still have to equip $\prod_{i \in I} X_i$ with a topology which has the following two properties:

- (i) the maps $g_i: \prod_{i \in I} X_i \rightarrow X_i$ are continuous,

³⁰¹Recall that the product $\prod_{i \in I} X_i$ is defined as the set of all maps $f: I \rightarrow \bigsqcup_{i \in I} X_i$ with the property that $f(i) \in X_i$ for all $i \in I$. Such a map f is often written as $(f(i))_{i \in I}$ and often we refer to $f(i)$ as the *i-component* of f . Furthermore we refer to the map

$$\begin{aligned} \prod_{i \in I} X_i &\rightarrow X_i \\ f &\mapsto f(i) \end{aligned}$$

as the *projection onto the i-component*. Note that if the X_i 's are groups, rings or modules, then the product inherits this algebraic structure by component-wise multiplication and addition.

- (ii) given a topological space Y' and continuous maps $g'_i: Y' \rightarrow X_i$ the map F , defined as in (D), is continuous.³⁰²

We set

$$\mathcal{B} := \left\{ \prod_{i \in I} U_i \subset \prod_{i \in I} X_i \mid \begin{array}{l} (1) \text{ each } U_i \text{ is open in } X_i \text{ and} \\ (2) \text{ there exist only finitely many } i\text{'s with } U_i \neq X_i \end{array} \right\}.$$

It is straightforward to verify that $\mathcal{B}$ has the basis property that we had introduced on page 40. We equip $\prod_{i \in I} X_i$ with the topology generated by this basis and we refer to this topology as the *product topology*.³⁰³ Now we need to verify that we found the correct topology on the direct limit.

- (i) It follows immediately from the definitions that the maps g_i are continuous with respect to the product topology.
- (ii) Suppose we are given Y and g'_i as above. We denote by F the map defined as above. We claim that F is continuous. By Lemma 1.23 it suffices to verify that the preimage of any set in $\mathcal{B}$ is open. So let $\prod_{i \in I} U_i$ be a set in $\mathcal{B}$. Then

$$F^{-1}\left(\prod_{i \in I} U_i\right) = \bigcap_{i \in I} (g'_i)^{-1}(U_i) = \bigcap_{U_i \neq X_i} \underbrace{(g'_i)^{-1}(U_i)}_{\text{open in } Y'} = \text{intersection of finitely many open subsets of } Y'.$$

This concludes the proof that F is continuous.

Now we can prove fairly easily that inverse limits exist in many categories that are of interest to us.

Proposition 13.5. *The inverse limit of any inverse system exists in the following categories:*

- (0) the category of sets,
- (1) the category of topological spaces,
- (2) the category of groups,
- (3) the category of topological groups,³⁰⁴
- (4) the category of abelian groups,

³⁰²On the product $\prod_{i \in I} X_i$ we could also consider the topology $\mathcal{T}$ that is generated by all sets of the form

$\prod_{i \in I} U_i$ with U_i open in X_i . This might be a more “natural” topology on $\prod_{i \in I} X_i$. But with this topology $\mathcal{T}$ the map F does not necessarily have to be continuous. The problem is that $\mathcal{T}$ has “too many” open sets.

³⁰³If the index set I is finite, i.e. if we consider the product of finitely many topological spaces, then this definition of the product topology agrees with the definition given on page 22.

³⁰⁴A *topological group* is a topological space X together with a group structure, such that the two maps

$$\begin{array}{ccc} X \times X & \rightarrow & X \\ (x, y) & \mapsto & x \cdot y \end{array} \quad \text{and} \quad \begin{array}{ccc} X & \rightarrow & X \\ x & \mapsto & x^{-1} \end{array}$$

are continuous. A *morphism between topological groups* X and Y is a homomorphism $f: X \rightarrow Y$ that is furthermore continuous. Examples of topological groups are given by $\text{SL}(n, \mathbb{R}) \subset \text{Mat}(n \times n, \mathbb{R}) \cong \mathbb{R}^{n^2}$ and $U(n) \subset \text{Mat}(n \times n, \mathbb{C}) \cong \mathbb{C}^{n^2} \cong \mathbb{R}^{4n^2}$. Furthermore $\det: \text{GL}(n, \mathbb{R}) \rightarrow \mathbb{R}^\times$ is a morphism in the category of

- (5) *the category of rings,*
- (6) *the category of R -modules, where R is a commutative ring.*

We will see that the proof of Proposition 13.5 is a relatively straightforward generalization of the examples (D) and (E) above.

Proof. Let $(I, \leq)$ be a preordered set and let $\mathcal{C}$ be one of the seven given categories. Furthermore let $(\{X_i\}_{i \in I}, \{f_{ji}: X_j \rightarrow X_i\}_{i \leq j})$ be an inverse system in the category $\mathcal{C}$. We define

$$\lim_{\leftarrow} X_i := \{(x_i)_{i \in I} \in \prod_{i \in I} X_i \mid f_{ji}(x_j) = x_i \text{ for all } i \leq j\}.$$

Furthermore we denote by $g_i: \lim_{\leftarrow} X_i \rightarrow X_i$ the obvious projection maps. It follows immediately from the definitions that for any $i \leq j$ we have $f_{ji} \circ g_j = g_i$. Furthermore, given an object Y' and morphisms $g'_i: Y' \rightarrow X_i$, $i \in I$ which satisfy the condition $f_{ji} \circ g'_j = g'_i$ for all $i \in I$ we define

$$F: Y' \rightarrow \lim_{\leftarrow} X_i$$

$$y \mapsto \left(\begin{array}{ccc} I & \rightarrow & \prod_{i \in I} X_i \\ i & \mapsto & g'_i(y) \end{array} \right).$$

It is straightforward to verify that $F(Y')$ lies in $\lim_{\leftarrow} X_i$ and that F is the unique map with the desired property. This shows that the inverse limit exists in the category of sets.³⁰⁵

For the remaining six categories we will use the same definition of the direct limit as a set and the same maps, but we now have to verify that we have objects and morphisms in the given category.

- (1) Suppose that we work in the category of topological spaces. We equip $\prod_{i \in I} X_i$ with the product topology that we had defined in Example (E) above. We equip $\lim_{\leftarrow} X_i$ with the subspace topology. The same argument as in Example (E) shows that the maps g_i and F are continuous.
- (2) In the category of groups spaces we view $\prod_{i \in I} X_i$ as a group via component-wise multiplication. Since the f_{ij} are group homomorphisms we see that $\lim_{\leftarrow} X_i$ is a subgroup of $\prod_{i \in I} X_i$. It is straightforward to verify that the above maps are all morphisms in the category of groups.
- (3) This case is a combination of (1) and (2).
- (4),(5),(6) These categories are dealt with the same way as (2).

For all seven categories we have now provided an inverse limit. □

topological groups. Also any group equipped with the discrete topology is a topological group and in this case any group homomorphism is also continuous.

³⁰⁵We will now see that we can use the “same” inverse limit for all the categories, this is very different from the construction of the direct limit for the same categories. There, in the proof of Proposition 13.1, we had to use several different construction.

Remark. We consider the important special case that the preordered set $(I, \leq) = (\mathbb{N}, \leq)$. Furthermore let

$$(\{X_n\}_{n \in \mathbb{N}}, \{f_{nm}: X_n \rightarrow X_m\}_{m \leq n})$$

be an inverse system over $(\mathbb{N}, \leq)$ in any of the above seven categories. Given $n \in \mathbb{N}_{\geq 2}$ we write $g_n := f_{n,n-1}: X_n \rightarrow X_{n-1}$. We have

$$\begin{aligned} \varprojlim X_i &= \{(x_1, x_2, x_3, \dots) \mid x_n \in X_n \text{ for all } n \in \mathbb{N} \text{ and } f_{nm}(x_n) = x_m \text{ for all } m \leq n\} \\ &= \{(x_1, x_2, x_3, \dots) \mid x_n \in X_n \text{ for all } n \in \mathbb{N} \text{ and } g_n(x_n) = x_{n-1} \text{ for all } n \in \mathbb{N}_{\geq 2}\} \\ &\quad \uparrow \end{aligned}$$

follows from $f_{nm} = f_{n,n-1} \circ \dots \circ f_{m+1,m} = g_n \circ \dots \circ g_{m+1}$

$$\begin{array}{ccc} & \vdots & \vdots \\ & \downarrow & \downarrow \\ & x_3 \in X_3 & \\ & \downarrow & \downarrow g_3 \\ & x_2 \in X_2 & \\ & \downarrow & \downarrow g_2 \\ & x_1 \in X_1 & \end{array}$$

= all infinite sequences of the form

This discussion shows that we can view of elements in the inverse limit as “upward going sequences”.

Now we return to our initial examples of inverse systems and their inverse limits.

Examples.

(A) It follows quite easily from the previous remark that

$$\varprojlim_{n \in \mathbb{N}} \mathbb{Z}^n = \{(x_1, x_2, x_3, \dots) \mid x_n \in \mathbb{Z}\} \cong \mathbb{Z}^{\mathbb{N}}.$$

Note that this is an uncountable group. Recall that on page 223 we had determined that

$$\varinjlim \mathbb{Z}^i = \mathbb{Z}^{(\mathbb{N})} = \{(x_1, x_2, \dots) \mid x_i \in \mathbb{Z} \text{ but only finitely many } x_i \text{ are non-zero}\}$$

which is a countable group.³⁰⁶ Thus we see that in this particular case the inverse limit is “much larger” than the direct limit.³⁰⁷

(B) Let p be a prime. The inverse limit $\widehat{\mathbb{Z}}_p := \varprojlim \mathbb{Z}_{p^i}$ of the rings $\mathbb{Z}_{p^i} = \mathbb{Z}/p^i\mathbb{Z}$ is called the *ring of the p -adic numbers*.³⁰⁸ The p -adic numbers play a very prominent role in many branches of pure mathematics.

³⁰⁶Why is $\mathbb{Z}^{(\mathbb{N})}$ countable?

³⁰⁷Note though that we are comparing limits of different system, first of a direct system and then of an inverse system.

³⁰⁸In number theory one often uses a different convention, namely one writes explicitly $\mathbb{Z}/p\mathbb{Z}$ for the quotient ring and one denotes the p -adic numbers by $\mathbb{Z}_p$. If the expression “ $\mathbb{Z}_p$ ” appears in a paper or a book it is a good idea to check the convention used by the author.

- (C) The inverse limit $\widehat{\mathbb{Z}} := \varprojlim \mathbb{Z}_n$ of the rings $\mathbb{Z}_n$, $n \in \mathbb{N}$, with the preorder on $\mathbb{N}$ defined by the divisibility relation $k|l$, is called the *ring of the profinite integers*. Using the Chinese Remainder Theorem one can show that

$$\widehat{\mathbb{Z}} \cong \prod_{\text{prime } p} \widehat{\mathbb{Z}}_p.$$

- (D) We let $(I, \leq) = (\mathbb{N}, \leq)$. Given $i \in \mathbb{N}$ we set $X_i := \mathbb{N}$ and for $i \leq j$ we define

$$\begin{aligned} f_{ji}: X_j = \mathbb{N} &\rightarrow X_i = \mathbb{N} \\ n &\mapsto n + (j - i). \end{aligned}$$

Then $(\{X_i = \mathbb{N}\}_{i \in \mathbb{N}}, \{f_{ji}\}_{j \leq i})$ forms an inverse system in the category of sets. It is an amusing exercise to show that in this case the inverse limit is the empty set.

- (E) Let X be a topological space. We denote by K the set of all compact subsets of X . We consider K with the preorder given by inclusion, i.e. we have $A \leq B$ if $A \subseteq B$. Given $A \in K$ we define³⁰⁹ $X_A := \pi_0(X \setminus A)$. Furthermore given $A \leq B$ we denote by $f_{AB}: \pi_0(X_B) \rightarrow \pi_0(X_A)$ the map induced by the inclusion $X_B \rightarrow X_A$. It is straightforward to verify that the sets $\{\pi_0(X_A)\}_{A \in K}$, together with the inclusion induced maps f_{AB} for $A \leq B$ define an inverse system in the category of sets. The elements of $\text{ends}(X) := \varprojlim \pi_0(X_A)$ are called the *ends of X* . In exercise sheet 12 we will determine the number of ends of $X = \mathbb{R}^n$.

Definition. Let $(\{0, x, y\}, \leq)$ be the push-pull set as defined on page 220, i.e. the preordered set with the non-trivial relations $0 \leq x$ and $0 \leq y$, and let $\mathcal{C}$ be a category. An inverse system in the category $\mathcal{C}$ over $(\{0, x, y\}, \leq)$ consists of three objects A_0, A_x, A_y , the identity morphisms $\text{id}_{A_0}, \text{id}_{A_x}, \text{id}_{A_y}$ and morphisms $f_x: A_x \rightarrow A_0$ and $f_y: A_y \rightarrow A_0$. We refer to such a system as a *pullback system* and often we arrange the morphisms in a diagram as follows

$$\begin{array}{ccc} & A_y & \\ & \downarrow f_y & \\ A_x & \xrightarrow{f_x} & A_0. \end{array}$$

If the inverse limit of such a pullback system exists, then we refer to it as the *pullback* of the *pullback system*.³¹⁰ The defining property of the pullback Z can be summarized in the

³⁰⁹Recall that on page 33, given a topological space Y we had defined $\pi_0(Y)$ as the set of path-components of Y . A map $f: Y \rightarrow Z$ between two topological spaces induces a map $f_*: \pi_0(Y) \rightarrow \pi_0(Z)$ in an obvious way.

³¹⁰It follows quite easily from the definitions or from the proof of Proposition 13.5 that the pullback is given by

$$Z = \{(x, y) \in A_x \times A_y \mid f_x(x) = f_y(y)\}.$$

following diagram

$$\begin{array}{ccccc}
 Z' & & \xrightarrow{g'_y} & & A_y \\
 & \searrow \text{!} & \nearrow & & \downarrow f_y \\
 & \text{pullback } Z & \xrightarrow{g_x} & & A_y \\
 & \downarrow g_y & & & \downarrow f_y \\
 & A_x & \xrightarrow{f_x} & & A_0.
 \end{array}$$

(Note: In the original image, there is a dashed arrow from Z' to $\text{pullback } Z$ labeled $\exists!$, and a curved arrow from Z' to A_x labeled g'_x .)

The following lemma will be proved in exercise sheet 12.

Lemma 13.6. *Let $p: X \rightarrow B$ be a covering of topological spaces, let C be a topological space and let $f: C \rightarrow B$ be a map. In the category of topological spaces we consider the pullback system*

$$\begin{array}{ccc}
 & X & \\
 & \downarrow p & \\
 C & \xrightarrow{f} & B
 \end{array}
 \quad \text{with the pullback} \quad
 \begin{array}{ccc}
 Y & \xrightarrow{F} & X \\
 q \downarrow & & \downarrow p \\
 C & \xrightarrow{f} & B.
 \end{array}$$

Then $q: Y \rightarrow C$ is also a covering.

We conclude this section on inverse limits with the following lemma.

Lemma 13.7. *Let $(I, \leq)$ be a preordered set and let $(\{X_i\}_{i \in I}, \{f_{ji}: X_j \rightarrow X_i\}_{i \leq j})$ be an inverse system of topological spaces. If all the X_i 's are compact and Hausdorff, then the inverse limit $\varprojlim X_i$ is also compact.³¹¹*

Example. Let p be a prime. The abelian group $\widehat{\mathbb{Z}}_p$ is the inverse limit of finite, hence compact groups. It follows from Proposition 13.9 that $\widehat{\mathbb{Z}}_p$ is also a compact topological group. On the other hand $\widehat{\mathbb{Z}}_p$ is infinite,³¹² which shows that the topology on $\widehat{\mathbb{Z}}_p$ is *not* the discrete topology.

The key ingredient in the proof of Lemma 13.7 is the following theorem.

Theorem 13.8. (Tychonoff's Theorem)³¹³ *The product of arbitrarily many compact topological spaces is again a compact topological space.*

³¹¹Note that this is very different from the situation of the direct limit. The direct limit of topological spaces is not necessarily compact, for example the direct limit of the direct system $[0, n]$, $n \in \mathbb{N}$ is given by the non-compact space $\bigcup_{n \in \mathbb{N}} [0, n] = \mathbb{R}$.

³¹²In fact $\widehat{\mathbb{Z}}_p$ is uncountable (why?).

³¹³Andrey Tikhonov (1906-1993) was a Russian mathematician. The spelling for the theorem is due to the fact that Tikhonov initially published in German and his name got transcribed from the Cyrillic alphabet to the Latin alphabet as Tychonoff.

If we take the product of only finitely many topological spaces, then we had obtained this result already in Analysis IV. For time reasons we cannot provide a proof of the general case of Tychonoff's Theorem. A proof of Tychonoff's Theorem is for example given in [J1, Chapter X].

Proof. Let $(I, \leq)$ be a preordered set and let $(\{X_i\}_{i \in I}, \{f_{ji}: X_j \rightarrow X_i\}_{i \leq j})$ be an inverse system of compact topological spaces that are Hausdorff. Given $i \in I$ we denote by $p_i: \prod_{j \in I} X_j \rightarrow X_i$ the projection map and we denote by $D_i = \{(x, x) \mid x \in X_i\} \subset X_i \times X_i$ the diagonal. As we had pointed out on page 14, the hypothesis that X_i is Hausdorff implies that the diagonal D_i is a closed subset of $X_i \times X_i$. By the proof of Proposition 13.5 we have

$$\begin{aligned} \varprojlim X_i &:= \{(x_i)_{i \in I} \in \prod_{i \in I} X_i \mid f_{lk}(x_l) = x_k \text{ for all } k \leq l\} \\ &= \bigcap_{k \leq l} \{x \in \prod_{i \in I} X_i \mid \underbrace{((f_{lk} \circ p_l) \times p_k)}_{\substack{(f_{lk} \circ p_l) \times p_k \text{ is a map} \\ \prod_{i \in I} X_i \rightarrow X_k \times X_k}}(x) \in D_k\} = \bigcap_{k \leq l} \underbrace{((f_{lk} \circ p_l) \times p_k)^{-1}(D_k)}_{\text{closed since } D_k \text{ closed}}. \end{aligned}$$

Thus we see that $\varprojlim X_i$ is the intersection of closed subsets of $\prod_{i \in I} X_i$, hence $\varprojlim X_i$ itself is a closed subset. But $\prod_{i \in I} X_i$ is compact by Tychonoff's Theorem 13.8, hence the closed subset $\varprojlim X_i$ is also compact. $\square$

13.5. The profinite completion of a group (*). ³¹⁴ Note that we can view any group as a topological group by equipping it with the discrete topology. With this topology a group is compact if and only if it is finite. Furthermore any homomorphism of groups equipped with the discrete topology is also a morphism in the category of topological groups.

Proposition 13.9. *Let $(I, \leq)$ be a preordered set and let $(\{G_i\}_{i \in I}, \{f_{ji}: G_j \rightarrow G_i\}_{i \leq j})$ be an inverse system of finite groups. Then the inverse limit $\varprojlim G_i$ exists in the category of topological groups and it is a compact topological group.*

Proof. It follows from Proposition 13.5 that the inverse limit $\varprojlim G_i$ exists in the category of topological groups. Since all the G_i 's are finite, in particular compact, it follows from Lemma 13.7 that the inverse limit $\varprojlim G_i$ is also compact. $\square$

Now let π be a group. We consider the set of all finite index normal subgroups $\{\pi_i\}_{i \in I}$. We equip the index set I with the preorder defined by the convention that $i \leq j$ if and only if $\pi_j \subset \pi_i$.³¹⁵ Given $i \leq j$ we denote by $\varphi_{ji}: \pi/\pi_j \rightarrow \pi/\pi_i$ the obvious projection map. Then $(\{\pi/\pi_i\}_{i \in I}, \{\varphi_{ji}: \pi/\pi_j \rightarrow \pi/\pi_i\}_{i \leq j})$ forms an inverse system. We refer to

$$\widehat{\pi} := \varprojlim \pi/\pi_i$$

³¹⁴This section is not part of the official lecture notes for Algebraic Topology I.

³¹⁵Note that we wrote $\pi_j \subset \pi_i$ instead of $\pi_i \subset \pi_j$.

as the *profinite completion* of π . By Proposition 13.9 the profinite completion $\widehat{\pi}$ is a compact topological group. It follows immediately from the definition of an inverse limit that there exists a unique homomorphism $\pi \rightarrow \widehat{\pi}$ such that for any i the following diagram commutes:

$$\begin{array}{ccc} \pi & \xrightarrow{\quad} & \widehat{\pi} \\ & \searrow & \searrow \\ & & \pi/\pi_i. \end{array}$$

Examples.

- (1) The profinite completion $\widehat{\mathbb{Z}}$ of the group $\mathbb{Z}$ equals precisely the profinite integers that we had encountered on page 244.
- (2) The profinite completion of a finite group G is the group itself, i.e. $\widehat{G} = G$.³¹⁶
- (3) Let $G = \mathbb{Q}$. This group has no proper subgroups of finite index.³¹⁷ It follows immediately from the definitions that $\widehat{\mathbb{Q}} = \{0\}$.

Now let $\varphi: A \rightarrow B$ be a group homomorphism. We denote by $\{B_i\}_{i \in I}$ the finite index normal subgroups of B . For $B_j \subset B_i$ we denote again by $p_{ji}: B/B_j \rightarrow B/B_i$ the projection map. For every $i \in I$ the preimage $\varphi^{-1}(B_i)$ is a finite index normal subgroup of A .³¹⁸ We now define $\psi_i: \widehat{A} \rightarrow A/\varphi^{-1}(B_i) \xrightarrow{\varphi} B/B_i$. Note that ψ_i is continuous since the first map is continuous by definition and the second map is a map between two topological spaces that are equipped with the discrete topology. It follows immediately from the definitions that for $B_j \subset B_i$ we have $p_{ji} \circ \psi_j = \psi_i$. It now follows from the definition of the inverse limit in the category of topological groups that there exists a unique continuous homomorphism $\widehat{\varphi}: \widehat{A} \rightarrow \widehat{B}$ such that for any i the following diagram commutes

$$\begin{array}{ccc} \widehat{A} & \xrightarrow{\widehat{\varphi}} & \widehat{B} \\ & \searrow & \searrow \\ & & B/B_i. \end{array} \quad \begin{array}{c} \psi_i \end{array}$$

It follows easily from the definitions and Proposition 13.9 that the maps

$$A \mapsto \widehat{A}$$

together with the maps

$$\begin{array}{ccc} \text{Mor}(A, B) & \rightarrow & \text{Mor}(\widehat{A}, \widehat{B}) \\ (\varphi: A \rightarrow B) & \mapsto & (\widehat{\varphi}: \widehat{A} \rightarrow \widehat{B}) \end{array}$$

define a functor from the category of groups to the category of compact topological groups.

³¹⁶Why is that?

³¹⁷Why not?

³¹⁸Why is it a finite index subgroup?

14. DECISION PROBLEMS

In the last chapters we had developed several techniques for determining the fundamental groups of topological spaces. In particular we managed to find finite presentations for the fundamental groups of large classes of spaces, e.g. graphs, surfaces, knot complements and mapping tori. In fact we now have enough tools to determine, with enough effort and perseverance, a presentation for the fundamental groups of most “reasonable” spaces. Furthermore we will see in Algebraic Topology II that the fundamental group of every compact manifold admits a *finite* presentation.

The question that now arises is, what information can we extract from a finite presentation of a group? Here are a couple of natural questions.

Question 14.1.

- (1) *Does there exist an algorithm that can decide whether or not a given finite presentation $\pi = \langle g_1, \dots, g_k \mid r_1, \dots, r_l \rangle$ represents the trivial group?*
- (2) *Does there exist an algorithm that can decide whether or not two given finite presentations $\pi = \langle g_1, \dots, g_k \mid r_1, \dots, r_l \rangle$ and $\Gamma = \langle h_1, \dots, h_m \mid s_1, \dots, s_n \rangle$ represent isomorphic groups?*³¹⁹
- (3) *Does there exist an algorithm that can determine, given*
 - (a) *a finite presentation $\pi = \langle g_1, \dots, g_k \mid r_1, \dots, r_l \rangle$ and*
 - (b) *a word w in $g_1, \dots, g_k$, i.e. an element $w \in \langle g_1, \dots, g_k \rangle$,**whether or not w represents the trivial element in the group π ?*³²⁰

Remark. The above question is formulated in a slightly informal way. After all, what is an “algorithm” supposed to be? The questions are made more precise in [CZ, Chapter 7].

On page 177 we had introduced the Tietze transformations on presentations. We had pointed out that two finite presentations for a given group are in fact related by a finite sequence of Tietze transformations. In particular, if $\pi = \langle g_1, \dots, g_k \mid r_1, \dots, r_l \rangle$ is a presentation for the trivial group, then we can turn the presentation after finitely many Tietze transformations into the trivial presentation $\langle \mid \rangle$. The problem is that we do not know how many Tietze transformations are required to turn a presentation of a trivial group into a trivial presentation. In particular at times it can be very hard to find the correct Tietze transformations.

Example. We consider the presentation

$$\pi = \langle a, b, c \mid a^3, b^3, c^4, ac = ca^{-1}, aba^{-1} = bcb^{-1} \rangle.$$

This is a presentation of the trivial group as can be seen as follows:³²¹

- (1) from $aba^{-1} \sim bcb^{-1}$ we get $(aba^{-1})^3 \sim (bcb^{-1})^3$,
- (2) since $a^{-1}a \sim 1$ and $b^{-1}b \sim 1$ it follows that $(aba^{-1})^3 \sim ab^3a^{-1}$ and $(bcb^{-1})^3 \sim bc^3b^{-1}$,

³¹⁹The question whether such an algorithm exists is often called the *isomorphism problem*.

³²⁰The question whether such an algorithm exists is often called the *word problem*.

³²¹For elements g, h of $\langle a, b, c \rangle$ we write $g \sim h$ if they represent the same element in π .

- (3) we have $b^3 \sim 1$, hence $ab^3a^{-1} \sim 1$, therefore we get from (1) and (2) that $bc^3b^{-1} \sim 1$ hence $c^3 \sim 1$,
- (4) we have $c^3 \sim 1$ but we also have the relation $c^4 \sim 1$, so $c \sim 1$,
- (5) since $aba^{-1} \sim bcb^{-1}$ it follows that $aba^{-1} \sim 1$ and so $b \sim 1$,
- (6) from $ac \sim ca^{-1}$ and $c \sim 1$ it follows that $a \sim a^{-1}$, hence $a^2 \sim 1$,
- (7) together with $a^3 \sim 1$ we obtain $a \sim 1$.
- (8) we now showed that a, b, c are all equivalent to 1, thus we see that the group defined by the presentation is the trivial group.

This example shows that if a presentation corresponds to the trivial group, then it can be surprisingly complicated to verify this statement.

Now suppose we are given a finite presentation and we have to decide whether or not it is a presentation of the trivial group. We can modify the presentation using Tietze transformations. If at some point we end up with the trivial presentation we know that the given presentation corresponds to the trivial group. But what happens if after many attempts at modifying the presentation using Tietze transformations we still do not have the trivial presentation. Does that mean that the presentation corresponds to a non-trivial group? Or does it just mean that we did not try long enough?

So the real question is, does there exist an algorithm, that given a presentation which defines a non-trivial group, actually verifies that the group is non-trivial?

Unfortunately the answer is quite sobering.

Theorem 14.2. *The answer to all three questions of Question 14.1 is no.*

Remark. Here “no” means that one can show that no such algorithm exists. This is a much stronger statement than saying that we do not know of an example of such an algorithm.

Proof.

- (1) Adyan [A] and Rabin [Ra] showed in 1955 that the answer to the first question is no.
- (2) An algorithm that performs (2) would also give an algorithm for determining whether or not a given presentation is isomorphic to the trivial group with the empty presentation $\langle \rangle$. But since the answer to (1) is no, it follows that the answer to (2) is also no.
- (3) If there did exist an algorithm that dealt with the third problem, then we could apply it in particular to the generators. Since a presentation presents the trivial group if and only if all generators are trivial, it follows that an algorithm for (3) would also give an algorithm for (1). Since the latter cannot exist, the former cannot exist either. $\square$

Remark. The statement of Theorem 14.2 that the answer to all three questions of Question 14.1 is no, is of course somewhat disappointing. But this raises the question whether there are interesting classes of groups for which the answer is actually yes. For example Mikhail Gromov [Gv]³²² introduced the notion of a “word hyperbolic group” for which the

³²²Mikhail Gromov (1943-) is a French-Russian mathematician famous for his many contributions to geometry and group theory. He was the first person to view groups as “geometric objects”, giving rise to the subject of “geometric group theory”.

answer to all three questions turns out to be yes. Examples of word hyperbolic groups are given by

- (1) finite groups,
- (2) free groups, fundamental groups of surfaces of genus ≥ 2 ,
- (3) fundamental groups of hyperbolic manifolds of any dimension,
- (4) fundamental groups of “Riemannian manifolds with negative sectional curvature”.

In fact in a precise sense a “random” finitely presented group is word hyperbolic, see e.g. [Ol] for more details.

The topic of word hyperbolic groups is a very important topic of mathematics in its own right. See e.g. [Lö] for an introduction to word hyperbolic groups.

Of course our main goal is not to classify groups, but our goal is to understand “reasonable” topological spaces. For example, given $n \in \mathbb{N}$ we would like to classify closed, orientable connected n -dimensional manifolds up to diffeomorphism. For $n = 1$ we did so in Proposition 1.33 and for $k = 2$ we saw the classification in Theorem 11.8. In particular we saw that in these dimensions only “few groups” appear as fundamental groups.

The situation is very different in higher dimensions. First of all we have the following proposition, which is actually fairly easy to prove, see e.g. [CZ, Theorem 5.1.1] for details.³²³

Proposition 14.3. *Given any finitely presented group π and any $n \geq 4$ there exists a closed orientable connected n -dimensional manifold with fundamental group π .*

Markov [Mar] used Proposition 14.3 and Theorem 14.2 to prove the following theorem.³²⁴

Theorem 14.4. *Let $n \geq 4$. Then there is no algorithm that can decide whether or not two closed orientable n -manifolds are diffeomorphic.*

This leaves us with the case of 3-dimensional manifolds. It turns out that in this dimension there does in fact exist such an algorithm.

Theorem 14.5. *There exists an algorithm that can decide whether or not two closed orientable connected 3-manifolds are diffeomorphic.*

In this case the algorithm is considerably more complicated than in the 1-dimensional and the 2-dimensional case. The proof that such an algorithm exists is due to the work of many mathematicians, first and foremost William Thurston and Grigori Perelman. The proof of the theorem was completed about 2003. It is impossible to mention here all the work that goes into proving Theorem 14.5. We refer to the survey paper [AFW, Theorem 4.27] for details and precise references.

³²³We will provide a proof in “Introduction to Knot Theory II”.

³²⁴Is Theorem 14.4 an immediate consequence of Proposition 14.3 and Theorem 14.2?

15. THE UNIVERSAL COVER OF TOPOLOGICAL SPACES

After spending a lot of time on determining the fundamental groups of topological spaces we now want to return to the study of covering maps. One key goal of this section is to address the following question.

Question 15.1. *Does every path-connected topological space B admit a path-connected covering $p: X \rightarrow B$ such that X is simply connected?*

We know of course that the answer is yes for the torus, the Klein bottle and the Möbius band. But does such a covering exist for the surface of genus 2? Or does it exist for the complement $S^3 \setminus K$ of a knot $K \subset S^3$? How about the wedge $S^1 \vee S^1$ of two circles?

15.1. Local properties of topological spaces. Before we can address Question 15.1 we need to recall the definition of a local property of topological space that we had already given in Section 1.6.

Definition. Let P be a property of topological spaces. We say a topological space X is *locally P* if given any $Q \in X$ and any neighborhood U of Q there exists an open neighborhood V of Q that is contained in U that has the property P .

Examples.

- (1) Every simply-connected topological space is by definition path-connected, thus every locally simply-connected topological space is also locally path-connected.
- (2) On page 35 we had seen that every open subset of $\mathbb{R}^n$ is locally simply-connected.
- (3) According to Lemma 1.25 every topological manifold is locally path-connected. In fact almost the same proof shows that every topological manifold is locally simply-connected.
- (4) It is straightforward to see that each topological graph is locally simply-connected. Similarly it is easy to verify that the wedge of two manifolds is locally simply-connected.
- (5) We consider

$$X := \mathbb{R}^2 \setminus \bigcup_{n=1}^{\infty} \frac{1}{n} \times [-1, 1],$$

with the usual subspace topology coming from $\mathbb{R}^2$. Then X is path-connected but not locally path-connected. Indeed, we consider the point $(0, 0) \in X$ and the neighborhood $U = (-\frac{1}{2}, \frac{1}{2}) \times (-\frac{1}{2}, \frac{1}{2}) \cap X$. It is straightforward to show that there does not exist a path-connected neighborhood V of $(0, 0)$ which is contained in U . We refer to Figure 140 for an illustration.

- (6) We consider

$$X := \mathbb{R}^2 \setminus \bigcup_{n=1}^{\infty} \left(\frac{1}{n}, 0\right),$$

with the subspace topology coming from $\mathbb{R}^2$. Then X is path-connected, locally path-connected but it is not locally simply connected. Indeed the origin admits no

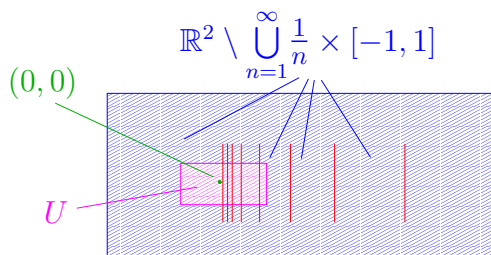


FIGURE 140.

neighborhood of the desired form, since any neighborhood of the origin contains “infinitely many holes”. We made this statement precise in exercise sheet 8.

15.2. Lifting maps to coverings. Let $p: (X, x_0) \rightarrow (B, b_0)$ be a covering of pointed topological spaces. Furthermore let $f: (Z, z_0) \rightarrow (B, b_0)$ be a map. Recall that on page 105 we had said that f *lifts*, if there exists a *lift* $\tilde{f}: (Z, z_0) \rightarrow (X, x_0)$, i.e. a map $\tilde{f}: Z \rightarrow X$ with $\tilde{f}(z_0) = x_0$ and which makes the following diagram commute

$$\begin{array}{ccc} & (X, x_0) & \\ \tilde{f} \nearrow & \downarrow p & \\ (Z, z_0) & \xrightarrow{f} & (B, b_0). \end{array}$$

For example, if $f: ([0, 1], 0) \rightarrow (B, b_0)$ is a map from the interval $[0, 1]$ to B , i.e. if f is a path with starting point b_0 , then we had seen in Proposition 6.11 that there exists a unique lift of f to a map $\tilde{f}: ([0, 1], 0) \rightarrow (X, x_0)$.

Example. Consider the following maps:

$$\begin{array}{ccc} & (S^1, 1) & \\ \text{??? lift } \tilde{f} \text{ ???} \nearrow & \downarrow p \text{ } z \mapsto z^3 & \\ (S^1, 1) & \xrightarrow{f} & (S^1, 1). \end{array}$$

Here the map p is a 3-fold covering. Does the map f lift to a map $\tilde{f}$?

To answer the question we need to study lifts of maps a little more carefully. Let $p: (X, x_0) \rightarrow (B, b_0)$ be a covering of pointed topological spaces and let $f: (Z, z_0) \rightarrow (B, b_0)$ be a map. Suppose that there exists a lift $\tilde{f}: (Z, z_0) \rightarrow (X, x_0)$. We then obtain the commutative diagram

$$\begin{array}{ccc} & (X, x_0) & \\ \tilde{f} \nearrow & \downarrow p & \\ (Z, z_0) & \xrightarrow{f} & (B, b_0) \end{array} \quad \begin{array}{l} \text{from the functoriality of the} \\ \text{fundamental group it follows} \\ \text{that } f_* = (p \circ \tilde{f})_* = p_* \circ \tilde{f}_*, \text{ i.e.} \\ \text{we get the commutative diagram} \end{array} \quad \begin{array}{ccc} & \pi_1(X, x_0) & \\ \tilde{f}_* \nearrow & \downarrow p_* & \\ \pi_1(Z, z_0) & \xrightarrow{f_*} & \pi_1(B, b_0). \end{array}$$

Thus it follows that if f lifts, then

$$\operatorname{im}(f_*) = \operatorname{im}((p \circ \tilde{f})_*) = \operatorname{im}(p_* \circ \tilde{f}_*) \subset \operatorname{im}(p_*).$$

So summarizing, if there exists a lift of f , then

$$\operatorname{im}(f_*: \pi_1(Z, z_0) \rightarrow \pi_1(B, b_0)) \subset \operatorname{im}(p_*: \pi_1(X, x_0) \rightarrow \pi_1(B, b_0)).$$

Example. In the above example we have $\pi_1(S^1, 1) = \mathbb{Z}$ and $\operatorname{im}(f_*) = 2\mathbb{Z}$ whereas we have $\operatorname{im}(p_*) = 3\mathbb{Z}$. So we see that f does not lift.

The following proposition says, that the above conclusion has a converse, for “reasonably” topological spaces. More precisely, we have the following proposition which, under a modest technical hypothesis, gives a complete criterion for when a map lifts.

Proposition 15.2. *Let $p: (X, x_0) \rightarrow (B, b_0)$ be a covering of pointed topological spaces. Furthermore let Z be a path-connected and locally path-connected topological space and let $f: (Z, z_0) \rightarrow (B, b_0)$ be a map. Then*

$$\begin{array}{ccc} & (X, x_0) & \\ \text{there exists a lift } \tilde{f} \nearrow & \downarrow p & \\ (Z, z_0) & \xrightarrow{f} (B, b_0) & \end{array} \iff \operatorname{im}(f_*) \subset \operatorname{im}(p_*).$$

Furthermore, if such a lift exists, then it is unique.

Example. Let $p: (X, x_0) \rightarrow (B, b_0)$ be a covering of pointed topological spaces.

- (1) The condition on the fundamental groups in Proposition 15.2 is automatically satisfied if Z is simply connected.
- (2) Now let $f: (S^n, s_0) \rightarrow (B, b_0)$ be a map. The topological manifold S^n is path-connected and locally path-connected. By Proposition 4.12 the sphere S^n is simply connected if $n \geq 2$. Thus it follows from (1) that there exists a uniquely determined lift $\tilde{f}: (S^n, s_0) \rightarrow (X, x_0)$ of the map $f: (S^n, s_0) \rightarrow (B, b_0)$.

In the proof of Proposition 15.2 we will need the following lemma.

Lemma 15.3. *Let $p: (X, x_0) \rightarrow (B, b_0)$ be a covering of pointed spaces, let γ be a path in B with starting point b_0 and let δ be another path in B with $\delta(0) = \gamma(1)$. We denote by $\tilde{\gamma}$ the lift of γ with starting point x_0 . Then*

$$\begin{array}{ccc} \text{endpoint of the lift of } \gamma * \delta & = & \text{endpoint of the lift of } \delta \\ \text{to the starting point } x_0 & & \text{to the starting point } \tilde{\gamma}(1). \end{array}$$

Proof. Let $p: (X, x_0) \rightarrow (B, b_0)$ be a covering of pointed spaces, let γ be a path in B with starting point b_0 and let δ be another path in B with $\delta(0) = \gamma(1)$. We denote by $\tilde{\gamma}$ the lift of γ with starting point x_0 and we denote by $\tilde{\delta}$ the lift of δ to the starting point $\tilde{\gamma}(1)$. Finally we denote by $\widetilde{\gamma * \delta}$ the lift of $\gamma * \delta$ to the starting point x_0 . For $t \in [0, 1]$ we write $\beta(t) = (\gamma * \delta)(\frac{1}{2}(t + 1))$ and $\tilde{\beta}(t) = \widetilde{(\gamma * \delta)}(\frac{1}{2}(t + 1))$. Note that by definition we have

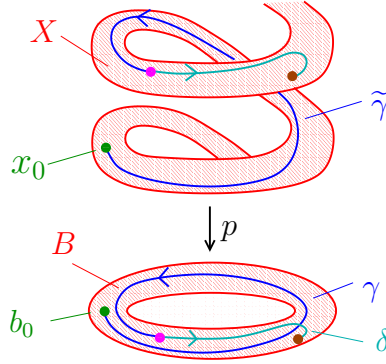


FIGURE 141. Illustration of Lemma 15.3

$\beta(t) = \delta(t)$ for $t \in [0, 1]$ and that $\tilde{\beta}$ and $\tilde{\delta}$ are both lifts of $\beta = \delta$ to the starting point $\beta(0) = \gamma(1) = \delta(0)$. Thus it follows that

$$\begin{aligned} \text{endpoint of } \widetilde{\gamma * \delta} &= (\widetilde{\gamma * \delta})(1) = \tilde{\beta}(1) = \tilde{\delta}(1) = \text{endpoint of } \tilde{\delta}. \\ &\quad \uparrow \\ &\quad \text{by Proposition 6.11} \end{aligned}$$

□

Now we can give the proof of Proposition 15.2.

Proof of Proposition 15.2. Let $p: (X, x_0) \rightarrow (B, b_0)$ be a covering of pointed topological spaces. Furthermore let Z be a path-connected and locally path-connected topological space and let $f: (Z, z_0) \rightarrow (B, b_0)$ be a map.

By the discussion on page 253 we only have to prove the “ $\Leftarrow$ ”-direction of the proposition. So suppose that we have

$$\text{im}(f_*: \pi_1(Z, z_0) \rightarrow \pi_1(B, b_0)) \subset \text{im}(p_*: \pi_1(X, x_0) \rightarrow \pi_1(B, b_0)).$$

We have to construct a lift $\tilde{f}: (Z, z_0) \rightarrow (X, x_0)$. We will do so as follows. Let $z \in Z$ be a point. Since Z is path-connected we can find a path $\alpha: [0, 1] \rightarrow Z$ with $\alpha(0) = z_0$ and $\alpha(1) = z$. According to Proposition 6.11 we can lift the path $f \circ \alpha: [0, 1] \rightarrow B$ to a path $\widetilde{(f \circ \alpha)}: [0, 1] \rightarrow X$ with starting point x_0 . We define

$$\tilde{f}(z) := \widetilde{(f \circ \alpha)}(1).$$

This definition is illustrated in Figure 142.

We have to verify the following statements:

- (1) the map $\tilde{f}: Z \rightarrow X$ is well-defined,
- (2) the map $\tilde{f}$ is continuous,
- (3) the map $\tilde{f}$ is a lift,
- (4) and the map $\tilde{f}$ is the unique lift.

We will do so in the following:

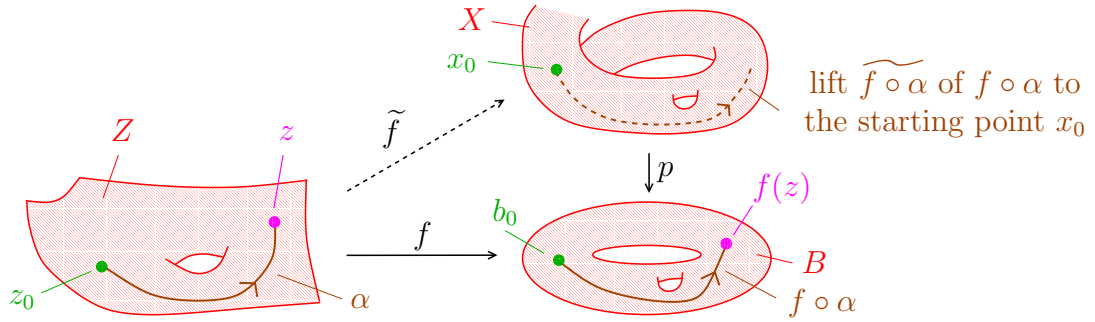


FIGURE 142.

- (1) We have to show that for $z \in Z$ the point $\tilde{f}(z) \in X$ is well-defined, i.e. independent of the choice of the path α . Thus let $\beta: [0, 1] \rightarrow Z$ be another path with $\beta(0) = z_0$ and $\beta(1) = z$. (This situation is illustrated in Figure 143.) It follows that

$$\begin{aligned}
 \widetilde{(f \circ \alpha)}(t=1) &= (\text{lift of } f \circ \alpha \text{ to the starting point } x_0)(t=1) \\
 &= (\text{lift of } (f \circ \alpha) * \overline{(f \circ \beta)} * (f \circ \beta) \text{ to the starting point } x_0)(t=1) \\
 &\quad \uparrow \\
 &\text{by Corollary 6.13 since } f \circ \alpha \text{ and } (f \circ \alpha) * \overline{(f \circ \beta)} * (f \circ \beta) \text{ are homotopic} \\
 &= (\text{lift of } (f \circ (\underbrace{\alpha * \beta}_{=: \gamma})) * (f \circ \beta) \text{ to the starting point } x_0)(t=1) \\
 &\quad \quad \quad =: \gamma, \text{ a loop in } (Z, z_0) \\
 &= (\text{lift of } f \circ \beta \text{ to the starting point } (\widetilde{f \circ \gamma})(1))(t=1) \\
 &\quad \uparrow \\
 &\text{Lemma 15.3, where } \widetilde{f \circ \gamma} \text{ denotes the lift of } f \circ \gamma \text{ to the starting point } x_0 \\
 &= (\text{lift of } f \circ \beta \text{ to starting point } x_0)(t=1) = \widetilde{(f \circ \beta)}(t=1). \\
 &\quad \uparrow \\
 &\text{by our hypothesis we have } [f \circ \gamma] = f_*([\gamma]) \in \text{im}(f_*) \subset p_*(\pi_1(X, x_0)), \\
 &\quad \text{hence } \widetilde{f \circ \gamma} \text{ is a loop in } x_0 \text{ by Lemma 6.15}^{325}
 \end{aligned}$$

Thus we have shown that the map $\tilde{f}$ is well-defined.

- (2) Now we want to show that $\tilde{f}$ is continuous. So let $z \in Z$. We pick a path α from z_0 to z . Since p is a covering there exists an open neighborhood V of $\tilde{f}(z)$ such that the restriction of p to V is a homeomorphism onto its image. We denote by $q: p(V) \rightarrow V$ the inverse map. Since Z is locally path-connected there exists an open

³²⁵Lemma 6.15 (1) says the following: if δ is a loop in (B, b_0) and if $\tilde{\delta}$ denotes the lift of δ to the starting point x_0 , then the following holds:

$$\tilde{\delta} \text{ is a loop in } (X, x_0) \iff [\delta] \text{ lies in } p_*(\pi_1(X, x_0)) \subset \pi_1(B, b_0).$$

We apply the lemma to $\delta = f \circ \gamma$.



Claim. On the open neighborhood U of z the map $\tilde{f}$ agrees with $q \circ f$.

$$\begin{aligned}\widetilde{f}(w) &= \text{endpoint of the lift of } f \circ (\alpha * \delta) = (f \circ \alpha) * (f \circ \delta) \text{ to the starting point } x_0 \\ &= \text{endpoint of the lift of } f \circ \delta \text{ to the starting point } \widetilde{f}(z)\end{aligned}$$

by Lemma 15.3 and the definition of $\widetilde{f}(z)$ as the endpoint of $\widetilde{f \circ \alpha}$

$$= \text{endpoint of } q \circ (f \circ \delta) = (q \circ f)(\delta(1)) = (q \circ f)(w).$$

↑
since $q = p^{-1}$ we know that $q \circ (f \circ \delta)$ is the lift of $f \circ \delta$ to the starting point $f(z)$



- (3) We need to show that $\tilde{f}$ is a lift, i.e. we need to show that $p \circ \tilde{f} = f$. So let $z \in Z$. Then

$$p(\tilde{f}(z)) = p(\tilde{\alpha}(1)) = \alpha(1) = z.$$

$\uparrow$

pick path α from z_0 to z and denote by $\tilde{\alpha}$ the lift of α

- (4) The uniqueness of the lift of paths, see Proposition 6.11, also gives the desired uniqueness statement for the lift of f . More precisely, suppose we are given another lift $\tilde{f}': Z \rightarrow X$ of f with $\tilde{f}'(z_0) = x_0$. Let $z \in Z$. Since Z is path-connected there exists a path $\alpha: [0, 1] \rightarrow Z$ from z_0 to z . Then $f \circ \alpha: [0, 1] \rightarrow B$ is a path and $\tilde{f} \circ \alpha$ and $\tilde{f}' \circ \alpha$ are both lifts of $f \circ \alpha$ to the starting point x_0 . By Proposition 6.11 we have $\tilde{f} \circ \alpha = \tilde{f}' \circ \alpha$. In particular the endpoints agree. But this implies that $\tilde{f}(z) = (\tilde{f} \circ \alpha)(1) = (\tilde{f}' \circ \alpha)(1) = \tilde{f}'(z)$. $\square$

In exercise sheet 13 we will prove the following corollary.

Corollary 15.4. *Let $\pi: X \rightarrow B$ be a covering of topological spaces. Then any open simply connected and locally path-connected subset $U \subset B$ is uniformly covered.*

15.3. Existence of covering spaces. Let $p: (X, x_0) \rightarrow (B, b_0)$ be a covering of pointed topological spaces. In Corollary 6.14 we had seen that

$$p_*: \pi_1(X, x_0) \rightarrow \pi_1(B, b_0)$$

is an injective map. Thus we can view $\pi_1(X, x_0)$ as a subgroup of $\pi_1(B, b_0)$.

Generalizing Question 15.1 one can now ask, whether to each subgroup Γ of $\pi_1(B, b_0)$ there exists a corresponding covering with $p_*(\pi_1(X, x_0)) = \Gamma$. We will see that this is indeed the case, at least under the rather technical assumption that the topological space B is locally simply connected. The following proposition gives in particular an affirmative answer to Question 15.1 under rather modest hypotheses.

Proposition 15.5. *Let Y be a path-connected and locally simply connected topological space, let $y_0 \in Y$ and let $\Gamma \subset \pi = \pi_1(Y, y_0)$ be a subgroup.*

- (1) *There exists a path-connected covering³²⁶ $p: (X, x_0) \rightarrow (Y, y_0)$ of pointed topological spaces such that $p_*(\pi_1(X, x_0)) = \Gamma$.*
- (2) *For the covering p from (1) we have*

$$[X : Y] = [\pi_1(Y, y_0) : \Gamma].$$

Example.

- (1) We consider $(Y, y_0) = (S^1, 1)$. We pick an identification³²⁷ $\pi_1(S^1, 1) = \mathbb{Z}$. Given $n \in \mathbb{N}$ we consider $\Gamma = n\mathbb{Z} \subset \mathbb{Z} = \pi_1(S^1, 1)$. Then the covering

$$\begin{aligned} p: S^1 &\rightarrow S^1 \\ z &\mapsto z^n \end{aligned}$$

³²⁶Hereby we say that a covering $p: X \rightarrow Y$ is path-connected if X is path-connected.

³²⁷This means that we pick an isomorphism $\pi_1(S^1, 1) \xrightarrow{\cong} \mathbb{Z}$ and then ignore it in the notation.

has the property that $p_*(\pi_1(S^1, 1)) = n\mathbb{Z}$. Another example is given by the covering

$$\begin{aligned} q: \mathbb{R}/n\mathbb{Z} &\rightarrow S^1 \\ t + n\mathbb{Z} &\mapsto e^{2\pi it}. \end{aligned}$$

Here we also have $\text{im}(q_*) = n\mathbb{Z}$.

- (2) Let Z be a simply connected topological space and let $z_0 \in Z$. Furthermore let π be a group that acts on Z continuously and discretely. We consider $Y = Z/\pi$, we denote by $q: Z \rightarrow Y = Z/\pi$ the projection map and we write $y_0 = q(z_0)$. Using the isomorphism of Theorem 6.16 we can identify π with $\pi_1(Y, y_0)$.

Now let $\Gamma \subset \pi = \pi_1(Y, y_0)$ be a subgroup. Then the group Γ also acts on Z and we can consider the projection map

$$p: X := Z/\Gamma \rightarrow Y := Z/\pi$$

We denote by x_0 the image of z_0 under the projection map $Z \rightarrow X = Z/\Gamma$. One can show easily that $p: X = Z/\Gamma \rightarrow Y = Z/\pi$ is a covering map with $p_*(\pi_1(X, x_0)) = \Gamma$. We summarize the situation in the following diagram

$$\begin{array}{ccc} Z & \searrow & \pi_1(Z, z_0) = \{e\} \\ q \downarrow & & \downarrow \\ Y = Z/\pi & \xleftarrow{p} X = Z/\Gamma & \pi_1(X, x_0) = \Gamma \\ & & \downarrow p_* \\ & & \pi_1(Y, y_0) = \pi. \end{array}$$

Proof of Proposition 15.5. Let Y be a path-connected and locally simply connected³²⁸ topological space, let $y_0 \in Y$ and let $\Gamma \subset \pi = \pi_1(Y, y_0)$ be a subgroup. We only need to show the existence of such a covering, the statement about the degree of the covering is an immediate consequence of Lemma 6.15 (3).

To get an idea for the construction, let us first consider the case that Γ is the trivial group. In this case we want to construct a covering $p: X \rightarrow Y$ such that X is simply-connected. Let us assume for a second, that we have such an X . Pick x_0 with $p(x_0) = y_0$. Then, somewhat similar to the statement of Theorem 6.16, we have a map

$$\begin{aligned} \Phi: X &\rightarrow \{\text{homotopy classes of paths starting in } y_0\} \\ x &\mapsto p \circ (\text{any path from } x \text{ to } x_0). \end{aligned}$$

In fact the proof of Theorem 6.16 comes close to showing that the map Φ is a bijection. Hence Φ is in fact a homeomorphism, provided we equip the right hand side with an appropriate topology.

Now that we are not given X , the idea is, that given (Y, y_0) we can define the right hand side of the above bijection, and with a suitable topology we will have found our X .

³²⁸Several times we will make use of the fact that a locally simply connected topological is in particular locally path-connected.

We first consider³²⁹

$$W := \{\text{all paths in } Y \text{ with starting point } y_0\}.$$

For $u, v \in W$ we write

$$u \sim v \iff u \text{ and } v \text{ have the same endpoint and } [u * \bar{v}] \in \Gamma.$$

Using the fact that $\Gamma \subset \pi_1(Y, y_0)$ is a subgroup one can show easily that $\sim$ is indeed an equivalence relation on W . The definitions of W and $\sim$ are illustrated in Figure 145. We write $X := W / \sim$ and $x_0 := [e_{y_0}]$, i.e. x_0 is the equivalence class of the constant path.

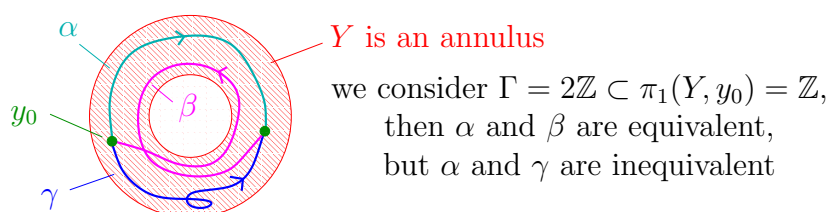


FIGURE 145.

Now we consider the map

$$\begin{aligned} p: X = W / \sim &\rightarrow Y \\ [v] &\mapsto v(1). \end{aligned}$$

Note that this map is well-defined, i.e. it is independent of the choice of the representative v .

What remains to be done is to verify that our construction gives the desired object. We still need to do the following steps:

- (1) we have to put a suitable topology on X ,
- (2) we have to show that with the topology from (1) the above map $p: X \rightarrow Y$ is continuous and open,
- (3) we need to prove that the map $p: X \rightarrow Y$ is a covering,
- (4) we have to show that $X = W / \sim$ with the topology from (1) is path-connected,
- (5) finally we need to show that we have the desired equality $p_*(\pi_1(X, x_0)) = \Gamma$.

We now perform these steps.

- (1) Let $x = [f] \in X = W / \sim$ and let V be an open path-connected neighborhood of $p(x) = f(1) \in Y$. We consider

$$U(x, V) := \{[f * u] \mid u \text{ is a path in } V \text{ with starting point } f(1)\} \subset X = W / \sim.$$

We denote by $\mathcal{B}$ the family of all such subsets of X . We have to verify that these sets satisfy the “basis property” that we had introduced on page 40.

³²⁹At a first reading of the proof of the proposition it is perhaps best to just think of the case that $Y = S^1 \times [-1, 1]$ and that $\Gamma \subset \pi_1(S^1)$ is the trivial group.

- (B1) For each $x \in X$ there exists an open path-connected neighborhood V .³³⁰ Clearly we have $x \in U(x, V)$. This shows that $\mathcal{B}$ satisfies (B1).
- (B2) Now let $x = [f]$ be a point in the intersection of $U(a, A)$ and $U(b, B)$. Since Y is in particular locally path-connected there exists an open path-connected neighborhood V of $f(1)$ with $V \subset A \cap B$. It is straightforward to see that $U(x, V) \subset U(a, A) \cap U(b, B)$. We refer to Figure 146 for an illustration.

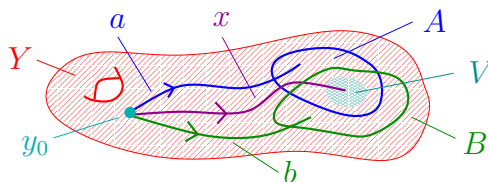


FIGURE 146.

We henceforth consider X as topological space with the topology generated by $\mathcal{B}$.

- (2) Now we want to show that the map $p: X = W/\sim \rightarrow Y$ is continuous and open. We start with the following observation: for every $x \in X$ and every open path-connected neighborhood V of $p(x)$ we have $p(U(x, V)) = V$.³³¹

We now turn to the actual proof of continuity. So let $A \subset Y$ be an open subset and let $x = [f] \in p^{-1}(A)$. It suffices to show that there exists an open subset B of Y such that $U(x, B) \subset p^{-1}(A)$. Since Y is locally path-connected, there exists an open neighborhood B of x that is contained in A and that is path-connected. By the above observation we have $p(U(x, B)) = B \subset A$, i.e. $U(x, B) \subset p^{-1}(A)$.

Finally we show that p is also open. By Lemma 1.24 it suffices to verify that images of the basis of the topology of X are open. But this statement we had just verified in the above observation.

- (3) Now we want to prove that $p: X = W/\sim \rightarrow Y$ is a covering map. So let $y \in Y$ be a point. We need to show that there exists a connected open neighborhood V of y such that the restriction of p to each component of $p^{-1}(V)$ is a homeomorphism.

Since Y is locally simply connected there exists an open simply connected neighborhood V of y . We claim that V already has the desired property.

Claim.

$$p^{-1}(V) = \bigsqcup_{\tilde{y} \in p^{-1}(y)} U(\tilde{y}, V).$$

It is clear that the right-hand side is contained in the left-hand side. Next we want to show that the left-hand side is contained in the right-hand side. So let $[g] \in p^{-1}(V)$.

³³⁰Hereby we use that Y is locally simply connected, in particular it is locally path-connected.

³³¹By definition we have $p(U(x, V)) \subset V$. But since V is path-connected we also have $V \subset p(U(x, V))$.

Hereby g is a path in Y from y_0 to a point $v \in V$. Since V is in particular path-connected there exists a path h in V from v to y . We then have

$$[g] = [g * h * \bar{h}] \in U(\underbrace{[g * h]}_{\in p^{-1}(y)}, V).$$

We refer to Figure 147 for an illustration.

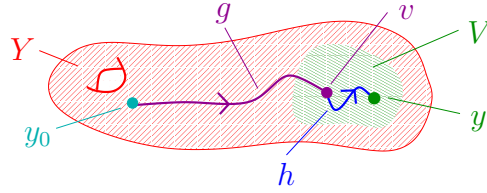


FIGURE 147.

It remains to show that the sets on the right-hand side of the claim are disjoint. So let $[v'], [v''] \in p^{-1}(y)$ and let $[g] \in U([v'], V) \cap U([v''], V)$. Then there exist paths u' and u'' in V with $[v' * u'] = [v'' * u'']$. It follows that

$$[v'] = [v' * u' * \bar{u}'] = [v'' * u'' * \bar{u}'] = [v''].$$

↑
 $u'' * \bar{u}'$ is null-homotopic since V is simply connected

Thus $U([v'], V) \cap U([v''], V)$ intersect only if $[v'] = [v'']$. We refer to Figure 148 for an illustration. This concludes the proof of the claim.

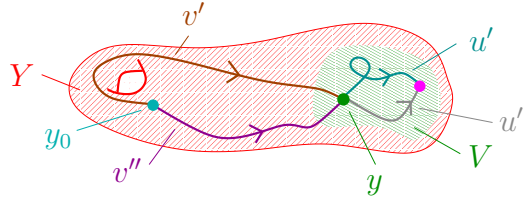


FIGURE 148.

Since the sets $U(\tilde{y}, V)$ are open and disjoint it now suffices to prove the following claim.

Claim. For each $\tilde{y} \in p^{-1}(y)$ the restriction of the projection map $p: X = W/\sim \rightarrow Y$ to the map $p: U(\tilde{y}, V) \rightarrow V$ is a homeomorphism.

So let $\tilde{y} \in p^{-1}(y)$. As we had just pointed out in (2), the map $p: U(\tilde{y}, V) \rightarrow V$ is surjective. From the fact that V is simply connected it also follows immediately that p is injective. Furthermore we had just verified in (2) that p is continuous and open. Therefore $p: U(\tilde{y}, V) \rightarrow V$ is a homeomorphism.

- (4) Now we want to show that $X = W/\sim$ is path-connected. We start out with the following claim.

Claim. Let $\alpha: [0, 1] \rightarrow Y$ be a path. Then

$$\begin{aligned} \tilde{\alpha}: [0, 1] &\rightarrow X = W/\sim \\ t &\mapsto \left[\begin{array}{ccc} [0, 1] & \rightarrow & Y \\ s & \mapsto & \alpha(ts) \end{array} \right]. \end{aligned}$$

is a continuous map.

By Lemma 1.23 it suffices to show that for each $U(x, V)$ in the basis $\mathcal{B}$ of the topology on X the preimage $\tilde{\alpha}^{-1}(U(x, V))$ is open in $[0, 1]$. So suppose we are given $U(x, V)$. We denote by $\{I_j\}_{j \in J}$ the path-components of $\alpha^{-1}(V)$. Since α is continuous each I_j is an open subset of $[0, 1]$. Note that for each $j \in J$ we either have³³² $\tilde{\alpha}(I_j) \subset U(x, V)$ or $\tilde{\alpha}(I_j) \cap U(x, V) = \emptyset$. It follows that

$$\tilde{\alpha}^{-1}(U(x, V)) = \text{union of all the } I_j \text{ with the property that } \tilde{\alpha}(I_j) \subset U(x, V).$$

Thus we see that $\tilde{\alpha}^{-1}(U(x, V))$ is a union of open subsets of $[0, 1]$, hence the preimage itself is open. This concludes the proof of the claim.

To show that X is path-connected it suffices to show that for every point $x \in X$ there exists a path in X from $x_0 = [e_{y_0}]$ to x . So let $x = [\alpha] \in X$. Then such a path is given by $\tilde{\alpha}: [0, 1] \rightarrow X$, where $\tilde{\alpha}$ is defined as in the claim. By the claim $\tilde{\alpha}$ is indeed continuous.

- (5) It remains to show that $p_*(\pi_1(X, x_0)) = \Gamma$. Let $g \in \pi_1(Y, y_0)$. We choose a representative $\alpha: [0, 1] \rightarrow Y$ of g . We consider the following path³³³ in $X = W/\sim$

$$\begin{aligned} \tilde{\alpha}: [0, 1] &\rightarrow X = W/\sim \\ t &\mapsto \left[\begin{array}{ccc} [0, 1] & \rightarrow & Y \\ s & \mapsto & \alpha(ts) \end{array} \right]. \end{aligned}$$

Note that $\tilde{\alpha}$ is evidently the lift of α to the starting point x_0 . Then we have

$$\begin{array}{ccc} g \in p_*(\pi_1(X, x_0)) & \Leftrightarrow & \tilde{\alpha} \text{ is a loop} \Leftrightarrow \text{in } X = W/\sim \text{ we have } \tilde{\alpha}(1) = x_0 = [e_{y_0}] \Leftrightarrow \alpha \in \Gamma. \\ \uparrow & & \uparrow \\ \text{Lemma 6.15 (1)} & & \text{definition of } \sim \text{ on } W \square \end{array}$$

Now the question arises, to what degree is the covering corresponding to the subgroup Γ unique. To answer that question we need an appropriate notion of two coverings being equivalent.

Definition. Let $p: X \rightarrow B$ and $q: Y \rightarrow B$ be two coverings of a topological space B . We say p and q are *equivalent*, if there exists a homeomorphism $\Phi: X \rightarrow Y$ such that the

³³²This might require a few minutes of thought.

³³³Note that this map is continuous by the claim in (4).

following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow[\cong]{\Phi} & Y \\ & \searrow p \quad \swarrow q & \\ & B. & \end{array}$$

Similarly we define the equivalence of coverings of pointed topological spaces.

Example. Given $n \in \mathbb{N}$ the two coverings

$$\begin{array}{ccc} p: S^1 & \rightarrow & S^1 \\ z & \mapsto & z^n \end{array} \quad \text{and} \quad \begin{array}{ccc} q: \mathbb{R}/n\mathbb{Z} & \rightarrow & S^1 \\ t + n\mathbb{Z} & \mapsto & e^{it} \end{array}$$

are equivalent. In fact the desired map $\Phi: S^1 \rightarrow \mathbb{R}/n\mathbb{Z}$ is given by $e^{2\pi it} \mapsto [nt]$.

Now we can formulate the following proposition:

Proposition 15.6. *Let B be a topological space, let $b_0 \in B$ and let $\Gamma \subset \pi_1(B, b_0)$ be a subgroup.*

- (1) *Let $p: (X, x_0) \rightarrow (B, b_0)$ be a path-connected and locally path-connected covering such that $p_*(\pi_1(X, x_0)) = \Gamma$. This covering has the following universal property: for any other path-connected covering $q: (Y, y_0) \rightarrow (B, b_0)$ with $\Gamma \subset q_*(\pi_1(Y, y_0))$ there exists a unique covering $r: (X, x_0) \rightarrow (Y, y_0)$ such that the following diagram commutes:*

$$\begin{array}{ccc} (X, x_0) & \xrightarrow{\exists! r} & (Y, y_0) \\ p \downarrow & \nearrow q & \\ (B, b_0) & & \end{array}$$

- (2) *If B is path-connected and locally simply connected, then there exists, up to equivalence, a unique path-connected covering $p: (X, x_0) \rightarrow (B, b_0)$ of pointed topological spaces such that $p_*(\pi_1(X, x_0)) = \Gamma$.*

Proof. Let B be a topological space and let $b_0 \in B$.

- (1) Let $p: (X, x_0) \rightarrow (B, b_0)$ and $q: (Y, y_0) \rightarrow (B, b_0)$ be two path-connected coverings such that $p_*(\pi_1(X, x_0)) \subset q_*(\pi_1(Y, y_0))$. We assume furthermore that X is locally path-connected. It follows from Proposition 15.2 that there exists a unique map $r: (X, x_0) \rightarrow (Y, y_0)$ which makes the following diagram commute

$$\begin{array}{ccc} & & (Y, y_0) \\ & \nearrow r & \downarrow q \\ (X, x_0) & \xrightarrow{p} & (B, b_0). \end{array}$$

It remains to show that r is a covering. This will be done in exercise sheet 13.

- (2) The existence of such a covering is just the statement of Proposition 15.5. The uniqueness statement is an immediate consequence of (1)³³⁴ and the usual proof of uniqueness for an object satisfying a universal property. We leave the details to the reader. $\square$

Corollary 15.7. *Let X be a path-connected and locally simply connected topological space and let $x_0 \in X$. Then the following holds:*

- (1) *There exists, up to equivalence, a unique path-connected covering*

$$p: (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$$

such that $\tilde{X}$ is simply connected.

- (2) *Let $q: (Y, y_0) \rightarrow (X, x_0)$ be another path-connected covering. Then there exists a unique covering $r: (\tilde{X}, \tilde{x}_0) \rightarrow (Y, y_0)$ such that the following diagram commutes:*

$$\begin{array}{ccc} (\tilde{X}, \tilde{x}_0) & \xrightarrow{r} & (Y, y_0) \\ p \downarrow & & \swarrow q \\ (X, x_0) & & \end{array}$$

Proof. Let X be a path-connected and locally simply connected topological space and let $x_0 \in X$.

- (1) We apply Proposition 15.6 to the trivial subgroup of $\pi_1(X, x_0)$ and we obtain, up to equivalence, a unique path-connected covering $p: (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$ such that $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ is the trivial group. But then it follows from Corollary 6.14 that $\tilde{X}$ is simply-connected.
- (2) The second statement is also an immediate consequence of Proposition 15.6. $\square$

Definition. Let X be a path-connected and locally simply connected topological space. We refer to the covering $p: \tilde{X} \rightarrow X$ such that $\tilde{X}$ is simply connected as the *universal covering* of X .³³⁵

Examples.

- (1) The map

$$\begin{array}{ccc} p: \mathbb{R} & \rightarrow & \mathbb{R}/\mathbb{Z} = S^1 \\ t & \mapsto & e^{2\pi it} \end{array}$$

is the universal covering of S^1 .

³³⁴If one does the argument carefully, then one sees, that here we use the following fact: if B is locally simply connected, then also every covering space of B is locally simply connected, in particular every covering space of B is locally path-connected.

³³⁵In the literature the universal covering of a topological space is usually indicated by the same symbol but decorated with a tilde, e.g. the universal covering of X is usually denoted by $\tilde{X}$.

(2) More generally, the map

$$\begin{aligned} p: \mathbb{R}^n &\rightarrow (S^1)^n \\ (t_1, \dots, t_n) &\mapsto (e^{2\pi i t_1}, \dots, e^{2\pi i t_n}) \end{aligned}$$

is the universal covering of the n -dimensional torus $(S^1)^n$.

(3) The projection map $p: S^3 \rightarrow S^3/\mathbb{Z}_p = L(p, q)$ is the universal covering of the lens space $L(p, q)$.

Remark. The universal covering is often also just called the *universal cover*. Also very often one suppresses the actual map p from the notation. For example, often one says that “ $\mathbb{R}^n$ is the universal cover of $(S^1)^n$ ”. The name “universal cover(ing)” stems from the fact, proved in Corollary 15.7, that the universal cover(ing) of a topological space X “covers” every other covering of X .

We have now shown that for every “reasonable” topological space X there exists a universal cover $p: \tilde{X} \rightarrow X$. So far we have given an abstract construction, but if possible we would like to visualize it. We have just given an explicit description of the universal cover of the n -dimensional torus $(S^1)^n$ and the lens spaces. We will now describe the universal covers of $X = S^1 \vee S^2$ and $X = S^1 \vee S^1$.

First we consider $X = S^1 \vee S^2$ where we identify $i \in S^1$ with $(0, 0, -1) \in S^2$. As on page 159 we consider the space

$$\tilde{X} = (\mathbb{R} \cup (\mathbb{Z} \times S^2)) / n \sim (n, (0, 0, -1)), \quad n \in \mathbb{Z}.$$

It is straightforward to see that the map

$$\begin{aligned} p: (\mathbb{R} \cup (\mathbb{Z} \times S^2)) / n \sim (n, (0, 0, -1)) &\rightarrow S^1 \vee S^2 \\ P &\mapsto \begin{cases} e^{i(2\pi P + \frac{\pi}{2})}, & \text{if } P \in \mathbb{R} \\ Q, & \text{if } P = (n, Q) \text{ with } n \in \mathbb{Z}, Q \in S^2 \end{cases} \end{aligned}$$

is well-defined and in fact a covering. We had seen on page 159 that $\tilde{X}$ is simply connected. Thus $p: \tilde{X} \rightarrow X$ is the universal cover of X .

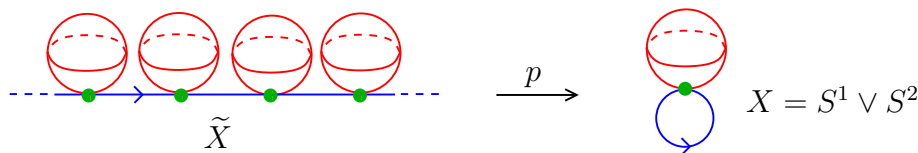


FIGURE 149.

Now we turn to $X = S^1 \vee S^1$. In the following we give a semi-rigorous description of the universal cover of $S^1 \vee S^1$ as a subset of $\mathbb{R}^2$. Iteratively we build a sequence of spaces Y_k , $k \in \mathbb{N}_0$ as follows:

(1) The space Y_0 is the cross centered at the origin where each segment has length 1.

- (2) Given Y_k we attach to each endpoint of Y_k three segments of length $\frac{1}{4^k}$ ³³⁶ such that each endpoint of Y_k now has a neighborhood that is a cross. We denote the resulting space by Y_{k+1} .

Finally we put $Y = \bigcup_{k \in \mathbb{N}_0} Y_k$. This construction is illustrated in Figure 150. We summarize

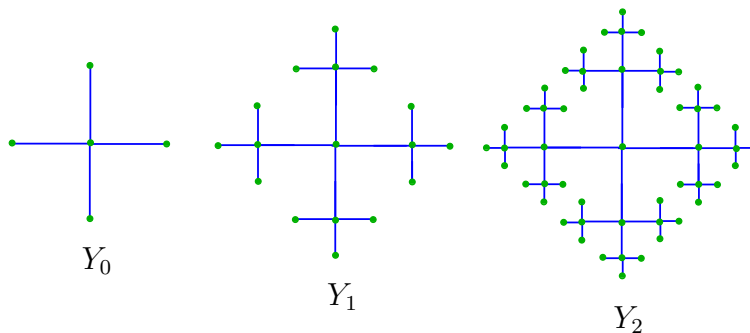


FIGURE 150.

some of the key properties of the construction in the following claim.

Claim.

- (1) Each Y_k is a tree.³³⁷
- (2) Y is simply connected.

Sketch of proof. First note that $\chi(Y_0) = 5 - 4 = 1$. For $k \geq 1$ we obtain Y_k from Y_{k-1} by adding three vertices and three edges, hence $\chi(Y_k) = \chi(Y_{k-1}) = \cdots = \chi(Y_0) = 1$. This shows that each Y_k is a tree. It follows from Proposition 7.7 that each Y_k is simply connected.

Given $k \in \mathbb{N}$ we denote by U_k the result of removing the 4^{k+1} endpoints of Y_k . Note that each U_k is open in Y_k , hence each U_k is open in Y . Furthermore each U_k deformation retracts to Y_{k-1} . Hence it follows from the fact that each Y_k is simply connected that also each U_k is simply connected.

It follows as in Lemma 9.5 that $Y = \bigcup_{k \in \mathbb{N}} Y_k = \bigcup_{k \in \mathbb{N}} U_k$ itself is simply connected. $\square$

Now we construct a covering $p: Y \rightarrow S^1 \vee S^1$. To distinguish the two copies of S^1 we name them A and B , which are wedged together along $a \in A$ and $b \in B$. We give both A and B the usual counterclockwise orientation. Furthermore we orient each edge of the infinite graph Y as indicated in Figure 151. We consider the map $p: Y \rightarrow A \vee B$ that is given as follows:

- (1) each vertex of the infinite graph Y gets sent to the point $a = b \in A \vee B$,

³³⁶In Figure 150 and 151 the lengths of the segments is $\frac{1}{2^k}$ since these are easier to draw.

³³⁷Recall that a tree is a graph with finitely many vertices V and edges E , such that the Euler characteristic $\chi(G) = \#V - \#E = 1$.

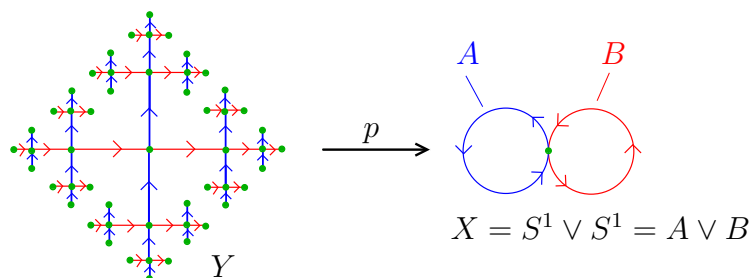


FIGURE 151.

- (2) each horizontal edge of Y gets sent in an orientation-preserving way to $A \setminus \{a\}$,
- (3) each vertical edge of Y gets sent in an orientation-preserving way to $B \setminus \{b\}$.

This map is illustrated in Figure 151. It is relatively straightforward to convince oneself that $p: Y \rightarrow X$ is in fact a covering.³³⁸ Since Y is simply connected, this is the universal cover of $X = A \vee B = S^1 \vee S^1$.

We have now determined explicitly the universal covers of several topological spaces. The surfaces of genus $g \geq 2$ are arguably one of the most interesting types of topological manifolds. Therefore it is natural to ask the following question.

Question 15.8.

- (1) *Given a surface Σ of genus $g \geq 2$ and a finite-index subgroup $\Gamma \subset \pi_1(\Sigma)$, what does the corresponding cover look like?*
- (2) *What does the universal cover of the surface of genus $g \geq 2$ look like?*

We will address these two questions in the following two sections. But for the first question we will obtain a complete answer only in Algebraic Topology II.

But before we turn to the covering spaces of surfaces we quickly recall that in Theorem 6.16 we had seen that if X is a simply connected topological space and if G is a group that acts continuously and discretely on X , then the fundamental group of the quotient space X/G is isomorphic to G . We will now show that, under modest extra hypotheses, the converse holds. More precisely, we have the following proposition.

Proposition 15.9. *Let Y be a path-connected, locally simply connected topological space. We write $\pi = \pi_1(Y)$. Then there exists a simply connected topological space X and a continuous and discrete action of π on X such that Y is homeomorphic to X/π .*

³³⁸We have to check that each point in $X = A \vee B$ admits an open neighborhood that is uniformly covered. It is clear that such a point exists for all points that are not the wedge point. For the wedge point itself the desired open neighborhood is given by small open intervals on A and B . Then each component of the preimage consists of a little cross with center P , with one horizontal edge going into P , one horizontal edge going out of P , one vertical edge going into P and one vertical edge going out of P .

Proof. Let Y be a path-connected, locally simply connected topological space. We pick a base point $y_0 \in Y$. In Proposition 15.5 we had already shown that Y admits a simply connected covering. We recall the explicit construction given in the proof of Proposition 15.5. Thus we consider

$$W := \{\text{all paths in } Y \text{ with starting point } y_0\}.$$

For $u, v \in W$ we write

$$u \sim v \quad :\Leftrightarrow \quad u \text{ and } v \text{ have the same endpoint and } [u * \bar{v}] = e \in \pi_1(Y, y_0)$$

and we put $X = W / \sim$. As on page 259, given $x = [f] \in X = W / \sim$ and an open path-connected neighborhood V of $f(1)$ we define

$$U(x, V) := \{[f * u] \mid u \text{ is a path in } V \text{ with starting point } f(1)\} \subset Y = W / \sim.$$

Furthermore, as in the proof of Proposition 15.5 we endow X with the topology generated by these sets $U(x, V)$.

Claim. The map

$$\begin{aligned} \pi_1(Y, y_0) \times X &\rightarrow X \\ ([g], [u]) &\mapsto [g * u] \end{aligned}$$

is a well-defined action.

We first show that the map is independent of the choice of u . So suppose that $u, v \in W$ are equivalent. Then $g * u$ and $g * v$ still have the same endpoint. Now we need to consider the second condition in the definition of $\sim$. We have

$$[(g * u) * (\overline{g * v})] = [g * u * \bar{v} * \bar{g}] = [g] * \underbrace{[u * \bar{v}]}_{=e \in \pi_1(Y, y_0)} * [g]^{-1} = e \in \pi_1(Y, y_0).$$

It is straightforward to see that the map does not depend on the choice of representative g . Finally we want to show that it is indeed an action. So let $[g], [h] \in \pi_1(Y, y_0)$. Then

$$[g] \cdot ([h] \cdot [u]) = [g] \cdot [h * u] = [g * (h * u)] = [(g * h) * u] = [g * h] \cdot [u] = ([g] \cdot [h]) \cdot [u].$$

This concludes the proof of the claim.

Now we need to show the following:

- (1) the group $\pi_1(Y, y_0)$ acts continuously and discretely on $X = W / \sim$,
- (2) there exists a homeomorphism $f: X / \pi_1(Y, y_0) \rightarrow Y$.

We do this in the following two steps:

- (1) It follows immediately from the definition of the topology on $X = W / \sim$ and from Lemma 1.23 that the action is continuous.

Now we need to show that the action is discrete. So let $x = [w] \in W / \sim$. Since W is locally simply connected we can pick a simply connected neighborhood V around the endpoint P of w . It suffices to show that for $[g] \in \pi_1(Y, y_0)$ we have

$$[g] \cdot U(x, V) \cap U(x, V) \neq \emptyset \quad \implies \quad [g] = e \in \pi_1(Y, y_0).$$

So suppose that $[g] \cdot U(x, V) \cap U(x, V) \neq \emptyset$. This means that there exist paths a and b in U , starting from P such that $g * w * a$ and $w * b$ have the same endpoint and they are homotopic. But then

$$[g] = [g] \cdot \overbrace{[w] \cdot [a * \bar{b}] * [w]^{-1}}^{=e \in \pi_1(Y, y_0)} = [(g * w * a) * \overline{(w * b)}] = e \in \pi_1(Y, y_0).$$

We refer to Figure 152 for an illustration.

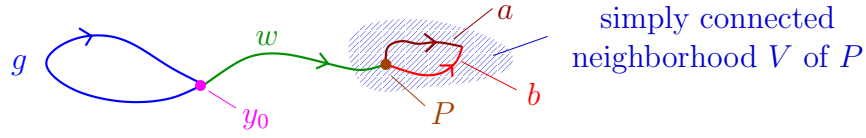


FIGURE 152.

(2) We consider the map

$$\begin{aligned} f: X/\pi_1(Y, y_0) &\rightarrow Y \\ [w] &\mapsto w(1). \end{aligned}$$

It is straightforward to see that this map is a bijection. It follows immediately from Lemma 1.7 that the map is continuous and it follows from Lemma 1.24 that the map is open. Thus it is a homeomorphism. $\square$

16. COVERING SPACES AND MANIFOLDS

16.1. Covering spaces of manifolds. In this section we show that covering spaces of (topological) manifolds are themselves (topological) manifolds.

Proposition 16.1. *Let $p: \tilde{X} \rightarrow X$ be a countable covering³³⁹ of a connected topological space X .*

- (1) *If X is an n -dimensional topological manifold (without boundary), then $\tilde{X}$ is also an n -dimensional topological manifold (without boundary).*
- (2) *If X admits a smooth atlas, then $\tilde{X}$ also admits a smooth atlas such that p is a local diffeomorphism,³⁴⁰ in particular such that p is a smooth map.*
- (3) *If X is an oriented manifold, then $\tilde{X}$ also admits an orientation such that $p: \tilde{X} \rightarrow X$ is orientation-preserving.*

In the proof of Proposition 16.1 we will need the following technical lemma.

Lemma 16.2. *Let $p: \tilde{X} \rightarrow X$ be a countable covering of a topological space X . If X is second-countable, then $\tilde{X}$ is also second-countable.*

Proof. Let $p: \tilde{X} \rightarrow X$ be a countable covering of a topological space X . Recall that we say $U \subset X$ is uniformly covered, if $p^{-1}(U)$ is the union of disjoint open subsets $\{\tilde{U}_i\}_{i \in I}$ with the property, that the restriction of p to each subset $\tilde{U}_i$ is a homeomorphism. If U is uniformly covered, then every open subset of U is also uniformly covered.

Let $\mathcal{B}$ be a countable basis for the topology of X . We denote by $\mathcal{U} = \{U_j\}_{j \in J}$ the family of all open subsets of X that are uniformly covered. Since p is a covering we have $\bigcup_{j \in J} U_j = X$. It follows from Lemma 1.22 that

$$\mathcal{B}' := \{B \in \mathcal{B} \mid \text{there exists a } j \in J \text{ with } B \subset U_j\}$$

is also a basis for the topology of X .

Now let $B \in \mathcal{B}'$. Since open subsets of uniformly covered subsets are uniformly covered we know that $p^{-1}(B)$ is the union of disjoint open sets B_i , $i \in I_B$ such that the restriction of p to each B_i is a homeomorphism. Since p is a countable covering the index set I_B is countable.

Claim.

$$\tilde{\mathcal{B}} := \{B_i \mid B \in \mathcal{B}' \text{ and } i \in I_B\}$$

is a basis for the topology of $\tilde{X}$.

³³⁹Recall that a covering $p: \tilde{X} \rightarrow X$ of a connected topological space is called countable if the index, i.e. the cardinality of the preimage of a point in X , is countable.

³⁴⁰Recall that a map $f: M \rightarrow N$ between two manifolds is called a *local diffeomorphism* if given any $P \in M$ there exists an open neighborhood U of P and open neighborhood V of $f(P)$ such that $f: U \rightarrow V$ is a diffeomorphism.

It is straightforward to see that $\tilde{\mathcal{B}}$ has the basis property. Every set in $\tilde{\mathcal{B}}$ is of course open in $\tilde{X}$. By the remark on page 41 it now suffices to show that given any open set W of $\tilde{X}$ and any point $x \in W$ there exists a set B_i in $\tilde{\mathcal{B}}$ with $x \in B \subset W$. So let $x \in W$. Since $\mathcal{B}'$ is a basis for the topology of X there exists a $B \in \mathcal{B}'$ with $p(x) \in B \subset p(W)$. But then there exists an $i \in I_B$ with $x \in B_i \subset W$. This completes the proof of the claim.

Since $\mathcal{B}'$ is countable and since each I_B is countable we see that $\tilde{\mathcal{B}}$ consists of countably many open sets. This shows that $\tilde{X}$ is second-countable. $\square$

Now we can provide the proof of Proposition 16.1.

Proof of Proposition 16.1. Let $p: \tilde{X} \rightarrow X$ be a countable covering of a connected topological space X .

- (1) First we note that it follows from the hypothesis that B is a topological manifold that B is Hausdorff. It follows from Lemma 6.4 (3) that $\tilde{X}$ is also Hausdorff. Similarly it follows from our hypothesis that B is a topological manifold and that p is a countable covering together with Lemma 16.2 that $\tilde{X}$ is second-countable.

Since $p: \tilde{X} \rightarrow X$ is a covering there exists a family of open subsets $\{U_i\}_{i \in I}$ of X such that each U_i is uniformly covered. Furthermore, since X is an n -dimensional manifold there exists an atlas $\{\Phi_j: V_j \rightarrow W_j\}_{j \in J}$ for X . After possibly replacing each V_j by the family $\{V_j \cap U_i\}_{i \in I}$ we can without loss of generality assume that each V_j is uniformly covered. Thus for each j the preimage $p^{-1}(V_j)$ is the union of disjoint open subsets $\{\tilde{V}_{ji}\}_{i \in I_j}$ with the property, that the restriction of p to each subset $\tilde{V}_{ji}$ is a homeomorphism. It is now clear that $\{\Phi_j \circ p: \tilde{V}_{ji} \rightarrow W_j \mid j \in J \text{ and } i \in I_j\}$ is an atlas for $\tilde{X}$.

Finally suppose that X has no boundary, by definition this means that X has no chart of type (i) in the sense of page 42. It is straightforward to see that this implies that $\tilde{X}$ does not have a chart of type (i) either.³⁴¹ Thus $\tilde{X}$ also has no boundary.

- (2) The transition maps of the atlas from (1) are precisely given by restrictions to open subsets of the transition maps of the atlas $\{\Phi_j: V_j \rightarrow W_j\}_{j \in J}$ for X . Thus if we start out with a smooth atlas for X , then the atlas we just constructed for $\tilde{X}$ is also smooth.

We also know that a covering map is a local homeomorphism. It follows immediately from the definitions that with this smooth atlas for $\tilde{X}$ these homeomorphisms are in fact diffeomorphisms.

- (3) The last statement follows easily from the fact that p is a local diffeomorphism. $\square$

Proposition 16.1 now raises the following question. Suppose we start out with a surface F of genus g and a finite index subgroup Γ of $\pi_1(F)$, then by Proposition 15.5 we obtain a corresponding finite covering $p: \tilde{F} \rightarrow F$. By Proposition 16.1 and Lemma 6.4 (4) the

³⁴¹Put differently, if $\tilde{X}$ has a chart of type (i), then using the fact that $p: \tilde{X} \rightarrow X$ is a local homeomorphism we can also easily construct a chart of type (i) for X .

topological space $\tilde{F}$ is again a connected, closed, orientable 2-dimensional manifold. From Theorem 11.9 we know that $\tilde{F}$ is again surface of some genus h . How can we determine h from g and Γ ?

For future reference we record the question.

Question 16.3. *Given a surface F of genus g and a finite index subgroup Γ of $\pi_1(F)$, what is the genus of the corresponding finite cover?*

It will take a while till we can give a satisfactory answer. But we want to discuss one example in detail. We consider the surface F of genus two that we view as the connect sum of two tori $S = S^1 \times S^1$ and $T = S^1 \times S^1$. We denote by u, v the standard generators for $\pi_1(S)$ given by the two copies of S^1 in each torus. Similarly we denote by x, y the standard generators for $\pi_1(T)$. We pick an embedded closed disk in S and an embedded closed disk in T which both lie in the exterior of the standard generators. We denote the corresponding open disks by D and E and we write $S^* = S \setminus D$ and $T^* = T \setminus E$. We build the connected sum $S \# T$ by gluing S^* to T^* along the boundary. Furthermore, by a slight abuse of notation, we denote by u, v and x, y also the generators of $\pi_1(S^*) = \langle u, v \rangle$ and $\pi_1(T^*) = \langle x, y \rangle$.

By the discussion on page 193 the abelianization of $\pi_1(S \# T) = \pi_1(S^* \cup_{\partial S^* = \partial T^*} T^*)$ is the free abelian group on u, v, x, y . Now we consider the epimorphism

$$\varphi: \pi_1(S \# T) \rightarrow \pi_1(S \# T) / [\pi_1(S \# T), \pi_1(S \# T)] \rightarrow \mathbb{Z}_3$$

given by $u \mapsto 1, v \mapsto 0, x \mapsto 0$ and $y \mapsto 0$. Our goal in the following discussion is to visualize the covering $p: \tilde{F} \rightarrow F = S \# T$ corresponding to $\ker(\varphi)$. Since φ is an epimorphism onto a group of order 3 we see that $\ker(\varphi)$ is a subgroup of index 3. By Lemma 6.15 this implies that p is a 3-fold covering.

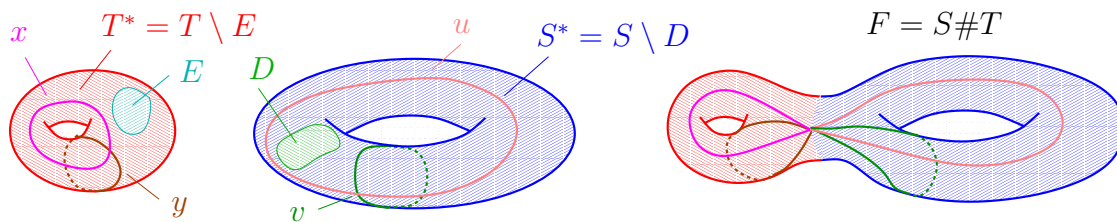


FIGURE 153.

We first consider the torus $S = S^1 \times S^1$ and furthermore we consider the epimorphism $\psi: \pi_1(S) = \langle u, v \mid [u, v] \rangle \rightarrow \mathbb{Z}_3$ given by $\psi(u) = 1$ and $\psi(v) = 0$. The covering corresponding

to $\ker(\psi) \subset \pi_1(S)$ is equivalent to the covering³⁴²

$$\begin{aligned} \tilde{S} = S^1 \times S^1 &\rightarrow S = S^1 \times S^1 \\ (a, b) &\mapsto (a^3, b). \end{aligned}$$

Now we consider the epimorphism $\psi^*: \pi_1(S^*) \rightarrow \mathbb{Z}_3$ given by $\psi^*(u) = 1$ and $\psi^*(v) = 0$. By the argument of Footnote 342 the covering corresponding to $\ker(\psi^*)$ is then given by $p: \tilde{S}^* := \tilde{S} \setminus p^{-1}(D) \rightarrow S^* = S \setminus D$. Recall that we perform the connected sum operation

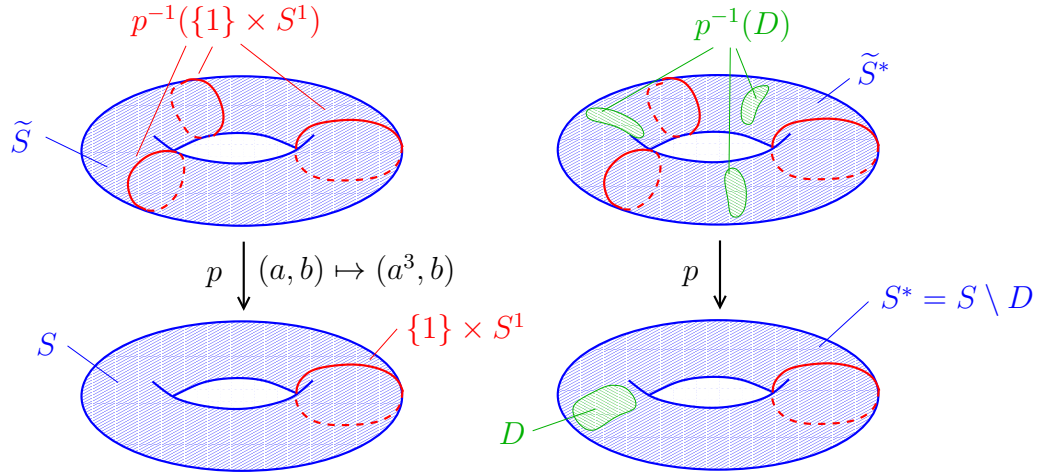


FIGURE 154.

$S \# T$ by gluing $T^* := T \setminus E$ to $S^* = S \setminus D$ to obtain $F = S \# T$. The same way we can also glue three copies of T^* , i.e. $T^* \times \{1, 2, 3\}$, to the three boundary components of $\tilde{S}^*$ to obtain a closed manifold $\tilde{F}$. The map

$$\begin{aligned} q: \tilde{F} &\rightarrow F \\ P &\mapsto \begin{cases} p(P), & \text{if } p \in \tilde{S}^*, \\ Q, & \text{if } P = Q \times \{n\} \text{ with } Q \in T^* \text{ and } n \in \{1, 2, 3\} \end{cases} \end{aligned}$$

is easily seen to be continuous. Similar to Footnote 342 one can now see that $q: \tilde{F} \rightarrow F$ is indeed the covering of F corresponding to $\ker(\varphi)$. This construction is illustrated in Figure 155 which also shows that $\tilde{F}$ is a surface of genus 4.

³⁴²Indeed, since $\tilde{S} \rightarrow S$ is a connected 3-fold covering we know from Lemma 6.15 that $p_*(\pi_1(\tilde{S}))$ is a subgroup of $\pi_1(S)$ of index three. Furthermore it is contained in the subgroup $\ker(\pi_1(S) \xrightarrow{\psi} \mathbb{Z}_3)$ which also has index three, thus $p_*(\pi_1(\tilde{S})) = \ker(\psi: \pi_1(S) \rightarrow \mathbb{Z}_3)$. But by Proposition 15.6 all path-connected coverings that correspond to the same subgroup are equivalent.

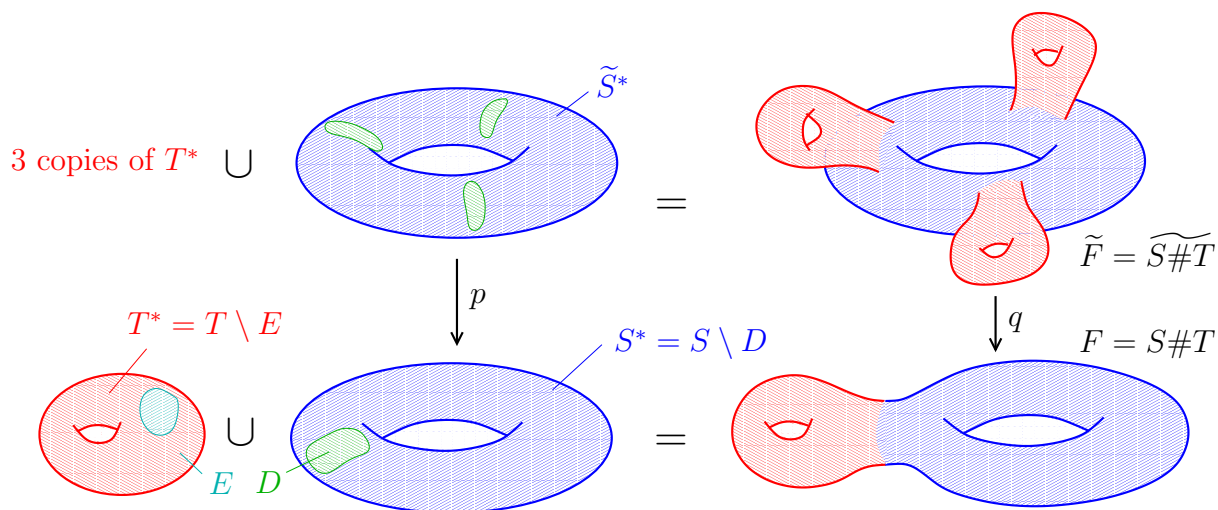


FIGURE 155.

16.2. The orientation cover of a non-orientable manifold. We had seen that many of our examples of non-orientable manifolds admit a 2-fold covering such that the covering space is an orientable manifold.

- (1) For example the projection map $p: S^2 \rightarrow \mathbb{RP}^2 = S^2/\pm 1$ shows that the real projective plane is covered by the sphere.
- (2) On page 94 we had considered the map

$$\begin{array}{ccc}
 \text{annulus} & & \text{Möbius band} \\
 p: \overbrace{\mathbb{R} \times [-1, 1]/(x, y) \sim (x + 1, y)} & \rightarrow & \overbrace{\mathbb{R} \times [-1, 1]/(x, y) \sim (x + \frac{1}{2}, 1 - y)} \\
 [(x, y)] & \rightarrow & [(x, y)]
 \end{array}$$

and we had seen in exercise sheet 7 that this is a 2-fold covering. Thus the non-orientable Möbius band admits a 2-fold covering such that the covering space is orientable. An alternative argument was also given on page 216.

- (3) Almost the same map as in (2) also shows that the torus is a 2-fold covering of the Klein bottle. An alternative argument was also given on page 216.

We will now see that any non-orientable manifold admits a 2-fold covering such that the covering space is orientable.

Proposition 16.4. *Let M be a connected non-orientable manifold. Then there exists a connected 2-fold covering $p: \widetilde{M} \rightarrow M$ such that $\widetilde{M}$ is orientable.*

We call $p: \widetilde{M} \rightarrow M$ an *orientation cover* of M . One can in fact show that all orientation covers of a connected non-orientable manifold are equivalent, but we will not make use of this fact.

Example. By Proposition 11.5 the connected sum $F = \Sigma_g \# \mathbb{RP}^2$ of the surface of genus g with the real projective plane is non-orientable. So according to Proposition 16.4 it admits a 2-fold covering $\tilde{F} \rightarrow F$ that is orientable. This raises the following question: what is the genus of $\tilde{F}$?

In the proof of Proposition 16.4 we will need the following lemma that we had already proved, with a slightly different language, in Analysis IV.

Lemma 16.5. *Let N be a connected manifold. If two orientations agree at a point, then they agree everywhere.*

Proof. Given two orientations on a manifold N it is straightforward to show that the set of points U where the orientations agree is open and that similarly that the set of points V where the orientations disagree is open. Since $N = U \sqcup V$ it follows from the hypothesis that N is connected that if $U \neq \emptyset$, then $U = N$. $\square$

Now we can provide the proof of Proposition 16.4.

Proof of Proposition 16.4. Let M be a connected non-orientable n -dimensional manifold. We consider the set

$$\widetilde{M} = \{(Q, O) \mid Q \in M \text{ and } O \text{ is an orientation of } T_Q M\}$$

together with the map

$$\begin{aligned} p: \widetilde{M} &\rightarrow M \\ (Q, O) &\mapsto Q. \end{aligned}$$

We now have to put a topology on $\widetilde{M}$. We use the following notation:

- (1) We denote by $O_{\mathbb{R}^n}$ the standard orientation of $\mathbb{R}^n$.
- (2) Given an orientation O for an n -dimensional vector space we denote by $-O$ the unique other orientation.
- (3) Given an isomorphism $\varphi: V \rightarrow W$ between vector spaces and given an orientation O on W we denote by φ^*O the corresponding orientation on V .

Given a chart $\Phi: U \rightarrow V$ of M and $\epsilon \in \{-, +\}$ we define³⁴³

$$B_\epsilon(\Phi) := \{(Q, \epsilon \cdot \Phi^*O_{\mathbb{R}^n}) \mid Q \in U\} \subset \widetilde{M}.$$

It is straightforward to show that the collection of these sets $B_\pm(\Phi)$ satisfies the basis properties (B1) and (B2) from page 40.³⁴⁴ We now endow $\widetilde{M}$ with the topology generated by the sets $B_\pm(\Phi)$. We leave it as an exercise to show that $p: \widetilde{M} \rightarrow M$ is continuous. Next we show that p is a 2-fold covering. So let $Q \in M$. We pick a connected chart³⁴⁵ $\Phi: U \rightarrow V$ around Q . Then $p^{-1}(U) = B_-(\Phi) \cup B_+(\Phi)$. It follows from Lemma 16.5 applied to the connected manifold U that $B_-(\Phi)$ and $B_+(\Phi)$ are disjoint. Furthermore it is

³⁴³Strictly speaking we should write $(d_P\Phi)^*O_{\mathbb{R}^n}$ instead of just $\Phi^*O_{\mathbb{R}^n}$.

³⁴⁴OK, so why does it satisfy (B2)?

³⁴⁵Here we say a chart $\Phi: U \rightarrow V$ is connected if U is connected.

straightforward to see that the two projections maps $p: B_{\pm}(\Phi) \rightarrow U$ are homeomorphisms. This shows that U is uniformly covered by two sets, i.e. p is a 2-fold covering.

In the proof of Proposition 16.1 we had seen that a smooth atlas for $\widetilde{M}$ is given by all maps $B_{\pm}(\Phi) \xrightarrow{p} U \xrightarrow{\Phi} V$ for all connected charts $\Phi: U \rightarrow V$ of M . In particular $\widetilde{M}$ is again an n -dimensional manifold and with this atlas the projection map $p: \widetilde{M} \rightarrow M$ is a local diffeomorphism.

Now we show that $\widetilde{M}$ is orientable. So let $(Q, O) \in \widetilde{M}$. The map $p: \widetilde{M} \rightarrow M$ is a local diffeomorphism. We now equip $T_{(Q, O)}\widetilde{M}$ with the orientation p^*O .³⁴⁶ By Footnote 53 it suffices to consider the continuity of the orientation with respect to the above atlas. So let $\Phi: U \rightarrow V$ be a connected chart of M . Let $\epsilon \in \{-, +\}$ and denote by $\Phi_{\epsilon}: B_{\pm}(\Phi) \rightarrow V$ the corresponding chart of $\widetilde{M}$. But then by construction we know that Φ_{ϵ} is orientation-preserving for all points if $\epsilon = +$ and it is orientation-reversing for all points if $\epsilon = -$.

Finally we show that $\widetilde{M}$ is connected. If it was not connected, then it would consist of at least two components $\{\widetilde{M}_i\}_{i \in I}$. Since M is connected the restriction of each $p: \widetilde{M} \rightarrow M$ to $p_i: \widetilde{M}_i \rightarrow M$ would also be a covering. Clearly $2 = [\widetilde{M} : M] = \sum_{i \in I} [\widetilde{M}_i : M]$. It follows that $\#I = 2$ and that each $p_i: \widetilde{M}_i \rightarrow M$ is a covering of degree one, i.e. each p_i is a homeomorphism. Since p_i is a local diffeomorphism it follows that $p_i: \widetilde{M}_i \rightarrow M$ is in fact a diffeomorphism. But that is not possible since M was assumed to be non-orientable whereas $\widetilde{M}_i$ is orientable. $\square$

Corollary 16.6.

- (1) *If M is a connected non-orientable manifold, then there exists an epimorphism $\pi_1(M) \rightarrow \mathbb{Z}_2$.*
- (2) *Any simply-connected manifold is orientable.*

Proof. The second statement of the corollary is an immediate consequence of the first statement. Thus it suffices to prove the first statement.

Let M be a connected non-orientable manifold. By Proposition 16.4 there exists a connected 2-fold covering $p: \widetilde{M} \rightarrow M$ such that $\widetilde{M}$ is orientable. It follows from Lemma 6.15 that $p_*(\pi_1(\widetilde{M}))$ is an index 2 subgroup of $\pi_1(M)$. In exercise sheet 14 we will show that every index 2 subgroup of a group π is normal. In particular $p_*(\pi_1(\widetilde{M}))$ is normal index 2 subgroup of $\pi_1(M)$. Thus we obtain an epimorphism $\pi_1(M) \rightarrow \Gamma := \pi_1(M)/p_*(\pi_1(\widetilde{M}))$. The group Γ has two elements, but any group with two elements is isomorphic to $\mathbb{Z}_2$. Thus we found the desired epimorphism $\pi_1(M) \rightarrow \mathbb{Z}_2$. $\square$

³⁴⁶Put differently, using p we can identify $T_{(Q, O)}\widetilde{M}$ with $T_Q M$ and we equip it with the orientation at hand, namely O .

17. COMPLEX MANIFOLDS

Eventually our goal will be to give an explicit description of the universal cover of the surfaces of genus $g \geq 2$. This will require a surprisingly long journey through complex manifolds, Riemannian manifolds and hyperbolic geometry. In this chapter we start out with a very short introduction to complex manifolds.

First we recall the definition of a holomorphic function.

Definition. Let $U \subset \mathbb{C}$ be an open subset. A function $f: U \rightarrow \mathbb{C}$ is *holomorphic* if for any $z_0 \in U$ the limit

$$\frac{d}{dz}f(z_0) := f'(z_0) := \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0} \in \mathbb{C}$$

exists.

Example. We had seen in Analysis III that polynomials, the exponential function, and more generally functions defined by power series are holomorphic. Moreover products, fractions, sums and compositions of holomorphic functions are again holomorphic.

In Analysis III we had shown that holomorphic functions have many surprising properties. We now recall three of the main results from Analysis III. These statements are also proved in any standard textbook on complex analysis, see e.g. [J2] or [La].

Theorem 17.1. (Liouville's Theorem) *Every bounded holomorphic map $f: \mathbb{C} \rightarrow \mathbb{C}$ is constant.*

Theorem 17.2. (Maximum Principle) *Let U be an open connected subset of $\mathbb{C}$ and let $f: U \rightarrow \mathbb{C}$ be a holomorphic function. If the function $|f|: U \rightarrow \mathbb{R}_{\geq 0}$ has a local maximum, then f is constant.*

Proposition 17.3. (Schwarz Reflection Principle) *Let U be an open subset of the upper half-plane $\{z \in \mathbb{C} \mid \operatorname{Im}(z) \geq 0\}$ and let $f: U \rightarrow \mathbb{C}$ be a continuous function with the following properties:*

- (1) f is holomorphic on $\mathring{U} = \{z \in U \mid \operatorname{Im}(z) > 0\}$,
- (2) f only assumes real values on $U \cap \mathbb{R}$.

We set $U' := \{\bar{z} \mid z \in U\}$, i.e. U' is the reflection of U in the x -axis. Then the function

$$\begin{aligned} \tilde{f}: U \cup U' &\rightarrow \mathbb{C} \\ z &\mapsto \begin{cases} f(z), & \text{if } z \in U, \\ \overline{f(\bar{z})}, & \text{if } z \in U', \end{cases} \end{aligned}$$

is holomorphic.

Given a topological manifold we now introduce the notion of a holomorphic atlas and a complex manifold. The definition is almost the same as for a smooth atlas and a manifold, except that we have to replace “smooth” by “holomorphic”.

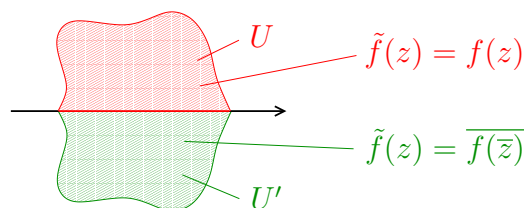


FIGURE 156. Illustration of the Schwarz reflection principle.

Definition.

- (1) Let $U \subset \mathbb{C}^m$ be open. We say that a map $f = (f_1, \dots, f_n): U \rightarrow \mathbb{C}^n$ is *holomorphic* if for any $i \in \{1, \dots, m\}$, $j \in \{1, \dots, n\}$ and any $(w_1, \dots, w_{i-1}, w_{i+1}, \dots, w_m) \in \mathbb{C}^{m-1}$ the map

$$\overbrace{\{z \in \mathbb{C} \mid (w_1, \dots, w_{i-1}, z, w_{i+1}, \dots, w_m) \in U\}}^{\text{open subset of } \mathbb{C}} \rightarrow \mathbb{C} \\ z \mapsto f_j(w_1, \dots, w_{i-1}, z, w_{i+1}, \dots, w_m)$$

is holomorphic

- (2) Let M be a $2n$ -dimensional topological manifold M without boundary. An atlas $\{\Phi_i: U_i \rightarrow V_i\}_{i \in I}$ for M is called *holomorphic* if for all $i, j \in I$ the transition map

$$\Phi_j \circ \Phi_i^{-1}: \underbrace{\Phi_i(U_i \cap U_j)}_{\text{open in } \mathbb{R}^{2n} = \mathbb{C}^n} \rightarrow \underbrace{\Phi_j(U_i \cap U_j)}_{\subset \mathbb{R}^{2n} = \mathbb{C}^n}$$

is holomorphic.

- (3) An *n -dimensional complex manifold* is a pair $(M, \mathcal{A})$ consisting of a $2n$ -dimensional topological manifold M without boundary, together with a holomorphic atlas for M .

Examples.

- (1) Let U be an open subset of $\mathbb{C}^n$. The identity map is an atlas for U and it is evidently a holomorphic atlas. Thus U is an n -dimensional complex manifold.
- (2) We consider the action

$$(\mathbb{Z} \oplus \mathbb{Z}i) \times \mathbb{C} \rightarrow \mathbb{C} \\ ((a + bi), z) \mapsto z + a + bi.$$

The same argument as in the proof of Proposition 1.28 shows that $\mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}i)$ is a 1-dimensional complex manifold. The topological space $\mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}i)$ is evidently diffeomorphic to the torus $\mathbb{R}^2/\mathbb{Z}^2 = S^1 \times S^1$. Thus we can view the torus as a complex manifold.

- (3) In exercise sheet 14 we will see that the complex projective space

$$\mathbb{CP}^n = (\mathbb{C}^{n+1} \setminus \{0\})/(\mathbb{C} \setminus \{0\}),$$

that we had introduced on page 25, is a complex n -dimensional manifold. As we had already pointed out on page 25, the one-dimensional projective space $\mathbb{CP}^1$ is

homeomorphic to S^2 . This shows that S^2 can be viewed as a 1-dimensional complex manifold.

We have now seen that the torus and the 2-dimensional sphere can be viewed as complex manifolds. The following proposition shows in particular that the Klein bottle and the real projective plane $\mathbb{RP}^2$ are *not* complex manifolds.

Proposition 17.4. *An n -dimensional complex manifold, viewed as a $2n$ -dimensional real manifold, is orientable.*

Sketch of proof. Let V be an n -dimensional complex vector space. We say that a set of vectors $\{v_1, \dots, v_n\}$ in V is *totally real* if the vectors $v_1, \dots, v_n, iv_1, \dots, iv_n$ form a basis of V , viewed as a $2n$ -dimensional real vector space. It is straightforward to show that every n -dimensional complex vector space admits a set $\{v_1, \dots, v_n\}$ of totally real vectors.

Claim. If $\{v_1, \dots, v_n\}$ and $\{w_1, \dots, w_n\}$ are two sets of totally real vectors for a complex vector space V , then the bases $\{v_1, \dots, v_n, iv_1, \dots, iv_n\}$ and $\{w_1, \dots, w_n, iw_1, \dots, iw_n\}$ are equivalent in the sense of page 49.

First we consider the case $n = 1$. In this case we are given two non-zero vectors v and w . Then $v = (a + ib)w$ for some $a + bi \neq 0$. The base change matrix from the real basis $\{v, iv\}$ to the real basis $\{w, iw\}$ is given by the real 2×2 -matrix

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

Evidently the determinant of this matrix is positive. This concludes the proof of the claim for $n = 1$. The general statement is basically just an exercise in linear algebra. This exercise though turns out to be slightly harder than one might expect.³⁴⁷ A proof is for example given in [CCL, Theorem 7.2.1]. This concludes the proof of the claim.

The claim shows that an n -dimensional complex vector space, viewed as a $2n$ -dimensional real vector space, has a canonical orientation. Now let M be an n -dimensional complex manifold. It is relatively straightforward to see that each tangent space $T_P M$ is canonically an n -dimensional complex vector space. We now equip it with the above canonical orientation. These orientations of the tangent spaces define a canonical orientation on the $2n$ -dimensional real manifold M . $\square$

We had just seen that the 2-sphere and the torus are complex manifolds and we had seen that non-orientable manifolds cannot be complex manifolds. The following question now arises:

³⁴⁷The problem is the following: Let A and B be two real $n \times n$ -matrices such that $\det(A + iB) \in \mathbb{C}$ is non-zero. Why does it follow that the determinant of the real matrix

$$\begin{pmatrix} A & -B \\ B & A \end{pmatrix}$$

is positive?

Question 17.5. *Let $g \geq 2$. Does the surface of genus g have the structure of a complex manifold?*

We will discuss this question later on.

Remark. It is often very hard to determine whether or not a given orientable even-dimensional manifold can also be a complex manifold. For example, Armand Borel and Jean-Pierre Serre [BS]³⁴⁸ showed in 1953 that for $n \neq 2, 6$ the n -dimensional sphere S^n cannot be a complex manifold. On the other hand it is still an open question whether S^6 is a complex manifold.³⁴⁹

Definition. We say that a map $f: M \rightarrow N$ between two complex manifolds is *holomorphic* if f is continuous and if for every chart $\Phi: U \rightarrow V$ in the holomorphic atlas of M and every chart $\Psi: W \rightarrow Y$ in the holomorphic atlas of N the map

$$\Phi(f^{-1}(W) \cap U) \xrightarrow{\Psi \circ f \circ \Phi^{-1}} Y$$

is holomorphic.

The following proposition will be proved in exercise sheet 14 using the Maximum Principle that we had formulated above.

Proposition 17.6. *Let M be a compact 1-dimensional complex manifold. Then any holomorphic function $f: M \rightarrow \mathbb{C}$ is constant.*

Remark. If $M = \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}i)$, then we can also give another, arguably easier proof. More precisely, let $f: \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}i) \rightarrow \mathbb{C}$ be a holomorphic function. Since $\mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}i)$ is compact the function f is bounded. We denote by $p: \mathbb{C} \rightarrow \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}i)$ the projection map. Then the composition $f \circ p: \mathbb{C} \rightarrow \mathbb{C}$ is a bounded holomorphic function, thus constant by the Liouville Theorem 17.1. Since p is a surjective it follows that f is also constant.

In the next definition we generalize the notion of (local) diffeomorphism in the obvious way to the complex setting.

Definition.

- (1) We say that a map $f: M \rightarrow N$ between two complex manifolds is a *biholomorphism* if f is a bijection and if both f and f^{-1} are holomorphic.
- (2) We say that a map $f: M \rightarrow N$ between two complex manifolds is a *local biholomorphism* if given any $P \in M$ there exists an open neighborhood U of P and an open neighborhood V of $f(P)$ such that $f: U \rightarrow V$ is a biholomorphism.

³⁴⁸Armand Borel (1923–2003) was a Swiss mathematician, Jean-Pierre Serre (*1926) is a French mathematician who was awarded a Fields medal in 1954.

³⁴⁹There is a recent paper by Michael Atiyah, who got a Fields medal in 1966, claiming that the answer is no, but it seems like most mathematicians in the field are not convinced by the argument.

Examples.

- (1) In some cases one can explicitly write down a biholomorphism. For example, for the upper half-plane $\mathbb{H} = \{x + iy \mid y > 0\}$ and the open disk $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ it is straightforward to verify that the maps

$$\begin{aligned} \Phi: \mathbb{H} &\rightarrow \mathbb{D} & \Psi: \mathbb{D} &\rightarrow \mathbb{H} \\ z &\mapsto \frac{z-i}{z+i} & w &\mapsto \frac{i+iw}{1-w} \end{aligned} \quad \text{and}$$

are biholomorphisms that are inverse to one-another.

- (2) The proof of Proposition 1.28 shows that $\mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$ has the structure of a complex 1-dimensional manifold in such a way that the projection map $\mathbb{C} \rightarrow \mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$ is a local biholomorphism.
- (3) Let $a, b \in \mathbb{C}$ be two complex numbers that are linearly independent over $\mathbb{R}$. We write $T(a, b) = \mathbb{C}/(\mathbb{Z}a + \mathbb{Z}b)$. The same argument as in (2) shows that this is a 1-dimensional complex manifold such that the projection map $p: \mathbb{C} \rightarrow T(a, b)$ is a local biholomorphism. Now suppose that $v, w \in \mathbb{C}$ are also two complex numbers that are linearly independent over $\mathbb{R}$. We can form the torus $T(v, w) = \mathbb{C}/(\mathbb{Z}v + \mathbb{Z}w)$. The tori $T(a, b)$ and $T(v, w)$ are diffeomorphic. But the argument of [Ba, p. 15] shows that the 1-dimensional complex manifolds $T(a, b)$ and $T(v, w)$ are biholomorphic only if $ab^{-1} = vw^{-1} \in \mathbb{C}$ or if $ab^{-1} = wv^{-1} \in \mathbb{C}$.

Lemma 17.7.

- (1) Let $p: \tilde{X} \rightarrow X$ be a countable covering of a complex manifold X . Then $\tilde{X}$ is also a complex manifold in such a way that p is a local biholomorphism.
- (2) Let $p: \tilde{X} \rightarrow X$ be a covering of complex manifolds such that p is a local biholomorphism. If Z is a complex manifold and if $f: Z \rightarrow X$ is a holomorphic map, then any lift $\tilde{f}: Z \rightarrow \tilde{X}$ is also holomorphic.

Proof. The first statement is proved almost the same way as Proposition 16.1 (2). The second statement is an immediate consequence of the fact that p is a local biholomorphism and that being holomorphic is a local condition. We leave the details to the reader. $\square$

In general it is hard to write down an explicit biholomorphism between two complex manifolds. One of the main sources of biholomorphisms is the Riemann mapping theorem that we had already formulated in Analysis III.

Theorem 17.8. (Riemann Mapping Theorem) *Let $U \subset \mathbb{C}$ be a simply connected open subset $U \neq \mathbb{C}$, then U is biholomorphic to $\mathbb{D}$.*

This theorem is proved in [J2] or alternatively in [La, Chapter X].

In many applications it is also useful to know that the biholomorphism from the Riemann Mapping Theorem extends to the boundary of U and $\mathbb{D}$. To state the corresponding theorem we need one more definition.

Definition.

- (1) We write $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ which we equip with the topology that we had defined on page 13.
- (2) Given $U \subset \overline{\mathbb{C}}$ we denote by $\overline{U}$ the closure of U in $\overline{\mathbb{C}}$.
- (3) A *Jordan curve* is the image of an injective map $S^1 \rightarrow \overline{\mathbb{C}}$.

The following theorem was proved in 1913 by Constantin Carathéodory.³⁵⁰ A proof is for example provided in [GM, Chapter I.3].

Theorem 17.9. (Carathéodory Theorem) *Let $U \subsetneq \mathbb{C}$ be a simply connected open subset U of $\mathbb{C}$ and let $\phi: U \rightarrow \mathbb{D}$ be a biholomorphism. If $\partial\overline{U}$ is a Jordan curve, then ϕ extends to a homeomorphism $\overline{U} \rightarrow \overline{\mathbb{D}}$.*

Example. We consider the open subset U of $\mathbb{C}$ that is shown in Figure 157. It is a simply connected open proper subset of $\mathbb{C}$ and the boundary is clearly a Jordan curve. Thus it follows from the Riemann Mapping Theorem and from Carathéodory's Theorem that there exists a homeomorphism $\phi: \overline{U} \rightarrow \overline{\mathbb{D}}$ which restricts to a biholomorphism $\phi: U \rightarrow \mathbb{D}$.

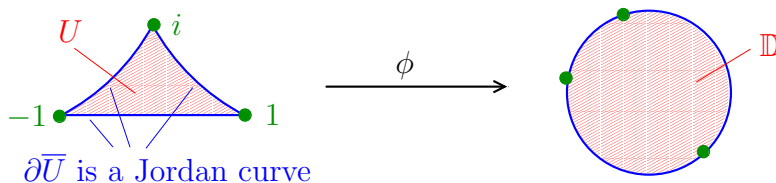


FIGURE 157.

³⁵⁰Constantin Carathéodory (1873-1950) was a Greek mathematician who was in particular a professor at the LMU in Munich.

18. HYPERBOLIC GEOMETRY

18.1. Hyperbolic space. We start out with the following definition.

Definition. Let V be a real vector space. A *symmetric form on V* is a map³⁵¹

$$g: V \times V \rightarrow \mathbb{R}$$

that is bilinear, i.e. for all $u, v, w \in V$ and $\mu, \lambda \in \mathbb{R}$ we have

$$\begin{aligned} g(\lambda u + \mu v, w) &= \lambda g(u, w) + \mu g(v, w) \\ g(u, \lambda v + \mu w) &= \lambda g(u, v) + \mu g(u, w) \end{aligned}$$

and that is symmetric, i.e. for all $v, w \in V$ we have

$$g(v, w) = g(w, v).$$

We say g is *positive-definite* if for all $v \in V \setminus \{0\}$ we have

$$g(v, v) > 0.$$

Now we can recall that following key definition from Analysis IV.

Definition. A *Riemannian structure* on a manifold M ³⁵² is a smooth³⁵³ map g which assigns to each point P a positive-definite symmetric form g_P on the tangent space $T_P M$.

In this chapter and the following chapter we are mostly interested in Riemannian structures on surfaces, and more specifically, in hyperbolic structures. We now recall the definition of hyperbolic space.

Definition.

(1) The *hyperbolic half-plane* is the manifold

$$\mathbb{H} = \{(x, y) \in \mathbb{R}^2 \mid y > 0\} = \{z = x + iy \in \mathbb{C} \mid y > 0\}$$

together with the Riemannian structure that assigns to the tangent space $T_z \mathbb{H} = \mathbb{R}^2$ at a point $z = x + iy \in \mathbb{H}$ the positive-definite symmetric form

$$\begin{aligned} g: \mathbb{R}^2 \times \mathbb{R}^2 &\rightarrow \mathbb{R} \\ (v, w) &\mapsto \frac{1}{y^2} \langle v, w \rangle \end{aligned}$$

where $\langle v, w \rangle$ denotes the usual scalar product on $\mathbb{R}^2$.

(2) A *Möbius transformation* of $\mathbb{H}$ is a map of the form³⁵⁴

$$\begin{aligned} \mathbb{H} &\rightarrow \mathbb{H} \\ z &\mapsto \frac{az + b}{cz + d} \end{aligned}$$

³⁵¹Sometimes such a structure is called a *bilinear symmetric form*, but we abbreviate this to *symmetric form*.

³⁵²If you are not comfortable with the notion of an “abstract manifold”, then for the time being you can think of “manifold” as a “submanifold of $\mathbb{R}^N$ ”.

³⁵³The notion of “smooth” in this context is made specific in Analysis IV.

³⁵⁴An elementary calculation shows that the image of a Möbius transformation $\mathbb{H} \rightarrow \mathbb{C}$ does indeed lie again in $\mathbb{H}$.

where $a, b, c, d \in \mathbb{R}$ with $ad - bc = 1$.

Example. In Figure 158 we see a decomposition of $\mathbb{H}$ into angels and devils. With respect to the hyperbolic metric all the angels have the same size and all the devils have the same size, even though they differ in the usual euclidean sense.



FIGURE 158.

Examples. We give three special types of Möbius transformations of $\mathbb{H}$:

- (1) given $r > 0$ the scalar multiplication $z \mapsto rz = \frac{\sqrt{r} \cdot z + 0}{0 \cdot z + \frac{1}{\sqrt{r}}}$ is a Möbius transformation,
- (2) given $d \in \mathbb{R}$ the horizontal translation $z \mapsto z + d = \frac{1 \cdot z + d}{0 \cdot z + 1}$ is a Möbiustransformation,
- (3) the map $z \mapsto -\frac{1}{z} = \frac{0 \cdot z + 1}{(-1) \cdot z + 0}$ is a Möbius transformation.

The third Möbius transformation is the composition of taking the inverse $z \mapsto \frac{1}{z}$ and the reflection $z \mapsto -z$ in the origin. We illustrate this map in Figure 159. For simplicity we refer to the map $z \mapsto -\frac{1}{z}$ as an inversion.

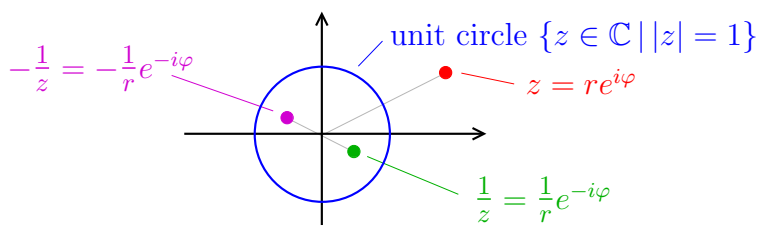


FIGURE 159.

Proposition 18.1. *Every Möbius transformation of $\mathbb{H}$ is the composition of scalar multiplications, horizontal translations and inversions.*

Proof. Let $a, b, c, d \in \mathbb{R}$ with $ad - bc = 1$ and denote by

$$\Phi(z) = \frac{az + b}{cz + d}$$

the corresponding Möbius transformation. If $c = 0$, then

$$\Phi(z) = \frac{a}{d}z + \frac{b}{d}$$

is the composition of the scalar multiplication $z \mapsto \frac{a}{d}z$ with the horizontal translation $z \mapsto \frac{b}{d} + z$. If $c \neq 0$, then

$$\Phi(z) = \frac{az + b}{cz + d} = \frac{acz + bc}{c^2z + dc} = \frac{acz + ad - 1}{c^2z + cd} = \frac{-1 + a(cz + d)}{c^2z + cd} = \frac{-1}{c^2z + dc} + \frac{a}{c}.$$

$\uparrow$
since $ad - bc = 1$

Thus we see that Φ is the composition of the following maps

$$z \mapsto c^2z, \quad z \mapsto z + dc, \quad z \mapsto \frac{-1}{z} \quad \text{und} \quad z \mapsto z + \frac{a}{c}.$$

□

Definition.

- (1) The *hyperbolic disk*³⁵⁵ is the manifold

$$\mathbb{D} = \{z \in \mathbb{C} \mid |z|^2 < 1\}$$

together with the Riemannian structure which assigns to the tangent space $T_z\mathbb{D} = \mathbb{R}^2$ at a point $z \in \mathbb{D}$ the positive-definite symmetric form

$$g_z: \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$$

$$(v, w) \mapsto \frac{2}{(1 - |z|^2)^2} \langle v, w \rangle$$

where, as above, $\langle v, w \rangle$ denotes the usual scalar product on $\mathbb{R}^2$.

- (2) A *Möbius transformation* of $\mathbb{D}$ is a map of the form³⁵⁶

$$\mathbb{D} \rightarrow \mathbb{D}$$

$$z \mapsto e^{i\theta} \cdot \frac{z - a}{1 - \bar{a}z}$$

for some $\theta \in \mathbb{R}$ and $a \in \mathbb{D}$.

Examples.

- (1) In Figure 160 we see three different decompositions of $\mathbb{D}$ into subsets. In each of the three cases these subsets have the same hyperbolic size, even though they differ in the usual euclidean sense.
- (2) If we set $a = 0$ in the definition of a Möbius transformation of $\mathbb{D}$, then we obtain precisely the rotation by the angle θ . Put differently, the Möbius transformations preserving 0 are precisely given by rotations around the origin.

³⁵⁵The hyperbolic disk is often also called the *Poincaré model for hyperbolic space*.

³⁵⁶An elementary calculation shows that the image of a Möbius transformation $\mathbb{D} \rightarrow \mathbb{C}$ does indeed lie again in $\mathbb{D}$.

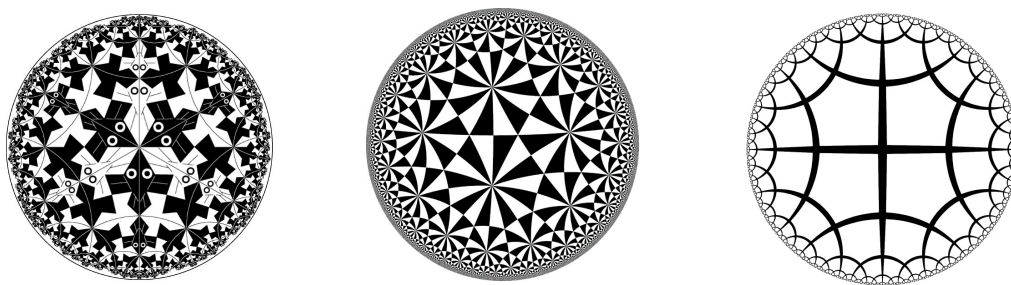


FIGURE 160.

Before we can state the next proposition we recall a few more definitions from Analysis IV.

Definition.

- (1) Given a smooth map $f: M \rightarrow N$ between manifolds and given $P \in M$ we denote by $df_P = f_*: T_P M \rightarrow T_{f(P)} N$ the differential, i.e. the induced homomorphism from the tangent space $T_P M$ to the tangent space $T_{f(P)} N$.
- (2) Let $f: M \rightarrow N$ be a local diffeomorphism. Given a Riemannian structure h on N we denote by f^*h the Riemannian structure on M that at each point $P \in M$ is given by $(f^*h)(v, w) = h(f_*v, f_*w)$.³⁵⁷
- (3) We say that a map $f: (M, g) \rightarrow (N, h)$ between Riemannian manifolds is a (*local*) *isometry* if f is a (local) diffeomorphism and if $g = f^*h$.
- (4) We say that Riemannian manifolds (M, g) and (N, h) are *isometric* if there exists an isometry from (M, g) to (N, h) .

The following proposition will allow us to go back and forth between the hyperbolic half-plane and the hyperbolic disk.

Proposition 18.2. *The maps*

$$\begin{array}{ccc} \Phi: \mathbb{H} & \rightarrow & \mathbb{D} \\ z & \mapsto & \frac{z-i}{z+i} \end{array} \quad \text{and} \quad \begin{array}{ccc} \Psi: \mathbb{D} & \rightarrow & \mathbb{H} \\ w & \mapsto & \frac{i+iw}{1-w} \end{array}$$

have the following properties:

- (1) Φ and Ψ are inverses to one another, in particular they are biholomorphisms,
- (2) they extend to homeomorphisms between $\overline{\mathbb{H}} = \{z \in \mathbb{C} \mid \operatorname{Im}(z) \geq 0\} \cup \{\infty\}$ ³⁵⁸ and $\overline{\mathbb{D}} = \{z \in \mathbb{C} \mid |z| \leq 1\}$,
- (3) they are orientation-preserving,³⁵⁹
- (4) they are isometries,

³⁵⁷We had shown in Analysis IV that f^*h is indeed a Riemannian structure on M . Note that in the proof, to show that f^*h is indeed non-degenerate, we need that f is a local diffeomorphism.

³⁵⁸Here $\overline{\mathbb{H}}$ has the same topology as defined on page 13.

³⁵⁹The hyperbolic half-plane $\mathbb{H}$ and the hyperbolic disk $\mathbb{D}$ are both subsets of $\mathbb{C} = \mathbb{R}^2$ and inherit from $\mathbb{C} = \mathbb{R}^2$ the canonical orientation.

(5) *they give a bijection between the Möbius transformations of $\mathbb{D}$ and $\mathbb{H}$.*³⁶⁰

Proof. All five statements follow from a straightforward calculation. The fact that Φ and Ψ are orientation-preserving isometries is precisely the statement of Satz 14.13 of Analysis IV where we gave the details of the calculation. $\square$

Remark. Proposition 18.2 says that $\mathbb{D}$ and $\mathbb{H}$ are two equivalent ways to describe hyperbolic space. In fact there are several other ways to describe hyperbolic space, for example we could also use the Beltrami-Klein model

https://en.wikipedia.org/wiki/Beltrami-Klein_model

or the hyperboloid model

https://en.wikipedia.org/wiki/Hyperboloid_model.

Some of the connections between these models are explained in the following video of Henry Segerman and Saul Schleimer:

https://www.youtube.com/watch?v=eGEQ_UuQtYs.

The following proposition summarizes some of the key properties of Möbius transformations.

Proposition 18.3.³⁶¹

- (1) *Compositions of Möbius transformations are again Möbius transformations and the inverse of a Möbius transformation is again a Möbius transformation.*
- (2) *Möbius transformations are biholomorphisms, they are isometries and they are orientation-preserving.*
- (3) *Möbius transformations act transitively, i.e. given any two points P and Q there exists a Möbius transformation which sends P to Q .*
- (4) *Given $P, Q \in \mathbb{D}$ and non-zero vectors $v \in T_P\mathbb{D}$ and $w \in T_Q\mathbb{D}$ with $\|v\|_g = \|w\|_g$ there exists a Möbius transformation ϕ with $\phi(P) = Q$ and $d\phi_P(v) = w$.*
- (5) *Let ϕ and ψ be two Möbius transformations. Suppose there exists a $P \in \mathbb{D}$ such that $\phi(P) = \psi(P)$ and such that $d\phi_P = d\psi_P$. Then $\phi = \psi$.*

Proof. By Proposition 18.2 it suffices to prove the statements for $\mathbb{D}$ or for $\mathbb{H}$, whichever is more convenient.

- (1),(2) The first two statements follow from a straightforward calculation, we refer to Lemma 14.1 and Lemma 14.3 of Analysis IV for details.

³⁶⁰More precisely, if α is a Möbius transformation of $\mathbb{H}$, then $\Phi \circ \alpha \circ \Psi$ is a Möbius transformation of $\mathbb{D}$ and conversely, if α is a Möbius transformation of $\mathbb{D}$, then $\Psi \circ \alpha \circ \Phi$ is a Möbius transformation of $\mathbb{H}$.

³⁶¹The statement of the proposition applies to Möbius transformations of $\mathbb{D}$ and to Möbius transformations of $\mathbb{H}$.

- (3) We prove this statement for $\mathbb{D}$. By (1) it suffices to show that given any $P \in \mathbb{D}$ there exists a Möbius transformation ψ with $\psi(P) = 0$. But such a Möbius transformation is given by

$$\psi(z) = \frac{z-P}{1-\overline{P}z}.$$

- (4) Using (3) we see that we only have to deal with the case that $P = Q = 0$. But then the desired Möbius transformation is given by a rotation.
- (5) By (3) there exists a Möbius transformation θ such that $\theta(0) = P$ and there exists a Möbius transformation ω such that $\omega(\phi(P)) = \omega(\psi(P)) = 0$. We consider the Möbius transformations $\phi' = \omega \circ \phi \circ \theta$ and $\psi' = \omega \circ \psi \circ \theta$. It follows from our hypothesis that both ϕ' and ψ' fix 0. As we had pointed out above, the only Möbius transformations of $\mathbb{D}$ that fix 0 are rotations.

By our hypothesis we have $d\phi'_0 = d\psi'_0$, which means that the rotations ϕ' and ψ' agree. But this implies that ϕ and ψ also agree. $\square$

Note that Möbius transformations extend in an obvious way uniquely to homeomorphisms of $\overline{\mathbb{D}} = \{z \in \mathbb{C} \mid |z| \leq 1\}$ and $\overline{\mathbb{H}} = \{z \in \mathbb{C} \mid \operatorname{Im}(z) \geq 0\} \cup \{\infty\}$. We will denote the extension of a Möbius transformation with the same symbol.

Lemma 18.4. *Let $\{P, Q, R\}$ and let $\{S, T, U\}$ be two sets of three distinct points on the boundary $\partial\overline{\mathbb{H}} = \mathbb{R} \cup \{\infty\}$. Then there exists a Möbius transformation ψ of $\mathbb{H}$ with $\psi(P) = S$ and $\psi(\{Q, R\}) = \{T, U\}$.*

Proof. Let P, Q, R be three distinct points on $\partial\overline{\mathbb{H}} = \mathbb{R} \cup \{\infty\}$. It suffices to show that there exists a Möbius transformation ψ with $\psi(P) = \infty$ and with $\psi(\{Q, R\}) = \{0, 1\}$. We proceed as follows:

- (1) We first claim that there exists a Möbius transformation α with $\alpha(P) = \infty$. Indeed, if $P = \infty$ then we just take $\alpha = \operatorname{id}$. Otherwise we can combine a horizontal translation with the inversion to obtain a Möbius transformation α with $\alpha(P) = \infty$. Given such α we write $Q' = \alpha(Q)$ and $R' = \alpha(R)$. Note that the points Q', R' lie on $\mathbb{R} = \overline{\mathbb{H}} \setminus \{\infty\}$. Without loss of generality we can assume that $Q' < R'$.
- (2) Next we pick a horizontal translation such that $\beta(Q') = 0$. Note that we still have $(\beta \circ \alpha)(P) = \beta(\infty) = \infty$. Also note that it follows from $Q' < R'$ that $\beta(R') > 0$.
- (3) Finally we pick a scalar multiplication γ such that $(\gamma \circ \beta)(R') = 1$. Note that γ fixes 0 and ∞ . Thus we see that $\psi = \gamma \circ \beta \circ \alpha$ has the desired property. $\square$

Definition. A *hyperbolic line* is a subset of $\mathbb{D}$ of one of the following two types:

- (1) it is the intersection of $\mathbb{D}$ with a euclidean line through the origin,
- (2) it is the intersection of $\mathbb{D}$ with a euclidean circle that intersects the circle $S^1 = \partial\overline{\mathbb{D}}$ orthogonally.

In each of the two cases we refer to the intersection of the euclidean object with $S^1 = \partial\overline{\mathbb{D}}$ as the *endpoints* of the hyperbolic line. We refer to Figure 161 for an illustration.

Lemma 18.5.

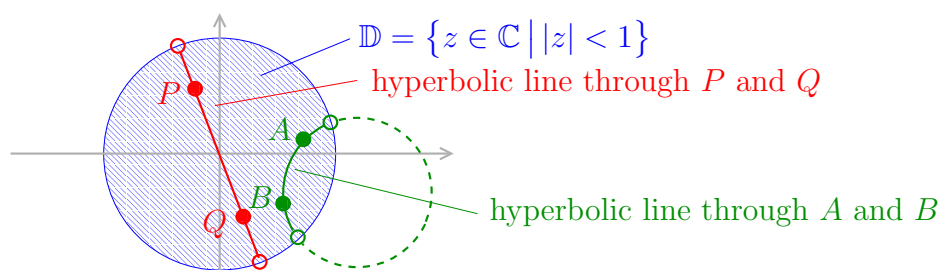


FIGURE 161.

- (1) For any two distinct points P and Q in $\mathbb{D}$ there exists a unique hyperbolic line that contains P and Q .
- (2) For any two distinct points P and Q in $\partial\mathbb{D}$ there exists a unique hyperbolic line with P and Q as endpoints.
- (3) Let P and Q be two distinct points in $\partial\mathbb{D}$. Every Möbius transformation ϕ sends the hyperbolic line with endpoints P and Q to the hyperbolic line with endpoints $\phi(P)$ and $\phi(Q)$.
- (4) Möbius transformations act transitively on hyperbolic lines, i.e. given any two hyperbolic lines g and h there exists a Möbius transformation ϕ with $\phi(g) = h$.

Proof. In principle one can prove the lemma directly in $\mathbb{D}$. But it is easier to prove it using $\mathbb{H}$. In $\mathbb{H}$ the hyperbolic lines are defined to be vertical lines $\{x + iy \mid y > 0\}$ and euclidean half-circles such that the origin lies on $\mathbb{R}$. It is straightforward to see that the diffeomorphisms of Proposition 18.2 define a bijection between the hyperbolic lines in $\mathbb{D}$ and in $\mathbb{H}$.

The corresponding statements for hyperbolic lines in $\mathbb{H}$ are straightforward to prove. The (easy) details are given in Lemmas 14.6 and 14.8 and Satz 14.7 in Analysis IV. $\square$

18.2. Angles in Riemannian manifolds. We continue our quick introduction to Riemannian manifolds and hyperbolic space with a short discussion of angles.

Definition.

- (1) Let g be a positive-definite symmetric form on a real vector space V . Given $v, w \in V \setminus \{0\}$ we define³⁶²

$$\text{angle between } v \text{ and } w := \text{unique } \alpha \in [0, \pi] \text{ with } g(v, w) = \cos(\alpha) \|v\| \|w\|.$$

- (2) Given a Riemannian manifold (M, g) and non-zero vectors $v, w \in T_P M$ we use the form g_P to define the angle between v and w .

Examples.

³⁶²We have $|g(v, w)| \leq \|v\| \|w\|$ by the Cauchy-Schwarz inequality.

- (1) Let U be an open subset of $\mathbb{R}^n$ and let $f: U \rightarrow \mathbb{R}_{>0}$ be a smooth function. For any $P \in U$ we can define a positive-definite symmetric form g_P on $T_P U = \mathbb{R}^n$ via

$$g_P(v, w) = f(P) \cdot \langle v, w \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the usual scalar product. A straightforward calculation shows that in this Riemannian structure the angle at any point is the usual euclidean angle in $\mathbb{R}^n$. This argument implies in particular to the Riemannian manifolds $\mathbb{D}$ and $\mathbb{H}$, in both cases the hyperbolic angle, defined by the hyperbolic Riemannian structure, is the same as the euclidean angle.

- (2) In Analysis IV we had seen that the sum of the interior angles of a triangle formed in $\mathbb{D}$ by hyperbolic lines is always less than π . This statement is illustrated in Figure 162.

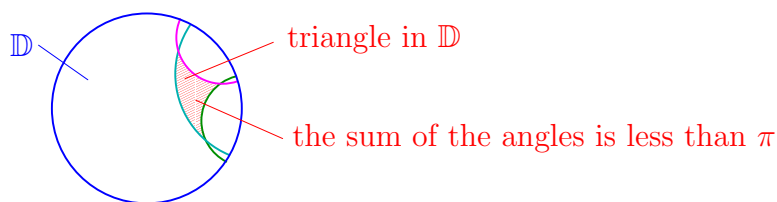


FIGURE 162.

The following lemma follows easily from the definitions and from Proposition 18.3 (2).

Lemma 18.6. *Local isometries, in particular Möbius transformations, preserve angles.*

Definition.

- (1) Let $P \in \mathbb{D}$ and let g be a hyperbolic line through P . Let Q be one of the two endpoints of g . We refer to the set of points between P and Q as a *ray emanating from P* .
- (2) Let $P \in \mathbb{D}$ and let r, s be two rays emanating from P . We refer to the angle between the tangent vectors of r and s at P as the *angle between the rays r and s* .

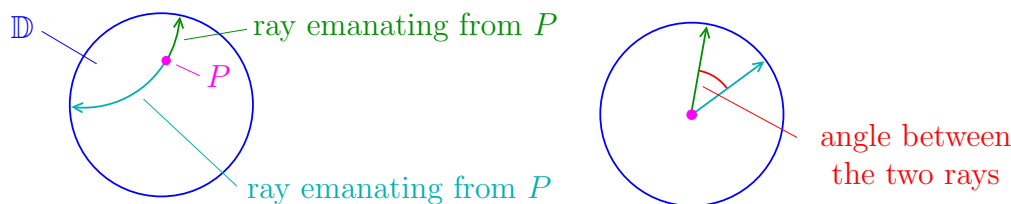


FIGURE 163.

Examples.

- (1) Given $\varphi \in [0, 2\pi)$ the set $S_\varphi := \{re^{i\varphi} \mid r \in [0, 1)\}$ is a ray emanating from 0.
- (2) It follows from the above discussion of angles in $\mathbb{D}$, that given $\varphi, \theta \in [0, 2\pi)$ the angle between the two rays S_φ and S_θ in $\mathbb{D}$ is the usual euclidean angle.

Lemma 18.7. *Let r, s be two rays emanating from $P \in \mathbb{D}$ and let r', s' be two rays emanating from $P' \in \mathbb{D}$. Then the following holds:*

$$\begin{aligned} \text{there is a Möbius transformation } \phi \\ \text{with } \phi(r \cup s) = r' \cup s' \end{aligned} \iff \text{angle between } r, s = \text{angle between } r', s'.$$

Furthermore, if such a Möbius transformation exists, then it is unique.

Proof. By Lemma 18.6 Möbius transformations preserve angles and by Proposition 18.3 Möbius transformations act transitively on $\mathbb{D}$. Therefore we can without loss of generality assume that $P = P' = 0$. The lemma follows from the observation, made on page 285, that Möbius transformations preserving 0 are precisely given by rotations around the origin. $\square$

18.3. The distance metric of a Riemannian manifold. We now recall some more definitions from Analysis IV.

Definition. Let (M, g) be a connected Riemannian manifold.

- (1) Given $v \in T_P M$ we write $\|v\|_g = \sqrt{g_P(v, v)}$. If g is understood from the context, then we just write $\|v\|$.
- (2) We say that a path $[a, b] \rightarrow M$ is *piecewise smooth* if there exists a subdivision $a = s_0 < s_1 < \dots < s_m = b$ such that for $i = 0, \dots, m-1$ the map $\gamma|_{[s_i, s_{i+1}]}$ is smooth. We define the *length of γ* as³⁶³

$$\ell_{(M, g)}(\gamma) := \sum_{i=0}^{m-1} \int_{t=s_i}^{t=s_{i+1}} \|\gamma'(t)\|_g dt.$$

If (M, g) is understood from the context, then we write $\ell(\gamma)$ instead of $\ell_{(M, g)}(\gamma)$.

- (3) Let P and Q be two points on M . We define³⁶⁴

$$d_{(M, g)}(P, Q) := \inf \{ \ell(\gamma) \mid \gamma \text{ is a piecewise smooth path in } M \text{ from } P \text{ to } Q \}.$$

If (M, g) is understood from the context, then we just write $d(P, Q)$.

We start our discussion of the distance function d on Riemannian manifolds with the following somewhat technical proposition.

Proposition 18.8. *Let (M, g) be a connected Riemannian manifold. Then the following hold:*

- (1) *the pair $(M, d_{(M, g)})$ is a metric space,*

³⁶³Recall that given a smooth curve $\delta: [a, b] \rightarrow M$ and given any $t \in [a, b]$ the curve defines, basically by definition of the tangent space $T_{\delta(t)} M$, an element in $T_{\delta(t)} M$. We denote this vector by $\delta'(t)$.

³⁶⁴Since M is connected it is by Lemma 1.25 also path-connected. We will see in exercise sheet 14 that we can connect any two points on a connected manifold by a piecewise smooth curve. This shows that $d_{(M, g)}(P, Q)$ is in fact defined.

- (2) the topology defined by the metric $d_{(M,g)}$ agrees with the topology of M ,
 (3) given any $P \in M$ and given any open neighborhood U of P there exists an $r > 0$ such that

$$B_r^{(M,g)}(P) := \{Q \in M \mid d_{(M,g)}(P, Q) < r\} \subset U.$$

Proof. Let (M, g) be a connected n -dimensional Riemannian manifold. We first show that $(M, d_{(M,g)})$ is a metric space.

It is clear that $d = d_{(M,g)}$ is symmetric. It follows easily from the definitions and the fact that the concatenation of two piecewise smooth paths is again piecewise smooth that d satisfies the triangle inequality.

Now let $P \neq Q$ be two distinct points in M . We need to show that $d(P, Q) > 0$. Since M is Hausdorff and since it is locally diffeomorphic to open subsets of $\mathbb{R}^n$ we can find a chart $\Phi: U \rightarrow B_{2r}(0)$ for P with $\Phi(P) = 0$ and such that $Q \notin U$. We write $\Psi = \Phi^{-1}$. Before we continue we introduce the following definitions:

- (1) We write $U_r = \Phi^{-1}(\{x \in \mathbb{R}^n \mid \|x\| = r\})$ and given a subset $I \subset [0, 2r]$ we write $U_I = \Phi^{-1}(\{x \in \mathbb{R}^n \mid \|x\| \in I\})$.
 (2) For a positive-definite bilinear form h on $\mathbb{R}^n$ we set

$$\|h\| := \min \{ \sqrt{h(v, v)} \mid v \in S^{n-1} \}.$$

Note that h is positive-definite, that $h: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ is continuous and that S^{n-1} is compact. It follows from Lemma 1.2 that the minimum exists and that $\|h\| > 0$. It is straightforward to see that for any $v \in \mathbb{R}^n$ we have

$$h(v, v) \geq \|h\| \cdot \|v\|_{\text{eucl}}.$$

- (3) We set

$$C := \min \{ \|(\Psi^*g)_P\| \mid P \in \overline{B_r(0)} \}.$$

Since $P \mapsto \|(\Psi^*g)_P\|$ is continuous and since $\overline{B_r(0)}$ is compact it follows from Lemma 1.2 that $C > 0$.

Claim. Let $\gamma: [0, 1] \rightarrow M$ be a piecewise smooth path from P to Q . There exists a $t \in [0, 1]$ with $\gamma(t) \in U_r$.

Suppose there is no $t \in (0, 1)$ with $\gamma(t) \in U_r$. We consider the sets $A = U_{[0,r]}$ and $B = M \setminus U_{[0,r]}$. The set A is clearly open. Furthermore it follows from Lemma 1.26 that $U_{[0,r]}$ is closed, hence B is open. Since $\gamma([0, 1]) \cap U_r = \emptyset$ we have $\gamma([0, 1]) = \gamma^{-1}(A) \sqcup \gamma^{-1}(B)$. But both sets $\gamma^{-1}(A)$ and $\gamma^{-1}(B)$ are open by the continuity of γ and both are non-empty since $0 \in \gamma^{-1}(A)$ and $1 \in \gamma^{-1}(B)$. But this contradicts the fact that $[0, 1]$ is connected. This concludes the proof of the claim.

Now it suffices to prove the following claim.

Claim. Let $\gamma: [0, 1] \rightarrow M$ be a piecewise smooth path from P to Q . Then $\ell_g(\gamma) \geq C \cdot r$.

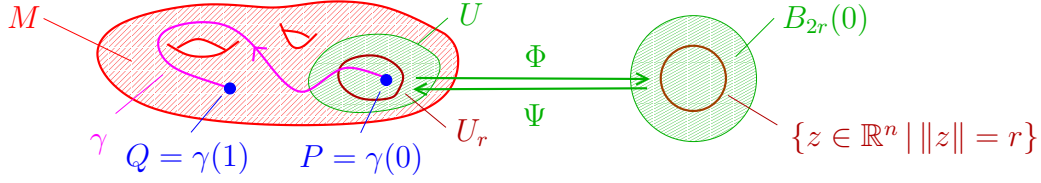


FIGURE 164. Illustration of the proof of Proposition 18.8

Let $T := \inf\{t \in (0, 1) \mid \gamma(t) \in U_r\}$. By the above claim this definition makes sense since we take the infimum of a non-empty set. By Lemma 1.26 the set U_r is a closed subset of M . Since γ is continuous we also have $\gamma(T) \in U_r$. We now have

$$\begin{aligned}
 \ell_g(\gamma) &\geq \ell_g(\gamma|_{[0, T]}) = \ell_{\Psi^*g}(\Phi \circ \gamma|_{[0, T]}) &&= \int_{t=0}^{t=T} \|(\Phi \circ \gamma)'(t)\|_{\Psi^*g} dt \\
 &\geq \int_{t=0}^{t=T} C \cdot \|(\Phi \circ \gamma)'(t)\|_{\text{eucl}} dt &&\geq C \cdot \left\| \int_{t=0}^{t=T} (\Phi \circ \gamma)'(t) dt \right\|_{\text{eucl}} \\
 &\quad \uparrow &&\quad \uparrow \\
 &\text{for any } v \in T_P \overline{B_r(0)} = \mathbb{R}^n \text{ we have} &&\text{Lemma 8.2 from Analysis II} \\
 &\|v\|_{(\Psi^*g)_P} \geq \|(\Psi^*g)_P \cdot v\|_{\text{eucl}} && \\
 &= C \cdot \underbrace{\|(\Phi \circ \gamma)(T) - (\Phi \circ \gamma)(0)\|_{\text{eucl}}}_{\|-\|_{\text{eucl}}=r} = C \cdot r.
 \end{aligned}$$

This concludes the proof of the claim and therefore the proof of statement (1) of the proposition.

Next we turn to the proof of statement (2). This means that we need to show that the topology defined by d is the same as the original topology of M . The proof is again somewhat technical and uses ideas similar to the proof of (1). We refer to [Le1, Lemma 6.2] for the proof.

Finally statement (3) is just an immediate consequence of the definitions and of statement (2). $\square$

Lemma 18.9. *Let (M, g) and (N, h) be two connected Riemannian manifolds. Furthermore let $f: (M, g) \rightarrow (N, h)$ be a local isometry. Then the following hold:*

(1) *For any piecewise smooth path $\gamma: [a, b] \rightarrow M$ we have*

$$\ell_{(N, h)}(f \circ \gamma) = \ell_{(M, g)}(\gamma).$$

(2) *For any $P, Q \in M$ we have*

$$d_{(N, h)}(f(P), f(Q)) \leq d_{(M, g)}(P, Q).$$

Proof. Let $f: (M, g) \rightarrow (N, h)$ be a local isometry. We start out with the following claim.

Claim.

(1) For a smooth path $\gamma: [c, d] \rightarrow M$ and $t \in (c, d)$ we have $\|(f \circ \gamma)'(t)\|_{(N, h)} = \|\gamma'(t)\|_{(M, g)}$.

- (2) For any piecewise smooth path $\gamma: [a, b] \rightarrow M$ we have $\ell_{(N,h)}(f \circ \gamma) = \ell_{(M,g)}(\gamma)$.

Now we provide the proof of the two statements.

- (1) The first statement is an immediate consequence of the definitions and the hypothesis that f is a local isometry.
 (2) By the additivity of the lengths of piecewise smooth paths it suffices to consider the case that $\gamma: [a, b] \rightarrow M$ is a smooth path. But in that case the equality follows immediately from the definitions and from (1).

This concludes the proof of the claim.

Now let P and Q be two points on M . For any piecewise smooth path p in M from P to Q we have

$$\begin{array}{ccccc} d_{(N,h)}(f(P), f(Q)) & \leq & \ell_{(N,h)}(f \circ p) & = & \ell_{(M,g)}(p). \\ & \uparrow & & \uparrow & \\ & \text{by definition of } d_{(N,h)} & & \text{by the claim} & \end{array}$$

It follows that the left-hand side is also less or equal than the infimum over all paths on the right-hand side. Thus $d_{(N,h)}(f(P), f(Q)) \leq d_{(M,g)}(P, Q)$. $\square$

18.4. The hyperbolic distance function. We recall the following definition from Analysis IV.

Definition. Let (M, g) be a Riemannian manifold. A curve $\gamma: [0, a] \rightarrow M$ is called a *geodesic* if the following conditions are satisfied:

- (1) γ is smooth,
- (2) $\|\gamma'(t)\|_g = 1$ für alle $t \in [0, a]$ and
- (3) $\ell(\gamma) = d_{(M,g)}(\gamma(0), \gamma(a))$.

The following proposition summarizes the key facts about the distance metric on $\mathbb{D}$.

Proposition 18.10.

- (1) Let $r \in (0, 1)$ and $\varphi \in \mathbb{R}$. In the Riemannian manifold $\mathbb{D}$ the path

$$\begin{array}{ccc} \gamma: [0, r] & \rightarrow & \mathbb{D} \\ t & \mapsto & te^{i\varphi} \end{array}$$

is, up to reparametrization, the unique shortest path from 0 to $re^{i\varphi}$.

- (2) We have

$$d_{(\mathbb{D},g)}(0, re^{i\varphi}) = \ell(\gamma) = \int_{t=0}^{t=r} \frac{\sqrt{2}}{1 - |\gamma(t)|^2} \|\gamma'(t)\|_{\text{eucl}} dt = \int_{t=0}^{t=r} \frac{\sqrt{2}}{1 - t^2} dt.$$

- (3) Given $\varphi \in \mathbb{R}$ we have

$$\lim_{r \rightarrow 1} d_{(\mathbb{D},g)}(0, re^{i\varphi}) = \infty.$$

- (4) Given any P and Q in $\mathbb{D}$ there is a unique geodesic from P to Q .

Proof. The first statement is a direct consequence of Satz 14.10 from Analysis IV. The second statement follows immediately from the first statement, and the third statement is a straightforward consequence of the second statement. For the fourth statement, note that it suffices to prove it for $P = 0$. In that case it is a straightforward consequence of (1). $\square$

Lemma 18.11. *Let (P, Q) and (S, T) be two pairs of distinct points on $\mathbb{D}$ such that $d(P, Q) = d(S, T)$. Then there exists a unique Möbius transformation ϕ with $\phi(P) = S$ and $\phi(Q) = T$.*

Example. Let $r \in (0, 1)$, $\varphi \in [0, 2\pi]$, $\psi \in [0, 2\pi]$ and $\gamma \in (0, \pi)$. We consider the pairs of points $P = re^{i\varphi}$, $Q = re^{i(\varphi+\gamma)}$ and $S = re^{i\psi}$, $T = re^{i(\psi-\gamma)}$. We then have

$$d(S, T) = d(T, S) = d(re^{i(\psi-\gamma)}, re^{i\psi}) = d(re^{i\varphi}, re^{i(\varphi+\gamma)}).$$

$\uparrow$
 since rotation by $\varphi + \gamma - \psi$ is an isometry

By Lemma 18.11 there exists a Möbius transformation ϕ with $\phi(P) = S$ and $\phi(Q) = T$. This Möbius transformation is illustrated in Figure 165. Recall that from Proposition 18.3 and Lemma 18.5 we know that Möbius transformations preserve hyperbolic lines and that they are orientation-preserving. In particular ϕ sends the hyperbolic line through P and Q to the hyperbolic line through S and T . Since ϕ is orientation-preserving it “swaps” the two sides of the hyperbolic line. We refer to Figure 165 for an illustration.

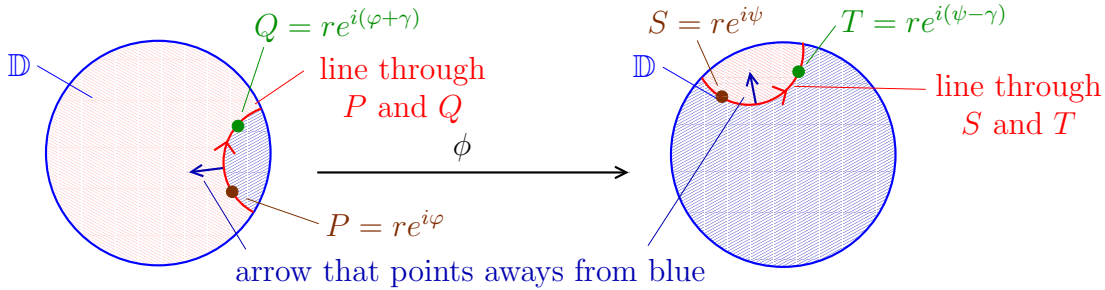


FIGURE 165.

Proof of Lemma 18.11. It follows from Proposition 18.3 (3) that, without loss of generality, we can assume that $P = 0$ and $S = 0$. We write $Q = re^{i\varphi}$ and $T = se^{i\psi}$. It is a direct consequence of Proposition 18.10 (1) and our hypothesis that

$$d(0, re^{i\varphi}) = d(P, Q) = d(S, T) = d(0, se^{i\psi})$$

that we have $r = s$. But this means that we can apply the rotation by the angle $\varphi - \psi$, which is a Möbius transformation, to turn (P, Q) into (S, T) . This proves the existence of ϕ . The uniqueness follows from the observation, that we had made already on page 285, that the only Möbius transformations that fix 0 are precisely rotations. $\square$

Lemma 18.12. *Let $P \in \mathbb{D}$ and $r > 0$. Let $S, T \in B_r(P)$. Then the image of the unique geodesic in $\mathbb{D}$ from S to T lies in $B_r(P)$.*

Proof. As usual, this time by Propositions 18.3, 18.5 and 18.10, it suffices to prove the lemma for $P = 0$. In that case it follows from an elementary argument. We leave the details of the proof as an exercise. We also refer to Figure 166 for an illustration. $\square$

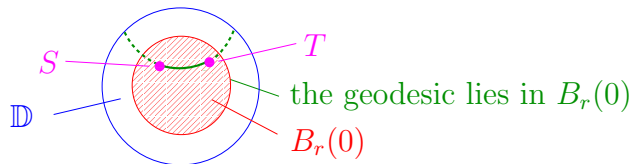


FIGURE 166.

18.5. Complete metric spaces. We conclude this chapter with a short discussion of complete metric spaces.

Definition.

- (1) We say that a metric space (X, d) is *complete* if every Cauchy sequence in (X, d) converges.
- (2) We say that a connected Riemannian manifold (M, g) is *complete* if the corresponding metric space $(M, d_{(M, g)})$ is complete.

Examples.

- (1) The metric space given by $\mathbb{R}^n$ and the euclidean metric is complete.
- (2) The metric space given by the open ball B^n and the euclidean metric is not complete. Indeed the sequence $a_n = 1 - \frac{1}{n}$ is a Cauchy sequence, but it does not converge in the open ball B^n . Similarly we see that for example the metric space $\mathbb{C} \setminus \{0, 1\}$ with the usual euclidean metric is not complete.

The following proposition gives a useful criterion for a metric space to be complete.

Proposition 18.13. *Every compact metric space is complete.*

Proof. Let (X, d) be a compact metric space and let $(a_n)_{n \in \mathbb{N}}$ be a Cauchy sequence in (X, d) . We want to show that the sequence converges, i.e. we want to show that

$$\exists x \in X \quad \forall \epsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n \geq N \quad d(a_n, x) < \epsilon.$$

Let us assume that no such x exists. Thus we negate the above statement and we obtain that

$$\forall x \in X \quad \exists \epsilon > 0 \quad \forall N \in \mathbb{N} \quad \exists n \geq N \quad d(a_n, x) \geq \epsilon.$$

Given $x \in X$ we pick $\epsilon_x > 0$ as above and we write $U_x := B_{\epsilon_x}(x)$. These open sets cover X . Since X is compact there exist $x_1, \dots, x_k$ such that $X = U_{x_1} \cup \dots \cup U_{x_k}$. Now let

$\epsilon := \min\{\epsilon_{x_1}, \dots, \epsilon_{x_k}, \frac{1}{3}d(x_i, x_j)\}$. We pick N as in the definition of a Cauchy sequence for the given $\epsilon > 0$. This means in particular that for all $m \geq N$ we have $d(a_N, a_m) < \epsilon$.

Since $X = U_{x_1} \cup \dots \cup U_{x_k}$ there exists an $i \in \{1, \dots, k\}$ with $a_N \in U_{x_i}$. It suffices to prove the following claim to obtain the desired contradiction.

Claim. For any $m \geq N$ we have $a_m \in U_{x_i}$.

Let $m \geq N$. Since $X = U_{x_1} \cup \dots \cup U_{x_k}$ there exists a $j \in \{1, \dots, k\}$ with $a_m \in U_{x_j}$. Suppose that $i \neq j$. Then we have

$$\begin{array}{ccccccc} 3\epsilon & \leq & d(x_i, x_j) & \leq & d(x_i, a_N) + d(a_N, a_m) + d(a_m, x_j) & < & \epsilon_{x_i} + \epsilon + \epsilon_{x_j} \leq 3\epsilon \\ & \uparrow & & \uparrow & & \uparrow & \\ & \text{definition of } \epsilon & & \text{triangle inequality} & & \text{since } a_N \in U_{x_i}, d(a_N, a_m) < \epsilon \text{ and } a_m \in U_{x_j} & \end{array}$$

Thus we obtain a contradiction to $i \neq j$, i.e. we have $a_m \in U_{x_i}$. This concludes the proof of the claim. $\square$

The Riemannian manifolds $\mathbb{D}$ and $\mathbb{H}$ are of course not compact, but nonetheless, as we will now see, we can use Proposition 18.13 to show that $\mathbb{D}$ and $\mathbb{H}$ are complete.

Lemma 18.14. *The Riemannian manifolds $\mathbb{D}$ and $\mathbb{H}$ are complete.*

Proof. By Proposition 18.2 the two Riemannian manifolds $\mathbb{D}$ and $\mathbb{H}$ are isometric. Therefore it suffices to show that $\mathbb{D}$ is complete. We denote by $d = d_{(\mathbb{D}, g)}$ the metric on $\mathbb{D}$ corresponding to the Riemannian structure. Let $\{a_n\}_{n \in \mathbb{N}}$ be a Cauchy sequence with respect to d . An elementary argument using the triangle inequality and the definition of a Cauchy sequence shows that there exists a $C \in \mathbb{R}$ such that $d(0, a_n) \leq C$ for all $n \in \mathbb{N}$.

It follows from Proposition 18.10 that there exists an $r \in (0, 1)$ such that all a_n are contained in the closed euclidean ball $\overline{B}_r^{\text{eucl}}(0)$. By the Heine-Borel Theorem this ball is a compact subset of $\mathbb{R}^2$, viewed with the usual topology. It follows from Proposition 18.8 that $\overline{B}_r^{\text{eucl}}(0)$ is also compact in the topology defined by d . We now see that $\{a_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in the compact metric space $(\overline{B}_r^{\text{eucl}}(0), d)$. It follows from Proposition 18.13 that the sequence converges. $\square$

19. THE UNIVERSAL COVER OF SURFACES

Throughout this chapter we mean by a *surface* a connected 2-dimensional manifold. For example the surface of genus g , $g \geq 0$, but also $\mathbb{C}$, $\mathbb{C} \setminus \{0, 1\}$ and the result of removing finitely many points from Σ_g are surfaces.

Definition. Let M be a surface. We say that a Riemannian structure g on M is *hyperbolic* if the following two conditions are satisfied:

- (1) (M, g) is locally isometric to $\mathbb{D}$, i.e. if given any $P \in M$ there exists an isometry $\Phi: U \rightarrow V$ from an open neighborhood U of P to an open subset V of $\mathbb{D}$,
- (2) (M, g) is complete.

If g is hyperbolic, then we refer to (M, g) as a hyperbolic surface.

Remark. In the definition of a hyperbolic structure we gave in Analysis IV we only demanded that (1) is satisfied. By Proposition 18.13 the condition (2) is automatically satisfied if M is a closed surface.

In this chapter we will pursue the following three goals:

- (1) we want to show that for any $g \geq 2$ the surface of genus g is a complex manifold,
- (2) we intend to prove that for any $g \geq 2$ the surface of genus g is hyperbolic,
- (3) given $g \geq 2$ we want to determine the universal cover of the surface of genus g .

19.1. Hyperbolic surfaces. In this section we will show that the surfaces of genus ≥ 2 are complex manifolds and that they are hyperbolic. In the subsequent subsection we will then consider more general types of surfaces.

Definition. Let M be a 2-dimensional topological manifold without boundary. A *Möbius structure* for M is a family of homeomorphisms $\{\Phi_i: U_i \rightarrow V_i\}_{i \in I}$ from open subsets of M to open subsets of $\mathbb{D}$ such that $\bigcup_{i \in I} U_i = M$ and such that for any $i, j \in I$ the transition map

$$\underbrace{\Phi_i(U_i \cap U_j)}_{\subset \mathbb{D}} \xrightarrow{(\Phi_i|_{U_i \cap U_j})^{-1}} U_i \cap U_j \xrightarrow{\Phi_j|_{U_i \cap U_j}} \underbrace{\Phi_j(U_i \cap U_j)}_{\subset \mathbb{D}}$$

is given by a Möbius transformation. Sometimes we refer to a manifold together with a Möbius structure as a *Möbius manifold*.

Example.

- (1) We will always view $\mathbb{D}$ as equipped with the Möbius structure that is given by the chart $\text{id}: \mathbb{D} \rightarrow \mathbb{D}$.
- (2) Any open subset of a Möbius manifold is also a Möbius manifold.

The following lemma says that Möbius structures are useful for solving two problems at once: we can use them to show that a manifold is a complex manifold and we can use them to show that a manifold has a hyperbolic structure.

Lemma 19.1. *Let M be a surface and let $\{\Phi_i: U_i \rightarrow V_i\}_{i \in I}$ be a Möbius structure for M . Then the following hold:*

- (1) *The charts form a holomorphic atlas for M , in particular M is a complex 1-dimensional manifold.*
- (2) (a) *The manifold M admits a unique Riemannian structure g such that all the charts in the atlas $\{\Phi_i: U_i \rightarrow V_i\}_{i \in I}$ are isometries.*
 (b) *If the Riemannian structure from (a) is complete, then (M, g) is hyperbolic.*

In the following we will always view a Möbius manifold as a complex 1-dimensional manifold and a Riemannian manifold with the structures coming from Lemma 19.1.

Proof. Let M be a surface and let $\{\Phi_i: U_i \rightarrow V_i\}_{i \in I}$ be a Möbius structure for M .

- (1) The first statement is an immediate consequence of the definitions and of the fact, proved in Proposition 18.3 (2), that Möbius transformations are biholomorphisms.
- (2) (a) We denote by g the Riemannian structure on $\mathbb{D}$. We define a Riemannian structure on M as follows. Let $P \in M$. We pick a chart $\Phi_i: U_i \rightarrow V_i$ from our Möbius structure and we define $h_P := \Phi_i^*(g_{\Phi_i(P)})$. Now suppose that we had picked a different chart Φ_j from the Möbius structure. The transition map from Φ_i to Φ_j is by definition a Möbius transformation, which is an isometry by Proposition 18.3 (2). It follows easily that Φ_j gives rise to the same definition of h_P . It is now straightforward to verify that this defines a Riemannian structure on M and that it has the desired properties.
 (b) This statement is an immediate consequence of (a) and of the definition of a hyperbolic surface. $\square$

The following proposition gives in particular an affirmative answer to Question 17.5.

Proposition 19.2. *Let $g \geq 2$. Then the following hold:*

- (1) *The surface of genus g admits a Möbius structure.*
- (2) *The surface of genus g is a complex 1-dimensional manifold.*
- (3) *The surface of genus g is hyperbolic.*

Proof. We prove the proposition for the case that $g = 2$. The general case is proved by almost the same argument.

We first show that the surface of genus 2 admits a Möbius structure. We recall that in Lemma 21.5 of Analysis IV we had shown that there exists an $r > 0$ such that the interior angle at any vertex of the hyperbolic octagon H_8 in $\mathbb{D}$, defined by $Q_k = re^{2\pi i k/16}$, $k = 1, 3, \dots, 15$, equals $\frac{\pi}{4}$.³⁶⁵ This hyperbolic octagon is illustrated in Figure 167.

³⁶⁵Here is a sketch for the proof: For $r \rightarrow 0$ the interior angle converges to the interior angle of a euclidean regular octagon, namely $\frac{3\pi}{4}$, whereas for $r \rightarrow \infty$ the interior angle converges to 0. The interior angle changes continuously with r (that is sort of clear, except that it is somewhat painful to provide a rigorous proof), so by the Intermediate Value Theorem there exists an r such that the interior angle equals $\frac{\pi}{4}$.

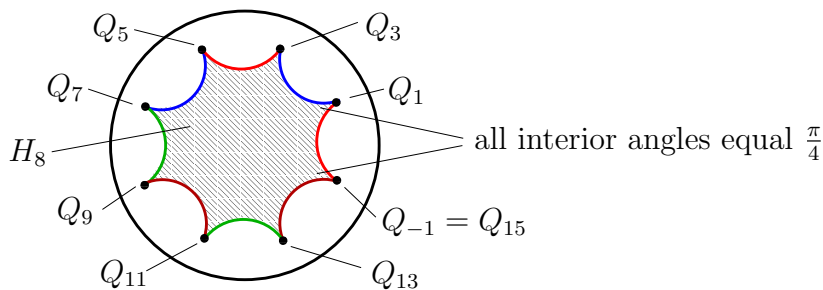


FIGURE 167.

Let $k \in \{0, 1, 4, 5\}$. By Lemma 18.11 and the example on page 295 there exists a unique Möbius transformation Φ_k with $\Phi_k(Q_{2k-1}) = Q_{2k+5}$ and $\Phi_k(Q_{2k+1}) = Q_{2k+3}$. The Möbius transformations Φ_0 and Φ_1 are illustrated in Figure 168.

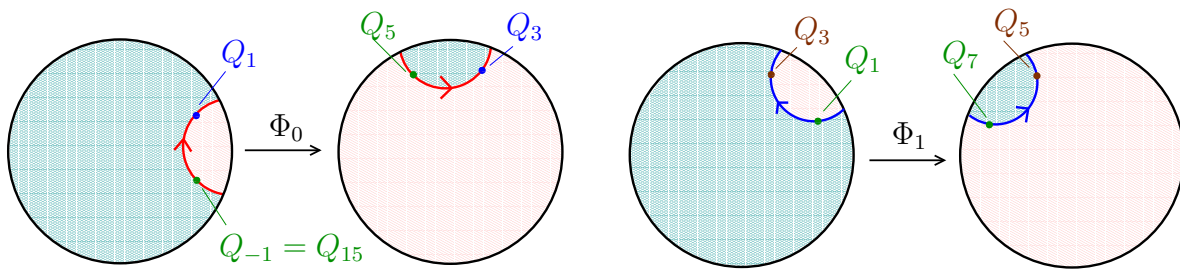


FIGURE 168.

Given $k \in \{0, 1, 4, 5\}$ we declare any point P on the edge connecting Q_{2k-1} and Q_{2k+1} to be equivalent to $\Phi_k(P)$. We denote by $\sim$ the equivalence relation that is generated by these equivalences. It is straightforward to see that $H_8/\sim$ is homeomorphic to the surface of genus 2 that we had defined on page 28 as $E_8/\sim$. In the following we denote by $p: H_8 \rightarrow H_8/\sim$ the obvious projection map.

We will now give an explicit Möbius structure for $H_8/\sim$. More precisely, given any $P \in H_8/\sim$ we will give a chart ψ_P such that the transition maps are given by Möbius transformations. So let $P \in H_8/\sim$.

- (1) If $P = p(z)$ for some $z \in \mathring{H}_8$, then the map

$$\begin{aligned} \psi_P: p(\mathring{H}_8) &\rightarrow \mathring{H}_8 \\ p(w) &\mapsto w \end{aligned}$$

is a chart around $P = p(z)$.

- (2) Now suppose that $P = p(z)$ where z lies in the interior of one of the eight edges of ∂H_8 . In the following we deal with the case that z lies on the edge from Q_{-1} to Q_1 .

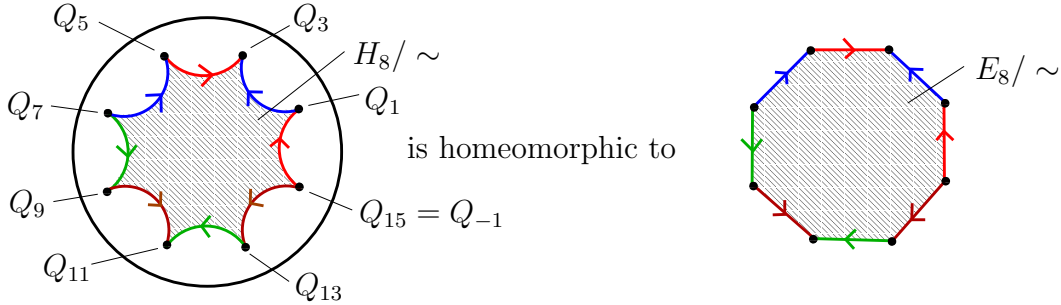


FIGURE 169.

We pick an $r > 0$ such that $B_{2r}(z)$ does not contain any of the vertices $Q_1, Q_3, \dots, Q_{15}$ and does not hit any of the other edges. Then we consider the map

$$\begin{aligned} \psi_P: p(B_r(z) \cap H_8) \cup p(\Phi_0(B_r(z)) \cap H_8) &\rightarrow B_r(z) \\ p(w) &\mapsto \begin{cases} w, & \text{if } w \in B_r(z) \cap H_8 \\ \Phi_0^{-1}(w), & \text{if } w \in \Phi_0(B_r(z)) \cap H_8. \end{cases} \end{aligned}$$

This map is a chart around $P = p(z)$. We refer to Figure 170 for an illustration of the definition of the map ψ_P .

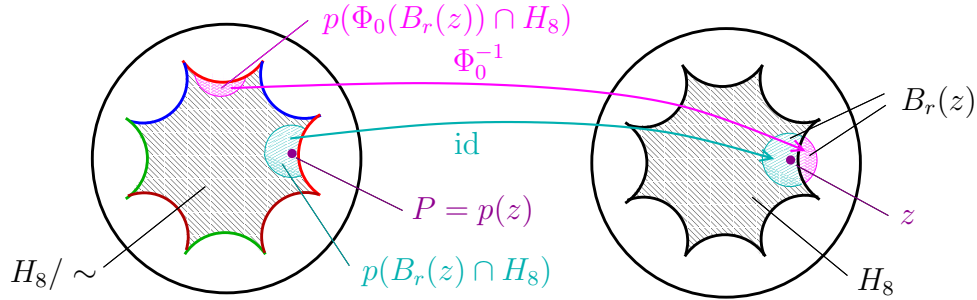


FIGURE 170.

- (3) Finally suppose that $P = p(Q_1) = \dots = p(Q_{15})$. We choose $r \in (0, \frac{1}{2}d(Q_{-1}, Q_1))$. Given $k \in \{1, 3, \dots, 15\}$ we define $\gamma(k)$ via the following table:

k	1	3	5	7	9	11	13	15
$\gamma(k)$	$\frac{3}{4}\pi$	π	$\frac{5}{4}\pi$	$\frac{1}{2}\pi$	$\frac{7}{4}\pi$	0	$\frac{1}{4}\pi$	$\frac{3}{2}\pi$.

Recall that we had chosen the octagon H_8 such that the angle at any vertex equals $\frac{\pi}{4}$. For $k = 1, 3, \dots, 15$ we can therefore apply Lemma 18.7 to find a unique Möbius transformation θ_k with the following two properties:

- (a) $\theta_k(Q_k) = 0$,

- (b) θ_k applied to the two rays emanating from Q_k equals the two rays $\{se^{i\gamma(k)} \mid s \geq 0\}$ and $\{se^{i(\gamma(k)+\frac{\pi}{4})} \mid s \geq 0\}$ emanating from 0.

We consider the map

$$\begin{aligned} \psi_P: p(B_r(Q_1) \cap H_8) \cup \cdots \cup p(B_r(Q_{15}) \cap H_8) &\rightarrow B_r(0) \\ p(z) &\mapsto \theta_k(z), \quad \text{if } z \in B_r(Q_k) \cap H_8. \end{aligned}$$

We leave it as an exercise to verify that ψ_P is well-defined and that it is indeed a homeomorphism. This map is a chart around $P = p(z)$. We refer to Figure 171 for an illustration of the definition of the map ψ_P .

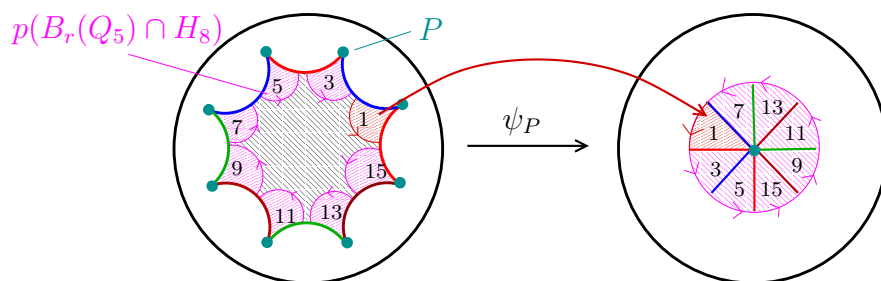


FIGURE 171.

It is straightforward to verify that all the transition maps between the above maps are given by Möbius transformations. We have thus shown that the maps $\{\psi_P\}_{P \in H_8}$ form a Möbius structure for $H_8/\sim$. This concludes the proof of (1).

Now we turn to the proof of (2). The existence of a Möbius structure together with Lemma 19.1 (2) shows immediately that the surface of genus 2 is a complex 1-dimensional manifold.

Finally we provide the proof of (3). In light of (1) and Lemma 19.1 (2) it suffices to show that the Riemannian manifold $(H_8/\sim, g)$ is complete. By Proposition 18.8 the topology on M defined by the metric $d_{(M,g)}$ agrees with the given topology of M . Since M is compact it now follows from Proposition 18.13 that (M, g) is complete. $\square$

19.2. More hyperbolic structures on the surfaces of genus $g \geq 2$ (*).³⁶⁶ We now want to study the question, how many hyperbolic metrics does a surface of genus g admit. To make this question more precise, we introduce the following definition.

Definition. Let M be a surface of genus $g \geq 2$ and let h, h' be two hyperbolic metrics on M . We say h and h' are *equivalent* if there exists an isometry from (M, h) to (M, h') .

This raises the question, whether a surface of genus $g \geq 2$ admits non-equivalent hyperbolic metrics. We return to the construction of a hyperbolic metric in Proposition 19.2. In our construction we started out with a regular $4g$ -gon with interior angle $\frac{2\pi}{4g}$ and we identified pairs of sides. Two facts were crucial to obtain a hyperbolic metric:

³⁶⁶This chapter is not part of the official lecture notes for Algebraic Topology I.

- (1) the sides were identified via an isometry,
- (2) the sum of the interior angles equals 2π .

Or put differently, any choice of a $4g$ -gon with side lengths $k_1, \dots, k_{4g}$ and interior angles $\alpha_1, \dots, \alpha_{4g}$ that satisfy the equations

$$k_1 = k_3, k_2 = k_4, \dots, k_{4g-3} = k_{4g-1}, k_{4g-2} = k_{4g} \quad \text{and} \quad \alpha_1 + \dots + \alpha_{4g} = 2\pi$$

gives rise to a hyperbolic metric on a topological space that is homeomorphic to the surface of genus g . It is of course not clear whether different parameters give rise to non-equivalent hyperbolic structures. One way to distinguish them, in principle at least, is to look at the shortest length of a closed curve that is not null-homotopic. We will not elaborate on this procedure.

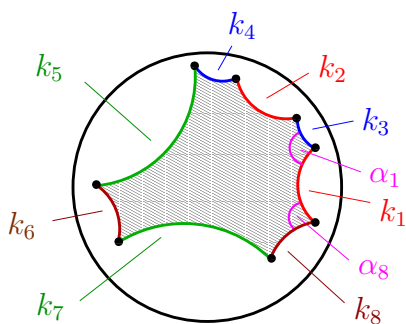


FIGURE 172.

How many such $4g$ -gons are there? On the one hand we have parameters $k_1, k_3, \dots, k_{4g-1}$ and $\alpha_1, \dots, \alpha_{4g}$. On the other hand we have the condition that $\alpha_1 + \dots + \alpha_{4g} = 2\pi$. Furthermore we want a $4g$ -gon, so the last angle and the length of the last edge are determined by the others. Thus we obtain two other conditions. Summarizing we have $2g + 4g = 6g$ parameters that need to satisfy 3 conditions. Of course it is not clear which of these Riemannian structures are isometric. This naive discussion suggests that the space of hyperbolic metrics is $(6g - 3)$ -dimensional. Surprisingly this is not too far off the correct answer, in fact in [BP, Chapter B.4] the following proposition is proved.

Proposition 19.3. *The space of hyperbolic metrics on a surface of genus $g \geq 2$ can be viewed as a manifold of dimension $6g - 6$.*

We will not make use of this proposition. The space of hyperbolic metric is called the *Teichmüller space*, more information can be found at

https://en.wikipedia.org/wiki/Teichmuller_space.

The theory is named after Oswald Teichmüller (1913-1943) who was a brilliant mathematician and an ardent Nazi, see

https://en.wikipedia.org/wiki/Oswald_Teichmuller

<http://www-history.mcs.st-andrews.ac.uk/Biographies/Teichmuller.html>

and

https://de.wikipedia.org/wiki/Oswald_Teichmueller.

19.3. More examples of hyperbolic surfaces. In this section we will generalize the discussion from the previous section to surfaces that are non-compact.

In the following let $Q_1 = 1, Q_2 = i, Q_3 = -1$ and $Q_0 = Q_4 = -i$. We denote by H_4 the closed, non-compact subset of $\mathbb{D}$ that is bounded by the four hyperbolic lines with endpoints Q_k, Q_{k+1} where $k = 0, 1, 2, 3$. We refer to Figure 174 for an illustration.

We consider the Möbius transformations³⁶⁷

$$\Phi_1(z) := (-i) \cdot \frac{z - \frac{-1+i}{2}}{1 - \frac{-1-i}{2}z} \quad \text{and} \quad \Phi_2(z) := (-i) \cdot \frac{z + \frac{-1-i}{2}}{1 + \frac{-1+i}{2}z}.$$

We record a few key properties of Φ_1 and Φ_2 in a lemma.

Lemma 19.4.

- (1) We have $\Phi_1(Q_2) = Q_4$ and $\Phi_1(Q_3) = Q_3$ and we have $\Phi_2(Q_2) = Q_4$ and $\Phi_1(Q_1) = 1$.
- (2) The restriction of Φ_1 to the hyperbolic line with endpoints Q_3 and Q_2 is given by reflection in the x -axis.³⁶⁸
- (3) The restriction of Φ_2 to the hyperbolic line with endpoints Q_1 and Q_2 is given by reflection in the x -axis.

Proof. The first statement follows immediately from plugging in the points. A straightforward, albeit slightly painful calculation shows that for any $\alpha \in \mathbb{R}$ we have

$$\Phi_1(-1 + i + e^{-i\alpha}) = -1 - i + e^{i\alpha}.$$

This proves the second statement. The third statement is proved by a similarly tedious calculation. $\square$

Remark. We consider the Möbius transformation Φ_1 in more detail. It sends the hyperbolic line with endpoints Q_3 and Q_2 to the hyperbolic line with endpoints Q_3 and Q_4 . In particular it sends the two halves determined by the hyperbolic lines to one-another. Which half gets sent to which half is determined by the fact that Φ_1 is orientation-preserving. We refer to Figure 173 for an illustration of the Möbius transformations Φ_1 and Φ_2 .

Given $k \in \{1, 2\}$ we declare any point P on the line with the endpoints Q_k and Q_{k+1} to be equivalent to $\Phi_{3-k}(P)$. We denote by $\sim$ the equivalence relation that is generated by these equivalences.

³⁶⁷Recall that a Möbius transformation of $\mathbb{D}$ is a map of the form $\psi(z) = e^{i\theta} \cdot \frac{z-a}{1-\bar{a}z}$ for some $\theta \in \mathbb{R}$ and $a \in \mathbb{D}$.

³⁶⁸The Möbius transformation Φ_1 sends the hyperbolic line with endpoints Q_3 and Q_2 to the hyperbolic line with endpoints Q_3 and Q_4 . There are many such Möbius transformations, but the Möbius transformation we picked has the extra property, that on the hyperbolic line it is the “obvious map”.

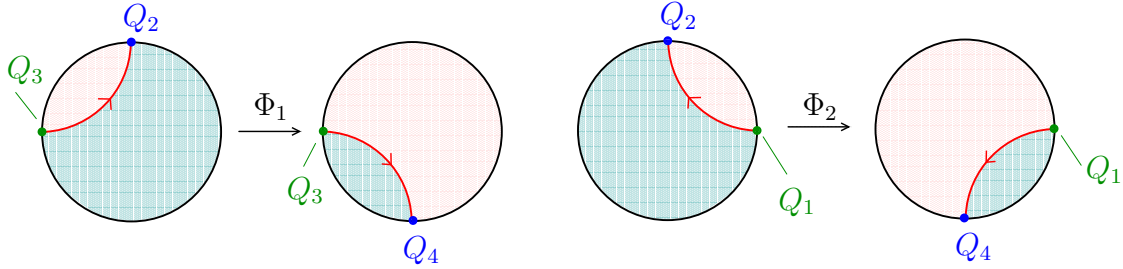


FIGURE 173.

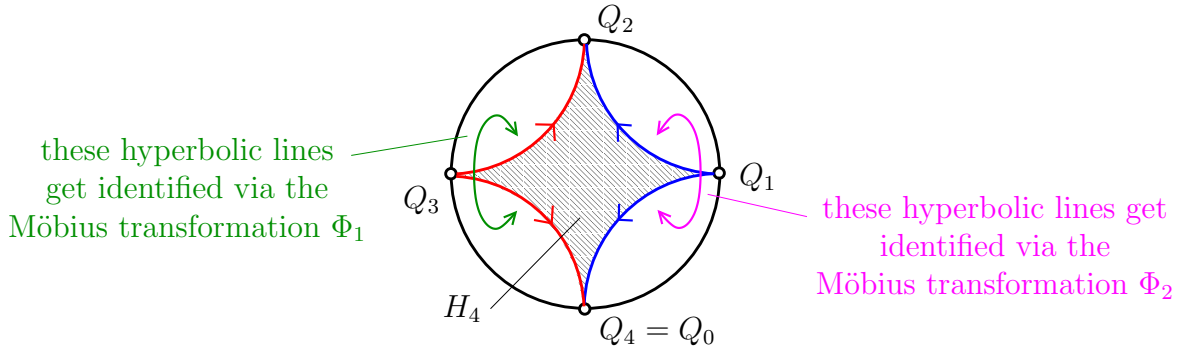


FIGURE 174.

Proposition 19.5.

- (1) *The topological space $H_4/\sim$ admits a Möbius structure. In particular it is a 1-dimensional complex manifold.*
- (2) *The Riemannian structure on $H_4/\sim$ coming from (1) and Lemma 19.1 is complete.*
- (3) *The topological space $H_4/\sim$ admits a hyperbolic metric.*

Proof. We start out with the proof of (1). We have to show that $H_4/\sim$ admits a Möbius structure. The following argument is almost identical to the proof of Proposition 19.2 (1).

We denote by $p: H_4 \rightarrow H_4/\sim$ the projection map. In the following let $P \in H_4/\sim$.

- (1) If $P = p(z)$ for some $z \in \mathring{H}_4$, then

$$\begin{array}{ccc} \psi_P: p(\mathring{H}_4) & \rightarrow & \mathring{H}_4 \\ p(w) & \mapsto & w \end{array}$$

is a chart around $P = p(z)$.

- (2) Now suppose that $P = p(z)$ where z lies on ∂H_4 . In the following we deal with the case that z lies on the edge from Q_1 to Q_2 . All other cases are dealt with almost the same way. We pick an $r > 0$ such that $B_{2r}(z)$ intersects no other component of ∂H_4 .

We consider the map

$$\begin{aligned} \psi_P: p(B_r(z) \cap H_4) \cup p(\Phi_2(B_r(z)) \cap H_4) &\rightarrow B_r(z) \\ p(w) &\mapsto \begin{cases} w, & \text{if } w \in B_r(z) \cap H_4 \\ \Phi_2^{-1}(w), & \text{if } w \in \Phi_1^{-1}(B_r(z)) \cap H_4. \end{cases} \end{aligned}$$

This map is a chart around $P = p(z)$.

The charts that we just constructed form an atlas for $H_4/\sim$. It follows immediately from the definitions that all transition maps are given by Möbius transformations. It follows that the maps ψ_P form a Möbius structure for $H_4/\sim$. As in the proof of Proposition 19.2 it follows that $H_4/\sim$ is a 1-dimensional complex manifold and that these charts satisfy the conditions of Lemma 19.1. This concludes the proof of (1).

We continue with the proof of (2). We have to show that the Riemannian structure on $H_4/\sim$ coming from (1) and Lemma 19.1 is complete. The subsequent argument is a variation on the proof of Proposition 18.13.

We denote by d the metric corresponding to the Riemannian structure on $H_4/\sim$. Let $\{a_n\}_{n \in \mathbb{N}}$ be a Cauchy sequence in $H_4/\sim$ with respect to d . As in the proof of Proposition 18.13 we see that there exists a $C \in \mathbb{R}$ such that $d(0, a_n) \leq C$ for all $n \in \mathbb{N}$.

Claim. There exists an $r \in [0, 1)$ such that for any $z \in H_4$ with $d(0, p(z)) \leq C$ we have $|z| \leq r$.

Given $s \in [0, 1)$ we define

$$g(s) := \int_{t=0}^{t=s} \frac{\sqrt{2}}{1-t^2} dt.$$

Clearly we have $\lim_{s \rightarrow 1} g(s) = \infty$. In particular there exists an $r \in [0, 1)$ such that $g(s) > C + 1$ for all $s \geq \frac{1}{2}r$.

We claim that r has the desired property. So let $z = x + iy \in H_4$ with $|z| > r$. Since H_4 is a subset of the square spanned by ± 1 and $\pm i$ we see that we have $|z| \leq |x| + |y|$. Put differently, we have $|x| \geq \frac{1}{2}r$ or $|y| \geq \frac{1}{2}r$. Without loss of generality we can assume that $|x| \geq \frac{1}{2}r$.

We need to show that $d(0, p(z)) \geq C + 1$. It suffices to show that for any piecewise smooth path $\gamma: [0, a] \rightarrow H_4/\sim$ with $\gamma(0) = 0$ and $\gamma(a) = p(z)$ we have $\ell(\gamma) \geq C + 1$.

We denote by

$$\begin{aligned} q: H_4/\sim &\rightarrow (-1, 1) \\ [x + iy] &\mapsto x \end{aligned}$$

the projection onto the real part. It follows from Lemma 19.4 that this map is indeed well-defined. A straightforward calculation shows that for all $t \in [0, a]$ we have

$$\|\gamma'(t)\|_{H_4/\sim} \geq \|(q \circ \gamma)'(t)\|_{\mathbb{D}}.$$

It follows that

$$\ell(\gamma) = \int_0^a \|\gamma'(t)\|_{H_4/\sim} dt \geq \int_0^a \|(q \circ \gamma)'(t)\|_{\mathbb{D}} dt \geq d_{(\mathbb{D},g)}(0, \overbrace{q(\gamma(a))}^{=x}) = g(|x|) \geq C + 1.$$

$\uparrow$ above inequality $\uparrow$ definition of $d_{(\mathbb{D},g)}$ $\uparrow$ Proposition 18.10 $\downarrow$ since $|x| \geq \frac{1}{2}r$

This concludes the proof of the claim.

By the above claim the sequence $\{a_n\}_{n \in \mathbb{N}}$ lies in the compact set $p(\{z \in H_4 \mid |z| \leq r\})$. But by Proposition 18.13 the metric space $p(\{z \in H_4 \mid |z| \leq r\})$ is complete, so the sequence $\{a_n\}_{n \in \mathbb{N}}$ converges in $p(\{z \in H_4 \mid |z| \leq r\})$, in particular it converges in $p(H_4) = H_4/\sim$. Thus we have proved that $H_4/\sim$ is complete. This concludes the proof of (2).

Finally statement (3) is an immediate consequence of the definition of a hyperbolic metric and of statement (2). $\square$

We had just seen in Proposition 19.5 that $H_4/\sim$ admits a Möbius structure, in particular it is a 1-dimensional complex manifold. It is relatively straightforward to show that $H_4/\sim$ is homeomorphic to the sphere minus three points, or equivalently, to $\mathbb{C} \setminus \{0, 1\}$. We now show the much stronger statement that $H_4/\sim$ is actually biholomorphic to $\mathbb{C} \setminus \{0, 1\}$.

Proposition 19.6. *There exists a biholomorphism $H_4/\sim \rightarrow \mathbb{C} \setminus \{0, 1\}$.*

Remark. We equip $H_4/\sim$ with the Riemannian metric g coming from the Möbius structure and we equip $\mathbb{C} \setminus \{0, 1\}$ with the usual euclidean Riemannian metric that we denote by h .

By Proposition 19.5 the metric space $(H_4/\sim, g)$ is complete whereas $(\mathbb{C} \setminus \{0, 1\}, h)$ is evidently not complete. Thus we see that the biholomorphism $H_4/\sim \rightarrow \mathbb{C} \setminus \{0, 1\}$ from Proposition 19.6 is *not* an isometry.

sphere minus three points

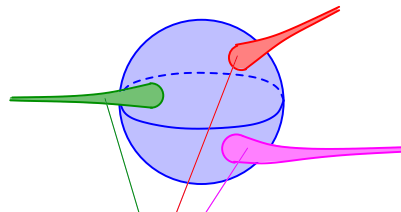
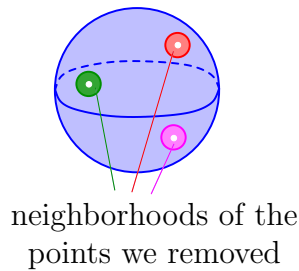


FIGURE 175.

Proof of Proposition 19.6. We denote by Δ the open subset of $\mathbb{D}$ bounded by the three hyperbolic lines with endpoints $-1, 1$ and i . (We refer to Figure 176 for an illustration of Δ .) It follows from applying the Riemann Mapping Theorem 17.8 together with Carathéodory's Theorem 17.9 that there exists a biholomorphism $\Psi: \Delta \rightarrow \mathbb{D}$ that extends to a homeomorphism $\Psi: \overline{\Delta} \rightarrow \overline{\mathbb{D}}$. We combine Ψ with the biholomorphism from $\mathbb{D}$ to $\mathbb{H}$ coming from

Proposition 18.2. This way we obtain a biholomorphism $\Phi: \Delta \rightarrow \mathbb{H}$ that extends to a homeomorphism $\Phi: \overline{\Delta} \rightarrow \overline{\mathbb{H}}$.

Using Lemma 18.4 we can, if necessary, compose Φ with a Möbius transformation of $\mathbb{H}$, to arrange that $\Phi(i) = \infty$ and that $\Phi(\{-1, 1\}) = \{0, 1\}$.

Note that Φ restricts to a homeomorphism $\Phi: \partial\Delta \rightarrow \partial\overline{\mathbb{H}} = \mathbb{R} \cup \{\infty\}$. By the above we have $\Phi(\partial\Delta) = \mathbb{R} \setminus \{0, 1\}$. Now we consider the map

$$\begin{aligned} \Psi: H_4 &\rightarrow \mathbb{C} \setminus \{0, 1\} \\ z &\mapsto \begin{cases} \Phi(z), & \text{if } z \in \overline{\Delta}, \\ \overline{\Phi(\bar{z})}, & \text{if } \bar{z} \in \overline{\Delta}, \end{cases} \end{aligned}$$

Note that by the above we have $f((-1, 1)) = (0, 1)$. Thus it follows from the Schwarz Reflection Principle, see Proposition 17.3, that the restriction of Ψ to the interior of H_4 is holomorphic. It follows from Lemma 19.4 that if P and Q are two equivalent points on ∂H_4 , then $\Psi(P) = \Psi(Q)$. Therefore the map Ψ factors through a map $H_4 / \sim \rightarrow \mathbb{C} \setminus \{0, 1\}$. Using once again the Schwarz Reflection Principle one can show that this map is in fact a biholomorphism. We leave the verification of this step as an exercise to the reader. $\square$

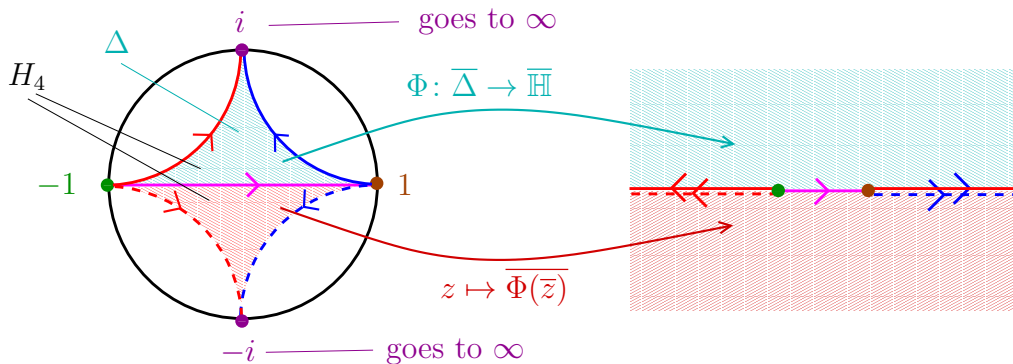


FIGURE 176. Illustration of the proof of Proposition 19.6

Definition. Given $g \in \mathbb{N}_0$ and $n \in \mathbb{N}_0$ we now denote by $F_{g,n}$ the surface that is obtained from the surface of genus g by removing n points. We refer to $F_{g,n}$ as the n -punctured surface of genus g .

For example the three-punctured sphere $F_{0,3}$ is the sphere with three points removed. As we had pointed out above, $F_{0,3}$ is diffeomorphic to $\mathbb{C} \setminus \{0, 1\}$. In particular, as we have just seen, the three-punctured sphere admits a hyperbolic structure. The following theorem says that “most” surfaces admit a hyperbolic structure.

Theorem 19.7. *Let $g \in \mathbb{N}_0$ and let $n \in \mathbb{N}_0$. If $2g + n - 2 > 0$, then $F_{g,n}$ admits a hyperbolic structure.*

Sketch of a proof. Let $g \in \mathbb{N}_0$ and let $n \in \mathbb{N}_0$. If $2g + n - 2 > 0$ then variations on the argument of Propositions 19.2 and 19.5 show that $F_{g,n}$ admits a hyperbolic metric. In Figure 177 we sketch the constructions needed to show that the once-punctured and the twice-punctured torus have a Möbius structure. The proof that the corresponding Riemannian manifolds are complete is similar to the proof of Proposition 19.5. $\square$

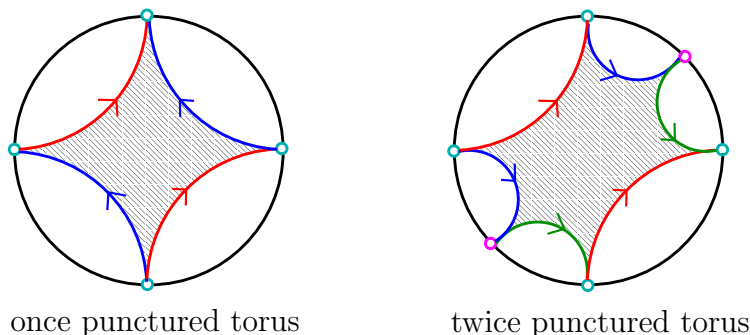


FIGURE 177.

19.4. The universal cover of surfaces.

Definition. Let $f: M \rightarrow N$ be a map between two Möbius manifolds.

- (1) We say that f is a *Möbius map* if for any chart $\Phi: U \rightarrow V$ in the Möbius structure of M and any chart $\Psi: X \rightarrow Y$ in the Möbius structure of N the restriction of the map

$$\underbrace{\Phi(U \cap f^{-1}(X))}_{\subset \mathbb{D}} \xrightarrow{\Psi \circ f \circ \Phi^{-1}} \underbrace{\Psi(X \cap f(U))}_{\subset \mathbb{D}}$$

to any component of $\Phi(U \cap f^{-1}(X))$ is given by a Möbius transformation.

- (2) We say that f is a *Möbius isomorphism*, if f is a *Möbius map*, if f is a bijection, and if f^{-1} is also a *Möbius map*.

The following lemma is an immediate consequence of the definitions and the fact, proved in Proposition 18.3, that Möbius transformations are biholomorphisms and isometries.

Lemma 19.8. *Let M and N be two Möbius manifolds and let $f: M \rightarrow N$ be a Möbius map. Then the following two statements hold:*

- (1) *the map f is a local biholomorphism,*
- (2) *the map f is a local isometry.*

The following theorem is the main result of this chapter.

Theorem 19.9. (Hadamard's Theorem) *Let $g \geq 2$, and let $M = H_{4g}/\sim$ be the surface of genus g with the Möbius structure constructed in Proposition 19.2. Then there exists a covering map $p: \mathbb{D} \rightarrow M$ with the following properties:*

- (1) p is a Möbius map,
- (2) p is a local isometry,
- (3) p is a local biholomorphism.

The hyperbolic disk $\mathbb{D}$ is of course simply connected. The theorem thus says that for any ≥ 2 the hyperbolic disk $\mathbb{D}$ is the universal cover of the surface of genus g . Thus we have now answered Question 15.8.

The proof of Theorem 19.9 will require the following two sections.

19.5. Proof of Theorem 19.9 I. In this section we will lay the technical foundations for the proof of Theorem 19.9.

In the following lemma we first summarize a few properties of Möbius maps.

Lemma 19.10.

- (1) If $U \subset \mathbb{D}$ is a connected subset and if $f: U \rightarrow \mathbb{D}$ is a Möbius map, then f is given by a Möbius transformation.
- (2) Let $f: M \rightarrow N$ be a map between two Möbius manifolds. If f is locally a Möbius map, then f is a Möbius map.
- (3) The composition of two Möbius maps is again a Möbius map.
- (4) A bijective Möbius map is a Möbius isomorphism.

Proof.

- (1) Let $U \subset \mathbb{D}$ be a connected subset and let $f: U \rightarrow \mathbb{D}$ be a Möbius map. It follows from Proposition 18.3 (4) that given $P \in U$ there exists a unique Möbius transformation ϕ_P such that $f = \phi_P$ in an open neighborhood of P . We consider the map

$$\begin{aligned} \phi: U &\rightarrow \text{set of all Möbius transformations} \\ P &\mapsto \phi_P. \end{aligned}$$

We equip the right-hand side of the map ϕ with the discrete topology. It follows from the definitions that this map is continuous. But then it follows from Lemma 1.13 that this map is constant. But that is exactly what we had tried to show.

- (2) This statement follows immediately from (1) and the definitions.
- (3) This statement is an immediate consequence of definitions and the fact, proved in Proposition 18.3, that the compositions of two Möbius transformations is again a Möbius transformation.
- (4) Let $f: M \rightarrow N$ be a bijective Möbius map. In Proposition 18.3 we had seen that the inverse of a Möbius transformation is again a Möbius transformation. It follows that f^{-1} is locally a Möbius map. But then it follows from (2), that f^{-1} is in fact a Möbius map. $\square$

Lemma 19.11. *Let U be an open connected subset of $\mathbb{D}$ and let M be a Möbius manifold. Furthermore let $\Phi, \Psi: U \rightarrow M$ be two Möbius maps.³⁶⁹ Suppose one of the following two conditions is satisfied:*

³⁶⁹Here we view $U \subset \mathbb{D}$ as equipped with the obvious Möbius structure.

- (1) *There exists an open non-empty subset of U on which Φ and Ψ agree, or*
- (2) *there exists a point $Q \in U$ with $\Phi(Q) = \Psi(Q)$ and a non-zero vector $v \in T_Q\mathbb{D}$ such that $d\Phi_Q(v) = d\Psi_Q(v)$,*

then the Möbius maps Φ and Ψ agree on all of U .

Proof. Let U be an open connected subset of $\mathbb{D}$, let M be a Möbius manifold and let $\Phi, \Psi: U \rightarrow M$ be two Möbius maps. We consider one more statement:

- (3) *There exists a point $Q \in U$ such that $\Phi(Q) = \Psi(Q)$ and such that $d\Phi_Q = d\Psi_Q$.*

Clearly we have (1) $\Rightarrow$ (3). We also have (2) $\Rightarrow$ (3). Indeed, the differentials $d\Phi_Q$ and $d\Psi_Q$ are homomorphisms between two 1-dimensional complex vector spaces. So if they agree on one non-zero vector, then they agree on all of the vector space $T_Q\mathbb{D}$.

So it suffices to show that if (3) is satisfied, then the maps Φ and Ψ agree on all of U . So suppose that there exists a point $Q \in U$ such that $\Phi(Q) = \Psi(Q)$ and such that $d\Phi_Q = d\Psi_Q$. We consider

$$Y := \{P \in U \mid \Phi(P) = \Psi(P) \text{ and } d\Phi_P = d\Psi_P\}.$$

By our hypothesis we know that Y contains Q , in particular Y is non-empty. Since Φ and Ψ are smooth it follows that Y is closed. It remains to show that Y is open.

So let $P \in Y$. We pick a map $\Theta: W \rightarrow \mathbb{D}$ from the Möbius structure of M such that W is an open neighborhood of $\Phi(P) = \Psi(P)$. We denote by V the component of $\Phi^{-1}(W) \cap \Psi^{-1}(W)$ that contains P . Since Φ and Ψ are Möbius maps it follows from Lemma 19.10 (3) that the maps $\Theta \circ \Phi: V \rightarrow \mathbb{D}$ and $\Theta \circ \Psi: V \rightarrow \mathbb{D}$ are also Möbius maps. Since V is connected it follows from Lemma 19.10 (1) that both maps are given by Möbius transformations. These two Möbius transformations agree at P and they have the same differential at P . It follows from Proposition 18.3 (5) that Φ and Ψ agree on an open neighborhood of P . This shows that Y is also open. $\square$

Definition. Let M be a Möbius manifold and let $Q \in M$ and $r > 0$. We say $B_r^M(Q)$ is *small* if there exists a Möbius isomorphism $\Omega: B_r^{\mathbb{D}}(0) \rightarrow B_r^M(Q)$ with $\Omega(0) = Q$.

Proposition 19.12. *Let M be a Möbius manifold and let $Q \in M$. Then there exists an $r > 0$ such that $B_r^M(Q)$ is small.*

In the proof of Proposition 19.12 we will make use of the following lemma.

Lemma 19.13. *Let (X, g) and (Y, h) be Riemannian manifolds. Let $Q \in X$ and $r > 0$. Let $f: B_{3r}^X(Q) \rightarrow (Y, h)$ be a local isometry and let $S, T \in B_r^X(Q)$. Then we have*

$$d_Y(f(S), f(T)) \leq d_X(S, T).$$

Proof. Let $S, T \in B_r^X(Q)$. Furthermore let $\epsilon > 0$. By the triangle inequality we have $d_X(S, T) < 2r$. In particular there exists a piecewise smooth path γ in X from S to T such that $\ell_X(\gamma) < 2r$ and with $\ell_X(\gamma) < d_X(S, T) + \epsilon$. Given any t we have

$$d_X(Q, \gamma(t)) \leq d_X(Q, S) + d_X(S, \gamma(t)) < r + \ell_X(\gamma) < 3r.$$

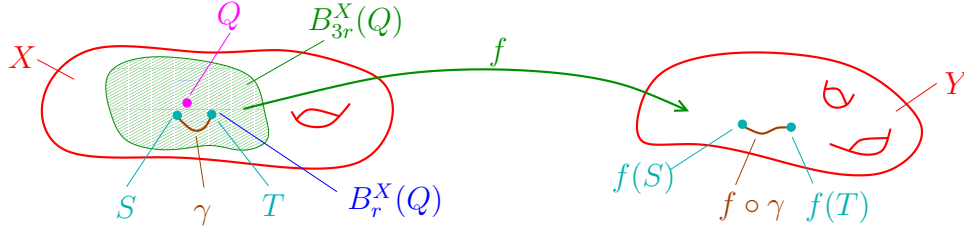


FIGURE 178. Illustration for Lemma 19.13.

Thus we have shown that the image of γ lies in $B_{3r}^X(Q)$. In particular the image of γ lies in the domain of f . Thus we see that

$$\begin{array}{ccc}
 d_Y(f(S), f(T)) & \leq & \ell_Y(f \circ \gamma) = \ell_X(\gamma) < d_X(S, T) + \epsilon. \\
 \uparrow & & \uparrow \\
 \text{since } f \circ \gamma \text{ is a path from } & & \text{by Lemma 18.9, since} \\
 f(S) \text{ to } f(T) & & f \text{ is a local isometry}
 \end{array}$$

Since this inequality holds for all $\epsilon > 0$ we see that $d_Y(f(S), f(T)) \leq d_X(S, T)$. $\square$

Now we can provide the proof of Proposition 19.12.

Proof of Proposition 19.12. Let M be a Möbius manifold and let $Q \in M$. We pick a map $\Phi: U \rightarrow V$ from the Möbius structure of M such that U is an open neighborhood of Q . By Proposition 18.3 (3), after possibly composing Φ with a Möbius transformation, we can furthermore assume that $\Phi(Q) = 0$. We write $\Omega := \Phi^{-1}$. By Proposition 18.8 (3) and by the openness of V there exists an $r > 0$ such that $B_{3r}^M(Q) \subset U$ and such that $B_{3r}^{\mathbb{D}}(0) \subset V$.³⁷⁰

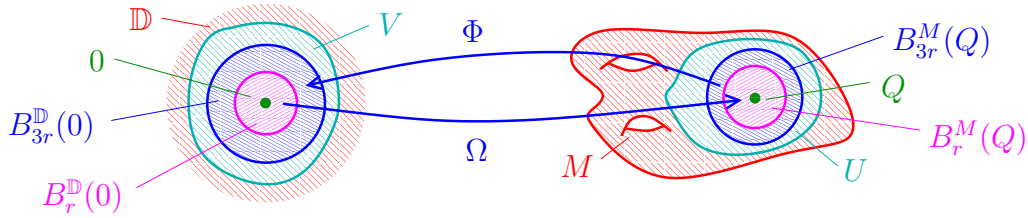


FIGURE 179. Illustration for the proof of Proposition 19.12.

It follows immediately from Lemmas 19.8 and 19.13 and from our choice of r that the following holds:

- (1) For any $S, T \in B_r^M(Q)$ we have $d_{\mathbb{D}}(\Phi(S), \Phi(T)) \leq d_M(S, T)$.
- (2) For any $S, T \in B_r^{\mathbb{D}}(0)$ we have $d_M(\Omega(S), \Omega(T)) \leq d_{\mathbb{D}}(S, T)$.

³⁷⁰Note the discrete factor 3 in the radius of the ball.

It follows immediately from these two inequalities, from $\Phi(Q) = 0$ and from the fact that $\Phi = \Omega^{-1}$ that Ω restricts to an isometry $\Omega: (B_r^{\mathbb{D}}(0), d_{\mathbb{D}}) \rightarrow (B_r^M(Q), d_M)$ of metric spaces which is furthermore a bijection. Furthermore, since Φ is a Möbius isomorphism it follows that the inverse map $\Omega = \Phi^{-1}: (B_r^{\mathbb{D}}(0), d_{\mathbb{D}}) \rightarrow (B_r^M(Q), d_M)$ is also a Möbius isomorphism. $\square$

Lemma 19.14. *Let M be a Möbius manifold. Let $Q \in M$ and let $r > 0$ such that $B_r^M(Q)$ is small. Let $\Phi: \mathbb{D} \rightarrow M$ be a Möbius map such that $B_r^M(Q) \subset \Phi(\mathbb{D})$. Then the following hold:*

- (1) *For any $X \in \Phi^{-1}(Q)$ the map Φ restricts to a Möbius isomorphism $B_r^{\mathbb{D}}(X) \rightarrow B_r^M(Q)$.*
- (2) *Let $s \leq \frac{r}{2}$ and let $Y \in \mathbb{D}$ such that $\Phi(Y) \in B_s^M(Q)$. Then there exists an $X \in \Phi^{-1}(Q)$ with $Y \in B_s^{\mathbb{D}}(X)$.*

Proof. Let M be a Möbius manifold. Let $Q \in M$ and let $r > 0$ such that $B_r^M(Q)$ is small. We pick a Möbius isomorphism $\Omega: B_r^{\mathbb{D}}(0) \rightarrow B_r^M(Q)$ as in the definition of “small”. Let $\Phi: \mathbb{D} \rightarrow M$ be a Möbius map such that $B_r^M(Q) \subset \Phi(\mathbb{D})$.

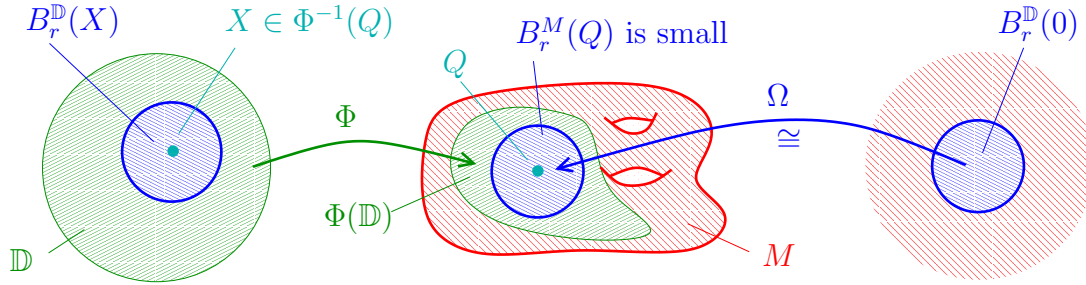


FIGURE 180. Illustration for the proof of Lemma 19.14 (1).

- (1) Let $X \in \Phi^{-1}(Q)$. It follows immediately from the fact that Möbius maps are local isometries and from Lemma 19.13 that $\Phi(B_r^{\mathbb{D}}(X)) \subset B_r^M(Q)$. Therefore we can consider the Möbius map $\Omega^{-1} \circ \Phi: B_r^{\mathbb{D}}(X) \rightarrow B_r^{\mathbb{D}}(0)$. By Lemma 19.10 this Möbius map is the restriction of a Möbius transformation. In particular it is a Möbius isomorphism. But since Ω is also a Möbius isomorphism it follows that $\Phi: B_r^{\mathbb{D}}(X) \rightarrow B_r^M(Q)$ is also a Möbius isomorphism.
- (2) Let $s \leq \frac{r}{2}$ and let $Y \in \mathbb{D}$ such that $\Phi(Y) \in B_s^M(Q)$, i.e. such that $d_M(Q, \Phi(Y)) < s$. By the triangle inequality we have

$$B_s^M(\Phi(Y)) \subset B_{2s}^M(Q) \subset B_r^M(Q).$$

The same argument as in (1) shows that the map Φ restricts to a Möbius isomorphism $B_s^{\mathbb{D}}(Y) \rightarrow B_s^M(\Phi(Y))$. We have $d_M(Q, \Phi(Y)) < s$, i.e. we have $Q \in B_s^M(\Phi(Y))$. Hence there exists an $X \in B_s^{\mathbb{D}}(Y)$ with $\Phi(X) = Q$. We refer to Figure 181 for an illustration. $\square$

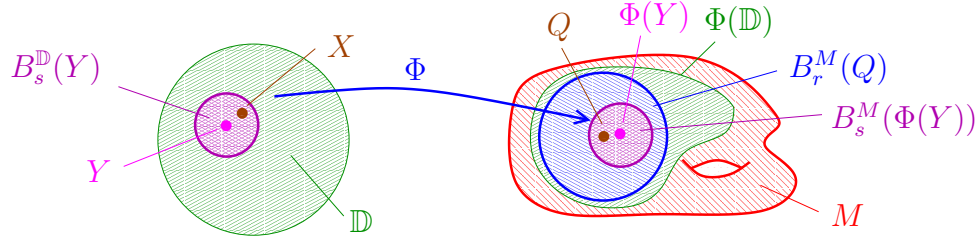


FIGURE 181. Illustration for the proof of Lemma 19.14 (2).

19.6. Proof of Theorem 19.9 II. In this section we will now finally provide the proof of Theorem 19.9. We will only cover the case $g = 2$, the proof of the general case is almost identical.

Throughout this section let $M = H_8 / \sim$ be the surface of genus 2 with the Möbius structure constructed in Proposition 19.2. In the following we will show that there exists a Möbius map $p: \mathbb{D} \rightarrow M$ that is in fact a covering map. It then follows from Lemma 19.8 that p is also a local isometry and a local biholomorphism.

For the remainder of the proof we adopt the following notation and conventions.

- (1) We view $\mathring{H}_8$ as a subset of $\mathbb{D}$ and also of $H_8 / \sim$.
- (2) Given $Z \in \mathbb{D}$ we denote by

$$\begin{aligned} \gamma_Z: [0, 1] &\rightarrow \mathbb{D} \\ t &\mapsto t \cdot Z \end{aligned}$$

the radial path from the origin $0 \in \mathbb{D}$ to Z .

- (3) We say that $U \subset \mathbb{D}$ is *convex* if it is convex in the usual euclidean sense, i.e. if given any $P, Q \in U$ the points $t \cdot P + (1 - t) \cdot Q$, $t \in [0, 1]$ also lie in U . On several occasions throughout the proof we will make use of the following elementary observations:
 - (a) convex sets are path-connected,
 - (b) the intersection of two convex sets is also convex,
in particular, combining (a) and (b) we get
 - (c) the intersection of two convex sets is path-connected.

Our first goal is to construct a suitable map $p: \mathbb{D} \rightarrow M = H_8 / \sim$. So let $Z \in \mathbb{D}$. We write $\gamma = \gamma_Z$. We say $t \in [0, 1]$ is *good* if there exists a convex open neighborhood U of $\gamma([0, t])$ and a map $\Psi: U \rightarrow M$ which has the following properties:

- (i) Ψ is a Möbius map,
- (ii) it restricts to the identity on $\mathring{H}_8 \cap U \subset M$.

We start out with the following lemma.

Lemma 19.15. *The point $t = 1$ is good.*

Proof. We set

$$T := \{t \in [0, 1] \mid t \text{ is good}\}.$$

We start out with the following observations:

- (1) We have that $0 \in T$ since the identity on the open neighborhood $\overset{\circ}{H}_8$ of 0 has the required properties.³⁷¹
- (2) Being good is clearly an open condition, hence $T \subset [0, 1]$ is open.

We want to show that $T = [0, 1]$. By the above, and since $[0, 1]$ is connected, it suffices to show that $s := \sup(T)$ is good.

Since $0 \in S$ and since T is open we see that $s > 0$. Therefore there exists an increasing sequence $\{t_n\}_{n \in \mathbb{N}}$ of good numbers $t_n \in [0, t)$ that converges to s . Since each t_n is good we can pick for each n a convex open neighborhood U_n of $\gamma([0, t_n])$ and a map $\Psi_n: U_n \rightarrow M$ that has the desired properties (i) and (ii).

Let $m, n \in \mathbb{N}$. The Möbius maps Ψ_m and Ψ_n agree on the non-empty open connected subset $\overset{\circ}{H}_8 \cap U_n \cap U_m$. It follows from Lemma 19.11 that the maps Ψ_m and Ψ_n agree on $U_n \cap U_m$.

By the continuity of γ the sequence $\{\gamma(t_n)\}_{n \in \mathbb{N}}$ converges to $\gamma(t)$, in particular it is a Cauchy sequence with respect to $d_{\mathbb{D}}$. It follows from Lemma 18.9 that $\{\Psi_n(\gamma(t_n))\}_{n \in \mathbb{N}}$ is a Cauchy sequence with respect to d_M .³⁷² It follows from Proposition 19.2 that the metric space $(M = H_8 / \sim, d_M)$ is complete. Therefore the Cauchy sequence $\{\Psi_n(\gamma(t_n))\}_{n \in \mathbb{N}}$ converges to a point $S \in M$.

It follows from Proposition 19.12 that there exists an $r > 0$ and a Möbius isomorphism $\Omega: B_r^{\mathbb{D}}(0) \rightarrow B_r^M(S)$ such that $\Omega(0) = S$. We pick an n such that $\Psi_n(\gamma(t_n)) \in B_r^M(S)$ and such that $d_{\mathbb{D}}(\gamma(t), \gamma(s)) < r$. We write $t = t_n$, $\Psi = \Psi_n$, $U = U_n$ and we write $X = \gamma(t)$. Let $v \in T_X \mathbb{D}$ be a non-zero vector. We let $w := d(\Omega^{-1} \circ \Psi)_X(v)$. The Möbius maps Ω and Ψ are in particular local isometries, therefore we have $\|w\| = \|v\|$. It follows from Proposition 18.3 (4) that there exists a Möbius transformation Θ with $\Theta(X) = \Omega^{-1}(\Psi(X))$ and such that $d\Theta_X(v) = w$.

The situation is illustrated in Figure 182. The Möbius maps $\Omega \circ \Theta: B_r^{\mathbb{D}}(X) \rightarrow M$ and $\Psi: U \rightarrow M$ agree at $X = \gamma(t)$ and we have $d(\Omega \circ \Theta)_X(v) = d\Psi_X(v)$. It follows from Lemma 19.11 that $\Omega \circ \Theta$ and Ψ agree on the path-connected set $B_r^{\mathbb{D}}(X) \cap U$.

Now we consider the map

$$\begin{aligned} \Phi: U \cup B_r^{\mathbb{D}}(\gamma(t)) &\rightarrow M \\ Q &\mapsto \begin{cases} \Psi(Q), & \text{if } Q \in U, \\ \Omega^{-1}(\Theta(Q)), & \text{if } Q \in B_r^{\mathbb{D}}(\gamma(t)). \end{cases} \end{aligned}$$

³⁷¹Hereby we use that for each point $Z \in \overset{\circ}{H}_8$ the identity map $\text{id}: \overset{\circ}{H}_8 \rightarrow \overset{\circ}{H}_8$ is part of the Möbius structure of M .

³⁷²Indeed, suppose we have $\epsilon > 0$. Since $\{\gamma(t_n)\}_{n \in \mathbb{N}}$ is a Cauchy sequence there exists an N such that $d_{\mathbb{D}}(\gamma(t_m), \gamma(t_n)) < \epsilon$ for all $m, n \geq N$. Now let $m, n \geq N$. Without loss of generality we have $m > n$. Then

$$\begin{aligned} d_M(\Psi_m(\gamma(t_m)), \Psi_n(\gamma(t_n))) &= d_M(\Psi_m(\gamma(t_m)), \Psi_m(\gamma(t_n))) \leq d_{\mathbb{D}}(\gamma(t_m), \gamma(t_n)) < \epsilon. \\ &\quad \uparrow \qquad \qquad \qquad \uparrow \\ &\text{since } \Psi_m \text{ and } \Psi_n \text{ agree on } \gamma|_{[0, t_n]} \qquad \qquad \text{Lemma 18.9} \end{aligned}$$

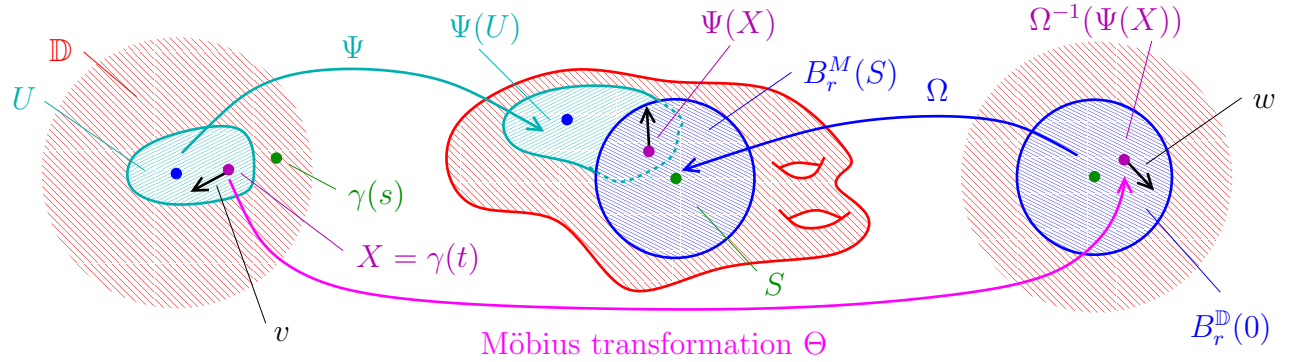


FIGURE 182.

By the above discussion this map is well-defined and it is locally a Möbius map. It follows from Lemma 19.10 (3) that this map is a Möbius map. It follows from $d_{\mathbb{D}}(\gamma(t), \gamma(s)) < r$ and the fact that γ is a geodesic in $\mathbb{D}$ that $\gamma([t, s]) \subset B_r^{\mathbb{D}}(\gamma(t))$, in particular $U \cup B_r^{\mathbb{D}}(\gamma(t))$ is an open subset of $\mathbb{D}$ that contains $\gamma([0, t]) \cup \gamma([t, s]) = \gamma([0, s])$. It is now straightforward to see that we can find a *convex* open subset W of $U \cup B_r^{\mathbb{D}}(\gamma(t))$ that contains $\gamma([0, s])$. The Möbius map $\Phi: W \rightarrow M$ now certifies that s is also good. $\square$

Thus we have now shown that there exists a convex open neighborhood U of $\gamma([0, 1])$ and a local isometry $\Psi: U \rightarrow H_8/\sim$ which agrees with the identity map on $U \cap \mathring{H}_8$. We define $p(Z) := \Psi(Z)$. By Lemma 19.11 this definition does not depend on the choice of U and Ψ .

Lemma 19.16. *The map $p: \mathbb{D} \rightarrow M$ is a Möbius map.*

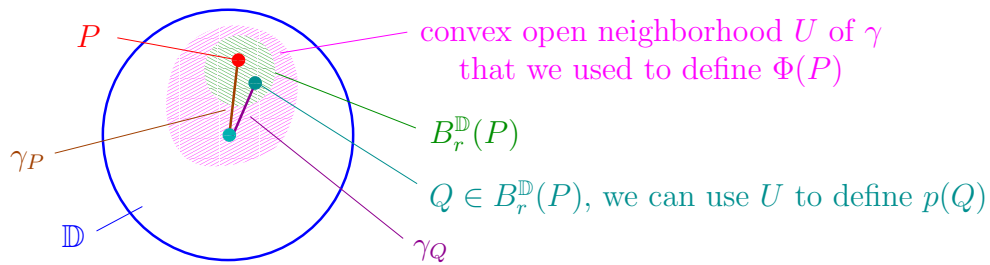


FIGURE 183. Illustration of the proof of Lemma 19.16.

Proof. By Lemma 19.10 (2) it suffices to show that p is locally a Möbius map. This means that it suffices to show that every point $P \in \mathbb{D}$ admits an open neighborhood W such that the restriction of p to W is a Möbius map. Thus let $P \in \mathbb{D}$. By the above claim there exists a convex open neighborhood U of $\gamma_P([0, 1])$ and a Möbius map $\Psi: U \rightarrow H_8/\sim$ which

restricts to the identity on $U \cap \overset{\circ}{H}_8$. Since U is open we can find an $r > 0$ such that for all $Q \in B_r^{\mathbb{D}}(P)$ the image of the radial path γ_Q from 0 to Q also lies in U . (We refer to Figure 183 for an illustration.) This implies that we can use the Möbius map $\Psi: U \rightarrow M$ to define p for any $Q \in B_r^{\mathbb{D}}(P)$. Put differently, $p = \Psi$ on $B_r^{\mathbb{D}}(P)$. We have thus shown that the restriction of p to the open neighborhood $W = B_r^{\mathbb{D}}(P)$ of P is a Möbius map. $\square$

The following lemma now concludes the proof of Theorem 19.9

Lemma 19.17. *The map $p: \mathbb{D} \rightarrow M = H_8/\sim$ is a covering map.*

Proof. First we show that the map $p: \mathbb{D} \rightarrow M = H_8/\sim$ is surjective. Recall that by construction p is the identity on $\overset{\circ}{H}_8$. Since p is continuous³⁷³ it follows that the restriction of p to $H_8 \rightarrow M = H_8/\sim$ is surjective.³⁷⁴

It remains to show that every $Q \in M$ admits a uniformly covered neighborhood. Let $Q \in M$. By Proposition 19.12 there exists an $r > 0$ such that $B_{2r}^M(Q)$ is small.³⁷⁵ We claim that $B_r^M(Q)$ is uniformly covered. More precisely, we make the following claims:

- (1) $p^{-1}(B_r^M(Q)) = \bigcup_{X \in p^{-1}(Q)} B_r^{\mathbb{D}}(X)$,
- (2) for $X \neq X' \in p^{-1}(Q)$ we have $B_r^{\mathbb{D}}(X) \cap B_r^{\mathbb{D}}(X') = \emptyset$,
- (3) given any $X \in p^{-1}(Q)$ the restriction of p to $B_r^{\mathbb{D}}(X) \rightarrow B_r^M(Q)$ is a homeomorphism.

We apply Lemma 19.14 to the map $p: \mathbb{D} \rightarrow M$. It gives us the following three facts:

- (a) If $Y \in \mathbb{D}$ satisfies $p(Y) \in B_r^M(Q)$, then there exists an $X \in p^{-1}(Q)$ with $Y \in B_r^{\mathbb{D}}(X)$.
- (b) For any $X \in p^{-1}(Q)$ the restriction of p to $B_r^{\mathbb{D}}(X) \rightarrow B_r^M(Q)$ is a Möbius isomorphism, in particular it is a homeomorphism.
- (c) For any $X \in p^{-1}(Q)$ the restriction of p to $B_{2r}^{\mathbb{D}}(X)$ is injective.

Now the desired properties (1) and (3) follow immediately from (a) and (b). It remains to prove (2). Suppose that we have $X \neq X' \in p^{-1}(Q)$ with $B_r^{\mathbb{D}}(X) \cap B_r^{\mathbb{D}}(X') \neq \emptyset$. We need to show $X = X'$. We pick $Y \in B_r^{\mathbb{D}}(X) \cap B_r^{\mathbb{D}}(X')$. The triangle inequality implies that $X' \in B_{2r}^{\mathbb{D}}(X)$. But p restricted to $B_{2r}^{\mathbb{D}}(X)$ is injective by (c). Hence $X = X'$. $\square$

19.7. Picard's Theorem. We also have the following variation on Theorem 19.9.

Proposition 19.18. *Let $H_4/\sim$ be the three-punctured sphere with the Möbius structure constructed in Proposition 19.5. Then there exists a covering map $p: \mathbb{D} \rightarrow H_4/\sim$ with the following properties:*

- (1) p is a Möbius map,
- (2) p is a local isometry,
- (3) p is a local biholomorphism.

³⁷³The map p is a Möbius map, in particular it is a local biholomorphism which implies that it is a local homeomorphism. Thus it follows from Lemma 1.20 that p itself is continuous.

³⁷⁴This is the only part of the proof where we actually use that we work with $H_8/\sim$, the remainder of the proof works for any manifold M with a Möbius structure such that the corresponding Riemannian manifold is complete.

³⁷⁵Note the factor 2.

Proof. The proof of Proposition 19.18 is almost the same as the proof of Theorem 19.9. In the proof of Theorem 19.9 we used that $H_8/\sim$ is complete. In the present case it follows from Proposition 19.5 (3) that $H_4/\sim$ is complete. The construction of the Möbius map $\Phi: \mathbb{D} \rightarrow H_4/\sim$ is verbatim the same as the construction of the Möbius map $\Phi: \mathbb{D} \rightarrow H_8/\sim$. The same argument as before shows that $\Phi: \mathbb{D} \rightarrow H_4/\sim$ is surjective and that it is a covering map. $\square$

Remark. We consider again the covering $p: \mathbb{D} \rightarrow H_8/\sim$ that we had just constructed. It follows from Corollary 15.4 that the interior of H_8 is uniformly covered. In particular we see that

$$\mathbb{D} \setminus \underbrace{p^{-1}(\partial H_8)}_{\text{“1-dimensional”}} = \bigsqcup \text{copies of } \mathring{H}_8.$$

Put differently, up to the “one-dimensional” subset $p^{-1}(\partial H_8)$ we can cover $\mathbb{D}$ by infinitely many disjoint copies of the open hyperbolic octagon. Such a decomposition is often called a *tessellation*. This tessellation is shown in Figure 184 on the left. The same argument also applies to the covering $\mathbb{D} \rightarrow H_4/\sim$ that we also had just constructed. The corresponding tessellation of $\mathbb{D}$ is shown in Figure 184 on the right.

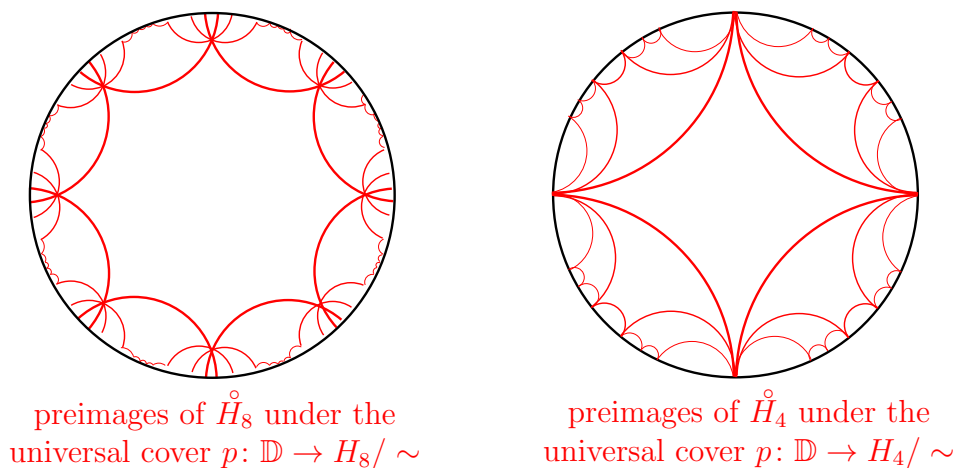


FIGURE 184.

At the end of the course Algebraic Topology I we return to complex analysis. There are different types of holomorphic functions defined on $\mathbb{C}$.

- (1) There are the (rather dull) constant functions.
- (2) There are holomorphic functions $f: \mathbb{C} \rightarrow \mathbb{C}$ that are surjective. For example it is an easy consequence of the fundamental theorem of algebra that all maps defined by non-constant polynomials are surjective.

- (3) There are also non-constant maps that are not surjective, for example the exponential function $z \mapsto \exp(z)$ is never zero³⁷⁶, in fact its image is $\mathbb{C} \setminus \{0\}$.

On the other hand we had seen in Analysis IV that the image of a non-constant holomorphic function on $\mathbb{C}$ is dense.³⁷⁷

We can now prove Picard's Theorem³⁷⁸ which is a significantly stronger statement than the above statement that the image is dense.

Theorem 19.19. (Picard's Theorem) *Let $f: \mathbb{C} \rightarrow \mathbb{C}$ be a non-constant holomorphic function. Then there exists at most one $z \in \mathbb{C}$ which does not lie in the image of f .*

Proof. Let $f: \mathbb{C} \rightarrow \mathbb{C}$ be a holomorphic function such that there exist two different complex numbers a, b that do not lie in the image of f . We need to show that f is constant. We consider the biholomorphism

$$\begin{aligned} \alpha: \mathbb{C} \setminus \{a, b\} &\rightarrow \mathbb{C} \setminus \{0, 1\} \\ z &\mapsto \frac{z-a}{b-a}. \end{aligned}$$

Furthermore we denote by $\beta: \mathbb{C} \setminus \{0, 1\} \rightarrow H_4/\sim$ the biholomorphism from Proposition 19.6. Now we consider the following diagram of maps

$$\begin{array}{ccccccc} & & & & \mathbb{D} & & \\ & & & & \downarrow \Phi & & \\ \mathbb{C} & \xrightarrow{f} & \mathbb{C} \setminus \{a, b\} & \xrightarrow[\cong]{\alpha} & \mathbb{C} \setminus \{0, 1\} & \xrightarrow[\cong]{\beta} & H_4/\sim \end{array}$$

where $\Phi: \mathbb{D} \rightarrow H_4/\sim$ is the covering map from Proposition 19.18. Recall that Φ is a Möbius map which implies by Lemma 19.8 that Φ is a local biholomorphism.

Since $\mathbb{C}$ is simply connected we can appeal to Proposition 15.2 to obtain a lift of the map $\beta \circ \alpha \circ f: \mathbb{C} \rightarrow H_4/\sim$ to the universal cover. More precisely, there exists a map $\widetilde{\beta \circ \alpha \circ f}: \mathbb{C} \rightarrow \mathbb{D}$ such that the following diagram commutes

$$\begin{array}{ccccccc} & & & & \mathbb{D} & & \\ & & & & \downarrow \Phi & & \\ \mathbb{C} & \xrightarrow{f} & \mathbb{C} \setminus \{a, b\} & \xrightarrow[\cong]{\alpha} & \mathbb{C} \setminus \{0, 1\} & \xrightarrow[\cong]{\beta} & H_4/\sim \end{array}$$

$\widetilde{\beta \circ \alpha \circ f}$

As we had pointed out in Lemma 17.7, it follows from the fact that $\widetilde{\beta \circ \alpha \circ f}$ is holomorphic and the fact that Φ is a local biholomorphism, that the map $\widetilde{\beta \circ \alpha \circ f}: \mathbb{C} \rightarrow \mathbb{D}$ is also holomorphic. But $\mathbb{D}$ is of course bounded. Therefore it follows from Liouville's

³⁷⁶Why not?

³⁷⁷More precisely, we had shown that if $f: \mathbb{C} \rightarrow \mathbb{C}$ is a holomorphic function, then given any $z \in \mathbb{C}$ and any $r > 0$ there exists a $w \in \mathbb{C}$ with $f(w) \in B_r(z)$.

³⁷⁸Emile Picard (1856-1941) was a French mathematician, we encountered him already in Analysis II, when we proved the Picard-Lindelöf Theorem.

Theorem 17.1 that $\widetilde{\beta \circ \alpha \circ f}$ is constant. But then $\beta \circ \alpha \circ f$ is also constant. Since α and β are biholomorphisms this implies that f itself is already constant. $\square$

Remark. In the literature Theorem 19.19 is sometimes referred to as Picard's Little Theorem, to distinguish it from Picard's Great Theorem which says that if z_0 is an essential singularity of a holomorphic function, then in any neighborhood of z_0 the function f assumes all, but possibly one, value *infinitely* many times. We refer to [C, Section 12] for details.

20. THE DECK TRANSFORMATION GROUP (*)

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Definition. Let $p: X \rightarrow B$ be a covering.

- (1) A *deck transformation* is a homeomorphism $d: X \rightarrow X$ such that $d(p(x)) = p(x)$ for all x , i.e. such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{d} & X \\ & \cong & \\ & p & \swarrow p \\ & & B. \end{array}$$

The group of deck transformations³⁸⁰ is called the *deck transformation group* of the covering $p: X \rightarrow B$ and we denote it by $D(p: X \rightarrow B)$.

- (2) We say that the covering is *regular* if for any $b \in B$ the deck transformation group D acts transitively on $p^{-1}(b)$, i.e. if given any two points x, x' with $p(x) = p(x') = b$ there exists a deck transformation d with $d(x) = x'$. Otherwise we say that the covering is *irregular*.

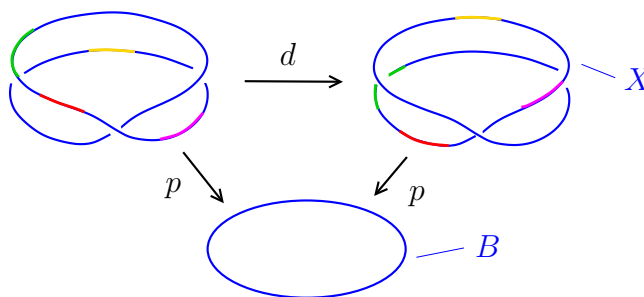


FIGURE 185. Schematic image of a deck transformation.

Example.

- (1) We consider the covering

$$\begin{aligned} p: S^1 &\rightarrow S^1 \\ z &\mapsto z^2. \end{aligned}$$

Then the map $d: S^1 \rightarrow S^1$ given by $d(z) := -z$ is a deck transformation. It is now clear that the covering is regular.

³⁷⁹This section is not part of the official lecture notes for Algebraic Topology I.

³⁸⁰Why do deck transformations form a group?

- (2) Let M be a non-orientable path-connected manifold. As in the proof of Proposition 16.4 we consider the covering

$$\begin{aligned} \widetilde{M} &:= \{(Q, O) \mid Q \in M \text{ and } O \text{ an orientation of } T_Q M\} \rightarrow M \\ (Q, O) &\mapsto Q. \end{aligned}$$

Then the map $d: \widetilde{M} \rightarrow \widetilde{M}$ given by $d((Q, O)) := (Q, -O)$ is a deck transformation of p . It is straightforward to see that the covering is regular.

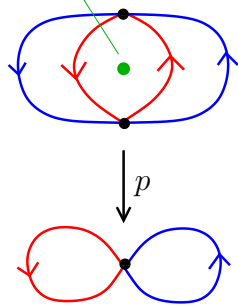
- (3) We consider again the first two coverings of graphs from Figure 60 which are also shown again in Figure 186. The covering maps are the ones suggested by the colors and the directions. We first make the following observations. A deck transformation d does the following:

- (a) it sends a vertex again to a vertex,
- (b) it sends a blue edge to a blue edge, preserving the orientation,
- (c) it sends a red edge to a red edge, preserving the orientation,
- (d) if e is an edge with endpoints P and Q , then $d(e)$ is an edge with endpoints $d(P)$ and $d(Q)$, in particular if e is a loop, then $d(e)$ is also a loop.

We now consider the two examples.

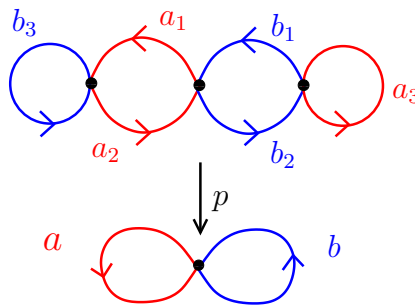
- (a) The example on the left admits one non-trivial deck transformation, namely the map that is given by “rotation by π ” around the center of the graph. The covering is now easily seen to be regular.
- (b) The loop a_3 in the graph on the right is the only red loop, so any deck transformation has to send it to itself. It follows that there can be no deck transformation that sends the vertex to the right to any of the other vertices.³⁸¹ Thus the covering on the right is irregular.

deck transformation is
given by rotation by π around P



regular covering

a deck transformation has to send the red loop a_3
to another loop, but a_3 is the only red loop



irregular covering

FIGURE 186.

³⁸¹Does this covering admit any deck transformation?

- (4) Finally we consider the 4-fold coverings illustrated in Figure 187. They are both regular coverings. For the graph on the left the deck transformation group is isomorphic to $\mathbb{Z}_4$, a generator is given by “rotation by $\frac{\pi}{2}$ ”. The deck transformation group on the right is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, here one $\mathbb{Z}_2$ summand is given by “rotation by π ” and the other one corresponds to “swapping the inner and the outer circle”.

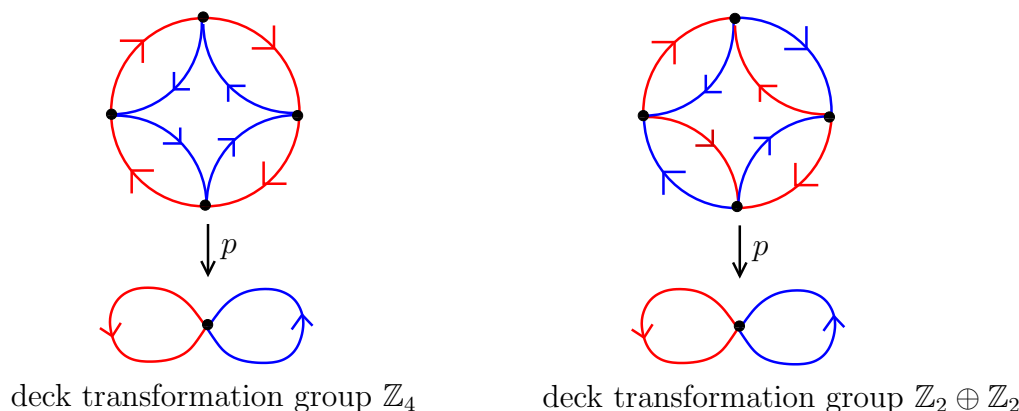


FIGURE 187.

Lemma 20.1. *Let $p: (X, x_0) \rightarrow (B, b_0)$ be a covering of pointed path-connected topological spaces and let d_1 and d_2 be two deck transformations of p . Then the following holds*

$$d_1 = d_2 \iff d_1(x_0) = d_2(x_0).$$

Proof. Evidently we only need to show the “ $\Leftarrow$ ”-direction. Thus let d be a deck transformation of p . One way of stating that d is a deck transformation is to state that the following diagram commutes:

$$\begin{array}{ccc} & (X, d(x_0)) & \\ & \uparrow d & \downarrow p \\ (X, x_0) & \xrightarrow{p} & (B, b_0). \end{array}$$

Put differently, the map d can be viewed as a lift of the map $(X, x_0) \rightarrow (B, b_0)$ to the covering $(X, d(x_0)) \rightarrow (B, b_0)$. By the uniqueness of lifts³⁸², that we had shown in Proposition 15.2, we now see that two deck transformations d_1 and d_2 of p with $d_1(x_0) = d_2(x_0)$ agree. $\square$

Example. We let $B = (S^1 \times \{a\} \sqcup S^1 \times \{b\}) / (1, a) \sim (1, b)$, i.e. B is the wedge of two circles and we consider

$$X = (S^1 \sqcup (S^1 \times \mathbb{Z}_5)) / (e^{2\pi i n/5} \sim (1, n)) \text{ for } n \in \mathbb{Z}_5,$$

³⁸²Here we use that X is path-connected.

i.e. X is a circle with five circles attached at the points $e^{2\pi ik/5}$, $k = 0, \dots, 4$. Then the map

$$p: (S^1 \sqcup (S^1 \times \mathbb{Z}_5)) / \sim \rightarrow G = (S^1 \times \{a\} \sqcup S^1 \times \{b\}) / (1, a) \sim (1, b)$$

that is given by $p(z) = (z^5, a)$ for $z \in S^1$ and $p((z, n)) = (z, b)$ for $(z, n) \in S^1 \times \mathbb{Z}_5$, is a covering map of degree five. This covering is illustrated in Figure 188.

We consider the map

$$\begin{aligned} \mathbb{Z}_5 &\rightarrow D(p: X \rightarrow B) \\ m &\mapsto \begin{pmatrix} X &\rightarrow X \\ z &\mapsto e^{2\pi im/5} z, & \text{for } z \in S^1, \\ (z, n) &\mapsto (z, m+n) & \text{for } (z, n) \in S^1 \times \mathbb{Z}_5 \end{pmatrix} \end{aligned}$$

In Figure 188 this corresponds to “rotation X by the angle $\frac{2\pi m}{5}$ ”. It is straightforward to show that this map is well-defined, i.e. the maps on the right are deck transformations, and using Lemma 20.1 one can easily see that this map is an isomorphism. Also one can now convince oneself easily that the covering $p: X \rightarrow B$ is regular.

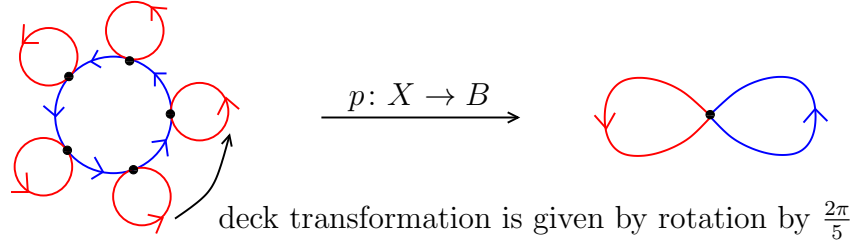


FIGURE 188.

Lemma 20.2. *Let X be a topological space on which a group G acts continuously and discretely. We consider the corresponding covering $p: X \rightarrow X/G$. Then $p: X \rightarrow X/G$ is regular and the map*

$$\begin{aligned} \Phi: G &\mapsto D(p: X \rightarrow X/G) \\ g &\mapsto \begin{pmatrix} X &\mapsto X \\ x &\mapsto gx \end{pmatrix} \end{aligned}$$

is an isomorphism of groups.

Proof. It is straightforward to see that the maps $x \mapsto gx$ are indeed deck transformations. It follows almost immediately that the covering $p: X \rightarrow X/G$ is regular. Also it follows easily from the axioms of a group action that Φ is a homomorphism of groups. Furthermore it is clear that Φ is a monomorphism.

It remains to show that Φ is an epimorphism. So let $d \in D(p: X \rightarrow X/G)$ be a deck transformation. We pick $x \in X$. Since $p(d(x)) = p(x)$ we have $d(x) = gx$ for some $g \in G$. But then it follows from Lemma 20.1 that $d(y) = gy$ for all $y \in X$. This shows that Φ is an epimorphism. $\square$

Lemma 20.3. *Any 2-fold covering of path-connected topological spaces is regular and the deck transformation group contains precisely two elements.*

Proof. Let $p: X \rightarrow B$ be a 2-fold covering of path-connected topological spaces. We leave it as an exercise to show that the map

$$\begin{aligned} d: X &\rightarrow X \\ x &\mapsto \text{the unique other element of } p^{-1}(p(x)) \end{aligned}$$

is continuous and that it is a deck transformation. It follows easily from Lemma 20.1 that any deck transformation of p is either the identity or it equals the above deck transformation d . $\square$

We recall the following important, albeit somewhat technical, definition.

Definition. Let G be a group and let $H \subset G$ be a subgroup. The *normalizer* $N(H)$ of the subgroup $H \subset G$ is defined as

$$N(H) := \{g \in G \mid g^{-1}Hg = H\}.$$

Remark.

- (1) By definition a subgroup H of G is normal if and only if $N(H) = G$.
- (2) It is straightforward to see that $N(H) \subset G$ is a subgroup of G . Furthermore it follows immediately from the definition of $N(H)$ that H is a normal subgroup of $N(H)$.
- (3) The normalizer $N(H)$ of $H \subset G$ is the largest subgroup of G which contains H as a normal subgroup.³⁸³

Example. We consider one last example in detail. Let Y be a path-connected and locally simply connected topological space and let $y_0 \in Y$. Furthermore let $\Gamma \subset \pi = \pi_1(Y, y_0)$ be a subgroup. By Proposition 15.5 there exists a path-connected covering $p: (X, x_0) \rightarrow (Y, y_0)$ of pointed topological spaces such that $p_*(\pi_1(X, x_0)) = \Gamma$. We recall the construction of X . We first considered

$$W := \{\text{all paths in } Y \text{ with starting point } y_0\}$$

and for $u, v \in W$ we wrote

$$u \sim v \quad :\Leftrightarrow \quad u \text{ and } v \text{ have the same endpoint and } [u * \bar{v}] \in \Gamma.$$

We defined $X = W / \sim$ and the projection map $p: W / \sim \rightarrow Y$ that is given by $[w] \mapsto w(1)$. We then showed that, with an appropriate topology on $W / \sim$, the map $p: W / \sim \rightarrow Y$ is a path-connected covering with $p_*(\pi_1(W / \sim, x_0)) = \Gamma$.

³⁸³As we had just pointed out, H is by definition a normal subgroup of $N(H)$. Furthermore, if A is a subgroup of G which contains H as a normal subgroup, then $A \subset N(H)$. In this sense $N(H)$ is the largest subgroup which contains H as a normal subgroup.

As in the proof of Proposition 15.9 we now would like to construct deck transformations by precomposing elements in W by a loop in y_0 . More precisely, let g be a loop in (Y, y_0) . We consider the map

$$\begin{aligned} W &\rightarrow W \\ w &\mapsto g * w \end{aligned}$$

We need to check that the map respects our equivalence relation $\sim$ on W . So suppose that $u, v \in W$ are equivalent. Then $g * u$ and $g * v$ still have the same endpoint. Now we need to consider the second condition. We have

$$[(g * u) * (\overline{g * v})] = [g * u * \bar{v} * \bar{g}] = [g] * \underbrace{[u * \bar{v}]}_{\in \Gamma} * [g]^{-1}.$$

In general it is not the case that the right hand side true lies again in Γ . But it is true if $[g]$ lies in the normalizer of Γ . As in the proof of Proposition 15.9 it is now straightforward to see that the map

$$\begin{aligned} \Psi: N(\Gamma)/\Gamma &\rightarrow D(p: X \rightarrow Y) \\ [g] &\mapsto \begin{pmatrix} W/\sim &\rightarrow W/\sim \\ [u] &\mapsto [g * u] \end{pmatrix} \end{aligned}$$

is a well-defined homomorphism. We will see in Proposition 20.4 that this in fact an isomorphism of groups.

Now we want to consider the general case of a covering. So let $p: (X, x_0) \rightarrow (B, b_0)$ be a covering of pointed path-connected topological spaces. We write $\pi := \pi_1(B, b_0)$ and $\Gamma := p_*(\pi_1(X, x_0))$.³⁸⁴ Now the question arises whether there is a connection between the groups π , Γ and $D(p: X \rightarrow B)$.

Similar to the proof of Theorem 6.16 we can consider the following map³⁸⁵

$$\begin{aligned} \Phi: D(p: X \rightarrow B) &\mapsto \pi/\Gamma \\ d &\mapsto [p \circ (\text{path in } X \text{ from } x_0 \text{ to } d(x_0))]. \end{aligned}$$

It is straightforward to see that this map is well-defined, i.e. it does not depend on the choice of the path.

The question now arises, whether this map Φ is a group isomorphism. But in general this is not the case, after all, if Γ is not normal in π , then π/Γ is not even a group. But we have the following proposition:

³⁸⁴Recall that by Corollary 6.14 the induced map $p_*: \Gamma \rightarrow \pi$ is injective, thus $\Gamma \cong \pi_1(X, x_0)$.

³⁸⁵We denote hereby with $[p \circ f]$ not only the equivalence class of the loop $p \circ f$ in π , but also its equivalence class π/Γ , since the notation $[p \circ f]\Gamma$ is too cumbersome.

Proposition 20.4. *Let $p: (X, x_0) \rightarrow (B, b_0)$ be a covering of pointed path-connected topological spaces. We write $\pi := \pi_1(B, b_0)$ and $\Gamma := p_*(\pi_1(X, x_0))$. The above map*

$$\begin{aligned} \Phi: D(p: X \rightarrow B) &\mapsto \pi/\Gamma \\ d &\mapsto [p \circ (\text{path in } X \text{ from } x_0 \text{ to } d(x_0))]. \end{aligned}$$

restricts to an isomorphism of groups

$$\Phi: D(p: X \rightarrow B) \xrightarrow{\cong} N(\Gamma)/\Gamma.$$

Proof. Let $p: (X, x_0) \rightarrow (B, b_0)$ be a covering of pointed path-connected topological spaces. We write $\pi := \pi_1(B, b_0)$, $\Gamma := p_*(\pi_1(X, x_0))$ and $D := D(p: X \rightarrow B)$.

- (A) First we have to show that $\Phi(D)$ lies indeed in $N(\Gamma)/\Gamma$. Thus let $d \in D$. We need to show that $\Phi(d)$ lies in $N(\Gamma)/\Gamma$. Thus let f be a loop in (X, x_0) and let g be a path in X from x_0 to $d(x_0)$. We have to verify that $[\overline{p \circ f}] \cdot [p \circ g] \cdot [p \circ f] \in \Gamma$. Indeed the following holds

$$[\overline{p \circ f}] \cdot [p \circ g] \cdot [p \circ f] = [\overline{p \circ f} * p \circ g * p \circ f] = [p \circ (\underbrace{\overline{f} * g * f}_{\text{loop in } (X, x_0)})] \in p_*(\pi_1(X, x_0)) = \Gamma.$$

- (B) Now we have to show that Φ is a group homomorphism. So let d and e be two deck transformations. We pick a path g from x_0 to $d(x_0)$ and we pick a path h from x_0 to $e(x_0)$. Then

$$\begin{array}{ccccccc} \Phi(d) \cdot \Phi(e) & = & [p \circ g] \cdot [p \circ h] & = & [p \circ (e \circ g)] \cdot [p \circ h] & = & [p \circ (e \circ g * h)] = \Phi(d \circ e). \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ \text{definition of } \Phi(e) \text{ and } \Phi(d) & & \text{since } p \circ e = p & & & & \text{since } g * (d \circ h) \text{ is a path} \\ & & & & & & \text{from } x_0 \text{ to } d(e(x_0)), \text{ see Figure 189} \end{array}$$

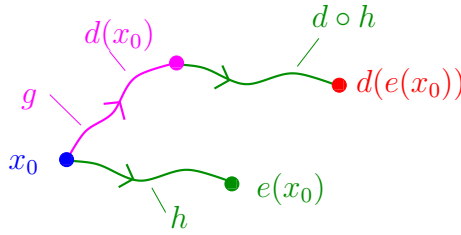


FIGURE 189.

- (C) We will now construct a map $\Psi: N(\Gamma)/\Gamma \rightarrow D$ in the inverse direction. Thus let $[f] \in N(\Gamma) \subset \pi = \pi_1(B, b_0)$ where f is a loop in (B, b_0) . Let x_1 be the endpoint of the lift $\tilde{f}$ of f to the starting point x_0 .³⁸⁶ We have $p(x_1) = p(\tilde{f}(1)) = f(1) = x_0$. Thus we

³⁸⁶According to Corollary 6.13 the definition of x_1 does not depend on the choice of the representative of $[f] \in N(\Gamma)$.

obtain the following diagram of maps of pointed topological spaces:

$$\begin{array}{ccc} & (X, x_1) & \\ & \downarrow p & \\ (X, x_0) & \xrightarrow{p} & (B, b_0). \end{array}$$

Then

$$\begin{array}{ccccc} p_*(\pi_1(X, x_1)) & = & \{ [p \circ (\widetilde{f} * g * \widetilde{f})] \mid [g] \in \pi_1(X, x_0) \} & = & [f] p_*(\pi_1(X, x_0)) [f]^{-1} \\ \uparrow & & & & \uparrow \\ \text{by Proposition 4.9 since } \widetilde{f} \text{ is a path from } x_0 \text{ to } x_1 & & & & \text{since } [f] \in N(\Gamma) \\ & & & & = p_*(\pi_1(X, x_0)). \end{array}$$

It follows from Proposition 15.2 that there exists a lift $\widetilde{p}: (X, x_0) \rightarrow (X, x_1)$ of p , i.e. there exists a map $\widetilde{p}$ such that the following diagram commutes:

$$\begin{array}{ccc} & (X, x_1) & \\ & \downarrow p & \\ (X, x_0) & \xrightarrow{p} & (B, b_0). \end{array}$$

The map $\widetilde{p}$ is now by definition a deck transformation.

We have thus constructed a map $N(\Gamma) \rightarrow D$. We still need to show that this map descends to a map $N(\Gamma)/\Gamma \rightarrow D$. So let $[f] \in N(\Gamma)$ and let $[g] \in \Gamma$. We denote by $\widetilde{f}$ the lift of f to the starting point x_0 . Similarly we define $\widetilde{g * f}$. By the above we only need to show that the endpoints of $\widetilde{f}$ and $\widetilde{g * f}$ agree. In fact we have

$$\begin{array}{ccccc} \widetilde{g * f}(1) & = & \text{endpoint of the lift of } f & = & \text{endpoint of the lift of } f & = & \widetilde{f}(1) \\ & \uparrow & \text{to the starting point } \widetilde{g}(1) & \uparrow & \text{to the starting point } x_0 & & \\ & \text{Lemma 15.3} & & \text{by Lemma 6.15 (2) (1) since } [g] \in p_*(\pi_1(X, x_0)) & & & \end{array}$$

- (D) We now want to show that Φ is a bijection. It suffices to show that $\Psi \circ \Phi = \text{id}$ and $\Phi \circ \Psi = \text{id}$.

First let $d \in D$. Then

$$\begin{aligned} \Psi(\Phi(d)) &= \Psi(f \text{ that lifts to a path from } x_0 \text{ to } d(x_0)) \\ &= \text{deck transformation that sends } x_0 \text{ to } d(x_0) = d \\ &\quad \uparrow \\ &\quad \text{Lemma 20.1} \end{aligned}$$

This shows that $\Psi \circ \Phi$ is a bijection.

Now let $g \in N(\Gamma)/\Gamma$. We pick a representative loop s in b_0 and we denote by $\tilde{s}$ the lift of s to the starting point x_0 . We denote by x_1 the endpoint of $\tilde{s}$. Then

$$\Phi(\Psi(g)) = [p \circ (\text{path in } X \text{ from } x_0 \text{ to } \overbrace{\Psi(g)(x_0)}^{=x_1})] \underset{\substack{\uparrow \\ \tilde{s} \text{ is a path from } x_0 \text{ to } x_1}}{=} [p \circ \tilde{s}] = [s] = g.$$

This shows that $\Phi \circ \Psi$ is a bijection. $\square$

In most applications we will use the following special case of Proposition 20.4.

Proposition 20.5. *Let X be a path-connected and locally simply connected topological space and let $x_0 \in X$. Let $\gamma: \pi_1(X, x_0) \rightarrow G$ be an epimorphism onto a group G . We denote by $p: (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$ the covering corresponding to $\ker(\gamma)$ that is given by Proposition 15.5 and that is unique up to covering equivalence by Proposition 15.6. Then the map*

$$\begin{aligned} \Phi: D(p: \tilde{X} \rightarrow X) &\rightarrow \pi_1(X, x_0)/p_*(\pi_1(\tilde{X}, \tilde{x}_0)) && \xrightarrow{\gamma} G. \\ d &\mapsto [p \circ (\text{path from } d(x_0) \text{ to } x_0)] \end{aligned}$$

is an isomorphism.

Proof. First note that $p_*(\pi_1(\tilde{X}, \tilde{x}_0)) = \ker(\gamma)$ is a normal subgroup of $\pi_1(X, x_0)$. Therefore the normalizer of $\Gamma := p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ equals $\pi_1(X, x_0)$. It now follows immediately from Proposition 20.4 that the homomorphism on the left is an isomorphism. The homomorphism on the right is an isomorphism since $p_*(\pi_1(\tilde{X}, \tilde{x}_0)) = \ker(\gamma)$. $\square$

The following lemma now gives a complete characterization of regular coverings.

Corollary 20.6. *Let $p: (X, x_0) \rightarrow (B, b_0)$ be a covering of pointed path-connected topological spaces. Then the following statements are equivalent:*

- (1) *The covering is regular.*
- (2) *The deck transformation group $D(p: X \rightarrow B)$ acts transitively on $p^{-1}(b_0) \subset X$.*
- (3) *The subgroup $p_*(\pi_1(X, x_0))$ is a normal subgroup of $\pi_1(B, b_0)$.*
- (4) *There exists a group G which acts discretely on X and there exists a homeomorphism $f: X/G \rightarrow B$ such that the following diagram commutes:*

$$\begin{array}{ccc} & X & \\ q \swarrow & & \searrow p \\ X/G & \xrightarrow{f} & B \end{array}$$

where $q: X \rightarrow X/G$ is the canonical projection map.

If p is a finite covering then the above statements are furthermore equivalent to

- (5) $\#D(p: X \rightarrow B) = [X : B]$.

Remark. Let $p: (X, x_0) \rightarrow (B, b_0)$ be a 2-fold covering of path-connected pointed topological spaces. In Lemma 20.3 we had seen that p is regular, which by Corollary 20.6 means that the index two subgroup $p_*(\pi_1(X, x_0))$ of $\pi_1(B, b_0)$ is normal. In fact we had already shown in exercise sheet 13 that any index 2 subgroup Γ of a group π is normal.³⁸⁷

Proof. Let $p: (X, x_0) \rightarrow (B, b_0)$ be a covering of pointed path-connected topological spaces. We write $\pi := \pi_1(B, b_0)$, $\Gamma := p_*(\pi_1(X, x_0))$ and $D = D(p: X \rightarrow B)$. The following four steps prove the equivalence of (1), (2) (3) and (4).

(1) $\Leftrightarrow$ (2) Recall that a covering is regular if for any $b \in B$ the group D acts transitively on $p^{-1}(b)$. Thus (1) $\Rightarrow$ (2) is trivial. Now suppose that d acts transitively on $p^{-1}(b_0)$. Let b be some other point in B . Since B is path-connected there exists a path s from b to b_0 .

Now let x, x' be two points in $p^{-1}(b)$. We denote by $\tilde{s}$ the lift of s to the starting point x and similarly we define $\tilde{s}'$. We denote by x_0 and x'_0 the endpoints of $\tilde{s}$ and $\tilde{s}'$. Since D acts transitively on $p^{-1}(b_0)$ there exists a deck transformation d with $d(x_0) = x'_0$. By the uniqueness of lifts to a starting point we then also have $d \circ \tilde{s} = \tilde{s}'$. But then we also have that d turns the starting point of $\tilde{s}$ into the starting point of $\tilde{s}'$, i.e. we have $d(x) = x'$.

(2) $\Leftrightarrow$ (3) We consider again the map

$$\begin{aligned} \Phi: D(p: X \rightarrow B) &\mapsto \pi/\Gamma \\ d &\mapsto [p \circ (\text{path in } X \text{ from } x_0 \text{ to } d(x_0))]. \end{aligned}$$

We first prove the following claim.

Claim. We have the following equivalence of statements:

$$D \text{ acts transitively on } p^{-1}(x_0) \subset X \quad \Leftrightarrow \quad \Phi: D \rightarrow \pi/\Gamma \text{ is surjective.}$$

First suppose that D acts transitively on $p^{-1}(x_0) \subset X$. Let $g \in \pi/\Gamma$ and pick a loop f that represents g . We denote by $\tilde{f}$ the lift of f to the starting point x_0 and we denote by $\tilde{x}_0$ the endpoint of $\tilde{f}$. Since D acts transitively there exists a deck transformation d with $d(x_0) = \tilde{x}_0$. Now

$$\begin{aligned} \Phi(d) = [p \circ (\text{path in } X \text{ from } x_0 \text{ to } d(x_0))] &= [p \circ (d \circ \tilde{f})] = [p \circ \tilde{f}] = [f] = g. \\ &\quad \uparrow \qquad \qquad \qquad \uparrow \\ &\text{since } \tilde{f} \text{ is a path from } x_0 \text{ to } d(x_0) \qquad \text{since } p \circ d = p \end{aligned}$$

Now suppose that $\Phi: D \rightarrow \pi/\Gamma$ is surjective. Let $\tilde{x}_0 \in p^{-1}(x_0)$. We pick a path $\tilde{f}$ from x_0 to $\tilde{x}_0$ and we denote by f the loop $p \circ \tilde{f}$. Since Φ is surjective there exists a deck transformation d with $\Phi(d) = [f]$. But then $d(x_0) = \tilde{x}_0$. This concludes the proof of the claim.

³⁸⁷We already made use of this fact in the proof of Corollary 16.6.

Now we have the following equivalences

$$\begin{array}{ccccc}
 D \text{ acts transitively on } p^{-1}(x_0) \subset X & \iff & \Phi: D \rightarrow \pi/\Gamma \text{ is surjective} & & \\
 & \iff & \pi = N(\Gamma) & \iff & \Gamma \text{ is normal in } \pi. \\
 \uparrow & & \uparrow & & \\
 & \text{Proposition 20.4} & & \text{by the definitions} &
 \end{array}$$

(1) $\Rightarrow$ (4) We will show that the deck transformation group D has the desired properties.

We first show that the deck transformation group acts discretely on X . So let $x \in X$. We write $b = p(x)$. Since $p: X \rightarrow B$ is a covering there exists a connected open neighborhood U of b that is uniformly covered. This means that the restriction of p to each component of $p^{-1}(U)$ is a homeomorphism. Now let V be the component of $p^{-1}(U)$ that contains x . Then $gV \cap V = \emptyset$ for each non-trivial g . This shows that the action of G is discrete.

The map $p: X \rightarrow B$ has the property that $p(gx) = p(x)$ for all $g \in D$. Therefore the map p factors through a map $f: X/G \rightarrow B$. The map f is clearly surjective. Furthermore it is injective since we suppose that D acts transitively on each $p^{-1}(b)$. Finally the map f is not only continuous but, as follows easily from Lemma 6.4 (1), it is also open. Thus f is a homeomorphism.

(4) $\Rightarrow$ (1) Now suppose that such a homeomorphism $f: X/G \rightarrow B$ exists. First note that for each $g \in G$ the map $x \mapsto gx$ not only defines a deck transformation of the covering $q: X \rightarrow X/G$ but also of the covering $p: X \rightarrow B$. Now let $b \in B$ and let $x, x' \in X$ with $p(x) = p(x')$. Since f is a homeomorphism we also have $q(x) = q(x')$. Hence there exists a $g \in G$ with $gx = x'$.

Now we suppose that p is a finite covering. We show the following equivalence:

(5) $\Leftrightarrow$ (3) We have the following equivalences:³⁸⁸

$$\begin{array}{ccccccc}
 & \text{Lemma 6.15 (3)} & & \text{Proposition 20.4} & & & \\
 & \downarrow & & \downarrow & & & \\
 \#D = [X : B] & \iff & \#D = [\pi : \Gamma] & \iff & [\Gamma : N(\Gamma)] = [\pi : \Gamma] & & \\
 & \iff & N(\Gamma) = \pi & \iff & \Gamma \subset \pi \text{ is a normal subgroup.} & \square &
 \end{array}$$

³⁸⁸Where do we actually use in the argument that the covering is finite?

21. RELATED CONSTRUCTIONS IN ALGEBRAIC GEOMETRY AND GALOIS THEORY (*)

³⁸⁹ In this short chapter we indicate some connections between the topics that we have covered so far to algebraic geometry and Galois theory. Since the author of these notes does not know much about either field the following discussion should be taken with a grain of salt. In particular for an expert the subsequent discussion will certainly look too simplistic.

21.1. The fundamental group of an algebraic variety (*).

Definition. Let $\mathbb{K}$ be a field.³⁹⁰

- (1) Given a set of multivariable polynomials S , i.e. given a subset $S \subset \mathbb{K}[t_1, \dots, t_n]$ we denote by

$$V(S) = \{(x_1, \dots, x_n) \in \mathbb{K}^n \mid f(x_1, \dots, x_n) = 0 \text{ for all } f \in S\}$$

the *vanishing set* of S , i.e. the set of all points in $\mathbb{K}^n$ where all the polynomials in S , viewed as maps $\mathbb{K}^n \rightarrow \mathbb{K}$, vanish.

- (2) We say $U \subset \mathbb{K}^n$ is *Zariski open*,³⁹¹ if there exists a subset S of $\mathbb{K}[t_1, \dots, t_n]$ such that $U = \mathbb{K}^n \setminus V(S)$.

Remark. Let S be a subset of $\mathbb{K}[t_1, \dots, t_n]$. Since the ring $\mathbb{K}[t_1, \dots, t_n]$ is Noetherian there exists a finite set $\{f_1, \dots, f_k\} \subset \mathbb{K}[t_1, \dots, t_n]$ such that $V(S) = V(f_1, \dots, f_k)$.

Lemma 21.1. *Let $\mathbb{K}$ be a field. The Zariski open subsets form a topology on $\mathbb{K}^n$.*

Remark.

- (1) The topology defined by the Zariski open subsets is of course called the *Zariski topology* on $\mathbb{K}^n$.
- (2) If $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, then every Zariski open subset is also open with respect to the usual topology on $\mathbb{R}^n$ or $\mathbb{C}^n = \mathbb{R}^{2n}$. But for $n \geq 1$ the converse does not hold, in fact “most” subsets that are open with respect to the usual topology are not open with respect to the Zariski topology.

Proof. Let $\mathbb{K}$ be a field.

- (1) The vanishing set of the empty set, i.e. $V(\emptyset)$ is all of $\mathbb{K}^n$, hence the empty set $\emptyset = \mathbb{K}^n \setminus V(\emptyset)$ is open.
- (2) The vanishing set of the constant polynomial $f = 1$ is the empty set, therefore $\mathbb{K}^n = \mathbb{K}^n \setminus V(1)$ is open.
- (3) We need to show that the intersection of finitely many Zariski open sets is again Zariski open. By induction it suffices to consider the case of two Zariski open sets.

³⁸⁹This chapter is not part of the official lecture notes for Algebraic Topology I.

³⁹⁰For example we could take $\mathbb{K} = \mathbb{Q}, \mathbb{R}, \mathbb{C}$ but $\mathbb{K}$ could also be a field of non-zero characteristic, e.g. for a prime p we could take $\mathbb{K} = \mathbb{F}_p$ or $\mathbb{K}$ could be the algebraic closure $\overline{\mathbb{F}}_p$ of $\mathbb{F}_p$.

³⁹¹Oscar Zariski (1899-1986) was a Russian-American mathematician working in algebraic geometry.

So suppose we are given Zariski open sets $X = \mathbb{K}^n \setminus V(S)$ and $Y = \mathbb{K}^n \setminus V(T)$. Then

$$\mathbb{K}^n \setminus (X \cap Y) = (\mathbb{K}^n \setminus X) \cup (\mathbb{K}^n \setminus Y) = V(S) \cup V(T) = V(\{s \cdot t\}_{s \in S, t \in T}).$$

- (4) We need to show that the union of Zariski open sets is again Zariski open. So suppose we are given a family of Zariski open sets $X_i = \mathbb{K}^n \setminus V(S_i)$, $i \in I$. Then

$$\mathbb{K}^n \setminus \bigcup_{i \in I} X_i = \bigcap_{i \in I} (\mathbb{K}^n \setminus X_i) = \bigcap_{i \in I} V(S_i) = V\left(\bigcup_{i \in I} S_i\right).$$

Thus we have verified that the Zariski open sets have all the properties of a topology. $\square$

Definition. Let $\mathbb{K}$ be a field.

- (1) A *variety over $\mathbb{K}$* is defined as the vanishing set of a set of multivariable polynomials.
- (2) The *Zariski topology on a variety $V \subset \mathbb{K}^n$* is defined as the subspace topology on V induced from the Zariski topology on $\mathbb{K}^n$.

Example. Let $\mathbb{K}$ be a field and let $f(s, t) = s^2 + t^2 - 1$. We define

$$S_{\mathbb{K}}^1 := V(f) = \{(x, y) \in \mathbb{K}^2 \mid f(x, y) = x^2 + y^2 - 1 = 0\} = \{(x, y) \in \mathbb{K}^2 \mid x^2 + y^2 = 1\}.$$

For $\mathbb{K} = \mathbb{R}$ we obtain the usual circle S^1 .

Any variety V is again a topological space, so in principle one could study the usual fundamental group. But in many cases, e.g. if the characteristic of $\mathbb{K}$ is non-zero, one will see that all continuous maps $S^1 \rightarrow V$ are constant, hence the usual fundamental group is trivial.

The correct analogue of a finite covering $p: X \rightarrow B$ between topological spaces for varieties is a finite étale morphism. The precise definition of a finite étale morphism is irrelevant for our purpose.³⁹² A heuristic fact is that varieties tend to have many finite étale morphisms, but there is no well-behaved analogue of an infinite covering in the context of varieties. In particular usually there is no concept of a “universal” étale morphism. The question is now, whether one can recover the notion of a fundamental group from the knowledge of finite covers.

Given a variety V we can consider all finite regular étale morphisms $p_i: W_i \rightarrow V$, $i \in I$. The corresponding deck transformations groups $D(p_i: W_i \rightarrow V)$, $i \in I$ are finite groups that form an inverse system. We can now define the étale fundamental group of V as

$$\pi_1^{\text{ét}}(V) := \varprojlim_i D(p_i: W_i \rightarrow V).$$

By Proposition 13.9 the étale fundamental group $\pi_1^{\text{ét}}(V)$ is naturally a compact topological group. In algebraic geometry the étale fundamental group of a variety plays a role similar to the role of the fundamental group of a topological space in topology.

If V is a complex variety without singularities, then the étale fundamental group is isomorphic to the profinite completion³⁹³ of the usual topological fundamental group $\pi_1(V)$,

³⁹²Furthermore it is unknown to the author of these lecture notes.

³⁹³Recall that on page 247 we had introduced the profinite-completion $\hat{\pi}$ of a group π .

i.e. we have

$$\pi_1^{\text{ét}}(V) \cong \widehat{\pi_1(V)}.$$

We refer to [Gk] for a proof of this statement.

More information on this topic can be found at

https://en.wikipedia.org/wiki/etale_fundamental_group

21.2. Galois theory (*). The theory of coverings of topological spaces has some similarities to the Galois theory of field extensions.

Throughout this section let K be a field of characteristic zero³⁹⁴ and let L/K be a field extension of K , i.e. L is a field which contains K as a subfield. The dimension of L as a K vector space is denoted by $[L : K]$. We consider the group

$$\text{Gal}(L/K) = \text{all field automorphisms of } L \text{ which are the identity on } K.$$

Put differently, a field automorphism $f: L \rightarrow L$ lies in $\text{Gal}(L/K)$ if and only if the following diagram commutes:

$$\begin{array}{ccc} & K & \\ \swarrow & & \searrow \\ L & \xrightarrow{f} & L. \end{array}$$

We consider two examples:

- (1) $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$ is a field extension with $[\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 2$ and $\text{Gal}(\mathbb{Q}(\sqrt{2})/\mathbb{Q})$ consists of the identity and the homomorphism $a + b\sqrt{2} \mapsto a - b\sqrt{2}$.
- (2) $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$ is a field extension with $[\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}] = 3$ and the group $\text{Gal}(\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q})$ consists only of the identity.

Given a field K we denote by $\overline{K}$ the algebraic closure of K . A field extension L/K is called *Galois*, if for each $f \in \text{Gal}(\overline{K}/K)$ we have $f(L) = L$.³⁹⁵ In the usual course on Galois theory one shows that L/K is Galois if and only if each polynomial in $K[x]$ which admits a zero in L splits over L , i.e. all other zeros of f lies also in L . For a Galois extension L/K we refer to $\text{Gal}(L/K)$ as the *Galois group* of the field extension.

We return to the above two examples:

- (1) $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$ is a Galois field extension.
- (2) $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$ is not a Galois field extension, since $\sqrt[3]{2}$ is a zero of $x^3 - 2$, but the complex zero $\sqrt[3]{2}e^{2\pi i/3}$ of f does not lie in $\mathbb{Q}(\sqrt[3]{2})$.³⁹⁶

If L/K is a finite extension, then the main theorem of Galois theory says that the following two statements are equivalent:

- (1) The field extension L/K is Galois.

³⁹⁴This restriction to characteristic zero is not necessary, but it simplifies the discussion.

³⁹⁵Strictly speaking this is the definition of a normal extension, but since K is a field of characteristic zero, a normal extension is already Galois.

³⁹⁶We put $z = \sqrt[3]{2}$ and $w = e^{2\pi i/3}$. It is a good exercise to show that $\mathbb{Q}(z, w)/\mathbb{Q}$ is a normal field extension with $[\mathbb{Q}(z, w) : \mathbb{Q}] = 6$. The group $\text{Gal}(\mathbb{Q}(z, w)/\mathbb{Q})$ is isomorphic to the permutation group S_3 .

$$(2) \# \text{Gal}(L/K) = [L : K].$$

The statement is formally very similar to the statement of Corollary 20.6 which says that for some finite covering $p: X \rightarrow B$ of topological spaces the following statements are equivalent:

(1) The covering $p: X \rightarrow B$ is regular.

$$(2) \#D(p: X \rightarrow B) = [X : B].$$

The analogy between Galois theory and covering theory goes considerably further. Many statements about the Galois group of a Galois extension have a corresponding statement for the deck transformation group of a normal covering. For example, given a finite extension L/K the (normal) subgroups of Galois group $\text{Gal}(L/K)$ correspond to (normal) intermediate fields and (normal) subgroups of the deck transformation group correspond to (regular) intermediate coverings.

Now let L/K be a Galois extension of infinite degree. All finite intermediate Galois extensions of K , i.e. all finite Galois extensions F/K with $F \subset L$, form a preordered set via the inclusion. The corresponding Galois groups form an inverse set. Indeed, given two finite Galois extensions F/K and G/K with $F \subseteq G$ the restriction of an automorphism of G to an automorphism of F defines a group homomorphism $\text{Gal}(G/K) \rightarrow \text{Gal}(F/K)$. Therefore we can form the inverse limit of these Galois groups. By [FJ, Section 1.3] there exists a canonical isomorphism of groups

$$\text{Gal}(L/K) = \varprojlim_{\substack{F/K \text{ finite} \\ \text{Galois extension}}} \text{Gal}(F/K).$$

On the right hand side we consider the inverse limit of an inverse system of finite groups. It follows from Proposition 13.9 that the right-hand side, and thus also the Galois group $\text{Gal}(L/K)$, is naturally a compact topological group. This topology on $\text{Gal}(L/K)$ is called the *Krull topology*.

Example. Let p be a prime. We denote by $\mathbb{F}_p$ the finite field with p elements and we denote by $\overline{\mathbb{F}}_p$ its algebraic closure. It is shown in [FJ, Section 1.5] that there exists a canonical isomorphism

$$\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p) \cong \widehat{\mathbb{Z}}$$

where $\widehat{\mathbb{Z}}$ denotes the profinite completion of $\mathbb{Z}$, or equivalently the profinite integers that we had introduced on page 244.

22. CW-COMPLEXES

In this chapter we will introduce CW-complexes. Initially the definition of a CW-complex might look rather confusing and artificial. But we will soon see that the notion of a CW-complex strikes the right balance between having good properties and being flexible enough to accommodate most topological spaces we are interested in are CW-complexes.

22.1. Definition of finite-dimensional CW-complexes and examples. We start out with the definition of a finite-dimensional CW-complex and its topological realization.

Definition.

- (a) A *CW-complex* X of dimension ≤ -1 is defined to be the empty set. Its *topological realization* $|X|$ is also defined to be the empty set.
- (b) We define inductively higher dimensional CW-complexes. Let $n \in \mathbb{N}_0$. A *CW-complex* X^n of dimension $\leq n$ consists of the following:
 - (a) a CW-complex X^{n-1} of dimension $\leq n-1$, with topological realization denoted by $|X^{n-1}|$,
 - (b) together with a set of maps $\{\varphi_i: S^{n-1} \rightarrow |X^{n-1}|\}_{i \in I_n}$.

The *topological realization* $|X^n|$ of X^n is defined as the topological space

$$|X^n| := \left(|X^{n-1}| \sqcup \bigsqcup_{i \in I} \overline{B^n} \right) / \sim$$

where for each $i \in I_n$ we use the map φ_i to identify the points on the corresponding copy of $S^{n-1} = \partial \overline{B^n}$ with the corresponding points in $|X^{n-1}|$.

- (c) A *CW-structure* for a topological space Y is a CW-complex X together with a homeomorphism $f: |X| \rightarrow Y$.

Remark.

- (1) Note that $\overline{B}^0$ consists of a point and that $S^{-1} = \emptyset$. It follows from the above definition that a CW-complex of dimension ≤ 0 is exactly the same as set of points with the discrete topology.
- (2) CW-complexes were introduced by John H. C. Whitehead³⁹⁷. The name comes from “C” for “closure-finite”, and the “W” for “weak topology”³⁹⁸. Some people also say it goes back to the initials C. W. of Whitehead.
- (3) In the notation we usually do not distinguish between the CW-complex and its topological realization, i.e. we usually denote the topological realization of a CW-complex X again by X .

Before we discuss a long list of examples of CW-complexes we want to introduce a few more definitions.

Definition. Let X be a CW-complex. We continue with the notation used in the previous definition

³⁹⁷John Henry Constantine Whitehead (1904-1960) was a British mathematician.

³⁹⁸We will explain later on what “closure-finite” and “weak topology” means.

- (1) The images of the $\overline{B}_i^k$, $i \in I_k$, in X are called the *k-dimensional cells*, or often short *k-cells*, of X .
- (2) The images of the B_i^k , $i \in I_k$, in X are called the *open k-cells* of X .
- (3) The maps $\varphi_i: S_i^{k-1} \rightarrow X$ are called the *attaching maps* of the *k-cells* and the maps $\overline{B}_i^k \rightarrow X$ are referred to as the *characteristic maps* of the *k-cells*.
- (4) The union of the cells of dimension $\leq k$ is called the *k-skeleton* of X and we denote it by X^k .
- (5) If X is non-empty, then we define the dimension $\dim(X) \in \mathbb{N}_0 \cup \{\infty\}$ as the maximal dimension of a cell.

Example. We consider the topological space which is sketched in Figure 190, namely

$$Y := [-1, 1] \times [-1, 1] \cup \{(x, y) \in \mathbb{R}^2 \mid x^2 + (y - 1)^2 = 1 \text{ and } y \geq 1\}.$$

The topological space X thus consists of a square together together with a semicircle.

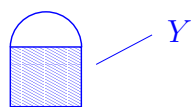


FIGURE 190.

Our goal is to find a CW-structure for X . First we consider the 0-dimensional CW-complex which consists of the 4 vertices of the square. Next we add a 1-cell. To do so we consider the map φ which sends $\partial \overline{B}^1 = \{\pm 1\}$ to the two points $\pm 1 \times -1$. The topological space $X^0 \cup_\varphi \overline{B}^1$ consists of the four points $\{\pm 1, \pm 1\}$ together with an edge which connects the lower two points. This topological space is sketched in the center of Figure 191. The

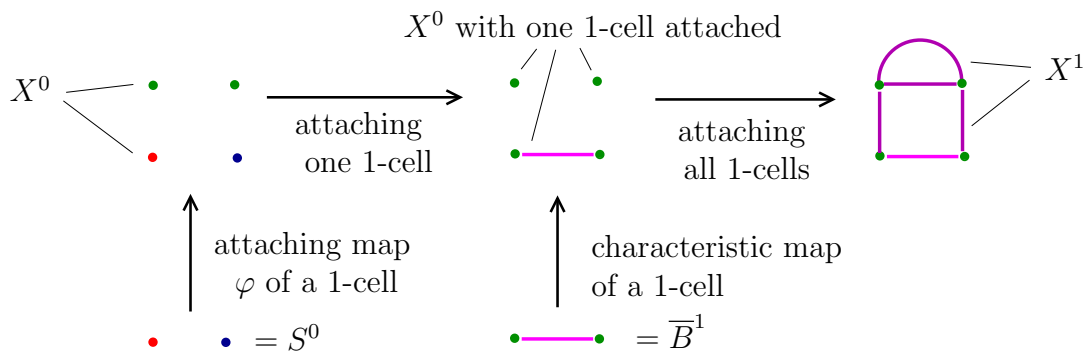


FIGURE 191. The CW-complex X^1 .

same way we attach four more 1-cells and we obtain the 1-dimensional CW-complex which is sketched on the right of Figure 191. Finally we attach a 2-cell to X^1 with the attaching

map shown in Figure 192 and we obtain a CW-complex X^2 whose topological realization is homeomorphic to Y .

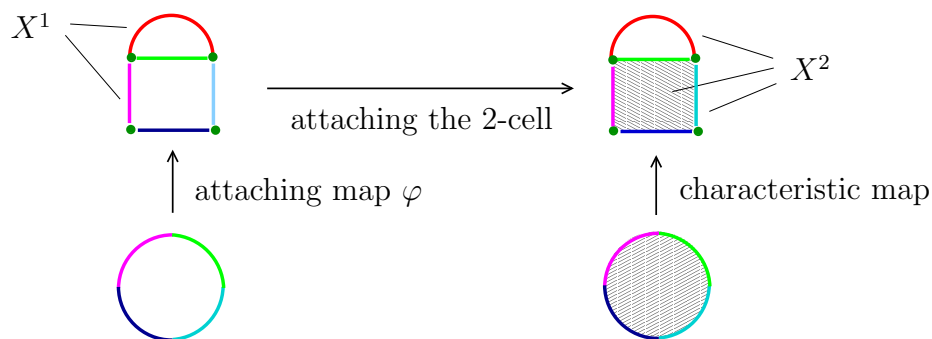


FIGURE 192. Attaching the 2-cell to the 1-skeleton X^1 .

We will now see that most of the topological spaces that we have encountered so far can be described as CW-complexes.

- (1) Any topological graph, as defined on page 38, is a one-dimensional CW-complex where the vertices correspond precisely to the 0-cells and the edges correspond precisely to the open 1-cells. In the following we refer to every one-dimensional CW-complex also as a topological graph. Hereby we refer to the 0-cells as the vertices and to the open 1-cells as the edges.³⁹⁹
- (2) As a special case of (1) we see that S^1 can be viewed as a CW-complex with one 0-cell and one 1-cell. Furthermore we can view the wedge $S^1 \vee S^1$ as a CW-complex with one 0-cell and two 1-cells. Finally $\mathbb{R}$ can be viewed as a CW-complex where the 0-cells are given by the integers $\mathbb{Z}$ and the 1-cells are given by the intervals $[k, k + 1]$ with $k \in \mathbb{Z}$. We refer to Figure 193 for an illustration.

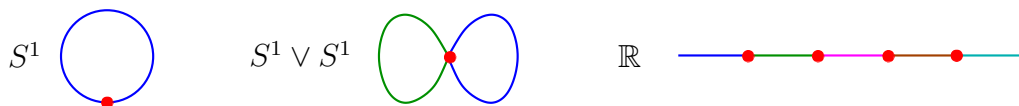


FIGURE 193.

- (3) Now we consider the torus $T = [0, 1] \times [0, 1] / \sim$. We denote by $p: [0, 1] \times [0, 1] \rightarrow T$ the obvious projection map. We can view T as a CW-complex with one 0-cell corresponding to the point $p(0, 0) = p(0, 1) = p(1, 0) = p(1, 1)$, two 1-cells corresponding to $p(0 \times [0, 1]) = p(1 \times [0, 1])$ and to $p([0, 1] \times 0) = p([0, 1] \times 1)$ and one 2-cell corresponding

³⁹⁹In fact, if X is a CW-complex with finitely many 0-cells and 1-cells, then it is relatively straightforward to show that there exists a graph G in the sense of page 38 such that G is homeomorphic to X .

to $p([0, 1] \times [0, 1])$. Note that since we have only one 0-cell there is a unique choice of attaching map for the 1-cells. Finally the attaching map for the 2-cell is illustrated in Figure 194.⁴⁰⁰ In Figure 194 we sketch the CW-structure of the torus and also the corresponding CW-structure of the torus viewed as a subset of $\mathbb{R}^3$.

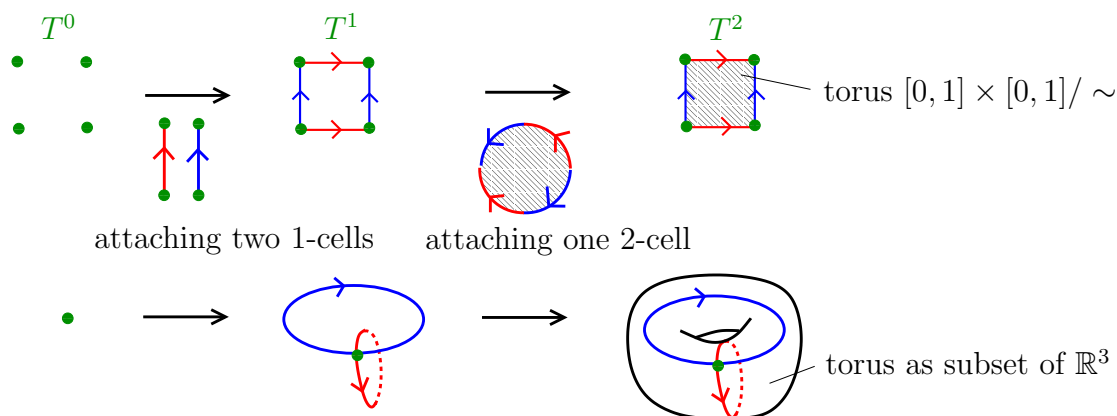


FIGURE 194. The torus as CW-complex.

- (4) Next we consider the Klein bottle $K = [0, 1] \times [0, 1] / \sim$. As usual we denote by $p: [0, 1] \times [0, 1] \rightarrow K$ the projection map. Similar to the torus we can view K as CW-complex with one 0-cell corresponding to $p(0, 0) = p(0, 1) = p(1, 0) = p(1, 1)$, two 1-cells corresponding to $p(0 \times [0, 1]) = p(1 \times [0, 1])$ and to $p([0, 1] \times 0) = p([0, 1] \times 1)$ and one 2-cell corresponding to $p([0, 1] \times [0, 1])$ with the attaching map sketched in Figure 195.⁴⁰¹

The 1-skeleta of the torus and the Klein bottle are both wedges of two circles. Furthermore both CW-complexes have precisely one 2-cell. The difference between the torus and the Klein bottle is that the attaching maps for the 2-cells are different.

- (5) In Figure 196 we show that we can endow the torus with different CW-structures.
 (6) In Figure 197 we sketch how we can view the projective plane $\mathbb{RP}^2$, the Möbius band and the surface of genus 2 as CW-complexes.
 (7) We conclude this discussion of surfaces with the 2-sphere. We can describe the 2-sphere as a CW-complex X as follows: We take a single 0-cell given by a point P and one

⁴⁰⁰Strictly speaking the attaching map is given by

$$\begin{aligned} S^1 &\mapsto T^1 \\ e^{it} &\mapsto \begin{cases} p(\frac{2}{\pi}t, 0), & \text{if } t \in [0, \frac{\pi}{2}] \\ p(1, \frac{2}{\pi}(t - \frac{\pi}{2})), & \text{if } t \in [\frac{\pi}{2}, \pi] \\ p(1 - \frac{2}{\pi}(\pi - t), 1), & \text{if } t \in [\pi, \frac{3\pi}{2}] \\ p(0, 1 - \frac{2}{\pi}(\frac{3\pi}{2} - t)), & \text{if } t \in [\frac{3\pi}{2}, 2\pi]. \end{cases} \end{aligned}$$

But writing out the attaching map with explicit formulas is clearly not very instructive.

⁴⁰¹Evidently, as in Footnote 400, one could spell out the attaching map with precise formulas, but there is no point in doing it.

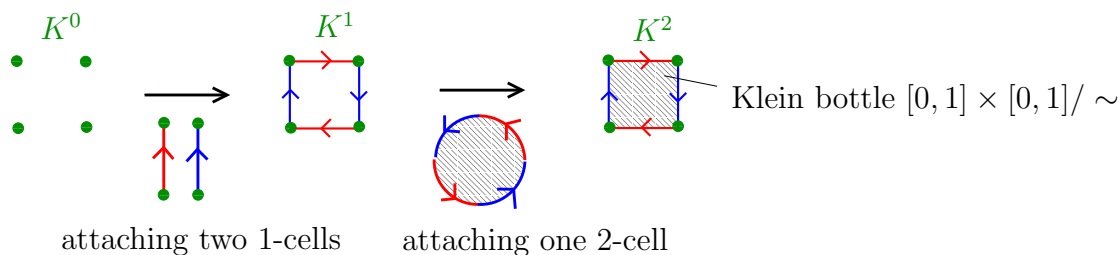


FIGURE 195. The Klein bottle as CW-complex.

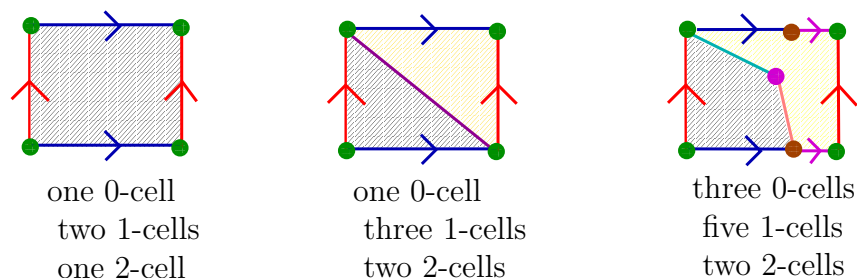


FIGURE 196. The torus with three different CW-structures.

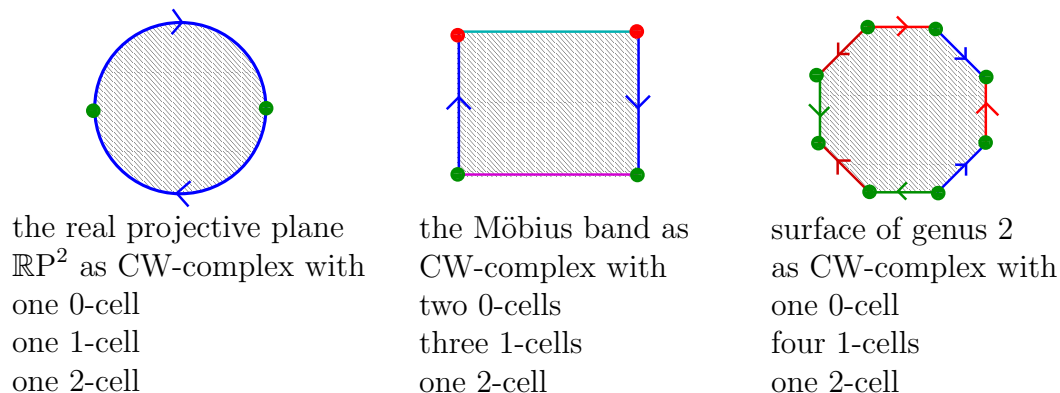


FIGURE 197.

2-cell where the attaching map $\varphi: S^1 \rightarrow X^0 = \{P\}$ sends the whole boundary S^1 to the point P . The topological realization of X is thus given by

$$X^0 \cup_{\varphi} \overline{B}^2 = \{P\} \cup_{P=\partial \overline{B}^2} \overline{B}^2 = \overline{B}^2 / S^1 \cong S^2.$$

$\uparrow$
 see page 23

The example of the 2-sphere can be generalized to spheres of arbitrary dimensions: given any $n \geq 1$ we can view S^n as a CW-complex with one 0-cell and one n -cell.

Definition. A *subcomplex* of a CW-complex X is a subset A with the property that the CW-structure on X restricts to a CW-structure on A .

Examples.

- (1) Given any CW-complex its n -skeleton X^n is a subcomplex of X . Furthermore, the union and also the intersection of arbitrarily many subcomplexes is again a subcomplex. Finally any component of a subcomplex is again a subcomplex.
- (2) Given a CW-complex X and $n \geq 1$ the subset $X^n \setminus X^{n-1}$, if non-empty, is *not* a subcomplex.
- (3) In Figure 198 we consider the topological space X that is given by removing an open disk from a torus. We see that X has the structure of a CW-structure such that the boundary ∂X is a CW-subcomplex.

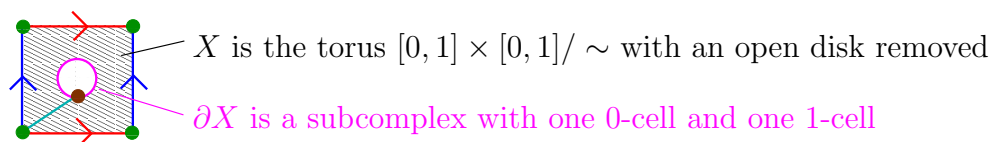


FIGURE 198.

We continue the discussion of CW-complexes with two more explicit examples, namely the real projective spaces and the complex projective spaces which we had already defined on page 25. In the following let $\mathbb{A} = \mathbb{R}$ or $\mathbb{A} = \mathbb{C}$. We denote by $\sim$ the equivalence relation on $\mathbb{A}^{n+1} \setminus \{0\}$ given by multiplication by an element in $\mathbb{A} \setminus \{0\}$ and we define

$$\mathbb{A}P^n := (\mathbb{A}^{n+1} \setminus \{0\}) / \sim.$$

In exercise sheet 14 of algebraic topology I we had seen that $\mathbb{A}P^n$ is a manifold. Given a point $(v_0, \dots, v_n) \in \mathbb{A}^{n+1} \setminus \{0\}$ we denote by $[v_0 : \dots : v_n]$ its equivalence class. For $k \leq n$ the map

$$\begin{aligned} \mathbb{A}P^k &\rightarrow \mathbb{A}P^n \\ [v_0 : \dots : v_k] &\mapsto [v_0 : \dots : v_k : 0 : \dots : 0] \end{aligned}$$

is easily seen to be an embedding and we use it to view $\mathbb{A}P^k$ as a submanifold of $\mathbb{A}P^n$.

In exercise sheet 1 we will give an explicit description of $\mathbb{R}P^n$ and $\mathbb{C}P^n$ as a CW-complex. In particular we will prove the following lemma.

Lemma 22.1. *Let $n \in \mathbb{N}_0$.*

- (1) *We can view $\mathbb{R}P^n$ as a CW-complex with the following properties:*
 - (a) *the CW-structure has exactly one cell in the dimensions $0, 1, \dots, n$ and no other cells,*
 - (b) *for each $k \leq n$ the k -skeleton of $\mathbb{R}P^n$ is given by $\mathbb{R}P^k$.*
- (2) *We can view $\mathbb{C}P^n$ as a CW-complex with the following properties:*

- (a) the CW-structure has exactly one cell in the dimensions $0, 2, \dots, 2n$ and no other cells,
- (b) for each $k \leq n$ the $2k$ -skeleton of $\mathbb{CP}^n$ is given by $\mathbb{CP}^k$.

We conclude this section with the following result which is proved in [Hi, Section 6.4], see also [Mi4, Theorem 3.5] or [Ni, pages 21 and 41] or [Mu2, Theorem 10.6]. The second statement follows from the argument of the proof of [Mc, Theorem 5.1].

Theorem 22.2. *Every compact connected n -dimensional manifold M admits a finite CW-structure with the following properties:*

- (1) the maximal dimension of a cell is n ,
- (2) there is precisely one cell of dimension n ,
- (3) the boundary ∂M is a subcomplex.

Remark. The theorem is stated for manifolds and not for topological manifolds. The situation for topological manifolds is more complicated and we will deal with it later on in algebraic topology III.

In the above discussion we have seen that arguably most of the topological spaces we are interested in can be viewed as CW-complexes. But note that not every topological space can be described as a CW-complex. For example consider the topological space X which consists of precisely two points P and Q and which is endowed with the trivial topology. (Recall that this means that the only open sets are the empty set and all of X .) Then we cannot view X as CW-complex since any CW-complex with finitely many points consists only of 0-cells together with the *discrete* topology. For the same reason $\{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\}$ is not a CW-complex.

22.2. Two topologies on $\mathbb{R}^\infty$. In the next section we want to introduce infinite-dimensional CW-complexes. Before we do so we want to get more acquainted with $\mathbb{R}^\infty$ and topologies on the set $\mathbb{R}^\infty$. First of all we write

$$\mathbb{R}^\infty := \mathbb{R}^{(\mathbb{N})} := \{(x_1, x_2, \dots) \mid \text{only finitely many } x_i \text{ are non-zero}\}.$$

As in the finite-dimensional case we can equip $\mathbb{R}^\infty$ with the euclidean metric⁴⁰²

$$d((x_1, x_2, \dots), (y_1, y_2, \dots)) := \sqrt{\sum_{i=1}^{\infty} |x_i - y_i|^2}.$$

As remarked on page 13, the metric d defines a topology on $\mathbb{R}^\infty$, more precisely

$$\mathcal{S} := \{U \subset \mathbb{R}^\infty \mid \text{for every } P \in U \text{ there exists an } \epsilon > 0 \text{ such that } B_\epsilon(P) \subset U\}$$

is a topology on $\mathbb{R}^\infty$.

This topology might feel like it is the “right topology” on $\mathbb{R}^\infty$. But the next two examples might sow some doubts.

⁴⁰²Note that the series on the right is in fact a finite sum, since all but finitely many terms are zero. It is again straightforward to verify that d is a metric.

Examples.

(1) We claim that the function

$$\begin{aligned}\alpha: \mathbb{R}^\infty &\rightarrow \mathbb{R} \\ (x_1, x_2, \dots) &\mapsto \sum_{i=1}^{\infty} 2^i \cdot x_i\end{aligned}$$

is not continuous with respect to $\mathcal{S}$. To show this claim we consider the open subset $U = (-1, 1) \subset \mathbb{R}$, then $\alpha^{-1}((-1, 1))$ contains the origin 0. We claim that there is no $\epsilon > 0$ with $B_\epsilon(0) \subset U$. Indeed, suppose such $\epsilon > 0$ exists. We pick $i \in \mathbb{N}$ with $\frac{1}{2^i} < \epsilon$ and we consider the point

$$Q = (0, \dots, 0, \frac{1}{2^i}, 0, \dots).$$

Then $Q \in B_\epsilon(0)$ but $\alpha(Q) = 1$, i.e. $Q \notin \alpha^{-1}((-1, 1))$.

(2) Let B be the infinite diagonal matrix where the diagonal entries are given by 2^i for $i = 1, 2, \dots$, i.e. let

$$B := \begin{pmatrix} 2 & 0 & 0 & \dots \\ 0 & 4 & 0 & \dots \\ 0 & 0 & 8 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We claim that the map

$$\begin{aligned}\beta: \mathbb{R}^\infty &\rightarrow \mathbb{R}^\infty \\ x &\mapsto B \cdot x\end{aligned}$$

is not continuous with respect to $\mathcal{S}$. Indeed, an argument similar to (1) shows that the preimage of the open set $U = B_1(0)$ is not open.

The maps α and β feel quite natural, so we might want to look around for a different topology on $\mathbb{R}^\infty$ for which these two maps are in fact continuous. Of course that would be achieved by equipping $\mathbb{R}^\infty$ with the discrete topology, but clearly that's not what we want. Another reasonable condition on the desired topology is that for any $n \in \mathbb{N}_0$ the inclusion map

$$\begin{aligned}\iota_n: \mathbb{R}^n &\hookrightarrow \mathbb{R}^\infty \\ (x_1, \dots, x_n) &\mapsto (x_1, \dots, x_n, 0, \dots)\end{aligned}$$

should also be continuous. This immediately eliminates the discrete topology as a contender for the “right topology”.

In fact this last condition leads us almost immediately to an alternative topology on $\mathbb{R}^\infty$. We had seen in the proof of Proposition 13.1 that in the category of topological spaces we have

$$\lim_{\rightarrow} \mathbb{R}^n = \mathbb{R}^\infty,$$

where $\mathbb{R}^\infty$ is equipped with the “weak topology”, i.e. with the topology

$$\mathcal{T} := \{U \subset \mathbb{R}^\infty \mid \text{for every } n \in \mathbb{N} \text{ the set } U \cap \mathbb{R}^n \text{ is open in } \mathbb{R}^n\}.$$

Put differently, a map $f: \mathbb{R}^\infty \rightarrow X$ to a topological space X is continuous with respect to $\mathcal{T}$ if and only if for each $n \in \mathbb{N}_0$ the map $\mathbb{R}^n \hookrightarrow \mathbb{R}^\infty \xrightarrow{f} X$ is continuous.

Examples.

- (1) By definition of $\mathcal{T}$ the inclusion map $\mathbb{R}^n \rightarrow \mathbb{R}^\infty$ is continuous with respect to $\mathcal{T}$.
- (2) The above map $\alpha: \mathbb{R}^\infty \rightarrow \mathbb{R}$ is continuous with respect to the topology $\mathcal{T}$. Indeed, for any $n \in \mathbb{N}$ the restriction $\mathbb{R}^n \rightarrow \mathbb{R}^\infty \rightarrow \mathbb{R}$ is a linear map, hence continuous.
- (3) In exercise sheet 2 we will see that $\mathcal{S} \subset \mathcal{T}$. This implies in particular that the identity map defines a continuous map $(\mathbb{R}^\infty, \mathcal{T}) \rightarrow (\mathbb{R}^\infty, \mathcal{S})$.

The following lemma shows in particular that the above map $\beta: \mathbb{R}^\infty \rightarrow \mathbb{R}^\infty$ is continuous with respect to the topology $\mathcal{T}$.⁴⁰³

Lemma 22.3. *Let $C = (c_{ij})_{i,j \in \mathbb{N}}$ be an infinite diagonal matrix. Then the map⁴⁰⁴*

$$\begin{aligned} \gamma: \mathbb{R}^\infty &\rightarrow \mathbb{R}^\infty \\ x &\mapsto C \cdot x \end{aligned}$$

is continuous with respect to the topology $\mathcal{T}$.

Proof. Let $U \in \mathcal{T}$. We need to show that $\gamma^{-1}(U)$ lies in $\mathcal{T}$. So given any $n \in \mathbb{N}$ we need to show that $\gamma^{-1}(U) \cap \mathbb{R}^n$ is an open subset of $\mathbb{R}^n$. So let $n \in \mathbb{N}$. Given $m \in \mathbb{N}$ we denote by $C_{m,n}$ the submatrix of C given by the top left $m \times n$ -submatrix of C and we denote by $\gamma_{m,n}: \mathbb{R}^n \rightarrow \mathbb{R}^m$ the map given by $x \mapsto C_{m,n} \cdot x$. We have⁴⁰⁵

$$\begin{aligned} \gamma^{-1}(U) \cap \mathbb{R}^n &= \{x \in \mathbb{R}^n \mid C \cdot x \in U\} = \{x \in \mathbb{R}^n \mid C \cdot x \in \bigcup_{m \in \mathbb{N}} (U \cap \mathbb{R}^m)\} \\ &= \bigcup_{m \in \mathbb{N}} \{x \in \mathbb{R}^n \mid C \cdot x \in U \cap \mathbb{R}^m\} = \bigcup_{m \in \mathbb{N}} \{x \in \mathbb{R}^n \mid C_{m,n} \cdot x \in U \cap \mathbb{R}^m\}. \\ &\quad \text{open in } \mathbb{R}^m \\ &= \bigcup_{m \in \mathbb{N}} \underbrace{\gamma_{m,n}^{-1}(\overbrace{U \cap \mathbb{R}^m}^{\text{open in } \mathbb{R}^m \text{ since } \gamma_{m,n} \text{ is continuous}})}_{\text{open in } \mathbb{R}^n \text{ since } \gamma_{m,n} \text{ is continuous}}. \end{aligned}$$

This shows that $\gamma^{-1}(U) \cap \mathbb{R}^n$ is a union of open subsets of $\mathbb{R}^n$, hence itself is open. $\square$

This discussion shows hopefully that the following convention is reasonable.

Convention. Unless we say something else we will view $\mathbb{R}^\infty$ as a topological space with the topology $\mathcal{T}$. Put differently, we think of $\mathbb{R}^\infty$ as $\mathbb{R}^\infty = \varinjlim \mathbb{R}^n$. Similarly we define $\mathbb{C}^\infty = \varinjlim \mathbb{C}^n$ as topological spaces.

In the future, the more we work with $\mathbb{R}^\infty$ and $\mathbb{C}^\infty$, the more we will see that the “direct limit topology” is in fact the “right” topology to work with.

⁴⁰³This example is less obvious. The topology $\mathcal{T}$ has more open sets than $\mathcal{S}$ which made it easier in (2) to argue that the map α is closed. But here in (3) it also makes it harder, since we have to check the continuity for more open sets on the right hand side.

⁴⁰⁴Does this also work for any infinite-square matrix?

⁴⁰⁵For each $k \in \mathbb{N}$ we view $\mathbb{R}^k$ as a subset of $\mathbb{R}^\infty$ via the inclusion ι_k .

22.3. Infinite-dimensional CW-complexes.

Definition.

- (a) A *CW-complex* X is defined as a sequence

$$X^{-1} := \emptyset \subset X^0 \subset X^1 \subset X^2 \subset \dots$$

of CW-complexes such that for each $m \leq n$ the CW-complex X^m is the m -skeleton of X^n . The cells of the X^n are called the cells of X .

- (b) We define the *topological realization* $|X|$ of such a CW-complex X as⁴⁰⁶

$$|X| := \varinjlim |X^n|.$$

As before, in the notation we usually do not distinguish between the notation for a CW-complex and its topological realization.

- (c) We define the dimension $\dim(X) \in \mathbb{N}_0 \cup \{\infty\}$ of a non-empty CW-complex X as the maximal dimension of a cell.⁴⁰⁷
- (d) We say a CW-complex is *finite* if it has only finitely many cells and we say it is *countable* if it has only countable many cells.
- (e) A *CW-structure* for a topological space Y is a CW-complex X together with a homeomorphism $f: X \rightarrow Y$.

Examples.

- (1) We consider the infinite sphere

$$S^\infty = \{x \in \mathbb{R}^\infty \mid d(x, x) = 1\} = \left\{ (x_1, x_2, \dots) \in \mathbb{R}^\infty \mid \sum_{i=1}^{\infty} x_i^2 = 1 \right\}$$

which equip with the subspace topology coming from $\mathbb{R}^\infty$. It follows easily from $\mathbb{R}^\infty = \varinjlim \mathbb{R}^n$ that $S^\infty = \varinjlim S^n$. We consider iteratively the spheres S^n as CW-complexes, where S^0 consists of two 0-cells and where S^n is built starting from S^{n-1} by attaching two n -cells (“the upper hemisphere and the lower hemisphere”) to S^{n-1} . We then obtain an ascending sequence

$$S^0 \subset S^1 \subset S^2 \subset \dots$$

of CW-complexes which satisfies precisely the required properties of a CW-complex as defined above. This shows that the sphere S^∞ is an infinite-dimensional CW-complex. We refer to Figure 199 for an illustration.⁴⁰⁸

- (2) Let $\mathbb{A} = \mathbb{R}$ or $\mathbb{A} = \mathbb{C}$. As in the case of spheres we can consider the infinite-dimensional projective space

$$\mathbb{A}P^\infty := (\mathbb{A}^\infty \setminus \{0\})/(\mathbb{A} \setminus \{0\}).$$

⁴⁰⁶This topology on a union of topological spaces is called the *weak topology*. This explains the “W” of the name CW-complex.

⁴⁰⁷Hereby we define $\dim(X) := \infty$ if X admits cells of arbitrarily large dimensions.

⁴⁰⁸If we equip S^∞ with the topology $\mathcal{S}$ coming from the euclidean metric on $\mathbb{R}^\infty$, then this argument does not work and it is not clear (and presumably wrong), that $(S^\infty, \mathcal{S})$ is a CW-complex.

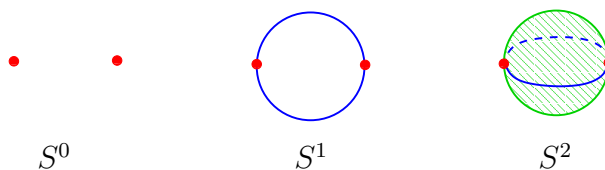


FIGURE 199.

FIGURE 200. An ascending sequence of CW-complexes $S^0 \subset S^1 \subset S^2 \dots$.

Given any $n \in \mathbb{N}$ we view $\mathbb{A}P^n$ as a subset of $\mathbb{A}P^\infty$ in the obvious way. It follows again easily from $\mathbb{A}P^\infty = \varinjlim \mathbb{A}P^n$ that $\mathbb{A}P^\infty = \varinjlim \mathbb{A}P^n$. By Lemma 22.1 we can therefore view $\mathbb{A}P^\infty$ as an infinite-dimensional CW-complex.

22.4. Properties of CW-complexes. In the previous section we had seen that, as promised, most examples of topological spaces that we are interested in are indeed CW-complexes. The following, somewhat technical, lemma shows that CW-complexes have, again as promised, many nice topological properties.

Lemma 22.4.

- (1) *Every point on a CW-complex is closed.*
- (2) *CW-complexes are Hausdorff.*
- (3) *Every cell of a CW-complex X is compact and it is a closed subset of X .*
- (4) *CW-complexes are locally contractible.⁴⁰⁹*
- (5) *CW-complexes are locally path-connected and locally simply connected.*
- (6) *A CW-complex is connected if and only if it is path-connected.*
- (7) *Every subcomplex A of a CW-complex admits an open neighborhood U such that A is a deformation retract of U .*

Remark.

- (1) In Lemma 22.4 (7) it is essential that A is a subcomplex. For example, if we consider $X = \mathbb{R}$ and $A = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\}$, then A does not admit an open neighborhood U such that A is a deformation retract of U .⁴¹⁰
- (2) It follows from Lemma 22.4 (3) that every finite CW-complex is compact.
- (3) Lemma 22.4 (5) sound somewhat technical, but it ensures that later on we can apply Proposition 15.5 to obtain coverings of CW-complexes.
- (4) It follows from Corollary 22.4 that every point on a CW-complex is simple in the sense of the definition on page 156.

⁴⁰⁹Recall that a topological space X is locally contractible if given any $x \in X$ and any open neighborhood U of x there exists a contractible neighborhood V of x that is contained in U . Hereby we say that V is contractible if V is homotopy equivalent to a topological space which consists of precisely one point.

⁴¹⁰Why not?

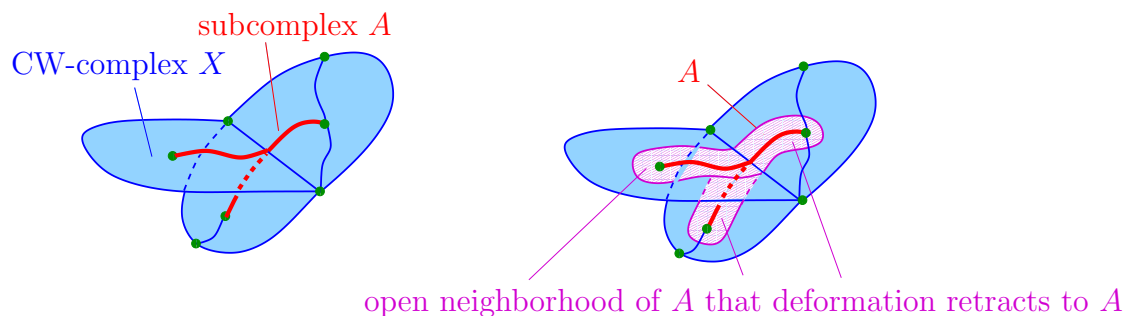


FIGURE 201. Illustration of Lemma 22.4 (7).

Proof (*).⁴¹¹⁴¹² We start out with the following notation. Given a subset A of $\overline{B}^n$ and $\epsilon > 0$ we denote by

$$N_\epsilon(A) := \{x \in \overline{B}^n \mid \text{there exists an } y \in A \text{ with } \|x - y\| < \epsilon\}$$

the ϵ -neighborhood around A . Furthermore given two subsets X and Y of $\overline{B}^n$ we write

$$d(X, Y) := \inf\{\|x - y\| \mid x \in X \text{ and } y \in Y\}.$$

Now let X be a CW-complex. We denote by Γ the set of cells of X and given $n \in \mathbb{N}_0$ we denote by Γ_n the set of n -cells. Given a cell γ we denote by Φ_γ the characteristic map.

Let $A \subset X$ be a subset and let ϵ be a function that assigns to each $\gamma \in \Gamma$ a positive number ϵ_γ . Given $n \in \mathbb{N}_0$ we define a subset $N_\epsilon^n(A)$ inductively as follows:

- (1) For $n = 0$ we define $N_\epsilon^0(A) := A \cap X^0$.
- (2) Suppose that $N_\epsilon^n(A)$ has already been defined. We define

$$N_\epsilon^{n+1}(A) := N_\epsilon^n(A) \cup \bigcup_{\gamma \in \Gamma_{n+1}} \Phi_\gamma \left(\underbrace{N_{\epsilon_\gamma}(\overbrace{\Phi_\gamma^{-1}(A)}^{\subset \overline{B}^{n+1}})}_{\text{radial coordinates on } \overline{B}^{n+1}} \cup \underbrace{(1 - \epsilon_\gamma, 1] \times \overbrace{\Phi_\gamma^{-1}(N_\epsilon^n(A))}^{\subset S^n}} \right).$$

It is straightforward to verify that each $N_\epsilon^n(A)$ is an open neighborhood of $A \cap X^n$ in X^n . The definition of $N_\epsilon^{n+1}(A)$ is illustrated in Figure 202. Finally we set $N_\epsilon(A) := \bigcup_{n \in \mathbb{N}} N_\epsilon^n(A)$.

It follows easily from the definition of the topology of a CW-complex that $N_\epsilon(A)$ is an open neighborhood of A in X . The definition of $N_\epsilon(A)$ is illustrated in Figure 203.

Now we turn to the proofs of the seven statements:

- (1) It follows easily from the definition of the topology on a CW-complex that every point is closed. We will fill in the details in exercise sheet 1.

⁴¹¹Proofs marked with (*) were only sketched or not done at all in the lectures. The proofs marked with (*) tend to be technical and not overly enlightening. It might make sense to browse through the proof to get an idea of what methods are used. Only if you have lots of time would I recommend to read the proofs in detail.

⁴¹²Since I did not do the proofs in the lectures the probability that there are mistakes is also higher.

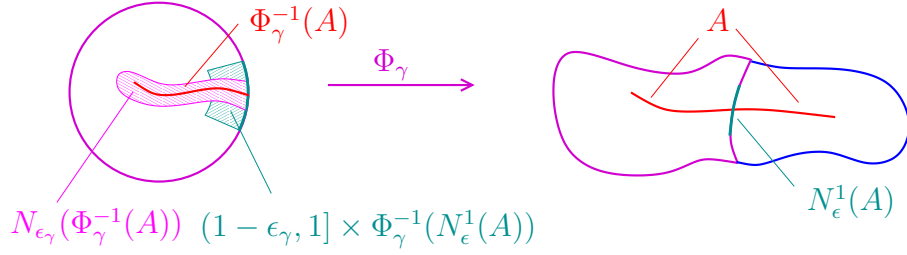


FIGURE 202.

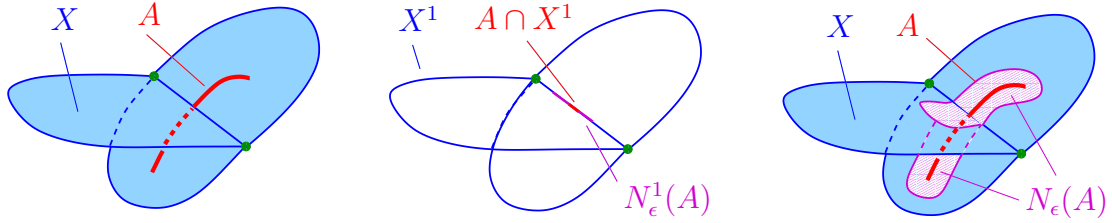


FIGURE 203.

- (2) We will prove the following statement: given any two disjoint closed subsets A and B of X there exist open disjoint neighborhoods of A and B . By (1) this implies that X is Hausdorff.

So let A and B be two disjoint closed subsets of X . We will find ϵ as above such that $N_\epsilon(A)$ and $N_\epsilon(B)$ are disjoint. We will define the values of ϵ inductively on the dimension of the cells. For 0-cells we take $\epsilon = 1$. Now suppose that we have already found disjoint open sets $N_\epsilon^n(A)$ and $N_\epsilon^n(B)$. Let γ be an $(n+1)$ -cell with characteristic map $\Phi_\gamma: \overline{B}^{n+1} \rightarrow X$.

Claim. We have

$$\begin{aligned} \mu_1 &:= d(\Phi_\gamma^{-1}(N_\epsilon^n(A)), \Phi_\gamma^{-1}(B)) > 0, \\ \mu_2 &:= d(\Phi_\gamma^{-1}(N_\epsilon^n(B)), \Phi_\gamma^{-1}(A)) > 0, \\ \mu_3 &:= d(\Phi_\gamma^{-1}(B), \Phi_\gamma^{-1}(A)) > 0. \end{aligned}$$

We first consider μ_1 . Suppose that $\mu_1 = 0$. If this was the case then there would exist a sequence of points in $\Phi_\gamma^{-1}(B)$ that converges to a point of $\Phi_\gamma^{-1}(B)$ in $\partial \overline{B}^{n+1}$ of distance zero from $\Phi_\gamma^{-1}(N_\epsilon^n(A))$. But this is impossible since $\Phi_\gamma^{-1}(N_\epsilon^n(B))$ is a neighborhood of $\Phi_\gamma^{-1}(B) \cap \partial \overline{B}^{n+1}$ in $\partial \overline{B}^{n+1}$ which is disjoint from $\Phi_\gamma^{-1}(N_\epsilon^n(A))$. The second inequality $\mu_2 > 0$ is proved exactly the same way. Finally we have $\mu_3 > 0$

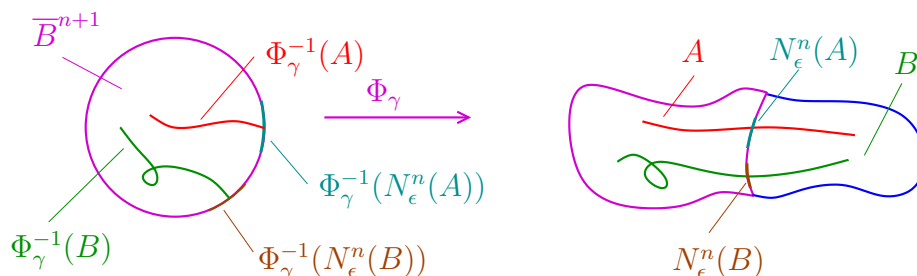


FIGURE 204.

since $\Phi_\gamma^{-1}(A)$ and $\Phi_\gamma^{-1}(B)$ are disjoint closed, hence compact, subsets of $\overline{B}^{n+1}$.⁴¹³ This concludes the proof of the claim.

We set $\epsilon_\gamma := \frac{1}{2} \min\{\mu_1, \mu_2, \mu_3\}$. We do this for all $(n+1)$ -cells γ . It then follows immediately from the definitions that $N_{\epsilon_\gamma}^{n+1}(A)$ and $N_{\epsilon_\gamma}^{n+1}(B)$ are disjoint.

- (3) Every k -cell is the image of the compact set $\overline{B}^k$, hence it is compact by Lemma 1.2. It follows from Lemma 1.1 and the fact, proved in (2), that X is Hausdorff that every k -cell is also a closed subset of X .
- (4) Let $x \in X$ and let U be an open neighborhood of x . We leave it as an exercise to show that there exists an ϵ such that $N_\epsilon(x)$ is contained in U .

It remains to show that $N_\epsilon(x)$ is contractible. We denote by m the unique natural number with $x \in X^m \setminus X^{m-1}$. It follows easily from the definitions that $N_\epsilon^m(x)$ is an open ball about x , in particular $N_\epsilon^m(x)$ deformation retracts onto x . Therefore it suffices to find a deformation retraction from $N_\epsilon(x)$ to $N_\epsilon^m(x)$.

First note that for $n > m$ we can construct a deformation retraction r_n from $N_\epsilon^n(x)$ to $N_\epsilon^{n-1}(x)$ by sliding outward along radial segments in the n -cells. We refer to Figure 205 for an illustration. A deformation retraction of $N_\epsilon(x)$ onto $N_\epsilon^m(x)$ is then obtained by performing the deformation retraction r_n from $N_\epsilon^n(x)$ to $N_\epsilon^{n-1}(x)$ during the “time-interval” $[\frac{1}{2^n}, \frac{1}{2^{n-1}}]$ where the points of $N_\epsilon(x) \setminus N_\epsilon^{n-1}(x)$ stay fixed during that time interval.

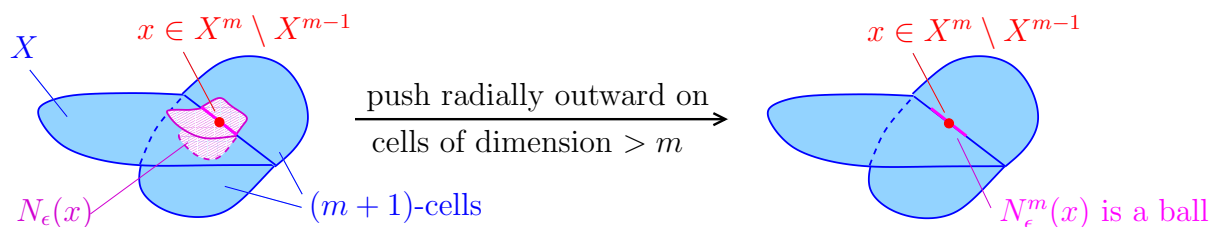


FIGURE 205.

⁴¹³Here we use the following general fact: if K and L are two compact disjoint subsets of $\mathbb{R}^n$, then $d(K, L) > 0$. How would you prove this?

- (5) This statement follows immediately from (4) since every contractible space is path-connected and simply connected.
- (6) By Lemma 1.19 a locally path-connected topological space is connected if and only if it is path-connected.
- (7) Let A be a subcomplex of X . We let $\epsilon \in (0, 1)$. We set $U = N_\epsilon(A)$. In each cell of $X \setminus A$ the set $N_\epsilon(A)$ is a product neighborhood of the boundary of this cell. So a deformation retraction of $N_\epsilon(A)$ onto A can be constructed as in (4). We leave the details to the reader. $\square$

Remark. A topological space X is called *normal* if any two disjoint closed subsets A and B of X are separated by neighborhoods, i.e. if there exist disjoint neighborhoods U of A and V of B . In the proof of Lemma 22.4 (2) we actually showed that CW-complexes are normal. In fact most spaces that we have encountered that are Hausdorff are also normal. For example $\mathbb{R}^n$ and subsets thereof are normal. But there are examples of topological spaces that are Hausdorff but that are not normal.⁴¹⁴

A famous lemma, proved by Urysohn, says that a normal topological space X has the property that given two disjoint closed subsets there exists a continuous function $f: X \rightarrow \mathbb{R}$ with $f|_A \equiv 0$ and $f|_B \equiv 1$. We refer to [J1, p. 109] for details. We will not make use of this rather neat fact.

Using Lemma 22.4 we can now prove the following very useful version of the Seifert–van Kampen Theorem 11.1. In view of Theorem 22.2 it can be viewed as a generalization of the Seifert–van Kampen Theorem 11.2 for manifolds.

Theorem 22.5. (Seifert–van Kampen for CW-complexes) *Let X be a CW-complex and let $X = A \cup B$ be a decomposition of X in two subcomplexes such that $A \cap B$ is path-connected. Let $x_0 \in A \cap B$. Then*

$$\pi_1(X, x_0) \cong \pi_1(A, x_0) *_{\pi_1(A \cap B, x_0)} \pi_1(B, x_0).$$

Examples.

- (1) Let X be the wedge of two circles. We can view X as a CW-complex with one 0-cell x_0 and two 1-cells a and b . Then $A = \{x_0, a\}$ and $B = \{x_0, b\}$ are subcomplexes, each of which is homeomorphic to S^1 . We refer to Figure 206 for an illustration. It follows, as on page 158, that

$$\begin{array}{ccccc} \pi_1(X, x_0) & \cong & \pi_1(A, x_0) *_{\pi_1(A \cap B, x_0)} \pi_1(B, x_0) & \cong & \langle a \rangle * \langle b \rangle = \langle a, b \rangle. \\ \uparrow & & & & \uparrow \\ \text{Theorem 22.5} & & & & \text{since } A \text{ and } B \text{ are homeomorphic to circles} \end{array}$$

- (2) We consider

$$K := \underbrace{S^1 \times S^1}_{=: A} \cup \underbrace{\overline{B}^2 \times \{1\}}_{=: B},$$

⁴¹⁴Fun exercise: try to find such an example.

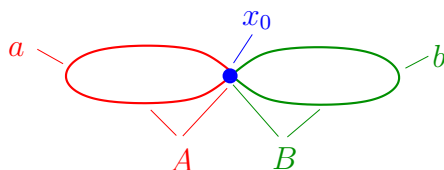


FIGURE 206.

i.e. K is given by a torus where the “central hole” is filled in with a disk. We refer to Figure 207 for an illustration.⁴¹⁵ In the above definition of K we already wrote K as the union of two subsets. It is quite straightforward to see that K can be given a CW-structure such that A and B are subcomplexes.⁴¹⁶ We pick the base point $x_0 = (1, 1)$. We obtain that

$$\begin{array}{ccccccc} \pi_1(K, x_0) & \cong & \pi_1(A, x_0) *_{\pi_1(A \cap B, x_0)} \pi_1(B, x_0) & \cong & \underbrace{\langle x, y | [x, y] \rangle *_{\langle s \rangle} \{e\}}_{\text{with } x \leftrightarrow s} & \cong & \langle y \rangle. \\ \uparrow & & & \uparrow & & \uparrow & \\ \text{Theorem 22.5} & & \text{since } A \cap B = S^1 \times \{1\} & & & \text{Lemma 10.8} & \end{array}$$

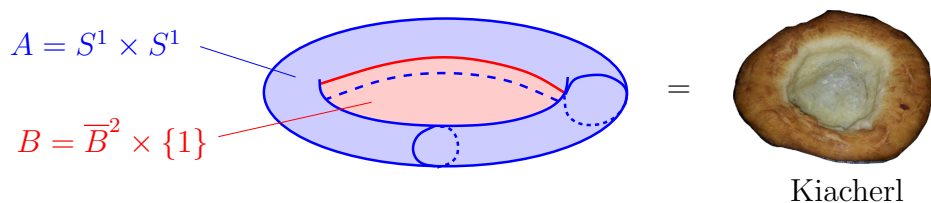


FIGURE 207.

Proof of Theorem 22.5 ().* ⁴¹⁷ Let X be a CW-complex and let $X = A \cup B$ be a decomposition of X in two subcomplexes such that $A \cap B$ is path-connected. Note that $A \cap B$ is also a subcomplex. Let $x_0 \in A \cap B$.

We use the notation from the proof of Lemma 22.4. We let $\epsilon \in (0, 1)$. We write $U = N_\epsilon(A)$ and $V = N_\epsilon(B)$. It follows easily from the definitions that $U \cap V = N_\epsilon(A \cap B)$. In the proof of Lemma 22.4 (7) we had seen that the path-connected set $A \cap B$ is a deformation retract of $U \cap V = N_\epsilon(A \cap B)$. This implies that $U \cap V$ is also path-connected. Therefore we can apply the Seifert–van Kampen Theorem 11.1 to the decomposition $X = U \cup V$. We

⁴¹⁵Here the name K stands for “kiacherl”.

⁴¹⁶How would you define the CW-structures?

⁴¹⁷The proof is very similar to the proof of the Seifert–van Kampen Theorem for manifolds 11.2.

obtain the following commutative diagram

$$\begin{array}{ccc}
 \pi_1(U, x_0) *_{\pi_1(U \cap V, x_0)} \pi_1(V, x_0) & \xrightarrow{\cong} & \pi_1(X, x_0) \\
 \uparrow & & \uparrow = \\
 \pi_1(A, x_0) *_{\pi_1(A \cap B, x_0)} \pi_1(B, x_0) & \longrightarrow & \pi_1(X, x_0),
 \end{array}$$

where the two vertical maps are induced by the inclusions. By Lemma 22.4 and Proposition 7.5 the three inclusion induced maps $\pi_1(A, x_0) \rightarrow \pi_1(U, x_0)$, $\pi_1(B, x_0) \rightarrow \pi_1(V, x_0)$ and $\pi_1(A \cap B, x_0) \rightarrow \pi_1(U \cap V, x_0)$ are isomorphisms. The top horizontal map is an isomorphism by the Seifert–van Kampen Theorem 11.1. It follows that the bottom horizontal map is also an isomorphism. $\square$

The following proposition is one of the key technical results about infinite CW-complexes.

Proposition 22.6. *Every compact (connected) subset of a CW-complex X is contained in a finite (connected) subcomplex.*

Remark. As we had remarked before, it follows from Lemma 22.4 (3) that every finite CW-complex is compact. It follows from Proposition 22.6 that the converse holds. Indeed, if X is a compact CW-complex, then it is by Proposition 22.6 contained in a finite subcomplex Y . But from $X \subset Y \subset X$ it follows that $X = Y$.

The proof of Proposition 22.6 will be a fairly straightforward consequence of the following rather technical lemma.

Lemma 22.7. *Let X be a CW-complex. Then the following statements hold:*

- (1) *Let $A \subset Y$ be a subset. If every open cell of Y contains at most one point of A , then A is closed and discrete.*
- (2) *If $C \subset X$ is a compact subset, then C is contained in a finite union of open cells.*
- (3) *Every cell of X is contained in a finite subcomplex of X .⁴¹⁸*

Proof of Lemma 22.7 ().* We start out with the following claim.

Claim. Let Y be a CW-complex. Then the statements (1), (2) and (3) are equivalent.

Let Y be a CW-complex. First we show (1) $\Rightarrow$ (2). Let $C \subset Y$ be a compact subset. For each cell that intersects C we pick a single point. We denote the resulting set of points by A . By (3) the set A is closed and discrete. As a subset of a compact set A is furthermore compact. Since A is compact and discrete we see that A is finite. But that means that C intersects only finitely many cells, i.e. C is contained in a finite union of open cells as claimed.

Now we will prove (2) $\Rightarrow$ (3). So let $C_\sigma \subset Y$ be an n -cell. We denote by U_σ the corresponding open cell. By Lemma 22.4 (3) and by definition of a cell we know that

⁴¹⁸This implies in particular that the *closure* of each open cell is contained in the union of *finitely* many open cells. This property is sometimes referred to as X being “closure finite” which explains the “C” in the name CW-complex.

$C_\sigma \setminus U_\sigma$ is a compact subset of the $(n-1)$ -skeleton Y^{n-1} . It follows from (2) that the subset $C_\sigma \setminus U_\sigma$ is contained in a finite union of open cells of dimension $\leq n-1$. By the same argument each of these lower-dimensional closed cells is again contained in a finite union of open cells of smaller dimension, except again for the cell itself. A straightforward induction argument now shows that we end up with finitely many cells which form a subcomplex of Y .

Next we will show (3) $\Rightarrow$ (1). Let $A \subset Y$ be a subset such that no two points lie in the same open cell. We consider the intersection of A with a closed cell. By (3) this intersection is contained in a finite subcomplex Y' . Since A has at most one point in common with any open cell this intersection $A \cap Y'$ must be finite and hence closed in $A \cap Y'$ by Lemma 22.4 (1). By Lemma 22.4 (3) the finite subcomplex Y' is a closed subset of X . Hence $A \cap Y'$ is a closed subset of X . For any point $x \in A$ the set $A \setminus \{x\}$ also satisfies the hypothesis for A , and so it must also be a closed subset of X . Hence $\{x\}$ is open in A , so A is discrete. This concludes the proof of the claim.

Now let X be a CW-complex. First note that statement (1) holds for X^0 by definition of the topology on $X^{(0)}$. Now suppose we know (2) for the n -skeleton X^n for some $n \in \mathbb{N}_0$. By claim we know that (2) and (3) also for X^n . But the proof of (2) $\Rightarrow$ (3) for a given n -cell only used (2) for subsets of X^{n-1} . Thus we actually have (3) for X^{n+1} . Thus we also have (1) for X^{n+1} . Summarizing we have all three statements for X^n for all n . But any cell lies in some X^n , and so we know (3) for X itself, and by the claim this implies that (1) and (2) also hold for X . $\square$

Now we can provide the proof of Proposition 22.6.

Proof of Proposition 22.6. Let X be a CW-complex and let $C \subset X$ be a compact subset. By Lemma 22.7 (2) the subset C is contained in a union of finitely many open cells. By Lemma 22.7 (3) each of these open cells is contained in a finite subcomplex. The union of these finitely many finite subcomplexes is a finite subcomplex Z which contains C . If C is connected, then we take the component of Z that contains C . As remarked on page 341, this is again a subcomplex. $\square$

In many situations the following proposition makes it possible to reduce the study of arbitrary CW-complexes to the study of finite CW-complexes.

Proposition 22.8. *Let X be a CW-complex with countably many cells. Then the following hold:*

- (a) *There exists a sequence $A_1 \subset A_2 \subset \dots$ of finite subcomplexes A_i of X such that $X = \bigcup_{i \in \mathbb{N}_0} A_i$.*
- (b) *If X is connected, then we can arrange in (a) that every A_i is connected.*

Proof. Let X be a CW-complex with countably many cells. We number them with $e_1, e_2, \dots$. By Lemma 22.7 (3) and Proposition 22.6 there exists for each i a finite connected subcomplex B_i that contains e_i . We define $A_1 := B_1$ and iteratively we define $A_i := A_{i-1} \cup B_i$. Clearly this sequence satisfies (a) and (b).

Now suppose that X is connected. We define the sequence by $A_1 := B_1$ and iteratively we define A_i as follows:

- (1) We pick a path $p: [0, 1] \rightarrow X$ from a point A_{i-1} to a point B_i .
- (2) Then we denote by A_i any finite connected subcomplex that contains the compact connected set $A_{i-1} \cup B_i \cup p([0, 1])$. Such a finite connected subcomplex exists by Lemma 22.7 (3) and Proposition 22.6.

This sequence clearly has the desired properties (a), (b) and (c). $\square$

We conclude this section with the following proposition that is a variation on Proposition 13.2.

Proposition 22.9. *Let X be a CW-complex and let $X_1 \subset X_2 \subset X_3 \subset \dots$ be a sequence of subcomplexes such that $X = \bigcup_{i \in \mathbb{N}} X_i$. Then for any $w \in X_1$ we have*

$$\pi_1(X) = \varinjlim \pi_1(X_i, w).$$

Examples.

- (1) We have⁴¹⁹

$$\begin{array}{ccccccc} & & & \uparrow & & \uparrow \cong & \\ & & & \pi_1(\mathbb{RP}^3) & & \mathbb{Z}_2 & \\ & & \uparrow & & \uparrow \cong & & \\ \pi_1(\mathbb{RP}^\infty) & = & \pi_1\left(\bigcup_{n \in \mathbb{N}} \mathbb{RP}^n\right) & = & \varinjlim & \pi_1(\mathbb{RP}^2) & = \varinjlim \uparrow \cong \cong \mathbb{Z}_2. \\ \uparrow & & \uparrow & & \uparrow & \uparrow & \\ \text{see page 345} & & \text{Proposition 22.9} & & \pi_1(\mathbb{RP}^1) & \mathbb{Z} & \end{array}$$

- (2) In exercise sheet 1 we will use Proposition 22.9 to show that there exists a CW-complex X with $\pi_1(X) \cong (\mathbb{Q}, +)$.

Proof. We start out with the following claim.

Claim. Every compact subset K is contained in some X_i .

It follows from Proposition 22.8 that K is contained in a finite subcomplex Y of X . From $X = \bigcup_{i \in \mathbb{N}} X_i$ and from the fact that each X_i is a subcomplex of X we obtain that every cell is contained in some X_i . But this implies that the finite subcomplex Y is also contained in some X_i . This concludes the proof of the claim.

The proof of Proposition 22.9 is now almost exactly the same as the proof of Proposition 13.2, except that we have to use the above observation to conclude that the image of every loop and of every homotopy in $X = \bigcup_{i \in \mathbb{N}} X_i$ lies already in some X_i . We leave the details to the reader. $\square$

⁴¹⁹By Corollary 6.17 we know that $\pi_1(\mathbb{RP}^n) \cong \mathbb{Z}_2$ for $n \geq 2$. But why is $\pi_1(\mathbb{RP}^1) \cong \mathbb{Z}$?

22.5. The Homotopy Extension Theorem. On several occasions we will need the following somewhat technical theorem.

Theorem 22.10. (Homotopy Extension Theorem) *Let X be a CW-complex, let Y be a topological space and let $f_0: X \rightarrow Y$ be map. Furthermore, suppose that $A \subset X$ is a subcomplex and suppose we are given a homotopy $G: A \times [0, 1] \rightarrow Y$ from $g_0 := f_0|_A$ to some map $g_1: A \rightarrow Y$. Then there exists a homotopy $F: X \times [0, 1] \rightarrow Y$ starting from f_0 such that the restriction $F|_{A \times [0, 1]}$ equals G .*

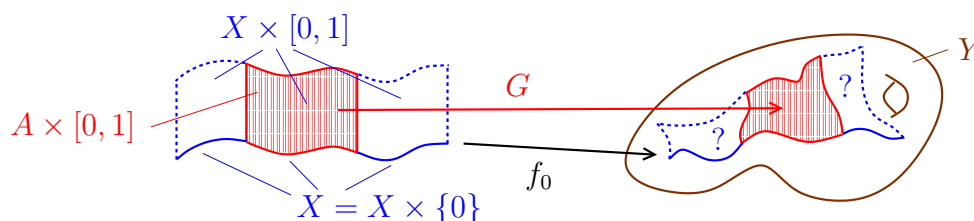


FIGURE 208. Illustration of the Homotopy Extension Theorem 22.10.

Remark. The statement of the Homotopy Extension Theorem does not hold for arbitrary subsets of X . For example, let $X = \mathbb{R}$ and $A = (0, 1)$. We write $Y = \mathbb{R} \times \{0\} \cup (0, 1) \times [0, 1]$ and let $f: X \rightarrow Y$ be the obvious inclusion map and let $G: (0, 1) \times [0, 1] \rightarrow Y$ be the obvious inclusion map. Then G cannot be extended to a homotopy on all of $\mathbb{R}$.

Before we can provide the proof of Theorem 22.10 we will need to formulate and prove two lemmas.

Lemma 22.11. *For any n there exists a retraction*

$$d: \underbrace{\overline{B}^n \times [0, 1]}_{\text{solid cylinder}} \rightarrow \underbrace{\overline{B}^n \times \{0\} \cup S^{n-1} \times [0, 1]}_{\text{cup}}.$$

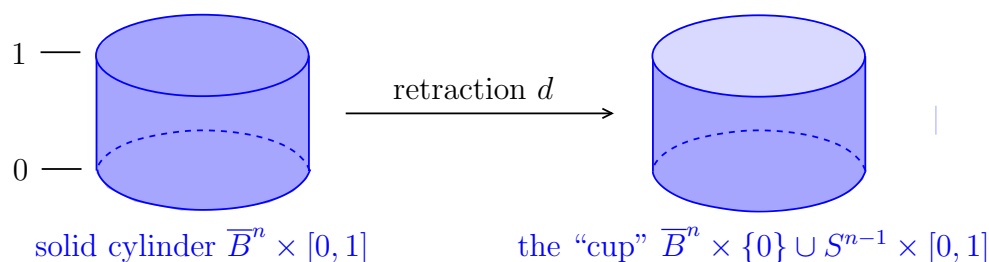


FIGURE 209.

Proof. It is straightforward to verify that the map

$$\begin{aligned} r: \overline{B}^n \times [0, 1] &\rightarrow \overline{B}^n \times \{0\} \cup S^{n-1} \times [0, 1] \\ P &\mapsto \text{the unique point on } \overline{B}^n \times \{0\} \cup S^{n-1} \times [0, 1] \text{ that} \\ &\quad \text{gets hit by the ray from } (0, 2) \in \mathbb{R}^n \times \mathbb{R} \text{ through } P. \end{aligned}$$

is a retraction. We refer to Figure 210 for an illustration. $\square$

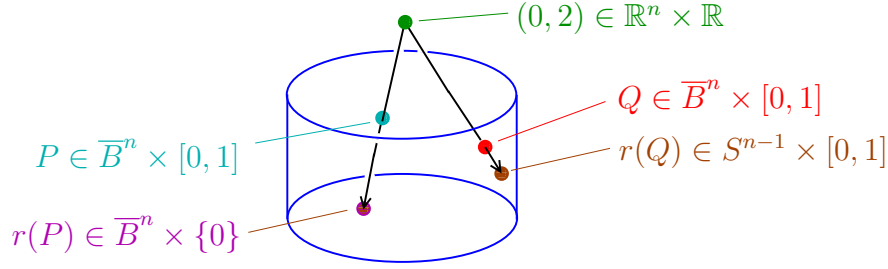


FIGURE 210.

Lemma 22.12. *Let X be a CW-complex and let A be a subcomplex. Then there exists a retraction*

$$H: X \times [0, 1] \rightarrow X \times \{0\} \cup A \times [0, 1].$$

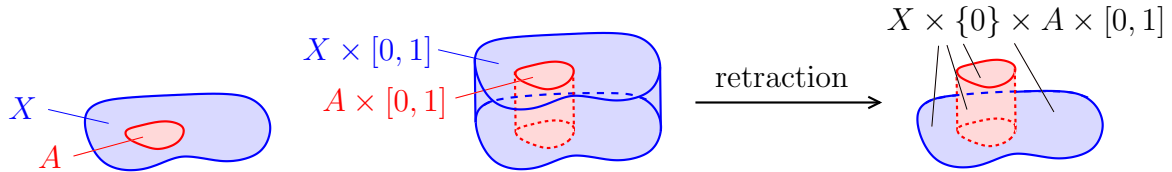


FIGURE 211. Illustration of Lemma 22.12.

Proof. Let X be a CW-complex and let A be a subcomplex. We only prove the lemma in the special case that X is finite-dimensional. The general case is proved in [Ha, Proposition 0.16]. We start out with the following claim.

Claim. For every $n \in \mathbb{N}_0$ there exists a retraction

$$H_n: X \times \{0\} \cup (X^n \cup A) \times [0, 1] \rightarrow X \times \{0\} \cup (X^{n-1} \cup A) \times [0, 1].$$

Let $n \in \mathbb{N}_0$. We denote by $\varphi_i: \overline{B}^n \rightarrow X$, $i \in I$, the characteristic maps of all the n -cells of X that do not lie in A . By Lemma 22.11 there exists a retraction d from $\overline{B}^n \times [0, 1]$ to $\overline{B}^n \times \{0\} \cup S^{n-1} \times [0, 1]$. Now we consider the map

$$\begin{aligned} H_n: X \times \{0\} \cup (X^n \cup A) \times [0, 1] &\rightarrow X \times \{0\} \cup (X^{n-1} \cup A) \times [0, 1] \\ (P, t) &\mapsto \begin{cases} \varphi_i(d(Q, t)), & \text{if } P = \varphi_i(Q) \text{ with } Q \in \overline{B}^n \text{ and } i \in I, \\ (P, t), & \text{otherwise.} \end{cases} \end{aligned}$$

This map is easily seen to be a retraction. This concludes the proof of the claim.

Now let $n \in \mathbb{N}_0$ be the dimension of the CW-complex X . Then the map

$$H := H_0 \circ H_1 \circ \cdots \circ H_n: \underbrace{X \times \{0\} \cup (X^n \cup A) \times [0, 1]}_{=X \times [0, 1]} \rightarrow \underbrace{X \times \{0\} \cup (X^{n-1} \cup A) \times [0, 1]}_{=X \times \{0\} \cup A \times [0, 1]}$$

is the desired retraction. We refer to Figure 212 for an illustration of the argument. $\square$

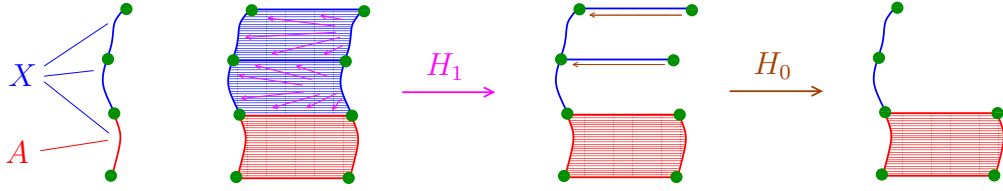


FIGURE 212.

Now we can easily provide a proof of Theorem 22.10.

Proof of Theorem 22.10. Let X be a CW-complex, let Y be a topological space and let $f_0: X \rightarrow Y$ be map. Furthermore, suppose that $A \subset X$ is a subcomplex and suppose we are given a homotopy $G: A \times [0, 1] \rightarrow Y$ from $g_0 := f_0|_A$ to some map $g_1: A \rightarrow Y$. By Lemma 22.12 there exists a retraction

$$H: X \times [0, 1] \rightarrow X \times \{0\} \cup A \times [0, 1].$$

It is now straightforward to see that the map

$$\begin{aligned} F: X \times [0, 1] &\xrightarrow{H} X \times \{0\} \cup A \times [0, 1] \rightarrow Y \\ (x, 0) &\mapsto f_0(x), & \text{for } x \in X, \\ (a, t) &\mapsto G(a, t), & \text{for } (a, t) \in A \times [0, 1] \end{aligned}$$

has the desired properties. $\square$

22.6. Fundamental groups of CW-complexes.

Definition. Given a finite CW-complex X we define its Euler characteristic $\chi(X)$ as follows:

$$\chi(X) := \sum_n (-1)^n \cdot \text{number of } n\text{-cells}.$$

Examples.

- (1) If G is a finite graph, then, according to the discussion on page 338, we can view G as a CW-complex. In this case the Euler characteristic agrees with the definition of the Euler characteristic of a graph that we had given on page 131.
- (2) For the CW-structures provided by Lemma 22.1 we have $\chi(\mathbb{R}P^n) = \frac{1}{2}(1 + (-1)^n)$ and $\chi(\mathbb{C}P^n) = n + 1$.

- (3) A short glance at the three examples in Figure 196 shows that these three different CW-structures for the torus have the same Euler characteristic. Later on we will find out whether or not that is a coincidence.

Proposition 22.13. *Let X be a connected 1-dimensional CW-complex and let $x_0 \in X$. Then the following hold:*

- (1) *The fundamental group $\pi_1(X, x_0)$ is a free group.*
- (2) *If X is a finite CW-complex, then $\pi_1(X, x_0)$ is a free group with $1 - \chi(X)$ generators.*

Proof. In Lemma 9.4 we had shown that if G is a finite connected graph, then $\pi_1(G)$ is a free group with $1 - \chi(G)$ generators. Verbatim the same proof applies to finite 1-dimensional CW-complexes, which proves (2).

Now we turn to the proof of (1). We will only provide the proof for the case that X has countably many cells. The general case is proved in [Ha, Proposition 1.A.1]. Thus let X be a connected countable 1-dimensional CW-complex and let $x_0 \in X$. If X is finite, then we are already done by the above argument. So suppose that X is infinite. Since X is connected this implies that X has infinitely many 1-cells. We start out with the following claim.

By Proposition 22.8 there exists a sequence of finite connected subcomplexes A_i , $i \in \mathbb{N}_0$ with the following properties:

- (a) each A_i is a subcomplex of A_{i+1} ,
- (b) $X = \bigcup_{i \in \mathbb{N}_0} A_i$.

By Proposition 4.9 we can assume without loss of generality that x_0 is a point in A_1 .

The proof of the following claim is a slight generalization of the proof of Lemma 9.4. We leave the details as an exercise to the reader.

Claim. Let C be a connected finite 1-dimensional CW-complex and let B be a connected subcomplex. Let $x \in B$. Suppose we are given a finite set b and an isomorphism $\phi: \pi_1(B, x) \cong \langle b \rangle$. Then there exists finite set c , disjoint from b , and an isomorphism $\psi: \pi_1(C, x) \cong \langle b \sqcup c \rangle$ such that the following diagram commutes:

$$\begin{array}{ccc} \pi_1(B, x) & \xrightarrow{\cong} & \langle b \rangle \\ \downarrow & & \downarrow \\ \pi_1(C, x) & \xrightarrow{\cong} & \langle b \sqcup c \rangle, \end{array}$$

where the left vertical map is the inclusion induced map and the right vertical map is the obvious map.

Iteratively we apply the claim to the above sequence $A_0, A_1, A_2, \dots$ of connected 1-dimensional CW-complexes and we obtain disjoint finite sets $a_0, a_1, a_2, \dots$ and identifications

$\pi_1(A_i, x_0) = \langle a_0 \sqcup \cdots \sqcup a_i \rangle$. We obtain that

$$\pi_1(X, x_0) \underset{\substack{\uparrow \\ \text{Proposition 22.9}}}{\cong} \lim_{\rightarrow} \pi_1(A_i, x_0) = \lim_{\rightarrow} \langle a_0 \sqcup \cdots \sqcup a_i \rangle = \underset{\substack{\uparrow \\ \text{as on page 223}}}{\cong} \langle \bigsqcup_{i \in \mathbb{N}} a_i \rangle.$$

We have thus shown that $\pi_1(X, x_0)$ is a free group. $\square$

The following definition is evidently inspired by the definition of a CW-complex.

Definition. Let X be a topological space and let $\varphi: S^{n-1} \rightarrow X$ be a map. We refer to $X \cup_{\varphi} \overline{B}^n$ as the *topological space X with an n -cell attached via the attaching map φ* . We refer to the image of $\overline{B}^n$ in $X \cup_{\varphi} \overline{B}^n$ as an n -cell.

The following proposition now tells us how attaching a cell affects the fundamental group of a given topological space.

Proposition 22.14. *Let X be a topological space, let $n \geq 2$, let $\varphi: S^{n-1} \rightarrow X$ be a map and let x_0 be a point in $\varphi(S^{n-1})$. Then the following hold:*

- (a) *If $n > 2$, then the inclusion induced map $\pi_1(X, x_0) \rightarrow \pi_1(X \cup_{\varphi} \overline{B}^n, x_0)$ is an isomorphism.*
- (b) *If $n = 2$, then $\varphi: S^1 \rightarrow X$ defines an element in $\pi_1(X, x_0)$ that we denote by $[\varphi]$, and the inclusion induced map $\pi_1(X, x_0) \rightarrow \pi_1(X \cup_{\varphi} \overline{B}^n, x_0)$ descends to an isomorphism*

$$\pi_1(X, x_0) / \langle\langle [\varphi] \rangle\rangle \xrightarrow{\cong} \pi_1(X \cup_{\varphi} \overline{B}^n, x_0).$$

Examples.

- (1) Let $X = S^1$ and let $n \in \mathbb{N}$. We attach to S^1 a 2-cell via the attaching map $\varphi: S^1 \rightarrow S^1$ that is given by $\varphi(z) = z^n$. Then⁴²⁰

$$\pi_1(S^1 \cup_{\varphi} \overline{B}^2, 1) \underset{\substack{\uparrow \\ \text{by Proposition 22.14}}}{=} \pi_1(S^1, 1) / \langle\langle [\varphi] \rangle\rangle = \underset{\substack{\uparrow \\ \text{see page 112}}}{=} \langle x \rangle / \langle x^n \rangle \cong \mathbb{Z}_n.$$

- (2) We attach a 2-cell to $X = \mathbb{R}^2$ via the attaching map $\text{id} = \varphi: S^1 \rightarrow S^1$ that is given by the identity. The resulting topological space is illustrated in Figure 213. This topological space is again simply connected by Proposition 22.14.

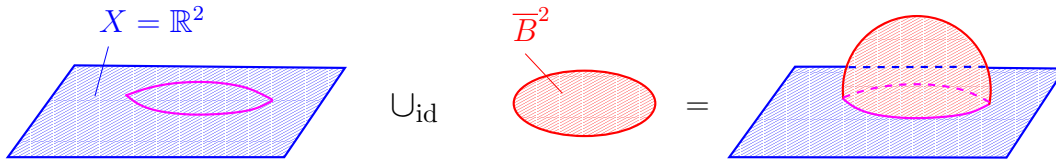


FIGURE 213.

⁴²⁰We leave it as an exercise to convince yourself that for $n = 2$ the topological space $S^1 \cup_{\varphi} \overline{B}^2$ is actually homeomorphic to $\mathbb{RP}^2$.

- (3) Using Propositions 22.13 and 22.14 we can now easily compute fundamental groups of finite CW-complexes. For example, as in Figure 214 we view the surface Σ of genus 2 as a CW-complex with one 0-cell P , four 1-cells x_1, y_1, x_2, y_2 and one 2-cell. We denote

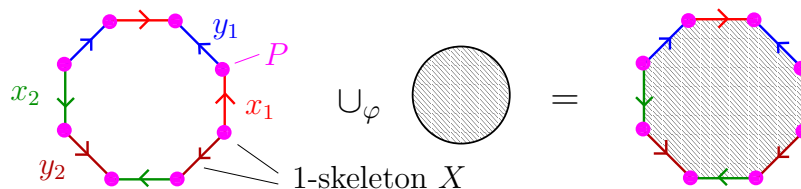


FIGURE 214.

by X the 1-skeleton of Σ and we denote by $\varphi: S^1 \rightarrow X$ the attaching map of the 2-cell. We have

$$\pi_1(\Sigma, P) = \pi_1(X \cup_{\varphi} \overline{B}^2, P) \underset{\substack{\uparrow \\ \text{by Proposition 22.14}}}{=} \pi_1(X, P) / \langle\langle [\varphi] \rangle\rangle \underset{\substack{\uparrow \\ \text{by Proposition 22.13 and since } [\varphi] = [x_1, y_1][x_2, y_2]}}{=} \langle x_1, y_1, x_2, y_2 \rangle / \langle\langle [x_1, y_1][x_2, y_2] \rangle\rangle.$$

Of course we had obtained the same result much earlier. The advantage of CW-complexes is that Proposition 22.14 absorbs many of the technical arguments we had to deal with in the proof of Proposition 11.3, i.e. in our initial calculation of $\pi_1(\Sigma, P)$.

Proof ()*. Let X be a topological space, let $n \geq 2$, let $\varphi: S^{n-1} \rightarrow X$ be a map and let x_0 be a point in $\varphi(S^{n-1})$. The key idea of the proof is to apply the Seifert–van Kampen Theorem 11.1 to the following two open⁴²¹ subsets of $X \cup_{\varphi} \overline{B}^n$

$$U = X \cup_{\varphi} \{z \in \overline{B}^n \mid \|z\| > \tfrac{1}{4}\} \quad \text{and} \quad V = \{z \in \overline{B}^n \mid \|z\| < \tfrac{3}{4}\}.$$

The remainder of the proof consists in showing that the resulting isomorphism implies the desired, a priori slightly different, isomorphism.

We pick a point $x_1 \in U \cap V$ and we pick a path in U from x_0 to x_1 . We refer to Figure 215 for an illustration. Note that V is a ball, hence $\pi_1(V, x_1)$ is the trivial group. We consider

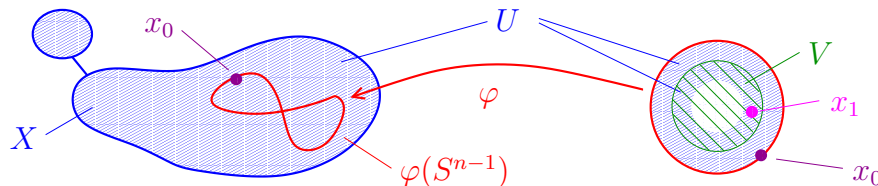


FIGURE 215.

⁴²¹Why are they open subsets?

the following diagram of maps

$$\begin{array}{ccc}
 \pi_1(X, x_0) & \xrightarrow{\quad} & \pi_1(X \cup_{\varphi} \overline{B}^n, x_0) \\
 \cong \downarrow & & = \downarrow \\
 \pi_1(U, x_0) & \xrightarrow{\quad} & \pi_1(X \cup_{\varphi} \overline{B}^n, x_0) \\
 \cong \downarrow & & \downarrow \cong \\
 \pi_1(U, x_1) & \xrightarrow{\quad} & \pi_1(X \cup_{\varphi} \overline{B}^n, x_1) \\
 \downarrow & & \downarrow = \\
 \pi_1(U, x_1) / \langle\langle \pi_1(U \cap V, x_1) \rangle\rangle & \xrightarrow{\quad} & \pi_1(X \cup_{\varphi} \overline{B}^n, x_1) \\
 \cong \downarrow \text{Lemma 10.8} & & \downarrow = \\
 \pi_1(U, x_1) *_{\pi_1(U \cap V, x_1)} \underbrace{\pi_1(V, x_1)}_{=\{e\}} & \xrightarrow[\text{Seifert-van Kampen}]{\cong} & \pi_1(X \cup_{\varphi} \overline{B}^n, x_1).
 \end{array}$$

We want to prove the following three statements:

- (i) the diagram commutes,
- (ii) all vertical maps except possibly the third vertical map on the left are isomorphisms,
- (iii) the bottom horizontal map is an isomorphism.

The two vertical maps on the second level are the isomorphisms induced by the choice of base point coming from the chosen path from x_0 to x_1 . All the other vertical and horizontal maps are induced by inclusions or they are the obvious algebraic map. It follows easily that the diagram commutes.

Most of the maps are obviously isomorphisms. The top left vertical map is an isomorphism by Lemma 7.4 and Proposition 7.5 since X is a deformation retract of U .⁴²² The bottom vertical map on the left is an isomorphism by Lemma 10.8.

Finally the bottom horizontal map is an isomorphism by the Seifert–van Kampen Theorem 11.1.

Now we consider the case $n \geq 3$. In this case $\pi_1(U \cap V, x_1)$ is trivial since $U \cap V$ is homotopy equivalent to S^{n-1} which has trivial fundamental group by Proposition 4.12. It is now clear that the one missing vertical map on the left is also an isomorphism. Since all vertical maps are now isomorphisms and since the bottom horizontal map is an isomorphism it follows that the top horizontal map is an isomorphism.

Finally we consider the case $n = 2$. In this case $U \cap V$ is homotopy equivalent to S^1 which implies that $\pi_1(U \cap V, x_1) \cong \mathbb{Z}$. Under the isomorphism $\Phi: \pi_1(X, x_0) \rightarrow \pi_1(U, x_1)$ this infinite cyclic subgroup corresponds to $\langle[\varphi]\rangle$. It is straightforward to see that we get a

⁴²²Why is it a deformation retract?

commutative diagram

$$\begin{array}{ccc}
 \pi_1(X, x_0) & \xrightarrow{\quad} & \pi_1(X \cup_{\varphi} \overline{B}^n, x_0) \\
 \downarrow & \searrow & \downarrow \\
 & \pi_1(X, x_0) / \langle\langle [\varphi] \rangle\rangle & \\
 \pi_1(U, x_1) / \langle\langle \pi_1(U \cap V, x_1) \rangle\rangle & \xrightarrow{\quad \cong \quad} & \pi_1(X \cup_{\varphi} \overline{B}^n, x_1).
 \end{array}$$

This concludes the proof of the proposition. $\square$

The following corollary says that every finitely presented group is the fundamental group of a finite CW-complex.

Corollary 22.15. *Given any finitely presented group π there exists a finite connected 2-dimensional CW-complex Y with $\pi_1(Y) \cong \pi$.*

Proof. Let $\pi = \langle g_1, \dots, g_k \mid r_1, \dots, r_m \rangle$ be a finitely presented group. We denote by X the wedge of k circles which we view as a CW-complex with one 0-cell x_0 and k 1-cells. Following the discussion on page 158 we can make the identification $\pi_1(X, x_0) = \langle g_1, \dots, g_k \rangle$. Now given $i = 1, \dots, m$ we pick a loop $\varphi_i: S^1 \rightarrow X$ in the base point x_0 such that $[\varphi_i]$ represents $r_i \in \pi_1(X, x_0) = \langle g_1, \dots, g_k \rangle$. We write $Y_0 = X$ and for $i = 1, \dots, m$ we iteratively define Y_i to be the 2-dimensional CW-complex that is given by attaching a 2-cell to Y_{i-1} with the attaching map φ_i . It follows immediately from Proposition 22.14, applied m times, that

$$\begin{aligned}
 \pi_1(Y_m, x_0) &\cong \langle g_1, \dots, g_k \rangle / \langle\langle [\varphi_1] \rangle\rangle \dots / \langle\langle [\varphi_m] \rangle\rangle \\
 &= \langle g_1, \dots, g_k \rangle / \langle\langle r_1, \dots, r_m \rangle\rangle = \langle g_1, \dots, g_k \mid r_1, \dots, r_m \rangle.
 \end{aligned}
 \quad \square$$

The following proposition says in particular that the fundamental group of a CW-complex is the same as the fundamental group of its 2-skeleton.

Proposition 22.16. *Let X be a connected CW-complex with countably many cells and let x_0 be a point in the 0-skeleton of X .*

- (1) *The inclusion induced map⁴²³ $\pi_1(X^1, x_0) \rightarrow \pi_1(X, x_0)$ is an epimorphism.*
- (2) *For any $k \geq 2$ the inclusion induced map $\pi_1(X^k, x_0) \rightarrow \pi_1(X, x_0)$ is an isomorphism.*
- (3) *If X has only finitely many cells in dimensions 0 and 1, then $\pi_1(X, x_0)$ is finitely generated.*
- (4) *If X has only finitely many cells in dimensions 0, 1 and 2, then $\pi_1(X, x_0)$ is finitely presented.*

Example. Let $X = \mathbb{RP}^\infty$ be the infinite real projective space with the CW-structure from page 345. For this CW-complex the 1-skeleton is $X^1 = \mathbb{RP}^1 \cong S^1$ and the 2-skeleton is $X^2 = \mathbb{RP}^2$. It follows from Proposition 22.16 (1) that $\mathbb{Z} \cong \pi_1(S^1) \rightarrow \pi_1(\mathbb{RP}^\infty)$ is an epimorphism and it follows from Proposition 22.16 (2) that $\mathbb{Z}_2 \cong \pi_1(\mathbb{RP}^2) \rightarrow \pi_1(\mathbb{RP}^\infty)$ is

⁴²³Recall that for $k \in \mathbb{N}$ we denote by X^k the k -skeleton of X , i.e. the union of all cells of dimension $\leq k$.

an isomorphism. This calculation of $\pi_1(\mathbb{RP}^\infty)$ is certainly easier than the argument from page 354.

Proof. We first suppose that X is a finite CW-complex. Then the first two statements follow immediately from iteratively applying Proposition 22.14 finitely many times. If X is an infinite CW-complex, but with countably many cells, then one can reduce the first two statements to the finite case using Propositions 22.8 and 22.9. We leave the details to the reader.

Now we turn to the proof of (3). By (1) it suffices to show that the fundamental group of a finite connected 1-dimensional CW-complex is finitely generated, but we had shown this in Proposition 22.13.

Finally we turn to the proof of (4). By (2) it suffices to show that a finite connected 2-dimensional CW-complex X has a finitely presented fundamental group. Let $x_0 \in X$. We denote by $\varphi_1, \dots, \varphi_k$ the attaching maps of the 2-cells. For $i = 1, \dots, k$ we pick a path p_i from x_0 to a point on $\varphi_i(S^1)$ and we write $\psi_i = p_i * \varphi_i * \bar{p}_i$. We then have

$$\begin{array}{ccc} \pi_1(X) & \cong & \pi_1(X^1)/\langle\langle [\psi_1] \rangle\rangle \dots / \langle\langle [\psi_k] \rangle\rangle = \pi_1(X^1)/\langle\langle [\psi_1], \dots, [\psi_k] \rangle\rangle. \\ \uparrow & & \uparrow \\ \text{Propositions 4.9 and 22.14} & & \text{Statement (3)} \end{array}$$

We have thus shown that $\pi_1(X)$ is a finitely presented group. $\square$

The following proposition is an immediate consequence of Proposition 22.16 and Theorem 22.2.

Proposition 22.17. *The fundamental group of any connected, compact manifold M is finitely presented.*

The following corollary is an immediate but useful consequence of Proposition 22.16 (1).

Corollary 22.18. *Let X be a connected CW-complex with countably many cells. If X has no 1-cells, then X is simply connected, i.e. $\pi_1(X) = 0$.*

Examples.

- (1) On page 340 we had pointed out that for $n \geq 2$ the sphere S^n admits a CW-structure with one 0-cell and one n -cell. From Corollary 22.18 we now recover the result from Proposition 4.12 that S^n is simply connected.
- (2) It follows from Lemma 22.1 and Corollary 22.18 that for any n we have $\pi_1(\mathbb{CP}^n) = 0$.

22.7. The Cellular Approximation Theorem.

Definition. A map $g: X \rightarrow Y$ between CW-complexes is called *cellular* if for each $n \in \mathbb{N}_0$ we have $g(X^n) \subset Y^n$.

Example. In general maps between CW-complexes are not cellular. For example consider the CW-complex $X = [0, 1]$ with two 0-cells and one 1-cell and we consider the CW-complex $Y = [0, 1] \times [0, 1]$ with four 0-cells, two 1-cells and one 2-cell. Then the diagonal map

$$\begin{aligned} g: X = [0, 1] &\rightarrow Y = [0, 1] \times [0, 1] \\ t &\mapsto (t, t) \end{aligned},$$

which is illustrated in Figure 217 on the left, is not a cellular map.

Before we can formulate the next proposition we need to introduce one more definition.

Definition. Let $f, g: X \rightarrow Y$ be two maps between topological spaces and let $A \subset X$ be a subset such that $f|_A = g|_A$.

- (1) A *homotopy between f and g rel A* is a homotopy $H: X \times [0, 1] \rightarrow Y$ between f and g such that $H(a, t) = f(a) = g(a)$ for all $t \in [0, 1]$ and all $a \in A$.
- (2) We say *f and g are homotopic rel A* , if such a homotopy rel A exists.

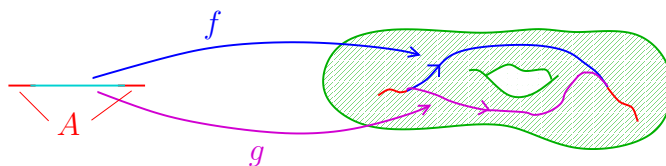


FIGURE 216. The maps f and g are homotopic, but they are not homotopic rel A .

Now we can formulate the main result of this section.

Theorem 22.19. (Cellular Approximation Theorem) *Let $g: X \rightarrow Y$ be a map between CW-complexes and let A be a subcomplex of X such that the restriction of g to A is a cellular map.⁴²⁴ Then g is homotopic rel A to a cellular map $\tilde{g}: X \rightarrow Y$.*

Example. We consider again the above example of a non-cellular map. More precisely, we consider again the above diagonal map

$$\begin{aligned} g: X = [0, 1] &\rightarrow Y = [0, 1] \times [0, 1] \\ t &\mapsto (t, t). \end{aligned}$$

We want to deform g into a cellular map, while keeping the end points fixed. Put differently, we want to find a homotopy rel $A = \{(0, 0), (1, 1)\}$ from g to a cellular map $\tilde{g}$. Such a map $\tilde{g}$ is given by

$$\begin{aligned} \tilde{g}: X = [0, 1] &\rightarrow Y = [0, 1] \times [0, 1] \\ t &\mapsto \begin{cases} (0, 2t), & \text{if } t \in [0, \frac{1}{2}] \\ (1, 2t - 1), & \text{if } t \in [\frac{1}{2}, 1] \end{cases} \end{aligned}$$

One way of stating this example is that we turn g into a cellular map $\tilde{g}$ by “pushing g off

⁴²⁴For example $f: X \rightarrow Y$ could be any map between CW-complexes, then the hypothesis is satisfied for A the empty set.

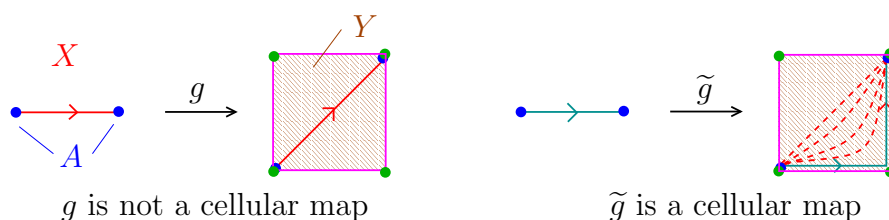


FIGURE 217.

the high-dimensional cell”. The proof of Cellular Approximation Theorem 22.19 is built on the same concept. But somewhat similar to the proof of Proposition 4.12, we have to worry about space-filling curves.

The following proposition is the key step towards proving the Cellular Approximation Theorem 22.19.

Proposition 22.20. *Let V and W be topological spaces, let $\psi: S^{n-1} \rightarrow V$ and $\varphi: S^{k-1} \rightarrow W$ be maps. Furthermore let*

$$h: V \cup_{\psi} \overline{B}^n \rightarrow W \cup_{\varphi} \overline{B}^k$$

be a map with $h(\psi(S^{n-1})) \subset W$. If $n < k$, then h is homotopic rel V to a map h' such that $h'(B^n)$ does not intersect B^k .

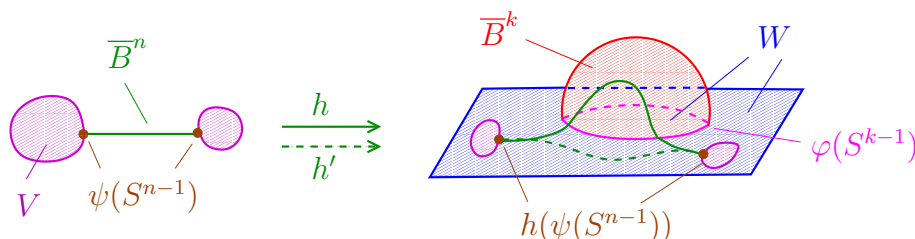


FIGURE 218. Illustration of Proposition 22.20.

The statement of Proposition 22.20 is fairly obvious, if there exists a point $P \in B^k$ that does not get hit by $h(B^n)$. Indeed, if such a point P exists, then we can use the deformation retraction $\overline{B}^n \setminus \{p\} \rightarrow S^{n-1}$ to push h on B^n off the ball B^k . Put in general such a point P does not exist, since even for $n < k$ there exist continuous surjective maps $B^n \rightarrow B^k$. To get around this problem requires a fairly hard, not particularly inspiring, argument which we will give later on in Section 22.8.

In the following we will use Proposition 22.20 to prove the Cellular Approximation Theorem 22.19.

Corollary 22.21. *Let Z be a finite n -dimensional CW-complex.*

- (1) Let $g: Z \rightarrow Y$ be a map to a k -dimensional CW-complex Y with $g(Z^{n-1}) \subset Y^{n-1}$. Let B be a subcomplex of Z with $g(B) \subset Y^n$. If $n < k$, then g is homotopic rel $Z^{n-1} \cup B$ to a map g' with $g'(Z) \subset Y^{n-1}$.
- (2) Let $g: Z \rightarrow Y$ be a map to a CW-complex Y such that $g(Z^{n-1}) \subset Y^{n-1}$. Let B be a subcomplex of Z with $g(B) \subset Y^n$. The map g is homotopic rel $Z^{n-1} \cup B$ to a map g' with $g'(Z) \subset Y^n$.

Now we can provide the proof of Proposition 22.20.

Proof. Let Z be a finite n -dimensional CW-complex. By Lemma 22.4 the CW-complex Z is compact. It follows from Proposition 22.6 that the image of the compact set Z in a CW-complex is contained in a finite subcomplex. Therefore we can assume in both (1) and (2) that Y is in fact a finite CW-complex.

- (1) Let $g: Z \rightarrow Y$ be a map to a finite k -dimensional CW-complex Y such that $g(Z^{n-1}) \subset Y^{n-1}$. Let B be a subcomplex of Z with $g(B) \subset Y^n$. We suppose that $n < k$. We write $Z = \widehat{Z} \cup e_1 \cup \cdots \cup e_r$ where $e_1, \dots, e_r$ are the n -cells such that the image is not contained in Y^{k-1} . We denote by $f_1, \dots, f_s$ the k -cells of Y . Note that by hypothesis on B the interiors of the cells $e_1, \dots, e_r$ are disjoint from B .

In the following we apply Proposition 22.20 precisely r times to get the desired homotopy. More precisely, we do the following:

- (a) We apply Proposition 22.20 to the map

$$h: (\widehat{Z} \cup e_1 \cup \cdots \cup e_{r-1}) \cup e_r \rightarrow (Y \cup f_1 \cup \cdots \cup f_{s-1}) \cup f_s$$

to obtain a homotopy rel $\widehat{Z} \supset Z^{n-1} \cup B$ to a map h_{rs} such that the image of the interior of e_r under the map h_{rs} does not intersect the interior of f_s . Next we apply Proposition 22.20 to the map

$$h_{rs}: (\widehat{Z} \cup e_1 \cup \cdots \cup e_{r-2} \cup e_r) \cup e_{r-1} \rightarrow (Y \cup f_1 \cup \cdots \cup f_{s-1}) \cup f_s.$$

Iterating this process we obtain a homotopy rel $\widehat{Z}$ to a map such that the image of the interiors of the cells $e_1, \dots, e_r$ does not intersect the interior of f_s .

- (b) Again iterating the process above we obtain a homotopy rel $\widehat{Z}$ to a map such that image of the interiors of the cells $e_1, \dots, e_r$ does not intersect the interior of any of the f_i . But that means that the image of Z lies in Y^{k-1} .
- (2) Let $g: Z \rightarrow Y$ be a map to a CW-complex Y such that $g(Z^{n-1}) \subset Y^{n-1}$ and let B be a subcomplex of Z with $g(B) \subset Y^n$. As we had pointed out in the beginning of the proof, we can assume that Y is finite-dimensional, we denote the dimension by k . But then statement (2) is just an immediate consequence of applying (1) iteratively $k - n$ times. $\square$

Now we turn to the proof of the Cellular Approximation Theorem 22.19.

Proof. Let X be a CW-complex and let A be a (possibly empty) subcomplex of X . Furthermore let Y be a CW-complex. We prove the Cellular Approximation Theorem 22.19

only in the special case that X has finitely many cells. The general case is dealt with in [Ha, Theorem 4.8].

Let $g: X \rightarrow Y$ be map.

- (1) We say an n -cell e of X is g -bad if $g(e)$ is not contained in Y^n .
- (2) If g is non-cellular, then we define

$$n(g) := \text{minimal dimension of a cell that is } g\text{-bad.}$$

If g is cellular, then we define $n(g) := \infty$.

Claim. Let $g: X \rightarrow Y$ be a non-cellular map such that the restriction of g to A is a cellular map. Then there exists a homotopy from g to a map g' rel A such that $n(g') > n(g)$.

We write $n = n(g)$. We apply Corollary 22.21 (2) to $g: Z := X^n \rightarrow Y$ and $B = A^n$.⁴²⁵ We obtain a homotopy G rel $X^{n-1} \cup A^n$ from $g: Z = X^n \rightarrow Y$ to a map $g': Z = X^n \rightarrow Y$ with $g'(X^n) \subset Y^n$. We can extend this homotopy to a homotopy G on $(X^n \cup A) \times [0, 1]$ via the trivial homotopy on $A \times [0, 1]$. Now we apply the Homotopy Extension Theorem 22.10 to obtain a homotopy rel $X^n \cup A$ from $g: X \rightarrow Y$ to a map $g': X \rightarrow Y$ such that $g'(X^n) \subset Y^n$. Note that by the choice of our homotopy we have $g'|_{X^{n-1}} = g|_{X^{n-1}}$. By definition of $n(g)$ the map $g'|_{X^{n-1}} = g|_{X^{n-1}}$ is cellular. Thus we see that $n(g') \geq n + 1 = n(g) + 1$. This concludes the proof of the claim.

Finally let $g: X \rightarrow Y$ be a non-cellular map such that the restriction of g to A is a cellular map.. We apply the claim iteratively as many times as possible to obtain a sequence of maps $g = g_1, g_2, g_3, \dots: X \rightarrow Y$, each of which is homotopic to g rel A , such that $n(g) = n(g_1) < n(g_2) < \dots$. Note that n can take on only finitely many values, namely $\{0, \dots, \dim(X), \infty\}$. This means that the sequence is finite which implies that the last map is cellular. $\square$

22.8. Proof of Proposition 22.20 (*).⁴²⁶ Before we can even think of proving Proposition 22.20 we need to introduce a few more definitions and we need to prove a rather technical lemma.

Definition.

- (1) A *closed half-space* in $\mathbb{R}^n$ is a set of the form $\{v \in \mathbb{R}^n \mid \langle v, w \rangle \geq 0\}$, for some fixed non-zero vector $w \in \mathbb{R}^n$.
- (2) A *convex polyhedron* in $\mathbb{R}^n$ is the intersection of finitely many closed half-spaces.
- (3) A *polyhedron* in $\mathbb{R}^n$ is the union of a finite number of convex polyhedra.⁴²⁷

⁴²⁵Note that by definition of n we have $g(X^{n-1}) \subset Y^{n-1}$.

⁴²⁶Here the (*) indicates once again that we do not provide the proof in the lecture. You are reading this section at your own risk.

⁴²⁷Note that the union of zero convex polyhedra is the empty set.

- (4) Given a polyhedron P a map $f: P \rightarrow \mathbb{R}^k$ is called *piecewise linear* if there exists a subdivision of P into finitely many convex polyhedra $Q_1, \dots, Q_m$ such that the restriction of f to each Q_i is affine linear.⁴²⁸

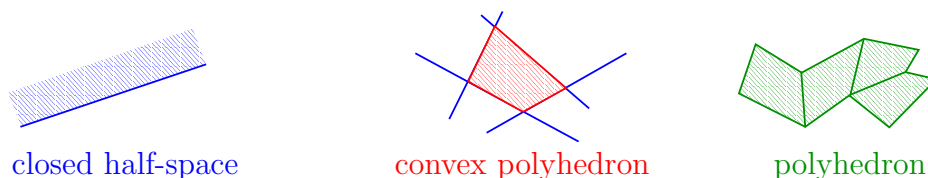


FIGURE 219.

The following, very technical lemma, will get us around the problem of space filling curves, i.e. it will get us around the problem that continuous maps $\overline{B}^n \rightarrow \overline{B}^k$ can be surjective even if $n < k$.

Lemma 22.22. *Let W be a topological space and let $\varphi: S^{k-1} \rightarrow W$ be map. Furthermore let $g: [0, 1]^n \rightarrow Z := W \cup_{\varphi} \overline{B}^k$ be a map. Then there exists a (possibly empty) polyhedron $K \subset [0, 1]^n$ and a map $\tilde{g}: [0, 1]^n \rightarrow Z$ with the following properties:*

- (1) *The map $\tilde{g}$ is homotopic to g rel $g^{-1}(W)$.*
- (2) *We have $\tilde{g}(K) \subset B^k$.*
- (3) *The restriction $\tilde{g}|_K: K \rightarrow B^k$ is piecewise linear.*
- (4) *There exists a non-empty open subset $U \subset B^k$ such that $\tilde{g}^{-1}(U) \subset K$.*

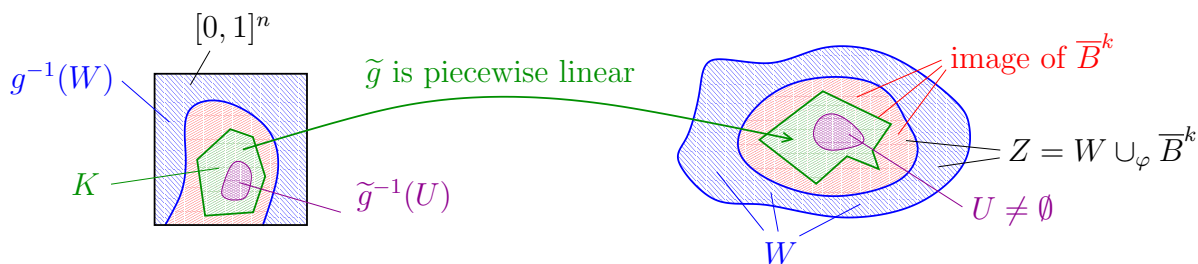


FIGURE 220. Illustration of Lemma 22.22.

Example. Suppose that in the setting of Lemma 22.22 there exists an open subset $U \subset B^k$ such that $g([0, 1]^n) \cap U = \emptyset$. Then the empty polyhedron $K = \emptyset$ and the map $\tilde{g} = g$ already have the desired property.

⁴²⁸A map $f: V \rightarrow \mathbb{R}^k$ from an open subset V of $\mathbb{R}^n$ is called *affine linear* if there exists a $P \in \mathbb{R}^k$ and a $k \times n$ -matrix A such that $f(v) = P + Av$ for all $v \in V$.

Proof. Let W be a topological space and let $\varphi: S^{k-1} \rightarrow W$ be map. Furthermore let $g: [0, 1]^n \rightarrow Z := W \cup_{\varphi} \overline{B}^k$ be a map. We write $D_1 := \overline{B}_{\frac{1}{4}}(0) \subset B^k$ and $D_2 := \overline{B}_{\frac{1}{2}}(0) \subset B^k$. The set $g^{-1}(D_2) \subset [0, 1]^n$ is closed and therefore compact. It follows from Analysis II Satz 3.12 that g is uniformly continuous on $g^{-1}(D_2)$.⁴²⁹ This implies that we can find an $\epsilon > 0$ with the following two properties:

- (i) For any $x, y \in g^{-1}(D_2)$ with $\|x - y\| < \epsilon$ we have $\|g(x) - g(y)\| < \frac{1}{8}$.
- (ii) We have⁴³⁰

$$\epsilon < \text{distance between } g^{-1}(D_1) \text{ and } [0, 1]^n \setminus g^{-1}(\overset{\circ}{D}_2).$$

We divide $[0, 1]^n$ into cubes such that the diameter of each cube is less than ϵ . We denote by K_1 the union of all the cubes that meet $g^{-1}(D_1)$ and we denote by K_2 the union of all the cubes meeting K_1 . By the second property of ϵ we have $K_2 \subset g^{-1}(D_2)$. We refer to Figure 221 for an illustration.

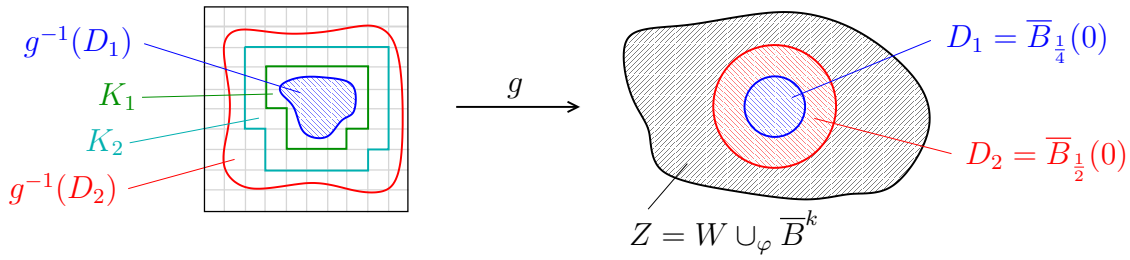


FIGURE 221.

We need one more definition: Suppose we are given points $P_1, \dots, P_{n+1} \in \mathbb{R}^n$ such that the vectors $P_2 - P_1, \dots, P_{n+1} - P_1$ are linearly independent. We refer to

$$\text{convex hull of } P_1, \dots, P_{n+1} = \left\{ \sum_{i=1}^{n+1} t_i P_i \mid \sum_{i=1}^{n+1} t_i = 1 \right\}$$

as a *simplex* in $\mathbb{R}^n$.⁴³¹ We will make use of the following trivial observation: if we are given $w_1, \dots, w_{n+1} \in \mathbb{R}^k$, then there exists a unique affine linear map Φ with $\Phi(P_i) = w_i$, namely

⁴²⁹Why can't we conclude that g is uniformly continuous on all of $[0, 1]^n$?

⁴³⁰Here recall that for two non-empty subsets A and B of $\mathbb{R}^n$ we refer to

$$d(A, B) := \{ \|P - Q\| \mid P \in A \text{ and } Q \in B \}$$

as the distance between A and B . If A and B are compact and disjoint, then $d(A, B) > 0$. (Why is that?)

⁴³¹Given a subset S of $\mathbb{R}^k$ we defined on page 18 the *convex hull* of S as the intersection of all convex subsets of $\mathbb{R}^k$ that contain S . As we had pointed out, if S is a finite subset of $\mathbb{R}^k$, then the *convex hull* of S is given by

$$\left\{ \sum_{s \in S} a_s \cdot s \mid a_s \in [0, 1] \text{ and } \sum_{s \in S} a_s = 1 \right\}.$$

the map that is given by

$$\Phi\left(\sum_{i=1}^{n+1} t_i P_i\right) := \sum_{i=1}^{n+1} t_i w_i.$$

We leave it as an exercise to show that each cube can be subdivided⁴³² into simplices. We also refer to Figure 222 for an illustration.

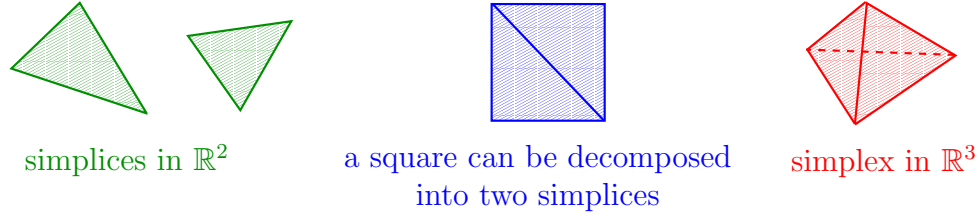


FIGURE 222.

Now we subdivide all the cubes in K_2 into simplices. By the above discussion there exists a map $h: K_2 \rightarrow \mathbb{R}^k$ that equals g on all the vertices of the simplices in the subdivision of the cubes.⁴³³ Furthermore we choose a map $\varphi: K_2 \rightarrow [0, 1]$ with $\varphi|_{\partial K_2} \equiv 0$ and $\varphi|_{K_1} \equiv 1$.⁴³⁴ Now we consider the homotopy⁴³⁵

$$\begin{aligned} F: [0, 1]^n \times [0, 1] &\rightarrow Z = W \cup_{\varphi} \overline{B}^k \\ (P, t) &\mapsto \begin{cases} (1 - t\varphi(P)) \cdot g(P) + t\varphi(P) \cdot h(P), & \text{if } P \in K_2 \\ g(P), & \text{otherwise.} \end{cases} \end{aligned}$$

This is a homotopy from g to a map $\tilde{g}: [0, 1] \rightarrow Z = W \cup_{\varphi} \overline{B}^k$ that restricts to the piecewise linear map h on K_1 .

Claim. The point $0 \in D_1$ does not lie in the image of $\tilde{g}(\overline{[0, 1]^n \setminus K_1})$.

We now provide the proof of the claim.

- (1) On $[0, 1]^n \setminus K_2$ we have $\tilde{g} = g$. By Property (ii) of ϵ we have $f^{-1}(D_1) \subset K_2$. It follows that the image of $[0, 1]^n \setminus K_2$ lies outside of D_1 .
- (2) Now let σ be a simplex of K_2 that does not lie in K_1 . Note that $K_2 \subset g^{-1}(D_2)$. It follows from Property (i) of ϵ that $g(\sigma)$ is contained in some ball B_{σ} of radius $\frac{1}{8}$. Since $g(\sigma) \subset B_{\sigma}$ and since B_{σ} is convex we also have $h(\sigma) \subset B_{\sigma}$. Again by the convexity of B_{σ} we have $\tilde{g}(\sigma) \subset B_{\sigma}$. We know that B_{σ} is not contained in D_1 since

⁴³²Here, by “subdivide” we mean that given any cube K there exist simplices $S_1, \dots, S_k$ such that the interiors of the S_i are disjoint and such that $K = S_1 \cup \dots \cup S_k$.

⁴³³Note that it follows from the uniqueness statement above that the map g is in fact continuous.

⁴³⁴Such a map can be constructed as follows: we assign the value 1 to all vertices in K_1 and we assign the value 0 to all other vertices in K_2 . By the above “trivial observation” this defines a map on K_2 with the desired property.

⁴³⁵Since $\varphi|_{\partial K_2} \equiv 0$ we see that F is indeed continuous.

σ contains points outside K_1 , hence outside $g^{-1}(D_1)$. The radius of B_σ is half that of D_1 , so it follows that 0 is not in B_σ , and hence 0 does not lie in $\tilde{g}(\sigma)$.

This concludes the proof of the claim.

Since $\tilde{g}(\overline{[0, 1]^n \setminus K_1})$ is compact it follows from the claim that there exists in fact an open neighborhood U of 0 that does not intersect the image of $[0, 1]^n \setminus K_1$. Put differently, we have $\tilde{g}^{-1}(U) \subset K_1$. It is now straightforward to verify that $K = K_1$, $\tilde{g}$ and U have the desired properties. $\square$

Lemma 22.23. *Let W be a topological space and let $\varphi: S^{k-1} \rightarrow W$ be the attaching map of a k -cell. Furthermore let $h: \overline{B}^n \rightarrow Z := W \cup_\varphi \overline{B}^k$ be a map with $h(S^{n-1}) \subset W$. If $n < k$, then h is homotopic rel S^{n-1} to a map h' that misses B^k .*

Proof. Let W be a topological space and let $\varphi: S^{k-1} \rightarrow W$ be the attaching map of a k -cell. Furthermore let $h: \overline{B}^n \rightarrow Z := W \cup_\varphi \overline{B}^k$ be a map with $h(S^{n-1}) \subset W$. We suppose that $n < k$. We have to show that h is homotopic rel S^{n-1} to a map h' that misses B^k .

By Lemma 1.4 there exists a homeomorphism $\Phi: [0, 1]^n \rightarrow \overline{B}^n$. So after precomposing h with Φ we might as well assume that the domain of h is $[0, 1]^n$, i.e. that h is a map $h: [0, 1]^n \rightarrow Z := W \cup_\varphi \overline{B}^k$ with $h(\partial[0, 1]^n) \subset W$.

We pick a polyhedron K , a map $\tilde{h}: [0, 1]^n \rightarrow Z$ and a non-empty open subset $U \subset B^n$ with $\tilde{h}^{-1}(U) \subset K$ as in the statement of Lemma 22.22. Since $n < k$ and since $\tilde{h}|_K: K \rightarrow B^k$ is piecewise linear we deduce that $\tilde{h}(K)$ cannot contain the open subset U of B^n .⁴³⁶ We pick a point $P \in U \setminus \tilde{h}(K)$. By statement (4) of Lemma 22.22 we know that P does not lie in the image of $\tilde{h}: [0, 1]^n \rightarrow Z$.

As we had pointed out on page 127, there exists a deformation retraction from $\overline{B}^k \setminus \{P\}$ to S^{k-1} . We can extend this deformation retraction via the identity map to a deformation retraction from $Z = W \cup_\varphi (\overline{B}^k \setminus \{P\})$ to W . If we compose $\tilde{h}$ with this deformation retraction we obtain a homotopy, rel S^{n-1} , from $\tilde{h}$ to a map h' that misses B^k . $\square$

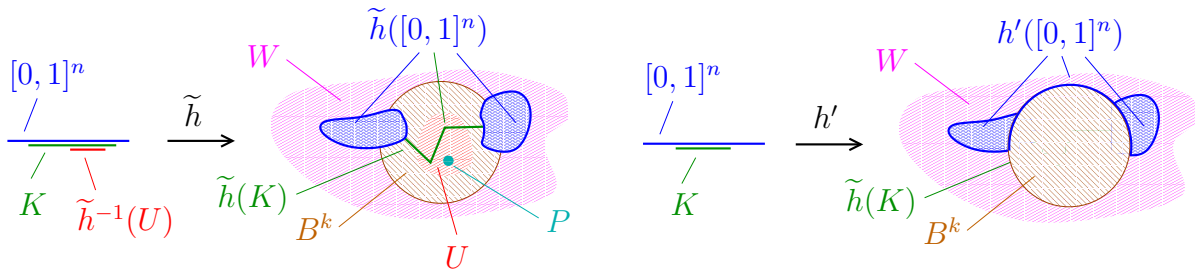


FIGURE 223. Illustration for the proof of Proposition 22.20.

⁴³⁶Indeed, since $\tilde{h}|_K: K \rightarrow B^k$ is piecewise linear and since $n < k$ we know that the image $\tilde{h}(K)$ is set of Lebesgue-measure zero in B^n whereas a non-empty open subset has positive Lebesgue-measure.

Now we can finally provide the proof of Proposition 22.20.

Proof of Proposition 22.20. Let V and W be topological spaces, let $\psi: S^{n-1} \rightarrow V$ and $\varphi: S^{k-1} \rightarrow W$ be maps. Furthermore let

$$g: V \cup_{\psi} \overline{B}^n \rightarrow W \cup_{\varphi} \overline{B}^k$$

be a map with $g(\psi(S^{n-1})) \subset W$. We suppose that $n < k$.

We denote by θ the obvious map $\overline{B}^n \rightarrow V \cup_{\psi} \overline{B}^n$, furthermore we write $h = g \circ \theta$. By our hypothesis we have $h(S^{n-1}) \subset W$. By Lemma 22.23 there exists a homotopy G rel S^{n-1} from h to a map $h': \overline{B}^n \rightarrow W$ that misses B^k . The desired homotopy is now given by⁴³⁷

$$\begin{aligned} F: (V \cup_{\psi} \overline{B}^n) \times [0, 1] &\rightarrow W \cup_{\varphi} \overline{B}^k \\ (P, t) &\mapsto \begin{cases} G(Q, t), & \text{if } P = \theta(Q), \text{ for some } Q \in \overline{B}^k, \\ P, & \text{otherwise.} \end{cases} \end{aligned} \quad \square$$

22.9. Coverings of CW-complexes.

Proposition 22.24. *Let $p: \tilde{X} \rightarrow X$ be a covering of a connected CW-complex X . We can equip $\tilde{X}$ with the structure of a CW-complex such that the following statements hold:*

- (1) *The map $p: \tilde{X} \rightarrow X$ is a cellular map.*
- (2) *For any $k \in \mathbb{N}_0$ we have⁴³⁸*

$$\#k\text{-cells of } \tilde{X} = [\tilde{X} : X] \cdot \#k\text{-cells of } X.$$

- (3) *The dimension of $\tilde{X}$ is the same as the dimension of X .*
- (4) *If X is a finite CW-complex and if p is a finite covering, then*

$$\chi(\tilde{X}) = [\tilde{X} : X] \cdot \chi(X).$$

Proof. Let $p: \tilde{X} \rightarrow X$ be a covering of a connected CW-complex X . We write $n := [\tilde{X} : X]$. Let $\psi: \overline{B}^k \rightarrow X$ be a characteristic map of a k -cell of X . We write $x = \psi(0)$. We consider the following diagram

$$\begin{array}{ccc} & & \tilde{X} \\ & \nearrow \tilde{\psi} & \downarrow p \\ \overline{B}^k & \xrightarrow{\psi} & X. \end{array}$$

Since $\overline{B}^k$ is simply-connected we can apply Proposition 15.2 which says in this context that for each $\tilde{x} \in p^{-1}(x)$ there exists a unique lift $\tilde{B}^k \rightarrow \tilde{X}$ with $\tilde{\psi}(0) = \tilde{x}$. By the definition on page 93, the degree $n = [\tilde{X} : X]$ of the covering equals the cardinality of $p^{-1}(x)$. Thus we see that ψ gives rise to n maps $\overline{B}^k \rightarrow \tilde{X}$. It is straightforward to verify that all these lifts

⁴³⁷Note that the map F is indeed continuous.

⁴³⁸Note that this is an equality of cardinalities. For example, if $[\tilde{X} : X]$ is infinite countable and if X has finitely many k -cells, then $\tilde{X}$ has infinitely many, but countably many, k -cells.

of the characteristic maps for the cells of X define a CW-structure for $\tilde{X}$. We leave the verification of the details as an exercise to the reader.

It follows immediately from the above construction and the definitions that the CW-structure we had just defined on $\tilde{X}$ has the desired properties (1) to (4). $\square$

Examples.

- (1) We equip S^1 with the CW-structure consisting of one 0-cell and one 1-cell. We denote by $p: \mathbb{R} \rightarrow S^1$, $t \mapsto e^{2\pi it}$ the universal covering of S^1 . If we apply the construction of the CW-structure on $\mathbb{R}$ given in the proof of Proposition 22.24, then we end up equipping $\mathbb{R}$ with the CW-structure where the 0-cells correspond to $\mathbb{Z}$ and the 1-cells are precisely the intervals $[k, k+1]$, with $k \in \mathbb{Z}$. We illustrate the CW-structure of $\mathbb{R}$ in Figure 224.

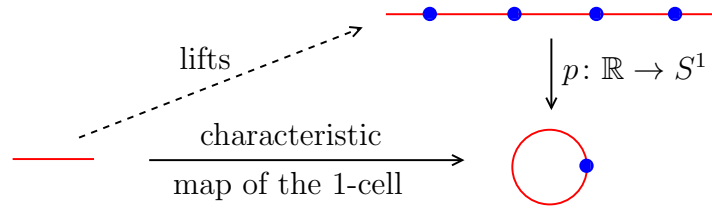


FIGURE 224.

- (2) Almost the same approach as in (1), applied to the universal cover $\mathbb{R}^2 \rightarrow S^1 \times S^1$, endows $\mathbb{R}^2$ with a CW-structure that is illustrated in Figure 225.

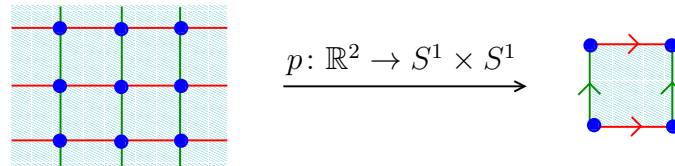


FIGURE 225.

- (3) In the proof of Hadamard's Theorem 19.9 we constructed an explicit covering map $\mathbb{D} \rightarrow H_8/\sim \cong E_8/\sim$. If we equip $E_8/\sim$ with the "usual" CW-structure then we obtain from Proposition 22.24 the CW-structure on $\mathbb{D}$ that is shown in Figure 226.

We conclude this discussion of CW-complexes with an application group theory. Given a free group F , i.e. given a group F that is isomorphic to the free group $\langle S \rangle$ generated by some set S we refer to the cardinality of S as the *rank* of F .⁴³⁹

⁴³⁹It follows from the discussions on the pages 180 and 139 that the rank of F is well-defined.

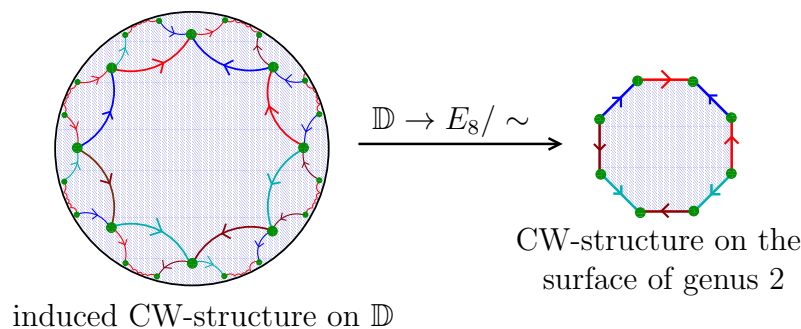


FIGURE 226.

Now we can formulate the following corollary to Proposition 22.24. The statement is purely group theoretic, but it is surprisingly hard to prove the statement without appealing to topological methods.⁴⁴⁰

Corollary 22.25. *Let F be a finitely generated free group.*

- (1) *Every subgroup of F is again a free group.*
- (2) *If G is a finite-index subgroup of F , then⁴⁴¹*

$$\text{rank}(G) - 1 = [F : G] \cdot (\text{rank}(F) - 1).$$

Examples.

- (1) If $F = \mathbb{Z}$ then the formula from Corollary 22.25 does indeed show that every finite-index subgroup of $\mathbb{Z}$ is again a free group of rank one, i.e. it every finite-index subgroup isomorphic to $\mathbb{Z}$.
- (2) Let $\alpha: F := \langle a, b \rangle \rightarrow \mathbb{Z}_3$ be the epimorphism defined given by $\alpha(a) = 1$ and $\alpha(b) = 0$. It follows from Corollary 22.25 that the kernel $G := \ker(\alpha)$ is a free group of rank $[F : G] \cdot (\text{rank}(F) - 1) + 1 = 3 \cdot 1 + 1 = 4$. In exercise sheet 2 we will determine four elements of $G = \ker(\alpha)$ that are generators for this free group.

Proof. Let F be a free group of rank m and let G be a subgroup of F . Let X be a 1-dimensional CW-complex with one 0-cell P and m 1-cells. By Proposition 22.13 we have $\pi_1(X, P) \cong F$.

By Lemma 22.4 (4) and Proposition 15.5 there exists a covering $p_*: (Y, Q) \rightarrow (X, P)$ of degree $[F : G]$ such that $p_*(\pi_1(Y, Q)) = G$. By Corollary 6.14 the map p_* is injective. Thus we see that $\pi_1(Y, Q)$ itself is isomorphic to G .

By Proposition 22.24 we can equip Y with the structure of a 1-dimensional CW-structure. We deduce from Proposition 22.13 that $G \cong \pi_1(Y, Q)$ is a free group.

⁴⁴⁰Try to prove it using only group theoretic methods!

⁴⁴¹Does the equality hold if G is infinite-index and if we interpret $[F : G] \cdot (\text{rank}(F) - 1)$, as usual, as $\infty \cdot (\text{rank}(F) - 1) = \infty$ for $F \not\cong \mathbb{Z}$?

Finally, suppose that G is in fact a finite-index subgroup of F . It follows from Proposition 22.24 that $\chi(Y) = [F : G] \cdot \chi(X)$. It is now a consequence of Proposition 22.13 that $\text{rank}(G) - 1 = [F : G] \cdot (\text{rank}(F) - 1)$. $\square$

Remark.

- (1) It is not hard to modify the proof of Corollary 22.25 to show that the subgroup of any free group (not necessarily finitely generated) is again a free group.
- (2) In principle one could have tried to prove Corollary 22.25 using topological graphs in the sense of page 38 instead of using CW-complexes. But then one would have had to show that coverings of topological graphs are again topological graphs. This would have been exceedingly painful⁴⁴², whereas, as we have just seen in Proposition 22.24, it is almost a triviality to show that coverings of 1-dimensional CW-complexes are again 1-dimensional CW-complexes. This shows that sometimes it pays to use more abstract language to prove statements even about “basic objects”.
- (3) Corollary 22.25 is a special case of the *Kurosh Subgroup Theorem*. This theorem says that if G is the product of groups $A_i, i \in I$ and if H is a subgroup of G , then H is a free product of a free group and groups $B_i, i \in I$ where each B_i is a (potentially trivial) group that is conjugate to a subgroup of A_i . We refer to [LS, Theorem IV.1.10] for details and a proof.

Example. Let G be a graph with one vertex and two edges v which we denote by s and t . By a slight abuse of notation we denote the corresponding elements in $\pi_1(G, v)$ by s and t as well. By Lemma 9.4 we can identify $\pi_1(G, v)$ with the free group $\langle s, t \rangle$ on the generators s and v . Now let $\phi: \pi_1(G, v) = \langle s, t \rangle \rightarrow \mathbb{Z}$ be the homomorphism that is given by $\phi(s) = 0$ and $\phi(t) = 1$. By Proposition 15.5 there exists a covering $q: (\tilde{G}, \tilde{v}) \rightarrow (G, v)$ with $q_*(\pi_1(\tilde{G}, \tilde{v})) = \ker(\phi)$, furthermore by Proposition 15.6 this covering is unique up to equivalence, in the sense of the definition on page 262.

We consider the infinite covering illustrated in Figure 60, which is very similar to the finite covering studied on pages 324.⁴⁴³ For $i \in \mathbb{Z}$ we denote by x_i the loop in $(\tilde{G}, \tilde{v})$ as illustrated in Figure 228. We denote by x_i also the corresponding element in $\pi_1(\tilde{G}, \tilde{v})$. It follows from Proposition 22.26 that $\pi_1(\tilde{G}, \tilde{v})$ equals the free group $\langle \dots x_{-2}, x_{-1}, x_0, x_1, x_2 \dots \rangle$. It is straightforward to see that

$$p_*(x_i) = t^i s t^{-i}.$$

⁴⁴²You can try to do this by hand.

⁴⁴³This covering can also be described rigorously. More precisely, we view G as the union of two copies A and B of S^1 glued along the common point 1. We define

$$\tilde{G} = (\mathbb{R} \sqcup (S^1 \times \mathbb{Z})) / (n \sim (1, n)) \text{ for } n \in \mathbb{Z}.$$

Then the projection map

$$p: (\mathbb{R} \sqcup (S^1 \times \mathbb{Z})) / \sim \rightarrow G = (S^1 \times \{A\} \sqcup S^1 \times \{B\}) / (1, A) \sim (1, B)$$

is given by $p(x) = (e^{2\pi i x}, A)$ for $x \in \mathbb{R}$ and $p((z, n)) = (z, B)$ for $(z, n) \in S^1 \times \mathbb{Z}$.

By Corollary 6.14 the homomorphism p_* is a monomorphism. Thus we see that $t^i s t^{-i}$, $i \in \mathbb{Z}$, generate an infinitely generated free subgroup of the finitely generated free group $\langle s, t \rangle$. Furthermore we see that $p_*(\pi_1(\tilde{G}, \tilde{v})) \subset \ker(\phi: \pi_1(G, v) \rightarrow \mathbb{Z})$. It is easy to see that the deck transformation group is isomorphic to $\mathbb{Z}$.

Now we claim that the coverings $p: \tilde{G} \rightarrow G$ and $q: \hat{G} \rightarrow G$ are equivalent. By Proposition 15.6 it suffices to show that $p_*(\pi_1(\tilde{G}, \tilde{v})) = q_*(\pi_1(\hat{G}, \hat{v}))$. We consider the following sequence of maps

$$\begin{array}{ccccccc} \mathbb{Z} & \cong & D(p: \tilde{G} \rightarrow G) & \cong & \pi_1(G, v)/p_*(\pi_1(\tilde{G}, \tilde{v})) & \twoheadrightarrow & \pi_1(G, v)/\ker(\phi: \pi_1(G, v) \rightarrow \mathbb{Z}) \xrightarrow[\cong]{\phi} \mathbb{Z} \\ \uparrow & & \uparrow & & \uparrow & & \\ \text{see above} & & \text{isomorphism of Proposition 20.5} & & \text{since } p_*(\pi_1(\tilde{G}, \tilde{v})) \subset \ker(\phi: \pi_1(G, v) \rightarrow \mathbb{Z}) & & \end{array}$$

Thus, reading from left to right, we get an epimorphism from $\mathbb{Z}$ onto itself. But any such epimorphism has to be an isomorphism. It follows that the third map is also an isomorphism, i.e. $p_*(\pi_1(\tilde{G}, \tilde{v})) = \pi_1(G, v)/\ker(\phi: \pi_1(G, v) \rightarrow \mathbb{Z})$, but the latter group is precisely $q_*(\pi_1(\hat{G}, \hat{v}))$. So we get the desired equality of subgroups.

deck transformation is given by translation by $n \in \mathbb{Z}$

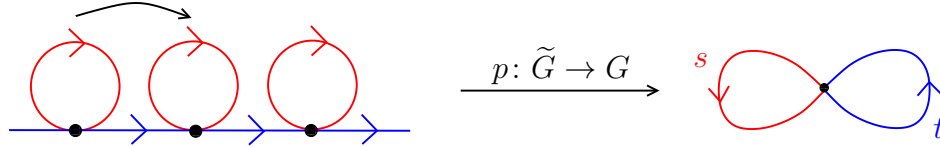


FIGURE 227.

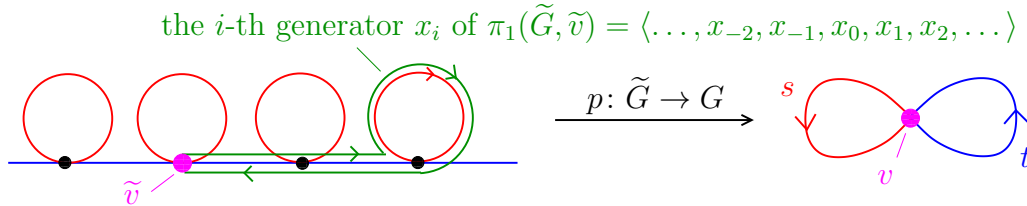


FIGURE 228.

22.10. Spanning trees of graphs (*). In Proposition 22.13 we had seen that the fundamental group of every connected 1-dimensional CW-complex is a free group. In this short section we will specify more precisely what are generators of this free group. To do so we need the following definition.

Definition. Let G be a connected graph. A *spanning tree* is a subcomplex T that is a tree and that contains all vertices of G .

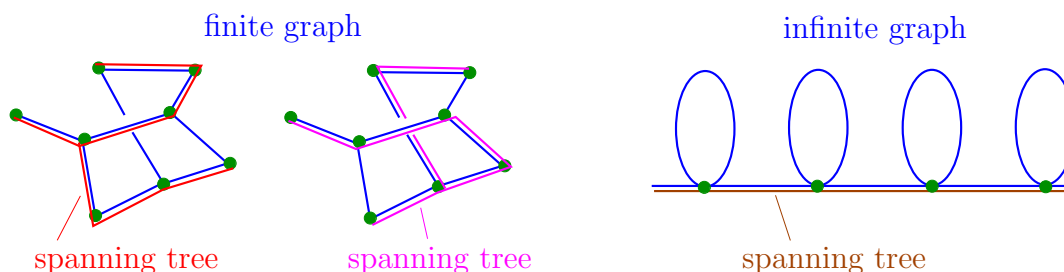


FIGURE 229.

In Figure 229 we give examples of spanning trees in graphs. We see in particular that in general there is no unique spanning tree. Now we can formulate the following generalization of Lemma 9.4.

Proposition 22.26. *Let G be a connected graph. Then the following holds:*

- (1) *The graph G admits a spanning tree T .*
- (2) *The group $\pi_1(G)$ is isomorphic to a free group where the cardinality of the generating set equals the cardinality of the set of edges in $G \setminus T$.*
- (3) *An explicit description of the isomorphism in (2) is given as follows. Pick a base point v in the spanning tree and for each cell e in $G \setminus T$ pick a path a_e from p to one of the endpoints and pick a path b_e from the other endpoint back to p ⁴⁴⁴ and let t_e be the loop given by first going along a_e , then going through e and then going back to the base point via b_e . Then the map*

$$\begin{aligned} \langle \{x_e\}_{e \text{ edge in } G \setminus T} \rangle &\rightarrow \pi_1(G, v) \\ x_e &\mapsto [t_e] \end{aligned}$$

is an isomorphism.

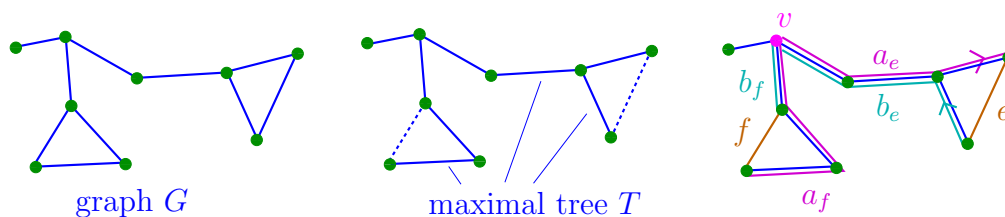


FIGURE 230.

Sketch of proof. We only provide an outline of the proof. The full details can be found on page 84 of [Ha]. Let G be a connected graph.

⁴⁴⁴The two paths exist since a tree is in particular connected.

- (1) We pick a vertex P on G . We define $V_0 = \{P\}$ and iteratively we define for $i \geq 1$ the set V_i to be the set of all vertices that are not contained in V_{i-1} but that are connected to a vertex in V_{i-1} by an edge. Now we define $T_0 = \{P\}$. Once T_{i-1} is defined we pick for each vertex in V_i a single edge that connects the vertex to a vertex in V_{i-1} . Then it is shown in [Ha, Proposition 1.A.1] that $\bigcup_{i \geq 0} T_i$ is a spanning tree.⁴⁴⁵

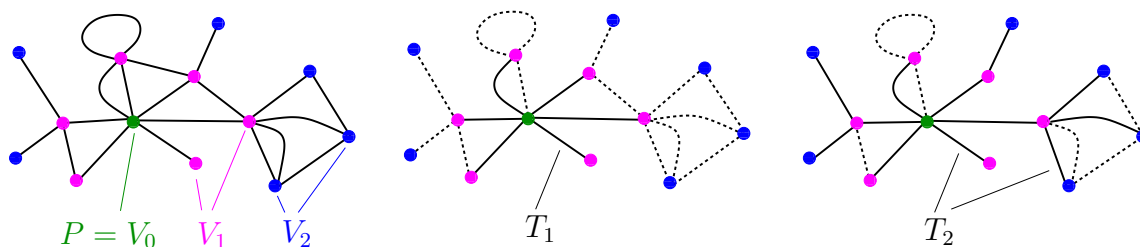


FIGURE 231.

- (2) If G is a finite graph, then the argument of the proof of Proposition 7.8 shows that the projection map $G \rightarrow G/T$ is a homotopy equivalence. The same statement also holds if G is an infinite graph, in this case the statement is proved in [Ha, Proposition 1.A.2]. The topological space G/T is again a graph with one vertex, given by the point $[T] \in G/T$ and an edge for each edge in $G \setminus T$. In particular G/T is homeomorphic to the wedge of circles, one for each edge in $G \setminus T$.⁴⁴⁶ It then follows from the straightforward generalization of Proposition 9.3 to the case of the wedge of arbitrarily many topological spaces that the fundamental group of such a wedge of circles is the free group, where the generators correspond precisely to the circles, i.e. the edges in $G \setminus T$. $\square$

⁴⁴⁵Since G is connected, every vertex lies in some V_i , hence each vertex lies in some T_i . It is also straightforward to see that each T_i is the deformation retract of T_{i+1} . Now one just needs to work a little harder to show that $T_0 = \{P\}$ is a deformation retract of T , i.e. T is contractible.

⁴⁴⁶On page 130 we had only defined the wedge of finitely many pointed topological spaces, it is clear how to generalize this definition to the case of infinitely many pointed topological spaces.

23. HIGHER HOMOTOPY GROUPS

After this somewhat technical chapter on CW-complexes we now return to our original goal, namely trying to distinguish topological spaces. Using the fundamental group we had established several non-trivial results, for example we had proved that

- (1) S^1 and S^k with $k \geq 2$ are not homeomorphic,
- (2) $\mathbb{R}^2$ and $\mathbb{R}^n$ with $n \geq 3$ are not homeomorphic,
- (3) surfaces of different genus are not homeomorphic,
- (4) we had shown that S^1 is not a deformation retract of $\overline{B}^2$,
- (5) the trefoil knot is not isotopic to the trivial knot.

On the other hand there are still many even rather elementary questions we cannot answer. For example at the moment we cannot show that for $k \neq l$ with $k, l \geq 2$ the spheres S^k and S^l are not homeomorphic and we cannot show that for $k \neq l$ with $k, l \geq 3$ the topological spaces $\mathbb{R}^k$ and $\mathbb{R}^l$ are not homeomorphic. Also, the somewhat mysterious lens spaces $L(p, q)$ still evaded our attempts at distinguishing them for a given p .

The problem with fundamental groups is that we only probe a topological space X with loops. The natural idea is now to go from using maps $S^1 \rightarrow X$ to studying maps $S^n \rightarrow X$ for $n \geq 2$. As we will see, this idea leads us naturally to the definition of the “higher homotopy groups” of a topological space.

23.1. Definition of the higher homotopy groups. Before we can introduce higher homotopy groups we need to introduce a few more definitions.

Definition. Let X and Y be topological spaces and let $A \subset X$ and $B \subset Y$ be subsets.

- (1) A map $f: (X, A) \rightarrow (Y, B)$ is defined as a map⁴⁴⁷ $f: X \rightarrow Y$ such that $f(A) \subset B$.

Now let $f, g: (X, A) \rightarrow (Y, B)$ be two maps.

- (2) A *homotopy between f and g* is a map⁴⁴⁸

$$H: X \times [0, 1] \rightarrow Y,$$

such that the following two conditions are satisfied:

- (a) $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for all $x \in X$, and
 - (b) $H(a, t) \in B$ for all $a \in A$ and all $t \in [0, 1]$.
- (3) If there exists a homotopy between f and g , then we say that f and g are *homotopic* and we write $f \simeq g$.

Notation. Throughout this chapter we write $I = [0, 1]$ for the unit interval. Given $n \in \mathbb{N}$ we denote by $I^n \subset \mathbb{R}^n$ the cube in $\mathbb{R}^n$ and we denote, as usual, by ∂I^n its boundary as a subset of $\mathbb{R}^n$.

⁴⁴⁷As usual all maps between topological spaces are understood to be continuous.

⁴⁴⁸For $A = B = \emptyset$ we get the usual notion of a homotopy between maps $f, g: X \rightarrow Y$.

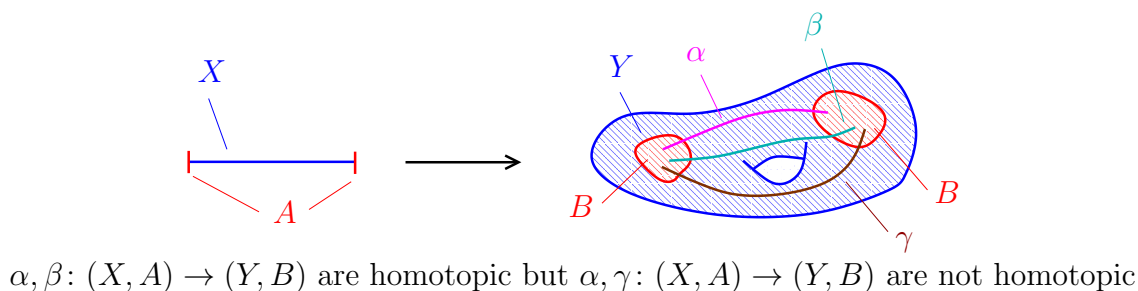


FIGURE 232.

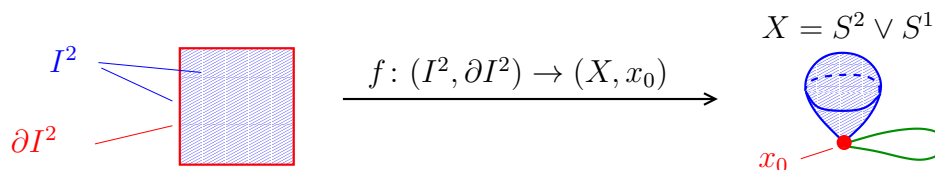
Now let X be a topological space and let $x_0 \in X$. With the above definitions we can write

$$\pi_1(X, x_0) = \text{homotopy classes of maps } (I, \partial I) \rightarrow (X, x_0).$$

This definition can be easily generalized to higher dimensions. More precisely, for any $n \in \mathbb{N}_0$ we define⁴⁴⁹

$$\pi_n(X, x_0) = \text{homotopy classes of maps } (I^n, \partial I^n) \rightarrow (X, x_0).$$

For $n \geq 1$ we refer to $\pi_n(X, x_0)$ as the n -th homotopy group.

FIGURE 233. Illustration of an element in $\pi_2(S^2 \vee S^1, x_0)$.

Note that $\pi_n(X, x_0)$ is a priori only a set. So to justify the name “homotopy group” we need to define a group structure on $\pi_n(X, x_0)$ which for $n = 1$ agrees with our previous definition of the group structure. Let $n \geq 1$ and let $f, g: (I^n, \partial I^n) \rightarrow (X, x_0)$ be two maps. We define

$$f * g: (I^n, \partial I^n) \rightarrow (X, x_0)$$

$$(x_1, \dots, x_n) \mapsto \begin{cases} f(2x_1, \dots, 2x_n), & \text{if } (x_1, \dots, x_n) \in [0, \frac{1}{2}]^n, \\ g(2x_1 - 1, \dots, 2x_n - 1), & \text{if } (x_1, \dots, x_n) \in [\frac{1}{2}, 1]^n, \\ x_0, & \text{otherwise.} \end{cases}$$

This map is illustrated in Figure 234.⁴⁵⁰ This is again a continuous map, and it is straight-

⁴⁴⁹This definition makes in particular sense for $n = 0$, we can thus consider $\pi_0(X, x_0)$. In Section 1.4 we had already defined $\pi_0(X)$ as the set of components of X . Does there exist a connection between these two objects? Or is the similarity in the notation just an unfortunate coincidence?

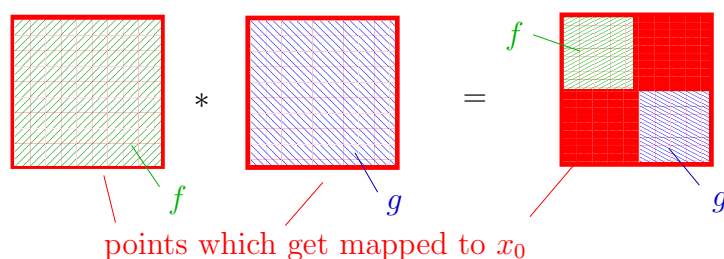


FIGURE 234.

forward to see that the homotopy class of $f * g$ only depends on the homotopy classes of f and g . Thus we can now define the following map:

$$\begin{aligned} \pi_n(X, x_0) \times \pi_n(X, x_0) &\rightarrow \pi_n(X, x_0) \\ ([f], [g]) &\mapsto [f] \cdot [g] := [f * g]. \end{aligned}$$

Note that for $n = 1$ this is precisely the same definition of the product structure that we had we already introduced in Chapter 4.2.

Now we can formulate the following proposition:

Proposition 23.1. *Let (X, x_0) be a pointed topological space and let $n \geq 1$. The map*

$$\begin{aligned} \pi_n(X, x_0) \times \pi_n(X, x_0) &\rightarrow \pi_n(X, x_0) \\ ([f], [g]) &\mapsto [f] \cdot [g] := [f * g] \end{aligned}$$

defines a group structure on $\pi_n(X, x_0)$.

As we will see, the proof of Proposition 23.1 is very similar to the proof of Proposition 4.6 and the proof of Proposition 4.5. To deal with the technical differences between the case $n = 1$ and the general case we need a few more definitions and one lemma.

Definition.

- (1) A *cuboid* is any subset of $\mathbb{R}^n$ of the form $[c_1, d_1] \times \cdots \times [c_n, d_n]$.
- (2) We say a cuboid $[c_1, d_1] \times \cdots \times [c_n, d_n]$ is *non-degenerate* if for all $i = 1, \dots, n$ we have $d_i > c_i$.

Let X be a topological space, let $x_0 \in X$ and let $f: I^n \rightarrow X$ be a map.

- (3) We say a cuboid $A \subset I^n$ is *boundary-trivial* if f sends all points on the boundary ∂A of A to x_0 .
- (4) Let $A = [a_1, b_1] \times \cdots \times [a_n, b_n] \subset I^n$ be a boundary-trivial cuboid and furthermore let $C = [c_1, d_1] \times \cdots \times [c_n, d_n]$ be a non-degenerate cuboid that is contained in A . We

⁴⁵⁰For mysterious reasons in all figures of this chapter the origin $(0, 0)$ is in the top left corner and the point $(1, 1)$ is the vertex at the bottom right.

define

$$f(A, C): I^n \rightarrow X$$

$$x = (x_1, \dots, x_n) \mapsto \begin{cases} f(x), & \text{if } x \notin A, \\ x_0, & \text{if } x \in A \setminus C, \\ f\left(a_1 + (x_1 - c_1) \frac{b_1 - a_1}{d_1 - c_1}, \dots, a_n + (x_n - c_n) \frac{b_n - a_n}{d_n - c_n}\right) & \text{if } x \in C. \end{cases}$$

The definition of $f(A, C)$ is sketched in Figure 235.

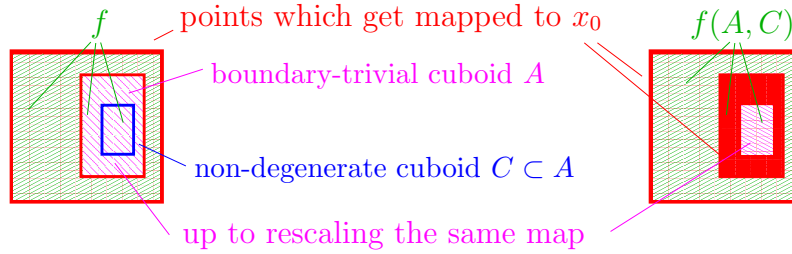


FIGURE 235.

Lemma 23.2. *Let (X, x_0) be a pointed topological space and let $f: (I^n, \partial I^n) \rightarrow (X, x_0)$ be a map. Furthermore let $A \subset I^n$ be a boundary-trivial cuboid and let C be a non-degenerate cuboid that is contained in A . Then the maps f and $f(A, C)$ from $(I^n, \partial I^n)$ to (X, x_0) are homotopic.*

Proof. Let $f: (I^n, \partial I^n) \rightarrow (X, x_0)$ be a map, let $A = [a_1, b_1] \times \dots \times [a_n, b_n] \subset I^n$ be a boundary-trivial cuboid and let $C = [c_1, d_1] \times \dots \times [c_n, d_n]$ be a non-degenerate cuboid that is contained in A . For $t \in [0, 1]$ we write

$$Q_t = [a_1(1-t) + c_1t, b_1(1-t) + d_1t] \times \dots \times [a_n(1-t) + c_nt, b_n(1-t) + d_nt].$$

Note that $Q_0 = A$ and $Q_1 = C$. It is now straightforward to see that the map

$$\begin{aligned} I^n \times [0, 1] &\rightarrow X \\ (x, t) &\mapsto F(A, C_t)(x) \end{aligned}$$

is a homotopy between f and $f(A, C)$. □

We can now provide the proof of Proposition 23.1

Proof of Proposition 23.1. The neutral element of the product structure is given by the equivalence class of the constant map

$$\begin{aligned} c_{x_0}: I^n &\rightarrow X \\ x &\mapsto x_0. \end{aligned}$$

Indeed, it follows immediately from Lemma 23.2, applied to $A = [0, 1]^n$ and $C = [0, \frac{1}{2}]^n$, that given any map $f: (I^n, \partial I^n) \rightarrow (X, x_0)$ we have $f * c_{x_0} \simeq f$. Basically the same argument shows also that $c_{x_0} * f \simeq f$.

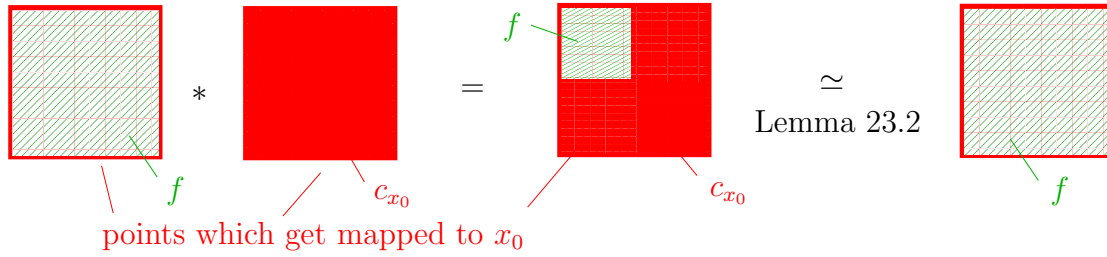


FIGURE 236.

Now consider three maps $f, g, h: (I^n, \partial I^n) \rightarrow (X, x_0)$. By applying Lemma 23.2 several times one can easily show that $(f * g) * h \simeq f * (g * h)$, see e.g. Figure 237. This proves that the multiplication on $\pi_n(X, x_0)$ is associative.

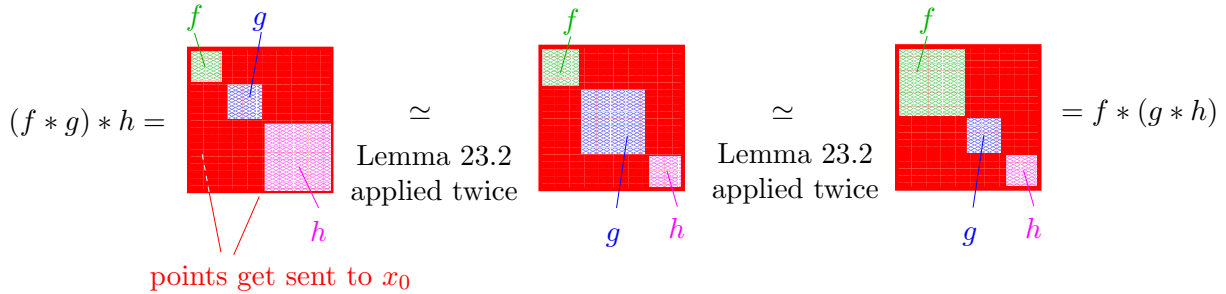


FIGURE 237.

Now we turn to the most interesting question, namely finding the inverse of a given element in $\pi_n(X, x_0)$. Thus suppose we are given a map $f: (I^n, \partial I^n) \rightarrow (X, x_0)$. We consider the map

$$\begin{aligned} \bar{f}: I^n &\rightarrow X \\ (x_1, \dots, x_n) &\mapsto f(1 - x_1, x_2, \dots, x_n). \end{aligned}$$

We claim that $[f] \cdot [\bar{f}] = [f * \bar{f}] = [c_{x_0}]$. Thus we need to show that $f * \bar{f}$ is homotopic to c_{x_0} . It follows from Lemma 23.2 that $f * \bar{f}$ is homotopic to the map

$$\begin{aligned} h: (I^n, \partial I^n) &\rightarrow (X, x_0) \\ (x_1, \dots, x_n) &\mapsto \begin{cases} f(2x_1, x_2, \dots, x_n), & \text{if } x_1 \in [0, \frac{1}{2}], \\ \bar{f}(2x_1 - 1, x_2, \dots, x_n), & \text{if } x_1 \in [\frac{1}{2}, 1]. \end{cases} \\ &= f(2 - 2x_1, x_2, \dots, x_n) \end{aligned}$$

Thus it suffices to show that h is homotopic to c_{x_0} . It is in fact straightforward to see that the map⁴⁵¹

$$H: I^n \times [0, 1] \rightarrow X$$

$$(x_1, \dots, x_n, t) \mapsto \begin{cases} f(2(x_1 - \frac{t}{2}), x_2, \dots, x_n), & \text{if } x_1 \in [\frac{t}{2}, \frac{1}{2}], \\ f(2 - 2(x_1 + \frac{t}{2}), x_2, \dots, x_n), & \text{if } x_1 \in (\frac{1}{2}, \frac{1+t}{2}], \\ x_0, & \text{otherwise} \end{cases}$$

is a homotopy from h to c_{x_0} . We refer to Figure 238 for an illustration of the argument. Basically the same argument also shows that $[f] \cdot [f] = [\bar{f} * f] = [c_{x_0}]$. $\square$

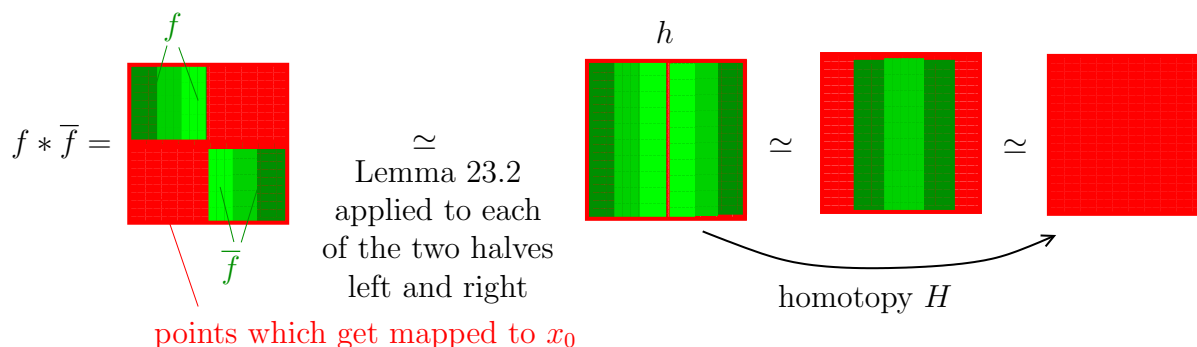


FIGURE 238.

Example. There is one non-empty topological space for which we can immediately compute the higher homotopy groups, namely the topological space consisting of a single point P . In this case the only map $I^n \rightarrow \{P\}$ is the constant map, hence we see that $\pi_n(\{P\}) = 0$ for any $n \geq 1$.

The higher homotopy groups $\pi_n(X, x_0)$ seem even more complicated than the fundamental group. Surprisingly at least their algebraic structure is much simpler.

Proposition 23.3. *For any pointed topological space (X, x_0) and any $n \geq 2$ the homotopy group $\pi_n(X, x_0)$ is abelian.*

Proof. Suppose we are given two maps $f, g: (I^n, \partial I^n) \rightarrow (X, x_0)$. We have to show that $f * g$ is homotopic to $g * f$. The homotopy is described in Figure 239 for $n = 2$. It is straightforward to write down an explicit homotopy between $f * g$ and $g * f$ by composing four instances of the homotopy provided by Lemma 23.2. We leave the details to the reader. $\square$

⁴⁵¹Even for $n = 1$ this homotopy is different from the homotopy used in Proposition 4.5 (3). Why is that?

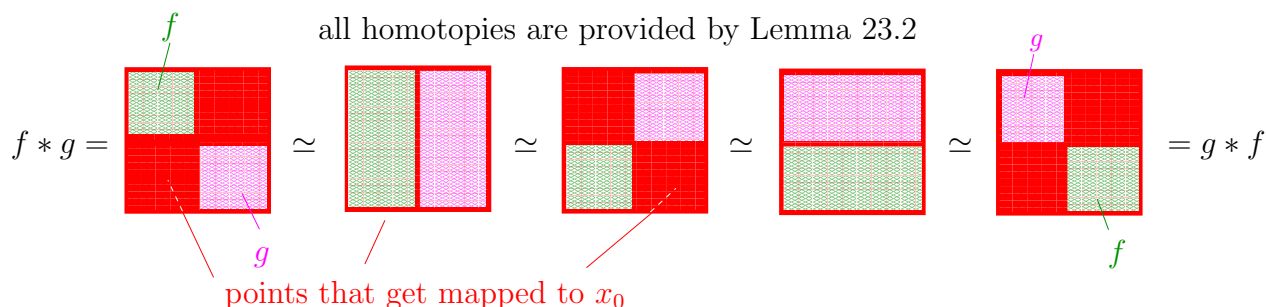


FIGURE 239. Illustration of the proof of Proposition 23.3.

Remark. Let $n \in \mathbb{N}$. There exists a homeomorphism

$$\begin{array}{ccccc} \Phi: I^n / \partial I^n & \cong & \overline{B}^n / S^{n-1} & \cong & S^n. \\ & \uparrow & & \uparrow & \\ & \text{see page 18} & & \text{see page 23} & \end{array}$$

Now let X be a topological space and let $x_0 \in X$. We write $*$ $:= \Phi(\partial I^n)$. Then the above homeomorphism Φ defines a one-to-one correspondence between maps

$$(I^n, \partial I^n) \mapsto (X, x_0)$$

and maps

$$(S^n, *) \rightarrow (X, x_0)$$

Thus we can view elements in $\pi_n(X, x_0)$ as homotopy classes of maps $(S^n, *) \rightarrow (X, x_0)$. This point of view is perhaps closer to our way of thinking about topological spaces, but as we have seen above, for technical purposes it is often much easier to work with the maps $(I^n, \partial I^n) \rightarrow (X, x_0)$. In Figure 240 we sketch the definition of the product $f * g$ of two maps $f, g: (S^n, *) \rightarrow (X, x_0)$.

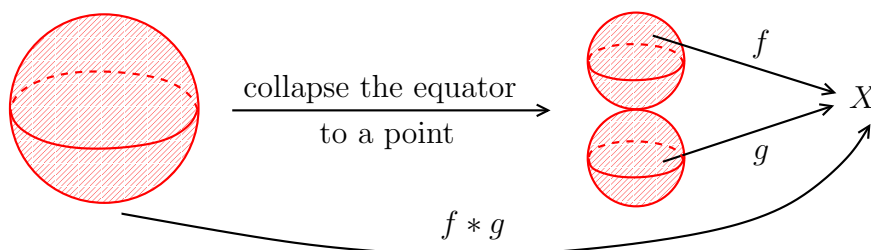


FIGURE 240.

23.2. Properties and calculations of the higher homotopy groups. Now we want to understand some of the basic properties of higher homotopy groups. Perhaps not surprisingly, many of the basic statements about the fundamental group can be easily generalized to the setup of higher homotopy groups.

For example the following proposition can be viewed a generalization of Proposition 4.9.

Proposition 23.4. *Let X be a topological space. Furthermore let x_0 and x_1 be two points in X which lie in the same component of X . Then the groups $\pi_n(X, x_0)$ and $\pi_n(X, x_1)$ are isomorphic.*

If X is a path-connected topological space and if we are only interested in the isomorphism type of the n -th homotopy group of X , then sometimes we just write $\pi_n(X)$, i.e. we suppress the base point from the notation.

Proof. Let X be a topological space and let x_0 and x_1 be two points which lie in the same component of X . By definition this means that x_0 and x_1 are connected through a path $p: [0, 1] \rightarrow X$. Let $f: (I^n, \partial I^n) \rightarrow (X, x_1)$ be a map. We consider the map

$$\begin{aligned} \tilde{f}: (I^n, \partial I^n) &\rightarrow (X, x_0) \\ (x_1, \dots, x_n) &\rightarrow \begin{cases} f(2x_1 - \frac{1}{2}, \dots, 2x_n - \frac{1}{2}), & \text{if } (x_1, \dots, x_n) \in [\frac{1}{4}, \frac{3}{4}]^n, \\ p(t), & \text{if } (x_1, \dots, x_n) \in \partial([\frac{t}{4}, \frac{4-t}{4}]^n) \text{ for } t \in [0, 1]. \end{cases} \end{aligned}$$

The definition of $\tilde{f}$ is also illustrated in Figure 241. It is now elementary to see that the

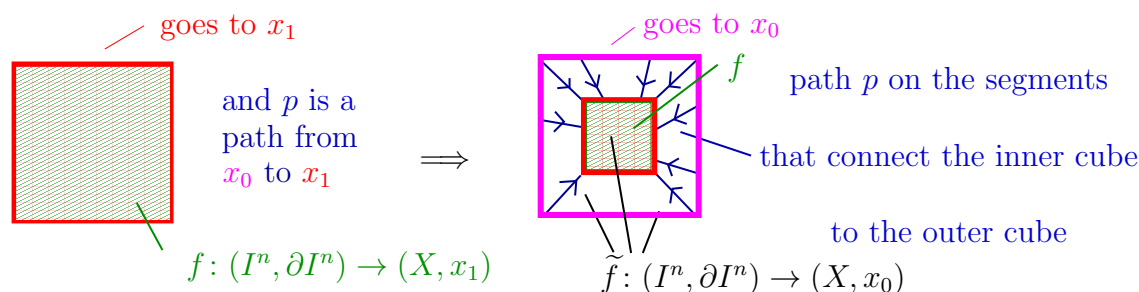


FIGURE 241. Illustration of the proof of Proposition 23.4.

map $f \mapsto \tilde{f}$ induces a group isomorphism from $\pi_n(X, x_1)$ to $\pi_n(X, x_0)$. We leave the details to the reader.⁴⁵² $\square$

Let $f: X \rightarrow Y$ be a map between two topological spaces and let $x_0 \in X$. We write $y_0 = f(x_0)$. Then one can easily verify that

$$\begin{aligned} f_*: \pi_n(X, x_0) &\rightarrow \pi_n(Y, y_0) \\ [\sigma: (I^n, \partial I^n) \rightarrow (X, x_0)] &\mapsto [f \circ \sigma: (I^n, \partial I^n) \xrightarrow{\sigma} (X, x_0) \xrightarrow{f} (Y, y_0)] \end{aligned}$$

⁴⁵²If we view elements in $\pi_1(X, x_i)$ as represented by maps $(S^1, *) \rightarrow (X, x_i)$, what does the corresponding map $\pi_1(X, x_1) \rightarrow \pi_1(X, x_0)$ look like?

is a well-defined map and that this map is a group homomorphism for every $n \geq 1$. As usual we say that f_* is the map induced by f .

It follows easily from the definitions that

$$\text{id}_* = \text{id} \quad \text{for all pointed pairs } (X, x_0),$$

and that

$$(f \circ g)_* = f_* \circ g_* \quad \text{for all maps } f: (X, x_0) \rightarrow (Y, y_0) \text{ and } g: (Y, y_0) \rightarrow (Z, z_0).$$

Summarizing we just observed that the following straightforward generalization of Proposition 5.1 holds.

Proposition 23.5. *Let $n \geq 1$, let $\mathcal{T}$ be the category of pointed topological spaces and let $\mathcal{G}$ be the category of groups. Then*

$$\begin{aligned} \text{Ob}(\mathcal{T}) &\rightarrow \text{Ob}(\mathcal{G}) \\ (X, x_0) &\mapsto \pi_n(X, x_0) \end{aligned}$$

together with the maps

$$\begin{aligned} \text{Mor}(\mathcal{T}) &\rightarrow \text{Mor}(\mathcal{G}) \\ (f: (X, x_0) \rightarrow (Y, y_0)) &\mapsto (f_*: \pi_n(X, x_0) \rightarrow \pi_n(Y, y_0)) \end{aligned}$$

is a covariant functor.

The same way as Proposition 7.5 one we can easily prove the following proposition:

Proposition 23.6. *Let $f: X \rightarrow Y$ be a map between topological spaces and let $x_0 \in X$ be a point. If f is a homotopy equivalence, then*

$$f_*: \pi_n(X, x_0) \rightarrow \pi_n(Y, f(x_0))$$

is an isomorphism.

Example. The topological space $\mathbb{R}^n$ is homotopy equivalent to a point P . Thus it follows from Proposition 23.6 and the discussion on page 384 that $\pi_k(\mathbb{R}^n) \cong \pi_k(P) \cong 0$. Exactly the same way, using the discussion on page 126, one can show that all the higher homotopy groups of convex subsets of $\mathbb{R}^n$ vanish, e.g. we have $\pi_k(\overline{B}^n) = \pi_k(B^n) = 0$.

In Proposition 4.12 we had seen that $\pi_1(S^n) = 0$ for $n \geq 2$. We can now prove the analogous statement for higher homotopy groups. More precisely, we have the following proposition.

Proposition 23.7. *For any natural numbers k and n with $k < n$ we have*

$$\pi_k(S^n) = 0.$$

Proof. Let $k < n$ be natural numbers. We write $P = (1, 0, \dots, 0) \in S^n$. We want to show that $\pi_k(S^n, P) = 0$. Hereby we take the point of view from page 385, i.e. we view elements of $\pi_k(S^n, P)$ as homotopy classes of maps $(S^k, *) \rightarrow (S^n, P)$. As on page 340 we consider the n -sphere S^n as a CW-complex with one 0-cell P and one n -cell. Similarly we view S^k as a CW-complex with one 0-cell $*$ and one k -cell. Let $g: (S^k, *) \rightarrow (S^n, P)$ be

a map. By the Cellular Approximation Theorem 22.19, applied to $g: S^k \rightarrow S^n$ and the subcomplex $\{*\} \subset S^k$, there exists a homotopy $F \text{ rel } *$ from the map g to a cellular map $\tilde{g}$. Since F is a homotopy $\text{rel } *$ we see that F also defines a homotopy $(S^k, *) \rightarrow (S^n, P)$, i.e. $[g] = [\tilde{g}] \in \pi_k(S^n, P)$.

Since $\tilde{g}$ is cellular we know that $\tilde{g}(S^k)$ lies in the k -skeleton of S^n . But since $k < n$ we see that the k -skeleton of our CW-structure on S^n consists only of the point P . Put differently, $\tilde{g}$ is the constant map that sends all points in S^k to P . We have thus shown that $\pi_k(S^n, P) = 0$. $\square$

The following proposition is also a straightforward generalization from the setting of fundamental groups to the setting of higher homotopy groups. The proof is verbatim the same as the proof of Proposition 22.9.

Proposition 23.8. *Let X be a CW-complex and let $X_1 \subset X_2 \subset X_3 \subset \dots$ be a sequence of subcomplexes such that $X = \bigcup_{i \in \mathbb{N}} X_i$. Then for any $w \in X_1$ we have*

$$\pi_n(X, w) = \lim_{\rightarrow} \pi_n(X_i, w).$$

Corollary 23.9. *For every n we have $\pi_n(S^\infty) = 0$.*

Proof. We equip S^∞ and all spheres S^k , $k \in \mathbb{N}$ with the CW-complex structure that we had defined on page 345. We have

$$\begin{array}{ccc} \text{by Proposition 23.8 and by definition} & \text{by Proposition 23.7 we have} & \\ \text{of the topology of } S^\infty & \pi_n(S^k) = 0 \text{ for } k > n & \\ \downarrow & \downarrow & \\ \pi_n(S^\infty) = \pi_n\left(\bigcup_{k \in \mathbb{N}} S^k\right) & = \lim_{\rightarrow} \pi_n(S^k) & = 0. \end{array} \quad \square$$

23.3. Covering spaces and higher homotopy groups. One of the main tools in studying fundamental groups was to study the relationship between the fundamental group of a topological space and coverings of the given topological space. Perhaps somewhat surprisingly, for $n \geq 2$ we have a completely different relationship between the n -th homotopy group and coverings.

Proposition 23.10. *Let $p: (X, x_0) \rightarrow (B, b_0)$ be a covering of pointed topological spaces. Let $n \geq 2$. Then $p_*: \pi_n(X, x_0) \rightarrow \pi_n(B, b_0)$ is an isomorphism.*

The following is an immediate, and very useful, corollary to Proposition 23.10.

Corollary 23.11. *Let X be a path-connected and locally simply connected topological space. Given any $n \geq 2$ we have⁴⁵³*

$$\pi_n(X) \cong \pi_n(\text{universal cover of } X).$$

We now present some examples for Corollary 23.11. We will consider a few more examples after the proof of Proposition 23.10.

⁴⁵³In Corollary 15.7 we had shown that every connected, path-connected and locally simply connected topological space X admits a universal covering $p: \tilde{X} \rightarrow X$, i.e. a covering such that $\tilde{X}$ is simply connected.

Examples.

- (1) Let $k \in \mathbb{N}$ and $n \geq 2$. We have

$$\begin{array}{ccccccc} \pi_n((S^1)^k) & = & \pi_1(\mathbb{R}^k/\mathbb{Z}^k) & \cong & \pi_n(\mathbb{R}^k) & = & 0. \\ & & \uparrow & & \uparrow & & \\ & & \text{Proposition 23.10} & & \text{page 387} & & \end{array}$$

In particular we see that all of the higher homotopy groups of S^1 and the torus $S^1 \times S^1$ vanish.

- (2) We consider $X = S^1 \vee S^2$. On page 265 we had determined its universal covering $\tilde{X}$. For convenience we picture it again in Figure 242. It follows from Proposition 23.10 that $\pi_2(S^1 \vee S^2)$ is isomorphic to $\pi_2(\tilde{X})$. At the moment we cannot determine $\pi_2(\tilde{X})$, but since $\tilde{X}$ has “infinitely many spheres” one might expect $\pi_2(\tilde{X})$ to be pretty big.

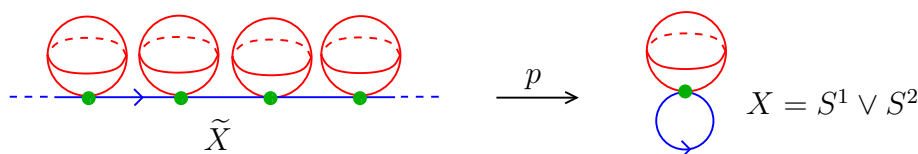


FIGURE 242.

- (3) One of our goals is still to complete the classification of lens spaces, in particular we would like to answer Question 6.7. So let $L(p, q)$ and $L(p, r)$ be two lens spaces. Then for any $n \geq 2$ we have

$$\begin{array}{ccccc} \pi_n(L(p, q)) & = & \pi_n(S^3) & = & \pi_n(L(p, r)). \\ \uparrow & & \uparrow & & \\ & & \text{Proposition 23.10} & & \end{array}$$

We do not know the higher homotopy groups of S^3 , but this argument shows that higher homotopy groups are useless for distinguishing lens spaces.

Proof of Proposition 23.10. Let $p: (X, x_0) \rightarrow (B, b_0)$ be a covering of pointed topological spaces and furthermore let $n \geq 2$. Our goal is to construct an inverse to the map $p_*: \pi_n(X, x_0) \rightarrow \pi_n(B, b_0)$.

We had seen on page 385 that we can think of elements in $\pi_n(B, b_0)$ as homotopy classes of maps $(S^n, *) \rightarrow (B, b_0)$. Thus let $f: (S^n, *) \rightarrow (B, b_0)$ be given. We are in the following

setting⁴⁵⁴

$$\begin{array}{ccc}
 (X, x_0) & & (X, x_0) \\
 \downarrow p & \xrightarrow[\text{since } \pi_1(S^n) = 0]{\text{by Proposition 15.2}} & \downarrow p \\
 (S^n, *) \xrightarrow{f} (B, b_0) & & (S^n, *) \xrightarrow{f} (B, b_0)
 \end{array}$$

exists
unique $\tilde{f}$

Furthermore it follows again from Proposition 15.2, applied to $Z = S^n \times [0, 1]$, that the lifts of two homotopic maps $(S^n, *) \rightarrow (B, b_0)$ are two homotopic maps $(S^n, *) \rightarrow (X, x_0)$.⁴⁵⁵ Summarizing we just constructed a map

$$\begin{aligned}
 \lambda: \pi_n(B, b_0) &\rightarrow \pi_n(X, x_0) \\
 [f] &\mapsto [\tilde{f}].
 \end{aligned}$$

Now we have to show that λ is an inverse to p_* . First of all, given any $[f] \in \pi_n(B, b_0)$ we have

$$(p_* \circ \lambda)([f]) = p_*([\tilde{f}]) = [p \circ \tilde{f}] = f.$$

Now let $[g] \in \pi_n(X, x_0)$. We then have

$$\begin{aligned}
 (\lambda \circ p_*)([g]) &= \lambda([p \circ g]) = [g]. \\
 &\quad \uparrow \\
 &\text{since } g \text{ is the unique lift of } p \circ g \text{ to } x_0.
 \end{aligned}$$

□

Now we give a few more applications of Proposition 23.10.

Examples.

- (1) Let Σ be a surface of genus ≥ 2 . In Theorem 19.9 we have seen that there exists a covering $p: \mathbb{D} \rightarrow \Sigma$ where $\mathbb{D} = B_1(0) \subset \mathbb{C}$. Since $\mathbb{D}$ is convex it follows from Proposition 23.10 and the discussion on page 387 that $\pi_n(\Sigma) \cong \pi_n(\mathbb{D}) = 0$ for all $n \geq 2$.

Together with the example on page 389 we have now shown that the higher homotopy groups of every surface of genus ≥ 1 vanish.

- (2) Let Σ be a surface of genus ≥ 1 and let $f: \Sigma \rightarrow \Sigma$ be a self-homeomorphism. We denote by $M(\Sigma, f)$ the mapping torus that we had introduced on page 212. For

⁴⁵⁴In Proposition 15.2 we need that the sphere S^n is of path-connected and locally path-connected, but there are no hypotheses on B or X .

⁴⁵⁵Here there is a little hidden subtlety: if $F: S^n \times [0, 1] \rightarrow B$ is a homotopy between two maps $f, g: (S^n, *) \rightarrow (B, b_0)$, then we can lift f, g to maps $\tilde{f}, \tilde{g}: (S^n, *) \rightarrow (X, x_0)$. Furthermore we can apply Proposition 15.2 to lift F to a map $\tilde{F}: S^n \times [0, 1] \rightarrow X$ such that $\tilde{F}(*, 0) = x_0$. We still need to show that $\tilde{F}(*, t) = x_0$ for all $t \in [0, 1]$. We know that for every $t \in [0, 1]$ we have $\tilde{F}(*, t) \in p^{-1}(b_0)$. By Lemma 6.4 the preimage $p^{-1}(b_0)$ is discrete. Since $[0, 1]$ is connected it follows that the map $p: [0, 1] \rightarrow p^{-1}(b_0)$ is constant by Lemma 1.13. But since it takes the value x_0 for $t = 0$ it takes the value x_0 for all $t \in [0, 1]$.

$n \geq 2$ we have

$$\begin{array}{ccccccc} \pi_n(M(\Sigma, f)) & \cong & \pi_n(\Sigma \times \mathbb{R}) & \cong & \pi_n(\Sigma) & = & 0. \\ \uparrow & & \uparrow & & \uparrow & & \\ \text{by Lemma 12.10} & & \text{Proposition 23.6} & & \text{previous example} & & \\ \text{and Proposition 23.10} & & & & & & \end{array}$$

Lemma 23.12. *Let X be a 1-dimensional CW-complex and let $x_0 \in X$ be base point. Then $\pi_n(X, x_0) = 0$ for every $n \geq 2$.*

Sketch of proof. Let X be a 1-dimensional CW-complex, let $x_0 \in X$ and let $n \geq 2$. We first assume that X is finite and connected. We write $k = 1 - \chi(X)$. We have

$$\begin{array}{ccccccc} \pi_n(X) & \cong & \pi_n\left(\bigvee_{i=1}^k S^1\right) & \cong & \pi_n\left(\text{universal cover of } \bigvee_{i=1}^k S^1\right) & \cong & \lim_{\rightarrow} \pi_n(\text{trees } T_n) = \lim_{\rightarrow} 0 = 0. \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ \text{Propositions 7.8 and 23.6} & & \text{Proposition 23.10} & & \text{Proposition 23.8 and Figure 243} & & \text{Proposition 7.7} \\ & & & & & & \text{and Proposition 23.6} \end{array}$$

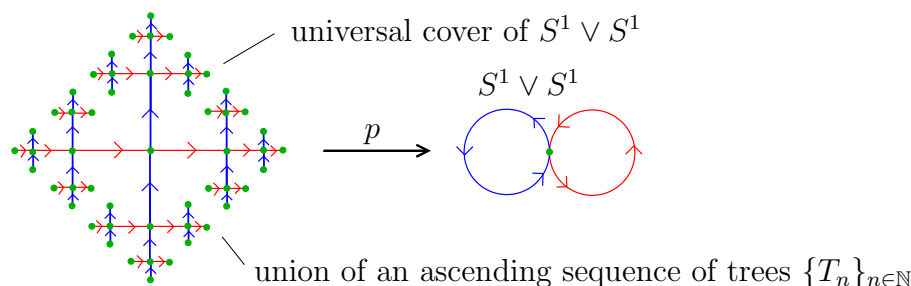


FIGURE 243.

Now suppose that X is an arbitrary 1-dimensional CW-complex. Let $[f] \in \pi_n(X, x_0)$. It follows from Proposition 22.6 that the image of f lies in a finite connected subcomplex Y of X . But by the above discussion $[f] = 0 \in \pi_1(Y, x_0)$, hence, by functoriality of π_n , we also have $[f] = 0 \in \pi_1(X, x_0)$. $\square$

23.4. Are there any higher homotopy groups that are non-trivial? We have now computed lots of higher homotopy groups, but so far the results have been rather dull, all the groups we have computed so far are trivial.

So are there any higher homotopy groups that are non-trivial? A natural candidate for a non-trivial group is surely $\pi_n(S^n)$. The following question naturally arises:

Question 23.13. *Let $n \geq 2$. Does the identity map $\text{id}: S^n \rightarrow S^n$ define a non-trivial element in the homotopy group $\pi_n(S^n, *)$?*

We can also ask, what are the higher homotopy groups of spheres, i.e. what can we say about $\pi_k(S^n)$ for $k > n$? On page 389 we had seen that all these groups are trivial for

$n = 1$. But what happens for $n > 1$? Can the higher homotopy groups of spheres be non-trivial?

To construct our first example of a potentially interesting map from a higher-dimensional to a lower-dimensional sphere we recall that in exercise sheet I.1 we had shown that $\mathbb{CP}^1$ is homeomorphic to S^2 . In the subsequent discussion we identify $\mathbb{CP}^1$ with S^2 . We consider the following map which is usually referred to as the *Hopf*⁴⁵⁶ map:

$$\begin{aligned} H: S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \mid z_1^2 + z_2^2 = 1\} &\rightarrow S^2 = \mathbb{CP}^1 \\ (z_1, z_2) &\mapsto [z_1 : z_2]. \end{aligned}$$

The homotopy class of H defines an element in $\pi_3(S^2)$.

Question 23.14. Does the Hopf map $H: S^3 \rightarrow S^2$ represent a non-trivial element in the homotopy group $\pi_3(S^2)$?

Remark. The definition of the Hopf map is very elegant, but unfortunately we pay for the elegance by loosing any intuition for what the map looks like. We will therefore give an alternative description of the Hopf map. To do so recall that in the proof of Lemma 12.1 we shown that $S^3 = (S^1 \times \overline{B}^2) \cup_{\varphi} (\overline{B}^2 \times S^1)$ where we identify the boundaries of the solid tori in the obvious way.

A straightforward calculation shows that for each $P \in S^2$ the preimage $H^{-1}(P)$ is a circle. Some of these circles are shown in Figure 244. In particular it is straightforward to show that the core curves $S^1 \times \{0\}$ and $\{0\} \times S^1$ are such circles and that on the torus $S^1 \times S^1$ the “diagonal circle” $\{(z, z) \mid z \in S^1\}$ is one of those preimages.

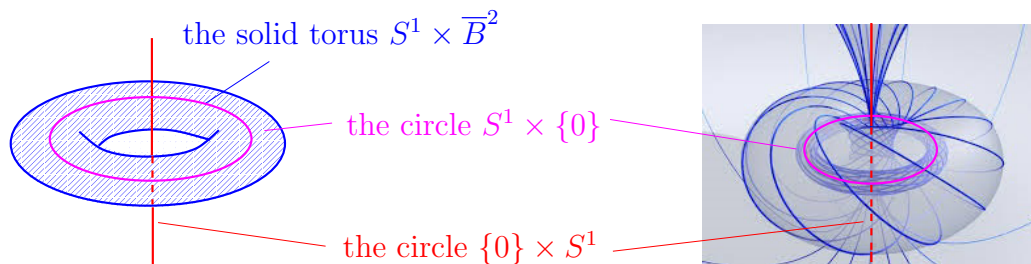


FIGURE 244.

We introduce one more construction of maps that represent potentially non-trivial elements in homotopy groups of spheres. Let X be a topological space. We define the *suspension* of X to be the topological space

$$\Sigma(X) := (X \times [-1, 1]) / \sim$$

where we identify all points in $X \times \{-1\}$ and we identify all points in $X \times \{1\}$. It is pretty clear from Figure 245 that the suspension of S^n is homeomorphic to S^{n+1} . In fact

⁴⁵⁶Heinz Hopf (1894-1971) was a German mathematician.

throughout the course we will use the following homeomorphism to identify $\Sigma(S^n)$ with S^{n+1} :

$$\begin{aligned}\Sigma(S^n) &\xrightarrow{\cong} S^{n+1} \subset \mathbb{R}^{n+1} \times \mathbb{R} \\ [(P, t)] &\mapsto (P \cos(\tfrac{\pi}{2}t), \sin(\tfrac{\pi}{2}t)).\end{aligned}$$

Furthermore given a map $f: X \rightarrow Y$ we denote by

$$\begin{aligned}\Sigma(f): \Sigma(X) &\rightarrow \Sigma(Y) \\ [(x, t)] &\mapsto [(f(x), t)]\end{aligned}$$

the *suspension* of f .

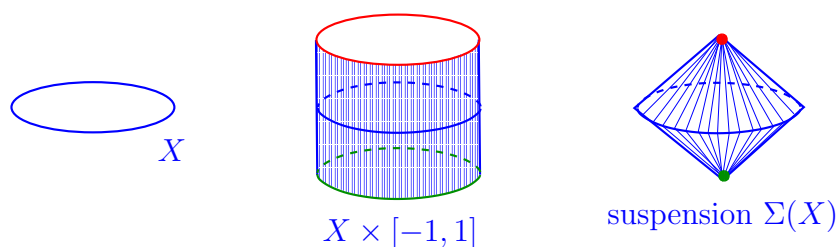


FIGURE 245.

Now we have the following elementary lemma.

Lemma 23.15. *Let $k, n \in \mathbb{N}$. We pick the base points $x_0 = (1, 0, \dots, 0) \in S^n$ and also $x_1 = (1, 0, \dots, 0) \in S^{n+1}$. Then the map*

$$\begin{aligned}\pi_k(S^n, x_0) &\rightarrow \pi_{k+1}(S^{n+1}, x_1) \\ [f: S^k \rightarrow S^n] &\mapsto [\Sigma(f): \Sigma(S^k) = S^{k+1} \rightarrow \Sigma(S^n) = S^{n+1}]\end{aligned}$$

is a well-defined group homomorphism.

Proof. It is straightforward to see that the map is well-defined. We leave it as an exercise to show that the map is in fact a group homomorphism.⁴⁵⁷ $\square$

We conclude this chapter with the following question:

Question 23.16. *Let $H: S^3 \rightarrow S^2$ be the Hopf map. Is the image of $[H] \in \pi_3(S^2)$ under the iterated suspension homomorphisms $\pi_3(S^2) \rightarrow \pi_{3+k}(S^{2+k})$ non-trivial?*

Unfortunately at the moment we do not have the tools to answer any of the above questions. In general it is very hard to compute higher homotopy groups as there are few computational techniques that can be used to show that higher homotopy groups are non-zero. One of the key problems is that there is no analogue to the Seifert-van Kampen Theorem 11.1. We will get to know a few techniques to compute higher homotopy groups of spheres, but we will cover these only in Algebraic Topology III, since the calculations will require the use of singular homology groups. But even then we will only be able to

⁴⁵⁷This exercise might in fact be rather painful.

determine some homotopy groups of spheres, in fact the full computation of homotopy groups of spheres is still an open problem.

This sorry state of affairs means that as of right now we cannot use higher homotopy groups to distinguish any of the topological spaces that we could not distinguish already using the fundamental group. In particular we still cannot show that for $k \neq l$ the topological spaces $\mathbb{R}^k$ and $\mathbb{R}^l$ are not homeomorphic. We will only be able to do so after developing a new tool, namely the singular homology groups, in the coming sections.

23.5. The Poincaré Conjecture. Before we bid a temporary farewell to the fundamental group and to higher homotopy groups we want to quickly discuss the Poincaré Conjecture.

Definition.

- (1) We say a topological space X is k -connected, if $\pi_i(X) = 0$ for $i = 0, \dots, k$.⁴⁵⁸
- (2) An n -dimensional manifold X that is closed and that is $\lfloor \frac{n}{2} \rfloor$ -connected is called an n -dimensional homotopy sphere.

Remark.

- (1) Since $\lfloor \frac{3}{2} \rfloor = 1$ we see that a 3-dimensional homotopy sphere is the same as a connected closed 3-dimensional manifold with $\pi_1(M) = 0$.
- (2) We have seen in Proposition 23.7 that each S^n is in fact a homotopy sphere. An n -dimensional homotopy sphere is thus a closed n -dimensional manifold that cannot be distinguished from the standard sphere S^n using the first $\lfloor \frac{n}{2} \rfloor$ homotopy groups.
- (3) The definition using $\lfloor \frac{n}{2} \rfloor$ -connected sounds slightly arbitrary, but we will see in Algebraic Topology III that an n -dimensional manifold that is $\lfloor \frac{n}{2} \rfloor$ -connected is in fact $(n-1)$ -connected. Furthermore we will see that a homotopy sphere cannot be distinguished from an ordinary sphere by any of the higher homotopy groups either.

One can now ask whether every homotopy sphere is S^n . Here one needs to be a little careful what one means by “is”. There are two reasonable interpretations. One could either interpret “is” as “is diffeomorphic to” or as “is homeomorphic to”. So we really have two, a priori different, questions that we can ask:

Question 23.17. Let $n \in \mathbb{N}$.

- (1) Is every n -dimensional homotopy sphere diffeomorphic to S^n ?
- (2) Is every n -dimensional homotopy sphere homeomorphic to S^n ?

Here an affirmative answer to the first question implies of course an affirmative answer to the second question. For $n = 1$ we had seen in Proposition 1.33 that the answer to the first question is “yes”. Furthermore in Theorem 11.8 we had seen that the answer to the first question is also “yes” for $n = 2$. In both these situations it was known by the 19th century that the answers are “yes”.

⁴⁵⁸For example, X is 0-connected if and only if it is path-connected. Furthermore, X is 1-connected if and only if it is simply connected.

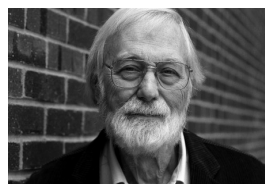
The above question was first raised in dimension $n = 3$ by Henri Poincaré in 1905. This special case became known as the Poincaré Conjecture in dimension three.

The first important result dealing with Question 23.17 was proved by John Milnor [Mi1] in 1956. In 1962 he obtained the Fields medal for proving the following theorem which shows that the above two questions raised in Question 23.17 are indeed two different questions.⁴⁵⁹

Theorem 23.18. (Milnor) *There exists a 7-dimensional manifold that is a homeomorphic to S^7 but that is not diffeomorphic to S^7 .*



Henri Poincaré (1854-1912)



John Milnor (*1931)

FIGURE 246.



FIGURE 247. The Fields medal

Remark. An n -dimensional manifold that is homeomorphic to S^n but not diffeomorphic to S^n is often called an *exotic n -sphere*. Milnor thus showed that in dimension 7 there exists an exotic 7-sphere. Michel Kervaire and John Milnor [KeM] showed that for $n \geq 5$ there exist at most finitely many, pairwise non-diffeomorphic, exotic n -spheres. Furthermore, in dimensions 5 and 6 there do not exist any exotic spheres. Somewhat earlier it had been shown in 1952 by Edwin Moise [Mo] that in dimension 3 there does not exist an exotic sphere.

In 1961 Stephen Smale proved the following theorem, for which he obtained the Fields medal in 1966.

Theorem 23.19. (Smale) *In dimensions $n \geq 5$ every homotopy sphere is homeomorphic to S^n .*

⁴⁵⁹The Fields medal is the most prestigious award in mathematics. It is awarded every four years to usually 2-4 mathematicians. One of the conditions is that you can only be awarded a Fields medal if you are under 40.

The next major result was obtained by Michael Freedman [Fr] in 1982. He was also awarded a Fields medal for his work.

Theorem 23.20. (Freedman) *Every 4-dimensional homotopy sphere is homeomorphic to S^4 .*

The proof by Freedman is so hard and specialized that only very few mathematicians have ever understood the proof.⁴⁶⁰

It took another 20 years before finally the original Poincaré Conjecture was dealt with. More precisely, in 2002 Grigori Perelman [Pe1, Pe2, Pe3, MTi] proved the following result.

Theorem 23.21. (Perelman) *Every 3-dimensional homotopy sphere is diffeomorphic to S^3 .*

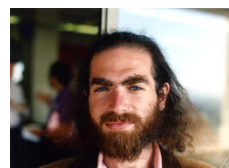
Perelman would have gotten the Fields medal in 2006 if he had not turned it down. Furthermore, the Poincaré Conjecture was one of the seven Millennium Problems⁴⁶¹ for which the Clay foundation awards a prize of one million dollars. This prize was also turned down by Perelman.



Stephen Smale (*1930)



Michael Freedman (*1951)



Grigori Perelman (*1966)

FIGURE 248.

What remains to be done is the following problem which is often referred to as the *smooth Poincaré Conjecture in dimension 4*.

Question 23.22. *Does there exist a 4-dimensional manifold that is homeomorphic to S^4 but that is not diffeomorphic to S^4 ?*

This question is considered to be one of the hardest problems in topology. Unfortunately there has been very little progress so far.

It might be surprising to see that the last open problem in this context is in dimension 4. But it turns out that the study of 4-dimensional manifolds is much harder than the study of manifolds in lower or higher dimensions.

The following theorem shows very clearly the special role played by 4-dimensional topology.

⁴⁶⁰I was told at some point that the total number of mathematicians who understood the proof is six, it is not clear whether at this moment anybody understands the proof.

⁴⁶¹

Theorem 23.23.

- (1) *For $n \neq 4$ every n -dimensional manifold homeomorphic to $\mathbb{R}^n$ is also diffeomorphic to $\mathbb{R}^n$.*
- (2) *In dimension 4 there exist uncountably many manifolds that are homeomorphic to $\mathbb{R}^4$ but that are pairwise non-diffeomorphic.*

The first statement was proved by John Stallings [St] in 1961. The second statement was proved by Cliff Taubes [Ta] in 1987. The latter result builds on the ground breaking work of Simon Donaldson [D] which he proved as a second year PhD student and for which he obtained a Fields medal in 1986.

24. THE HOMOLOGY GROUPS OF A TOPOLOGICAL SPACE

One of our main goals is to show that for $m \neq n$ the spheres S^m and S^n are not homeomorphic. We have seen that for most cases fundamental groups are of no use. Furthermore, it might be that in principle higher homotopy groups can distinguish such spheres, but since we cannot compute higher homotopy groups, those are of no use either. So we need a new approach to finding invariants that can distinguish spheres.

In this chapter we want to introduce a new functor from the category of topological spaces to an algebraic category, more precisely, we will introduce a functor from the category of topological spaces to the category of abelian groups.

24.1. Singular chains.

Definition.

- (1) Given k points $P_1, \dots, P_k$ in $\mathbb{R}^n$ such that $P_2 - P_1, \dots, P_k - P_1$ are linearly independent we refer to the convex hull of $P_1, \dots, P_k$ as a k -simplex.
- (2) Given $n \in \mathbb{N}_0$ we refer to the convex hull of the standard basis vectors $e_1, \dots, e_{n+1}$ of $\mathbb{R}^{n+1}$, i.e. to

$$\Delta^n := \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid x_0 + \dots + x_n = 1 \text{ and } x_i \geq 0 \text{ for all } i = 0, \dots, n\}$$

as the *standard n -simplex*.

- (3) Given the standard n -simplex Δ^n we define its boundary as⁴⁶²

$$\partial\Delta^n = \bigcup_{i=0}^n \{(t_0, \dots, t_n) \in \Delta^n \mid t_i = 0\}.$$

Example. The standard 0-simplex Δ^0 is just the point $\{1\} \in \mathbb{R}$, the standard 1-simplex Δ^1 equals the segment in $\mathbb{R}^2$ from $(1, 0)$ to $(0, 1)$ and the standard 2-simplex Δ^2 is the triangle in $\mathbb{R}^3$ spanned by the vertices $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$. We refer to Figure 249 for an illustration of the standard simplices in dimension 0, 1 and 2 and for an illustration of the boundary of the standard 2-simplex.

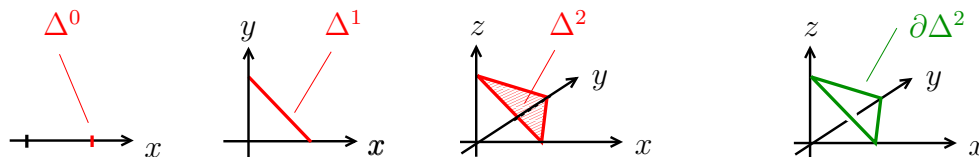


FIGURE 249.

For the record we state the following elementary lemma.

⁴⁶²The definition of the “boundary” of the standard n -simplex Δ^n is the “obvious one”. Nonetheless note that if we had taken the original definition of the boundary of a subset A in some subspace X as introduced on page 14 and applied it to $\Delta^n \subset \mathbb{R}^{n+1}$, then the boundary of Δ^n in $\mathbb{R}^{n+1}$ would be all of Δ^n .

Lemma 24.1. *The map*

$$\begin{aligned} \Delta^n &\xrightarrow{\cong} \left\{ (x_1, \dots, x_n) \in [0, 1]^n \mid \sum_{i=1}^n x_i \in [0, 1] \right\} \xrightarrow{\text{Lemma 1.4}} \overline{B}^n \\ (t_0, \dots, t_n) &\mapsto (t_0, \dots, t_{n-1}). \end{aligned}$$

is a homeomorphism that restricts to a homeomorphism from $\partial\Delta^n$ to S^{n-1} .

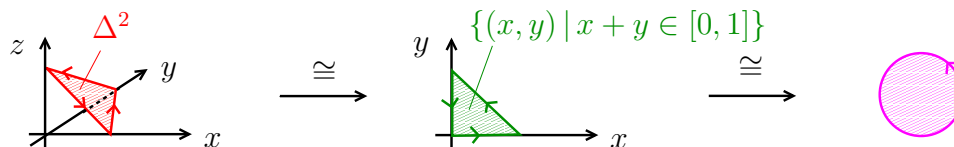


FIGURE 250. Illustration of the homeomorphism from Lemma 24.1.

Definition. An r -dimensional face of the standard n -simplex is defined as a subset of the form

$$\{(t_0, \dots, t_n) \in \Delta^n \mid t_{d_1} = t_{d_2} = \dots = t_{d_{n+1-r}} = 0\},$$

where $d_1 < d_2 < \dots < d_{n+1-r} \in \{0, \dots, n\}$ are fixed. We illustrate the definition in Figure 251 on the left.

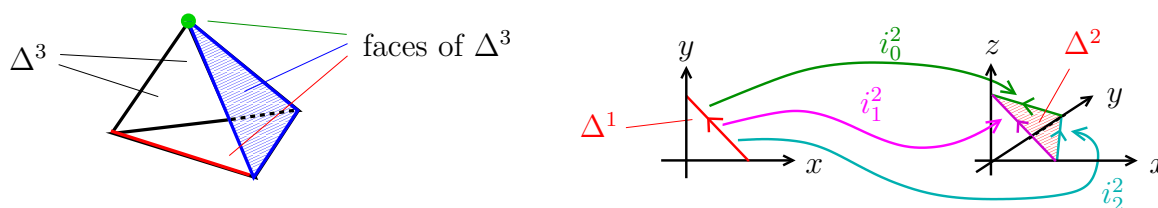


FIGURE 251.

Remark.

- (1) The boundary $\partial\Delta^n$ of the standard n -simplex is the union of all $(n-1)$ -dimensional faces of Δ^n .
- (2) Every r -dimensional face of Δ^r is homeomorphic to the standard r -simplex. For later we fix homeomorphism in the case of $(n-1)$ -dimensional faces. More precisely, for $j \in \{0, \dots, n\}$ we consider the map

$$\begin{aligned} i_j^n: \Delta^{n-1} &\rightarrow \Delta^n \\ (t_0, \dots, t_{n-1}) &\mapsto (t_0, \dots, t_{j-1}, 0, t_j, \dots, t_{n-1}). \end{aligned}$$

It is straightforward to see that these maps are homeomorphisms. The maps $i_0^2, i_1^2, i_2^2: \Delta^1 \rightarrow \Delta^2$ are illustrated in Figure 251 on the right.

The first idea for the definition of homology groups is now the following. In the definition of the fundamental group $\pi_1(X, x_0)$ we had considered all maps $f: [0, 1] \rightarrow X$ with the extra condition $f(0) = f(1) = x_0$. Using this extra condition we could concatenate loops to eventually end up with a group structure. Now given $n \in \mathbb{N}_0$ we want to consider all maps from Δ^n to X with no extra condition on the boundary. This time there is no reasonable way how two such maps can be composed to give a new map. Since we nonetheless want the notion of a group we turn the set of all maps $\Delta^n \rightarrow X$ into a group by “brute force”, i.e. by considering the free abelian group generated by these maps.

Definition. Let X be a topological space and let $n \in \mathbb{N}_0$.

- (1) A *singular n -simplex* is a map $\sigma: \Delta^n \rightarrow X$.
- (2) We define

$$C_n(X) := \text{free abelian group generated by the set of singular } n\text{-simplices in } X.$$

For $n \in \mathbb{Z}_{<0}$ we set $C_n(X) = 0$.

- (3) We refer to an element in $C_n(X)$ as a *singular n -chain* in X .

Examples.

- (1) For $n = 0$ a singular n -simplex is a map from Δ^0 to X . Such a map is uniquely determined by the corresponding point in X . Therefore frequently we will identify singular 0-simplices with points in X .
- (2) For points $z_0, \dots, z_k \in \mathbb{R}^n$ we consider the map

$$\begin{aligned} [z_0, \dots, z_k]: \Delta^k &\rightarrow \mathbb{R}^n \\ (t_0, \dots, t_k) &\mapsto \sum_{j=0}^k t_j \cdot z_j. \end{aligned}$$

The map $[z_0, \dots, z_k]$ thus sends the standard k -simplex Δ^k to the simplex spanned by $z_0, \dots, z_k$ in $\mathbb{R}^n$. Put differently, $[z_0, \dots, z_k]$ is a singular k -simplex in $\mathbb{R}^n$.

Remark. Let X be a topological space. By the discussion on page 137 we can view every element in $C_n(X)$, i.e. every singular n -chain, as a linear combination $a_1\sigma_1 + \dots + a_k\sigma_k$ where $a_1, \dots, a_k \in \mathbb{Z}$ and where $\sigma_1, \dots, \sigma_k$ are singular n -simplices.

Now let $f: X \rightarrow Y$ be a map between topological spaces. For each n we consider

$$\begin{aligned} f_*: C_n(X) &\rightarrow C_n(Y) \\ \sum_{i=1}^k a_i \cdot \sigma_i &\mapsto \sum_{i=1}^k a_i \cdot (f \circ \sigma_i). \end{aligned}$$

It is easy to verify that $X \mapsto C_n(X)$ and $f \mapsto f_*$ defines a functor from the category of topological spaces to the category of abelian groups.

Have we made any progress towards our goal of distinguishing topological spaces? Not really. For example, if X is a manifold of dimension ≥ 1 , then given any $n \in \mathbb{N}_0$ there exist uncountable many maps $\Delta^n \rightarrow X$, i.e. the group $C_n(X)$ is a free abelian group with uncountably many generators. Such groups are totally useless for distinguishing non-homeomorphic topological spaces.

24.2. Definition of the homology groups of a topological space. Before we give the definition of homology groups it is perhaps helpful to recall the definition of de Rham cohomology.

Let M be a manifold. As in Analysis IV we write

$$\Omega^n(M) := \{\text{smooth } n\text{-forms on } M\}.$$

The isomorphism type of this vector space is an invariant of the diffeomorphism type of the manifold M , but it is rather useless, since for $1 \leq n \leq \dim(M)$ this is a vector space of uncountable dimension. To cut $\Omega^n(M)$ into something much smaller we had defined linear maps $d_n: \Omega^n(M) \rightarrow \Omega^{n+1}(M)$ with $d_{n+1} \circ d_n = 0$. Finally we defined the n -th de Rham cohomology group of M as

$$H_{\text{dR}}^n(M) := \ker\{d_n: \Omega^n(M) \rightarrow \Omega^{n+1}(M)\} / \text{im}\{d_{n-1}: \Omega^{n-1}(M) \rightarrow \Omega^n(M)\}.$$

We are now in a similar situation: to a topological space X we associated the free abelian group $C_n(X)$ which is “too big”. The idea now is to introduce a “differential” as for the case of differential forms. But in this case there is no good way to associate to a map $\Delta^n \rightarrow X$ a new map $\Delta^{n+1} \rightarrow X$. But by restricting the map $\Delta^n \rightarrow X$ to each of the $(n-1)$ -dimensional faces of Δ^n which are, as we have seen, homeomorphic to the standard $(n-1)$ -simplex, we can associate to a singular n -simplex $\sigma: \Delta^n \rightarrow X$ a sum of singular $(n-1)$ -simplices, i.e. a singular $(n-1)$ -chain. To do this we will use the maps $i_k^n: \Delta^{n-1} \rightarrow \Delta^n$, $k = 0, \dots, n$ from page 399 to identify the standard $(n-1)$ -simplex with the $(n-1)$ -dimensional faces of Δ^n . We can now define

$$\tilde{\partial}_n \sigma := \sum_{j=0}^n \sigma \circ i_j^n.$$

By Lemma 8.1 this defines a map $\tilde{\partial}_n: C_n(X) \rightarrow C_{n-1}(X)$. But it is quite straightforward to see that for this definition the composition of the maps $\tilde{\partial}_n: C_n(X) \rightarrow C_{n-1}(X)$ and $\tilde{\partial}_{n-1}: C_{n-1}(X) \rightarrow C_{n-2}(X)$ is not zero⁴⁶³. But we can rectify this problem by cleverly introducing signs. This leads us to the following definition.

Definition. Let X be a topological space and let $\sigma: \Delta^n \rightarrow X$ be a singular n -simplex. The *boundary* $\partial_n \sigma \in C_{n-1}(X)$ of σ is defined as

$$\partial_n \sigma := \sum_{j=0}^n (-1)^j \cdot \sigma \circ i_j^n.$$

By Lemma 8.1 the map $\sigma \mapsto \partial_n \sigma$ extends uniquely to a linear map

$$\begin{aligned} \partial_n: C_n(X) &\rightarrow C_{n-1}(X) \\ \sum_{i=1}^k a_i \sigma_i &\mapsto \sum_{i=1}^k a_i \cdot \partial \sigma_i, \end{aligned}$$

⁴⁶³Indeed, let S be a set and let $\mathbb{Z}^{(S)}$ be the free abelian group generated by S . We define $\epsilon: \mathbb{Z}^{(S)} \rightarrow \mathbb{Z}$ to be the linear map that assigns to each $s \in S$ the value 1. It follows easily from the definitions that for any $\sigma: \Delta^n \rightarrow X$ we have $\epsilon(\partial_n \sigma) = n + 1$ and $\epsilon((\tilde{\partial}_{n-1} \circ \tilde{\partial}_n)(\sigma)) = n \cdot (n + 1)$.

which we refer to as the n -th *boundary map*. If it is clear with what dimension we are working with, then we just write ∂ instead of ∂_n .

Examples.

- (1) For any singular 1-simplex $\sigma: \Delta^1 \rightarrow X$ we have⁴⁶⁴

$$\partial\sigma = (-1)^0 \cdot \sigma \circ i_0^1 + (-1)^1 \cdot \sigma \circ i_1^1 = \sigma(0, 1) - \sigma(1, 0).$$

Put differently, a singular 1-simplex can be viewed as a map σ from the “interval” Δ^1 to X . The boundary of σ is thus the “endpoint” $\sigma(0, 1)$ “minus” the “starting point” $\sigma(1, 0)$. Hereby we view points in X again as singular 0-simplices. We sketch the definition of a singular 1-simplex in Figure 252.

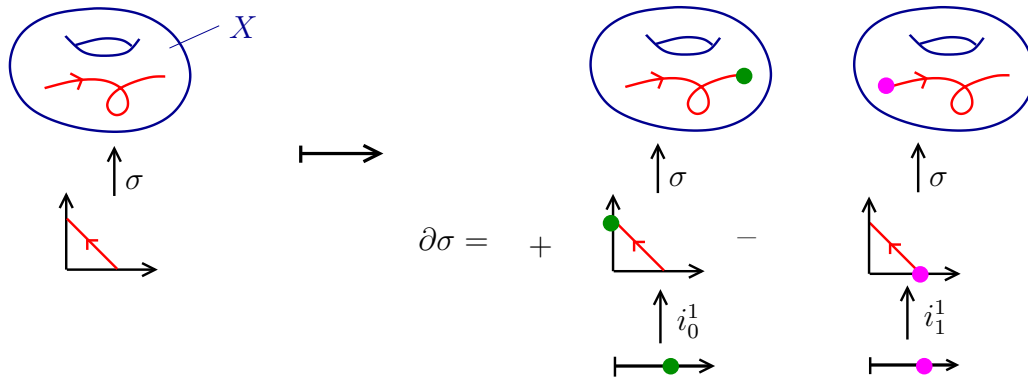


FIGURE 252. Boundary of a singular 1-simplex.

- (2) In Figure 253 we sketch the boundary of a singular 2-simplex.

As promised, these boundary maps have the following fundamental property:

Proposition 24.2. *Let X be a topological space. For any $n \in \mathbb{N}_0$ the composition*

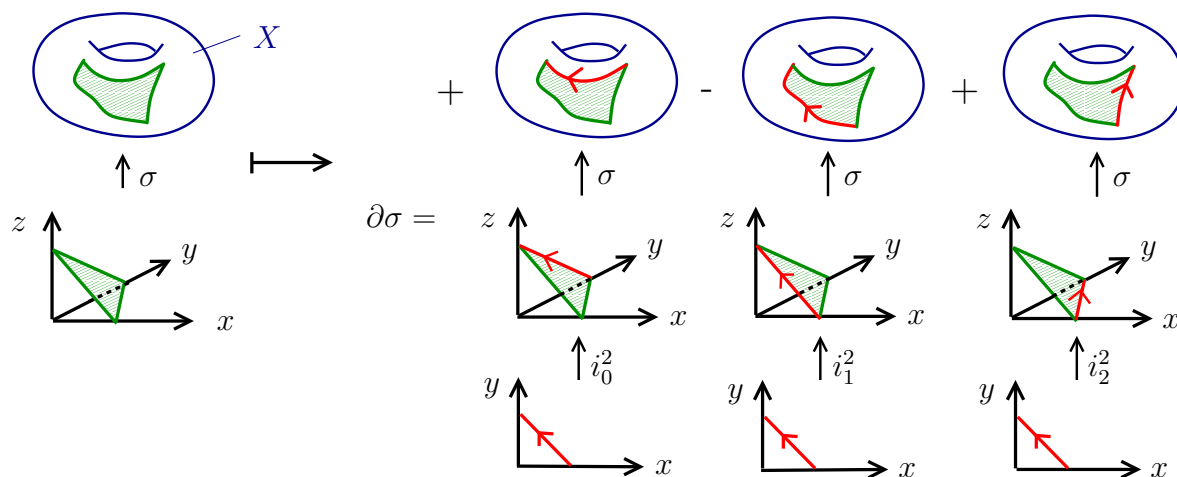
$$\partial_{n-1} \circ \partial_n: C_n(X) \rightarrow C_{n-2}(X)$$

is the zero map.

Proof. It suffices to show that $\partial_{n-1}(\partial_n(\sigma)) = 0$ for every singular n -simplex $\sigma: \Delta^n \rightarrow X$. Thus let $\sigma: \Delta^n \rightarrow X$ be a singular n -simplex. Then it follows immediately from the definitions that

$$\begin{aligned} \partial_{n-1}(\partial_n(\sigma)) &= \partial_{n-1} \left(\sum_{j=0}^n (-1)^j \cdot \sigma \circ i_j^n \right) = \sum_{k=0}^{n-1} (-1)^k \sum_{j=0}^n (-1)^j \cdot (\sigma \circ i_j^n) \circ i_k^{n-1} \\ &= \sum_{0=k < j \leq n} (-1)^{k+j} \cdot \sigma \circ (i_j^n \circ i_k^{n-1}) + \sum_{0=j \leq k \leq n-1} (-1)^{k+j} \cdot \sigma \circ (i_j^n \circ i_k^{n-1}) =: (*). \end{aligned}$$

⁴⁶⁴Here we again use the convention that given a point $x \in X$ we denote by x also the corresponding singular 0-simplex.



Now we want to show that each expression $\sigma \circ (i_j^n \circ i_k^{n-1})$ appears precisely twice, but with opposite signs. From the definitions it follows easily that for $j \leq k$ we have⁴⁶⁵

$$i_i^n \circ i_k^{n-1} = i_{k+1}^n \circ i_i^{n-1}.$$

$$\begin{aligned}
(*) &= \sum_{0=k < j \leq n} (-1)^{k+j} \cdot \sigma \circ (i_j^n \circ i_k^{n-1}) + \sum_{0=j \leq k \leq n-1} (-1)^{k+j} \cdot \sigma \circ (i_{k+1}^n \circ i_j^{n-1}) \\
&\quad \uparrow \\
&\text{above calculation of } i_j^n \circ i_k^{n-1} \\
&= \sum_{0=k < j \leq n} (-1)^{k+j} \cdot \sigma \circ (i_j^n \circ i_k^{n-1}) + \sum_{0=j < l \leq n} (-1)^{l+j-1} \cdot \sigma \circ (i_l^n \circ i_j^{n-1}) = 0. \\
&\quad \uparrow \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \uparrow \\
&\text{substitution } l = k + 1 \qquad \qquad \qquad \text{all summands left and right cancel} \qquad \square
\end{aligned}$$

Definition. Let X be a topological space and $n \in \mathbb{N}_0$. We define the n -th singular homology group of X as the quotient group⁴⁶⁶

$$H_n(X) := \ker\{\partial_n: C_n(X) \rightarrow C_{n-1}(X)\} / \operatorname{im}\{\partial_{n+1}: C_{n+1}(X) \rightarrow C_n(X)\}.$$

For $n \in \mathbb{Z}_{\leq 0}$ we extend this definition to $H_n(X) := 0$. Throughout the lecture notes we will mostly shorten the name “singular homology” to “homology”.⁴⁶⁷

⁴⁶⁵Indeed, if $j \leq k$, then the zero-entries of $(i_j^n \circ i_k^{n-1})(t_0, \dots, t_{n-1})$ are the entries j and $k + 1$.

⁴⁶⁶It follows from Proposition 24.2 that we have the following inclusion:

$$\operatorname{im}\{\partial_{n+1}: C_{n+1}(X) \rightarrow C_n(X)\} \subset \ker\{\partial_n: C_n(X) \rightarrow C_{n-1}(X)\}.$$

Therefore it does indeed make sense to consider the quotient.

⁴⁶⁷Later on we will also introduce “cellular homology”. At that point it will be useful to remember that our present homology groups have a more precise name, namely singular homology groups.

One of the key goals of this course is to develop tools for computing homology groups of topological spaces. In particular we want to compute the homology groups for all spheres. In the subsequent discussions it is helpful to introduce a few more definitions.

Definition. Let X be a topological space.

- (1) We say $c \in C_n(X)$ is an n -cycle, if $\partial c = 0$.
- (2) If c is an n -cycle, then we refer to $[c] \in H_n(X)$ as the *homology class* of c .
- (3) We say two singular n -chains c and d are *homologous* if there exists a singular $(n+1)$ -chain e with $\partial e = c - d$. In particular, if c and d are cycles, then they are homologous if they represent the same homology class.
- (4) An n -cycle c is called *null-homologous*, if $[c] = 0 \in H_n(X)$.
- (5) Given a singular 1-simplex $\sigma: \Delta^1 \rightarrow X$ we denote by $\bar{\sigma}$ the singular 1-simplex that is given by $\bar{\sigma}(t, 1-t) = \sigma(1-t, t)$. Put differently, $\bar{\sigma}$ is given by reversing the orientation of σ .

The following lemma will be proved in exercise sheet 3.

Lemma 24.3. Let X be a topological space and let $\sigma: \Delta^1 \rightarrow X$ be a singular 1-simplex.

- (1) If σ is a constant map, then σ is null-homologous.
- (2) The singular 1-chains $-\sigma$ and $\bar{\sigma}$ are homologous.
- (3) Let $s \in (0, 1)$ and consider the singular 1-simplices

$$\alpha: \Delta^1 \rightarrow X \quad \text{and} \quad \beta: \Delta^1 \rightarrow X$$

$$(t, 1-t) \mapsto \sigma(st, 1-st) \quad \text{and} \quad (t, 1-t) \mapsto \sigma(s+t(1-s), 1-(s+t(1-s))).$$

Then the singular 1-chains σ and $\alpha + \beta$ are homologous.

The second part of the lemma says that for a singular 1-simplex the geometric operation of reversing the orientation is (up to the boundary of a simplicial 2-chain) the same as algebraically inverting the sign. Furthermore, the third part of the lemma says that for a singular 1-simplex (again up to the boundary of a simplicial 2-chain) it does not make a difference whether or not we split the singular 1-simplex into two singular 1-simplices.

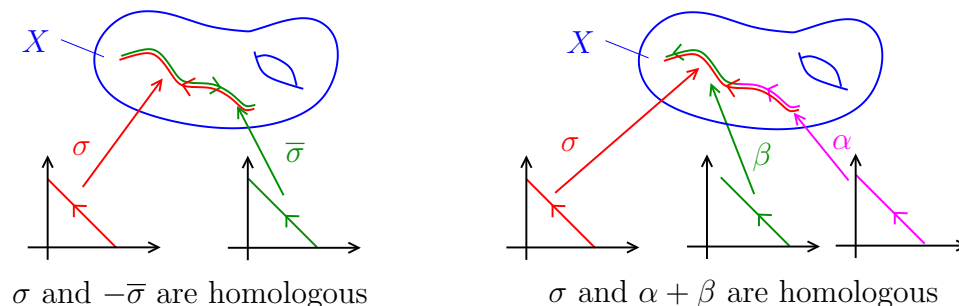


FIGURE 254. Illustration of Lemma 24.3.

Examples.

- (1) We consider Figure 255. We see three singular chains on the torus, and it is straightforward to show that all three singular chains are cycles. Now the question arises,

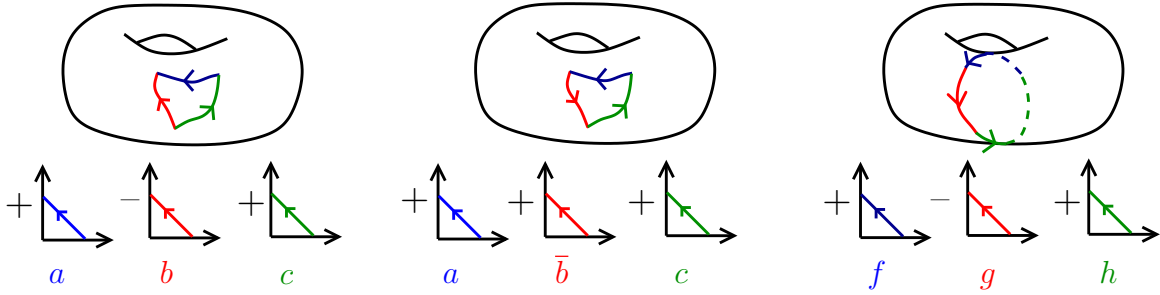


FIGURE 255. Three examples of 1-cycles on the torus.

which of these cycles are in fact null-homologous. In Figure 1 we see that the first cycle $a - b + c$ in Figure 255 is the boundary of a singular 2-simplex D . One can easily

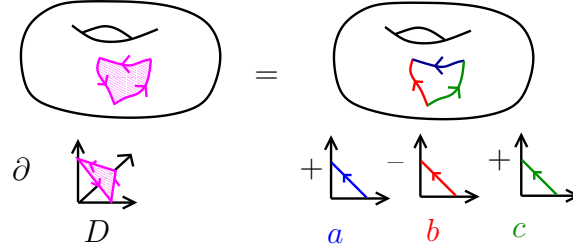


FIGURE 256. The first cycle in Figure 255 is the boundary of a 2-dimensional singular chain.

see that the second cycle, $a + \bar{b} + c$ in Figure 255 is *not* the boundary of a singular 2-simplex. On the other hand by Lemma 24.3 we know that $b + \bar{b}$ is a null-homologous cycle, i.e. there exists an $E \in C_2(X)$ such that $b + \bar{b} = \partial E \in C_1(X)$. Thus it follows that

$$a + \bar{b} + c = a + \partial E - b + c = a - b + c + \partial E = \partial D + \partial E = \partial(D + E),$$

i.e. $a + \bar{b} + c$ is null-homologous.

It is not clear whether the third cycle in Figure 255 is the boundary of a 2-dimensional singular chain. We will see later that this not the case. But it will take us quite some time to develop the techniques which will allow us to prove this statement.

- (2) In Figure 257 we show the torus $T = [0, 1] \times [0, 1] / \sim$ and two singular simplices $\alpha: \Delta^2 \rightarrow T$ and $\beta: \Delta^2 \rightarrow T$ that are given by affine linear maps.⁴⁶⁸ It is straightforward to verify that $\alpha - \beta$ is a cycle, hence it represents an element in $H_2(T)$. We will later see whether or not this element is zero in $H_2(T)$.

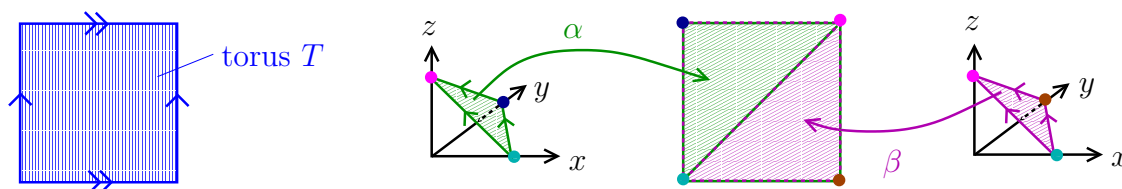


FIGURE 257.

- (3) In exercise sheet 3 we will try to generalize the approach taken for the torus to other 2-dimensional manifolds, namely the sphere, the Klein bottle and the projective space $\mathbb{R}P^2$.

24.3. First calculations of homology groups. Given a topological space the corresponding homology groups are defined as the quotient of two abelian groups which in general will be infinitely generated. Somewhat surprisingly we will see in the coming chapters that the homology groups are nonetheless very frequently finitely generated abelian groups.

We start out our computations of homology groups with the 0-th homology of a path-connected topological space. To state the calculation we first need to formulate and prove the following lemma.

Lemma 24.4. *Given a topological space X the map*

$$\begin{aligned} \epsilon: H_0(X) &\rightarrow \mathbb{Z} \\ \left[\sum_{i=1}^k a_i \sigma_i \right] &\mapsto \sum_{i=1}^k a_i \end{aligned}$$

is well-defined. In the following we refer to this map as the augmentation map.

Proof. Let X be a topological space. We have

$$\begin{aligned} H_0(X) &= \ker\{\partial_0: C_0(X) \rightarrow C_{-1}(X)\} / \operatorname{im}\{\partial_n: C_1(X) \rightarrow C_0(X)\} \\ &= C_0(X) / \operatorname{im}\{\partial_n: C_1(X) \rightarrow C_0(X)\}. \\ &\quad \uparrow \\ &\text{since } C_{-1}(X) = 0 \end{aligned}$$

⁴⁶⁸More precisely, α is the unique affine linear map with $\alpha(1, 0, 0) = [(0, 0)]$, $\alpha(0, 1, 0) = [(1, 0)]$ and $\alpha(0, 0, 1) = [(1, 1)]$ whereas β is the unique affine linear map with $\beta(1, 0, 0) = [(0, 0)]$, $\beta(0, 1, 0) = [(0, 1)]$ and $\beta(0, 0, 1) = [(1, 1)]$.

This observation shows that we have to show that $\epsilon(\partial c) = 0$ for every $c \in C_1(X)$. Thus let $c = \sum_{i=1}^n b_i \sigma_i \in C_1(X)$. Then

$$\epsilon(\partial c) = \epsilon\left(\sum_{i=1}^n b_i \partial \sigma_i\right) \underset{\substack{\text{for every singular 1-simplex } \sigma \text{ we} \\ \text{have } \partial \sigma = \sigma(0, 1) - \sigma(1, 0), \text{ see page 402}}}{=} \epsilon\left(\sum_{i=1}^n b_i \cdot \sigma_i(0, 1) - b_i \cdot \sigma_i(1, 0)\right) \underset{\text{by definition of } \epsilon}{=} \sum_{i=1}^n b_i - \sum_{i=1}^n b_i = 0.$$

□

Now we can formulate the following proposition.

Proposition 24.5. *Let X be a path-connected non-empty topological space and let P be a point in X . Then the augmentation map*

$$\epsilon: H_0(X) \rightarrow \mathbb{Z}$$

is an isomorphism where the inverse is given by the map

$$\begin{aligned} \iota: \mathbb{Z} &\rightarrow H_0(X) \\ n &\mapsto n \cdot [P]. \end{aligned}$$

In particular we have $H_0(X) \cong \mathbb{Z}$.

Proof. Let X be a non-empty path-connected topological space. Let P be a point in X . It is clear that $\epsilon \circ \iota = \text{id}_{\mathbb{Z}}$, in particular the homomorphism ϵ is surjective. It remains to show that ϵ is injective.

Thus let $c \in C_0(X)$ with $\epsilon(c) = 0$. By definition of $C_0(X)$ we can find points $Q_1, \dots, Q_k$ in X and $a_1, \dots, a_k \in \mathbb{Z}$ such that $c = \sum_{i=1}^k a_i \cdot Q_i$. We choose a point P in X . Since X is path-connected we can find, for $i = 1, \dots, k$, a path from P to Q_i . Put differently, for each $i = 1, \dots, k$ we can find a singular 1-simplex σ_i with $\partial \sigma_i = Q_i - P$. Then we have

$$\partial \left(\sum_{i=1}^k a_i \sigma_i \right) = \sum_{i=1}^k a_i \partial \sigma_i = \sum_{i=1}^k a_i (Q_i - P) = \sum_{i=1}^k a_i Q_i - \underbrace{\left(\sum_{i=1}^k a_i \right)}_{=\epsilon(c)=0} \cdot P = \sum_{i=1}^k a_i Q_i.$$

This implies that $[c] = 0 \in H_0(X)$. Thus we have shown that ϵ is injective. □

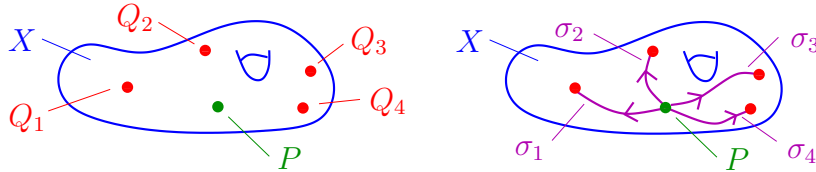


FIGURE 258. Illustration of the proof of Proposition 24.5.

In exercise sheet 4 we will prove the following lemma.

Lemma 24.6. *Let X be a topological space which consists of a single point. Then the following holds*

$$H_n(X) \cong \begin{cases} \mathbb{Z}, & \text{if } n = 0, \\ 0, & \text{if } n \neq 0. \end{cases}$$

The calculation of $H_0(X)$ in Proposition 24.5 and the calculation of the homology of a point in Lemma 24.6 are the only calculations of homology groups of path-connected topological spaces that one can do “by hand”. For all other topological spaces we will need to work significantly harder.

24.4. Algebraic chain complexes. Before we continue with the discussion of homology groups it is convenient to introduce several algebraic definitions. The following definitions are just the obvious algebraic abstractions of the definitions that we had already introduced.

Definition.

- (1) An (algebraic) chain complex (C_*, ∂_*) is a sequence

$$\cdots \rightarrow C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots \rightarrow C_1 \xrightarrow{\partial_1} C_0 \rightarrow 0$$

of maps between abelian groups such that for every i we have $\partial_{i-1} \circ \partial_i = 0$. As before we introduce the convention that $C_n = 0$ for $n \in \mathbb{Z}_{<0}$.

- (2) We define the n -th homology group of the chain complex precisely the same way as we defined the homology of a topological space, namely we set

$$H_n(C) := \ker\{\partial_n: C_n \rightarrow C_{n-1}\} / \operatorname{im}\{\partial_{n+1}: C_{n+1} \rightarrow C_n\}.$$

- (3) As before we refer to elements in the kernel of $\partial_n: C_n \rightarrow C_{n-1}$ as *cycles*. Furthermore we say that a cycle $z \in C_n$ is *null-homologous* if $[z] = 0 \in H_n(C)$.

Now we introduce a new notion, namely the notion of a chain map between chain complexes.

Definition. A chain map $f: C_* \rightarrow D_*$ between two chain complexes⁴⁶⁹ consists of a family $\{f_n: C_n \rightarrow D_n\}_{n \in \mathbb{N}_0}$ of maps such that for each n the following equality holds:⁴⁷⁰

$$f_{n-1} \circ \partial_n = \partial_n \circ f_n.$$

Put differently the maps f_n have the property that the following diagram commutes:

$$\begin{array}{ccccccc} \cdots & \xrightarrow{\partial} & C_{n+1} & \xrightarrow{\partial_{n+1}} & C_n & \xrightarrow{\partial_n} & C_{n-1} \xrightarrow{\partial_{n-1}} \cdots \\ & & \downarrow f_{n+1} & & \downarrow f_n & & \downarrow f_{n-1} \\ \cdots & \xrightarrow{\partial} & D_{n+1} & \xrightarrow{\partial_{n+1}} & D_n & \xrightarrow{\partial_n} & D_{n-1} \xrightarrow{\partial_{n-1}} \cdots \end{array}$$

⁴⁶⁹As usual we suppress bits from the notation which are understood from the context. In this instance we suppress the notation for the boundary maps in the chain complexes. Put differently, instead of saying “let (C_*, ∂_*) be a chain complex” we say simply “let C_* be a chain complex”.

⁴⁷⁰More precisely, for any $c \in C$ we have $f_{n-1}(\partial_n(c)) = \partial_n(f_n(c))$ where on the left-hand side we consider the boundary map $C_n \rightarrow C_{n-1}$, and on the right-hand side we consider the boundary map $D_n \rightarrow D_{n-1}$ which we also denote by ∂_n . Usually it is obvious from the context which boundary map we consider.

Remark. On many occasions we will use the elementary observation that the composition of two chain maps $f: C_* \rightarrow D_*$ and $g: D_* \rightarrow E_*$ defines a chain map $g \circ f: C_* \rightarrow E_*$.

Now we can prove the following lemma:

Lemma 24.7. *Let $f: C_* \rightarrow D_*$ be a chain map between chain complexes C_* and D_* . Then the map*

$$\begin{array}{ccc} f_*: H_n(C_*) & \rightarrow & H_n(D_*) \\ [c] & \mapsto & [f_n(c)] \end{array}$$

is well-defined.

In the following we refer to the map $f_*: H_n(C) \rightarrow H_n(D)$ as the *induced map on homology*.

Proof. Let $c \in C_n$ be a cycle. First we have to show that $f_n(c)$ is again a cycle. Indeed we have

$$\begin{array}{ccccc} \partial_n(f_n(c)) & = & f_{n-1}(\partial_n c) & = & f_n(0) = 0. \\ & \uparrow & & \uparrow & \\ & \text{definition of a chain map} & & \text{since } c \text{ is a cycle} & \end{array}$$

Now let $c, d \in C_n$ be two cycles which represent the same element in $H_n(C)$. We have to show that $[f_n(c)] = [f_n(d)]$. Indeed we have

$$\begin{array}{ccccccc} [f_n(c)] - [f_n(d)] & = & [f_n(c - d)] & = & [f_n(\partial_{n+1}(e))] & = & [\partial_{n+1}(f_{n+1}(e))] = 0 \in H_n(D). \\ & & \uparrow & & \uparrow & & \\ & & \text{since } c, d \text{ are homologous there} & & \text{definition of a chain map} & & \\ & & \text{exists an } e \in C_{n+1} \text{ with } \partial_{n+1}e = c - d & & & & \square \end{array}$$

We can state the above results more formally. More precisely, we consider the category $\mathcal{C}$ of chain complexes which is given by the objects

$$\text{Ob}(\mathcal{C}) := \text{all chain complexes},$$

and for $C_*, D_* \in \text{Ob}(\mathcal{C})$ the morphisms are given by

$$\text{Mor}(C_*, D_*) := \text{all chain maps from } C_* \text{ to } D_*.$$

The following lemma follows easily from Lemma 24.7 and from the definitions:

Lemma 24.8. *For each $n \in \mathbb{N}_0$ the maps*

$$C_* \mapsto H_n(C_*)$$

and

$$(f: C_* \rightarrow D_*) \mapsto (f_*: H_n(C_*) \rightarrow H_n(D_*))$$

define a functor from the category of chain complexes to the category of abelian groups.

24.5. The functoriality of homology groups. Let $f: X \rightarrow Y$ be a map between topological spaces. We had already seen that for each n the map f induces a map

$$\begin{aligned} f_*: C_n(X) &\rightarrow C_n(Y) \\ \sum_{i=1}^k a_i \cdot \sigma_i &\mapsto \sum_{i=1}^k a_i \cdot f \circ \sigma_i. \end{aligned}$$

We can now formulate the following lemma:

Lemma 24.9. *The maps*

$$\text{topological space } X \mapsto C_*(X)$$

and

$$(f: X \rightarrow Y) \mapsto (f_*: C_*(X) \rightarrow C_*(Y))$$

define a functor from the category of topological spaces to the category of chain complexes.

Proof. Let $f: X \rightarrow Y$ be a map between topological spaces and let $\sigma: \Delta^n \rightarrow X$ be a singular n -simplex. We have

$$f_*(\partial\sigma) = f_*\left(\sum_{j=0}^n (-1)^j \cdot \sigma \circ i_j^n\right) = \sum_{j=0}^n (-1)^j \cdot f \circ (\sigma \circ i_j^n) = \sum_{j=0}^n (-1)^j \cdot \underbrace{(f \circ \sigma) \circ i_j^n}_{=f_*\sigma} = \partial(f_*\sigma).$$

Thus we have proved that $f_*: C_*(X) \rightarrow C_*(Y)$ is a chain map. $\square$

In particular, using Lemma 24.7 we immediately obtain the following lemma.

Lemma 24.10. *Let $f: X \rightarrow Y$ be a map between topological spaces. Then the map*

$$\begin{aligned} f_*: H_n(X) &\rightarrow H_n(Y) \\ \left[\sum_{i=1}^k a_i \cdot \sigma_i \right] &\mapsto \left[\sum_{i=1}^k a_i \cdot (f \circ \sigma_i) \right] \end{aligned}$$

is well-defined.

Example. Let $f: X \rightarrow Y$ be a map between two path-connected topological spaces. It follows easily from Proposition 24.5 and the definitions that the map $f_*: H_0(X) \rightarrow H_0(Y)$ is an isomorphism.

The combination of Lemmas 24.9 and 24.8 now gives us the following lemma.

Lemma 24.11. *For each $n \in \mathbb{N}_0$ the map*

$$X \mapsto H_n(X)$$

together with the map

$$(f: X \rightarrow Y) \mapsto \left(\begin{array}{ccc} f_*: H_n(X) & \rightarrow & H_n(Y) \\ \left[\sum_{i=1}^k a_i \cdot \sigma_i \right] & \mapsto & \left[\sum_{i=1}^k a_i \cdot (f \circ \sigma_i) \right] \end{array} \right)$$

defines a functor from the category of topological spaces to the category of abelian groups.

24.6. Direct products and direct sums (*). In this short section we recall the definition of the direct product and the direct sum of a family of abelian groups.

Let $G_i, i \in I$ be a family of abelian groups. We define the *direct product* as

$$\prod_{i \in I} G_i := \left\{ \text{all maps } g: I \rightarrow \bigsqcup_{i \in I} G_i \text{ such that for all } i \in I \text{ we have } g(i) \in G_i \right\}.$$

Given such a map we usually write $g_i := g(i)$ for $i \in I$ and we denote such a map by $\{g_i\}_{i \in I}$. The direct product is again a group with respect to the group structure

$$\begin{aligned} \prod_{i \in I} G_i \times \prod_{i \in I} G_i &\rightarrow \prod_{i \in I} G_i \\ ((a_i)_{i \in I}, (b_i)_{i \in I}) &\mapsto (a_i + b_i)_{i \in I}. \end{aligned}$$

Furthermore we consider the *direct sum*

$$\bigoplus_{i \in I} G_i := \left\{ (g_i)_{i \in I} \in \prod_{i \in I} G_i \mid \text{there exists only finitely many } i \in I \text{ with } g_i \neq e \right\}.$$

The same way one can also define the direct product respectively the direct sum of rings, algebras, vector spaces, modules and so on.

Example. Let $I = \mathbb{Z}$ and for each $i \in I = \mathbb{Z}$ we pick $G_i = \mathbb{R}$. Then

$$\prod_{i \in \mathbb{N}} \mathbb{R} = \text{all real-valued sequences } i \mapsto a_i$$

and

$$\bigoplus_{i \in \mathbb{N}} \mathbb{R} = \text{all real-valued sequences with only finitely many non-zero terms.}$$

In general the direct product is “significantly larger” than the direct sum. For example the direct product $\prod_{i \in \mathbb{N}} \mathbb{Z}$ is uncountable while the direct sum $\bigoplus_{i \in \mathbb{N}} \mathbb{Z}$ is countable.⁴⁷¹

For some element $g_j \in G_j$, following the discussion on page 137, we sometimes denote by g_j also the element in $\bigoplus_{i \in I} G_i$ that is given by $i \mapsto e$ if $i \neq j$ and $j \mapsto g_j$. In particular we can view each G_j as a subgroup of $\bigoplus_{i \in I} G_i$. Furthermore for $i_1, \dots, i_l \in I$ and $g_{i_j} \in G_{i_j}$ we can consider the corresponding element

$$g_{i_1} + \dots + g_{i_l} \in \bigoplus_{i \in I} G_i.$$

It follows immediately from the definitions that any element of $\bigoplus_{i \in I} G_i$ is of that form.

Finally let $G_i, i \in I$ be a family of abelian groups, let H be an abelian group and let $\varphi_i: G_i \rightarrow H, i \in I$ be a family of homomorphisms. We refer to

$$\begin{aligned} \bigoplus_{i \in I} \varphi_i &: \bigoplus_{i \in I} G_i \rightarrow H \\ g_{i_1} + \dots + g_{i_l} &\mapsto \varphi_{i_1}(g_{i_1}) + \dots + \varphi_{i_l}(g_{i_l}) \end{aligned}$$

⁴⁷¹Why is that the case?

as the direct sum of the homomorphisms φ_i , $i \in I$.

24.7. The homology groups of a direct sum. Now we return to the study of chain complexes. Let C_* and D_* be two chain complexes. The same way as we introduced the direct sum of abelian groups we can also define the direct sum of chain complexes. The following lemma follows easily from the definitions.

Lemma 24.12. *Let $\{C_a\}_{a \in A}$ be a family of chain complexes. Then the inclusion maps $C_a \rightarrow \bigoplus_{a \in A} C_a$ induce an isomorphism*

$$\bigoplus_{a \in A} H_n(C_a) \xrightarrow{\cong} H_n\left(\bigoplus_{a \in A} C_a\right).$$

Now we can formulate and prove the following lemma.

Lemma 24.13. *Let X be a topological space with path-components X_a , $a \in A$. Given $a \in A$ we denote by $i_a: X_a \rightarrow X$ the inclusion map. Then the map*

$$\bigoplus_{a \in A} i_a: \bigoplus_{a \in A} C_*(X_a) \rightarrow C_*(X)$$

is an isomorphism of chain complexes. In particular for each $n \in \mathbb{N}_0$ the map

$$\bigoplus_{a \in A} i_{a*}: \bigoplus_{a \in A} H_n(X_a) \rightarrow H_n(X)$$

is an isomorphism.

The lemma says in particular that the homology groups of a topological space X are the direct sum of the homology groups of the path-components of X . Put differently, this shows that in most cases it suffices to consider the homology groups of path-connected topological spaces.

Example. One of our goals is to calculate the homology groups of the spheres. We can now calculate the homology groups of the 0-dimensional sphere $S^0 = \{\pm 1\}$. More precisely, by Lemmas 24.6 and 24.13 we have

$$H_n(S^0) \cong \begin{cases} \mathbb{Z}^2, & \text{if } n = 0, \\ 0, & \text{if } n \neq 0. \end{cases}$$

We will later reduce the computation of the homology groups of all other spheres to this calculation.

Proof. Since the standard simplex Δ^n is path-connected, the image of a map $\sigma: \Delta^n \rightarrow X$ lies in a path-component X_a . This shows that

$$\bigsqcup_{a \in A} \text{singular } n\text{-simplices in } X_a = \text{singular } n\text{-simplices in } X.$$

It follows that the map

$$\begin{aligned} \bigoplus_{a \in A} i_a: \bigoplus_{a \in A} C_n(X_a) &\rightarrow C_n(X) \\ \left(\sum_{i=1}^k c_i(\sigma: \Delta^n \rightarrow X_{a_i}) \right) &\mapsto \left(\sum_{i=1}^k c_i(\sigma: \Delta^n \rightarrow X_{a_i}) \right) \end{aligned}$$

is an isomorphism of free abelian groups. From the definitions it follows immediately that this is a chain map. The second statement of the lemma now follows from Lemma 24.12. $\square$

The following corollary is an immediate consequence of Proposition 24.5 and Lemma 24.12.

Corollary 24.14. *Let X be a topological space. As on page 33 we denote by $\pi_0(X)$ the set of path-components of X . Then*

$$H_0(X) \cong \mathbb{Z}^{(\pi_0(X))}.$$

In particular, if X consists of n path-components with $n \in \mathbb{N}$, then

$$H_0(X) \cong \mathbb{Z}^n.$$

25. HOMOLOGY AND HOMOTOPIES

It was not overly difficult to introduce the homology groups of a topological space. We also proved several straightforward properties of homology groups. But so far we are unable to calculate any interesting homology group. In this section we will show in particular that homotopy equivalent topological spaces have isomorphic homology groups. That does not yet lead to the calculation of non-trivial homology groups, but this result will play an important role in the calculation of the homology groups of spheres later on.

25.1. Chain homotopies. In this section we introduce a criterion that ensures that two chain maps $f, g: C_* \rightarrow D_*$ induce the same maps on the corresponding homology groups.

Definition. Let $f, g: C_* \rightarrow D_*$ be chain maps between chain complexes C_* and D_* . A *chain homotopy* $P = \{P_n\}_{n \in \mathbb{N}_0}$ between f and g consists of family $\{P_n: C_n \rightarrow D_{n+1}\}_{n \in \mathbb{N}_0}$ such that for each $n \in \mathbb{N}_0$ we have

$$\partial_{n+1} \circ P_n + P_{n-1} \circ \partial_n = f_n - g_n.$$

If there exists a chain homotopy between chain maps f and g , then we say that f and g are *chain homotopic* and we write $f \simeq g$.

For the definition of a chain homotopy and the subsequent discussion of chain homotopies it is sometimes helpful to keep in mind the following diagram that combines all objects involved in the definition:

$$\begin{array}{ccccccc}
 \xrightarrow{\partial_{n+2}} & C_{n+1} & \xrightarrow{\partial_{n+1}} & C_n & \xrightarrow{\partial_n} & C_{n-1} & \xrightarrow{\partial_{n-1}} \\
 & \downarrow f_*, g_* & \swarrow P_n & \downarrow f_*, g_* & \swarrow P_{n-1} & \downarrow f_*, g_* & \\
 \xrightarrow{\partial_{n+2}} & D_{n+1} & \xrightarrow{\partial_{n+1}} & D_n & \xrightarrow{\partial_n} & D_{n-1} & \xrightarrow{\partial_{n-1}}
 \end{array}$$

The following basic lemma will be proved in exercise sheet 4.

Lemma 25.1. *Let C_* and D_* be two chain complexes.*

- (1) *Chain homotopy defines an equivalence relation on the set of all chain maps from C_* to D_* . In particular, if $f, g, h: C_* \rightarrow D_*$ are chain maps with $f \simeq g$ and $g \simeq h$, then $f \simeq h$.*
- (2) *Let $f, g: C_* \rightarrow D_*$ be chain maps, and let $e: B_* \rightarrow C_*$ and $h: D_* \rightarrow E_*$ be two other chain maps. Then*

$$f \simeq g \implies f \circ e \simeq g \circ e \quad \text{and} \quad h \circ f \simeq h \circ g.$$

Now we can formulate the following lemma.

Lemma 25.2. *Let $f, g: C_* \rightarrow D_*$ be two chain maps between chain complexes. Then*

$$f \simeq g \implies f_* = g_*: H_n(C) \rightarrow H_n(D) \text{ for all } n \in \mathbb{N}_0.$$

Proof. Let $P = \{P_n\}_{n \in \mathbb{N}_0}$ be a chain homotopy between f and g . Then for any cycle $z \in C_n$ we get the equalities

$$f(z) - g(z) = (f - g)(z) = (\partial_{n+1}P_n + P_{n-1}\partial_n)(z) = (\partial_{n+1}P_n)(z) + P_{n-1}\underbrace{\partial_n z}_{=0} = \partial P_n z,$$

i.e. $f(z)$ and $g(z)$ are homologous. $\square$

Definition. Let C_* and D_* be two chain complexes.

- (1) A chain map $f: C_* \rightarrow D_*$ is called a *homotopy equivalence between C_* and D_** , if it admits a chain homotopy inverse, i.e. a chain map $g: D_* \rightarrow C_*$ with $g \circ f \simeq \text{id}_{C_*}$ and $f \circ g \simeq \text{id}_{D_*}$.
- (2) If there exists a homotopy equivalence between C_* and D_* , then we say that the chain complexes C_* and D_* are *homotopy equivalent* and we write $C_* \simeq D_*$.

Remark. It is again straightforward to show that “homotopy equivalence” of chain complexes is indeed an equivalence relation.

For later we record the following straightforward corollary to Lemma 25.2.

Corollary 25.3. *If $f: C_* \rightarrow D_*$ is a homotopy equivalence between chain complexes, then the induced maps $f_*: H_n(C) \rightarrow H_n(D)$ are isomorphisms.*

25.2. Homology and homotopic maps. The name *chain homotopy* introduced in the previous section suggests that this concept is related to the notion of a homotopy between two topological spaces. As we will see shortly, this is indeed the case. For the reader's convenience we recall that if $f, g: X \rightarrow Y$ are two maps between topological spaces, then a *homotopy between the maps f and g* is a map

$$F: X \times [0, 1] \rightarrow Y,$$

such that

$$F(x, 0) = f(x) \text{ and } F(x, 1) = g(x) \text{ for all } x \in X.$$

If there exists a homotopy between f and g , then we say that f and g are *homotopic* and we write $f \simeq g$.

Proposition 25.4. *Let X and Y be two topological spaces and let $f, g: X \rightarrow Y$ be two maps. If f and g are homotopic, then the following hold:*

- (1) *the induced maps f_* and g_* from $C_*(X)$ to $C_*(Y)$ are chain homotopic,*
- (2) *for each $n \in \mathbb{N}_0$ we have $f_* = g_*: H_n(X) \rightarrow H_n(Y)$.*

Remark. The proposition can be stated more succinctly as follows:

$$f \simeq g: X \rightarrow Y \implies f_* \simeq g_*: C_*(X) \rightarrow C_*(Y) \implies f_* = g_*: H_n(X) \rightarrow H_n(Y).$$

Proof. Let $f, g: X \rightarrow Y$ be two maps between topological spaces and let $F: X \times [0, 1] \rightarrow Y$ be a homotopy between the maps f and g . Our goal is to find a chain homotopy between f_* and g_* . The second statement about the induced maps on homology then follows from Lemma 25.2.

Thus let $\sigma: \Delta^n \rightarrow X$ be a singular n -simplex. We need to find $P(\sigma) \in C_{n+1}(Y)$ and $P(\partial\sigma) \in C_n(Y)$ such that

$$\partial P(\sigma) = -P(\partial\sigma) + f_*(\sigma) - g_*(\sigma).$$

Note that $F \circ (\sigma \times \text{id}_{[0,1]})$ defines a map $\Delta^n \times [0, 1] \rightarrow Y$. We can now rewrite the above desired equality as follows:

$$\partial P(\sigma) = -P(\partial\sigma) + \underbrace{\text{map } F \circ (\sigma \times \text{id}_{[0,1]}) \text{ restricted to } \Delta^n \times \{0\}}_{=\Delta^n} - \underbrace{\text{map } F \circ (\sigma \times \text{id}_{[0,1]}) \text{ restricted to } \Delta^n \times \{1\}}_{=\Delta^n}.$$

To get an idea we consider Figure 259. We make the following trivial observation: we have

$$\partial(\Delta^n \times [0, 1]) = \partial\Delta^n \times [0, 1] \cup \Delta^n \times \{0\} \cup \Delta^n \times \{1\}.$$

Comparing the desired equality of maps and the equality of sets suggests that each map $P_k: C_k(X) \rightarrow C_{k+1}(Y)$ should take the role of replacing a singular k -simplex $\mu: \Delta^k \rightarrow X$ by the map $F \circ (\mu \times \text{id}_{[0,1]}): \Delta^k \times [0, 1] \rightarrow Y$. This is a good idea, but the problem is that $\Delta^k \times [0, 1]$ is not a $(k+1)$ -simplex. The idea now is to write $\Delta^k \times [0, 1]$ as the union of $(k+1)$ -simplices. In Figure 260 we see for example that we can view $\Delta^1 \times [0, 1]$ as the union of the triangles with the vertices v_0, w_0, w_1 respectively v_0, v_1, w_1 .

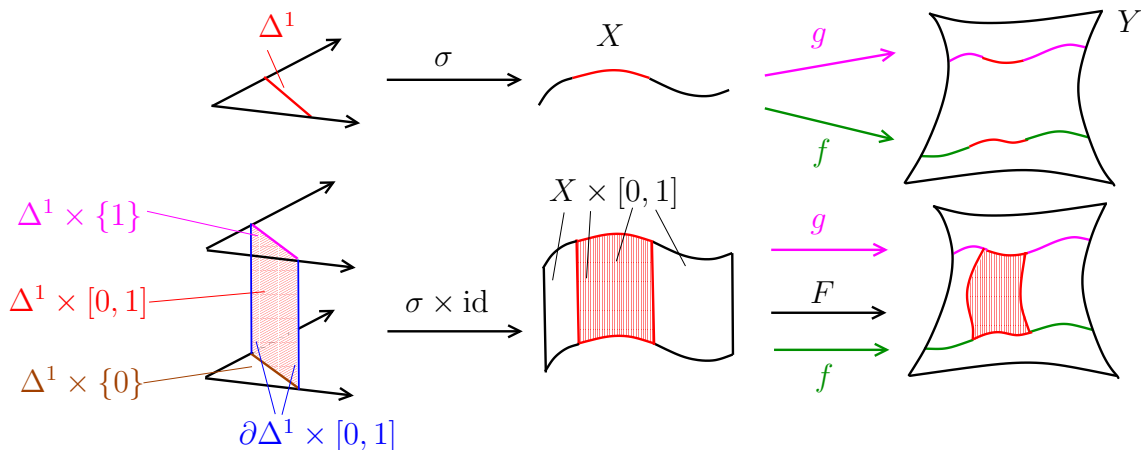


FIGURE 259.

To generalize this observation we have to introduce some more notation. For $i = 0, \dots, n$ we write

$$v_i := (0, \dots, 1, 0, \dots, 0) \times \{0\} \in \mathbb{R}^{n+1} \times \{0\},$$

and

$$w_i := (0, \dots, 1, 0, \dots, 0) \times \{1\} \in \mathbb{R}^{n+1} \times \{1\}.$$

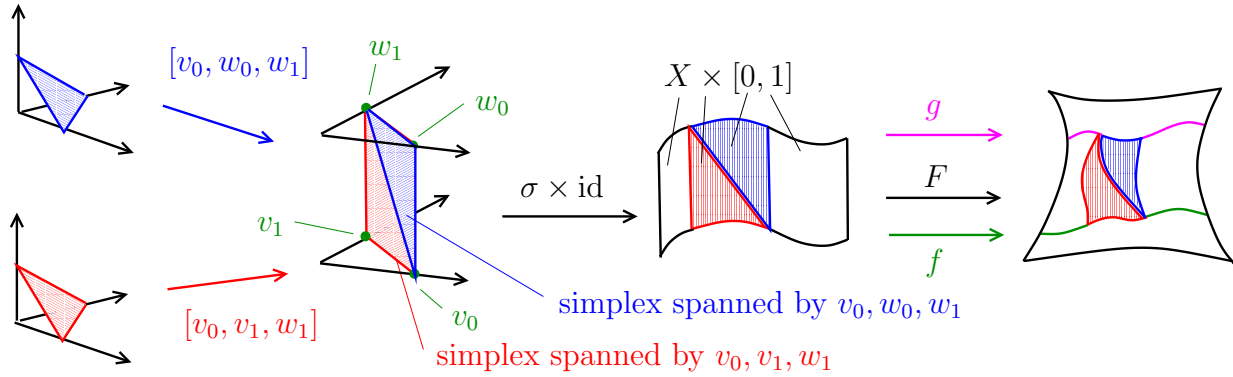


FIGURE 260.

The points $v_0, \dots, v_n$ thus span the simplex $\Delta^n \times 0$ and the points $v_0, \dots, v_n$ span the simplex $\Delta^n \times 1$. Furthermore for each i the points $v_0, \dots, v_i, w_i, \dots, w_n$ span a $(n+1)$ -simplex that is a subset of $\Delta^n \times [0, 1]$.⁴⁷²

For $a_1, \dots, a_{k+1} \in \Delta^n \times [0, 1]$ we consider, as on page 400, the map

$$\begin{aligned} [a_0, \dots, a_k] : \Delta^k &\rightarrow \Delta^n \times [0, 1] \\ (\lambda_0, \dots, \lambda_k) &\mapsto \sum_{j=0}^k \lambda_j a_j. \end{aligned}$$

The map $[a_0, \dots, a_k]$ thus sends the standard k -simplex Δ^k to the simplex spanned by $a_0, \dots, a_k$ in $\Delta^n \times [0, 1]$.

We define

$$P_n : C_n(X) \rightarrow C_{n+1}(Y)$$

to be the map that is given by

$$(\sigma : \Delta^n \rightarrow X) \mapsto \sum_{j=0}^n (-1)^j \cdot F \circ (\sigma \times \text{id}_{[0,1]}) \circ \underbrace{[v_0, \dots, v_j, w_j, \dots, w_n]}_{\Delta^{n+1} \rightarrow \Delta^n \times [0,1]}.$$

We claim that the maps form indeed the desired chain homotopy between the chain maps f_* and g_* . More precisely, we want to prove the following claim.

Claim. For every $n \in \mathbb{N}_0$ we have

$$\partial_{n+1} P_n + P_{n-1} \partial_n = f_* - g_*.$$

⁴⁷²It is a straightforward exercise to show that the union of these $(n+1)$ -dimensional simplices is precisely $\Delta^n \times [0, 1]$. We will not make use of this fact.

Proof of the claim ()*.⁴⁷³ For a singular n -simplex $\sigma: \Delta^n \rightarrow X$ we have

$$\begin{aligned}\partial P_n(\sigma) &= \partial \left(\sum_{j=0}^n (-1)^j \cdot F \circ (\sigma \times \text{id}) \circ [v_0, \dots, v_j, w_j, \dots, w_n] \right) \\ &= \sum_{k=0}^{n+1} (-1)^k \sum_{j=0}^n (-1)^j \cdot F \circ (\sigma \times \text{id}) \circ [v_0, \dots, v_j, w_j, \dots, w_n] \circ i_k^{n+1}.\end{aligned}$$

It is straightforward to see that⁴⁷⁴

$$[v_0, \dots, v_j, w_j, \dots, w_n] \circ i_k^{n+1} = \begin{cases} [v_0, \dots, \widehat{v_k}, \dots, v_j, w_j, \dots, w_n], & \text{if } k \leq j, \\ [v_0, \dots, v_j, w_j, \dots, \widehat{w_{k-1}}, \dots, w_n], & \text{if } k > j. \end{cases}$$

Thus it follows that⁴⁷⁵

$$\begin{aligned}\partial P_n(\sigma) &= \sum_{0 \leq k \leq j \leq n} (-1)^k (-1)^j \cdot F \circ (\sigma \times \text{id}) \circ [v_0, \dots, \widehat{v_k}, \dots, v_j, w_j, \dots, w_n] \\ &\quad + \sum_{n+1 \geq k > j \geq 0} (-1)^k (-1)^j \cdot F \circ (\sigma \times \text{id}) \circ [v_0, \dots, v_j, w_j, \dots, \widehat{w_{k-1}}, \dots, w_n].\end{aligned}$$

We now split the sums into the cases $k = j$ and $k < j$, and into the cases $k = j + 1$ and $k > j + 1$. We then see that

$$\begin{aligned}\partial P_n(\sigma) &= \sum_{\substack{k=0 \\ n+1}}^n (-1)^{2k} \cdot F \circ (\sigma \times \text{id}) \circ [v_0, \dots, v_{k-1}, w_k, \dots, w_n] \\ &\quad + \sum_{\substack{k=1 \\ n+1}}^n (-1)^{2k+1} \cdot F \circ (\sigma \times \text{id}) \circ [v_0, \dots, v_{k-1}, w_k, \dots, w_n] \\ &\quad + \sum_{0 \leq k < j \leq n} (-1)^k (-1)^j \cdot F \circ (\sigma \times \text{id}) \circ [v_0, \dots, \widehat{v_k}, \dots, v_j, w_j, \dots, w_n] \\ &\quad + \sum_{n+1 \geq k > j+1 \geq 1} (-1)^k (-1)^j \cdot F \circ (\sigma \times \text{id}) \circ [v_0, \dots, v_j, w_j, \dots, \widehat{w_{k-1}}, \dots, w_n].\end{aligned}$$

⁴⁷³We did not do the proof of the claim in the lectures. The proof does not require any new ideas. The difficulty in proving Proposition 25.4 lies in coming up with the correct definition of the maps P_n . The verification that these maps have the desired property, i.e. the proof of the claim, is a relatively straightforward argument.

⁴⁷⁴The first case, $k \leq j$ is slightly easier: one has to figure out, where the h -th vertex e_h of the standard n -simplex gets sent to. If $h < k$, then i_k^{n+1} sends the h -th vertex again to the h -th vertex, and $[v_0, \dots, v_j, w_j, \dots, w_n]$ sends this vertex to v_h . If $h \geq k$, then i_k^{n+1} sends the h -th vertex again to the $(h+1)$ -th vertex, and $[v_0, \dots, v_j, w_j, \dots, w_n]$ sends this vertex to v_{h+1} . If $k \leq j$ this corresponds just to the map $[v_0, \dots, \widehat{v_k}, \dots, v_j, w_j, \dots, w_n]$.

The second case, $k > j$ is proved precisely the same way.

⁴⁷⁵Hereby we use the usual notation that $\widehat{v_k}$ that we do not plug in $\widehat{v_k}$, the same applies to $\widehat{w_{k-1}}$.

Now we consider the first two sums. The summands cancel each other, except for the summands for $k = 0$ and $k = n + 1$. Thus we obtain that

$$\begin{aligned}
 (25.1) \quad \partial P_n(\sigma) &= F \circ (\sigma \times \text{id}) \circ [w_0, \dots, w_n] \\
 &\quad - F \circ (\sigma \times \text{id}) \circ [v_0, \dots, v_n] \\
 &\quad + \sum_{0 \leq k < j \leq n+1} (-1)^k (-1)^j \cdot F \circ (\sigma \times \text{id}) \circ [v_0, \dots, \widehat{v}_k, \dots, v_i, w_i, \dots, w_n] \\
 &\quad + \sum_{n+1 \geq k > j+1 \geq 1} (-1)^k (-1)^j F \circ (\sigma \times \text{id}) \circ [v_0, \dots, v_i, w_i, \dots, \widehat{w}_k, \dots, w_n].
 \end{aligned}$$

The first summand is just $g \circ \sigma = g_*(\sigma)$ and the second is $-f \circ \sigma = -f_*(\sigma)$. It remains to show that the remaining summand equals $-P_{n-1}\partial(\sigma)$.

This we can show through an explicit calculation. More precisely, we have

$$\begin{aligned}
 P_{n-1}\partial(\sigma) &= P_{n-1} \left(\sum_{s=0}^{n-1} (-1)^s \sigma \circ i_s^n \right) \\
 &= \sum_{r=0}^n (-1)^r \sum_{s=0}^n (-1)^s \cdot F \circ (\sigma \circ i_s^n \times \text{id}) \circ [v_0, \dots, v_r, w_r, \dots, w_{n-1}].
 \end{aligned}$$

It follows easily from the definitions that

$$i_s^n \circ [v_0, \dots, v_r, w_r, \dots, w_{n-1}] = \begin{cases} [v_0, \dots, \widehat{v}_s, \dots, v_{r+1}, w_{r+1}, \dots, w_n], & \text{if } s \leq r, \\ [v_0, \dots, v_r, w_r, \dots, \widehat{w}_{s-1}, \dots, w_n], & \text{if } s > r. \end{cases}$$

Thus we obtain that

$$\begin{aligned}
 (25.2) \quad P_{n-1}\partial(\sigma) &= \sum_{0 \leq s \leq r \leq n-1} (-1)^r (-1)^s \cdot F \circ (\sigma \times \text{id}) \circ [v_0, \dots, \widehat{v}_s, \dots, v_{r+1}, w_{r+1}, \dots, w_n] \\
 &\quad + \sum_{n \geq s > r \geq 0} (-1)^r (-1)^s \cdot F \circ (\sigma \times \text{id}) \circ [v_0, \dots, v_r, w_r, \dots, \widehat{w}_{s-1}, \dots, w_n].
 \end{aligned}$$

If in the first term of (25.2) we perform the substitutions $s = k$ and $r = j - 1$ and if in the second term of (25.2) we perform the substitutions $s = k - 1$, $r = j - 1$, then we see immediately that these terms are just the negative of the corresponding terms in (25.1).

Thus we have proved the claim that $\partial P_n = f_* - g_* - P_{n-1}\partial$. $\square$

We recall the following definition from page 125.

Definition. Let X be and Y topological spaces.

- (1) A map $f: X \rightarrow Y$ is a *homotopy equivalence between X and Y* , if there exists a homotopy inverse to f , i.e. a continuous map $g: Y \rightarrow X$ with $g \circ f \simeq \text{id}_X$ and $f \circ g \simeq \text{id}_Y$.
- (2) If there exists a homotopy equivalence between X and Y , then we say that the topological spaces X and Y are *homotopy equivalent* and we write $X \simeq Y$.

- (3) We say X is *contractible*, if X is homotopy equivalent to $Y = \{*\}$, i.e. if X is homotopy equivalent to a topological space which consists of precisely one point.

Now we obtain the following corollary to Proposition 25.4. This corollary mirrors the fact, proved in Propositions 7.5 and 23.6, that homotopy equivalences induce isomorphisms of fundamental groups and higher homotopy groups.

Corollary 25.5. *Let X be and Y topological spaces and let $n \in \mathbb{N}_0$.*

- (1) *If $f: X \rightarrow Y$ is a homotopy equivalence, then the induced map $f_*: H_n(X) \rightarrow H_n(Y)$ is an isomorphism.*
- (2) *If X and Y are homotopy equivalent, then $H_n(X) \cong H_n(Y)$.*
- (3) *If X is contractible, then $H_0(X) \cong \mathbb{Z}$ and $H_n(X) = 0$ for $n \geq 1$.*
- (4) *If X is a deformation retract of Y , then the inclusion induced map $H_n(X) \rightarrow H_n(Y)$ is an isomorphism.*

Proof.

- (1) Let $f: X \rightarrow Y$ be a homotopy equivalence and let $g: Y \rightarrow X$ be a homotopy inverse of f , i.e. g is a map such that $f \circ g \simeq \text{id}_Y$ and $g \circ f \simeq \text{id}_X$. It follows from Proposition 25.4 and the functoriality of homology that $f_* \circ g_* = (f \circ g)_* = \text{id}_{H_n(Y)}$ and similarly $g_* \circ f_* = \text{id}_{H_n(X)}$. But that means precisely that $f_*: H_n(X) \rightarrow H_n(Y)$ is an isomorphism.
- (2) This statement follows immediately from (1).
- (3) This statement follows immediately from (2) and Lemma 24.6.
- (4) This statement follows immediately from (1) and Lemma 7.4. □

Examples.

- (1) On page 126 we had seen that every convex subset of $\mathbb{R}^n$, e.g. the open disk B^n , the closed disk $\overline{B}^n$, any simplex or $\mathbb{R}^n$, is contractible. Thus it follows from Corollary 25.5, that

$$H_i(\text{convex subset of } \mathbb{R}^n) \cong H_i(B^n) \cong H_i(\overline{B}^n) \cong H_i(\mathbb{R}^n) \cong \begin{cases} \mathbb{Z}, & \text{if } i = 0, \\ 0, & \text{if } i > 0. \end{cases}$$

- (2) In exercise sheet 4 we will use Corollary 25.5 to show that if A is a compact convex subset of $\mathbb{R}^n$ and if $x \in A$, then the inclusion $\mathbb{R}^n \setminus A \rightarrow \mathbb{R}^n \setminus \{x\}$ induces isomorphisms $H_i(\mathbb{R}^n \setminus A) \rightarrow H_i(\mathbb{R}^n \setminus \{x\})$ for all i .

Remark. Given a map $f: X \rightarrow Y$ between topological spaces we denote by $[f]$ the homotopy equivalence class of f , and we denote by $[X, Y]$ the set of equivalence classes of maps from X to Y . As we had pointed out on page 125,

$$\begin{aligned} \text{Ob}(\mathcal{H}) &:= \text{all topological spaces,} \\ \text{Mor}(X, Y) &:= [X, Y] \end{aligned}$$

together with the composition

$$\begin{aligned} \text{Mor}(X, Y) \times \text{Mor}(Y, Z) &\rightarrow \text{Mor}(X, Z) \\ ([f], [g]) &\mapsto [f \circ g] \end{aligned}$$

$$X \mapsto H_n(X)$$
$$[f] \mapsto f_*: H_n(X) \rightarrow H_n(Y)$$
$$\begin{array}{ccc}
\text{category } \mathcal{T} \text{ of} & \xrightarrow{X \mapsto X} & \text{homotopy} \\
\text{topological spaces} & & \text{category } \mathcal{H} \\
& \searrow X \mapsto H_n(X) & \swarrow X \mapsto H_n(X) \\
& \text{category of abelian} & \\
& \text{groups.} &
\end{array}$$

26. LONG EXACT SEQUENCES AND THE HOMOLOGY OF QUOTIENT SPACES

In this section, given a topological space X and a “well-behaved” subset A we will relate the homology groups of X , A and the quotient space X/A via a long exact sequence. We will use this long exact sequence to fulfill our dream of calculating the homology groups of all spheres. Unfortunately it will take a very long time to prove the existence of this long exact sequence.

26.1. Long exact sequences. In this chapter we will formulate and prove a result which will allow us to determine the homology groups of spheres. But before we can formulate the result we still need to recall and introduce a few more definitions.

Definition. Let X be a topological space. We denote by $\epsilon: H_0(X) \rightarrow \mathbb{Z}$ the augmentation map. We define the n -th reduced homology of X as

$$\tilde{H}_n(X) := \begin{cases} H_n(X), & \text{if } n > 0, \text{ and} \\ \ker\{\epsilon: H_0(X) \rightarrow \mathbb{Z}\}, & \text{if } n = 0. \end{cases}$$

The following elementary lemma summarizes a few properties of reduced homology.

Lemma 26.1.

(1) *For every non-empty topological space X we have*

$$H_n(X) \cong \begin{cases} \tilde{H}_0(X) \oplus \mathbb{Z}, & \text{if } n = 0, \\ H_n(X), & \text{if } n \neq 0. \end{cases}$$

(2) *If X is a topological space with $n \in \mathbb{N}$ path-components, then $\tilde{H}_0(X) \cong \mathbb{Z}^{n-1}$.*

(3) *If $f: X \rightarrow Y$ is a map between two topological spaces, then the induced map $f_*: H_0(X) \rightarrow H_0(Y)$ restricts to a map $f_*: \tilde{H}_0(X) \rightarrow \tilde{H}_0(Y)$. In particular the maps*

$$X \mapsto \tilde{H}_0(X)$$

and

$$(f: X \rightarrow Y) \mapsto (f_*: \tilde{H}_0(X) \rightarrow \tilde{H}_0(Y))$$

define a functor from the category of topological spaces to the category of abelian groups.

Remark.

(1) It follows from Corollary 25.5 together with Lemma 26.1 (2) that the reduced homology group of contractible spaces vanish. In particular for any $k \in \mathbb{N}_0$ we have

$$\tilde{H}_n(\Delta^k) = \tilde{H}_n(B^k) = \tilde{H}_n(\overline{B}^k) = \tilde{H}_n(\mathbb{R}^k) = 0.$$

(2) We will often use Lemma 26.1 without explicitly mentioning it. For example we will refer to earlier results about homology groups and state the corresponding results for reduced homology groups.

Proof.

- (1) Let X be a non-empty topological space. By the definition of $\tilde{H}_n(X)$ we only have to deal with the case $n = 0$. We pick a point $P \in X$. We denote by $i: \mathbb{Z} \rightarrow H_0(X)$ the map that is given by $i(n) = [n \cdot P]$. Evidently we have $\epsilon \circ i = \text{id}_{\mathbb{Z}}$. It follows easily from this observation and from $H_0(X) = \ker(\epsilon)$ that the map⁴⁷⁶

$$\begin{aligned} \tilde{H}_0(X) \oplus \mathbb{Z} &\rightarrow H_0(X) \\ c \oplus n &\mapsto c + i(n) \end{aligned}$$

is an isomorphism.

- (2) This statement follows immediately from (1), Corollary 24.14 and Lemma 8.6 (1).
 (3) Let $f: X \rightarrow Y$ be a between two topological spaces. It is a straightforward consequence of the definition of the augmentation maps that the following diagram commutes:

$$\begin{array}{ccc} H_0(X) & \xrightarrow{f_*} & H_0(Y) \\ & \searrow \epsilon & \swarrow \epsilon \\ & \mathbb{Z} & \end{array}$$

It follows immediately that the induced map $f_*: H_0(X) \rightarrow H_0(Y)$ restricts to a map $f_*: \tilde{H}_0(X) \rightarrow \tilde{H}_0(Y)$. $\square$

We need two more definitions before we can formulate one of the key theorems of Algebraic Topology II.

Definition. Let X be a topological space and let $A \subset X$ be a subset. We say A is *well-behaved* if there exists an open neighborhood U of A in X such that A is a deformation retract of U .

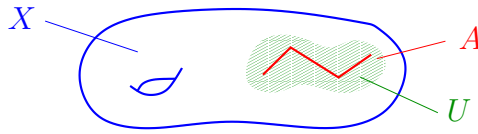


FIGURE 261. Illustration of a well-behaved subset.

Examples.

- (1) The sphere S^{n-1} is a well-behaved subset of the closed ball $\overline{B}^n$. In fact a deformation retraction is given by the map

$$\begin{aligned} (\overline{B}^n \setminus \{0\}) \times [0, 1] &\rightarrow \overline{B}^n \setminus \{0\} \\ (x, t) &\mapsto \left(1 - t \frac{1 - \|x\|}{\|x\|}\right) \cdot x. \end{aligned}$$

⁴⁷⁶Why is this map an epimorphism?

- (2) We can generalize the first example: It follows easily from Collar Neighborhood Theorem 2.6 that every boundary component of a manifold is well-behaved.
- (3) Finally we had seen in Lemma 22.4 (7) that every subcomplex A of a CW-complex X is well-behaved.

Definition. A sequence of group homomorphisms

$$\dots A_{n+1} \xrightarrow{f_{n+1}} A_n \xrightarrow{f_n} A_{n-1} \xrightarrow{f_{n-1}} A_{n-2} \rightarrow \dots$$

is called *exact*, if for each n we have

$$\ker(f_n) = \operatorname{im}(f_{n+1}).$$

Remark.

- (1) An exact sequence satisfies in particular that $f_n \circ f_{n+1} = 0$ for all $n \in \mathbb{N}_0$.
- (2) We have the following equivalences of statements:

$$\begin{aligned} \text{the sequence } 0 \rightarrow A \xrightarrow{f} B \quad & \text{is exact} \iff f \text{ is injective} \\ \text{the sequence } A \xrightarrow{f} B \rightarrow 0 \quad & \text{is exact} \iff f \text{ is surjective} \\ \text{the sequence } 0 \rightarrow A \xrightarrow{f} B \rightarrow 0 \quad & \text{is exact} \iff f \text{ is an isomorphism} \end{aligned}$$

- (3) Sometimes we also use the following observations:

$$\begin{aligned} \text{the sequence } 0 \rightarrow A \rightarrow B \xrightarrow{f} C \text{ is exact} & \implies A \cong \ker(f) \\ \text{the sequence } A \xrightarrow{f} B \rightarrow C \rightarrow 0 \text{ is exact} & \implies C \cong \operatorname{coker}(f) \end{aligned}$$

Now we can formulate the following theorem.

Theorem 26.2. *Let X be a topological space and let $A \subset X$ be a non-empty closed well-behaved subset. We denote by $i: A \rightarrow X$ the inclusion map and we denote by $p: X \rightarrow X/A$ the projection map. Then for each n there exists a homomorphism $\partial: \tilde{H}_n(X/A) \rightarrow \tilde{H}_{n-1}(A)$ such that the sequence*

$$\begin{array}{ccccccc} & & & \dots & \xrightarrow{p_*} & \tilde{H}_{n+1}(X/A) & \xrightarrow{\partial} \\ \rightarrow & \tilde{H}_n(A) & \xrightarrow{i_*} & \tilde{H}_n(X) & \xrightarrow{p_*} & \tilde{H}_n(X/A) & \xrightarrow{\partial} \\ \rightarrow & \tilde{H}_{n-1}(A) & \xrightarrow{i_*} & \dots & & & \end{array}$$

is exact.

It will take a while to prove Theorem 26.2. Therefore we will first consider several applications which will convince us that proving Theorem 26.2 is worth the effort.

26.2. The homology groups of spheres. Using Theorem 26.2 we can now determine the homology groups of spheres and we get a nice and satisfactory result.

Proposition 26.3. *For each n we have*

$$\tilde{H}_k(S^n) \cong \begin{cases} \mathbb{Z}, & \text{if } k = n, \text{ and} \\ 0, & \text{if } k \neq n. \end{cases}$$

The combination of Lemma 26.1 and Proposition 26.3 says that for any $n > 0$ we have

$$H_k(S^n) \cong \begin{cases} \mathbb{Z}, & \text{if } k = 0 \text{ or } k = n, \text{ and} \\ 0, & \text{if } k \neq 0, n. \end{cases}$$

Furthermore we have

$$H_k(S^0) \cong \begin{cases} \mathbb{Z}^2, & \text{if } k = 0, \text{ and} \\ 0, & \text{if } k \neq 0. \end{cases}$$

Corollary 26.4. *For $k \neq l$ the spheres S^k and S^l are not homeomorphic.*⁴⁷⁷

So we have finally managed to distinguish spheres of different dimensions. Our initial attempt had been to use higher homotopy groups to prove that result. The definition of higher homotopy groups is arguably much more natural than the definition of homology groups, but the latter groups are significantly easier to compute.

Proof of Proposition 26.3 assuming Theorem 26.2. We prove the proposition by induction on n . If $n = 0$, then $S^n = S^0$ consists of precisely two points, and the proposition follows from Lemmas 24.6, 24.13 and 26.1.

Now we suppose that the statement of the proposition holds for $n - 1$. On page 23 we had remarked that S^n is homeomorphic to $\overline{B}^n/S^{n-1}$. We had just seen on page 423 that S^{n-1} is a well-behaved subset of $\overline{B}^n \setminus \{0\}$. Therefore we can apply Theorem 26.2 to $X = \overline{B}^n$ and the closed subset $A = S^{n-1}$ and we obtain the following exact sequence

$$\begin{array}{ccccccc} \dots & \longrightarrow & \tilde{H}_{k+1}(\overline{B}^n) & \xrightarrow{p_*} & \tilde{H}_{k+1}(\overline{B}^n/S^{n-1}) & \xrightarrow{\partial} & \\ \rightarrow & \tilde{H}_k(S^{n-1}) & \xrightarrow{i_*} & \tilde{H}_k(\overline{B}^n) & \xrightarrow{p_*} & \tilde{H}_k(\overline{B}^n/S^{n-1}) & \xrightarrow{\partial} \\ \rightarrow & \tilde{H}_{k-1}(S^{n-1}) & \xrightarrow{i_*} & \dots & & & \end{array}$$

From the induction hypothesis, from the homeomorphism $S^n \cong \overline{B}^n/S^{n-1}$ and from the fact that $\tilde{H}_i(\overline{B}^n) = 0$ for all i (obtained on page 422) we now see that the above long exact sequence is of the following form:

$$\begin{array}{ccccccc} \dots & \longrightarrow & 0 & \xrightarrow{p_*} & \tilde{H}_{k+1}(S^n) & \xrightarrow{\partial} & \\ \rightarrow & \tilde{H}_k(S^{n-1}) & \xrightarrow{i_*} & 0 & \xrightarrow{p_*} & \tilde{H}_k(S^n) & \xrightarrow{\partial} \\ \rightarrow & \tilde{H}_{k-1}(S^{n-1}) & \xrightarrow{i_*} & \dots & & & \end{array}$$

In particular we obtain for each k an exact sequence

$$0 \rightarrow \tilde{H}_{k+1}(S^n) \xrightarrow{\partial} \tilde{H}_k(S^{n-1}) \rightarrow 0.$$

But this means that the maps ∂ are isomorphisms, and using our induction hypothesis we obtain the desired result that

$$\tilde{H}_k(S^n) = \tilde{H}_{k-1}(S^{n-1}) \cong \begin{cases} \mathbb{Z}, & \text{if } k - 1 = n - 1, \text{ i.e. if } k = n \text{ and} \\ 0, & \text{if } k \neq n. \end{cases} \quad \square$$

⁴⁷⁷We know from Analysis II that for $k \neq l$ an open subset of $\mathbb{R}^k$ cannot be diffeomorphic to an open subset of $\mathbb{R}^l$. Via charts this implied that S^k and S^l are not *diffeomorphic*.

In Lemma 1.12 we had used elementary arguments to show that $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^2$. Furthermore, in Lemma 7.6 we had used the fundamental group to prove that $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^n$ for $n \geq 3$. Now we can finally show that for any $k \neq l$ the topological spaces $\mathbb{R}^k$ and $\mathbb{R}^l$ are not homeomorphic.

Proposition 26.5. *For any $k \neq l$ the topological spaces $\mathbb{R}^k$ and $\mathbb{R}^l$ are not homeomorphic.*

Proof. Clearly we only have to deal with the case $k, l \geq 1$. Let $k, l \in \mathbb{N}$ and suppose there exists a homeomorphism $f: \mathbb{R}^k \rightarrow \mathbb{R}^l$. Then we obtain that

$$\begin{array}{ccccccc}
 & & \text{Corollary 25.5} & & & & \text{Corollary 25.5} \\
 & & \downarrow & & & & \downarrow \\
 \mathbb{Z} & \cong & \tilde{H}_{k-1}(S^{k-1}) & \cong & \tilde{H}_{k-1}(\mathbb{R}^k \setminus \{0\}) & \cong & \tilde{H}_{k-1}(\mathbb{R}^{l-1} \setminus \{f(0)\}) \cong \tilde{H}_{k-1}(S^{l-1}). \\
 \uparrow & & & & \uparrow & & \\
 \text{Proposition 26.3} & & & & \text{since } f_* \text{ is a homeomorphism} & &
 \end{array}$$

So we have $\tilde{H}_{k-1}(S^{l-1}) \cong \mathbb{Z}$. But by Proposition 26.3 that is only possible if $k = l$. $\square$

In Lemma 5.2 we had used the fundamental group to show that S^1 is not a retract of $\overline{B}^2$. Precisely the same argument, together with Proposition 26.3, now gives us the following proposition.

Proposition 26.6. *For any $n \in \mathbb{N}$ the sphere $S^{n-1} = \partial \overline{B}^{n-1}$ is not a retract of the closed ball $\overline{B}^n$.*

Remark. I found the following amusing description of Proposition 26.6 online:

“Suppose a shark jumps into a shoal of fish (a kind of big ball). The small fishes start escaping in all directions towards the border of the shoal, where the fishes stand still. Yet they escape with a certain disposition to follow a continuous flow, as they usually do, since everybody tends to follow its neighborhoods. But since there is no continuous retraction to the boundary, somebody doesn’t know where to go, and stay there for a moment, much to the shark’s satisfaction. There is also a 2D version, with a wolf entering into a herd of sheep. ”

Proof. Let us suppose there exists a retraction $r: \overline{B}^n \rightarrow S^{n-1}$. We denote by $i: S^{n-1} \rightarrow \overline{B}^n$ the inclusion map. We consider the following diagram

$$\begin{array}{ccc}
 & \tilde{H}_{n-1}(\overline{B}^n) & \\
 i_* \nearrow & & \searrow r_* \\
 \tilde{H}_{n-1}(S^{n-1}) & \xrightarrow{(r \circ i)_*} & \tilde{H}_{n-1}(S^{n-1}).
 \end{array}$$

This diagram commutes since it follows from the functoriality of the homology groups that $r_* \circ i_* = (r \circ i)_*$. Since r is a retraction it follows that $r \circ i = \text{id}_{S^{n-1}}$, i.e. $(r \circ i)_* = (\text{id}_{S^{n-1}})_*$ is the identity map on $\tilde{H}_{n-1}(S^{n-1})$ which, by Proposition 26.3, is isomorphic to $\mathbb{Z}$. The lower map is thus an isomorphism, but the upper map factorizes through the trivial group,

i.e. the composition of i_* and r_* cannot be an isomorphism of $\mathbb{Z}$. Thus we have obtained a contradiction. $\square$

Theorem 26.7. (Brouwer⁴⁷⁸ Fixed Point Theorem) *Every map $f: \overline{B}^n \rightarrow \overline{B}^n$ admits a fixed point, i.e. for every map $f: \overline{B}^n \rightarrow \overline{B}^n$ there exists a point $x \in \overline{B}^n$ such that $f(x) = x$.*⁴⁷⁹

Proof. Suppose there exists a map $f: \overline{B}^n \rightarrow \overline{B}^n$ without a fixed point. Since f admits no fixed points there exists for each x precisely one ray which starts at $f(x)$ and which goes through x . We consider the following map which is illustrated in Figure 262:

$$\begin{aligned} \Phi: \overline{B}^n &\rightarrow \overline{B}^n \\ x &\mapsto \text{the intersection point of } S^{n-1} \text{ with the} \\ &\quad \text{uniquely determined ray from } f(x) \text{ to } x. \end{aligned}$$

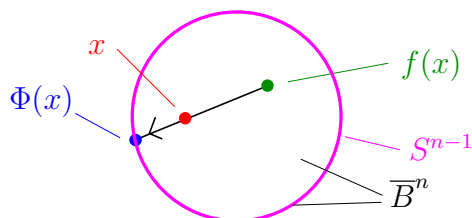


FIGURE 262.

It is elementary to show that Φ is continuous.⁴⁸⁰ It is evident that $\Phi(x) = x$ for all $x \in S^{n-1}$ and that $\Phi(x) \in S^{n-1}$ for all $x \in \overline{B}^n$. The map Φ is thus a retraction from $\overline{B}^n$ to S^{n-1} . But according to Proposition 26.6 this not possible. Thus we have obtained a contradiction. $\square$

The Brouwer Fixed Point Theorem pops up many times in mathematics. It also has applications beyond mathematics:

- (1) The board game “Hex” was invented in 1942 by Piet Hein⁴⁸¹ and it was re-invented in 1948 by John Nash⁴⁸² The game is illustrated in Figure 263 and an online version

⁴⁷⁸Luitzen Brouwer (1881-1966) was a Dutch mathematician.

⁴⁷⁹How can one prove the statement for $n = 1$ using elementary methods from Analysis I?

⁴⁸⁰Here is the precise argument: given $x \in \overline{B}^n$ the ray r_x emanating from x and through $f(x)$ is given by $r + x(t) = x + t(f(x) - x)$ for $t \in \mathbb{R}_{\geq 0}$. For any x there exists a unique $t_x \geq 0$ with $\|r_x(t_x)\| = 1$, and $\Phi(x) = r_x(t_x)$. It suffices to show that $x \mapsto t_x$ is continuous. One can determine t_x through the quadratic equation

$$1 = \|r_x(t_x)\|^2 = \langle r_x(t_x), r_x(t_x) \rangle = 2\langle x, f(x) - x \rangle + \|f(x) - x\|^2 = \langle x + t_x(f(x) - x), x + t_x(f(x) - x) \rangle$$

and the quadratic formula for polynomials of degree two which is continuous in the coefficients.

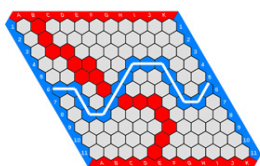
⁴⁸¹Piet Hein (1905-1996) was a Danish mathematician.

⁴⁸²John Nash (1928-2015) was an American mathematician who was awarded the Nobel prize in economics in 1994. The movie “a beautiful mind” was based on his struggles with schizophrenia.

can be found at

<http://www.lutanho.net/play/hex.html>

David Gale [Ga] showed in 1979 that the statement that the board game “Hex” cannot end in a draw, is equivalent to the Brouwer Fixed-Point Theorem. We also refer to [Gh, p. 102] for a discussion of Hex and the Brouwer Fixed-Point Theorem.



11 × 11 Hex gameboard showing
a winning configuration for Blue

FIGURE 263.

- (2) The Brouwer Fixed Point Theorem can also be used to show the existence of a “Nash equilibrium” in game theory. We refer to the following website for more information:

https://en.wikipedia.org/wiki/Nash_equilibrium.

26.3. Basic homological algebra. Now that we have made use of Theorem 26.2 we also need to provide a proof of Theorem 26.2. In this chapter we will first develop several algebraic methods which later on will be used in the proof of the existence of the exact sequence of Theorem 26.2.

Definition. A *short exact sequence* is an exact sequence of the form

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0,$$

which consists of at most three non-trivial groups. A *short exact sequence of chain complexes*

$$0 \rightarrow A_* \xrightarrow{i} B_* \xrightarrow{p} C_* \rightarrow 0$$

consists of chain complexes A_*, B_*, C_* and chain maps $i: A_* \rightarrow B_*$ and $p: B_* \rightarrow C_*$ such that for each n the maps

$$0 \rightarrow A_n \xrightarrow{i} B_n \xrightarrow{p} C_n \rightarrow 0$$

form a short exact sequence.

Remark. Throughout the subsequent discussion in the following pages it is helpful to keep in mind the following commutative diagram of short exact sequences:

$$\begin{array}{ccccccccc}
 0 & \longrightarrow & A_{n+1} & \xrightarrow{i} & B_{n+1} & \xrightarrow{p} & C_{n+1} & \longrightarrow & 0 \\
 & & \downarrow \partial_{n+1} & & \downarrow \partial_{n+1} & & \downarrow \partial_{n+1} & & \\
 0 & \longrightarrow & A_n & \xrightarrow{i} & B_n & \xrightarrow{p} & C_n & \longrightarrow & 0 \\
 & & \downarrow \partial_n & & \downarrow \partial_n & & \downarrow \partial_n & & \\
 0 & \longrightarrow & A_{n-1} & \xrightarrow{i} & B_{n-1} & \xrightarrow{p} & C_{n-1} & \longrightarrow & 0 \\
 & & \downarrow \partial_{n-1} & & \downarrow \partial_{n-1} & & \downarrow \partial_{n-1} & & \\
 0 & \longrightarrow & A_{n-2} & \xrightarrow{i} & B_{n-2} & \xrightarrow{p} & C_{n-2} & \longrightarrow & 0.
 \end{array}$$

Now suppose that we are given such a short exact sequence of chain complexes. We will define a map

$$\partial_n: H_n(C) \rightarrow H_{n-1}(A)$$

as follows:

- (1) Let $z \in H_n(C)$.
- (2) We pick $c \in C_n$ with $z = [c]$.
- (3) By the surjectivity of p we have $c = p(b)$ for some $b \in B_n$.
- (4) We have

$$\begin{array}{ccccccc}
 p(\partial b) & = & \partial p(b) & = & \partial c & = & 0. \\
 & \uparrow & & & \uparrow & & \\
 & \text{since the diagram commutes} & & & \text{since } c \text{ is a cycle} & &
 \end{array}$$

- (5) Since the second horizontal row in the above diagram is exact it follows immediately that $p(\partial b) = i(a)$ for some uniquely determined $a \in A_{n-1}$.
- (6) We put $\partial_n(z) := [a]$.

We refer to this map $\partial_n: H_n(C) \rightarrow H_{n-1}(A)$ as the *connecting homomorphism*.

The following lemma summarizes two of the key properties of the connecting homomorphisms.

Lemma 26.8.

- (1) Let $0 \rightarrow A_* \rightarrow B_* \rightarrow C_* \rightarrow 0$ be a short exact sequence of chain complexes. The connecting homomorphism $\partial_n: H_n(C) \rightarrow H_{n-1}(A)$ is well-defined and it is indeed a homomorphism.
- (2) If

$$\begin{array}{ccccccccc}
 0 & \longrightarrow & A_* & \xrightarrow{i} & B_* & \xrightarrow{p} & C_* & \longrightarrow & 0 \\
 & & \downarrow a & & \downarrow b & & \downarrow c & & \\
 0 & \longrightarrow & A'_* & \xrightarrow{i} & B'_* & \xrightarrow{p} & C'_* & \longrightarrow & 0
 \end{array}$$

is a commutative diagram of chain maps where the horizontal sequences are exact, then the following diagram commutes for every $n \in \mathbb{N}$:

$$\begin{array}{ccc} H_n(C) & \xrightarrow{\partial_n} & H_{n-1}(A) \\ \downarrow c_* & & \downarrow a_* \\ H_n(C') & \xrightarrow{\partial_n} & H_{n-1}(A') \end{array}$$

The proof of this lemma and the following proposition are done by a “diagram chase”. This means that one needs to keep the commutative diagram on 429 in mind and one has to use on several occasions that the vertical maps are chain maps and that the horizontal sequences are exact. In both cases the proofs are quite elementary.

Proof.

The proof below is elementary but not very readable. It is much easier to prove the proposition on your own than to try to read the proof.

- (1) We start out with the following claim.

Claim. The map

$$\partial_n: H_n(C) \rightarrow H_{n-1}(A)$$

is well-defined.

Let $z \in H_n(C)$. We denote by c , b and a the elements as in the definition of ∂_n . We need to show that the element a is indeed a cycle and that the definition of $\partial_n([c])$ does not depend on the choices of c and b .

The proofs of these two statements is not particularly difficult and it is done via “diagram chasing”. First we note that

$$\begin{array}{c} i(\partial a) = \partial i(a) = \partial(\partial b) = 0. \\ \uparrow \\ \text{since the diagram commutes} \end{array}$$

Since i is injective we obtain that $\partial a = 0$, i.e. a is indeed a cycle. Now we show that the homology class $[a]$ depends only on the homology class $[c]$.

- (a) If we chose a different element in B_n , then it follows from the fact that the horizontal sequences are exact that this element would be of the form $b + i(a')$ for some $a' \in A_n$. But then we obtain the following equality:

$$\begin{array}{c} \text{since the diagram commutes} \\ \downarrow \\ \partial(b + i(a')) = \partial b + \partial(i(a')) = \underbrace{\partial b}_{=i(a)} + i(\partial a') = i(a + \partial a'). \end{array}$$

Since $[a] = [a + \partial a'] \in H_{n-1}(A)$ this does not change the homology class.

- (b) If we picked an other representative for $[c]$, then this representative would be of the form $c + \partial c'$ for some $c' \in C_{n+1}$. Since the map $p: B_n \rightarrow C_n$ is an epimorphism there exists a $b' \in B_{n+1}$ with $p(b') = c'$. It follows that

$$\begin{array}{ccccccc} p(b + \partial b') & = & p(b) + p(\partial b') & = & c + \partial p(b') & = & c + \partial c'. \\ & & \uparrow & & & & \\ & & \text{since the diagram commutes} & & & & \end{array}$$

Since $\partial(b + \partial b') = \partial b$ we have ended up with the same element in B_{n-1} as before.

This concludes the proof of the claim.

Claim. The map

$$\partial_n: H_n(C) \rightarrow H_{n-1}(A)$$

is a homomorphism.

Thus let $[c_1]$ and $[c_2]$ be given. For $j = 1, 2$ we choose b_j as in this definition of ∂_n , and we denote by a_j the uniquely determined element with $i(a_j) = \partial b_j$. Then we also have $p(b_1 + b_2) = c_1 + c_2$ and $i(a_1 + a_2) = i(a_1) + i(a_2) = \partial b_1 + \partial b_2 = \partial(b_1 + b_2)$. It follows that

$$\partial_n([c_1 + c_2]) = [a_1 + a_2] = [a_1] + [a_2] = \partial_n([c_1]) + \partial_n([c_2]).$$

Thus we have proved the claim.

- (2) The second statement follows easily from the definitions. We leave the verification of the details as an exercise to the reader. $\square$

Now we can formulate the following proposition which explains the name “connecting homomorphism”.

Proposition 26.9. *Let*

$$0 \rightarrow A_* \xrightarrow{i} B_* \xrightarrow{p} C_* \rightarrow 0$$

be a short exact sequence of chain complexes. Then the sequence

$$\dots \rightarrow H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{p_*} H_n(C) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(B) \xrightarrow{p_*} \dots$$

is exact.

Remark.

- (1) In the following we refer to the exact sequence of Proposition 26.9 as the *long exact sequence in homology* corresponding to the short exact sequence of chain complexes.
- (2) Let

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_* & \xrightarrow{i} & B_* & \xrightarrow{p} & C_* & \longrightarrow & 0 \\ & & \downarrow a & & \downarrow b & & \downarrow c & & \\ 0 & \longrightarrow & A'_* & \xrightarrow{i} & B'_* & \xrightarrow{p} & C'_* & \longrightarrow & 0 \end{array}$$

be a commutative diagram of chain maps where the horizontal sequences are exact. It follows from Lemma 26.8 that the following diagram commutes:

$$\begin{array}{ccccccccc}
 \dots & \longrightarrow & H_n(A) & \xrightarrow{i_*} & H_n(B) & \xrightarrow{p_*} & H_n(C) & \xrightarrow{\partial} & H_{n-1}(A) & \longrightarrow & \dots \\
 & & \downarrow a_* & & \downarrow b_* & & \downarrow c_* & & \downarrow a_* & & \\
 \dots & \longrightarrow & H_n(A') & \xrightarrow{i_*} & H_n(B') & \xrightarrow{p_*} & H_n(C') & \xrightarrow{\partial} & H_{n-1}(A') & \longrightarrow & \dots
 \end{array}$$

Put differently, we have defined a functor from the category of short exact sequences of chain complexes⁴⁸³ to the category⁴⁸⁴ of long exact sequences of abelian groups.

(3) Let

$$\begin{array}{ccccccccc}
 0 & \longrightarrow & A & \xrightarrow{\varphi} & B & \xrightarrow{\psi} & C & \longrightarrow & 0 \\
 & & \downarrow a & & \downarrow b & & \downarrow c & & \\
 0 & \longrightarrow & A' & \xrightarrow{\varphi'} & B' & \xrightarrow{\psi'} & C' & \longrightarrow & 0
 \end{array}$$

be a commutative diagram of homomorphisms between abelian groups. In exercise sheet 5 we will show that there exists an exact sequence

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \ker(a) & \xrightarrow{\varphi} & \ker(b) & \xrightarrow{\psi} & \ker(c) \\
 & & & & & & \searrow \\
 & & & & & & \text{coker}(a) & \xrightarrow{\varphi'} & \text{coker}(b) & \xrightarrow{\psi'} & \text{coker}(c) & \longrightarrow & 0.
 \end{array}$$

This statement is often called the “snake lemma”.

Proof. Proof of Proposition 26.9]

The proof below is elementary but not very readable. It is much easier to prove the proposition on your own than to try to read the proof.

Let

$$0 \rightarrow A_* \xrightarrow{i} B_* \xrightarrow{p} C_* \rightarrow 0$$

be a short exact sequence of chain complexes. Then the sequence

$$\dots \rightarrow H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{p_*} H_n(C) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(B) \xrightarrow{p_*} \dots$$

is exact. We examine the exactness at the three groups $H_{n-1}(A)$, $H_n(B)$ and $H_n(C)$. Put differently, at each of the three groups we want to show the following two statements:

⁴⁸³Here the objects are short exact sequences of chain complexes and morphisms between two short exact sequences $0 \rightarrow A_* \rightarrow B_* \rightarrow C_* \rightarrow 0$ and $0 \rightarrow A'_* \rightarrow B'_* \rightarrow C'_* \rightarrow 0$ are given by a triple of chain maps $a: A_* \rightarrow A'_*$, $b: B_* \rightarrow B'_*$ and $c: C_* \rightarrow C'_*$ such that the following diagram commutes:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & A_* & \longrightarrow & B_* & \longrightarrow & C_* \longrightarrow 0 \\
 & & \downarrow a & & \downarrow b & & \downarrow c \\
 0 & \longrightarrow & A'_* & \longrightarrow & B'_* & \longrightarrow & C'_* \longrightarrow 0.
 \end{array}$$

⁴⁸⁴Here the objects are chain complexes of abelian groups and morphisms are chain maps.

- (1) the image of the map to the left is contained in the image of the map to the right, or equivalently, the composition of the map to the left and the map to the right is zero,
- (2) the kernel of the map to the right is contained in the image of the map to the left.

Now we perform these steps:

- (A) First let $[c] \in H_n(C)$. We want to show that $i(\partial[c]) = 0$. We choose b with $p(b) = c$ and denote by $a \in A_{n-1}$ the element with $i(a) = \partial b$. Then we have

$$\begin{array}{ccccccc} i(\partial[c]) & = & i([a]) & = & [\partial b] & = & 0 \in H_{n-1}(B). \\ & & \uparrow & & & & \\ & & \text{definition of } \partial: H_n(C) \rightarrow H_{n-1}(A) & & & & \end{array}$$

Now let $[a] \in \ker(i_*)$. Then $i(a) = \partial b$ for some $b \in B_n$. It follows immediately from the definitions that $\partial[p(b)] = [a]$.

- (B) From $p \circ i = 0$ it follows that $p_* \circ i_* = 0$, i.e. $\text{im}(i_*) \subset \ker(p_*)$. Conversely let $b \in \ker(p_*)$. Then $p(b) = \partial c'$ for some $c' \in C_{n+1}$. Since p is surjective, there exists a $b' \in B_{n+1}$ with $p(b') = c'$. From the commutativity it follows that $p(\partial b') = \partial p(b') = \partial c' = p(b)$. Thus it follows that $p(b - \partial b') = 0$, i.e. $b - \partial b' = i(a)$ for some $a \in A_n$. But then it follows that $[b] = [b - \partial b'] = [i(a)] \in \text{im}(i_*)$.
- (C) Let $[b] \in H_n(B)$. We have to show that $\partial(p_*([b])) = 0$. In the definition of $\partial p_*([b])$ we choose some $b \in B$. Since $\partial b = 0$ it follows immediately that $\partial p_*([b]) = 0$. Finally let $[c] \in \ker(\partial: H_n(C) \rightarrow H_{n-1}(A))$. We choose $b \in B_n$ with $p(b) = c$. Then there exists a uniquely determined $a \in A_{n-1}$ with $i(a) = \partial b$. Since we have $[a] = \partial[c] = 0 \in H_{n-1}(A)$ there exists an $a' \in A_n$ with $\partial a' = a$. Then it follows that

$$\partial(b - i(a')) = \partial b - \partial i(a') = \partial b - i(\partial a') = \partial b - i(a) = 0,$$

i.e. $b - i(a')$ is a cycle. On the other hand we have $p(b - i(a')) = p(b) - (p \circ i)(a') = p(b)$.

Thus we see that $p_*([b - i(a')]) = [c]$. We have thus shown that $\ker(\partial) \subset \text{im}(p_*)$.

Thus we have proved the claim and therefore we have also completed the proof of the proposition. $\square$

The proof of the following lemma is also done via “diagram chasing”. We leave the proof as an exercise.

Lemma 26.10. (Five-Lemma) *Let*

$$\begin{array}{ccccccccc} A & \xrightarrow{i} & B & \xrightarrow{j} & C & \xrightarrow{k} & D & \xrightarrow{l} & E \\ \downarrow a & & \cong \downarrow b & & \downarrow c & & \cong \downarrow d & & \downarrow e \\ A' & \xrightarrow{i'} & B' & \xrightarrow{j'} & C' & \xrightarrow{k'} & D' & \xrightarrow{l'} & E' \end{array}$$

*be a commutative diagram of abelian groups such that the horizontal sequences are exact. If a is surjective, if b and d are isomorphisms, and if e is injective, then c is an isomorphism.*⁴⁸⁵

⁴⁸⁵In many applications it suffices to remember the following slightly weaker statement: if the two vertical maps on the left and right are isomorphisms, then the middle map is also an isomorphism.

26.4. Relative homology groups. In this section we will introduce the relative homology groups which will play an important role in the proof of Theorem 26.2 and also in the remainder of this course.

Definition. Let (X, A) be a pair topological space.⁴⁸⁶ We view each chain group $C_n(A)$ as a subgroup of $C_n(X)$. Thus we can define

$$C_n(X, A) := C_n(X)/C_n(A)$$

and we consider the boundary map⁴⁸⁷

$$\begin{aligned} \partial_n: C_n(X, A) &\rightarrow C_{n-1}(X, A) \\ [c] &\mapsto [\partial_n c]. \end{aligned}$$

We refer to

$$\dots C_{n+1}(X, A) \xrightarrow{\partial_{n+1}} C_n(X, A) \xrightarrow{\partial_n} C_{n-1}(X, A) \xrightarrow{\partial_{n-1}} \dots$$

as the chain complex of the pair (X, A) .⁴⁸⁸ Given any $n \in \mathbb{N}_0$ we define the n -th relative homology group of (X, A) as follows:

$$H_n(X, A) := H_n(C_*(X, A)).$$

Examples.

- (1) If $A = \emptyset$, then for all $n \in \mathbb{N}_0$ we have $C_n(\emptyset) = 0$ and $C_n(X, \emptyset) = C_n(X)$, i.e. we have $H_n(X, \emptyset) = H_n(X)$. In the following we will freely go back and forth between the notations $H_n(X, \emptyset)$ and $H_n(X)$.
- (2) We let $X = \mathbb{R}^2$ and $A = B_1(-2) \cup B_1(2)$. Let $\sigma: \Delta^1 \rightarrow X$ be a singular 1-simplex with $\sigma(0, 1) \in B_1(2)$ and $\sigma(1, 0) \in B_1(-2)$. (We refer to Figure 264 for an illustration.) Then we have

$$\partial\sigma = \underbrace{\sigma(0, 1)}_{\in B_1(2)} - \underbrace{\sigma(1, 0)}_{\in B_1(-2)}.$$

This shows that $\partial\sigma$ lies in $C_0(A)$. Put differently, the singular 1-simplex σ , viewed as an element in $C_1(X, A)$, is a cycle and therefore it represents an element in $H_1(X, A)$. Later on we will see whether or not $[\sigma] \in H_1(X, A)$ is non-trivial.

Now let $f: (X, A) \rightarrow (Y, B)$ be a map of pairs of topological space, i.e. a map $f: X \rightarrow Y$ with $f(A) \subset B$. Then f induces maps

$$\begin{aligned} f_*: H_n(X, A) &\rightarrow H_n(Y, B) \\ \left[\sum_{j=1}^n a_j \sigma_j \right] &\rightarrow \left[\sum_{j=1}^n a_j (f \circ \sigma_j) \right]. \end{aligned}$$

⁴⁸⁶Recall that according to the definition on page 81 this means that X is a topological space and $A \subset X$ is a subset.

⁴⁸⁷The map ∂_n is well-defined, since for any $c \in C_n(A) \subset C_n(X)$ the boundary ∂c lies obviously in $C_{n-1}(A) \subset C_{n-1}(X)$.

⁴⁸⁸Since $\partial_{n-1} \circ \partial_n: C_n(X) \rightarrow C_{n-2}(X)$ is the zero map it follows that $\partial_{n-1} \circ \partial_n: C_n(X, A) \rightarrow C_{n-2}(X, A)$ is also the zero map.

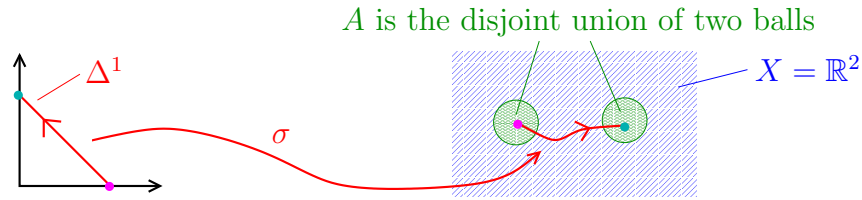


FIGURE 264.

It is straightforward to show that these maps are well-defined.⁴⁸⁹

Remark. If (Y, B) is a pair of topological spaces and if $X \subset Y$ and $A \subset B$, then we have the obvious inclusion map $(X, A) \rightarrow (Y, B)$ and we obtain the induced maps $H_n(X, A) \rightarrow H_n(Y, B)$ in relative homology. If it is clear from the context that we are dealing with such an inclusion induced map, then on many occasions we will not explicitly point out that we are working with the inclusion induced maps.

Now we consider the homology of a topological space relative to a point. We will provide the proof in exercise sheet 5.

Lemma 26.11. *Given a topological space X , a point $x_0 \in X$ and $n \in \mathbb{N}_0$ we consider the map*

$$\begin{array}{ccccccc} \tilde{H}_n(X) & \longrightarrow & H_n(X) & \xrightarrow{=} & H_n(X, \emptyset) & \longrightarrow & H_n(X, \{x_0\}) \\ & & & & \searrow & & \\ & & & & \text{=: } \Phi_{X, x_0} & & \end{array}$$

Then the following holds:

(1) The map

$$\Phi_{X, x_0}: \tilde{H}_n(X) \rightarrow H_n(X, \{x_0\})$$

is an isomorphism.

⁴⁸⁹We refer to the category that is given by

$$\begin{array}{ll} \text{Ob}(\mathcal{P}) & := \text{all pairs of topological spaces,} \\ \text{Mor}((X, A), (Y, B)) & := \text{all maps } (X, A) \rightarrow (Y, B) \end{array}$$

with the usual composition of maps as the *category of topological pairs*. Then for each $n \in \mathbb{N}_0$ the maps

$$(X, A) \rightarrow H_n(X, A)$$

and

$$(f: (X, A) \rightarrow (Y, B)) \mapsto (f_*: H_n(X, A) \rightarrow H_n(Y, B))$$

define a functor from the category of topological pairs to the category of groups.

⁴⁹⁰Recall that, as defined on page 379, we say that two maps $f, g: (X, A) \rightarrow (Y, B)$ between pairs of topological spaces are homotopic, if there exists a map $F: X \times [0, 1] \rightarrow Y$ such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$ for all $x \in X$, and $f(a) \in B$ for all $a \in A$ and all $t \in [0, 1]$. It follows easily from Corollary 26.13 below, the Five-Lemma 26.10 and the homotopy invariance of the ordinary homology groups, that homotopic maps induce the same maps $f_*, g_*: H_n(X, A) \rightarrow H_n(Y, B)$.

- (2) For any map $f: X \rightarrow Y$ of topological spaces and any $a \in X$ we have the following commutative diagram

$$\begin{array}{ccc} \tilde{H}_n(X) & \xrightarrow{f_*} & \tilde{H}_n(Y) \\ \Phi_{X,x_0} \downarrow & & \downarrow \Phi_{Y,f(x_0)} \\ H_n(X, \{x_0\}) & \xrightarrow{f_*} & H_n(Y, \{f(x_0)\}). \end{array}$$

Before we can state the next proposition we need to introduce the notion of a triple of topological spaces which is defined in almost the same fashion as a pair of topological spaces.

Definition.

- (1) A *triple of topological spaces* is a triple (X, B, A) where X is a topological space and $A \subset B \subset X$ are subsets.
- (2) A map $f: (X, B, A) \rightarrow (X', B', A')$ between triples of topological spaces is a continuous map $f: X \rightarrow X'$ such that $f(B) \subset B'$ and $f(A) \subset A'$.

Proposition 26.12. *Let X be a topological space and let $A \subset B \subset X$ be two subsets. We denote by $i: (B, A) \rightarrow (X, A)$ and by $p: (X, A) \rightarrow (X, B)$ the obvious maps. Then the following statements hold:*

- (1) For each $n \in \mathbb{N}_0$ the map⁴⁹¹⁴⁹²

$$\begin{aligned} \partial_n: H_n(X, B) &\rightarrow H_{n-1}(B, A) \\ \left[\sum_{j=1}^k a_j \cdot (\sigma_j: \Delta^n \rightarrow X) \right] &\mapsto \left[\sum_{j=1}^k a_j \cdot \partial \sigma_j \right] \end{aligned}$$

is well-defined.

- (2) The maps

$$\dots \rightarrow H_n(B, A) \xrightarrow{i_*} H_n(X, A) \xrightarrow{p_*} H_n(X, B) \xrightarrow{\partial_n} H_{n-1}(B, A) \rightarrow \dots$$

form an exact sequence.

- (3) The map in (2) which assigns to a triple (X, B, A) of topological spaces a long exact sequence of homology groups is functorial.⁴⁹³

⁴⁹¹The fact that here the symbol “ ∂_n ” denotes the boundary map of singular chains and the desired map $H_n(X, B) \rightarrow H_{n-1}(B, A)$ is an unfortunate problem in our choice of notation.

⁴⁹²Hereby we use that every homology class in $H_n(X, B)$ has a representative in $C_n(X, B) = C_n(X)/C_n(B)$, so it can be represented by a singular n -chain in X .

⁴⁹³More precisely, given a map $f: (X, B, A) \rightarrow (X', B', A')$ between triples of topological spaces the following diagram commutes

$$\begin{array}{ccccccc} \dots & \longrightarrow & H_n(B, A) & \longrightarrow & H_n(X, A) & \longrightarrow & H_n(X, B) \xrightarrow{\partial} H_{n-1}(B, A) \longrightarrow \dots \\ & & \downarrow f_* & & \downarrow f_* & & \downarrow f_* \\ \dots & \longrightarrow & H_n(B', A') & \longrightarrow & H_n(X', A') & \longrightarrow & H_n(X', B') \xrightarrow{\partial} H_{n-1}(B', A') \longrightarrow \dots \end{array}$$

The long exact sequence of Proposition 26.12 (2) is sometimes called the *long exact sequence of the triple* (X, B, A) and we refer to ∂_n as the *connecting homomorphism* of the triple (X, B, A) .

Proof. It follows immediately from the definitions and the third isomorphism theorem⁴⁹⁴ that

$$0 \rightarrow C_*(B, A) \xrightarrow{i_*} C_*(X, A) \xrightarrow{p_*} C_*(X, B) \rightarrow 0$$

is a short exact sequence of chain complexes.

Claim. The maps ∂_n introduced in Statement (1) of the propositions are precisely the connecting homomorphisms, as defined on page 429, of the above short exact sequence of chain complexes. In particular the maps ∂_n are well-defined.

The claim follows basically immediately from the definitions. But for completeness' sake we give some details. We consider the following commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & C_n(B, A) & \longrightarrow & C_n(X, A) & \longrightarrow & C_n(X, B) \longrightarrow 0 \\ & & \downarrow \partial & & \downarrow \partial & & \downarrow \partial \\ 0 & \longrightarrow & C_{n-1}(B, A) & \longrightarrow & C_{n-1}(X, A) & \longrightarrow & C_{n-1}(X, B) \longrightarrow 0. \end{array}$$

Let $z \in H_n(X, A)$. We will now determine the image of z under the connecting homomorphism $H_n(X, B) \rightarrow H_{n-1}(B, A)$ which during the proof of the claim we denote by ∂^c .

We pick a representative $\sum_{j=1}^k a_j \cdot (\sigma_j: \Delta^n \rightarrow X)$ in $C_n(X)$. Note that $\sum_{j=1}^k a_j \cdot (\sigma_j: \Delta^n \rightarrow X)$ also defines an element in $C_n(X, A)$. We then take the boundary of the above singular n -chain under the boundary map $\partial_n: C_n(X, A) \rightarrow C_{n-1}(X, A)$ and we end up with the singular n -chain $\sum_{j=1}^k a_j \cdot \partial \sigma_j \in C_{n-1}(X, A)$. As we have seen in the proof of Lemma 26.8 this singular n -chain lies in fact in $C_{n-1}(B, A) \subset C_{n-1}(X, A)$. The homology class thereof is by definition precisely $\partial^c(z) \in H_{n-1}(B, A)$. Summarizing we have shown that

$$\partial^c(z) = \left[\sum_{j=1}^k a_j \cdot \partial \sigma_j \right].$$

This shows that the map defined in (1) agrees with ∂^c . This concludes the proof of the claim.

The fact that the stated sequence of homology groups is exact is now an immediate consequence of Proposition 26.9. Finally the functoriality of the long exact sequence is an immediate corollary to the remark on page 431. $\square$

For convenience we record two special cases of Proposition 26.12 in the following corollary.

Corollary 26.13. *Let X be a topological space and let $B \subset X$ be a subset. Then there exists a long exact sequence*

$$\dots \rightarrow H_n(B) \xrightarrow{i_*} H_n(X) \xrightarrow{p_*} H_n(X, B) \xrightarrow{\partial} H_{n-1}(B) \rightarrow \dots$$

⁴⁹⁴The third isomorphism theorem implies that $(C_*(X)/C_*(A))/(C_*(B)/C_*(A)) \cong C_*(X)/C_*(B)$.

Furthermore, if B is non-empty then there exists also a long exact sequence

$$\dots \rightarrow \tilde{H}_n(B) \xrightarrow{i_*} \tilde{H}_n(X) \xrightarrow{p_*} H_n(X, B) \xrightarrow{\partial} \tilde{H}_{n-1}(B) \rightarrow \dots$$

Both long exact sequences define a functor from the category of pairs of topological spaces that we had defined on page 81, to the category of chain complexes.

The long exact sequence of Corollary 26.12 is sometimes called the *long exact sequence (in reduced homology) of the pair (X, B)* .

Proof. The first statement follows from Proposition 26.12 applied to $A = \emptyset$ and the second statement follows from Proposition 26.12 applied to $A = \{a\}$ a point in B . $\square$

Example. We let $X = \mathbb{R}^2$ and $A = B_1(-2) \cup B_1(2)$. We want to determine the relative homology groups of (X, A) . We denote by $i: A \rightarrow X$ the inclusion map. By Corollary 26.13 we have a long exact sequence

$$H_2(X) \rightarrow H_2(X, A) \rightarrow H_1(A) \xrightarrow{i_*} H_1(X) \rightarrow H_1(X, A) \rightarrow H_0(A) \rightarrow H_0(X) \rightarrow H_0(X, A) \rightarrow 0.$$

By Corollary 25.5 (3) together with Lemma 24.13 we have $H_i(X) = H_i(\mathbb{R}^2) = 0$ and also $H_i(A) = H_i(B_1(-2)) \oplus H_i(B_1(2)) = 0$ for $i \geq 1$. Thus we see from the above long exact sequence that $H_i(X, A) = 0$ for $i \geq 2$. We denote by $\epsilon: H_0(\mathbb{R}^2) \rightarrow \mathbb{Z}$ the augmentation map that is an isomorphism by Proposition 24.5. Similarly we denote by $\mu_1: H_0(B_1(-2)) \rightarrow \mathbb{Z}$ and $\mu_2: H_0(B_1(2)) \rightarrow \mathbb{Z}$ the isomorphisms that are given by the augmentation maps. We obtain the following commutative diagram⁴⁹⁵

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_1(X, A) & \xrightarrow{\partial} & H_0(A) & \xrightarrow{i_*} & H_0(X) & \longrightarrow & H_0(X, A) & \longrightarrow & 0 \\ & & \downarrow = & & \downarrow \mu_1 \oplus \mu_2 \cong & & \downarrow \cong \epsilon & & \downarrow = & & \\ 0 & \longrightarrow & H_1(X, A) & \longrightarrow & \mathbb{Z}^2 & \longrightarrow & \mathbb{Z} & \longrightarrow & H_0(X, A) & \longrightarrow & 0. \end{array}$$

We want to determine the matrix representing the homomorphism $\mathbb{Z}^2 \rightarrow \mathbb{Z}$ at the bottom. We do so in the following claim.

Claim. The homomorphism $\epsilon \circ i_* \circ (\mu_1 \oplus \mu_2)^{-1}: \mathbb{Z}^2 \rightarrow \mathbb{Z}$ is given by the matrix $\begin{pmatrix} 1 & 1 \end{pmatrix}$.

We pick a point $P \in B_1(-2)$ and we pick a point $Q \in B_1(2)$. It follows from Proposition 24.5 and Lemma 24.13 that the map

$$\begin{aligned} \Psi: \mathbb{Z}^2 &\rightarrow H_0(A) = H_0(B_1(-2)) \oplus H_0(B_1(2)) \\ (m, n) &\mapsto [m \cdot P + n \cdot Q] \end{aligned}$$

is the inverse to $\mu_1 \oplus \mu_2$. Now it is easy to verify that $\epsilon \circ i_* \circ (\mu_1 \oplus \mu_2)^{-1} = \epsilon \circ i_* \circ \Psi$ applied to $(1, 0)$ and $(0, 1)$ gives in both cases the value 1. So the map is described by the matrix $\begin{pmatrix} 1 & 1 \end{pmatrix}$. This concludes the proof of the claim.

⁴⁹⁵Why is the map $\mathbb{Z}^2 \rightarrow \mathbb{Z}$ given by the matrix $\begin{pmatrix} 1 & 1 \end{pmatrix}$?

Since the top sequence in the above commutative diagram is exact and since the vertical maps are isomorphisms we see that the bottom sequence is also exact. It is now straightforward to see that

$$H_1(X, A) \cong \ker \{ \mathbb{Z}^2 \xrightarrow{\begin{pmatrix} 1 & 1 \end{pmatrix}} \mathbb{Z} \} \cong \mathbb{Z} \quad \text{and} \quad H_0(X, A) \cong \operatorname{coker} \{ \mathbb{Z}^2 \xrightarrow{\begin{pmatrix} 1 & 1 \end{pmatrix}} \mathbb{Z} \} \cong 0.$$

Now we return to the situation of the example on page 434. Let $\sigma: \Delta^1 \rightarrow X$ be a singular 1-simplex with $\sigma(0, 1) \in B_1(2)$ and $\sigma(1, 0) \in B_1(-2)$. We have

$$\begin{aligned} ((\mu_1 \oplus \mu_2) \circ \partial)[\sigma] &= ((\mu_1 \oplus \mu_2)([\partial\sigma]) = (\mu_1 \oplus \mu_2)(\underbrace{[\sigma(0, 1)]}_{\in B_1(2)} - \underbrace{[\sigma(1, 0)]}_{\in B_1(-2)}) = (1, -1). \\ &\quad \uparrow \\ &\quad \text{description of } \partial \text{ in Proposition 26.12} \end{aligned}$$

Since $(1, -1)$ is a generator of the kernel of $\mathbb{Z}^2 \xrightarrow{\begin{pmatrix} 1 & 1 \end{pmatrix}} \mathbb{Z}$ we see that $[\sigma]$ is in fact a generator of $H_1(X, A) \cong \mathbb{Z}$. In particular it is non-zero.

We conclude this section with the following corollary.

Corollary 26.14. *Let X be a topological space.*

- (1) *Let $A \subset B$ be subsets of X . If A is a deformation retract of B , then for any $n \in \mathbb{N}_0$ the inclusion induced map*

$$H_n(X, A) \rightarrow H_n(X, B)$$

is an isomorphism.

- (2) *If A is a deformation retract of X , then $H_n(X, A) = 0$ for all $n \in \mathbb{N}_0$.*

Proof. Let X be a topological space.

- (1) Let $A \subset B$ be a deformation retract of B . We consider the following commutative diagram

$$\begin{array}{ccccccccc} H_n(A) & \longrightarrow & H_n(X) & \longrightarrow & H_n(X, A) & \longrightarrow & H_{n-1}(A) & \longrightarrow & H_{n-1}(X) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H_n(B) & \longrightarrow & H_n(X) & \longrightarrow & H_n(X, B) & \longrightarrow & H_{n-1}(B) & \longrightarrow & H_{n-1}(X), \end{array}$$

where the vertical maps are induced by the inclusions of topological spaces and the horizontal maps are given by the long exact sequences of the pairs (X, A) and (X, B) from⁴⁹⁶ Corollary 26.13. The second and the fifth vertical map are both evidently isomorphisms. Since A is a deformation retract of B we obtain from Corollary 25.5 that the first and the fourth map are isomorphisms. We conclude with the Five-Lemma 26.10 that the middle vertical map is also an isomorphism.

- (2) The second statement follows from the first statement applied to $B = X$ and the trivial observation that $H_n(X, X) = 0$ for all $n \in \mathbb{N}_0$. $\square$

⁴⁹⁶Note that here we use not only the existence of the two long exact sequences in homology but we also use the functoriality of the long exact sequence.

The statement of Corollary 26.13 looks already very similar to the statement of Theorem 26.2 that we actually want to prove. Now we “only” have to find the connection between the groups $H_n(X, A)$ and $H_n(X/A)$. The notations already look very similar, and they do so for good reasons, since in both cases “one ignores what lies in A ”. Nonetheless, proving the precise relationship will still require some substantial piece of work.

26.5. The Excision Theorem. The key step in finding the connection between the groups $H_n(X, A)$ and $H_n(X/A)$ is given by the following Excision Theorem 26.15.

Theorem 26.15. (Excision Theorem) *Let X be a topological space and let $Z \subset A \subset X$ be subsets such that the closure of Z is contained in the interior of A . Then the inclusion $(X \setminus Z, A \setminus Z) \rightarrow (X, A)$ induces for each $n \in \mathbb{N}_0$ an isomorphism*

$$H_n(X \setminus Z, A \setminus Z) \xrightarrow{\cong} H_n(X, A).$$

The proposition thus says in particular that if we are given a pair of topological spaces (X, A) and if we excise a subset Z from the interior of A , then this does not affect the relative homology groups.

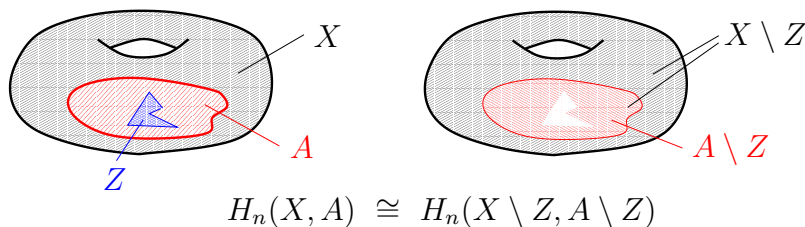


FIGURE 265. Illustration of the Excision Theorem 26.15.

Definition. Let X be a topological space and let $A \subset X$ be a non-empty subset. We denote by $p: (X, A) \rightarrow (X/A, A/A)$ the obvious projection map. For each $n \in \mathbb{N}_0$ we refer to

$$H_n(X, A) \xrightarrow{p_*} H_n(X/A, A/A) \xleftarrow[\cong]{\text{Lemma 26.11}} \tilde{H}_n(X/A)$$

as the *natural homomorphism* $H_n(X, A) \rightarrow \tilde{H}_n(X/A)$.

Proposition 26.16. *Let X be a topological space and let $A \subset X$ be a non-empty closed well-behaved subset. Then for each $n \in \mathbb{N}_0$ the natural homomorphism $H_n(X, A) \rightarrow \tilde{H}_n(X/A)$ is an isomorphism.*

Proof of Proposition 26.16 assuming the Excision Theorem 26.15. Let X be a topological space and let $A \subset X$ be a non-empty closed well-behaved subset. Since A is well-behaved there exists an open neighborhood U of A in X such that A is a deformation retract of U .

Now we consider the following diagram

$$\begin{array}{ccccc}
 H_n(X, A) & \longrightarrow & H_n(X, U) & \longleftarrow & H_n(X \setminus A, U \setminus A) \\
 \downarrow p_* & & \downarrow p_* & & \downarrow p_* \\
 H_n(X/A, A/A) & \longrightarrow & H_n(X/A, U/A) & \longleftarrow & H_n(X/A \setminus A/A, U/A \setminus A/A),
 \end{array}$$

hereby all the horizontal maps are induced by inclusions of pairs of topological spaces and all the vertical maps arise from the projection map $p: X \rightarrow X/A$. It follows easily from the definitions that the diagram commutes.

The map we are interested in is the vertical map on the left. The right vertical map is an isomorphism since⁴⁹⁷

$$X \setminus A = X/A \setminus A/A$$

and

$$U \setminus A = U/A \setminus A/A,$$

i.e. the map $(X/A \setminus A/A, U/A \setminus A/A) \rightarrow (X \setminus A, U \setminus A)$ is in particular a homeomorphism of pairs.

Our goal is to show that the vertical map on the left map is an isomorphism. This follows from the commutativity of the above diagram and from the following claim:

Claim. All horizontal maps are isomorphisms.

It follows from the Excision Theorem 26.15 that both the top horizontal and the bottom horizontal map on the right-hand side are isomorphisms. Since A is deformation retract of U we obtain from Corollary 26.14 that the top left horizontal map is an isomorphism. Exactly the same way one can show that the lower-left horizontal map is an isomorphism.⁴⁹⁸ Thus we have proved the claim. $\square$

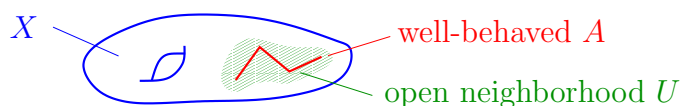


FIGURE 266. Illustration for the proof of Proposition 26.16.

Proof of Theorem 26.2 assuming the Excision Theorem 26.15. Let X be a topological space and let $A \subset X$ be a non-empty closed well-behaved subset. By Corollary 26.13 there exists a long exact sequence

$$\dots \rightarrow \tilde{H}_n(A) \xrightarrow{i_*} \tilde{H}_n(X) \xrightarrow{p_*} H_n(X, A) \xrightarrow{\partial} \tilde{H}_{n-1}(A) \rightarrow \dots$$

By Proposition 26.16 we have $H_n(X, A) \cong \tilde{H}_n(X/A)$ which gives us the desired long exact sequence. $\square$

⁴⁹⁷Note that these are really two equalities of sets.

⁴⁹⁸Hereby we use implicitly that A/A is a deformation retract of U/A . Why is this the case?

26.6. The proof of the Excision Theorem 26.15: the idea. We are still not done with the proof of Theorem 26.2 since we still have to prove the Excision Theorem 26.15. In this section we want to come up with the idea for the proof of the Excision Theorem 26.15. The full details of the proof will then be given in the following section.

Thus let X be a topological space and let $Z \subset A \subset X$ be two subsets such that the closure of Z is contained in the interior of A . Our goal is to show that the inclusion $(X \setminus Z, A \setminus Z) \rightarrow (X, A)$ induces for every $n \in \mathbb{N}_0$ an isomorphism

$$H_n(X \setminus Z, A \setminus Z) \rightarrow H_n(X, A).$$

Let us attempt to prove that the map $H_n(X \setminus Z, A \setminus Z) \rightarrow H_n(X, A)$ is surjective. We take an element in $H_n(X, A)$ and we choose a representative $a_1\sigma_1 + \cdots + a_r\sigma_r$ where $\sigma_i: \Delta^n \rightarrow X$, $i = 1, \dots, r$ are singular n -simplices in X .

We make the following two basic observations:

- (1) If the singular n -simplices $\sigma_1, \dots, \sigma_r$ lie in $X \setminus Z$,⁴⁹⁹ then $[a_1\sigma_1 + \cdots + a_r\sigma_r]$ evidently lies in the image of the inclusion induced map $H_n(X \setminus Z, A \setminus Z) \rightarrow H_n(X, A)$.
- (2) If a singular n -simplex lies entirely in A , then it represents the zero element in the quotient group $C_n(X, A) = C_n(X)/C_n(A)$.

The combination of these two observations proves the following lemma.

Lemma 26.17. *Let X be a topological space and let $Z \subset A$ be two subsets of X . Let $c = a_1\sigma_1 + \cdots + a_r\sigma_r$ be a cycle in X such that each σ_i lies entirely in A or it lies entirely in $X \setminus Z$. Then $[c]$ lies in the image of $H_n(X \setminus Z, A \setminus Z) \rightarrow H_n(X, A)$.*

Unfortunately not every cycle in $C_n(X)$ satisfies the hypothesis of the lemma. But we will see in the next section that we can subdivide each singular simplex into smaller singular simplices that lie entirely in A or lie entirely in $X \setminus Z$.

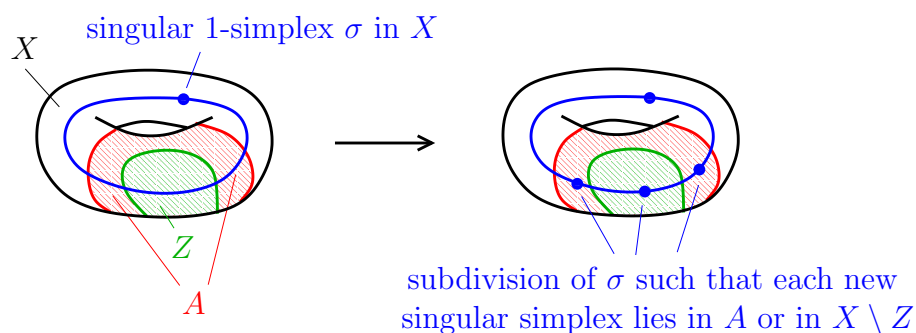


FIGURE 267. Illustration of the key idea of the proof of the Excision Theorem 26.15.

⁴⁹⁹Given a topological space X and a subset A we say that a singular simplex $\sigma: \Delta^n \rightarrow X$ lies in A if the image of σ lies in A .

26.7. The proof of the Excision Theorem 26.15: the full details. In this chapter we will now turn the vague idea of the last section into a rigorous proof of the Excision Theorem 26.15. This will take a surprisingly long time.

Thus let again X be a topological space. A *covering* of X is a family $\mathcal{U} = \{U_i\}_{i \in I}$ of subsets of X such that

$$X = \bigcup_{i \in I} \overset{\circ}{U}_i.$$

Given a covering $\mathcal{U} = \{U_i\}_{i \in I}$ we consider the following subgroup of $C_n(X)$:

$$C_n^{\mathcal{U}}(X) := \left\{ \sum_{j=1}^k a_j \sigma_j \mid \text{for each } j \text{ there exists a } U_i \in \mathcal{U} \text{ such that the image of } \sigma_j \text{ lies in } U_i \right\}.$$

It follows quite easily from the definitions that the restriction of usual the boundary map $\partial_n: C_n(X) \xrightarrow{\partial} C_{n-1}(X)$ to $C_n^{\mathcal{U}}(X)$ restricts to a map $\partial_n: C_n^{\mathcal{U}}(X) \xrightarrow{\partial} C_{n-1}^{\mathcal{U}}(X)$. Since this map is the restriction of the usual boundary map it follows in particular that the composition $\partial_n \circ \partial_{n+1}: C_{n+1}^{\mathcal{U}}(X) \xrightarrow{\partial} C_n^{\mathcal{U}}(X)$ is the zero map. We write

$$\begin{aligned} H_n^{\mathcal{U}}(X) &:= H_n(C_*^{\mathcal{U}}(X), \partial_*) \\ &= \ker \{ \partial_n: C_n^{\mathcal{U}}(X) \rightarrow C_{n-1}^{\mathcal{U}}(X) \} / \operatorname{im} \{ \partial_{n+1}: C_{n+1}^{\mathcal{U}}(X) \rightarrow C_n^{\mathcal{U}}(X) \}. \end{aligned}$$

The homology groups $H_n^{\mathcal{U}}(X)$ are thus defined by singular chains which are “small” in the sense that the image of each singular simplex lies in some U_i .

The following proposition says that we can determine the homology of a topological space through such “small” singular chains. As we will see later, this proposition is at the heart of many methods for computing homology groups.

Proposition 26.18. *Let X be a topological space and let $\mathcal{U} = \{U_i\}_{i \in I}$ be a covering of X . Then the inclusion map*

$$C_*^{\mathcal{U}}(X) \rightarrow C_*(X)$$

induces for every $n \in \mathbb{N}_0$ an isomorphism⁵⁰⁰

$$H_n^{\mathcal{U}}(X) \xrightarrow{\cong} H_n(X).$$

The idea of the proof of Proposition 26.18 is the one we already described in the previous section. We want to replace systematically each singular n -simplex $\sigma: \Delta^n \rightarrow X$ by a singular n -chain of the form $\sum_{j=1}^r a_j \cdot \sigma_j$ such that the image of each singular simplex σ_j lies in some $U_i \in \mathcal{U}$. As we have seen before in Lemma 24.3 in the 1-dimensional case, an appropriate subdividing procedure does not change the homology class. The systematic approach we are going to use is given by “barycentric subdivision”. The key idea is sketched in Figure 268 for the case $n = 2$.

Before we can execute this idea we have to introduce a few more definitions.

⁵⁰⁰In fact a stronger statement holds: the inclusion map $C_*^{\mathcal{U}}(X) \rightarrow C_*(X)$ is a homotopy equivalence which then implies by Corollary 25.3 that for every $n \in \mathbb{N}_0$ the induced map $H_n^{\mathcal{U}}(X) \xrightarrow{\cong} H_n(X)$ is an isomorphism. We will not make use of this stronger statement. This stronger statement is proved in [Ha, Proposition 2.21].

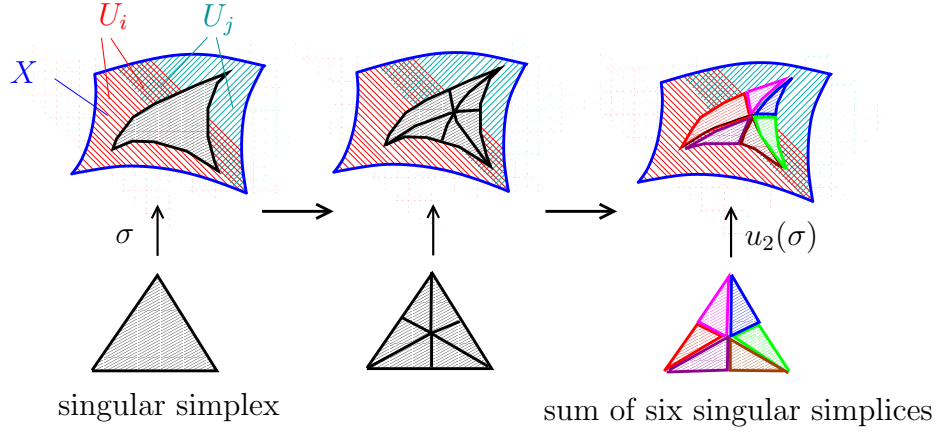


FIGURE 268. Barycentric decomposition of a singular 2-simplex.

- (1) We denote by $B_k := (\frac{1}{k+1}, \dots, \frac{1}{k+1}) \in \Delta^k$ the *barycenter*⁵⁰¹ of the standard n -simplex Δ^n .
- (2) Given a singular l -simplex $\mu: \Delta^l \rightarrow \Delta^k$ we consider the singular $(l+1)$ -simplex $c(\mu)$ which is defined as follows:⁵⁰²

$$c(\mu): \Delta^{l+1} \rightarrow \Delta^k$$

$$(t_0, \dots, t_{l+1}) \mapsto \begin{cases} \mu\left(\overbrace{\frac{1}{t_0 + \dots + t_l} \cdot (t_0, \dots, t_l)}^{\in \Delta^l}\right) \cdot (1 - t_{l+1}) + B_k \cdot t_{l+1}, & \text{if } t_{l+1} < 1, \\ B_k, & \text{if } t_{l+1} = 1. \end{cases}$$

The singular $(l+1)$ -simplex $c(\mu)$ thus corresponds to the “cone” which is given by the singular l -simplex μ and the barycenter B_k . By Lemma 8.1 this map $\mu \rightarrow c(\mu)$ defines a linear map $C_l(\Delta^k) \rightarrow C_{l+1}(\Delta^k)$ that we also denote by c and which we call the *cone map*.

- (3) For $n = 0, 1, 2, 3, \dots$ we define inductively a map $u_n: C_n(\Delta^k) \rightarrow C_n(\Delta^k)$ as follows:
 - (a) $u_0: C_0(\Delta^k) \rightarrow C_0(\Delta^k)$ is the identity.
 - (b) Suppose $u_{n-1}: C_{n-1}(\Delta^k) \rightarrow C_{n-1}(\Delta^k)$ is already defined, then we define

$$u_n: C_n(\Delta^k) \rightarrow C_n(\Delta^k)$$

$$\sum_{l=1}^r a_l \sigma_l \mapsto (-1)^n \sum_{l=1}^r a_l \cdot c\left(\underbrace{u_{n-1}(\partial \sigma_l)}_{\in C_{n-1}(\Delta^k)}\right).$$

We refer to u_n as the n -th *subdivision map*. The definition of u_1 is illustrated in Figure 270.

⁵⁰¹The word “barycenter” comes from the Greek word “barys” which means heavy. In physics the barycenter of an object is the center of mass, in our case the barycenter of the standard n -simplex is the center of mass if we view Δ^k as a physical object of uniform density.

⁵⁰²It is straightforward to verify that $c(\mu)$ is indeed continuous.

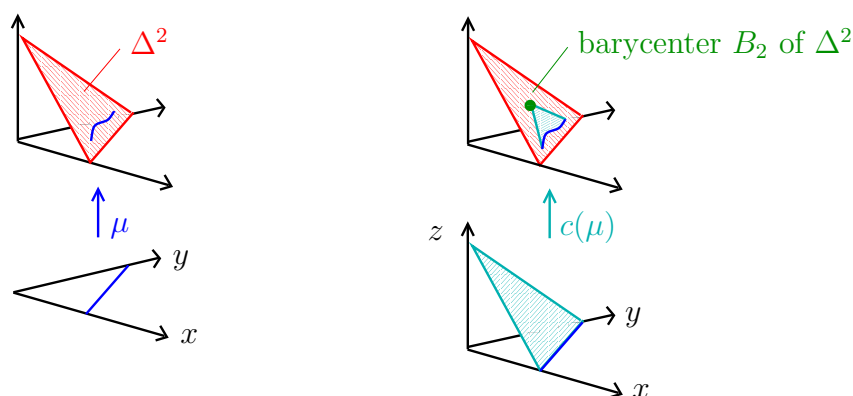


FIGURE 269. Illustration of the cone map.

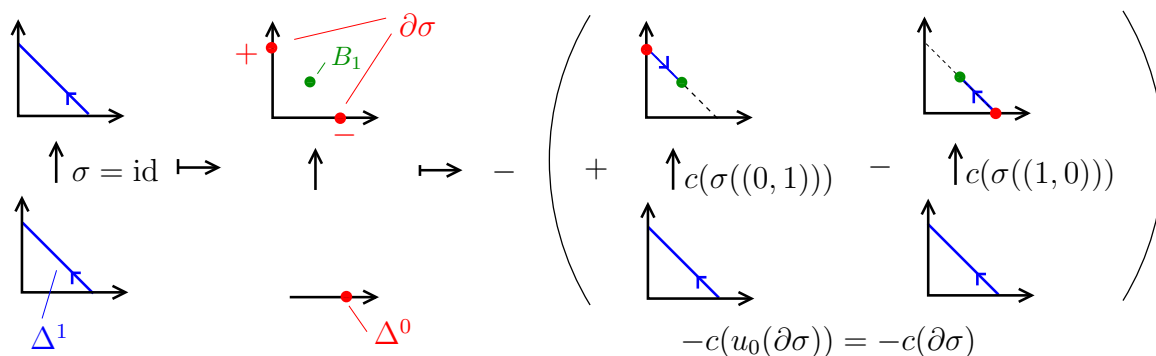


FIGURE 270. Illustration of the definition of first subdivision map.

- (4) Now we now want to define the subdivision map for singular n -simplices in an arbitrary topological space. We first make the observation that a singular n -simplex $\sigma: \Delta^n \rightarrow X$ is also given by $\sigma \circ \text{id}_{\Delta^n}: \Delta^n \xrightarrow{\text{id}} \Delta^n \rightarrow X$. Put differently, $\sigma = \sigma_*(\text{id}_{\Delta^n})$. Therefore we consider

$$\begin{aligned}
 u_n: C_n(X) &\rightarrow C_n(X) \\
 \sum_{l=1}^r a_l \cdot (\sigma_l: \Delta^n \rightarrow X) &\mapsto \sum_{l=1}^r a_l \cdot (\sigma_l)_* \underbrace{(u_n(\text{id}_{\Delta^n}))}_{\in C_n(\Delta^n)} \\
 &\quad \underbrace{\hspace{10em}}_{\in C_n(X)}
 \end{aligned}$$

The map u_n is illustrated for $n = 1$ in Figure 270. For $n = 2$ we give a more schematic sketch in Figure 268 without considering signs and orientations.

Before we can continue we need to prove several technical lemmas about the subdivision maps.

Lemma 26.19. *For any map $f: X \rightarrow Y$ between topological spaces and any $n \in \mathbb{N}_0$ we have*

$$f_* \circ u_n = u_n \circ f_*: C_n(X) \rightarrow C_n(Y).$$

Proof. Let $\sigma: \Delta^n \rightarrow X$ be a singular simplex. Then

$$\begin{array}{ccccccc} f_*(u_n(\sigma)) & = & f_*(\sigma_*(u_n(\text{id}_{\Delta^n}))) & = & (f \circ \sigma)_*(u_n(\text{id}_{\Delta^n})) & = & u_n(f_*\sigma). \\ \uparrow & & \uparrow & & \uparrow & & \\ \text{by definition of } u_n(\sigma) & & \text{since } (f \circ \sigma)_* = f_* \circ \sigma_* & & \text{by definition of } u_n(f_*\sigma) & & \\ & & \text{as maps } C_n(\Delta^n) \rightarrow C_n(Y) & & & & \end{array}$$

Since the desired equality holds for all generators of $C_n(X)$ it also holds for all of $C_n(X)$. $\square$

Lemma 26.20. *Let X be a topological space. The maps $u_n: C_n(X) \rightarrow C_n(X)$, $n \in \mathbb{N}_0$, form a chain map from the chain complex $(C_*(X), \partial)$ to itself.*

The statement of Lemma 26.20 is equivalent to the statement that for any singular n -simplex σ in X we have

$$u_{n-1}(\partial\sigma) = \partial(u_n(\sigma)).$$

Put differently, Lemma 26.20 says that it does not make a difference whether we first take the boundary and then subdivide, or whether we first subdivide and then take the boundary. This statement is sketched in Figure 271. The picture is quite convincing, and in fact the proof of the lemma is not very hard either.

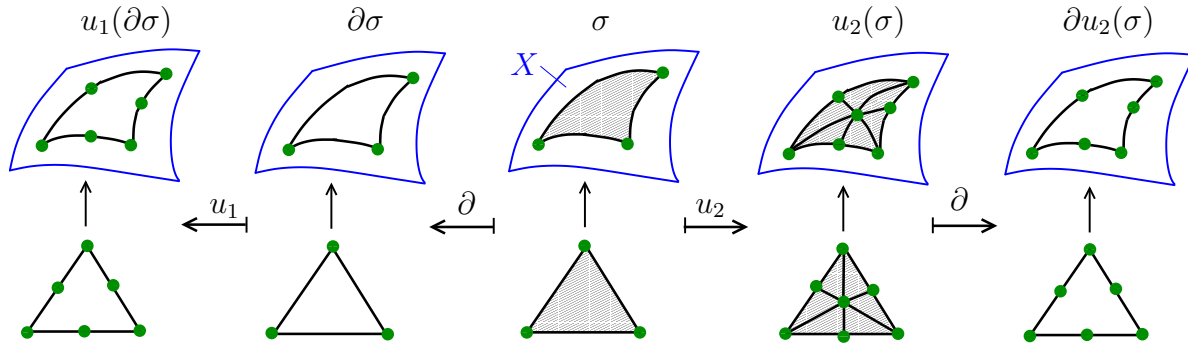


FIGURE 271. Illustration of Lemma 26.20.

Proof of Lemma 26.20 ().*⁵⁰³ We adopt and recall the following notation:

- (1) we denote the barycenter of Δ^n by b ,

⁵⁰³We will not do this proof in the lectures. The proof is somewhat lengthy but straightforward. The cleverness hereby does not lie in the proof, but in the cleverly chosen definitions, which turn the verification into a simple calculation.

- (2) as on page 400, given $x_0, \dots, x_k \in \Delta^n$ we denote by $[x_0, \dots, x_k]$ the singular k -chain $\Delta^k \rightarrow \Delta^n$ that is given by⁵⁰⁴

$$\begin{aligned} \Delta^k &\rightarrow \Delta^n \\ \sum_{i=0}^k t_i e_{i+1} &\mapsto \sum_{i=0}^k t_i x_i, \end{aligned}$$

- (3) for $x_0, \dots, x_k \in \Delta^n$ we write

$$[x_0, \dots, \widehat{x}_i, \dots, x_k] := [x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_k].$$

We make the following two observations.

- (A) For any $x_0, \dots, x_k \in \Delta^n$ we have by definition

$$c([x_0, \dots, x_k]) = [x_0, \dots, x_k, b].$$

- (B) For any $w_0, \dots, w_l \in \Delta^n$ we have by definition

$$\begin{aligned} \partial(c([w_0, \dots, w_l])) &= \partial([w_0, \dots, w_l, b]) \\ &= \sum_{j=0}^l (-1)^j [w_0, \dots, \widehat{w}_j, \dots, w_{l-1}, b] + (-1)^{l+1} [w_0, \dots, w_l] \\ &\quad \uparrow \\ &\quad \text{definition of } \partial \\ &\quad \downarrow \\ &= \partial[w_0, \dots, w_l, b] + (-1)^{l+1} [w_0, \dots, w_l]. \end{aligned}$$

By the linearity of c and ∂ we obtain for any $\sigma \in C_l(\Delta_n)$ the equality

$$\partial(c(\sigma)) = \partial\sigma + (-1)^{l+1}\sigma.$$

After these preparations we can now start with the actual proof of the lemma. The proof will be broken into two claims.

Claim. For any $k \in \{0, \dots, n\}$ and any $x_0, \dots, x_k \in \Delta^n$ we have

$$\partial(u_k([x_0, \dots, x_k])) = u_{k-1}(\partial[x_0, \dots, x_k]).$$

We will prove this claim by induction on k . The case $k = 0$ is trivial. So now suppose we have already shown the case $k - 1$. We then have

$$\begin{aligned} &\text{definition of } u_k, \text{ see page 444} \\ &\quad \downarrow \\ \partial(u_k([x_0, \dots, x_k])) &= (-1)^k \cdot \partial(c(u_{k-1}(\partial[x_0, \dots, x_k]))) \\ &= (-1)^k \cdot \partial(u_{k-1}(\partial[x_0, \dots, x_k])) + (-1)^k \cdot (-1)^k \cdot u_{k-1}(\partial[x_0, \dots, x_k]) \\ &\quad \uparrow \\ &\quad \text{statement (B) applied to } \sigma = u_{k-1}([x_0, \dots, x_k]) \\ &= (-1)^k \cdot u_{k-2}(\underbrace{\partial(\partial[x_0, \dots, x_k])}_{=0}) + u_{k-1}(\partial[x_0, \dots, x_k]) \\ &\quad \uparrow \\ &\quad \text{induction hypothesis} \\ &= u_{k-1}(\partial[x_0, \dots, x_k]). \end{aligned}$$

⁵⁰⁴It is straightforward to show that this map does indeed take values in Δ^n .

This concludes the proof of the claim.

The following claim is just a reformulation of the statement of the lemma.

Claim. For any singular n -simplex σ in X we have $u_{n-1}(\partial\sigma) = \partial(u_n(\sigma))$.

So let σ be a singular n -simplex in X . Then we have

$$\begin{array}{ccccccc}
 & \text{by definition} & & \text{since } \sigma_* \text{ is a chain map} & & & \\
 & \downarrow & & \downarrow & & & \\
 \partial(u_n(\sigma)) & = \partial\sigma_*(u_n(\text{id}_{\Delta^n})) & = & \partial\sigma_*(u_n(\text{id}_{\Delta^n})) & & & \\
 & = \sigma_*(u_n(\partial\text{id}_{\Delta^n})) & = & u_n(\sigma_*(\partial\text{id}_{\Delta^n})) & = & u_{n-1}(\partial\sigma_*(\text{id}_{\Delta^n})) & = u_{n-1}(\partial\sigma). \\
 & \uparrow & & \uparrow & & \uparrow & \\
 & \text{by the previous claim} & & \text{Lemma 26.19} & & \text{since } \sigma_* \text{ is a chain map} &
 \end{array}$$

This concludes the proof of the claim. $\square$

Lemma 26.21. *Let X be a topological space. The chain maps*

$$u_n: C_n(X) \rightarrow C_n(X)$$

are chain homotopic to the identity.

Proof. Showing that the chain maps $(u_n)_{n \in \mathbb{N}}$ are chain homotopic to the identity requires us to find maps $P_n: C_n(X) \rightarrow C_{n+1}(X)$, $n \in \mathbb{N}_0$ such that for each $n \in \mathbb{N}_0$ the following equality holds:

$$\partial_{n+1} \circ P_n = -P_{n-1} \circ \partial_n + u_n - \text{id}.$$

First we consider the case that X is a convex subset of some $\mathbb{R}^m$. Recall that we defined $C_{-1}(X) = 0$, accordingly we define $P_{-1}: C_{-1}(X) \rightarrow C_0(X)$ to be the zero map. Let us suppose that we have already found maps $P_{-1}, \dots, P_{n-1}$ with the desired properties.

We want to define the next map $P_n: C_n(X) \rightarrow C_{n+1}(X)$. This map has to have the property that for each singular n -simplex σ we have

$$\partial_{n+1}(P_n(\sigma)) = -(P_{n-1} \circ \partial_n)(\sigma) + (u_n - \text{id})(\sigma).$$

Put differently, to define $P_n(\sigma)$ we have to find a singular $(n+1)$ -chain $E_\sigma \in C_{n+1}(X)$ with

$$\partial E_\sigma = -(P_{n-1} \circ \partial)(\sigma) + (u_n - \text{id})(\sigma),$$

so that we can then define $P_n(\sigma) := E_\sigma$. Thus let $\sigma: \Delta^n \rightarrow X$ be a singular n -simplex.

(1) For $n = 0$ we have

$$\underbrace{(P_{-1} \circ \partial_0)}_{=0} + \underbrace{(u_0 - \text{id})}_{=\text{id}}(\sigma) = 0 \in C_0(X),$$

and we put $E_\sigma := 0 \in C_1(X)$.

(2) Now suppose that $n > 0$. Since X is convex we know from the discussion on page 420 that $H_n(X) = 0$. Therefore to show that an element $c \in C_n(X)$ is the boundary of

some element in $C_{n+1}(X)$ it suffices to show that $\partial c = 0$. In our specific situation we calculate

$$\begin{aligned} & \text{by Lemma 26.20 the subdivision maps form a chain map, i.e. } \partial \circ u_n = u_{n-1} \circ \partial \\ & \quad \downarrow \\ \partial((-P_{n-1}\partial_n + u_n - \text{id})(\sigma)) &= -\partial P_{n-1}\partial\sigma + u_{n-1}\partial\sigma - \partial\sigma \\ &= \underbrace{(-\partial P_{n-1} + u_{n-1} - \text{id})}_{=P_{n-2}\partial}(\partial\sigma) = P_{n-2}\underbrace{\partial\partial\sigma}_{=0} = 0. \end{aligned}$$

As we had just remarked, this implies that there exists an $E_\sigma \in C_{n+1}(X)$ with $\partial(E_\sigma) = (P_{n-1}\partial_n - u_{n-1} - \text{id})(\sigma)$.

We now denote by $P_n: C_n(X) \rightarrow C_{n+1}(X)$ the map which assigns to each singular n -simplex σ the element $E_\sigma \in C_{n+1}(X)$. Then it follows from the definition of E_σ that

$$\partial_{n+1} \circ P_n + P_{n-1} \circ \partial_n = u_n - \text{id}$$

as maps $C_n(X) \rightarrow C_n(X)$. This concludes the proof of Lemma 26.21 if X is a convex subset of some $\mathbb{R}^m$.

Now we deal with the general case. So let X be a topological space. We consider

$$\begin{aligned} P_n: C_n(X) &\rightarrow C_{n+1}(X) \\ \sum_{l=1}^r a_l \cdot (\sigma_l: \Delta^n \rightarrow X) &\mapsto \sum_{l=1}^r a_l \cdot (\sigma_l)_* \underbrace{(P_n(\text{id}_{\Delta^n}))}_{\substack{\uparrow \\ \text{defined since } \Delta^n \text{ is convex}}}. \end{aligned}$$

It is straightforward to see that this is a chain homotopy from the subdivision map to the identity map. $\square$

Definition. Given a topological space X and $m \in \mathbb{N}_0$ we denote by u_n^m the m -fold composition of $u_n: C_n(X) \rightarrow C_n(X)$.

Lemma 26.22. *For each topological space X and every $m \in \mathbb{N}$ the maps $\{u_n^m\}_{n \geq 0}$ form a chain map $C_*(X) \rightarrow C_*(X)$ that is chain homotopic to the identity.*

Proof. In Lemma 26.20 we had already seen that the maps $\{u_n\}_{n \geq 0}$ form a chain map $C_*(X) \rightarrow C_*(X)$. As we had remarked on page 409, the composition of chain maps is again a chain map. Thus for any $m \in \mathbb{N}_0$ the maps $\{u_n^m\}_{n \geq 0}$ are also chain maps.

By Lemma 26.21 we have $u_* \simeq \text{id}$. By Lemma 25.1 (2) this implies $u_*^2 \simeq u_*$, and by Lemma 25.1 (1) this implies $u_*^2 \simeq \text{id}$. An induction argument now shows that $u_*^m \simeq \text{id}$ for all m . $\square$

Definition. Given a singular chain c we can write it uniquely as $c = a_1\sigma_1 + \cdots + a_s\sigma_s$ with $a_1, \dots, a_s \in \mathbb{Z} \setminus \{0\}$ and where the σ_i are pairwise different. We then say $\sigma_1, \dots, \sigma_s$ are the *pieces* of c .

If $\sigma: \Delta^n \rightarrow X$ is a singular n -simplex, then the pieces of $u_n(\sigma)$ are evidently “smaller” than the pieces of σ , and by iteratively applying u_n one can hope to make the resulting

pieces “arbitrary small”. In the following two lemmas we will make it precise what we mean by “arbitrary small”. Before we formulate the first lemma we need to recall and introduce several definitions.

Definition.

- (1) Given a bounded subset A of $\mathbb{R}^{n+1}$ the diameter is defined as

$$d(A) := \sup \{ \|a - b\| \mid a, b \in A \} \in \mathbb{R}_{\geq 0}.$$

- (2) Given a singular simple $\sigma: \Delta^n \rightarrow X$ we denote by $|\sigma|$ the image of σ .

We can now formulate the following lemma.

Lemma 26.23. *For any $x_0, \dots, x_k \in \mathbb{R}^n$ and any piece τ of $u_k([x_0, \dots, x_k])$ we have⁵⁰⁵*

$$d(|\tau|) \leq \frac{n}{n+1} \cdot d(|[x_0, \dots, x_k]|).$$



FIGURE 272. Illustration of Lemma 26.23.

In the proof of Lemma 26.23 we will need the following elementary⁵⁰⁶ geometric lemma.

Lemma 26.24. *Let $y_0, \dots, y_k \in \mathbb{R}^n$.*

- (1) *We have the equality*

$$d(|[y_0, \dots, y_k]|) = \max \{ \|y_i - y_j\| \mid i, j = 1, \dots, k \}.$$

- (2) *We denote by $b = \frac{1}{k} \sum_{i=0}^k y_i$ the barycenter of $y_0, \dots, y_k$. Then for any $i = 0, \dots, k$ we have*

$$\|y_i - b\| \leq d(|[y_0, \dots, y_k]|).$$

Proof of Lemma 26.24 ().*

- (1) The inequality “ $\geq$ ” follows immediately from the definitions and the observation that each y_i lies in

$$|[y_0, \dots, y_k]| = \left\{ \sum_{i=0}^k t_i \cdot y_i \mid t_i \in [0, 1] \text{ and } \sum_{i=0}^k t_i = 1 \right\}.$$

⁵⁰⁵Recall the definition on page 400 of the map $[x_0, \dots, x_k]: \Delta^k \rightarrow \mathbb{R}^n$.

⁵⁰⁶Here “elementary” means that the statement and the proof are basically high school level mathematics which does not mean that the proofs are entirely obvious.

We turn to the proof of the reverse inequality “ $\leq$ ”. So we pick $a, b \in [y_0, \dots, y_k]$. By definition we can write $a = \sum_{i=0}^k s_i y_i$ and $b = \sum_{i=0}^k t_i y_i$ with $s_i, t_i \in [0, 1]$ and $\sum_{i=0}^k s_i = \sum_{i=0}^k t_i = 1$. We then have

$$\begin{aligned}
 & \text{since } \sum_{i=0}^k t_i = 1 \\
 & \downarrow \\
 \|a - b\| &= \left\| a - \sum_{i=0}^k t_i y_i \right\| = \left\| \sum_{i=0}^k t_i (a - y_i) \right\| \leq \sum_{i=0}^k \|t_i (a - y_i)\| \\
 & \leq \sum_{i=0}^k t_i \cdot \max_{i=0, \dots, k} \{ \|a - y_i\| \} = \max_{i=0, \dots, k} \{ \|a - y_i\| \} \leq \max_{i, j=0, \dots, k} \{ \|y_j - y_i\| \} \\
 & \quad \uparrow \quad \quad \quad \uparrow \\
 & \quad \text{since } \sum_{i=0}^k t_i = 1 \quad \quad \text{same argument applied to } a = \sum_{j=0}^k s_j y_j
 \end{aligned}$$

(2) We denote by b_i the barycenter of $y_0, \dots, \widehat{y_i}, \dots, y_k$, i.e.

$$b_i = \frac{1}{k} \sum_{j \neq i} y_j.$$

We then have

$$b = \frac{1}{k+1} y_i + \frac{k}{k+1} b_i.$$

It follows that

$$\begin{aligned}
 \|b - y_i\| &= \left\| \frac{1}{k+1} y_i + \frac{k}{k+1} b_i - y_i \right\| = \left\| \frac{k}{k+1} b_i - \frac{k}{k+1} y_i \right\| \\
 &= \frac{k}{k+1} \|b_i - y_i\| \leq \frac{k}{k+1} d([y_0, \dots, y_k]).
 \end{aligned}$$

Thus we have obtained the desired inequality. $\square$

Now we are ready to provide the proof of Lemma 26.23.

Proof of Lemma 26.23 ().* We need to show that for any $x_0, \dots, x_k \in \mathbb{R}^n$ and any piece τ of $u_k([x_0, \dots, x_k])$ we have

$$d(|\tau|) \leq \frac{n}{n+1} \cdot d([x_0, \dots, x_k]).$$

We prove the claim by induction on k . For $k = 0$ all the diameters are zero, so the inequality holds trivially. Now suppose that the claim holds for $k - 1$. Let $x_0, \dots, x_k \in \Delta^n$ and denote by $b = \frac{1}{k} \sum_{i=0}^k x_i$ the barycenter of $x_0, \dots, x_k$. We denote by $\sigma_1, \dots, \sigma_r$ the pieces of $\partial[x_0, \dots, x_k]$.⁵⁰⁷

Now let τ be a piece of $u_k([x_0, \dots, x_k])$. By the definition of the subdivision map given on page 445 we see that $|\tau| = [w_0, \dots, w_{k-1}, b]$ where $[w_0, \dots, w_{k-1}]$ is a piece of $u_{k-1}(\sigma_s)$ for some $s \in \{1, \dots, r\}$. By Lemma 26.24 (1) it now suffices to prove the following claim.

Claim.

(1) For any $i, j = 0, \dots, k - 1$ we have $\|w_i - w_j\| \leq \frac{k}{k+1} d([x_0, \dots, x_k])$.

⁵⁰⁷Why does it not say that the number of pieces is $k + 1$?

(2) For any $i = 0, \dots, k-1$ we have $\|w_i - b\| \leq \frac{k}{k+1} d([x_0, \dots, x_k])$.

We now provide a proof of these two inequalities. First let $i, j \in \{0, \dots, k-1\}$. Then

$$\begin{aligned} \|w_i - w_j\| &\leq d([w_0, \dots, w_{k-1}]) \leq \frac{k-1}{k} d(\sigma_s) \leq \frac{k}{k+1} d([x_0, \dots, x_k]). \\ &\quad \uparrow \qquad \qquad \qquad \uparrow \\ &\quad \text{induction hypothesis} \quad \text{since } |\sigma_s| \subset |[x_0, \dots, x_k]| \text{ and } \frac{k-1}{k} \leq \frac{k}{k+1} \\ &\quad \text{applied to } \sigma_s \end{aligned}$$

Now let $i \in \{0, \dots, k-1\}$. We then have

$$\begin{aligned} \|w_i - b\| &\leq \frac{k}{k+1} d([w_0, \dots, w_{k-1}]) \leq \frac{k}{k+1} d([x_0, \dots, x_k]) \\ &\quad \uparrow \qquad \qquad \qquad \uparrow \\ &\quad \text{Lemma 26.24 (2)} \qquad \text{since } [w_0, \dots, w_{k-1}] \subset [x_0, \dots, x_k] \end{aligned}$$

This concludes the proof of the claim. $\square$

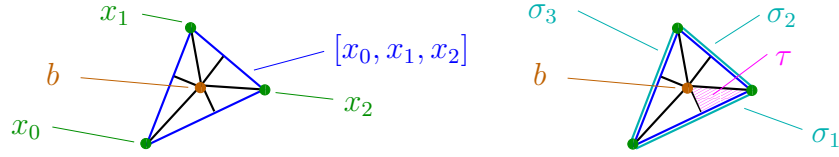


FIGURE 273. Illustration of the proof of Lemma 26.23.

Lemma 26.25. *Let X be a topological space and let $\mathcal{U} = \{U_i\}_{i \in I}$ be a covering of X . For every singular n -simplex $\sigma: \Delta^n \rightarrow X$ there exists an m such that $u_n^m(\sigma)$ lies in $C_n^{\mathcal{U}}(X)$.*

Proof. Let X be a topological space, let $\mathcal{U} = \{U_i\}_{i \in I}$ be a covering of X and let $\sigma: \Delta^n \rightarrow X$ be a singular n -simplex. By Lebesgue's Lemma 1.5 there exists a $\delta > 0$ such that for any subset A of Δ^n with diameter $d(A) < \delta$ there exists an $i \in I$ such that $A \subset \sigma^{-1}(\overset{\circ}{U}_i)$.

We pick an $m \in \mathbb{N}$ such that $(\frac{n}{n+1})^m < \delta$. We have

$$\begin{aligned} u_n^m(\sigma) &= u_n^m(\sigma_*(\text{id}_{\Delta^n})) = \sigma_*(u_n^m(\text{id}_{\Delta^n})) \stackrel{\text{by Lemma 26.23 and } (\frac{n}{n+1})^m < \delta}{=} \sigma_* \left(\begin{array}{c} \text{simplicial chain such that the image} \\ \text{has diameter less than } \delta \end{array} \right) \\ &\stackrel{\text{Lemma 26.19}}{\downarrow} \stackrel{\uparrow}{=} \sigma_* \left(\begin{array}{c} \text{simplicial chain such that the image} \\ \text{of each piece lies in some } \sigma^{-1}(\overset{\circ}{U}_i) \end{array} \right). \\ &\qquad \qquad \qquad \uparrow \\ &\qquad \qquad \text{by definition of } \delta \end{aligned}$$

But this means precisely that $u_n^m(\sigma) \in C_n^{\mathcal{U}}(X)$. $\square$

Now we can prove Proposition 26.18.

Proof of Proposition 26.18. Let X be a topological space and let $\mathcal{U} = \{U_i\}_{i \in I}$ be a covering of X . We want to show that the inclusion map $\iota_*: C_*^{\mathcal{U}}(X) \rightarrow C_*(X)$ induces for each n an isomorphism

$$\iota_*: H_n^{\mathcal{U}}(X) \xrightarrow{\cong} H_n(X).$$

We first show that the map $\iota_*: H_n^{\mathcal{U}}(X) \xrightarrow{\cong} H_n(X)$ is surjective. Let $z \in H_n(X)$ be given. We choose a cycle $c = \sum_{j=1}^r a_j \sigma_j \in C_n(X)$ which represents $z \in H_n(X)$. It follows from Lemma 26.25, applied to the singular n -simplices $\sigma_1, \dots, \sigma_r$, that there exists an $m \in \mathbb{N}$ such that $u_n^m(c) \in C_n^{\mathcal{U}}(X)$. By Lemma 26.22 the maps $\{u_n^m\}_{n \in \mathbb{N}_0}$ form a chain map that is chain homotopic to the identity. By definition this means that there exists a chain homotopy $P_k: C_k(X) \rightarrow C_{k+1}(X)$, $k \in \mathbb{N}_0$. Thus we obtain that

$$\begin{aligned} z = [c] &= [u_n^m(c) - \overbrace{P_{n-1}\partial_n(c)}^{=0} - \partial_{n+1}P_n(c)] = [u_n^m(c) - \partial_{n+1}P_n(c)] = \overbrace{[u_n^m(c)]}^{\in \iota_*(H_n^{\mathcal{U}}(X))} \\ &\quad \uparrow \qquad \qquad \qquad \text{since } \partial_{n+1}P_n + P_{n-1}\partial_n = u_n^m - \text{id} \qquad \qquad \qquad \uparrow \qquad \text{equality in } H_n(X) \end{aligned}$$

Thus we have shown that $\iota_*: H_n^{\mathcal{U}}(X) \xrightarrow{\cong} H_n(X)$ is surjective.

The argument that the map $\iota_*: H_n^{\mathcal{U}}(X) \xrightarrow{\cong} H_n(X)$ is injective is very similar. Indeed, let $[c] \in H_n^{\mathcal{U}}(X)$ be an element such that $[c] = 0 \in H_n(X)$. This means that there exists an $e \in C_{n+1}(X)$ with $\partial e = c$. From Lemma 26.25 it follows, as above, that there exists an $m \in \mathbb{N}$ with $u_{n+1}^m(e) \in C_{n+1}^{\mathcal{U}}(X)$. We again choose a chain homotopy $\{P_k\}_{k \in \mathbb{N}_0}$ between the chain maps $\{u_k^m\}_{k \in \mathbb{N}_0}$ and id . Then we have

$$\begin{aligned} c = \partial e &= \partial(u_{n+1}^m(e) - \partial P_{n+1}(e) - P_n \partial e) = \underbrace{\partial u_{n+1}^m(e)}_{\in C_{n+1}^{\mathcal{U}}(X)} - \underbrace{\partial P_n(c)}_{\in C_{n+1}^{\mathcal{U}}(X)} \\ &\quad \uparrow \\ &\quad \text{since } \partial P_{n+1} + P_n \partial = u_{n+1}^m - \text{id} \end{aligned}$$

Thus have we have shown that c is indeed the boundary of a $(n+1)$ -dimensional singular chain, in $C_{n+1}^{\mathcal{U}}(X)$ i.e. $[c] = 0 \in H_n^{\mathcal{U}}(X)$. Put differently, we have now proved that $\iota_*: H_n^{\mathcal{U}}(X) \xrightarrow{\cong} H_n(X)$ is injective. $\square$

Using Proposition 26.18 we can now prove the Excision Theorem 26.15.

Proof of the Excision Theorem 26.15. Let X be a topological space and let $Z \subset A \subset X$ be subsets such that the closure of Z is contained in the interior of A . We have to show that the inclusion $(X \setminus Z, A \setminus Z) \rightarrow (X, A)$ induces for each $n \in \mathbb{N}_0$ an isomorphism

$$H_n(X \setminus Z, A \setminus Z) \xrightarrow{\cong} H_n(X, A).$$

We put $B := X \setminus Z$. Note that $A \cap B = A \setminus Z$. It follows from our hypothesis on Z and A that X is the union of the interiors of A and B , put differently, the two sets $\{A, B\}$ is a covering of X .

For each n we consider the following commutative diagram of abelian groups:⁵⁰⁸

$$\begin{array}{ccccccc} 0 & \longrightarrow & C_n(A) & \longrightarrow & C_n^{\{A,B\}}(X) & \longrightarrow & C_n^{\{A,B\}}(X)/C_n(A) \longrightarrow 0 \\ & & \downarrow = & & \downarrow \hookrightarrow & & \downarrow \\ 0 & \longrightarrow & C_n(A) & \longrightarrow & C_n(X) & \longrightarrow & C_n(X, A) \longrightarrow 0. \end{array}$$

Note that by definition the horizontal sequences are in fact short exact sequences of chain complexes. We first make the observation that it follows immediately from the definition of $C_n^{\{A,B\}}(X)$ that the inclusion

$$C_n(B) \rightarrow C_n^{\{A,B\}}(X)$$

induces a chain map

$$C_n(B, A \cap B) \rightarrow C_n^{\{A,B\}}(X)/C_n(A)$$

that is in fact an isomorphism. Thus in the above diagram we can replace the chain complex $C_n^{\{A,B\}}(X)/C_n(A)$ by $C_n(B, A \cap B) = C_n(X \setminus Z, A \setminus Z)$.

Now we consider the long exact sequence of homology groups arising from the above short exact sequences of chain complexes. From the remark on page 431 we obtain the following commutative diagram

$$\begin{array}{ccccccccccc} \dots & \longrightarrow & H_n(A) & \longrightarrow & H_n^{\{A,B\}}(X) & \longrightarrow & \overbrace{H_n(B, A \cap B)}^{=H_n(X \setminus Z, A \setminus Z)} & \xrightarrow{\partial} & H_{n-1}(A) & \longrightarrow & H_{n-1}^{\{A,B\}}(X) & \longrightarrow \dots \\ & & \downarrow = & & \downarrow & & \downarrow & & \downarrow = & & \downarrow & \\ \dots & \longrightarrow & H_n(A) & \longrightarrow & H_n(X) & \longrightarrow & H_n(X, A) & \xrightarrow{\partial} & H_{n-1}(A) & \longrightarrow & H_{n-1}(X) & \longrightarrow \dots \end{array}$$

From Proposition 26.18 it follows that the second and fifth vertical map are isomorphisms. Since the first and fourth map are evidently isomorphisms it now follows from the Five-Lemma 26.10 that the middle map is also an isomorphism. $\square$

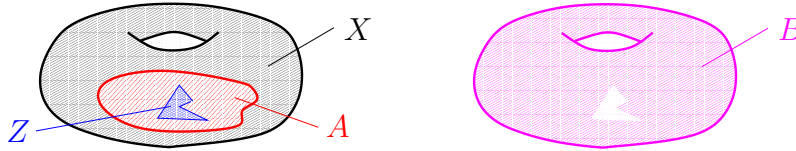


FIGURE 274. Illustration for the proof of the Excision Theorem 26.15.

We conclude this section with the following variation of the Excision Theorem 26.15.

⁵⁰⁸Here we use the definition from page 443 which in this context says that

$$C_n^{\{A,B\}}(X) := \{\sigma + \tau \mid \sigma \in C_n(A) \text{ and } \tau \in C_n(B)\}.$$

Theorem 26.26. *Let X be a topological space and let $Z \subset A \subset X$ be two subsets. We suppose that there exists a subset $U \subset Z$ such that the closure of U is contained in the interior of A and such that following statements hold:*

- (1) $X \setminus Z$ is a deformation retract of $X \setminus U$
- (2) $A \setminus Z$ is a deformation retract of $A \setminus U$.

Then the inclusion $(X \setminus Z, A \setminus Z) \rightarrow (X, A)$ induces for each $n \in \mathbb{N}_0$ an isomorphism

$$H_n(X \setminus Z, A \setminus Z) \xrightarrow{\cong} H_n(X, A).$$

The hypotheses of Theorem 26.26 might sound complicated, but in practice these conditions are often satisfied. For example we can now prove the following variation on the Excision Theorem 26.15 that we will use on many occasions.

Theorem 26.27. (Excision Theorem for Manifolds) *Let X be an m -dimensional manifold and let A and B be two m -dimensional submanifolds such that $A \cap B$ is the union of some boundary components of A and also the union of some boundary components of B . Then for every $n \in \mathbb{N}_0$ the inclusion induced map*

$$H_n(B, A \cap B) \xrightarrow{\cong} H_n(X, A)$$

is an isomorphism.

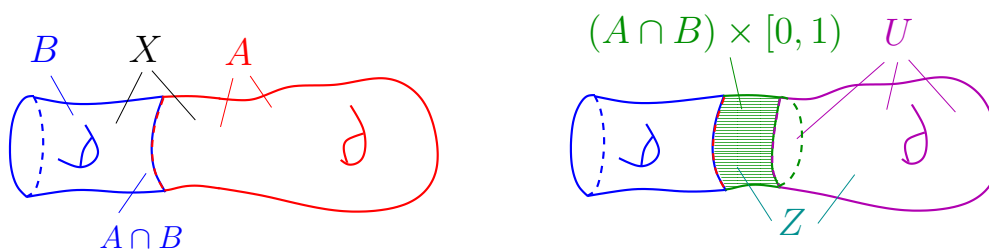


FIGURE 275. Illustration of Theorem 26.27.

Proof of Theorem 26.27 using Theorem 26.26. By the Collar Neighborhood Theorem 2.6 applied the manifold A admits a collar neighborhood of ∂A , i.e. there exists an embedding $f: \partial A \times [0, 1) \rightarrow A$ that is the identity map on $\partial A \times \{0\} = \partial A$. We set $Z = A \setminus (A \cap B)$ and $U = B \setminus f((A \cap B) \times [0, \frac{1}{2}))$. We make the following two observations:

- (1) The obvious deformation retraction from $(A \cap B) \times [0, \frac{1}{2})$ to $(A \cap B) \times \{0\} = A \cap B$ gives rise to a deformation retraction $f((A \cap B) \times [0, \frac{1}{2}))$ to $f(A \cap B)$ which in turn extends to a deformation retractions from $X \setminus U$ to $X \setminus Z$ and from $A \setminus U$ to $A \setminus Z$.
- (2) We have $(X \setminus Z, A \setminus Z) = (B, A \cap B)$.

The statement of Theorem 26.27 now follows immediately from Theorem 26.26. $\square$

Proof of Theorem 26.26. We consider the following commutative diagram of inclusion induced maps

$$\begin{array}{ccc} H_n(X \setminus Z, A \setminus Z) & \longrightarrow & H_n(X, A) \\ \downarrow & \nearrow \cong & \\ H_n(X \setminus U, A \setminus U) & & \end{array}$$

The diagonal map is an isomorphism by the Excision Theorem 26.15. Thus it suffices to show that the vertical map is an isomorphism.

We consider the following commutative diagram:

$$\begin{array}{ccccccccc} H_n(A \setminus Z) & \longrightarrow & H_n(X \setminus Z) & \longrightarrow & H_n(X \setminus Z, A \setminus Z) & \longrightarrow & H_{n-1}(A \setminus Z) & \longrightarrow & H_{n-1}(X \setminus Z) \\ \downarrow = & & \downarrow & & \downarrow & & \downarrow = & & \downarrow \\ H_n(A \setminus Z) & \longrightarrow & H_n(X \setminus U) & \longrightarrow & H_n(X \setminus U, A \setminus Z) & \longrightarrow & H_{n-1}(A \setminus Z) & \longrightarrow & H_{n-1}(X \setminus U) \\ \downarrow & & \downarrow = & & \downarrow & & \downarrow & & \downarrow = \\ H_n(A \setminus U) & \longrightarrow & H_n(X \setminus U) & \longrightarrow & H_n(X \setminus U, A \setminus U) & \longrightarrow & H_{n-1}(A \setminus U) & \longrightarrow & H_{n-1}(X \setminus U). \end{array}$$

Hereby the horizontal maps are pieces of the long exact sequences of the various pairs of topological spaces provided by Corollary 26.13. Furthermore the vertical maps are induced by the various inclusion maps. On the upper level the first and the fourth vertical maps are the identity. Furthermore it follows from the hypothesis that $X \setminus Z$ is a deformation retract of $X \setminus U$ and from Corollary 26.14 that the second and the fifth vertical map are also isomorphisms. It follows from the Five-Lemma 26.10 that the upper middle vertical map is also an isomorphism.

Using the hypothesis that $A \setminus Z$ is a deformation retract of $A \setminus U$ one can show exactly the same way that the lower middle vertical map is also an isomorphism. Combining the two middle vertical maps we obtain the desired isomorphism. $\square$

26.8. Explicit generators of homology groups. We have now finally proved the Excision Theorem 26.15, which in turn means that we have now given the complete proofs of Proposition 26.3 which says that for any $n \in \mathbb{N}_0$ we have

$$\tilde{H}_k(S^n) \cong \begin{cases} \mathbb{Z}, & \text{if } k = n, \text{ and} \\ 0, & \text{if } k \neq n. \end{cases}$$

Thus we have determined the homology groups of all finite-dimensional spheres. In this chapter, given $n \in \mathbb{N}_0$, we want to determine an explicit singular n -chain which is a generator of $\tilde{H}_n(S^n)$.

We start out with the following elementary lemma that we will use explicitly and implicitly on many occasions.

Lemma 26.28. *Let $n \in \mathbb{N}_0$. We have*

$$\tilde{H}_k(\overline{B}^n, S^{n-1}) \cong \begin{cases} \mathbb{Z}, & \text{if } k = n, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. We consider the long exact sequence in reduced homology of the pair $(\overline{B}^n, S^{n-1})$:

$$\begin{array}{ccccccc} \dots & \rightarrow & \underbrace{\widetilde{H}_k(\overline{B}^n)}_{=0, \text{ see page 422}} & \rightarrow & H_k(\overline{B}^n, S^{n-1}) & \xrightarrow{\partial} & \widetilde{H}_{k-1}(S^{n-1}) \rightarrow \underbrace{\widetilde{H}_{k-1}(\overline{B}^n)}_{=0, \text{ see page 422}} \rightarrow \dots \end{array}$$

The statement of the lemma is now an immediate consequence of Proposition 26.3. $\square$

We continue with the following lemma.

Lemma 26.29. *For any $n \in \mathbb{N}_0$ the following two statements hold:*

- (1) $H_n(\Delta^n, \partial\Delta^n) \cong \mathbb{Z}$.
- (2) *The identity map $\text{id}: \Delta^n \rightarrow \Delta^n$ represents a generator of $H_n(\Delta^n, \partial\Delta^n) \cong \mathbb{Z}$.*

Proof. The first statement is an immediate consequence of Lemmas 26.28 and 24.1. Now we turn to the proof of the second statement. We prove the second statement by induction on n . It follows easily from Proposition 24.5 that the statement holds for $n = 0$. Now we suppose that the desired statement holds for $n - 1$ with $n \in \mathbb{N}$.

Our next goal is to find a connection between the two homology groups $H_n(\Delta^n, \partial\Delta^n)$ and $H_{n-1}(\Delta^{n-1}, \partial\Delta^{n-1})$. We introduce the following notation:

- (1) We consider

$$\Lambda := \bigcup_{i=0}^{n-1} \{(t_0, \dots, t_n) \in \Delta^n \mid t_i = 0\},$$

i.e. Λ is the union of all $(n - 1)$ -dimensional faces of Δ^n except for the face given by $t_n = 0$. It is elementary⁵⁰⁹ to show that Λ is a deformation retract of Δ^n . We refer to Figure 276 for an illustration.

- (2) We write

$$\mathring{\Lambda} := \bigcup_{i=1}^n \{(t_0, \dots, t_n) \in \Lambda \mid t_i = 0 \text{ and } t_0 > 0\},$$

the set $\mathring{\Lambda}$ is thus the “interior”⁵¹⁰ of Λ .

In the following we view $\mathbb{R}^n$ as the subset of $\mathbb{R}^{n+1}$ given by setting the last coordinate equal to zero. This point of view allows us to view Δ^{n-1} as a subset of Δ^n . Note that with this convention we have $\partial\Delta^{n-1} \subset \Lambda$.

Claim. We consider the maps

$$H_n(\Delta^n, \partial\Delta^n) \xrightarrow{\partial} H_{n-1}(\partial\Delta^n, \Lambda) \xleftarrow{i_*} H_{n-1}(\Delta^{n-1}, \partial\Delta^{n-1}).$$

⁵⁰⁹When we say that it is “elementary to show something” this usually means that one can show the statement using basic methods without the need to use advanced results. In this case one can write down a deformation retraction “by hand”, but it is not entirely easy to do so. We leave it as an exercise to write down such a deformation retraction.

⁵¹⁰Why is “interior” put in quotes?

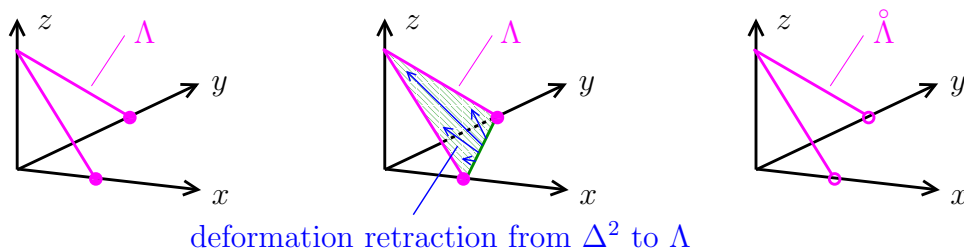
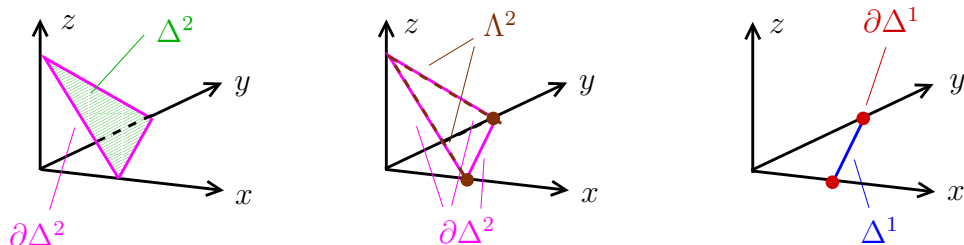


FIGURE 276.

Hereby the map on the left is the connecting homomorphism in the long exact sequence of the triple $(\Delta^n, \partial\Delta^n, \Lambda)$ and the right hand map is induced by the inclusion. Then the following holds:

- (1) Both maps are isomorphisms.
- (2) $\partial([\text{id}_{\Delta^n}]) = (-1)^n \cdot i_*([\text{id}_{\Delta^{n-1}}])$.

FIGURE 277. Illustration of the pairs $(\Delta^n, \partial\Delta^n)$, $(\partial\Delta^n, \Lambda)$ and $(\Delta^{n-1}, \partial\Delta^{n-1})$.

It is clear that a proof of the claim together with the induction hypothesis implies that $[\text{id}_{\Delta^n}]$ is a generator of $H_n(\Delta^n, \partial\Delta^n)$. In particular a proof of the claim implies Lemma 26.29.

Now we turn to the proof of the claim. Of the two maps in the claim we first consider the map ∂ on the left. It follows from the exact sequence of the triple $(\Delta^n, \partial\Delta^n, \Lambda)$ that the following sequence is exact:

$$\underbrace{H_n(\Delta^n, \Lambda)}_{= 0 \text{ by Corollary 26.14 since } \Lambda^n \text{ is a def. retract of } \Delta^n} \xrightarrow{p} H_n(\Delta^n, \partial\Delta^n) \xrightarrow{\partial} H_{n-1}(\partial\Delta^n, \Lambda) \xrightarrow{i_*} \underbrace{H_{n-1}(\Delta^n, \Lambda)}_{= 0 \text{ by Corollary 26.14 since } \Lambda^n \text{ is a def. retract of } \Delta^n}.$$

This shows that $H_n(\Delta^n, \partial\Delta^n) \xrightarrow{\partial} H_{n-1}(\partial\Delta^n, \Lambda)$ is an isomorphism. Furthermore we have

$$\begin{array}{ccccccc} \partial([\text{id}_{\Delta^n}]) & = & [\partial \text{id}_{\Delta^n}] & = & \left[\underbrace{\sum_{j=0}^{n-1} (-1)^j \text{id} \circ i_j^n}_{\text{lies in } C_{n-1}(\Lambda)} + (-1)^n \cdot \underbrace{\text{id} \circ i_n^n}_{=\text{id}_{\Delta^{n-1}}} \right] & = & (-1)^n \cdot [\text{id}_{\Delta^{n-1}}]. \\ \uparrow & & & & & \uparrow & \\ \text{Proposition 26.12} & & & & & \text{since we work in } H_{n-1}(\partial\Delta^n, \Lambda) & \end{array}$$

Now we turn our attention to the right-hand map in the statement of the claim. It follows from Theorem 26.26, applied to $X = \partial\Delta^n$, $A = \Lambda$, $Z = \mathring{\Delta}^n$ and $U = (0, \dots, 0, 1)^{511}$ that the inclusion induced map

$$H_{n-1}(\Delta^{n-1}, \partial\Delta^{n-1}) = H_{n-1}(\underbrace{\partial\Delta^n \setminus \mathring{\Delta}^n}_{=\Delta^{n-1}}, \underbrace{\Lambda \setminus \mathring{\Delta}^n}_{=\partial\Delta^{n-1}}) \xrightarrow{i_*} H_{n-1}(\partial\Delta^n, \Lambda)$$

is an isomorphism. Furthermore we evidently have $i_*([\text{id}_{\Delta^{n-1}}]) = [\text{id}_{\Delta^{n-1}}]$.

Thus we have proved that both maps in the claim are isomorphisms and that

$$\partial([\text{id}_{\Delta^n}]) = (-1) \cdot [\text{id}_{\Delta^{n-1}}] = (-1)^n \cdot i_*([\text{id}_{\Delta^{n-1}}]).$$

This concludes the proof of the claim and thus also of the lemma. $\square$

We now want to find explicit generators of $H_n(S^n) \cong \mathbb{Z}$ for $n \geq 1$. We consider the map $\alpha: \Delta^n \rightarrow S^n$ that is given by the concatenation of the following two maps

$$\begin{array}{c} \Delta^n \xrightarrow{\text{Lemma 24.1}} \overline{B}^n \rightarrow S^n \subset \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R} \\ x \mapsto (x, \sqrt{1 - \|x\|^2}). \end{array}$$

This map restricts to a homeomorphism $\Delta^n \rightarrow S_{\geq 0}^n = \{(x_1, \dots, x_{n+1}) \in S^n \mid x_{n+1} \geq 0\}$ and it sends $\partial\Delta^n$ to $S^{n-1} = \{(x_1, \dots, x_{n+1}) \in S^n \mid x_{n+1} = 0\}$. The definition of α is also sketched in Figure 278.

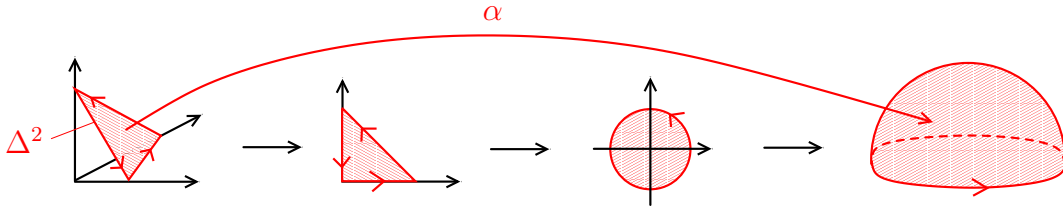


FIGURE 278.

We now denote by $\beta: \Delta^n \rightarrow S^n$ the map which is the composition of α with reflection in the hyperplane defined by $x_{n+1} = 0$. The maps $\alpha: \Delta^n \rightarrow S^n$ and $\beta: \Delta^n \rightarrow S^n$ both define singular n -chains in S^n . Note that α and β agree on $\partial\Delta^n$. Both maps are sketched in Figure 279.

Lemma 26.30. *Let $n \geq 1$. The above singular n -chain $\alpha - \beta$ is a cycle and it represents a generator of $\tilde{H}_n(S^n) = H_n(S^n)$.*

⁵¹¹One can easily show that U has the property required in Theorem 26.26.

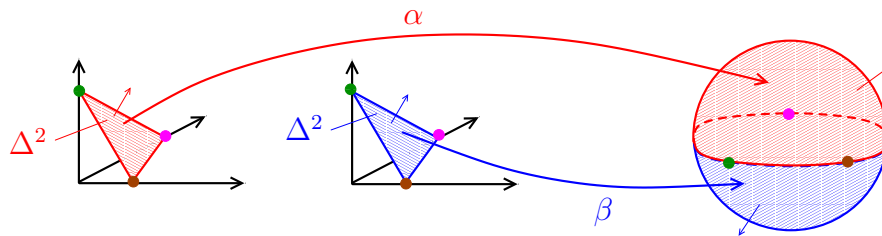


FIGURE 279.

Definition.

- (1) We refer to the generator from Lemma 26.30 as the *standard generator of $H_n(S^n)$* .⁵¹²
- (2) Slightly more generally, if I is an interval and x is a point in I , then the inclusion $S^n \times \{x\} \rightarrow S^n \times I$ is a homotopy equivalence which implies by Corollary 25.5 that $H_n(S^n \times \{x\}) \rightarrow H_n(S^n \times I)$ is an isomorphism. We refer to the image of the standard generator of $H_n(S^n \times \{x\}) = H_n(S^n)$ as the *standard generator of $H_n(S^n \times I)$* .⁵¹³
- (3) The *standard generator of $\tilde{H}_n(\overline{B}^n, S^{n-1})$* is defined to be the unique element that gets sent to the standard generator of $\tilde{H}_{n-1}(S^{n-1})$ under the connecting homomorphism $H_n(\overline{B}^n, S^{n-1}) \rightarrow \tilde{H}_{n-1}(S^{n-1})$ of the long exact sequence in reduced homology of the pair $(\overline{B}^n, S^{n-1})$.

Proof. We start out with the following claim.

Claim. The singular n -chain $\alpha - \beta$ is a cycle.

We denote by ρ the reflection in the hyperplane defined by $x_{n+1} = 0$. We have

$$\begin{array}{ccccccc}
 & & & & \text{since } \alpha(\partial\Delta^n) \subset \mathbb{R}^n \times 0 \text{ we have } \partial\alpha \in C_{n-1}(\mathbb{R}^n \times 0) & & \\
 & & & & \downarrow & & \\
 \partial(\alpha - \beta) & = & \partial(\alpha - \rho_*(\alpha)) & = & (\text{id} - \rho_*)(\partial\alpha) & = & 0. \\
 & & \uparrow & & \uparrow & & \\
 & & \text{since } \rho_* \text{ is a chain map} & & \text{since } \rho = \text{id on } \mathbb{R}^n \times 0 \subset \mathbb{R}^{n+1} & &
 \end{array}$$

This concludes the proof of the claim.

⁵¹²It follows easily from Lemma 24.3 and the definitions that the two definitions of standard generator of $H_1(S^1)$ agree.

⁵¹³It follows from Corollary 25.5 that the definition of the standard generator of $H_n(S^n \times I)$ does not depend on the choice of x .

We denote by

$$S_{\geq 0}^n := \{(x_1, \dots, x_{n+1}) \in S^n \mid x_{n+1} \geq 0\}$$

and

$$S_{\leq 0}^n := \{(x_1, \dots, x_{n+1}) \in S^n \mid x_{n+1} \leq 0\}$$

the two “hemispheres”. Both hemispheres are homomorphic to the closed ball $\overline{B}^n$, in particular they are contractible which implies by the discussion on page 422 that their reduced homology groups vanish. We also consider the “equator”

$$S_{=0}^n := \{(x_1, \dots, x_{n+1}) \in S^n \mid x_{n+1} = 0\}.$$

Next we consider the following sequence of maps

isomorphism by the long exact sequence in reduced homology of the pair $(S^n, S_{\leq 0}^n)$	isomorphism by the Excision Theorem for Manifolds 26.27	isomorphism since α is a homeomorphism
$\downarrow$	$\downarrow$	$\downarrow$
$\tilde{H}_n(S^n) \rightarrow H_n(S^n, S_{\leq 0}^n)$	$\xleftarrow{i_*} H_n(S_{\geq 0}^n, S_{=0}^n)$	$\xleftarrow{\alpha_*} H_n(\Delta^n, \partial\Delta^n)$
$[\alpha - \beta] \mapsto [\alpha - \beta] = [\alpha]$	$\mapsto [\alpha]$	$\mapsto [\text{id}_{\Delta^n}]$
	$\uparrow$ since $\beta = 0$ in $C_n(S^n, S_{\leq 0}^n)$	

So we have seen that all maps are isomorphisms. Furthermore, by Lemma 26.29 we know that $[\text{id}_{\Delta^n}]$ is a generator of $H_n(\Delta^n, \partial\Delta^n)$. But $[\text{id}_{\Delta^n}]$ and $[\alpha - \beta]$ get sent to the same element in $H_n(S^n, S_{\leq 0}^n)$, therefore $[\alpha - \beta]$ must also be generator of $\tilde{H}_n(S^n)$. $\square$

In the following lemma we give an alternative description of the standard generator of $H_1(S^1)$.

Lemma 26.31. *The singular 1-simplex*

$$\begin{aligned} \sigma: \Delta^1 &\rightarrow S^1 \\ (1-t, t) &\mapsto e^{2\pi i t} \end{aligned}$$

is a cycle that represents the standard generator of $H_1(S^1) \cong \mathbb{Z}$.



FIGURE 280. Two representatives of the standard generator of $H_1(S^1)$.

Proof. We denote by $\alpha, \beta: \Delta^1 \rightarrow S^1$ the singular 1-simplices in the definition of the standard generator of $H_1(S^1)$. We have

$$\begin{array}{ccccc} [\alpha - \beta] & = & [\alpha + \bar{\beta}] & = & [\sigma] \in H_1(S^1). \\ \uparrow & & \uparrow & & \\ & \text{by Lemma 24.3 (2) and (3)} & & & \end{array}$$

We refer to Figure 280 for an illustration. □

Remark. By Lemma 26.31 we know that there exists a singular 1-simplex σ that represents a generator of $H_1(S^1)$. In exercise sheet 6 we will discuss the question, for which $n \in \mathbb{N}$ does there exist a singular n -simplex $\sigma: \Delta^n \rightarrow S^n$ that represents a generator of $H_n(S^n)$.

26.9. Applications to topological manifolds. In this last section, at the very end of this very long chapter, we will apply some of the results that we have obtained to the study of topological manifolds.

First of all we can now generalize Proposition 11.10 from the case of 2-dimensional topological manifolds to the higher-dimensional case.

Proposition 26.32. *Let X be an n -dimensional topological manifold with $\partial X \neq \emptyset$. Then $Y := \partial X$ is an $(n-1)$ -dimensional topological manifold with $\partial Y = \emptyset$.*

Proof. The proof is formally exactly the same as the proof of Proposition 11.10 where we had proved the statement for $n = 2$. We just need to replace the fundamental group by the $(n-1)$ -st reduced homology groups and we need to replace the properties of the fundamental group we used in the proof of Proposition 11.10 by the corresponding properties of the reduced homology groups. In particular we need to replace the fact that $\pi_1(S^1) \neq 0$ by the corresponding statement, given in Proposition 26.3, that $\tilde{H}_{n-1}(S^{n-1}) = \mathbb{Z}$ and we need to replace the fact that the fundamental group of a star-shaped subset of $\mathbb{R}^2$ has trivial fundamental group by the fact, obtained in Corollary 25.5, that the $(n-1)$ -st reduced homology group of a star-shaped subset of $\mathbb{R}^n$ vanishes. We leave the details to the reader. □

Lemma 26.33. *Let M be a k -dimensional topological manifold. Then for any $P \in M$ and any $l \in \mathbb{N}_0$ we have*

$$H_l(M, M \setminus \{P\}) \cong \begin{cases} \mathbb{Z}, & \text{if } P \in M \setminus \partial M \text{ and } l = k, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Let P be a point on a k -dimensional topological manifold M . First consider the case that $P \in M \setminus \partial M$. It follows from the definition of ∂M , given on page 42, that we

can pick a chart $\Phi: U \rightarrow B^k$ for P with $\Phi(P) = 0$. We then have

$$\begin{array}{ccccc}
 & \text{isomorphism since } \Phi \text{ is a homeomorphism} & & & \\
 & \downarrow \Phi_* & & & \\
 H_l(M, M \setminus \{P\}) & \leftarrow H_l(U, U \setminus \{P\}) & \xrightarrow{\Phi_*} & H_l(B^k, B^k \setminus \{0\}) & \xrightarrow{\partial} \tilde{H}_{l-1}(B^k \setminus \{0\}) \leftarrow \tilde{H}_{l-1}(S^{k-1}). \\
 \uparrow & & & \uparrow & \uparrow \\
 \text{isomorphism by the} & \text{isomorphism by the long exact} & & \text{isomorphism since} & \\
 \text{Excision Theorem 26.15 applied} & \text{sequence in reduced homology} & & S^{k-1} \text{ is homotopy} & \\
 \text{to } X = M, A = M \setminus \{P\}, Z = M \setminus U & \text{of the pair } (B^k, B^k \setminus \{0\}) \text{ and the} & & \text{equivalent to } B^k \setminus \{0\} & \\
 & \text{fact that } \tilde{H}_*(B^k) = 0 \text{ by page 422} & & &
 \end{array}$$

For $P \in M \setminus \partial M$ the statement now follows from the calculation of the reduced homology groups of spheres given in Proposition 26.3.

The case that $P \in \partial M$ is dealt with in a very similar way. The proof of this case is an exercise in exercise sheet 6. $\square$

Example. Let M be a k -dimensional topological manifold and let $P \in M \setminus \partial M$. In the following we want to give an example of an explicit singular k -simplex that represents a generator for $H_k(M, M \setminus \{P\}) \cong \mathbb{Z}$. We pick a chart $\Phi: U \rightarrow B^k$ around P with $\Phi(P) = 0$ and we pick an affine linear map $\sigma: \Delta^k \rightarrow B^k$ such that 0 lies in the image of the interior $\overset{\circ}{\Delta}^k := \Delta^k \setminus \partial \Delta^k$ of Δ^k . Then a variation on the proof of Lemma 26.33 together with Lemma 26.29 shows that $\Phi^{-1} \circ \sigma: \Delta^k \rightarrow M$ represents a generator of $H_k(M, M \setminus \{P\})$. We leave it as an exercise to fill in the details. We refer to Figure 281 for an illustration.

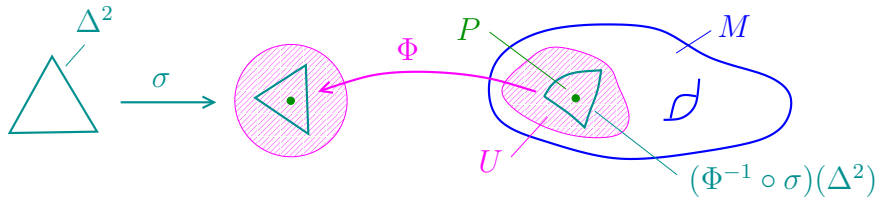


FIGURE 281.

We will prove the following corollary in exercise sheet 6.⁵¹⁴

Corollary 26.34. *If M is a non-empty topological manifold, then $M \neq \partial M$, i.e. there exists a point in $M \setminus \partial M$.*

Proposition 26.35. *Let M be an m -dimensional non-empty topological manifold and let N be an n -dimensional topological manifold. If M and N are homeomorphic, then $m = n$.*

It follows in particular from Proposition 26.35 that the dimension of a non-empty topological manifold is well-defined.⁵¹⁵

⁵¹⁴One might think that the statement of the corollary is obvious, but it does not follow from the definition of a topological manifold. One really does need to use Lemma 26.33.

⁵¹⁵We had already pointed out in Footnote 477 that we had known basically since Analysis II that manifolds of different dimensions cannot be *diffeomorphic*.

Proof. Let $f: M \rightarrow N$ be a homeomorphism between a non-empty m -dimensional topological manifold M and a topological manifold N . By Corollary 26.34 we can find a point $P \in M \setminus \partial M$. Then

$$\begin{array}{ccc} \mathbb{Z} & \cong & H_m(M, M \setminus \{P\}) & \xrightarrow{\cong} & H_m(N, N \setminus \{f(P)\}) \\ \uparrow & & & & \uparrow \\ & \text{Lemma 26.33} & & & \text{since } f \text{ is a homeomorphism} \end{array}$$

It now follows from Lemma 26.33 that N is also m -dimensional. □

27. THE DEGREE OF A SELF-MAP OF A SPHERE

Now that we know the homology groups of spheres, we want to study the maps on homology induced by maps $f: S^n \rightarrow S^n$. This is interesting in its own right, but it will also play a major role later on when we will try to determine the homology groups of CW-complexes.

Definition. Let $n > 0$ and let $f: S^n \rightarrow S^n$ be map. Since $H_n(S^n)$ is isomorphic to $\mathbb{Z}$ we know that the induced map $f_*: H_n(S^n) \rightarrow H_n(S^n)$ is given by multiplication by a uniquely determined integer $\deg(f) \in \mathbb{Z}$ which we call the *degree of f* .

Remark. Let $n > 0$ and let $f: S^n \rightarrow S^n$ be a map. If $a \in H_n(S^n)$ is non-trivial, then the degree $\deg(f)$ is uniquely determined by the equality $f_*(a) = \deg(f) \cdot a$.

Example. We consider the map $f: S^1 \rightarrow S^1$ that is illustrated in Figure 282 on the left.⁵¹⁶ We denote by $\tau: \Delta^1 \rightarrow S^1$ the singular 1-simplex that is given by $\tau(1-t, t) = e^{2\pi it}$. By Lemma 26.31 we know that $[\tau] \in H_1(S^1)$ is a generator.

We pick singular 1-simplices $\sigma_1, \dots, \sigma_5$ as in the middle of Figure 282 and we denote by $\gamma: \Delta^1 \rightarrow S^1$ the constant singular 1-simplex that sends all points to 1. Then

$$\begin{aligned}
 & \text{by Lemma 24.3 (3)} \\
 & \downarrow \\
 f_*([\tau]) &= f_*([\sigma_1 + \dots + \sigma_5]) = [\underbrace{\sigma_1 \circ f}_{=\gamma} + \underbrace{\sigma_2 \circ f}_{=\tau} + \underbrace{f \circ f}_{=\gamma} + \underbrace{f \circ \sigma_4}_{=\bar{\tau}} + \underbrace{\sigma_5 \circ f}_{=\tau}] \\
 &= [2 \cdot \tau + \bar{\tau} + 2 \cdot \gamma] = [\tau] = \underbrace{(+1)}_{=: \deg(f)} \cdot [\tau].
 \end{aligned}$$

$\uparrow$
 Lemma 24.3 (2)

Summarizing we have shown that $\deg(f) = +1$. With almost the same approach we can

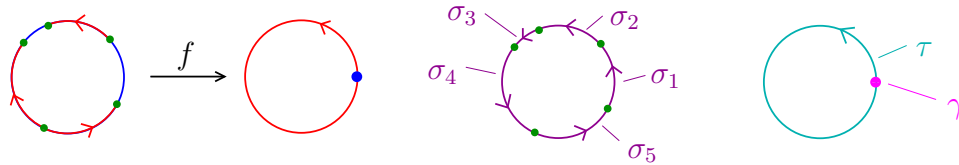


FIGURE 282.

calculate the degrees of the maps that are illustrated in Figure 283.

The same arguments that we used in discussion of the examples gives us the following lemma. We leave the details as an exercise to the reader.

Lemma 27.1. *For any $k \in \mathbb{Z}$ the degree of the map*

$$\begin{aligned}
 S^1 &\rightarrow S^1 \\
 z &\mapsto z^k
 \end{aligned}$$

⁵¹⁶More precisely, given an “interval” on S^1 going from $e^{i\alpha}$ to $e^{i\beta}$ we mean the singular 1-simplex that is given by $(1-t, t) \mapsto e^{i(\alpha \cdot (1-t) + \beta \cdot t)}$.

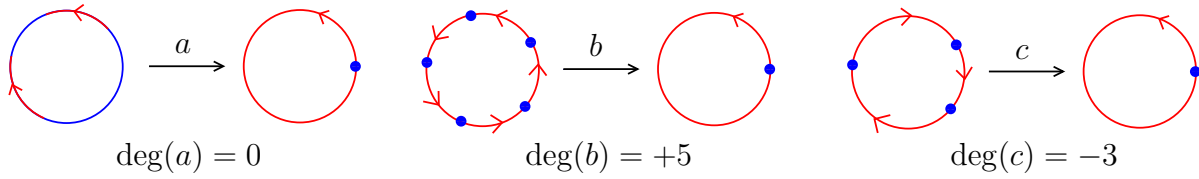


FIGURE 283.

equals k .

Proof. We denote by $\tau: \Delta^1 \rightarrow S^1$ the singular 1-simplex that is given by $\tau(1-t, t) = e^{2\pi it}$. By Lemma 26.31 we know that $[\tau] \in H_1(S^1)$ is a generator.

Now we first consider the case $k \geq 1$. For $j = 0, \dots, k-1$ we denote by $\sigma_j: \Delta^1 \rightarrow S^1$ the singular 1-simplex that is given by $(1-t, t) \mapsto e^{2\pi i(\frac{t+j}{k})}$. Then

$$f_*([\tau]) = f_*([\sigma_0 + \dots + \sigma_{k-1}]) = [\overbrace{f \circ \sigma_0}^{=\tau} + \dots + \overbrace{f \circ \sigma_{k-1}}^{=\tau}] = k \cdot [\tau].$$

$\uparrow$
 by Lemma 24.3 (3)

The case $k < 0$ is proved in a similar fashion using Lemma 24.3 (2). Finally the case $k = 0$ follows from the fact that the constant map $S^1 \rightarrow S^1$, $z \mapsto 1$ factors through a point which by Lemma 24.6 has trivial first homology. $\square$

In the following lemma we summarize several properties of the degree of the self-map of a sphere.

Lemma 27.2. *Let $f, g: S^n \rightarrow S^n$ be two maps. Then the following hold:*

- (1) $\deg(\text{id}_{S^n}) = 1$,
- (2) if f is not surjective, then $\deg(f) = 0$,
- (3) if f is homotopic to g , then $\deg(f) = \deg(g)$,
- (4) $\deg(f \circ g) = \deg(f) \cdot \deg(g)$,
- (5) if f is a self-homeomorphism of S^n , then $\deg(f^{-1}) = \deg(f)$,
- (6) if f is the reflection in a hyperplane, then $\deg(f) = -1$,
- (7) if f is the reflection in the origin i.e. if $f(x) = -x$ for all $x \in S^n$, then we have $\deg(f) = (-1)^{n+1}$,

Proof.

- (1) Since $(\text{id}_{S^n})_* = \text{id}_{H_n(S^n)}$ we know that $\deg(\text{id}_{S^n}) = 1$.
- (2) Suppose that the map $f: S^n \rightarrow S^n$ misses the point $P \in S^n$. We consider the commutative diagram

$$\begin{array}{ccc}
 & S^n \setminus \{P\} & \\
 f \nearrow & & \searrow i \\
 S^n & \xrightarrow{f} & S^n
 \end{array}
 \quad \text{which gives rise to the commutative diagram} \quad
 \begin{array}{ccc}
 & H_n(S^n \setminus \{P\}) & \\
 f_* \nearrow & & \searrow i_* \\
 H_n(S^n) & \xrightarrow{f_*} & H_n(S^n)
 \end{array}$$

Since $S^n \setminus \{P\}$ is homeomorphic to B^n we know that it is contractible which implies by Corollary 25.5 that $H_n(S^n \setminus \{P\}) = 0$. But this means that $f_*: H_n(S^n) \rightarrow H_n(S^n)$ factors through the zero group, i.e. f_* is the zero map.

- (3) If f is homotopic to g , then it follows from Proposition 25.4 that $\deg(f) = \deg(g)$.
 (4) We consider the following diagram

$$\begin{array}{ccccc} H_n(S^n) & \xrightarrow[\cdot \deg(f)]{f_*} & H_n(S^n) & \xrightarrow[\cdot \deg(g)]{g_*} & H_n(S^n) \\ & \searrow & & \nearrow & \\ & & (f \circ g)_* & & \end{array}$$

This diagram commutes since $(f \circ g)_* = f_* \circ g_*$. It follows that the curved map is given by multiplication by $\deg(f) \cdot \deg(g)$, i.e. $\deg(f \circ g) = \deg(f) \cdot \deg(g)$.

- (5) This statement follows immediately from (1) and (4).
 (6) We consider again the singular chain

$$\begin{aligned} \alpha: \Delta^n &\xrightarrow{\cong} \left\{ (x_1, \dots, x_n) \in [0, 1]^n \mid \sum_{i=1}^n x_i \in [0, 1] \right\} \xrightarrow{\cong} \overline{B}^n \rightarrow S^n. \\ (t_0, \dots, t_n) &\mapsto (t_0, \dots, t_{n-1}) \qquad \qquad \qquad x \mapsto (x, \sqrt{1 - \|x\|^2}). \end{aligned}$$

that we had already considered on page 459. Let ρ be the reflection in the hyperplane V defined by $x_{n+1} = 0$. We write $\beta = \rho \circ \alpha$. In Lemma 26.30 we had seen that the singular n -chain $\alpha - \beta$ is a generator of $H_n(S^n)$.

We then have

$$\rho_*([\alpha - \beta]) = [\rho \circ \alpha] - [\rho \circ \beta] = [\beta] - \underbrace{[\rho \circ \rho \circ \alpha]}_{=\text{id}} = [\beta] - [\alpha] = -[\alpha - \beta].$$

The map ρ_* is therefore given by multiplication with -1 . This shows that reflection in V has degree -1 .

Now let f be the reflection in some other hyperplane W of $\mathbb{R}^{n+1}$. We pick an orthogonal matrix $A \in O(n+1)$ with $A \cdot W = V$.⁵¹⁷ Then

$$\begin{array}{ccccc} \deg(f) & = & \deg(\text{reflection in } W) & = & \deg(\text{reflection in } A^{-1} \cdot V) \\ & = & \deg(A^{-1} \circ \text{reflection in } V \circ A) & = & \deg(\text{reflection in } V) = -1. \\ \uparrow & & & \uparrow & \uparrow \\ \text{see also Figure 284} & & \text{follows from (1), (4) und (5)} & & \text{see above} \\ & & \text{since } A \text{ defines homeomorphism of } S^n & & \end{array}$$

- (7) The reflection in the origin is the composition of the reflections in the $(n+1)$ hyperplanes $H_i := \{(t_1, \dots, t_{n+1}) \in \mathbb{R}^{n+1} \mid t_i = 0\}$, $i = 1, \dots, n+1$. The desired statement thus follows from (4) and (6). $\square$

⁵¹⁷Why does such an orthogonal matrix A exist? The statement is very similar to Satz 3.11 from Analysis IV.

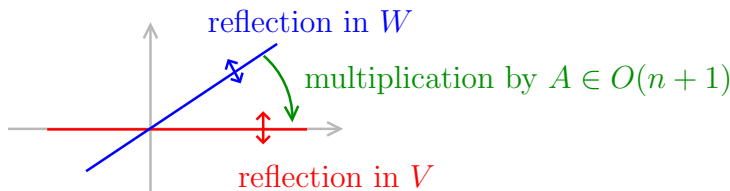


FIGURE 284. Illustration for the proof of Lemma 27.2 (6).

Lemma 27.3. *Let $A \in O(n+1)$ be an orthogonal matrix. Then*

$$\deg \left(\begin{array}{ccc} S^n & \rightarrow & S^n \\ P & \mapsto & A \cdot P \end{array} \right) = \det(A).$$

Proof. Let $A \in O(n+1)$ be an orthogonal matrix. We give two slightly different proofs that $\deg(A) = \det(A)$.

- (1) It is a standard fact proved in Linear Algebra, see e.g. [Cu, Theorem 33.1], that the map $P \mapsto A \cdot P$ is the composition of reflections in finitely many hyperplanes $H_1, \dots, H_k$.⁵¹⁸ We see that

$$\begin{array}{ccccccc} \deg(A) & = & \deg(H_1) \cdots \deg(H_k) & = & (-1)^k & = & \det(H_1) \cdots \det(H_k) = \det(A). \\ \uparrow & & \uparrow & & \uparrow & & \\ \text{Lemma 27.2 (4)} & & \text{Lemma 27.2 (6)} & & \text{determinant of reflection} = -1 & & \end{array}$$

- (2) For the second proof we first assume that $\det(A) = 1$, i.e. that $A \in SO(n+1)$. On page 33 we had seen that $SO(n+1)$ is path-connected, so there exists a map $h: [0, 1] \rightarrow SO(n+1)$ with $h(0) = \text{id}$ and $h(1) = A$. But then $F(x, t) = h(t) \cdot x$ defines a homotopy from id to multiplication by A . It follows from Lemma 27.2 (1) and (3) that $\deg(A) = 1$.

Now suppose that $\det(A) \neq 1$. Then $\det(A) = -1$ and if we compose A with the map h that is given by reflection in a hyperplane, then multiplicativity of determinants shows that $\det(h \circ A) = \det(h) \cdot \det(A) = (-1)^2 = 1$. From the above we see that $\deg(h \circ A) = 1$. From Lemma 27.2 (4) and (6) we easily obtain that $\deg(A) = -1$. $\square$

We have now seen that the circle admits self-maps of arbitrary degrees and that every sphere admits self-maps of degree $-1, 0, 1$. It is natural to ask whether there are also maps on higher-dimensional spheres that have other degrees. The combination of Lemma 27.1 and the following lemma shows that in fact every $k \in \mathbb{Z}$ is realized as the degree of every high-dimensional sphere.

Lemma 27.4. *Let $f: S^n \rightarrow S^n$ be a map with $n \geq 1$. Following the discussion on page 465, we denote by $\Sigma(f): \Sigma(S^n) \rightarrow \Sigma(S^n)$ the suspension of f and we make the identification*

⁵¹⁸For example, a matrix in $O(2)$ corresponds to rotation by an angle α . But rotation by α is the same as the composition of the reflection in the line defined by the angle $\varphi = \frac{\alpha}{2}$ and the reflection in the line defined by the angle $\varphi = \alpha$.

$\Sigma(S^n) = S^{n+1}$. Then

$$\deg(\Sigma(f): S^{n+1} \rightarrow S^{n+1}) = \deg(f: S^n \rightarrow S^n).$$

Example. Let $k \in \mathbb{Z}$. We view S^2 as a subset of $\mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$. We consider the map

$$\begin{aligned} g: S^2 &\rightarrow S^2 \\ (z, w) &\mapsto (z^k, w). \end{aligned}$$

Under the identification $S^2 = \Sigma(S^1)$ from page 465 the map g equals $\Sigma(z \mapsto z^k)$. Thus it follows from Lemmas 27.1 and 27.4 that $\deg(g) = \deg(\Sigma(z \mapsto z^k)) = k$.

Proof. As usual we denote by $S_{\geq 0}^{n+1}$ and $S_{\leq 0}^{n+1}$ the upper and lower hemisphere and we identify $S_{\geq 0}^{n+1} \cap S_{\leq 0}^{n+1}$ with S^n . We consider the following diagram

$$\begin{array}{ccccccc} \tilde{H}_{n+1}(S^{n+1}) & \xrightarrow{\cong} & H_{n+1}(S^{n+1}, S_{\leq 0}^{n+1}) & \xleftarrow{\cong} & H_{n+1}(S_{\geq 0}^{n+1}, S^n) & \xrightarrow[\partial_n]{\cong} & \tilde{H}_n(S^n) \\ \cdot \deg(\Sigma(f)) \downarrow \Sigma(f)_* & & \downarrow \Sigma(f)_* & & \downarrow \Sigma(f)_* & & \cdot \deg(f) \downarrow f_* \\ \tilde{H}_{n+1}(S^{n+1}) & \xrightarrow{\cong} & H_{n+1}(S^{n+1}, S_{\leq 0}^{n+1}) & \xleftarrow{\cong} & H_{n+1}(S_{\geq 0}^{n+1}, S^n) & \xrightarrow[\partial_n]{\cong} & \tilde{H}_n(S^n) \end{array}$$

where the two leftmost maps are induced by the obvious maps between (pairs) of topological spaces. We make the following observations and justifications:

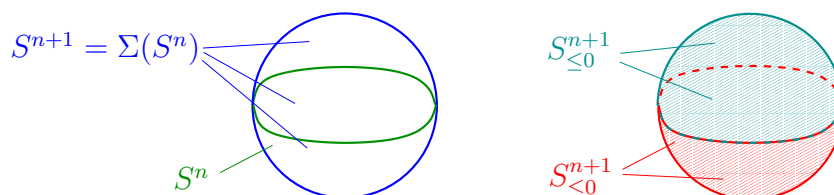


FIGURE 285. Illustration for the proof of Lemma 27.4 (6).

- (1) in both rows the leftmost horizontal maps are isomorphisms by the long exact sequence in reduced homology of the pair $(S^{n+1}, S_{\leq 0}^{n+1})$, see Corollary 26.13, and the observation that the reduced homology of $S_{\leq 0}^{n+1} \cong \overline{B}^{n+1}$ vanishes by Corollary 25.5,
- (2) in both rows the middle horizontal maps are isomorphisms by the Excision Theorem for Manifolds 26.27,
- (3) the same argument as in (1) shows that in both rows the right horizontal maps are isomorphisms,
- (4) the two rectangles on the left commute by the functoriality of homology groups,
- (5) the rectangle on the right commutes by the functoriality of the connecting homomorphism ∂ , see e.g. Footnote 493, and by the observation that by definition $\Sigma(f)|_{S^n} = f$.

The vertical homomorphism on the right is given by multiplication by $\deg(f)$. Since the diagram commutes and since all horizontal maps are isomorphisms we see that the vertical homomorphism on the left is also given by multiplication by $\deg(f)$, i.e. we obtain the desired equality we have $\deg(\Sigma(f)) = \deg(f)$. $\square$

Theorem 27.5. *Let $f: S^n \rightarrow S^n$ be a map. If $\deg(f) \neq (-1)^{n+1}$, then f has a fixed-point.*

Example. Theorem 27.5 says in particular that every map $f: S^2 \rightarrow S^2$ of degree one has a fixed-point. Together with Lemma 27.2 this implies that every map $f: S^2 \rightarrow S^2$ that is homotopic to the identity has a fixed-point. This means that we cannot “deform” the identity to a map that is fixed point free. This is in contrast to the case $n = 1$ where we can “deform” the identity into the map that is given by rotation by a small angle, which is of course a fixed-point free map.

Proof. Let $f: S^n \rightarrow S^n$ be a map which admits no fixed point. This means that for all $x \in S^n$ we have $f(x) \neq x$. We need to show that $\deg(f) = (-1)^{n+1}$. We will do this by showing that id is homotopic to $-\text{id}$.

Our hypothesis implies that $(1-t)f(x) \neq tx$ for all $t \in [0, 1]$ and all $x \in S^n$.⁵¹⁹ Therefore we can consider the map

$$\begin{aligned} F: S^n \times [0, 1] &\rightarrow S^n \\ (x, t) &\mapsto \frac{(1-t)f(x) - tx}{\|(1-t)f(x) - tx\|}. \end{aligned}$$

This map is a homotopy between the maps $x \mapsto f(x)$ and $x \mapsto -x$. Thus it follows that

$$\begin{array}{ccccc} \deg(f) & = & \deg(-\text{id}) & = & (-1)^{n+1}. \\ \uparrow & & \uparrow & & \\ \text{Lemma 27.2 (3)} & & \text{Lemma 27.2 (7)} & & \end{array}$$

$\square$

We now recall several definitions from Analysis II and IV.

Definition. Let $M \subset \mathbb{R}^n$ be a submanifold and let $P \in M$.

- (1) A vector $v \in \mathbb{R}^n$ is called a *tangent vector to M in the point P* if there exists a smooth curve

$$\begin{aligned} \gamma: (-\epsilon, \epsilon) &\rightarrow M \\ t &\mapsto \gamma(t) \end{aligned}$$

such that $\gamma(0) = P$ and $\gamma'(0) = v$.

- (2) The set of all tangent vectors at the point P is called the *tangent space* $T_P M$.
 (3) A *vector field* on M is a continuous map $v: M \rightarrow \mathbb{R}^n$ such that for any $P \in M$ we have $v(P) \in T_P M$.

Examples.

⁵¹⁹This case be seen as follows: if $t \neq \frac{1}{2}$ we have $\|(1-t)f(x)\| = 1-t \neq t = \|tx\|$. On the other hand, for $t = \frac{1}{2}$ we have $(1-t)f(x) = \frac{1}{2}f(x) \neq \frac{1}{2}x = tx$.

- (1) We had seen in Analysis II that the tangent space of the sphere S^n at a point P is given as follows:

$$T_P S^n = \{v \in \mathbb{R}^{n+1} \mid P \cdot v = 0\}.$$

- (2) For an odd-dimensional sphere S^{2k-1} a vector field is given by⁵²⁰

$$\begin{aligned} v: S^{2k-1} &\rightarrow \mathbb{R}^{2k} \\ (x_1, x_2, \dots, x_{2k-1}, x_{2k}) &\mapsto (-x_2, x_1, \dots, -x_{2k}, x_{2k-1}). \end{aligned}$$

This vector field has the property that it is nowhere vanishing. In the case of the 1-sphere S^1 this is just the vector field of constant length 1 that points in the “counterclockwise direction”.

- (3) For even-dimensional sphere S^{2k} a vector field is similarly given by

$$\begin{aligned} v: S^{2k} &\rightarrow \mathbb{R}^{2k+1} \\ (x_1, x_2, \dots, x_{2k-1}, x_{2k}, x_{2k+1}) &\mapsto (-x_2, x_1, \dots, -x_{2k}, 0). \end{aligned}$$

This vector field vanishes at the “north pole” $(0, \dots, 0, 1)$ and it also vanishes at the “south pole” $(0, \dots, 0, -1)$. Both types of vector fields are sketched in Figure 286.

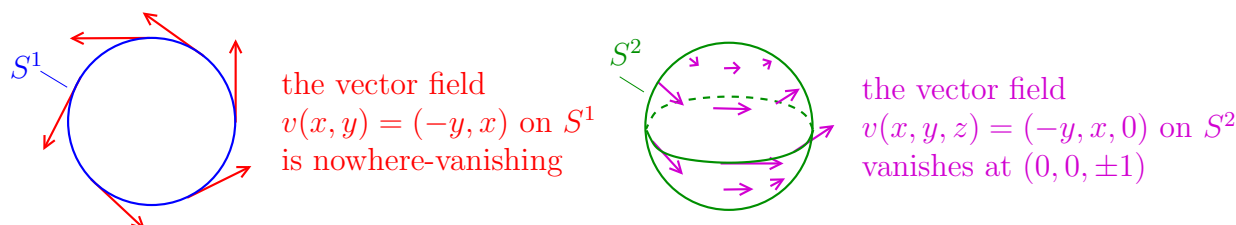


FIGURE 286.

One can now ask whether it is possible to find a nowhere vanishing vector field on even-dimensional spheres. The following proposition says that this is not possible.

Theorem 27.6. *Every vector field on an even-dimensional sphere S^{2k} vanishes on at least one point.*⁵²¹

Remark.

- (1) For $n = 2$ we get the following “real life application”: on the surface of earth there always exists at least one point where the wind does not blow at all. This fact is illustrated at the following website

<https://earth.nullschool.net/>

⁵²⁰Indeed, for any $P \in S^{2k-1}$ we have $P \cdot v(P) = 0$, i.e. $v(P)$ does indeed lie in the tangent space $T_P S^{2k-1}$.

⁵²¹On S^{2k} we had so far found vector fields which vanish at *two* points. Can we find a vector field on S^{2k} which vanishes on only *one* point? We will discuss this question in exercise sheet 6.

- (2) In the literature Theorem 27.6 is known as the “hairy ball theorem” since the theorem says that “you can’t comb a hairy ball flat without creating a cowlick”.
- (3) In German Theorem 27.6 is known as the “Satz vom (gekämmten) Igel”, for the same reason as the “hairy ball theorem”.

Proof. Let v be a vector field on the sphere S^n which is nowhere-vanishing. We have to show that n is odd. We consider the vector field w on S^n that is given by rescaling v to length one, i.e. which is given by

$$\begin{aligned} S^n &\rightarrow \mathbb{R}^{n+1} \\ x &\mapsto w(x) := \frac{1}{\|v(x)\|} v(x). \end{aligned}$$

Furthermore we consider the map

$$\begin{aligned} F: S^n \times [0, 1] &\mapsto S^n \\ (x, t) &\mapsto \cos(\pi t)x + \sin(\pi t)w(x). \end{aligned}$$

The map does indeed take values in S^n , since

$$\begin{aligned} (\cos(\pi t)x + \sin(\pi t)w(x)) \cdot (\cos(\pi t)x + \sin(\pi t)w(x)) &= \cos(\pi t)^2 \|x\|^2 + \sin(\pi t)^2 \|w(x)\|^2 = 1. \\ &\quad \uparrow \qquad \qquad \qquad \uparrow \\ &\text{since } w(x) \cdot x = 0 \qquad \text{since } \|x\| = \|w(x)\| = 1 \end{aligned}$$

For $t = 0$ the map F is the identity and for $t = 1$ the map F is $-\text{id}$. Put differently, the map F is a homotopy between the two maps id and $-\text{id}$. It follows that

$$\begin{array}{ccccc} 1 & = & \deg(\text{id}) & = & \deg(-\text{id}) & = & (-1)^{n+1}. \\ \uparrow & & \uparrow & & \uparrow & & \\ \text{Lemma 27.2 (1)} & & \text{Lemma 27.2 (3)} & & \text{Lemma 27.2 (7)} & & \end{array}$$

But this means that n is odd. □

28. THE MAYER–VIETORIS SEQUENCE AND ITS APPLICATIONS

In this chapter we will state and prove the Mayer–Vietoris sequence for homology groups. It makes it possible to compute the homology groups of a topological space from the homology groups of appropriate subspaces. The role of the Mayer–Vietoris sequence in homology theory is similar to the role of the Seifert–van Kampen Theorem 11.1 for fundamental groups. The Mayer–Vietoris sequence will give us the tool to compute the homology groups of many topological spaces that we are interested in. In particular we will determine the homology groups of $\mathbb{RP}^2$, the torus and the Klein bottle.

28.1. Split exact sequences. We had already seen in Theorem 26.2 that homology groups of different topological spaces are sometimes related by a long exact sequence of homology groups. The Mayer–Vietoris sequence that we will introduce later on in this chapter is another instance of such a long exact sequence. Very often one ends up in the situation that one knows the homology groups of two out of three topological spaces and we would like to determine the homology groups of the third topological space from the long exact sequence.

Algebraically the question is as follows: suppose we are given an exact sequence

$$\dots \rightarrow A_n \rightarrow B_n \rightarrow C_n \rightarrow A_{n-1} \rightarrow \dots$$

of abelian groups where we know the groups A_* and C_* , can we determine the groups B_* from this long exact sequence? The same question of course also arises when the roles of A, B and C are switched.

In this section we will study the simplest case, namely the case of a short exact sequence

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0.$$

If we know A and C , can we read “read off” B from this short exact sequence? In general the answer is no. For example the short exact sequence

$$0 \rightarrow \mathbb{Z}_2 \longrightarrow B \longrightarrow \mathbb{Z}_2 \rightarrow 0$$

could be of the form

$$0 \rightarrow \mathbb{Z}_2 \xrightarrow{x \mapsto (x,0)} \mathbb{Z}_2 \oplus \mathbb{Z}_2 \xrightarrow{(x,y) \mapsto y} \mathbb{Z}_2 \rightarrow 0$$

but it could also be of the form⁵²²

$$0 \rightarrow \mathbb{Z}_2 \xrightarrow{x \mapsto 2x} \mathbb{Z}_4 \xrightarrow{x \mapsto x} \mathbb{Z}_2 \rightarrow 0.$$

In this short section we will give a criterion for when the naive guess $B \cong A \oplus C$ is actually the correct answer.

Definition. We say that a short exact sequence

$$0 \rightarrow A \xrightarrow{i} B \xrightarrow{p} C \rightarrow 0$$

of abelian groups *splits*, if there exists a *splitting* $s: C \rightarrow B$, i.e. a homomorphism $s: C \rightarrow B$ such that $p \circ s = \text{id}_C$.

⁵²²Are there any other possibilities for the isomorphism type of B ?

For example the short exact sequence

$$0 \rightarrow \mathbb{Z} \xrightarrow{k \mapsto (k,0)} \mathbb{Z} \oplus \mathbb{Z}_n \xrightarrow{(k,l) \mapsto l} \mathbb{Z}_n \rightarrow 0$$

splits, in fact a splitting is given by the map $b \mapsto (0, b)$. On the other hand the short exact sequence

$$0 \rightarrow \mathbb{Z} \xrightarrow{k \mapsto nk} \mathbb{Z} \rightarrow \mathbb{Z}_n \rightarrow 0.$$

does not split.⁵²³

Lemma 28.1. *Let*

$$0 \rightarrow A \xrightarrow{i} B \xrightarrow{p} C \rightarrow 0$$

be a short exact sequence of abelian groups. If C is a free abelian group, then the short exact sequence splits.

Proof. Let $\{c_i\}_{i \in I}$ be a basis of the free abelian group C . For each $c_i \in C$ we choose a $b_i \in B$ with $p(b_i) = c_i$. By Lemma 8.1 there exists a unique homomorphism $s: C \rightarrow B$ with $s(c_i) = b_i$ for all i . We have $(p \circ s)(c_i) = c_i$ for each c_i . But then it follows from the uniqueness part of Lemma 8.1 that $p \circ s = \text{id}_C$. The homomorphism s is thus indeed a splitting. $\square$

The following lemma gives two useful criteria for a short exact sequence to split.

Lemma 28.2. *Let*

$$0 \rightarrow A \xrightarrow{i} B \xrightarrow{p} C \rightarrow 0$$

be a short exact sequence of abelian groups. Then the following statements are equivalent:

- (1) *the short exact sequence splits,*
- (2) *there exists a homomorphism $t: B \rightarrow A$ such that $t \circ i = \text{id}_A$,*
- (3) *there exists an isomorphism $\Phi: B \rightarrow A \oplus C$ such that the following diagram commutes:*

$$\begin{array}{ccccccc} & & & B & & & \\ & & i \nearrow & \downarrow \cong \Phi & \searrow p & & \\ 0 & \longrightarrow & A & & & C & \longrightarrow 0 \\ & & \searrow & \downarrow & \nearrow & & \\ & & & A \oplus C & & & \end{array}$$

where the lower left diagonal map is the obvious inclusion and the lower right diagonal map is the obvious projection map.

Proof.

- (1) $\Rightarrow$ (2) Let $s: C \rightarrow B$ be a splitting, i.e. let s be a homomorphism such that $p \circ s = \text{id}_C$. We need to find an appropriate homomorphism $t: B \rightarrow A$. Since the sequence

⁵²³Why does it not split?

is exact, our task is basically the same as finding an appropriate homomorphism $B \rightarrow \ker(p)$. Let $b \in B$. We have

$$p(b - s(p(b))) = p(b) - \underbrace{(p \circ s)}_{= \text{id}}(p(b)) = p(b) - p(b) = 0.$$

By the exactness of the short exact sequence there exists a unique $a \in A$ with $i(a) = b - s(p(b))$. We define $t(b) := a$. It follows easily from the fact that p and s are homomorphisms that the map $t: B \rightarrow A$ is a homomorphism. It remains to show that $t \circ i = \text{id}_A$. Let $a \in A$. We have to show that $t(i(a)) - a = 0$. Since i is injective it suffices to show that $i(t(i(a))) - i(a) = 0$. We have

$$\begin{array}{ccccc} i(t(i(a))) - i(a) & = & (i(a) - s(p(i(a)))) - i(a) & = & i(a) - i(a) = 0. \\ \uparrow & & & & \uparrow \\ \text{since } t(y) \text{ is the unique element} & & & & \text{since } p \circ i = 0 \\ \text{with } i(t(y)) = y - s(p(y)) & & & & \end{array}$$

(2) $\Rightarrow$ (3) Let $t: B \rightarrow A$ be a homomorphism such that $t \circ i = \text{id}_A$. We define

$$\begin{aligned} \Phi: B &\rightarrow A \oplus C \\ b &\mapsto t(b) \oplus p(b). \end{aligned}$$

It follows easily from the exactness of the initial sequence that this map is in fact an isomorphism.⁵²⁴ It is straightforward to verify that Φ has all the other desired properties.

(3) $\Rightarrow$ (1) Now suppose we are given an isomorphism $\Phi: B \rightarrow A \oplus C$ such that the given diagram commutes. It follows immediately from the definitions and given properties that the map

$$C \xrightarrow{\alpha \rightarrow (0, c)} A \oplus C \xrightarrow{\Phi^{-1}} B$$

is a splitting. □

In many applications we only need the following corollary that is an immediate consequence of Lemmas 28.1 and 28.2.

Corollary 28.3. *Let*

$$0 \rightarrow A \xrightarrow{i} B \xrightarrow{p} C \rightarrow 0$$

be a short exact sequence of abelian groups. Then the following hold:

- (1) *If C is a free-abelian group, then $B \cong A \oplus C$.*
- (2) *Suppose A and C are free abelian groups. If X is a basis for A and if $Y \subset B$ is a subset such that $p(Y)$ is a basis for C , then $X \cup Y$ is a basis for the free-abelian group B .*

Proof. The first statement is an immediate consequence of Lemmas 28.1 and 28.2. The second statement follows easily from Lemma 28.2 (1) $\Rightarrow$ (3). We leave the details as an exercise to the reader. □

⁵²⁴In fact the argument is basically the same as in the proof of Lemma 26.1 (1).

The above discussion deals with the case of a short exact sequence. The following lemma allows us to break a long exact sequence into shorter exact sequences.

Lemma 28.4. *Let*

$$\dots \rightarrow A_{n+2} \xrightarrow{f_{n+2}} A_{n+1} \xrightarrow{f_{n+1}} A_n \xrightarrow{f_n} A_{n-1} \xrightarrow{f_{n-1}} A_{n-2} \rightarrow \dots$$

be an exact sequence of abelian groups. Then for any n the sequences

$$0 \rightarrow \operatorname{coker} \{A_{n+2} \xrightarrow{f_{n+2}} A_{n+1}\} \xrightarrow{f_{n+1}} A_n \xrightarrow{f_n} A_{n-1} \xrightarrow{f_{n-1}} A_{n-2} \rightarrow \dots$$

and

$$\dots \rightarrow A_{n+2} \xrightarrow{f_{n+2}} A_{n+1} \xrightarrow{f_{n+1}} A_n \xrightarrow{f_n} \ker \{A_{n-1} \xrightarrow{f_{n-1}} A_{n-2}\} \rightarrow 0$$

are also exact. In particular

$$0 \rightarrow \operatorname{coker} \{A_{n+2} \xrightarrow{f_{n+2}} A_{n+1}\} \xrightarrow{f_{n+1}} A_n \xrightarrow{f_n} \ker \{A_{n-1} \xrightarrow{f_{n-1}} A_{n-2}\} \rightarrow 0$$

is a short exact sequence.

Proof. The statements follow immediately from the definitions. $\square$

Example. A typical application is that if we are given an exact sequence

$$\dots A_{n+2} \xrightarrow{f_{n+2}} A_{n+1} \xrightarrow{f_{n+1}} A_n \xrightarrow{f_n=0} A_{n-1} \xrightarrow{f_{n-1}} A_{n-2} \xrightarrow{f_{n-2}} A_{n-3} \dots$$

of abelian groups such that the map f_n is the zero map, then the exact sequence can be split “into the two halves” along that zero map, i.e.

$$0 \rightarrow A_{n-1} \xrightarrow{f_{n-1}} A_{n-2} \xrightarrow{f_{n-2}} A_{n-3} \dots$$

and

$$\dots A_{n+2} \xrightarrow{f_{n+2}} A_{n+1} \xrightarrow{f_{n+1}} A_n \rightarrow 0$$

are also exact sequences. In the remainder of this course we will frequently use this fact without explicitly citing Lemma 28.4.

28.2. The Mayer–Vietoris sequence. In the following theorem and throughout the subsequent discussion we will use the following notation: given a topological space D and a subset $C \subset D$, we denote the corresponding inclusion map $C \rightarrow D$ by i_C .

Theorem 28.5. (Mayer–Vietoris)⁵²⁵ *Let X be a topological space and let $A, B \subset X$ be subsets such that $X = \mathring{A} \cup \mathring{B}$. Then the following two statements hold:*

(1) *For each $n \in \mathbb{N}$ there exists a unique homomorphism*

$$\partial_n: H_n(X) \rightarrow H_{n-1}(A \cap B)$$

⁵²⁵Walther Mayer (1887–1948) and Leopold Vietoris (1891–2002) were two Austrian mathematicians. Vietoris was the longest living Austrian man ever.

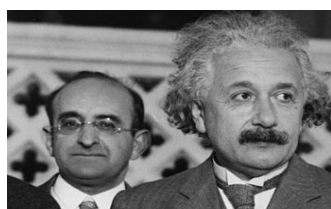
which has the following property: for every $a \in C_n(A)$ and $b \in C_n(B)$ such that $a + b \in C_n(X)$ is a cycle we have⁵²⁶

$$\partial_n([a + b]) = [\partial a] = -[\partial b] \in H_{n-1}(A \cap B).$$

(2) The following sequence is exact:

$$\cdots \rightarrow H_n(A \cap B) \xrightarrow{i_{A \cap B} \oplus -i_{A \cap B}} H_n(A) \oplus H_n(B) \xrightarrow{i_A + i_B} H_n(X) \xrightarrow{\partial_n} H_{n-1}(A \cap B) \rightarrow \cdots$$

Totally analogous statements also hold if we replace the homology groups of $A \cap B, A, B$ and X by the corresponding reduced homology groups.



Walther Mayer (1887-1948)
with some other person



Leopold Vietoris (1891-2002) at 110

FIGURE 287.

The long exact sequence in Theorem 28.5 is usually called the Mayer–Vietoris sequence of the decomposition $X = A \cup B$.

Proof. Let X be a topological space and let $A, B \subset X$ be subsets such that $X = \mathring{A} \cup \mathring{B}$. As a reminder, following the notation from page 443 we write

$$C_n^{\{A, B\}}(X) := \{a + b \mid a \in C_n(A) \text{ and } b \in C_n(B)\}.$$

The inclusion maps induce the following sequence of chain maps:

$$0 \rightarrow C_*(A \cap B) \xrightarrow{i_{A \cap B} \oplus -i_{A \cap B}} C_*(A) \oplus C_*(B) \xrightarrow{i_A + i_B} C_*^{\{A, B\}}(X) \rightarrow 0.$$

Claim. The above sequence of chain maps is exact.

We prove the claim in the following four steps:

- (1) It is clear that each of the maps $C_*(A \cap B) \rightarrow C_*(A)$ and $C_*(A \cap B) \rightarrow C_*(B)$ is already injective, in particular the map $i_{A \cap B} \oplus -i_{A \cap B}$ is injective.
- (2) It is an immediate consequence of the definition of $C_*^{\{A, B\}}(X)$ that the map $i_A + i_B$ is surjective.
- (3) It follows immediately from the definitions that the composition of both maps is the zero map,⁵²⁷ i.e. $\text{im}(i_{A \cap B} \oplus -i_{A \cap B}) \subset \ker(i_A + i_B)$.

⁵²⁶Here in the expression ∂a we mean by ∂ the boundary map $\partial: C_n(A) \rightarrow C_{n-1}(A)$. In our case we will see that $\partial a \in C_{n-1}(A \cap B) \subset C_{n-1}(A)$. A similar statement holds for ∂b .

⁵²⁷Note that here we use the minus-sign that appears in the first map.

- (4) It remains to show that $\ker(i_A + i_B) \subset \text{im}(i_{A \cap B} \oplus -i_{A \cap B})$. Thus let $x \in C_n(A)$ and $y \in C_n(B)$ such that $(i_A + i_B)(x + y) = 0$. By definition this means that x and y , viewed as elements in $C_n(X)$, satisfy $x = -y$. This means that x lies in $C_n(A)$ and also in $C_n(B)$. Put differently, $x \in C_n(A) \cap C_n(B) = C_n(A \cap B)$. But then we have $(i_{A \cap B} \oplus -i_{A \cap B})(x) = x + y$.

Thus we have proved the claim.

For each $n \in \mathbb{N}$ we denote by

$$p_n: H_n^{\{A,B\}}(X) \rightarrow H_{n-1}(A \cap B)$$

the connecting homomorphism of the above short exact sequence of chain complexes, as defined on page 429. We deduce from Proposition 26.9 that

$$\cdots \rightarrow H_n(A \cap B) \xrightarrow{i_{A \cap B} \oplus -i_{A \cap B}} H_n(A) \oplus H_n(B) \xrightarrow{i_A + i_B} H_n^{\{A,B\}}(X) \xrightarrow{p_n} H_{n-1}(A \cap B) \rightarrow \cdots$$

is an exact sequence. It follows from Proposition 26.18, together with our hypothesis that $X = \overset{\circ}{A} \cup \overset{\circ}{B}$, that for every $n \in \mathbb{N}_0$ the inclusion map

$$C_*^{\{A,B\}}(X) \rightarrow C_*(X)$$

induces an isomorphism

$$\Phi_n: H_n^{\{A,B\}}(X) \xrightarrow{\cong} H_n(X).$$

For each $n \in \mathbb{N}_0$ we now write $\partial_n := p_n \circ \Phi_n^{-1}$.

It remains to show that these maps have the two desired properties stated in the theorem:

- (2) It is clear from the above discussion that the sequence given in (2) is indeed exact.
 (1) We leave it as an exercise to verify that it follows from the definition of the connecting homomorphism p_n on page 429 that for every $a \in C_n(A)$ and $b \in C_n(B)$ such that $a + b \in C_n(X)$ is a cycle we have

$$\partial_n([a + b]) = p_n([a + b]) = [\partial a] = -[\partial b] \in H_{n-1}(A \cap B).$$

Furthermore, the isomorphism $\Phi_n: H_n^{\{A,B\}}(X) \xrightarrow{\cong} H_n(X)$ shows that every element in $H_n(X)$ is of the form $[a + b]$ for some $a \in C_n(A)$ and $b \in C_n(B)$, which implies that ∂_n is uniquely determined by the above rule.

We leave it as an exercise to verify that the analogous statements also hold if we replace the homology groups of $A \cap B$, A , B and X by the corresponding reduced homology groups. $\square$

We conclude this section with the following variation on the Mayer–Vietoris Theorem 28.5. This theorem might sound rather unnatural, but it will play an important role in the proof of the Poincaré Duality Theorem in Algebraic Topology III.

Theorem 28.6. (The inverted Mayer–Vietoris Theorem) *Let X be a topological space and let U and V be open subsets. Then for all $n \in \mathbb{N}_0$ there exists a homomorphism $\partial_n: H_n(X, U \cup V) \rightarrow H_{n-1}(X, U \cap V)$ such that*

$$\cdots \rightarrow H_n(X, U \cap V) \xrightarrow{i \oplus -i} H_n(X, U) \oplus H_n(X, V) \xrightarrow{i+i} H_n(X, U \cup V) \xrightarrow{\partial} H_{n-1}(X, U \cap V) \rightarrow \cdots$$

is an exact sequence.

Proof (*). As in the proof of Theorem 28.5 we have a short exact sequence

$$0 \rightarrow C_n(X, U \cap V) \rightarrow C_n(X, U) \oplus C_n(X, V) \rightarrow C_n(X)/C_n^{\{U, V\}}(U \cup V) \rightarrow 0.$$

Thus we get a corresponding long exact sequence of homology groups. It suffices to show that the homology groups of the chain complex on the right are precisely the homology groups of the pair $(X, U \cup V)$. To do so we consider the following commutative pair of short exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & C_*^{\{U, V\}}(U \cup V) & \longrightarrow & C_*(X) & \longrightarrow & C_*(X)/C_*^{\{U, V\}}(U \cup V) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & C_*(U \cup V) & \longrightarrow & C_*(X) & \longrightarrow & C_*(X, U \cup V) \longrightarrow 0. \end{array}$$

We obtain the corresponding long exact sequence of homology groups:

$$\begin{array}{ccccccc} \dots \longrightarrow & H_n^{\{U, V\}}(U \cup V) & \longrightarrow & H_n(X) & \longrightarrow & H_n(C_*(X)/C_*^{\{U, V\}}(U \cup V)) & \longrightarrow & H_{n-1}^{\{U, V\}}(U \cup V) \longrightarrow \dots \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \dots \longrightarrow & H_n(U \cup V) & \longrightarrow & H_n(X) & \longrightarrow & H_n(X, U \cup V) & \longrightarrow & H_{n-1}(U \cup V) \longrightarrow \dots \end{array}$$

It follows from Proposition 26.18 that the first and fourth vertical map are an isomorphism. The second and fifth map are the identity. Hence it follows from the Five-Lemma 26.10 that the third vertical map is also an isomorphism. $\square$

28.3. Applications of the Mayer–Vietoris sequence. As a first example we want to determine the homology groups of the 2-dimensional projective space $\mathbb{RP}^2$.

Lemma 28.7. *We have*

$$H_n(\mathbb{RP}^2) \cong \begin{cases} 0, & \text{if } n \geq 2, \\ \mathbb{Z}_2, & \text{if } n = 1, \\ \mathbb{Z}, & \text{if } n = 0. \end{cases}$$

Proof. In the proof of Lemma 28.7 we use the description from page 67 of $\mathbb{RP}^2$ as

$$\mathbb{RP}^2 = \overline{B}^2 / \sim \quad \text{where } x \sim y \text{ if } x, y \in \partial \overline{B}^2 \text{ and } x = -y.$$

In the following we denote by p the projection map $\overline{B}^2 \rightarrow \overline{B}^2 / \sim$. We consider the open subsets

$$A := p(\{z \in \overline{B}^2 \mid |z| > \frac{1}{4}\}) \quad \text{and} \quad B := p(\{z \in \overline{B}^2 \mid |z| < \frac{3}{4}\}).$$

Evidently we have $\mathring{A} \cup \mathring{B} = A \cup B = \mathbb{RP}^2$. From Mayer–Vietoris Theorem 28.5 we obtain the following long exact sequence of homology groups:

$$\dots \rightarrow \tilde{H}_2(A) \oplus \tilde{H}_2(B) \rightarrow \tilde{H}_2(\mathbb{RP}^2) \xrightarrow{\partial} \tilde{H}_1(A \cap B) \rightarrow \tilde{H}_1(A) \oplus \tilde{H}_1(B) \rightarrow \tilde{H}_1(\mathbb{RP}^2) \rightarrow \dots$$

Now we have to collect all the available information on the reduced homology groups of A , B and $A \cap B$. We make the following observations:

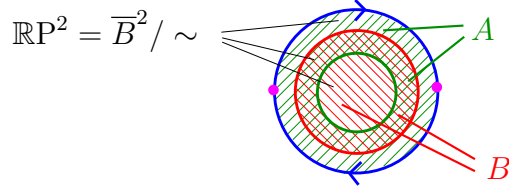


FIGURE 288. Illustration for the proof of Lemma 28.7.

- (1) We had remarked on page 422 that the reduced homology of a disk vanishes, i.e. we have $\tilde{H}_n(B) = 0$ for all $n \in \mathbb{N}_0$.
- (2) The map $j: S^1 \rightarrow A \cap B = p(\{z \in \overline{B^2} \mid |z| \in (\frac{1}{4}, \frac{3}{4})\})$ given by $z \mapsto \frac{1}{2}z$ is a homotopy equivalence, since $p(\frac{1}{2}S^1)$ is a deformation retract of $A \cap B$. It follows from Corollary 25.5 (2) and Proposition 26.3 that $\tilde{H}_n(A \cap B) = 0$ for $n \neq 1$ and that $j_*: \tilde{H}_1(S^1) \rightarrow \tilde{H}_1(A \cap B)$ is an isomorphism.
- (3) We consider the retraction

$$\begin{aligned} r: A &\rightarrow \{z \in \partial \overline{B^2}\} / z \sim -z \\ p(se^{i\varphi}) &\mapsto e^{i\varphi} \end{aligned}$$

which is easily seen to be a homotopy equivalence and we also consider the homeomorphism

$$\begin{aligned} f: \{z \in \partial \overline{B^2}\} / z \sim -z &\rightarrow S^1 \\ z &\mapsto z^2. \end{aligned}$$

As in (2) it follows that $\tilde{H}_n(A) = 0$ for $n \neq 1$ and that $(f \circ r)_*: \tilde{H}_1(A) \rightarrow \tilde{H}_1(S^1)$ is an isomorphism.

The above Mayer–Vietoris sequence is thus in our case of the following form:

$$\begin{array}{ccccccc} \dots \rightarrow 0 \rightarrow \tilde{H}_3(\mathbb{RP}^2) \rightarrow 0 \rightarrow 0 \rightarrow \tilde{H}_2(\mathbb{RP}^2) \xrightarrow{\partial} \tilde{H}_1(A \cap B) & \xrightarrow{i_*} & \tilde{H}_1(A) & \xrightarrow{i_*} & \tilde{H}_1(\mathbb{RP}^2) \rightarrow 0 \\ & j_* \uparrow \cong & & \cong \downarrow (f \circ r)_* & \\ & H_1(S^1) & \xrightarrow[(\cdot \deg(f \circ r \circ i \circ j)_*)]{(f \circ r \circ i \circ j)_*} & H_1(S^1). & \end{array}$$

We obtain immediately that $\tilde{H}_n(\mathbb{RP}^2) = 0$ for $n \geq 3$. Furthermore it follows immediately from the definitions that the map $f \circ r \circ i \circ j: S^1 \rightarrow S^1$ is given by $z \mapsto z^2$ which implies by Lemma 27.1 that $\deg(f \circ r \circ i \circ j) = 2$. Thus we obtain the following exact sequence

$$0 \rightarrow \tilde{H}_2(\mathbb{RP}^2) \rightarrow \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \rightarrow \tilde{H}_1(\mathbb{RP}^2) \rightarrow 0.$$

It follows from the discussion on page 424 that

$$\tilde{H}_2(\mathbb{RP}^2) \cong \ker \{\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}\} = 0 \quad \text{and} \quad \tilde{H}_1(\mathbb{RP}^2) \cong \operatorname{coker} \{\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}\} = \mathbb{Z}_2. \quad \square$$

The following proposition is similar in vein to Proposition 9.3.

Proposition 28.8. *Let A and B be two topological spaces⁵²⁸ and let $a \in A$ and $b \in B$ be simple points.⁵²⁹⁵³⁰ Then the inclusion maps induce for every $n \in \mathbb{N}_0$ an isomorphism*

$$\tilde{H}_n(A) \oplus \tilde{H}_n(B) \rightarrow \tilde{H}_n(A \vee B).$$

Proof ()*. We write $X = A \vee B$ and $x_0 = \{a, b\} \in A \vee B$. We pick an open neighborhood C in A of a such that $\{a\}$ is a deformation retract of A . Furthermore we pick an open neighborhood D of b in B such that b is a deformation retract of B . We consider $U := A \cup D$

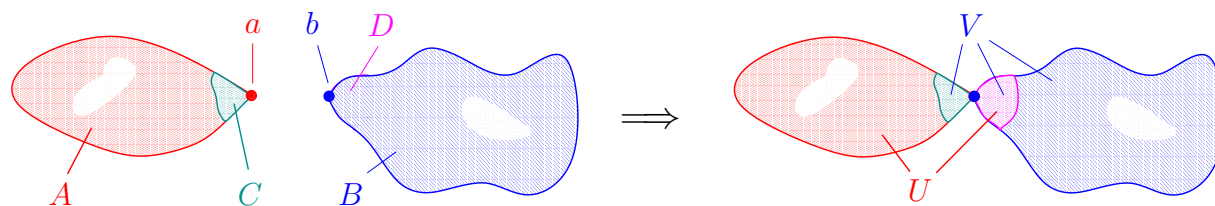


FIGURE 289. Schematic image for the proof of Proposition 28.8.

and $V := B \cup C$. We consider the following diagram of maps

$$\begin{array}{ccccccc} & & \tilde{H}_n(A) \oplus \tilde{H}_n(B) & & & & \\ & & \cong \downarrow & & \downarrow \cong & & \\ \dots \longrightarrow & \tilde{H}_n(U \cap V) & \longrightarrow & \tilde{H}_n(U) \oplus \tilde{H}_n(V) & \longrightarrow & \tilde{H}_n(X) & \longrightarrow \tilde{H}_{n-1}(U \cap V) \longrightarrow \dots \end{array}$$

We make the following observations:

- (1) The bottom sequence is exact by the Mayer–Vietoris Theorem 28.5.
- (2) The intersection $U \cap V$ deformation retracts to x_0 , in particular we obtain from Corollary 25.5 that $\tilde{H}_n(U \cap V) = \tilde{H}_{n-1}(U \cap V) = 0$.
- (3) The vertical maps are isomorphisms by Corollary 25.5 since A is a deformation retract of U and since B is a deformation retract of V .

The combination of these three observations gives the desired proof of the proposition. $\square$

28.4. The Mayer–Vietoris Theorem for CW-complexes. When we studied fundamental groups we first proved the Seifert–van Kampen Theorem 11.1 for the decomposition of a topological space X into two *open* subsets. Later on we showed in Theorem 22.5 that an analogous statement holds if we decompose a CW-complex into subcomplexes.

In a similar fashion we now state and prove the “CW-complex version” of the Mayer–Vietoris Theorem 28.5.

⁵²⁸Note that in contrast to Proposition 9.3 we no longer assume that A and B are path-connected.

⁵²⁹Recall that according to the definition on page 156 a point x in a topological space X is called *simple*, if there exists an open neighborhood U of x such that $\{x\}$ is a deformation retract of U . Put differently, $\{x\}$ is well-behaved.

⁵³⁰Later on we will use the fact that by Corollary 22.4 (4) every point on a CW-complex is simple.

Theorem 28.9. (Mayer–Vietoris Theorem for CW-complexes) *Let X be a CW-complex and let A and B be two subcomplexes of X such that $X = A \cup B$.*

- (1) *For each $n \in \mathbb{N}$ there exists a unique homomorphism $\partial_n: H_n(X) \rightarrow H_{n-1}(A \cap B)$ which has the following properties: If $a \in C_n(A)$ and $b \in C_n(B)$ are such that $a + b \in C_n(X)$ is a cycle, then*

$$\partial_n([a + b]) = [\partial a] = -[\partial b] \in H_{n-1}(A \cap B).$$

- (2) *The following sequence is exact:*

$$\cdots \rightarrow H_n(A \cap B) \xrightarrow{i_{A \cap B} \oplus -i_{A \cap B}} H_n(A) \oplus H_n(B) \xrightarrow{i_A + i_B} H_n(X) \xrightarrow{\partial_n} H_{n-1}(A \cap B) \rightarrow \cdots$$

A totally analogous statement also holds if we replace the homology groups of $A \cap B$, A , B and X by the corresponding reduced homology groups.

Proof ().* Let X be a CW-complex and let A and B be two subcomplexes of X . As was shown in the proof of Lemma 22.4, there exist open neighborhoods U of A and V of B such that the following statements hold:

- (1) A is a deformation retract of U ,
- (2) B is a deformation retract of V , and
- (3) $A \cap B$ is a deformation retract of $U \cap V$.

Now we consider the following diagram

$$\begin{array}{ccccccc} H_n(A \cap B) & \xrightarrow{i \oplus -i} & H_n(A) \oplus H_n(B) & \xrightarrow{i+i} & H_n(X) & & H_{n-1}(A \cap B) \\ \downarrow & & \downarrow & & \downarrow = & & \downarrow \\ \cdots \rightarrow H_n(U \cap V) & \xrightarrow{i \oplus -i} & H_n(U) \oplus H_n(V) & \xrightarrow{i+i} & H_n(X) & \xrightarrow{\partial_n} & H_{n-1}(U \cap V) \rightarrow \cdots \end{array}$$

where the bottom sequence is given by the Mayer–Vietoris Theorem 28.5. We make the following observations:

- (1) the diagram commutes since all the relevant maps are given by inclusions,
- (2) the vertical maps are isomorphisms by Corollary 25.5.

The theorem is now an immediate consequence of the Mayer–Vietoris Theorem 28.5. $\square$

28.5. The homology groups of the torus and the Klein bottle. In this section we will determine the homology groups of the torus and the Klein bottle.

Lemma 28.10. *Let $T = S^1 \times S^1$ be the torus, then*

$$H_n(T) \cong \begin{cases} \mathbb{Z}, & \text{if } n = 0, 2, \\ \mathbb{Z}^2, & \text{if } n = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Throughout the proof we identify the torus T with $S^1 \times \mathbb{R}/4\mathbb{Z}$ in the obvious way. As usual we denote by $p: S^1 \times \mathbb{R} \rightarrow T = S^1 \times \mathbb{R}/4\mathbb{Z}$ the projection map. In the following we write $A := p(S^1 \times ([0, 1] \cup [3, 4]))$ and $B := p(S^1 \times [1, 3])$. Evidently we have $T = A \cup B$.

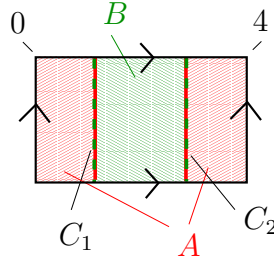


FIGURE 290.

Furthermore we write $C_1 := p(S^1 \times \{1\})$ and $C_2 := p(S^1 \times \{3\})$. Evidently we have $A \cap B = C_1 \cup C_2$.

It is straightforward to verify that T admits a CW-structure such that A and B are subcomplexes. Therefore we can apply the Mayer–Vietoris Theorem 28.9 for CW-complexes and we obtain the following long exact sequence in homology⁵³¹

$$\begin{array}{ccccccc}
 & & & \dots & \rightarrow & H_2(T) & \xrightarrow{\partial} \\
 \rightarrow & \begin{array}{c} H_1(C_1) \\ H_1(C_2) \end{array} & \xrightarrow{\begin{pmatrix} i_{C_1} & i_{C_2} \\ -i_{C_1} & -i_{C_2} \end{pmatrix}} & \begin{array}{c} H_1(A) \\ H_1(B) \end{array} & \xrightarrow{i_A+i_B} & H_1(T) & \xrightarrow{\partial} \\
 \rightarrow & \begin{array}{c} H_0(C_1) \\ H_0(C_2) \end{array} & \xrightarrow{\begin{pmatrix} i_{C_1} & i_{C_2} \\ -i_{C_1} & -i_{C_2} \end{pmatrix}} & \begin{array}{c} H_0(A) \\ H_0(B) \end{array} & \xrightarrow{i_A+i_B} & H_0(T) & \rightarrow 0.
 \end{array}$$

By Lemma 26.31, for each interval I and each point $x \in I$ the map $(1-t, t) \rightarrow (e^{2\pi it}, x)$ is a singular chain that represents a generator of $H_1(S^1 \times I) \cong \mathbb{Z}$. Now we use this generator to identify the groups $H_1(A)$, $H_1(B)$, $H_1(C_1)$ and $H_1(C_2)$ with $\mathbb{Z}$. The topological spaces A, B, C_1 and C_2 are path-connected, so we can identify their 0-th homology groups with $\mathbb{Z}$ using the isomorphism from Proposition 24.5. Thus we obtain the following Mayer–Vietoris sequence

$$\begin{array}{ccccccc}
 & & & 0 & \rightarrow & H_2(T) & \xrightarrow{\partial} \\
 \rightarrow & \begin{array}{c} \mathbb{Z} \\ \mathbb{Z} \end{array} & \xrightarrow{\begin{pmatrix} i_{C_1} & i_{C_2} \\ -i_{C_1} & -i_{C_2} \end{pmatrix}} & \begin{array}{c} \mathbb{Z} \\ \mathbb{Z} \end{array} & \xrightarrow{i_A+i_B} & H_1(T) & \xrightarrow{\partial} \\
 \rightarrow & \begin{array}{c} \mathbb{Z} \\ \mathbb{Z} \end{array} & \xrightarrow{\begin{pmatrix} i_{C_1} & i_{C_2} \\ -i_{C_1} & -i_{C_2} \end{pmatrix}} & \begin{array}{c} \mathbb{Z} \\ \mathbb{Z} \end{array} & \xrightarrow{i_A+i_B} & H_0(T) & \rightarrow 0.
 \end{array}$$

⁵³¹Hereby we write

$$\begin{array}{c} H_i(C_1) \\ H_i(C_2) \end{array} \text{ instead of } H_i(C_1) \oplus H_i(C_2) \quad \text{and} \quad \begin{array}{c} H_i(A) \\ H_i(B) \end{array} \text{ instead of } H_i(A) \oplus H_i(B).$$

The reason is that we think of direct sums as column vectors so that we can use the usual matrix notation for describing the maps between the direct sums.

To determine the homology groups $H_2(T)$ and $H_1(T)$ we now need to understand the maps in the above long exact sequence. It follows immediately from the choice of the identifications of the 0-th and first homology groups that all the inclusion maps induce the identity. Put differently, we obtain the following long exact sequence:

$$\begin{array}{ccccccc} & & 0 & \rightarrow & H_2(T) & \xrightarrow{\partial} & \\ & & & & & & \\ \rightarrow & \mathbb{Z} & \xrightarrow{\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}} & \mathbb{Z} & \xrightarrow{(1 \ 1)} & H_1(T) & \xrightarrow{\partial} \\ & \mathbb{Z} & & \mathbb{Z} & & & \\ \rightarrow & \mathbb{Z} & \xrightarrow{\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}} & \mathbb{Z} & \xrightarrow{(1 \ 1)} & \mathbb{Z} & \rightarrow H_0(T). \end{array}$$

It follows from the discussion on page 424 that

$$H_2(T) \cong \ker \left(\mathbb{Z} \xrightarrow{\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}} \mathbb{Z} \right) = \mathbb{Z} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cong \mathbb{Z}.$$

Using Lemma 28.4 we obtain the following short exact sequence:

$$0 \rightarrow \underbrace{\operatorname{coker} \left(\mathbb{Z} \xrightarrow{\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}} \mathbb{Z} \right)}_{\cong \begin{pmatrix} \mathbb{Z} \\ 0 \end{pmatrix}} \rightarrow H_1(T) \rightarrow \underbrace{\ker \left(\mathbb{Z} \xrightarrow{\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}} \mathbb{Z} \right)}_{\cong \mathbb{Z} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix}} \rightarrow 0.$$

Thus we obtain a short exact sequence

$$0 \rightarrow \begin{pmatrix} \mathbb{Z} \\ 0 \end{pmatrix} \rightarrow H_1(T) \rightarrow \mathbb{Z} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} \rightarrow 0.$$

It follows from Corollary 28.3 that $H_1(T) \cong \mathbb{Z}^2$. Finally it follows from the above long exact sequence and the discussion on page 424 that⁵³²

$$H_0(T) \cong \operatorname{coker} \left(\mathbb{Z} \xrightarrow{\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}} \mathbb{Z} \right) \cong \mathbb{Z}.$$

This concludes the proof of the lemma. $\square$

Now we also want to determine explicit generators for $H_1(T) \cong \mathbb{Z}^2$ and $H_2(T) \cong \mathbb{Z}$. We will do so in the following lemma.

Lemma 28.11. *Let $T = S^1 \times S^1$ be the torus, then the following hold.⁵³³*

- (1) *The singular 1-simplices c and d shown in Figure 291 on the left are generators for $H_1(T) \cong \mathbb{Z}^2$.*

⁵³²Of course we already knew by Proposition 24.5 that $H_0(T) \cong \mathbb{Z}$.

⁵³³Here the singular simplices in Figure 291 are understood to be given by the obvious affine maps $\Delta^n \rightarrow [0, 1] \times [0, 1] / \sim$.

- (2) The singular 2-chain $\alpha - \beta$ that is given by the singular 2-simplices α and β shown in Figure 291 on the right is a generator for $H_2(T) \cong \mathbb{Z}$.⁵³⁴

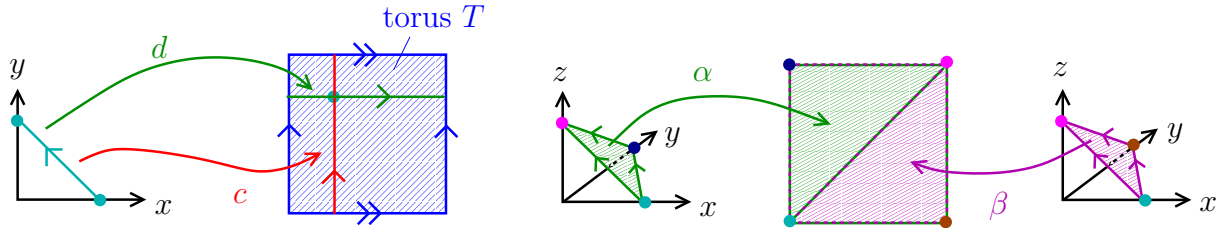


FIGURE 291.

Proof. We continue with the notation that we had introduced in the proof of Lemma 28.10. Recall that this means that we identify the torus T with the product $S^1 \times \mathbb{R}/4\mathbb{Z}$, we denote by $p: S^1 \times \mathbb{R} \rightarrow T = S^1 \times \mathbb{R}/4\mathbb{Z}$ the projection map, we write $A := p(S^1 \times ([0, 1] \cup [3, 4]))$, $B := p(S^1 \times [1, 3])$ and we write $C_1 := p(S^1 \times \{1\})$ and $C_2 := p(S^1 \times \{3\})$.

- (1) Let c be the standard generator of $H_1(B)$ and let a and b be the singular simplices in A and B that are indicated in Figure 292. We consider the following short exact

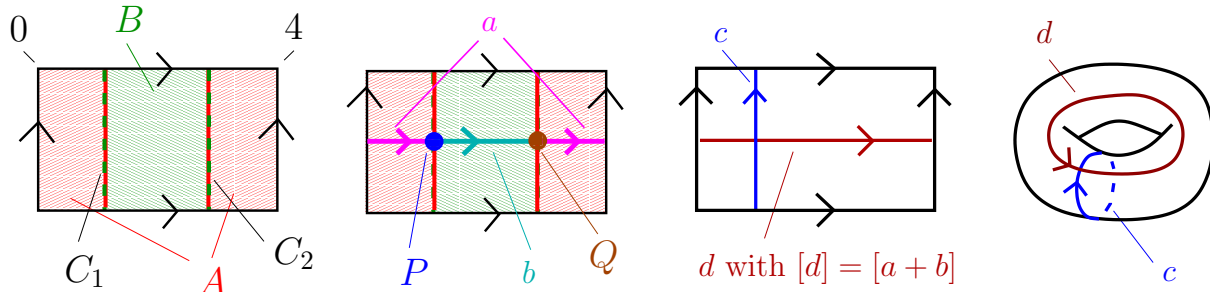


FIGURE 292.

sequence that we had obtained in the proof of Lemma 28.10:

$$0 \rightarrow \text{coker} \begin{pmatrix} H_1(C_1) = \mathbb{Z} & \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} & H_1(A) = \mathbb{Z} \\ H_1(C_2) = \mathbb{Z} & & H_1(B) = \mathbb{Z} \end{pmatrix} \rightarrow H_1(T) \xrightarrow{\partial_1} \ker \begin{pmatrix} H_0(C_1) = \mathbb{Z} & \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} & H_0(A) = \mathbb{Z} \\ H_0(C_2) = \mathbb{Z} & & H_0(B) = \mathbb{Z} \end{pmatrix} \rightarrow 0.$$

It is clear that $c = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \begin{matrix} H_1(C_1) = \mathbb{Z} \\ H_1(C_2) = \mathbb{Z} \end{matrix}$ represents a generator of the group on the left-hand side. Furthermore note that $a + b$ is a cycle in T with $a \in C_1(A)$ and $b \in C_1(B)$,

⁵³⁴This singular 2-chain is precisely the one that we had already considered in Figure 257. In particular this answers our question from page 406, the cycle $\alpha - \beta$ does indeed represent a non-trivial element in $H_2(T)$.

thus we have

$$\begin{array}{c} \partial_1([a+b]) = [\partial a] = [P-Q] = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \in \begin{array}{l} H_1(C_1) = \mathbb{Z} \\ H_1(C_2) = \mathbb{Z} \end{array} \\ \uparrow \\ \text{see the definition of } \partial_1 \text{ in Theorem 28.9 (1)} \end{array}$$

Thus we see that $\partial_1([a+b])$ is a generator of the group on the right-hand side of the above short exact sequence. It follows from Corollary 28.3 (2) that $[c]$ and $[a+b]$ generate the free abelian group $H_1(T) \cong \mathbb{Z}^2$. The desired statement now follows from the observation that Lemma 24.3 (3) implies that $[a+b] = [d]$.

- (2) Let α and β be the two singular 2-simplices that are shown in Figure 291 on the right. As we will see, the proof that $\alpha - \beta$ represents a generator of $H_2(T)$ is quite similar to the proof of Lemma 26.30.

First note that it is straightforward to verify that the singular 2-chain $\alpha - \beta$ is indeed a cycle. We denote by A and B the images of $\alpha: \Delta^2 \rightarrow T$ and $\beta: \Delta^2 \rightarrow T$. (We refer to Figure 293 for an illustration.) We consider the following maps

$$\begin{array}{ccccccc} H_2(T) & & & & & & H_2(\Delta^2, \partial\Delta^2) \\ \downarrow & \searrow [\alpha-\beta] \mapsto [\alpha-\beta] & & \swarrow [\text{id}_{\Delta^2}] \mapsto [\alpha] & & & \downarrow \cong \\ H_2(T/B) & \longrightarrow & \underbrace{H_2(T/B, B/B)}_{\text{here } [\alpha-\beta] = [\alpha]} & \xleftarrow{\cong} & H_2(A/\partial A, \partial A/\partial A) & \xleftarrow[\cong]{\alpha_*} & H_2(\Delta^2/\partial\Delta^2, \partial\Delta^2/\partial\Delta^2). \end{array}$$

We make the following observations:

- (a) By Lemma 26.29 we know that $[\text{id}_{\Delta^2}] \in H_2(\Delta^2, \partial\Delta^2) \cong \mathbb{Z}$ is a generator.
- (b) The two horizontal maps on the right are induced by homeomorphisms of pairs, hence they are isomorphisms.
- (c) The vertical homomorphism on the right is an isomorphism by Proposition 26.16 and Lemma 26.11.
- (d) The statements (a), (b) and (c) imply that $[\alpha]$ represents a generator of the group $H_2(T/B, B/B) \cong \mathbb{Z}$.
- (e) The singular 2-simplex $\beta: \Delta^2 \rightarrow B$ is zero in $C_2(T/B, B/B)$. Therefore we have $[\alpha - \beta] = [\alpha] \in H_2(T/B, B/B)$.
- (f) The statements (d) and (e) imply that the image of $[\alpha - \beta] \in H_2(T) \cong \mathbb{Z}$ under the map $H_2(T) \rightarrow H_2(T/B, B/B)$ is a generator of $H_2(T/B, B/B) \cong \mathbb{Z}$. But this implies that $[\alpha - \beta]$ is a generator of $H_2(T) \cong \mathbb{Z}$. $\square$

In exercise sheet 7 we will determine the homology groups of the Klein bottle. More precisely, using the Mayer–Vietoris Theorem 28.9 for CW-complexes we will prove the following lemma.

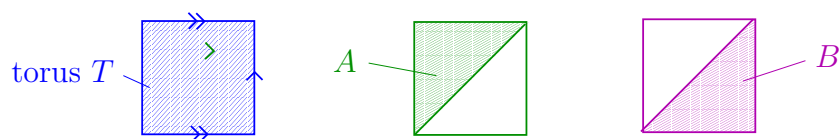


FIGURE 293. Illustration for the proof of Lemma 28.11 (2).

Lemma 28.12. *We denote by K the Klein bottle. Then*

$$H_n(K) \cong \begin{cases} \mathbb{Z}, & \text{if } n = 0, \\ \mathbb{Z} \oplus \mathbb{Z}_2, & \text{if } n = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore we can determine explicitly two generators of $H_1(K)$. More precisely, we consider Figure 294. There the singular 1-simplex d generates the $\mathbb{Z}_2$ -summand and the singular 1-simplex c generates a $\mathbb{Z}$ -summand of $H_1(K)$.⁵³⁵ We leave the verification of this statement as an exercise to the reader.

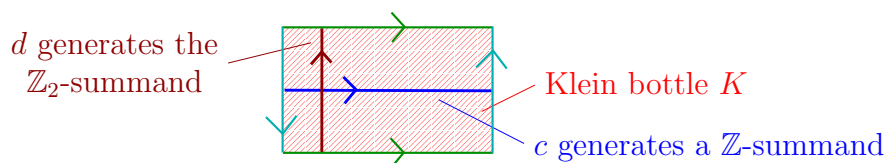


FIGURE 294.

Remark. With some effort one could now use the Mayer–Vietoris Theorem 28.9 to compute the homology groups of the surfaces of genus ≥ 2 . We will not do so, since we will soon get to know an alternative approach to calculating homology groups, with which we can easily compute these homology groups.⁵³⁶

28.6. The homology groups of a knot complement. Finally we want to determine the first homology group of the complement of a knot. We recall the relevant definitions from pages 58 and 2.3.

- (1) A *knot* is a 1-dimensional submanifold of S^3 that is diffeomorphic to S^1 .
- (2) Let K be an oriented knot in S^3 . By the Tubular Neighborhood Theorem 2.2 there exists an orientation-preserving embedding $F: S^1 \times \overline{B}^2 \rightarrow S^3$ with $\Phi(S^1 \times \{0\}) = K$. For any $* \in S^1$ we refer to the oriented submanifold $F(\{*\} \times S^1)$ as a *meridian* of K .

⁵³⁵It is straightforward to see that c generates an element of infinite order since the “vertical projection” $K \rightarrow [0, 1]/\sim$ sends c to a generator of $H_1([0, 1]/\sim) = H_1(S^1)$.

⁵³⁶Of course it is a great exercise to determine the homology groups of the surfaces of genus ≥ 2 with our present knowledge.

We refer to Figure 295 for an illustration of an oriented knot and a meridian. We also introduce one new definition:

- (3) Let M be a manifold and let C be an oriented curve in M , i.e. C is an oriented 1-dimensional submanifold for which there exists an orientation-preserving diffeomorphism $\Phi: S^1 \rightarrow C$. Sometimes, by a slight abuse of notation, we denote by C also the element in $H_1(M)$ that is given by the image of the standard generator of $H_1(S^1)$ under the map $H_1(S^1) \xrightarrow{\Phi_*} H_1(C) \rightarrow H_1(M)$.⁵³⁷

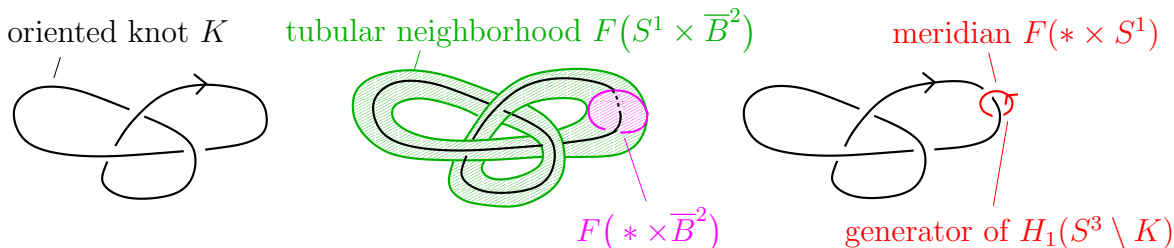


FIGURE 295.

Lemma 28.13. *Let $K \subset S^3$ be a knot. Then*⁵³⁸

$$H_0(S^3 \setminus K) \cong \mathbb{Z} \quad \text{and} \quad H_1(S^3 \setminus K) \cong \mathbb{Z}.$$

Furthermore every meridian is a generator of $H_1(S^3 \setminus K)$.

Later on we will compute the other homology groups of $S^3 \setminus K$.

Remark. The result of Lemma 28.13 is easy to state but also slightly disappointing since it shows that the first homology groups cannot be used to distinguish different knots. In contrast to this fact we had seen in Proposition 12.4 and Lemma 12.5 that the fundamental group can be used to distinguish the unknot from the trefoil.

Proof. Let $K \subset S^3$ be a knot. By the Tubular Neighborhood Theorem 2.2 there exists an embedding $\Phi: S^1 \times \overline{B}^2 \rightarrow S^3$ with $\Phi(S^1 \times \{0\}) = K$. In the following we write $K \times \overline{B}^2$ instead of $\Phi(S^1 \times \overline{B}^2)$. It is clear that the interiors of $S^3 \setminus K$ and $K \times \overline{B}^2$ cover S^3 . Therefore we can apply the Mayer–Vietoris Theorem 28.5 to $S^3 = (S^3 \setminus K) \cup (K \times \overline{B}^2)$ and we obtain the following exact sequence

$$\begin{array}{ccccccc} \overbrace{H_2(S^3)}^{=0} & \longrightarrow & H_1(K \times (\overline{B}^2 \setminus 0)) & \longrightarrow & H_1(S^3 \setminus K) \oplus H_1(K \times \overline{B}^2) & \longrightarrow & \overbrace{H_1(S^3)}^{=0} \longrightarrow \\ & & \longrightarrow & & \longrightarrow & & \\ & & \tilde{H}_0(K \times (\overline{B}^2 \setminus 0)) & \longrightarrow & \tilde{H}_0(S^3 \setminus K) \oplus \tilde{H}_0(K \times \overline{B}^2) & \longrightarrow & \underbrace{\tilde{H}_0(S^3)}_{=0} \longrightarrow 0. \end{array}$$

⁵³⁷It follows from Lemma 12.6 and Proposition 25.4 that, up to a sign, the homology class does not depend on the choice of the orientation-preserving diffeomorphism Φ .

⁵³⁸By Proposition 24.5 the first statement just says that $S^3 \setminus K$ is path-connected. This seems obvious from the pictures, but it is nonetheless non-trivial to prove.

Now we have to study the homology groups of the various spaces in more detail.

- (1) We denote by $\mu = \Phi(* \times S^1)$ the meridian and we define $\lambda = \Phi(S^1 \times *)$. It follows from Lemma 28.11 that μ and λ is a basis for $H_1(\Phi(S^1 \times S^1))$. The inclusion $S^1 \times S^1 \rightarrow K \times (\overline{B}^2 \setminus 0)$ is a homotopy equivalence. It follows from the above, Proposition 24.5 and Corollary 25.5 that $H_1(K \times (\overline{B}^2 \setminus 0)) \cong \mathbb{Z} \cdot \mu \oplus \mathbb{Z} \cdot \lambda$ and that $\tilde{H}_0(K \times (\overline{B}^2 \setminus 0)) \cong \mathbb{Z}$.
- (2) We write $\gamma = K \times \{0\}$. The inclusion $K \times \{0\} \rightarrow K \times \overline{B}^2$ is a homotopy equivalence. Similar to (1) we see that $H_1(K \times \overline{B}^2) \cong \mathbb{Z} \cdot \gamma$ and that $\tilde{H}_0(K \times \overline{B}^2) \cong \mathbb{Z}$.
- (3) Under the inclusion induced map $H_1(\tilde{K \times (\overline{B}^2 \setminus 0)}) = \mathbb{Z} \cdot \mu \oplus \mathbb{Z} \cdot \lambda \rightarrow H_1(K \times \overline{B}^2) = \mathbb{Z} \cdot \gamma$ we have $\mu \mapsto 0$ ⁵³⁹ and we have $\lambda \mapsto \gamma$.⁵⁴⁰

Thus we obtain an exact sequence of the following form

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z} \cdot \mu \oplus \mathbb{Z} \cdot \lambda & \xrightarrow[\cong]{\begin{smallmatrix} \mu \mapsto (\mu, 0) \\ \lambda \mapsto (\lambda, \gamma) \end{smallmatrix}} & H_1(S^3 \setminus K) \oplus \mathbb{Z} \cdot \gamma & \longrightarrow & 0 \longrightarrow \\ & & \longrightarrow 0 & & \longrightarrow H_0(S^3 \setminus K) \oplus 0 & \longrightarrow & 0 \longrightarrow 0. \end{array}$$

Now it follows immediately from Corollary 28.3 and Lemma 8.6 (1) that $H_1(S^3 \setminus K) \cong \mathbb{Z}$ and that $\tilde{H}_0(S^3 \setminus K) \cong 0$.⁵⁴¹ Furthermore it follows easily that $\mu \in H_1(S^3 \setminus K) = H_1(S^3 \setminus K)$ is a generator.⁵⁴² $\square$

Remark. In the proof of Lemma 28.13 we used the Mayer–Vietoris Theorem 28.5 in a different way than in the previous examples. More precisely, instead of decomposing $S^3 \setminus K$ into smaller pieces, we used that for a decomposition $X = U \cup V$ the Mayer–Vietoris sequence not only relates the homology groups of U, V and $U \cap V$ to the homology groups of X , but also that it relates the homology groups of U to the homology groups of $V, U \cap V$ and X .

28.7. The homology groups of a mapping torus (*). On page 212 we had defined the mapping torus of a self-homeomorphism and in Proposition 12.12 we had determined the fundamental group of the mapping torus. In this section we will determine the homology groups of a mapping torus.

For convenience we recall the definition of the mapping torus of a self-homeomorphism. Let X be a topological space and let $f: X \rightarrow X$ be a homeomorphism. We refer to

$$M(X, f) := X \times [0, 1] / (x, 0) \sim (f(x), 1)$$

as the *mapping torus* of (X, f) .

⁵³⁹This can be seen as follows: the map $H_1(* \times S^1) \rightarrow H_1(K \times \overline{B}^2)$ factors through $H_1(* \times \overline{B}^2) = 0$.

⁵⁴⁰Why is that?

⁵⁴¹Can we use the same approach to determine $H_2(S^3 \setminus K)$?

⁵⁴²Why does this follow?

Proposition 28.14. *Let X be a topological space and let $f: X \rightarrow X$ be a homeomorphism. Then for any $n \in \mathbb{N}_0$ we have a short exact sequence⁵⁴³*

$$0 \rightarrow \operatorname{coker}(H_n(X) \xrightarrow{f_* - \operatorname{id}} H_n(X)) \xrightarrow{i} H_n(M(X, f)) \rightarrow \ker(H_{n-1}(X) \xrightarrow{f_* - \operatorname{id}} H_{n-1}(X)) \rightarrow 0,$$

where the map i is induced by the inclusion $X = X \times \{0\} \rightarrow M(X, f)$.

Example. First let $X = S^1$ and let $f: S^1 \rightarrow S^1$ be the self-homeomorphism that is given by $f(z) = \bar{z} = z^{-1}$. We had seen on page 212 that the mapping torus $M(S^1, f)$ is homeomorphic to the Klein bottle. We make the standard identifications $H_0(S^1) = \mathbb{Z}$ and $H_1(S^1) = \mathbb{Z}$. It follows from the discussion on page 410 the induced map $f_*: H_0(S^1) \rightarrow H_0(S^1)$ is the identity map. Furthermore we had shown in Lemma 27.1 that the induced homomorphism $f_*: \mathbb{Z} = H_1(S^1) \rightarrow H_1(S^1) = \mathbb{Z}$ is given by multiplication by -1 . The short exact sequences of Proposition 28.14 are therefore of the form

$$\begin{aligned} n = 2: & \quad 0 \rightarrow H_2(M(S^1, f)) \rightarrow \ker\{\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}\} \rightarrow 0 \\ n = 1: & \quad 0 \rightarrow \operatorname{coker}\{\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}\} \rightarrow H_1(M(S^1, f)) \rightarrow \ker\{\mathbb{Z} \xrightarrow{\cdot 0} \mathbb{Z}\} \rightarrow 0 \end{aligned}$$

and

$$n = 0: \quad 0 \rightarrow \operatorname{coker}\{\mathbb{Z} \xrightarrow{\cdot 0} \mathbb{Z}\} \rightarrow H_0(M(S^1, f)) \rightarrow 0.$$

Using Corollary 28.3 we now deduce that

$$H_2(M(S^1, f)) = 0, \quad H_1(M(S^1, f)) \cong \mathbb{Z}_2 \oplus \mathbb{Z} \quad \text{and} \quad H_0(M(S^1, f)) = 0.$$

This confirms, as it should, the calculation that we had made in Lemma 28.12.

Proof. The proof of the proposition is quite similar to the calculation of the homology groups of the torus.⁵⁴⁴ Let X be a topological space and let $f: X \rightarrow X$ be a homeomorphism. We compute the homology of the following space that is clearly homeomorphic to the mapping torus:

$$M := X \times [0, 10] / (x, 0) \sim (f(x), 1).$$

We denote by $p: X \times [0, 10] \rightarrow M$ the obvious projection map. Furthermore we write $A := p(S^1 \times ([0, 4] \cup (6, 10])) \subset M$ and $B := p(S^1 \times (2, 8)) \subset M$. Evidently the interiors of A and B cover all of M . Finally we write $C_1 := p(X \times (2, 4))$ and $C_2 := p(X \times (6, 5))$. Thus we have $A \cap B = C_1 \cup C_2$. Finally, given $t \in [0, 10]$ we denote by

$$\begin{aligned} i_t: X &\rightarrow p(X \times \{t\}) \\ x &\mapsto p(x, t) \end{aligned}$$

the obvious inclusion map. We write $i_{3,A}$ for the composition of i_3 with the inclusion map $p(X \times \{3\}) \rightarrow A$. We denote analogous compositions by the obvious variations in the notation.

The following claim is a straightforward consequence of Proposition 25.4. We leave the details as an exercise to the reader.

⁵⁴³Recall that $H_{-1}(X)$ is defined to be the trivial group.

⁵⁴⁴In fact if one reads the subsequent proof with $X = S^1$ and $f = \operatorname{id}$ in mind, then one obtains almost the same argument as for the torus.

Claim. For any $n \in \mathbb{N}_0$ the following two diagrams commute

$$\begin{array}{ccc} H_n(X) & \xrightarrow{\text{id}} & H_n(X) \\ & \searrow i_{3,B} & \swarrow i_{7,B} \\ & H_n(B) & \end{array} \quad \text{and} \quad \begin{array}{ccc} H_n(X) & \xrightarrow{f_*^{-1}} & H_n(X) \\ & \searrow i_{3,A} & \swarrow i_{7,A} \\ & H_n(B) & \end{array}$$

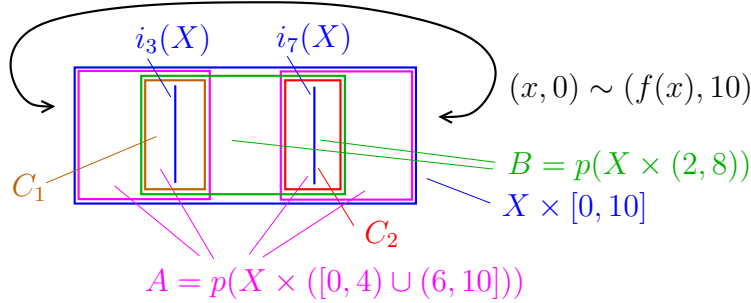


FIGURE 296.

We consider the following diagram:⁵⁴⁵

$$\begin{array}{ccccccc} \rightarrow & \begin{array}{c} H_n(C_1) \\ H_n(C_2) \end{array} & \xrightarrow{\begin{pmatrix} i_{C_1} & i_{C_2} \\ -i_{C_1} & -i_{C_2} \end{pmatrix}} & \begin{array}{c} H_n(A) \\ H_n(B) \end{array} & \xrightarrow{(i_A \ i_B)} & H_n(M) & \xrightarrow{\partial} & \begin{array}{c} H_{n-1}(C_1) \\ H_{n-1}(C_2) \end{array} & \xrightarrow{\begin{pmatrix} i_{C_1} & i_{C_2} \\ -i_{C_1} & -i_{C_2} \end{pmatrix}} & \begin{array}{c} H_{n-1}(A) \\ H_{n-1}(B) \end{array} & \rightarrow \\ & \uparrow \begin{pmatrix} i_3 & 0 \\ 0 & i_7 \end{pmatrix} & & \uparrow \begin{pmatrix} i_3 & 0 \\ 0 & i_7 \end{pmatrix} & = & \uparrow \begin{pmatrix} i_3 & 0 \\ 0 & i_7 \end{pmatrix} & & \uparrow \begin{pmatrix} i_3 & 0 \\ 0 & i_7 \end{pmatrix} & & \uparrow \begin{pmatrix} i_3 & 0 \\ 0 & i_7 \end{pmatrix} \\ \rightarrow & \begin{array}{c} H_n(X) \\ H_n(X) \end{array} & \xrightarrow{\begin{pmatrix} \text{id} & f_* \\ -\text{id} & -\text{id} \end{pmatrix}} & \begin{array}{c} H_n(X) \\ H_n(X) \end{array} & \xrightarrow{(i_A \ i_B)} & H_n(M) & \xrightarrow{\partial} & \begin{array}{c} H_{n-1}(X) \\ H_{n-1}(X) \end{array} & \xrightarrow{\begin{pmatrix} \text{id} & f_* \\ -\text{id} & -\text{id} \end{pmatrix}} & \begin{array}{c} H_{n-1}(X) \\ H_{n-1}(X) \end{array} & \rightarrow \\ & \uparrow \begin{pmatrix} \text{id} \\ -\text{id} \end{pmatrix} & & \uparrow \begin{pmatrix} \text{id} \\ 0 \end{pmatrix} & = & \uparrow \begin{pmatrix} \text{id} \\ -\text{id} \end{pmatrix} & & \uparrow \begin{pmatrix} \text{id} \\ 0 \end{pmatrix} & & \uparrow \begin{pmatrix} \text{id} \\ 0 \end{pmatrix} \\ \rightarrow & H_n(X) & \xrightarrow{\text{id} - f_*} & H_n(X) & \xrightarrow{i} & H_n(M) & \xrightarrow{\partial} & H_{n-1}(X) & \xrightarrow{\text{id} - f_*} & H_{n-1}(X) & \rightarrow 0. \end{array}$$

The above diagram arose as follows:

- (1) On top we wrote down the Mayer–Vietoris sequence from Theorem 28.5 for the decomposition $M = A \cup B$, which is of course exact.
- (2) We observed that the topological spaces A, B, C_1, C_2 are homotopy equivalent to X , more precisely, the maps $i_3: X \rightarrow A, i_3: X \rightarrow C_1, i_7: X \rightarrow C_2$ and $i_7: X \rightarrow B$ are homotopy equivalences. It follows from Corollary 25.5 that the induced maps on homology are isomorphisms. In particular the vertical maps between the first and the second horizontal sequence are isomorphisms.

⁵⁴⁵Hereby we use the same notation as on page 483.

- (3) We wanted the top part of the diagram to commute. Since the top horizontal maps and the vertical maps were given we had to find the appropriate maps for the second horizontal sequence. For the first square to commute we needed to find maps $a, b, c, d: H_n(X) \rightarrow H_n(X)$ such that

$$\underbrace{\begin{pmatrix} i_{C_1,A} & i_{C_2,A} \\ -i_{C_1,B} & -i_{C_2,B} \end{pmatrix}}_{=\begin{pmatrix} i_{3,A} & i_{7,A} \\ -i_{3,B} & -i_{7,B} \end{pmatrix}} \underbrace{\begin{pmatrix} i_{3,C_1} & 0 \\ 0 & i_{7,C_2} \end{pmatrix}}_{=\begin{pmatrix} i_{3,A} \circ a & i_{3,A} \circ b \\ i_{7,B} \circ c & i_{7,B} \circ d \end{pmatrix}} = \begin{pmatrix} i_{3,A} & 0 \\ 0 & i_{7,B} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

It follows from the above claim that a solution is given by $a = \text{id}, b = f_*, c = -\text{id}$ and $d = -\text{id}$.

- (4) It is straightforward to see that the bottom squares in the above diagram commute. We leave it as an exercise to verify that the bottom sequence is also exact.

As we had just pointed out, the bottom horizontal sequence is exact. The proposition is thus an immediate consequence of Lemma 28.4. $\square$

We can now also easily prove the following lemma.

Lemma 28.15. *Let X be a topological space. Then for any $n \in \mathbb{N}_0$ we have an isomorphism*

$$H_n(X \times S^1) \cong H_n(X) \oplus H_{n-1}(X).$$

Example. Applying Lemma 28.15 iteratively one can show that for $k = 0, \dots, n$ we have

$$H_k(n\text{-dimensional torus } (S^1)^n) = \text{free abelian group of rank } \binom{n}{k}$$

and that all other homology groups vanish.

For more examples please check out the following link

http://www.uni-regensburg.de/Fakultaeten/nat_Fak_I/friedl/mapping-tori.txt

Proof. Let $n \in \mathbb{N}_0$. We identify $X \times S^1$ with $M(X, f := \text{id})$. By Proposition 28.14 we have a short exact sequence

$$0 \rightarrow \text{coker}(H_n(X) \xrightarrow{f_* - \text{id}} H_n(X)) \xrightarrow{i} H_n(M(X, f)) \rightarrow \ker(H_{n-1}(X) \xrightarrow{f_* - \text{id}} H_{n-1}(X)) \rightarrow 0,$$

where the map i is induced by the inclusion $X = X \times \{0\} \rightarrow M(X, f)$. In our case $f_* - \text{id}_* = 0$, so we obtain the short exact sequence

$$0 \rightarrow H_n(X) \xrightarrow{i} H_n(X \times S^1) \rightarrow H_{n-1}(X) \rightarrow 0$$

where the map i is induced by the inclusion $X = X \times \{1\} \rightarrow X \times S^1$. The inclusion map $X \rightarrow X \times S^1$ admits the splitting $p: X \times S^1 \rightarrow X$ that is given by projection onto the first factor. In particular $p_* \circ i_* = (p \circ i)_* = \text{id}$. It follows from Lemma 28.2 that the above short exact sequence splits and we obtain the desired result. $\square$

We conclude this long chapter with the following lemma.

Lemma 28.16. *Let X be a path-connected topological space and let $f: X \rightarrow X$ be a homeomorphism. Then there exists a short exact sequence*

$$0 \rightarrow \operatorname{coker}(H_1(X) \xrightarrow{f_* - \operatorname{id}} H_1(X)) \xrightarrow{i} H_1(M(X, f)) \rightarrow \mathbb{Z} \rightarrow 0$$

and we have an isomorphism

$$H_1(M(X, f)) \cong \mathbb{Z} \oplus \operatorname{coker}(H_n(X) \xrightarrow{f_* - \operatorname{id}} H_n(X)).$$

Proof. It follows from the discussion on page 410 that in this case $f_* - \operatorname{id}: H_0(X) \rightarrow H_0(X)$ is the zero map. Thus we obtain from Proposition 28.14 the short exact sequence

$$0 \rightarrow \operatorname{coker}(H_1(X) \xrightarrow{f_* - \operatorname{id}} H_1(X)) \xrightarrow{i} H_1(M(X, f)) \rightarrow \mathbb{Z} \rightarrow 0.$$

Since $\mathbb{Z}$ is free-abelian it follows from Corollary 28.3 that

$$H_1(M(X, f)) \cong \mathbb{Z} \oplus \operatorname{coker}(H_n(X) \xrightarrow{f_* - \operatorname{id}} H_n(X)).$$

□

29. CELLULAR HOMOLOGY

We had seen in Propositions 22.13 and 22.14 that it is quite easy to compute the fundamental group of a finite CW-complex. In this section we will develop a relatively straightforward method for computing the homology groups of a finite CW-complex. We will use this method for finally determining the homology groups of the surfaces of genus ≥ 3 and all the projective spaces $\mathbb{RP}^n$ and $\mathbb{CP}^n$.

Before we can present the algorithm for computing homology groups of CW-complexes we will need to discuss the homology groups of a nested sequence of topological spaces. In particular this will allow us to reduce many arguments on homology groups of CW-complexes to the case of finite CW-complexes.

29.1. The homology groups of a nested sequence of topological spaces. In Propositions 13.2 and 23.8 we had seen that in favorable situations the homotopy groups of a union of a nested sequence of topological spaces is the direct limit of the corresponding fundamental groups. We will now see that a similar statement holds for homology groups.

Proposition 29.1. *Let X be a topological space and let $X_i, i \in \mathbb{N}$ be a nested⁵⁴⁶ sequence of open subsets such that $X = \bigcup_{i \in \mathbb{N}} X_i$. Then for any $n \in \mathbb{N}_0$ the inclusion induced maps $H_n(X_i) \rightarrow H_n(X)$ induce an isomorphism*

$$\lim_{\rightarrow} H_n(X_i) \xrightarrow{\cong} H_n(X).$$

Proposition 29.1 will be an immediate consequence of the next two lemmas. Before we can state the first of these two lemmas we recall that in Proposition 13.1 we had seen that the direct limit of any direct system exists in the category of abelian groups. Almost the same argument as in Proposition 13.1 shows that the same conclusion holds in the category of chain complexes. Now we can formulate the following lemma.

Lemma 29.2. *Let X be a topological space and let $X_i, i \in \mathbb{N}$ be a nested sequence of open subsets such that $X = \bigcup_{i \in \mathbb{N}} X_i$. Then the chain complexes $C_*(X_i), i \in \mathbb{N}$ together with the inclusion induced chain maps $C_*(X_i) \rightarrow C_*(X_j)$ for $i \leq j$ form a direct system of chain complexes.⁵⁴⁷ Furthermore the inclusion induced maps $C_*(X_i) \rightarrow C_*(X)$ induce an isomorphism*

$$\lim_{\rightarrow} C_*(X_i) \xrightarrow{\cong} C_*(X)$$

of chain complexes.

Proof. Given a topological space Y and $n \in \mathbb{N}_0$ we denote by $S_n(Y)$ the set of singular n -simplices in Y . The proof of Proposition 13.2 shows that the conditions (1) and (2) imply that

$$S_n(X) = \bigcup_{i \in \mathbb{N}} S_n(X_i).$$

⁵⁴⁶Recall that this means that for each $i \in \mathbb{N}$ we have $X_i \subset X_{i+1}$.

⁵⁴⁷We refer to page 221 for the definition of a direct system.

As in the example on page 223 we now see that

$$\varinjlim C_n(X_i) = \varinjlim \mathbb{Z}^{(S_n(X_i))} = \mathbb{Z}^{(S_n(X))} = C_n(X).$$

We leave it as an exercise to verify that the inclusion maps also induce an isomorphism of chain complexes. $\square$

The second lemma required for the proof of Proposition 29.1 is purely algebraic.

Lemma 29.3. *Let $(\{C_i\}_{i \in I}, \{f_{ij}\}_{i \leq j})$ be a direct system of chain complexes. Then given any $n \in \mathbb{N}_0$ the maps $H_n(C_i) \rightarrow H_n(\varinjlim C_i)$ induce an isomorphism*

$$\varinjlim H_n(C_i) \xrightarrow{\cong} H_n(\varinjlim C_i).$$

Proof. The lemma is proved quite easily once one unravels all the definitions. We leave it as an exercise to prove the statement, alternatively we refer to [Lü, Lemma 6.49] for a proof. $\square$

In many, arguably most, situations it is more convenient to work with sequences of CW-complexes. We have the following variation on Proposition 29.1.

Proposition 29.4. *Let X be a CW-complex and let $X_i, i \in \mathbb{N}$ be a nested sequence of subcomplexes such that $X = \bigcup_{i \in \mathbb{N}} X_i$. Then for every $n \in \mathbb{N}_0$ the inclusion induced maps $H_n(X_i) \rightarrow H_n(X)$ induce an isomorphism*

$$\varinjlim H_n(X_i) \xrightarrow{\cong} H_n(X).$$

Proof. Let $\epsilon \in (0, 1)$. Given $i \in \mathbb{N}$ we define, with the notation of the proof of Lemma 22.4, $U_i = N_\epsilon(X_i)$. By definition the U_i form a nested sequence of open subsets of X such that $\bigcup_{i \in \mathbb{N}} U_i = \bigcup_{i \in \mathbb{N}} X_i = X$. By the proof of Lemma 22.4 we know that X_i is a deformation retract of U_i . We now see that

$$\begin{array}{ccccc} \varinjlim H_n(X_i) & \xrightarrow{\cong} & \varinjlim H_n(U_i) & \xrightarrow{\cong} & H_n(X) \\ \uparrow & & \uparrow & & \uparrow \\ & \text{by Corollary 25.5, since} & & \text{Proposition 29.1} & \\ & \text{each } X_i \text{ is a deformation of } U_i & & & \end{array}$$

$\square$

We conclude this short section with a few examples:

Examples.

(1) For any $n \in \mathbb{N}_0$ we have

$$\begin{array}{ccccccc} H_n(S^\infty) & = & \varinjlim_{k \in \mathbb{N}} H_n(S^k) & = & \varinjlim_{k \in \mathbb{N}} \begin{cases} \mathbb{Z}, & \text{if } n = k, 0, \\ 0, & \text{else} \end{cases} & = & \begin{cases} \mathbb{Z}, & \text{if } n = 0, \\ 0, & \text{else.} \end{cases} \\ \uparrow & & \uparrow & & \uparrow & & \\ \text{by Proposition 29.4,} & & \text{Proposition 26.3} & & & & \\ \text{where we view } S^\infty \text{ as a} & & & & & & \\ \text{CW-complex as on page 345} & & & & & & \end{array}$$

- (2) As many times before we consider the real line with infinitely many 2-dimensional spheres attached. More precisely, we consider

$$X := (\mathbb{R} \cup S^2 \times \mathbb{Z}) / \sim$$

where given $i \in \mathbb{Z}$ we identify $i \in \mathbb{R}$ with $((0, 0, -1), i) \in S^2 \times \mathbb{Z}$. We view X as a CW-complex in the obvious way. Furthermore, given $k \in \mathbb{N}$ we consider the subcomplex

$$X_k := ([-k, k] \cup S^2 \times \{-k, \dots, k\}) / \sim.$$

Note that X_k is the interval $[-k, k]$ with $2k + 1$ spheres attached. (We refer to Figure 297 for an illustration.) We have

$$H_n(X) \underset{\substack{\uparrow \\ \text{Proposition 29.4}}}{=} \varinjlim H_n(X_k) \underset{\substack{\uparrow \\ \text{see exercise sheet 7}}}{=} \varinjlim \begin{cases} \mathbb{Z}, & \text{if } n = 0, \\ \mathbb{Z}^{2k+1}, & \text{if } n = 2, \\ 0, & \text{else} \end{cases} = \begin{cases} \mathbb{Z}, & \text{if } n = 0, \\ \mathbb{Z}^\infty, & \text{if } n = 2, \\ 0, & \text{else.} \end{cases} \underset{\substack{\uparrow \\ \text{see page 223}}}{=}$$

Hereby recall that

$$\mathbb{Z}^\infty := \mathbb{Z}^{(\mathbb{N})} = \{(x_1, x_2, \dots) \mid x_i \in \mathbb{Z} \text{ but only finitely many } x_i\text{'s are non-zero}\}.$$

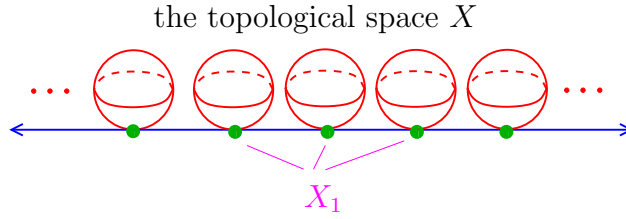


FIGURE 297.

We conclude this section with the following proposition.

Proposition 29.5. *Let $\{A_k\}_{k \in K}$ be a countable family of connected CW-complexes. Given $j \in K$ we denote by*

$$i_j: A_j \rightarrow \bigvee_{k \in K} A_k \quad \text{respectively} \quad p_j: \bigvee_{k \in K} A_k \rightarrow A_j$$

the obvious inclusion map respectively the obvious projection map. Then for every $n \in \mathbb{N}_0$ the map

$$\bigoplus_{k \in K} p_k: \tilde{H}_n\left(\bigvee_{k \in K} A_k\right) \rightarrow \bigoplus_{k \in K} \tilde{H}_n(A_k)$$

is an isomorphism where the inverse is given by

$$\bigoplus_{k \in K} i_k: \bigoplus_{k \in K} \tilde{H}_n(A_k) \rightarrow \tilde{H}_n\left(\bigvee_{k \in K} A_k\right)$$

Proof. If K is finite, then the proposition is an immediate consequence of Propositions 28.8. Now suppose that K is infinite. Since K is countable we can assume that $K = \mathbb{N}$. Then for every $n \in \mathbb{N}_0$ we have

$$\begin{array}{ccccccc} \tilde{H}_n\left(\bigvee_{k \in \mathbb{N}} A_k\right) & \xrightarrow{\cong} & \lim_{l \in \mathbb{N}} \tilde{H}_n\left(\bigvee_{k=1}^l A_k\right) & \xrightarrow{\cong} & \lim_{l \in \mathbb{N}} \bigoplus_{k=l}^{\infty} \tilde{H}_n(A_k) & \xrightarrow{\cong} & \bigoplus_{k \in \mathbb{N}} \tilde{H}_n(A_k). \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ \text{Proposition 29.4} & & \text{Proposition 28.8} & & \text{as on page 223} & & \end{array}$$

We leave it as an exercise to verify that the isomorphism is indeed given by $\bigoplus_{k \in K} p_k$ and that the inverse is the map $\bigoplus_{k \in K} i_k$. $\square$

29.2. The definition of cellular homology. Now we turn, as promised, to the study of homology groups of CW-complexes. We need to recall two definitions:

- (1) First recall that in Lemma 24.1 we gave a homeomorphism $f: (\Delta^n, \partial\Delta^n) \rightarrow (\overline{B}^n, \partial\overline{B}^n)$. This homeomorphism together with Lemma 26.29 gives us the following isomorphism:

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{\cong} & H_n(\Delta^n, \partial\Delta^n) \xrightarrow{f_*} H_n(\overline{B}^n, \partial\overline{B}^n) \\ 1 & \mapsto & [\text{id}_{\Delta^n}] \end{array}$$

In the following we will usually identify $H_n(\overline{B}^n, \partial\overline{B}^n)$ with $\mathbb{Z}$ using the above isomorphism.

- (2) Let X be a topological space and let $A \subset X$ be a non-empty subset. Following the discussion on page 440 we refer to

$$H_n(X, A) \xrightarrow{\quad} H_n(X/A, A/A) \xleftarrow[\cong]{\text{Lemma 26.11}} \tilde{H}_n(X/A)$$

as the natural homomorphism $H_n(X, A) \rightarrow \tilde{H}_n(X/A)$. We will often write down this map without explicitly referring to it as being the natural homomorphism.

Now we can formulate one of the key lemmas regarding homology groups of CW-complexes.

Lemma 29.6. *For every CW-complex X with countably many cells the following statements hold:*

- (1) *Denote by I_n the set of n -cells and given $i \in I_n$ denote by $\Phi_i: \overline{B}_i^n \rightarrow X^n$ the corresponding characteristic map. Then for every $k \in \mathbb{N}_0$ all the maps⁵⁴⁸*

$$\bigoplus_{i \in I_n} H_k(\overline{B}_i^n, \partial\overline{B}_i^n) \rightarrow \bigoplus_{i \in I_n} \tilde{H}_k(\overline{B}_i^n / \partial\overline{B}_i^n) \xrightarrow{\bigoplus \Phi_i} \tilde{H}_k(X^n / X^{n-1}) \leftarrow H_k(X^n, X^{n-1})$$

are isomorphisms. In particular the following holds:

- (a) *For $k \neq n$ we have $H_k(X^n, X^{n-1}) = 0$.*
 (b) *The group $H_n(X^n, X^{n-1})$ is a free abelian group where the rank equals the cardinality of the set of n -cells in the CW-structure.*

⁵⁴⁸Here the maps to the left and right are examples of the aforementioned natural homomorphism

- (2) For $k > n$ we have $H_k(X^n) = 0$.
 (3) For $k < n$ the inclusion $i: X^n \rightarrow X$ induces an isomorphism $i_*: H_k(X^n) \xrightarrow{\cong} H_k(X)$.
 (4) The inclusion $i: X^n \rightarrow X$ induces an epimorphism $i_*: H_n(X^n) \twoheadrightarrow H_n(X)$.

Remark. The statements (3) and (4) of Lemma 29.6 can be viewed as “homology analogues” of Proposition 22.16.

Proof. Let X be a CW-complex with countably many cells.

- (1) We have

$$\begin{array}{ccccccc}
 & \text{isomorphism by} & & \text{isomorphism by} & & & \\
 & \text{Proposition 26.16} & & \text{Proposition 29.5} & & & \\
 \bigoplus_{i \in I_n} H_k(\overline{B}_i^n, \partial \overline{B}_i^n) & \xrightarrow{\downarrow} & \bigoplus_{i \in I_n} \tilde{H}_k(\overline{B}_i^n / \partial \overline{B}_i^n) & \xrightarrow{\downarrow} & \tilde{H}_k\left(\bigvee_{i \in I_n} \overline{B}_i^n / \partial \overline{B}_i^n\right) & & \\
 & & & & \rightarrow \tilde{H}_k(X^n / X^{n-1}) & \xleftarrow{\quad} & H_k(X^n, X^{n-1}). \\
 & & & \uparrow & & \uparrow & \\
 & & \text{since the characteristic maps induce a} & & & \text{isomorphism by} & \\
 & & \text{homeomorphism } \bigvee_{i \in I_n} \overline{B}_i^n / \partial \overline{B}_i^n \rightarrow X^n / X^{n-1} & & & \text{Proposition 26.16} & \\
 & & & & & \text{and Lemma 22.4 (7)} &
 \end{array}$$

The remaining statements are now an immediate consequence of Lemma 26.28.

- (2) We fix $k \in \mathbb{N}_0$. We want to show that $H_k(X^n) = 0$ for $n = 0, \dots, k-1$. First of all we have $H_k(X^0) = 0$ by Proposition 24.5 and Lemma 24.13.⁵⁴⁹ Now assume that we already know that $H_k(X^n) = 0$ for some $n < k-1$. We want to show that $H_k(X^{n+1}) = 0$. We consider the long exact sequence of the pair (X^{n+1}, X^n) :

$$\begin{array}{ccccccc}
 \dots \rightarrow H_{k+1}(X^{n+1}, X^n) \rightarrow \underbrace{H_k(X^n)}_{=0} \rightarrow H_k(X^{n+1}) \rightarrow \underbrace{H_k(X^{n+1}, X^n)}_{=0 \text{ by (1) since } n < k-1} \rightarrow \dots \\
 \text{ i.e. since } k \neq n+1
 \end{array}$$

Thus we obtain that indeed $H_k(X^{n+1}) = 0$.

- (3) We fix $k \in \mathbb{N}_0$. Let $k > m$. This time we consider the following excerpt from the long exact sequence of the pair (X^{m+1}, X^m) :

$$\begin{array}{ccccccc}
 \dots \rightarrow \underbrace{H_{k+1}(X^{m+1}, X^m)}_{=0 \text{ by (1) since } k < m} \rightarrow H_k(X^m) \rightarrow H_k(X^{m+1}) \rightarrow \underbrace{H_k(X^{m+1}, X^m)}_{=0 \text{ by (1) since } k < m} \rightarrow \dots
 \end{array}$$

This implies in particular that for each $k \leq n \leq l$ we have inclusion induced isomorphisms $H_k(X^n) \xrightarrow{\cong} H_k(X^{n+1}) \xrightarrow{\cong} \dots \xrightarrow{\cong} H_k(X^l)$. In particular, if X is finite-dimensional, then claim (3) follows immediately from this observation. The infinite-dimensional case follows from the finite-dimensional case together with Proposition 29.4.

⁵⁴⁹Hereby we use that X^0 has the discrete topology.

- (4) By (3) it suffices to show that $H_n(X^{n+1}) \rightarrow H_n(X)$ is an isomorphism. To show this we consider the following excerpt from long exact sequence of the pair (X^{n+1}, X^n) :

$$\dots \rightarrow H_{n+1}(X^{n+1}, X^n) \rightarrow H_n(X^n) \rightarrow H_n(X^{n+1}) \rightarrow \underbrace{H_n(X^{n+1}, X^n)}_{=0 \text{ by (1)}} \rightarrow \dots$$

We see that the inclusion induced map $H_n(X^n) \rightarrow H_n(X^{n+1})$ is indeed an epimorphism. $\square$

Definition. Given a CW-complex X , we denote by $d = d_n$ the map

$$H_n(X^n, X^{n-1}) \xrightarrow{\partial_n} H_{n-1}(X^{n-1}) \xrightarrow{j_{n-1}} H_{n-1}(X^{n-1}, X^{n-2})$$

$\xRightarrow{=:d=d_n}$

where ∂_n is the connecting map in the long exact sequence of the pair (X^n, X^{n-1}) and where j_{n-1} is the map induced by the projection $C_{n-1}(X^{n-1}) \rightarrow C_{n-1}(X^{n-1}, X^{n-2})$. We refer to d_n as the *cellular boundary map*.⁵⁵⁰

The name “cellular boundary map” already suggests that the maps d_n have the following important property:

Lemma 29.7. *For all $n \in \mathbb{N}_0$ we have $d_n \circ d_{n+1} = 0$.*

Proof. We consider the following commutative diagram of maps

$$\begin{array}{ccccc} & & H_n(X^n) & & \\ & \nearrow \partial_{n+1} & \searrow j_n & & \\ H_{n+1}(X^{n+1}, X^n) & \xrightarrow{d_{n+1}} & H_n(X^n, X^{n-1}) & \xrightarrow{d_n} & H_{n-1}(X^{n-1}, X^{n-2}) \\ & & \searrow \partial_n & \nearrow j_{n-1} & \\ & & H_{n-1}(X^{n-1}) & & \end{array}$$

The map $\partial_n \circ j_n: H_n(X^n) \rightarrow H_{n-1}(X^{n-1})$ is the composition of two successive maps in the long exact sequence of the pair (X^n, X^{n-1}) , i.e. the map $\partial_n \circ j_n$ is the zero map. Since the diagram commutes we obtain that $d_n \circ d_{n+1} = 0$. $\square$

Definition. Let X be a CW-complex.

- (1) We write $C_*^{\text{CW}}(X) := H_n(X^n, X^{n-1})$ and we refer to $(C_*^{\text{CW}}(X), d_n)$ as the *cellular chain complex* of the CW-complex X .⁵⁵¹
- (2) We denote the homology groups of the cellular chain complex by $H_n^{\text{CW}}(X)$ and we refer to these groups as the *cellular homology groups of the CW-complex X* .

⁵⁵⁰It follows easily from the definitions that d_n is precisely the connecting homomorphism of the long exact sequence of the triple (X^n, X^{n-1}, X^{n-2}) .

⁵⁵¹It follows from Lemma 29.7 that $(H_n(X^n, X^{n-1}), d_n)$ is indeed a chain complex.

Example. We consider the n -sphere S^n for $n \geq 2$. On page 340 we had shown that we can view S^n as a CW-complex with one 0-cell and one n -cell. The cellular chain complex $C_*^{\text{CW}}(X)$ is thus of the form

$$0 \rightarrow \underbrace{\mathbb{Z}}_{=C_n^{\text{CW}}(X)} \rightarrow 0 \rightarrow \dots \rightarrow 0 \rightarrow \underbrace{\mathbb{Z}}_{=C_0^{\text{CW}}(X)} \rightarrow 0.$$

Since $n \geq 2$ both the $\mathbb{Z}$'s are separated by at least one zero group. The cellular boundary maps are therefore necessarily the zero maps and we obtain that

$$H_i^{\text{CW}}(X) \cong \begin{cases} \mathbb{Z}, & \text{if } i = 0, n, \\ 0, & \text{otherwise.} \end{cases}$$

These are of course isomorphic to the “usual” homology groups of S^n . In the next section we will see that this is not a coincidence.

29.3. The relationship between cellular and singular homology. Now we can formulate and prove the following proposition.

Proposition 29.8. *For each CW-complex X with countably many cells there exists an isomorphism*

$$H_n(X) \xrightarrow{\cong} H_n^{\text{CW}}(X).$$

*More precisely, there exists a uniquely determined isomorphism $\Phi_X: H_n(X) \xrightarrow{\cong} H_n^{\text{CW}}(X)$ with the property that the following diagram commutes.*⁵⁵²

$$\begin{array}{ccccc} & & H_n(X) & & \\ & \nearrow & & \searrow \Phi & \\ H_n(X^n) & & & & H_n^{\text{CW}}(X). \\ & \searrow j_n & \ker(d_n) & \nearrow & \end{array}$$

Remark. We recall that a map $f: X \rightarrow Y$ between CW-complexes is called *cellular*, if for each $n \in \mathbb{N}_0$ we have $f(X^n) \subset Y^n$. We denote by CW the category of CW-complexes where the objects are CW-complexes and the morphisms are cellular maps. It follows immediately from the definitions that a cellular map induces a chain map⁵⁵³

$$f_*: C_*^{\text{CW}}(X) \rightarrow C_*^{\text{CW}}(Y)$$

and hence also maps

$$f_*: H_*^{\text{CW}}(X) \rightarrow H_*^{\text{CW}}(Y)$$

on cellular homology. It is now straightforward to show that for each $n \in \mathbb{N}_0$ the maps

$$\begin{aligned} X &\mapsto H_n^{\text{CW}}(X) \\ (f: X \rightarrow Y) &\mapsto (f_*: H_n^{\text{CW}}(X) \rightarrow H_n^{\text{CW}}(Y)) \end{aligned}$$

⁵⁵²Here we again denote by $j_n: H_n(X^n) \rightarrow H_n(X^n, X^{n-1})$ the obvious map. In the long exact sequence of the pair (X^n, X^{n-1}) we see that the image of $H_n(X^n)$ in $H_n(X^n, X^{n-1})$ lies in the kernel of the connecting homomorphism ∂_n . This implies that the map $j_n: H_n(X^n) \rightarrow H_n(X^n, X^{n-1})$ takes values in $\ker(d_n)$.

⁵⁵³Why is the induced map indeed a chain map?

define a functor from the category $\mathcal{CW}$ of CW-complexes to the category $\mathcal{A}$ of abelian groups. Furthermore, it follows easily from the definitions that for every cellular map $f: X \rightarrow Y$ and any $n \in \mathbb{N}_0$ the following diagram commutes

$$\begin{array}{ccc} H_n(X) & \xrightarrow{f_*} & H_n(Y) \\ \Phi_X \downarrow \cong & & \cong \downarrow \Phi_Y \\ H_n^{\text{CW}}(X) & \xrightarrow{f_*} & H_n^{\text{CW}}(Y) \end{array}$$

where Φ_X and Φ_Y denote the isomorphisms from Proposition 29.8.

Example. Even though the cellular boundary maps in the cellular chain complex are still somewhat mysterious we can already use the proposition to do some calculations. As an example, we return to the n -dimensional complex projective space $\mathbb{CP}^n$ which we had defined on page 25 as

$$\mathbb{CP}^n := (\mathbb{C}^{n+1} \setminus \{0\}) / (\mathbb{C} \setminus \{0\}).$$

In Lemma 22.1 we had shown that we can view $\mathbb{CP}^n$ as a CW-complex which admits exactly one cell in dimensions $0, 2, \dots, 2n$ and which admits no other cells. In particular the cellular chain complex is of the form

$$0 \rightarrow \underbrace{C_{2n}^{\text{CW}}(\mathbb{CP}^n)}_{\cong \mathbb{Z}} \rightarrow \underbrace{C_{2n-1}^{\text{CW}}(\mathbb{CP}^n)}_{=0} \rightarrow \cdots \rightarrow \underbrace{C_2^{\text{CW}}(\mathbb{CP}^n)}_{\cong \mathbb{Z}} \rightarrow \underbrace{C_1^{\text{CW}}(\mathbb{CP}^n)}_{=0} \rightarrow \underbrace{C_0^{\text{CW}}(\mathbb{CP}^n)}_{\cong \mathbb{Z}} \rightarrow 0.$$

Since every other group is the zero map it follows that all boundary maps are also zero. In particular the homology groups equal the cellular chain groups. Thus we have shown that

$$H_k(\mathbb{CP}^n) \underset{\substack{\uparrow \\ \text{Proposition 29.8}}}{\cong} H_k^{\text{CW}}(\mathbb{CP}^n) \cong C_k^{\text{CW}}(\mathbb{CP}^n) \cong \begin{cases} \mathbb{Z}, & \text{if } k = 0, 2, 4, \dots, 2n, \\ 0, & \text{otherwise.} \end{cases}$$

It follows from the discussion on page 346 together with Lemma 22.1 that we can view the infinite-dimensional complex projective space $\mathbb{CP}^\infty$ as a CW-complex that has precisely one cell in each even dimension and no cells in odd dimensions. The same argument as above now shows that

$$H_k(\mathbb{CP}^\infty) \cong H_k^{\text{CW}}(\mathbb{CP}^\infty) \cong C_k^{\text{CW}}(\mathbb{CP}^\infty) \cong \begin{cases} \mathbb{Z}, & \text{if } k \text{ is even,} \\ 0, & \text{if } k \text{ is odd.} \end{cases}$$

Proof of Proposition 29.8. We add a couple of maps to the commutative diagram that we had already encounter in the proof of Lemma 29.7:

$$\begin{array}{ccccccc}
 & & & & & & 0 \\
 & & & & & \nearrow & \\
 & & & H_n(X^{n+1}) & & & \\
 & & \nearrow & & \searrow & & \\
 0 & \searrow & H_n(X^n) & \xrightarrow{j_n} & H_n(X^n, X^{n-1}) & \xrightarrow{d_n} & H_{n-1}(X^{n-1}, X^{n-2}) \\
 & \nearrow \partial_{n+1} & & \searrow \partial_n & & \nearrow j_{n-1} & \\
 H_{n+1}(X^{n+1}, X^n) & \xrightarrow{d_{n+1}} & H_n(X^n, X^{n-1}) & \xrightarrow{d_n} & H_{n-1}(X^{n-1}, X^{n-2}) & & \\
 & & & & \searrow \partial_n & & \\
 & & & & H_{n-1}(X^{n-1}) & \xrightarrow{j_{n-1}} & \\
 & & & & \nearrow & & \\
 & & & & 0 & &
 \end{array}$$

Claim. All the diagonal sequences in the above commutative diagram are exact.

From Lemma 29.6 (1) and (2) we know that

$$H_n(X^{n+1}, X^n) = 0, \quad H_n(X^{n-1}) = 0 \quad \text{and} \quad H_{n-1}(X^{n-2}) = 0.$$

It thus follows that the diagonal sequences in the above commutative diagram are just the long exact sequences of the corresponding pairs of topological spaces. In particular all the diagonal sequences are exact. This concludes the proof of the claim.

Now we consider the inclusion induced maps

$$\begin{array}{ccccc}
 H_n(X^n)/\text{im}(\partial_{n+1}) & \xrightarrow{\cong} & H_n(X^{n+1}) & \xrightarrow{\cong} & H_n(X). \\
 \uparrow & & \uparrow & & \\
 \text{isomorphism since the left} & & \text{isomorphism by} & & \\
 \text{diagonal sequence is exact} & & \text{Lemma 29.6 (3)} & &
 \end{array}$$

We use this map to make the identification $H_n(X) = H_n(X^n)/\text{im}(\partial_{n+1})$. Next we prove the following claim.

Claim. The map $j_n: H_n(X^n) \rightarrow H_n(X^n, X^{n-1})$ induces an isomorphism

$$\Phi: H_n(X) = H_n(X^n)/\text{im}(\partial_{n+1}) \rightarrow \ker(d_n)/\text{im}(d_{n+1}) = H_n^{\text{CW}}(X).$$

We will prove this claim once again by a “diagram chase”.⁵⁵⁴

- (1) Since $\partial_n \circ j_n = 0$ it follows that $j_n(H_n(X^n)) \subset \ker(d_n)$.
- (2) The map j_{n-1} is injective, i.e. $\ker(d_n) = \ker(\partial_n) = \text{im}(j_n)$, i.e. the map j_n induces an epimorphism

$$H_n(X^n) \rightarrow \ker(d_n).$$

Since the left triangle commutes it also induces an epimorphism

$$H_n(X^n)/\text{im}(\partial_{n+1}) \rightarrow \ker(d_n)/\text{im}(d_{n+1}).$$

⁵⁵⁴It is arguably best not to read the remainder of the proof but to go on the chase on one's own.

- (3) If an element $j_n(c) \in H_n(X^n, X^{n-1})$ lies in the image of $d_{n+1} = j_n \circ \partial_{n+1}$, then $j_n(c) = j_n(\partial_{n+1}(d))$ for some $d \in H_{n+1}(X^{n+1}, X^n)$. Since j_n is injective it follows that $c = \partial_{n+1}(d)$. Thus we have now shown that the map

$$H_n(X^n)/\text{im}(\partial_{n+1}) \rightarrow \ker(d_n)/\text{im}(d_{n+1})$$

is injective.

This concludes the proof of the claim.

By construction the isomorphism Φ has the property that the following diagram commutes:

$$\begin{array}{ccccc} & & H_n(X) & & \\ & \nearrow & & \searrow \Phi & \\ H_n(X^n) & & & & H_n^{\text{CW}}(X). \\ & \searrow j_n & \ker(d_n) & \nearrow & \end{array}$$

Note that the top-left diagonal map is an epimorphism by Lemma 29.6 (4). This implies immediately that Φ is uniquely determined. $\square$

On page 501 we had already that Proposition 29.8 can be useful, even if we cannot determine the boundary maps d_n . Another, more theoretical application is, that the cellular chain complex is “much smaller” than the singular chain complex. For example, if X is a CW-complex with finitely many cells, then the chain group $C_*^{\text{CW}}(X)$ in the cellular chain complex are *finitely generated* free abelian groups. This is in contrast to the singular chain groups $C_*(X)$ which are in almost all cases free abelian groups of uncountable rank.⁵⁵⁵

Using this observation we can now easily prove the following proposition. It can be viewed as an analogue to Proposition 22.16.

Proposition 29.9. *Let X be a CW-complex with countably many cells. The following statements hold:*

- (1) *If X contains precisely $d \in \mathbb{N}_0$ cells of dimension n , then $H_n(X)$ is generated by at most d elements.*
- (2) *If X consists only of finitely many cells, then all homology groups of X are finitely generated groups.*
- (3) *If X is finite-dimensional, then*

$$H_k(X) = 0 \text{ for all } k > \dim(X).$$

- (4) *If X is an n -dimensional CW-complex, then $H_n(X)$ is a free abelian group.*

Example. It follows from Proposition 29.9 that for any graph G we have $H_i(G) = 0$ for every $i \geq 2$ and that $H_1(G)$ is a free-abelian group.

Proof. Let X be a CW-complex X with countably many cells.

⁵⁵⁵For which topological spaces and which n is the singular chain complex $C_n(X)$ finitely generated? Can you give a complete answer?

- (1) If X contains precisely d cells of dimension n , then it follows from Lemma 29.6 that $C_*^{\text{CW}}(X) \cong \mathbb{Z}^d$. By Lemma 8.6 any subgroup U of $\mathbb{Z}^d$ is a free abelian group with $\text{rank}(U) \leq d$. Thus it follows that

$$H_n(X) \cong \underbrace{\ker \{d_n: C_n^{\text{CW}}(X) \rightarrow C_{n-1}^{\text{CW}}(X)\}}_{\text{free abelian group of rank } \leq d} / \text{im} \{d_{n+1}: C_{n+1}^{\text{CW}}(X) \rightarrow C_n^{\text{CW}}(X)\}$$

is generated by at most d elements.

- (2) This statement follows immediately from (1).
 (3) This statement is in fact already a consequence of Lemma 29.6 (2).
 (4) Now suppose that X is an n -dimensional CW-complex X . We have

$$\begin{array}{ccccc} H_n(X) & \cong & H_n^{\text{CW}}(X) & = & \ker \{d_n: C_n^{\text{CW}}(X) \rightarrow C_{n-1}^{\text{CW}}(X)\}, \\ & \uparrow & & \uparrow & \\ & \text{Proposition 29.8} & & \text{we have } C_{n+1}^{\text{CW}}(X) = 0 \text{ since } X \text{ is } n\text{-dimensional} & \end{array}$$

i.e. $H_n(X)$ is a subgroup of the free abelian group $C_n^{\text{CW}}(X)$. It follows from Lemma 8.2 that the group $H_n(X)$ itself is a free abelian group. $\square$

In Theorem 22.2 we had seen that every compact n -dimensional manifold admits a CW-structure with finitely many cells and such that the maximal dimension of a cell is n . If we combine this result with Proposition 29.9 we obtain immediately the following useful result.

Proposition 29.10. *Let M be a compact n -dimensional manifold. Then the following holds:*

- (1) *All homology groups are finitely generated abelian groups.*
 (2) *For any $k > n$ we have $H_k(M) = 0$.*

Remark.

- (1) Note that the statement of Proposition 29.10 is in contrast to the situation for higher homotopy groups. As formulated in Question 23.14 we have the strong suspicion that $\pi_3(S^2) \neq 0$.
 (2) Thus we have now shown that for a CW-complex X the singular homology groups $H_n(X)$ are isomorphic to the cellular homology groups $H_n^{\text{CW}}(X)$. One might now ask, whether one can generalize this statement, namely whether the singular chain complex $(C_*(X), \partial_*)$ and the cellular chain complex (C_*^{CW}, d_*) are in fact chain homotopic. This is indeed the case. But the proof requires more homological algebra than we have at the moment at our disposal. Therefore we postpone the proof to a later section in Algebraic Topology II.

29.4. The cellular boundary maps. If we want to use Proposition 29.8 to calculate homology groups of CW-complexes we need to develop tools for computing the cellular boundary maps d_n in the cellular chain complex. Let X be a CW-complex. Throughout this section we will use the following notation:

- (1) For $n \in \mathbb{N}_0$ we denote by I_n the set of n -cells of X .
- (2) Given an n -cell α we denote by $\varphi_\alpha: S_\alpha^{n-1} \rightarrow X^{n-1}$ the corresponding attaching map and we denote by $\Phi_\alpha: \overline{B}_\alpha^n \rightarrow X^n$ the corresponding characteristic map.
- (3) Given $n \in \mathbb{N}_0$ we denote by

$$\Psi_n: \bigvee_{\alpha \in I_n} \overline{B}_\alpha^n / \partial \overline{B}_\alpha^n \rightarrow X^n / X^{n-1}$$

the map that is given by

$$x \mapsto \Phi_\alpha(x), \quad \text{if } x \in \overline{B}_\alpha^n.$$

It is straightforward to verify that Ψ_n is a homeomorphism.

- (4) Given $\beta \in I_n$ we denote by

$$i_\beta: \overline{B}_\beta^n / \partial \overline{B}_\beta^n \rightarrow \bigvee_{\alpha \in I_n} \overline{B}_\alpha^n / \partial \overline{B}_\alpha^n \quad \text{and} \quad p_\beta: \bigvee_{\alpha \in I_n} \overline{B}_\alpha^n / \partial \overline{B}_\alpha^n \rightarrow \overline{B}_\beta^n / \partial \overline{B}_\beta^n,$$

the obvious inclusion map respectively the obvious projection map. By Proposition 29.5 the maps

$$\bigoplus_{\beta \in I} H_n(\overline{B}_\beta^n / \partial \overline{B}_\beta^n) \xrightleftharpoons[\oplus i_\beta]{\oplus p_\beta} H_n\left(\bigvee_{\beta \in I_n} \overline{B}_\beta^n / \partial \overline{B}_\beta^n\right).$$

are inverses of one-another.

- (5) Given $\alpha \in I_n$ we denote, by a slight abuse of notation, the standard generator of $H_n(\overline{B}_\alpha^n, \partial \overline{B}_\alpha^n) \cong \mathbb{Z}$, as defined on page 460, by α . Put differently, we make the identification $H_n(\overline{B}_\alpha^n, \partial \overline{B}_\alpha^n) = \mathbb{Z} \cdot \alpha$.

It follows from the discussion above and from Lemma 29.6 and its proof that all the maps in the following sequence are isomorphisms:

$$\begin{array}{c} \bigoplus_{\alpha \in I_n} \overbrace{H_n(\overline{B}_\alpha^n, \partial \overline{B}_\alpha^n)}^{=\mathbb{Z} \cdot \alpha} \xrightarrow{\cong} \bigoplus_{\alpha \in I_n} \tilde{H}_n(\overline{B}_\alpha^n / \partial \overline{B}_\alpha^n) \xrightarrow{\cong} \tilde{H}_n(\bigvee_{\alpha} \overline{B}_\alpha^n / \partial \overline{B}_\alpha^n) \xrightarrow[\Psi_n]{\cong} \tilde{H}_n(X^n / X^{n-1}) \xleftarrow[\cong]{} H_n(X^n, X^{n-1}). \\ \xrightarrow{\hspace{10cm} =: \Omega_n \hspace{10cm}} \end{array}$$

By another abuse of notation we often denote $\Omega_n(\alpha)$ also by α . Put differently, we identify $H_n(X^n, X^{n-1})$ with the free abelian group generated by the set I_n of all n -cells.

Our goal now is to determine the matrix of the cellular boundary map

$$d_n: \underbrace{H_n(X^n, X^{n-1})}_{\substack{\text{free abelian group} \\ \text{generated by } I_n}} \xrightarrow{=C_n^{\text{CW}}(X)} \underbrace{H_{n-1}(X^{n-1}, X^{n-2})}_{\substack{\text{free abelian group} \\ \text{generated by } I_{n-1}}} \xrightarrow{=C_{n-1}^{\text{CW}}(X)} \dots$$

corresponding to the bases given by the cells. As a warm-up with start out with the case $n = 1$.

Lemma 29.11. *Let X be a CW-complex with countably many cells. We adopt all the notations introduced above.*

(1) *For a 1-cell α with attaching map $\varphi_\alpha: \overline{B}^1 \rightarrow X^0$ we have*

$$d_1(\alpha) = \varphi_\alpha(1) - \varphi_\alpha(-1).$$

(2) *If X has precisely one 0-cell, then the cellular boundary map d_1 is the zero map.*

Proof. The second statement of the lemma is evidently a consequence of the first statement. Therefore it suffices to prove the first statement. So let α be a 1-cell with characteristic map $\Phi_\alpha: \overline{B}^1 \rightarrow X^0$. We have to determine the map

$$\begin{array}{ccc} \mathbb{Z} \cdot \alpha \rightarrow \bigoplus_{j \in I_1} H_1(\overline{B}_j^1, \partial \overline{B}_j^1) & \xrightarrow{\Omega_1} & H_1(X^1, X^0) \\ & \downarrow d_1 & \\ & H_0(X^0, \underbrace{X^{-1}}_{=\emptyset}) & \xrightarrow{\Omega_0^{-1}} \bigoplus_{i \in I_0} H_0(\overline{B}_i^0, \underbrace{\partial \overline{B}_i^0}_{=\emptyset}). \end{array}$$

Since $X^{-1} = \emptyset$ we see that in the present case we have $d_1 = \partial_1$, i.e. the cellular boundary map d_1 is the connecting homomorphism $H_1(X^1, X^0) \rightarrow H_0(X^0)$ of the long exact sequence of the pair (X^1, X^0) . It follows immediately from the definitions of the connecting homomorphism, as defined in Proposition 26.12 (1), that $d_1(\alpha) = \varphi_\alpha(1) - \varphi_\alpha(-1)$. $\square$

Example. We consider the graph G that is given in Figure 298 with two vertices β_1, β_2 and four edges $\alpha_1, \dots, \alpha_4$. We view it as a CW-complex in the obvious way. According to

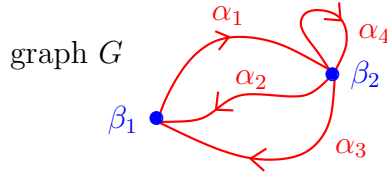


FIGURE 298.

Lemma 29.11 the cellular chain complex of X is of the following form

$$0 \rightarrow C_1^{\text{CW}}(G) = \mathbb{Z}^4 \xrightarrow{\begin{array}{c} \alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4 \\ \beta_1 \quad \begin{pmatrix} 1 & -1 & -1 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix} \\ \beta_2 \end{array}} C_0^{\text{CW}}(G) = \mathbb{Z}^2 \rightarrow 0.$$

Thus we obtain from Proposition 29.8 that

$$H_1(G) \cong \ker \left(\mathbb{Z}^4 \xrightarrow{\begin{pmatrix} 1 & -1 & -1 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}} \mathbb{Z}^4 \right) \cong \mathbb{Z}^3 \quad \text{and} \quad H_0(G) \cong \text{coker} \left(\mathbb{Z}^4 \xrightarrow{\begin{pmatrix} 1 & -1 & -1 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}} \mathbb{Z}^4 \right) \cong \mathbb{Z}.$$

After this example we return to the discussion of the cellular boundary maps. Now let $n > 1$. We want to determine the cellular boundary map

$$d_n: H_n(X^n, X^{n-1}) \rightarrow H_{n-1}(X^{n-1}, X^{n-2}).$$

Given an n -cell α and an $(n-1)$ -cell β we consider the map

$$S_\alpha^{n-1} = \partial \overline{B}_\alpha^n \xrightarrow{\varphi_\alpha} X^{n-1} \xrightarrow{q} X^{n-1}/X^{n-2} \xrightarrow[\cong]{\Psi_{n-1}^{-1}} \bigvee_{j \in I_{n-1}} \overline{B}_j^{n-1} / \partial \overline{B}_j^{n-1} \xrightarrow{p_\beta} \overline{B}_\beta^{n-1} / \partial \overline{B}_\beta^{n-1} \xrightarrow[\cong]{\Theta} S_\beta^{n-1}$$

where $q: X^{n-1} \rightarrow X^{n-1}/X^{n-2}$ is the quotient map and where $\Theta: \overline{B}^n / \partial \overline{B}^n \xrightarrow{\cong} S^{n-1}$ denotes the homeomorphism from page 23. Since S_α^{n-1} and S_β^{n-1} are copies of S^{n-1} we can consider the degree of the map as defined on page 465.

We will now see that this degree is just the $\beta \times \alpha$ -entry of the matrix that represents the cellular boundary map d_n with respect to the bases given by all n -cells and all $(n-1)$ -cells. More precisely we have the following proposition:

Proposition 29.12. *Let X be a CW-complex. We use all the notation that we introduced above. Then for each n -cell α we have*

$$d_n(\alpha) = \sum_{\beta \in I_{n-1}} d_{\alpha, \beta} \cdot \beta$$

where

$$d_{\alpha, \beta} := \text{degree of the map } S_\alpha^{n-1} \xrightarrow{\Theta \circ p_\beta \circ \Psi_{n-1}^{-1} \circ q \circ \varphi_\alpha} S_\beta^{n-1}.$$

Proof. We consider the following commutative⁵⁵⁶⁵⁵⁷ diagram

$$\begin{array}{c}
 \begin{array}{c}
 \bigoplus_{i \in I_n} H_n(\overline{B}_i^n, \partial \overline{B}_i^n) \xrightarrow{\cong} \bigoplus_{i \in I_n} \tilde{H}_n(\overline{B}_i^n / \partial \overline{B}_i^n) \xrightarrow{\cong} \tilde{H}_n(X^n / X^{n-1}) \xrightarrow{\cong} H_n(X^n, X^{n-1}) \\
 \uparrow \scriptstyle H_n(\overline{B}_\alpha^n, \partial \overline{B}_\alpha^n) \quad \searrow \scriptstyle \cong \quad \nearrow \scriptstyle \Psi_n \cong \\
 \tilde{H}_n(\overline{B}_\alpha^n / \partial \overline{B}_\alpha^n) \xrightarrow{\cong} \tilde{H}_n\left(\bigvee_{i \in I_n} \overline{B}_i^n / \partial \overline{B}_i^n\right) \xrightarrow{\cong} \tilde{H}_n(X^n / X^{n-1}) \\
 \downarrow \scriptstyle \partial \quad \downarrow \scriptstyle \partial \quad \downarrow \scriptstyle \partial \\
 \tilde{H}_{n-1}(S_\alpha^{n-1}) \xrightarrow{\cdot d_{\alpha, \beta}} \tilde{H}_{n-1}(S_\beta^{n-1}) \xrightarrow{\cong} \tilde{H}_{n-1}(\overline{B}_\beta^{n-1} / \partial \overline{B}_\beta^{n-1}) \xrightarrow{\cong} \tilde{H}_{n-1}(X^{n-1} / X^{n-2}) \xrightarrow{\cong} H_{n-1}(X^{n-1}, X^{n-2}) \\
 \uparrow \scriptstyle \Theta_* \quad \uparrow \scriptstyle p_\beta \quad \uparrow \scriptstyle \Psi_{n-1}^{-1} \quad \uparrow \scriptstyle q \quad \uparrow \scriptstyle j_{n-1} \\
 \tilde{H}_{n-1}(S_\beta^{n-1}) \xrightarrow{\cdot d_{\alpha, \beta}} \tilde{H}_{n-1}(S_\beta^{n-1}) \xrightarrow{\cong} \tilde{H}_{n-1}(\overline{B}_\beta^{n-1} / \partial \overline{B}_\beta^{n-1}) \xrightarrow{\cong} \tilde{H}_{n-1}(X^{n-1} / X^{n-2}) \xrightarrow{\cong} H_{n-1}(X^{n-1}, X^{n-2}) \\
 \downarrow \scriptstyle \partial \quad \downarrow \scriptstyle \partial \quad \downarrow \scriptstyle \partial \quad \downarrow \scriptstyle \partial \quad \downarrow \scriptstyle \partial \\
 H_{n-1}(\overline{B}_\beta^{n-1}, \partial \overline{B}_\beta^{n-1}) \xrightarrow{\cong} \tilde{H}_{n-1}(\overline{B}_\beta^{n-1} / \partial \overline{B}_\beta^{n-1}) \xrightarrow{\cong} \tilde{H}_{n-1}(X^{n-1} / X^{n-2}) \xrightarrow{\cong} H_{n-1}(X^{n-1}, X^{n-2})
 \end{array}
 \end{array}$$

Ω_n (top arc), Ω_{n-1}^{-1} (bottom arc), d_n (right vertical arrow), ∂ (left vertical arrows), Θ_* (diagonal arrow), p_β (diagonal arrow), Ψ_{n-1}^{-1} (diagonal arrow), q (diagonal arrow), j_{n-1} (diagonal arrow).

⁵⁵⁶Why is the diagram commutative?

⁵⁵⁷The strange shape of the diagram is due to space constraints.

The proposition now follows directly from the definitions and the commutativity of the above diagram. We leave it as an exercise to the reader to unravel all the definitions and to digest the diagram. $\square$

Example.

- (1) We consider the surface Σ of genus 2. As usual we view Σ as a CW-complex with one 0-cell P , four 1-cells a, b, c, d and one 2-cell F . We denote by $\varphi: S^1 \rightarrow \Sigma^1$ the attaching map of the 2-cell F . (We refer to Figure 299 for an illustration.) It follows from the discussion on page 465 that the degree of the map $p_a \circ \Psi_1^{-1} \circ q \circ \varphi: S_F^1 \rightarrow S_a^1$ is zero. The same conclusion holds for all other 1-cells. It follows from Proposition 29.12 that

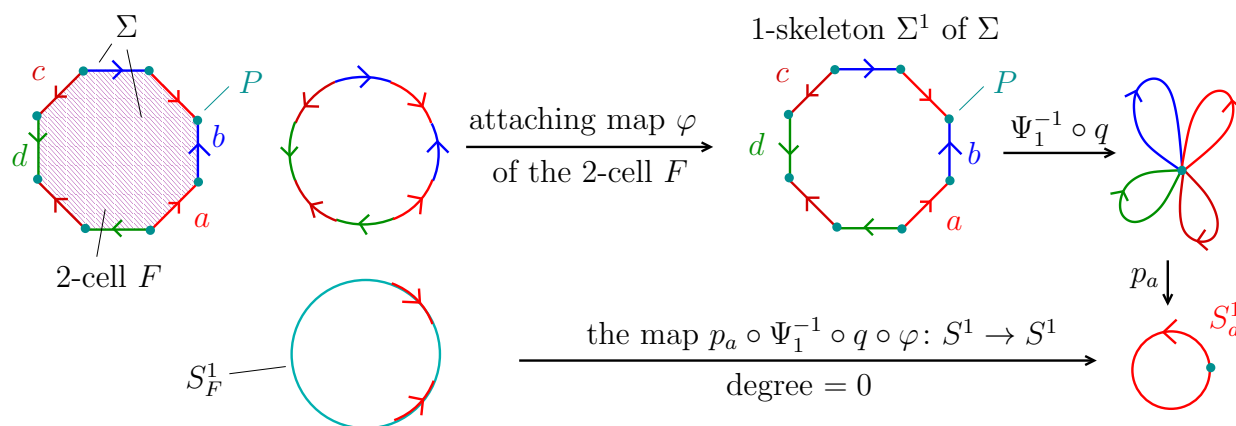


FIGURE 299.

the cellular boundary map d_2 in the cellular chain complex of Σ is zero. Since Σ has only one 0-cell we know from Lemma 29.11 that the cellular boundary map d_1 is also zero. Thus we see that

$$\begin{array}{ccccc}
 H_n(\Sigma) & \cong & H_n^{\text{CW}}(\Sigma) & = & C_n^{\text{CW}}(\Sigma) & \cong & \begin{cases} 0, & \text{if } n \geq 3, \\ \mathbb{Z}, & \text{if } n = 2, \\ \mathbb{Z}^4, & \text{if } n = 1, \\ \mathbb{Z}, & \text{if } n = 0. \end{cases} \\
 \uparrow & & \uparrow & & & & \\
 \text{Proposition 29.8} & & \text{since all the cellular} & & & & \\
 & & \text{boundary maps are zero} & & & &
 \end{array}$$

In Figure 300 on the left-hand side we show closed curves a, b, c, d . It follows fairly easily from the above calculation and the explicit isomorphism of Proposition 29.8 that these curves represent a basis for $H_1(\Sigma) \cong \mathbb{Z}^4$. These curves in turn are isotopic to the curves a', b', c', d' on the right-hand side, which therefore, by Proposition 25.4, also represent a basis for $H_1(\Sigma) \cong \mathbb{Z}^4$.

- (2) As a reality check we also consider once again the Klein bottle K . We view K as a CW-complex with one 0-cell P , two 1-cells a, b and one 2-cell F . (We refer to

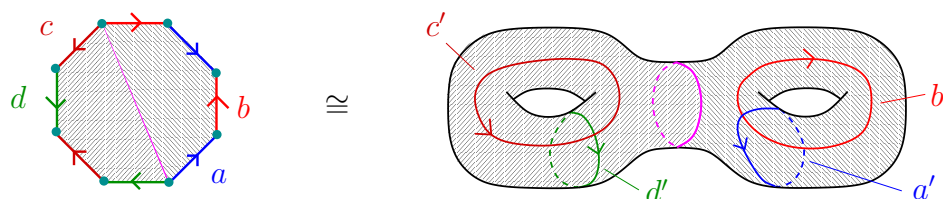


FIGURE 300.

Figure 301 for an illustration.) It follows from Proposition 29.12, the discussion on page 465 and Lemma 29.11 that the cellular chain complex is of the form

$$0 \rightarrow \mathbb{Z} \xrightarrow{\begin{pmatrix} -2 \\ 0 \end{pmatrix}} \mathbb{Z}^2 \xrightarrow{\begin{pmatrix} 0 & 0 \end{pmatrix}} \mathbb{Z} \rightarrow 0.$$

Therefore we see that $H_2(K) = 0$, $H_1(K) \cong \mathbb{Z}_2 \oplus \mathbb{Z}$ and $H_0(K) \cong \mathbb{Z}$. This is of course the same result that we had already obtained in Lemma 28.12.

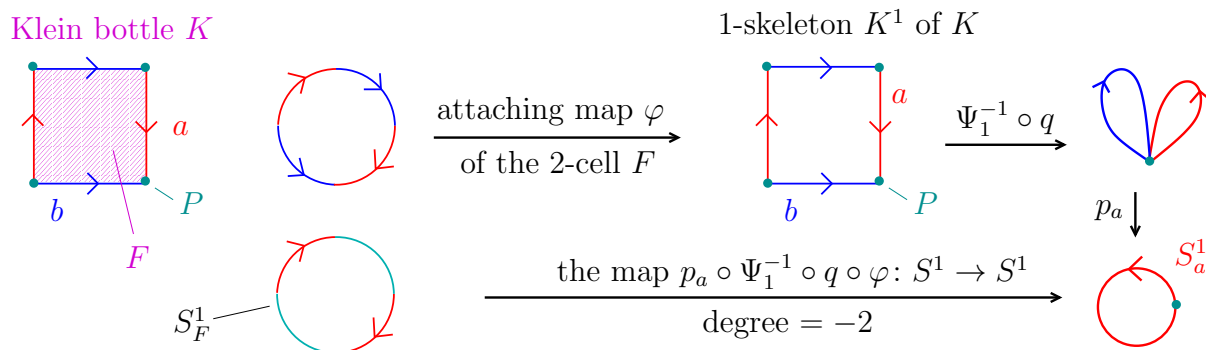


FIGURE 301.

29.5. The homology groups of 2-dimensional manifolds. Recall that given $g, m \in \mathbb{N}_0$ we write⁵⁵⁸

$\Sigma_{g,m} :=$ the surface of genus g minus m open disks

and for $k \in \mathbb{N}$ and $m \in \mathbb{N}_0$ we write

$R_{k,m} :=$ the manifold $k \cdot \mathbb{RP}^2$ minus m open disks.

In Theorem 11.12 we had seen that every connected, compact 2-dimensional topological manifold is homeomorphic to either $\Sigma_{g,m}$ for unique $g, m \in \mathbb{N}_0$ or to $R_{k,m}$ for unique $k \in \mathbb{N}$ and $m \in \mathbb{N}_0$.

Our goal in this section will be to determine the homology groups of $\Sigma_{g,n}$ and $R_{k,n}$. Before we consider the general case we will consider the special case $\Sigma_{2,1}$ in great detail.

⁵⁵⁸Also recall the definition on page 201 what we mean by a “surface minus m open disks”.

Example. We consider $\Sigma_{2,1}$ and we equip it with the CW-structure that is illustrated in Figure 302. This CW-structure has two 0-cells P, Q , six 1-cells a, b, c, d, w, z and it has one 2-cell F . Given $i = 1, 2$ we denote by A_i the matrix that represents the cellular boundary

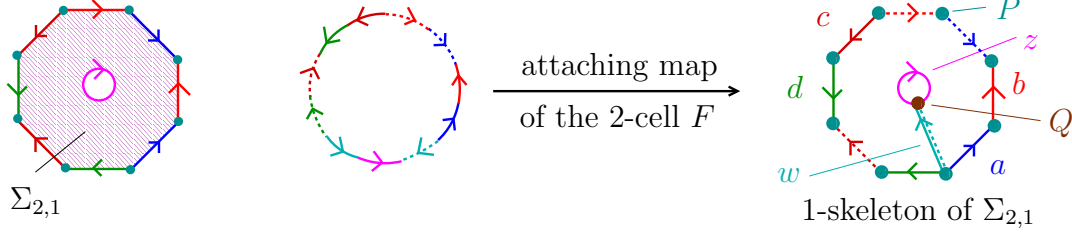


FIGURE 302.

map $d_i: C_i^{\text{CW}}(\Sigma_{2,1}) \rightarrow C_{i-1}^{\text{CW}}(\Sigma_{2,1})$ with respect to the given bases.

	F									
The argument on page 508 shows that	a	$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$	It follows from Lemma 29.11 that we have	P	a	b	c	d	w	z
	b			$\begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$						
	c									
	d									
	w									
	z									
$A_2 =$		Q								

Some elementary linear algebra shows that

$$H_n(\Sigma_{2,1}) \cong H_n^{\text{CW}}(\Sigma_{2,1}) = H_n\left(0 \rightarrow \mathbb{Z} \xrightarrow{A_2} \mathbb{Z}^6 \xrightarrow{A_1} \mathbb{Z}^2 \rightarrow 0\right) \cong \begin{cases} 0, & \text{if } n \geq 3, \\ 0, & \text{if } n = 2, \\ \mathbb{Z}^4, & \text{if } n = 1, \\ \mathbb{Z}, & \text{if } n = 0. \end{cases}$$

$\uparrow$
 Proposition 29.8

From this discussion we get in fact a more precise statement. Namely, by the choice of the CW-structure on $\Sigma_{2,1}$ we have that the map inclusion map $\partial\Sigma_{2,1} \rightarrow \Sigma_{2,1}$ is in fact a cellular map. Therefore it follows from the remark on page 501 that the following diagram commutes

$$\begin{array}{ccc}
 H_1(\partial\Sigma_{2,1}) & \xrightarrow{f_*} & H_1(\Sigma_{2,1}) \\
 \cong \downarrow & & \downarrow \cong \\
 \mathbb{Z} = H_1^{\text{CW}}(\Sigma_{2,1}) & \xrightarrow[\substack{f_* \\ [z] \mapsto [z]}]{} & H_1^{\text{CW}}(\Sigma_{2,1}) = \mathbb{Z}^4
 \end{array}$$

where the vertical maps denote the isomorphisms from Proposition 29.8. The 1-cell z is null-homologous in $C_*^{\text{CW}}(\Sigma_{2,1})$ since we have just seen that it is the boundary of the 2-cell F . Thus we see that the bottom horizontal map is the zero map, but that implies that the top horizontal map is also the zero map.

After the calculations that we just did it we can now easily prove the following lemma.

Lemma 29.13.

- (1) Let $g \in \mathbb{N}_0$. Then the inclusion induced map

$$H_1(\partial\Sigma_{g,1}) \rightarrow H_1(\Sigma_{g,1})$$

is the zero map.

- (2) Let M be a manifold and let C be an oriented curve in M . If there exists a $g \in \mathbb{N}_0$ and a map $\psi: \Sigma_{g,1} \rightarrow M$ such that $\psi(\partial\Sigma_{g,1}) = C$, then $C = 0 \in H_1(M)$.⁵⁵⁹

Example. Loosely speaking Lemma 29.13 (2) says that if a curve C is the boundary of an orientable compact surface in M , then C is null-homologous. For example in Figure 303 we see a curve C on a surface M . Since C is the boundary of a compact orientable subsurface contained in M we see that $C = 0 \in H_1(M)$.

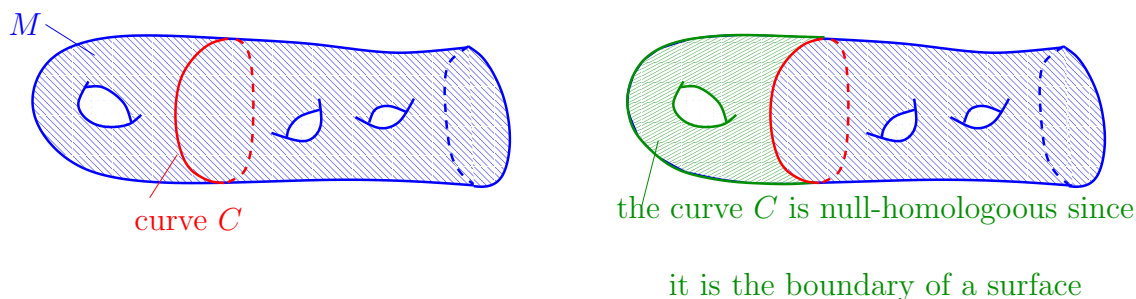


FIGURE 303.

Proof.

- (1) For $g = 1$ we had proved the statement in the previous example. The general case is proved entirely the same way.
- (2) Now let M be a manifold and let C be an oriented curve in M . Suppose there exists a $g \in \mathbb{N}_0$ and a map $\psi: \Sigma_{g,1} \rightarrow M$ such that $\psi(\partial\Sigma_{g,1}) = C$. We get the commutative diagram

$$\begin{array}{ccccc}
 & & H_1(C) & & \\
 & \nearrow & & \searrow & \\
 H_1(S^1) & & & & H_1(M). \\
 & \searrow & & \nearrow & \\
 & H_1(\partial\Sigma_{g,1}) & \xrightarrow{\quad} & H_1(\Sigma_{g,1}) & \xrightarrow{\psi_*}
 \end{array}$$

By the first statement the bottom horizontal map is the zero map. But that shows that the homology class represented by C is also zero. $\square$

⁵⁵⁹Following the convention of page 488 the curve C defines a homology class in $H_1(M)$ that we also denote by C .

In the following proposition we give the calculation for the homology groups of all connected, compact 2-dimensional topological manifolds.

Proposition 29.14. *For any $g \in \mathbb{N}_0$ we have*

$$H_n(\Sigma_g) \cong \begin{cases} 0, & \text{if } n \geq 3, \\ \mathbb{Z}, & \text{if } n = 2, \\ \mathbb{Z}^{2g}, & \text{if } n = 1, \\ \mathbb{Z}, & \text{if } n = 0. \end{cases} \quad \begin{array}{l} \text{Furthermore} \\ \text{for } m \geq 1 \\ \text{we have} \end{array} \quad H_n(\Sigma_{g,m}) \cong \begin{cases} 0, & \text{if } n \geq 3, \\ 0, & \text{if } n = 2, \\ \mathbb{Z}^{2g+m-1}, & \text{if } n = 1, \\ \mathbb{Z}, & \text{if } n = 0. \end{cases}$$

For any $k \in \mathbb{N}_0$ we have

$$H_n(k \cdot \mathbb{RP}^2) \cong \begin{cases} 0, & \text{if } n \geq 3, \\ 0, & \text{if } n = 2, \\ \mathbb{Z}_2 \oplus \mathbb{Z}^{k-1}, & \text{if } n = 1, \\ \mathbb{Z}, & \text{if } n = 0. \end{cases} \quad \begin{array}{l} \text{Furthermore} \\ \text{for } m \geq 1 \\ \text{we have} \end{array} \quad H_n(R_{k,m}) \cong \begin{cases} 0, & \text{if } n \geq 3, \\ 0, & \text{if } n = 2, \\ \mathbb{Z}^{k+m-1}, & \text{if } n = 1, \\ \mathbb{Z}, & \text{if } n = 0. \end{cases}$$

Proof. The calculation of the homology groups $\Sigma_{g,m}$ is a slight generalization of the arguments that we provided above and on page 508. We leave the details to the reader.

We will deal with the homology of $R_{k,m}$ in exercise sheet 9. The calculation is evidently just a variation on the arguments we provided for Σ_2 and $\Sigma_{2,2}$. The basis for the calculation is the description of $\mathbb{RP}^2 \# \mathbb{RP}^2$ as a CW-complex as illustrated in Figure 304. $\square$

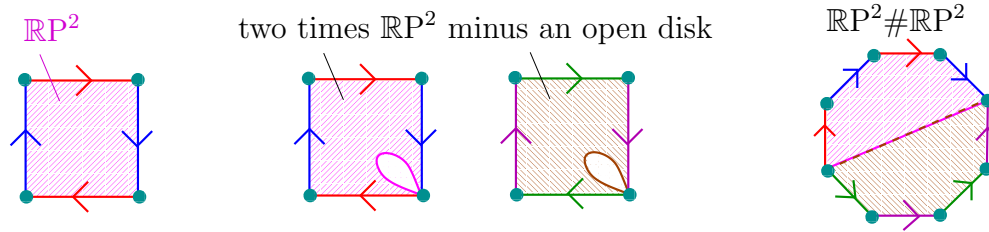


FIGURE 304.

Remark. A careful reading of the Proof of Proposition 29.14 gives us some extra information. First of all we endow $\Sigma_{g,m}$ with an orientation. The boundary components of $\Sigma_{g,m}$ then inherit an orientation. By the discussion on page 488 the boundary components, viewed as oriented curves, then define elements $c_1, \dots, c_m \in H_1(\Sigma_{g,m})$. Then there exists an exact sequence

$$0 \rightarrow \mathbb{Z} \xrightarrow{1 \mapsto \sum c_i} \bigoplus_{i=1}^m \mathbb{Z} \cdot c_i \rightarrow H_1(\Sigma_{g,m}) \rightarrow H_1(\Sigma_g) \rightarrow 0.$$

$\uparrow$
inclusion induced map

As we had seen in Figure 300, a basis for $H_1(\Sigma_g)$ is given by a pair of curves for each “hole”. The above exact sequence shows that a basis for the homology group $H_1(\Sigma_{g,m}) \cong \mathbb{Z}^{2g+m-1}$ is obtained by a pair of curves for each “hole” together with boundary curves for all except one boundary component. This statement is illustrated in Figure 305.

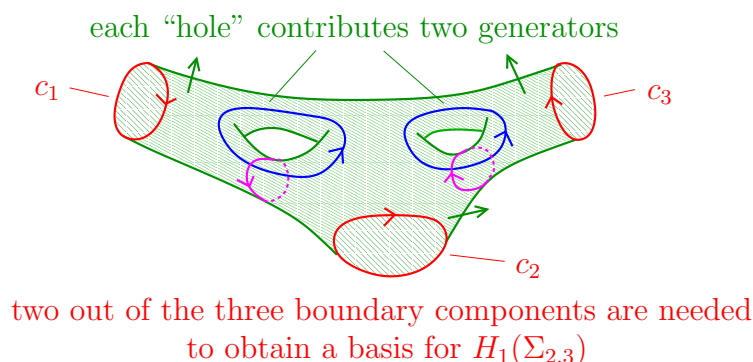


FIGURE 305.

We conclude this section with the surface Σ_∞ of infinite genus that we had already encountered on page 230. Using a variation on the proof of Lemma 13.3 one can show, using the above calculations and Proposition 29.1, that

$$H_n(\Sigma_\infty) \cong \begin{cases} 0, & \text{if } n \geq 2, \\ \mathbb{Z}^\infty, & \text{if } n = 1, \\ \mathbb{Z}, & \text{if } n = 0. \end{cases}$$

We leave the details as an exercise to the reader.

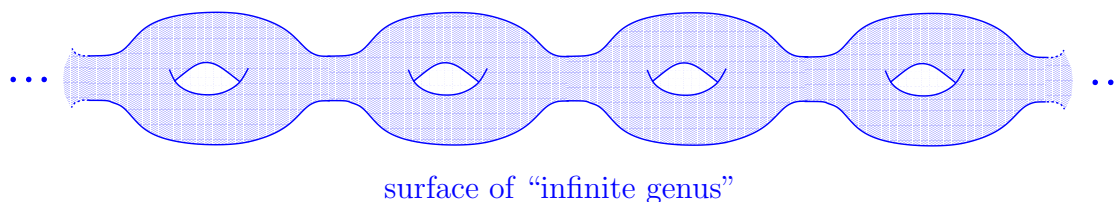


FIGURE 306.

29.6. The local degree. To determine the cellular boundary maps in the cellular chain complex of more complicated CW-complexes, e.g. for the real projective space $\mathbb{RP}^n$, we need to introduce a new method for computing the degree of a map $f: S^n \rightarrow S^n$. This method for computing degrees can be applied in most “reasonable” cases.

Convention. Let $n \geq 1$. We use the standard generator of $H_n(S^n) \cong \mathbb{Z}$, as defined on page 460, to make the identification $H_n(S^n) = \mathbb{Z}$. Now let $x \in S^n$ be a point and let U be a neighborhood of x in S^n . We have isomorphisms

$$\begin{array}{ccccc} \mathbb{Z} = H_n(S^n) & \xrightarrow{\cong} & H_n(S^n, S^n \setminus \{x\}) & \xleftarrow{\cong} & H_n(U, U \setminus \{x\}) \\ \uparrow & & \uparrow & & \uparrow \\ & \text{isomorphism by the long exact} & & \text{isomorphism by the} & \\ & \text{of the pair } (S^n, S^n \setminus \{x\}) & & \text{Excision Theorem 26.15} & \end{array}$$

Throughout this section will use these isomorphisms to identify the above homology groups with $\mathbb{Z}$.

Definition.

- (1) Let $\Phi: \mathbb{Z} \rightarrow \mathbb{Z}$ be a homomorphism. It is given by multiplication by a unique integer which we refer to as the *degree* $\deg(\phi)$.
- (2) We say a map $f: X \rightarrow Y$ between two topological spaces is *nice at a point* $x \in X$ if there exists a neighborhood U of x such that for all $x' \neq x$ in U we have $f(x') \neq f(x)$. We refer to such a neighborhood as a *nice neighborhood of* x .
- (3) Let $n \geq 1$, let $f: S^n \rightarrow S^n$ be a map and let $x \in S^n$ be a nice point. Let U be a nice neighborhood of x and let V be a neighborhood of $f(x)$ such that $f(U) \subset V$. The *local degree of* f *at the point* x is defined as⁵⁶⁰

$$\deg f|_x := \deg \left(f_*: \underbrace{H_n(U, U \setminus \{x\})}_{=\mathbb{Z}} \rightarrow \underbrace{H_n(V, V \setminus \{f(x)\})}_{=\mathbb{Z}} \right).$$

Example. We consider the functions

$$\begin{array}{lll} f: \mathbb{R} \rightarrow \mathbb{R} & \text{and} & g: \mathbb{R} \rightarrow \mathbb{R} \\ t \mapsto t^2 & & t \mapsto \sin(t) \end{array} \quad \text{and} \quad \begin{array}{ll} h: \mathbb{R} \rightarrow \mathbb{R} \\ t \mapsto \begin{cases} 0, & \text{if } t = 0, \\ t \sin(\frac{1}{t}), & \text{if } t \neq 0. \end{cases} \end{array}$$

For the function f every point on $\mathbb{R}$ is nice, in fact for all $x \in \mathbb{R}$ we can take $U = \mathbb{R}$.⁵⁶¹ For the function g also every point x on $\mathbb{R}$ is nice, in fact for all points $x \in \mathbb{R}$ we can take $U = (x - \frac{\pi}{4}, x + \frac{\pi}{4})$. Finally it is straightforward to see that for the function h the point $x = 0$ is not nice.

Lemma 29.15. *The local degree $\deg f|_x$ of a map $f: S^n \rightarrow S^n$ at a nice point $x \in S^n$ is well-defined.*

Proof. Let $n \geq 1$, let $f: S^n \rightarrow S^n$ be a map and let $x \in S^n$ be a nice point. Let U, U' be nice neighborhoods of x and let V, V' be neighborhoods of $f(x)$ such that $f(U) \subset V$ and such that $f(U') \subset V'$. After replacing U' by $U \cap U'$ and replacing V' by $V \cap V'$ we can without loss of generality assume that $U' \subset U$ and $V' \subset V$. We consider the following commutative diagram

$$\begin{array}{ccc} H_n(U', U' \setminus \{x\}) & \xrightarrow{f_*} & H_n(V', V' \setminus \{f(x)\}) \\ \downarrow \cong & & \downarrow \cong \\ H_n(U, U \setminus \{x\}) & \xrightarrow{f_*} & H_n(V, V \setminus \{f(x)\}). \end{array}$$

The vertical maps are isomorphisms by the Excision Theorem 26.15. It follows immediately that the degree of the top map agrees with the degree of the bottom map. $\square$

⁵⁶⁰Note that f induces a map of pairs $(U, U \setminus \{x\}) \rightarrow (V, V \setminus \{f(x)\})$ since U is nice.

⁵⁶¹Note that there is no problem at $x = 0$.

Lemma 29.16. *Let $n \geq 1$, let $f: S^n \rightarrow S^n$ be a map and let $x \in S^n$. If f is a homeomorphism, then for every nice point $x \in S^n$ we have*

$$\deg f|_x = \deg f.$$

Proof. From the hypothesis that f is a homeomorphism we obtain that $U = S^n$ is a nice neighborhood for x . We take $V = S^n$ and we consider the following diagram

$$\begin{array}{ccc} H_n(S^n) & \xrightarrow{f_*} & H_n(S^n) \\ \downarrow \cong & & \downarrow \cong \\ H_n(S^n, S^n \setminus \{x\}) & \xrightarrow{f_*} & H_n(S^n, S^n \setminus \{f(x)\}) \end{array}$$

The statement now follows immediately from the definitions. $\square$

The following lemma can often be used to determine the local degree of a map.

Lemma 29.17. *Let $f: S^n \rightarrow S^n$ be a map with $n \geq 1$ and let $x \in S^n$.*

(1) *If f is a local homeomorphism around x , then x is nice and*

$$\deg f|_x = \pm 1.$$

(2) *If f is a local diffeomorphism around x , then x is nice and*

$$\deg f|_x = \begin{cases} 1, & \text{if } f \text{ is locally orientation-preserving at } x, \\ -1, & \text{if } f \text{ is locally orientation-reversing at } x. \end{cases}$$

Example. We consider the map $f: S^1 \rightarrow S^1$ given by $(x, y) \mapsto (x, |y|)$. It follows from Lemma 29.17 (2) that for every point on the open upper semicircle the local degree is $+1$ and that for every point on the open lower semicircle the local degree is -1 . In exercise sheet 9 we will prove that the local degree at the points ± 1 is in both cases zero.

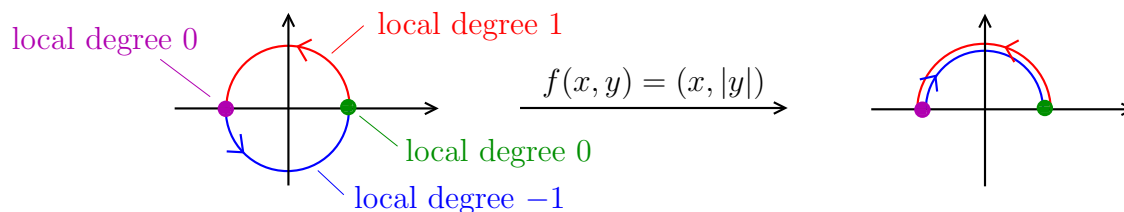


FIGURE 307.

Proof. Let $f: S^n \rightarrow S^n$ be a map with $n \geq 1$ and let $x \in S^n$.

(1) Suppose that f is a local homeomorphism around the point x . Recall that this means that there exists an open neighborhood U of x and an open neighborhood V of $f(x)$ such that $f: U \rightarrow V$ is a homeomorphism. It is clear that U is nice. Since the map f restricts to a homeomorphism $(U, U \setminus \{x\}) \rightarrow (V, V \setminus \{f(x)\})$ of pairs we see that the induced map $H_n(U, U \setminus \{x\}) \rightarrow H_n(V, V \setminus \{f(x)\})$ is an isomorphism, which implies immediately that $\deg f|_x = \pm 1$.

- (2) We only sketch the proof of (2). We leave it as an exercise to fill in the details. First of all we introduce the following conventions and definitions:

- (a) Let U of $\mathbb{R}^n$ be an open subset that contains 0. We use the obvious isomorphisms

$$\begin{array}{ccccccc} H_{n-1}(S^{n-1}) & \xrightarrow{\cong} & H_{n-1}(\mathbb{R}^n \setminus 0) & \xrightarrow{\cong} & H_n(\mathbb{R}^n, \mathbb{R}^n \setminus 0) & \xleftarrow{\cong} & H_n(U, U \setminus 0) \\ & & \uparrow & & & & \uparrow \\ & & \text{exact sequence of the} & & \text{Excision Theorem 26.15} & & \\ & & \text{pair } (\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) & & & & \end{array}$$

and the standard generator of $H_{n-1}(S^{n-1})$ to identify $H_{n-1}(U, U \setminus 0)$ with $\mathbb{Z}$.

- (b) Given an open subset U of $\mathbb{R}^n$ and a map $f: U \rightarrow \mathbb{R}^n$ with $f(0) = 0$ we define

$$\deg(f) = \deg\left(f_*: \underbrace{H_n(U, U \setminus 0)}_{=\mathbb{Z}} \rightarrow \underbrace{H_n(\mathbb{R}^n, \mathbb{R}^n \setminus 0)}_{=\mathbb{Z}}\right).$$

- (c) Given a matrix $A \in \text{GL}(n, \mathbb{R})$ we write $\deg(A)$ for the degree of the map $\mathbb{R}^n \setminus 0 \rightarrow \mathbb{R}^n \setminus 0$ that is given by multiplication by A .

Using orientation-preserving charts one sees easily that it suffices to prove the following claim.

Claim. Let $f: U \rightarrow V$ be a diffeomorphism between two open subsets of $\mathbb{R}^n$ such that $f(0) = 0$. Then

$$\deg(f) = \begin{cases} 1, & \text{if } f \text{ is orientation-preserving,} \\ -1, & \text{if } f \text{ is orientation-reversing.} \end{cases}$$

We start out with the following four observations:

- (a) Since f is a diffeomorphism we know from Analysis II that the differential D_0f is an invertible matrix. Furthermore we know that f is orientation-preserving if $\det(D_0f) > 0$ and we know that f is orientation-reversing if $\det(D_0f) < 0$.
- (b) In exercise sheet 9 we will show that there exists an open neighborhood U' of 0, that is contained in U , such that the restrictions of f and D_0f to maps $(U', U' \setminus 0) \rightarrow (\mathbb{R}^n, \mathbb{R}^n \setminus 0)$ are homotopic. In particular these two maps have the same degree by Proposition 25.4.
- (c) If A and B are two matrices in $\text{GL}(n, \mathbb{R})$ that lie in the same path-component of $\text{GL}(n, \mathbb{R})$, then the corresponding maps $(\mathbb{R}^n, \mathbb{R}^n \setminus 0) \rightarrow (\mathbb{R}^n, \mathbb{R}^n \setminus 0)$ are homotopic.⁵⁶² In particular they have the same degree by Proposition 25.4.
- (d) We denote by R the matrix corresponding to the reflection in the hyperplane defined by $x_n = 0$. By the discussion on page 33 a matrix $A \in \text{GL}(n, \mathbb{R})$ lies in the same path-component of $\text{GL}(n, \mathbb{R})$ as the identity matrix id if $\det(A) > 0$ and it lies in the same path-component of $\text{GL}(n, \mathbb{R})$ as R if $\det(A) < 0$.

⁵⁶²Why is that?

Summarizing we have

$$\begin{array}{ccccc} \deg(f) & = & \deg(D_0 f) & = & \begin{cases} \deg(\text{id}), & \text{if } f \text{ or.-pres.} \\ \deg(R), & \text{if } f \text{ or.-rev.} \end{cases} & = & \begin{cases} 1, & \text{if } f \text{ or.-pres.} \\ -1, & \text{if } f \text{ or.-rev.} \end{cases} \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ \text{by (b)} & & \text{by (a), (c) and (d)} & & \text{similar argument as in Lemma 27.2 (6)} \end{array}$$

This concludes the proof of the claim. $\square$

Using the following proposition we can now determine the degree of a map from local degrees.

Proposition 29.18. *Let $f: S^n \rightarrow S^n$ be a map where $n > 0$ and let $y \in S^n$ such that $f^{-1}(y)$ consists of finitely many points $x_1, \dots, x_m$.⁵⁶³ Then⁵⁶⁴*

$$\deg f = \sum_{i=1}^m \deg f|_{x_i}.$$

Proof ()*. We choose disjoint open neighborhoods $U_1, \dots, U_m$ for $x_1, \dots, x_m$ and we choose an open neighborhood V of y such that $f(U_i) \subset V$ for $i = 1, \dots, m$. We consider the following diagram:

$$\begin{array}{ccccc} H_n(S^n) & \xrightarrow{f_*} & H_n(S^n) & & \\ \downarrow & \searrow j & \downarrow \cong & & \\ \bigoplus_{i=1}^m H_n(S^n, S^n \setminus x_i) & \xleftarrow{\oplus p_i} & H_n(S^n, S^n \setminus f^{-1}(y)) & \xrightarrow{f_*} & H_n(S^n, S^n \setminus y) \\ \cong \uparrow & \nearrow \oplus k_i & & & \cong \uparrow q \\ \bigoplus_{i=1}^m H_n(U_i, U_i \setminus x_i) & \xrightarrow{f_*} & H_n(V, V \setminus y). \end{array}$$

Hereby the maps k_i and p_i are induced by inclusions. One sees easily that this diagram is commutative.

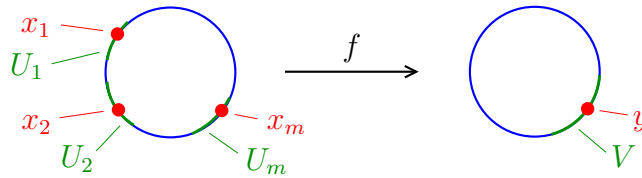


FIGURE 308. Illustration for the proof of Proposition 29.18.

⁵⁶³Does there always exist such a y ?

⁵⁶⁴The hypothesis implies immediately that all the points $x_1, \dots, x_m$ are nice, hence the local degrees are defined.

It follows from the Excision Theorem 26.15 that the map⁵⁶⁵

$$\bigoplus_{i=1}^m k_i : \bigoplus_{i=1}^m H_n(U_i, U_i \setminus x_i) \xrightarrow{\cong} H_n(S^n, S^n \setminus f^{-1}(y))$$

is an isomorphism. From the commutativity of the lower triangle it follows immediately that the map

$$\bigoplus_{i=1}^m p_i : H_n(S^n, S^n \setminus f^{-1}(y)) \rightarrow \bigoplus_{i=1}^m H_n(S^n, S^n \setminus x_i)$$

is also an isomorphism. Recall that identify $\mathbb{Z} = H_n(S^n, S^n \setminus x_i) = H_n(U_i, U_i \setminus x_i) = H_n(V, V \setminus y)$. The commutativity of the triangles and the choice of the identifications with $\mathbb{Z}$ implies that $(p_i \circ j)(1) = (p_i \circ k_i)(1)$ for $i = 1, \dots, m$. Thus we see that

$$\sum_{i=1}^m (p_i \circ j)(1) = \sum_{i=1}^m (k_i \circ j)(1).$$

In particular it follows that

$$((p_1 \circ j)(1), \dots, (p_m \circ j)(1)) = ((p_1 \circ k_1)(1), \dots, (p_m \circ k_m)(1)).$$

Since $\bigoplus_{i=1}^m p_i$ is an isomorphism, we now obtain that

$$j(1) = \sum_{i=1}^m k_i(1).$$

Now we see that⁵⁶⁶

$$\begin{aligned} & \begin{array}{l} \text{by the definition of } \deg f \text{ and} \\ \text{by the chosen identifications with } \mathbb{Z} \end{array} \\ \deg f & \xrightarrow{\downarrow} (q^{-1} \circ f_*)(j(1_{H_n(S^n)})) = (q^{-1} \circ f_*)\left(\sum_{i=1}^m k_i(1)\right) \\ &= \sum_{i=1}^m (q^{-1} \circ f_* \circ k_i)(1_{H_n(U_i, U_i \setminus x_i)}) \underset{\substack{\uparrow \\ \text{by the commutativity of the} \\ \text{lower rectangle}}}{=} \sum_{i=1}^m f_*(1_{H_n(U_i, U_i \setminus x_i)}) \underset{\substack{\uparrow \\ \text{by the definition of } \deg f|_{x_i} \\ \text{and the identifications with } \mathbb{Z}}}{=} \sum_{i=1}^m \deg f|_{x_i}. \end{aligned} \quad \square$$

Examples.

⁵⁶⁵More precisely, it follows from the following commutative diagram

$$\begin{array}{ccc} \bigoplus_{i=1}^m H_n(U_i, U_i \setminus x_i) & \xrightarrow{\cong} & H_n\left(\bigcup_i U_i, \bigcup_i U_i \setminus x_i\right) \\ & \searrow & \swarrow \cong \\ & H_n(S^n, S^n \setminus f^{-1}(y)) & \end{array} \quad \begin{array}{l} \\ \text{Excision Theorem 26.15} \end{array}$$

that the left diagonal map is an isomorphism.

⁵⁶⁶Hereby we write $1_{H_n(S^n)}$ to indicate $1 \in H_n(S^n)$ and we use the same notation for the other spaces.

- (1) The map $f: S^1 \rightarrow S^1$ given by $f(x, y) = (x, |y|)$ that is illustrated in Figure 307 has degree zero by Lemma 27.2. Taking preimages of points in S^1 one sees that in each case the sum of the local degrees of the preimages is zero.
- (2) In Figure 309 we sketch a map $f: S^1 \rightarrow S^1$ and we choose several points $y \in S^1$. We

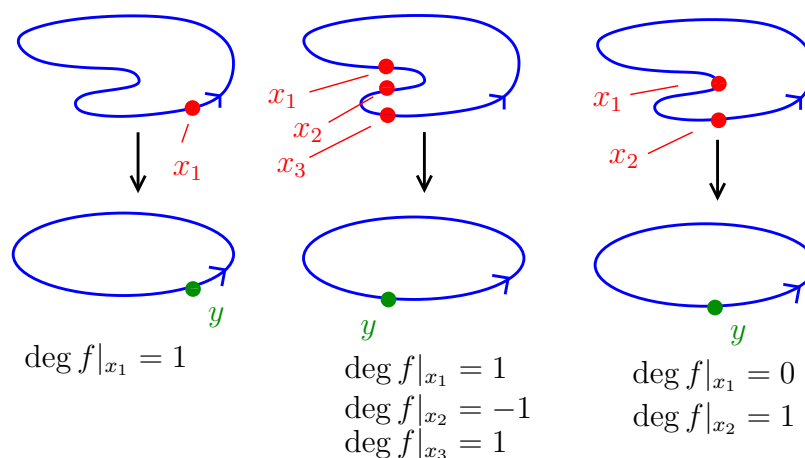


FIGURE 309.

use Lemma 29.17 to determine the various degrees at the given points.⁵⁶⁷ We see that the number of preimages and the degrees at the points in the preimages vary, but in all three cases the sum of the degrees at the points in the preimage is the same.

- (3) We consider the map

$$\begin{aligned} f: S^1 &\rightarrow S^1 \\ z &\mapsto z^m \end{aligned}$$

We had already determined in Lemma 27.1 that $\deg(f) = m$. We will now show that the degree can also easily be determined using Proposition 29.18.

First let $m \neq 0$. Then we choose $y = 1$. The preimages of y are given by $x_j = e^{2\pi i j/m}$ where $j = 1, \dots, m$. It follows from Lemma 29.17 (2) that for each i we have

$$\deg f|_{x_i} = \begin{cases} 1, & \text{if } m > 0, \\ -1, & \text{if } m < 0. \end{cases}$$

Thus it follows from Proposition 29.18 that $\deg f = |m| \cdot \text{sign}(m) = m$. If $m = 0$, then $\deg f = 0$. The easiest way to see this is to take $y = -1$, since in this case the preimage is the empty set.

We can now finally determine the homology groups of the real projective spaces.

⁵⁶⁷Why is in the third example the degree $\deg f|_{x_1}$ equal to zero?

Lemma 29.19. *For any $n \in \mathbb{N}_0$ we have⁵⁶⁸*

$$H_k(\mathbb{RP}^n) \cong \begin{cases} \mathbb{Z}, & \text{if } k = 0, \\ \mathbb{Z}_2, & \text{if } k \text{ is odd and } k \neq n, \\ 0, & \text{if } k \text{ is even and } k \neq 0, \\ \mathbb{Z}, & \text{if } k = n \text{ and } n \text{ is odd.} \end{cases}$$

Proof (*). As a reminder we had defined

$$\mathbb{RP}^n = S^n / \sim \quad \text{where } x \sim -x \text{ for all } x \in S^n.$$

In Lemma 22.1 we had already seen that we can view $\mathbb{RP}^n$ as a CW-complex with the following two properties:

- (1) the CW-structure has exactly one cell in the dimensions $0, 1, \dots, n$ and no other cells,
- (2) for each $k \leq n$ the k -skeleton of $\mathbb{RP}^n$ is given by $\mathbb{RP}^k$.

We now recall the definition of this CW-structure. For each $k \in \mathbb{N}_0$ we define

$$\begin{aligned} \varphi_k: S^{k-1} = \partial \overline{B}^k &\rightarrow \mathbb{RP}^{k-1} \\ (x_1, \dots, x_k) &\mapsto [(x_1, \dots, x_k)]. \end{aligned}$$

Then we obtain a homeomorphism

$$\begin{aligned} \mathbb{RP}^{k-1} \cup_{\varphi_k} \overline{B}^k &\xrightarrow{\cong} \mathbb{RP}^k \\ x &\mapsto \begin{cases} (x, 0), & \text{if } x \in \mathbb{RP}^{k-1} \\ [(x, \sqrt{1 - \|x\|^2})], & \text{if } x \in \overline{B}^k. \end{cases} \end{aligned}$$

Thus we see inductively that we can view $\mathbb{RP}^n$ as a CW-complex with a single cell in the dimensions $0, \dots, n$ where the attaching maps are given precisely by the maps $\varphi_1, \dots, \varphi_n$. As usual we denote the characteristic map of the k -cell, which in this case is the obvious map $\overline{B}^k \rightarrow \mathbb{RP}^k$, by Φ_k .

Claim. For each $k = 1, \dots, n$ the cellular boundary map

$$d_k: \mathbb{Z} = C_k^{\text{CW}}(X) \rightarrow C_{k-1}^{\text{CW}}(X) = \mathbb{Z}$$

is given by multiplication by $1 + (-1)^k$.

Let $k \in \{1, \dots, n\}$. We denote by $\Theta: \overline{B}^{k-1} / \partial \overline{B}^{k-1} \rightarrow S^{k-1}$ the homeomorphism from page 23 and we denote by $q: \mathbb{RP}^{k-1} \rightarrow \mathbb{RP}^{k-1} / \mathbb{RP}^{k-2}$ the projection map. By Proposition 29.12 it suffices to show that the degree of the map

$$\begin{array}{ccccccc} S^{k-1} & \xrightarrow{\varphi_k} & \mathbb{RP}^{k-1} & \xrightarrow{q} & \mathbb{RP}^{k-1} / \mathbb{RP}^{k-2} & \xrightarrow{\Phi_{k-1}^{-1}} & \overline{B}^{k-1} / \partial \overline{B}^{k-1} \xrightarrow{\Theta} S^{k-1} \\ & & & & & & \searrow \scriptstyle \text{=: } \Omega \\ & & & & & & \end{array}$$

⁵⁶⁸For $n = 2$ we thus obtain, as we should, the same result as in Section 28.3.

equals $1 + (-1)^k$. (We refer to Figure 310 for an illustration of the map.) We pick the point $y = (0, \dots, 0, 1) \in S^{k-1}$. The preimages of y under the map Ω are $x_1 = (0, \dots, 0, 1)$ and $x_2 = (0, \dots, 0, -1)$. We leave it as an exercise to verify that⁵⁶⁹ $\deg \Omega|_{x_1} = 1$ and that $\deg \Omega|_{x_2} = (-1)^n$. The desired statement now follows from Proposition 29.18. This concludes the proof of the claim.

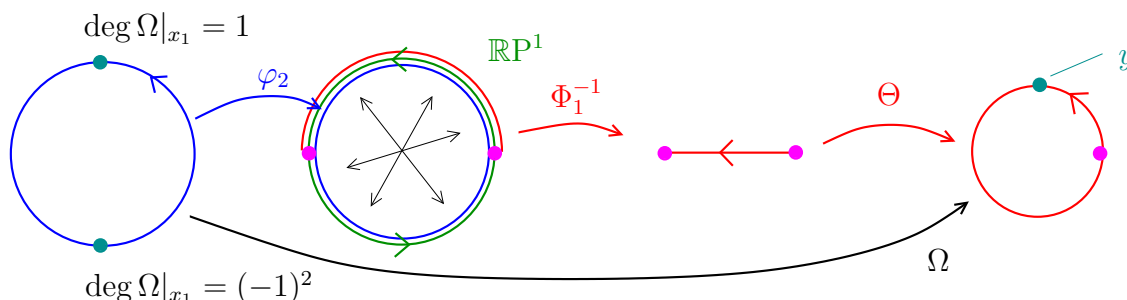


FIGURE 310. Illustration from the proof of Lemma 29.19.

It follows from the claim that the cellular chain complex of X is of the form

$$0 \rightarrow \mathbb{Z} \xrightarrow{1+(-1)^n} \mathbb{Z} \dots \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \rightarrow 0.$$

It is now straightforward to determine the cellular homology groups which, by Proposition 29.8, are isomorphic to the singular homology groups of $\mathbb{RP}^n$. $\square$

Remark. Using the approach of the proof of Lemma 29.19, applied to the infinite CW-complex $\mathbb{RP}^\infty$, or alternatively using $\mathbb{RP}^\infty = \bigcup_n \mathbb{RP}^n$ and Proposition 29.4, we see that

$$H_k(\mathbb{RP}^\infty) \cong \begin{cases} \mathbb{Z}, & \text{if } k = 0, \\ \mathbb{Z}_2, & \text{if } k \text{ is odd,} \\ 0, & \text{if } k \text{ is even and } k > 0. \end{cases}$$

Remark. We have now seen that the homology groups of CW-complexes can be computed fairly easily using cellular chain complexes. One might ask, why did we introduce singular chain complexes, if it is so much easier to do calculations with cellular chain complexes. There are two related answers:

- (1) we also want to study topological spaces that are not CW-complexes,
- (2) it is much harder to prove some statements using cellular chain complexes, for example it is very difficult, if not impossible, to show that homeomorphic CW-complexes have isomorphic homology groups, just using the definition of cellular chain complexes.

⁵⁶⁹It is not entirely easy to struggle through all the orientation conventions and all the maps, but it can indeed be done.

A general slogan in topology is that for calculations it is usually best to work with “small objects”, whereas for proving general properties, it is often best to work with “big objects”.

30. THE JORDAN CURVE THEOREM

We start out with the following proposition.

Proposition 30.1.

- (1) For every injective map $h: \overline{B}^k \rightarrow S^n$ we have

$$\tilde{H}_i(S^n \setminus h(\overline{B}^k)) = 0 \quad \text{for all } i \in \mathbb{N}_0.$$

- (2) For every injective map $h: S^k \rightarrow S^n$ we have

$$\tilde{H}_i(S^n \setminus h(S^k)) \cong \begin{cases} \mathbb{Z}, & \text{if } i = n - k - 1, \\ 0, & \text{otherwise.} \end{cases}$$

Example.

- (1) Let $K \subset S^3$ be a knot, i.e. let K be a 1-dimensional submanifold of $S^3 = \mathbb{R}^3 \cup \{\infty\}$ that is diffeomorphic to S^1 . This means in particular that there exists an embedding $h: S^1 \rightarrow S^3$ with $h(S^1) = K$. In Lemma 28.13 we had already determined that $H_1(S^3 \setminus K) \cong \mathbb{Z}$. We can now compute the higher homology groups as well. More precisely, the combination of Proposition 30.1 (2) and Lemma 26.1 says that

$$H_i(S^3 \setminus K) = H_i(S^3 \setminus h(S^1)) \cong \begin{cases} \mathbb{Z}, & \text{if } i = 0, 1, \\ 0, & \text{otherwise.} \end{cases}$$

- (2) In Proposition 30.1 we do not assume that h is in any way smooth. For example there are injective maps $h: S^1 \rightarrow S^2$ of infinite length, see e.g.

https://en.wikipedia.org/wiki/Koch_snowflake,

to which the proposition nonetheless applies.

- (3) In Figure 311 we see the image of an injective map $\varphi: [0, 1] \rightarrow S^3 = \mathbb{R}^3 \cup \{\infty\}$ and we see a closed curve $C \subset S^3 \setminus \varphi([0, 1])$. It follows from Proposition 30.1 (1) that $H_1(S^3 \setminus \varphi([0, 1])) = 0$, in particular C is null-homologous in $H_1(S^3 \setminus \varphi([0, 1])) = 0$.⁵⁷⁰ But on [Sti, p. 152] it is shown that the curve C represents in fact a non-trivial element in $\pi_1(S^3 \setminus \varphi([0, 1]))$.

To make the exposition more readable we prove the two parts of Proposition 30.1 separately.

Proof of Proposition 30.1 (1). We write $I = [-1, 1]$. By Lemma 1.4 there exists a homeomorphism $I^k \xrightarrow{\cong} \overline{B}^k$. Therefore in the statement of Proposition 30.1 (1) we can replace $\overline{B}^k$ by I^k . So we now want to show that for any injective map $h: I^k \rightarrow S^n$ we have

$$\tilde{H}_i(S^n \setminus h(I^k)) = 0 \quad \text{for all } i \in \mathbb{N}_0.$$

⁵⁷⁰In Lemma 29.13 we had seen that the boundary curve of an orientable surface with one boundary component is null-homologous. In exercise sheet 9 we will see that the closed curve C is in fact the boundary of an orientable surface that is embedded in $S^3 \setminus \varphi([0, 1])$.

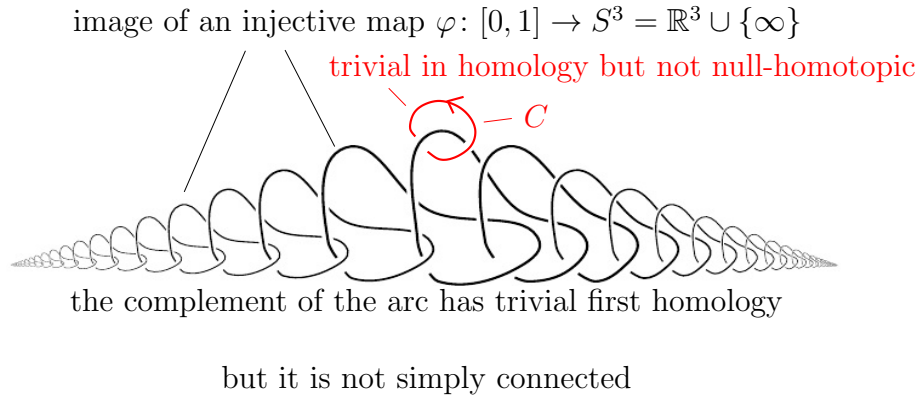


FIGURE 311.

We will prove this statement by induction on k . We start out with $k = 0$. So let $h: I^0 = \{0\} \rightarrow S^n$ be an injective map. It follows from the stereographic projection given on page 17 that $S^n \setminus h(\{0\})$ is homeomorphic to $\mathbb{R}^n$. Hence we obtain from Corollary 25.5 (3) that $\tilde{H}_*(S^n \setminus h(\{0\})) = 0$.

We now do the induction step. We assume we know the desired statement for all injective maps $I^{k-1} \rightarrow S^n$. Now let $h: I^k \rightarrow S^n$ be an injective map. We suppose that the desired conclusion does not hold, i.e. we suppose that $\tilde{H}_j(S^n \setminus h(I^k)) \neq 0$ for some j . We pick a non-trivial element $g_0 \in \tilde{H}_j(S^n \setminus h(I^k))$.

We write $Q = I^{k-1}$, $J_0 = I$ and we write

$$A_1 = S^n \setminus h(Q \times [0, \tfrac{1}{2}]) \quad \text{and} \quad A_2 = S^n \setminus h(Q \times [\tfrac{1}{2}, 1]).$$

Since h is injective we have

$$A_1 \cap A_2 = S^n \setminus h(I^k) \quad \text{and} \quad A_1 \cup A_2 = S^n \setminus h(Q \times \{0\}).$$

From Lemma 1.2 (3) we know that A_1 and A_2 are open subsets of $A_1 \cup A_2$. Therefore we

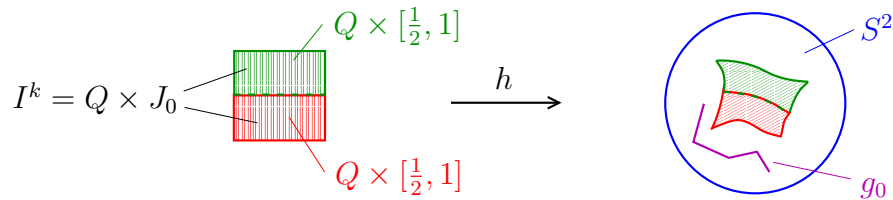


FIGURE 312. Illustration for the proof of Proposition 30.1 (1).

can apply the Mayer-Vietoris Theorem 28.5 and we obtain the long exact sequence

$$\dots \rightarrow \underbrace{\tilde{H}_{j+1}(A_1 \cup A_2)}_{= 0 \text{ by induction}} \rightarrow \tilde{H}_j(A_1 \cap A_2) \rightarrow \tilde{H}_j(A_1) \oplus \tilde{H}_j(A_2) \rightarrow \underbrace{\tilde{H}_i(A_1 \cup A_2)}_{= 0 \text{ by induction}} \rightarrow \dots$$

So we see that the middle map is an isomorphism. Thus there exists an $i \in \{1, 2\}$ such that the image of g_0 is non-trivial in $\tilde{H}_j(A_i)$. If $i = 1$, then we set $J_1 := [0, \frac{1}{2}]$, otherwise we set $J_1 := [\frac{1}{2}, 1]$. We denote by

$$\varphi_1: \tilde{H}_j(S^n \setminus h(Q \times J_0)) \rightarrow \tilde{H}_j(S^n \setminus h(Q \times J_1))$$

the inclusion induced map. By the above exact sequence φ_1 is an epimorphism. We denote by ρ_1 the inverse of the induced isomorphism

$$\tilde{H}_j(S^n \setminus h(Q \times J_0)) / \ker(\varphi_1) \rightarrow \tilde{H}_j(S^n \setminus h(Q \times J_1)).$$

We now iterate this process by running the same argument with J_0 replaced by J_1 and so on. For each $m \in \mathbb{N}$ we can find an interval J_m of length $\frac{1}{2^m}$ that is contained in J_{m-1} such that for the inclusion induced homomorphism

$$\varphi_m: \tilde{H}_j(S^n \setminus h(Q \times J_{m-1})) \rightarrow \tilde{H}_j(S^n \setminus h(Q \times J_m))$$

we have that $g_m := \varphi_m(g_{m-1})$ is non-zero. We define ρ_m the same way from φ_m as we had obtained ρ_1 from φ_1 .

By the definition of the direct limit we have the following commutative diagram:⁵⁷¹

$$\begin{array}{ccc}
 \vdots & & \\
 \uparrow \varphi_3 & & \\
 \tilde{H}_j(S^n \setminus h(Q \times J_2)) & \xrightarrow{\quad} & \lim \tilde{H}_j(S^n \setminus h(Q \times J_m)) \\
 \uparrow \varphi_2 & \searrow \rho_2 \circ \rho_1 & \uparrow \\
 \tilde{H}_j(S^n \setminus h(Q \times J_1)) & \xrightarrow{\quad} & \tilde{H}_j(S^n \setminus h(Q \times I)) \\
 \uparrow \varphi_1 & \searrow \rho_1 & \downarrow \\
 \tilde{H}_j(S^n \setminus h(Q \times J_0)) & \xrightarrow{\text{image of } g_0 \text{ is non-zero}} & \bigcup_{m \in \mathbb{N}} \ker(\varphi_m \circ \cdots \circ \varphi_1)
 \end{array}$$

Since each J_m has half the length of J_{m-1} we conclude that the intersection $\bigcap_{m \in \mathbb{N}} J_m$ contains a unique point $t \in [0, 1]$.⁵⁷² We have

$$\begin{array}{ccccccc}
 \lim_{\rightarrow} \tilde{H}_j(S^n \setminus h(Q \times J_m)) & \cong & \tilde{H}_j\left(\bigcap_{m \in \mathbb{N}} S^n \setminus h(Q \times J_m)\right) & = & \tilde{H}_j(S^n \setminus h(Q \times \{t\})) & = & 0. \\
 \uparrow & & \uparrow & & \uparrow & & \uparrow \\
 \text{by Proposition 29.1 since the} & & \text{since } h \text{ is injective} & & \text{by induction} & & \\
 \text{subsets } S^n \setminus h(Q \times J_m) \text{ are open} & & & & \text{since } Q \times \{t\} \cong I^{k-1} & &
 \end{array}$$

⁵⁷¹Note that we have

$$\ker(\varphi_1) \subset \ker(\varphi_2 \circ \varphi_1) \subset \ker(\varphi_3 \circ \varphi_2 \circ \varphi_1) \subset \dots$$

It follows that the union of these subgroups is again a subgroup.

⁵⁷²Why does this follow? What results from Analysis I do we need to draw that conclusion?

Proof. Let C be a subset of $\mathbb{R}^2$ that is homeomorphic to S^1 . We pick an injective map $h: S^1 \rightarrow S^2 = \mathbb{R}^2 \cup \{\infty\}$ with $h(S^1) = C$. We now see that

$$\begin{array}{ccccccc} \tilde{H}_0(\mathbb{R}^2 \setminus C) & = & \tilde{H}_0(\mathbb{R}^2 \setminus h(S^1)) & \cong & \tilde{H}_0(S^2 \setminus h(S^1)) & \cong & \mathbb{Z} \\ & & \uparrow & & \uparrow & & \\ & & \text{see exercise sheet 7} & & \text{Proposition 30.1 (2)} & & \end{array}$$

It follows from Lemma 26.1 that $\mathbb{R}^2 \setminus C$ has precisely two path-components. $\square$

Proposition 30.1, together with Lemma 26.1, says in particular, that if $h: S^{n-1} \rightarrow S^n$ is an injective map, then

$$H_i(S^n \setminus h(S^{n-1})) \cong \begin{cases} \mathbb{Z}^2, & \text{if } i = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Put differently, homologically the topological space $S^n \setminus h(S^{n-1})$ looks like the disjoint union of two open n -dimensional balls. One might now suspect that $S^n \setminus h(S^{n-1})$ is in fact homeomorphic to the union of two open n -dimensional balls.

This is in fact true in dimension $n = 2$. More precisely Arthur Schönflies⁵⁷⁷ proved in 1908 the following theorem. We refer to [Tn, Section 3] for a proof of the theorem.⁵⁷⁸

Theorem 30.3. (Schönflies Theorem) *Let $h: S^1 \rightarrow S^2$ be an injective map. Then h extends to a homeomorphism of S^2 , i.e. there exists a homeomorphism $H: S^2 \rightarrow S^2$ such that $H|_{S^1} = h$. In particular both components of $S^2 \setminus h(S^1)$ are homeomorphic to open disks.*

It is natural to ask whether the conclusion of the Schönflies Theorem also holds in higher dimensions. But this hope breaks down already one dimension higher. More precisely, James Alexander⁵⁷⁹ showed in 1924 that there exists an injective map $S^2 \rightarrow \mathbb{R}^3$ such that neither of the two components of the complement of $h(S^2)$ is simply connected. The image $h(S^2)$ is called the *Alexander horned sphere*. It is illustrated in Figure 313. A more detailed description of the Alexander horned sphere and the proof that the complement is not simply connected is given in [Al] and also [Ha, p. 170].

We have now seen that injective maps $S^2 \rightarrow S^3$ can be pretty “wild”. The map that defines the Alexander horned sphere is continuous, but it is certainly not smooth. So what happens if we restrict ourselves to smooth maps? There in fact the following theorem holds.

Theorem 30.4. (Generalized smooth Schönflies Theorem) *Let $h: S^{n-1} \rightarrow S^n$ be an embedding. If $n \neq 4$, then there exists a diffeomorphism $H: S^n \rightarrow S^n$ with $H|_{S^{n-1}} = h$.*

⁵⁷⁷Arthur Schönflies (1853–1928) was a German mathematician.

⁵⁷⁸More precisely, in [Tn, Section 3] a proof for the following statement is given:
Let $g: S^1 \rightarrow \mathbb{R}^2$ be an injective map. Then g extends to a homeomorphism of $\mathbb{R}^2$.

We leave it as an exercise to deduce Theorem 30.3 from that statement.

⁵⁷⁹James Alexander (1888–1971) was an American mathematician. He also introduced the Alexander polynomial of a knot.

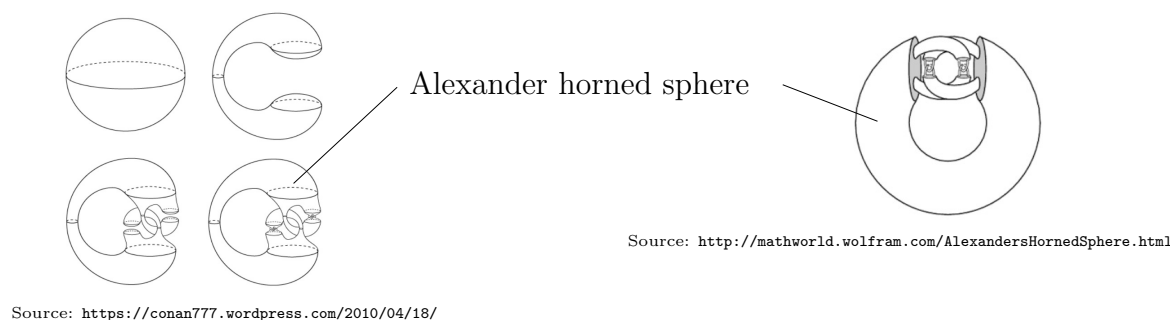


FIGURE 313.

For $n = 2$ the statement follows fairly easily from the classification of smooth 2-dimensional manifolds that we had given in Theorem 11.12, see e.g. [Hi, p. 207]. For $n = 3$ this statement was proved by James Alexander [Al2], see also [Ha3, Chapter 1] for an exposition of the proof. For $n \geq 5$ this statement was independently proved by Barry Mazur [Maz] in 1959 and by Morton Brown [Bw] in 1960.

The case $n = 4$ is missing in the Generalized Schönflies Theorem. In fact it is still unknown whether the statement holds for $n = 4$.

Question 30.5. *Is it true that given any embedding $h: S^3 \rightarrow S^4$ there exists a diffeomorphism $H: S^4 \rightarrow S^4$ with $H|_{S^3} = h$?*

This question is very much open and there is not even a consensus whether one expects the answer to be yes or no. If you can answer this question you would almost certainly obtain a Fields medal.⁵⁸⁰

We conclude this section with two more consequences of Proposition 30.1.

Theorem 30.6. (Invariance of Domain Theorem) *Let U be an open subset of $\mathbb{R}^n$ and let $h: U \rightarrow \mathbb{R}^n$ be an injective map. Then $h(U)$ is also an open subset of $\mathbb{R}^n$.⁵⁸¹*

Remark. In Satz 6.14 of Analysis III we had proved the “Open Mapping Theorem”, which says that if $G \subset \mathbb{C}$ is an open subset and if $f: G \rightarrow \mathbb{C}$ is a non-constant *holomorphic* function, then $f(G)$ is also open. In fact, one can deduce the “Open Mapping Theorem” from the Invariance of Domain Theorem 30.6 since by [J2, Satz 10] every non-constant holomorphic function on an open subset of $\mathbb{C}$ is locally the power of an injective function.

In the proof of Theorem 30.6 we need the following elementary lemma.

Lemma 30.7. *Let $U \subset \mathbb{R}^n$ be a subset. If U is open in $\mathbb{R}^n$, then every path-component of U is also an open subset of $\mathbb{R}^n$.*

⁵⁸⁰But hurry up, you can only get a Fields medal if you are under 40.

⁵⁸¹How can one prove the theorem for $n = 1$ using only the methods from Analysis I?

Proof. Let C be a path-component of the subset $U \subset \mathbb{R}^n$. Let $P \in C$. Since U is open there exists an $r > 0$ such that $B_r(P) \subset U$. But all points in $B_r(P)$ are path-equivalent to P , hence $B_r(P) \subset C$. $\square$

Now we can provide the proof of Theorem 30.6.

Proof of Theorem 30.6. By the discussion on page 17 we can view $\mathbb{R}^n$ as an open subset of $\mathbb{R}^n \cup \{\infty\} = S^n$. So we can view h as a map from U to S^n and we need to show that $h(U)$ is an open subset of S^n .

Let $x \in U$. We pick a closed n -ball B in U that contains x in the interior. It suffices to show that $h(B \setminus \partial B) \subset h(U)$ is open in S^n . Note that $S^n \setminus h(\partial B)$ is open by Lemma 1.2 (3). By Lemma 30.7 it therefore suffices to show that $h(B \setminus \partial B)$ is a path-component of $S^n \setminus h(\partial B)$. We will do so in the following claim.

Claim. The topological space $S^n \setminus h(\partial B)$ has precisely two path-components namely the subsets $h(B \setminus \partial B)$ and $S^n \setminus h(B)$.

By Proposition 30.1 (2) and Lemma 26.1 we know that

$$S^n \setminus h(\partial B) = h(B \setminus \partial B) \sqcup (S^n \setminus h(B))$$

has two path-components. It suffices to show that the two sets on the right-hand side are path-connected. The first set $h(B \setminus \partial B)$ is path-connected since it is the image of the path-connected topological space $B \setminus \partial B$, while the second set $S^n \setminus h(B)$ is path-connected by Proposition 30.1 (1) and Lemma 26.1. This concludes the proof of the claim. $\square$

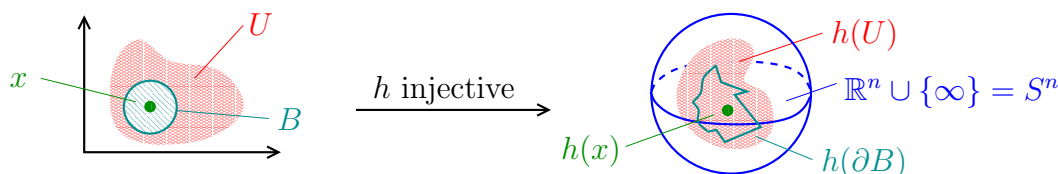


FIGURE 314. Illustration for the proof of Theorem 30.6.

Corollary 30.8. *Let M be a closed n -dimensional topological manifold and let N be a connected n -dimensional topological manifold without boundary. Let $h: M \rightarrow N$ be a map.*

- (1) *If $h: M \rightarrow N$ is locally injective, then h is also surjective.*
- (2) *If h is injective, then h is already a homeomorphism.*

Example. Corollary 30.8 implies in particular that there exists no injective map from the closed manifold S^n into the manifold $\mathbb{R}^n$.⁵⁸²

⁵⁸²How can you show this for $n = 1$, just using methods from Analysis I?

Proof. The second statement of the corollary is an immediate consequence of the first statement and Proposition 1.3. So it suffices to prove the first statement.

Let $h: M \rightarrow N$ be a locally injective map from a closed n -dimensional topological manifold M to a connected n -dimensional topological manifold N without boundary.

The image $h(M)$ is closed in N since M is compact and since N is Hausdorff. Since N is connected it suffices to show that $h(M)$ is also open in N .

So let $Q \in h(M)$. We pick $P \in M$ with $h(P) = Q$. We pick a chart $\Phi: U \rightarrow V$ around P and we pick a chart $\Psi: W \rightarrow X$ around Q . After possibly replacing U by $U \cap h^{-1}(W)$ we can assume that $h(U) \subset W$. Since h is locally injective we can, after possibly shrinking U even more, assume that $h|_U$ is injective.

From Theorem 30.6 we know that $(\Psi \circ h \circ \Phi^{-1})(V)$ is an open subset of $\mathbb{R}^n$. Since Ψ is a homeomorphism we see that $h(\Phi^{-1}(V)) = h(U)$ is an open neighborhood of Q that is contained in $h(M)$. $\square$

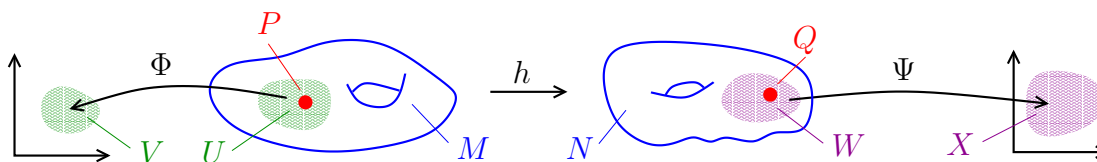


FIGURE 315. Illustration for the proof of Corollary 30.8.

31. TOPOLOGICAL ROBOTICS

In this chapter we will first study the matrix groups $SO(3)$ and $SU(2)$, viewed as topological spaces. In particular we will determine $\pi_1(SO(3))$ and we will use this result to give an application to topological robotics.

31.1. The matrix groups $SO(3)$ and $SU(2)$. We start out with the following lemma.

Lemma 31.1. *The set*

$$SO(n) = \{A \in GL(n, \mathbb{R}) \mid AA^t = \text{id} \text{ and } \det(A) = 1\}$$

of orthogonal matrices is a closed $\frac{1}{2}n(n-1)$ -dimensional submanifold of $M(n \times n, \mathbb{R}) = \mathbb{R}^{n^2}$ and the set of unitary matrices

$$SU(n) = \{A \in GL(n, \mathbb{C}) \mid A\bar{A}^t = \text{id} \text{ and } \det(A) = 1\}$$

is a closed $(n^2 - 1)$ -dimensional submanifold of $M(n \times n, \mathbb{C}) = \mathbb{C}^{n^2} = \mathbb{R}^{4n^2}$.

Proof. Both statements follow from a slight generalization of the Regular Value Theorem from Analysis II. We refer to [J3, p. 11] for details. $\square$

Lemma 31.1 implies in particular that we can view $SO(3)$ and $SU(2)$ as 3-dimensional manifolds. Whenever we have new examples of manifolds it is natural to ask whether they are “new” manifolds or “old” manifolds in disguise. In this case we can give a perhaps surprisingly simple answer.

Theorem 31.2. *We denote by $p: S^3 \rightarrow \mathbb{RP}^3 = S^3/\pm 1$ the usual projection map. Then there exists a commutative diagram*

$$\begin{array}{ccc} S^3 & \xrightarrow[\cong]{f} & SU(2) \\ p \downarrow & & \downarrow q \\ \mathbb{RP}^3 & \xrightarrow[\cong]{g} & SO(3), \end{array}$$

where f, g and q have the following properties:

- (1) *the maps f and g are diffeomorphisms,*
- (2) *the map q is a 2-fold covering map.*

Remark. The fact that $SU(2)$ is the universal covering of $SO(3)$ plays an important role in theoretical physics and it is essential in understanding the notion of “spin” of an electron.⁵⁸³

In the proof of Theorem 31.2 we will need the following technical lemma.

Lemma 31.3. *Let $q: M \rightarrow N$ be a map between two n -dimensional manifolds. Suppose the following conditions are satisfied:*

- (1) *the map q is surjective,*

⁵⁸³At least that is a slogan that I remember from the days when I studied physics.

- (2) the map q is locally injective, i.e. for each $P \in M$ there exists a neighborhood U such that $q|_U$ is injective,
 (3) for each point $Q \in N$ the preimage $q^{-1}(Q)$ is a finite set.

Then q is a covering map.

Proof ()*. Since q is surjective we only have to show that each $Q \in N$ admits an open neighborhood that is uniformly covered. So let $Q \in N$. We denote its preimages by $P_1, \dots, P_r$. Let $i \in \{1, \dots, m\}$. By (2) we can pick a closed ball B_i around P_i such that $q: B_i \rightarrow N$ is injective. Since the manifold N is in particular Hausdorff it follows from Proposition 1.3 that $q: B_i \rightarrow q(B_i)$ is a homeomorphism. It follows from the proof of Corollary 30.8 that $q(\mathring{B}_i)$ is an open neighborhood of Q . It is now straightforward to see that $U := \bigcap_{i=1}^m q(\mathring{B}_i)$ is a uniformly covered open neighborhood of Q . $\square$

Now we can provide the proof of Theorem 31.2.

Proof of Theorem 31.2. In the following we define the maps f, q and g and we show that they have the desired properties.

- (f) One can easily show that the map

$$\begin{aligned} f: S^3 &\rightarrow SU(2) \\ (z, w) &\mapsto \begin{pmatrix} z & -\bar{w} \\ w & \bar{z} \end{pmatrix} \end{aligned}$$

is a bijection. It follows from Proposition 1.3 and the fact that S^3 is compact that the map f is in fact a homeomorphism.

- (q) We consider the subset $T \subset \{A \in SU(2) \mid \operatorname{tr}(A) = 0\}$.⁵⁸⁴ Note that T is a 3-dimensional real vector space. We equip it with the basis given by the so-called “Pauli matrices”⁵⁸⁵

$$E_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \text{and } E_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and we equip T with the inner product given by $(A, B) \mapsto \langle A, B \rangle := \frac{1}{2} \operatorname{tr}(A\bar{B}^t)$. The basis defines an isomorphism $\Phi: \mathbb{R}^3 \rightarrow T$ and it is straightforward to verify that this isomorphism defines in fact an isometry from $\mathbb{R}^3$ equipped with the usual Euclidean inner product to $(T, \langle, \rangle)$.⁵⁸⁶ In particular Φ induces an isomorphism $SO(3) \rightarrow \operatorname{Isom}(T)$. We use this isomorphism to identify these two groups. It is straightforward to verify that the map

$$\begin{aligned} q: SU(2) &\rightarrow \operatorname{Isom}(T) = SO(3) \\ A &\mapsto (B \mapsto ABA^{-1}) \end{aligned}$$

⁵⁸⁴Recall that the trace $\operatorname{tr}(A)$ of a matrix is defined as the sum of all the diagonal entries.

⁵⁸⁵Wolfgang Pauli (1900–1958) was Swiss-American theoretical physicist and one of the pioneers of quantum physics.

⁵⁸⁶Put differently, for $i, j \in \{1, 2, 3\}$ we have $\langle E_i, E_j \rangle = \delta_{ij}$.

is well-defined, i.e. for each $A \in SU(2)$ the map $T \rightarrow T$ given by $B \mapsto ABA^{-1}$ is in fact an isometry with respect to the inner product $\langle \cdot, \cdot \rangle$. It follows easily from the definition that q is a group homomorphism. Finally it is straightforward to show that for $A, A' \in SU(2)$ we have

$$q(A) = q(A') \iff A = -A'.$$

Thus we see that q is locally injective. But then it follows from Corollary 30.8 and the fact that $SU(2)$ and $SO(3)$ are closed 3-dimensional manifolds that q is in fact also a surjection. Now it follows from Lemma 31.3 that q is a 2-fold covering map.

(g) For any $(z, w) \in S^3$ we have $q(f(-z, -w)) = q(-f(z, w)) = q(f(z, w))$. Thus we see that the map

$$\begin{aligned} g: \mathbb{RP}^3 &\rightarrow SO(3) \\ [(z, w)] &\mapsto q(f(z, w)) \end{aligned}$$

is well-defined. It follows immediately from the fact, obtained in (2), that q is surjective, that g is also surjective. Similarly it is straightforward to show, using (2), that the map g is injective. Since $\mathbb{RP}^3$ is compact it follows from Proposition 1.3 that g is in fact a homeomorphism.

Finally we note that it follows easily from the definitions that $g \circ p = q \circ f$, i.e. the diagram given in the theorem commutes. $\square$

From Theorem 31.2 and Corollary 6.17 we obtain immediately the following, perhaps initially somewhat surprising, corollary.

Corollary 31.4. *We have*

$$\pi_1(SO(3)) \cong \mathbb{Z}_2.$$

Remark. The fact that $\pi_1(SO(3)) \cong \mathbb{Z}_2$ lies at the heart of the “belt trick” or the “plate trick” for which videos and information can be found here:

https://en.wikipedia.org/wiki/Plate_trick

https://www.youtube.com/watch?v=JaIR-cWk_-o

<https://www.youtube.com/watch?v=JDJKfs3HqRg>

The mathematics is hereby as follows:

- (1) First of all note that a matrix in $SO(3)$ is precisely given by a choice of three orthogonal vectors v_1, v_2, v_3 of $\mathbb{R}^3$.
- (2) Now suppose we are given a belt with a buckle to the right, a red edge and a green edge, with a blue top side and a pink lower side.
- (3) Any point on a the twisted belt defines a triple of orthogonal vector: the first vector points towards the buckle, the second one points towards the red edge and the third is orthogonal to the buckle and it is attached to the top side of the belt.

- (4) When we walk on the center line of the belt from the buckle towards the other end of the belt then we obtain a path $[0, 1] \rightarrow SO(3)$. In particular, if the end of the belt is in the same plane as the buckle, then the path defined by the belt is in fact a loop in $SO(3)$.
- (5) Now we give a belt, that is fixed at the buckle, a full twist. As in (4) this defines an element $\gamma \in \pi_1(SO(3))$.
- (6) The element $\gamma^2 \in \pi_1(SO(3))$ is given by doing another full twist along the buckle. Now one can untwist the belt while keeping the end in the same plane, meaning we can find a path-homotopy from γ^2 to the

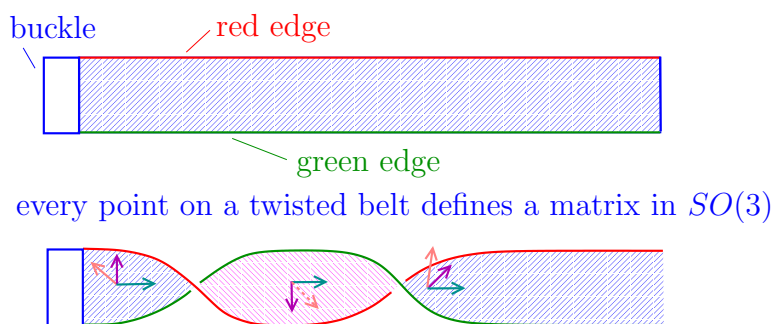


FIGURE 316.

This raises immediately the following question.

Question 31.5. *Let $n \geq 4$. What is $\pi_1(SO(n))$?⁵⁸⁷⁵⁸⁸*

We will return to this question later on in Algebraic Topology III.

31.2. Topological robotics. In this section we want to discuss a “real-life” application of Corollary 31.4. Let us assume that we are given a solid ball and we are allowed to perform the following operations:

- (1) first we can rotate the ball around the z -axis by an angle $\gamma \in S^1$,
- (2) then we can rotate the ball around the y -axis by an angle $\beta \in S^1$, finally
- (3) we can rotate the ball around the x -axis by an angle $\alpha \in S^1$.

Using these three operations, in that order, we can rotate the ball into any position. More precisely, we have the following lemma which we will prove in exercise sheet 9.

Lemma 31.6. *The map*

$$\begin{aligned} \Phi: S^1 \times S^1 \times S^1 &\rightarrow SO(3) \\ (\alpha, \beta, \gamma) &\mapsto \begin{array}{c} \text{rotation by } \alpha \\ \text{around the } x\text{-axis} \end{array} \circ \begin{array}{c} \text{rotation by } \beta \\ \text{around the } y\text{-axis} \end{array} \circ \begin{array}{c} \text{rotation by } \gamma \\ \text{around the } z\text{-axis} \end{array} \end{aligned}$$

⁵⁸⁷Recall that we had pointed out on page 33 that $SO(n)$ is path-connected.

⁵⁸⁸What is $\pi_1(SO(2))$?

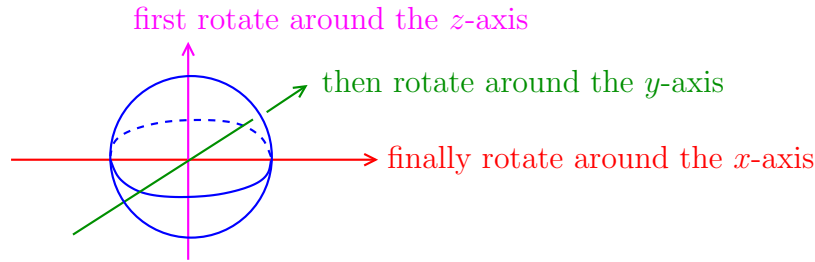


FIGURE 317.

is a surjection.⁵⁸⁹

This lemma gives rise to the following practical problem, given a matrix $P \in SO(3)$ can we continuously find parameters $\alpha(P), \beta(P), \gamma(P) \in S^1$ such that these parameters give back the matrix P . More precisely, we have the following question.

Question 31.7. *Does there exist a continuous map $\Psi: SO(3) \rightarrow S^1 \times S^1 \times S^1$ such that $\Phi \circ \Psi = \text{id}_{SO(3)}$?*

It turns out that the answer is negative. More precisely, we have the following proposition.

Proposition 31.8. *There is no continuous map $\Psi: SO(3) \rightarrow S^1 \times S^1 \times S^1$ such that $\Phi \circ \Psi = \text{id}_{SO(3)}$.*

Proof. Let us suppose there exists a continuous map $\Psi: SO(3) \rightarrow S^1 \times S^1 \times S^1$ such that $\Phi \circ \Psi = \text{id}_{SO(3)}$. Then we get the following commutative diagram

$$\begin{array}{ccc}
 & \overbrace{\pi_1(S^1 \times S^1 \times S^1)}^{\cong \mathbb{Z}^3} & \\
 \Psi_* \nearrow & & \searrow \Phi_* \\
 \underbrace{\pi_1(SO(3))}_{\cong \mathbb{Z}_2} & \xrightarrow{\text{id}_*} & \underbrace{\pi_1(SO(3))}_{\cong \mathbb{Z}_2}
 \end{array}$$

By Corollary 31.4 we have that $\pi_1(SO(3)) \cong \mathbb{Z}_2$ and from Corollary 6.17 we know that $\pi_1(S^1 \times S^1 \times S^1) \cong \mathbb{Z}^3$. Since $\mathbb{Z}_2$ is a finite group and since $\mathbb{Z}^3$ is torsion-free we see that the homomorphism $\Psi_*: \pi_1(SO(3)) \rightarrow \pi_1(S^1 \times S^1 \times S^1)$ is the trivial map. But that is not possible since the horizontal map is an isomorphism of a non-trivial group. $\square$

This proposition falls into the area of “topological robotics”, more information on this example and on topological robotics can be found in [Gh, p. 76]. Other aspects of topological robotics are discussed in the monograph [Fa].

⁵⁸⁹The map is also continuous. Why is that?

We conclude this chapter with a somewhat unrelated but fun topic that also appears in topological robotics.

- (1) We recall that an abstract graph G is a triple (V, E, φ) where V is a non-empty set, E is a set and φ is a map

$$\varphi: E \rightarrow \{\text{subsets of } V \text{ with one or two elements}\}.$$

The elements of V are called *vertices of G* and the elements of E are called the *edges of G* . Furthermore, given $e \in E$ the points in $\varphi(e)$ are called the *endpoints of e* .

- (2) Given $n \in \mathbb{N}$ we define the *linear graph of length n* to be the graph $G(n)$ which is given by the vertex set $V = \{0, \dots, n\}$ and the edge set $\{e_0, \dots, e_{n-1}\}$ with $\varphi(e_i) = \{i, i+1\}$ for $i = 0, \dots, n-1$. This graph is illustrated in Figure 318.

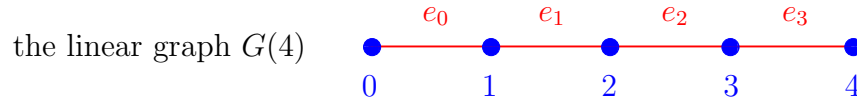


FIGURE 318.

Definition.

- (1) A *planar linkage* is a triple $(G = (V, E), \varphi, \psi)$ consisting of a finite abstract graph $G = (V, E)$, a function $\varphi: E \rightarrow \mathbb{R}_{>0}$ and a map $\psi: W \rightarrow \mathbb{C} = \mathbb{R}^2$ where W is a subset of V . Put differently, φ assigns a length to each edge and ψ represents a subset of the vertices by points in $\mathbb{C}$.
- (2) A *configuration* of a planar linkage $(G = (V, E), \varphi, \psi)$ is a map $f: V \rightarrow \mathbb{R}^2 = \mathbb{C}$ such that $f|_W = \psi$ and such that for every edge e the distance between the endpoints of e is $\varphi(e)$.⁵⁹⁰
- (3) The set $C(G = (V, E), \varphi, \psi)$ of all configurations is called the *configuration space* of the linkage. We have an injective map

$$\begin{aligned} C(G = (V, E), \varphi, \psi) &\rightarrow \mathbb{C}^{|V|} := \text{maps}(V \rightarrow \mathbb{C}) \\ f &\mapsto (v \mapsto f(v)) \end{aligned}$$

thus we can view the configuration space $C(G = (V, E), \varphi, \psi)$ as a topological space.

In some cases the topological shape of the configuration space of a planar linkage is easy to describe. More precisely, we have the following lemma which is a straightforward consequence of the definitions and Proposition 1.3. We leave the proof as an exercise.

⁵⁹⁰In particular if e has only one endpoint, then we need $\varphi(e) = 0$.

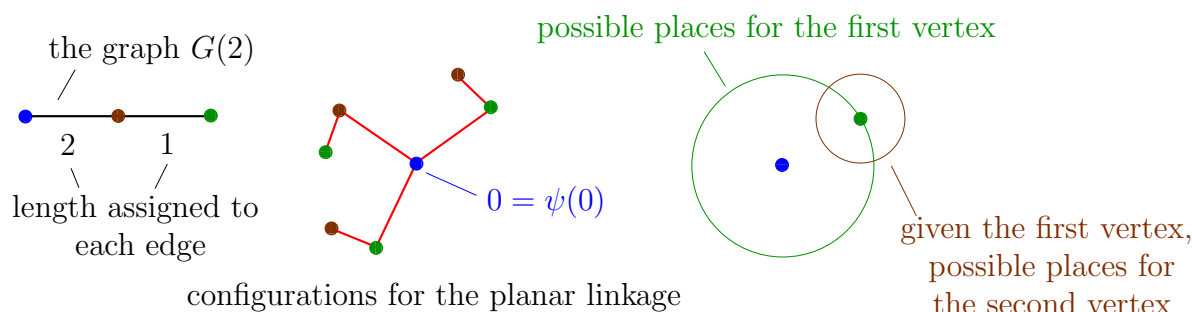


FIGURE 319.

Lemma 31.9. *Let $G(m)$ be the linear graph with m edges. We consider the planar linkage $(G(m), \varphi, \psi(0) = 0)$. For $i = 1, \dots, m$ we write $r_i = \varphi(e_i)$. Then the map*

$$(S^1)^m = S^1 \times \dots \times S^1 \rightarrow C(G(m), \varphi, \psi) \\ (z_1, \dots, z_m) \mapsto \begin{pmatrix} \{0, \dots, m\} \rightarrow \mathbb{C} \\ k \mapsto \sum_{i=1}^k r_i z_i \end{pmatrix}$$

is a homeomorphism.

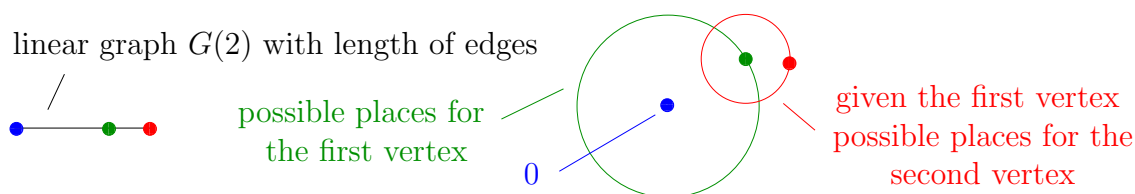


FIGURE 320. Sketch for the proof of Lemma 31.9.

For general linkages the topology of the configuration spaces are much harder to describe than for free planar linkages.

- (1) Given $r_1, \dots, r_m > 0$ we denote by $C(r_1, \dots, r_m, P, Q)$ the configuration space corresponding to the planar linkage defined by $G(m)$, $\varphi(e_i) = r_i$ and by the conditions that $\psi(0) = P$ and $\psi(m) = Q$.

(a) Using elementary arguments one can easily show that

$$C(r_1, r_2, P, Q) = \begin{cases} \emptyset, & \text{if } r_1 + r_2 < d(P, Q), \\ \text{one point,} & \text{if } r_1 + r_2 = d(P, Q), \\ \text{two points,} & \text{if } r_1 + r_2 > d(P, Q). \end{cases}$$

- (b) Kevin Walker [Wk, p. 35], see also [Gh, p. 12], showed in 1985 in his bachelor thesis that the surfaces of genus $0, 1, \dots, 4$ are homeomorphic to configuration spaces of the type $(r_1, r_2, r_3, r_4, P, Q)$ for suitable values of $r_1, \dots, r_4$ and points P and Q .

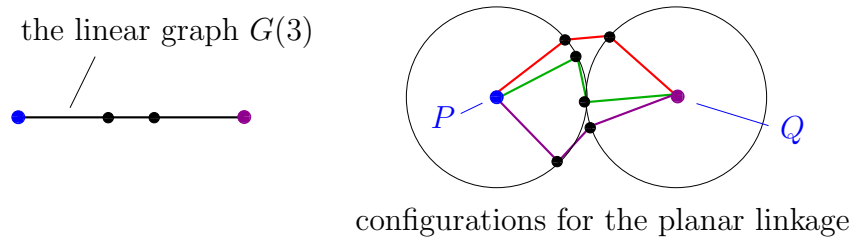


FIGURE 321.

- (2) Michael Kapovich and John Millson [KaM] showed that given any compact manifold M there exists a planar linkage such that each component of the configuration space is homeomorphic to M .

32. THE FIRST HOMOLOGY GROUP AND THE FUNDAMENTAL GROUP

In this section we want to relate the first homology group and the fundamental group of a path-connected topological space X . Both are defined by mapping 1-dimensional manifolds into X . So it is reasonable to expect that there is a relationship. But clearly in general they cannot be the same, since the first homology group is by definition abelian whereas we had seen on many occasions that fundamental groups can be non-abelian.

32.1. The Hurewicz homomorphism. Let X be a topological space. Throughout this section we denote by Φ the following bijection:

$$\begin{aligned} \Phi: \{\text{paths } [0, 1] \rightarrow X\} &\rightarrow \{\text{singular 1-simplices in } X\} \\ (\gamma: [0, 1] \rightarrow X) &\mapsto \left(\begin{array}{c} \Delta^1 \rightarrow X \\ (1-t, t) \mapsto \gamma(t) \end{array} \right). \end{aligned}$$

We start out with the following lemma.

Lemma 32.1. *Let X be a pointed topological space. If f and g are two paths in X such that the endpoint of f agrees with the starting point of g , then $\Phi(f) + \Phi(g)$ and $\Phi(f * g)$ are homologous.*

Proof. The proof of the lemma is sketched in Figure 322. Filling in the details of the proof is going to be a proof in exercise sheet 7. $\square$

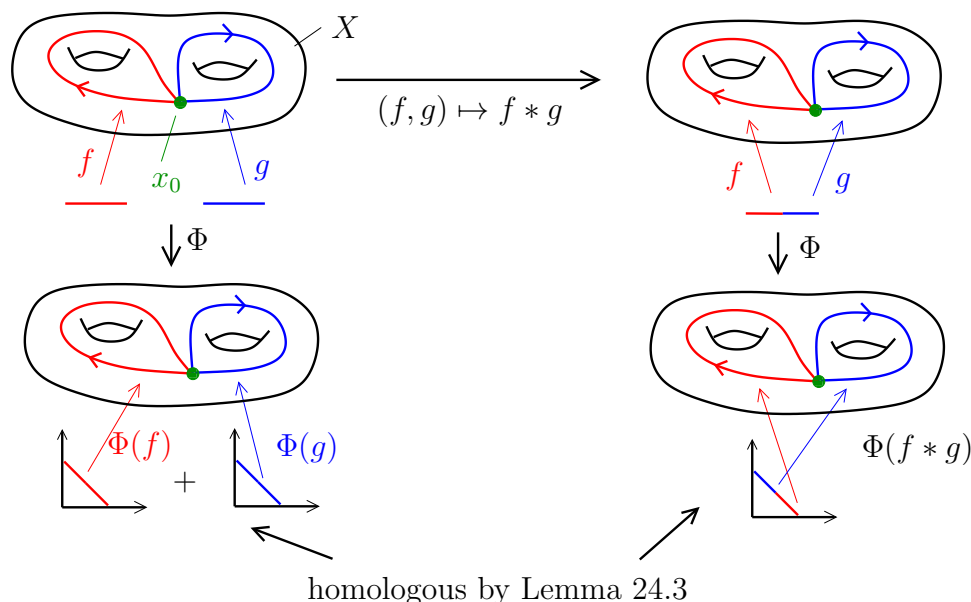


FIGURE 322. Illustration of the proof of Lemma 32.1.

Now we can easily prove the following proposition.

Proposition 32.2. *Let (X, x_0) be a pointed topological space.*

- (1) *If γ is a loop in (X, x_0) , then $\Phi(\gamma) \in C_1(X)$ is a cycle.*
- (2) *The map Φ induces a well-defined map*

$$\begin{aligned} \Phi_{(X, x_0)}: \pi_1(X, x_0) &\rightarrow H_1(X) \\ [\gamma] &\mapsto [\Phi(\gamma)]. \end{aligned}$$

- (3) *The map $\Phi_{(X, x_0)}: \pi_1(X, x_0) \rightarrow H_1(X)$ is a group homomorphism.*

Proof.

- (1) The first statement follows immediately from the definitions and the description of $\partial: C_1(X) \rightarrow C_0(X)$ given on page 402.
- (2) We denote by $\alpha: \Delta^1 \rightarrow [0, 1]$ the homeomorphism that is given by $\alpha(1 - t, t) = t$. Now suppose that $f, g: [0, 1] \rightarrow X$ are two homotopic loops in x_0 . Then

$$\begin{array}{ccccccc} [\Phi(f)] & = & [f \circ \alpha \circ \text{id}_{\Delta^1}] & = & (f \circ \alpha_*)([\text{id}_{\Delta^1}]) & = & (g \circ \alpha_*)([\text{id}_{\Delta^1}]) = \dots = [\Phi(g)]. \\ \uparrow & & & & \uparrow & & \\ \text{since } \Phi(f) \text{ and } f \circ \alpha \circ \text{id}_{\Delta^1} & \text{are} & & & \text{by Proposition 25.4} & & \\ & \text{the same map} & & & \text{since } f \text{ and } g \text{ are homotopic} & & \end{array}$$

- (3) The third statement is an immediate consequence of Lemma 32.1. □

Definition. Given a pointed topological space (X, x_0) we refer to the homomorphism

$$\Phi_{(X, x_0)}: \pi_1(X, x_0) \rightarrow H_1(X)$$

that we had just constructed as the *Hurewicz homomorphism*.⁵⁹¹

It is clear that the Hurewicz homomorphism is in general neither injective nor surjective:

- (1) The Hurewicz homomorphism is in general not injective since $\pi_1(X, x_0)$ is in many cases a non-abelian group whereas $H_1(X)$ is an abelian group, and it is not possible that a non-abelian group is a subgroup of an abelian group.
- (2) The Hurewicz homomorphism is in general not surjective, since $\pi_1(X, x_0)$ depends only on the path-component of x_0 , while $H_1(X)$ “sees” all path-components of X . But as we will see in the next lemma, this turns out to be the only obstacle to Φ being surjective.

Lemma 32.3. *Let X be a topological space and let $x_0 \in X$. If X is path-connected, then the Hurewicz homomorphism*

$$\Phi: \pi_1(X, x_0) \rightarrow H_1(X)$$

is surjective.

Proof. We first point out that the map

$$\begin{aligned} \Phi: \{\text{paths } [0, 1] \rightarrow X\} &\rightarrow \{\text{singular 1-simplices in } X\} \\ (\gamma: [0, 1] \rightarrow X) &\mapsto \left(\begin{array}{ccc} \Delta^1 & \rightarrow & X \\ (1 - t, t) & \mapsto & \gamma(t) \end{array} \right) \end{aligned}$$

⁵⁹¹ Witold Hurewicz (1904–1956) was a Polish mathematician.

is a bijection. We denote its inverse map by Ψ .

We pick a homology class $z \in H_1(X)$. Let $c = \sum_{i=1}^k a_i \sigma_i$ be a cycle in $C_1(X)$ that represents z . It follows from Lemma 24.3 that, after possibly replacing a singular chain σ_i by $\bar{\sigma}_i$, we can assume that all the coefficients a_i are positive. Furthermore, since we do not require that the σ_i 's are different, we can in fact suppose that $a_1 = \cdots = a_k = 1$.

Since X is path-connected we can choose for each $i \in \{1, \dots, k\}$ a path p_i from the base point x_0 to the starting point $\sigma_i(1, 0)$ of σ_i . As usual we denote by $\bar{p}_i$ the path with opposite orientation. According to Lemma 24.3 the singular chain $\Phi(p_i) + \Phi(\bar{p}_i) = \Phi(p_i) + \overline{\Phi(p_i)}$ is a null-homologous cycle for each i . Since $\partial c = 0$ we see that the “endpoints” of the σ_i appear exactly the same number of times as the “starting points” of the σ_i . This means that there exists a permutation $s \in S_k$ such that for each i the “endpoint” $\sigma_i(0, 1)$ is just the “starting point” $\sigma_{s(i)}(1, 0)$ of $\sigma_{s(i)}$. We obtain the following equality in $H_1(X)$:

$$\begin{aligned}
 c = \sum_{i=1}^k \sigma_i &= \sum_{i=1}^k \sigma_i + \sum_{i=1}^k \Phi(p_i) + \sum_{i=1}^k \Phi(\bar{p}_i) = \sum_{i=1}^k (\Phi(p_i) + \Phi(\Psi(\sigma_i)) + \Phi(\bar{p}_{s(i)})) \\
 &= \sum_{i=1}^k \underbrace{\Phi(p_i \cdot \Psi(\sigma_i) \cdot \bar{p}_{s(i)})}_{\text{loop in } (X, x_0)} = \underbrace{\Phi\left(\prod_{i=1}^k p_i \cdot \Psi(\sigma_i) \cdot \bar{p}_{s(i)}\right)}_{\text{loop in } (X, x_0)}.
 \end{aligned}$$

by Lemma 32.1
by Lemma 32.1

Thus we have shown that $z = [c] \in \Phi(\pi_1(X, x_0))$. □

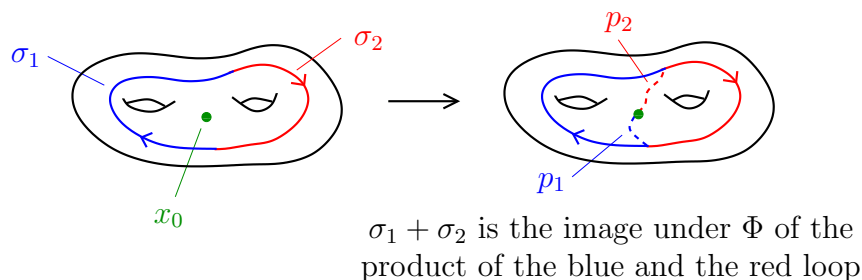


FIGURE 323. Sketch for the proof that Φ is surjective.

In the following we want to study to what degree the Hurewicz homomorphism fails to be injective. Before we can do so we need to recall some definitions and results from Section 10.3.

Definition. Let π be a group.

- (1) For $x, y \in \pi$ we denote by $[x, y] := xyx^{-1}y^{-1}$ the *commutator* of x and y .

(2) We refer to

$$[\pi, \pi] := \langle\langle \{[x, y] \mid x, y \in \pi\} \rangle\rangle = \left\{ \prod_{i=1}^n [x_i, y_i] \mid x_1, y_1, \dots, x_n, y_n \in \pi \right\}$$

as the *commutator subgroup* of π .

(3) Furthermore we refer to $\pi_{\text{ab}} := \pi/[\pi, \pi]$ as the *abelianization* of π .

In Section 10.3 we had seen that the abelianization is “the largest abelian quotient” of π . More precisely, if $\alpha: \pi \rightarrow H$ is a homomorphism from π to an abelian group H , then Proposition 10.6 says that there exists a unique homomorphism $\beta: \pi_{\text{ab}} := \pi/[\pi, \pi] \rightarrow H$ such that the following diagram commutes:

$$\begin{array}{ccc} \pi & \xrightarrow{\quad} & \pi_{\text{ab}} := \pi/[\pi, \pi] \\ & \searrow \alpha & \downarrow \beta \\ & & H. \end{array}$$

In particular, given a pointed topological space (X, x_0) the corresponding Hurewicz homomorphism $\Phi_{(X, x_0)}: \pi_1(X, x_0) \rightarrow H_1(X)$ induces a homomorphism $\pi_1(X, x_0)_{\text{ab}} \rightarrow H_1(X)$ that we also denote by $\Phi_{(X, x_0)}$ and that we also refer to as the Hurewicz homomorphism. Now we can formulate the following proposition.

Proposition 32.4. *Let (X, x_0) be a pointed topological space. If X is path-connected, then the Hurewicz homomorphism*

$$\Phi_{(X, x_0)}: \pi_1(X, x_0)_{\text{ab}} \rightarrow H_1(X)$$

is an isomorphism.

Examples.

(1) If we compare our calculations of fundamental groups with our calculations of homology groups then we see that we had encountered this isomorphism many times.

(a) In Corollary 6.17 we had shown that $\pi_1((S^1)^n) \cong \mathbb{Z}^n$ whereas on page 492 we had seen that $H_1((S^1)^n) \cong \mathbb{Z}^n$.

(b) For $n \geq 2$ we had seen in Corollary 32.5 that $\pi_1(\mathbb{RP}^n) \cong \mathbb{Z}_n$ and in Lemma 28.7 we had calculated that $H_1(\mathbb{RP}^n) \cong \mathbb{Z}_2$.

(c) In Proposition 11.3 we had shown that

$$\pi_1(\text{Klein bottle}) \cong \langle x, y \mid yxyx^{-1} \rangle.$$

The abelianization of that group is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}_2$ which is the first homology of the Klein bottle that we had determined in Lemma 28.12.

(d) In Proposition 11.3 we had shown that for any $g \in \mathbb{N}$ we have

$$\pi_1(\text{surface of genus } g) \cong \langle x_1, y_1, \dots, x_g, y_g \mid [x_1, y_1] \cdots [x_g, y_g] \rangle.$$

The abelianization of this group is isomorphic to $\mathbb{Z}^{2g}$ which is precisely the first homology group that we had determined in Proposition 29.14.

- (e) Let X be a path-connected topological space and let $f: X \rightarrow X$ be a homeomorphism. On page 493 we had seen that there exists a short exact sequence

$$0 \rightarrow \operatorname{coker}(H_1(X) \xrightarrow{f_* - \operatorname{id}} H_1(X)) \xrightarrow{i} H_1(M(X, f)) \rightarrow \mathbb{Z} \rightarrow 0.$$

On the other hand we had seen in Proposition 12.12 and the remark on page 218 that there exists an isomorphism $\psi: \pi_1(M(X, f)) \cong \pi_1(X) \rtimes \mathbb{Z}$ where the action of $1 \in \mathbb{Z}$ on $\pi_1(X)$ is induced by f . Therefore we can put $\pi_1(M(X, f))$ into a short exact sequence

$$0 \rightarrow \pi_1(X) \rightarrow \pi_1(M(X, f)) \rightarrow \mathbb{Z} \rightarrow 0$$

We now apply the abelianization functor to these groups and homomorphisms. We obtain the exact sequence⁵⁹²

$$\pi_1(X)_{\operatorname{ab}} \rightarrow \pi_1(M(X, f))_{\operatorname{ab}} \rightarrow \mathbb{Z} \rightarrow 0$$

which, using Proposition 32.4 we can rewrite as

$$H_1(X) \rightarrow H_1(M(X, f)) \rightarrow \mathbb{Z} \rightarrow 0.$$

This is, fortunately, consistent with the above short exact sequence.

- (2) We also obtain some new information. Together with Lemma 28.13 we have now shown that given any knot $K \subset S^3$ the abelianization of $\pi_1(S^3 \setminus K)$ is isomorphic to $\mathbb{Z}$.

Proof. Let (X, x_0) be a pointed path-connected topological space. We write $\pi := \pi_1(X, x_0)$. It follows from Lemma 32.3 that the Hurewicz homomorphism $\Phi_{(X, x_0)}: \pi_{\operatorname{ab}} \rightarrow H_1(X)$ is surjective. It remains to show that $\Phi_{(X, x_0)}$ is injective.

In the following we will use the bijection Φ from the proof of Lemma 32.3 to identify paths $[0, 1] \rightarrow X$ with singular 1-simplices.

So let $f: S^1 \rightarrow X$ be a loop in (X, x_0) such that $[f] = 0 \in H_1(X)$. This means that there exists a 2-chain $\sum_{j=1}^s n_j \sigma_j$ in X whose boundary is the singular 1-simplex f . Furthermore we can assume that all n_j are either -1 or 1 . Given $j \in \{1, \dots, s\}$ and $k \in \{0, 1, 2\}$ we write⁵⁹³

$$\mu_{(j,k)} = \sigma_j \circ i_k^2 \quad \text{and} \quad n_{(j,k)} := (-1)^k \cdot n_j.$$

⁵⁹²We had seen in Algebraic Topology I that if $\varphi: A \rightarrow B$ is an epimorphism, then the induced map $A_{\operatorname{ab}} \rightarrow B_{\operatorname{ab}}$ is also an epimorphism. We had also seen that there exist monomorphisms $\varphi: A \rightarrow B$ such that the induced map $A_{\operatorname{ab}} \rightarrow B_{\operatorname{ab}}$ is not a monomorphism.

⁵⁹³Recall that for $j \in \{0, \dots, n\}$ we denote by i_j^n the map

$$\begin{aligned} i_j^n: \Delta^{n-1} &\rightarrow \Delta^n \\ (t_0, \dots, t_{n-1}) &\mapsto (t_0, \dots, t_{j-1}, 0, t_j, \dots, t_{n-1}). \end{aligned}$$

Without loss of generality we can assume that $\mu_{(1,0)} = f$ and that $n_{(1,0)} = 1$. We write $A = \{1, \dots, s\} \times \{0, 1, 2\} \setminus \{(1, 0)\}$. We have the following equality in $C_1(X)$:

$$f = \partial \left(\sum_{j=1}^s n_j \sigma_j \right) \underset{\substack{\uparrow \\ \text{definition of } \partial}}{=} \sum_{j=1}^s \sum_{k=0}^2 n_{(j,k)} \cdot \mu_{j,k} = \mu_{(1,0)} + \sum_{a \in A} n_a \cdot \mu_a = f + \sum_{a \in A} n_a \cdot \mu_a.$$

Since all n_a 's are equal to ± 1 we see that there exists a decomposition $A = A' \sqcup A''$ and a bijection $\psi: A' \rightarrow A''$ such that for all $a \in A'$ we have $\mu_{\psi(a)} = \mu_a$ and $n_{\psi(a)} = -n_a$. We denote the inverse map $A'' \rightarrow A'$ by ψ as well.

We make the following choices and introduce the following definitions:

- (1) For each point P in X that is a starting or an endpoint of some μ_a we pick a path β_P from x_0 to P . For $P = x_0$ we just pick the constant path.
- (2) Given $\mu_{(i,j)}$ we write

$$\widehat{\mu}_{(i,j)} := \beta_{\text{starting point of } \mu_{(i,j)}} * \mu_{(i,j)} * \overline{\beta}_{\text{endpoint of } \mu_{(i,j)}}.$$

We refer to Figure 324 for an illustration.

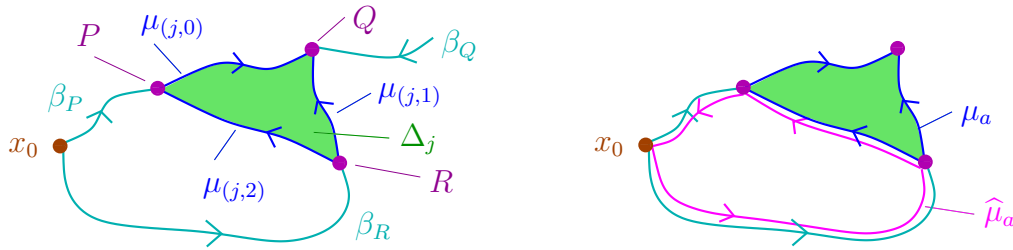


FIGURE 324. Illustration for the proof of Proposition 32.4.

We make the following elementary observation: for any (i, j) we have

$$\widehat{\mu}_{(i,j)}^{-1} := \beta_{\text{starting point of } \bar{\mu}_{(i,j)}} * \bar{\mu}_{(i,j)} * \overline{\beta}_{\text{endpoint of } \bar{\mu}_{(i,j)}}.$$

In π_{ab} we have the following equalities:⁵⁹⁴

$$\begin{aligned}
e &= \prod_{j=1}^s \beta_{\text{starting point of } \mu_{(j,0)}} * (\mu_{(j,0)} * \bar{\mu}_{(j,1)} * \mu_{(j,2)})^{n_j} * \bar{\beta}_{\text{starting point of } \mu_{(j,0)}} \\
&\quad \uparrow \\
&\text{since } \mu_{(j,0)} * \bar{\mu}_{(j,1)} * \mu_{(j,2)} \text{ is null-homotopic} \\
&= \prod_{j=1}^s \hat{\mu}_{(j,0)}^{n_{(j,0)}} \cdot \hat{\mu}_{(j,1)}^{n_{(j,1)}} \cdot \hat{\mu}_{(j,2)}^{n_{(j,2)}} &= \prod_{a \in \{1, \dots, k\} \times \{0, 1, 2\}} \hat{\mu}_a^{n_a} \\
&\quad \uparrow \\
&\text{follows from inserting null-homotopic paths } \bar{\beta}_P * \beta_P \\
&\quad \text{together with the above equality (*)} \\
&= f \cdot \prod_{a \in A'} \hat{\mu}_a^{n_a} \cdot \hat{\mu}_{\psi(a)}^{n_{\psi(a)}} &= f \cdot \prod_{a \in A'} \hat{\mu}_a^{n_a} \cdot \hat{\mu}_a^{-n_a} = f. \\
&\quad \uparrow &\quad \uparrow \\
&\text{since } \{1, \dots, k\} \times \{0, 1, 2\} = (1, 0) \sqcup A' \sqcup \psi(A') &\text{for } a \in A' \text{ we have } \mu_{\psi(a)} = \mu_a \text{ and } n_a = -n_a. \\
&\text{and since } \mu_{(1,0)} = f
\end{aligned}$$

This completes the proof of the proposition.⁵⁹⁵ □

For most topological spaces for which we had computed the fundamental group we had also determined the homology groups. But there is one exception, namely the lens spaces. From Corollary 6.17 and Proposition 32.4 we obtain immediately the following corollary.

Corollary 32.5. *For any coprime $p, q \in \mathbb{N}$ we have*

$$H_1(\text{lens space } L(p, q)) \cong \mathbb{Z}_p.$$

Proposition 32.4 is perhaps a mixed blessing. On the one hand it allows us to use our techniques developed in Algebraic Topology I to determine first homology groups. On the other hand it tells us that all the information that is contained in the first homology group is already contained in the fundamental group. So we do not get any new information.

32.2. Natural transformations. We conclude the chapter with some more category theory. We have now seen that we can assign to a pointed topological (X, x_0) the groups $\pi_1(X, x_0)$ and $H_1(X)$ and that there exists a corresponding Hurewicz homomorphism $\Phi_X: \pi_1(X, x_0) \rightarrow H_1(X)$. If we consider the fundamental group and the first homology group as functors, then we can write the connection between these two functors in a more precise way. But before we can do so we need to introduce the notion of a *natural transformation* between two functors.

Definition. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories and let $F, G: \mathcal{C} \rightarrow \mathcal{D}$ be two functors. A *natural transformation between the functors F and G* assigns to each $X \in \text{Ob}(\mathcal{C})$ a morphism $\Phi_X: F(X) \rightarrow G(X)$ such that for each morphism $f: X \rightarrow Y$ in $\mathcal{C}$ the following diagram

⁵⁹⁴In the notation we do not distinguish between loops and elements they represent in π_{ab} .

⁵⁹⁵Where did we actually use that we work with π_{ab} instead of π ?

commutes:

$$\begin{array}{ccc} F(X) & \xrightarrow{F_*(f)} & F(Y) \\ \Phi_X \downarrow & & \downarrow \Phi_Y \\ G(X) & \xrightarrow{G_*(f)} & G(Y). \end{array}$$

If for every $X \in \text{Ob}(\mathcal{C})$ the morphism $\Phi_X: F(X) \rightarrow G(X)$ is an isomorphism, then we refer to the natural transformation as a *natural isomorphism*. Given $X \in \text{Ob}(\mathcal{C})$ we often say that $F(X)$ and $G(X)$ are *naturally* isomorphic.⁵⁹⁶

Remark. The notion of a natural transformation can also be summarized in the following diagram:

$$\begin{array}{ccccc} & & F(X) & \xrightarrow{F_*(f)} & F(Y) \\ & \nearrow & \downarrow \Phi_X & \searrow & \downarrow \Phi_Y \\ X & \xrightarrow{f} & & & Y \\ & \searrow & G(X) & \xrightarrow{G_*(f)} & G(Y) \end{array}$$

Here the wiggly arrows are not maps but they are just meant to remind us which objects a functor associates to a morphism $f: X \rightarrow Y$.

We will now see that we have encountered natural transformations and natural isomorphisms much more often than one might have initially thought.

Examples.

- (1) We denote by $\mathcal{P}$ the category of pointed topological spaces and we denote by $\mathcal{A}$ the category of abelian groups. Let $n \in \mathbb{N}_0$. Lemma 26.11 (2) says that the isomorphisms from Lemma 26.11 (1) give a natural transformation from the functor

$$\begin{array}{ccc} \mathcal{P} & \rightarrow & \mathcal{A} \\ (X, x_0) & \mapsto & H_n(X, x_0) \end{array} \quad \text{to the functor} \quad \begin{array}{ccc} \mathcal{P} & \rightarrow & \mathcal{A} \\ (X, x_0) & \mapsto & \tilde{H}_n(X). \end{array}$$

Put differently, the relative homology groups $H_n(X, x_0)$ and the reduced homology group $\tilde{H}_n(X)$ are naturally isomorphic.

- (2) Given a pair (X, A) of topological spaces we had defined on page 440 what we had called a natural homomorphism $H_n(X, A) \rightarrow \tilde{H}_n(X/A)$. These natural homomorphisms define a natural transformation from the functor

$$\begin{array}{ccc} \mathcal{P} & \rightarrow & \mathcal{A} \\ (X, A) & \mapsto & H_n(X, A) \end{array} \quad \text{to the functor} \quad \begin{array}{ccc} \mathcal{P} & \rightarrow & \mathcal{A} \\ (X, A) & \mapsto & \tilde{H}_n(X/A). \end{array}$$

⁵⁹⁶In the literature the words “natural isomorphism” and “canonical isomorphism” are often used indiscriminately, but whereas the expression “natural isomorphism” has a precise meaning, the expression “canonical isomorphism” is not a well-defined mathematical concept.

- (3) Given a topological space A the *cone on A* is defined as $CA := A \times [0, 1]/A \times 1$. Note that the “tip” $P = [A \times 1]$ of the cone is a deformation retract of CA . Furthermore, given a map $f: A \rightarrow X$ between topological spaces the corresponding mapping cone is defined as

$$M(f: A \rightarrow X) := X \cup CA / \sim \text{ where } f(a) \sim (a, 0) \text{ for all } a \in A.$$

Finally, given a pair (X, A) of topological spaces we denote by $i: A \rightarrow X$ the inclusion map and have isomorphisms

$$\begin{array}{ccccccc} \underbrace{\widetilde{H}_n(M(i: A \rightarrow X))}_{= \widetilde{H}_n(M(i: A \rightarrow X))} & \cong & H_n(X \cup CA / \sim, CA) & \cong & H_n(X \cup CA / \sim \setminus P, CA \setminus P) & \xrightarrow{\cong} & H_n(X, A) \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ \text{by Corollary 26.13 since } CA \text{ is contractible} & & \text{Excision Theorem 26.15} & & \text{extension of Corollary 26.14} \end{array}$$

These isomorphisms define a natural isomorphism from the functor

$$\begin{array}{ccc} \mathcal{P} & \rightarrow & \mathcal{A} \\ (X, A) & \mapsto & \widetilde{H}_n(M(i: A \rightarrow X)) \end{array} \quad \text{to the functor} \quad \begin{array}{ccc} \mathcal{P} & \rightarrow & \mathcal{A} \\ (X, A) & \mapsto & H_n(X, A). \end{array}$$

This observation can be viewed as a generalization of (1). Thus, if we want, we can always replace relative homology of a pair (X, A) of topological spaces by the ordinary homology of the mapping cone of the inclusion $A \rightarrow X$.

- (4) Let $\mathcal{V}$ be the category of real vector spaces and consider the functor F given by⁵⁹⁷

$$\text{real vector space } V \mapsto (V^*)^*$$

and

$$(f: V \rightarrow W) \mapsto ((f^*)^*: (V^*)^* \rightarrow (W^*)^*)$$

For each real vector V we denote by Φ_V the isomorphism

$$\begin{array}{ccc} V & \rightarrow & (V^*)^* \\ v & \mapsto & \left(\begin{array}{ccc} V^* & \rightarrow & \mathbb{R} \\ (\varphi: V \rightarrow \mathbb{R}) & \mapsto & \varphi(v) \end{array} \right). \end{array}$$

It is straightforward to verify that these maps define a natural transformation from the identity functor to the functor F .

- (5) Let $\mathcal{G}$ be the category of groups. Given a group G we denote by

$$\begin{array}{ccc} \Phi_G: G & \rightarrow & G_{\text{ab}} = G/[G, G] \\ g & \mapsto & [g] \end{array}$$

the projection map. It is straightforward to verify that these maps Φ_G define a natural transformation from the identity functor $\mathcal{G} \rightarrow \mathcal{G}$ to the abelianization functor $\mathcal{G} \rightarrow \mathcal{G}$.

⁵⁹⁷As usual, given a real vector space V we denote by $V^* = \text{Hom}_{\mathbb{R}}(V, \mathbb{R})$ the dual space. Therefore $(V^*)^*$ is the dual space of the dual space of V . The functor given by $V \mapsto V^*$ is contravariant, but the “double functor” $V \mapsto (V^*)^*$ is again covariant.

- (6) Given $n \in \mathbb{N}_0$ we consider the functor C_n from the category of topological spaces to the category of abelian groups which is given by

$$\text{topological space } X \mapsto C_n(X)$$

and

$$(f: X \rightarrow Y) \mapsto (f_*: C_n(X) \rightarrow C_n(Y)).$$

Given a topological space X we denote by $\Phi_X: C_n(X) \rightarrow C_n(X)$ the map that is given by the n -th subdivision map that we had defined on page 445. It follows from Lemma 26.19 that these maps define a natural transformation from the functor C_n to itself.

- (7) Let $\mathcal{Q}$ be the category of pairs of topological spaces that we had introduced on page 81 and let $\mathcal{A}$ be the category of abelian groups. Let $n \in \mathbb{N}$. We consider the two functors F and G that are given by

$$\begin{aligned} F(X, A) &:= H_n(X, A) \\ (f: (X, A) \rightarrow (Y, B)) &\mapsto (f_*: H_n(X, A) \rightarrow H_n(Y, B)) \end{aligned}$$

and

$$\begin{aligned} G(X, A) &:= H_{n-1}(A) \\ (f: (X, A) \rightarrow (Y, B)) &\mapsto (f_*: H_{n-1}(A) \rightarrow H_{n-1}(B)). \end{aligned}$$

Given each pair (X, A) of topological spaces we denote by $\Phi_{X,A}$ the connecting homomorphism $\partial: H_n(X, A) \rightarrow H_{n-1}(A)$. It follows from the remark on page 431 that these homomorphisms define a natural transformation from F to G .

- (8) Recall that we denote by $\mathcal{CW}$ the category of CW-complexes where the objects are CW-complexes and the morphisms are cellular maps. On page 501 we had seen that cellular homology groups define a functor from $\mathcal{CW}$ to $\mathcal{A}$. Given a CW-complex X and $n \in \mathbb{N}_0$ we had constructed Proposition 29.8 a uniquely determined isomorphism $H_n(X) \rightarrow H_n^{\text{CW}}(X)$. In fact these isomorphisms are natural in the sense that they define a natural transformation from the functor

$$\begin{array}{ccc} \mathcal{CW} & \rightarrow & \mathcal{A} \\ X & \mapsto & H_n(X) \end{array} \quad \text{to the functor} \quad \begin{array}{ccc} \mathcal{CW} & \rightarrow & \mathcal{A} \\ X & \mapsto & H_n^{\text{CW}}(X). \end{array}$$

We leave it as an exercise to verify this statement. We will not make use of this fact hence we will not bother to prove it.

Now that we have mastered the notion of a natural transformation we can formulate the following proposition.

Proposition 32.6. *The Hurewicz homomorphisms*

$$\Phi_{(X, x_0)}: \pi_1(X, x_0) \rightarrow H_1(X)$$

define a natural transformation between from the functor⁵⁹⁸

$$\begin{array}{ccc} \mathcal{P} & \rightarrow & \mathcal{G} \\ (X, x_0) & \mapsto & \pi_1(X, x_0) \end{array}$$

to the functor

$$\begin{array}{ccc} \mathcal{P} & \rightarrow & \mathcal{G} \\ (X, x_0) & \mapsto & H_1(X). \end{array}$$

The proof of the proposition follows easily from the definitions. We leave the details to the reader. As an illustration of Proposition 32.6 we prove the following corollary.

Corollary 32.7. *Let $p: Y \rightarrow X$ be a finite-index covering of path-connected topological spaces. Then the cokernel of the map*

$$p_*: H_1(Y) \rightarrow H_1(X)$$

is finite.

Proof. We pick a base point y_0 of Y and we write $x_0 = p(y_0)$. We consider the following diagram of maps:

$$\begin{array}{ccccc} \pi_1(Y, y_0) & \xrightarrow{p_*} & \pi_1(X, x_0) & & \\ & \searrow & \downarrow \Phi_X & \searrow & \\ & & \pi_1(Y, y_0)_{\text{ab}} & \xrightarrow{p_*} & \pi_1(X, x_0)_{\text{ab}} \\ & \swarrow \cong & \downarrow \Phi_Y & \swarrow \cong & \\ & & H_1(Y) & \xrightarrow{p_*} & H_1(X). \end{array}$$

Hereby it follows from Proposition 32.4 that the lower diagonal maps are isomorphisms. The “big rectangle in the back” commutes by Proposition 32.6. The upper parallelogram commutes by the discussion on page 547. Since the lower diagonal maps are isomorphisms it follows that the lower parallelogram also commutes.

By Lemma 6.15 and by our hypothesis that p is a finite covering we know that $p_*(\pi_1(Y, y_0))$ is a finite index subgroup of $\pi_1(X, x_0)$, i.e. $p_*: \pi_1(Y, y_0) \rightarrow \pi_1(X, x_0)$ has a finite cokernel. It follows⁵⁹⁹ that $p_*: \pi_1(Y, y_0)_{\text{ab}} \rightarrow \pi_1(X, x_0)_{\text{ab}}$ also has a finite cokernel. From the commutativity of the lower parallelogram it follows that the map $p_*: H_1(Y) \rightarrow H_1(X)$ also has a finite cokernel $\square$

⁵⁹⁸Recall that we denote by $\mathcal{P}$ the category of pointed topological spaces and that we denote by $\mathcal{G}$ the category of groups.

⁵⁹⁹Why does this follow?

32.3. The Hurewicz homomorphism in higher dimensions. Proposition 32.4 also raises the question whether there is a connection between higher homotopy groups and higher homology groups.

Definition. Let X be a topological space, let $x_0 \in X$ and let $n \geq 1$. We denote by $\omega_n \in H_n(S^n)$ the standard generator as defined on page 460. We refer to the map

$$\begin{aligned} \Phi_{X,x_0}: \pi_n(X, x_0) &\rightarrow H_n(X) \\ [f: S^n \rightarrow X] &\mapsto f_*(\omega_n) \end{aligned}$$

as the *Hurewicz map*.⁶⁰⁰

Remark. Using the bijection given on page 68 and using Lemma 26.31 one can easily show that for $n = 1$ this definition of the Hurewicz map agrees with the definition given on page 540.

In the next lemma we show that the Hurewicz map is in fact a homomorphism.

Lemma 32.8. *Let X be a topological space, let $x_0 \in X$ and let $n \geq 1$. The Hurewicz map*

$$\Phi = \Phi_{X,x_0}: \pi_n(X, x_0) \rightarrow H_n(X)$$

is an isomorphism.

Proof ()*. As usual we write $I = [0, 1]$. By Lemmas 1.4 and 24.1 there exists a homeomorphism $\varphi: (\Delta^n, \partial\Delta^n) \rightarrow (I^n, \partial I^n)$. It follows from Lemma 26.29 that $\varphi: \Delta^n \rightarrow I^n$ represents a generator of $H_n(I^n, \partial I^n)$. We now write $\omega := [\varphi] \in H_n(I^n, \partial I^n)$.

Using the bijection from page 385 we see that it suffices to prove the following claim.

Claim. Let $f, g: (I^n, \partial I^n) \rightarrow (X, x_0)$ be two maps. Then

$$[f_*(\omega)] + [g_*(\omega)] = [(f * g)_*(\omega)] \in H_n(X, x_0) = H_n(X).$$

We write $J = I^n$, $J_1 = I^{n-1} \times [0, \frac{1}{2}]$ and $J_2 = I^{n-1} \times [\frac{1}{2}, 1]$. Furthermore we write $K = \partial J \cup I^{n-1} \times \{\frac{1}{2}\}$. We refer to Figure 325 for an illustration. For $k = 1, 2$ we denote by $i_k: (J_k, \partial J_k) \rightarrow (J, K)$ the inclusion map and we denote by $\varphi_k: (J_k, \partial J_k) \rightarrow (J, \partial J)$ the obvious homeomorphisms given by stretching the last coordinate.

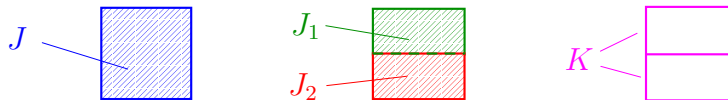


FIGURE 325. Illustration for the proof of Lemma 32.8.

⁶⁰⁰It follows from Proposition 25.4 that the Hurewicz homomorphism is indeed well-defined.

The map $f * g: (J, \partial J) \rightarrow (X, x_0)$ factors through a map $(J, K) \rightarrow (X, x_0)$ that we also denote by $f * g$. Thus we have the following commutative diagram

$$\begin{array}{ccccc}
 & & (J_1, \partial J_1) & & \\
 & \nearrow \varphi_1 & \downarrow i_1 & \searrow f \circ \varphi_1^{-1} & \\
 (J, \partial J) & \xrightarrow{i} & (J, K) & \xrightarrow{f * g} & (X, x_0) \\
 & \searrow \varphi_2 & \uparrow i_2 & \nearrow g \circ \varphi_2^{-1} & \\
 & & (J_2, \partial J_2) & &
 \end{array}$$

For $k = 1, 2$ we write $\omega_k = (\varphi_k)_*(\omega)$. Thus it suffices to show that $i_{1*}(\omega_1) + i_{2*}(\omega_2) = i_*(\omega)$.

We consider the following commutative diagram⁶⁰¹

$$\begin{array}{ccccc}
 & & H_n(J_1, \partial J_1) & \xrightarrow{k_1} & H_n(J, \partial J_1 \cup J_2) \\
 & \nearrow \varphi_1 & \downarrow i_1 & \nearrow j_1 & \\
 H_n(J, \partial J) & \xrightarrow{i} & H_n(J, K) & \xrightarrow{j_2} & H_n(J, J_1 \cup \partial J_2) \\
 & \searrow \varphi_2 & \uparrow i_2 & \nearrow j_2 & \\
 & & H_n(J_2, \partial J_2) & \xrightarrow{k_2} &
 \end{array}$$

where all maps, except for φ_1 and φ_2 , are induced by the obvious inclusion maps. We have

$$\begin{aligned}
 (j_1)(i(\omega)) &= (k_1 \circ \varphi_1)(\omega) = j_1(i_1(\omega_1)) = j_1(i_1(\omega_1)) + j_1(i_2(\omega_2)) \\
 &\quad \uparrow \\
 &\text{since } j_1 \circ i_2 = 0
 \end{aligned}$$

and similarly we show that $(j_2)(i(\omega)) = j_2(i_1(\omega_1)) + j_2(i_2(\omega_2))$.

It remains to show that $\ker(j_1) \cap \ker(j_2) = 0$. We consider the following diagram

$$\begin{array}{ccccc}
 & & & & H_n(J, \partial J_1 \cup J_2) \\
 & & & \nearrow j_1 & \\
 & & H_n(J, K) & \searrow j_2 & \\
 & \nearrow & \uparrow i_2 & \nearrow k_2 & \\
 H_n(J, \partial J_1 \cup J_2) & \xleftarrow{\cong} & H_n(J_2, \partial J_2) & \xrightarrow{\cong} & H_n(J, J_1 \cup \partial J_2)
 \end{array}$$

The bottom two horizontal maps are hereby isomorphisms by the Excision Theorem 26.26. The diagonal sequence is exact by Proposition 26.12. It now follows from some mild diagram chasing that $\ker(j_1) \cap \ker(j_2) = 0$. $\square$

The following lemma gives in particular an affirmative answer to Question 23.13.

Lemma 32.9. *For every $n \in \mathbb{N}$ the Hurewicz homomorphism*

$$\Phi: \pi_n(S^n, *) \rightarrow H_n(S^n) \cong \mathbb{Z}$$

⁶⁰¹To simplify the notation we denote the induced map by the same symbol as the map between pairs of topological spaces, i.e. we drop the subscript $*$ from the notation.

sends $[\text{id}_{S^n}: S^n \rightarrow S^n] \in \pi_n(S^n)$ to generator of $H_n(S^n)$, in particular the Hurewicz homomorphism is epimorphism.

Proof. We denote by $\omega \in H_n(S^n)$ the standard generator. By definition of the Hurewicz homomorphism we have

$$\Phi(\underbrace{[\text{id}_{S^n}: S^n \rightarrow S^n]}_{\in \pi_n(S^n, *)}) = (\text{id}_{S^n})_*([\omega]) = [\omega].$$

Thus we see that the generator of $H_n(S^n) \cong \mathbb{Z}$ lies in the image of the Hurewicz homomorphism. But this implies that the Hurewicz homomorphism $\Phi: \pi_n(S^n, *) \rightarrow H_n(S^n)$ is an epimorphism. $\square$

It is natural to ask whether the Hurewicz homomorphism

$$\pi_n(X, x_0) \rightarrow H_n(X)$$

is always an isomorphism for $n \geq 2$. But browsing through our examples shows that the Hurewicz homomorphism is in general neither a monomorphism nor an isomorphism:

- (1) In Lemma 28.10 we had shown that $H_2(T) \cong \mathbb{Z}$ is non-zero whereas on page 389 we had shown that $\pi_2(T) = 0$. This shows that the Hurewicz homomorphism $\pi_2(T) \rightarrow H_2(T)$ is not an epimorphism.
- (2) We consider $X = S^1 \vee S^2$. It follows immediately from Propositions 26.3 and 28.8 that $H_2(S^1 \vee S^2) \cong \mathbb{Z}$. On the other hand we know by Proposition 23.10 and the discussion on page 265 that $\pi_2(S^1 \vee S^2)$ is isomorphic to $\pi_2(\tilde{X})$ where $\tilde{X}$ is the real line “with infinitely 2-spheres attached”, see e.g. Figure 242 for an illustration. Using the projections onto the various spheres and using Lemma 32.9 one can now show that $\pi_2(\tilde{X})$ admits an epimorphism onto $\mathbb{Z}^{(\mathbb{N})}$.⁶⁰² It is now clear that there is no monomorphism from $\pi_2(X) = \pi_2(\tilde{X})$ into $H_2(X) \cong \mathbb{Z}$. In particular the Hurewicz homomorphism, whatever it is precisely, cannot be a monomorphism.

⁶⁰²The proof of this statement is a slight variation on the proof of exercise 1 (a) in exercise sheet 3.

33. SIMPLICIAL COMPLEXES AND HOMOLOGY GROUPS OF MANIFOLDS

In this section we introduce the notion of a simplicial complex. Simplicial complexes can be viewed as special classes of CW-complexes with particularly nice properties. We will then see that every manifold admits the structure of a simplicial complex and we will use this fact to obtain several results on homology groups of manifolds. The eternal problem in topology is to strike the right balance between rigor and geometric intuition. In this chapter we will at times be slightly less rigorous than in the other chapters.

Later on, in Algebraic Topology III we will reprove most of the results from this section more generally for topological manifolds. These proofs will then be very rigorous but much less intuitive.

33.1. Simplicial complexes. In the following we recall several definitions from page 398.

Definition.

- (1) Given $n \in \mathbb{N}_0$ we refer to

$$\Delta^n := \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid x_0 + \dots + x_n = 1 \text{ and } x_i \geq 0 \text{ for all } i = 0, \dots, n\}$$

as the *standard n -simplex*.

- (2) A *face* of codimension r of the standard n -simplex is defined as a subset of the form

$$\{(t_0, \dots, t_n) \in \Delta^n \mid t_{d_1} = t_{d_2} = \dots = t_{d_r} = 0\},$$

where $d_1 < d_2 < \dots < d_r \in \{0, \dots, n\}$ are fixed. We refer to Figure 251 for an illustration.

- (3) The boundary $\partial\Delta^n$ of the standard n -simplex is defined as the union of all faces of Δ^n of codimension 1.

We also introduce one new definition:

- (4) We refer to

$$\overset{\circ}{\Delta}^n := \Delta^n \setminus \partial\Delta^n$$

as the *interior* of Δ^n . This is a submanifold of $\mathbb{R}^{n+1}$ and we equip it with the orientation which is given by the rule that a basis $\{v_1, \dots, v_n\}$ for $T_P \overset{\circ}{\Delta}^n$ is positive if the basis $\{v_1, \dots, v_n, (1, \dots, 1)\}$ is a positive basis for $\mathbb{R}^{n+1}$.

- (5) Let F be the face Δ^n that is given by setting $t_i = 0$. We equip the interior of F with the orientation which is given by the rule that a basis $\{v_1, \dots, v_{n-1}\}$ for $T_P F$ is positive if the basis $\{v_1, \dots, v_{n-1}, e_{i+1}, (1, \dots, 1)\}$ is a positive basis for $\mathbb{R}^{n+1}$.

Remark. The homeomorphism $\Delta^n \rightarrow S_{\geq 0}^n = \{(x_1, \dots, x_{n+1}) \in S^n \mid x_{n+1} \geq 0\}$ that is given in page 459 and that is illustrated in Figure 278 restricts to an orientation-preserving diffeomorphism $\overset{\circ}{\Delta}^n \rightarrow S_{> 0}^n$.

Next we introduce the notion of a simplicial precomplex.

Definition.

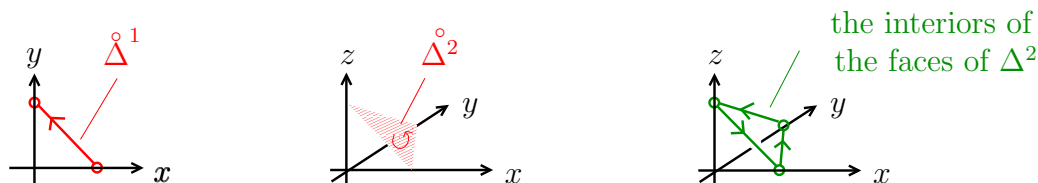


FIGURE 326.

- (1) A *simplicial precomplex* is a topological space X that is Hausdorff together with a family of injective maps $\varphi_i: \Delta^{n_i} \rightarrow X$, $i \in I$.⁶⁰³
- (2) We refer to the image of a map $\varphi_i: \Delta^{n_i} \rightarrow X$ as an n_i -dimensional *simplex* of X .⁶⁰⁴ The image of a face of Δ^{n_i} is called a *face of the simplex*, the image of $\partial\Delta^{n_i}$ is called the *boundary of the simplex*, and the image of the interior $\overset{\circ}{\Delta}^{n_i}$ is called the *interior of the simplex*.
- (3) The *dimension* $\dim(X) \in \mathbb{N}_0 \cup \{\infty\}$ of a non-empty simplicial precomplex X is defined as the maximum dimension of a simplex.
- (4) A *subcomplex* of a simplicial precomplex X is defined as a union of simplices of X .

Now we can introduce the notion of a simplicial complex, which is what we are really interested in.

Definition. A *simplicial complex* is a simplicial precomplex $(X, \{\varphi_i: \Delta^{n_i} \rightarrow X\}_{i \in I})$ such that the following conditions are satisfied:

- (1) For each $x \in X$ there exists a simplex such that x lies in the interior of the simplex.
- (2) The face of any simplex of X is again a simplex of X .
- (3) If the images of φ_i and φ_j intersect, then there exist faces F_i of Δ^{n_i} and F_j of Δ^{n_j} of codimension ≥ 1 such that the following two conditions are satisfied:
 - (a) we have $\varphi_i(\Delta^{n_i}) \cap \varphi_j(\Delta^{n_j}) = \varphi_i(F_i) = \varphi_j(F_j)$,
 - (b) the map $\varphi_j^{-1} \circ \varphi_i: F_i \rightarrow F_j$ is an affine homeomorphism.⁶⁰⁵
- (4) A subset $U \subset X$ is open if and only if for every $i \in I$ the preimage $\varphi_i^{-1}(U)$ is open in Δ^{n_i} .⁶⁰⁶

⁶⁰³Note that it follows from Proposition 1.3 that each map $\varphi_i: \Delta^{n_i} \rightarrow X$ is a homeomorphism onto its image.

⁶⁰⁴It follows from Footnote 603 together with Proposition 26.35 that the definition of the dimension of a simplex is well-defined. More precisely, it follows that for two injective maps $\varphi: \Delta^k \rightarrow X$ and $\psi: \Delta^l \rightarrow X$ with $k \neq l$ the images have to be different.

⁶⁰⁵Note that F_i respectively F_j are subsets of $\mathbb{R}^{n_i}$ respectively $\mathbb{R}^{n_j}$. An *affine homeomorphism* between F_i and F_j is an affine, linear bijective map from F_i to F_j , i.e. it is a map of form $v \mapsto A \cdot v + P$ for some invertible matrix A and vector P that restricts to a bijection from F_i to F_j .

⁶⁰⁶From the continuity of the maps φ_i , $i \in I$ it follows that if $U \subset X$ is open, then for all $i \in I$ the preimage $\varphi_i^{-1}(U)$ is open in Δ^{n_i} . The statement of (4) is thus that the converse also holds.

Put slightly more naively, a simplicial complex is a topological space which is “built out of simplices glued along faces”. We refer to Figure 327 for an illustration of examples and non-examples of simplicial complexes.

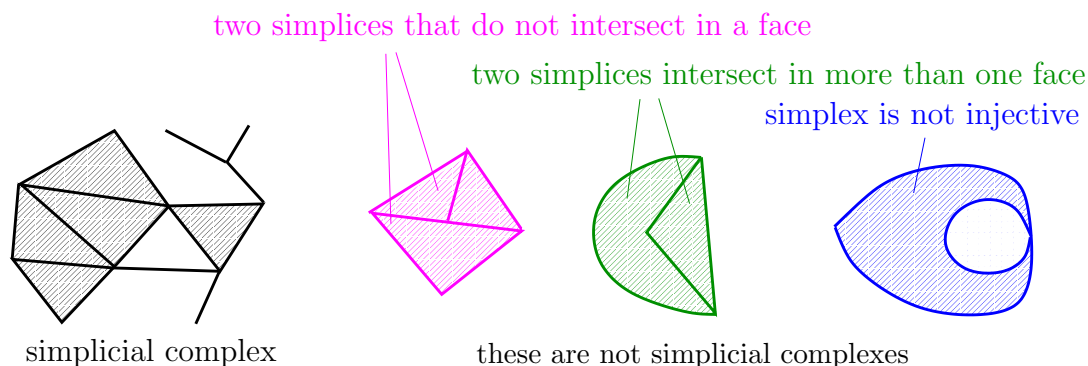


FIGURE 327.

We leave the straightforward proof of the following lemma as an exercise to the reader.

Lemma 33.1. *Let X be a simplicial complex.*

- (1) *For each $x \in X$ there exists a unique simplex such that x lies in the interior of the simplex.*
- (2) *The boundary of an n -simplex is contained in a subcomplex of dimension $\leq n - 1$.*

Recall that in Lemma 1.4, that given any $n \in \mathbb{N}_0$, we had written down a specific homeomorphism $(\overline{B}^n, S^{n-1}) \rightarrow (\Delta^n, \partial\Delta^n)$. If we now precompose the maps $\Delta^n \rightarrow X$ of a simplicial complex with these homeomorphisms we easily obtain the following lemma from Lemma 33.1 (2). We leave it as an exercise to the reader to fill in all the details.

Lemma 33.2. *Any simplicial complex can also be viewed as CW-complex where the n -simplices of the simplicial complex correspond to the n -cells of the CW-complex.*

It is straightforward to see that every 1-dimensional CW-complex is also a 1-dimensional simplicial complex. On the other hand not every CW-complex is necessarily a simplicial complex. For example, if $n \geq 2$ and if we consider the CW-complex with one 0-cell and one n -cell, then this CW-complex is not a simplicial complex.

33.2. Homology groups of manifolds.

Definition. Let X be a Hausdorff space. A *simplicial structure* for X consists of a family of injective maps $\varphi_i: \Delta^{n_i} \rightarrow X$, $i \in I$ such that $(X, \{\varphi_i\}_{i \in I})$ is a simplicial complex.⁶⁰⁷

The following proposition is proved in [Mu2, Theorem 10.6].

⁶⁰⁷In the literature a simplicial structure is also often called a *triangulation*.

Proposition 33.3. *Let M be an n -dimensional manifold. Then admits M an n -dimensional simplicial structure $\{\varphi_i: \Delta^{n_i} \rightarrow M\}_{i \in I}$ with the following properties:*

- (1) *for each i the restriction of φ_i to $\varphi_i: \overset{\circ}{\Delta}^{n_i} \rightarrow M$ is an embedding,⁶⁰⁸*
- (2) *the boundary ∂M is a subcomplex.*

As we had just pointed out in Lemma 33.2, every simplicial complex is also a CW-complex. Thus we can view Proposition 33.3 as a strengthening of Theorem 22.2.

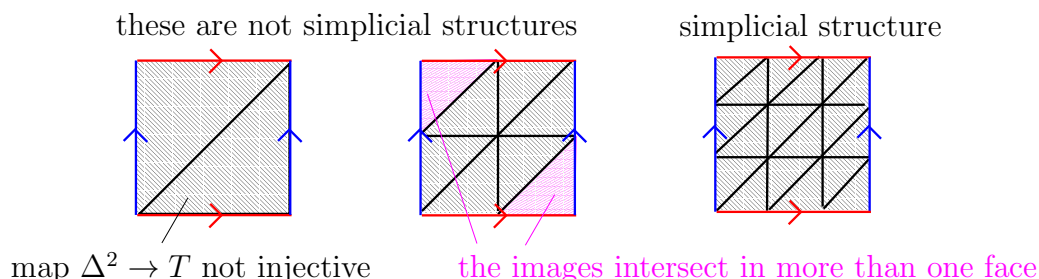


FIGURE 328. The torus as simplicial complex.

Remark. A simplicial structure for a surface can be viewed as an approximation by triangles. Such simplicial structures play an important role in computer visualizations. In this context one usually talks of triangulations. We refer to Figure 329 for more illustrations.

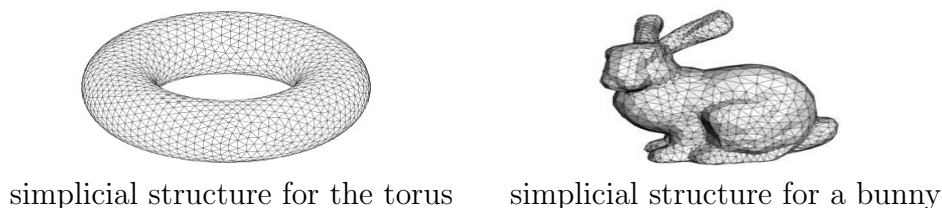


FIGURE 329.

Definition. Let X be a finite simplicial complex and let c be a k -simplex of X . We say c has *order* r , if there exist precisely r simplices of dimension $k + 1$ that contain c in the boundary.

Lemma 33.4. *Let M be an n -dimensional manifold that is equipped with a simplicial structure and let c be a $(n - 1)$ -dimensional simplex of M . Then one of the following holds:*

- (1) *Either c lies in the boundary of M , in this case the order of c equals one,*
- (2) *or the interior c lies in the interior of M , in this case the order is equal to two.*

⁶⁰⁸Note that given n the standard-simplex $\Delta^n \subset \mathbb{R}^{n+1}$ is not a submanifold of $\mathbb{R}^{n+1}$ (why not?) but the interior $\overset{\circ}{\Delta}^n$ is an n -dimensional submanifold of $\mathbb{R}^{n+1}$.

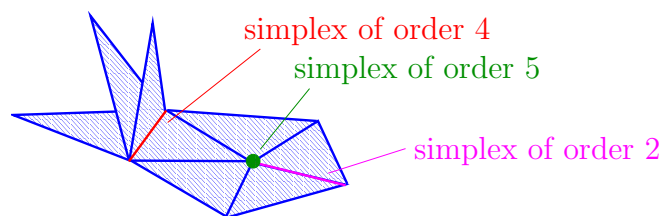


FIGURE 330.

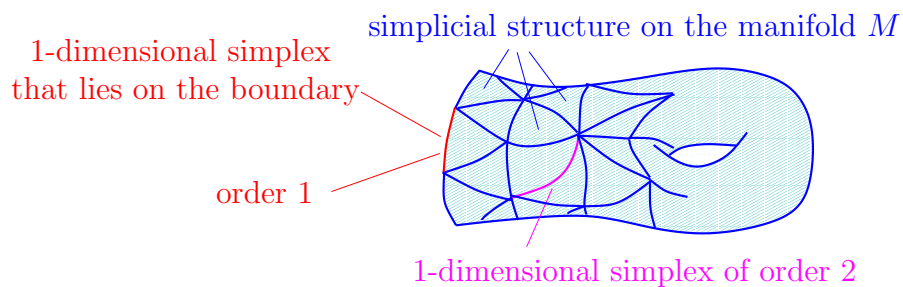


FIGURE 331. Illustration of Lemma 33.4.

Sketch of a proof for Lemma 33.4. Let M be an n -dimensional manifold that is equipped with a simplicial structure and let c be a $(n - 1)$ -dimensional simplex of M . We start out with the following claim.

Claim. Let P be a point in the interior of c .

- (1) If P lies on the boundary of M , then c has order one.
- (2) If P lies in the interior of M , then c has order two.

Suppose that c has order k . We write

$$X_k := \{(z, w) \mid z = re^{2\pi ij/k}, r \in [0, 1), j \in \{0, \dots, k-1\} \text{ and } w \in \mathbb{R}^{n-2} \text{ with } \|w\| < 1\}.$$

We sketch X_k for $k = 1, 2, 3$ in Figure 332. Since P lies in the interior of c it follows⁶⁰⁹

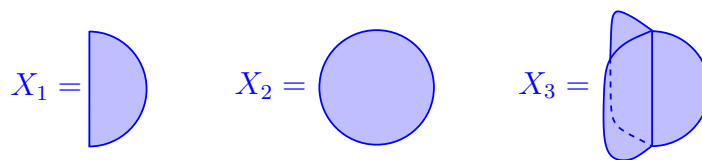


FIGURE 332.

that there exists an open neighborhood U of P and a homeomorphism $\Phi: U \rightarrow X_k$ with

⁶⁰⁹This step requires some thought.

$\Phi(P) = 0$. We now see that

$$\begin{array}{ccccccc}
 H_{n-1}(M, M \setminus \{P\}) & \cong & H_{n-1}(U, U \setminus \{P\}) & \xrightarrow{\Phi} & H_{n-1}(X_k, X_k \setminus \{0\}) & \cong & \mathbb{Z}^{k-1}. \\
 \uparrow & & & & & \uparrow & \\
 & & \text{Excision Theorem 26.15} & & & \text{see exercise sheet 10} &
 \end{array}$$

The desired statement now follows from Lemma 26.33. This concludes the proof of the claim.

Now we turn to the actual proof of the lemma. First suppose that the interior of c contains a point on the boundary of M . Then it follows from the claim that the order of c equals one. Since ∂M is a subcomplex of M it follows from Lemma 33.1 (1) that all of c is contained in the boundary of M .

Now suppose that the interior of c does not contain a point that lies on the boundary of M , i.e. we suppose that the interior c lies in the interior of M . It follows from the claim that the order of c equals two. $\square$

In Proposition 29.10 we had already seen that for a given n -dimensional compact manifold M all homology groups are finitely generated abelian groups and that for $k > n$ we have $H_k(M) = 0$. Now we can prove the following proposition which gives us more information on homology groups of manifolds.⁶¹⁰

Proposition 33.5. *Let M be a connected n -dimensional manifold. Then the following holds⁶¹¹*

$$H_n(M) \cong \begin{cases} \mathbb{Z}, & \text{if } M \text{ is orientable and closed,} \\ 0, & \text{otherwise.} \end{cases}$$

Examples.

- (1) In Question 1.35 we had asked whether $\mathbb{RP}^{2n}$ is orientable. We can now answer this question in the negative. More precisely, it follows from Proposition 33.5 and the calculation of the homology of real projective spaces in Lemma 29.19 that $\mathbb{RP}^{2n}$ is non-orientable.
- (2) On page 99 we had pointed out that every lens space $L(p, q)$ is an orientable closed compact 3-dimensional manifold. It follows from Proposition 33.5 that $H_3(L(p, q)) \cong \mathbb{Z}$. Unfortunately the calculation does not help us in our goal of answering Question 6.7.

*Sketch of a proof for Proposition 33.5.*⁶¹² Let M be a connected n -dimensional manifold. We pick an n -dimensional simplicial structure $\{\varphi_i: \Delta^{n_k} \rightarrow M\}_{k \in K}$ for M as in Proposition 33.3. We now introduce some notation and conventions.

⁶¹⁰Note that Proposition 33.5 is consistent with the calculation of H_2 of 2-dimensional manifolds that we had given in Proposition 29.14.

⁶¹¹Recall that a manifold is called closed if it is compact and if it has no boundary.

⁶¹²A careful reading of the proof shows that the argument has several little gaps. We leave it as an exercise to find all gaps and to fill them. Alternatively you can wait for a later chapter.

- (1) We denote by $I \subset K$ the subset corresponding to the $(n-1)$ -simplices. Given $i \in I$ we write $X_i = \varphi_i(\Delta^{n-1})$.
- (2) Similarly we denote by $J \subset K$ the subset corresponding to the n -simplices of M . Given $j \in J$ we write $Y_j = \varphi_j(\Delta^n)$.
- (3) By Lemma 33.2 we can view M as a CW-complex, where the cells correspond to the simplices.
- (4) It follows from Proposition 22.6 and (3) that M is compact if and only if I is finite.⁶¹³
- (5) Given any k we view $\overset{\circ}{\Delta}^k$ and we equip it with the orientation which is given by the rule, that a basis $\{v_1, \dots, v_k\}$ for $T_P \overset{\circ}{\Delta}^k$ is positive if the basis $\{v_1, \dots, v_k, (1, \dots, 1)\}$ is a positive basis for $\mathbb{R}^{k+1}$.
- (6) Using the orientations from (5) and the maps φ_i and ψ_j we view the interiors of $X_i, i \in I$ as oriented $(n-1)$ -dimensional submanifolds of M and we view the interiors of $Y_j, j \in J$, as oriented n -dimensional submanifolds of M .⁶¹⁴
- (7) Given $i \in I$ and $j \in J$ we denote by a_{ij} the coefficient of X_i in $d_n(Y_j)$ where $d_n: C_n^{\text{CW}}(M) \rightarrow C_{n-1}^{\text{CW}}(M)$ denotes the cellular boundary map.

We start out with the following observation.

$$(*) \quad \begin{array}{ccc} H_n(M) & \cong & H_n^{\text{CW}}(M) = \ker \{C_n^{\text{CW}}(M) \xrightarrow{\partial} C_{n-1}^{\text{CW}}(M)\} = \text{cycles in } C_n^{\text{CW}}(M). \\ \uparrow & & \uparrow \\ \text{Proposition 29.8} & & M \text{ has no simplices of dimension } n+1, \text{ hence } C_{n+1}^{\text{CW}}(M) = 0 \end{array}$$

We continue with the following claim which is the key ingredient to the proof of Proposition 33.5.

Claim. Let $i \in I$. Then the following hold:

- (1) If the simplex X_i lies on the boundary of M , then there exists precisely one j with $a_{ij} \neq 0$ and we have $a_{ij} = \pm 1$.
- (2) If the simplex X_i does not lie the boundary of M , then there exist precisely two j_1, j_2 with $a_{i,j_1} \neq 0$ and $a_{i,j_2} \neq 0$. Furthermore we have $a_{i,j_1} = \pm 1$ and $a_{i,j_2} = \pm 1$. Finally, we have $a_{i,j_1} = -a_{i,j_2}$ if and only if the orientations of Y_{j_1} and Y_{j_2} agree along X_i .

Most of the statements of the claim follow immediately from Lemma 33.4 and from the definitions. It only remains to prove the statement regarding the orientations.

So suppose that an $(n-1)$ -simplex X_i lies on the face F of an n -simplex Y_j . We denote by $\varphi: \Delta^{n-1} \rightarrow M$ and $\psi: \Delta^n \rightarrow M$ the corresponding maps from the simplicial structure. An elementary, albeit painful, argument shows that, with the sign conventions from page 553, we have the following equalities:

$$a_{ij} = \begin{cases} +1, & \text{if } \varphi^{-1} \circ \psi: F \rightarrow \Delta^{n-1} \text{ is orientation-preserving,} \\ -1, & \text{if } \varphi^{-1} \circ \psi: F \rightarrow \Delta^{n-1} \text{ is orientation-reversing.} \end{cases}$$

⁶¹³Here we use the fact that in an n -dimensional manifold every point lies in an n -simplex. Why is that?

⁶¹⁴Here we use that the restrictions of the maps φ_i to Δ^{n_i} are embeddings.

Now suppose we have two simplices Y_{j_1} and Y_{j_2} that meet along X_i . Then it is straightforward to see that the orientations of Y_{j_1} and Y_{j_2} agree along X_i if and only if they induce the opposite orientations on X_i . We refer to Figure 333 for an illustration. This concludes the proof of the claim.

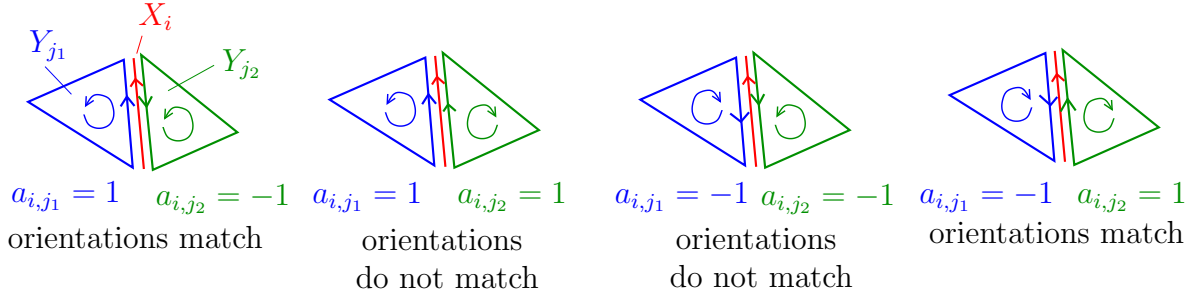


FIGURE 333.

Now we suppose that M is orientable and closed. Since M is compact we know that I is finite. We choose an orientation for M . Given $i \in I$ we define

$$\epsilon_i := \begin{cases} +1, & \text{if the orientations of } M \text{ and } Y_i \text{ match,} \\ -1, & \text{otherwise.} \end{cases}$$

It follows immediately from the claim that

$$c := \sum_{i \in I} \epsilon_i \cdot \psi_i \in C_n^{\text{CW}}(M)$$

is an n -cycle in $C_n^{\text{CW}}(M)$. Now suppose that we are given another cycle $d = \sum_{i \in I} d_i \cdot \psi_i$. We pick $i_0 \in I$ and we define $m \in \mathbb{Z}$ via $d_{i_0} = m \cdot \epsilon_{i_0}$. We set $I' := \{i \in I \mid d_i = m \cdot \epsilon_i\}$. It follows from the claim and the hypothesis that M is connected that $I' = I$. Thus we see that $d = m \cdot c$. Together with the above observation (*) we now see that $H_n(M)$ is isomorphic to $\mathbb{Z}$. This concludes the proof if M is orientable and closed.

Now suppose that $H_n(M) \neq 0$. We need to show that M is orientable and closed. By the observation (*) there exists a non-zero cycle $d = \sum_{i \in I} d_i \cdot \psi_i$. We pick $i_0 \in I$ with $a_{i_0} \neq 0$. We see $I' = \{i \in I \mid |a_i| = |a_{i_0}|\}$. It follows from the above claim that $M' = \bigcup_{i \in I'} Y_i$ is a closed submanifold of M . Since M is connected we see that $M' = M$. But this implies that M is closed.

Given $i = 0, \dots, n$ we denote by M^n the union of all n -simplices. We can now orient M as follows:

- (1) Given a point $P \in \overset{\circ}{y}_i$ we now endow it with the orientation of Y_i if $a_i > 0$, otherwise we endow it with the opposite orientation. This defines an orientation on $M^n \setminus M^{n-1}$.
- (2) It follows again from the claim that the orientations from (1) extend to orientations on the interiors of $(n-1)$ -simplices. This defines an orientation on $M^n \setminus M^{n-2}$.

- (3) Finally let P be a point on M^{n-2} . We pick a chart $\Phi: U \rightarrow V$ around P such that V is an open ball. Since $M^{n-2} \cap U$ is of codimension 2 we see that $U \setminus (U \cap M^{n-2})$ is connected.⁶¹⁵ It follows that Φ is either orientation-preserving for all $Q \in U \setminus (U \cap M^{n-2})$ or Φ is orientation-reversing for all $Q \in U \setminus (U \cap M^{n-2})$. We can now use Φ to define an orientation for all points in U that agrees with the given orientation on $U \setminus (U \cap M^{n-2})$. $\square$

In a very similar fashion to Proposition 33.5 we can also prove the following proposition.

Proposition 33.6. *Let M be a connected n -dimensional manifold. Then the following holds:*

$$H_n(M, \partial M) \cong \begin{cases} \mathbb{Z}, & \text{if } M \text{ is orientable and compact,} \\ 0, & \text{otherwise.} \end{cases}$$

We have just seen that for a connected, orientable, compact n -dimensional manifold M we have $H_n(M, \partial M) \cong \mathbb{Z}$. In the following we will see that a choice of an orientation determines a generator $[M]$ of $H_n(M, \partial M)$.

In Lemma 24.1 we had given a homeomorphism $\Phi: \Delta^n \rightarrow \overline{B}^n$. We recall that in Lemma 26.29 we had shown that id_{Δ^n} represents a generator of $H_n(\Delta^n, \partial \Delta^n)$. We now write

$$[\overline{B}^n] := [\text{id}_{\Delta^n}].$$

We can now formulate the following proposition.

Proposition 33.7. *Let M be connected, oriented, compact n -dimensional manifold. Then there exists a unique generator $[M]$ of $H_n(M, \partial M)$ with the following property: For any orientation-preserving embedding $\Phi: \overline{B}^n \rightarrow M$ with $\Phi(\overline{B}^n) \subset M \setminus \partial M$ we have*

$$\Phi([\overline{B}^n]) = \text{the image of } [M] \text{ under the map } H_n(M, \partial M) \rightarrow H_n(M, M \setminus \Phi(\overline{B}^n)).$$

Proof. To simplify the notation we prove the proposition only in the closed case. We pick a simplicial structure for M and we define the cycle $c \in C_n^{\text{CW}}(M, \partial M)$ as in the proof of Proposition 33.5.

We pick an orientation preserving embedding map $\Phi: \overline{B}^n \rightarrow M$ such that $\Phi(\overline{B}^n)$ lies in the interior of an n -simplex y of M . We consider the following commutative diagram of

⁶¹⁵This statement requires an argument that we are skipping

homomorphisms:

$$\begin{array}{ccccc}
 H_n(\overline{B}^n, \partial \overline{B}^n) & & & & \\
 \cong \downarrow \Phi_* & & & & \\
 H_n(\Phi(\overline{B}^n), \Phi(\partial \overline{B}^n)) & \xrightarrow{\cong} & H_n(y, y \setminus \Phi(B^n)) & \xrightarrow{\cong} & H_n(M, M \setminus \Phi(B^n)) \longleftarrow H_n(M, \partial M) \\
 & \cong \uparrow (*) & & \cong \uparrow & \cong \uparrow \\
 & H_n(y, \partial y) & \xrightarrow{\cong} & H_n(M, M \setminus \overset{\circ}{y}) & \longleftarrow H_n(M, \partial M) \\
 & \cong \uparrow & & \cong \uparrow & \cong \uparrow \\
 & H_n^{\text{CW}}(y, \partial y) & \xrightarrow{\cong} & H_n^{\text{CW}}(M, M \setminus \overset{\circ}{y}) & \longleftarrow H_n^{\text{CW}}(M, \partial M). \\
 & & \xleftarrow{[c] \mapsto [y]} & &
 \end{array}$$

Here are the justification for some of the arrows and the claim that the diagram commutes:

- (1) The horizontal isomorphisms on the right are given by the Excision Theorem 26.15.
- (2) The bottom vertical maps are the isomorphisms from Proposition 29.8.
- (3) The lower two squares commute by the remark on page 548.
- (4) It follows from the long exact sequence in homology of the triple $(y, y \setminus \Phi(B^n), \partial y)$, the fact that ∂y is a deformation retract of $y \setminus \Phi(B^n)$ and Corollary 25.5 (4) that the homomorphism $(*)$ is an isomorphism.
- (5) We denote by $[y] \in H_n^{\text{CW}}(y, \partial y)$ the generator which corresponds to the one n -cell of y .
- (6) It follows immediately from the definition of c that the image of $[c]$ under the map $H_n^{\text{CW}}(M, \partial M) \rightarrow H_n^{\text{CW}}(y, \partial y)$ is precisely $[y]$.

The upshot is that $[c]$ gets sent to a generator of $H_n(M, M \setminus \Phi(B^n))$ and $[\overline{B}^n]$ also gets sent to a generator. Hence there exists an $\epsilon \in \{-1, 1\}$ such that $\epsilon \cdot [c]$ and $[\overline{B}^n]$ get sent to the same element. We now claim that

$$[M] := \epsilon \cdot [c]$$

has the desired property.

We have already shown that $[M]$ has the desired property for the given map Φ . Now let $\Psi: \overline{B}^n \rightarrow M$ be another such map. By Theorem 2.8 there exists a diffeomorphism f of M with the following two properties:

- (1) the map f is homotopic rel ∂M to the identity,
- (2) $f \circ \Phi = \Psi$.

It follows from (1) and from Proposition 25.4 that the map $f_*: H_n(M, \partial M) \rightarrow H_n(M, \partial M)$ is the identity. We consider the following commutative diagram

$$\begin{array}{ccccc}
 & & H_n(M, M \setminus \Phi(\overline{B}^n)) & \longleftarrow & H_n(M, \partial M) \\
 & \nearrow \Phi_* & \downarrow f_* & & \downarrow f_* = \text{id} \\
 H_n(\overline{B}^n, \partial \overline{B}^n) & & \begin{array}{c} \Phi_*([\overline{B}^n]) \\ \downarrow \\ \Psi_*([\overline{B}^n]) \end{array} & & \begin{array}{c} [M] \\ \downarrow \\ [M] \end{array} \\
 & \searrow \Psi_* & \downarrow & & \\
 & & H_n(M, M \setminus \Phi(\overline{B}^n)) & \longleftarrow & H_n(M, \partial M).
 \end{array}$$

It now follows immediately from this diagram that $[M]$ also has the desired property for Ψ . $\square$

Definition. Let M be connected, oriented, compact n -dimensional manifold. We refer to the uniquely determined element $[M] \in H_n(M, \partial M)$ from Proposition 33.7 as the *fundamental class* of M .

Example. We endow S^n with the usual orientation, i.e given $P \in S^n$ and a basis $v_1, \dots, v_n$ of $T_P M$ we say it is a positive basis if $P, v_1, \dots, v_n$ is a positive basis for $\mathbb{R}^{n+1}$. We denote by $\Omega: \overline{B}^n/S^{n-1} \rightarrow S^n$ the homeomorphism from page 23. Valiantly struggling through the definitions shows that $[S^n]$ equals the image of $[\overline{B}^n] \in H_n(\overline{B}^n, S^{n-1})$ under the composition of the maps $H_n(\overline{B}^n, S^{n-1}) \rightarrow H_n(\overline{B}^n/S^{n-1}) \xrightarrow{\Omega_*} H_n(S^n)$.

Given an oriented manifold M we denote, as usual, by $-M$ the same manifold but with opposite orientation. We can now formulate the following lemma that will be proved in exercise sheet 8.

Lemma 33.8. *For any connected, oriented, compact n -dimensional manifold M we have*

$$[-M] = -[M] \in H_n(M).$$

Let M be a compact n -dimensional manifold with boundary. We had already seen that ∂M is a closed $(n-1)$ -dimensional manifold. If M is oriented, then we had endowed ∂M in Analysis IV with an orientation as follows: Let $P \in \partial M$. We denote by $w \in T_P M$ the normal vector to ∂M which points into the “interior” of M . Now let $v_1, \dots, v_{n-1}$ be a basis of $T_P(\partial M)$. We say that $v_1, \dots, v_{n-1}$ is a positive basis of $T_P(\partial M)$ if $w, v_1, \dots, v_{n-1}$ is a positive basis of $T_P M$. We can now formulate the next proposition.

Proposition 33.9. *Let M be a compact oriented n -dimensional manifold with boundary components $N_1, \dots, N_k$. Then the boundary map*

$$\partial: H_n(M, \partial M) \rightarrow H_{n-1}(\partial M) = \bigoplus_{i=1}^k H_{n-1}(N_i)$$

of the long exact sequence in homology of the pair $(M, \partial M)$ has the property that

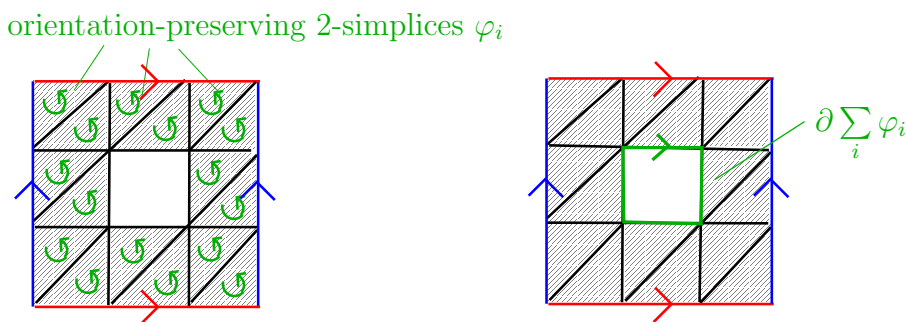
$$[M] \mapsto [N_1] + \dots + [N_k].$$

Sketch of proof. By Proposition 33.3 the manifold M admits a simplicial structure such that ∂M is a subcomplex. We now use this simplicial structure to define the n -cycle c with $[c] = [M]$. By restricting c to the faces that lie on ∂M we obtain cycles $d_i \in C_{n-1}^{\text{CW}}(N_i)$, $i = 1, \dots, k$. As in the proof of Propositions 33.3 Now we consider the following diagram:

$$\begin{array}{ccc}
 H_n^{\text{CW}}(M) & \xrightarrow{[c] \mapsto [d_1] + \dots + [d_k]} & \bigoplus_{i=1}^k H_{n-1}^{\text{CW}}(N_i) \\
 \downarrow [c] \mapsto [M] \cong & & \cong \downarrow [d_i] \mapsto [N_i] \\
 H_n(M) & \xrightarrow{\quad} & \bigoplus_{i=1}^k H_{n-1}(N_i).
 \end{array}$$

This diagram commutes by the remark on page 548. The proposition is now an immediate consequence. $\square$

Example. In Figure 334 we consider the surface F that is given removing an open disk from a torus. We illustrate a simplicial structure with oriented 2-simplices. The sum of the oriented 2-simplices is a generator of $H_2(F)$, and the boundary of the sum is a generator of $H_1(\partial F)$.



surface of genus one with one boundary component viewed as a simplicial complex

FIGURE 334.

For the record we state the following elementary corollary to Proposition 33.9.

Corollary 33.10. *Let M be a compact oriented n -dimensional manifold with one boundary component. Then the inclusion induced map*

$$i_*: H_{n-1}(\partial M) \rightarrow H_{n-1}(M)$$

is the zero map.

Proof. We consider the following piece of the long exact sequence in homology of the pair $(M, \partial M)$:

$$H_n(M, \partial M) \xrightarrow{[M] \mapsto [\partial M]} \underbrace{H_{n-1}(\partial M)}_{=\mathbb{Z} \cdot [\partial M]} \xrightarrow{i_*} H_{n-1}(M).$$

Since ∂M is by hypothesis connected it follows from Proposition 33.9 that the map on the left is an epimorphism. By the exactness of the sequence it follows that the map on the right is the zero map. $\square$

The following proposition gives us some extra information about homology groups of manifolds with boundary.

Proposition 33.11. *Let M be a compact n -dimensional manifold with non-empty boundary. Then M is homotopy equivalent to a $(n - 1)$ -dimensional CW-complex with finitely many cells. In particular $H_{n-1}(M)$ is a free abelian group.*

Examples.

- (1) The Möbius band and the annulus are both homotopy equivalent to S^1 .
- (2) Given $g \in \mathbb{N}_0$ and $k \in \mathbb{N}_0$ we denote as usual by $\Sigma_{g,k}$ the surface of genus g with k open disks removed. In Figure 335 we sketch a deformation retraction from $\Sigma_{2,2}$ to the graph of Euler characteristic -4 .

More generally, almost the same argument shows that $\Sigma_{g,k}$ with $k \geq 1$ admits a deformation retraction to a graph with k vertices and $2g + 2(k - 1)$ edges, i.e. a graph of Euler characteristic $k - (2g + 2(k - 1)) = -2g - k + 2$. Using Proposition 7.8 we also see that $\Sigma_{g,k}$ is homotopy equivalent to the wedge of $2g + k - 1$ circles.

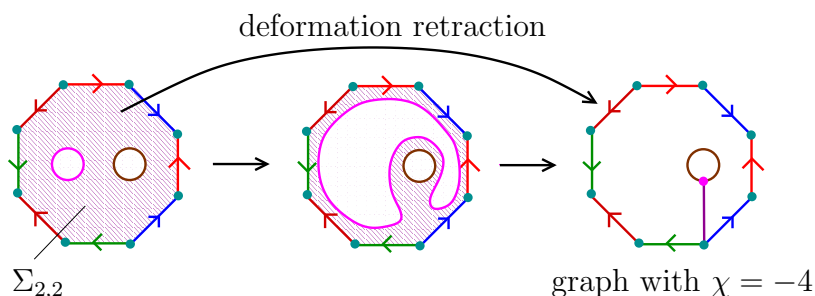


FIGURE 335.

Proof. Let M be a compact n -dimensional manifold with non-empty boundary. By Proposition 33.3 we know that M admits a simplicial structure such that ∂M is a subcomplex. From Lemma 33.4 it follows that M contains an $(n - 1)$ -simplex of order one.

Given a subcomplex W of M we denote by $n(W)$ the number of n -simplices in W . We can now formulate the following claim.

Claim. Suppose V is a subcomplex of M with $n(W) > 0$ that contains at least one $(n-1)$ -simplex of order one. Then there exists a subcomplex W of V with the following three properties:

- (1) W is a deformation retract of V ,
- (2) $n(W) < n(V)$,
- (3) the simplicial complex W contains at least one $(n-1)$ -simplex of order one. For each k there exists a simplicial complex U in M with k simplices the dimension n such that U is a deformation retract of M and such that U is an $(n-1)$ -simplex the order one contains.

Let V be a subcomplex of M with $n(W) > 0$ that contains a $(n-1)$ -simplex c of order one. Let d be the n -dimensional simplex of W with $c \subset \partial d$. We denote by c' the union of all faces of d except for c . As on page 457 one can easily show that c' is a deformation retract of d . We denote by W the union of all simplices of U except for d . Since c' is a deformation retract of d we see that W is a deformation retract of V . Clearly we have $n(W) = n(V) - 1$. Finally note that all the $(n-1)$ -simplices in $c' \subset W$ have order one. This concludes the proof of the claim.

We apply this claim to M and obtain a sequence $M = V \supset V_1 \supset V_k$ of subcomplexes, where each subcomplex V_i has no n -simplices and where each is a deformation retract of M . It follows from Corollary 25.5 (4) and Proposition 29.9 that $H_{n-1}(M)$ is a free abelian group. $\square$

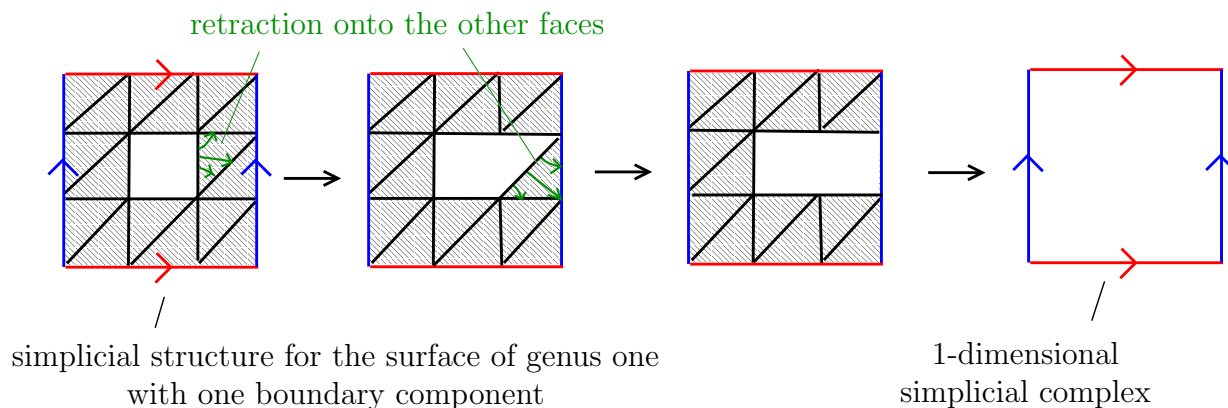


FIGURE 336. Illustration of the proof of Proposition 33.11.

Recall that on page 56 we had introduced the connected sum $M \# N$ of two oriented connected manifolds M and N . In Proposition 9.9 we had shown that $\pi_1(M \# N)$ is the free product of $\pi_1(M)$ and $\pi_1(N)$. We now have a very similar result for the homology groups of $M \# N$.

Proposition 33.12. *Let M and N be two closed, oriented, connected n -dimensional manifolds. Then for any $k \in \mathbb{N}_0$ we have*

$$H_k(M \# N) \cong \begin{cases} H_k(M) \oplus H_k(N), & \text{if } k \neq 0, n, \\ \mathbb{Z}, & \text{if } k = 0, n. \end{cases}$$

Moreover, for $k \neq 0, n$ the inclusion maps induce isomorphisms

$$H_k(M) \oplus H_k(N) \xleftarrow{\cong} H_k(M \setminus B^n) \oplus H_k(N \setminus B^n) \xrightarrow{\cong} H_k(M \# N).$$

+++++++= make the proof look more like the proof of Proposition 47.4 ++++++

Proof. Let M and N be oriented two closed, oriented, connected n -dimensional manifolds. We pick an orientation-preserving embedding $\varphi: \overline{B}^n \rightarrow M \setminus \partial M$ and we pick an orientation-reversing embedding $\psi: \overline{B}^n \rightarrow N \setminus \partial N$. Recall that

$$M \# N := (M \setminus \varphi(B^n)) \sqcup (N \setminus \psi(B^n)) / \{\varphi(P) = \psi(P) \mid P \in S^{n-1}\}.$$

We will prove the corresponding statement for reduced homology, i.e. we will show that for any $k \in \mathbb{N}_0$ we have

$$\tilde{H}_k(M \# N) = \begin{cases} \tilde{H}_k(M) \oplus \tilde{H}_k(N), & \text{if } k \neq n, \\ \mathbb{Z}, & \text{if } k = n. \end{cases}$$

To simplify the notation we write $S^{n-1} = \varphi(S^{n-1}) = \psi(S^{n-1})$ and similarly we write $B^n = \varphi(B^n) = \psi(B^n)$. It follows from Theorem 22.2 and from the Mayer-Vietoris Theorem 28.9 that there exists a long exact sequence

$$\rightarrow \tilde{H}_k(S^{n-1}) \xrightarrow{i \oplus -i} \tilde{H}_n(M \setminus B^n) \oplus \tilde{H}_n(N \setminus B^n) \xrightarrow{i+i} \tilde{H}_k(M \# N) \xrightarrow{\partial} \tilde{H}_{k-1}(S^{n-1}) \rightarrow \dots$$

where i always stands for the corresponding inclusion induced map. By Proposition 33.6 we have $\tilde{H}_n(M \setminus B^n) = \tilde{H}_n(N \setminus B^n) = 0$. Furthermore by Proposition 26.3 we have $\tilde{H}_n(S^n) \cong \mathbb{Z}$ and $\tilde{H}_k(S^n) = 0$ for $k \neq n$. The above Mayer-Vietoris sequence thus becomes

$$0 \rightarrow \tilde{H}_n(M \# N) \rightarrow \tilde{H}_{n-1}(S^{n-1}) \rightarrow \tilde{H}_{n-1}(M \setminus B^n) \oplus \tilde{H}_{n-1}(N \setminus B^n) \rightarrow \tilde{H}_{n-1}(M \# N) \rightarrow 0.$$

Since S^{n-1} is the unique boundary component of $M \setminus B^n$ we deduce from Corollary 33.10 that the inclusion induced map $\tilde{H}_{n-1}(S^{n-1}) \rightarrow \tilde{H}_{n-1}(M \setminus B^n)$ is the zero map. The same way we that the inclusion induced map $\tilde{H}_{n-1}(S^{n-1}) \rightarrow \tilde{H}_{n-1}(N \setminus B^n)$ is also the zero map.

It now follows from the above long exact sequence that $\tilde{H}_n(M \# N) \cong \tilde{H}_{n-1}(S^{n-1}) \cong \mathbb{Z}$. Furthermore it follows that for any $k < n$ the inclusion maps induce an isomorphism $\tilde{H}_k(M \setminus B^n) \oplus \tilde{H}_k(N \setminus B^n) \rightarrow \tilde{H}_k(M \# N)$.

But the same argument, applied to the decomposition $M = (M \setminus B^n) \cup \overline{B}^n$ and to the decomposition $N = (N \setminus B^n) \cup \overline{B}^n$ also shows that the inclusions induce isomorphisms

$$\widetilde{H}_k(M \setminus B^n) \oplus \overbrace{\widetilde{H}_k(\overline{B}^n)}^{=0} \xrightarrow{\cong} \widetilde{H}_k(M)$$

and

$$\widetilde{H}_k(M \setminus B^n) \oplus \underbrace{\widetilde{H}_k(\overline{B}^n)}_{=0} \xrightarrow{\cong} \widetilde{H}_k(M).$$

We have thus verified all the desired statements. $\square$

Examples.

(1) On page 501 we had shown that

$$H_k(\mathbb{CP}^2) \cong \begin{cases} \mathbb{Z}, & \text{if } k = 0, 2, 4, \\ 0, & \text{otherwise.} \end{cases}$$

It follows from Proposition 33.12 that

$$H_k(\mathbb{CP}^2 \# \mathbb{CP}^2) \cong \begin{cases} \mathbb{Z}, & \text{if } k = 0, 4, \\ \mathbb{Z} \oplus \mathbb{Z}, & \text{if } k = 2, \\ 0, & \text{otherwise.} \end{cases}$$

(2) Let $T = S^1 \times S^1$ be the torus. In Figure 35 we had seen that the connected sum $T \# T$ is the surface Σ_2 of genus 2. More generally, the same argument shows that the connected sum of g copies of the torus T is the surface Σ_g of genus g . Together with the calculation of the homology groups of the torus in Lemma 28.10 we now see that

$$H_k(\Sigma_g) \cong H_k(\underbrace{T \# \dots \# T}_{g \text{ copies}}) \cong \begin{cases} \mathbb{Z}, & \text{if } k = 0, 2, \\ \mathbb{Z}^{2g}, & \text{if } k = 1, \\ 0, & \text{if } k > 2. \end{cases}$$

This agrees of course with the result obtained in Proposition 29.14.

Proposition 33.13. *Let M be a closed, oriented, connected n -dimensional manifold. Then the following hold:*

(1) *If $\varphi: \overline{B}^n \rightarrow M$ is an embedding, then the inclusion induced map*

$$H_k(M \setminus \varphi(B^n)) \rightarrow H_k(M)$$

is an isomorphism for any $k < n$.

(2) *The homology group $H_{n-1}(M)$ is a free abelian group.*

Proof. The proof of the first statement is a slight variation on the proof of Proposition 33.12. We will do the proof in exercise sheet 10. The second statement is an immediate consequence of the first statement and of Proposition 33.11. $\square$

Remark. In this section we have obtained many results on the homology groups of manifolds. It is natural to ask whether the statements also hold for topological manifolds. In the above arguments we made use of the fact that manifolds have a simplicial structure. This is in general not the case for topological manifolds. In a later chapter we will reprove the above statements for topological manifolds using a different approach. This new approach will be more general, but also more abstract and less intuitive.

33.3. Representing homology classes by manifolds.

Definition. Let M be a manifold.

- (1) If N is a closed, connected, oriented submanifold of M , then we denote by $[N]$ the image of the fundamental class $[N]$ in $H_n(M)$ under the inclusion induced map $H_n(N) \rightarrow H_n(M)$.
- (2) We say that a homology class $z \in H_n(M)$ is represented by a submanifold if there exists an n -dimensional closed, connected, oriented submanifold N such that $[N] = z \in H_n(M)$.

Examples.

- (1) We first consider the torus $T = S^1 \times S^1$. Under the usual isomorphism $H_1(S^1 \times S^1) \cong \mathbb{Z}^2$ the basis element $(1, 0)$ is presented by the oriented submanifold $S^1 \times \{1\}$ and the basis element $(0, 1)$ is represented by the oriented submanifold $\{1\} \times S^1$. This situation is illustrated in Figure 338 on the left.

Next we consider the homology class $(2, 0)$. As shown in Figure 338 on the right we can represent it by a disconnected submanifold. But can we also represent it by a connected submanifold?

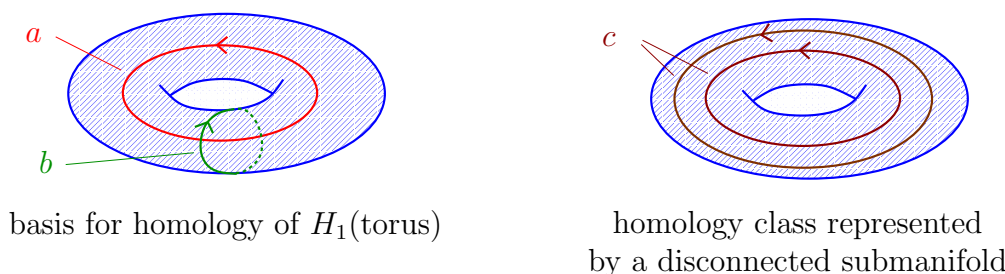


FIGURE 337.

- (2) We consider the surface Σ_2 of genus 2. As in Figure 35 we view Σ_2 as the connected sum $T \# T$ of two tori. If we use the calculation on page 568 to determine an isomorphism $H_1(\Sigma_2) \cong \mathbb{Z}^4$, then the basis elements are represented by the submanifolds a_1, b_1, a_2, b_2 that are illustrated in Figure 338.^{616 617}

⁶¹⁶Another point of view is that the submanifolds a_1, b_1, a_2, b_2 are given by deforming the curves x_1, y_1, x_2, y_2 that are shown in Figure 112.

⁶¹⁷What do these submanifolds look like in the formal picture of the surface of genus 2 as the quotient $E_8 / \sim$ of the euclidean octagon?

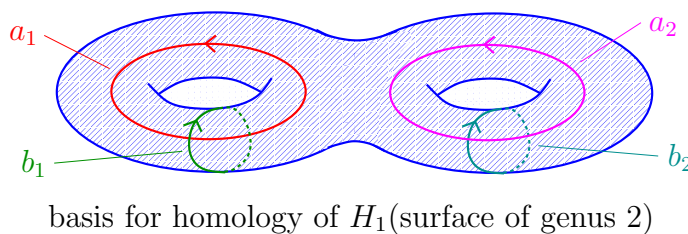


FIGURE 338.

- (3) We consider again the complex projective space $\mathbb{CP}^n$. We had seen on page 341 that for $k \leq n$ the map

$$\begin{aligned} \mathbb{CP}^k &\rightarrow \mathbb{CP}^n \\ [v_0 : \dots : v_k] &\mapsto [v_0 : \dots : v_k : 0 : \dots : 0] \end{aligned}$$

is an embedding. We use this map to view $\mathbb{CP}^k$ as a submanifold of $\mathbb{CP}^n$. It follows from the discussion on page 501 and from Lemma 22.1 that

$$H_k(\mathbb{CP}^n) \cong \begin{cases} \mathbb{Z} \cdot [\mathbb{CP}^i], & \text{if } k = 2i \text{ for some } i \in \{0, \dots, n\}, \\ 0, & \text{otherwise.} \end{cases}$$

Put differently, the generators of the non-trivial homology groups of the complex projective space $\mathbb{CP}^n$ are represented by the complex projective spaces $\mathbb{CP}^k$, $k = 0, \dots, n$.

If X is a topological space that is not a manifold then it does not make sense to ask whether a homology class is represented by a submanifold. But given a homology class $c \in H_n(X)$ it still makes sense to ask whether there exists a map from a manifold such that the image of the fundamental class equals c . This leads to the following definition.

Definition. Let X be a topological space and let $z \in H_n(X)$ be a homology class. We say z can be realized by a manifold if there exists an n -dimensional closed, connected, oriented manifold M and a map $f: M \rightarrow X$ such that $f_*([M]) = z$.

Example. If X is a connected topological space, then it follows from Lemma 32.3 that any class in $H_1(X)$ is realized by a manifold. Furthermore a relatively straightforward argument, see [Cl, p. 3] shows that also every class in $H_2(X)$ is realized by a manifold.

The following theorem was first proved by René Thom [Tm] in 1954⁶¹⁸ (See also [Ru, Theorem IV.7.37]). The proof of the theorem goes well beyond what we can do at the moment. We will not make use of the theorem.

Theorem 33.14.

- (1) *If X is a connected topological space, then every homology class of dimension ≤ 6 can be realized by a manifold.*

⁶¹⁸Ren Thom (1923-2002) was a French mathematician who received the Fields Medal in 1958.

- (2) Given any $n \geq 7$ there exists a topological space X and a homology class in $H_n(X)$ that is not realized by a manifold.
- (3) Given any n there exists a $k \in \mathbb{N}$ such that for any topological space X and any homology class $z \in H_n(X)$ the class $k \cdot z$ can be realized by a manifold.

33.4. The degree of a map between oriented manifolds.

Definition. Let $f: M \rightarrow N$ be a map between two closed, oriented, connected n -dimensional manifolds. We denote by $\deg(f) \in \mathbb{Z}$ the unique integer with

$$f_*([M]) = \deg(f) \cdot [N].$$

We refer to $\deg(f)$ as the *degree* of f .

Example. For a map $f: S^n \rightarrow S^n$ the above degree is exactly the same as the degree that we had defined on page 465.

Lemma 33.15. *Let $f: L \rightarrow M$ and $g: M \rightarrow N$ be two maps between two closed, oriented, connected n -dimensional manifolds. Then*

$$\deg(g \circ f) = \deg(g) \cdot \deg(f).$$

Proof. We have

$$(g \circ f)_*([L]) = g_*(f_*([L])) = g_*(\deg(f) \cdot [L]) = \deg(g) \cdot \deg(f) \cdot [L].$$

This shows that $\deg(g \circ f) = \deg(g) \cdot \deg(f)$. □

Proposition 33.16. *Let M be a closed n -dimensional oriented manifold. Then for any $k \in \mathbb{Z}$, there exists a map $f: M \rightarrow S^n$ of degree k .*

Proof. Let M be a closed n -dimensional oriented manifold and let $k \in \mathbb{Z}$. By Lemmas 27.1 and 27.4 there exists a map $g: S^n \rightarrow S^n$ of degree k . It follows from this fact and the combination of Proposition 33.16 and Lemma [multiplicativity-of-degrees](#) that it suffices to show that there exists a map $f: M \rightarrow S^n$ of degree 1. We will show the existence in the remainder of the proof.

We pick an orientation-preserving embedding $\Phi: \overline{B}^n \rightarrow M$. This map descends to a homeomorphism $\overline{B}^n/S^{n-1} \rightarrow M/(M \setminus \Phi(B^n))$ that we also denote by Φ . We write $\Psi := \Phi^{-1}: M/(M \setminus \Phi(B^n)) \rightarrow \overline{B}^n/S^{n-1}$. We denote by $\Omega: \overline{B}^n/S^{n-1} \rightarrow S^n$ the homeomorphism from page 23. We denote by f the map given by

$$M \rightarrow M/(M \setminus \Phi(B^n)) \xrightarrow{\Psi} \overline{B}^n/S^{n-1} \xrightarrow{\Omega} S^n.$$

We claim that this map is indeed a map of degree one.

We consider the following diagram of homomorphisms

$$\begin{array}{ccccc}
 H_n(M) & \longrightarrow & H_n(M, M \setminus \Phi(B^n)) & \xleftarrow{\Phi_*} & H_n(\overline{B}, S^{n-1}) \\
 & & \cong \downarrow & & \downarrow \cong \\
 & & \tilde{H}_n(M/(M \setminus \Phi(B^n))) & \xleftarrow{\Phi_*} & \tilde{H}_n(\overline{B}/S^{n-1}) \\
 & & & \searrow \Psi_* & \downarrow \Omega_* \\
 & \searrow f_* & & & H_n(S^n).
 \end{array}$$

We make the following remarks:

- (1) The vertical isomorphisms are given by the composition of the isomorphisms provided by Proposition 26.16 (which we can apply by Theorem 22.2 and Lemma 22.4 (6)) and by Lemma 26.11. The square commutes by Lemma 26.11.⁶¹⁹
- (2) By definition of $[M]$ we have

$$\Phi([\overline{B}^n]) = \text{the image of } [M] \text{ under the map } H_n(M) \rightarrow H_n(M, M \setminus \Phi(B^n)).$$

- (3) We had seen on page 563 that the image of $[\overline{B}^n]$ under the composition of the right vertical maps equals $[S^n]$.

It is now straightforward to see that combining all these facts implies that $f_*([M]) = [S^n]$. Thus we have shown that f is a map of degree one. $\square$

The following theorem says that the degree of a map is a complete invariant up to homotopy. We will not make use of this theorem.

Theorem 33.17. (Hopf Theorem) *Let M be a closed n -dimensional oriented manifold and let $f, g: M \rightarrow S^n$ be two maps. Then*

$$\deg(f) = \deg(g) \iff \text{the maps } f \text{ and } g \text{ are homotopic.}$$

Proof. The “ $\Leftarrow$ ”-direction is an immediate consequence of Proposition 25.4. The “ $\Rightarrow$ ”-direction is much harder, it was initially proved by Heinz Hopf whom we already encountered before. We refer to [OR, p. 191] or [Mi2, p. 51] for a proof. $\square$

In the following discussion, given $g \in \mathbb{N}_0$ we denote by Σ_g the surface of genus g .

Lemma 33.18. *For any $g \geq k$ there exists a degree-one map from Σ_g to Σ_k*

Proof. The case $k = 0$ follows immediately from Proposition 33.16. In the following we will cover the case $g = 2$ and $k = 1$. All other cases will be a variation of the subsequent argument and we leave them as an exercise to the reader. We denote by Σ the surface of genus 2 and we denote by T the torus.

We consider the maps $\Phi: \overline{B}^2 \rightarrow \Sigma$, $\Psi: \overline{B}^2 \rightarrow T$ and $\mu: \Sigma \rightarrow T$ that are illustrated in Figure 54. These maps can be chosen such that $\Psi = \mu \circ \Phi$. We had already encountered the map μ on page 573. We will follow the construction of Proposition 33.16 to obtain a

⁶¹⁹What about the role of Proposition 26.16? Is there something to check when we want to show that the diagram commutes?

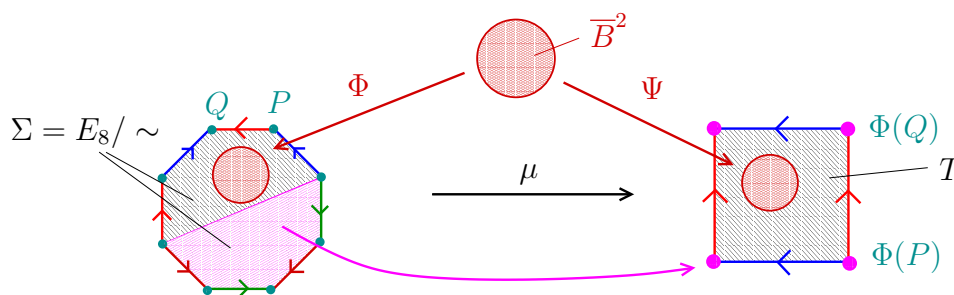


FIGURE 339.

degree one map $f: \Sigma \rightarrow S^2$ from Φ and to obtain a degree one map $\mu: T \rightarrow S^2$ from Φ . It follows from $\Psi = \mu \circ \Phi$ and the explicit constructions that $f = \mu \circ g$. We obtain the following commutative diagram

$$\begin{array}{ccc}
 H_n(\Sigma) & \xrightarrow{\mu_*} & H_2(T) \\
 \searrow f_* & & \swarrow g_* \\
 [\Sigma] \rightarrow [S^2] & & [T] \rightarrow [S^2] \\
 & H_2(S^2) &
 \end{array}$$

The diagonal maps are isomorphisms with $f_*([\Sigma]) = [S^2]$ and $g_*([T]) = [S^2]$. It follows from $f_* = \mu_* \circ g_*$ that $\mu_*([\Sigma]) = [T]$. We have thus shown that $\mu: \Sigma \rightarrow T$ is a degree-one map. $\square$

Proposition 33.19. *Let $p: \tilde{N} \rightarrow N$ be a finite covering of closed, oriented, connected manifolds. If p is orientation-preserving, then*

$$\deg(p) = [\tilde{N} : N].$$

Proof. This proposition follows from a generalization of Proposition 29.18 to the degrees of maps between closed, oriented, connected manifolds. We leave it as a slightly challenging exercise to fill in the details. $\square$

Proposition 33.20. *Let $f: M \rightarrow N$ be a map between closed, oriented, connected n -dimensional manifolds. If $\deg(f) = \pm 1$, then $f_*: \pi_1(M) \rightarrow \pi_1(N)$ is an epimorphism.*

The statement of Proposition 33.20 is somewhat surprising in so far as it takes as input fairly simple information, namely the induced map on n -th homology, which is an abelian group, and gives as output information about fundamental groups, which are usually much harder to deal with. Also it is a priori not clear why the homology in the maximal dimension should have any impact on fundamental groups.

Proof. Let $f: M \rightarrow N$ be a map between closed, oriented, connected n -dimensional manifolds. We fix a base point $P \in M$ and we write $Q = f(P)$. Furthermore we write $\pi = \pi_1(N, Q)$ and $\Gamma = f_*(\pi_1(M, P))$.

By Proposition 15.5⁶²⁰ there exists a path-connected covering $p: (\tilde{N}, \tilde{Q}) \rightarrow (N, Q)$ such that $p_*(\pi_1(\tilde{N}, \tilde{Q})) = \Gamma$ and such that $[\tilde{N} : N] = [\Gamma : \pi]$. By Proposition 16.1 we can view $\tilde{N}$ as an oriented manifold such that $p: \tilde{N} \rightarrow N$ is orientation preserving and a local diffeomorphism. By Proposition 15.2 the map $f: (M, P) \rightarrow (N, Q)$ lifts to a map $\tilde{f}: (M, P) \rightarrow (\tilde{N}, \tilde{Q})$.

First suppose that Γ is a subgroup of infinite index. It follows from $[\tilde{N} : N] = [\Gamma : \pi]$ and Lemma 6.4 (5) that $\tilde{N}$ is not compact. By Proposition 33.5 we therefore have $H_n(\tilde{N}) = 0$. But this shows that $f_* = p_* \circ \tilde{f}_*: H_n(M) \rightarrow H_n(N)$ factors through the trivial group. Hence $\deg(f) = 0$. But this is a contradiction to our hypothesis that $\deg(f) = \pm 1$.

So now we know that Γ is in fact a finite-index subgroup. It follows from $[\tilde{N} : N] = [\Gamma : \pi]$ and Lemma 6.4 (4) that $\tilde{N}$ is compact. We now have

$$\begin{array}{ccccccc} \pm 1 & = & \deg(f) & = & \deg(\tilde{f}) \cdot \deg(p) & = & \deg(\tilde{f}) \cdot [\tilde{N} : N] = \deg(\tilde{f}) \cdot [\Gamma : \pi]. \\ & & \uparrow & & \uparrow & & \\ & & \text{Lemma 33.15} & & \text{Proposition 33.19} & & \end{array}$$

This shows that $[\Gamma : \pi] = 1$, i.e. $\Gamma = \pi$. □

We had just seen in Lemma 33.18 that for any $g \geq k$ there exists a degree-one map from Σ_g to Σ_k . We can now prove the following converse statement.

Corollary 33.21. *If there exists a degree-one map from Σ_g to Σ_k , then $g \geq k$.*

Proof. Suppose that there exists a degree-one map from Σ_g to Σ_k . By Proposition 33.20 we obtain an epimorphism $\pi_1(\Sigma_g) \rightarrow \pi_1(\Sigma_k)$. In particular we obtain an epimorphism from the abelianization $\pi_1(\Sigma_g)$ onto the abelianization of $\pi_1(\Sigma_k)$. By Proposition 11.3 the former is isomorphic to $\mathbb{Z}^{2g}$ and the latter is isomorphic to $\mathbb{Z}^{2k}$. It follows from Lemma 8.5 that $g \geq k$. □

q It follows from Lemma 33.18 that there exists a degree-one map $S^2 \times S^2 \rightarrow S^4$. We conclude this chapter with the following two questions.

Question 33.22. *Does there exist a degree-one map from the 4-dimensional sphere S^4 to $S^2 \times S^2$?*

Question 33.23. *Does there exist a degree-one map from $S^1 \times S^2$ to $\mathbb{RP}^3$?*

⁶²⁰Hereby we use that according to the discussion on pages 251 any manifold is locally simply connected.

34. THE EULER CHARACTERISTIC

34.1. The definition of the Euler characteristic. Let X be a finite CW-complex, i.e. let X be a CW-complex with finitely many cells. Recall that on page 357 we had defined the Euler characteristic $\chi(X)$ as follows:

$$\chi(X) = \sum_n (-1)^n \cdot \text{number of } n\text{-cells}.$$

A short glance on the examples in Figure 196 shows that the three different CW-structures for the torus provided there have the same Euler characteristic.

The following proposition says that this is no coincidence since the Euler characteristic is in fact a topological invariant.

Proposition 34.1. *Let X be a finite CW-complex, then⁶²¹*

$$\chi(X) = \sum_n (-1)^n \cdot \text{rank}(H_n(X)).$$

In the following, given a topological space X we refer to

$$b_n(X) := \text{rank}(H_n(X))$$

as the n -th Betti number of X . These numbers are named after the Italian mathematician Enrico Betti (1823-1892)⁶²² who had studied an early version of homology groups. With this definition we can now reformulate Proposition 34.1 and we obtain that for any finite CW-complex the following equality holds:

$$\chi(X) = \sum_n (-1)^n \cdot b_n(X).$$

Examples.

- (1) It follows either from the CW-structure of the n -dimensional sphere S^n with one 0-cell and one n -cell given on page 340 or from the calculation of the homology groups in Proposition 26.3 that

$$\chi(S^n) = 1 + (-1)^n = \begin{cases} 0, & \text{if } n \text{ odd,} \\ 2, & \text{if } n \text{ even.} \end{cases}$$

- (2) It follows from Proposition 29.14 that for a surface Σ_g of genus g we have

$$\chi(\Sigma_g) = 2 - 2g.$$

⁶²¹As a reminder, if G is a finitely generated abelian group, then it follows from the classification of finitely generated abelian groups, see Theorem 8.3, that G is isomorphic to $\mathbb{Z}^r \oplus T$, where T is finite. We then call r the *rank* of G . Put differently, let $T \subset G$ be the torsion subgroup, then T/G is isomorphic to $\mathbb{Z}^r$ for an r , and this r is just the rank of G .

⁶²²

Proof. We denote by (C_n, d_n) the cellular chain complex of X . By definition the rank of C_n equals the number of n -cells in X . Furthermore it follows from Proposition 29.8 that the homology groups of the chain complex (C_n, ∂) are isomorphic to the singular homology groups of X .

Therefore it suffices to prove the following claim:

Claim. Let

$$0 \rightarrow C_k \xrightarrow{d_k} C_{k-1} \xrightarrow{d_{k-1}} \dots C_1 \xrightarrow{d_1} C_0 \rightarrow 0$$

be a chain complex of finitely generated abelian groups. Then the following equality holds:

$$\sum_{n=0}^k (-1)^n \operatorname{rank} C_n = \sum_{n=0}^k (-1)^n \operatorname{rank} H_n(C_*).$$

Before we turn to the actual proof of the claim we recall that if

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

is a exact sequence of finitely generated abelian groups, then Lemma 8.5 (2) says that⁶²³

$$\operatorname{rank} B = \operatorname{rank} A + \operatorname{rank} C.$$

For the chain complex we now write for each n

$$Z_n := \ker(d_n), \quad B_n := \operatorname{im}(d_{n+1}) \quad \text{and} \quad H_n = Z_n/B_n.$$

These groups form the following two types of short exact sequences

$$0 \rightarrow Z_n \rightarrow C_n \rightarrow B_{n-1} \rightarrow 0,$$

and

$$0 \rightarrow B_n \rightarrow Z_n \rightarrow H_n \rightarrow 0.$$

Thus it follows that

$$\operatorname{rank} C_n = \operatorname{rank} Z_n + \operatorname{rank} B_{n-1},$$

and

$$\operatorname{rank} Z_n = \operatorname{rank} B_n + \operatorname{rank} H_n.$$

⁶²³The statement is analogous to the dimension formula for vector spaces which says that if $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is an exact sequence of finite-dimensional vector spaces, then $\dim(B) = \dim(A) + \dim(C)$.

Now we see that

$$\begin{aligned}
 \sum_{n=0}^k (-1)^n \operatorname{rank} C_n &= \sum_{n=0}^k (-1)^n (\operatorname{rank} Z_n + \operatorname{rank} B_{n-1}) \\
 &= \sum_{n=0}^k (-1)^n (\operatorname{rank} B_n + \operatorname{rank} H_n + \operatorname{rank} B_{n-1}) \\
 &= \sum_{n=0}^k (-1)^n \operatorname{rank} H_n + \sum_{n=0}^k (-1)^n \operatorname{rank} B_n + \sum_{n=0}^k (-1)^n \operatorname{rank} B_{n-1} \\
 &= \sum_{n=0}^k (-1)^n \operatorname{rank} H_n + \sum_{n=0}^k (-1)^n \operatorname{rank} B_n - \sum_{m=0}^k (-1)^m \operatorname{rank} B_m \\
 &= \sum_{n=0}^k (-1)^n \operatorname{rank} H_n.
 \end{aligned}$$

□

We conclude this section with the following lemma.

Lemma 34.2. *Let*

$$0 \rightarrow A_k \rightarrow A_{k-1} \rightarrow \dots \rightarrow A_1 \rightarrow A_0 \rightarrow 0$$

be a long exact sequence of abelian groups. Then

$$\sum_n (-1)^n \operatorname{rank} A_n = 0.$$

Proof. By hypothesis the homology groups of the chain complex vanishes. Thus it follows immediately from the claim in the proof of Proposition 34.1 that the alternating sums of the ranks of the groups A_n is zero. □

34.2. Properties of the Euler characteristic. The Euler characteristic is an invariant which “sees less” than the homology groups. On the other hand we will now see that it behaves better than the homology groups under several natural operations.

Lemma 34.3. *Let X be and Y be two finite CW-complexes, then*

$$\chi(X \times Y) = \chi(X) \cdot \chi(Y).$$

Example. For any CW-complex X we have $\chi(S^1 \times X) = \chi(S^1) \cdot \chi(X) = 0$.

Proof. Let $\{e_\alpha\}_{\alpha \in A}$ be the characteristic maps of X and let $\{f_\beta\}_{\beta \in B}$ be the characteristic maps of Y . We identify in the following for each choice of k, l the topological space $B^k \times B^l$ with B^{k+l} .⁶²⁴ Now we can view $X \times Y$ as a CW-complex, where for each n we have

$$(X \times Y)_n = \bigcup_{k+l=n} X_k \times Y_l,$$

⁶²⁴More precisely, we pick once and for all a homeomorphism $B^k \times B^l \rightarrow B^{k+l}$, which exists by the discussion on page 18, and we use it to identify $B^k \times B^l$ with B^{k+l} .

and where the cells in $X \times Y$ are just the products of the cells in X and Y . More precisely, for some k -cell of X with characteristic map e_α and an l -cell of Y with characteristic map e_β we obtain a $k + l$ -cell of $X \times Y$ with characteristic map

$$\begin{aligned} B^{k+l} = B^k \times B^l &\rightarrow X \times Y \\ (x, y) &\mapsto (e_\alpha(x), e_\beta(y)). \end{aligned}$$

Furthermore each cell in $X \times Y$ is of this form. Thus we have that

$$\#n\text{-cells of } X \times Y = \sum_{k+l=n} \#k\text{-cells of } X \cdot \#l\text{-cells of } Y.$$

In exercise sheet 9 we will deduce from this formula the desired statement

$$\chi(X \times Y) = \chi(X) \cdot \chi(Y). \quad \square$$

The following lemma makes it possible to determine the Euler characteristic of a CW-complex from suitable decompositions.

Lemma 34.4. *Let $X = Y \cup Z$ be a decomposition of a finite CW-complex into two sub-complexes Y and Z . Then the following equality holds*

$$\chi(X) = \chi(Y) + \chi(Z) - \chi(Y \cap Z).$$

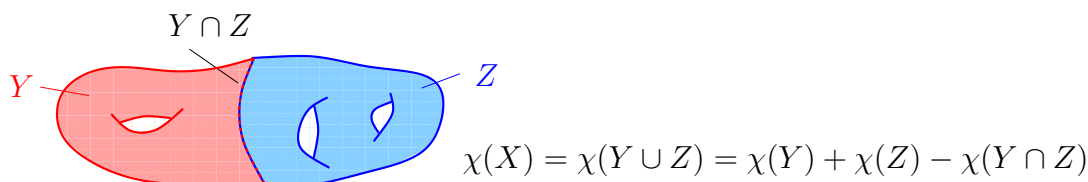


FIGURE 340.

This lemma follows from applying Lemma 34.2 and Proposition 34.1 to the long exact sequence of homology groups that we obtain by applying the Mayer-Vietoris Theorem 28.5 to the decomposition $X = Y \cup Z$. We now give an alternative, totally elementary proof of the lemma.

Proof. For a finite CW-complex W we denote by $n(W)$ the number of n -cells. In our case we have

$$\begin{aligned} &\text{since } X \text{ is the disjoint union of } Y \setminus (Y \cap Z), Z \setminus (Y \cap Z) \text{ and } Y \cap Z \\ &\quad \downarrow \\ n(X) &= n(Y \setminus (Y \cap Z)) + n(Z \setminus (Y \cap Z)) + n(Y \cap Z) \\ &= (n(Y) - n(Y \cap Z)) + (n(Z) - n(Y \cap Z)) + n(Y \cap Z) \\ &= n(Y) + n(Z) - n(Y \cap Z). \end{aligned}$$

By performing an alternating sum over all n we obtain immediately that

$$\chi(X) = \chi(Y) + \chi(Z) - \chi(Y \cap Z).$$

We have thus proved the desired equality. □

In Lemma 28.13 we had shown that for a knot $K \subset S^3$ we have $H_0(S^3 \setminus K) \cong \mathbb{Z}$ and $H_1(S^3 \setminus K) \cong \mathbb{Z}$. Using Lemma 34.4 we can now finally determine all the other homology groups of the knot complement $S^3 \setminus K$.

Lemma 34.5. *Let $K \subset S^3$ be a knot. Then the following holds*

$$H_0(S^3 \setminus K) \cong \mathbb{Z}, \quad H_1(S^3 \setminus K) \cong \mathbb{Z} \quad \text{and} \quad H_i(S^3 \setminus K) = 0 \text{ for } i \geq 2.$$

In exercise sheet we will determine the homology groups of the complement of a link.

Proof. Let $K \subset S^3$ be a knot. As we had just mentioned, in Lemma 28.13 we had shown that we have $H_0(S^3 \setminus K) \cong H_1(S^3 \setminus K) \cong \mathbb{Z}$.

As in the proof of Lemma 28.13 we apply the Tubular Neighborhood Theorem 2.2 to show that there exists an embedding $\Phi: S^1 \times \overline{B}^2 \rightarrow S^3$ with $\Phi(S^1 \times \{0\}) = K$. We write $Y := S^3 \setminus \Phi(S^1 \times \overline{B}^2)$. Note that Y is a deformation retract⁶²⁵ of $S^3 \setminus K$ and therefore Y is homotopy equivalent to $S^3 \setminus K$ by Lemma 7.4. By Corollary 25.5 it therefore suffices to determine the homology groups of Y .

Since Y is a compact oriented n -dimensional manifold with non-empty boundary it follows from Proposition 33.11 that $H_2(Y)$ is torsion-free and that $H_i(Y) = 0$ for $i \geq 3$. Thus it now suffices to show that $\text{rank}(H_2(Y)) = 0$.

By Theorem 22.2 there exists a CW-structure for S^3 such that Y is a subcomplex. We put $Z := \Phi(S^1 \times \overline{B}^2)$. Then $Y \cup Z = S^3$ and $Y \cap Z$ is the 2-torus $\Phi(S^1 \times S^1)$. We now have

$$\begin{aligned} \text{since } H_0(Y) \cong H_1(Y) \cong \mathbb{Z} \text{ and } H_i(Y) = 0 \text{ for } i \geq 3 \\ \downarrow \\ \text{rank}(H_2(Y)) &= \chi(Y) = \chi(Y \cup Z) - \chi(Z) + \chi(Y \cap Z) \\ &\quad \uparrow \\ &\quad \text{Lemma 34.4} \\ &= \chi(S^3) - \underbrace{\chi(S^1 \times \overline{B}^2)}_{= 0 \text{ by Lemma 34.3}} - \underbrace{\chi(S^1 \times S^1)}_{= 0 \text{ by Lemma 34.3}} = 0. \end{aligned} \quad \square$$

The following lemma is an immediate consequence of Proposition 33.12.

Lemma 34.6.

(1) *If M and N are two compact oriented n -manifolds, then*

$$\chi(M \# N) = \chi(M) + \chi(N) + \begin{cases} -2, & \text{if } n \text{ is even,} \\ 0, & \text{if } n \text{ is odd.} \end{cases}$$

⁶²⁵A deformation retraction is given by the map

$$\begin{aligned} (S^3 \setminus K) \times [0, 1] &\rightarrow Y = S^3 \setminus \Phi(S^1 \times \overline{B}^2) \\ (x, t) &\mapsto \begin{cases} x, & \text{if } y \in Y = S^3 \setminus \Phi(S^1 \times \overline{B}^2), \\ \Phi(z, ((1-t)r + t)e^{i\alpha}), & \text{if } y = \Phi(z, re^{i\alpha}) \in \Phi(S^1 \times \overline{B}^2). \end{cases} \end{aligned}$$

(2) If $k \in \mathbb{N}$, then

$$\chi(k \cdot \mathbb{RP}^2) = 1 - k.$$

Examples.

- (1) In Figure 35 we had seen that the connected sum of two tori is the surface of genus two. By Lemma 34.6 we obtain that⁶²⁶

$$\chi(\text{surface of genus two}) = \chi(\text{torus}) + \chi(\text{torus}) - 2 = -2.$$

We had obtained the same result on page 575 using a different method.

- (2) On page 197 we had seen that $\mathbb{RP}^2 \# \mathbb{RP}^2$ is homeomorphic to the Klein bottle. Thus we have shown that the Euler characteristic of the Klein bottle is 0.⁶²⁷

Let $p: \tilde{X} \rightarrow X$ be a k -fold covering of path-connected topological spaces, then it follows from Lemma 6.15 (3) that $p_*(\pi_1(\tilde{X}))$ is a subgroup of index k of $\pi_1(X)$. There is no analogous statement for homology groups. For example the projection map $p: S^2 \rightarrow S^2 / \sim = \mathbb{RP}^2$ is a 2-fold covering, but $H_2(S^2) = \mathbb{Z}$ and $H_2(\mathbb{RP}^2) = 0$. In Proposition 38.5 we had shown that for any n the map $p_*: H_k(\tilde{X}; \mathbb{Q}) \rightarrow H_k(X; \mathbb{Q})$ is an epimorphism but the above example shows that p_* does not have to be injective.

It is therefore perhaps surprising that we had already seen in Proposition 22.24 that the Euler characteristic behaves well under finite coverings. For convenience we recall, a slightly simplified version, of the statement of Proposition 22.24.

Proposition 22.24. *Let $p: \tilde{Y} \rightarrow Y$ be a finite covering of a finite connected CW-complex Y . Then $\tilde{Y}$ is also a finite CW-complex with*

$$\chi(\tilde{Y}) = [\tilde{Y} : Y] \cdot \chi(Y).$$

Examples.

- (1) We consider again the 2-fold covering $p: S^2 \rightarrow \mathbb{RP}^2$. In this case we have by the calculation on page 480 or alternatively in Lemma 29.19 that

$$\begin{array}{lll} H_0(S^2) \cong \mathbb{Z}, & & H_0(\mathbb{RP}^2) \cong \mathbb{Z}, \\ H_1(S^2) \cong 0, & \text{and} & H_1(\mathbb{RP}^2) \cong \mathbb{Z}_2, \\ H_2(S^2) \cong \mathbb{Z}, & & H_2(\mathbb{RP}^2) \cong 0, \\ H_i(S^2) \cong 0, \text{ for } i \geq 2, & & H_i(\mathbb{RP}^2) \cong 0, \text{ for } i \geq 2. \end{array}$$

Thus we see that the homology groups are quite different. Nonetheless we observe that $\chi(S^2) = 2$ is indeed twice $\chi(\mathbb{RP}^2) = 1$.

⁶²⁶How many arguments can you find in the lecture notes that show that the Euler characteristic of the torus is zero?

⁶²⁷For that statement one can also find easily many alternative proofs in the lecture notes.

- (2) Let p, q be coprime natural numbers. We consider the corresponding lens space $L(p, q)$. By definition $L(p, q)$ is finitely covered by S^3 . It follows from Proposition 22.24 that $\chi(L(p, q)) = 0$. We already know from Proposition 24.5 that $H_0(L(p, q)) \cong \mathbb{Z}$, from Corollary 32.5 that $H_1(L(p, q)) \cong \mathbb{Z}_p$ and from page 558 that $H_3(L(p, q)) \cong \mathbb{Z}$. Also we know from Proposition 29.10 that $H_n(L(p, q)) = 0$ for $n \geq 4$. So at the moment we are still missing $H_2(L(p, q))$. It now follows from Proposition 34.1 that
- $$\text{rank } H_2(L(p, q)) = \chi(L(p, q)) - \text{rank } H_0(L(p, q)) + \text{rank } H_1(L(p, q)) + \text{rank } H_3(L(p, q)) = 0.$$
- Furthermore we know from Proposition 33.13 that $H_2(L(p, q))$ is a free abelian group. Therefore we see that $H_2(L(p, q)) = 0$. Summarizing we have shown that

$$H_n(L(p, q)) \cong \begin{cases} \mathbb{Z}, & \text{if } n = 0, 3 \\ 0, & \text{if } n = 2 \text{ or } n > 3 \\ \mathbb{Z}_p, & \text{if } n = 1. \end{cases}$$

This shows that homology groups cannot distinguish between two lens spaces of the form $L(p, q)$ and $L(p, r)$. In particular we have made no progress on Question 6.7.

- (3) Let F be a closed surface of genus g . Furthermore let $p: \tilde{F} \rightarrow F$ be a k -fold covering. By Theorem 11.9 the oriented 2-dimensional manifold $\tilde{F}$ is diffeomorphic to precisely one surface of genus $\tilde{g}$. By the remark on page 575 we have

$$\chi(F) = 2 - 2g \quad \text{and} \quad \chi(\tilde{F}) = 2 - 2\tilde{g}.$$

On the other it follows from Proposition 22.24 that

$$\chi(\tilde{F}) = k \cdot \chi(F).$$

If we put these equalities together we obtain that

$$\text{genus of } \tilde{F} = k \cdot \text{genus of } F + 1 - k.$$

This finally gives an answer to Question 16.3.

In Question 11.4 we had asked whether the fundamental group of the surface of genus g is isomorphic to the free group F_{2g} on $2g$ generators. We can now answer this question in the negative.

Proposition 34.7. *Let $g \geq 2$. Then the group*

$$\pi_1(\text{surface of genus } g) \cong \langle x_1, y_1, \dots, x_g, y_g \mid [x_1, y_1] \cdots [x_g, y_g] \rangle$$

is not isomorphic to the free group F_{2g} on $2g$ generators.

Proof. Let F be a closed surface of genus g . We write $\pi = \pi_1(F)$. Let $\alpha: \pi_1(F) \rightarrow \mathbb{Z}_k$ be an epimorphism for some $k \geq 2$.⁶²⁸ We write $\tilde{\pi} := \ker(\alpha)$. We denote by $p: \tilde{F} \rightarrow F$ the covering with $p_*(\pi_1(\tilde{F})) = \tilde{\pi}$ which is given by Proposition 15.5. By Lemma 6.15 (3) this is a k -fold covering. By the discussion on page 581 we know that

$$\text{genus of } \tilde{F} = k \cdot \text{genus of } F + 1 - k = k \cdot (g - 1) + 1.$$

⁶²⁸Why does such an epimorphism exist?

In particular it follows from Proposition 11.3 that $\tilde{\pi}_{\text{ab}} \cong \mathbb{Z}^{2k(g-1)+2}$.

Now suppose π is a free group. Then it would follow from Corollary 22.25 that $\tilde{\pi}$ is also a free group with

$$\text{rank}(\tilde{\pi}) = k \cdot (\text{rank}(\pi) - 1) + 1 = k(2g - 1) + 1.$$

In particular we would get that $\tilde{\pi}_{\text{ab}} \cong \mathbb{Z}^{k(2g-1)+1}$. But $k(2g - 1) + 1 \neq 2k(g - 1) + 2$. So we obtained a contradiction. $\square$

34.3. Groups acting on spheres. We recall that a group G acts discretely on a topological space X , if for each $x \in X$ there exists an open neighborhood U such that $U \cap gU = \emptyset$ for all $g \neq e$. For example for each $k \in \mathbb{N}$ the action

$$\begin{aligned} \mathbb{Z}_k \times S^1 &\rightarrow S^1 \\ (l, z) &\mapsto e^{2\pi i l/k} \cdot z \end{aligned}$$

is discrete. We can easily generalize this example. More precisely, let S^{2n-1} be an odd-dimensional sphere. We view S^{2n-1} as subset of $\mathbb{R}^{2n} = \mathbb{C}^n$. Then

$$\begin{aligned} \mathbb{Z}_k \times S^{2n-1} &\rightarrow S^{2n-1} \\ (l, (z_1, \dots, z_n)) &\mapsto (e^{2\pi i l/k} z_1, \dots, e^{2\pi i l/k} z_n) \end{aligned}$$

is easily seen to be a discrete action of $\mathbb{Z}_k$ auf S^{2n-1} .

It is less clear what groups can act on even-dimensional spheres. The group $\mathbb{Z}_2 \cong \{\pm 1\}$ acts of course discretely on any sphere. The following proposition says that $\mathbb{Z}_2$ is the only group that can act non-trivially on an even-dimensional sphere. This proposition and the above example give a complete answer to Question 6.8.

Proposition 34.8. *If a group G acts freely and continuously on S^{2n} , then G is either trivial or $G \cong \mathbb{Z}_2$.*

Proof. Let G be a group that acts freely and continuously on S^{2n} . By Lemma 6.5 the action is also discrete. By Proposition 6.9 the projection map $S^{2n} \rightarrow S^{2n}/G$ is a covering of degree $|G|$. From Proposition 22.24 and the calculation of the Euler characteristic on page 424 we obtain that

$$2 = \chi(S^{2n}) = |G| \cdot \chi(S^{2n}/G).$$

It follows that $|G|$ divides two, i.e. $|G| = 1$ or $|G| = 2$. In first case the group G is the trivial group and in the second case we have $G \cong \mathbb{Z}_2$.⁶²⁹ $\square$

Example. Let Σ be the surface of genus three. By the remark on page 575 we have $\chi(\Sigma) = 6$. The proof of Proposition 34.8 shows that the order of any group acting discretely and continuously on Σ has to have an order that divides 6.

In Figure 341 we sketch a discrete and continuous action of the group $\mathbb{Z}_3$ on Σ . In exercise sheet 10 we will give a formal proof that the group $\mathbb{Z}_3$ admits a discrete and continuous action on Σ . Furthermore we will discuss the question whether the groups $\mathbb{Z}_2, \mathbb{Z}_6$ and S_3 can act discretely and continuously on Σ .

⁶²⁹Why is every group with two elements isomorphic to $\mathbb{Z}_2$?

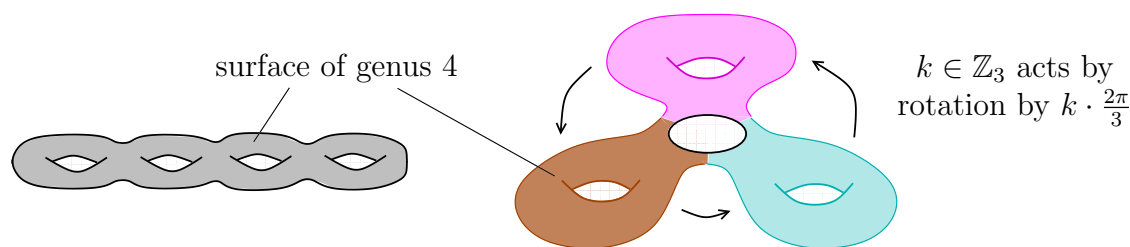


FIGURE 341.

34.4. Graphs. First recall that on page 338 we had defined a topological graph to be a 1-dimensional CW-complex. Furthermore we refer to the 0-cells as the vertices of the graph and we refer to the 1-cells as the edges of the graph. The Euler characteristic of a finite graph G is thus given by

$$\chi(G) = \# \text{vertices} - \# \text{edges}.$$

In particular this matches the original definition that we had given on page 131.

In the following we say that a graph G is a *tree*, if G is contractible.⁶³⁰ Using the results of this chapter we can now easily prove the following lemma:

Lemma 34.9. *Let G be a connected graph with e vertices and k edges. Then*

$$H_0(G) \cong \mathbb{Z}, \quad H_1(G) \cong \mathbb{Z}^{k-e+1} \quad \text{and} \quad H_i(G) = 0 \text{ for } i \geq 2.$$

Example. In Figure 342 we see a connected graph G with 22 vertices and 24 edges. It follows from Lemma 34.9 that its first homology group is isomorphic to $\mathbb{Z}^3$. In Figure 342 we sketched three cycles which represent a generating set for $H_1(G) \cong \mathbb{Z}^3$. But there are infinitely many other choices of generating sets.⁶³¹

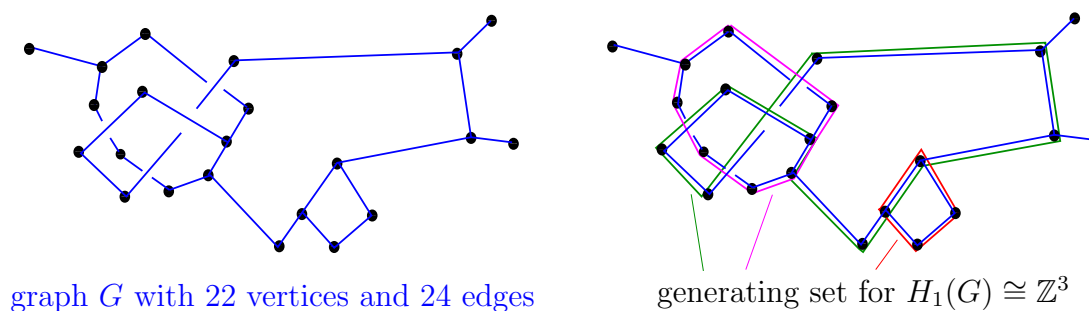


FIGURE 342.

⁶³⁰On page 131 we had defined a *tree* to be a finite connected graph with Euler characteristic 1. In Proposition 7.8 and Lemma 9.4 we had seen that a finite graph is contractible if and only if its Euler characteristic equals 1. Thus we see that our new definition of a tree agrees with the old definition of a tree for finite graphs.

⁶³¹What are other examples of generating sets for $H_1(G)$?

Remark. Lemma 34.9 is consistent with the calculation of the fundamental group in Lemma 9.4 and the fact that by Proposition 32.4 the abelianization of the fundamental group is isomorphic to the first homology group.

Proof. Let G be a connected graph with e vertices and k edges. Now that we have developed so many techniques for computing homology groups it is perhaps not surprising that we can provide two different proofs for the lemma.

- (1) First of all, from Proposition 24.5 and Lemma 22.4 it follows immediately that $H_0(G) \cong \mathbb{Z}$. From Proposition 29.9 it follows furthermore that $H_i(G) = 0$ for $i \geq 2$ and that $H_1(G)$ is torsion-free.

From Proposition 34.1 we obtain that

$$\chi(X) = \sum_i (-1)^i \operatorname{rank}(H_i(X)).$$

Putting everything together we see that

$$1 - \operatorname{rank}(H_1(G)) = \chi(X) = \#0\text{-cells} - \#1\text{-cells} = e - k.$$

Since $H_1(G)$ is torsion-free we obtain that $H_1(G) \cong \mathbb{Z}^{k-e+1}$.

- (2) An alternative proof for Lemma 34.9 is given as follows: we have

$$\begin{array}{ccccccc} \tilde{H}_k(G) & \cong & \tilde{H}_k(\text{wedge of } 1 - \chi(G) \text{ circles}) & \cong & \bigoplus_{i=1}^{1-\chi(G)} \tilde{H}_k(S^1) & \cong & \begin{cases} \mathbb{Z}^{1-\chi(G)}, & \text{if } k = 1, \\ 0, & \text{otherwise.} \end{cases} \\ \uparrow & & \uparrow & & \uparrow & & \\ \text{Proposition 7.8 (3) and Corollary 25.5} & & \text{Proposition 28.8} & & \text{Proposition 26.3} & & \end{array}$$

It is a matter of taste which of the two proofs is nicer. $\square$

34.5. The Lefschetz Fixed Point Theorem (*). ⁶³² The following theorem was proved by Solomon Lefschetz in the 1920s. ⁶³³

Theorem 34.10. (Lefschetz Fixed Point Theorem) *Let X be a finite CW-complex (e.g. a compact manifold) and let $f: X \rightarrow X$ be a map. If* ⁶³⁴

$$\tau(f) := \sum_{n \in \mathbb{N}} (-1)^n \cdot \operatorname{tr}(f_*: H_n(X) \rightarrow H_n(X)) \neq 0,$$

then f has a fixed point, i.e. there exists an $x \in X$ with $f(x) = x$.

A proof for the Lefschetz Fixed Point Theorem is given in [Ha, Theorem 2.C.3].

Examples.

⁶³²This section is not part of the official lecture notes for Algebraic Topology II.

⁶³³Solomon Lefschetz (1884-1972) was an American mathematician.

⁶³⁴Let H be a finitely generated abelian group and let $\varphi: H \rightarrow H$ be a homomorphism. We write $\overline{H} := H/\text{torsion}$. The homomorphism φ defines a homomorphism $\overline{\varphi}: \overline{H} \rightarrow \overline{H}$. By the classification of finitely generated abelian groups, see Theorem 8.3, there exists an isomorphism $\psi: \overline{H} \cong \mathbb{Z}^k$. We define $\operatorname{tr}(\varphi)$ to be the trace of the homomorphism $\mathbb{Z}^k \xrightarrow{\psi^{-1}} \overline{H} \xrightarrow{\overline{\varphi}} \overline{H} \xrightarrow{\psi} \mathbb{Z}^k$. As in linear algebra one can easily show that this definition does not depend on the choice of ψ .

- (1) Suppose X is a finite connected CW-complex with $H_i(X) = 0$ for $i \geq 1$, for example we could take $X = \overline{B}^n$. It follows from the discussion on page 410 that for any map $f: X \rightarrow X$ we have $\tau(f) = 1$. It follows from the Lefschetz Fixed Point Theorem that f has a fixed point. This discussion shows that the Brouwer Fixed Point Theorem 26.7 can be viewed as a special case of the Lefschetz Fixed Point Theorem.
- (2) We consider the CW-complex S^n . If $f: S^n \rightarrow S^n$ is a map that is homotopic to the identity, then it follows from Proposition 25.4 that $\tau(f) = \tau(\text{id}) = 1 + (-1)^n$.
 - (a) If n is even, then $\tau(f) = 2$ and we obtain from the Lefschetz Fixed Point Theorem that f has a fixed point.
 - (b) On the other hand, if n is odd, then f does not necessarily have a fixed point. For example the rotation map $f: S^1 \rightarrow S^1$ given by $f(e^{it}) = e^{i(t+\frac{1}{2})}$ is homotopic to the identity, but it has no fixed points.

35. APPLICATIONS OF THE EULER CHARACTERISTIC

In this chapter we show that the fairly simple concept of the Euler characteristic can be used to give answers to several questions about 3-dimensional objects.

35.1. Building a leather football. In this section we will discuss the shapes of footballs and the platonic solids. The results of this section will not be used later on and we will present some of the material in a slightly informal, somewhat less rigorous style.

First we recall that an n -gon on the 2-dimensional sphere S^2 is called *regular* if the isometry group of the sphere acts transitively on the set of vertices and the set of faces of the n -gon. In particular all interior angles are the same. For example the classical leather football is made out of regular pentagons and hexagons.

Now we want to determine how many pentagons and hexagons are required to make a foot ball. So suppose a football is made out of p spherical regular pentagons and h spherical regular hexagons where the vertices of the pentagons are precisely the vertices of hexagons. This endows S^2 with a CW-structure. Each 1-cell bounds two 2-cells. In Analysis IV we had seen that the interior angle of a spherical pentagon and a spherical hexagon are larger than the corresponding euclidean angles. Therefore we see that that interior angles are larger than⁶³⁵ $\pi - \frac{2\pi}{5} = \frac{3}{5}\pi$. So at any vertex there are precisely three hexagons and pentagons that meet. Therefore we see that

$$\begin{aligned}\#2\text{-cells} &= p + h \\ \#1\text{-cells} &= \frac{1}{2}(5p + 6h) \\ \#0\text{-cells} &= \frac{1}{3}(5p + 6h).\end{aligned}$$

By Proposition 34.1 and the above calculation of the Euler characteristic of S^2 we now have

$$2 = \chi(S^2) = \sum_{n=0}^2 (-1)^n \cdot \#n\text{-cells} = \frac{1}{3}(5p + 6h) - \frac{1}{2}(5p + 6h) + (p + h) = \frac{1}{6}p.$$

Thus we see that a football has to have precisely 12 pentagons. The number of hexagons cannot be determined. The standard football has 20 hexagons, as pictured in Figure 343. But in principle one could also build a football with zero hexagons, see Figure 344 and one obtains a spherical dodecahedron. One could also build with many hexagons, but they would be much less round since they would have bigger flat parts.

The 1-skeleton of this CW-structure were in fact already “invented by nature”, more precisely, there exists a molecule, called the Buckminsterfullerene C_{60} built out of 60 carbon atoms which has exactly the same shape as a football. More information can be found at

<https://en.wikipedia.org/wiki/Buckminsterfullerene>

⁶³⁵Where does that formula come from?

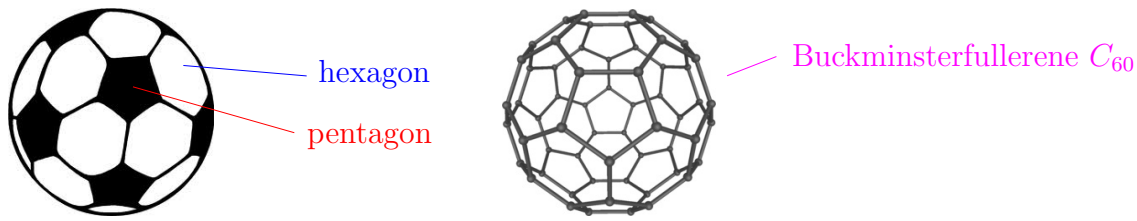


FIGURE 343.

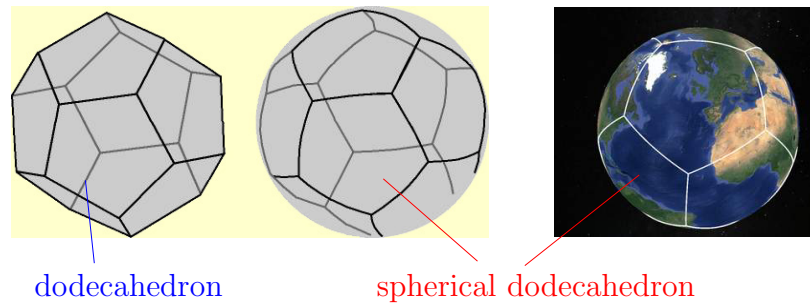


FIGURE 344.

35.2. Platonic solids. Recall that on page 18 we had defined the *convex hull* of a subset $S \subset \mathbb{R}^n$ as the intersection of all convex subsets of $\mathbb{R}^k$ that contain S . Since the intersection of convex sets is again convex we see that the convex hull of S is a convex subset of $\mathbb{R}^k$.

Definition.

- (1) A *polyhedron* is the convex hull of finitely many points in $\mathbb{R}^k$.⁶³⁶⁶³⁷
- (2) The dimension of a polyhedron is the minimal dimension of an affine subspace of $\mathbb{R}^k$ that contains the polyhedron.

In exercise sheet 10 we will prove the following lemma.

Lemma 35.1. *If P is a polyhedron in $\mathbb{R}^k$, then there exists a unique finite set V such that P is the convex hull of V , but such that for any proper subset W of V , P is not the convex hull of W .*

Definition. Let P be a polyhedron in $\mathbb{R}^k$.

- (1) We refer to the finite set of Lemma 35.1 as the *vertices of the polyhedron P* .

⁶³⁶In the literature a polyhedron is often also called a polytope.

⁶³⁷Recall that we had seen on page 18 that the convex hull of points $P_1, \dots, P_n \in \mathbb{R}^k$ is given by the set

$$\left\{ \sum_{i=1}^n t_i P_i \mid t_1, \dots, t_n \in \mathbb{R}_{\geq 0} \text{ and } \sum_{i=1}^n t_i = 1 \right\}.$$

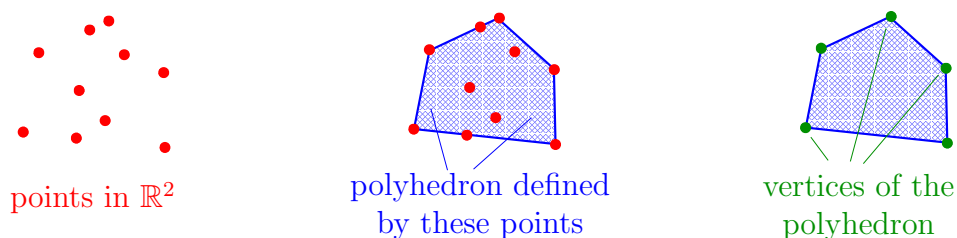


FIGURE 345. Illustration of Lemma 35.1.

- (2) If P is a k -dimensional polyhedron, then an *edge* of the polyhedron is the convex hull of two vertices that has the property that it lies on ∂P .
- (3) If P is a k -dimensional polyhedron, then an m -dimensional *face* of the polyhedron is the convex hull of m vertices that has the property that it lies on ∂P .
- (4) An edge is a 1-dimensional face and sometimes we refer to a 2-dimensional face just as a face, without specifying the dimension.
- (5) The *valence* of a vertex is defined as the number of edges that contain the vertex.
- (6) A *rotational symmetry* of P is a matrix $A \in O(k)$ such that $A \cdot P = P$. The *rotational symmetry group* $S(P) \subset O(k)$ is defined as the group of all rotational symmetries.
- (7) The polyhedron P is called *regular* if for any $m \in \{0, \dots, k\}$ the rotational symmetry group $S(P)$ acts transitively on the set of m -dimensional faces.
- (8) We say that two polyhedra P and Q in $\mathbb{R}^k$ are *equivalent* if there exists an $r > 0$ and an orthogonal matrix $A \in O(k)$ such that $P = A \cdot (r \cdot Q)$.

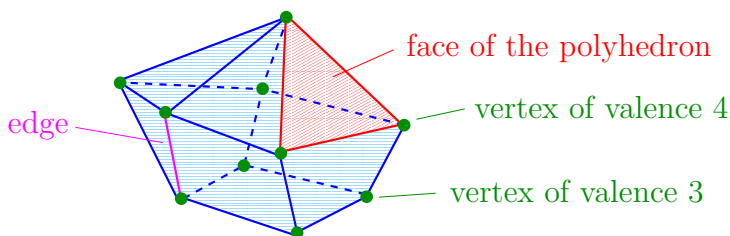


FIGURE 346.

Example.

- (1) The standard n -simplex Δ^n is an n -dimensional polyhedron in $\mathbb{R}^{n+1}$ with vertex set $\{(1, 0, \dots, 0), (0, 1, 0, \dots, 0), \dots, (0, \dots, 0, 1)\}$. Each vertex has valence n .
- (2) Let $n \in \mathbb{N}_{\geq 3}$ and $r > 0$. The convex hull of the points $r, re^{2\pi i/n}, \dots, re^{2\pi i(n-1)/n}$ in $\mathbb{C} = \mathbb{R}^2$ is a regular polyhedron with vertices $r, re^{2\pi i/n}, \dots, re^{2\pi i(n-1)/n}$, where each vertex has valence 2. It is quite easy to see that up to equivalence these are all regular 2-dimensional polyhedra in $\mathbb{R}^2$.

- (3) We consider the cube $[-1, 1]^3 \subset \mathbb{R}^3$. In [Ar, p. 38] it is shown that $S([-1, 1]^3)$ is isomorphic to S_4 . More precisely, we denote by D the set of the diagonals in the cube, i.e. the segments in $\mathbb{R}^3$ spanned by opposite vertices of the cube. Note that D has precisely four elements. Then the map

$$\begin{aligned} S([-1, 1]^3) &\rightarrow S_4 \cong \text{permutation group of } D \\ A &\mapsto \begin{pmatrix} D & \rightarrow & D \\ d & \mapsto & A \cdot d \end{pmatrix} \end{aligned}$$

is shown in [Ar, p. 38] to be an isomorphism.⁶³⁸ It is clear that the cube is a regular polyhedron. Since it has six sides it is in our context also referred to as *regular hexahedron*.

- (4) The polyhedron spanned by $\{(\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1)\}$ is a regular 3-dimensional polyhedron where each vertex has valence 4.

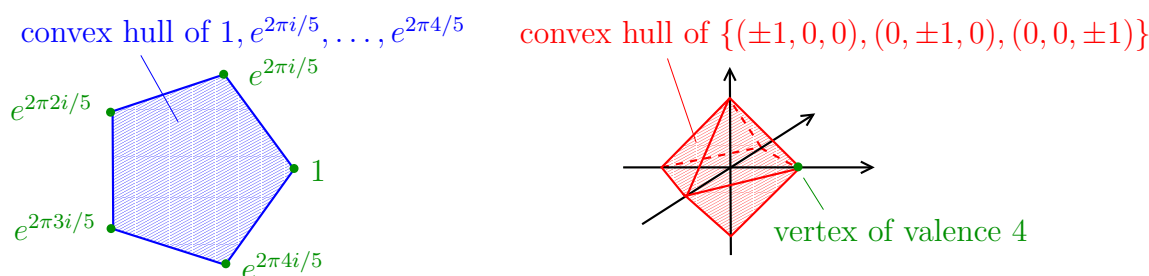


FIGURE 347.

- (5) Finally we consider the regular dodecahedron D shown in Figure 348.⁶³⁹ We can inscribe five cubes $C_1, \dots, C_5$ into the regular dodecahedron. Then the map

$$\begin{aligned} S(D) &\rightarrow A_5 \cong \text{positive permutations of } \{C_1, \dots, C_5\} \\ A &\mapsto (C_i \mapsto A \cdot C_i) \end{aligned}$$

is shown in [Ar, p. 40] to be an isomorphism.

Definition. A *Platonic solid* is a regular 3-dimensional polyhedron in $\mathbb{R}^3$.

Theorem 35.2. *Up to rotation and stretching the only regular polyhedra in $\mathbb{R}^3$ are the platonic solids illustrated in Figure 349.⁶⁴⁰ They have the following types of faces and*

⁶³⁸It is a good exercise to verify this statement.

⁶³⁹The fact that a regular dodecahedron really exists is shown rigorously in [Cx, Section 6.7].

⁶⁴⁰A precise definition of the platonic solids in Figure 349 is given in [Cx, Section 1.2].

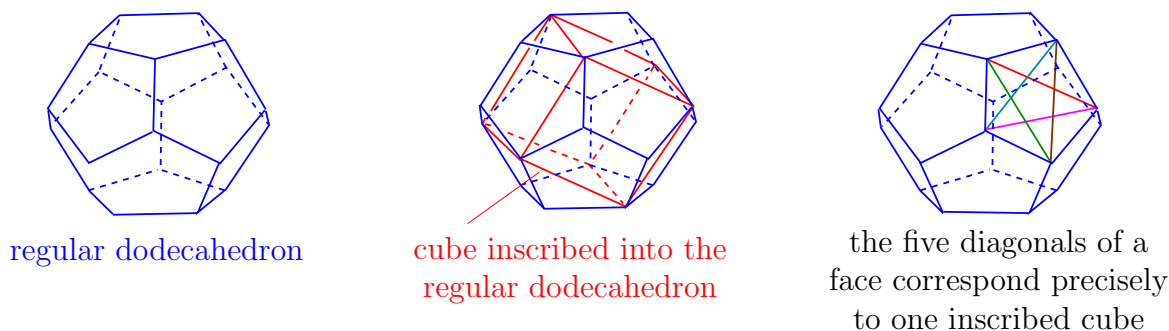


FIGURE 348.

valences and they have the following rotational symmetry groups:

	tetrahedron	hexahedron	octahedron	dodecahedron	icosahedron
number of faces	4	6	8	12	20
shape of face	3-gon	4-gon	3-gon	5-gon	3-gon
valence of vertex	4	3	4	3	5
rotational symmetry group	A_4	S_4	S_4	A_5	A_5

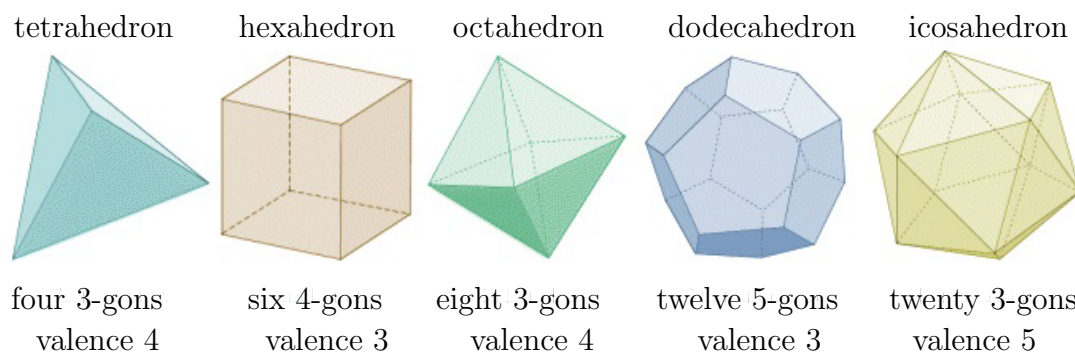


FIGURE 349. The five platonic solids.

Proof. We will not provide a full proof for the theorem. We refer to [Cx, Section 6.7] for a full proof of the first three rows and we refer to [Ar, p. 40] or alternatively to [Aa, Chapter 5.5] for a proof of the statement regarding the rotational symmetry groups.⁶⁴¹ But we will show that any platonic solid fits into the table of the theorem.

⁶⁴¹We had sketched the proofs for the regular hexahedron and the regular dodecahedron above.

So let P be a regular polyhedron. Using Lemma 1.4 one can easily show that P admits a CW-structure such that

$$\begin{aligned} \# \text{ of 0-cells} &= \# \text{ of vertices,} \\ \# \text{ of 1-cells} &= \# \text{ of edges,} \\ \# \text{ of 2-cells} &= \# \text{ of faces,} \\ \# \text{ of 3-cells} &= 1. \end{aligned}$$

We first consider the case that all its faces are triangles. We denote by k the number triangles. We consider the projection of the vertices and edges of the polyhedron onto the sphere, see Figure 350. The interior angle is greater than the Euclidean angle of an equilateral triangle, so it is greater than $\frac{\pi}{3}$. This shows that at a vertex it is only possible that 3, 4 or 5 triangles meet.

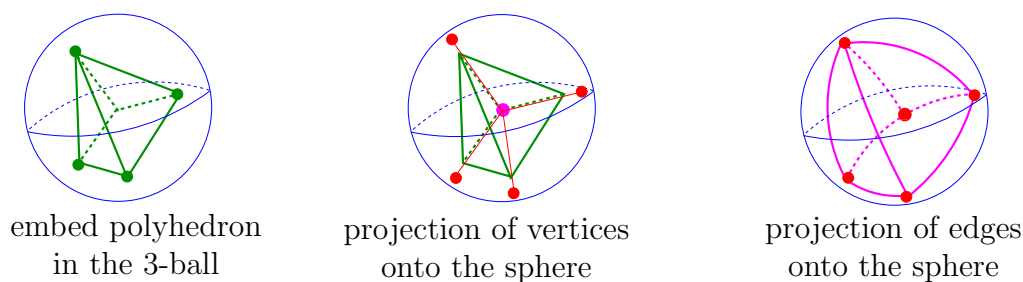


FIGURE 350.

Let us first consider the case that three triangles meet at a vertex. Then we have

$$\#2\text{-cells} = k, \quad \#1\text{-cells} = \frac{1}{2} \cdot 3k \quad \text{and} \quad \#0\text{-cells} = \frac{1}{3} \cdot 3k.$$

Since the Euler characteristic equals two we get

$$2 = \chi(S^2) = \sum_{n=0}^2 (-1)^n \cdot \#n\text{-cells} = k - \frac{3}{2}k + k = \frac{1}{2}k.$$

It follows that $k = 4$. Almost the same calculation shows that if four triangles meet at a vertex, then $k - \frac{3}{2}k + \frac{3}{4}k = 2$, i.e. $k = 8$. Finally if five triangles meet at a vertex, then we see that $k - \frac{3}{2}k + \frac{3}{5}k = 2$, i.e. $k = 20$.

If the polyhedron is built out of squares, then the same argument shows that the valence of a vertex can only be three and the same calculation shows that there have to be precisely six faces. Finally if the polyhedron is built out of pentagons, then as above we see that the valence of a vertex can only be three and that there have to be precisely twelve faces. $\square$

On page 588 we had just seen that there exist infinitely many equivalence classes of regular polyhedra in $\mathbb{R}^2$. On the other hand we have just seen in Theorem 35.2 that in $\mathbb{R}^3$ there are only finitely many, namely precisely five equivalence classes of regular polyhedra in $\mathbb{R}^3$. This begs the question, what is the number in higher dimensions? It turns out that

in $\mathbb{R}^4$ there are six equivalence classes of regular polyhedra, whereas in all higher dimensions there are only three equivalence classes of polyhedra. A nice exposition of these facts is given on the following website:

<http://math.ucr.edu/home/baez/platonic.html>

35.3. Homology spheres.

Definition. We say that an n -dimensional topological manifold X is a *homology n -sphere* if for any $k \in \mathbb{N}_0$ we have $H_k(X) \cong H_k(S^n)$.

It follows from the classification of 1-dimensional topological manifolds that we had stated in Proposition 1.29 and Theorem 11.8 and the calculation of homology groups in Proposition 29.14 that for $n = 1, 2$ every homology n -sphere is homeomorphic to the standard sphere $n = 2$. In 1900 Henri Poincaré⁶⁴² claimed that the same also holds for $n = 3$, i.e. the claim that every homology 3-sphere is in fact homeomorphic to S^3 . Shortly afterwards, in 1904, Henri Poincaré himself found a counterexample to this claim. More precisely, he proved the following proposition.

Proposition 35.3. *There exists a closed, orientable 3-dimensional manifold M that is a homology 3-sphere but such that $\pi_1(M)$ is a non-trivial group.*

After proving Proposition 35.3 Henri Poincaré formulated the following conjecture.

Conjecture 35.4. (Poincaré Conjecture) *If M is a topological 3-dimensional homology sphere that is simply connected, then M is homeomorphic to S^3 .*

We will discuss generalizations of the Poincaré Conjecture and the status of the original conjecture and of the generalizations later on in Algebraic Topology III.

Nowadays there are many ways to prove Proposition 35.3. In the following we will give Poincaré's example of a such a manifold.

In Theorem 35.2 we had seen that we can view A_5 as a subgroup of $SO(3)$. In Theorem 31.2 we had shown that there exists a group homomorphism $q: S^3 = SU(2) \rightarrow SO(3)$ where the kernel is given by $\pm \text{id}$. We write $\Gamma := q^{-1}(A_5)$. Since A_5 has $\frac{1}{2}5! = 60$ elements and since q has a kernel with two elements we see that Γ is a group with 120 elements.

Proposition 35.5. *The topological space $M := S^3/\Gamma$ is a 3-dimensional homology sphere with $\pi_1(M) \cong \Gamma$ a non-trivial group.*

The manifold M is called the *Poincaré homology sphere*. It was initially discovered by Henri Poincaré in 1904 by a different construction. The Poincaré homology sphere is ubiquitous in topology and can be described in many different ways. We refer to [KSc] for altogether eight different ways to construct the Poincaré homology sphere.

In the proof of Proposition 35.5 we will make use of the following lemma.

⁶⁴²Henri Poincaré (1854-1912) was a French mathematician. He can be viewed as the “father of algebraic topology”, he first introduced the fundamental group and the homology groups of topological spaces. We had encountered Henri Poincaré before, the disk model $\mathbb{D}$ for hyperbolic space goes back to Poincaré and it is often called the “Poincaré model of hyperbolic space”.

Lemma 35.6. *Let M be an orientable closed 3-dimensional manifold. If $H_1(M) = 0$, then M is an integral homology sphere.*

Proof. Let M be an orientable closed 3-dimensional manifold with $H_1(M) = 0$. By Proposition 33.5 it remains to show that $H_2(M) = 0$. By Proposition 33.13 we know that $H_2(M)$ is a free abelian group. So it suffices to show that $b_2(M) = 0$. We have

$$\begin{array}{ccccccc} b_2(M) & = & \chi(M) & = & \frac{1}{120} \cdot \chi(S^3) & = & 0. \\ & & \uparrow & & \uparrow & & \uparrow \\ & \text{since } b_0(M) = b_3(M) = 1 & & \text{Proposition 22.24} & & \text{page 575} & \\ & \text{and } b_1(M) = 0 & & & & & \end{array}$$

□

Now we can provide the proof of Proposition 35.5.

Proof of Proposition 35.5. It follows from Proposition 1.28 that $M := S^3/\Gamma$ is a 3-dimensional manifold and it follows from Proposition 4.12 and Theorem 6.16 that $\pi_1(M) \cong \Gamma$.

It remains to show that M or a 2-fold cover of M is a homology sphere. By Lemma 35.6 it suffices to prove the following claim.

Claim. We either have $H_1(M) = 0$ or there exists a 2-fold cover M' of M with $H_1(M') = 0$.

By Proposition 32.4 it suffices to show that the abelianization of Γ is trivial. Since Γ is a finitely generated group this is equivalent to showing that Γ does not admit an epimorphism onto a non-trivial abelian group. By construction the group Γ fits into a short exact sequence

$$0 \rightarrow \mathbb{Z}_2 \xrightarrow{\rho} \Gamma \rightarrow A_5 \rightarrow 0.$$

By Proposition 10.6 we get an induced exact sequence

$$\mathbb{Z}_2 \xrightarrow{\rho} \Gamma_{\text{ab}} \rightarrow (A_5)_{\text{ab}} \rightarrow 0.$$

From the algebra course we know that the abelianization of A_5 is trivial. This means So now there are two possibilities:

- (1) Either $\Gamma_{\text{ab}} = 0$, in which case we are done.
- (2) Or the map $\mathbb{Z}_2 \xrightarrow{\rho} \Gamma_{\text{ab}}$ is an isomorphism.⁶⁴³ In this case the map $\Gamma \rightarrow \Gamma_{\text{ab}} \xrightarrow{\rho^{-1}} \mathbb{Z}_2$ is a splitting of $\rho: \mathbb{Z}_2 \rightarrow \Gamma$. The argument of Lemma 28.2 shows that this implies that $\Gamma \cong \mathbb{Z}_2 \times A_5$. In particular A_5 is a subgroup of $\pi_1(M) = \Gamma$ of index two. By Proposition 15.5 there exists a 2-fold covering M' of M with fundamental group A_5 . But then $H_1(M') = (A_5)_{\text{ab}} = 0$.

This concludes the proof of the claim. □

⁶⁴³In fact this case does not occur, see e.g. [KSc] for details. But I don't know a quick argument which proves that $\Gamma_{\text{ab}} = 0$.

35.4. Planar graphs. We recall several definitions that we had made along the way:

- (1) On page 37 we had defined an *abstract graph* G is a triple (V, E, φ) where V is a non-empty set, E is a set and φ is a map

$$\varphi: E \rightarrow \{\text{subsets of } V \text{ with one or two elements}\}.$$

The elements of V are called *vertices of G* and the elements of E are called the *edges of G* .

- (2) On page 338 we had defined a topological graph to be a one-dimensional CW-complex.
- (a) We refer to the 0-cells as the vertices and the 1-cells as the edges of the topological graph.
 - (b) Given a topological the *underlying abstract graph* is defined as the abstract graph $(V, E, \varphi: V \rightarrow E)$ given by setting V to be the set vertices and E the set of edges and given $e \in E$ we define $\varphi(e)$ to be the vertices that lie on e .

Examples.

- (1) For $n \in \mathbb{N}$ we define the *complete graph* K_n as the abstract graph with vertex set $V = \{v_1, \dots, v_n\}$ and edge set $E = \{\{v_i, v_j\} \mid i, j \in \{1, \dots, n\} \text{ with } i \neq j\}$. Furthermore we define $\varphi: E \rightarrow V$ via $\varphi(e) = e$. In Figure 352 we show topological realizations of the complete graphs K_1, K_2, K_3, K_4 and K_5 .

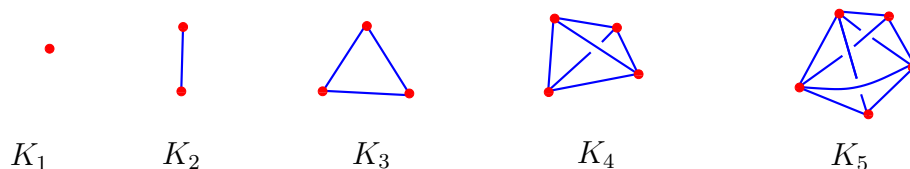


FIGURE 351.

- (2) For $m, n \in \mathbb{N}$ we define the *complete bipartite graph* $K_{m,n}$ as the abstract graph with vertex set $V = \{x_1, \dots, x_m\} \cup \{y_1, \dots, y_n\}$ and edge set $E = \{\{x_i, y_j\} \mid i \in \{1, \dots, m\} \text{ and } j \in \{1, \dots, n\}\}$. Furthermore we again define $\varphi: E \rightarrow V$ via $\varphi(e) = e$. In Figure 352 we show topological realizations of the complete bipartite graphs $K_{2,3}$, $K_{1,5}$ and $K_{3,3}$.

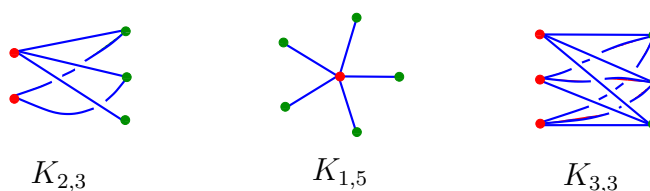


FIGURE 352.

Definition.

- (1) We say a finite topological graph G is *planar* if there exists an embedding from G into $\mathbb{R}^2$, by this we mean an injective continuous map $G: \mathbb{R}^2$ such that the restriction to the interior of every edge is smooth.
- (2) We say a finite abstract graph is *planar* if it admits a planar topological realization.

Example. The above graphs K_1, K_2, K_3 and $K_{1,5}$ are planar. The above topological realizations of K_4 and $K_{2,3}$ are not planar. But as we see in Figure 353 both graphs admit planar realizations.

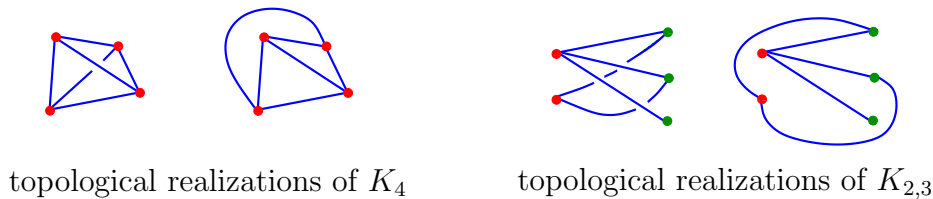


FIGURE 353.

Now the following question arises.

Question 35.7. *Are the graphs K_5 and $K_{3,3}$ also planar?*

To answer that question we need to introduce one more definition: A graph G is called *simple* if the following two conditions are satisfied:

- (1) the graph G has no loops, i.e. if there is no edge where the two endpoints agree,
- (2) the graph G does not have multiple edges, i.e. there are no two edges with the same endpoints.

Clearly the complete graphs K_n and the complete bipartite graphs $K_{n,m}$ are simple.

Proposition 35.8. *Let G be a simple graph with v vertices and e edges. If G is planar, then*

$$e \leq 3v - 6.$$

Sketch of proof. Let G be a simple graph with v vertices and e edges which admits a planar realization. By abuse of notation we denote the planar realization by G as well. Without loss of generality we can assume that G lies in $B := B_1(0)$.

We now view G as a subset of the sphere $S^2 = \overline{B}/S^1$. We refer to the closures of the components of $S^2 \setminus G$ as *faces* and we denote by f the number of faces. We make the following observation:

- (1) The vertices, edges and faces define a CW-structure for $S^2 = \overline{B}/S^1$.⁶⁴⁴

⁶⁴⁴Here is the reason why it is the sketch of a proof: we do not prove that each face is in fact homeomorphic to a closed disk. This statement is “clear from the picture”, but it is rather painful to give a rigorous proof.

- (2) Since G is simple each face is bounded by at least three edges. Each edge is the boundary of two faces. Hence we get $3f \leq 2e$.

We now see that

$$\begin{array}{ccccccc} 2 & = & \chi(S^2) & = & v - e + f & \leq & v - e + \frac{2}{3}e = v - \frac{1}{3}e. \\ \uparrow & & & & \uparrow & & \\ & & \text{see page 575} & & \text{since } 3f \leq 2e & & \end{array}$$

Solving for e we get the desired inequality $e \leq 3v - 6$. $\square$

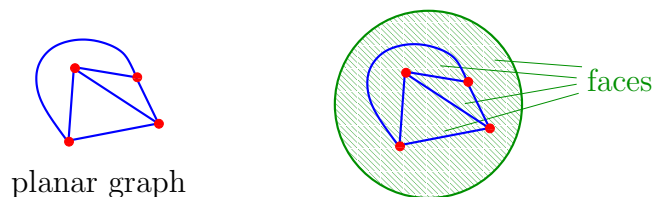


FIGURE 354.

It follows immediately from Proposition 35.8 and an elementary calculation that K_5 and $K_{3,3}$ are not planar. This gives a negative answer to Question 35.7. Evidently, if a graph G contains K_5 or $K_{3,3}$ as a subgraph, then G itself can also not be planar.

Remark. John Conway and Cameron Gordon [CG] showed in 1984 that the image of any embedding of K_7 in $\mathbb{R}^3$ contains a non-trivial knot. More precisely, for any embedding $h: K_7 \rightarrow \mathbb{R}^3$ the image $h(K_7)$ contains a closed loop C that is knotted in the sense that $\pi_1(\mathbb{R}^3 \setminus C)$ is not isomorphic to $\mathbb{Z}$.

Recall that the *valence* of a vertex is defined as the number of edges that contain the vertex.

- (1) If a simple graph admits a vertex v of valence one, then v is the endpoint of a single edge e . If we delete v and e we obtain a new graph topological realization that is homotopy equivalent to the previous one.
- (2) If a simple graph admits a vertex v of valence two, then v is the endpoint of the two edges e_1 and e_2 . We can combine e_1 and e_2 into one edge and eliminate v . This way we obtain a new graph with a homeomorphic topological realization.

This shows that for the most part we can restrict ourselves to simple graphs where every vertex has valence at least three.

If we restrict ourselves to simple graphs such that every vertex has valence at least three, then the graphs K_5 and $K_{3,3}$ turn out to be the only obstruction to being planar. More precisely the following theorem was proved by Kuratowski ⁶⁴⁵ in 1930. We refer to [BM, Chapter 10.5] for a proof.

⁶⁴⁵Kazimierz Kuratowski (1896-1980) was a Polish mathematician.

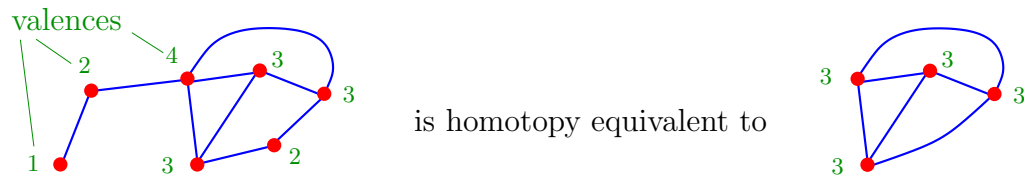


FIGURE 355.

Theorem 35.9. (Kuratowski's Reduction Theorem) *Let G be a finite simple graph such that every vertex has valence at least three. Then the following holds:*

$$G \text{ is planar} \iff G \text{ contains neither } K_5 \text{ nor } K_{3,3}.$$

36. HOMOLOGY WITH COEFFICIENTS

36.1. Chain complexes over rings. Let R be a commutative ring.⁶⁴⁶ A *chain complex* (C_*, ∂_*) of R -modules is a sequence

$$\rightarrow C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \dots C_1 \xrightarrow{\partial_1} C_0 \rightarrow 0$$

of homomorphisms between R -modules such that for all i we have $\partial_{i-1} \circ \partial_i = 0$. We define as usual the n -th homology of the chain complex as

$$H_n(C) := \ker\{\partial_n: C_n \rightarrow C_{n-1}\} / \operatorname{im}\{\partial_{n+1}: C_{n+1} \rightarrow C_n\}.$$

Hereby $H_n(C)$ is the quotient of two R -modules, thus it is itself an R -module.

Most of the results which we have proved for chain complexes of free abelian groups hold, with the obvious modifications, also for chain complexes of R -modules. In particular one can easily convince oneself that Lemma 24.7, Lemma 25.2, Corollary 25.3 and Propositions 26.9 and Lemma 26.8 also hold in this more general context.

36.2. The tensor product of abelian groups. Let A and B abelian groups. We define the *tensor product* $A \otimes B$ as follows:

$$A \otimes B := \left\{ \begin{array}{l} \text{free abelian group which is generated by all} \\ \text{symbols of the form } a \otimes b \text{ with } a \in A, b \in B \end{array} \right\} / \sim$$

where for all $a, a' \in A$ and $b, b' \in B$ we have

$$\begin{aligned} (a + a') \otimes b &\sim a \otimes b + a' \otimes b, \\ a \otimes (b + b') &\sim a \otimes b + a \otimes b'. \end{aligned}$$

For $a \in A$ and $b \in B$ we denote the image of $a \otimes b$ in $A \otimes B$ again by $a \otimes b$. Any element in $A \otimes B$ is thus of the form $\sum_{i=1}^n a_i \otimes b_i$. It follows immediately from the definition of the tensor product that it is bilinear, i.e. for any $a, a' \in A$ and $b, b' \in B$ we have

$$\begin{aligned} (a + a') \otimes b &= a \otimes b + a' \otimes b \in A \otimes B \\ a \otimes (b + b') &= a \otimes b + a \otimes b' \in A \otimes B. \end{aligned}$$

We start out with the following lemma.

Lemma 36.1. *For any abelian group A the map*

$$\begin{aligned} \Phi: \mathbb{Z} \otimes A &\rightarrow A \\ \sum_{i=1}^k n_i \otimes a_i &\mapsto \sum_{i=1}^k n_i a_i \end{aligned}$$

is well-defined and it is an isomorphism.

⁶⁴⁶In this course we mean by a *ring* an abelian group with a composition which is associative and distributive and which admits a multiplicatively neutral element. We do not assume that a ring is commutative.

Proof. It is straightforward to verify that Φ is a well-defined group homomorphism. We consider the map

$$\begin{aligned}\Psi: A &\rightarrow \mathbb{Z} \otimes A \\ a &\mapsto 1 \otimes a.\end{aligned}$$

Now it suffices to show that Φ and Ψ inverse to one another. It is clear that $\Phi \circ \Psi = \text{id}_A$. It remains to show that $\Psi \circ \Phi = \text{id}_{\mathbb{Z} \otimes A}$. Since $\mathbb{Z} \otimes A$ is generated by elements of the form $n \otimes a$ it suffices to show that

$$(\Psi \circ \Phi)(n \otimes a) = n \otimes a \in \mathbb{Z} \otimes A$$

for all $n \in \mathbb{Z}$ and $a \in A$. Thus let $n \in \mathbb{Z}$ and $a \in A$. We first consider the case that $n > 0$. Then the following holds

$$\begin{array}{ccccccc}(\Psi \circ \Phi)(n \otimes a) & = & 1 \otimes na & = & 1 \otimes \sum_{i=1}^n a & = & \sum_{i=1}^n 1 \otimes a = \left(\sum_{i=1}^n 1\right) \otimes a = n \otimes a. \\ & \uparrow & & & \uparrow & & \uparrow \\ & \text{by definition of } \Phi \text{ and } \Psi & & & \text{since the tensor product is bilinear} & & \end{array}$$

The case $n = 0$ is trivial and the case $n < 0$ is treated in a very similar way. $\square$

For abelian groups A, B, C and $A_i, i \in I$ one can now easily prove the following statements:

(1) The map

$$\begin{aligned}A \otimes B &\rightarrow B \otimes A \\ \sum_{i=1}^n a_i \otimes b_i &\mapsto \sum_{i=1}^k b_i \otimes a_i\end{aligned}$$

is a well-defined isomorphism.

(2) The map

$$\begin{aligned}\left(\bigoplus_{i \in I} A_i\right) \otimes B &\rightarrow \bigoplus_{i \in I} A_i \otimes B \\ \left(\sum_{i=1}^n a_i\right) \otimes b &\mapsto \sum_{i=1}^n a_i \otimes b\end{aligned}$$

is a well-defined isomorphism.

(3) The map

$$\begin{aligned}(A \otimes B) \otimes C &\rightarrow A \otimes (B \otimes C) \\ \sum_{j=1}^n (a_j \otimes b_j) \otimes c_j &\mapsto \sum_{j=1}^k a_j \otimes (b_j \otimes c_j)\end{aligned}$$

is a well-defined isomorphism. This isomorphism allows us to write $A \otimes B \otimes C$ unambiguously.

(4) In exercise sheet 10 we will see that for each $k \in \mathbb{N}_0$ the map

$$\begin{aligned}\mathbb{Z}_k \otimes A &\rightarrow A/kA \\ \sum_{i=1}^m n_i \otimes a_i &\mapsto \sum_{i=1}^m n_i a_i\end{aligned}$$

is a well-defined isomorphism. In particular for $k = 0$ we obtain the isomorphism

$$\begin{aligned} \mathbb{Z} \otimes A &\rightarrow A \\ \sum_{i=1}^m n_i \otimes a_i &\mapsto \sum_{i=1}^m n_i a_i. \end{aligned}$$

- (5) Let A be a finitely generated abelian group of rank r . By the classification of finitely generated abelian groups, see Theorem 8.3, we have $A \cong \mathbb{Z}^r \oplus T$ where T is the direct sum of finitely many finite cyclic groups. It follows from (2) and (4) that for $S = \mathbb{Q}$, $S = \mathbb{R}$ or $S = \mathbb{C}$ we have

$$A \otimes S \cong S^r.$$

If $f: A \rightarrow A'$ and $g: B \rightarrow B'$ are homomorphisms between abelian groups, then it is straightforward to verify that the map $f \otimes g$ defined by

$$\begin{aligned} f \otimes g: A \otimes B &\rightarrow A' \otimes B' \\ \sum_{i=1}^n a_i \otimes b_i &\mapsto \sum_{i=1}^n f(a_i) \otimes g(b_i) \end{aligned}$$

is also a homomorphism. For future reference we record the following lemma:

Lemma 36.2. *For any homomorphisms $f': A' \rightarrow A''$ and $g': B' \rightarrow B''$ between abelian groups we have*

$$(f \otimes g) \circ (f' \otimes g') = (f \circ f') \otimes (g \circ g').$$

An important special case is hereby the case that $B = R$ is a commutative ring. Then the map

$$\begin{aligned} (A \otimes R) \times R &\mapsto A \otimes R \\ \left(\sum_{i=1}^n a_i \otimes b_i, r \right) &\mapsto \sum_{i=1}^n a_i \otimes b_i r \end{aligned}$$

defines an R -module structure on $A \otimes R$. For example, if A is an abelian group, then $A \otimes \mathbb{Q}$ is a rational vector space. If A is in fact finitely generated, then it is a straightforward consequence of the classification of finitely generated groups, see Theorem 8.3, that

$$\dim_{\mathbb{Q}}(A \otimes \mathbb{Q}) = \text{rank}(A).$$

This statement will be proved in detail in exercise sheet 10.

Furthermore, if $f: A \rightarrow B$ is a homomorphism of abelian groups, then the map

$$\begin{aligned} f \otimes \text{id}: A \otimes R &\rightarrow B \otimes R \\ \sum_i a_i \otimes r_i &\mapsto \sum_i f(a_i) \otimes r_i \end{aligned}$$

is easily seen to be a homomorphism of R -modules. Now one can also show easily that

$$\begin{aligned} A &\mapsto A \otimes R \\ (f: A \rightarrow B) &\mapsto (f \otimes \text{id}: A \otimes R \rightarrow B \otimes R) \end{aligned}$$

defines a functor from the category of abelian group to the category of R -modules.

Remark. One can introduce the tensor product more generally for modules over an arbitrary commutative ring. More precisely, let R be a commutative ring and let V and W be two R -modules. We say that an R -homomorphism $f: V \times W \rightarrow X$ is *bilinear*, if for all $v, v' \in V$, $w, w' \in W$ and $r \in R$ the following holds:

$$\begin{aligned} f(v + v', w) &= f(v, w) + f(v', w), & \text{and} & & f(v, w + w') &= f(v, w) + f(v, w') \\ f(rv, w) &= rf(v, w) & & & f(v, rw) &= rf(v, w). \end{aligned}$$

We say a bilinear map $f: V \times W \rightarrow X$ has the *universal bilinear property*, if for every bilinear map $g: V \times W \rightarrow Y$ there exists a *unique* R -homomorphism $\varphi: X \rightarrow Y$ such that the following diagram commutes:

$$\begin{array}{ccc} V \times W & \xrightarrow{f} & X \\ & \searrow g & \downarrow \varphi \text{ unique} \\ & & Y. \end{array}$$

In exercise sheet 10 we will see that following statements hold:

- (a) There exists a bilinear map $f: V \times W \rightarrow X$ which has the universal bilinear property.
- (b) If $f: V \times W \rightarrow X$ and $f': V \times W \rightarrow X'$ have the universal bilinear property, then there exists a unique isomorphism $\varphi: X \rightarrow X'$ of R -modules such that $\varphi(f(v, w)) = f'(v, w)$ for all $v \in V$ and $w \in W$.

We denote by $V \otimes_R W$ the, up to isomorphism, uniquely determined R -module that satisfies (a) and (b). Note that we have a uniquely determined map $V \times W \rightarrow V \otimes_R W$ defined by $(v, w) \mapsto v \otimes w$. If $R = \mathbb{Z}$, then we obtain the tensor product of abelian groups that we have defined above, and we write $V \otimes W$ instead of $V \otimes_{\mathbb{Z}} W$.

We conclude this section with the following lemma.

Lemma 36.3. *Let*

$$0 \rightarrow A \xrightarrow{i} B \xrightarrow{p} C \rightarrow 0$$

be a short exact sequence and let G be an abelian group. If the short exact sequence splits, then

$$0 \rightarrow A \otimes G \xrightarrow{i \otimes \text{id}} B \otimes G \xrightarrow{p \otimes \text{id}} C \otimes G \rightarrow 0$$

is exact.

Proof. According to Lemma 28.2 we can without loss of generality suppose that the short exact sequence is of the form

$$0 \rightarrow A \xrightarrow{a \mapsto (a, 0)} A \oplus C \xrightarrow{(a, c) \mapsto c} C \rightarrow 0.$$

If we tensor this short exact sequence with G , then using $(A \oplus C) \otimes G = A \otimes G \oplus C \otimes G$ we obtain the sequence

$$0 \rightarrow A \otimes G \xrightarrow{\sum_i a_i \otimes g_i \mapsto \left(\sum_i a_i \otimes g_i, 0 \right)} A \otimes G \oplus C \otimes G \xrightarrow{\left(\sum_i a_i \otimes g_i, \sum_j c_j \otimes g_j \right) \mapsto \sum_j c_j \otimes g_j} C \otimes G \rightarrow 0,$$

which is evidently exact. $\square$

36.3. The tensor product of a chain complex with an abelian group. Let

$$\rightarrow C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \dots C_1 \xrightarrow{\partial_1} C_0 \rightarrow 0$$

be an algebraic chain complex and let G be an abelian group. Then we can tensor the chain complex with G , i.e. we consider the following sequence of G -modules and G -module homomorphisms:

$$\rightarrow C_n \otimes G \xrightarrow{\partial_n \otimes \text{id}} C_{n-1} \otimes G \xrightarrow{\partial_{n-1} \otimes \text{id}} \dots C_1 \otimes G \xrightarrow{\partial_1 \otimes \text{id}} C_0 \otimes G \rightarrow 0.$$

For all i we have

$$(\partial_{i-1} \otimes \text{id}) \circ (\partial_i \otimes \text{id}) = (\partial_{i-1} \circ \partial_i) \otimes \text{id} = 0 \otimes \text{id} = 0,$$

$\uparrow$
Lemma 36.2

i.e. the above sequence of maps is again a chain complex.

Now let X be a topological space. For an abelian group G we define

$$C_*(X; G) := C_*(X) \otimes G$$

and we denote by

$$H_k(X; G) := H_k(C_*(X; G), \partial)$$

the corresponding homology groups. We then call $H_k(X; G)$ the k -th homology group of X with G -coefficients. We define the notions of “cycle”, “homologous” and “null-homologous” in exactly the same way as on page 404.

Example. We consider an electric circuit as a finite graph X where at the vertices we have resistors or capacitors. Kirchhoff’s current rule, see e.g.

https://en.wikipedia.org/wiki/Kirchhoff's_circuit_laws

states that

At any node (vertex) in an electrical circuit, the sum of currents flowing into that node is equal to the sum of currents flowing out of that node,

or equivalently

The algebraic sum of currents in a network meeting at a point is zero.

Put differently, a current defines a cycle in $C_1(X; \mathbb{R})$.⁶⁴⁷

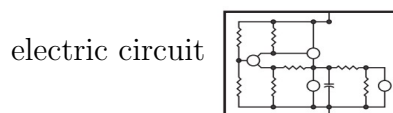


FIGURE 356.

⁶⁴⁷This observation and the figure are due to Ghrist’s excellent book on applied topology [Gh].

We start out with the following two observations. Let X be a topological space.

- (1) If the coefficients are given by a commutative ring R , e.g. if $R = \mathbb{Z}_n$ or $R = \mathbb{Q}, \mathbb{R}, \mathbb{C}$, then each $C_n(X; R) = C_n(X) \otimes R$ is an R -module, each boundary map is an R -module homomorphism, and each homology group $H_n(X; R)$ is also an R -module.
- (2) From property (4) of the tensor product stated on page 600 it follows that we have natural isomorphisms $C_*(X; \mathbb{Z}) \cong C_*(X)$ and $H_*(X; \mathbb{Z}) \cong H_*(X)$. In the following we will identify these groups and we will go back and forth between these two notations, depending on which one is more convenient.

Most results for homology groups of topological spaces carry over, with the obvious modifications, to the theory of homology groups with R -coefficients. For example we have the following statements:

- (1) A map $f: X \rightarrow Y$ between topological spaces induces a chain map of R -homomorphism $C_*(X; R) \rightarrow C_*(Y; R)$, which in turn, for each k , induces an R -homomorphism $f_*: H_k(X; R) \rightarrow H_k(Y; R)$. Furthermore the k -th homology group with R -coefficients defines a functor from the category of topological spaces to the category of R -module.
- (2) If X is path-connected, then the argument of Proposition 24.5 implies that $H_0(X; R) = R$.
- (3) We had seen in Proposition refprop:homotopic maps that if $f, g: X \rightarrow Y$ are homotopic two maps between topological spaces, then the f_* and g_* are chain homotopic maps $C_*(X) \rightarrow C_*(Y)$. Then the maps $C_*(X) \otimes R \rightarrow C_*(Y) \otimes R$ are also chain homotopic, and we see that

$$f_* = g_*: H_n(X; R) \rightarrow H_n(Y; R).$$

- (4) For a pair (X, A) of topological space we can also tensor the relative chain complex $C_*(X, A)$ with R to obtain the chain complex $C_*(X, A; R) := C_*(X, A) \otimes R$. The corresponding homology groups are the relative homology groups $H_k(X, A; R)$.
- (5) Given a triple (X, B, A) of topological spaces we also obtain a short exact sequence of chain complexes $0 \rightarrow C_*(B, A; R) \rightarrow C_*(X, A; R) \rightarrow C_*(X, B; R) \rightarrow 0$. By applying the obvious version of Proposition 26.9 to this short exact sequence we obtain the following long exact sequence of homology groups with R -coefficients:

$$\dots \rightarrow H_n(B, A; R) \xrightarrow{i_*} H_n(X, A; R) \xrightarrow{p_*} H_n(X, B; R) \xrightarrow{\partial} H_{n-1}(B, A; R) \rightarrow \dots$$

- (6) We have an obvious version of the Excision Theorem 26.15 and the Mayer-Vietoris Theorem 28.5 for homology groups with R -coefficients.
- (7) For a CW-complexes X we consider $C_*^{\text{CW}}(X; R) := C_*^{\text{CW}}(X) \otimes R$ and the corresponding cellular homology groups $H_*^{\text{CW}}(X; R)$. Then we have an isomorphism $H_*^{\text{CW}}(X; R) \cong H_*(X)$.
- (8) If X is a finite CW-complex and if $\mathbb{F}$ is a field, then we will see in exercise sheet 10 that

$$\chi(X) = \sum_i (-1)^i \dim_{\mathbb{F}}(H_i(X; \mathbb{F})).$$

We also have the following elementary lemma.

Lemma 36.4. *The homology groups are “functorial in the coefficients”, i.e. if $\varphi: A \rightarrow B$ is a homomorphism of abelian groups, then we get induced maps $\varphi_*: H_n(X; A) \rightarrow H_n(X; B)$. In fact these maps define a natural transformation from “homology with A -coefficients” to “homology with B -coefficients”, i.e. given a map $f: X \rightarrow Y$ we get a commutative diagram*

$$\begin{array}{ccc} H_n(X; A) & \xrightarrow{f_*} & H_n(Y; A) \\ \downarrow \varphi_* & & \downarrow \varphi_* \\ H_n(X; B) & \xrightarrow{f_*} & H_n(Y; B). \end{array}$$

Proof. The lemma follows immediately from the definitions. $\square$

Remark. Why should we study the homology groups of a topological space with R -coefficients? There are several answers one can give:

- (1) At times it is easier to work over a field instead of working over the ring $\mathbb{Z}$. For example, if we only want to determine the Euler characteristic of a CW-complex we can now use a field, even a very simple one like $\mathbb{F}_2$, instead of working with the integers.
- (2) The calculations become much simpler if we work over a finite field. For example if we work over the field $R = \mathbb{F}_2$ with two elements, then we do not need to worry about signs.
- (3) In Analysis IV we had used differential forms to associate to a manifold M its de Rham cohomology groups $H_{\text{dR}}^k(M)$ ⁶⁴⁸ which are real vector spaces. For each k we can now also study the homology groups $H_k(M; \mathbb{R})$ which are also real vector spaces. The notations $H_{\text{dR}}^k(M)$ and $H_k(M; \mathbb{R})$ look similar and the names “de Rham cohomology” and “homology” also sound similar. We will study the relationship between these two objects later on in the course.

One can now ask whether there exists a connection between the several homology groups exist. For example one could ask, whether $H_k(X; R) = H_k(X) \otimes R$. But this is not the case. For instance consider the real projective plane $\mathbb{RP}^2$ with the CW-structure with one 0-cell, of one 1-cell and of one 2-cell that we had introduced on page 520. In the proof of Lemma 29.19 we had seen that the cellular chain complex is given by

$$0 \rightarrow \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \rightarrow 0,$$

i.e. $H_2(\mathbb{RP}^2) = 0$, $H_1(\mathbb{RP}^2) = \mathbb{Z}_2$ and $H_0(\mathbb{RP}^2) = \mathbb{Z}$. On the other hand, if we tensor the above cellular chain complex with $\mathbb{F}_2$ tensor, then we obtain the chain complex

$$0 \rightarrow \mathbb{F}_2 \xrightarrow{\cdot 2=0} \mathbb{F}_2 \xrightarrow{0} \mathbb{F}_2 \rightarrow 0,$$

⁶⁴⁸In Analysis IV we had written $H^k(M)$ for the k -th de Rham cohomology groups. We write $H_{\text{dR}}^k(M)$ instead to remember the source of the definition.

i.e. $H_2(\mathbb{RP}^2; \mathbb{F}_2) = \mathbb{F}_2$, $H_1(\mathbb{RP}^2; \mathbb{F}_2) = \mathbb{F}_2$ and $H_0(\mathbb{RP}^2; \mathbb{F}_2) = \mathbb{F}_2$. Thus we see that

$$\mathbb{F}_2 = H_2(X; \mathbb{F}_2) \neq H_2(X) \otimes \mathbb{F}_2 = 0.$$

It is worth noting that even though the homology groups for $\mathbb{Z}$ -coefficients and $\mathbb{F}_2$ -coefficients are quite different, in both cases the alternating sums of the dimensions and ranks give, as promised in Proposition 34.1 and in (8) above, the Euler characteristic. Indeed, we have

$$\text{rank}(H_0(X)) - \text{rank}(H_1(X)) + \text{rank}(H_2(X)) = 1 - 0 + 0 = 1,$$

and

$$\dim_{\mathbb{F}_2}(H_0(X; \mathbb{F}_2)) - \dim_{\mathbb{F}_2}(H_1(X; \mathbb{F}_2)) + \dim_{\mathbb{F}_2}(H_2(X; \mathbb{F}_2)) = 1 - 1 + 1 = 1.$$

In the following chapter we will see that, despite this example, one can always deduce the homology groups $H_*(X; R)$ from the homology groups $H_*(X)$. But the connection between these homology groups will turn out to be less naive than one might have thought initially.

36.4. Free resolutions and the G -torsion of an abelian group. As a reminder, we denote by $\mathcal{C}$ the category of chain complexes, hereby the objects are given by

$$\text{Ob}(\mathcal{C}) := \text{all chain complexes},$$

and the morphisms are given by

$$\text{Mor}(C_*, D_*) := \text{all chain maps of } C_* \text{ to } D_*.$$

Let $F: \mathcal{C} \rightarrow \mathcal{C}$ be a functor.

(1) We say, F is *exact*, if for every exact sequence

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{j} C \longrightarrow 0$$

of abelian groups the sequence

$$0 \longrightarrow F(A) \xrightarrow{F(i)} F(B) \xrightarrow{F(j)} F(C) \longrightarrow 0$$

is also exact.

(2) We say, F is *left-exact*, if for every exact sequence

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{j} C$$

of abelian groups the sequence

$$0 \longrightarrow F(A) \xrightarrow{F(i)} F(B) \xrightarrow{F(j)} F(C)$$

is also exact.

(3) We say, F is *right-exact*, if for every exact sequence

$$A \xrightarrow{i} B \xrightarrow{j} C \longrightarrow 0$$

of abelian groups the sequence

$$F(A) \xrightarrow{F(i)} F(B) \xrightarrow{F(j)} F(C) \longrightarrow 0$$

is also exact.

If a functor is left-exact and right-exact, then the functor is evidently also exact.

In the previous chapter we had already implicitly that tensoring with an abelian group G is in general not left-exact. For example for any $k \in \mathbb{N}$ the sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot k} \mathbb{Z} \longrightarrow \mathbb{Z}_k$$

is exact, but after tensoring with $\otimes \mathbb{Z}_k$ we obtain the sequence

$$0 \longrightarrow \mathbb{Z}_k \xrightarrow[=0]{\cdot k} \mathbb{Z}_k \longrightarrow \mathbb{Z}_k$$

that is not exact. But we will now see that tensoring with an abelian group G is always right-exact.

Lemma 36.5. *Let*

$$A \xrightarrow{i} B \xrightarrow{j} C \rightarrow 0$$

be an exact sequence of abelian groups. Then for each abelian group the sequence

$$A \otimes G \xrightarrow{i \otimes \text{id}} B \otimes G \xrightarrow{j \otimes \text{id}} C \otimes G \rightarrow 0$$

is also exact.

Proof. By Lemma 36.2 we have $(i \otimes \text{id}) \circ (j \otimes \text{id}) = (i \circ j) \otimes \text{id} = 0$, i.e. the composition of the maps in the lower sequence is the zero map. Since $j: B \rightarrow C$ is surjective it follows easily that the map $j \otimes \text{id}: B \otimes G \rightarrow C \otimes G$ is also surjective.⁶⁴⁹ Thus it remains to show that the induced map

$$j \otimes \text{id}: B \otimes G / \text{im}\{i \otimes \text{id}: A \otimes G \rightarrow B \otimes G\} \rightarrow C \otimes G$$

is an isomorphism. Since, as we had just seen, the map is surjective, it suffices to show that the map is injective. We will do so by providing a right-inverse for $j \otimes \text{id}$, i.e. we will provide a map φ with $\varphi \circ (j \otimes \text{id}) = \text{id}$.

We consider the map

$$\begin{aligned} \varphi: C \times G &\rightarrow B \otimes G / \text{im}\{i \otimes \text{id}: A \otimes G \rightarrow B \otimes G\} \\ (c, g) &\mapsto b \otimes g \quad \text{where } j(b) = c. \end{aligned}$$

The map is well-defined, since for any $b, b' \in B$ with $j(b) = j(b') = c$ we have $j(b - b') = 0$. From the exactness of the sequence at B it follows that $b - b' = i(a)$ for an $a \in A$. Then it follows that

$$b \otimes g - b' \otimes g = (b - b') \otimes g = i(a) \otimes g \in \text{im}\{i \otimes \text{id}: A \otimes G \rightarrow B \otimes G\}.$$

Thus we have proved that φ is well-defined.

⁶⁴⁹Indeed, if $\sum_{i=1}^n c_i \otimes g_i$ is an element in $C \otimes G$, then there exist by our hypothesis $b_i \in B$ with $j(b_i) = c_i$,

but then we have $(j \otimes \text{id})\left(\sum_{i=1}^n b_i \otimes g_i\right) = \sum_{i=1}^n c_i \otimes g_i$.

Since φ is linear in C and in G the map φ induces a map

$$\begin{aligned} C \otimes G &\rightarrow B \otimes G / \text{im}\{i \otimes \text{id}: A \otimes G \rightarrow B \otimes G\} \\ \sum_{i=1}^k c_i \otimes g_i &\mapsto \sum_{i=1}^k \varphi(c_i, g_i). \end{aligned}$$

The map is evidently a right-inverse to the map

$$j \otimes \text{id}: B \otimes G / \text{im}\{i \otimes \text{id}: A \otimes G \rightarrow B \otimes G\} \rightarrow C \otimes G. \quad \square$$

Now we want to determine more precisely, to what degree tensoring with G is not left exact. Before we can do so we need to recall the definition of a free abelian group that we had already given on page 137 and we need to introduce one new definition.

- (1) Let S be a non-empty set. We refer to

$\mathbb{Z}^{(S)} :=$ all maps from S to $\mathbb{Z}$ which are non-zero for only finitely many $s \in S$

as the free abelian group generated by S .

- (2) A group G is called *free abelian* if there exists a set S and an isomorphism $G \rightarrow \mathbb{Z}^{(S)}$.⁶⁵⁰

- (3) A *free resolution* of an abelian group H is an exact sequence

$$\dots F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \rightarrow 0$$

where the F_i are free abelian groups.

Examples.

- (1) Let G be a *finitely generated* abelian group. By the classification of finitely generated abelian groups, see Theorem 8.3 there exist $r \in \mathbb{N}_0$, non-zero natural numbers $a_1, \dots, a_k$ and an isomorphism

$$\Phi: G \xrightarrow{\cong} \mathbb{Z}^r \oplus \bigoplus_{i=1}^k \mathbb{Z}_{a_i}.$$

Thus we obtain a free resolution

$$0 \rightarrow \mathbb{Z}^k \xrightarrow{0 \oplus \begin{pmatrix} a_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & a_k \end{pmatrix}} \mathbb{Z}^r \oplus \mathbb{Z}^k \rightarrow \underbrace{\mathbb{Z}^r \oplus \bigoplus_{i=1}^k \mathbb{Z}_{a_i}}_{\cong G} \rightarrow 0.$$

In most cases when we want to work with explicit examples this resolution will turn out to be the most useful resolution.

⁶⁵⁰It follows easily from the definitions that group G is free abelian if there exists a subset $\mathcal{B} = \{b_i\}_{i \in I} \subset G$ with following property: For any abelian group A and each choice of elements $a_i \in A, i \in I$ there exists a unique homomorphism $\varphi: G \rightarrow A$ such that $\varphi(b_i) = a_i$ for all $i \in I$. We call a such subset $\mathcal{B}$ a basis of G .

- (2) Before we give the second example of a free resolution we recall the following: if M is a group, then by Lemma 8.1 there exists a unique homomorphism $\alpha(M): \mathbb{Z}^{(M)} \rightarrow M$ such that for each $m \in M$ we have $\alpha(M)(m) = m$. Evidently this map is an epimorphism.

Now let H be an abelian group. We consider

$$\mathbb{Z}^{(H)} \xrightarrow{\alpha(H)} H \rightarrow 0.$$

The group $\mathbb{Z}^{(H)}$ is free abelian and $\alpha(H)$ is an epimorphism. In particular the sequence is exact. We now set $H_1 := \ker(\mathbb{Z}^{(H)} \xrightarrow{\alpha(H)} H)$. Then

$$H_1 \rightarrow \mathbb{Z}^{(H)} \xrightarrow{\alpha(H)} H \rightarrow 0$$

is exact. Now we consider

$$\mathbb{Z}^{(H_1)} \xrightarrow{\alpha(H_1)} \mathbb{Z}^{(H)} \xrightarrow{\alpha(H)} H \rightarrow 0.$$

This sequence is exact and both the groups $\mathbb{Z}^{(H_1)}$ and $\mathbb{Z}^{(H)}$ are free abelian. We now set $H_2 := \ker(\mathbb{Z}^{(H_1)} \xrightarrow{\alpha(H_1)} \mathbb{Z}^{(H)})$ and we continue this procedure. If we iterate this procedure we obtain a free resolution of H .⁶⁵¹ Note that this resolution is unambiguously defined, we did not make any choices along the way. Therefore we refer to this resolution of H as the *canonical resolution of H* .

The second example provides the proof for the following lemma.

Lemma 36.6. *Every abelian group admits a free resolution.*

Remark. One can easily generalize all the definitions and results to R -modules. More precisely, let R again be a commutative ring. We define a free R -module precisely the same way as we defined a free abelian group. A *free resolution* of an R -module M is an exact sequence

$$\dots \rightarrow F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} M \rightarrow 0$$

where each F_i is a free R -module. The proof of Lemma 36.6 now shows, with basically no modifications, that every R -module admits a free resolution.

Now we prove the following lemma:

Lemma 36.7. *Let F_* be a free resolution of an abelian group H and let F'_* be a free resolution of an abelian group H' . For any homomorphism $\alpha: H \rightarrow H'$ there exist maps $\alpha_i: F_i \rightarrow F'_i$ such that the following diagram commutes:*

$$\begin{array}{ccccccccc} \dots & \longrightarrow & F_2 & \xrightarrow{f_2} & F_1 & \xrightarrow{f_1} & F_0 & \xrightarrow{f_0} & H & \longrightarrow & 0 \\ & & \downarrow \alpha_2 & & \downarrow \alpha_1 & & \downarrow \alpha_0 & & \downarrow \alpha & & \\ \dots & \longrightarrow & F'_2 & \xrightarrow{f'_2} & F'_1 & \xrightarrow{f'_1} & F'_0 & \xrightarrow{f'_0} & H' & \longrightarrow & 0. \end{array}$$

⁶⁵¹Note that this procedure does not stop, i.e. it produces an infinite resolution.

Furthermore the maps α_i , viewed as a chain map between the horizontal chain complexes⁶⁵² are, up to chain homotopy, uniquely determined.

Proof. First we define the map α_0 . We choose a basis $\mathcal{B}_0$ of the free abelian group F_0 . For each $x_i \in \mathcal{B}_0$ we choose an $x'_i \in F'_0$ with $f'_0(x'_i) = \alpha(f_0(x_i))$. This is possible, since f'_0 is surjective. By Lemma 8.1 there exists a unique map $\alpha_0: F_0 \rightarrow F'_0$ with $\alpha_0(x_i) = x'_i$ for all $i \in I$. It is now evident that the right square commutes.

Now we turn to the definition of the map α_1 . We choose a basis $\mathcal{B}_1$ of the free abelian group F_1 . Let $x_i \in \mathcal{B}_1$. It follows from the commutativity of the square on the right that

$$f'_0((\alpha_0 \circ f_1)(x)) = \alpha((f_0 \circ f_1)(x)) = 0.$$

Since the lower sequence is exact there exists an $x'_i \in F'_1$ with $f'_1(x'_i) = (\alpha_0 \circ f_1)(x_i)$. Again by Lemma 8.1 there exist now a unique map $\alpha_1: F_1 \rightarrow F'_1$ with $\alpha_1(x_i) = x'_i$ for all $i \in I$. It is now evident that the “second square from the right” also commutes. The same way we now define the further maps α_i for $i \geq 2$.

Now let $\alpha_i, \alpha'_i: F_i \rightarrow F'_i$ be two chain maps which extend $\alpha: H \rightarrow H'$. We have to find a chain homotopy from α_i to α'_i . The maps $\beta_i = \alpha_i - \alpha'_i$ are an extension of the zero map $\beta = 0: H \rightarrow H'$. It suffices therefore to find a chain homotopy λ from $\beta_i = \alpha_i - \alpha'_i$ to the zero. The construction of the chain homotopy is very similar to the construction of the maps α_i in the first part of the proof.

We recall in the following diagram the type of map which we have to construct:

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & F_2 & \xrightarrow{f_2} & F_1 & \xrightarrow{f_1} & F_0 & \xrightarrow{f_0} & H & \longrightarrow & 0 \\ & \searrow & \downarrow \beta_2 & \swarrow \lambda_2 & \downarrow \beta_1 & \swarrow \lambda_1 & \downarrow \beta_0 & \swarrow \lambda_0 & \downarrow \beta & \swarrow \lambda_{-1} & \downarrow \beta=0 \\ \cdots & \longrightarrow & F'_2 & \xrightarrow{f'_2} & F'_1 & \xrightarrow{f'_1} & F'_0 & \xrightarrow{f'_0} & H' & \longrightarrow & 0. \end{array}$$

We put $\lambda_{-1} = 0$. First we need to find a map $\lambda_0: F_0 \rightarrow F'_1$ such that $\beta_0 = f'_1 \circ \lambda_0$. For each $x \in \mathcal{B}_0$ we choose an $x' \in F'_1$ with $f'_1(x') = \beta_0(x)$. The existence of such x' follows from the observation that $f'_0(\beta_0(x)) = \beta(f_0(x)) = 0$ and from the exactness of the lower sequence.

Now we suppose that we have already found maps $\lambda_0, \dots, \lambda_{i-1}$ which have the desired property that

$$(36.1) \quad f'_j \circ \lambda_{j-1} + \lambda_{j-2} \circ f_{j-1} = \beta_{j-1} \quad \text{for } j = 0, \dots, i.$$

Now we choose again a basis $\mathcal{B}_i$ of the free abelian group F_i . We have assign to x an element $x' \in F'_{i+1}$ which satisfies the equation

$$f'_{i+1}(x') = \beta_i(x) - \lambda_{i-1}(f_i(x)).$$

⁶⁵²Note that an exact sequence is in particular a chain complex.

From the exactness of the lower sequence it follows that $\text{im}(f'_{i+1}) = \ker(f'_i)$. Thus it suffices to show that $f'_i \circ (\beta_i - \lambda_{i-1} \circ f_i) = 0$. We have now but indeed that

$$\begin{aligned}
 f'_i \circ (\beta_i - \lambda_{i-1} \circ f_i) &= f'_i \circ \beta_i - f'_i \circ \lambda_{i-1} \circ f_i = \beta_{i-1} \circ f_i - f'_i \circ \lambda_{i-1} \circ f_i \\
 &\quad \uparrow \\
 &\quad \text{since the } \beta_i \text{'s are a chain map, i.e. } f'_i \circ \beta_i = \beta_{i-1} \circ f_i \\
 &= (\beta_{i-1} - f'_i \circ \lambda_{i-1}) \circ f_i = \lambda_{i-2} \circ f_{i-1} \circ f_i = 0. \\
 &\quad \uparrow \qquad \qquad \qquad \uparrow \\
 &\quad \text{by (36.1)} \qquad \qquad \qquad \text{since } f_{i-1} \circ f_i = 0
 \end{aligned}$$

□

Using Lemma 36.7 we can now prove the following lemma:

Lemma 36.8. *Let*

$$\dots F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \rightarrow 0$$

and

$$\dots F'_2 \xrightarrow{f'_2} F'_1 \xrightarrow{f'_1} F'_0 \xrightarrow{f'_0} H \rightarrow 0$$

be free resolutions of an abelian group H . Then there exists for each abelian group G a canonical isomorphism

$$H_n(F_* \otimes G) \xrightarrow{\cong} H_n(F'_* \otimes G).$$

Proof. It follows from Lemma 36.7 that there exist maps $\alpha_i: F_i \rightarrow F'_i$ such that the following diagram commutes:

$$\begin{array}{ccccccccccc}
 \dots & \longrightarrow & F_2 & \xrightarrow{f_2} & F_1 & \xrightarrow{f_1} & F_0 & \xrightarrow{f_0} & H & \longrightarrow & 0 \\
 & & \downarrow \alpha_2 & & \downarrow \alpha_1 & & \downarrow \alpha_0 & & \downarrow \text{id} & & \\
 \dots & \longrightarrow & F'_2 & \xrightarrow{f'_2} & F'_1 & \xrightarrow{f'_1} & F'_0 & \xrightarrow{f'_0} & H & \longrightarrow & 0
 \end{array}$$

where the maps α_i , viewed as chain maps between the horizontal chain complexes, are uniquely determined up to chain homotopy. We tensor this commutative diagram now with G and we obtain the following commutative diagram:

$$\begin{array}{ccccccccccc}
 \dots & \longrightarrow & F_2 \otimes G & \xrightarrow{f_2 \otimes \text{id}} & F_1 \otimes G & \xrightarrow{f_1 \otimes \text{id}} & F_0 \otimes G & \xrightarrow{f_0 \otimes \text{id}} & H \otimes G & \longrightarrow & 0 \\
 & & \downarrow \alpha_2 & & \downarrow \alpha_1 & & \downarrow \alpha_0 & & \downarrow \text{id} & & \\
 \dots & \longrightarrow & F'_2 \otimes G & \xrightarrow{f'_2 \otimes \text{id}} & F'_1 \otimes G & \xrightarrow{f'_1 \otimes \text{id}} & F'_0 \otimes G & \xrightarrow{f'_0 \otimes \text{id}} & H \otimes G & \longrightarrow & 0.
 \end{array}$$

The maps then induce the maps

$$\alpha_*: H_n(F_* \otimes G) \rightarrow H_n(F'_* \otimes G).$$

The map is uniquely determined, since the chain maps are uniquely determined up to chain homotopy.

Now we have to still show that the maps $\alpha_*: H_n(F_* \otimes G) \rightarrow H_n(F'_* \otimes G)$ are isomorphisms. We will do this by construction an inverse map to the map α_* . We apply Lemma 36.7 to

the resolutions F'_* and F_* (i.e. we swap the roles of the resolutions) and we obtain the lower half of the following commutative diagram.

$$\begin{array}{ccccccccc}
 \dots & \longrightarrow & F_2 & \xrightarrow{f_2} & F_1 & \xrightarrow{f_1} & F_0 & \xrightarrow{f_0} & H & \longrightarrow & 0 \\
 & & \downarrow \alpha_2 & & \downarrow \alpha_1 & & \downarrow \alpha_0 & & \downarrow \text{id} & & \\
 \dots & \longrightarrow & F'_2 & \xrightarrow{f'_2} & F'_1 & \xrightarrow{f'_1} & F'_0 & \xrightarrow{f'_0} & H & \longrightarrow & 0 \\
 & & \downarrow \beta_2 & & \downarrow \beta_1 & & \downarrow \beta_0 & & \downarrow \text{id} & & \\
 \dots & \longrightarrow & F_2 & \xrightarrow{f_2} & F_1 & \xrightarrow{f_1} & F_0 & \xrightarrow{f_0} & H & \longrightarrow & 0.
 \end{array}$$

The maps $\beta_i \circ \alpha_i$ now give the following commutative diagram:

$$\begin{array}{ccccccccc}
 \dots & \longrightarrow & F_2 & \xrightarrow{f_2} & F_1 & \xrightarrow{f_1} & F_0 & \xrightarrow{f_0} & H & \longrightarrow & 0 \\
 & & \downarrow \beta_2 \circ \alpha_2 & & \downarrow \beta_1 \circ \alpha_1 & & \downarrow \beta_0 \circ \alpha_0 & & \downarrow \text{id} & & \\
 \dots & \longrightarrow & F_2 & \xrightarrow{f_2} & F_1 & \xrightarrow{f_1} & F_0 & \xrightarrow{f_0} & H & \longrightarrow & 0.
 \end{array}$$

On the other hand the maps $\text{id}: F_i \rightarrow F_i$ define a commutative diagram. From Lemma 36.7 it now follows that the maps $\beta_i \circ \alpha_i$ are chain homotopic to the identity. Therefore the maps $(\beta_i \circ \alpha_i) \otimes \text{id}: F_i \otimes G \rightarrow F_i \otimes G$ are chain homotopic to the identity. Thus it follows that $\beta_* \circ \alpha_*: H_n(F_* \otimes G) \rightarrow H_n(F_* \otimes G)$ are the identity maps. The same way one shows that the maps $\alpha_* \circ \beta_*$ are the identity maps. Thus we have proved that the maps α_* are isomorphisms. $\square$

Definition. Let G be and H abelian groups. We denote by

$$\dots \rightarrow F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \rightarrow 0$$

the canonical resolution from page 608. We define the n -th G -torsion group of H as

$$\text{Tor}_n(H, G) := H_n(F_* \otimes G).$$

The following lemma is an immediate consequence of Lemma 36.8.

Lemma 36.9. *Let G be and H abelian groups and let be*

$$\dots \rightarrow F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \rightarrow 0$$

any resolution of H . Then there exists a canonical isomorphism

$$\text{Tor}_n(H, G) \xrightarrow{\cong} H_n(F_* \otimes G).$$

In many situations we only want to determine the isomorphism type of $\text{Tor}_n(H, G)$ where H is a finitely generated abelian group. In such situations it is usually much more convenient to work with the resolution given on page 607.

Let H, H' and G be abelian groups and let $\alpha: H \rightarrow H'$ be a homomorphism. It follows from Lemma 36.7 that α induces for each n a homomorphism

$$\alpha_*: \text{Tor}_n(H, G) \rightarrow \text{Tor}_n(H', G).$$

One can now easily show that the maps given by

$$\begin{aligned} H &\mapsto \operatorname{Tor}_n(H, G) \\ (\alpha: H \rightarrow H') &\mapsto (\alpha_*: \operatorname{Tor}_n(H, G) \rightarrow \operatorname{Tor}_n(H', G)) \end{aligned}$$

define a covariant functor from the category of abelian groups to the category of abelian groups.

The following lemma shows that the 0-th torsion groups are objects that we are already familiar with.

Lemma 36.10. *Let G and H be abelian groups, then there exists a natural isomorphism*

$$\operatorname{Tor}_0(H, G) \xrightarrow{\cong} H \otimes G.$$

In the remainder of this course we will use the isomorphism from Lemma 36.10 to identify the groups $\operatorname{Tor}_0(H, G)$ and $H \otimes G$.

Proof. Let G and H be abelian groups. Let

$$\dots F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \rightarrow 0$$

be the canonical resolution of H . By definition we have

$$\operatorname{Tor}_0(H, G) = H_0(\dots F_1 \otimes G \xrightarrow{f_1 \otimes \operatorname{id}} F_0 \otimes G \rightarrow 0) = \operatorname{coker}(F_1 \otimes G \xrightarrow{f_1 \otimes \operatorname{id}} F_0 \otimes G).$$

On the other hand, if we tensor the above exact sequence with G , then it follows from Lemma 36.5 that the “right hand part” of the sequence is still exact i.e. the sequence

$$F_1 \otimes G \xrightarrow{f_1 \otimes \operatorname{id}} F_0 \otimes G \xrightarrow{f_0 \otimes \operatorname{id}} H \otimes G \rightarrow 0$$

is exact. We now obtain the following isomorphism

$$\begin{aligned} \operatorname{Tor}_0(H, G) &= \operatorname{coker}(F_1 \otimes G \xrightarrow{f_1 \otimes \operatorname{id}} F_0 \otimes G) = (F_0 \otimes G) / \operatorname{im}(f_1 \otimes \operatorname{id}) \\ &= (F_0 \otimes G) / \ker(f_0 \otimes \operatorname{id}) \xrightarrow[\cong]{f_0 \otimes \operatorname{id}} \operatorname{im}(f_0 \otimes \operatorname{id}: F_0 \otimes G \rightarrow H \otimes G) \\ &= H \otimes G. \end{aligned}$$

It is straightforward to verify that these isomorphisms define in fact a natural isomorphism from the functor $H \mapsto \operatorname{Tor}_0(H, G)$ to the functor $H \mapsto H \otimes G$. $\square$

Remark. Without any major modifications one can carry through the above discussion for modules over a commutative ring R . Indeed, if M and N are modules over R , then we choose a free resolution

$$\dots F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} M \rightarrow 0$$

and we define

$$\operatorname{Tor}_n^R(M, N) := H_n(F_* \otimes_R N).$$

⁶⁵³Recall that natural isomorphism means that we have a natural isomorphism from the functor $H \mapsto \operatorname{Tor}_0(H, G)$ to the functor $H \mapsto H \otimes G$.

This definition does not depend on the choice of the free resolution. For each n the map $N \mapsto \operatorname{Tor}_n^R(M, N)$ defines a functor from the category of R -modules to itself. Furthermore given any two R -modules M and N we have a natural isomorphism $\operatorname{Tor}_0^R(M, N) = M \otimes_R N$.

The following lemma shows that we do not need to worry about higher torsion groups for abelian groups.

Lemma 36.11. *For any two abelian groups G and H we have $\operatorname{Tor}_n(H, G) = 0$ for $n \geq 2$.*

Proof. We first prove the lemma if H is finitely generated. On page 607 we had seen that H admits a free resolution of length 1, i.e. a free resolution of the form

$$0 \rightarrow F_1 \rightarrow F_0 \rightarrow H \rightarrow 0.$$

But then it follows from the definitions that $\operatorname{Tor}_n(H, G) = 0$ for $n \geq 2$.

Now suppose that H is any abelian group. We have the exact sequence

$$0 \rightarrow \ker(\alpha(H): \mathbb{Z}^{(H)} \rightarrow H) \rightarrow \mathbb{Z}^{(H)} \xrightarrow{\alpha(H)} H \rightarrow 0.$$

where $\alpha(H)$ is defined as in the proof of Lemma 36.6. It follows from Lemma 8.2 that the group $\ker(\alpha(H): \mathbb{Z}^{(H)} \rightarrow H)$ as the subgroup of the free abelian group $\mathbb{Z}^{(H)}$ is itself free abelian. So in this general case we also found a free resolution of length 1, which gives us again that $\operatorname{Tor}_n(H, G) = 0$ for $n \geq 2$. $\square$

Given two abelian groups H and G we now write $\operatorname{Tor}(H, G) := \operatorname{Tor}_1(H, G)$ and we refer to this group as the G -torsion of H .⁶⁵⁴ In the light of Lemma 36.8 one can say that the G -torsion group “measures”, to what degree the tensor functor $- \otimes G$ is not left-exact.

Now we summarize several properties of $\operatorname{Tor}(H, G)$ in the following lemma.

Lemma 36.12. *Let $G, H, H_i, i \in I$ and $G_j, j \in J$ be abelian groups. Then the following holds:*

- (1) $\operatorname{Tor}(\bigoplus_i H_i, G) \cong \bigoplus_i \operatorname{Tor}(H_i, G)$,
- (2) $\operatorname{Tor}(H, \bigoplus_j G_j) \cong \bigoplus_j \operatorname{Tor}(H, G_j)$,
- (3) $\operatorname{Tor}(H, G) = 0$, if H is a torsion-free abelian group,⁶⁵⁵
- (4) $\operatorname{Tor}(H, G) = 0$, if G is a torsion-free abelian group,
- (5) if T denotes the torsion subgroup⁶⁵⁶ of H , then $\operatorname{Tor}(H, G) = \operatorname{Tor}(T, G)$,
- (6) for all $n \in \mathbb{N}$ we have an isomorphism $\operatorname{Tor}(\mathbb{Z}_n, G) \cong \ker\{G \xrightarrow{n} G\}$,

⁶⁵⁴For a free resolution $0 \rightarrow F_1 \xrightarrow{i} F_0 \xrightarrow{j} H \rightarrow 0$ we have

$$\operatorname{Tor}(H, G) = \ker\{F_1 \otimes G \xrightarrow{i \otimes \operatorname{id}} F_0 \otimes G\}.$$

⁶⁵⁵A group H is called torsion-free if H has no element of finite order. Every free abelian group is torsion-free, but not every torsion-free abelian group is a free abelian group. For example $\mathbb{Q}$ is a torsion-free abelian group, but we will see in exercise sheet 11 that $\mathbb{Q}$ is not a free abelian group.

⁶⁵⁶The torsion subgroup of an abelian group is defined as the subgroup of all elements of finite order.

(7) for all $n, m \in \mathbb{N}$ we have $\text{Tor}(\mathbb{Z}_n, \mathbb{Z}_m) \cong \mathbb{Z}_{\gcd(n, m)}$.

Using this lemma one can usually determine the G -torsion of an abelian group H without any problems. For example it follows from (3) that $\text{Tor}(H, \mathbb{Q}) = \text{Tor}(H, \mathbb{R}) = 0$ for all abelian groups H . Furthermore we will show in exercise sheet 11 that if G and H are finitely generated abelian groups, then the torsion group $\text{Tor}(H, G)$ is finite.⁶⁵⁷

Proof.

- (1) We choose for each i a free resolution F_*^i of H_i . Then $\bigoplus_i F_*^i$ is a free resolution of $\bigoplus_i H_i$, and the statement follows from Lemmas 24.12 and 36.9.
- (2) We pick a free resolution F_* for H . The statement follows easily from the canonical isomorphisms

$$F_i \otimes \bigoplus_j G_j \cong \bigoplus_j F_i \otimes G_j.$$

- (3) If H is a free abelian group, then $0 \rightarrow F_0 := H \xrightarrow{\text{id}} H \rightarrow 0$ is already a free resolution, and it is clear that $\text{Tor}(H, G) = 0$. Later on we will only use the special case that H is a free abelian group. Therefore we do not provide the proof of the general case of a torsion-free abelian group which is not free. We refer to [Ha, p. 265] for the proof.
- (4) We pick a free resolution $0 \rightarrow F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H$ for H . First we consider the case that G is a free abelian group. In this case we have $G \cong \bigoplus_I \mathbb{Z}$ for an index set I . It follows from (2) that it suffices to show that $\text{Tor}(H, \mathbb{Z}) = 0$. If we tensor the above free resolution with $\mathbb{Z}$, we obtain (up to a canonical isomorphism), the same exact sequence. In particular the first homology group of $F_* \otimes \mathbb{Z} \cong F_*$ vanishes.

Now we no longer suppose that G is free abelian, we just suppose that G is torsion-free. We denote by $\{H_i\}_{i \in I}$ the set of finitely generated subgroups of G . We write $i \leq j$ if and only if $G_i \subset G_j$. Together with the inclusion maps the groups $\{G_i\}_{i \in I}$ form a directed system. We have

$$\begin{aligned} \text{Tor}(H, G) &= \text{Tor}(H, \varinjlim G_i) = H_1(0 \rightarrow F_1 \otimes \varinjlim G_i \rightarrow F_0 \otimes \varinjlim G_i) \\ &= H_1(\varinjlim (0 \rightarrow F_1 \otimes G_i \rightarrow F_0 \otimes G_i \rightarrow 0)) \\ &\quad \uparrow \\ &\text{since the } F_i \text{ are free tensor product commutes with direct limit} \\ &= \varinjlim H_1(0 \rightarrow F_1 \otimes G_i \rightarrow F_0 \otimes G_i \rightarrow 0) = 0. \\ &\quad \uparrow \qquad \qquad \qquad = 0 \text{ since } G_i \text{ is a free abelian group} \\ &\quad \text{Lemma 29.3} \end{aligned}$$

- (5) If H is a finitely generated abelian group, then we can write H as a direct sum $F \oplus T$, where T is the torsion subgroup and where F is a free abelian group. It

⁶⁵⁷This fact explains the name “torsion group”.

follows from (1) and (2) that

$$\mathrm{Tor}(H, G) \cong \mathrm{Tor}(F \oplus T, G) \cong \mathrm{Tor}(F, G) \oplus \mathrm{Tor}(T, G) = \mathrm{Tor}(T, G).$$

For the proof in the case that H is not finitely generated we once again refer to [Ha, p. 265] for a proof.

(6) This statement follows easily from the definitions, if we use the free resolution

$$0 \rightarrow \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \rightarrow \mathbb{Z}_n \rightarrow 0$$

for $H = \mathbb{Z}_n$.

(7) The last statement follows immediately from (5). $\square$

36.5. The Universal Coefficient Theorem. Now we can formulate and prove the following theorem:

Theorem 36.13. (The Universal Coefficient Theorem) *Let (C_n, ∂_n) be a chain complex of free abelian groups and let G be an abelian group. Then for each n there exists a short exact sequence of the form*

$$0 \rightarrow H_n(C) \otimes G \xrightarrow{i_C} H_n(C; G) \xrightarrow{p_C} \mathrm{Tor}(H_{n-1}(C), G) \rightarrow 0.$$

Hereby the map i_C is given by

$$\begin{aligned} H_n(C) \otimes G &\rightarrow H_n(C; G) \\ \sum_{i=1}^k [c_i] \otimes g_i &\mapsto \sum_{i=1}^k [c_i \otimes g_i]. \end{aligned}$$

Proof. Let (C_n, ∂_n) be a chain complex of free abelian groups. For each n we consider $Z_n := \ker(\partial_n)$ and $B_n := \mathrm{im}(\partial_{n+1})$. We then obtain the following diagram

$$\begin{array}{ccccccc} & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & Z_n & \longrightarrow & C_n & \xrightarrow{\partial_n} & B_{n-1} \longrightarrow 0 \\ & & \downarrow 0 & & \downarrow & & \downarrow 0 \\ 0 & \longrightarrow & Z_{n-1} & \longrightarrow & C_{n-1} & \xrightarrow{\partial_{n-1}} & B_{n-2} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \end{array}$$

One sees easily that this diagram is commutative and that the horizontal sequences are exact. There we can view the left and the right vertical columns as chain complexes. In other words, the above diagram is a short exact sequence of chain complexes.

Now let G be an abelian group. We tensor the above short exact sequence of chain complexes with G and we obtain the following diagram:

$$\begin{array}{ccccccc}
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & Z_n \otimes G & \longrightarrow & C_n \otimes G & \xrightarrow{\partial_n \otimes \text{id}} & B_{n-1} \otimes G \longrightarrow 0 \\
 & & \downarrow 0 & & \downarrow \partial_n \otimes \text{id} & & \downarrow 0 \\
 0 & \longrightarrow & Z_{n-1} \otimes G & \longrightarrow & C_{n-1} \otimes G & \xrightarrow{\partial_{n-1} \otimes \text{id}} & B_{n-2} \otimes G \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow
 \end{array}$$

This is again a sequence of chain complexes.

The groups B_n , as subgroups of free abelian groups, are according to Lemma 8.2 again free abelian groups. It follows from Lemmas 28.1 and 36.3 that the horizontal sequences are exact. Therefore we can apply Proposition 26.9. Since the vertical boundary maps on the left and on the right are the zero maps, we obtain the following long exact sequence

$$\rightarrow B_n \otimes G \rightarrow Z_n \otimes G \rightarrow H_n(C; G) \xrightarrow{\partial} B_{n-1} \otimes G \rightarrow Z_{n-1} \otimes G \rightarrow \dots$$

It follows easily from the definition of the boundary map of the long exact sequence that the boundary map $B_n \otimes G \rightarrow Z_n \otimes G$ is the map that is induced by inclusion $i_n: B_n \rightarrow Z_n$.⁶⁵⁸

Now we obtain the following short exact sequence

$$0 \rightarrow \text{coker}(i_n \otimes \text{id}) \rightarrow H_n(C; G) \rightarrow \ker(i_{n-1} \otimes \text{id}) \rightarrow 0.$$

We still have to show that the expressions left and right coincide with the desired expressions.

We consider the exact sequence

$$0 \rightarrow B_n \xrightarrow{i_n} Z_n \rightarrow H_n(C) \rightarrow 0.$$

It follows again from Lemma 8.2 that B_n and Z_n are free groups, i.e. this the above exact sequence is a free resolution of $H_n(C)$. It follows from the definition of $\text{Tor}_0(H_n(C), G)$ and from Lemma 36.10 that

$$\text{coker}(i \otimes \text{id}) = \text{Tor}_0(H_n(C), G) = H_n(C) \otimes G.$$

On the other hand the sequence

$$0 \rightarrow B_{n-1} \xrightarrow{i_{n-1}} Z_{n-1} \rightarrow H_{n-1}(C) \rightarrow 0$$

is a free resolution of $H_{n-1}(C)$. Thus it follows from the definition of the G -torsion of $H_{n-1}(C)$ that

$$\ker(i_{n-1} \otimes \text{id}) = \text{Tor}(H_{n-1}(C), G).$$

□

⁶⁵⁸Why is this the case?

Remark. Let R be a commutative ring and let (C_*, ∂_*) be a chain complex of R -modules. Furthermore let M be another R -module. We consider the chain complex $C_* \otimes_R M$ of R -modules. One can ask whether one can determine the homology groups of $C_* \otimes_R M$ from the torsion groups $\text{Tor}_k^R(H_l(C), M)$. If R is a principal ideal domain (usually abbreviated by PID), then the almost the same statement as in the Universal Coefficient Theorem 36.13 holds. On the other hand, if R is not a principal ideal, then the higher torsion groups are in general non-zero and the problem becomes much more subtle. The torsion-terms and the homology with M -coefficients are then related by a “spectral sequence”, see e.g. [Rt2, Theorem 10.90] for details.

Remark. We have now seen that we can assign to each chain complex of free abelian groups, each abelian group G and each $n \in \mathbb{N}_0$ a short exact sequence

$$0 \rightarrow H_n(C) \otimes G \xrightarrow{i_C} H_n(C; G) \xrightarrow{p_C} \text{Tor}(H_{n-1}(C), G) \rightarrow 0.$$

The question now arises, whether this assignment can be viewed as a functor between suitably chosen categories. This is indeed the case: Let $\mathcal{C}$ be the category of chain complexes and let $\mathcal{S}$ be the category of short exact sequences, where the objects are short exact sequences of and where the morphisms are chain maps. Then any abelian group G and any $n \in \mathbb{N}_0$ give rise to a functor $\mathcal{C} \rightarrow \mathcal{S}$ where we assign to a chain complex C the short exact sequence

$$0 \rightarrow H_n(C) \otimes G \rightarrow H_n(C; G) \rightarrow \text{Tor}(H_{n-1}(C), G) \rightarrow 0,$$

and where we assign to a chain map $f: C \rightarrow D$ the diagram⁶⁵⁹

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_n(C) \otimes G & \xrightarrow{i_C} & H_n(C; G) & \xrightarrow{p_C} & \text{Tor}(H_{n-1}(C), G) \longrightarrow 0 \\ & & \downarrow f_* & & \downarrow f_* & & \downarrow f_* \\ 0 & \longrightarrow & H_n(D) \otimes G & \xrightarrow{i_D} & H_n(D; G) & \xrightarrow{p_D} & \text{Tor}(H_{n-1}(D), G) \longrightarrow 0. \end{array}$$

The vertical maps are hereby induced by the chain map $f: C \rightarrow D$. It follows directly from the definitions, that this diagram commutes. We leave the verification of this statement as an exercise to the reader.

The Universal Coefficient Theorem 36.13 is in a sense slightly disappointing, in so far as it says that the homology groups of $C \otimes G$ do not contain more formation than was already contained in the homology groups of C . But this is not surprising since we had already seen in Proposition 36.19 that in this most cases the homology groups determine the chain homotopy type of a chain complexes C_* , and therefore they also determine the groups $H_n(C; G)$.

Our goal had been to determine the homology groups $H_n(C; G)$ only in terms of the original groups $H_k(C)$. But in general, given a short exact sequence, it is not possible to

⁶⁵⁹All the vertical maps are denoted by “ f_* ”, but how are they actually defined?

determine the middle group of a short exact sequence from the groups to the left and the right. For example, the short exact sequence

$$0 \rightarrow \mathbb{Z} \longrightarrow B \longrightarrow \mathbb{Z}_n \rightarrow 0$$

could really be of the form

$$0 \rightarrow \mathbb{Z} \xrightarrow{k \mapsto (k,0)} \mathbb{Z} \oplus \mathbb{Z}_n \xrightarrow{(k,l) \mapsto l} \mathbb{Z}_n \rightarrow 0$$

but it could also be of the form⁶⁶⁰

$$0 \rightarrow \mathbb{Z} \xrightarrow{k \mapsto nk} \mathbb{Z} \longrightarrow \mathbb{Z}_n \rightarrow 0.$$

On the other hand we had seen in Lemma 28.2 that in of a short exact sequence

$$0 \rightarrow A \xrightarrow{i} B \xrightarrow{p} C \rightarrow 0,$$

which splits, the middle group B is the direct sum of the “outer” groups A and C . The question arises, whether the short exact sequence in the Universal Coefficient Theorem 36.13 splits. This is fortunately the case:

Proposition 36.14. *Let (C_n, ∂_n) be a chain complex of free abelian groups and let G be an abelian group. Then the short exact sequence*

$$0 \rightarrow H_n(C) \otimes G \xrightarrow{i} H_n(C; G) \rightarrow \operatorname{Tor}(H_{n-1}(C), G) \rightarrow 0$$

splits. In particular there exists an isomorphism

$$H_n(C; G) \cong H_n(C) \otimes G \oplus \operatorname{Tor}(H_{n-1}(C), G).$$

Proof. According to Lemma 28.2 it suffices to construct a map $t: H_n(C; G) \rightarrow H_n(C) \otimes G$ such that $t \circ i = \operatorname{id}_{H_n(C)}$.

We first consider the short exact sequence

$$0 \rightarrow Z_n \rightarrow C_n \rightarrow B_{n-1} \rightarrow 0.$$

The group $B_{n-1} \subset C_{n-1}$ is a subgroup of a free abelian group, thus it is itself a free abelian group by Lemma 8.2. It follows from Lemma 28.1 that the above short exact sequence splits. In particular there exists according to Lemma 28.2 a map $p: C_n \rightarrow Z_n$ such that p is the identity on $Z_n \subset C_n$. We denote the composition of p with the projection $Z_n \rightarrow H_n(C) = Z_n/B_n$ by p_n . Now we obtain the following diagram:

$$\begin{array}{ccccccc} \dots & \longrightarrow & C_{n+1} & \xrightarrow{\partial_{n+1}} & C_n & \xrightarrow{\partial_n} & C_{n-1} \xrightarrow{\partial_{n-1}} \dots \\ & & \downarrow p_{n+1} & & \downarrow p_n & & \downarrow p_{n-1} \\ \dots & \longrightarrow & H_{n+1} & \xrightarrow{0} & H_n & \xrightarrow{0} & H_{n-1} \xrightarrow{0} \dots \end{array}$$

The lower sequence is evidently a chain complex. The vertical maps form a chain map: Indeed, since let $c \in C_n$. The ‘down-then-right’ map is evidently the zero map. On the other hand we have $\partial_n(c_n) \subset B_{n-1} \subset Z_{n-1} \subset C_{n-1}$, i.e. $p_n(\partial_n(c)) = \partial_n(c) = 0 \in Z_{n-1}/B_{n-1}$.

⁶⁶⁰Are there any other possibilities for B besides $B = \mathbb{Z}$ and $B = \mathbb{Z}_n$?

We now tensor the above diagram with G and we obtain the following commutative diagram of chain complexes.

$$\begin{array}{ccccccc}
 \dots & \longrightarrow & C_{n+1} \otimes G & \xrightarrow{\partial_{n+1} \otimes \text{id}} & C_n \otimes G & \xrightarrow{\partial_n \otimes \text{id}} & C_{n-1} \otimes G \xrightarrow{\partial_{n-1} \otimes \text{id}} \dots \\
 & & \downarrow & & \downarrow & & \downarrow \\
 \dots & \longrightarrow & H_{n+1} \otimes G & \xrightarrow{0} & H_n \otimes G & \xrightarrow{0} & H_{n-1} \otimes G \xrightarrow{0} \dots
 \end{array}$$

The induced maps on the homology of these two chain complexes now give us a map

$$t: H_n(C; G) = H_n(C_* \otimes G) \rightarrow H_n(\text{lower sequence}) = H_n \otimes G.$$

It is now straightforward to verify that $t \circ i = \text{id}_{H_n(C)}$. $\square$

Remark. Let

$$0 \rightarrow A \xrightarrow{i} B \xrightarrow{p} C \rightarrow 0$$

be a short exact sequence. If the short exact sequence splits, i.e. if there exists a homomorphism $s: C \rightarrow B$ with $p \circ s = \text{id}_C$, then usually there exists more than one splitting. We consider for example the short exact sequence

$$0 \rightarrow \mathbb{Z} \xrightarrow{a \mapsto (a, 0)} \mathbb{Z}^2 \xrightarrow{(a, c) \mapsto c} \mathbb{Z} \rightarrow 0.$$

Then for each $a \in \mathbb{Z}$ the map $c \mapsto (ac, c)$ is a splitting of the exact sequence. We had seen in Proposition 36.14 that the sequence

$$0 \rightarrow H_n(C) \otimes G \xrightarrow{i} H_n(C; G) \rightarrow \text{Tor}(H_{n-1}(C), G) \rightarrow 0$$

of the Universal Coefficient Theorem 36.13 splits, but there exists no canonical isomorphism

$$H_n(C; G) \cong H_n(C) \otimes G \oplus \text{Tor}(H_{n-1}(C), G).$$

We refer to the remark at the lower end of page 264 of Hatcher: Algebraic Topology for more information.

We conclude this section with the following lemma.

Lemma 36.15. *Let $f: S^n \rightarrow S^n$ be a map and let $k \in \mathbb{N}$. Then the induced map*

$$f_*: H_n(S^n; \mathbb{Z}_k) \rightarrow H_n(S^n; \mathbb{Z}_k)$$

is given by multiplication by $\deg(f)$.

Proof. By the functoriality of the short exact sequence of the Universal Coefficient Theorem 36.13, see page 617 we have the following commutative diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & H_n(S^n) \otimes \mathbb{Z}_k & \longrightarrow & H_n(S^n; \mathbb{Z}_k) & \longrightarrow & \text{Tor}(H_{n-1}(S^n), \mathbb{Z}_k) \longrightarrow 0 \\
 & & \downarrow f_* \otimes \text{id} & & \downarrow f_* & & \downarrow f_* \\
 0 & \longrightarrow & H_n(S^n) \otimes \mathbb{Z}_k & \longrightarrow & H_n(S^n; \mathbb{Z}_k) & \longrightarrow & \text{Tor}(H_{n-1}(S^n), \mathbb{Z}_k) \longrightarrow 0.
 \end{array}$$

Note that $H_{n-1}(S^n)$ is torsion-free, hence $\text{Tor}(H_{n-1}(S^n), \mathbb{Z}_k) = 0$ by Lemma 36.12. It follows from the exactness of the horizontal sequences that the horizontal maps on the

left are isomorphisms. The vertical map on the left is given by multiplication by $\deg(f)$ by the definition of $\deg(f)$. It now follows that the middle vertical map is also given by multiplication by $\deg(f)$. $\square$

36.6. Applications to topological spaces. If (X, A) is a pair of topological spaces, then we can apply the results from the previous chapter to the chain complex $C_*(X, A)$. From the Universal Coefficient Theorem 36.13 and from Proposition 36.14 we immediately obtain the following theorem:

Theorem 36.16. (Universal Coefficient Theorem) *Let (X, A) be a pair of topological spaces and let G be an abelian group. Then there exists for each n a short exact sequence*

$$0 \rightarrow H_n(X, A) \otimes G \rightarrow H_n(X, A; G) \rightarrow \operatorname{Tor}(H_{n-1}(X, A), G) \rightarrow 0,$$

and this short exact sequence splits. In particular there exists an isomorphism

$$H_n(X, A; G) \cong H_n(X, A) \otimes G \oplus \operatorname{Tor}(H_{n-1}(X, A), G).$$

Example. We consider again the real projective space $\mathbb{RP}^n$. In Lemma 29.19 we had seen that

$$H_k(\mathbb{RP}^n) \cong \begin{cases} \mathbb{Z}, & \text{if } k = 0, \\ \mathbb{Z}_2, & \text{if } k \text{ is odd and } k < n, \\ 0, & \text{if } k \text{ is even and } k \leq n, \\ \mathbb{Z}, & \text{if } k = n \text{ and } n \text{ is odd.} \end{cases}$$

It follows from Universal Coefficient Theorem 36.16 and Lemma 36.12 that for any even number $2k \in \{0, \dots, n\}$ we have

$$H_{2k}(\mathbb{RP}^n; \mathbb{F}_2) \cong H_{2k}(\mathbb{RP}^n) \otimes \mathbb{F}_2 \oplus \operatorname{Tor}(H_{2k-1}(\mathbb{RP}^n), \mathbb{F}_2) \cong 0 \otimes \mathbb{F}_2 \oplus \operatorname{Tor}(\mathbb{F}_2, \mathbb{F}_2) = \mathbb{F}_2$$

whereas for $2k+1 \in \{0, \dots, n\}$ an odd number we have

$$H_{2k+1}(\mathbb{RP}^n; \mathbb{F}_2) \cong H_{2k+1}(\mathbb{RP}^n) \otimes \mathbb{F}_2 \oplus \operatorname{Tor}(H_{2k}(\mathbb{RP}^n), \mathbb{F}_2) \cong \mathbb{F}_2 \otimes \mathbb{F}_2 \oplus \operatorname{Tor}(\mathbb{Z}, \mathbb{F}_2) = \mathbb{F}_2.$$

Also it follows easily that $H_k(\mathbb{RP}^n) = 0$ for $k > n$. Summarizing we obtain the following easy-to-remember result

$$H_k(\mathbb{RP}^n; \mathbb{F}_2) \cong \begin{cases} \mathbb{F}_2, & \text{if } k = 0, \dots, n, \\ 0, & \text{otherwise.} \end{cases}$$

Fortunately this calculation agrees with the calculation in Chapter 36.3.

The combination of Universal Coefficient Theorem 36.16 and Lemma 36.12 (4) gives us the following lemma.

Corollary 36.17. *Let (X, A) be a pair of topological spaces and let R be a subring of $\mathbb{C}$. Then for each n the map*

$$\begin{aligned} H_n(X, A; \mathbb{Z}) \otimes R &\rightarrow H_n(X, A; R) \\ \sum_{i=1}^n [c_i] \otimes r_i &\mapsto \left[\sum_{i=1}^n c_i \otimes r_i \right] \end{aligned}$$

is an isomorphism.

Let (X, A) be a pair of topological spaces and let R be a subring of $\mathbb{C}$. Since the map $H_n(X, A; \mathbb{Z}) \otimes R \rightarrow H_n(X, A; R)$ is functorial we can and we will in the following identify the groups $H_n(X, A; \mathbb{Z}) \otimes R$ and $H_n(X, A; R)$ using the isomorphism from Corollary 36.17.

From the functoriality of the short exact sequence of the Universal Coefficient Theorem 36.13, see page 617, it follows furthermore that the short exact sequences in Universal Coefficient Theorem 36.16 and the isomorphism in Corollary 36.17 commute with induced maps of topological spaces. For example for every map $f: (X, A) \rightarrow (Y, B)$ between pairs of topological spaces we obtain the following *commutative* diagram of short exact sequences:

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_n(X, A) \otimes G & \longrightarrow & H_n(X, A; G) & \longrightarrow & \text{Tor}(H_{n-1}(X, A), G) \longrightarrow 0 \\ & & \downarrow f_* & & \downarrow f_* & & \downarrow f_* \\ 0 & \longrightarrow & H_n(Y, B) \otimes G & \longrightarrow & H_n(Y, B; G) & \longrightarrow & \text{Tor}(H_{n-1}(Y, B), G) \longrightarrow 0. \end{array}$$

36.7. The singular chain complex and the cellular chain complexes are chain homotopic (*). In this short section we return to the relationship between cellular and singular chain complexes of a CW-complex. More precisely, let X be a CW-complex. In Proposition 29.8 we had shown that for any $n \in \mathbb{N}_0$ there exists a natural isomorphism $H_n(X) \rightarrow H_n^{\text{CW}}(X)$. Picking up the discussion from page 504 we can now generalize this statement. More precisely we have the following proposition.

Proposition 36.18. *Let X be a CW-complex which in each dimension has only finitely many cells. Then the singular chain complex $C_*(X)$ is chain homotopic to the cellular chain complex $C_*^{\text{CW}}(X)$.*

This proposition follows immediately from Proposition 29.8 and the following algebraic proposition.

Proposition 36.19. *Let C_* and C'_* be two free⁶⁶¹ chain complexes. Suppose that for each $n \in \mathbb{N}_0$ we are given an isomorphism $\varphi_n: H_n(C_*) \rightarrow H_n(C'_*)$, then there exists a chain map $C_* \rightarrow C'_*$, which induces the maps φ_n , and which is a homotopy equivalence.*

The statement of Proposition 36.19 is going to be an immediate consequence of the next two lemmas. Before we can state the first lemma we need to introduce one more definition.

Definition. We say that a chain complex is *basic* if it is of the form $0 \rightarrow E_n \xrightarrow{e_n} E_{n-1} \rightarrow 0$ where E_n and E_{n-1} are free abelian groups and where e_n is injective.

The first lemma implies Proposition 36.19 in the special case that both chain complexes are basic.

Lemma 36.20. *Suppose that we have two basic chain complexes $0 \rightarrow E_n \xrightarrow{e_n} E_{n-1} \rightarrow 0$ and $0 \rightarrow F_n \xrightarrow{f_n} F_{n-1} \rightarrow 0$. Then for every isomorphism*

$$\varphi: H_{n-1}(E_*) = \text{coker}(e_n) \rightarrow \text{coker}(f_n) = H_{n-1}(F_*)$$

⁶⁶¹A chain complex (D_*, ∂_*) is called *free*, if all the groups D_n are free abelian groups.

there exists a chain map $E_* \rightarrow F_*$ that induces that isomorphism and that is a homotopy equivalence.

Proof. By Lemma 36.7, applied twice, there exist maps $\alpha_i: E_i \rightarrow F_i$, $i = n-1, n$ and there exist maps $\beta_i: F_i \rightarrow E_i$, $i = n-1, n$ such that the diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & E_n & \xrightarrow{e_n} & E_{n-1} & \xrightarrow{p} & \operatorname{coker}(f_n) \longrightarrow 0 \\
 & & \downarrow \alpha_n & & \downarrow \alpha_{n-1} & & \downarrow \varphi \\
 0 & \longrightarrow & F_n & \xrightarrow{f_n} & F_{n-1} & \xrightarrow{q} & \operatorname{coker}(f_n) \longrightarrow 0 \\
 & & \downarrow \beta_n & & \downarrow \beta_{n-1} & & \downarrow \varphi^{-1} \\
 0 & \longrightarrow & E_n & \xrightarrow{e_n} & E_{n-1} & \xrightarrow{p} & \operatorname{coker}(e_n) \longrightarrow 0
 \end{array}$$

commutes. From the second part of Lemma 36.7 it follows that $\beta_* \circ \alpha_* \simeq \operatorname{id}_{E_*}$ and that $\alpha_* \circ \beta_* \simeq \operatorname{id}_{F_*}$. Put differently, α_* is a chain homotopy equivalence. $\square$

The next lemma says that we can decompose every free chain complex into a direct sum of basic chain complexes. Proposition 36.19 is an immediate consequence of Lemma 36.20 and this lemma.

Lemma 36.21. *Let (C_*, ∂_*) be a free chain complex. Then there exist basic chain complexes (F_*^n, ∂_*) , $n \in \mathbb{N}_0$ such that*

$$C_* \cong \bigoplus_{n \in \mathbb{N}_0} F_*^n.$$

Proof. This concludes the proof of the claim. Let (C_*, ∂_*) be a chain complex. For $n \in \mathbb{N}_0$ we write $Z_n = \ker(\partial_n: C_n \rightarrow C_{n-1})$ and $B_n = \operatorname{im}(\partial_n: C_n \rightarrow C_{n-1})$. Note that we have a short exact sequence

$$0 \rightarrow Z_n \rightarrow C_n \xrightarrow{\partial_n} B_{n-1} \rightarrow 0.$$

Since C_{n-1} is free abelian it follows from Lemma 8.2 that the subgroup B_{n-1} is also free abelian. It follows immediately from Lemmas 28.1 and 28.2 that there exists an isomorphism $\Phi_n: C_n \rightarrow Z_n \oplus B_{n-1}$ such that the following diagram commutes:

$$\begin{array}{ccccccc}
 & & & C_n & & & \\
 & \nearrow i & & \downarrow \cong & \searrow \partial_n & & \\
 0 & \longrightarrow & Z_n & & & B_{n-1} & \longrightarrow 0 \\
 & \searrow & & \downarrow \Phi_n & \nearrow & & \\
 & & & Z_n \oplus B_{n-1} & & &
 \end{array}$$

where the lower left diagonal map is the obvious inclusion and the lower right diagonal map is the obvious projection map. Given $n \in \mathbb{N}_0$ we now denote by F_*^n the chain complex that is given by

$$0 \rightarrow B_{n-1} \hookrightarrow Z_{n-1} \rightarrow 0.$$

As before it follows from Lemma 8.2 that this is a basic chain complex. It is straightforward to verify that

$$\bigoplus_{n \in \mathbb{N}_0} \Phi_n : C_* \rightarrow \bigoplus_{n \in \mathbb{N}_0} F_*^n$$

is an isomorphism of chain complexes.

□

37. THE HOMOLOGY GROUPS OF PRODUCTS

Let X be and Y two topological spaces. In this chapter we want to study the question, what is the connection between the homology groups of X and Y on the one side and the homology groups of the product topological space $X \times Y$ on the other side.

37.1. The tensor product of chain complexes. First we want to find some connection between the chain complexes of X , Y and $X \times Y$. To do so we need the notion of the tensor product of chain complexes. Therefore we start out with a short excursion into homological algebra, i.e. into the study of chain complexes.

Let $\mathcal{C} = (C_*, \partial_*)$ and $\mathcal{C}' = (C'_*, \partial'_*)$ be two chain complexes. We define the tensor product $\mathcal{C} \otimes \mathcal{C}'$ of the chain complexes as follows: The n -th chain group is

$$(\mathcal{C} \otimes \mathcal{C}')_n := \bigoplus_{p+q=n} C_p \otimes C'_q$$

and we consider the linear maps $\bar{p}: (\mathcal{C} \otimes \mathcal{C}')_n \rightarrow (\mathcal{C} \otimes \mathcal{C}')_{n-1}$ which are uniquely determined by

$$\bar{p}(c_p \otimes c'_q) = \partial c_p \otimes c'_q + (-1)^p c_p \otimes \partial' c'_q$$

for $c_p \in C_p$ and $c'_q \in C'_q$. Then

$$\begin{aligned} \bar{p}(\bar{p}(c_p \otimes c'_q)) &= \bar{p}(\underbrace{\partial c_p}_{\in C_{p-1}} \otimes c'_q) + \bar{p}((-1)^p c_p \otimes \partial' c'_q) \\ &= \partial \partial c_p \otimes c'_q + (-1)^{p-1} \partial c_p \otimes \partial' c'_q + (-1)^p \partial c_p \otimes \partial' c'_q + (-1)^p c_p \otimes \partial' \partial' c'_q \\ &= (-1)^{p-1} \partial c_p \otimes \partial' c'_q - (-1)^{p-1} \partial c_p \otimes \partial' c'_q \\ &= 0. \end{aligned}$$

Thus we see that $(\mathcal{C} \otimes \mathcal{C}', \bar{p})$ is a chain complex.⁶⁶² We illustrate the definition of the tensor product of chain complexes in Figure 357.

$$\begin{array}{ccccc} C_0 \otimes C'_2 & \xleftarrow{\partial_1 \otimes \text{id}} & C_1 \otimes C'_2 & \xleftarrow{\partial_2 \otimes \text{id}} & C_2 \otimes C'_2 \\ \downarrow \text{id} \otimes \partial'_2 & & \downarrow -\text{id} \otimes \partial'_2 & & \downarrow \text{id} \otimes \partial'_2 \\ C_0 \otimes C'_1 & \xleftarrow{\partial_1 \otimes \text{id}} & C_1 \otimes C'_1 & \xleftarrow{\partial_2 \otimes \text{id}} & C_2 \otimes C'_1 \\ \downarrow \text{id} \otimes \partial'_1 & & \downarrow -\text{id} \otimes \partial'_1 & & \downarrow \text{id} \otimes \partial'_1 \\ C_0 \otimes C'_0 & \xleftarrow{\partial_1 \otimes \text{id}} & C_1 \otimes C'_0 & \xleftarrow{\partial_2 \otimes \text{id}} & C_2 \otimes C'_0 \end{array}$$

FIGURE 357. The tensor product of chain complexes.

⁶⁶²Note that we made use of the extra-term $(-1)^p$ in the definition of the boundary map. If we had defined the boundary map “naively” as $\bar{p}(c_p \otimes c'_q) = \partial c_p \otimes c'_q + c_p \otimes \partial' c'_q$, then the result would not have been a chain complex.

We conclude this section with the following lemma. We leave the elementary proof as an exercise.

Lemma 37.1.

- (1) Given two chain maps $f: \mathcal{C} \rightarrow \mathcal{D}$ and $f': \mathcal{C}' \rightarrow \mathcal{D}'$ the induced map

$$\begin{aligned} f \otimes f': \mathcal{C} \otimes \mathcal{C}' &\rightarrow \mathcal{D} \otimes \mathcal{D}' \\ \sum_{i=1}^n c_i \otimes c'_i &\mapsto \sum_{i=1}^n f(c_i) \otimes f'(c'_i) \end{aligned}$$

is also a chain map.

- (2) If $f, g: \mathcal{C} \rightarrow \mathcal{D}$ and $f', g': \mathcal{C}' \rightarrow \mathcal{D}'$ are chain homotopic maps, then the induced maps

$$f \otimes f', g \otimes g': \mathcal{C} \otimes \mathcal{C}' \rightarrow \mathcal{D} \otimes \mathcal{D}'$$

are also chain homotopic.

- (3) If $f: \mathcal{C} \rightarrow \mathcal{D}$ and $f': \mathcal{C}' \rightarrow \mathcal{D}'$ are homotopy equivalences, then the induced map

$$f \otimes f': \mathcal{C} \otimes \mathcal{C}' \rightarrow \mathcal{D} \otimes \mathcal{D}'$$

is also a homotopy equivalence.

37.2. The product of CW-complexes. Now we consider the special case that X and Y are two CW-complexes. In order to avoid certain topological technicalities we suppose in the following that given any dimension the CW-complexes X and Y have only finitely many cells.

Now we would like to view the product $X \times Y$ as a CW-complex. Using the homeomorphism from Lemma 1.4 we identify in the following the n -dimensional disk B^n with the n -dimensional cube $Q^n = [0, 1]^n$. For $p + q = n$ we evidently have $Q^n = Q^p \times Q^q$. The idea is now to define a CW-structure on $X \times Y$ such that the product of a p -cell in X with a q -cell in Y corresponds to a $(p + q)$ -cell in $X \times Y$.

In the following we want to execute this idea rigorously. We first make the observation that

$$\partial(Q^n) = \partial Q^p \times Q^q \cup Q^p \times \partial Q^q.$$

Now we define the CW-structure auf $X \times Y$ inductively as follows:

- (1) We put $(X \times Y)^0 = X^0 \times Y^0$.
- (2) Now we suppose that the $(n - 1)$ -skeleton $(X \times Y)^{(n-1)}$ has already been defined. We consider a p -cell e in X which is given by a characteristic map $\varphi: Q^p \rightarrow X^n$ and a $(n - p)$ -cell f in Y which is given by a characteristic map $\psi: Q^q \rightarrow Y^n$. Then we denote by $e \times f$ the n -cell in $X \times Y$ which is given by the attaching map⁶⁶³

$$\begin{aligned} \partial(Q^n) = \partial Q^p \times Q^q \cup Q^p \times \partial Q^q &\rightarrow (X \times Y)^{n-1} \\ (x, y) &\mapsto (\varphi(x), \psi(y)). \end{aligned}$$

We then denote by $(X \times Y)^n$ the union of $(X \times Y)^{n-1}$ with all such n -dimensional product cells.

⁶⁶³Why does the image of this attaching map lie in $(X \times Y)^{n-1}$?

Now we can view $X \times Y$ as a CW-complex where each cell in $X \times Y$ is the product, in the above sense, of a cell in X and of a cell in Y . It follows that for each n there is a canonical isomorphism⁶⁶⁴

$$C_n^{\text{CW}}(X \times Y) = \bigoplus_{p=0}^n C_p^{\text{CW}}(X) \otimes C_{n-p}^{\text{CW}}(Y).$$

Of course we want more than just an isomorphisms of the groups in this corresponding chain complexes. What we really want is an isomorphism of chain complexes, i.e. we want to relate the isomorphisms to the boundary maps. Now let e be a p -cell in X and let f be a q -cell in Y . Purely viewed as a set we had seen that

$$\partial(Q^{p+q}) = \partial Q^p \times Q^q \cup Q^p \times \partial Q^q.$$

But we really have to “take the boundary” of maps. A careful analysis of the signs involved shows that

$$\underbrace{\partial(e \times f)}_{\in C_{p+q-1}^{\text{CW}}(X \times Y)} = \underbrace{\partial e \times f}_{\in C_{p-1}^{\text{CW}}(X) \otimes C_q^{\text{CW}}(Y)} + (-1)^p \underbrace{e \times \partial f}_{\in C_p^{\text{CW}}(X) \otimes C_{q-1}^{\text{CW}}(Y)}.$$

This statement is illustrated in Figure 358.

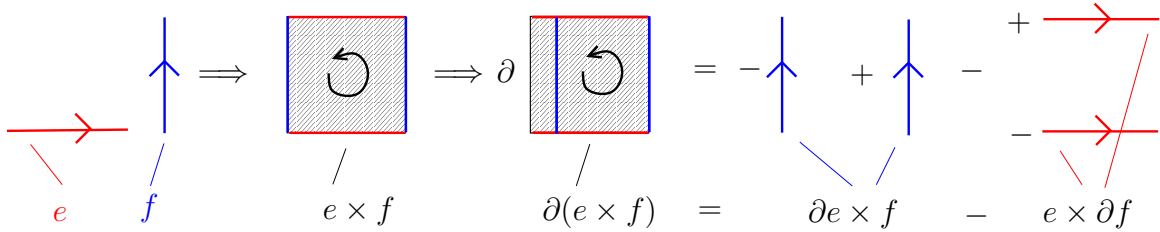


FIGURE 358. The boundary of the product of two 1-cells.

Thus we have “proved” the following proposition :

Proposition 37.2. *Let X be and Y two CW-complexes with finitely many cells in each dimension. Then there exists an isomorphism*

$$\Phi_{XY}: C_*^{\text{CW}}(X) \otimes C_*^{\text{CW}}(Y) \xrightarrow{\cong} C_*^{\text{CW}}(X \times Y)$$

of cellular chain complexes.

⁶⁶⁴Hereby we have following simple algebraic fact used: if M and N two sets, then we have a canonical isomorphism

$$\mathbb{Z}^{(M \times N)} = \mathbb{Z}^{(M)} \otimes \mathbb{Z}^{(N)},$$

where, as always, given a set W denote by $\mathbb{Z}^{(W)}$ the free abelian group generated by W .

37.3. The Eilenberg-Zilber Theorem. The Eilenberg-Zilber Theorem is the analogue of Proposition 37.2 for the singular chain complexes of arbitrary topological spaces.

Theorem 37.3. (Eilenberg-Zilber) *Let X be and Y two topological spaces. Then there exists chain maps*

$$C_*(X) \otimes C_*(Y) \xrightarrow{\mu} C_*(X \times Y)$$

and

$$C_*(X \times Y) \xrightarrow{\nu} C_*(X) \otimes C_*(Y),$$

which are chain homotopy-inverses. The maps μ and ν are furthermore natural in X and Y .⁶⁶⁵ In particular we have an isomorphism

$$H_*(X \times Y) \cong H_*(C_*(X) \otimes C_*(Y)).$$

Proof. In the following, given two chain complexes C_* and D_* we write $C_* \simeq D_*$ if they are homotopy equivalent.

We first consider the case that X and Y are CW-complexes which have only finitely many cells in each dimension. Then we have

$$\begin{array}{ccccccc} C_*(X) \otimes C_*(Y) & \simeq & C_*(X) \otimes C_*(Y) & \simeq & C_*^{\text{CW}}(X \times Y) & \simeq & C_*(X \times Y) \\ & \uparrow & & \uparrow & & \uparrow & \\ & \text{Proposition 36.18 and Lemma 37.1} & & \text{Proposition 37.2} & & \text{Proposition 36.18} & \end{array}$$

□

37.4. The Künneth-formula for chain complexes. From the previous chapter follows that we can find a connection between the homology groups of the chain complexes $\mathcal{C}$, $\mathcal{C}'$ and $\mathcal{C} \otimes \mathcal{C}'$. We consider hereby the following map:

$$\begin{aligned} \Theta: H_p(\mathcal{C}) \otimes H_q(\mathcal{C}') &\rightarrow H_{p+q}(\mathcal{C} \otimes \mathcal{C}') \\ \sum_{i=1}^r [c_i] \otimes [c'_i] &\mapsto \sum_{i=1}^r [c_i \otimes c'_i]. \end{aligned}$$

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Now we can formulate the following proposition.

⁶⁶⁵Here natural means that for maps $f: X \rightarrow X'$ and $g: Y \rightarrow Y'$ we obtain commutative diagrams

$$\begin{array}{ccc} C_*(X) \otimes C_*(Y) & \xrightarrow{\mu} & C_*(X \times Y) \\ \downarrow f_* \otimes f'_* & & \downarrow (f \times f')_* \\ C_*(X') \otimes C_*(Y') & \xrightarrow{\mu} & C_*(X' \times Y') \end{array} \quad \text{and} \quad \begin{array}{ccc} C_*(X \times Y) & \xrightarrow{\nu} & C_*(X) \otimes C_*(Y) \\ \downarrow (f \times f')_* & & \downarrow f_* \otimes f'_* \\ C_*(X' \times Y') & \xrightarrow{\nu} & C_*(X') \otimes C_*(Y'). \end{array}$$

⁶⁶⁶The map is well-defined. First, if c_i and c'_i cycle, then

$$\bar{p}(c_i \otimes c'_i) = \partial c_i \otimes c'_i + (-1)^p c_i \otimes \partial c'_i = 0$$

i.e. $c_i \otimes c'_i \in CC \otimes \mathcal{C}'$ is a cycle. Second, if $c_i + \partial d_i$ is an other representative of $[c_i]$ then

$$(c_i + \partial d_i) \otimes c'_i = c_i \otimes c'_i + \bar{p}(d_i \otimes c'_i),$$

Proposition 37.4. (Künneth-Formula for Chain Complexes)⁶⁶⁷ *Let $\mathcal{C}$ and $\mathcal{C}'$ be two chain complexes. If $\mathcal{C}$ is free, i.e. if all groups C_n are free abelian groups, then there exists a short exact sequence*

$$0 \rightarrow \bigoplus_{p+q=n} H_p(\mathcal{C}) \otimes H_q(\mathcal{C}') \xrightarrow{\Theta} H_n(\mathcal{C} \otimes \mathcal{C}') \rightarrow \bigoplus_{p+q=n-1} \text{Tor}(H_p(\mathcal{C}), H_q(\mathcal{C}')) \rightarrow 0.$$

If $\mathcal{C}'$ is a chain complex which consists of a single group G , i.e. if it is of the form $0 \rightarrow G \rightarrow 0$, then we recover the Universal Coefficient Theorem 36.13 as a special case of the Künneth-formula. In fact, as we will see, the proof of the Künneth-formula is quite similar to the proof of the Universal Coefficient Theorem 36.13.

Proof. We first consider the special case that all boundary maps $C_n \rightarrow C_{n-1}$ are the zero map. In this case we have $H_n(\mathcal{C}) = C_n$ for all n . It follows furthermore that $\bar{p}c_p \otimes c'_q = (-1)^p c_p \otimes \partial c'_q$. Thus the chain complex $\mathcal{C} \otimes \mathcal{C}'$ is the direct sum of the chain complexes $C_p \otimes \mathcal{C}'$. Since $\mathcal{C}$ is free, it follows furthermore that each chain complex $C_p \otimes \mathcal{C}'$ is just the direct sum of several copies of the chain complex $\mathcal{C}'$. It follows that

$$H_n(C_p \otimes \mathcal{C}') \cong C_p \otimes H_{n-p}(\mathcal{C}') = H_p(\mathcal{C}) \otimes H_{n-p}(\mathcal{C}').$$

In particular we then have

$$H_n(\mathcal{C} \otimes \mathcal{C}') \cong \bigoplus_{p+q=n} H_p(\mathcal{C}) \otimes H_q(\mathcal{C}').$$

In Lemma 36.12 we had seen that the torsion group $\text{Tor}(A, B)$ vanishes, if A is a torsion-free group. Thus we obtain the desired short exact sequence.

Now we turn to the general case, i.e. we no longer assume that the boundary maps $C_n \rightarrow C_{n-1}$ are zero. The following proof is quite similar to the proof of the Universal Coefficient Theorem 36.13.

We consider again

$$Z_p = \ker\{\partial_p: C_p \rightarrow C_{p-1}\},$$

and

$$B_p = \text{im}\{\partial_{p+1}: C_{p+1} \rightarrow C_p\}.$$

We then denote by $\mathcal{Z}$ the chain complex which is given by the groups Z_p , where all the boundary maps are the zero maps. The same way obtain from the groups B_p a chain complex $\mathcal{B}$. As in the proof of the Universal Coefficient Theorem 36.13 we obtain a short exact sequence of chain maps

$$0 \rightarrow \mathcal{Z} \rightarrow \mathcal{C} \xrightarrow{\partial} \mathcal{B} \rightarrow 0.$$

and precisely the same way, if $c'_i + \partial d'_i$ is an other representative of $[c'_i]$ then

$$c_i \otimes (c'_i + \partial d'_i) = c_i \otimes c'_i + (-1)^p \bar{p}(c_i \otimes d'_i).$$

⁶⁶⁷Hermann Künneth (1892-1975) was a German mathematician.

Furthermore we see again that the groups B_{n-1} are free abelian groups and that therefore for each C'_q the tensor product of the above exact sequence with C'_q is again an exact sequence. Therefore it follows that

$$0 \rightarrow \mathcal{Z} \otimes \mathcal{C}' \rightarrow \mathcal{C} \otimes \mathcal{C}' \rightarrow \mathcal{B} \otimes \mathcal{C}' \rightarrow 0$$

is a short exact sequence of chain maps.

From Proposition 26.9 we thus obtain the following long exact sequence⁶⁶⁸

$$\dots \rightarrow H_n(\mathcal{Z} \otimes \mathcal{C}') \rightarrow H_n(\mathcal{C} \otimes \mathcal{C}') \rightarrow H_{n-1}(\mathcal{B} \otimes \mathcal{C}') \rightarrow H_{n-1}(\mathcal{Z} \otimes \mathcal{C}') \rightarrow \dots$$

The “boundary map” $H_{n-1}(\mathcal{B} \otimes \mathcal{C}') \xrightarrow{\partial} H_{n-1}(\mathcal{Z} \otimes \mathcal{C}')$ from Proposition 26.9 is hereby, as in the proof of the Universal Coefficient Theorem 36.13, induced by the inclusion $\mathcal{B} \rightarrow \mathcal{Z}$.

Since the boundary maps in $\mathcal{B}$ and $\mathcal{Z}$ are the zero maps it follows from the special case that we had considered in the beginning that we obtain the following long exact sequence:

$$\dots \xrightarrow{i_n} \bigoplus_{j=0}^n Z_n \otimes H_{n-j}(\mathcal{C}') \rightarrow H_n(\mathcal{C} \otimes \mathcal{C}') \rightarrow \bigoplus_{j=0}^{n-1} B_j \otimes H_{n-j-1}(\mathcal{C}') \xrightarrow{i_{n-1}} \bigoplus_{j=0}^{n-1} Z_j \otimes H_{n-j-1}(\mathcal{C}') \rightarrow \dots$$

Thus we obtain a short exact sequence

$$0 \rightarrow \operatorname{coker}(i_n) \rightarrow H_n(\mathcal{C} \otimes \mathcal{C}') \rightarrow \ker(i_{n-1}) \rightarrow 0.$$

Now we want to show that the expressions left and right correspond to the desired expressions. The proof of this claim precisely the same as the proof of the corresponding statement in the proof of the Universal Coefficient Theorem 36.13.

We now carry out the argument in more detail. As in the proof of the Universal Coefficient Theorem 36.13 we see that for each $j \in \{0, \dots, n-1\}$ the sequence

$$0 \rightarrow B_j \rightarrow Z_j \rightarrow H_j(\mathcal{C}) \rightarrow 0$$

is a free resolution of $H_j(\mathcal{C})$. In particular it follows from the definitions and from Lemma 36.10 that for each l we have

$$\operatorname{coker}(i_j: B_j \otimes H_l(\mathcal{C}') \xrightarrow{i_j} Z_j \otimes H_l(\mathcal{C}')) = H_j(\mathcal{C}) \otimes H_l(\mathcal{C}'),$$

and

$$\ker(i_j: B_j \otimes H_l(\mathcal{C}') \xrightarrow{i_j} Z_j \otimes H_l(\mathcal{C}')) = \operatorname{Tor}_1(H_j(\mathcal{C}), H_l(\mathcal{C}')).$$

Thus we see that

$$\begin{aligned} \operatorname{coker}(i_n) &= \operatorname{coker} \left(\bigoplus_{j=0}^n B_j \otimes H_{n-j}(\mathcal{C}') \rightarrow \bigoplus_{j=0}^n Z_j \otimes H_{n-j}(\mathcal{C}') \right) \\ &= \bigoplus_{j=0}^n \operatorname{coker} (i_j: B_j \otimes H_{n-j}(\mathcal{C}') \xrightarrow{i_j} Z_j \otimes H_{n-j}(\mathcal{C}')) = \bigoplus_{j=0}^n H_j(\mathcal{C}) \otimes H_{n-j}(\mathcal{C}'). \end{aligned}$$

⁶⁶⁸The index goes down by one in the map $H_n(\mathcal{C} \otimes \mathcal{C}') \rightarrow H_{n-1}(\mathcal{B} \otimes \mathcal{C}')$, since the map is induced by the short exact sequence $0 \rightarrow Z_p \rightarrow C_p \rightarrow B_{p-1} \rightarrow 0$.

The same way one shows that

$$\begin{aligned} \ker(i_{n-1}) &= \ker\left(\bigoplus_{j=0}^{n-1} B_j \otimes H_{n-j-1}(\mathcal{C}') \rightarrow \bigoplus_{j=0}^{n-1} Z_j \otimes H_{n-j-1}(\mathcal{C}')\right) \\ &= \bigoplus_{j=0}^{n-1} \ker(i_j: B_j \otimes H_{n-j-1}(\mathcal{C}') \xrightarrow{i_j} Z_j \otimes H_{n-j-1}(\mathcal{C}')) = \bigoplus_{j=0}^{n-1} \operatorname{Tor}(H_j(\mathcal{C}), H_{n-j}(\mathcal{C}')). \end{aligned}$$

□

Remark. The short exact sequence in the Künneth-Theorem is natural. More precisely, if $f: \mathcal{C} \rightarrow \mathcal{D}$ and $f': \mathcal{C}' \rightarrow \mathcal{D}'$ are chain maps, then we obtain a commutative diagram of short exact sequences:⁶⁶⁹

$$\begin{array}{ccccccc} 0 \longrightarrow & \bigoplus_{p+q=n} H_p(\mathcal{C}) \otimes H_q(\mathcal{C}') & \xrightarrow{\Theta} & H_n(\mathcal{C} \otimes \mathcal{C}') & \longrightarrow & \bigoplus_{p+q=n} \operatorname{Tor}(H_p(\mathcal{C}), H_q(\mathcal{C}')) & \longrightarrow 0 \\ & \downarrow f_* \otimes f'_* & & \downarrow (f \otimes f')_* & & \downarrow \operatorname{Tor}(f_*, f'_*) & \\ 0 \longrightarrow & \bigoplus_{p+q=n} H_p(\mathcal{D}) \otimes H_q(\mathcal{D}') & \xrightarrow{\Theta} & H_n(\mathcal{D} \otimes \mathcal{D}') & \longrightarrow & \bigoplus_{p+q=n} \operatorname{Tor}(H_p(\mathcal{D}), H_q(\mathcal{D}')) & \longrightarrow 0. \end{array}$$

We now return to the initial question, what is the relationship between the homology groups of the topological spaces X, Y and $X \times Y$. The Eilenberg-Zilber Theorem 37.3 together with the Künneth-formula for chain complexes from Proposition 37.4 now gives us the following proposition:

Proposition 37.5. (The Künneth-Formula for Topological Spaces) *Let X be and Y two topological spaces. Then there exists a short exact sequence*

$$0 \rightarrow \bigoplus_{p+q=n} H_p(X) \otimes H_q(Y) \xrightarrow{\Theta} H_n(X \times Y) \rightarrow \bigoplus_{p+q=n-1} \operatorname{Tor}(H_p(X), H_q(Y)) \rightarrow 0,$$

and this short exact sequence is natural with respect to the topological spaces X and Y .

Remark. The short exact sequence in Proposition 37.5 splits, but this splitting is not natural. We refer to [Mu1, Theorem 5.3] for details.

Examples.

- (1) Let X be a topological space. We want to determine the homology groups of $S^1 \times X$. The homology groups of S^1 are of course $\mathbb{Z}$ in the dimensions 0 and 1. Since for each abelian group A we have $\mathbb{Z} \otimes A = A$ and $\operatorname{Tor}(\mathbb{Z}, A) = 0$ it now follows from the Künneth-formula that there exists an isomorphism

$$H_{n-1}(X) \oplus H_n(X) \rightarrow H_n(S^1 \times X).$$

This is of course precisely the same result as in Lemma 28.15.

⁶⁶⁹The torsion group $\operatorname{Tor}(-, -)$ is “functorial in both entries”, i.e. if $\varphi: A \rightarrow B$ and $\psi: C \rightarrow D$ maps, then there exists an induced map $\operatorname{Tor}(\varphi, \psi): \operatorname{Tor}(A, B) \rightarrow \operatorname{Tor}(C, D)$.

- (2) Let X and Y be two path-connected topological spaces. Then it follows from the Künneth-Formula 37.5 for Topological Spaces, Proposition 24.5 and Lemma 36.12 that

$$H_1(X \times Y) \cong H_1(X) \oplus H_1(Y).$$

This is consistent with the product formula for fundamental groups proved in Proposition 6.18 and with the Hurewicz isomorphism of Proposition 32.4.

It follows immediately from the Künneth-formula that

$$H_k(S^2 \times S^4) \cong \begin{cases} \mathbb{Z}, & \text{if } k = 0, 2, 4, \\ 0, & \text{otherwise.} \end{cases}$$

But by the calculation on page 501 these are precisely the homology groups of the complex projective space $\mathbb{CP}^3$. Furthermore it follows from Propositions 4.12 and 6.18 that $S^2 \times S^2$ is simply connected and it follows from the discussion on page 363 that $\mathbb{CP}^3$ is also simply connected. So all the invariants we are aware of, and that we know how to compute⁶⁷⁰ agree for $S^2 \times S^4$ and $\mathbb{CP}^3$. This raises the following question which will answer in Algebraic Topology III.

Question 37.6. *Are the topological spaces $S^2 \times S^4$ and $\mathbb{CP}^3$ homeomorphic?*

In a very similar fashion we see that

$$H_k(S^2 \times S^2) \cong \begin{cases} \mathbb{Z}, & \text{if } k = 0, 4, \\ \mathbb{Z} \oplus \mathbb{Z}, & \text{if } k = 2, \\ 0, & \text{otherwise.} \end{cases}$$

But by the calculation on page 568 these are precisely the homology groups of the connected sum $\mathbb{CP}^2 \# \mathbb{CP}^2$. As above, both topological spaces are furthermore simply-connected. So this raises the following question.

Question 37.7. *Are the topological spaces $S^2 \times S^2$ and $\mathbb{CP}^2 \# \mathbb{CP}^2$ homeomorphic?*

We conclude this chapter with the following proposition.

Proposition 37.8. *Let M and N be manifolds and let $\varphi \in H_k(M)$ and $\psi \in H_l(M)$. If φ is realized by a closed oriented k -dimensional submanifold A and if ψ is realized by a closed oriented l -dimensional submanifold B , then $\Theta(\varphi \times \psi) \in H_{k+l}(M \times N)$ is realized by the closed oriented $(k + l)$ -dimensional submanifold $A \times B$.*

Example. If we identify $H_2(S^2 \times S^2)$ with $\mathbb{Z}^2$ via the Künneth-formula, then the element $(1, 0)$ is represented by the submanifold $S^2 \times \{*\}$ and the element $(0, 1)$ is represented by the submanifold $\{*\} \times S^2$.

⁶⁷⁰As of right now we do not know the higher homotopy groups of these two manifolds, but these will not turn out to be very useful.

Proof. Let M and N be manifolds. Suppose $\varphi \in H_k(M)$ is realized by a closed oriented k -dimensional submanifold A and that $\psi \in H_l(M)$ is realized by a closed oriented l -dimensional submanifold B . We denote by $i: A \rightarrow M$ and by $j: B \rightarrow N$ the inclusion maps.

By the naturality of the Künneth-formula^{37.5} for topological spaces we have the following commutative diagram

$$\begin{array}{ccccccc}
 0 \longrightarrow & \bigoplus_{p+q=k+l} & H_p(A) \otimes H_q(B) & \xrightarrow{\Theta} & H_{k+l}(A \times B) & \longrightarrow & \bigoplus_{p+q=k+l} \operatorname{Tor}(H_p(A), H_q(A)) \longrightarrow 0 \\
 & \downarrow i_* \otimes j_* & & & \downarrow (i \times j)_* & & \downarrow \operatorname{Tor}(i_*, j_*) \\
 0 \longrightarrow & \bigoplus_{p+q=k+l} & H_p(M) \otimes H_q(N) & \xrightarrow{\Theta} & H_{k+l}(M \times N) & \longrightarrow & \bigoplus_{p+q=k+l} \operatorname{Tor}(H_p(M), H_q(M)) \longrightarrow 0.
 \end{array}$$

It follows from Propositions 29.10 and 33.5 together with Lemma 36.12 that this simplifies to

$$\begin{array}{ccc}
 0 \longrightarrow & \mathbb{Z} \cdot [A] \otimes \mathbb{Z} \cdot [B] & \xrightarrow{\Theta} \mathbb{Z}[A \times B] \longrightarrow 0 \\
 & \downarrow \begin{array}{c} [A] \otimes [B] \\ \downarrow \\ \varphi \otimes \psi \end{array} & \downarrow (i \times j)_* \\
 & H_k(M) \otimes H_l(N) & \xrightarrow{\Theta} H_{k+l}(M \times N).
 \end{array}$$

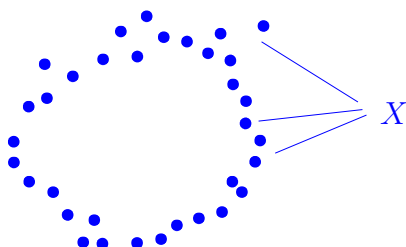
The top horizontal map is therefore an isomorphism and we see that $[A] \otimes [B]$ which is a generator of $\mathbb{Z} \cdot [A] \otimes \mathbb{Z} \cdot [B] \cong \mathbb{Z}$ gets sent to a generator of $\mathbb{Z} \cdot [A \times B]$, i.e. it gets sent to $\pm[A \times B]$. It follows that $\Theta(\varphi \otimes \psi)$ is represented by $A \times B$ with an appropriate orientation for $A \times B$.⁶⁷¹ $\square$

⁶⁷¹We leave it as an exercise to check all the sign conventions to realize that it is actually realized by $A \times B$ with the usual product orientation.

38. APPLICATIONS OF HOMOLOGY GROUPS

38.1. Persistent homology. In this section we outline the idea behind “persistent homology”. Persistent homology was introduced by Gunnar Carlsson [Ca] about 2006. More information and a much more careful treatment of persistent homology can for example be found in [EH]. Some extra information can also be found in [Gh].

We consider Figure 359. What do we see? If we forget about topology for a second, then we would say that we see a set X of dots that form a loop.



dots that appear to the eye to form a loop

FIGURE 359.

If we view X as a topological space, then it is just a finite discrete subset of $\mathbb{R}^2$. In particular its homology groups are non-zero only in dimension 0.

Now given $r \in [0, \infty)$ we denote by $X(r)$ the subset of $\mathbb{R}^2$ that is given by replacing each point $x \in X$ by the closed ball $\overline{B}_r(x)$ of radius r around x . The homeomorphism type of $X(r)$ changes with the radius r . For example in Figure 360 we see that at $r = 1$ all but one ball are joined up and we see that $H_1(X(1)) \cong \mathbb{Z}^3$. We show three cycles c_1, c_2, c_3 in $X(1)$ which form a basis for $H_1(X(1))$. Two of the generators, namely c_1, c_2 are formed “by accident”, namely they arise from balls which are close by. This get captured by the fact that they become null-homologous in $H_1(X(2))$ whereas c_3 represents a non-trivial element for “much longer”. Put differently, c_3 is a persistent element in homology, hence it captures global information.

Summarizing we can study the topological spaces $X(r)$ for $r \in [0, \infty)$ and for any $r < s$ we have an inclusion induced map $H_n(X(r)) \rightarrow H_n(X(s))$. If there is a non-zero element in $H_n(X(r))$ which stays non-zero for a “long time”, then we say it is persistent and we hope that it captures something interesting.

The same approach can be applied to any finite subset of some $\mathbb{R}^n$. Pretty much every data one has is a finite subset of some $\mathbb{R}^n$, so one can study its persistent homology. The real question is, whether one captures something significant or not. For example persistent homology was applied in [SHP]⁶⁷² to study the “The topological shape of Brexit”. I have

⁶⁷²Here is an easy way to tell that that paper was not written by mathematicians: In a mathematics paper the authors are invariably listed according to the alphabetic order of their last names. In most other sciences this is not the case.

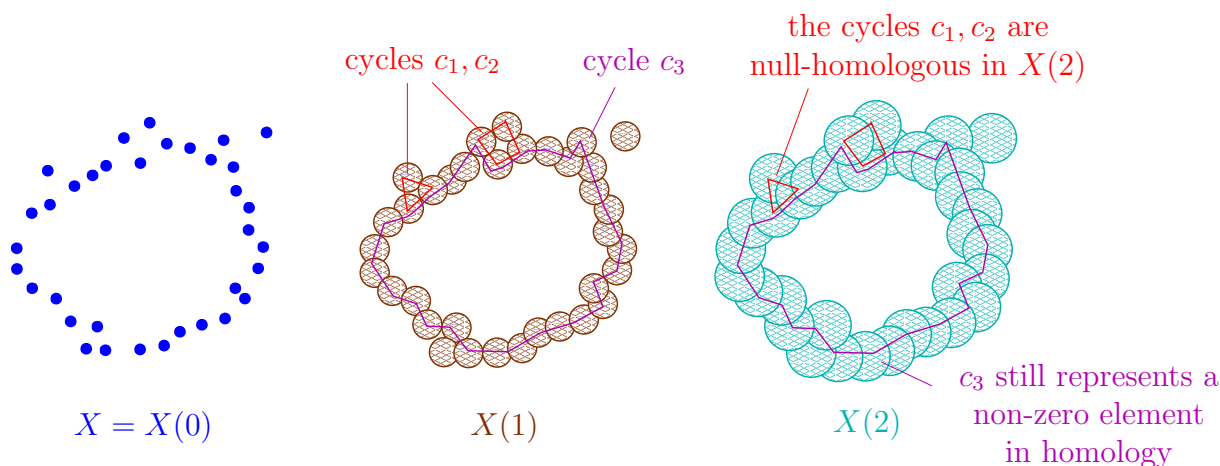


FIGURE 360.

my doubts whether one can really obtain anything meaningful about the politics of Brexit using persistent homology.

38.2. Division algebras. An *algebra* over a field $\mathbb{F}$ is an $\mathbb{F}$ -vector space A together with a bilinear map

$$\begin{aligned} A \times A &\rightarrow A \\ (v, w) &\mapsto v \cdot w. \end{aligned}$$

- (1) We say that an algebra is *commutative* if for any $a, b \in A$ we have $a \cdot b = b \cdot a$.
- (2) We say that an algebra is *associative* if for any $a, b, c \in A$ we have $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.
- (3) We say that the algebra *has an identity* if there exists an element $1 \in A$ with $1 \neq 0$ such that $1 \cdot a = a \cdot 1 = a$ for all $a \in A$.
- (4) A *division algebra* is an algebra such that for any $a \neq 0$ and $b \in A$ there exists an $x \in A$ with $ax = b$ and a $y \in A$ with $ya = b$.

Examples.

- (1) The complex numbers $\mathbb{C}$ with the usual multiplication are an algebra over $\mathbb{R}$ that is commutative and that is a division algebra.
- (2) The cross product

$$\begin{aligned} \mathbb{R}^3 \times \mathbb{R}^3 &\rightarrow \mathbb{R}^3 \\ \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \times \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} &\mapsto v \times w := \begin{pmatrix} v_2 w_3 - w_2 v_3 \\ -(v_1 w_3 - w_1 v_3) \\ v_1 w_2 - w_1 v_2 \end{pmatrix} \end{aligned}$$

is an algebra. Clearly we have $v \times w = -w \times v$, hence the algebra is non-commutative.

The geometric interpretation of the cross product is given as follows: for $v, w \in \mathbb{R}^3$ the cross product $v \times w$ is the unique vector that satisfies the following three properties:

- (a) the vector is orthogonal to v and w ,
- (b) the length equals the area of the parallelogram spanned by v and w ,
- (c) we have $\det(v w v \times w) \geq 0$.⁶⁷³

It follows easily from the geometric interpretation that $\mathbb{R}^3$ with the cross product is a division algebra that does not admit an identity.

- (3) The quaternions $\mathbb{H}$ are defined as the 4-dimensional real vector space $\mathbb{R}^4$ spanned by $1, i, j, k$ and the multiplication which is determined by

$$i^2 = j^2 = k^2 = -1, \text{ and } 1i = i, 1j = j, 1k = k \text{ and } ij = k, jk = i, ki = j.$$

The quaternions are a non-commutative associative division algebra with identity.

- (4) The Cayley octonions $\mathbb{O}$ are an 8-dimensional division algebra with identity that is neither commutative nor associative. It is defined as $\mathbb{O} = \mathbb{H} \oplus \mathbb{H}$ with the multiplication

$$(z_1, z_2) \cdot (w_1, w_2) := (z_1 w_1 - \bar{w}_2 z_2, z_2 \bar{w}_1 + w_2 z_1)$$

where given $w = a + bi + cj + dk \in \mathbb{H}$ we write $\bar{w} = a - bi - cj - dk$.

Lemma 38.1. *Let A be a finite-dimensional division algebra. Then the following two statements are equivalent:*

- (1) A is a division algebra,
- (2) A has no zero-divisors, i.e. for any $a, b \in A$ with $a \cdot b = 0$ we have $a = 0$ or $b = 0$.

Proof. Let $a \in A$ be non-zero. Then the following holds:

$$\begin{aligned} \text{for any } x \in A \text{ there exists } y \in A \text{ with } ax = y &\Leftrightarrow \text{the map } A \rightarrow A, x \mapsto ax \text{ is surjective} \\ &\Leftrightarrow \text{the map } A \rightarrow A, x \mapsto ax \text{ is injective} \\ &\quad \uparrow \\ &\text{since } x \mapsto ax \text{ is a homomorphism of finite-dimensional } \mathbb{R}\text{-vector spaces} \\ &\Leftrightarrow \text{for any } b \neq 0 \text{ we have } ab \neq 0. \end{aligned}$$

The lemma now follows easily from this observation and the definitions. $\square$

Theorem 38.2. *Any finite-dimensional division algebra over $\mathbb{R}$ which is commutative and which has an identity is isomorphic either to $\mathbb{R}$ or to $\mathbb{C}$.*

Remark. Note that we do not assume that the division algebra is associative. It follows relatively easily from the fact that $\mathbb{C}$ is algebraically closed field that a finite-dimensional division algebra over $\mathbb{R}$ which is commutative, which has an identity and which is associative is isomorphic either to $\mathbb{R}$ or to $\mathbb{C}$.

Proof. Let $(A, *)$ be a finite-dimensional division algebra over $\mathbb{R}$ which is commutative and which has an identity. Since A is a finite-dimensional vector space we can, after applying an isomorphism, assume that $A = \mathbb{R}^n$ for some $n \in \mathbb{N}$.

⁶⁷³If v and w are non-zero this just means that v, w and $v \times w$ form a positive basis for $\mathbb{R}^3$.

We first consider the case $n = 1$. In that case the algebra structure is given by $x * y = xay$ for some $a \in \mathbb{R}$. Since the algebra is a division algebra we have $a \neq 0$. Then the map $\varphi(x) = \frac{1}{a}x$ defines an isomorphism from $(\mathbb{R}, *)$ to $(\mathbb{R}, \cdot)$.

Now suppose that $n \geq 2$. We first want to show that $n = 2$. To do this we consider the map

$$\begin{aligned} f: S^{n-1} &\rightarrow S^{n-1} \\ x &\mapsto f(x) := \frac{x*x}{|x*x|}. \end{aligned}$$

This is indeed well-defined, since for $x \neq 0$ it follows from our hypothesis on A and from Lemma 38.1 that $x*x \neq 0$. The map f is continuous since the multiplication map $\mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is bilinear, hence continuous.⁶⁷⁴

We have $f(-x) = f(x)$ for all $x \in \mathbb{R}^n$, hence f descends to a map $\bar{f}: \mathbb{RP}^{n-1} \rightarrow S^{n-1}$.

Claim. The map $\bar{f}: \mathbb{RP}^{n-1} \rightarrow S^{n-1}$ is injective.

So let $[x], [y] \in \mathbb{RP}^{n-1} = S^{n-1} / \sim$. We have

$$\begin{aligned} \bar{f}([x]) = \bar{f}([y]) &\implies f(x) = f(y) \implies x * x = \alpha^2 y * y \text{ for } \alpha = \sqrt{|x * x| |y * y|} \\ &\implies 0 = x * x - \alpha^2 y * y = (x + \alpha y) * (x - \alpha y) \\ &\quad \uparrow \\ &\text{since “.” is commutative and bilinear} \\ &\implies x = \pm \alpha y \implies x = \pm y \implies [x] = [y]. \\ &\quad \uparrow \quad \quad \uparrow \\ &\text{Lemma 38.1} \quad \quad \text{since } x, y \in S^{n-1} \end{aligned}$$

This concludes the proof of the claim.

Since $\bar{f}: \mathbb{RP}^{n-1} \rightarrow S^{n-1}$ is an injective map from a compact space to a Hausdorff space it must be a homeomorphism onto its image. Since S^{n-1} is connected and since $\mathbb{RP}^{n-1}$ is a closed manifold it follows from Corollary 30.8 that $\bar{f}: \mathbb{RP}^{n-1} \rightarrow S^{n-1}$ is in fact a homeomorphism, in particular the fundamental groups are isomorphic. But it follows from Proposition 4.12 and Corollary 6.17 that this is only possible if $n = 2$.

It remains to show that $(\mathbb{R}^2, *)$ is isomorphic to $\mathbb{C}$. We start out with the following claim.

Claim. There exists a $k \in A \setminus \mathbb{R} \cdot 1$ with $k * k = c \cdot 1$ for some $c \in \mathbb{R}_{<0}$.

We first pick a random element $j \in A \setminus \mathbb{R} \cdot 1$. Then 1 and j are a basis for $\mathbb{R}^2$ and we can write $j * j = a + bj$ for some $a, b \in \mathbb{R}$. Then $(j - \frac{b}{2}) * (j - \frac{b}{2}) = a + \frac{1}{4}b^2$. We set $k = j - \frac{b}{2}$ and $c = a + \frac{1}{4}b^2$. We still need to show that $c < 0$. Suppose that $c \geq 0$. Then $k * k = c$ implies that $(k + \sqrt{c} \cdot 1) * (k - \sqrt{c} \cdot 1) = 0$. From Lemma 38.1 we conclude that $k = \pm \sqrt{c} \cdot 1$. But this is not possible since k is linearly independent from 1. This concludes the proof of the claim.

⁶⁷⁴Indeed, the fact that the map is bilinear implies that there exists a real $n \times n$ -matrix P such for any $v, w \in \mathbb{R}^n$ we have $v \cdot w = v^T P w$, hence the map is continuous.

One can now easily verify that

$$\begin{aligned} A &\rightarrow \mathbb{C} \\ x + yk &\mapsto x + y\frac{i}{\sqrt{-k}} \end{aligned}$$

is an isomorphism of algebras. □

Theorem 38.3.

- (1) *Any finite-dimensional division algebra over $\mathbb{R}$ which is associative and which has an identity is isomorphic to $\mathbb{R}$, $\mathbb{C}$ or to $\mathbb{H}$.*
- (2) *Any finite-dimensional division algebra over $\mathbb{R}$ which has an identity is of dimension 1, 2, 4 or 8.*

Proof.

- (1) The first statement was originally proved by Frobenius in 1878. A short self-contained prove is for example given in [Pa2].
- (2) The second statement was first proved using topological methods by Kervaire [Ke] and Milnor [Mi3, Corollary 1] building on work of Bott [Bo]. Full details for the proof can also be found in [Ha2, Theorem 2.16]. As of right now there is no purely algebraic proof for the statement.⁶⁷⁵ □

On page 341, given $n \in \mathbb{N}_0$, we had introduced the real projective space $\mathbb{R}P^n$ and the complex projective spaces $\mathbb{C}P^n$. Almost the same definition can also be used to define the *quaternionic projective spaces* $\mathbb{H}P^n$. As in Lemma 22.1 one can show that $\mathbb{H}P^n$ admits a CW-structure that has exactly one cell in the dimensions $0, 4, 8, \dots, 4n$ and that has no other cells. It follows in particular that its Euler characteristic is $n + 1$ and the argument of the proof of Proposition 23.7 shows that $\mathbb{H}P^n$ is 3-connected.

Since the octonions are not associative the definition on page 341 cannot be applied to the octonions. Nonetheless, one can make sense of the *octonionic projective spaces* $\mathbb{O}P^n$ for $n = 1, 2$. These are closed $8n$ -dimensional manifolds that admit a CW-structure with a unique cells in dimensions $0, 8, 2n$ and no other cells. As above we see that $\mathbb{O}P^n$ is 7-connected. We refer to [Bz, Lc] for details.

In particular we see that $\mathbb{O}P^2$ is a closed 16-dimensional manifold that is 7-connected and that has odd Euler characteristic.

The following question is, to the best my knowledge, open:

Question 38.4. *Does there exist an 8-connected closed manifold with odd Euler characteristic?*

⁶⁷⁵One might ask whether any finite-dimensional division algebra over $\mathbb{R}$ which has an identity is isomorphic to $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ to $\mathbb{O}$. I could not find a reference for the statement. The statement holds under the stronger hypothesis that the algebra is “normed” or “alternativ”.

38.3. The transfer map. Before we can give the next application of homology theory, namely the Borsuk-Ulam Theorem, we need to introduce and study the transfer map in homology corresponding to a finite covering.

Let $p: \tilde{X} \rightarrow X$ be a finite k -fold covering. Let $\sigma: \Delta^n \rightarrow X$ be a singular simplex. Since Δ^n is simply connected it follows from Proposition 15.2 that there exist precisely k different lifts of σ to $\tilde{X}$, i.e. there exist precisely k different singular simplices $\tilde{\sigma}_1, \dots, \tilde{\sigma}_k: \Delta^n \rightarrow \tilde{X}$ with $p \circ \tilde{\sigma}_i = \sigma$ for $i = 1, \dots, k$. We write

$$p^*(\sigma) := \sum_{i=1}^k \tilde{\sigma}_i.$$

Given any commutative ring R we refer to the map

$$\begin{aligned} p^*: C_*(X; R) &\rightarrow C_*(\tilde{X}; R) \\ \sum_{i=1}^n \sigma_i \otimes r_i &\mapsto \sum_{i=1}^n p^*(\sigma_i) \otimes r_i \end{aligned}$$

as the *transfer map* of the covering. It is straightforward to verify that the transfer map is in fact a chain map.

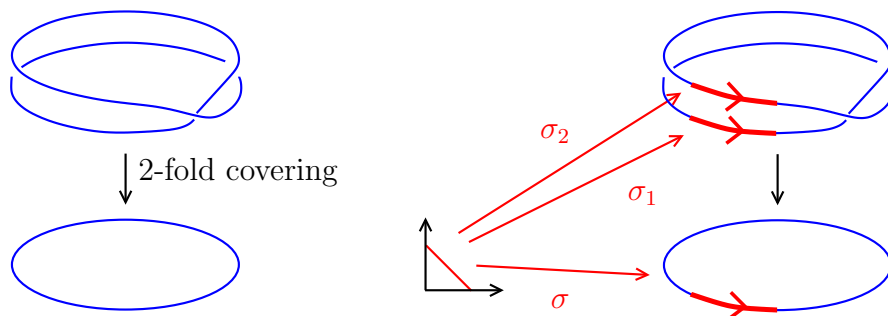


FIGURE 361.

Proposition 38.5. *Let $p: \tilde{X} \rightarrow X$ be a finite covering of degree k and let $n \in \mathbb{N}_0$. Then the following hold:*

- (1) *For any commutative ring R the map $p_* \circ p^*: H_n(\tilde{X}; R) \rightarrow H_n(X; R)$ is given by multiplication by k .*
- (2) *The map $p_*: H_n(\tilde{X}; \mathbb{Q}) \rightarrow H_n(X; \mathbb{Q})$ is an epimorphism.*

Remark. If $p: \tilde{X} \rightarrow X$ is a finite covering of path-connected topological spaces, then we had seen in Corollary 32.7 that the map $p_*: H_1(\tilde{X}) \rightarrow H_1(X)$ has finite cokernel. This implies in particular⁶⁷⁶ that $p_*: H_1(\tilde{X}; \mathbb{Q}) \rightarrow H_1(X; \mathbb{Q})$ is an epimorphism. Therefore we can view Proposition 38.5 (2) as a generalization of Corollary 32.7.

⁶⁷⁶Why does this follow?

Note that the conclusion of Proposition 38.5 does not hold if we drop the condition that the covering is finite. For example we have an infinite covering $p: \mathbb{R} \rightarrow S^1$ but $0 = H_1(\mathbb{R}; \mathbb{Q}) \xrightarrow{p_*} H_1(S^1; \mathbb{Q}) = \mathbb{Q}$ is not an epimorphism.

Proof. Let $p: \tilde{X} \rightarrow X$ be a finite k -fold covering.

- (1) Let R be a commutative ring. We denote by $p^*: C_n(X; R) \rightarrow C_n(\tilde{X}; R)$ the transfer map. Let $\sigma: \Delta^m \rightarrow X$ be a singular simplex. We denote the corresponding lifts by $\tilde{\sigma}_1, \dots, \tilde{\sigma}_k$. Then

$$(p_* \circ p^*)(\sigma) = p_*(\tilde{\sigma}_1 + \dots + \tilde{\sigma}_k) = \underbrace{p \circ \tilde{\sigma}_1 + \dots + p \circ \tilde{\sigma}_k}_{k \text{ copies}} = \underbrace{\sigma + \dots + \sigma}_{k \text{ copies}} = k \cdot \sigma.$$

We thus see that $p^* \circ p_*: C_m(X; R) \rightarrow C_m(X; R)$ is multiplication by k for any m . Since p_* and p^* are chain maps we see that the map

$$H_n(X; R) \xrightarrow{p^*} H_n(\tilde{X}; R) \xrightarrow{p_*} H_n(X; R)$$

is also given by multiplication by k .

- (2) Since multiplication by k is an isomorphism of any rational vector space we obtain from (1) that $p_* \circ p^*$ is an isomorphism. In particular the second map, namely p_* , is an epimorphism. $\square$

Proposition 38.6. *Let $p: \tilde{X} \rightarrow X$ be a 2-fold covering. Then there exists a long exact sequence*

$$\dots \rightarrow H_n(X; \mathbb{Z}_2) \xrightarrow{\tau} H_n(\tilde{X}; \mathbb{Z}_2) \xrightarrow{p_*} H_n(X; \mathbb{Z}_2) \xrightarrow{\partial} H_{n-1}(X; \mathbb{Z}_2) \rightarrow \dots$$

Proof. By Proposition 26.9 it suffices to show that the following sequence of chain maps

$$0 \rightarrow C_*(X; \mathbb{Z}_2) \xrightarrow{\tau} C_*(\tilde{X}; \mathbb{Z}_2) \xrightarrow{p_*} C_*(X; \mathbb{Z}_2) \rightarrow 0$$

is in fact exact. By Lemma 20.3 there exists a unique non-trivial covering map $t: \tilde{X} \rightarrow \tilde{X}$. By a slight abuse of notation we denote the induced maps on $C_n(\tilde{X})$ and $C_n(\tilde{X}; \mathbb{Z}_2)$ by t as well. By Proposition 15.2 we can pick for each singular cell σ of X a lift $\tilde{\sigma}$ to $\tilde{X}$. We make the following observations:

- (1) given any singular n -simplex σ we have $p_*(\tilde{\sigma}) = \sigma$,
- (2) given any singular n -simplex σ we have $\tau(\sigma) = \tilde{\sigma} + t\tilde{\sigma}$,
- (3) given any singular simplex μ of $\tilde{X}$ we have $p_*(t\mu) = p_*(\mu)$,
- (4) any singular n -simplex μ of $\tilde{X}$ is of the form $\mu = \tilde{\sigma}$ or $\mu = t\tilde{\sigma}$ for some singular n -simplex σ of X .

Now we turn to the proof of the exactness of the above sequence.

- (1) We first show that the map $p_*: C_n(\tilde{X}; \mathbb{Z}) \rightarrow C_n(X; \mathbb{Z}_2) = C_n(X) \otimes \mathbb{Z}_2$ is surjective. So let $c = \sigma_1 \otimes b_1 + \dots + \sigma_k \otimes b_k \in C_n(X)$. It follows from (1) that

$$p(\tilde{\sigma}_1 \otimes b_1 + \dots + \tilde{\sigma}_k \otimes b_k) = \sigma_1 \otimes b_1 + \dots + \sigma_k \otimes b_k = c.$$

- (2) Now we want to show that $\tau: C_n(X; \mathbb{Z}_2) \rightarrow C_n(\tilde{X}; \mathbb{Z}_2)$ is injective. Let $c \in C_n(X; \mathbb{Z}_2)$ be non-trivial. We can write $c = \sigma_1 \otimes 1 + \cdots + \sigma_k \otimes 1$ where $\sigma_1, \dots, \sigma_k$ are pairwise different singular simplices in X . Then it follows from (2)

$$\tau(c) = (\tilde{\sigma}_1 + t\tilde{\sigma}_1) \otimes 1 + \cdots + (\tilde{\sigma}_k + t\tilde{\sigma}_k) \otimes 1$$

is also non-zero since the singular simplices $\tilde{\sigma}_1, t\tilde{\sigma}_1, \dots, \tilde{\sigma}_k, t\tilde{\sigma}_k$ are also pairwise different.

- (3) Next we show that $\text{im}(\tau) \subset \ker(p_*)$. For any singular n -simplex σ of $\tilde{X}$ we have by (3) above that⁶⁷⁷ $(\tau \circ p_*)(\sigma) = p(\tilde{\sigma} + t\tilde{\sigma}) = 2\sigma$, but any multiple of two is zero in $C_*(X; \mathbb{Z}_2)$.
- (4) Finally we have to show that $\ker(p_*) = \text{im}(\tau)$. So let $c \in \ker(p_*)$. By (4) we can write

$$c = (\tilde{\sigma}_1 \otimes a_1 + t\tilde{\sigma}_1 \otimes b_1) \cdots + (\tilde{\sigma}_k \otimes a_k + t\tilde{\sigma}_k \otimes b_k)$$

where $\sigma_1, \dots, \sigma_k$ are pairwise different. By (3) we have

$$p(c) = \sigma_1 \otimes (a_1 + b_1) + \cdots + \sigma_k \otimes (a_k + b_k).$$

But this implies that $a_i = -b_i$ for $i = 1, \dots, k$. But since we work over $\mathbb{Z}_2$ we now have $a_i = -b_i$ for $i = 1, \dots, k$. It now follows from (2) that

$$c = \tau(\sigma_1 \otimes a_1 + \cdots + \sigma_k \otimes a_k),$$

as desired. □

38.4. The Borsuk-Ulam Theorem and the Ham-Sandwich Theorem. Our goal in this section is to prove the following generalization of Theorem 6.21 to the higher-dimensional situation.

Theorem 38.7. (Borsuk-Ulam) *For every map $f: S^n \rightarrow \mathbb{R}^n$ there exists a pair of antipodal points x and $-x$ in S^n with $f(x) = f(-x)$.*

The key ingredient in the proof of Theorem 38.7 is the following proposition.

Proposition 38.8. *Let $f: S^n \rightarrow S^n$ be a map. If $f(-x) = -f(x)$ for all $x \in S^n$, then $\deg(f)$ is odd.*

Proof. In the following we prove the proposition for $n > 1$. The proof for $n = 1$ is almost identical.

We now intend to apply Proposition 38.6 to the 2-fold covering $p: S^n \rightarrow \mathbb{RP}^n$. Before we do so we recall from Lemma 29.19 that

$$H_k(S^n; \mathbb{F}_2) \cong \begin{cases} \mathbb{F}_2, & \text{if } k = 0 \text{ or } k = n, \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad H_k(\mathbb{RP}^n; \mathbb{F}_2) \cong \begin{cases} \mathbb{F}_2, & \text{if } k = 0, \dots, n, \\ 0, & \text{otherwise.} \end{cases}$$

⁶⁷⁷That is of course basically the same argument as in the proof of Proposition 38.5.

The long exact sequence in homology from Proposition 38.6 now looks as follows:

$$\begin{array}{ccccccc}
 0 \longrightarrow H_n(\mathbb{RP}^n; \mathbb{Z}_2) & \xrightarrow{\tau} & H_n(S^n; \mathbb{Z}_2) & \xrightarrow{p_*} & H_n(\mathbb{RP}^n; \mathbb{Z}_2) & \xrightarrow{\cong} & H_{n-1}(\mathbb{RP}^n; \mathbb{Z}_2) \longrightarrow 0 \longrightarrow \dots \\
 & & \dots \longrightarrow & 0 & \longrightarrow & H_i(\mathbb{RP}^n; \mathbb{Z}_2) & \xrightarrow{\cong} H_{i-1}(\mathbb{RP}^n; \mathbb{Z}_2) \longrightarrow 0 \longrightarrow \dots \\
 & & \dots \longrightarrow & 0 & \longrightarrow & H_1(\mathbb{RP}^n; \mathbb{Z}_2) & \xrightarrow{\cong} H_0(\mathbb{RP}^n; \mathbb{Z}_2) \xrightarrow{0} \longrightarrow \\
 & & & & & \xrightarrow{0} H_0(S_n; \mathbb{Z}_2) & \xrightarrow{p_*} H_0(\mathbb{RP}^n; \mathbb{Z}_2) \longrightarrow 0.
 \end{array}$$

Here the isomorphisms are a straightforward consequence of the calculation of the homology groups and the fact that the sequence is exact.

Now let $f: S^n \rightarrow S^n$ be a map which satisfies $f(-x) = -f(x)$ for all $x \in S^n$. This map descends to a well-defined map

$$\begin{array}{ccc}
 \bar{f}: \mathbb{RP}^n & \rightarrow & \mathbb{RP}^n \\
 [x] & \mapsto & [f(x)].
 \end{array}$$

Claim. The diagram

$$\begin{array}{ccccccc}
 \longrightarrow H_k(\mathbb{RP}^n; \mathbb{Z}_2) & \xrightarrow{\tau} & H_k(S^n; \mathbb{Z}_2) & \xrightarrow{p_*} & H_k(\mathbb{RP}^n; \mathbb{Z}_2) & \xrightarrow{\partial} & H_{k-1}(\mathbb{RP}^n; \mathbb{Z}_2) \longrightarrow \\
 \downarrow \bar{f}_* & & \downarrow f_* & & \downarrow \bar{f}_* & & \downarrow f_* \\
 \longrightarrow H_k(\mathbb{RP}^n; \mathbb{Z}_2) & \xrightarrow{\tau} & H_k(S^n; \mathbb{Z}_2) & \xrightarrow{p_*} & H_k(\mathbb{RP}^n; \mathbb{Z}_2) & \xrightarrow{\partial} & H_{k-1}(\mathbb{RP}^n; \mathbb{Z}_2) \longrightarrow
 \end{array}$$

commutes.

By the remark on page 431 it suffices to show that the following diagram commutes:

$$\begin{array}{ccccccc}
 0 \longrightarrow C_k(\mathbb{RP}^n; \mathbb{Z}_2) & \xrightarrow{\tau} & C_k(S^n; \mathbb{Z}_2) & \xrightarrow{p_*} & C_k(\mathbb{RP}^n; \mathbb{Z}_2) & \longrightarrow & 0 \\
 \downarrow \bar{f}_* & & \downarrow f_* & & \downarrow \bar{f}_* & & \\
 0 \longrightarrow H_k(\mathbb{RP}^n; \mathbb{Z}_2) & \xrightarrow{\tau} & H_k(S^n; \mathbb{Z}_2) & \xrightarrow{p_*} & H_k(\mathbb{RP}^n; \mathbb{Z}_2) & \longrightarrow & 0.
 \end{array}$$

Here the right-hand square commutes since $p \circ f = \bar{f} \circ p$. The left-hand square commutes since for a singular k -simplex $\sigma: \Delta^k \rightarrow \mathbb{RP}^n$ with lifts $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ the two lifts of $\bar{f}\sigma$ are $f \circ \tilde{\sigma}_1$ and $f \circ \tilde{\sigma}_2$ since f takes antipodal points to antipodal points. This concludes the proof of the claim.

It follows from the long exact sequence in the beginning of the proof and from the above claim that for any $k \in \{0, \dots, n-1\}$ we have a commutative diagram

$$\begin{array}{ccc}
 H_{k+1}(\mathbb{RP}^n, \mathbb{Z}_2) & \xrightarrow{\cong} & H_k(\mathbb{RP}^n, \mathbb{Z}_2) \\
 \downarrow \bar{f}_* & & \downarrow \bar{f}_* \\
 H_{k+1}(\mathbb{RP}^n, \mathbb{Z}_2) & \xrightarrow{\cong} & H_k(\mathbb{RP}^n, \mathbb{Z}_2).
 \end{array}$$

The vertical map is an isomorphism for $k = 0$. Since the horizontal maps are isomorphisms it now follows from an elementary induction argument that the maps $\bar{f}_*: H_k(\mathbb{RP}^n, \mathbb{Z}_2) \rightarrow$

$H_k(\mathbb{RP}^n, \mathbb{Z}_2)$ are isomorphisms for $k = 1, \dots, n$. Furthermore we have a commutative diagram

$$\begin{array}{ccc} H_n(\mathbb{RP}^n, \mathbb{Z}_2) & \xrightarrow[\cong]{\partial} & H_n(S^n, \mathbb{Z}_2) \\ \downarrow \bar{\tau}_* & & \downarrow f_* \\ H_n(\mathbb{RP}^n, \mathbb{Z}_2) & \xrightarrow[\cong]{\tau_*} & H_n(S^n, \mathbb{Z}_2). \end{array}$$

Thus we see that $f_*: H_n(S^n, \mathbb{Z}_2) \rightarrow H_n(S^n, \mathbb{Z}_2)$ is an isomorphism. But by Lemma 36.15 this implies that the degree of f is odd. $\square$

Now we turn to the proof of the Borsuk-Ulam Theorem 38.7.

Proof of Theorem 38.7. Let $g: S^n \rightarrow \mathbb{R}^n$ be a map. Suppose there exists no pair of antipodal points x and $-x$ in S^n with $g(x) = g(-x)$. Put differently, suppose that for any $x \in S^n$ we have $g(x) \neq -g(-x)$. We consider the map

$$\begin{aligned} f: S^n &\rightarrow S^{n-1} \\ x &\mapsto f(x) := \frac{g(x) - g(-x)}{|g(x) - g(-x)|}. \end{aligned}$$

Evidently we have $f(x) = -f(-x)$ for all $x \in S^{n-1}$. From Proposition 38.8 it follows that $\deg(f|_{S^{n-1}})$ is odd.

Now we consider the map

$$\begin{aligned} \Phi: S^{n-1} \times [0, 1] &\rightarrow S^n \\ (x, t) &\mapsto (\underbrace{x \cdot \sqrt{1-t^2}}_{\in \mathbb{R}^n}, t). \end{aligned}$$

This defines a homotopy from $\Phi(-, 0) = f|_{S^{n-1}}$ to the constant map $\Phi(-, 1)$. But the former map has odd degree and the latter map has degree zero. But homotopic maps have the same degree by Lemma 27.2 (3). We have thus obtained a contradiction. $\square$

In the following we will state and prove several consequences of the Borsuk-Ulam Theorem 38.7.

Proposition 38.9. *Let $A_1, \dots, A_{n+1}$ be closed subsets of S^n with $A_1 \cup \dots \cup A_{n+1} = S^n$. Then there exists an $i \in \{1, \dots, n+1\}$ such that A_i contains antipodal points.*

Example. For $n = 1$ we obtain that if $S^1 = A \cup B$ with A and B closed, then A or B contains antipodal points. How can one prove this “naively”?

Proof. Given a subset A of S^n and $x \in S^n$ we write

$$d(x, A) := \inf\{d(x, a) \mid a \in A\},$$

where $d(x, a)$ denotes the distance function on S^n given by the minimal length of a path connecting x and a . Note that $d(x, A)$ is continuous in x . Now we consider the function

$$\begin{aligned} f: S^n &\rightarrow \mathbb{R}^n \\ x &\mapsto (d(x, A_1), \dots, d(x, A_n)). \end{aligned}$$

By the Borsuk-Ulam Theorem 38.7 there exists an x with $f(x) = f(-x)$. We first suppose that there exists an $i \in \{1, \dots, n\}$ with $d(x, A_i) = 0$. From $f(x) = f(-x)$ it follows that $d(-x, A_i) = 0$ as well. Since A_i is closed it follows that $x, -x$ lie in A_i .

Now suppose that $d(x, A_i) \neq 0$ for $i = 1, \dots, n$. Since $A_1 \cup \dots \cup A_{n+1} = S^n$ it follows that x and $-x$ lie in A_{n+1} . $\square$

Before we can state the next consequence of the Borsuk-Ulam Theorem 38.7 we need to introduce one more definition.

Definition. Let $A \subset \mathbb{R}^n$ be a measurable subset⁶⁷⁸ of finite volume. We say a hyperplane $H \subset \mathbb{R}^n$ *bisects* A if for the two half-spaces V, V' determined by H we have $\text{vol}(A \cap V) = \text{vol}(A \cap V')$.

Theorem 38.10. (Ham-Sandwich Theorem) *Let $A_1, \dots, A_n$ be measurable subsets of $\mathbb{R}^n$. Then there exists an $(n-1)$ -dimensional hyperplane in $\mathbb{R}^n$ that bisects $A_1, \dots, A_n$.*

Remark. The Ham-Sandwich Theorem gets its name from the following application. Suppose we are given a sandwich with an upper piece of bread A_1 , some ham A_2 and a lower piece of bread A_3 . Then using a knife one can cut the sandwich into two pieces such that each piece of bread and the ham are bisected. This situation is, more or less convincingly, illustrated in Figure 362.

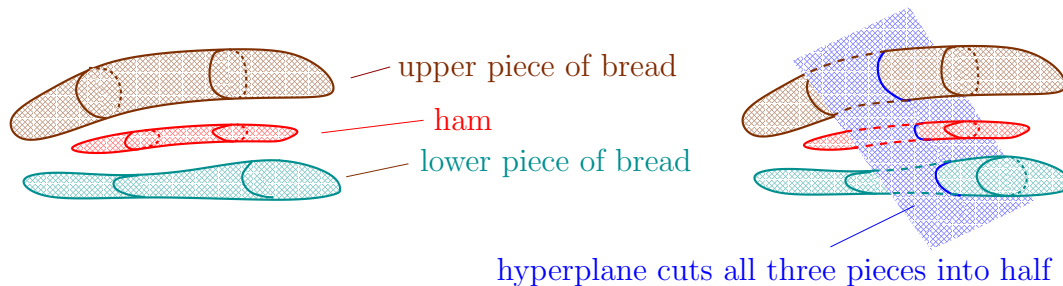


FIGURE 362.

In the proof of the Ham-Sandwich Theorem we will need the following result from measure theory. It is Proposition 6.2 (6) from Analysis III.

Proposition 38.11. *Let $\{X_n\}_{n \in \mathbb{N}}$ be a sequence of measurable subsets of $\mathbb{R}^n$ such that for each $n \in \mathbb{N}$ we have $X_n \subset X_{n+1}$. Then*

$$\lim_{n \rightarrow \infty} \text{vol}(X_n) = \text{vol}\left(\bigcup_{n \in \mathbb{N}} X_n\right).$$

Now we can provide the proof of the Ham-Sandwich Theorem.

⁶⁷⁸Here “measurable” means Lebesgue-measurable in the sense of Analysis III.

Proof of the Ham-Sandwich Theorem. Let $A_1, \dots, A_n$ be measure subsets of $\mathbb{R}^n$. Without loss of generality we can assume that the volume of each A_i is positive. Given a point $P \in S^{n-1}$ and $t \in \mathbb{R}$ we write

$$H_{P,t} := \{tP + w \mid P \cdot w = 0\}$$

and

$$\begin{aligned} V_{P,t}^+ &:= \{tP + w \mid P \cdot w \geq 0\} \\ V_{P,t}^- &:= \{tP + w \mid P \cdot w \leq 0\}. \end{aligned}$$

Put differently, $H_{P,t}$ is the hyperplane through $t \cdot P$ orthogonal to P , and $V_{P,t}^\pm$ are the two half-space determined by $H_{P,t}$. Note that $V_{-P,-t}^\pm = V_{P,t}^\mp$. On several occasions we will use, without comment, that for a subset $A \subset \mathbb{R}^n$ of finite volume we have the equality

$$\begin{aligned} \text{vol}(A \cap V_{P,t}^\pm) &= \text{vol}(A \cap (\mathbb{R}^n \setminus V_{P,t}^\mp)) = \text{vol}(A) - \text{vol}(V_{P,t}^\mp). \\ &\quad \uparrow \\ &\text{since } H_{P,t} = V_{P,t}^+ \cap V_{P,t}^- \text{ is a set of measure zero} \end{aligned}$$

We consider the function

$$\begin{aligned} \mathbb{R} &\rightarrow \mathbb{R}_{\geq 0} \\ t &\mapsto \text{vol}(A_n \cap V_{P,t}^+). \end{aligned}$$

This function is continuous⁶⁷⁹ and we have

$$\begin{aligned} \lim_{t \rightarrow \infty} \text{vol}(A_n \cap V_{P,t}^+) &= \text{vol}\left(\bigcup_{t \in \mathbb{Z}} (A_n \cap V_{P,t}^+)\right) = \text{vol}(A_n). \\ &\quad \uparrow \\ &\text{Proposition 38.11} \end{aligned}$$

and

$$\begin{aligned} \lim_{t \rightarrow -\infty} \text{vol}(A_n \cap V_{P,t}^+) &= \text{vol}(A_n) - \underbrace{\lim_{t \rightarrow -\infty} \text{vol}(A_n \cap V_{P,t}^-)}_{= \text{vol}(A_n) \text{ by Proposition 38.11}} = 0. \end{aligned}$$

We deduce from the Intermediate Value Theorem that

$$T := \{t \in \mathbb{R} \mid \text{vol}(A_n \cap V_{P,t}^+) = \tfrac{1}{2} \text{vol}(A_n)\}.$$

is a non-empty set.⁶⁸⁰ It is straightforward to see that it is a connected subset of $\mathbb{R}$, i.e. it is an interval. Since $\text{vol}(A_n) > 0$ we know that T is a finite interval. We define

$$t(P) := \tfrac{1}{2}(\inf(T) + \sup(T)).$$

Note that $t(-P) = -t(P)$. Furthermore note that the function $S^{n-1} \rightarrow \mathbb{R}$ given by $P \mapsto t(P)$ is continuous.⁶⁸¹

⁶⁷⁹This requires at least a little bit of thought since we do not assume that A is bounded, we only assume it has finite volume.

⁶⁸⁰One might think that T consists of just a point, but if A_n is disconnected, then it can happen that T is in fact a interval that is not just a point.

⁶⁸¹Again this requires a little bit of thought.

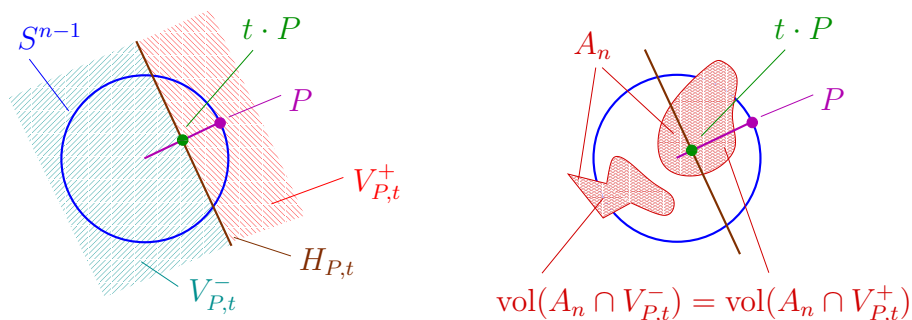


FIGURE 363. Illustration for the proof of the Ham-Sandwich Theorem 38.10

Now we consider the map

$$\begin{aligned} f: S^{n-1} &\rightarrow \mathbb{R}^{n-1} \\ P &\mapsto (\text{vol}(A_1 \cap V_{P,t(P)}^+), \dots, \text{vol}(A_{n-1} \cap V_{P,t(P)}^+)). \end{aligned}$$

The previous two continuity arguments (which we skipped) show that f is continuous. By the Borsuk–Ulam Theorem 38.7, there exists a point $P \in S^{n-1}$ with $f(P) = f(-P)$. We claim that the hyperplane $H_{P,t(P)}$ has the desired property.

By definition of $t(P)$ the hyperplane $H_{P,t(P)}$ bisects A_n . For $i = 1, \dots, n-1$ we furthermore have

$$\begin{aligned} \text{vol}(A_i \cap V_{P,t(P)}^+) &= \text{vol}(A_i \cap V_{-P,t(-P)}^+) = \text{vol}(A_i \cap V_{-P,-t(P)}^+) = \text{vol}(A_i \cap V_{P,t(P)}^-) \\ &\quad \uparrow \qquad \qquad \qquad \uparrow \qquad \qquad \qquad \uparrow \\ &\quad \text{since } f(P) = f(-P) \qquad \text{since } t(-P) = -t(P) \qquad \text{since } V_{Q,s}^+ = V_{-Q,-s}^- \end{aligned}$$

This shows that the hyperplane $H_{P,t(P)}$ also bisects $A_1, \dots, A_{n-1}$. $\square$

Remark. The proof for the existence of the hyperplane that bisects the given sets $A_1, \dots, A_n$ is non-constructive, in the sense that even with some flexibility in mathematical rigorosity one cannot turn it into an algorithm that given $A_1, \dots, A_n$ finds the hyperplane.

39. THE HOMOLOGY GROUPS OF TOPOLOGICAL MANIFOLDS

39.1. The topology of topological manifolds. In earlier sections we had discussed the homology groups of manifolds at length. The key tool hereby was that compact manifolds are finite CW-complexes and that every manifold can be viewed as a simplicial complex. The situation is more complicated for topological manifolds. The following theorem can be viewed as a weak version of Theorem 22.2 for topological manifolds.⁶⁸²

Theorem 39.1. *Every compact topological manifold is homotopy equivalent to a finite CW-complex.*

Proof. Olof Hanner [Hr, Theorem 3.3] showed in 1950 that every topological manifold is an “absolute neighborhood retract” and James West [We] showed in 1977 that every compact “absolute neighborhood retract” is homotopy equivalent to a compact CW-complex.

An alternative proof is given by Rob Kirby and Larry Siebenmann [KS, KS2]. □

Of course it would be much nicer if every compact topological manifold was in fact a simplicial complex or at least a CW-complex. In the following we summarize what is known about the existence of simplicial structures and of CW-structures on topological manifolds.

- (1) Tibor Radó [Rd] (see also [MTh, Theorem 3.1.1]) showed in 1926 that every compact 2-dimensional topological manifold has a simplicial structure, in particular it has a CW-structure. Edwin Moise [Mo] proved the analogous result for 3-dimensional topological manifolds.
- (2) Rob Kirby and Larry Siebenmann [KS2, Essay III.2] showed that every compact topological manifold of dimension ≥ 6 has a CW-structure. In 1982 Frank Quinn [Qu] extended this result to the case of 5-dimensional manifolds.
- (3) Andrew Casson, see [AM, page xvi], showed in the 1980s that there exist closed topological 4-dimensional manifolds that do not have a simplicial structure.
- (4) Ciprian Manolescu [Man] showed in 2013 that for every $n \geq 5$ there exists a closed n -dimensional topological manifold that does not admit a simplicial structure.
- (5) It seems to be unknown whether every compact 4-dimensional topological manifold has a CW-structure.
- (6) Using [W1, Corollary 5.1] and the computations of homology groups of topological manifolds one can prove the following refinement of Theorem 39.1:
 - (a) Every closed m -dimensional topological manifold is homotopy equivalent to a finite m -dimensional CW-complex.
 - (b) Every connected m -dimensional topological manifold with non-empty boundary is homotopy equivalent to a finite $(m - 1)$ -dimensional CW-complex.

⁶⁸² Theorem 22.2 says that every compact n -dimensional manifold M admits a CW-structure with finitely many cells, such that the maximal dimension of a cell is n . Given an n -dimensional compact topological manifold one can ask whether it is homotopy equivalent to a finite CW-complex such that the maximal dimension of a cell is n . I did not find a reference for such a statement, so I am not sure whether it holds.

We will not make use of these two statements since some of the results we will prove later on are actually used in the proof of the above two statements.

In this section we will see that many of the results that we had stated and proved for manifolds also hold for topological manifolds. We start out with the following proposition which is the generalization of Propositions 22.17 and 29.10 (1) to the case of topological manifolds.

Proposition 39.2. *Let M be a compact topological manifold. Then the following hold:*

- (1) *The group $\pi_1(M)$ is finitely presented.*
- (2) *All homology groups $H_k(M)$ are finitely generated abelian groups.*
- (3) *All relative homology groups $H_k(M, \partial M)$ are finitely generated abelian groups.*

Proof. The first two statements of the proposition are an immediate consequence of Theorem 39.1 together with Proposition 22.16, Corollary 25.5 and Proposition 29.9. The third part is proved in exercise sheet 1. $\square$

On page 357 we had introduced the Euler characteristic of a finite CW-complex X and we had shown in Proposition 34.1 that the Euler characteristic is determined by the Betti numbers of X . It is not known whether a compact topological manifold admits the structure of a finite CW-complex. Therefore we use the formula from Proposition 34.1 to define the Euler-characteristic of a compact topological manifold.

Definition. Given a compact topological manifold X we define its *Euler characteristic* of X to be

$$\chi(X) := \sum_n (-1)^n \cdot b_n(X).$$

It follows from Proposition 39.2 that the Euler characteristic is defined, i.e. it is a finite sum. Furthermore, if X does in fact admit the structure of a CW-complex, then it follows from Proposition 34.1 that the two definitions of the Euler characteristic agree.

It turns out that the formulas we proved before for the Euler characteristic of CW-complexes also hold for topological manifolds. More precisely, we have the following proposition that is the analogue of Lemmas 34.3 and 34.4 and of Proposition 22.24 in our new context.

Proposition 39.3.

- (1) *Let X be and Y be two compact topological manifolds, then*

$$\chi(X \times Y) = \chi(X) \cdot \chi(Y).$$

- (2) *Let $X = Y \cup Z$ be a decomposition of a compact topological manifold into two compact submanifolds Y and Z such that $\partial Y = \partial Z$ is a submanifold of X . Then the following equality holds*

$$\chi(X) = \chi(Y) + \chi(Z) - \chi(Y \cap Z).$$

- (3) Let X be a compact topological manifold and let $p: \tilde{X} \rightarrow X$ be a finite covering. Then

$$\chi(\tilde{X}) = [\tilde{X} : X] \cdot \chi(X).$$

Sketch of proof.

- (1) This equality can be proved using the Künneth-Formula 37.4 for Chain Complexes. We will fill in the details in exercise sheet 3.
- (2) We will not make use of this statement and we will not provide a proof thereof. We will discuss the strategy for proving the statement in exercise sheet 8.
- (3) Let X be a compact topological manifold and let $p: \tilde{X} \rightarrow X$ be a finite covering. Without loss of generality we can assume that X and $\tilde{X}$ are connected. By Theorem 39.1 there exists a homotopy equivalence $f: X \rightarrow Y$, where Y is a finite CW-complex. By Proposition 15.5 there exists a connected covering $q: \tilde{Y} \rightarrow Y$ such that $\pi_1(\tilde{Y}) = f_*(\pi_1(\tilde{X}))$. Using Proposition 15.2 one can now show that $f: X \rightarrow Y$ lifts to map $\tilde{f}: \tilde{X} \rightarrow \tilde{Y}$ and that this map is again a homotopy equivalence. We now obtain that

$$\begin{array}{ccccc} \chi(\tilde{X}) & = & \chi(\tilde{X}) & = & [\pi_1(\tilde{X}) : \pi_1(X)] \cdot \chi(X) & = & [\tilde{Y} : Y] \cdot \chi(Y). \\ & \uparrow & & \uparrow & & \uparrow & \\ & \text{since } X \text{ and } Y \text{ are} & & \text{Proposition 22.24} & & \text{since } \tilde{X} \text{ and } \tilde{Y} \text{ are} & \\ & \text{homotopy equivalent} & & \text{and Lemma 6.15} & & \text{homotopy equivalent} & \square \end{array}$$

The following theorem is the generalization of the Collar Neighborhood Theorem 39.4 to topological manifolds. We refer to [Vi, Theorem Appendix II.7] or [Ha, p. 253] for a proof.

Theorem 39.4. (Topological Collar Neighborhood Theorem) *Let M be a compact topological manifold. Then there exists a map $\Phi: \partial M \times [0, 1] \rightarrow M$ with the following three properties:*

- (1) *the map Φ is a homeomorphism onto its image,*
- (2) *the image of Φ is a neighborhood of ∂M ,*
- (3) *for all $P \in \partial M$ we have $\Phi(P, 0) = P$.*

For later we record the following corollary to the Topological Collar Neighborhood Theorem 39.4.

Corollary 39.5. *Let M be a connected n -dimensional topological manifold. If ∂M is non-empty, then M is homotopy equivalent to a non-compact n -dimensional topological manifold.*

Sketch of proof. We pick a map $\Phi: \partial M \times [0, 1] \rightarrow M$ as in the conclusion of the Topological Collar Neighborhood Theorem 39.4. We write $W := M \setminus \Phi(\partial M \times [0, 1])$. It is straightforward to see that “pushing the collar $\partial M \times [0, 1]$ into $M \times \{1\}$ ” defines deformation retraction from $M = W \cup \Phi(\partial M \times [0, 1])$ to W . Similarly one shows that W is a deformation retract of $M' := W \cup \Phi(\partial M \times [0, \frac{1}{2}])$. It follows from Lemmas 7.4 and 7.3 that M and M' are homotopy equivalent. But M' is a non-compact n -dimensional topological manifold. $\square$

For later we also record the following generalization of the Excision Theorem 26.27 for Manifolds to the more general situation of topological manifold.

Theorem 39.6. (Excision Theorem for Topological Manifolds) *Let X be an m -dimensional topological manifold and M and N be two m -dimensional submanifolds such that $M \cap N$ is the union of some boundary components of M and such that $M \cap N$ is the union of some boundary components of N . Then for any $n \in \mathbb{N}_0$ the inclusion $(M, M \cap N) \rightarrow (X, N)$ induces an isomorphism*

$$H_n(M, M \cap N) \xrightarrow{\cong} H_n(X, N).$$

Proof. The proof of Theorem 39.6 is identical to the proof of Theorem 26.27, except that we have to replace the Collar Neighborhood Theorem 2.6 by the Topological Collar Neighborhood Theorem 39.4. $\square$

Definition. Given a topological manifold M we refer to

$$DM := (M \times \{1\})_{\partial M \times \{1\} = \partial M \times \{2\}}(M \times \{2\})$$

as the *double* of M . We view M as a submanifold of DM via the inclusion $P \mapsto [(P, 1)]$.

Example. The double of M is given by taking two copies of M and identifying the boundary points “left and right” in the obvious way. For example, the double of the torus minus two open disks is the surface of genus 3. We refer to Figure 364 for an illustration. In exercise

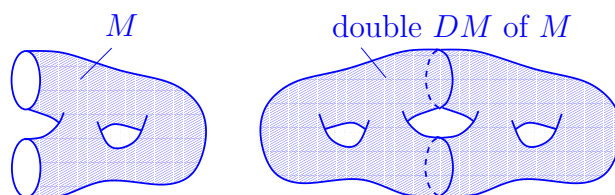


FIGURE 364.

sheet 1 we will determine the double of the Möbius band.

Using the Collar Neighborhood Theorem 39.4 and the Topological Collar Neighborhood Theorem 39.4 one can easily prove the following lemma.

Lemma 39.7. *Let M be a (topological) n -dimensional manifold, then the following statements hold:*

- (1) *The double DM is also (topological) n -dimensional manifold.*
- (2) *The boundary of the double DM is empty.*

Lemma 39.7 says in particular that we can turn any topological manifold into a manifold without boundary by doubling. In particular, if the original (topological) manifold is compact, then the double is a closed (topological) manifold. The doubling trick is often useful

when one tries to extend results on closed (topological) manifolds to compact (topological) manifolds.

The following lemma relates the algebraic invariants of M to the invariants of DM .

Lemma 39.8. *Let M be a topological manifold and let R be a commutative ring.*

- (1) *For any i the inclusion induced map $H_i(M; R) \rightarrow H_i(DM; R)$ is a monomorphism.*
- (2) *If ∂M is connected, then the inclusion induced map $\pi_1(M; R) \rightarrow \pi_1(DM; R)$ is a monomorphism.*

Proof. We denote by $i: M = M \times \{1\} \rightarrow DM$ the inclusion and we denote by

$$\begin{aligned} r: DM = (M \times \{1\} \cup M \times \{2\}) / \sim &\rightarrow M \\ (P, i) &\mapsto P \end{aligned}$$

the map which “folds” the double DM back onto M . Clearly we have $r \circ i = \text{id}_M$. Both statements now follow from the observation that $r_* \circ i_* = \text{id}_*$, i.e. i_* needs to be a monomorphism. $\square$

39.2. Orientations of topological manifolds. We start out with the following elementary lemma.

Lemma 39.9. *Let R be a commutative ring.*

- (1) *For any $s \in (0, \infty)$ we have*

$$\tilde{H}_k(S_s^{n-1}(0); R) \cong \begin{cases} R, & \text{if } k = n - 1, \\ 0, & \text{otherwise.} \end{cases}$$

- (2) *Let V be an open subset of $\mathbb{R}^n$ with $0 \in V$. Let $r > 0$ such that $B_r^n(0) \subset V$. Then for any $s \in (0, r)$ the inclusion maps induce isomorphisms*

$$\tilde{H}_k(S_s^{n-1}(0); R) \xrightarrow{\cong} H_k(B_r^n(0), B_r^n(0) \setminus \{0\}) \xrightarrow{\cong} H_k(V, V \setminus \{0\}).$$

- (3) *Let M be an n -dimensional topological manifold and let $x \in M \setminus \partial M$. Then*

$$H_k(M, M \setminus \{x\}; R) \cong \tilde{H}_k(S^n; R) \cong \begin{cases} R, & \text{if } k = n, \\ 0, & \text{otherwise.} \end{cases}$$

Remark. A topological space X that has the property that for each $x \in X$ we have $H_k(X, X \setminus \{x\}) \cong \tilde{H}_k(S^{n-1})$ is often called a *homology manifold*. Such topological spaces are thus generalizations of manifold. It is a fun exercise to determine whether or not the line with two zeros is a homology manifold.

Proof. Let R be a commutative ring.

- (1) The first statement is proved the same way as

- (2) Let V be an open subset of $\mathbb{R}^n$ with $0 \in V$. Let $r > 0$ such that $B_r^n(0) \subset V$ and let $s \in (0, r)$. Then we have isomorphisms

$$\begin{array}{ccccc} \tilde{H}_k(S_s^{n-1}(0); R) & \xrightarrow{\cong} & \tilde{H}_k(B_r^n(0) \setminus \{0\}) & \xrightarrow{\cong} & H_{k+1}(B_r^n(0), B_r^n(0) \setminus \{0\}) & \xrightarrow{\cong} & H_{k+1}(V, V \setminus \{0\}). \\ \uparrow & & \uparrow & & \uparrow & & \\ \text{since } S_s^{n-1}(0) \text{ is a deformation} & & \text{by the long exact sequence} & & \text{Excision Theorem 26.15} & & \\ \text{retract of } B_r^n(0) \setminus \{0\} & & \text{of the pair } B_r^n(0), B_r^n(0) \setminus \{0\} & & & & \\ & & \text{and since } \tilde{H}_k(B_r^n(0)) = 0 & & & & \end{array}$$

- (3) Let M be an n -dimensional topological manifold and let $x \in M \setminus \partial M$. We pick a chart $\Phi: U \rightarrow V$ around x such that $\Phi(x) = 0$. Then

$$\begin{array}{ccc} \tilde{H}_k(M, M \setminus \{x\}; R) \cong H_k(U, U \setminus \{x\}; R) & \xrightarrow{\Phi_k} & H_k(V, V \setminus \{0\}; R) \cong \begin{cases} R, & \text{if } k = n, \\ 0, & \text{otherwise.} \end{cases} \\ \uparrow & & \uparrow \\ \text{Excision Theorem 26.15} & & \text{by (1) and (2)} \quad \square \end{array}$$

Definition. Let M be an n -dimensional topological manifold and let R be a commutative ring.

- (1) An R -section for M at $x \in M$ is an element $\mu_x \in H_n(M, M \setminus \{x\}; R) \cong R$.
- (2) An R -section for M is a choice for each $x \in M \setminus \partial M$ of an R -section μ_x such that the following “continuity” condition is satisfied: for each $x \in M$ there exists an open neighborhood U of x in $M \setminus \partial M$ and an element $\mu_U \in H_n(M, M \setminus U; R)$ such that for each x the image of μ_U under the map

$$H_n(M, M \setminus U; R) \rightarrow H_n(M, M \setminus \{x\}; R)$$

equals μ_x . If μ_x is a generator for each x , then we refer to an R -section as an R -orientation.

- (3) We denote by $\Gamma_R(M)$ the set of all R -sections of M . Note that $\Gamma_R(M)$ is an R -module in an obvious way.

In exercise sheet 1 we will prove the following lemma.

Lemma 39.10. *Let M be a connected n -dimensional topological manifold.*

- (1) *Let R be a commutative ring and let $\{\alpha_x\}_{x \in M}$ and $\{\beta_x\}_{x \in M}$ be two R -sections for M . If there exists an x with $\alpha_x = \beta_x$, then the two R -sections agree.*
- (2) *Let $\{\alpha_x\}_{x \in M}$ is a $\mathbb{Z}$ -section for M . Suppose there exists a $y \in M$, an $n \in \mathbb{Z}$ and a $\beta_y \in H_n(M, M \setminus \{y\})$ such that $\alpha_y = n \cdot \beta_y$, then there exists a $\mathbb{Z}$ -section $\{\beta_x\}_{x \in M}$ such that $\alpha_x = n \cdot \beta_x$ for all $x \in M$.*

Definition. Let M be an n -dimensional topological manifold and let R be a commutative ring.

- (1) An R -section for M at $x \in M$ that is a generator of $H_n(M, M \setminus \{x\}; R) \cong R$ is called R -orientation for M at $x \in M$.
- (2) An R -section for M that is a generator at each $x \in M$ is called an R -orientation.
- (3) The topological manifold M is called R -orientable if it admits an R -orientation.

- (4) A map $f: M \rightarrow N$ between two R -oriented n -dimensional topological manifolds is called *orientation-preserving* if for any $x \in M \setminus \partial M$ the image of the local orientation at x equals the local orientation at $f(x)$.

If $R = \mathbb{Z}$, then we just say *local orientation at a point*, *orientation* and *orientable*.

Example. It follows immediately from Lemma 39.10 that an orientable connected topological manifold admits precisely two orientations.

In exercise sheet 2 we will use the Topological Collar Neighborhood Theorem 39.4 to prove the following lemma.

Lemma 39.11. *Let M be a connected topological manifold and let R be a ring. Then*

$$M \text{ is } R\text{-orientable} \iff \text{the double } DM \text{ is } R\text{-orientable}.$$

Furthermore, if M is equipped with an R -orientation, then DM admits an R -orientation such that the inclusion map $M \rightarrow DM$ is orientation preserving.

Proposition 39.12.

- (1) *Let M be an n -dimensional manifold. Then*

$$\begin{array}{ccc} M \text{ is orientable in} & & M \text{ is } R\text{-orientable for} \\ \text{the sense of Analysis IV} & \iff & \text{every commutative ring } R. \end{array}$$

- (2) *Every topological manifold is $\mathbb{F}_2$ -orientable.*

In the proof of Proposition 39.12 we will need the following lemma.

Lemma 39.13. *Let $\Psi: V \rightarrow V'$ be a diffeomorphism between two open connected neighborhoods of $0 \in \mathbb{R}^n$. Let $r \in (0, \infty)$ such that $B_r^n(0) \subset V \cap V'$. Then⁶⁸³*

$$\deg \left(\underbrace{H_n(V, V \setminus \{0\})}_{=H_n(B_r^n(0), B_r^n(0) \setminus \{0\})=\mathbb{Z}} \xrightarrow{\Psi_*} \underbrace{H_n(V', V' \setminus \{0\})}_{=H_n(B_r^n(0), B_r^n(0) \setminus \{0\})=\mathbb{Z}} \right) = \begin{cases} 1, & \text{if } \Psi \text{ is orientation-preserving} \\ -1, & \text{if } \Psi \text{ is orientation-reversing.} \end{cases}$$

Sketch of proof. Let $A_0 = d\Psi_0$ be the differential of Ψ at 0. By Lemma 39.9 it suffices to prove the claim for some $r > 0$. Since Ψ is differentiable in 0 there exists an $r > 0$ such that there exists a homotopy from $\Psi: S_r^n \rightarrow \mathbb{R}^n \setminus \{0\}$ to the map $\Psi: A_0 = d\Psi_0: S_r^n \rightarrow \mathbb{R}^n \setminus \{0\}$. Furthermore, by the Gram-Schmidt process there exists an orthogonal matrix A_1 and a path $t \mapsto A_t \in \text{GL}(n, \mathbb{R})$ from A_0 to A_1 . It follows that⁶⁸⁴

$$\begin{aligned} \deg(\Psi: H_n(V, V \setminus \{0\}) \rightarrow H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})) &= \deg(A_0: H_n(V, V \setminus \{0\}) \rightarrow H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})) \\ &= \deg(A_1: H_n(V, V \setminus \{0\}) \rightarrow H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})) \\ &= \deg(A_1: H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \rightarrow H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})) \\ &= \begin{cases} +1, & \text{if } \Psi \text{ is orientation-preserving,} \\ -1, & \text{if } \Psi \text{ is orientation-reversing.} \end{cases} \end{aligned}$$

⁶⁸³Here we use the inclusion maps and Lemma 39.9 to make the identifications $H_n(V, V \setminus \{0\}) = H_n(B_r^n(0), B_r^n(0) \setminus \{0\}) = R$ and the same for v' .

⁶⁸⁴Here we use the isomorphisms from Lemma 39.9 to identify the various homology groups with $\mathbb{Z}$.

Proof. Let M be an n -dimensional, connected, orientable topological manifold and let R be a commutative ring. We pick an orientation $\{\mu_x\}_{x \in M}$ for M . For each $x \in M$ we denote by $\tilde{\mu}_x \in H_n(M|x; R)$ the image of $\mu_x \otimes 1$ under the map $H_n(M|x) \otimes R \rightarrow H_n(M|x; R)$. It is straightforward to verify that these define an R -orientation for M . $\square$

Lemma 39.16. *Let M be a topological manifold.*

- (1) *Let R be a commutative ring such that M is R -orientable. Then for any $x \in M$ the map*

$$\begin{aligned} \Phi: \Gamma_R(M) &\rightarrow H_n(M|x; R) \cong R \\ \{\alpha_y\}_{y \in M} &\mapsto \alpha_x \end{aligned}$$

is an isomorphism of R -modules.

- (2) *If M is non-orientable, then $\Gamma_{\mathbb{Z}}(M) = 0$.*

Proof. Let M be an n -dimensional connected topological manifold and let R be a commutative ring.

- (1) Suppose M is R -orientable. We pick an orientation $\{\mu_y\}_{y \in M}$ for M . By definition every element in $H_n(M|x; R)$ is of the form $r \cdot \mu_x$ for some $r \in R$. We consider the map

$$\begin{aligned} \Psi: H_n(M|x; R) &\rightarrow \Gamma_R(M) \\ r \cdot \mu_x &\mapsto (y \mapsto r \cdot \mu_y). \end{aligned}$$

Evidently we have $\Phi \circ \Psi = \text{id}$. By Lemma 39.10 we know that $\Psi \circ \Phi$ is also the identity.

- (2) Suppose that M admits a non-zero $\mathbb{Z}$ -section $\{\alpha_x\}_{x \in M}$. We need to show that M is orientable. We pick $y \in M$ with $\alpha_y \neq 0$. We pick a generator $\beta_y \in H_n(M, M \setminus \{y\})$. Then $\alpha_y = n \cdot \beta_y$ for some $n \in \mathbb{Z}$. By Lemma 39.10 there exists a $\mathbb{Z}$ -section $\{\beta_x\}_{x \in M}$ such that $\alpha_x = n \cdot \beta_x$ for all $x \in M$. It follows now easily that $\{\beta_x\}_{x \in M}$ is an orientation for M . $\square$

39.3. The fundamental class of closed topological manifolds. The following is the key theorem regarding homology groups of closed topological manifolds.

Theorem 39.17. *Let M be a connected closed n -dimensional topological manifold and let R be a commutative ring. Then the following hold:*

- (1) *If M is R -orientable, then for all $x \in M$ the map*

$$H_n(M; R) \rightarrow H_n(M, M \setminus \{x\}; R)$$

is an isomorphism.

- (2) *If M is not orientable, then $H_n(M) = 0$.*
 (3) *We have $H_i(M; R) = 0$ for $i > n$.*

Remark.

- (1) Note that the second statement of Theorem 39.17 only deals with integer coefficients. In [Ha, Theorem 3.26] it is proved that if M is not R -orientable, then for any $x \in M$ the map $H_n(M; R) \rightarrow H_n(M|x; R) \cong R$ is injective with image $\{r \in R \mid 2r = 0\}$. We will not make use of this statement.
- (2) The last statement of Theorem 39.17 is a generalization of Proposition 29.10 (2) from the case of manifolds to closed topological manifolds. It also gives a proof for Proposition 29.10 (2) that does not rely on Theorem 22.2.

The following proposition follows easily from Theorem 39.17. The proof is an exercise in exercise sheet 1.

Proposition 39.18. *Let M be a connected closed n -dimensional topological manifold and let R be a commutative ring.*

- (1) *If we are given an R -orientation $\{\mu_x\}_{x \in M}$ for M , then there exists a unique element $[M] \in H_n(M; R)$ with the property that for all $x \in M$ the image of $[M]$ under the map $H_n(M; R) \xrightarrow{\cong} H_n(M, M \setminus \{x\}; R)$ equals μ_x .*
- (2) *If $[M] \in H_n(M; R)$ has the property that for all $x \in M$ the image of $[M]$ under the map $\rho_x: H_n(M; R) \xrightarrow{\cong} H_n(M, M \setminus \{x\}; R)$ is a generator, then $\{\rho_x\}_{x \in X}$ defines an R -orientation for M .*

Definition. Let M be a closed n -dimensional topological manifold and let R be a commutative ring. An R -fundamental class of M is an element $[M] \in H_n(M; R)$ with the property that for all $x \in M$ the image of $[M]$ under the map $H_n(M; R) \xrightarrow{\cong} H_n(M, M \setminus \{x\}; R)$ is a generator of $H_n(M, M \setminus \{x\}; R)$. For $R = \mathbb{Z}$ we just say *fundamental class*.

Remark.

- (1) By Proposition 39.18 an R -orientation determines and is in turn uniquely determined by an R -fundamental class.
- (2) If M is a connected closed n -dimensional manifold, then we had seen in Proposition 39.12 that an orientation of M also determines a $\mathbb{Z}$ -orientation. It follows easily from the definitions that the fundamental class defined on page 563 and in Proposition 39.18 agree.
- (3) If $[M]$ is a fundamental class for $R = \mathbb{Z}$, then for any other ring R the image of $[M]$ under the map $H_n(M; \mathbb{Z}) \rightarrow H_n(M; R)$ is an R -fundamental class.

Before we provide the proof of Theorem 39.24 we introduce the following helpful notation.

Notation. Let M be a topological manifold, let R be a commutative ring R and let $X \subset M$ be a subset.

(1) We write $H_n(M|X; R) := H_n(M, M \setminus X; R)$.

(2) Given a homology class $\alpha \in H_n(M; R)$ we denote by α_X the image of α in $H_n(M|X; R)$.

Given a point $x \in M$ we use the obvious abbreviations $H_n(M|x; R) := H_n(M|\{x\}; R)$ and $\alpha_x := \alpha_{\{x\}}$.

The key to the proof of Theorem 39.24 is the following lemma.

Lemma 39.19. *Let M be an n -dimensional topological manifold and let $A \subset M$ be a compact subset. Let R be a commutative ring. Then the following hold:*

- (1) *If $\{\alpha_x\}_{x \in M}$ is an R -section for M , then there exists a unique class $\alpha_A \in H_n(M|A; R)$ such that for each $x \in A$ the image of μ_A in $H_n(M|x; R)$ agrees with α_x .*
- (2) *For $i > n$ we have $H_i(M|A; R) = 0$.*

Before we provide the proof of Lemma 39.19 we show that it implies fairly easily Theorem 39.19.

Proof of Theorem 39.19 using Lemma 39.19. Let M be a connected closed n -dimensional topological manifold and let R be a commutative ring. We apply Lemma 39.19 to the special case $A = M$. Then statement (c) of Theorem 39.19 follows immediately from statement (b) of Lemma 39.19.

Recall that we denote by $\Gamma_R(M)$ the set of all R -sections of M . The map

$$\begin{array}{ccc} H_n(M; R) & \mapsto & \Gamma_R(M) \\ \alpha & \mapsto & (x \mapsto \alpha_x) \\ & & \uparrow \\ & & \text{image of } \alpha \text{ under the map } H_n(M; R) \rightarrow H_n(M|x; R) \end{array}$$

is an R -module homomorphism. It follows from Lemma 39.19 that this map is an isomorphism. The desired statements (1) and (2) now follow from Lemma 39.16. $\square$

Now we turn to the proof of Lemma 39.19.

Proof of Lemma 39.19. The coefficient ring R does not play a role in the proof of Lemma 39.19, therefore we suppress it from the notation.

The proof of Lemma 39.19 now proceeds in four steps.

- (1) We first claim that if the statement is true for compact sets A and B and for the intersection $A \cap B$, then it is also true for $A \cup B$. We consider the inverted Mayer–Vietoris Sequence from Theorem 28.6

$$\begin{array}{l} = 0, \text{ by hypothesis on } A \cap B \\ \overbrace{H_n(M|A \cap B)} \rightarrow H_n(M|A \cup B) \xrightarrow{\Phi} H_n(M|A) \oplus H_n(M|B) \xrightarrow{\Psi} H_n(M|A \cap B) \rightarrow \dots \\ \alpha \mapsto (\alpha, -\alpha) \qquad (\alpha, \beta) \mapsto \alpha + \beta \end{array}$$

Now we can prove the desired statements.

- (b) For $i > n$ the group $H_i(M|A \cup B)$ is sandwiched, by our hypothesis on A, B and $A \cap B$, between groups that are zero, thus we see that $H_i(M|A \cup B) = 0$ for $i > n$.

- (a) Now suppose we are given a section $\{\alpha_x\}_{x \in M}$. We denote by α_A, α_B and $\alpha_{A \cap B}$ the corresponding relative homology classes given by our hypothesis. The images of α_A and α_B in $H_n(M|A \cap B)$ have obviously the defining property of $\alpha_{A \cap B}$, by the uniqueness of $\alpha_{A \cap B}$ these images thus equal $\alpha_{A \cap B}$. By the exactness of the above sequence we see that $(\alpha_A, -\alpha_B) = \Phi(\alpha_{A \cup B})$ for some $\alpha_{A \cup B} \in H_n(M|A \cup B)$. If $x \in A$, then the image of $\alpha_{A \cup B}$ in $H_n(M|x)$ is the image of α_A in $H_n(M|x)$, hence it is α_x . The same argument applies to $x \in B$.

It remains to prove the uniqueness of $\alpha_{A \cup B}$. If $\beta \in H_n(M|A \cup B)$ has the same property, then $\alpha_{A \cup B} - \beta$ gets sent to 0 for all $H_n(M|x)$ with $x \in A \cup B$. But by the uniqueness of α_A and α_B this difference gets sent to zero in $H_n(M|A)$ and also in $H_n(M|B)$. By the exactness of the sequence we see that $\alpha_{A \cup B} - \beta$ is already zero.

- (2) Next we reduce the statement to the case $\mathbb{R}^n$. So suppose that the desired statement holds for every compact subset B of $\mathbb{R}^n$. Now let M be an n -dimensional topological manifold. It follows from excision that the statement also holds for every compact subset B of M which is contained in the domain of a chart. Now let A be a compact subset of M . It can be written as the union of finitely many compact sets $A_1, \dots, A_m$, each of which is contained in the domain of a chart.⁶⁸⁵ The statement now follows by induction on m together with (1).
- (3) Now let A be a convex compact subset of $\mathbb{R}^n$. For $x \in A$ we consider the following commutative diagram of long exact sequences

$$\begin{array}{ccccccc}
 \dots & \longrightarrow & H_i(\mathbb{R}^n \setminus A) & \longrightarrow & H_i(\mathbb{R}^n) & \longrightarrow & H_i(\mathbb{R}^n, \mathbb{R}^n \setminus A) \longrightarrow H_{i-1}(\mathbb{R}^n \setminus A) \longrightarrow \dots \\
 & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\
 \dots & \longrightarrow & H_i(\mathbb{R}^n \setminus \{x\}) & \longrightarrow & H_i(\mathbb{R}^n) & \longrightarrow & H_i(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\}) \longrightarrow H_{i-1}(\mathbb{R}^n \setminus \{x\}) \longrightarrow \dots
 \end{array}$$

The vertical map on the left and right are isomorphisms since A is convex, see the example on page 420. The second and the fifth vertical map are the identity. It follows from the 5-Lemma that the third vertical map is also an isomorphism.

- (4) Let A be a union of convex compact sets $A_1, \dots, A_n$ of $\mathbb{R}^n$. We know from (3) that the statement holds for the A_i . Since intersections of convex sets are again convex the statement also holds for arbitrary non-empty intersections of the A_i . It follows from a straightforward induction argument, using (1), that the desired statement also holds for A .
- (5) Now let $A \subset \mathbb{R}^n$ be an arbitrary compact set. We pick a closed ball B that contains A . We denote by $\alpha_B \in H_n(\mathbb{R}^n|B)$ the class constructed in (3). Then the image α_A of the class α_B under the map $H_n(\mathbb{R}^n|B) \rightarrow H_n(\mathbb{R}^n|A)$ has the desired property for all $x \in A$. To complete the proof of (1) we still need to prove the uniqueness of α_A .

We need to show that if $\beta \in H_n(\mathbb{R}^n|A)$ is a class such that $\beta_x = 0$ for all $x \in A$, then β is already zero. So let $\beta \in H_n(\mathbb{R}^n|A)$. We pick a relative cycle $z \in C_n(\mathbb{R}^n, \mathbb{R}^n \setminus A)$ that represents β . We denote by $C \subset \mathbb{R}^n \setminus A$ the union of the images of the singular simplices appearing in ∂C . Since C and A are compact and since $A \subset \mathbb{R}^n \setminus C$ we see that the distance from A to C is positive, i.e.

$$\rho := \inf\{d(a, c) \mid a \in A \text{ and } c \in C\}$$

is positive. Since A is compact we can cover A by finitely many closed balls of radius $< \rho$. We denote by K the union of these balls. We refer to Figure 365 for an illustration. By the choice of ρ we have $K \subset C$. It follows from our hypothesis on β and from (4) that $[z] = 0 \in H_n(\mathbb{R}^n|K)$. But this shows that the image of $[z]$ under the map $H_n(\mathbb{R}^n|K) \rightarrow H_n(\mathbb{R}^n|A)$ is also zero. But this image is precisely β .

Finally we show that $H_i(\mathbb{R}^n|A) = 0$ for $i > n$. So let $\beta \in H_i(\mathbb{R}^n|A)$. We pick a relative cycle $z \in C_n(\mathbb{R}^n, \mathbb{R}^n \setminus A)$ that represents β . We define C , ρ and K as

⁶⁸⁵Why is that possible?

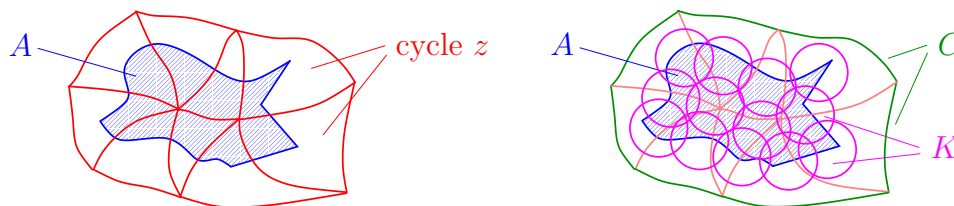


FIGURE 365. Illustration for the proof of Lemma 39.19.

above. By (4) we have $[z] = 0 \in H_i(\mathbb{R}^n|K)$. But as before this implies that also $\beta = 0 \in H_i(\mathbb{R}^n|A)$. $\square$

In the following proposition we consider homology groups of non-compact topological manifolds.

Proposition 39.20. *Let M be a connected n -dimensional topological manifold and let R be a commutative ring. If M is non-compact, then $H_i(M; R) = 0$ for $i \geq n$.*

Proof. Let M be a connected n -dimensional topological manifold, let R be a commutative ring and let $i \geq n$. Let $\alpha \in H_i(M; R)$. We need to show that $\alpha = 0$. We represent α by a cycle $z = \sum_{i=1}^m \sigma_i \otimes r_i$. The union of the images of $\sigma_1, \dots, \sigma_m$ is a compact subset K of M . There exists an open subset $U \subset M$ that contains K and such that the closure $\bar{U}$ is compact.⁶⁸⁶ We write $V = M \setminus \bar{U}$. We consider the long exact sequence of the triple $(M, U \cup V, V)$, see Corollary 26.13, it fits in to the following commutative diagram

$$\begin{array}{ccccc} H_{i+1}(M, U \cup V; R) & \longrightarrow & H_i(U \cup V, V; R) & \longrightarrow & H_i(M, V; R) \\ & & \cong \uparrow & & \uparrow \\ & & H_i(U; R) & \longrightarrow & H_i(M; R). \end{array}$$

Here the left vertical map is an isomorphism since $U \cup V$ is the disjoint union of the open sets U and V .

We first consider the case $i > n$. Note that $U \cup V$ and V are complements of compact sets in M . It follows from Lemma 39.19 that the modules $H_{i+1}(M, U \cup V; R)$ and $H_i(M, V; R)$ are zero. It follows that $H_i(U; R) = 0$, but z represents an element in $H_i(U; R)$. It follows that $[z] = 0$ in $H_i(M; R)$.

Now we consider the case $i = n$. In this case $[z] \in H_n(M; R)$ defines an R -section of M via the map $x \mapsto [z]_x$. Note that $[z]_x = 0$ for any point outside of the compact set K . Since M is non-compact such points exist. But since M is connected it follows from Lemma 39.10 that in fact $[z]_x \in H_n(M|x; R)$ is zero for all $x \in M$. It follows from Lemma 39.19 that $[z] = 0$ in $H_n(M, V; R)$. By Lemma 39.19 the top left term of the above diagram is zero. Therefore it follows from the above commutative diagram that if $[z] = 0$ in $H_n(M, V; R)$, then we also have $[z] = 0$ in $H_n(U; R)$, which in turn implies that $[z] = 0$ in $H_n(M; R)$. $\square$

⁶⁸⁶Why does such a U exist?

The following proposition now generalizes Proposition 33.5 from manifolds to topological manifolds.

Proposition 39.21. *Let M be a connected n -dimensional topological manifold and let R be a commutative ring. Then the following holds*

$$H_n(M; R) \cong \begin{cases} R, & \text{if } M \text{ is } R\text{-orientable and closed,} \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Let M be a connected n -dimensional topological manifold and let R be a commutative ring. Most of the desired statements follow immediately from Theorem 39.17.

It suffices to show that if M not closed, then $H_n(M; R) = 0$. So assume that M is not closed. This means that M is non-compact or M has non-empty boundary. In the former case it follows from Proposition 39.20 that $H_n(M; R) = 0$. In the latter case we had seen in Corollary 39.5 that M is homotopy equivalent to a non-compact connected n -dimensional topological manifold. It follows from Corollary 25.5 and the above that $H_n(M; R) = 0$. $\square$

Remark. Let $X = S^1 \cup \{*\}$ the topological space that is given by the circle “with two 1’s”, where the topology is defined in the same way as we defined the topology on the “line with two 0’s” on page 14. It is straightforward to see that X is locally homeomorphic to an open subset of $\mathbb{R}$. It is not a manifold since it is not Hausdorff. In exercise sheet 10 of Algebraic Topology II we had seen that $H_1(S^1) \cong \mathbb{Z}^2$. This shows that in the proof of Proposition 39.21 we must have used at some point that manifolds are Hausdorff. Where did we make use of this fact?

The first part of the following proposition generalizes the result of Proposition 33.13 from manifolds to topological manifolds.

Proposition 39.22. *Let M be a closed, connected n -dimensional topological manifold.*

- (1) *If M is orientable, then $H_{n-1}(M) \cong \mathbb{Z}^k$ for some $k \in \mathbb{N}_0$.*
- (2) *If M is non-orientable, then $H_{n-1}(M) \cong \mathbb{Z}_2 \oplus \mathbb{Z}^k$ for some $k \in \mathbb{N}_0$.*

Proof. Let M be a closed, connected n -dimensional topological manifold. By Proposition 39.2 we know that $H_{n-1}(M)$ is a finitely generated abelian group.

We first consider the case that M is orientable. By the classification of finitely generated abelian groups, see Theorem 8.3, it suffices to show for every prime p that $H_{n-1}(M)$ has no p -torsion. By Lemma 36.12 this means that it suffices to show that $\text{Tor}(H_{n-1}(M; \mathbb{Z}), \mathbb{F}_p) = 0$ for all primes p .

So let p be a prime. It follows from Lemma 39.15 that M is also $\mathbb{F}_p$ -orientable. We now have

$$\begin{array}{ccccccc} \mathbb{F}_p & \cong & H_n(M; \mathbb{F}_p) & \cong & H_n(M; \mathbb{Z}) \otimes \mathbb{F}_p & \oplus & \text{Tor}(H_{n-1}(M; \mathbb{Z}), \mathbb{F}_p) \\ & & \uparrow & & \uparrow & & \uparrow \end{array}$$

Proposition 39.18 Universal Coefficient Theorem 36.16 by Proposition 39.21 we have $H_n(M; \mathbb{Z}) \cong \mathbb{Z}$

It follows from Lemma 8.6 that $\text{Tor}(H_{n-1}(M; \mathbb{Z}), \mathbb{F}_p) = 0$.

Now we turn to the case that M is non-orientable. By Proposition 39.14 there exists a connected 2-fold covering $q: \widetilde{M} \rightarrow M$ such that $\widetilde{M}$ is orientable. We consider the maps

$$\begin{array}{ccccc} H_{n-1}(M; \mathbb{Z}) & \xrightarrow{q^*} & H_{n-1}(\widetilde{M}; \mathbb{Z}) & \xrightarrow{q_*} & H_{n-1}(M; \mathbb{Z}) \\ & & \searrow \scriptstyle \cdot 2 & \nearrow & \\ & & & & \end{array}$$

Here q_* denotes of course the map induced by the projection q and q^* denotes the transfer map that we had introduced on page 638. By Proposition 38.5 the composition of q^* and q_* is given by multiplication by 2. Put differently, the above diagram commutes.

Since $\widetilde{M}$ is orientable we obtain from the case (1) that we had just proved that $H_{n-1}(\widetilde{M}; \mathbb{Z})$ is torsion-free. In particular all torsion-elements of $H_{n-1}(M; \mathbb{Z})$ lie in the kernel of q^* , in particular all torsion-elements of $H_{n-1}(M; \mathbb{Z})$ lie in the kernel of multiplication by 2. This is only possible if $H_{n-1}(M; \mathbb{Z}) \cong (\mathbb{Z}_2)^r \oplus \mathbb{Z}^k$ for some $r \in \mathbb{N}_0$ and some $k \in \mathbb{N}_0$.

It remains to show that $r = 1$. In light of Lemma 36.12 it suffices to show that $\text{Tor}(H_{n-1}(M; \mathbb{Z}), \mathbb{F}_2) \cong \mathbb{F}_2$. The argument is very similar to the argument in the orientable case. More precisely, we have

$$\begin{array}{ccccc} \mathbb{F}_2 \cong H_n(M; \mathbb{F}_2) & \cong & H_n(M; \mathbb{Z}) \otimes \mathbb{F}_2 \oplus \text{Tor}(H_{n-1}(M; \mathbb{Z}), \mathbb{F}_2) & \cong & 0 \oplus \text{Tor}(H_{n-1}(M; \mathbb{Z}), \mathbb{F}_2). \\ \uparrow & & \uparrow & & \uparrow \\ \text{by Proposition 39.12} & & \text{Universal Coefficient Theorem 36.16} & & \text{by Proposition 39.21} \\ \text{and Proposition 39.18} & & & & \text{we have } H_n(M; \mathbb{Z}) \cong \mathbb{Z} \end{array} \quad \square$$

We conclude this section with the following corollary.

Corollary 39.23. *Let M be a closed, connected n -dimensional topological manifold and let p be a prime. If M is non-orientable, then the following holds*

$$H_n(M; \mathbb{Z}_p) \cong \begin{cases} \mathbb{Z}_2, & \text{if } p = 2, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. The corollary is an immediate consequence of the Universal Coefficient Theorem 36.16, Propositions 39.21 and 39.22 and of Lemma 36.12. $\square$

Remark. It follows from Lemma 39.15, Proposition 39.21 and Corollary 39.23 that for every closed, connected n -dimensional topological manifold M we have $H_n(M; \mathbb{F}_2) \cong \mathbb{F}_2$. We denote the unique non-trivial element in $H_n(M; \mathbb{F}_2)$ by $[M]_{\mathbb{F}_2}$.

Given a topological space X and a homology class $z \in H_n(X; \mathbb{F}_2)$ Thom [Tm] (see also [Ru, Theorem IV.7.33]) showed that there exists an n -dimensional closed, connected manifold M and a map $f: M \rightarrow X$ such that $f_*([M]_{\mathbb{F}_2}) = z$. This can be viewed as the $\mathbb{F}_2$ -analogue of Theorem 33.14.

39.4. The fundamental class of topological manifolds with boundary. In this section we want to generalize Proposition 33.7 from manifolds to topological manifolds.

Theorem 39.24. *Let M be a connected, compact n -dimensional topological manifold and let R be a commutative ring. Then the following hold:*

- (1) If M is R -orientable, then $H_n(M, \partial M; R) \cong R$. Furthermore if we are given an orientation $\{\mu_x\}_{x \in M \setminus \partial M}$, then there exists a unique element $[M] \in H_n(M, \partial M; R)$ such that for each $x \in M \setminus \partial M$ we have $[M]_x = \mu_x$.
- (2) If M is not orientable, then $H_n(M, \partial M) = 0$.

Proof. Let M be a connected, compact n -dimensional topological manifold and let R be a commutative ring. If M has no boundary, then we are done by Theorem 39.19.

So suppose that $\partial M \neq \emptyset$. We denote again by DM the double of M . Here we think of $DM = M \cup M'$ where M' is another copy of M . We write $\overset{\circ}{M}' := M' \setminus \partial M'$. We consider the maps

$$\begin{array}{ccccccc}
 \underbrace{H_n(\overset{\circ}{M}'; R)}_{= 0 \text{ by Prop. 39.21}} & \rightarrow & H_n(DM; R) & \rightarrow & H_n(DM, \overset{\circ}{M}'; R) & \rightarrow & H_{n-1}(\overset{\circ}{M}'; R) \xrightarrow[\text{by 39.8}]{\text{injective}} H_{n-1}(DM; R) \\
 & & \searrow \Phi & & \uparrow \cong & & \text{Excision Theorem 39.6} \\
 & & & & H_n(M, \partial M; R), & &
 \end{array}$$

where on the top we have the exact sequence of the pair $(DM, \overset{\circ}{M}')$. It follows from the information provided in the diagram that the map $\Phi: H_n(DM; R) \rightarrow H_n(M, \partial M; R)$ is an isomorphism. Now we turn to the proof of the two statements.

- (2) If M is not orientable, then it follows from Lemma 39.11 that the double DM is not orientable either. It follows from Theorem 39.17 that $H_n(DM) = 0$, hence we see that $H_n(M, \partial M) = 0$.
- (1) Now suppose that M is R -oriented. It follows again from Lemma 39.11 that the double DM admits an R -orientation such that the inclusion map $M \rightarrow DM$ is orientation-preserving. We denote by $[DM]$ the R -fundamental class of M given by Proposition 39.18. We write $[M] := \Phi([DM])$. We claim that $[M]$ has the desired property. So let $x \in M \setminus \partial M$. We denote the generator of $H_n(M, M \setminus \{x\}; R)$ corresponding to the orientation of M by μ_x and we denote the generators of $H_n(DM, DM \setminus \{x\}; R)$ corresponding to the orientation of DM by σ_x . We consider the following diagram

$$\begin{array}{ccccc}
 H_n(DM; R) & \xrightarrow{[DM] \mapsto \sigma_x} & H_n(DM, DM \setminus \{x\}; R) & & \\
 \downarrow [DM] \mapsto [M] \cong & \searrow & \uparrow \mu_x \mapsto \sigma_x \text{ excision isom.} & & \\
 & H_n(DM, \overset{\circ}{M}'; R) & & & \\
 \uparrow \cong \text{ excision} & & & & \\
 H_n(M, \partial M; R) & \xrightarrow{\quad} & H_n(M, M \setminus \{x\}; R). & &
 \end{array}$$

Here all maps are the obvious inclusion induced maps. Note that the diagram commutes. It now follows immediately that the bottom horizontal map is an isomorphism and that it sends $[M]$ to μ_x . The uniqueness of $[M]$ follows immediately from the fact that the bottom horizontal map is an isomorphism. $\square$

We now generalize the definition of R -fundamental class from the closed case to the case of topological manifold with boundary.

Definition. Let M be a compact n -dimensional topological manifold and let R be a commutative ring. An R -fundamental class of M is an element $[M] \in H_n(M, \partial M; R)$ with the property that for all $x \in M \setminus \partial M$ the image of $[M]$ under the map $H_n(M, \partial M; R) \xrightarrow{\cong} H_n(M, M \setminus \{x\}; R)$ is a generator of $H_n(M, M \setminus \{x\}; R)$. For $R = \mathbb{Z}$ we just say *fundamental class*.

Lemma 39.25. *Let M be a compact m -dimensional topological manifold, let $W \subset M$ be a compact m -dimensional topological submanifold and let R be a commutative ring. We write $\overset{\circ}{W} = W \setminus \partial W$. Then the following two statements hold:*

- (1) *The inclusion induced map $j_*: H_m(W, \partial W; R) \rightarrow H_m(M, M \setminus \overset{\circ}{W}; R)$ is an isomorphism.*
- (2) *If $[M]$ is an R -fundamental class, then the image of $[M]$ under the map*

$$H_m(M, \partial M; R) \rightarrow H_m(M, M \setminus \overset{\circ}{W}) \xrightarrow{j_*^{-1}} H_m(W, \partial W; R)$$

is an R -fundamental class for W .

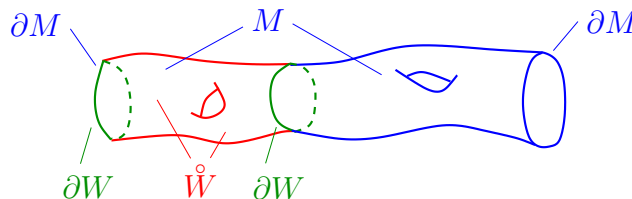


FIGURE 366. Illustration for the proof of Lemma 39.25.

The lemma says that the orientation of a manifold M defines an orientation on every compact submanifold W of codimension zero. We refer to the orientation on W as the *induced orientation*.

Proof. The first statement is an immediate consequence of the Excision Theorem 39.6 for Topological Manifolds. The second statement follows easily from the definitions and the usual Excision Theorem 26.15. We leave the details to the reader. $\square$

The following theorem is a generalization of Proposition 33.6 from manifolds to topological manifolds.

Theorem 39.26. *Let M be a compact n -dimensional topological manifold and let R be a commutative ring. We consider the boundary map*

$$\partial: H_n(M, \partial M; R) \rightarrow H_{n-1}(\partial M; R)$$

of the long exact sequence of the pair. If $[M]$ is an R -fundamental class for M , then $\partial([M]) \in H_{n-1}(\partial M; R)$ is an R -fundamental class for ∂M .

Remark. Theorem 39.26 says in particular that if M is an R -oriented compact n -dimensional topological manifold, then its boundary is an R -oriented manifold and the orientation of M induces canonically an R -orientation on the boundary.

Proof. Let M be a compact n -dimensional topological manifold. In the proof we drop the commutative ring R from the notation since it plays no role in the argument. We apply the Topological Collar Neighborhood Theorem 39.4 to obtain an open neighborhood $\partial M \times [0, 2)$ of ∂M in M .

Now let $x \in \partial M$. We pick a closed n -dimensional ball $B \subset M$ such that x lies in the interior $\mathring{B}$ of B . We write $E = B \times [0, 1] \subset \partial M \times [0, 1] \subset M$. We also pick a point $y \in \mathring{E} = \mathring{B} \times (0, 1)$. We consider the following diagram of maps:

$$\begin{array}{ccccc}
 & & H_n(M, M \setminus \{y\}) & & \\
 & \nearrow \cong & & \nwarrow \cong & \\
 H_n(M, \partial M) & \xrightarrow{\quad} & H_n(M, M \setminus \mathring{E}) & \xleftarrow{\cong} & H_n(E, \partial E) \\
 \partial \downarrow & & \partial \downarrow & & \partial \downarrow \cong \\
 H_{n-1}(\partial M) & \xrightarrow{\quad} & H_{n-1}(M \setminus \mathring{E}) & \xleftarrow{\quad} & H_{n-1}(\partial E) \\
 \downarrow & & \downarrow & & \downarrow \cong \\
 H_{n-1}(\partial M, \partial M \setminus \{x\}) & \xrightarrow{\cong} & H_{n-1}(M \setminus \mathring{E}, (M \setminus \mathring{E}) \setminus \{x\}) & \xleftarrow{\cong} & H_{n-1}(\partial E, \partial E \setminus \{x\}).
 \end{array}$$

We make the following comments regarding the diagram:

- (1) All maps are either inclusion induced maps or the boundary maps in the long exact sequences of a pair.
- (2) All rectangles and the top triangle commute.
- (3) The horizontal maps that are marked as isomorphisms are isomorphisms by the Excision Theorem 39.6 for Topological Manifolds. The top left diagonal map is an isomorphism since M is orientable. The top right diagonal map is an isomorphism since $\mathring{E}$ is homotopy equivalent to the point $\{y\}$.
- (4) The two vertical maps on the right are isomorphism since E is homeomorphic to a closed n -ball.
- (5) It follows from the top part of the triangle that the image of $[M]$ in $H_n(E, \partial E)$ is a generator.
- (6) The right two vertical maps are isomorphisms, hence the image of $[M]$ in $H_{n-1}(\partial E, \partial E \setminus \{x\})$ is a generator.
- (7) Since the bottom two horizontal maps are isomorphisms and since the diagram commutes we now see that the image of $\partial[M] \in H_{n-1}(M)$ in $H_{n-1}(\partial M, \partial M \setminus \{x\})$ is a generator.

Summarizing we have thus shown, as desired, that $\partial[M] \in H_{n-1}(M)$ is a fundamental class for ∂M . $\square$

We summarize most of the results we have obtained in the previous in the following proposition.

Proposition 39.27. *Let M be a connected n -dimensional topological manifold and let $R = \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ or $R = \mathbb{C}$. Then the following statements hold:*

- (1) *For $i > n$ we have $H_i(M; R) = 0$ and $H_i(M, \partial M; R) = 0$.*
- (2)

$$H_n(M, \partial M; R) \cong \begin{cases} R, & \text{if } M \text{ is orientable and compact,} \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore

$$H_n(M; R) \cong \begin{cases} R, & \text{if } M \text{ is orientable and closed,} \\ 0, & \text{otherwise.} \end{cases}$$

- (3) *If M is oriented, then there exists a canonical generator $[M]$ of $H_n(M, \partial; R)$, called the fundamental class, that has the following two properties:*
 - (a) *For any $x \in M \setminus \partial M$ the image of $[M]$ in $H_n(M, M \setminus \{x\}; R)$ is a generator.*
 - (b) *The class $\partial[M] \in H_{n-1}(\partial M; R)$ is a fundamental class of ∂M .*

Proof. The proposition basically follows from combining Theorem 39.24, Theorem 39.26 and Proposition 39.20.

The only bit missing is to show that if M is non-orientable, then $H_n(M; R) = 0$ and $H_n(M, \partial; R) = 0$. We had seen that $H_n(M; \mathbb{Z}) = 0$ and $H_n(M, \partial; \mathbb{Z}) = 0$. But since R is torsion-free it follows from the Universal Coefficient Theorem 36.16 together with Lemma 36.12 that we also have $H_n(M; R) = 0$ and $H_n(M, \partial; R) = 0$. $\square$

39.5. Chiral and amphichiral manifolds.

Definition. Let M be a compact, oriented topological manifold.

- (1) We say that a self-homeomorphism $f: M \rightarrow M$ is *orientation-preserving* if $f_*([M]) = [M]$, otherwise we say that f is *orientation-reversing*.
- (2) We say that M is *amphichiral* if M admits an orientation-reversing self-homeomorphism, otherwise we say that M is *chiral*.

Examples.

- (1) It follows from Lemma 26.30 that the map

$$\begin{aligned} \rho: S^n &\rightarrow S^n \\ (x_0, \dots, x_{n-1}, x_n) &\mapsto (x_0, \dots, x_{n-1}, -x_n), \end{aligned}$$

i.e. the map that is given by reflection in the $x_n = 0$ hyperplane, is orientation-reversing. This shows that the sphere S^n is amphichiral.

- (2) Let M be a compact, oriented topological manifold. It follows from (1) and the naturality of the Künneth-Formula 37.5 for Topological Spaces that the map

$$\begin{aligned} S^n \times M &\rightarrow S^n \times M \\ (x, P) &\mapsto (\rho(x), P), \end{aligned}$$

is orientation-reversing. This shows that the product of the sphere S^n with any compact, oriented topological manifold is amphichiral.

- (3) It follows from (2) that the torus $S^1 \times S^1$ is amphichiral. In fact with some more effort one can show that for any $g \in \mathbb{N}$ the surface of genus g is amphichiral. Pictorially an orientation-preserving self-homeomorphism is given by “reflection in the yz -plane”, see Figure 39.5.⁶⁸⁷

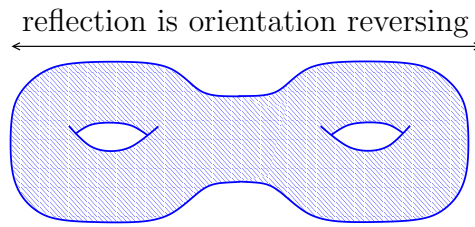


FIGURE 367.

This raises the following question.

Question 39.28.

- (1) *Are there compact, oriented topological manifolds that are chiral?*
- (2) *For example, are lens spaces chiral?*

⁶⁸⁷What is an example of an orientation-reversing self-homeomorphism in the “octagon-picture” of a surface of genus 2?

40. THE COHOMOLOGY GROUPS OF A TOPOLOGICAL SPACE

40.1. Dual cochain complexes. Throughout this chapter we denote by G an abelian group. Now let $f: A \rightarrow B$ be a group homomorphism. Then induces f a map

$$\begin{aligned} f^*: \operatorname{Hom}(B, G) &\rightarrow \operatorname{Hom}(A, G) \\ (\varphi: B \rightarrow G) &\mapsto \left(\begin{array}{ccc} A & \rightarrow & G \\ a & \mapsto & \varphi(f(a)). \end{array} \right) \end{aligned}$$

For example, if A is a subgroup of B and if $i: A \rightarrow B$ is the inclusion map, then for each homomorphism $\varphi: B \rightarrow G$ the map $i^*\varphi: A \rightarrow G$ is just the restriction of φ to $A \subset B$.

It follows immediately from the definitions that

$$(\operatorname{id}_A)^* = \operatorname{id}_{\operatorname{Hom}(A, G)}, \quad \text{for all abelian groups } A$$

and

$$(g \circ f)^* = f^* \circ g^* \quad \text{for all homomorphisms } f: A \rightarrow B \text{ and } g: B \rightarrow C.$$

Put differently we see that the maps

$$\begin{aligned} A &\mapsto \operatorname{Hom}(A, G) \\ (f: A \rightarrow B) &\mapsto (f^*: \operatorname{Hom}(B, G) \rightarrow \operatorname{Hom}(A, G)) \end{aligned}$$

define a *contravariant* functor from the category of abelian groups to the category of abelian groups.

Now suppose we are given a chain complex

$$(C_*, \partial_*): \quad \rightarrow C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \dots C_1 \xrightarrow{\partial_1} C_0 \rightarrow 0.$$

We obtain the following sequence of maps

$$\rightarrow \operatorname{Hom}(C_n, G) \xleftarrow{\partial_n^*} \operatorname{Hom}(C_{n-1}, G) \xleftarrow{\partial_{n-1}^*} \dots \operatorname{Hom}(C_1, G) \xleftarrow{\partial_1^*} \operatorname{Hom}(C_0, G) \rightarrow 0.$$

We call this sequence of maps the cochain complex dual to (C_*, ∂_*) .⁶⁸⁸ Now we define the n -th cohomology group of the chain complex (C_*, ∂_*) with G -coefficients as follows:

$$H^n(C; G) := \ker\{\partial_{n+1}^*: \operatorname{Hom}(C_n, G) \rightarrow \operatorname{Hom}(C_{n+1}, G)\} / \operatorname{im}\{\partial_n^*: \operatorname{Hom}(C_{n-1}, G) \rightarrow \operatorname{Hom}(C_n, G)\}.$$

If G is a commutative ring, for example if $G = \mathbb{F}_p, G = \mathbb{Z}, G = \mathbb{Q}$ or $G = \mathbb{R}$, then the boundary maps are module homomorphisms, in particular the cohomology group $H^n(C; G)$ is a module over the commutative ring G .

⁶⁸⁸A cochain complex is a sequence

$$\leftarrow D_n \xleftarrow{\delta_n} D_{n-1} \xleftarrow{\delta_{n-1}} \dots D_1 \xleftarrow{\delta_1} D_0 \rightarrow 0$$

of maps between abelian groups such that for all i holds $\delta_{i+1} \circ \delta_i = 0$. Many of the definitions for chain complexes can be used almost verbatim for cochain complexes. For example we say that the cochain complex *exact* if for each n we have

$$\ker(\delta_n) = \operatorname{im}(\delta_{n-1}).$$

Similarly we have the notion of a cochain map and it is straightforward to see that cochain complexes together with the cochain maps form a category, called the *category of cochain complexes*.

Many of the statements about chain complexes and their homology groups hold in a slightly modified version also for cohomology groups. For example, if $f: C_* \rightarrow D_*$ is a chain map between chain complexes C_* and D_* , then for each n the map

$$\begin{aligned} f^*: H^n(D; G) &\rightarrow H^n(C; G) \\ [\varphi: D_n \rightarrow G] &\mapsto [\varphi \circ f_*: C_n \rightarrow G] \end{aligned}$$

is well-defined. Furthermore for each n the maps

$$\begin{aligned} C_* &\mapsto H^n(C; G) \\ f &\mapsto f^* \end{aligned}$$

define a contravariant functor from the category of chain complexes to the category of abelian groups. Furthermore, similar to Proposition 26.9 one can show that if

$$0 \rightarrow A_* \xrightarrow{i} B_* \xrightarrow{p} C_* \rightarrow 0$$

is a short exact sequence of chain complexes, then for each n there exists a map

$$\delta_n: H^n(A; G) \rightarrow H^{n+1}(C; G),$$

such that

$$\dots \rightarrow H^n(C; G) \xrightarrow{p^*} H^n(B; G) \xrightarrow{i^*} H^n(A; G) \xrightarrow{\delta} H^{n+1}(C; G) \xrightarrow{p^*} H_{n+1}(B; G) \xrightarrow{i^*} \dots$$

forms an exact sequence forms.⁶⁸⁹

40.2. The singular cohomology group of a topological space. Now let X be a topological space and let again G be an abelian group. We write

$$C^n(X; G) := \text{Hom}(C_n(X), G) \quad \text{and} \quad \delta_n := \partial_n^*.$$

We obtain the cochain complex

$$\dots \xleftarrow{\delta_{n+2}} C^{n+1}(X; G) \xleftarrow{\delta_{n+1}} C^n(X; G) \xleftarrow{\delta_n} C^{n-1}(X; G) \xleftarrow{\delta_{n-1}} \dots \xleftarrow{\delta_1} C^0(X; G) \leftarrow 0.$$

Now we define the n -th singular cohomology group of X with coefficients in G ⁶⁹⁰ as follows

$$\begin{aligned} H^n(X; G) &:= H^n(C_*(X); G) \\ &= \ker\{\delta_{n+1}: C^n(X; G) \rightarrow C^{n+1}(X; G)\} / \text{im}\{\delta_n: C^{n-1}(X; G) \rightarrow C^n(X; G)\}. \end{aligned}$$

If $G = \mathbb{Z}$, then we simply write $H^n(X) = H^n(X; \mathbb{Z})$. If $f: X \rightarrow Y$ is a map between topological spaces, then we had already seen that f induces a chain map $f_*: C_*(X) \rightarrow C_*(Y)$. It follows from the discussion of the previous chapter that f induces a map $f^*: H^n(Y; G) \rightarrow H^n(X; G)$. Furthermore the maps

$$\begin{aligned} X &\mapsto H^n(X; G) \\ (f: X \rightarrow Y) &\mapsto (f^*: H^n(Y; G) \rightarrow H^n(X; G)) \end{aligned}$$

⁶⁸⁹The fact that the maps go in “the wrong direction” is a never-ending source of confusion.

⁶⁹⁰In most cases we drop the adjective ‘singular’ and we just talk of the n -th cohomology group of X with coefficients in G .

define a contravariant functor from the category of topological spaces to the category of abelian groups. (If G is a commutative ring, then this is in fact a functor to the category of G -modules.)

Before we develop the theory of cohomology groups in detail we first want to consider an explicit example.

Example. Let X be a path-connected topological space. What is $H_0(X; G)$? To answer this question we need to consider the “right hand end” of the chain complex of $C_*(X)$ and of the cochain complex $\text{Hom}(C_*(X), G)$:

$$C_1(X) \xrightarrow{\partial_1} C_0(X) \rightarrow 0 \quad \text{and} \quad \text{Hom}(C_1(X), G) \xleftarrow{\partial_1^*} \text{Hom}(C_0(X), G) \rightarrow 0.$$

Thus it follows that

$$H^0(X; G) = \ker \left\{ \begin{array}{ccc} \text{Hom}(C_0(X), G) & \rightarrow & \text{Hom}(C_1(X), G) \\ (f: C_0(X) \rightarrow G) & \mapsto & (f \circ \partial_1: C_1(X) \rightarrow G) \end{array} \right\}.$$

Since the singular 1-simplices form a basis of $C_1(X)$ we can rewrite $H^0(X; G)$ as follows:

$$H^0(X; G) = \{f: C_0(X) \rightarrow G \mid f(\partial\sigma) = 0 \text{ for all singular 1-simplices } \sigma: \Delta^1 \rightarrow X\}.$$

As a reminder, we identify the singular 0-simplices in X with the points in X . An element in $\text{Hom}(C_0, G)$ is uniquely determined by the values on the singular 0-simplices. Put differently, we can view an element in $\text{Hom}(C_0, G)$ as function from X to G . For a singular 1-simplex $\sigma: \Delta^1 \rightarrow X$ we therefore obtain that $f(\partial\sigma)$ equals the difference between the values at the starting point and at the endpoint of σ . Since we demand that $f(\partial\sigma) = 0$ for all $\sigma: \Delta^1 \rightarrow X$, and since X is path-connected, follows, that f has to assign all points in X the *same* value in G . Thus we see that $H^0(X; G) \cong G$.

Many of the statements about homology groups can be modified in a straightforward way to give corresponding statements about cohomology:

- (1) Homotopic maps $f, g: X \rightarrow Y$ induce the same maps

$$f^* = g^*: H^n(Y; G) \rightarrow H^n(X; G).$$

- (2) Let X be a topological space with finitely many path-components $X_1, \dots, X_k$, then a slight generalization of the above argument shows that

$$H^n(X; G) := \bigoplus_{i=1}^k H^n(X_i; G).$$

In particular, if X is a path-connected topological space, then $H^0(X; G) \cong G$. If X has infinitely many path-components, then the situation is more difficult. We will deal with this case in exercise sheet 12.

- (3) If (X, A) is a pair of topological spaces, then we define the relative cohomology group as

$$H^n(X, A; G) := H^n(C_*(X, A); G)$$

If $f: (X, A) \rightarrow (Y, B)$ is a map of pairs of topological space, i.e. a map $f: X \rightarrow Y$ with $f(A) \subset B$, then f induces maps

$$f^*: H^n(Y, B; G) \rightarrow H^n(X, A; G).$$

Furthermore, given a pair (X, A) of topological spaces we obtain a long exact sequence

$$\dots H^n(X, A; G) \xrightarrow{p^*} H^n(X; G) \xrightarrow{i^*} H^n(A; G) \xrightarrow{\delta} H^{n+1}(X, A; G) \dots$$

hereby the maps are given as follows:

- (a) p^* is map induced by the projection $C_*(X) \rightarrow C_*(X, A)$,
- (b) i^* is the map induced by the inclusion $C_*(A) \rightarrow C_*(X)$,
- (c) the “coboundary-map” $\delta: H^n(A; G) \rightarrow H^{n+1}(X, A; G)$ is given by the map

$$\begin{aligned} H^n(A; G) &\rightarrow H^{n+1}(X, A; G) \\ [f: C_n(A) \rightarrow G] &\mapsto [f \circ \partial: C_{n+1}(X, A) \rightarrow G] \end{aligned}$$

- (4) (Excision Theorem for cohomology groups) Let $Z \subset A \subset X$ be subsets such that the closure of Z is contained in the interior of A . Then the inclusion

$$i: (X \setminus Z, A \setminus Z) \rightarrow (X, A)$$

induces for each $n \in \mathbb{N}_0$ an isomorphism

$$i^*: H^n(X, A; G) \xrightarrow{\cong} H^n(X \setminus Z, A \setminus Z; G).$$

- (5) (Mayer–Vietoris sequence for cohomology group) Let X be a topological space and let $A, B \subset X$ be subsets such that $X = \text{int}(A) \cup \text{int}(B)$. Then there exists a long exact sequence

$$\dots \rightarrow H^n(X; G) \xrightarrow{i^* \oplus i^*} H^n(A; G) \oplus H^n(B; G) \xrightarrow{i^* \oplus -i^*} H^n(A \cap B; G) \rightarrow H^{n+1}(X; G) \rightarrow \dots$$

where the maps i^* are induced by the restriction to the various subspaces.

- (6) Let X be a CW-complex. As a reminder, we call

$$C_*^{\text{CW}}(X): \dots \xrightarrow{d_{n+1}} H_n(X^n, X^{n-1}) \xrightarrow{d_n} H_{n-1}(X^{n-1}, X^{n-2}) \xrightarrow{d_{n-2}} \dots$$

the *cellular chain complex*. We refer to

$$H_{\text{CW}}^n(X; G) := H^n(C_*^{\text{CW}}(X), G)$$

as the *n-th cellular cohomology group of X with G-coefficients*. As in Proposition 29.8 there exists again a canonical isomorphism

$$H_{\text{CW}}^n(X; G) \cong H^n(X; G).$$

For details of the proofs we refer to pages 199 to 204 of [Ha].

The last statement says that we can determine the cohomology groups of a topological space using the cellular chain complex. We will make use of this fact in the following chapter to determine the cohomology groups of several spaces.

40.3. Examples of cohomology groups. We first consider the n -sphere with $n > 0$. As we had seen on page 340 we can view S^n as a CW-complex with one 0-cell and one other cell of dimension n , and the boundary maps are the zero maps. Thus the cellular chain complex is of the form

$$0 \rightarrow \mathbb{Z} \xrightarrow{0} \cdots \xrightarrow{0} \mathbb{Z} \rightarrow 0.$$

For any abelian group G the dual cochain complex is then given by

$$0 \rightarrow \operatorname{Hom}(\mathbb{Z}, G) \xleftarrow{0} \cdots \xleftarrow{0} \operatorname{Hom}(\mathbb{Z}, G) \rightarrow 0.$$

The evaluation map

$$\begin{aligned} \epsilon: \operatorname{Hom}(\mathbb{Z}, G) &\rightarrow G \\ f &\mapsto f(1) \end{aligned}$$

is an isomorphism between G and $\operatorname{Hom}(\mathbb{Z}, G)$. The above dual cochain complex is thus of the form

$$0 \rightarrow G \xleftarrow{0} \cdots \xleftarrow{0} G \leftarrow 0.$$

Thus we see that

$$H^k(S^n; G) \cong \begin{cases} G, & \text{if } k = 0, n, \\ 0, & \text{otherwise.} \end{cases}$$

Now we consider the closed surface $\Sigma = \Sigma_g$ of genus g . As we pointed out on page 508, we can view it as a CW-complex with one 0-cell, $2g$ 1-cells and one 2-cell. Furthermore we had seen on page 508 that the cellular chain complex is of the form

$$0 \rightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z}^{2g} \xrightarrow{0} \mathbb{Z} \rightarrow 0.$$

Therefore it follows that for any abelian group G we have

$$H^2(\Sigma_g; G) \cong G, \quad H^1(\Sigma_g; G) \cong G^{2g} \quad \text{and} \quad H^0(\Sigma_g; G) \cong G.$$

Now recall that given $g \in \mathbb{N}_0$ and $m \in \mathbb{N}_0$ we denote by $\Sigma_{g,m}$ the result of removing m open disks from the surface of genus g . On page 565 we had argued that $\Sigma_{g,m}$ for $m \geq 1$ is homotopy equivalent to the wedge of $2g + m - 1$ circles. Using the fact that homotopy equivalent topological spaces have isomorphic cohomology groups we can use the wedge of $2g + m - 1$ circles to compute the cohomology groups of $\Sigma_{g,m}$. Using the CW-structure for the wedge of circles with one 0-cell we get almost immediately that

$$H^2(\Sigma_{g,m}; G) \cong 0, \quad H^1(\Sigma_{g,m}; G) \cong G^{2g+m-1} \quad \text{and} \quad H^0(\Sigma_{g,m}; G) \cong G.$$

We see that for $G = \mathbb{Z}$ these cohomology groups are precisely the same as the homology groups that we had determined in Proposition 29.14.

In the previous examples the cohomology groups with G -coefficients coincide with the homology groups with G -coefficients. Now we consider the 2-dimensional projective space $\mathbb{RP}^2$. In the proof of Lemma 29.19 we had seen that there exists a CW-structure on $\mathbb{RP}^2$ such that the corresponding cellular chain complex is of the following form:

$$0 \rightarrow \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \rightarrow 0.$$

The corresponding dual cochain complex is thus of the form

$$0 \rightarrow \operatorname{Hom}(\mathbb{Z}, G) \xleftarrow{(\cdot)^*} \operatorname{Hom}(\mathbb{Z}, G) \xleftarrow{0} \operatorname{Hom}(\mathbb{Z}, G) \rightarrow 0.$$

Using the aforementioned evaluation $\epsilon: \operatorname{Hom}(\mathbb{Z}, G) \xrightarrow{\cong} G$ we see that this cochain complex is isomorphic to the following cochain complex:

$$0 \rightarrow G \xleftarrow{\cdot^2} G \xleftarrow{0} G \rightarrow 0.$$

For $G = \mathbb{Z}$ we thus obtain that

$$H^2(\mathbb{RP}^2) \cong \mathbb{Z}_2, \quad H^1(\mathbb{RP}^2) \cong 0 \quad \text{and} \quad H^0(\mathbb{RP}^2) \cong \mathbb{Z}.$$

On the other hand we had seen on page 480 that

$$H_2(\mathbb{RP}^2) = 0, \quad H_1(\mathbb{RP}^2) \cong \mathbb{Z}_2 \quad \text{and} \quad H_0(\mathbb{RP}^2) \cong \mathbb{Z}.$$

Therefore in this case the homology groups and the cohomology groups of $\mathbb{RP}^2$ with $\mathbb{Z}$ -coefficients do *not* coincide.

To add to the confusion we also compute the cohomology of $\mathbb{RP}^2$ with $\mathbb{F}_2$ -coefficients. The corresponding dual cellular cochain complex is then given by

$$0 \rightarrow \mathbb{F}_2 \xleftarrow{\cdot^2=0} \mathbb{F}_2 \xleftarrow{0} \mathbb{F}_2 \rightarrow 0$$

and we see that

$$H^2(\mathbb{RP}^2; \mathbb{F}_2) \cong \mathbb{F}_2, \quad H^1(\mathbb{RP}^2; \mathbb{F}_2) \cong \mathbb{F}_2 \quad \text{and} \quad H^0(\mathbb{RP}^2; \mathbb{F}_2) \cong \mathbb{F}_2.$$

If we compare this result with the calculation on page 605 of the homology of $\mathbb{RP}^2$ with $\mathbb{F}_2$ -coefficients we see that we get the same result. More precisely, the homology groups and the cohomology groups of $\mathbb{RP}^2$ with $\mathbb{F}_2$ -coefficients coincide.

40.4. The cohomology groups of a direct system of topological spaces. In Propositions 13.2 and 29.1 we had seen that in favorable situations the fundamental group and the homology groups of a union of topological spaces are the direct limit of the corresponding fundamental groups and homology groups.

We recall that Proposition 29.1 was a consequence of the following two lemmas.

Lemma 29.2. *Let $X_i, i \in \mathbb{N}$ be a sequence of topological spaces such that the following holds:*

- (1) *the sequence is nested, i.e. for each $i \in \mathbb{N}$ we have $X_i \subset X_{i+1}$, and*
- (2) *each X_i is open in X_{i+1} .*

We write $X = \bigcup_{i \in \mathbb{N}} X_i$. Then the inclusion induced maps $C_*(X_i) \rightarrow C_*(X)$ induce an isomorphism

$$\lim_{\rightarrow} C_*(X_i) \rightarrow C_*(X)$$

of chain complexes.

The second lemma was purely algebraic.

Lemma 29.3. *Let $(\{C_i\}_{i \in \mathbb{N}}, \{f_{ij}\}_{i \leq j})$ be a direct system of chain complexes. Then given any $n \in \mathbb{N}_0$ the maps $H_n(C_i) \rightarrow H_n(\varinjlim C_i)$ induce an isomorphism*

$$\varinjlim H_n(C_i) \xrightarrow{\cong} H_n(\varinjlim C_i).$$

The last lemma can be summarized as follows, for $j < k$ in I we have

$$\begin{array}{ccc} H_n(C_k) & & \\ \uparrow & \searrow & \nearrow \\ & \varinjlim H_n(C_i) \xrightarrow{\cong} H_n(\varinjlim C_i) & \\ & \nearrow & \nwarrow \\ H_n(C_j) & & \end{array}$$

Cohomology is a contravariant functor, this implies that the maps $C_i \rightarrow \varinjlim C_i$ induce maps $H^n(\varinjlim C_i) \rightarrow H^n(C_i)$. Put differently, for any $j < k$ we are in the following situation

$$\begin{array}{ccc} & & H^n(C_k) \\ & \nearrow & \downarrow \\ H^n(\varinjlim C_i) & \xleftarrow{\quad} \varprojlim H^n(C_i) & \\ & \searrow & \downarrow \\ & & H^n(C_j) \end{array}$$

where the dotted arrow exists by the definition of the inverse limit of the groups $H^n(C_i)$. The question is whether the dotted arrow is in fact an isomorphism.

It turns out that this is not necessarily the case. To formulate the condition when it is an isomorphism we need the following definition.

Definition. We say a directed system of groups $(\{D_i\}_{i \in \mathbb{N}}, \{\varphi_{ij}\}_{i \geq j})$ satisfies the *Mittag-Leffler condition* if given any $i \in \mathbb{N}$ there exists an $n \geq i$ such that for all $r, s \geq n$ we have

$$\text{im}(f_{ri}: D_r \rightarrow D_i) = \text{im}(f_{si}: D_s \rightarrow D_i).$$

Example. If all the homomorphisms in the directed system are epimorphisms, then the directed system satisfies the Mittag-Leffler condition.

Now we can formulate the following lemma.

Lemma 40.1. *Let $(\{C_i\}_{i \in \mathbb{N}}, \{f_{ij}\}_{i \leq j})$ be a direct system of chain complexes over $\mathbb{N}$. If for each n the cochain groups $\text{Hom}(C_i, \mathbb{Z})$ satisfy the Mittag-Leffler condition, then there exists a short exact sequence^{691 692}*

$$0 \rightarrow \varprojlim^1 H^{n-1}(C_i) \rightarrow H^n(\varinjlim C_i) \rightarrow \varprojlim H^n(C_i) \rightarrow 0.$$

Furthermore, if the inverse system $H^{n-1}(C_i)$ satisfies the Mittag-Leffler condition, then $\varprojlim^1 H^{n-1}(C_i) = 0$, in particular in this case we have an isomorphism

$$H^n(\varinjlim C_i) \xrightarrow{\cong} \varprojlim H^n(C_i).$$

Proof. The lemma can be proved without too many troubles in an elementary fashion. The most difficult part was to find the right condition on the complexes. We refer to [Wb, p. 84] or alternatively to [Lü, Lemma 6.51] for a reference for the existence of the short exact sequence. See [Mas3, Lemma A.17] for a proof of the second statement. $\square$

The following proposition can be viewed as the analogue of Proposition 29.4 for cohomology groups.

Proposition 40.2. *Let X be a CW-complex and let $X_i, i \in \mathbb{N}$ be a sequence of CW-subcomplexes such that for each $i \in \mathbb{N}$ we have $X_i \subset X_{i+1}$ and such that $X = \bigcup_{i \in \mathbb{N}} X_i$. Then for any $n \in \mathbb{N}_0$ we have a short exact sequence*

$$0 \rightarrow \varprojlim^1 H^{n-1}(X_i; \mathbb{Z}) \rightarrow H^n(X; \mathbb{Z}) \rightarrow \varprojlim H^n(X; \mathbb{Z}) \rightarrow 0.$$

Furthermore, if the inverse system $H^{n-1}(C_i)$ satisfies the Mittag-Leffler condition, then we have an isomorphism

$$H^n(X) \xrightarrow{\cong} \varprojlim H^n(X_i).$$

Proof. Let X be a CW-complex and let $X_i, i \in \mathbb{N}$ be a sequence of CW-subcomplexes such that for each $i \in \mathbb{N}$ we have $X_i \subset X_{i+1}$ and such that $X = \bigcup_{i \in \mathbb{N}} X_i$. For each $j > i$ we have that $C_n(X_i)$ is a subsummand of $C_n(X_j)$. This implies that $\text{Hom}(C_n(X_j); \mathbb{Z}) \rightarrow \text{Hom}(C_n(X_i); \mathbb{Z})$ is an epimorphism. The chain groups thus satisfy the Mittag-Leffler condition. The combination of Lemmas 29.2 and 40.1 now gives us the desired statement. $\square$

We conclude this section with a few examples:

Examples.

⁶⁹¹Given an inverse system $\{H_i\}_{i \in \mathbb{N}}$ of abelian groups there exists in general an abelian group $\varprojlim^1 A_i$. It plays the same role as the Tor groups do for tensoring and the Ext groups do for taking a dual, namely they measure to what degree taking cohomology does not commute with direct limits. A formal definition of $\varprojlim^1 A_i$ is given in [Ha, p. 313]. For our purpose one can take the short exact sequence as the definition, namely it is the kernel of the map $H^n(\varinjlim C_i) \rightarrow \varprojlim H^n(C_i)$.

⁶⁹²Note also the discrete “1” in the notation on the left.

- (1) As on page 496 we consider the real line with infinitely many 2-dimensional spheres attached, i.e. we consider

$$X := (\mathbb{R} \cup S^2 \times \mathbb{Z}) / \sim$$

where given $i \in \mathbb{Z}$ we identify $i \in \mathbb{R}$ with $((0, 0, -1), i) \in S^2 \times \mathbb{Z}$. We view X as a CW-complex in the obvious way. Furthermore, given $k \in \mathbb{N}$ we consider the subcomplex

$$X_k := ([-k, k] \cup S^2 \times \{-k, \dots, k\}) / \sim.$$

We leave it as an exercise to verify that

$$H^n(X_k) = \begin{cases} \mathbb{Z}, & \text{if } n = 0, \\ \mathbb{Z}^{2k+1}, & \text{if } n = 2, \\ 0, & \text{else.} \end{cases}$$

and that for each k the direct system of cohomology groups $\{H^n(X_k)\}_{k \in \mathbb{N}}$ satisfies the Mittag-Leffler condition. We obtain that

$$H^n(X) \underset{\substack{\uparrow \\ \text{Proposition 40.2}}}{=} \varprojlim H^n(X_k) = \varprojlim \begin{cases} \mathbb{Z}, & \text{if } n = 0, \\ \mathbb{Z}^{2k+1}, & \text{if } n = 2, \\ 0, & \text{else} \end{cases} = \begin{cases} \mathbb{Z}, & \text{if } n = 0, \\ \mathbb{Z}^{(\mathbb{N})}, & \text{if } n = 2, \\ 0, & \text{else.} \end{cases} \underset{\substack{\uparrow \\ \text{see page 243}}}{=}$$

Hereby recall that

$$\mathbb{Z}^{\mathbb{N}} = \{(x_1, x_2, \dots) \mid x_i \in \mathbb{Z}\}.$$

On 496 we had seen that $H_2(X) \cong \mathbb{Z}^{(\mathbb{N})}$. In this case we see that $H_2(X) \not\cong H^2(X)$ since the former group is countable whereas the latter group is uncountable.

- (2) We consider again the surface Σ_∞ of infinite genus that we had already encountered on pages 230 and 513. Using a variation on example (1) and the proof of Lemma 13.3 one can show that

$$H^n(\Sigma_\infty) \cong \begin{cases} 0, & \text{if } n \geq 2, \\ \mathbb{Z}^{\mathbb{N}}, & \text{if } n = 1, \\ 0, & \text{if } n = 0. \end{cases}$$

We leave the details as an exercise to the reader.

- (3) We recall that in Lemma 22.1 we had shown that we can view $\mathbb{CP}^n$ as a CW-complex which admits exactly one cell in dimensions $0, 2, \dots, 2n$ and which admits no odd-cells. By the discussion on page 669 we can use the cellular chain complex to compute the cohomology of $\mathbb{CP}^n$ and we obtain that

$$H^k(\mathbb{CP}^n; \mathbb{Z}) \cong \begin{cases} G, & \text{if } k = 0, 2, 4, \dots, 2n, \\ 0, & \text{otherwise.} \end{cases}$$

It follows from this calculation and from Proposition 40.2 that

$$H^k(\mathbb{CP}^\infty; \mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & \text{if } k \text{ even,} \\ 0, & \text{if } k \text{ odd.} \end{cases}$$

41. THE UNIVERSAL COEFFICIENT THEOREM FOR COHOMOLOGY GROUPS

41.1. The Ext group. The question now arises how we can explain the erratic relationship between homology groups and cohomology groups that we had seen in the previous example. Furthermore it would be interesting to know whether one can determine the cohomology with G -coefficients from the homology with $\mathbb{Z}$ -coefficients.

These questions are purely in the realm of algebra. More precisely, let (C_*, ∂_*) be a chain complex and let G an abelian group. We want to find a connection between the cohomology group $H^*(C; G)$ and the homology groups $H_*(C)$. This problem, and later on its solution, is very similar to the problem we faced in Chapter 36.

As a reminder, we denote by $\mathcal{C}$ the category of chain complexes. We now denote by $\mathcal{C}^*$ the category of cochain complexes. On page 605 we had defined what it means for a covariant functor $F: \mathcal{C} \rightarrow \mathcal{C}$ to be exact, left-exact and right-exact. The same way we can introduce these notions for contravariant functors $F: \mathcal{C} \rightarrow \mathcal{C}^*$. For example a contravariant functor $F: \mathcal{C} \rightarrow \mathcal{C}^*$ is called *left-exact*⁶⁹³ if for all exact sequences

$$A \xrightarrow{i} B \xrightarrow{j} C \longrightarrow 0$$

of abelian groups the sequence

$$0 \longrightarrow F(C) \xrightarrow{F(j)} F(B) \xrightarrow{F(i)} F(A)$$

is also exact. Furthermore a contravariant functor $F: \mathcal{C} \rightarrow \mathcal{C}^*$ is called *right-exact*, if for all exact sequences

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{j} C$$

the sequence

$$F(C) \xrightarrow{F(j)} F(B) \xrightarrow{F(i)} F(A) \longrightarrow 0$$

is also exact.

We had seen that the covariant functor $- \otimes G$ is right-exact but that in general it is not left-exact. With the contravariant functor $\text{Hom}(-, G)$ the situation is precisely the converse. In exercise sheet 13 we will show that this functor is left-exact. On the other hand if we consider the chain complex

$$0 \rightarrow \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{p} \mathbb{Z}_2$$

together with $G = \mathbb{F}_2$, then the corresponding cochain complex is of the form

$$\mathbb{F}_2 \xrightarrow[p^*]{\cong} \mathbb{F}_2 \xrightarrow[=0]{(\cdot 2)^*} \mathbb{F}_2 \rightarrow 0.$$

Since this cochain complex is not exact it follows that the functor $\text{Hom}(-, \mathbb{F}_2)$ is not a right-exact functor.

⁶⁹³Admittedly one could also refer to this definition as “right”-exact, since the direction changes. But the definition we give is the one which is commonly used in the literature.

In Chapter 36.4 we had introduced the torsion groups $\text{Tor}(G, H)$ which “measure” to what degree the functor $- \otimes G$ fails to be a left-exact functor. The same way we now introduce the so called Ext-group $\text{Ext}(G, H)$ which “measure” to what degree the functor $\text{Hom}(-, G)$ fails to be right-exact.

Definition. Let G be and H abelian groups. We denote by

$$\dots F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \rightarrow 0$$

the canonical resolution from page 608. We define the n -th G -torsion group of H as n -th G -ext-group of H as

$$\text{Ext}_n(H, G) := H^n(F_*; G).$$

As in Chapter 36.4 one can now show that the isomorphism type of this group does not depend on the choice of the free resolution. The 0-th Ext-group is, similar to the 0-th tor group, an “old acquaintance”. More precisely, in exercise sheet 13 we will show that

$$\text{Ext}_0(H, G) = \text{Hom}(H, G).$$

As we had pointed out in the proof of Lemma 36.11 any abelian group H admits a free resolution of length 2:

$$0 \rightarrow F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \rightarrow 0.$$

In particular we immediately obtain that $\text{Ext}_n(H, G) = 0$ for $n \geq 2$. We refer to

$$\text{Ext}(H, G) := \text{Ext}_1(H, G)$$

as the G -Ext-group of H . The properties of the Ext-group are very similar to the properties of the torsion group.

Lemma 41.1. *Let G and H be abelian groups. The following statements hold:*

(1) *If $H_1, \dots, H_k$ are abelian groups, then*

$$\text{Ext}\left(\bigoplus_{i=1}^k H_i, G\right) \cong \bigoplus_{i=1}^k \text{Ext}(H_i, G).$$

(2) *If $G_1, \dots, G_k$ are abelian groups, then*

$$\text{Ext}\left(H, \bigoplus_{i=1}^k G_i\right) \cong \bigoplus_{i=1}^k \text{Ext}(H, G_i),$$

(3) *$\text{Ext}(H, G) = 0$, if H is a free abelian group.*

(4) *For all n we have an isomorphism $\text{Ext}(\mathbb{Z}_n, G) \cong G/nG$,*

(5) *For $n, m \neq 0$ we have $\text{Ext}(\mathbb{Z}_n, \mathbb{Z}_m) \cong \mathbb{Z}_{\text{gcd}(n, m)}$.*

(6) *For a finitely generated (torsion?????) abelian group G we have $\text{Ext}(G, \mathbb{Q}/\mathbb{Z}) = 0$.*

Remark. The first two statements of the lemma also generalize to the case of infinitely groups, but some care needs to be taken. More precisely, the following holds:

(1') If $H_i, i \in I$ is a family of abelian groups, then

$$\operatorname{Ext}\left(\bigoplus_{i \in I} H_i, G\right) \cong \prod_{i \in I} \operatorname{Ext}(H_i, G).$$

(2') If H is a finitely generated abelian group and if $G_i, i \in I$ is a family of abelian groups, then

$$\operatorname{Ext}\left(H, \bigoplus_{i \in I} G_i\right) \cong \prod_{i \in I} \operatorname{Ext}(H, G_i).$$

In both cases the statement follows from Lemma 40.1, applied to I with the trivial order, and the determination of the inverse limit of abelian groups on page 240. In (2') the condition that H is finitely generated is necessary to ensure that the Mittag-Leffler condition is satisfied. We leave it as an exercise to verify the details.

Proof. The first five statements are proved almost the same way as we had proved Lemma 36.12. We leave the details as an exercise to the reader.

Finally we turn to the last statement. ++++++ $\mathbb{Q}/\mathbb{Z}$ is injective, whatever that means ++++++ □

We had seen that $\operatorname{Tor}(H, G) = 0$ if G is a torsion-free abelian group. For the Ext -group the corresponding statement does not hold. Indeed, using the classification of finitely generated abelian groups, see Theorem 8.3, and the above properties (1), (3) and (4) one can easily prove the following lemma.

Lemma 41.2. *Let H be a finitely generated abelian group. Then⁶⁹⁴⁶⁹⁵*

$$\operatorname{Ext}(H, \mathbb{Z}) \cong \text{torsion-subgroup of } H.$$

Now let $\alpha: H \rightarrow H'$ be a homomorphism between abelian groups and let G be an abelian group. Then it follows from Lemma 36.7 and from the definition of the G -torsion that α induces a homomorphism

$$\alpha^*: \operatorname{Ext}(H', G) \rightarrow \operatorname{Ext}(H, G).$$

It is now straightforward to show that

$$\begin{aligned} H &\mapsto \operatorname{Ext}(H, G) \\ (\alpha: H \rightarrow H') &\mapsto (\alpha^*: \operatorname{Ext}(H', G) \rightarrow \operatorname{Ext}(H, G)) \end{aligned}$$

is a *contravariant* functor from the category of abelian groups to the category of abelian groups. On the other hand, if $\beta: G \rightarrow G'$ is a homomorphism between abelian groups then

⁶⁹⁴This isomorphism is course not canonical.

⁶⁹⁵Here recall that the torsion-subgroup of an abelian group H is the subgroup given by all elements of finite order. (Why is that a subgroup?)

one sees easily that β induces a homomorphism

$$\beta_*: \text{Ext}(H, G) \rightarrow \text{Ext}(H, G')$$

Furthermore one can easily prove that

$$\begin{aligned} G &\mapsto \text{Ext}(H, G) \\ (\beta: G \rightarrow G') &\mapsto (\beta_*: \text{Ext}(H, G) \rightarrow \text{Ext}(H, G')) \end{aligned}$$

is a *covariant* functor from the category of abelian groups to the category of abelian groups.

We have one more general result for calculating Ext-groups, before we can state the result we need one more definition.

Definition. An abelian group G is called *divisible* if for any $g \in G$ and $n \in \mathbb{N}$ there exists an $h \in G$ with $h^n = g$.

Examples.

- (1) The groups $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ are divisible.
- (2) Every quotient of a divisible group is divisible, for example $\mathbb{Q}/\mathbb{Z}$ is divisible.
- (3) The group $\mathbb{Z}$ is not divisible, and the only finite group that is divisible is the trivial group.

Proposition 41.3. *Let G and H be abelian groups. If G is divisible, then $\text{Ext}(H, G) = 0$.*

In the proof of Proposition 41.3 it is useful to introduce the following notion.

Definition. An abelian group G is called *injective* if for all abelian groups A and B and injective homomorphism $\varphi: B \rightarrow A$ and homomorphisms $\beta: B \rightarrow G$ there exists a homomorphism $\alpha: A \rightarrow G$ such that the following diagram commutes

$$\begin{array}{ccccc} 0 & \longrightarrow & B & \xrightarrow{\varphi} & A \\ & & \beta \downarrow & \swarrow \alpha & \\ & & G & & \end{array}$$

The following lemma shows why we are interested in injective groups.

Lemma 41.4. *Let G and H be abelian groups. If G is injective, then $\text{Ext}(H, G) = 0$.*

Proof. Let H be an abelian group. As we had pointed out in the proof of Lemma 36.11 the abelian group H admits a free resolution of length 2:

$$0 \rightarrow F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\varphi_0} H \rightarrow 0.$$

By definition $\text{Ext}(H, G)$ is the cokernel of the map

$$\text{Hom} \left(\begin{array}{c} F_1 \\ \downarrow \\ G \end{array} \right) \xleftarrow{\varphi_1} \text{Hom} \left(\begin{array}{c} F_0 \\ \downarrow \\ G \end{array} \right).$$

But by definition of G being divisible this map is in fact an epimorphism. Thus we have shown that $\text{Ext}(H, G) = 0$. $\square$

To prove Proposition 41.3 it remains to prove the following lemma.

Lemma 41.5. *An abelian group that is divisible is also injective.*

Remark. In exercise sheet 2 we will prove the converse, namely that an injective abelian group is also divisible.

Proof. Let G be an abelian group that is divisible. Suppose we are given a diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & B & \xrightarrow{\varphi} & A \\ & & \beta \downarrow & \swarrow ?\alpha? & \\ & & G & & \end{array}$$

Since φ is a monomorphism we can assume that B is in fact a subgroup of A and that φ is the inclusion map.

We now consider all pairs (C_i, γ_i) where C_i is a subgroup of A which contains B and where $\gamma_i: C_i \rightarrow G$ is a homomorphism such that $\gamma_i|_B = \beta$. This family of pairs is non-empty since (B, β) satisfies the required conditions.

We define $(C_i, \gamma_i) < (C_j, \gamma_j)$ if $C_i \subset C_j$ and if $\gamma_j|_{C_i} = \gamma_i$. We can apply Zorn's Lemma⁶⁹⁶ to this family and we obtain a maximal pair (C_m, γ_m) . It remains to prove the following claim.

Claim. We have $C_m = A$.

Suppose that this is not the case. Then there exists an $a \in A \setminus C_m$. We denote by $\langle\langle C_m, a \rangle\rangle$ the subgroup of A generated by C_m and a . To obtain a contradiction it suffices to show that there exists a homomorphism $\delta: \langle\langle C_m, a \rangle\rangle \rightarrow G$ that extends γ_m . We distinguish two cases:

- (1) If there exists no $n \in \mathbb{N}$ such that $a^k \in C_m$ then we extend γ_m to $\langle\langle C_m, a \rangle\rangle$ by setting $\delta(a) = 0$.
- (2) Otherwise there exists a minimal $n \in \mathbb{N}$ such that $a^k \in C_m$. We write $g = \gamma_m(a^n)$. Since G is divisible there exists an $h \in G$ such that $h^n = g$. We extend γ_m to $\langle\langle C_m, a \rangle\rangle$ by setting $\delta(a) = h$.

In both cases it is straightforward to see that δ is a well-defined homomorphism. This concludes the proof of the claim and thus of the lemma. $\square$

⁶⁹⁶Recall that Zorn's Lemma says the following:

Zorn's Lemma. *Suppose a partially ordered set $(P, \leq)$ has the property that every chain has an upper bound in P . Then the set P contains at least one maximal element.*

We also recall the definitions in the statement of Zorn's Lemma:

- (1) A partially ordered set $(P, \leq)$ is a set together with a preorder that has the property that $a \leq b$ and $b \leq a$ implies that $a = b$.
- (2) A chain in P is a subset Q such that for each $q, q' \in Q$ we have $q \leq q'$ or $q' \leq q$.
- (3) A maximal element of subset Q is an element $q_m \in Q$ such that $q \leq q_m$ for all $q \in Q$.

We refer to [Ci, Theorem 4.3.4] for a proof of Zorn's Lemma. Perhaps it is more fun to look at Zorn's Lemma in The Simpsons, see

http://www.simonsingh.net/Simpsons_Mathematics/zorns-lemma/

41.2. The Universal Coefficient Theorem for Cohomology Groups. Let (C_n, ∂_n) be a chain complex of free abelian groups and let G be an abelian group. We want to explore the connection between the cohomology group of C and the usual homology groups. For example we have the evaluation map which is given by⁶⁹⁷

$$\begin{aligned} \text{ev}: H^n(C; G) &\rightarrow \text{Hom}(H_n(C), G) \\ [f: C_n \rightarrow G] &\mapsto \left(\begin{array}{ccc} H_n(C) & \rightarrow & G \\ [c] & \mapsto & f(c) \end{array} \right). \end{aligned}$$

It follows easily from the examples in the previous chapter that this map is in general not an isomorphism.

The connection between homology groups and cohomology group is given by the following theorem.

Theorem 41.6. (The Universal Coefficient Theorem for Cohomology Groups)

Let (C_n, ∂_n) be a chain complex of free abelian groups and let G be an abelian group. Then for each n there exists a short exact sequence

$$0 \rightarrow \text{Ext}(H_{n-1}(C), G) \rightarrow H^n(C; G) \xrightarrow{\text{ev}} \text{Hom}(H_n(C), G) \rightarrow 0.$$

This exact sequence splits and there exists therefore an isomorphism

$$H^n(C; G) \cong \text{Ext}(H_{n-1}(C), G) \oplus \text{Hom}(H_n(C), G).$$

The proof of the theorem is very similar to the proof of Theorem 41.6. We refer to [Ha, Chapter 3.1] for the details of the proof.

Remark.

- (1) The short exact sequence is natural, in this sense that if $f: C \rightarrow D$ is a chain map then the induced maps

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Ext}(H_{n-1}(D), G) & \longrightarrow & H^n(D; G) & \xrightarrow{\text{ev}} & \text{Hom}(H_n(D), G) \longrightarrow 0 \\ & & \downarrow f^* & & \downarrow f^* & & \downarrow f^* \\ 0 & \longrightarrow & \text{Ext}(H_{n-1}(C), G) & \longrightarrow & H^n(C; G) & \xrightarrow{\text{ev}} & \text{Hom}(H_n(C), G) \longrightarrow 0 \end{array}$$

form a commutative diagram. In particular, if (X, A) is a pair of topological spaces then we obtain the following commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Ext}(H_{n-2}(A), G) & \longrightarrow & H^{n-1}(A; G) & \xrightarrow{\text{ev}} & \text{Hom}(H_{n-1}(A), G) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \text{Ext}(H_{n-1}(X, A), G) & \longrightarrow & H^n(X, A; G) & \xrightarrow{\text{ev}} & \text{Hom}(H_n(X, A), G) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \text{Ext}(H_{n-1}(X), G) & \longrightarrow & H^n(X; G) & \xrightarrow{\text{ev}} & \text{Hom}(H_n(X), G) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \text{Ext}(H_{n-1}(A), G) & \longrightarrow & H^n(A; G) & \xrightarrow{\text{ev}} & \text{Hom}(H_n(A), G) \longrightarrow 0. \end{array}$$

⁶⁹⁷It is straightforward to verify that this map is well-defined.

(2) There exist no “canonical” isomorphism

$$H^n(C; G) \cong \text{Ext}(H_{n-1}(C), G) \oplus \text{Hom}(H_n(C), G)$$

in this sense that one cannot associate to chain complexes and groups an isomorphism can such that a chain map $f: C \rightarrow D$ always gives rise to a commutative diagram.

Example. We consider again the projective space $P := \mathbb{RP}^2$. As a reminder, we had shown on page 480 that $H_2(P) = 0$, $H_1(P) = \mathbb{Z}_2$ and $H_0(P) = \mathbb{Z}$. It follows from Theorem 41.6 and from the properties of the Ext-group that we had listed on page 676 that

$$\begin{aligned} H^2(P; \mathbb{Z}) &\cong \text{Hom}(H_2(P), \mathbb{Z}) \oplus \text{Ext}(H_1(P), \mathbb{Z}) \cong \text{Hom}(0, \mathbb{Z}) \oplus \text{Ext}(\mathbb{Z}_2, \mathbb{Z}) = 0 \oplus \mathbb{Z}_2 = \mathbb{Z}_2 \\ H^1(P; \mathbb{Z}) &\cong \text{Hom}(H_1(P), \mathbb{Z}) \oplus \text{Ext}(H_0(P), \mathbb{Z}) \cong \text{Hom}(\mathbb{Z}_2, \mathbb{Z}) \oplus \text{Ext}(\mathbb{Z}, \mathbb{Z}) = 0 \oplus 0 = 0 \\ H^0(P; \mathbb{Z}) &\cong \text{Hom}(H_0(P), \mathbb{Z}) \cong \text{Hom}(\mathbb{Z}, \mathbb{Z}) = \mathbb{Z}. \end{aligned}$$

and

$$\begin{aligned} H^2(P; \mathbb{F}_2) &\cong \text{Hom}(H_2(P), \mathbb{F}_2) \oplus \text{Ext}(H_1(P), \mathbb{F}_2) \cong \text{Hom}(0, \mathbb{F}_2) \oplus \text{Ext}(\mathbb{F}_2, \mathbb{F}_2) = 0 \oplus \mathbb{F}_2 \cong \mathbb{F}_2 \\ H^1(P; \mathbb{F}_2) &\cong \text{Hom}(H_1(P), \mathbb{F}_2) \oplus \text{Ext}(H_0(P), \mathbb{F}_2) \cong \text{Hom}(\mathbb{F}_2, \mathbb{F}_2) \oplus \text{Ext}(\mathbb{F}_2, \mathbb{Z}) = \mathbb{F}_2 \oplus 0 = \mathbb{F}_2 \\ H^0(P; \mathbb{F}_2) &\cong \text{Hom}(H_0(P), \mathbb{F}_2) \cong \text{Hom}(\mathbb{Z}, \mathbb{F}_2) = \mathbb{F}_2. \end{aligned}$$

Fortunately this coincides with the calculations from Chapter 40.3.

Example. We return to the study of lens spaces. On page 581 we had seen that

$$H_n(L(P, q)) \cong \begin{cases} \mathbb{Z}, & \text{if } n = 0, 3 \\ 0, & \text{if } n = 2 \text{ or } n > 3 \\ \mathbb{Z}_p, & \text{if } n = 1. \end{cases}$$

As in the previous example we can now use Theorem 41.6 to compute the cohomology groups of $L(p, q)$. We leave it as an exercise to the reader to verify that⁶⁹⁸

$$H^n(L(P, q)) \cong \begin{cases} \mathbb{Z}, & \text{if } n = 0, 3 \\ 0, & \text{if } n = 1 \text{ or } n > 3 \\ \mathbb{Z}_p, & \text{if } n = 2. \end{cases}$$

This shows that cohomology groups also cannot distinguish between two lens spaces of the form $L(p, q)$ and $L(p, r)$. In particular we have made no progress on Question 6.7 since the introduction of the fundamental group. All the extra invariants, namely homology and cohomology groups have proved useless.

Example. In the following let X be a homotopy equivalent to a finite CW-complex. Furthermore let $G = \mathbb{R}$. In the previous chapter we had seen that the Ext-group vanishes for $G = \mathbb{R}$. Thus it follows from Theorem 41.6 and the properties of Ext-groups that we had

⁶⁹⁸Note that the $\mathbb{Z}_p$ shifted from $n = 1$ in homology to $n = 2$ in cohomology.

listed on page 676 that⁶⁹⁹

$$H^n(X; \mathbb{R}) \cong \operatorname{Hom}(H_n(X); \mathbb{R}) \cong \operatorname{Hom}(H_n(X)/\operatorname{torsion}; \mathbb{R}) \cong \operatorname{Hom}(\mathbb{Z}^{b_i(X)}; \mathbb{R}) \cong \mathbb{R}^{b_1(X)}.$$

It follows furthermore from Theorem 36.13 and from Lemma 36.12 that

$$H_n(X; \mathbb{R}) \cong H_n(X) \otimes \mathbb{R} \cong (H_n(X)/\operatorname{torsion}) \otimes \mathbb{R} \cong \mathbb{Z}^{b_i(X)} \otimes \mathbb{R} \cong \mathbb{R}^{b_1(X)}.$$

Thus we see that the homology groups and the cohomology group with $\mathbb{R}$ -coefficients are real vector spaces of the same dimension, in particular they are isomorphic.⁷⁰⁰ Note though that there is no canonical isomorphism from $H_n(X; \mathbb{R})$ to $H^n(X; \mathbb{R})$.⁷⁰¹ Exactly the same conclusion holds if we replace $\mathbb{R}$ by any other subfield of $\mathbb{C}$, e.g. $\mathbb{C}$ itself or $\mathbb{C}$.

Proposition 41.7. *Let M be a closed connected n -dimensional topological manifold. Then*

$$H^n(M; \mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & \text{if } M \text{ is orientable,} \\ \mathbb{Z}_2, & \text{if } M \text{ is not orientable.} \end{cases}$$

Proof. The proposition is an immediate consequence of the Universal Coefficient Theorem 41.6 for Cohomology Groups, Lemma 41.1, Proposition 39.21 and Proposition 39.22. $\square$

Finally we record the following useful consequence of the Universal Coefficient Theorem 41.6 for Cohomology Groups:

Proposition 41.8. *Let X be a path-connected topological space, let $x_0 \in X$ and let G be an abelian group. Then the Hurewicz homomorphism $\pi_1(X, x_0) \rightarrow H_1(X)$ from page 540 and the evaluation map $\operatorname{ev}: H^1(X; G) \rightarrow \operatorname{Hom}(H_1(X); G)$ induce isomorphisms*

$$H^1(X; G) \xrightarrow{\cong} \operatorname{Hom}(H_1(X); G) \xrightarrow{\cong} \operatorname{Hom}(\pi_1(X, x_0), G).$$

Proof. We have $H_0(X) \cong \mathbb{Z}$. By Lemma 41.1 (3) we have $\operatorname{Ext}(H_0(X), G) = 0$. Therefore the first map is an isomorphism by the Universal Coefficient Theorem 41.6 for Cohomology Groups.

The second map is an isomorphism by Propositions 10.6 (2) and Proposition 32.4. $\square$

We conclude this section with two lemmas that relate the dimensions of homology and cohomology groups.

Lemma 41.9. *Let X be a topological space and let $k \in \mathbb{N}_0$ such that $H_k(X; \mathbb{Z})$ and $H_{k-1}(X; \mathbb{Z})$ are finitely generated. Then for $S = \mathbb{Q}$ or $R = \mathbb{R}$ the vector spaces $H_k(X; S)$ and $H^k(X; S)$ have the same dimension.*

⁶⁹⁹As a reminder, the i -th Betti number of X is defined as $b_i(X) := \operatorname{rank}(H_i(X))$. Since X is assumed to be homotopy equivalent to a finite CW-complex it follows from Corollary 25.5 and Proposition 29.9 that the i -th Betti number is finite.

⁷⁰⁰Does this conclusion also hold if we drop the condition that X is homotopy equivalent to a finite CW-complex?

⁷⁰¹If V is any finite-dimensional real vector space, then $V^* := \operatorname{Hom}_{\mathbb{R}}(V, \mathbb{R})$ is a real vector space with $\dim(V^*) = \dim(V)$. In particular V and V^* are isomorphic real vector spaces. But there is no canonical isomorphism from V to V^* .

Proof. Let X be a topological space, let $k \in \mathbb{N}_0$ such that $H_k(X; \mathbb{Z})$ is finitely generated and let $S = \mathbb{Q}$ or $R = \mathbb{R}$. Since $H_k(X; \mathbb{Z})$ is finitely generated we can, by the classification of finitely generated abelian groups, make the identification $H_k(X; \mathbb{Z}) = \mathbb{Z}^r \oplus T$ where T is a torsion group. We then have

$$\begin{array}{ccccccc} \dim_S(H_k(X; S)) & = & \dim_S(H_k(X; \mathbb{Z}) \otimes S) & = & \dim_S((\mathbb{Z}^r \oplus T) \otimes S) & = & \dim_S(S^r) = r \\ & \uparrow & & & \uparrow & & \\ & \text{Theorem 36.13 and Lemma 36.12 (4)} & & & \text{see page 600} & & \end{array}$$

and

$$\begin{array}{ccccccc} \dim_S(H^k(X; S)) & = & \dim_S(\text{Hom}_{\mathbb{Z}}(H^k(X; \mathbb{Z}), S)) & = & \dim_S(\text{Hom}_{\mathbb{Z}}(\mathbb{Z}^r \oplus T), S) & = & r \\ & \uparrow & & & \uparrow & & \\ & \text{Theorem 41.6 and Proposition 41.3} & & & \text{since } S \text{ is torsion-free} & & \square \end{array}$$

The following lemma will be proved in exercise sheet 2.

Lemma 41.10. *Let X be a topological space and let $k \in \mathbb{N}_0$ such that $H_k(X; \mathbb{Z})$ and $H_{k-1}(X; \mathbb{Z})$ are finitely generated. Then for each prime p the vector spaces $H_k(X; \mathbb{F}_p)$ and $H^k(X; \mathbb{F}_p)$ have the same dimension.*

41.3. The cohomology group of product topological spaces. In this chapter we want to consider the following question.

Question 41.11. *Let X be and Y two topological spaces. What is the connection between the cohomology group of X , Y and of $X \times Y$?*

As a reminder, the Eilenberg-Zilber Theorem 37.3 says that there exists a natural homotopy equivalence

$$C_*(X) \otimes C_*(Y) \rightarrow C_*(X \times Y).$$

To answer the above question we now have to address the following algebraic question: what is the connection between the cohomology groups of two chain complexes $\mathcal{C}$, $\mathcal{C}'$ and cohomology groups of the tensor product $\mathcal{C} \otimes \mathcal{C}'$? Let R be a ring. We consider the linear map:

$$\begin{aligned} \Theta: \text{Hom}(C_p, R) \times \text{Hom}(C'_q, R) &\rightarrow \text{Hom}(\mathcal{C} \otimes \mathcal{C}', R) \\ (\varphi_p, \varphi'_q) &\mapsto \left(\begin{array}{l} \bigoplus_{i,j} C_i \otimes C'_j \rightarrow C_p \otimes C'_q \rightarrow R \\ \sum_k \sum_{i,j} c_i^k \otimes (c'_j)^k \mapsto \sum_k c_p^k \otimes (c'_q)^k \mapsto \sum_k \underbrace{\varphi_p(c_p^k) \cdot \varphi'_q((c'_q)^k)}_{\text{product in the ring } R} \end{array} \right). \end{aligned}$$

It is straightforward to verify that this map induces a well-defined map on cohomology group, i.e. that the map

$$\begin{aligned} \Theta: H^p(\mathcal{C}; R) \otimes H^q(\mathcal{C}'; R) &\rightarrow H^{p+q}(\mathcal{C} \otimes \mathcal{C}'; R) \\ ([\varphi_p], [\varphi'_q]) &\mapsto [\Theta(\varphi_p, \varphi'_q)] \end{aligned}$$

is well-defined. Furthermore one can show easily that Θ is natural⁷⁰² with respect to $\mathcal{C}$ and $\mathcal{C}'$.

Now we can formulate the following proposition.

Proposition 41.12. (The Künneth-formula for cochain complexes) *Let $\mathcal{C}$ and $\mathcal{C}'$ be chain complexes. If $\mathcal{C}$ is free and if all homology groups of $\mathcal{C}$ are finitely generated, then there exists a short exact sequence⁷⁰³*

$$0 \rightarrow \bigoplus_{p+q=n} H^p(\mathcal{C}) \otimes H^q(\mathcal{C}') \xrightarrow{\Theta} H^n(\mathcal{C} \otimes \mathcal{C}') \rightarrow \bigoplus_{p+q=n+1} \text{Tor}(H^p(\mathcal{C}), H^q(\mathcal{C}')) \rightarrow 0$$

which is natural with respect to $\mathcal{C}$ and $\mathcal{C}'$.

The proof of this proposition is similar to the proof of Proposition 37.4 and therefore we skip the proof. A detailed proof is given in [Mu1, Chapter 60].

For us the most important application of the previous theorem is the cohomology group of products of topological spaces. The following proposition follows immediately from the Eilenberg-Zilber Theorem 37.3 and Proposition 41.12.

Proposition 41.13. (The Künneth-formula for cohomology groups) *Let X and Y be two topological spaces. If all the homology groups of X are finitely generated, then there exists a short exact sequence*

$$0 \rightarrow \bigoplus_{p+q=n} H^p(X) \otimes H^q(Y) \xrightarrow{\Theta} H^n(X \times Y) \rightarrow \bigoplus_{p+q=n+1} \text{Tor}(H^p(X), H^q(Y)) \rightarrow 0$$

which is natural with respect to the topological spaces X and Y .

We want to use the Künneth-formula 41.13 for cohomology groups to determine the cohomology groups of $S^2 \times S^4$. Before we do so it is convenient to introduce the Kronecker pairing and to prove two lemmas.

Definition. Let C_* be a chain complex of a abelian groups and let R be a ring. We refer to the map

$$\begin{aligned} \langle \cdot, \cdot \rangle: H^n(C; R) \times H_n(C \otimes R) &\rightarrow R \\ ([\delta], [c \otimes r]) &\mapsto \langle [\delta], [c \otimes r] \rangle := \delta(c) \cdot r \end{aligned}$$

as the *Kronecker pairing*.⁷⁰⁴

⁷⁰²In this case this means that for chain maps $f: \mathcal{C} \rightarrow \mathcal{D}$ and $f': \mathcal{C}' \rightarrow \mathcal{D}'$ the following diagram commutes:

$$\begin{array}{ccc} H^p(\mathcal{D}; R) \otimes H^q(\mathcal{D}'; R) & \xrightarrow{\Theta} & H^{p+q}(\mathcal{D} \otimes \mathcal{D}'; R) \\ \downarrow f^* \otimes (f')^* & & \downarrow (f \otimes f')^* \\ H^p(\mathcal{C}; R) \otimes H^q(\mathcal{C}'; R) & \xrightarrow{\Theta} & H^{p+q}(\mathcal{C} \otimes \mathcal{C}'; R). \end{array}$$

⁷⁰³The fact that in the short exact sequence the torsion group appears is *not* a typo, even though one might have initially have thought that perhaps Ext-groups will appear in the statement of the proposition.

⁷⁰⁴If $R = \mathbb{Z}$, then $\langle [\delta], [c] \rangle = \text{ev}([\delta])([c])$ where ev denotes the evaluation map $H^n(C) \rightarrow \text{Hom}(H_n(C), \mathbb{Z})$ introduced on page 680.

The following lemma follows immediately from the definitions.

Lemma 41.14. *Let $f: X \rightarrow Y$ be a map between two topological spaces, let R be a commutative ring, let $c \in H_n(X; R)$ and let $\delta \in H^n(Y; R)$, then*

$$\langle f^*(\delta), c \rangle = \langle \delta, f_*(c) \rangle.$$

Lemma 41.15. *Let M be a connected, oriented, closed n -dimensional topological manifold. Then $H^n(M) \cong \mathbb{Z}$ and there exists a unique element $[M]^* \in H^n(M)$ with $\langle [M]^*, [M] \rangle = 1$.*

Definition. Let M be a connected, oriented, closed n -dimensional topological manifold. We refer to the uniquely determined element $[M]^*$ from Lemma 41.15 as the *dual fundamental class* of M .

Proof. It follows from and the Universal Coefficient Theorem 41.6 for Cohomology Groups together with Proposition 39.22 that the map

$$H^n(M) \xrightarrow{\text{ev}} \text{Hom}(H_n(M), \mathbb{Z})$$

is an isomorphism. Furthermore, by Theorem 39.17 and the definition of a fundamental class we have $H_n(M) \cong \mathbb{Z} \cdot [M]$. Hence there exists a unique element $[M]^* \in H^n(M)$ with $\langle [M]^*, [M] \rangle = 1$. $\square$

Now we turn to promised example $S^2 \times S^4$. We denote by $p: S^2 \times S^4 \rightarrow S^2$ and by $q: S^2 \times S^4 \rightarrow S^4$ the two obvious projection maps. Let $a \in H^2(S^2; \mathbb{Z}) \cong \mathbb{Z}$ and $b \in H^2(S^4; \mathbb{Z}) \cong \mathbb{Z}$ be generators. In exercise sheet 3 we will use the Künneth-formula 41.13 for cohomology groups to show that⁷⁰⁵

$$H^k(S^2 \times S^4; \mathbb{Z}) = \begin{cases} \mathbb{Z} \cdot 1, & \text{if } k = 0, \\ \mathbb{Z} \cdot p^*([S^2]^*), & \text{if } k = 2, \\ \mathbb{Z} \cdot q^*([S^4]^*), & \text{if } k = 4, \\ \mathbb{Z} \cdot [S^2 \times S^4]^*, & \text{if } k = 0, \\ 0, & \text{otherwise.} \end{cases}$$

⁷⁰⁵It follows almost immediately from the Künneth-formula 41.13 for cohomology groups that $H^k(S^2 \times S^4; \mathbb{Z}) \cong \mathbb{Z}$ and $H^k(S^2 \times S^4; \mathbb{Z}) \cong \mathbb{Z}$. But why are $p^*([S^2]^*)$ and $q^*([S^4]^*)$ generators?

42. COHOMOLOGY WITH COMPACT SUPPORT

42.1. Direct limits of directed systems. In this chapter we will at some point use the direct limit of a direct system. This time we will not work with the directed set $(\mathbb{N}, \leq)$ but with more complicated directed sets. We use the opportunity to recall the definitions and we will prove two straightforward lemmas.

First of all, a *directed set* is a set X together with a relation $\leq$ on X which satisfies the following three conditions:

- (1) the relation is reflexive, i.e. for any $x \in X$ we have $x \leq x$.
- (2) the relation is transitive, i.e. for any $x, y, z \in X$ we have

$$x \leq y \text{ and } y \leq z \implies x \leq z.$$

- (3) for any $x, y \in X$ there exists a $z \in X$ with $x \leq z$ and $y \leq z$.

A *direct system of abelian groups over the directed set* $(I, \leq)$ is a family of objects $\{X_i\}_{i \in I}$ in $\mathcal{C}$ indexed by I , together with a family of morphisms $\{f_{ij}: X_i \rightarrow X_j\}$ for all $i, j \in I$ with $i \leq j$ such that the following two conditions are satisfied:

- (1) $f_{ii} = \text{id}_{X_i}$ for all $i \in I$,
- (2) $f_{ik} = f_{jk} \circ f_{ij}$ for all $i, j, k \in I$ with $i \leq j \leq k$.

On page 226 we had shown that the direct limit exists, more precisely we had shown that

$$\varinjlim X_i := \left(\bigsqcup_{i \in I} X_i \right) / \sim \text{ where } x \sim f_{ij}(x) \text{ for all } i \leq j, x \in X_i$$

is a direct limit. Hereby we view $\varinjlim X_i$ as a group as follows: if we are given $a \in X_i$ and $b \in X_j$, by definition of a directed set there exists $k \in I$ with $i \leq k$ and $j \leq k$, and we define

$$[a] + [b] := \underbrace{[f_{ik}(a)]}_{\in X_k} + \underbrace{[f_{jk}(b)]}_{\in X_k} \in \varinjlim X_i.$$

The first lemma is an immediate consequence of the above construction of the direct limit.

Lemma 42.1. *Let $(I, \leq)$ be a directed set and let $(\{G_i\}_{i \in I}, \{f_{ij}: G_i \rightarrow G_j\}_{i \leq j})$ be a direct system of abelian groups over I . Then the following two statements hold:*

- (1) *Every element in $\varinjlim G_i$ lies in the image of one of the maps $G_j \rightarrow \varinjlim G_i$.*
- (2) *Let $a_j \in G_j$. The image of a_j in $\varinjlim G_i$ is zero if and only if there exists a $k \geq j$ such that $f_{jk}(a_j) = 0 \in G_k$.*
- (3) *Let $a_j \in G_j$ and let $a_k \in G_l$. The images of a_j and a_k in $\varinjlim G_i$ agree if and only if there exists an l with $l \geq j$ and $l \geq k$ such that $f_{jl}(a_j) = f_{kl}(a_k) \in G_l$.*
- (4) *If all maps $f_{ij}: G_i \rightarrow G_j$ are isomorphisms, then for any j we have $G_j \cong \varinjlim G_i$.*

The second lemma is useful for simplifying calculations.

Lemma 42.2. *Let $(I, \leq)$ be a directed set and let $(\{G_i\}_{i \in I}, \{f_{ij}: G_i \rightarrow G_j\}_{i \leq j})$ be a direct system of abelian groups over I . Let J be a subset of I with the property.*

- (1) *There exists a unique map*

$$\varinjlim_{j \in J} G_j \longrightarrow \varinjlim_{i \in I} G_i$$

such that for any $j \in J$ the following diagram commutes:

$$\begin{array}{ccc} & G_j & \\ \swarrow & & \searrow \\ \varinjlim_{j \in J} G_j & \longrightarrow & \varinjlim_{i \in I} G_i. \end{array}$$

- (2) *If J has the property that given any $i \in I$ there exists a $j \in J$ with $i \leq j$, then the above map*

$$\varinjlim_{j \in J} G_j \longrightarrow \varinjlim_{i \in I} G_i$$

is an isomorphism.

The first part of the lemma is just an immediate consequence of the definition of a direct limit. We will prove the second part of the lemma in exercise sheet 2.

42.2. The definition of cohomology with compact support. In this section we introduce “cohomology with compact support”, which is an interesting variation on the usual cohomology groups. Later on it will be a key tool in proving the Poincaré Duality Theorem, which is the most important result regarding homology and cohomology groups of topological manifolds.

We start out with the following definition.

Definition. Let X be a topological space and let G be an abelian group. We say that a cochain $\varphi \in C^i(X; G)$ has *compact support* if there exists a compact set $K \subset X$ such that for $\varphi(c)$ for all singular simplices in $X \setminus K$, i.e. if φ vanishes on the image of $C_i(X \setminus K) \rightarrow C_i(X)$.

Lemma 42.3. *Let X be a topological space and let G be an abelian group.*

- (1) *A cochain $\varphi \in C^i(X; G)$ has compact support if and only if there exists a compact subset K such that φ lies in the image of the map $C^i(X, X \setminus K; G) \rightarrow C^i(X; G)$.*
- (2) *The set of cochains that have compact support form a subcomplex $(C_c^*(X; G); \delta)$ of the cochain complex $(C^*(X; G); \delta)$.*

Proof. Let X be a topological space and let G be an abelian group. The first statement of the lemma follows immediately from the definitions.

We turn to the second statement. Given a subset $A \subset X$ we say that a cochain $\varphi \in C^i(X; G)$ *vanishes on A* if $\varphi(c) = 0$ for all singular i -simplices in A , i.e. if it vanishes on the image of $C_i(A) \rightarrow C_i(X)$.

Claim. Let $A, B \subset X$ be two subsets.

- (1) If $\varphi \in C^i(X; G)$ vanishes on A and if $\psi \in C^i(X; G)$ vanishes on A , then for any $r, s \in \mathbb{Z}$ the linear combination $r\varphi + s\psi \in C^i(X; G)$ vanishes on $A \cup B$.
- (2) If $\varphi \in C^i(X; G)$ vanishes on A , then $\delta\varphi \in C^{i+1}(X; G)$ also vanishes on A .

The first statement is obvious. Now we turn to the proof of the second statement. Suppose we are given a cochain $\varphi \in C^i(X; G)$ that vanishes on A . We need to show that $\delta\varphi \in C^{i+1}(X; G)$ also vanishes on A . So let c be a singular $(i+1)$ -simplex in A . Then

$$(\delta\varphi)(c) = \varphi(\partial c) = 0.$$

↑
the image of each summand of ∂c also lies in A

This concludes the proof of the claim.

The lemma is now an immediate consequence of the claim and the observation that given two compact subsets $K, L \subset X$ we have that $(X \setminus K) \cap (X \setminus L) = X \setminus (K \cup L)$ is again the complement of a compact subset of X . $\square$

Definition. Let X be a topological space and let G be an abelian group. We define the n -th cohomology group with compact support $H_c^n(X; G)$ to be the cohomology of the cochain complex given by cochains with compact support, i.e.

$$H_c^n(X; G) := \ker\{\delta_{n+1}: C_c^n(X; G) \rightarrow C_c^{n+1}(X; G)\} / \operatorname{im}\{\delta_n: C_c^{n-1}(X; G) \rightarrow C_c^n(X; G)\}.$$

Remark.

- (1) If X is a compact topological space, then the cohomology groups with compact support are just the usual cohomology groups.
- (2) On page 244 we had introduced the set of ends(X) of a topological space X . It follows immediately from the definitions and the discussion on page 668 that

$$H_c^0(X; \mathbb{Z}) \cong \mathbb{Z}^{\operatorname{ends}(X)}.$$

- (3) If $f: X \rightarrow Y$ is a homeomorphism, then f evidently induces for each $i \in \mathbb{N}_0$ an isomorphism $H^i(Y; G) \rightarrow H^i(X; G)$.
- (4) In exercise sheet 2 we will show that for any path-connected non-compact topological space X and any abelian group G we have $H_c^0(X; G) = 0$.
- (5) We will discuss the question whether cohomology with compact support is functorial in exercise sheet 2.

Given a topological space X we consider the set $\mathcal{K}(X)$ of all compact subsets of X . The set $\mathcal{K}(X)$ together with the relation “ $\subset$ ” defined by inclusion is a directed set. Given two compact subsets $K \subset L$ and $i \in \mathbb{N}_0$ we get induced maps $H^i(X, X \setminus K; G) \rightarrow H^i(X, X \setminus L; G)$.⁷⁰⁶ These cohomology groups now form a direct system over the directed set $\mathcal{K}(X)$

⁷⁰⁶The most difficult part about cohomology is to figure out which way the maps go. In this case $K \subset L$ implies $X \setminus L \subset X \setminus K$ which means that we have a map of pairs $(X, X \setminus L) \rightarrow (X, X \setminus K)$ given by inclusion, which by the discussion on page 669 implies that we do indeed get an induced map $H^i(X, X \setminus K; G) \rightarrow H^i(X, X \setminus L; G)$.

and we can form the direct limit

$$\varinjlim_{\mathcal{K}(X)} H^i(X, X \setminus K; G).$$

For compact subsets $L \subset M$ it follows from Lemma 42.3 that we have maps

$$\begin{array}{ccccc} H^i(X, X \setminus M; G) & & & & \\ \uparrow & \searrow \psi_M & & & \\ & \varinjlim_{\mathcal{K}(X)} H^i(X, X \setminus K; G) & \xrightarrow[\Phi]{\exists!} & & H_c^i(X; G) \\ & \nearrow \psi_L & & & \\ H^i(X, X \setminus L; G) & & & & \end{array}$$

The bigger triangle is easily seen to commute. Since this is the case for all compact subsets $L \subset M$ we get the induced dotted arrow.

The following proposition says that the dotted arrow is in fact an isomorphism. This gives in particular a reformulation of the concept of cohomology with compact support.

Proposition 42.4. *Let X be a topological space and let G be an abelian group. Then the above map*

$$\Phi: \varinjlim_{\mathcal{K}(X)} H^i(X, X \setminus K; G) \rightarrow H_c^i(X; G)$$

is an isomorphism.

Proof. We need to show that the map Φ in the proposition is a bijection.

We first show that it is surjective. So let $[\varphi] \in H_c^i(X; G)$. By definition this means that there exists a compact subset K such that φ vanishes on $C_i(X \setminus K)$, i.e. φ lies in the image of $C^i(X, X \setminus K; G) \rightarrow C^i(X; G)$ which in turn implies that $[\varphi]$ lies in the image of $H^i(X, X \setminus K; G)$. But then $[\varphi]$ also lies in the image of $\varinjlim_{\mathcal{K}(X)} H^i(X, X \setminus K; G)$.

Now we show that the map is injective. So suppose we are given an element c in the kernel of Φ . By Lemma 42.1 (1) it lies in the image of some $H^i(X, X \setminus L; G)$. We pick a representative $\varphi \in C^i(X, X \setminus L; G)$. Since it represents the zero class in $H_c^i(X; G)$ there exists a compactly supported $(i-1)$ -cocycle η with $\delta\eta = \varphi$. We pick a compact subset M of X such that η vanishes on $X \setminus M$. It then also vanishes on $X \setminus (M \cup L)$. But this shows that $[\varphi]$ represents the zero class in $H^i(X, X \setminus (M \cup L); G)$. It follows from Lemma 42.1 (2) that $\psi_L([\varphi]) = c$ is zero. $\square$

We now turn to a key example, namely we consider the cohomology $H_n^c(\mathbb{R}^n; \mathbb{Z})$. We have

$$\begin{array}{ccccc}
 & & \text{Lemma 42.2} & & \vdots \\
 & & \downarrow & & H^k(\mathbb{R}^n, \mathbb{R}^n \setminus \overline{B}_2(0); \mathbb{Z}) \\
 H_c^k(\mathbb{R}^n; \mathbb{Z}) & = & \varinjlim_{K(\mathbb{R}^n)} H^k(\mathbb{R}^n, \mathbb{R}^n \setminus K; \mathbb{Z}) & = & \varinjlim_{s \in \mathbb{N}} H^k(\mathbb{R}^n, \mathbb{R}^n \setminus \overline{B}_1(0); \mathbb{Z}) \\
 & & & & \uparrow \\
 & & & & H^k(\mathbb{R}^n, \mathbb{R}^n \setminus \overline{B}_0(0); \mathbb{Z}) \\
 & \cong & H^k(\mathbb{R}^n, \mathbb{R}^n \setminus \overline{B}_0(0); \mathbb{Z}) & \cong & \begin{cases} \mathbb{Z}, & \text{if } k = n, \\ 0, & \text{otherwise.} \end{cases} \\
 \uparrow & & \uparrow & & \\
 \text{by Corollary 26.14 and Lemma 42.1} & & \text{same argument as in the proof of Lemma 39.9} & &
 \end{array}$$

We draw two interesting conclusions from this example:

- (1) This example shows that cohomology with compact support can be used to show that the topological spaces $\mathbb{R}^n$ and $\mathbb{R}^m$ are not homeomorphic for $n \neq m$.
- (2) The example also shows that cohomology with compact support, in contrast to all other invariants we have studied so far, is not the same for homotopy equivalent topological spaces.

In exercise sheet 2 we will compute cohomology with support for several other topological spaces. In particular we will use cohomology with compact support to show that the topological spaces

$$X := \overline{B}_1(0) \cup x\text{-axis} \quad \text{and} \quad Y := \overline{B}_1(0) \cup x\text{-axis} \cup y\text{-axis}$$

that are illustrated in Figure 42.2 are not homeomorphic.

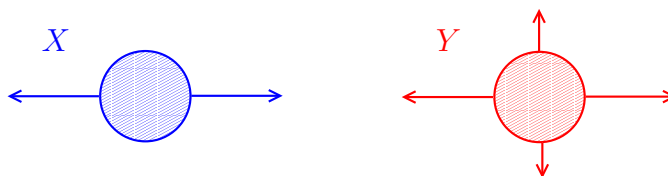


FIGURE 368.

In the previous examples cohomology with compact support turns out to be “bigger” than ordinary cohomology. On the other hand, let us once again consider the topological space

$$X := (\mathbb{R} \cup S^2 \times \mathbb{Z}) / \sim$$

where given $i \in \mathbb{Z}$ we identify $i \in \mathbb{R}$ with $((0, 0, -1), i) \in S^2 \times \mathbb{Z}$. We will see in exercise sheet 2 that

$$H_c^2(X) \cong \mathbb{Z}^{(\mathbb{N})},$$

in particular $H_c^2(X)$ is countable, whereas on page 674 we had seen that

$$H^2(X) \cong \mathbb{Z}^{\mathbb{N}}$$

is uncountable.

We leave the proof of the following proposition as an exercise to the reader.

Proposition 42.5. *Let M be a topological manifold and let G be an abelian group. We write $\mathring{M} := M \setminus \partial M$. For any compact subset $K \subset \mathring{M}$ we have two inclusion induced maps*

$$\begin{array}{c} H^k(\mathring{M}, \mathring{M} \setminus K; G) \xleftarrow{\cong} H^k(M, M \setminus K; G) \rightarrow H^k(M, M \setminus \partial M; G). \\ \uparrow \\ \text{isomorphism by excision for cohomology groups} \end{array}$$

These maps give rise to a homomorphism

$$H_c^k(\mathring{M}; G) = \varinjlim_{K \in \mathcal{K}(\mathring{M})} H^k(\mathring{M}, \mathring{M} \setminus K; G) \rightarrow H^k(M, M \setminus \partial M; G).$$

If M is compact, then this homomorphism is an isomorphism.

Example. We have

$$\begin{array}{ccccc} H_c^k(\mathbb{R}^n; \mathbb{Z}) & \cong & H_c^k(B^n; \mathbb{Z}) & \cong & H^k(\overline{B}^n, S^{n-1}; \mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & \text{if } k = n, \\ 0, & \text{otherwise.} \end{cases} \\ \uparrow & & \uparrow & & \uparrow \\ \text{Lemma 1.4} & & \text{Proposition 42.5} & & \text{same argument as in the proof of Lemma 39.9} \end{array}$$

Thus we recover the above example. In fact the proof of Proposition 42.5 is a generalization of the above calculation.

Finally we return to a more theoretic statement. Suppose U is an open subset of a topological space V , then for each compact subset $K \subset U$ the inclusion $i: U \rightarrow V$ induces a map

$$i^*: H^n(V, V \setminus K) \rightarrow H^n(U, U \setminus K)$$

is an isomorphism by excision for cohomology groups, see page 669. But this implies that for each compact subset $K \subset U$ we have a map

$$(i^*)^{-1}: H^n(U, U \setminus K) \rightarrow H^n(V, V \setminus K).$$

But since $\mathcal{K}(U) \subset \mathcal{K}(V)$ this, together with Lemma 42.2 implies that we also have an induced map

$$i_*: H_c^n(U) = \varinjlim H^n(U, U \setminus K) \rightarrow \varinjlim H^n(V, V \setminus K) = H_c^n(V).$$

Note that, confusingly, the induced map on cohomology with compact support “goes the same direction” as the inclusion map $i: U \rightarrow V$.

We have the following proposition, which can be viewed as an analogue of Proposition 29.1.

Proposition 42.6. *Let $X_i, i \in \mathbb{N}$ be a sequence of topological spaces such that the following holds:*

- (1) *the sequence is nested, i.e. for each $i \in \mathbb{N}$ we have $X_i \subset X_{i+1}$, and*
- (2) *each X_i is open in X_{i+1} .*

We write $X = \bigcup_{i \in I} X_i$. Then for any $n \in \mathbb{N}_0$ the inclusion induced maps $H_c^n(X_i) \rightarrow H_c^n(X)$ induce an isomorphism

$$\lim_{\longrightarrow} H_c^n(X_i) \xrightarrow{\cong} H_c^n(X).$$

We leave the proof as an exercise to the reader.

43. THE DE RHAM-COHOMOLOGY AND SINGULAR COHOMOLOGY

In this chapter we want to find the connection between the de Rham-cohomology groups from Analysis IV and the singular cohomology groups from the present course. We first recall the definition and the properties of alternating forms and the wedge product, and then afterwards the definition of the de Rham-cohomology of a manifold.

43.1. Alternating forms and the wedge-product. An *alternating k -form on a vector space V* is a map

$$\begin{aligned}\omega: V^k &\rightarrow \mathbb{R} \\ (v_1, \dots, v_k) &\mapsto \omega(v_1, \dots, v_k)\end{aligned}$$

with the following properties:

(i) ω is linear in each argument, i.e. for all $v, v' \in V$ and $\lambda \in \mathbb{R}$ we have

$$\begin{aligned}\omega(\dots, v + v', \dots) &= \omega(\dots, v, \dots) + \omega(\dots, v', \dots) \\ \omega(\dots, \lambda v, \dots) &= \lambda \cdot \omega(\dots, v, \dots),\end{aligned}$$

(ii) swapping two arguments flips the signs, i.e. for any $v, v' \in V$ we have

$$\omega(\dots, v, \dots, v', \dots) = -\omega(\dots, v', \dots, v, \dots).$$

We denote the vector space of alternating k -forms on V by $\wedge^k V^*$. A 0-form on V is by definition a real number, i.e. $\wedge^0 V^* = \mathbb{R}$.

Example.

- (1) An alternating 1-form is simply a linear map $V \rightarrow \mathbb{R}$. In particular $\wedge^1 V^* = V^* := \text{Hom}(V, \mathbb{R})$ is the dual space of V .
- (2) For $V = \mathbb{R}^n$ we denote by dx_i the 1-form which is given by

$$dx_i(v_1, \dots, v_n) = v_i.$$

Put differently, $dx_1, \dots, dx_n$ is the basis that is dual to the standard basis $e_1, \dots, e_n$ of $\mathbb{R}^n$.

- (3) If $V = \mathbb{R}^n$, then for each $\lambda \in \mathbb{R}$ the map

$$\begin{aligned}(\mathbb{R}^n)^n &\rightarrow \mathbb{R} \\ (v_1, \dots, v_n) &\mapsto \det(v_1 \dots v_n)\end{aligned}$$

is an alternating n -form and one can show that each alternating n -form on $\mathbb{R}^n$ is of this form, i.e. $\wedge^n(\mathbb{R}^n)^* \cong \mathbb{R}$.

We now recall the wedge product of 1-forms. In the following let V be an n -dimensional real vector space and let $\varphi_1, \dots, \varphi_k \in V^*$ be 1-forms. We consider the map

$$\begin{aligned}\varphi_1 \wedge \dots \wedge \varphi_k: V^k &\rightarrow \mathbb{R} \\ (v_1, \dots, v_k) &\mapsto \det \begin{pmatrix} \varphi_1(v_1) & \dots & \varphi_1(v_k) \\ \vdots & & \vdots \\ \varphi_k(v_1) & \dots & \varphi_k(v_k) \end{pmatrix}.\end{aligned}$$

In Analysis IV we had seen that this is an alternating k -form. We refer to this alternating k -form as the *wedge product* of $\varphi_1, \dots, \varphi_k$.

Now let $\varphi_1, \dots, \varphi_n$ be a basis for V^* and let $k \in \{0, \dots, n\}$. In Analysis IV course we had proved that the wedge products

$$\varphi_{i_1} \wedge \dots \wedge \varphi_{i_k} \quad \text{with} \quad i_1 < i_2 < \dots < i_k$$

form a basis of $\wedge^k V^*$. Furthermore we had seen that $\wedge^k V^* = 0$ for $k > \dim(V)$.

Now we can introduce the wedge product of general alternating forms. Let $\omega \in \wedge^k V^*$ and $\sigma \in \wedge^l V^*$. Then by the above we can write

$$\omega = \sum_{i_1 < \dots < i_k} a_{i_1, \dots, i_k} \varphi_{i_1} \wedge \dots \wedge \varphi_{i_k}$$

and

$$\sigma = \sum_{j_1 < \dots < j_l} b_{j_1, \dots, j_l} \varphi_{j_1} \wedge \dots \wedge \varphi_{j_l}.$$

Now we define

$$\omega \wedge \sigma := \sum_{i_1 < \dots < i_k} \sum_{j_1 < \dots < j_l} a_{i_1, \dots, i_k} b_{j_1, \dots, j_l} \varphi_{i_1} \wedge \dots \wedge \varphi_{i_k} \wedge \varphi_{j_1} \wedge \dots \wedge \varphi_{j_l}.$$

One can show easily that $\omega \wedge \sigma$ is an alternating $k+l$ -form. The following lemma, that we had proved in Analysis IV, summarizes some of the key properties of the wedge product of alternating forms. It is also straightforward to deduce the statements of the lemma from the definitions.

Lemma 43.1. *Let V be a real vector space.*

(1) *For any $\omega \in \wedge^k V^*$ and $\sigma \in \wedge^l V^*$ we have*

$$\omega \wedge \sigma = (-1)^{kl} \sigma \wedge \omega.$$

(2) *The map*

$$\wedge^k V^* \times \wedge^l V^* \rightarrow \wedge^{k+l} V^*$$

is bilinear⁷⁰⁷ and distributive.

(3) *For every $\omega \in \wedge^k V^*$, $\sigma \in \wedge^l V^*$ and $\tau \in \wedge^m V^*$ we have*

$$(\omega \wedge \sigma) \wedge \tau = \omega \wedge (\sigma \wedge \tau).$$

⁷⁰⁷A map $\Theta: U \times V \rightarrow W$ between R -modules is called *bilinear*, if for all $u, u' \in U$, $v, v' \in V$ and $r \in R$ we have

$$\begin{aligned} \Theta(u + u', v) &= \Theta(u, v) + \Theta(u', v) & \text{and} & & \Theta(ru, v) &= r\Theta(u, v), & \text{and} \\ \Theta(u, v + v') &= \Theta(u, v) + \Theta(u, v') & \text{and} & & \Theta(u, rv) &= r\Theta(u, v). \end{aligned}$$

43.2. The de Rham-cohomology of a manifold. What are cohomology groups good for? The universal coefficient theorem 41.6 for cohomology group says that the cohomology group do not contain more information than the homology groups. Furthermore cohomology group have the “disadvantage” that the induced maps go the “wrong” direction which often leads to total confusion.

In this chapter we want to sketch one answer: the cohomology groups with $\mathbb{R}$ -coefficients of a manifold correspond to the de Rham-cohomology groups which we had defined in Analysis IV using differential forms.

We recall several definitions and results from Analysis IV. In the following let M be an n -dimensional manifold. A k -form on M assigns to each point $P \in M$ an alternating k -form on the tangent space $T_P M$. Using charts one can introduce the notions of continuity and smoothness of k -forms. We refer to Analysis IV for details.

We then write

$$\Omega^k(M) := \{\text{smooth } k\text{-forms on } M\}.$$

If $f: M \rightarrow N$ is a smooth map between manifolds, then we had shown in Analysis IV that f induces for each k a map⁷⁰⁸

$$\begin{aligned} f^*: \Omega^k(N) &\rightarrow \Omega^k(M) \\ \omega &\mapsto (f^*\omega: (v_1, \dots, v_k) \in T_P M \mapsto \omega_{f(P)}(df(v_1), \dots, df(v_k))). \end{aligned}$$

We had introduced differential forms in Analysis IV since these are the “right” objects on a manifold that can be integrated. Somewhat more precisely, if M is a compact oriented k -dimensional manifold and if $\omega \in \Omega^k(M)$, then we had introduced the Integral

$$\int_M \omega \in \mathbb{R}.$$

Once again we refer to Analysis IV for details.

In Analysis IV we had furthermore introduced the differential⁷⁰⁹

$$d_k: \Omega^k(M) \rightarrow \Omega^{k+1}(M).$$

The following proposition, that was proved in Analysis IV, summarizes some of the key properties of the differential.

Proposition 43.2.

⁷⁰⁸It is clear that $f^*\omega$ is a k -form on M , what we did have to show is that $f^*\omega$ is in fact a smooth k -form.

⁷⁰⁹Here is a quick summary of the definition. Using a partition of unity it suffices to define the differential for k -forms on an open subset $U \subset \mathbb{R}^n$. We consider a k -form

$$\omega = \sum_{i_1 < \dots < i_k} f_{i_1 \dots i_k} dx_{i_1} \wedge \dots \wedge dx_{i_k}$$

where $f_{i_1 \dots i_k}$ are real-valued smooth functions on U . The differential of ω is then defined as the $(k+1)$ -form

$$d\omega := \sum_{i_1 < \dots < i_k} df_{i_1 \dots i_k} \wedge dx_{i_1} \wedge \dots \wedge dx_{i_k}.$$

- (1) Let M be a manifold. Then for any k the map $d \circ d: \Omega_k(N) \rightarrow \Omega_{k+2}(N)$ is the zero map.
- (2) Let $f: M \rightarrow N$ be a smooth map between two manifolds. Then for any k -form ω on N we have

$$d(f^*\omega) = f^*(d\omega).$$

Put differently, f^* is a cochain map.

One of the key results in Analysis IV was Stokes' Theorem:

Theorem 43.3. (Stokes' Theorem) *If M is a compact oriented k -dimensional manifold with boundary ∂M and if $\omega \in \Omega^{k-1}(M)$, then⁷¹⁰*

$$\int_M d\omega = \int_{\partial M} \omega.$$

Let M be a manifold. As we had just pointed out in Proposition 43.2 the differentials $d_k: \Omega^k(M) \rightarrow \Omega^{k+1}(M)$ have the property that $d_{l+1} \circ d_l = 0$. Thus we obtain a cochain complex

$$0 \rightarrow \Omega^0(M) \xrightarrow{d_0} \Omega^1(M) \xrightarrow{d_1} \Omega^2(M) \xrightarrow{d_2} \dots$$

The k -th de Rham cohomology group of M is now defined as

$$H_{\text{dR}}^k(M) := \ker\{d_k: \Omega^k(M) \rightarrow \Omega^{k+1}(M)\} / \text{im}\{d_{k-1}: \Omega^{k-1}(M) \rightarrow \Omega^k(M)\}.$$

Note that $H_{\text{dR}}^k(M)$ is the quotient of real vector spaces, i.e. $H_{\text{dR}}^k(M)$ is itself a real vector space.

The following proposition summarizes some of the results that we had proved in Analysis IV.

Proposition 43.4. *Let M be an n -dimensional compact manifold. Then the following hold:*

- (1) $\dim(H_{\text{dR}}^0(M))$ equals the number of components of M ,
- (2) $H_{\text{dR}}^k(M) = 0$ for $k > n$.
- (3) If $f: M \rightarrow N$ is a smooth map to another manifold, then

$$\begin{array}{ccc} f^*: H_{\text{dR}}^k(N) & \rightarrow & H_{\text{dR}}^k(M) \\ [\omega] & \mapsto & [f^*\omega] \end{array}$$

is a well-defined map.

- (4) If M is orientable, then $H_{\text{dR}}^n(M) \neq 0$.

Here the first statement followed fairly easily from the definitions and an elementary argument. The second statement is an immediate consequence of the fact that for any vector space V we have $\wedge^k V^* = 0$ for $k > \dim(V)$, see page 694. The third statement is an immediate consequence of Proposition 43.2. The last statement was a consequence of Stokes' Theorem and a straightforward calculation.

⁷¹⁰Here the right-hand side is understood to be zero if $\partial M = \emptyset$.

We now denote by $\mathcal{M}$ the category of manifolds which is given by

$$\begin{aligned}\mathrm{Ob}(\mathcal{M}) &:= \text{all manifolds,} \\ \mathrm{Mor}(M, N) &:= \text{all smooth maps from } M \text{ to } N.\end{aligned}$$

It is straightforward to verify that the maps

$$\begin{aligned}M &\mapsto H_{\mathrm{dR}}^k(M) \\ (f: M \rightarrow N) &\mapsto (f^*: H_{\mathrm{dR}}^k(N) \rightarrow H_{\mathrm{dR}}^k(M))\end{aligned}$$

define a contravariant functor from the category of manifolds to the category $\mathcal{V}$ of real vector spaces.

43.3. The connection between the two cohomology theories. The real singular cohomology $H^n(M; \mathbb{R})$ and the de Rham-cohomology $H_{\mathrm{dR}}^n(M)$ define thus contravariant functors from the category $\mathcal{M}$ of manifolds to the category of real vector spaces. The notation suggests that both functors are related. The few statements we know about de Rham cohomology which we summarized in Proposition 43.4 all hold also for singular cohomology. But is that just a coincidence, or is there some deeper connection?

Thus let M be a manifold. Can we find any connection between $H_{\mathrm{dR}}^n(M)$ and $H^n(M; \mathbb{R})$? We will now see that any element $H_{\mathrm{dR}}^n(M)$ as well as any element $H^n(M; \mathbb{R})$ assign to a closed⁷¹¹ n -dimensional oriented submanifold N of M a real number.

- (1) First suppose we are given a de Rham-cohomology class $[\omega]$. Then we can integrate the n -form ω on the oriented n -submanifold N .⁷¹² and obtain the real number $\int_N \omega \in \mathbb{R}$. We now show that this real number does not depend on the choice of representative. Indeed, suppose $\omega = \tau + d\sigma$ is another representative, then

$$\begin{aligned}\int_N \omega &= \int_N (\tau + d\sigma) = \int_N \tau + \int_N d\sigma = \int_N \tau + \underbrace{\int_{\partial N} \sigma}_{=0 \text{ since } \partial N = \emptyset} = \int_N \tau.\end{aligned}$$

$\uparrow$
 Stokes' Theorem 43.3

- (2) Now we are given a singular cohomology class $[\varphi: C_n(M) \rightarrow \mathbb{R}]$. We denote by $[N] \in H_n(N)$ the fundamental class of N and we consider the real number $\varphi([N])$. One can show that this definition does depend on the choice of the representatives φ .

The above analogy can be somewhat generalized. Let N an oriented n -dimensional manifold. If $f: N \rightarrow M$ is a smooth map then we integrate the pullback $f^*\omega$ on N and we can apply the cohomology class $f^*\varphi$ to $[N]$. In both cases we obtain a real number.

This discussion makes it perhaps slightly more convincing that there should be a connection between these two cohomology groups. In fact the following theorem holds.

⁷¹¹As a reminder, a manifold is called closed, if it is compact and if its boundary is empty.

⁷¹²More precisely, we consider the restriction of ω to N , this is an n -form on an oriented closed n -dimensional manifold. As we had pointed out above we can now integrate this form and we obtain a real number.

Theorem 43.5. *Let M be a manifold. Then for each n there exists an isomorphism*

$$\Phi: H_{\text{dR}}^n(M) \xrightarrow{\cong} H^n(M; \mathbb{R})$$

such that for every closed n -dimensional N , every smooth map $f: N \rightarrow M$ and every class $[\omega] \in H_{\text{dR}}^n(M)$ we have

$$\int_{f^*\omega} N = \underbrace{\Phi([\omega])}_{\in H^n(M; \mathbb{R})} \underbrace{([f_*(N)])}_{\in H_n(M; \mathbb{R})}.$$

Remark. Let M be a manifold. It follows from Theorem 43.5 and the Universal Coefficient Theorem 41.6 for Cohomology Groups together with Proposition 41.3 that for any n we have an isomorphism $H_{\text{dR}}^n(M) \cong \text{Hom}(H_n(M; \mathbb{Z}), \mathbb{R})$.

By now it should be clear that we want not only “some random” isomorphisms, but we want isomorphisms that respect smooth maps between manifolds. To make this precise we recall the notion of a natural transformation that we had already introduced on page 546.

Let $\mathcal{C}$ and $\mathcal{D}$ categories and let $F, G: \mathcal{C} \rightarrow \mathcal{D}$ be two functors. A *natural transformation between F and G* assigns to each $X \in \text{Ob}(\mathcal{C})$ a morphism $\Phi_X: F(X) \rightarrow G(X)$ such that for each morphism $f: X \rightarrow Y$ in $\mathcal{C}$ the following diagram commutes:

$$\begin{array}{ccc} F(X) & \xrightarrow{F_*(f)} & F(Y) \\ \Phi_X \downarrow & & \downarrow \Phi_Y \\ G(X) & \xrightarrow{G_*(f)} & G(Y). \end{array}$$

With this definition we can now formulate the following proposition.

Theorem 43.6. *There exists a natural transformation between the functors*

$$\begin{array}{ccc} \mathcal{M} & \rightarrow & \mathcal{V} \\ M & \mapsto & H_{\text{dR}}^n(M) \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{M} & \rightarrow & \mathcal{V} \\ M & \mapsto & H^n(M; \mathbb{R}). \end{array}$$

The proof of Theorems 43.5 and 43.6 goes well beyond what we can cover in this course. The proof is for example given in [Wa, p. 205]. In [Wa] the equivalence of de Rham cohomology and singular cohomology is proved by relating both to another type of cohomology theory, namely sheaf cohomology. An alternative, more direct proof is given in [Lü, Kapitel 15].

We now use Theorem 43.5 to prove the following proposition.

Proposition 43.7. *Let M be a connected manifold, let P be a base point and furthermore let $\phi \in H^1(M; \mathbb{Z}) = \text{Hom}(\pi_1(M, P), \mathbb{Z})$.⁷¹³ Then there exists a smooth map $g: M \rightarrow S^1$ with $g(P) = 1$ such that*

$$g_* = \phi: \pi_1(M, P) \rightarrow \pi_1(S^1, 1) = \mathbb{Z}.$$

⁷¹³Here we use the isomorphisms from Proposition 41.8 to identify the groups $H^1(M; \mathbb{Z})$ and $\text{Hom}(\pi_1(M), \mathbb{Z})$.

Sketch of proof. Let M be a connected manifold. We fix a base point $P \in M$. Let $\phi \in H^1(M; \mathbb{Z})$. By Theorem 43.5 there exists a closed 1-form ω on M such that for every smooth map $f: S^1 \rightarrow M$ we have

$$\int_{S^1} f^* \omega = \phi(f_*([S^1])).$$

Since every piecewise smooth map $S^1 \rightarrow M$ can be approximated by smooth maps $S^1 \rightarrow M$ we see that the same conclusion also holds for maps $S^1 \rightarrow M$ that are piecewise smooth.⁷¹⁴ Below we will use the following consequence: let $\delta: [0, 1] \rightarrow M$ be a piecewise smooth loop in P , i.e. we have $\delta(0) = \delta(1)$. Then

$$\int_{[0,1]} \delta^* \omega = \phi(\underbrace{[\delta]}_{\in \pi_1(M, P)}).$$

We define

$$\begin{aligned} g: M &\rightarrow \mathbb{R}/\mathbb{Z} \\ Q &\mapsto \int_{[0,1]} \gamma^* \omega, \quad \text{where } \gamma \text{ is a piecewise smooth path}^{715} \text{ from } P \text{ to } Q. \end{aligned}$$

We claim that the map g , followed by the diffeomorphism $\mathbb{R}/\mathbb{Z} \rightarrow S^1$ given by $[t] \mapsto e^{2\pi i t}$ has the required properties. It remains to prove the following three statements:

- (1) The map g is well-defined.
- (2) The map g is smooth.
- (3) We have $g_* = \phi: \pi_1(M, P) \rightarrow \pi_1(S^1, 1) = \mathbb{Z}$.

Now we turn to the proofs of the statements.

- (1) Let Q in M . Suppose we are given two piecewise smooth paths β and γ from P to Q . Then

$$\begin{array}{ccccccc} \int_{[0,1]} \beta^* \omega & - & \int_{[0,1]} \gamma^* \omega & = & \int_{[0,1]} (\beta * \gamma)^* \omega & = & \phi([\beta * \gamma]) \in \mathbb{Z} \\ & & & & \uparrow & & \uparrow \\ & & & & \text{follows from the definitions} & & \text{by the above discussion} \end{array}$$

This shows that g is indeed well-defined.

- (2) Let $Q \in M$. We need to show that g is smooth in Q . We pick a path γ from P to Q . We pick a chart $\Phi: U \rightarrow B_r(0)$ around Q with $\Phi(Q) = 0$. Given a point $R \in U$ we connect it to Q via a radial path coming from the open ball $B_r(0)$ and we concatenate it with the path γ . Using these paths it is now relatively straightforward to show that g is smooth in U . We leave the verification of the details to the reader.

⁷¹⁴This statement can be proved in an elementary, albeit slightly painful way. We leave it as an exercise to fill in the technical details.

⁷¹⁵We had seen in exercise sheet 14 of Algebraic Topology I that any two points on a connected manifold can be connected by a piecewise smooth path.

- (3) We pick a random element in $\pi_1(M, P)$ and we represent it by a piecewise smooth loop $\delta: [0, 1] \rightarrow M$. We make the usual identification $\pi_1(\mathbb{R}/\mathbb{Z}, 0) = \pi_1(S^1, 1) = \mathbb{Z}$. Then

$$\begin{aligned} & \text{by Theorem 6.16} \\ & \downarrow \\ g_*([\delta]) &= \text{endpoint of the lift of } g \circ \delta \text{ to the universal cover } \mathbb{R} \text{ of } \mathbb{R}/\mathbb{Z} \\ &= \text{endpoint of the path } t \mapsto \int_{[0,t]} \delta^* \omega = \int_{[0,1]} \delta^* \omega = \phi([\delta]). \end{aligned}$$

We have thus shown that $g_* = \phi: \pi_1(M, P) \rightarrow \pi_1(S^1, 1) = \mathbb{Z}$. $\square$

43.4. The wedge product of differential forms. For a vector space V we had already introduced on page 694 the wedge product

$$\wedge^k V^* \times \wedge^l V^* \rightarrow \wedge^{k+l} V^*.$$

Using charts we can for each manifold M also define a map

$$\wedge: \Omega^k(M) \times \Omega^l(M) \rightarrow \Omega^{k+l}(M).$$

We refer to Analysis IV for details. In Analysis IV we had also shown that for any k -form ω and any l -form σ on a manifold we have

$$d(\omega \wedge \sigma) = d\omega \wedge \sigma + (-1)^k \omega \wedge d\sigma.$$

This formula together with the fact that $d^2 = 0$ has the following surprising consequence: The map

$$\begin{aligned} \wedge: H_{\text{dR}}^k(M) \times H_{\text{dR}}^l(M) &\rightarrow H_{\text{dR}}^{k+l}(M) \\ ([\omega], [\sigma]) &\mapsto [\omega] \wedge [\sigma] := [\omega \wedge \sigma] \end{aligned}$$

is well-defined. It follows from Lemma 43.1 that the map is bilinear and distributive. We write⁷¹⁶

$$H_{\text{dR}}(M) := \bigoplus_{i=0}^{\infty} H_{\text{dR}}^i(M).$$

If $x \in H_{\text{dR}}(M)$ lies in a summand $H_{\text{dR}}^d(M)$, then we call x *homogeneous* and we refer to $\dim(x) := d$ as the *dimension of x* . We consider the following map

$$\begin{aligned} H_{\text{dR}}(M) \times H_{\text{dR}}(M) &\rightarrow H_{\text{dR}}(M) \\ \left(\sum_{i=0}^{\infty} x_i, \sum_{j=0}^{\infty} y_j \right) &\mapsto \sum_{i,j=0}^{\infty} x_i \wedge y_j. \end{aligned}$$

It follows from Lemma 43.1 that the map is associative. The constant function⁷¹⁷ $f: M \rightarrow \mathbb{R}$, $x \mapsto 1$, is furthermore a multiplicatively neutral element. Thus we see that $H_{\text{dR}}(M)$

⁷¹⁶Since the cohomology groups vanish beyond the dimension of M this is in fact a finite direct sum of vector spaces.

⁷¹⁷As a reminder: zero-dimensional alternating forms on a manifold M are exactly the same as real-valued functions on M .

together with this product map forms a ring.⁷¹⁸ The zero element in this ring is hereby given by the zero function on M .

If $f: M \rightarrow N$ is a map between manifolds then one can now show that for any choice of $\omega \in \Omega^k(N)$ and $\sigma \in \Omega^l(N)$ the following equality holds:

$$(f^*\omega \wedge f^*\sigma) = f^*(\omega \wedge \sigma).$$

The map f thus induces a *ring homomorphism* $f^*: H_{\text{dR}}(N) \rightarrow H_{\text{dR}}(M)$. In fact the maps

$$\begin{aligned} M &\mapsto H_{\text{dR}}^k(M) \\ (f: M \rightarrow N) &\mapsto (f^*: H_{\text{dR}}^k(N) \rightarrow H_{\text{dR}}^k(M)) \end{aligned}$$

define a contravariant functor from the category of manifolds to the category of rings.

Example. We consider the 1-form dx and dy on $\mathbb{R}^2$ which are given by⁷¹⁹ $dx(v, w) = v$ and $dy(v, w) = w$. Furthermore we identify the 2-torus T with the quotient space $\mathbb{R}^2/\mathbb{Z}^2$ and we denote the projection $\mathbb{R}^2 \rightarrow T = \mathbb{R}^2/\mathbb{Z}^2$ by p . We had seen on page 248 of Analysis IV that dx and dy induce 1-forms on T which we also denote by dx and dy .⁷²⁰ The 1-form dx and dy are closed, i.e. $d(dx) = d(dy) = 0$.⁷²¹ It follows furthermore from Stokes' Theorem 43.3 that dx and dy define non-trivial elements in $H_{\text{dR}}^1(T)$.⁷²² If we integrate the 2-form $dx \wedge dy$ on T with the standard orientation, then we had seen on page 248 of Analysis IV that we obtain 1. It follows in particular from Stokes' Theorem 43.3 that $dx \wedge dy$ is not exact i.e. there exists no 1-form ω such that $dx \wedge dy = d\omega$. Thus it follows that $dx \wedge dy$ represents a non-trivial element in $H_{\text{dR}}^2(T)$.

It follows in particular that dx and dy define non-trivial elements in $H_{\text{dR}}^1(T)$ and that

$$[dx] \wedge [dy] = [dx \wedge dy] \neq -[dx \wedge dy] = [dy \wedge dx] = [dy] \wedge [dx].$$

Thus we see that $H_{\text{dR}}(T)$ is a non-commutative ring.

We have just in the example seen that in general the de Rham cohomology ring is not commutative. We want to study the (non-) commutativity somewhat more carefully. Let φ be an alternating k -form and let ψ be an alternating l -form. Then it follows from the

⁷¹⁸Hereby we mean by a ring an abelian group together with a product structure which is associative, distributive and that admits a neutral element. We do not demand that a ring is commutative.

⁷¹⁹Hereby we identify $T_P\mathbb{R}^2$ as always with $\mathbb{R}^2$.

⁷²⁰More precisely, we had shown that there exists a unique 1-form dx on T such that for each point $P \in \mathbb{R}^2$ and $v \in T_P\mathbb{R}^2$ we have $dx_P(v) = dx_{p(P)}(p_*(v))$. Of course the same statement also holds for dy .

⁷²¹This is clear for dx and dy viewed as 1-forms on $\mathbb{R}^2$. But the projection map $p: \mathbb{R}^2 \rightarrow T = \mathbb{R}^2/\mathbb{Z}^2$ is a local diffeomorphism. Since the differential commutes with smooth maps we see that also that $d(dx) = d(dy) = 0$ on $\mathbb{R}^2/\mathbb{Z}^2$.

⁷²²Indeed, if we integrate dx along the closed oriented submanifold $X := \mathbb{R}/\mathbb{Z} \times 0 \in T = \mathbb{R}^2/\mathbb{Z}^2$, then we obtain $\int_X dx = 1$. On the other hand, if $dx = d\omega$ for some 0-form on T , then we would get from Stokes' Theorem 43.3 that $\int_X dx = \int_X d\omega = \int_{\partial X} \omega$, but this is zero, since $\partial X = \emptyset$.

definition of the wedge product, and we will prove it in exercise sheet 13, that

$$\varphi \wedge \psi = (-1)^{kl} \psi \wedge \varphi.$$

It follows direct from the definitions that for homogeneous elements a and b in $H_{\text{dR}}(M)$ the following equality holds:

$$a \wedge b = (-1)^{\dim(a) \dim(b)} b \wedge a.$$

This discovery of a ring structure on de Rham cohomology raises the question whether one can also define such a ring structure using singular cohomology and whether such a ring structure exists on singular cohomology for more general spaces. We will answer these questions in the next chapter.

44. THE CUP-PRODUCT AND THE CAP-PRODUCT

In this chapter we will introduce the cup-product on cohomology groups. This cup-product is perhaps the key reason why cohomology groups are of great interest.

44.1. The definition of the cup-product. The discussion in Chapter 43.4 suggests that there should exist a ring structure on the cohomology group of a manifold, and perhaps also, on the cohomology group of any topological space. Now let X be a topological space and let R be a commutative ring. We consider

$$H^*(X; R) := \bigoplus_{i=0}^{\infty} H^i(X; R).$$

We want to define a map

$$H^*(X; R) \times H^*(X; R) \rightarrow H^*(X; R)$$

which turns the abelian group $H^*(X; R)$ into a ring.⁷²³ It follows from the Eilenberg-Zilber Theorem 37.3 and the discussion on page 683 that for each p, q there exists a natural⁷²⁴ map

$$\Theta: H^p(X; R) \otimes H^q(X; R) \rightarrow H^{p+q}(X \times X; R).$$

We can “add up” these maps for all p, q and we obtain a natural map

$$\Theta: H^*(X; R) \otimes H^*(X; R) \rightarrow H^*(X \times X; R),$$

which is bilinear.⁷²⁵ This is nice, but we really would like to end up in the cohomology of X and not in the cohomology of $X \times X$. Furthermore, as of right now the same discussion works for homology groups consider and we obtain a bilinear map

$$\Theta: H_*(X; R) \otimes H_*(X; R) \rightarrow H_*(X \times X; R).$$

We now have to find a way how to go back from the topological space $X \times X$ to the original space X . In general there does not exist an “interesting”⁷²⁶ map $X \times X \rightarrow X$. But

⁷²³Hereby we want of course a ring structure which is non-zero for at least some topological spaces.

⁷²⁴In this case *natural* means that for every map $f: X \rightarrow Y$ the following diagram commutes:

$$\begin{array}{ccc} H^p(Y; R) \otimes H^q(Y; R) & \xrightarrow{\Theta} & H^{p+q}(Y \times Y; R) \\ \downarrow f^* \times f^* & & \downarrow (f \times f)^* \\ H^p(X; R) \otimes H^q(X; R) & \xrightarrow{\Theta} & H^{p+q}(X \times X; R). \end{array}$$

⁷²⁵It follows easily from the definition of $\Theta: H^*(X; R) \otimes H^*(X; R) \rightarrow H^*(X \times X; R)$ that the map is bilinear.

⁷²⁶The projection from $X \times X$ onto one of the two factors does not count as “interesting”, since it does not “mix” the two factors.

there does exist an interesting map in the opposite direction, namely the diagonal map

$$\begin{aligned} d: X &\rightarrow X \times X \\ x &\mapsto (x, x). \end{aligned}$$

This diagonal map d induces a map

$$d_*: H_*(X; R) \rightarrow H_*(X \times X; R)$$

on homology, and, much more interesting in our context, it also induces a map

$$d^*: H^*(X \times X; R) \rightarrow H^*(X; R).$$

on cohomology. Now we define the *cup-product*⁷²⁷ on the cohomology as the composition

$$H^p(X; R) \otimes H^q(X; R) \xrightarrow{\Theta} H^{p+q}(X \times X; R) \xrightarrow{d^*} H^{p+q}(X; R).$$

For $a \in H^p(X; R)$ and $b \in H^q(X; R)$ we denote the image of (a, b) by $a \cup b \in H^{p+q}(X; R)$ and we refer to $a \cup b$ as the *cup-product of a and b* . All these maps put together give us a map

$$\cup: H^*(X; R) \otimes H^*(X; R) \rightarrow H^*(X; R),$$

which is evidently bilinear.

For calculations of homology and cohomology groups we had seen that often it is more convenient to work with cellular homology, since cellular chain complexes are much smaller than singular chain complexes. We will now introduce the cup-product using cellular chain complexes. More precisely, let X be a CW-complex which has only finitely many cells in each dimension. By the Cellular Approximation Theorem 22.19 there exists a cellular approximation $d^{\text{CW}}: X \rightarrow X \times X$ of the diagonal map $d: X \rightarrow X \times X$. In Proposition 37.2 we had seen that there exists a canonical isomorphism

$$\Phi_{XX}: C_*^{\text{CW}}(X) \otimes C_*^{\text{CW}}(X) \xrightarrow{\cong} C_*^{\text{CW}}(X \times X)$$

of chain complexes. Given a commutative ring R we denote by

$$\Theta: H_{\text{CW}}^*(X; R) \otimes H_{\text{CW}}^*(X; R) \rightarrow H_{\text{CW}}^*(X \times X; R)$$

the induced map on cohomology groups. The cup-product on cellular cohomology is then defined as the composition

$$H_{\text{CW}}^*(X; R) \otimes H_{\text{CW}}^*(X; R) \xrightarrow{\Theta} H_{\text{CW}}^*(X \times X; R) \xrightarrow{d_{\text{CW}}^*} H_{\text{CW}}^*(X; R).$$

It follows easily from the definitions that, using the canonical isomorphism $H_{\text{CW}}^*(X; R) \cong H^*(X; R)$ from page 669, the two definitions of the cup-product agree for CW-complexes.

In the following proposition we summarize some of the key properties of the cup-product.

Proposition 44.1. *Let X be a topological space and R be a commutative ring. Then the following statements hold:*

⁷²⁷The name “cup-product” comes from the notation ‘ $a \cup b$ ’ for the composition since $\cup$ looks like a cup.

- (1) The abelian group $H^*(X; R)$ together with the composition

$$\cup: H^*(X; R) \otimes H^*(X; R) \rightarrow H^*(X; R)$$

is a ring where the multiplicative neutral element is represented by the map $f: C_0(X) \rightarrow \mathbb{Z}$ which assigns to each point in X the value 1.

- (2) For homogeneous elements⁷²⁸ a and b in $H^*(M; R)$ we have that

$$a \cup b = (-1)^{\dim(a) \dim(b)} b \cup a.$$

Remark. The cup-product is not commutative, but only commutative up to a sign that depends on the dimensions of a and b . This type of commutativity is sometimes referred to as “supercommutativity”.

Proof. Let X be a topological space. To simplify the notation we only consider the case $R = \mathbb{Z}$.

- (1) It follows from the bilinearity of $H^p(X) \otimes H^q(X) \xrightarrow{\cup} H^{p+q}(X \times X)$ that the cup-product is distributive.
- (2) It follows easily from the definitions that the map ++++++ is that so? clearer with Lemma 44.6 ++++++ $f: C_0(X) \rightarrow \mathbb{Z}$ which assigns to each point in X the value 1 represents an element in $H^0(X; R)$ which is a neutral element with respect to the cup-product.
- (3) We prove the associativity of the cup-product only in the special case that X is a CW-complex which admits only finitely many cells in each dimension. For the remainder of the proof we mean by a CW-complex a CW-complex which has in each dimension only finitely many cells.

As a reminder, for two CW-complexes U and V we introduced in Chapter 37.2 an isomorphism

$$\Phi_{UV}: C_*^{\text{CW}}(U) \otimes C_*^{\text{CW}}(V) \xrightarrow{\cong} C_*^{\text{CW}}(U \times V)$$

introduced. Now let X, Y and Z CW-complexes. It follows easily from the definitions that the following diagram commutes:

$$\begin{array}{ccc} C_*^{\text{CW}}(X) \otimes C_*^{\text{CW}}(Y) \otimes C_*^{\text{CW}}(Z) & \xrightarrow{\text{id}_X \otimes \Phi_{YZ}} & C_*^{\text{CW}}(X) \otimes C_*^{\text{CW}}(Y \times Z) \\ \downarrow \Phi_{XY} \otimes \text{id}_Z & & \downarrow \Phi_{X, Y \times Z} \\ C_*^{\text{CW}}(X \times Y) \otimes C_*^{\text{CW}}(Z) & \xrightarrow{\Phi_{X \times Y, Z}} & C_*^{\text{CW}}(X \times Y \times Z). \end{array}$$

We now apply this observation to $X = Y = Z$, then one obtains relatively easily from the definition of the cellular cup-product that it is associative.

⁷²⁸Homogeneous elements in $H^*(M; R)$ are of course defined the same way as we defined homogeneous elements in the de Rham-cohomology of a manifold.

(4) We consider the maps

$$\begin{aligned} w_{pq}: C_p(X) \otimes C_q(Y) &\rightarrow C_q(Y) \otimes C_p(X) \\ \sum_{i=1}^r c_i \otimes d_i &\mapsto \sum_{i=1}^r d_i \otimes c_i. \end{aligned}$$

The map is evidently an isomorphism of abelian groups. On the other hand it follows from the definition of the differential in the tensor product of chain complexes⁷²⁹ that the maps w_{pq} do *not* define a chain map $C_*(X) \otimes C_*(Y) \rightarrow C_*(Y) \otimes C_*(X)$. But one can show easily that the maps $(-1)^{pq} w_{pq}$ define a chain map $C_*(X) \otimes C_*(Y) \rightarrow C_*(Y) \otimes C_*(X)$. $\square$

From the definitions and the naturality of the maps in the definition of the cup-product one can easily deduce the following lemma.

Lemma 44.2. *Let $f: X \rightarrow Y$ be a map between topological spaces and let R be a commutative ring. Then*

$$f^*: H^*(Y; R) \rightarrow H^*(X; R)$$

is a ring homomorphism, i.e. for any $a \in H^k(Y; R)$ and $b \in H^l(Y; R)$ we have

$$f^*(a) \cup f^*(b) = f^*(a \cup b).$$

Let R be a commutative ring. One can now easily verify that the maps

$$\begin{aligned} X &\mapsto H^*(X; R) \\ (f: X \rightarrow Y) &\mapsto (f^*: H^*(Y; R) \rightarrow H^*(X; R)) \end{aligned}$$

define a contravariant functor from the category $\mathcal{T}$ of topological spaces to the category $\mathcal{R}$ of rings.⁷³⁰ As a reminder, we denote by $\mathcal{M}$ the category of manifolds and we denote by $\mathcal{V}$ the category of real vector spaces. Then

$$\begin{aligned} M &\mapsto H_{\text{dR}}^k(M) \\ (f: M \rightarrow N) &\mapsto (f^* H_{\text{dR}}^k(N) \rightarrow H_{\text{dR}}^k(M)) \end{aligned}$$

is a contravariant functor from the category of manifolds to the category of rings.

If we assign to a manifold its k -th real cohomology groups, then we had seen in Theorem 43.6 that it is isomorphic to the k -th de Rham-cohomology group. Now we can generalize Theorem 43.6.

⁷²⁹As a reminder, for $c_p \in C_p$ and $c'_q \in C'_q$ the boundary map is given by

$$\bar{p}(c_p \otimes c'_q) = \partial c_p \otimes c'_q + (-1)^p c_p \otimes \partial' c'_q.$$

⁷³⁰The category $\mathcal{R}$ of rings is of course given as follows:

$$\begin{aligned} \text{Ob}(\mathcal{R}) &:= \text{all rings,} \\ \text{Mor}(R, S) &:= \text{all ring homomorphisms of } R \text{ to } S. \end{aligned}$$

Theorem 44.3. *There exists a natural transformation between the functors*

$$\begin{array}{ccc} \mathcal{M} & \rightarrow & \mathcal{R} \\ M & \mapsto & H_{\text{dR}}^*(M) \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{M} & \rightarrow & \mathcal{R} \\ M & \mapsto & H^*(M; \mathbb{R}) \end{array}$$

The proof of Theorem 44.3 is of course also well beyond the scope of this course. We refer to [Wa, Theorem 5.45] for a proof of the theorem.

44.2. Examples of the cup-product. In this section we want to determine the cup-product of several manifolds. We start out with determining the cup-product on the 2-dimensional torus. Thus let $T = S_a^1 \times S_b^1$ be the 2-dimensional torus. We consider $S_a^1 = S^1 = [0, 1]/\sim$ as a CW-complex with one 0-cell $*$ and one 1-cell a , precisely the same way we consider $S_b^1 = S^1$ as a CW-complex with one 0-cell $*$ and one 1-cell b . We consider $S_a^1 \times S_b^1$ as a CW-complex with the product CW-structure, i.e. $S_a^1 \times S_b^1$ has one 0-cell $* \times *$, two 1-cells $a \times *$ and $* \times b$, and one 2-cell $a \times b$. If there is no danger of confusion we denote for sake of simplicity the point $* \times *$ with $*$ and we denote the 1-cell $a \times *$ by a and we denote $* \times b$ by b . The boundary map in the corresponding cellular chain complexes are the zero maps, i.e. the cellular chain complex is of the following form:⁷³¹

$$C_*(S_a^1 \times S_b^1): 0 \rightarrow \mathbb{Z} \cdot (a \times b) \xrightarrow{0} \mathbb{Z} \cdot a \oplus \mathbb{Z} \cdot b \xrightarrow{0} \mathbb{Z} \cdot * \rightarrow 0.$$

We now denote by $a^*, b^* \in \text{Hom}(C_1(T), \mathbb{Z})$ the basis that is dual to the base $a, b \in C_1(T)$. Put differently a^* and b^* are the homomorphisms defined by

$$a^*(a) = 1 \text{ and } a^*(b) = 0 \text{ and } b^*(a) = 0 \text{ and } b^*(b) = 1.$$

Since the boundary maps in the cellular chain complex are the zero maps the boundary maps in the dual complex $\text{Hom}(C_*(T), \mathbb{Z})$ are also the zero maps. In particular a^* and b^* are cocycles and a^* and b^* form a basis of $H^1(T)$.

The same way we denote by $(a \times b)^*: C_2(T) \rightarrow \mathbb{Z}$ the map which is given by $(a \times b)^*(a \times b) = 1$. Then as above $(a \times b)^*$ is a generator of $H^2(T) \cong \mathbb{Z}$. Therefore if we want to determine the cup-product $x \cup y$ for $x, y \in H_1(T)$ it suffices to determine $(x \cup y)(a \times b)$. Thus we want to determine the following matrix:

$$\begin{pmatrix} (a^* \cup a^*)(a \times b) & (a^* \cup b^*)(a \times b) \\ (b^* \cup a^*)(a \times b) & (b^* \cup b^*)(a \times b) \end{pmatrix}.$$

Now we consider

$$T_1 = T = S_{a_1}^1 \times S_{b_1}^1 \quad \text{and} \quad T_2 = T = S_{a_2}^1 \times S_{b_2}^1,$$

and we use the same notation as above for the cells of $S_{a_i}^1$ and $S_{b_i}^1$.

The diagonal map $d: T \rightarrow T \times T = T_1 \times T_2$ is not a cellular map. Therefore we need to replace d by a cellular map g which is homotopic to d . By the Cellular Approximation

⁷³¹In this chapter we work throughout with the cellular chain complexes, for sake of simplicity we denote these chain complexes by $C_*(T)$ instead of $C_*^{\text{CW}}(T)$.

Theorem 22.19 the diagonal map $d_a: S_a^1 \rightarrow S_{a_1}^1 \times S_{a_2}^1$ is homotopic to the cellular map

$$\begin{aligned} d_a^{\text{CW}}: S_a^1 = [0, 1]/\sim &\rightarrow S_{a_1}^1 \times S_{a_2}^1 = [0, 1]/\sim \times [0, 1]/\sim \\ t &\mapsto \begin{cases} (0, 2t), & \text{if } t \in [0, \frac{1}{2}] \\ (1, 2t - 1), & \text{if } t \in [\frac{1}{2}, 1] \end{cases} \end{aligned}$$

We define d_b^{CW} precisely the same way. Then $d^{\text{CW}} = d_a^{\text{CW}} \times d_b^{\text{CW}}: T \rightarrow T \times T$ is a cellular approximation of the diagonal map $d: T \rightarrow T \times T$. We obtain that

$$(f_a \times f_b)(a \times b) = (a_1 + a_2) \times (b_1 + b_2).$$

It follows from the definitions that

$$\begin{aligned} (a^* \cup a^*)(a \times b) &= \Phi_{TT}(a^* \otimes a^*)(d^{\text{CW}}(a \times b)) \\ &= (a_1 \times a_2)^*((a_1 + a_2) \times (b_1 + b_2)) = 1 \times 0 = 0. \end{aligned}$$

The same way one sees that

$$\begin{aligned} (b^* \cup b^*)(a \times b) &= \Phi_{TT}(b^* \otimes b^*)(d^{\text{CW}}(a \times b)) \\ &= (b_1 \times b_2)^*((a_1 + a_2) \times (b_1 + b_2)) = 0, \end{aligned}$$

and

$$\begin{aligned} (a^* \cup b^*)(a \times b) &= \Phi_{TT}(a^* \otimes b^*)(d^{\text{CW}}(a \times b)) \\ &= (a_1 \times b_2)^*((a_1 + a_2) \times (b_1 + b_2)) = 1. \end{aligned}$$

Furthermore we have

$$\begin{aligned} (b^* \cup a^*)(a \times b) &= \Phi_{TT}(b^* \otimes a^*)(d^{\text{CW}}(a \times b)) \\ &= (b_1 \times a_2)^*((a_1 + a_2) \times (b_1 + b_2)) = -1. \end{aligned}$$

Thus we see that with respect to the bases $\{a, b\}$ for $H_1(T)$ and $\{a \times b\}$ for $H_2(T)$ the cup-product is represented by the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Thus we have shown that the cup-product in general non-zero and that it is in general not commutative.⁷³²

More generally we have the following lemma.

Lemma 44.4. *Let $n \in \mathbb{N}$. For $i = 1, 2$ we denote by $p_i: S^1 \times S^1 \rightarrow S^1$ the projection onto the i -th term. With respect to the basis $p_1^*([S^n]^*), p_2^*([S^n]^*)$ for $H^n(S^n \times S^n)$ and $[S^n \times S^n]^*$ for $H^{2n}(S^n \times S^n)$ the cup-product is represented by the matrix*

$$\begin{pmatrix} 0 & 1 \\ (-1)^n & 0 \end{pmatrix}.$$

Proof. We had just provided the proof for $n = 1$. The higher dimensions are proved with an almost identical argument. We leave the details to the reader. $\square$

⁷³²On page 272 of Analysis IV we studied the wedge-product on the de Rham-cohomology of the torus $S^1 \times S^1$. We showed that $H_{\text{dR}}^1(S^1 \times S^1)$ has dimension at least 2 (we had no tools for showing that the dimension is precisely two) and we showed that on this subspace $\mathbb{R}^2$ the wedge-product was given by the above matrix.

The cohomology rings contain more information than the homology or the cohomology groups. For example the 2-torus and $S^1 \wedge S^1 \wedge S^2$ have the same homology groups and the same cohomology groups. We had just seen that the cup-product in the first case is non-zero, whereas we will see in exercise sheet 8 that it vanishes in the second case.⁷³³

In Proposition 33.16 we had seen that there exists a degree-one map from $S^2 \times S^2$ onto the 4-dimensional sphere S^4 . In Question 33.22 we had asked whether there exists a degree-one map from the 4-dimensional sphere to $S^2 \times S^2$. We can now answer this question in the negative.

Lemma 44.5. *There is no map $f: S^4 \rightarrow S^2 \times S^2$ of non-zero degree, i.e. there is no map such that $f_*([S^4]) \in H_4(S^2 \times S^2)$ is non-trivial.*

Proof. Thus let $f: S^4 \rightarrow S^2 \times S^2$ be a map. We had just proved in Lemma 44.4 that there exist elements $a, b \in H^2(S^2 \times S^2)$ such that $a \cup b = [S^2 \times S^2]^* \in H^4(S^2 \times S^2)$. It follows that

$$\begin{array}{rclcl}
 \deg(f) & = & \langle [S^2 \times S^2]^*, [f_*([S^4])] \rangle & = & \langle f^*([S^2 \times S^2]^*), [S^4] \rangle \\
 & \uparrow & & \uparrow & \\
 & \text{since } \langle [S^2 \times S^2]^*, [S^2 \times S^2] \rangle = 1 & \text{Lemma 41.14} & & \\
 & = & \langle f^*(a \cup b), [S^4] \rangle & = & \langle f^*a \cup f^*b, [S^4] \rangle = 0. \\
 & \uparrow & \uparrow & & \\
 & \text{by Lemma 44.2} & \text{since } f^*a, f^*b \in H^2(S^4) = 0 & & \square
 \end{array}$$

Remark. In Question 33.23 we had asked whether there exists a degree-one map from $S^1 \times S^2$ to $\mathbb{RP}^3$. We cannot use the argument of Lemma 44.5 to answer this question, since there are no non-trivial cup-products on $\mathbb{RP}^3$.

44.3. More on the cup-product. Now we will give an alternative description of the cup-product. To do we recall some notation from page 400.

Notation. For $i = 0, \dots, n$ we write

$$v_i := (0, \dots, 1, 0, \dots, 0) \in \mathbb{R}^{n+1}.$$

Furthermore for $a_1, \dots, a_{k+1} \in \Delta^n$ we consider the map

$$\begin{aligned}
 [a_1, \dots, a_{k+1}] : \Delta^k &\rightarrow \Delta^n \\
 (\lambda_1, \dots, \lambda_{k+1}) &\mapsto \sum_{j=1}^{k+1} \lambda_j a_j.
 \end{aligned}$$

The map $[a_1, \dots, a_{k+1}]$ thus sends the standard k -simplex Δ^k to the simplex “spanned by $a_1, \dots, a_{k+1}$ ” in Δ^n .

⁷³³Of course we do not need to use cup-products to distinguish the 2-torus and $S^1 \wedge S^1 \wedge S^2$, this can already be done using fundamental groups. Nonetheless it is encouraging to see that we found a new structure.

We start out with the following lemma that gives an alternative calculation of the cup-product.⁷³⁴

Lemma 44.6. *Let X be a topological space and let R be a commutative ring. Then the cup-product on X is given by*

$$\begin{aligned} H^p(X; R) \times H^q(X; R) &\rightarrow H^{p+q}(X; R) \\ ([\varphi], [\psi]) &\mapsto \left[\begin{array}{ccc} C_{p+q}(X) & \rightarrow & R \\ \sigma & \mapsto & \varphi(\sigma \circ [v_0, \dots, v_p]) \cdot \psi(\sigma \circ [v_0, \dots, v_p]) \end{array} \right]. \end{aligned}$$

Proof. We consider the map

$$\begin{aligned} \Delta: C_n(X) &\rightarrow C_*(X) \otimes C_*(X) \\ \sigma &\mapsto \sum_{p+q=n} \sigma \circ [v_0, \dots, v_p] \otimes \sigma \circ [v_p, \dots, v_n] \end{aligned}$$

which is often called the Alexander-Whitney diagonal approximation. On [Br, p. 328] or [DK, p. 65] it is shown that Δ is chain homotopic to the map d_* induced by the diagonal map $d: X \rightarrow X \times X$. Now let $\varphi \in C^p(X; R)$, $\psi \in C^q(X; R)$ be cocycles and let $\sigma \in C_{p+q}(X)$ be a cycle. Then

$$\begin{aligned} &\text{since } d \text{ and } \Delta \text{ are chain-homotopic and since } \varphi, \psi \text{ are cocycles} \\ &\quad \downarrow \\ (\varphi \cup \psi)(\sigma) &= (\varphi \otimes \psi)(d_*(\sigma)) = (\varphi \otimes \psi)(d_*(\sigma)) \\ &= (\varphi \otimes \psi) \left(\sum_{p+q=n} \sigma \circ [v_0, \dots, v_p] \otimes \sigma \circ [v_p, \dots, v_n] \right) \\ &= \varphi(\sigma \circ [v_0, \dots, v_p]) \cdot \psi(\sigma \circ [v_p, \dots, v_n]). \quad \square \end{aligned}$$

Now we will extend the cup-product to the case of relative cohomology groups. More precisely, let X be a topological space, let A and B be subsets of X and let R be a commutative ring. We define the *cup-product* as

$$\begin{aligned} H^p(X, A; R) \times H^q(X, B; R) &\rightarrow H^{p+q}(X, A \cup B; R) \\ ([\varphi], [\psi]) &\mapsto \left[\begin{array}{ccc} C_{p+q}(X) & \rightarrow & R \\ \sigma & \mapsto & \varphi(\sigma \circ [v_0, \dots, v_p]) \cdot \psi(\sigma \circ [v_0, \dots, v_p]) \end{array} \right]. \end{aligned}$$

It follows from Lemma 44.6 that for $A = B = \emptyset$ we get the same cup-product as we had introduced on page 704.

We leave it as an exercise to verify that this cup-product is well-defined. This cup-product is clearly bilinear. Furthermore, if $A = B$, then one can again verify that $\cup$ defines a ring structure on $H^*(X, A; R)$.

The following lemma is a straightforward generalization of Lemma 44.2. We leave the proof as an exercise to the reader.

⁷³⁴In fact in some books, e.g. [Ha], the formula for cup-product in the lemma is given as the definition of the cup-product.

Lemma 44.7. *Let $f: (X, A) \rightarrow (Y, B)$ be a map between pairs of topological spaces and let R be a commutative ring. Then for any $p \in H^k(Y, B; R)$ and $q \in H^l(Y, B; R)$ we have*

$$f^*(p) \cup f^*(q) = f^*(p \cup q).$$

We conclude the discussion of cup-products with the following elementary lemma.

Lemma 44.8. *Let X be a topological space, let A and B be subsets of X and let R be a commutative ring.*

(1) *The following diagram commutes*

$$\begin{array}{ccc} H^k(X, A; R) \times H^l(X, B; R) & \xrightarrow{\cup} & H^{k+l}(X, A \cup B; R) \\ \downarrow & & \downarrow \\ H^k(X; R) \times H^l(X; R) & \xrightarrow{\cup} & H^{k+l}(X; R). \end{array}$$

(2) *If $\alpha \in H^k(X; R)$ lies in the image of $H^k(X, A; R)$ and if $\beta \in H^l(X; R)$ lies in the image of $H^l(X, B; R)$ and if $A \cup B = X$, then $\alpha \cup \beta = 0$.*

Proof. The first statement follows immediately from the definitions. The second statement is an immediate consequence of the first statement since $H^{k+l}(X, A \cup B; R) = H^{k+l}(X, X; R) = 0$. $\square$

44.4. The definition of the cap-product. The formula in Lemma 44.6 also suggests that one can define other types of products. More precisely, let X be a topological space and let R be a commutative ring. We now introduce a pairing that relates homology and cohomology groups of X . Given $k \leq l$ we define

$$\cap: C^k(X; R) \times C_l(X; R) \rightarrow C_{l-k}(X; R)$$

by the assignment

$$(\varphi, \sigma) \mapsto \varphi \cap \sigma := \underbrace{\varphi(\sigma \circ [v_0, \dots, v_k])}_{\in R} \cdot \underbrace{\sigma \circ [v_k, \dots, v_l]}_{\in C_{l-k}(X; R)}$$

for $\varphi \in C^k(X; R)$ and $\sigma: \Delta^k \rightarrow X$ and by extending this map bilinearly over R to all elements in $C^k(X; R) \times C^l(X; R)$. We start out with the following elementary lemma.

Lemma 44.9. *Let X be a topological space and let R be a commutative ring. Furthermore let $\varphi \in C^k(X; R)$ and $\sigma: \Delta^k \rightarrow X$. Then*

$$\partial(\varphi \cap \sigma) = (-1)^k(\varphi \cap \partial\sigma - \delta\varphi \cap \sigma).$$

Proof. The lemma follows from an elementary calculation. Indeed, we have

$$\begin{aligned}
 \varphi \cap \partial \sigma &= \sum_{i=0}^k (-1)^i \varphi(\sigma \circ [v_0, \dots, \widehat{v}_i, \dots, v_{k+1}]) \cdot \sigma \circ [v_{k+1}, \dots, v_l] \\
 &\quad + \sum_{i=k+1}^l (-1)^i \varphi(\sigma \circ [v_0, \dots, v_k]) \cdot \sigma \circ [v_l, \dots, \widehat{v}_i, \dots, v_l] \\
 \delta \varphi \cap \sigma &= \sum_{i=0}^{k+1} (-1)^i \varphi(\sigma \circ [v_0, \dots, \widehat{v}_i, \dots, v_{k+1}]) \cdot \sigma \circ [v_{k+1}, \dots, v_l] \\
 \partial(\varphi \cap \sigma) &= \sum_{i=k}^l (-1)^{i-l} \varphi(\sigma \circ [v_0, \dots, v_k]) \cdot \sigma \circ [v_l, \dots, \widehat{v}_i, \dots, v_l].
 \end{aligned}$$

It is now straightforward to verify that the stated equality holds. $\square$

The following lemma says that “ $\cap$ ” descends to a map on homology and cohomology.

Lemma 44.10. *Let X be a topological space and let R be a commutative ring. Then the map*

$$\begin{aligned}
 H^k(X; R) \times H_l(X; R) &\rightarrow H_{l-k}(X; R) \\
 [(\varphi, \sigma)] &\mapsto [\varphi \cap \sigma]
 \end{aligned}$$

is well-defined.

We refer to the pairing from Lemma 44.10 as the *cap-product*⁷³⁵

Proof. The lemma now follows easily from Lemma 44.9. Indeed, let φ be a cocycle and let σ be a cycle.

- (1) The claim implies that $\varphi \cap \sigma$ is a cycle, i.e. it represents a homology class.
- (2) If $\sigma' = \sigma + \partial \tau$ is another representative of $[\sigma]$, then

$$\begin{aligned}
 \varphi \cap \sigma - \varphi \cap \sigma' &= \varphi \cap (\sigma - \sigma') = \varphi \cap \partial \tau = \pm \partial(\varphi \cap \tau). \\
 &\quad \uparrow \\
 &\text{by the claim and since } \delta \varphi = 0
 \end{aligned}$$

This shows that $\varphi \cap \sigma$ and $\varphi \cap \sigma'$ represent the same homology class, i.e. the cap-product does not depend on the choice of the representative of the homology class.

- (3) The proof that the cap-product does not depend on the choice of the cohomology class is basically the same as the proof of (2). $\square$

For future applications we state the special case $k = l$ in the cap-product in the following lemma.

Lemma 44.11. *Let X be a topological space. The composition*

$$H^k(X; \mathbb{Z}) \times H_k(X; \mathbb{Z}) \xrightarrow{\cap} H_0(X; \mathbb{Z}) \xrightarrow{\text{augmentation}} \mathbb{Z}$$

⁷³⁵The name “cap-product” comes from the notation ‘ $a \cap b$ ’ for the composition since $\cap$ looks like, just a little bit, like the “a” in cup.

is precisely the Kronecker pairing. In particular, if X is path-connected, and if we make the usual identification $H_0(X; R) = R$ given by the augmentation map, see Proposition 24.5, then for any $\varphi \in H^k(X; R)$ and $\sigma \in H_k(X; R)$ we have

$$\varphi \cap \sigma = \langle \varphi, \sigma \rangle \in R.$$

Let X be a topological space, let A be a subset, let $k \leq l$ and let R again be a commutative ring. It is straightforward to verify that the formula for the cap-product also induces well-defined bilinear maps

$$\begin{aligned} H^k(X; R) \times H_l(X, A; R) &\xrightarrow{\cap} H_{l-k}(X, A; R) \\ H^k(X, A; R) \times H_l(X, A; R) &\xrightarrow{\cap} H_{l-k}(X; R). \end{aligned}$$

In fact more generally, if we are given a subset C of X which can be decomposed as $C = A \sqcup B$ where A and B are a union of components of C , then there exists also a well-defined bilinear map

$$H^k(X, A; R) \times H_l(X, A \cup B; R) \xrightarrow{\cap} H_{l-k}(X, B; R).$$

We refer to all these maps also as cap-products.

Now suppose we are given a map $f: X \rightarrow Y$. We get induced maps on homology and cohomology which can be summarized in the following diagram.

$$\begin{array}{ccc} H^k(X) \times H_l(X) & \xrightarrow{\cap} & H_{l-k}(X) \\ f^* \uparrow & & \downarrow f_* \\ H^k(Y) \times H_l(Y) & \xrightarrow{\cap} & H_{l-k}(Y). \end{array}$$

The diagram guides us to the statement of the functoriality property of the cap-product.

Lemma 44.12. *Let $f: (X, A) \rightarrow (Y, B)$ be map between pairs of topological spaces, let $k \leq l$ and let R be a commutative ring.⁷³⁶ Then for any $\varphi \in H^k(Y, B; R)$ and $\sigma \in H_l(X, A; R)$ we have*

$$\varphi \cap f_*(\sigma) = f_* \underbrace{(f^*(\varphi) \cap \sigma)}_{\in H_{l-k}(X; R)} \in H_{l-k}(Y; R).$$

If $k = l$ and if X and Y are path-connected, then

$$\varphi \cap f_*(\sigma) = f^*(\varphi) \cap \sigma \in \underbrace{H_0(X; R) = R = H_0(Y; R)}_{\substack{\text{identification from} \\ \text{Proposition 24.5}}}.$$

Proof. The equality follows immediately from the definitions. We leave the details to the skeptical reader. $\square$

The following lemma relates the cup-product to the cap-product.

⁷³⁶In most applications we have $A = B = \emptyset$.

Lemma 44.13. *Let X be a topological space, let R be a commutative ring, let $\varphi \in C^k(X; R)$, $\psi \in C^l(X; R)$ and let $\sigma \in C_n(X; R)$. If $k + l \geq n$, then⁷³⁷*

$$\varphi \cap (\psi \cap \sigma) = (\psi \cup \varphi) \cap \sigma.$$

Proof. We have

see notation on page 709

$$\begin{aligned} \varphi \cap (\psi \cap \sigma) & \quad \downarrow \\ &= \varphi \cap (\psi(\sigma \circ [v_0, \dots, v_l]) \cdot \sigma \circ [v_l, \dots, v_n]) \\ &= \psi(\sigma \circ [v_0, \dots, v_l]) \cdot \varphi \cap (\sigma \circ [v_l, \dots, v_n]) \\ &= \psi(\sigma \circ [v_0, \dots, v_l]) \cdot \varphi(\sigma \circ [v_l, \dots, v_{l+k}]) \cdot \sigma \circ [v_{l+k}, \dots, v_n] \\ &= (\psi \cup \varphi) \cap \sigma. \\ & \quad \uparrow \end{aligned}$$

Lemma 44.6

□

It is perhaps not immediately clear what the cap-product is good for. But we will see in the next chapter that it is essential for the formulation of the Poincaré Duality Theorem.

⁷³⁷Note that on the right hand side it says $(\psi \cup \varphi) \cap \sigma$ instead of $(\varphi \cup \psi) \cap \sigma$. The difference between these two expression is, by Proposition 44.1, just a factor of $(-1)^{kl}$. In fact if one picks a different different sign conventions for the cap-product this sign goes away. This approach is taken in [Br, Chapter 5].

45. POINCARÉ-DUALITY

45.1. The Poincaré Duality Theorem. The following theorem is by far the most important theorem regarding homology and cohomology groups of topological manifolds.⁷³⁸

Theorem 45.1. (Poincaré Duality Theorem) *Let M be a compact n -dimensional topological manifold and let R be a commutative ring. Furthermore let $\partial M = A \sqcup B$ be a decomposition of ∂M such that A and B are a union of components of ∂M .⁷³⁹ If M is R -oriented, i.e. if we have an R -fundamental class $[M] \in H_n(M, \partial M; R)$, then for each $k \in \mathbb{N}_0$ the map*

+++++ later on use PD for the inverse map ++++++

$$\begin{array}{ccc} \text{PD}: H^k(M, A; R) & \xrightarrow{\cong} & H_{n-k}(M, B; R) \\ \sigma & \mapsto & [M] \cap \sigma \end{array}$$

is an isomorphism.

Remark.

- (1) For topological manifolds with non-empty boundary the Poincaré Duality Theorem is sometimes also called the *Poincaré-Lefschetz Duality Theorem*.
- (2) One can generalize the theorem a little bit in the following sense: we can work with any decomposition of ∂M into $(n-1)$ -dimensional submanifolds A and B such that $\partial M = A \cup B$ and $A \cap B$ equals the boundary of ∂A and ∂B . We refer to [Ha, Theorem 3.43] for details.

We will now give a few immediate consequences. Afterwards we will first provide a “pictorial” proof for closed manifolds, before we give a rigorous, but less pictorial proof. In the coming chapters we will then use the Poincaré Duality Theorem 45.1 to prove many more statements.

For many applications it suffices to work with the following slightly less technical version of Poincaré Duality Theorem 45.1.

Theorem 45.2. (Poincaré Duality Theorem) *Let M be a compact, n -dimensional topological manifold. If M is orientable, then for each $k \in \mathbb{N}_0$ and every commutative ring*

⁷³⁸The theorem, in a simpler version first goes back to Henri Poincaré whom we had already encountered twice before.

⁷³⁹In most applications we have $A = \partial M$ and $B = \emptyset$ or alternatively $A = \emptyset$ and $B = \partial M$.

R there exist isomorphisms

$$\text{PD}: H^k(M; R) \xrightarrow{\cong} H_{n-k}(M, \partial M; R)$$

and

$$\text{PD}: H^k(M, \partial M; R) \xrightarrow{\cong} H_{n-k}(M; R)$$

Furthermore for $R = \mathbb{Q}$ and $R = \mathbb{R}$ we have⁷⁴⁰

$$\dim_R(H^k(M; R)) = \dim_R(H^{n-k}(M, \partial M; R))$$

and

$$\dim_R(H_k(M; R)) = \dim_R(H_{n-k}(M, \partial M; R)).$$

Proof. The first part of the theorem is an immediate consequence of Poincaré Duality Theorem 45.1 applied to the case that $R = \mathbb{Z}$ and to $A = \emptyset$ and $B = \partial M$ respectively $A = \partial M$ and $B = \emptyset$. Now suppose that $R = \mathbb{Q}$ or $R = \mathbb{R}$. In this case it follows from Lemma 41.9 that for each i we have

$$\dim_R(H_i(M; R)) = \dim_R(H^i(M; R))$$

and

$$\dim_R(H_i(M, \partial M; R)) = \dim_R(H^i(M, \partial M; R)).$$

The desired equalities of dimensions now follows immediately from the first part of the theorem. $\square$

The following corollary gives a new proof of Proposition 33.13 and Proposition 39.22 (1).

Corollary 45.3. *Let M be a compact, orientable, connected n -dimensional topological manifold. Then the homology group $H_{n-1}(M)$ is a free abelian group.*

Proof. We have

$$\begin{array}{ccc} H_{n-1}(M) & \cong & H^1(M, \partial M) & \cong & \text{Hom}(H_1(M, \partial M), \mathbb{Z}) \oplus \text{Ext}(\overbrace{H_0(M, \partial M)}^{\cong \mathbb{Z}}, \mathbb{Z}) \\ & \uparrow & & \uparrow & \\ & \text{Poincaré Duality Theorem 45.2} & & \text{Universal Coefficient Theorem 41.6} & \\ & \cong & \text{Hom}(H_1(M, \partial M), \mathbb{Z}) & \cong & \text{Hom}(H_1(\mathbb{Z}^k \oplus T), \mathbb{Z}) \cong \mathbb{Z}^k. \\ & \uparrow & & \uparrow & \\ & \text{Lemma 41.1 (3)} & & \text{by Proposition 39.2 the group } H_1(M, \partial M) \text{ is finitely generated,} & \\ & & & \text{hence it is isomorphic to } \mathbb{Z}^k \oplus T \text{ for some torsion-group } T & \square \end{array}$$

In Proposition 39.12 we had shown that every topological manifold is $\mathbb{F}_2$ -orientable. This now allows us to prove the following theorem.

⁷⁴⁰The statement implies of course that $H^k(M; R)$ and $H^{n-k}(M, \partial M; R)$ are isomorphic. We prefer to state the result in terms of dimensions to stress that the isomorphism is not natural.

Theorem 45.4. (Poincaré Duality Theorem with $\mathbb{F}_2$ -coefficients) *Let M be a compact, n -dimensional topological manifold. For each $k \in \mathbb{N}_0$ there exist isomorphisms*

$$\text{PD}: H^k(M; \mathbb{F}_2) \xrightarrow{\cong} H_{n-k}(M, \partial M; \mathbb{F}_2)$$

and

$$\text{PD}: H^k(M, \partial M; \mathbb{F}_2) \xrightarrow{\cong} H_{n-k}(M; \mathbb{F}_2)$$

Furthermore we have⁷⁴¹

$$\dim_{\mathbb{F}_2}(H^k(M; \mathbb{F}_2)) = \dim_{\mathbb{F}_2}(H^{n-k}(M, \partial M; \mathbb{F}_2))$$

and

$$\dim_{\mathbb{F}_2}(H_k(M; \mathbb{F}_2)) = \dim_{\mathbb{F}_2}(H_{n-k}(M, \partial M; \mathbb{F}_2)).$$

Proof. The proof is basically the same as the proof of Theorem 45.2. We just need to replace $R = \mathbb{Q}, \mathbb{R}$ by $R = \mathbb{F}_2$ and we need to replace Lemma 41.9 by Lemma 41.10. $\square$

Example. We consider the real projective space $\mathbb{R}P^n$. On page 620 we had determined that

$$H_k(\mathbb{R}P^n; \mathbb{F}_2) \cong \begin{cases} \mathbb{F}_2, & \text{if } k = 0, \dots, n, \\ 0, & \text{otherwise.} \end{cases}$$

So the homology groups with $\mathbb{F}_2$ -coefficients are in particular symmetric, which is exactly what the Poincaré Duality Theorem 45.4 with $\mathbb{F}_2$ -coefficients predicts.

Corollary 45.5. *The Euler characteristic of every odd-dimensional closed topological manifold is zero.*

Proof. Let M be a closed topological manifold of dimension $2n + 1$. Then

$$\begin{aligned} \chi(M) &= \sum_{i=0}^n (-1)^i \dim_{\mathbb{F}_2}(H_i(M, \mathbb{F}_2)) + \sum_{i=n+1}^{2n+1} (-1)^i \underbrace{\dim_{\mathbb{F}_2}(H_i(M, \mathbb{F}_2))}_{= \dim_{\mathbb{F}_2}(H_{2n+i-1}(M; \mathbb{F}_2)) \text{ by Theorem 45.4}} \\ &\quad \uparrow \text{see page 603} \\ &= \sum_{i=0}^n (-1)^i \dim_{\mathbb{F}_2}(H_i(M, \mathbb{F}_2)) + \sum_{j=0}^n \underbrace{(-1)^{2n+1-j}}_{=-(-1)^j} \dim_{\mathbb{F}_2}(H_j(M, \mathbb{F}_2)) = 0. \\ &\quad \uparrow \text{substitution } j = 2n + 1 - i \text{ in the second sum} \end{aligned}$$

$\square$

In exercise sheet 3 we will prove the following corollary to Corollary 45.5.

Corollary 45.6. *Let M be connected closed 3-dimensional manifold. If M is non-orientable, then $H_1(M; \mathbb{Z})$ is infinite.*

Note that in Corollary 16.6 we showed that if N is a non-orientable manifold, then $\pi_1(N)$, and hence by Proposition 32.4 also $H_1(N; \mathbb{Z})$, admits an epimorphism onto $\mathbb{Z}_2$. For closed 3-dimensional manifolds Corollary 45.6 is thus a considerable strengthening of Corollary 16.6.

+++++ = also, for 3-manifold have $\chi(M) = \frac{1}{2}\chi(\partial M)$, furthermore for every boundary component F of M we have that the rank of the image of $H_1(F) \rightarrow H_1(M)$ is at least half of the rank of $H_1(F)$. +++++

45.2. A proof of a weaker version of the Poincaré Duality Theorem 45.1. In this section we will provide a proof for the following weaker version Poincaré Duality Theorem 45.1.

Theorem 45.7. (Poincaré Duality Theorem) *Let M be a closed oriented n -dimensional⁷⁴² manifold. Then for each $k \in \mathbb{N}_0$ there exists an isomorphism*

$$\text{PD}: H_k(M; \mathbb{Z}) \xrightarrow{\cong} H^{n-k}(M; \mathbb{Z}).$$

The proof we provide for Theorem 45.7 has the great advantage that it is very pictorial and one “can see” what is happening. Also, much later, when we introduce a new invariant called Reidemeister torsion, the method of the proof of Theorem 45.7 will be useful. In the next section we will give a rigorous, more formal proof of Poincaré Duality Theorem 45.7, therefore we will take the liberty of skipping a few steps in the following proof.

Sketch of the proof of Theorem 45.1. Let M be a closed oriented n -dimensional manifold. According to Proposition 33.3 the manifold M admits a simplicial structure X .⁷⁴³ As a reminder, this means in particular that M is the union of simplices (in particular M is the union of images of injective maps $\sigma: \Delta^m \rightarrow M$).

We first recall several definitions for the standard-simplex Δ^n :

- (1) A *face* of codimension r of the standard m -simplex is a subset of the form

$$\{(t_0, \dots, t_m) \in \Delta^m \mid t_{d_1} = t_{d_2} = \dots = t_{d_r} = 0\},$$

where $d_1 < d_2 < \dots < d_r \in \{0, \dots, m\}$ are fixed.

- (2) We denote by $B_m := (\frac{1}{m+1}, \dots, \frac{1}{m+1}) \in \Delta^m$ the *barycenter* of the standard m -simplex Δ^m .

- (3) If $P_0, \dots, P_k$ are points in the simplex Δ^m , then we denote by

$$\langle P_0, \dots, P_k \rangle := \left\{ \sum_{i=0}^k t_i P_i \mid \sum_{i=0}^k t_i = 1 \text{ and } t_i \geq 0 \text{ for } i = 0, \dots, k \right\}$$

the convex hull of $P_0, \dots, P_k$ in Δ^m . Recall that this means that $\langle P_0, \dots, P_k \rangle$ is the smallest convex subset of Δ^m which contains the points $P_0, \dots, P_k$.

Now we use the notation that we introduced for the standard-simplices in the obvious way for the simplices of M .⁷⁴⁴ We introduce a few more definitions.

- (1) If σ is a face of a simplex σ of M , then we write $\sigma \leq \tau$. If $\sigma \neq \tau$, then we write $\sigma < \tau$. It follows hereby from the definition a simplicial complex that a face of a simplex $\sigma \subset M$ is again a simplex of M .
- (2) We denote the barycenter of a simplex σ by $\underline{\sigma}$.

⁷⁴²Note that here we do not allow topological manifolds.

⁷⁴³As a reminder, this means that there exists a simplicial complex X and a homeomorphism $X \rightarrow M$. In the following we will suppress the homeomorphism in the notation, i.e. we will view M itself as a simplicial complex.

⁷⁴⁴For example given a simplex $\sigma: \Delta^m \rightarrow M$ we refer to the point $\sigma(B_n)$ as the barycenter of the simplex σ .

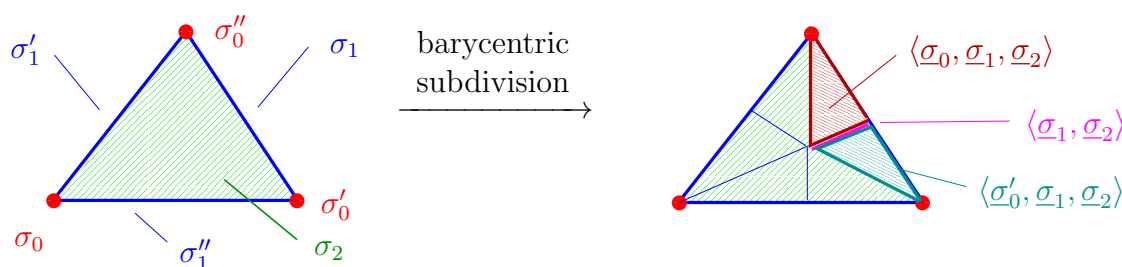
Note that for a sequence $\sigma_0 < \sigma_1 < \cdots < \sigma_k$ of faces of σ the set $\langle \underline{\sigma}_0, \dots, \underline{\sigma}_k \rangle$ is a k -dimensional simplex in σ .

Claim. Let σ be a simplex in M . The set of all simplices of the form

$$\langle \underline{\sigma}_0, \dots, \underline{\sigma}_k \rangle \quad \text{where} \quad \sigma_0 < \sigma_1 < \cdots < \sigma_k \leq \sigma$$

forms a triangulation of σ .

We refer to this subdivision of σ as the *barycentric subdivision* of σ . The proof the claim is elementary and we leave it as an exercise to the reader. The statement of the claim is illustrated in Figure 369.



2-dimensional simplex σ_2 with
three 0-dimensional faces $\sigma_0, \sigma_0', \sigma_0''$ and three 1-dimensional faces $\sigma_1, \sigma_1', \sigma_1''$

FIGURE 369. Barycentric subdivision of a 2-simplex σ_2 .

Now we apply the barycentric subdivision to each simplex of M and we obtain a new simplicial complex which we call *the barycentric subdivision of M* . For a k -simplex σ of M we consider

$$\sigma^d := \bigcup_{\sigma = \sigma_0 < \sigma_1 < \cdots < \sigma_k} \langle \underline{\sigma}_0, \underline{\sigma}_1, \dots, \underline{\sigma}_k \rangle.$$

We refer to σ^d as the *cell dual to σ* . Furthermore we consider

$$\partial \sigma^d := \bigcup_{\sigma < \sigma_1 < \cdots < \sigma_k} \langle \underline{\sigma}_0, \underline{\sigma}_1, \dots, \underline{\sigma}_k \rangle \subset \sigma^d.$$

We illustrate σ^d and $\partial \sigma^d$ in Figure 370. We see in Figure 370 that in both examples the dual cell is a disk and that $\partial \sigma^d$ is the boundary of σ^d . The following claim says that this is not a coincidence.

Claim. The pair $(\sigma^d, \partial \sigma^d)$ is homeomorphic to $(\overline{B}^{n-k}, S^{n-k})$.

In the proof of the claim one needs to use that M is a closed n -dimensional manifold. A proof of the claim can for example be found in [RS, Chapter 2].

In Figure 370 we see two simplices σ and τ , where $\sigma \subset \partial \tau$. In the dual cells this inclusion is reversed and we have $\tau^d \subset \partial \sigma^d$. Again this is not a coincidence, in fact the following claim holds in general.

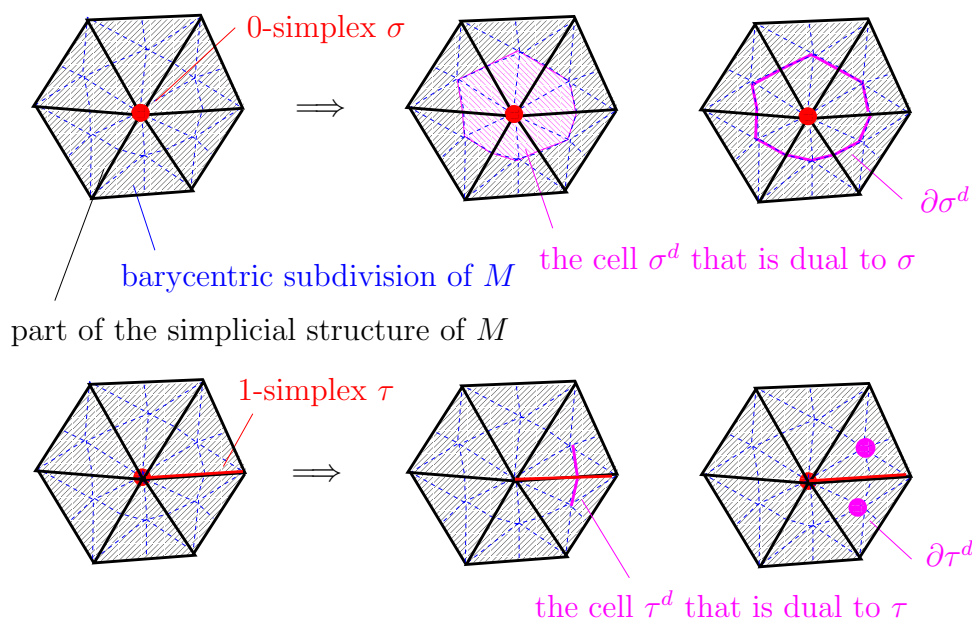


FIGURE 370. Dual cells for some 0-simplex and a 1-simplex.

Claim. Let σ and τ be two simplices of M . Then the following holds

$$\sigma \subset \partial\tau \text{ of codimension } c \iff \tau^d \subset \partial\sigma^d \text{ of codimension } c.$$

Let σ be a simplex in the boundary of τ , i.e. σ is a face of τ . Then each sequence $\tau = \tau_0 < \tau_1 < \dots < \tau_k$ can be extended to a sequence $\sigma < \tau = \tau_0 < \tau_1 < \dots < \tau_k$. In particular we have $\langle \underline{\sigma}_0, \dots, \underline{\sigma}_k \rangle \subset \langle \underline{\sigma}, \underline{\sigma}_0, \dots, \underline{\sigma}_k \rangle$, thus $\tau^d \subset \partial\sigma^d$. Thus we see that τ^d is a cell in the boundary of σ^d . The statement regarding the codimension also follows easily from the argument. The converse statement is proved exactly the same way. Thus we have proved the claim.

As a reminder, if σ is a k -dimensional simplex of M , then we denote by σ not only the subset of M , but we denote by σ also the corresponding map $\sigma: \Delta^k \rightarrow M$. Therefore we denote by $\sigma^d: \overline{B}^{n-k} \rightarrow M$ the characteristic map for $\sigma^d \subset M$, where we choose the map such that at the point $\underline{\sigma}$ the positive basis of σ together with a positive basis of σ^d form a positive basis of M . Now one can easily show that these cells form a CW-structure for M . We denote this CW-complex by Y and we call it the CW-complex structure that is *dual* to X , or just *the dual CW-complex structure*.

If $\sigma_1, \dots, \sigma_l$ denote the k -simplices of X , then $\sigma_1^d, \dots, \sigma_l^d$ are the $(n-k)$ -cells of Y . The cells $\sigma_1^d, \dots, \sigma_l^d$ form a basis of $C_{n-k}^{\text{CW}}(Y)$. We now denote the dual basis of $\text{Hom}(C_{n-k}^{\text{CW}}(Y), \mathbb{Z})$

by $\text{PD}(\sigma_1), \dots, \text{PD}(\sigma_l)$. Put differently, $\text{PD}(\sigma_i)$ is the map given by

$$\begin{aligned} \text{PD}(\sigma_i): C_{n-k}^{\text{CW}}(Y) &\rightarrow \mathbb{Z} \\ \sum_{j=1}^l a_j \sigma_j^d &\mapsto a_i. \end{aligned}$$

Claim. The maps

$$\begin{aligned} \text{PD}: C_k^{\text{CW}}(X) &\rightarrow \text{Hom}(C_{n-k}^{\text{CW}}(Y), \mathbb{Z}) \\ \sum_{i=1}^m a_i \sigma_i &\mapsto \sum_{i=1}^m a_i \text{PD}(\sigma_i) \end{aligned}$$

are isomorphisms such that for each k the following diagram commutes:

$$\begin{array}{ccc} C_k^{\text{CW}}(X) & \xrightarrow{\text{PD}} & \text{Hom}(C_{n-k}^{\text{CW}}(Y), \mathbb{Z}) \\ \downarrow \partial & & \downarrow \partial^* \\ C_{k-1}^{\text{CW}}(X) & \xrightarrow{\text{PD}} & \text{Hom}(C_{n-k+1}^{\text{CW}}(Y), \mathbb{Z}). \end{array}$$

If $\sigma_1, \dots, \sigma_l$ are the k -simplices of X , then by definition $\text{PD}(\sigma_1), \dots, \text{PD}(\sigma_l)$ form a basis for $\text{Hom}(C_{n-k}^{\text{CW}}(Y), \mathbb{Z})$. Thus it follows that the maps

$$\begin{aligned} \text{PD}: C_k^{\text{CW}}(X) &\rightarrow \text{Hom}(C_{n-k}^{\text{CW}}(Y), \mathbb{Z}) \\ \sum_{i=1}^m a_i \sigma_i &\mapsto \sum_{i=1}^m a_i \text{PD}(\sigma_i) \end{aligned}$$

are isomorphisms. Now we still need to show that the diagram commutes.

As a reminder, we had shown that for simplices σ and τ of M we have

$$\sigma \subset \partial\tau \text{ of codimension } c \iff \tau^d \subset \partial\sigma^d \text{ of codimension } c.$$

It follows relatively easily from the definitions that the desired diagram with $\mathbb{F}_2$ -coefficients commutes, i.e. the diagram

$$\begin{array}{ccc} C_k^{\text{CW}}(X; \mathbb{F}_2) & \xrightarrow{\text{PD}} & \text{Hom}(C_{n-k}^{\text{CW}}(Y; \mathbb{F}_2), \mathbb{F}_2) \\ \downarrow \partial & & \downarrow \partial^* \\ C_{k-1}^{\text{CW}}(X; \mathbb{F}_2) & \xrightarrow{\text{PD}} & \text{Hom}(C_{n-k+1}^{\text{CW}}(Y; \mathbb{F}_2), \mathbb{F}_2) \end{array}$$

commutes. Now one can show that one can pick consistent orientations for the simplices of X and the cells of Y such that diagram commutes with $\mathbb{Z}$ -coefficients. The proof of this statement is given in [Tu, p. 73]. this concludes the proof of the claim.

It follows from the claim and the definitions that PD induces an isomorphism

$$H_k^{\text{CW}}(X) \xrightarrow{\cong} H_{\text{CW}}^{n-k}(Y).$$

By Proposition 29.8 and the discussion on page 669 the homology groups and the cohomology groups of M are isomorphic to the cellular homology and cohomology groups. Thus we obtain the following isomorphisms:

$$H_k(M) = H_k^{\text{CW}}(X) \cong H_{\text{CW}}^{n-k}(Y) = H_{\text{CW}}^{n-k}(M). \quad \square$$

45.3. The proof of the Poincaré Duality Theorem 45.1. In order to simplify the notation a little bit we will prove the following slightly simplified version of Poincaré Duality Theorem 45.1. The general case can be proved basically the same way.

Theorem 45.8. *Let M be a closed oriented n -dimensional topological manifold with corresponding fundamental class $[M] \in H_n(M; \mathbb{Z})$, then for each $k \in \mathbb{N}_0$ the map*

$$\begin{array}{ccc} \text{PD}: H^k(M; \mathbb{Z}) & \xrightarrow{\cong} & H_{n-k}(M; \mathbb{Z}) \\ \sigma & \mapsto & [M] \cap \sigma \end{array}$$

is an isomorphism.

The key idea is to follow the strategy of the proof of Lemma 39.19, namely we want to reduce the proof of Theorem 45.8 to the case of smaller submanifolds and to recover the statement for the whole manifold by a suitable version of a Mayer–Vietoris sequence.

To do so we recall the notation we had introduced on page 655. Namely, given a topological manifold M and a subset $X \subset M$ we write $H_n(M|X) := H_n(M, M \setminus X)$.

To carry out the above strategy we need a notion version of Poincaré duality for non-compact manifolds. So let M be an oriented n -dimensional topological manifold. We do not assume that M is compact. Given a compact subset $K \subset M$ there exists by Lemma 39.19 a unique class $\mu_K \in H_n(M|K)$ such that the restriction to each point in K corresponds to the given orientation.

Lemma 45.9. *There exists a unique map*

$$\text{PD}_M: H_c^k(M) \rightarrow H_{n-k}(M)$$

such that for each compact subset $K \subset M$ the following diagram commutes

$$\begin{array}{ccc} H^k(M|K) & \xrightarrow{\quad} & H_{n-k}(M) \\ & \searrow & \nearrow \text{PD}_M \\ & H_c^k(M) & \end{array}$$

Proof. So let $x \in H_c^k(M)$. By Proposition 42.4 and Lemma 42.1 there exists a compact subset K and a class $x_K \in H^k(M|K)$ such that x is the image of x_K under the map $H^k(M|K) \rightarrow H_c^k(M)$. We define

$$\text{PD}(x) := x_K \cap \mu_K.$$

We only need to show that this definition does not depend on the choice of K and x_K .

So let L be another compact subset and let $x_L \in H^k(M|L)$ such that x is the image of x_L under the map $H^k(M|L) \rightarrow H_c^k(M)$. After possibly replacing L by $K \cup L$ we can without loss of generality assume that $K \subset L$. After possibly replacing L by an even bigger compact set can, by Lemma 42.1 (3), assume that the image of x_K in $H^k(M|L)$ agrees with x_L .

We denote by $i: X \setminus L \rightarrow X \setminus K$ the inclusion map. As we just pointed out we have $i^*(x_K) = x_L$. It follows that

$$\begin{array}{ccccccc} x_K \cap \mu_K & = & x_K \cap i^*(\mu_L) & = & i_*(x_K) \cap \mu_L & = & x_L \cap \mu_L. \\ \uparrow & & \uparrow & & \uparrow & & \\ \text{uniqueness statement} & & \text{Lemma 44.12} & & \text{see above} & & \\ \text{of Lemma 39.19} & & & & & & \end{array}$$

□

We will prove the following generalization of Theorem 45.8

Theorem 45.10. *Let M be an oriented n -dimensional topological manifold. Then for each $k \in \mathbb{N}_0$ the map*

$$\text{PD}_M: H_c^k(M) \xrightarrow{\cong} H_{n-k}(M)$$

from Lemma 45.9 is an isomorphism.

In the proof of Theorem 45.10 we will need the following lemma that we will prove in exercise sheet 5.

Lemma 45.11. *Let M be an oriented n -dimensional topological manifold and let U, V be open subsets with $M = U \cup V$. Then given any compact subsets $C \subset M$ there exist compact subsets $K \subset U$ and $L \subset V$ such that $C \subset K \cup L$.*

The key to the proof of Theorem 45.10 is the following lemma.

Lemma 45.12. *Let M be an oriented n -dimensional topological manifold that is the union of two open sets U and V . Then there exists a diagram*

$$\begin{array}{ccccccc} \dots \longrightarrow & H_c^k(U \cap V) & \longrightarrow & H_c^k(U) \oplus H_c^k(V) & \longrightarrow & H_c^k(M) & \longrightarrow H_c^{k+1}(U \cap V) \longrightarrow \dots \\ & \downarrow \text{PD}_{U \cap V} & & \downarrow \text{PD}_U \oplus -\text{PD}_V & & \downarrow \text{PD}_M & & \downarrow \text{PD}_{U \cap V} \\ \dots \longrightarrow & H_{n-k}(U \cap V) & \longrightarrow & H_{n-k}(U) \oplus H_{n-k}(V) & \longrightarrow & H_{n-k}(M) & \longrightarrow & H_{n-k-1}(U \cap V) \longrightarrow \dots \end{array}$$

with the following properties:

- (1) *the diagram commutes up to sign,*
- (2) *the lower exact sequence is the Mayer–Vietoris sequence for $M = U \cup V$,*
- (3) *the upper horizontal sequence is also exact.*

Proof. Let $K \subset U$ and $L \subset V$ be compact subsets. We consider the following diagram

$$\begin{array}{ccccccc} \dots \longrightarrow & H^k(M|K \cap L) & \longrightarrow & H^k(M|K) \oplus H^k(M|L) & \longrightarrow & H^k(M|K \cup L) & \longrightarrow \dots \\ & \downarrow \cong & & \downarrow \cong & & \downarrow & \\ & H^k(U \cap V|K \cap L) & & H^k(U|K) \oplus H^k(V|L) & & & \\ & \downarrow \cap \mu_{K \cap L} & & \downarrow \cap \mu_K \oplus \cap -\mu_L & & \downarrow \cap \mu_{K \cup L} & \\ \dots \longrightarrow & H_{n-k}(U \cap V) & \longrightarrow & H_{n-k}(U) \oplus H_{n-k}(V) & \longrightarrow & H_{n-k}(M) & \longrightarrow \dots \end{array}$$

where the lower horizontal sequence is the Mayer–Vietoris sequence corresponding to $M = U \cup V$ and the upper horizontal sequence is the inverted Mayer–Vietoris Sequence from

Theorem 28.6 corresponding to $M \setminus (K \cap L) = (M \setminus K) \cup (M \setminus L)$. The two vertical maps on the top left are isomorphisms by excision for cohomology groups, see page 669.

Claim. The above diagram commutes up to a sign which does not depend on K and L .

Before we prove the claim we show that it implies Theorem 45.10.

We take the limit of the above diagram over the directed set

$$I = \{(K, L) \mid K \text{ compact subset of } U \text{ and } L \text{ compact subset of } V\}.$$

We make the following remarks regarding this new diagram:

- (1) By the functoriality of direct limits, and since the signs in the commutativity do not depend on K and L , the new diagram also commutes up to sign.
- (2) Every compact subset of $U \cap V$ is of the form $K \cap L$, it follows from Lemma 42.2 that

$$\lim_{\substack{\longrightarrow \\ (K, L) \in I}} H^k(U \cap V \mid K \cap L) = \lim_{\substack{\longrightarrow \\ C \in \mathcal{K}(U \cap V)}} H^k(U \cap V \mid C) = H_c^k(U \cap V).$$

- (3) It follows from Lemmas 42.2 and 45.11 that

$$\lim_{\substack{\longrightarrow \\ (I, L) \in I}} H^k(M \mid K \cup L) = \lim_{\substack{\longrightarrow \\ C \in \mathcal{K}(U \cap V)}} H^k(M \mid C) = H_c^k(M).$$

(4) It follows from Lemma 29.3 that the direct limit of exact sequences is again exact. All these remarks put together show that the direct limit of the above diagram gives rise precisely to the diagram that we claimed exists. This concludes the proof that the claim implies the lemma.

Now we turn to the proof the claim. It is straightforward to show that the two rectangles shown in the diagram commute. What is not so trivial is that the rectangle, not really shown, involving the boundary and coboundary maps commute. Put differently, we need to show that the following diagram commutes:

$$\begin{array}{ccc} H^k(M \mid K \cup L) & \xrightarrow{\delta} & H^{k+1}(M \mid K \cap L) \xrightarrow{\cong} H^{k+1}(U \cap V \mid K \cap L) \\ \downarrow \cap \mu_{K \cup L} & & \downarrow \cap \mu_{K \cap L} \\ H_{n-k}(M) & \xrightarrow{\partial} & H_{n-k-1}(U \cap V). \end{array}$$

We write $A = M \setminus K$ and $B = M \setminus L$. The map δ is the coboundary map in the inverted Mayer-Vietoris sequence, i.e. it is the coboundary map in the short exact sequence of cochain complexes

$$0 \rightarrow C^*(M, A + B) \rightarrow C^*(M, A) \oplus C^*(M, B) \rightarrow C^*(M, A \cap B) \rightarrow 0,$$

where $C^*(M, A + B)$ is defined as the cochains on M that vanish on chains in A and on chains in B . To evaluate the coboundary map δ on a cohomology class represented by a cocycle $\varphi \in C^*(M, A \cap B)$ we need to write φ as $\varphi = \varphi_A - \varphi_B$ for some $\varphi_A \in C^*(M, A)$ and some $\varphi_B \in C^*(M, B)$. Then $\delta(\varphi)$ is represented by the cocycle $\delta\varphi_A = \delta\varphi_B \in C^*(M, A + B)$,

where the equality $\delta\varphi_A = \delta\varphi_B$ comes from the fact that φ is a cocycle, which implies that $\delta\varphi = \delta\varphi_A - \delta\varphi_B = 0$. Similarly the boundary map ∂ in the homology Mayer–Vietoris sequence is obtained by representing an element of $H_i(M)$ by a cycle z that is a sum of chains $z_U \in C_i(U)$ and $z_V \in C_i(V)$, and then $\partial([z]) = [\partial z_U]$.

Since the open sets $U \setminus L$, $U \cap V$ and $V \setminus K$ cover M we can apply barycentric subdivision to any representative α of $\mu_{K \cup L}$ such that $\alpha = \alpha_{U \setminus L} + \alpha_{U \cap V} + \alpha_{V \setminus K}$ of chains in $U \setminus L$, $U \cap V$ and $V \setminus K$. We refer to Figure 371 for an illustration. The chain $\alpha_{U \cap V}$ represents $\mu_{K \cap L}$ since the two other chains $\alpha_{U \setminus L}$ and $\alpha_{V \setminus K}$ lie in the complement of $K \cap L$, hence they vanish in $H_n(M|K \cap L) \cong H_n(U \cap V|K \cap L)$. Similarly $\alpha_{U \setminus L} + \alpha_{U \cap V}$ represents μ_K .

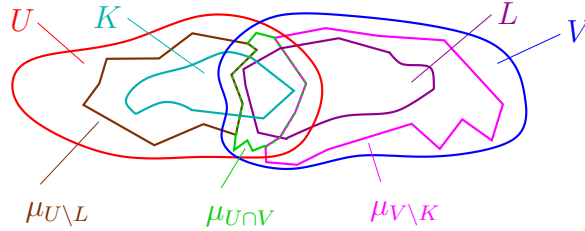


FIGURE 371.

Now return to the above rectangle. Let φ be a cocycle representing an element in $H^k(M|K \cup L)$.

- (1) We first go counter-clockwise around the rectangle. Under the map δ the element $[\varphi]$ element gets sent to the cohomology class $\delta\varphi_A$. This element then gets sent to $\delta\varphi_A \cap \alpha_{U \cap V} \in H_{n-k-1}(U \cap V)$. But this represents the same homology class as $\varphi_A \cap \partial\alpha_{U \cap V}$ since

$$\begin{aligned} \partial(\varphi_A \cap \alpha_{U \cap V}) &= (-1)^k(\varphi_A \cap \partial\alpha_{U \cap V} - \delta\varphi_A \cap \alpha_{U \cap V}) \\ &\quad \uparrow \\ &\text{by Lemma 44.9} \end{aligned}$$

and since $\varphi_A \cap \alpha_{U \cap V}$ is a chain in $U \cap V$.

- (2) Now we go clockwise around the triangle. Then $[\varphi]$ gets first mapped to $[\varphi \cap \alpha]$. To determine $\partial[\varphi \cap \alpha]$ in the Mayer–Vietoris sequence we write $\varphi \cap \alpha$ as the sum of a chain U and of a chain V as follows:

$$\varphi \cap \alpha = (\varphi \cap \alpha_{U \setminus L}) + (\varphi \cap \alpha_{U \cap V} + \varphi \cap \alpha_{V \setminus K}).$$

By the definition of the boundary map ∂ in the Mayer–Vietoris sequence we obtain the element $[\partial(\varphi \cap \alpha_{U \setminus L})] \in H_{n-k-1}(U \cap V)$.

Now we need to show that the two results are the same. We have

$$\begin{array}{ccccc}
 \partial(\varphi \cap \alpha_{U \setminus L}) & = & (-1)^k \varphi \cap \partial \alpha_{U \setminus L} & = & (-1)^k \varphi_A \cap \partial \alpha_{U \setminus L} \\
 \uparrow & & \uparrow & & \\
 \text{by Lemma 44.9 and} & & \text{we have } \varphi_B \cap \partial \alpha_{U \setminus L} = 0 & & \text{since } \varphi_B \text{ is zero} \\
 \text{since } \delta \varphi = 0 & & \text{on chains in } B = M \setminus L & & \\
 & & = & & (-1)^k \varphi_A \cap \partial \alpha_{U \cap V}.
 \end{array}$$

Here the last equality comes from the fact that $\varphi_A \cap \partial(\alpha_{U \setminus L} + \alpha_{U \cap V}) = 0$ since $\partial(\alpha_{U \setminus L} + \alpha_{U \cap V})$ is a chain in $U \setminus K$ by the above observation that $\alpha_{U \setminus L} + \alpha_{U \cap V}$ represents μ_K and since φ_A vanishes on chains in $A = M \setminus K$.

This concludes the proof that the above rectangle commutes, up to a sign that only depends on k . $\square$

Now we turn to the actual proof of Theorem 45.10.

Proof of Theorem 45.10. Let M be an oriented n -dimensional topological manifold. We want to show that for each $k \in \mathbb{N}_0$ the map

$$\text{PD}_M: H_c^k(M) \rightarrow H_{n-k}(M)$$

from Lemma 45.9 is an isomorphism.

We first make two preliminary observations:

- (A) Suppose M is the union of two open sets U and V such that PD_U, PD_V and $\text{PD}_{U \cap V}$ are isomorphisms. It follows from Lemma 45.12 and the Five-Lemma 26.10 that PD_M is also an isomorphism.
- (B) Suppose M is the union of a nested sequence $U_1 \subset U_2 \subset \dots$ such that each $\text{PD}_{U_i}: H_c^k(U_i) \rightarrow H_{n-k}(U_i)$ is an isomorphism. We take the direct limit over $i \in \mathbb{N}$ and we apply Propositions 42.6 and 29.1 and we obtain that $\text{PD}_M: H_c^k(M) \rightarrow H_{n-k}(M)$ is indeed an isomorphism.

Now we turn to the actual proof of the duality statement.

- (1) First we consider the case M is an open convex subset of $\mathbb{R}^n$. We use Lemmas 1.4 and 24.1 to identify $\mathbb{R}^n$ with the interior of Δ^n .

As in the proof of Proposition 42.5 one can show that there exists an isomorphism $H_c^k(\overset{\circ}{\Delta}^n) \cong H^k(\Delta^n, \partial \Delta^n)$. In fact one can show easily that there exists a commutative diagram

$$\begin{array}{ccc}
 H_c^k(\overset{\circ}{\Delta}^n) & \xrightarrow{\text{PD}} & H_{n-k}(\overset{\circ}{\Delta}^n) \\
 \downarrow \cong & & \downarrow \cong \\
 H^k(\Delta^n, \partial \Delta^n) & \xrightarrow{\cap [\text{id}_{\Delta^n}]} & H_{n-k}(\Delta^n).
 \end{array}$$

Here $\text{id}_{\Delta^n}: \Delta^n \rightarrow \Delta^n$ is $++++++$ replace by $\overline{B}^n$??????

- (2) Now suppose that M is an open subset of $\mathbb{R}^n$ that is the union of n finitely many open convex subsets of $\mathbb{R}^n$. We proof the claim by induction on n .

By (1) we know that the statement holds for $n = 1$. Now suppose it holds for all unions of $n - 1$ open convex subsets of $\mathbb{R}^n$. Let M be the union of convex open subsets $U_1, \dots, U_n$. We write $V = U_1 \cup \dots \cup U_{n-1}$. By induction the statement holds for V and also for $V \cap U_n = (U_1 \cap U_n) \cup \dots \cup (U_{n-1} \cap U_n)$. Hence by (A) the statement also holds for $M = U_1 \cup \dots \cup U_n$.

- (3) Now suppose that M is an arbitrary open subset of $\mathbb{R}^n$. We can write M as the union of infinite-countably many bounded convex subsets $U_i, i \in \mathbb{N}$.⁷⁴⁵ For $i \in \mathbb{N}$ we write $V_i = \bigcup_{j < i} U_j$. Then the statement holds for all V_i by (2). But then the statement follows for $M = \bigcup_{i \in \mathbb{N}} V_i$ by (B).
- (4) Finally let M be any topological n -dimensional manifold. Since M is locally homeomorphic to open subsets of $\mathbb{R}^n$ and since M is second-countable we can cover M by infinite-countably many open subsets $U_i, i \in \mathbb{N}$, each of which is homeomorphic to an open subset of $\mathbb{R}^n$. By (3) the duality holds for all the U_i . The same argument as in (1) and (2) then shows that the duality also holds for M . $\square$

⁷⁴⁵For example the U_i could be chosen to be open balls.

46. THE CUP-PRODUCT AND ALGEBRAIC INTERSECTION NUMBERS

46.1. The cup-product of topological manifolds.

Definition. Let R be a commutative ring and let V, W and S be R -modules.⁷⁴⁶

- (1) A *pairing* on $V \times W$ is a map

$$V \times W \rightarrow S$$

that is bilinear, i.e. that is linear in V and it is linear in W . If $V = W$, then we usually refer to a pairing as a *form*.

- (2) We say that a pairing

$$\langle \cdot, \cdot \rangle: V \times W \rightarrow S$$

is *non-degenerate* if the induced maps

$$\begin{array}{ccc} V & \rightarrow & \text{Hom}_R(W, S) \\ v & \mapsto & \left(\begin{array}{ccc} W & \rightarrow & S \\ w & \mapsto & \langle v, w \rangle \end{array} \right) \end{array} \quad \text{and} \quad \begin{array}{ccc} W & \rightarrow & \text{Hom}_R(V, S) \\ w & \mapsto & \left(\begin{array}{ccc} V & \rightarrow & S \\ v & \mapsto & \langle v, w \rangle \end{array} \right) \end{array}$$

are a monomorphisms. We say the pairing is *non-singular* if the above map is an isomorphism.

Example. Let $A \in M(n \times n, R)$ be a square matrix. Then the pairing

$$\begin{array}{ccc} R^n \times R^n & \rightarrow & R \\ (v, w) & \mapsto & v^t A w \end{array}$$

is non-degenerate if and only if $\det(A) \neq 0$ and it is non-singular if and only if $\det(A)$ is a unit in R .

The following lemma is straightforward to prove.

Lemma 46.1. *Let R be a commutative ring, let V and W be free R -modules of rank one, let v and w be generators of V respectively W and let*

$$\langle \cdot, \cdot \rangle: V \times W \rightarrow R$$

be a form. Then the following holds

$$\text{the pairing is non-singular} \iff \langle v, w \rangle \text{ is a unit in } R.$$

Definition. Let M be a compact n -dimensional topological manifold and let R be a commutative ring. If M is R -oriented, i.e. if we are given R -fundamental class $[M] \in H_n(M, \partial M; R)$ then for any k we refer to the pairing⁷⁴⁷

$$\begin{array}{ccc} H^k(M, \partial M; R) \times H^{n-k}(M; R) & \rightarrow & R \\ \varphi \cup \psi & \mapsto & \underbrace{(\varphi \cup \psi)}_{\in H^n(M, \partial M; R)}([M]) \end{array}$$

⁷⁴⁶In most cases we have $S = R$, but later on we will see one instance where $S \neq R$.

⁷⁴⁷Note that we use here the cup-product on relative cohomology that we had introduced on page 710.

as a *cup-product pairing* of M .

Proposition 46.2. *Let M be a closed n -dimensional topological manifold, let R be a commutative ring and suppose we are given an R -fundamental class $[M] \in H_n(M; R)$.*

(1) *Let $k \in \mathbb{N}_0$. The cup-product pairing*

$$\begin{aligned} H^k(M; \mathbb{Z}) \times H^{n-k}(M; \mathbb{Z}) &\rightarrow \mathbb{Z} \\ \varphi \cup \psi &\mapsto \langle \varphi \cup \psi, [M] \rangle \end{aligned}$$

is non-singular. If $H_{k-1}(M; \mathbb{Z})$ is in fact torsion-free, then the above pairing is non-singular.

(2) *If R is a field, then for any $k \in \mathbb{N}_0$ the cup-product pairing*

$$\begin{aligned} H^k(M; R) \times H^{n-k}(M; R) &\rightarrow R \\ \varphi \cup \psi &\mapsto \langle \varphi \cup \psi, [M] \rangle \end{aligned}$$

is non-singular.

Remark. Let M be a closed n -dimensional topological manifold and suppose we are given a fundamental class $[M] \in H_n(M; \mathbb{Z})$. One can easily modify the proof of Proposition 46.2 to show that for any $k \in \mathbb{N}_0$ the induced pairing

$$\begin{aligned} H^k(M; \mathbb{Z})/\text{torsion} \times H^{n-k}(M; \mathbb{Z})/\text{torsion} &\rightarrow \mathbb{Z} \\ \varphi \cup \psi &\mapsto \langle \varphi \cup \psi, [M] \rangle \end{aligned}$$

is non-singular.

Proof. First we consider the case $R = \mathbb{Z}$. We consider the map

$$\begin{array}{ccc} H^k(M; \mathbb{Z}) & \xrightarrow{\text{ev}} & \text{Hom}_{\mathbb{Z}}(H_k(M; \mathbb{Z}), \mathbb{Z}) \xrightarrow{(\cap[M])^*} \text{Hom}_{\mathbb{Z}}(H^{n-k}(M), \mathbb{Z}). \\ \varphi \mapsto & \xrightarrow{\hspace{10em}} & \begin{array}{ccc} H^{n-k}(M; \mathbb{Z}) & \rightarrow & \mathbb{Z} \\ \psi & \mapsto & \langle \varphi, \psi \cap [M] \rangle. \end{array} \end{array}$$

The second map is the dual map corresponding to the map $H^{n-k}(M; \mathbb{Z}) \xrightarrow{\cap[M]} H_k(M; \mathbb{Z})$. We have

$$\begin{array}{ccccc} \langle \varphi, \psi \cap [M] \rangle & = & \varphi \cap (\psi \cap \varphi)([M]) & = & (\psi \cup \varphi)([M]). \\ \uparrow & & \uparrow & & \\ & \text{see Lemma 44.11} & & \text{Lemma 44.13} & \end{array}$$

So it suffices to show that the above composition of two maps is an isomorphism. The first map is an isomorphism by the Universal Coefficient Theorem 41.6 for Cohomology Groups together with Lemma 41.1 (3) and our hypothesis that $H_{k-1}(M; \mathbb{Z})$ is torsion-free⁷⁴⁸. Furthermore the second map is an isomorphism by the Poincaré Duality Theorem 45.1. This concludes the proof for the ring $R = \mathbb{Z}$.

⁷⁴⁸Since $H_{k-1}(M; \mathbb{Z})$ is finitely generated this implies that $H_{k-1}(M; \mathbb{Z})$ is in fact a free abelian group.

We now turn to the case that R is a field. We consider the map

$$\begin{aligned} \text{ev}: H^k(M; R) &\rightarrow \text{Hom}_R(H_k(M; R), R) \\ \varphi &\mapsto \left(\begin{array}{ccc} H_k(M; R) & \rightarrow & R \\ \left[\sum_{i=1}^s \sigma_i \otimes r_i \right] & \mapsto & \sum_{i=1}^s \varphi(\sigma_i) \cdot r_i. \end{array} \right) \end{aligned}$$

As above it suffices to show that the map

$$H^k(M; R) \xrightarrow{\text{ev}_R} \text{Hom}_R(H_k(M; R), R) \xrightarrow{(\cap[M])^*} \text{Hom}_R(H^{n-k}(M), R).$$

is an isomorphism. In fact, as above, by the Poincaré Duality Theorem 45.1 it only remains to prove that the map ev is an isomorphism.

This follows from the following purely algebraic lemma that we will prove in exercise sheet 4.

Claim. Let C_* be a chain complex of free abelian groups and let R be a field. Then the map

$$\begin{aligned} \text{ev}: H^k(C_*; R) &\rightarrow \text{Hom}_R(H_k(C_*; R), R) \\ \varphi &\mapsto \left(\begin{array}{ccc} H_k(C_*; R) & \rightarrow & R \\ \left[\sum_{i=1}^s \sigma_i \otimes r_i \right] & \mapsto & \sum_{i=1}^s \varphi(\sigma_i) \cdot r_i. \end{array} \right) \end{aligned}$$

is an isomorphism. □

We now turn to the calculation of the cup-product pairing on $S^2 \times S^4$. We denote by $p: S^2 \times S^4 \rightarrow S^2$ and by $q: S^2 \times S^4 \rightarrow S^4$ the two obvious projection maps. Let $1 \in H^0(S^2 \times S^4; \mathbb{Z})$ be the neutral element from Proposition 44.1, let $a \in H^2(S^2; \mathbb{Z}) \cong \mathbb{Z}$, $b \in H^2(S^4; \mathbb{Z}) \cong \mathbb{Z}$ and $c \in H^2(S^2 \times S^4; \mathbb{Z}) \cong \mathbb{Z}$ be generators. On page 685 we had used the Künneth-formula 41.13 for cohomology groups to show that

$$H^*(S^2 \times S^4; \mathbb{Z}) = \mathbb{Z} \cdot 1 \oplus \mathbb{Z} \cdot p^*(a) \oplus \mathbb{Z} \cdot q^*(b) \oplus \mathbb{Z} \cdot c.$$

Lemma 46.3. *The multiplication table for the cup-product on*

$$H^*(S^2 \times S^4; \mathbb{Z}) = \mathbb{Z} \cdot 1 \oplus \mathbb{Z} \cdot p^*(a) \oplus \mathbb{Z} \cdot q^*(b) \oplus \mathbb{Z} \cdot c$$

is given by

$\cup$	1	$p^*(a)$	$q^*(b)$	c
1	1	$p^*(a)$	$q^*(b)$	c
$p^*(a)$	$p^*(a)$	0	$\epsilon \cdot c$	0
$q^*(b)$	$q^*(b)$	$\epsilon \cdot c$	0	0
c	c	0	0	0

where $\epsilon \in \{-1, 1\}$.

Proof. Since $1 \in H^0(S^2 \times S^4; \mathbb{Z})$ is the neutral element with respect to the cup-product we only need to determine the colored terms.

- (1) The green-colored zeroes follow from the observation that for $k + l > 6$ the cup-product $H^k(S^2 \times S^4; \mathbb{Z}) \cup H^l(S^2 \times S^4; \mathbb{Z})$ takes values in $H^{k+l}(S^2 \times S^4; \mathbb{Z}) = 0$.

- (2) We turn to the blue-colored entries. It follows from Proposition 46.2 (1) together with Lemma 46.1 that $q^*(b) \cup p^*(a) = \epsilon c$ for some $\epsilon \in \{-1, 1\}$.
- (3) We have

$$\begin{array}{ccccccc}
 p^*(a) \cup p^*(a) & = & p^*(a \cup a) & = & p^*(0) & = & 0. \\
 \uparrow & & \uparrow & & & & \\
 & \text{Lemma 44.2} & & \text{since } a \cup a \in H^4(S^2; \mathbb{Z}) = 0 & & & \square
 \end{array}$$

We recall that in Lemma 22.1 we had shown that we can view $\mathbb{CP}^n$ as a CW-complex which admits exactly one cell in dimensions $0, 2, \dots, 2n$ and which admits no odd-cells. Furthermore for $k \in \{1, \dots, n-1\}$ the k -skeleton on $\mathbb{CP}^n$ is $\mathbb{CP}^k$. By the discussion on page 669 we can use the cellular chain complex to compute the cohomology of $\mathbb{CP}^n$ and we obtain, as on page 674, that

$$H^k(\mathbb{CP}^n; \mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & \text{if } k = 0, 2, 4, \dots, 2n, \\ 0, & \text{otherwise.} \end{cases}$$

Now we have the tools to determine the cup-product of each complex-projective space.

Proposition 46.4. *Let $n \in \mathbb{N}$. For $k = 0, \dots, n$ there exists a generator x^k of $H^{2k}(\mathbb{CP}^n; \mathbb{Z}) \cong \mathbb{Z}$ such that for any $k, l \in \{0, \dots, n\}$ we have*

$$x^k \cup x^l = \begin{cases} x^{k+l}, & \text{if } k+l \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

Remark.

- (1) Basically the same argument as in the proof of Proposition 46.4 also allows us to compute cohomology ring for the infinite complex projective space $\mathbb{CP}^\infty$ that we had introduced on page 345. On page 674 we had seen that

$$H^k(\mathbb{CP}^\infty; \mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & \text{if } k \text{ even,} \\ 0, & \text{if } k \text{ odd.} \end{cases}$$

The proof of Proposition 46.4 can easily be modified to show that for every $k \in \mathbb{N}_0$ there exists a generator x^k of $H^{2k}(\mathbb{CP}^\infty; \mathbb{Z}) \cong \mathbb{Z}$ such that for any $k, l \in \{0, \dots, n\}$ we have

$$x^k \cup x^l = x^{k+l}.$$

- (2) We can summarize the calculation of the cup-product of $\mathbb{CP}^n$ and of $\mathbb{CP}^\infty$ as follows: we have ring isomorphisms

$$(H^*(\mathbb{CP}^n; \mathbb{Z}), \cup) \cong \mathbb{Z}[x]/(x^{n+1})$$

and

$$(H^*(\mathbb{CP}^\infty; \mathbb{Z}), \cup) \cong \mathbb{Z}[x].$$

Proof. We prove the lemma by induction on n . The statement clearly holds for $n = 1$. Now suppose it holds for some $n - 1$. We denote the corresponding elements in the cohomology of $\mathbb{CP}^{n-1}$ by x^j , $j = 0, \dots, n - 1$.

We view $\mathbb{CP}^{n-1}$ as a subset of $\mathbb{CP}^n$. We denote by $i: \mathbb{CP}^{n-1} \rightarrow \mathbb{CP}^n$ the inclusion map. It follows easily from the above discussion of the cellular chain complexes that the map $i^*: H^j(\mathbb{CP}^n; \mathbb{Z}) \rightarrow H^j(\mathbb{CP}^{n-1}; \mathbb{Z})$ is an isomorphism for $j = 0, \dots, 2n-1$. For $j = 1, \dots, n-1$ we write $y^j := (i^*)^{-1}(x^j) \in H^{2j}(\mathbb{CP}^n; \mathbb{Z})$. Furthermore we define $y^0 \in H^0(\mathbb{CP}^n)$ to be the neutral element with respect to the cup-product and we set $y^n := y^1 \cup y^{n-1}$. We claim that $y^0, y, y^2, \dots, y^n$ have the desired properties. First note that it follows from Proposition 46.2 (1) together with Lemma 46.1 that y^n is a generator of $H^{2n}(\mathbb{CP}^n; \mathbb{Z})$.

Now let $k, l \in \{0, \dots, n\}$. If $k = 0$ or $l = 0$, then one of the terms is the neutral element and there is nothing to prove. Now suppose that $k, l \geq 1$. If $k + l < n$, then we have

$$\begin{array}{ccccccc} y^k \cup y^l & = & i^*(x^k) \cup i^*(x^l) & = & i^*(x^k \cup x^l) & = & i^*(x^{k+l}) = y^{k+l}. \\ \uparrow & & \uparrow & & \uparrow & & \\ \text{by definition} & & \text{Lemma 44.2} & & \text{by induction} & & \end{array}$$

If $k = 1$ and $l = n-1$ we have

$$\begin{array}{ccc} y^1 \cup y^{n-1} & = & y^n \\ \uparrow & & \\ \text{by definition of } y^n & & \end{array}$$

If $k + l = n$ and $k \geq 2$ we have

$$\begin{array}{ccccccc} y^k \cup y^{n-k} & = & (y^1 \cup y^{k-1}) \cup y^{n-k} & = & y^1 \cup (y^1 \cup y^{n-k}) & = & y^1 \cup y^{n-1} = y^n. \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ \text{by the above} & & \text{by Proposition 44.1} & & \text{by the above} & & \text{by definition} \\ \text{since } 1 + k - 1 < n & & \cup \text{ is associative} & & \text{since } 1 + n - k < n & & \end{array}$$

Finally if $k + l > n$, then $y^k \cup y^l = 0$ since $H^{2(k+l)}(\mathbb{CP}^n; \mathbb{Z}) = 0$. □

In Question 37.6 we had asked whether the topological spaces $S^2 \times S^4$ and $\mathbb{CP}^3$ are homeomorphic. We can now answer the question in the negative.

Corollary 46.5. *The topological spaces $S^2 \times S^4$ and $\mathbb{CP}^3$ are not homeomorphic.*

Proof. In Lemma 46.3 we have seen that the cup-product

$$H^2(S^2 \times S^4; \mathbb{Z}) \times H^2(S^2 \times S^4; \mathbb{Z}) \rightarrow H^4(S^2 \times S^4; \mathbb{Z})$$

is zero, whereas we had seen in Proposition 46.4 that the cup-product

$$H^2(\mathbb{CP}^3; \mathbb{Z}) \times H^2(\mathbb{CP}^3; \mathbb{Z}) \rightarrow H^4(\mathbb{CP}^3; \mathbb{Z})$$

is non-zero. □

In Question 39.28 we had asked whether there exist compact, oriented topological manifolds that are chiral. We will now see that the complex projective space $\mathbb{CP}^2$ is chiral.

Before we do so we recall that on page 25 we had seen that $\mathbb{CP}^n$ is naturally a complex 2-dimensional manifold. Furthermore the proof of Proposition 17.4 implies that any complex manifold comes with a natural orientation. Putting these two statements together implies that $\mathbb{CP}^n$ comes with a canonical orientation, in particular it comes with a canonical fundamental class $[\mathbb{CP}^n] \in H^{2n}(\mathbb{CP}^n)$.

We can now answer Question 39.28 in the affirmative.

Corollary 46.6. *The manifold $\mathbb{CP}^2$ is chiral, i.e. there does not exist a self-homeomorphism $f: \mathbb{CP}^2 \rightarrow \mathbb{CP}^2$ with $f_*([\mathbb{CP}]) = -[\mathbb{CP}^2]$.*

Remark. In exercise sheet 4 we will discuss for which n the complex projective space $\mathbb{CP}^n$ is chiral.

Proof. Let $x \in H^2(\mathbb{CP}^2; \mathbb{Z})$ and $y \in H^4(\mathbb{CP}^2; \mathbb{Z})$ be the generators from Proposition 46.4 with $x \cup x = y$. Since y and $[\mathbb{CP}^2]^*$ are generators of $H^4(\mathbb{CP}^2; \mathbb{Z})$ we know that $y = \epsilon [\mathbb{CP}^2]^*$ for some $\epsilon \in \{-1, 1\}$.

Let $f: \mathbb{CP}^2 \rightarrow \mathbb{CP}^2$ be a self-homeomorphism. We want to show that $f_*([\mathbb{CP}^2]) = [\mathbb{CP}^2] \in H_4(\mathbb{CP}^2; \mathbb{Z})$, or equivalently,⁷⁴⁹ that $f^*([\mathbb{CP}^2]^*) = [\mathbb{CP}^2]^* \in H^4(\mathbb{CP}^2; \mathbb{Z})$.

We now see that

$$\begin{aligned}
 f^*([\mathbb{CP}^2]^*) &= f^*(\epsilon \cdot y) = \epsilon \cdot f^*(x \cup x) &&= \epsilon \cdot (f^*(x) \cup f^*(x)) &&= \epsilon \cdot (\eta \cdot x \cup \eta \cdot x) \\
 &&&\uparrow &&\uparrow \\
 &&&\text{Lemma 44.2} &&f^*(x) \text{ is also a generator of } H^2(\mathbb{CP}^2; \mathbb{Z}), \\
 &&&&&\text{hence } f^*(x) = \eta \cdot x \text{ for some } \eta \in \{-1, 1\} \\
 &&&= \epsilon \cdot \eta^2 \cup (x \cup x) &&= \epsilon \cdot y = [\mathbb{CP}^2]^* \\
 &&\uparrow &&\uparrow \\
 &&\cup \text{ is} &&\text{since } \eta \in \{-1, 1\}
 \end{aligned}$$

□

Finally we consider the real projective space $\mathbb{RP}^n$. Using the CW-structure provided on page 520 one can easily show that

$$H^k(\mathbb{RP}^n; \mathbb{F}_2) \cong \begin{cases} \mathbb{F}_2, & \text{if } k = 0, 1, \dots, n, \\ 0, & \text{otherwise.} \end{cases}$$

The cup-product on $\mathbb{RP}^n$ looks somewhat similar to the cup-product of each complex-projective space.

Proposition 46.7. *Let $n \in \mathbb{N}$. For $k = 0, \dots, n$ we denote by x^k the unique non-trivial element of $H^k(\mathbb{RP}^n; \mathbb{F}_2) \cong \mathbb{F}_2$. Then for any $k, l \in \{0, \dots, n\}$ we have*

$$x^k \cup x^l = \begin{cases} x^{k+l}, & \text{if } k+l \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

Remark. We can summarize the calculation of the cup-product of $\mathbb{RP}^n$ as follows: we have a ring isomorphism

$$(H^*(\mathbb{RP}^n; \mathbb{F}_2), \cup) \cong \mathbb{F}_2[x]/(x^{n+1}).$$

As for the case of complex projective spaces we can generalize this to the infinite real projective space $\mathbb{RP}^\infty$, this time we obtain a ring isomorphism

$$(H^*(\mathbb{RP}^\infty; \mathbb{F}_2), \cup) \cong \mathbb{F}_2[x].$$

Proof. The proof is very similar to the proof of Proposition 46.7. The biggest difference is that we have to apply Proposition 46.2 (2) for the field $R = \mathbb{F}_2$ instead of making use of Proposition 46.2 (1). We leave the details to the reader. □

⁷⁴⁹Why are these two statements equivalent?

On page 45 we had given a slightly indirect argument using Proposition 1.28 to show $\mathbb{RP}^n = S^n/\{\pm 1\}$ is an n -dimensional manifold. We now give an explicit atlas for $\mathbb{RP}^n$. For $i = 1, \dots, n+1$ we consider the map

$$\begin{aligned} \Psi_i: \mathbb{R}^n &\rightarrow \mathbb{RP}^n \\ (x_1, \dots, x_n) &\mapsto [x_1 : \dots : x_{i-1} : 1 : x_i : \dots : x_n]. \end{aligned}$$

We write $\Phi_i := \Psi_i^{-1}: \Psi_i(\mathbb{R}^n) \rightarrow \mathbb{R}^n$. Then $\Phi_1, \dots, \Phi_{n+1}$ are easily seen to be an atlas. One can ask whether there exists a simpler atlas for $\mathbb{RP}^n$. For example one could ask whether one can cover $\mathbb{RP}^n$ by fewer than $n+1$ open balls. The next propositions shows that this is *not* the case.

Proposition 46.8. *The complex projective space $\mathbb{CP}^n$ cannot be covered by fewer than $n+1$ open subsets that are homeomorphic to open balls.*

Proof. Let $U_1, \dots, U_k$ be open subsets in $\mathbb{RP}^n$ with $U_1 \cup \dots \cup U_k = \mathbb{CP}^n$ such that each U_j is homeomorphic to an open ball. We need to show that $k \geq n+1$.

Let $x \in H^1(\mathbb{RP}^n; \mathbb{F}_2)$ be the generator. For $j = 1, \dots, k$ we denote by $i_j: H^1(\mathbb{RP}^n, U_j; \mathbb{F}_2) \rightarrow H^1(\mathbb{RP}^n; \mathbb{F}_2)$ the obvious map. It follows from the long exact sequence of cohomology groups corresponding to the pair $(\mathbb{RP}^n, U_j)$ and the fact that U_j is homeomorphic to an open ball that i_j is an isomorphism. We write $x_j := i_j^{-1}(x) \in H^1(\mathbb{RP}^n, U_j; \mathbb{F}_2)$.

By Lemma 44.8 we have the following commutative diagram

$$\begin{array}{ccc} H^1(\mathbb{RP}^n, U_1; \mathbb{F}_2) \times \dots \times H^1(\mathbb{RP}^n, U_k; \mathbb{F}_2) & \xrightarrow{\cup} & H^k(\mathbb{RP}^n, \overbrace{U_1 \cup \dots \cup U_k}^{=\mathbb{RP}^n}; \mathbb{F}_2) \\ \downarrow (x_1, \dots, x_k) & & \downarrow \\ \downarrow (x, \dots, x) & & \downarrow \\ H^1(\mathbb{RP}^n; \mathbb{F}_2) \times \dots \times H^1(\mathbb{RP}^n; \mathbb{F}_2) & \xrightarrow{\cup} & H^k(\mathbb{RP}^n; \mathbb{F}_2). \end{array}$$

Since $U_1 \cup \dots \cup U_k = \mathbb{RP}^n$ we see that group on the top-right is zero. But this implies that the cup-product $x_1 \cup \dots \cup x_k = 0$. Since the diagram commutes this implies that $x^k = x \cup \dots \cup x = 0$. But by Proposition 46.7 this is only possible if $k \geq n+1$. $\square$

Remark. Basically the same proof as in Proposition 46.8, together with the calculation from Proposition 46.7, can be used to show that the complex projective space $\mathbb{CP}^n$ cannot also not be covered by fewer than $n+1$ open subsets that are homeomorphic to open balls.

46.2. Cup-products and degree-one maps. Let $f: M \rightarrow N$ be a map between closed oriented connected n -dimensional manifolds. For the remainder of the section it is convenient to keep the following diagram of maps in mind:

$$\begin{array}{ccc} H_k(M) & \begin{array}{c} \xrightarrow{\text{PD}_M} \\ \xleftarrow{\cap[M]} \end{array} & H^{n-k}(M) \\ f_* \downarrow & & \uparrow f^* \\ H_k(N) & \begin{array}{c} \xrightarrow{\text{PD}_N} \\ \xleftarrow{\cap[N]} \end{array} & H^{n-k}(N). \end{array}$$

We refer to

$$\begin{aligned} f^!: H_k(N; \mathbb{Z}) &\rightarrow H_k(M; \mathbb{Z}) \\ \sigma &\mapsto f^*(\text{PD}_N(\sigma)) \cap [M] \end{aligned}$$

as the *Umkehr map* of f .⁷⁵⁰

We recall that on page 571 we had defined the degree of such a map f as the unique number $\deg(f) \in \mathbb{Z}$ with $f_*([M]) = \deg(f) \cdot [N]$. We had obtained the following results:

- (1) In Proposition 33.16 we had shown that for any closed n -dimensional oriented manifold M there exists a degree-one map $f: M \rightarrow S^n$.
- (2) In Proposition 33.20 we had seen that if $\deg(f) = \pm 1$, then $f_*: \pi_1(M) \rightarrow \pi_1(N)$ is an epimorphism.
- (3) In Lemma 44.5 we had used cup-products to show that is no map $f: S^4 \rightarrow S^2 \times S^2$ of non-zero degree.

We now add the following proposition to our list on results on degree-one maps.

Proposition 46.9. *Let $f: M \rightarrow N$ be a map between closed oriented n -dimensional manifolds that is of degree 1. Then for every $k \in \mathbb{N}_0$ the following holds:*

- (1) *the map $f_* \circ f^!$ is the identity, in particular the map $f^!: H_k(N) \rightarrow H_k(M)$ splits,*
- (2) *the image $f^!(H_k(N))$ is a subsummand of $H_k(M)$, i.e. there exists a subgroup Q of $H_k(M)$ such that $H_k(M) = f^!(H_k(N)) \oplus Q$.*

In Question 33.23 we asked whether there exists a degree-one map from $S^1 \times S^2$ to $\mathbb{RP}^3$. The group $H_1(\mathbb{RP}^3; \mathbb{Z}) \cong \mathbb{Z}_2$ is not a subsummand of $H_1(S^1 \times S^2) \cong \mathbb{Z}$, thus we can finally give a negative answer to Question 33.23

Proof. Let $f: M \rightarrow N$ be a map between closed oriented n -dimensional manifolds that is of degree 1. Let $\sigma \in H_k(N)$. We have

$$\begin{aligned} (f_* \circ f^!)(\sigma) &= f_*(f^*(\text{PD}_N(\sigma)) \cap [M]) &= \text{PD}_N(\sigma) \cap f_*([M]) &= \text{PD}_N(\sigma) \cap [N] &= [\sigma]. \\ &\quad \uparrow &\quad \uparrow &\quad \uparrow \\ &\text{Lemma 44.13} &\text{since } \deg(f) = 1 &\text{definition of } \text{PD}_N \end{aligned}$$

The second statement follows from combining the first statement with Lemma 28.2. $\square$

⁷⁵⁰This map is indeed called “Umkehr map” in English. The name stems from the fact that it is map on homology that “goes the wrong way”.

46.3. Algebraic intersection numbers of submanifolds. The following definition is a variation on the definition that we had given on page 53 and that generalizes the definition on page 58.

Definition. Let M be a manifold.

- (1) Let W, W' be two submanifolds. We say that W and W' are *smoothly isotopic*, if there exists a *smooth isotopy* from W to W' , i.e. a smooth map

$$\begin{aligned} H: W \times [0, 1] &\rightarrow M \\ (z, t) &\mapsto H(z, t) \end{aligned}$$

such that the following hold:

- (a) For each $P \in W$ we have $H(P, 0) = P$.
 (b) The map

$$\begin{aligned} W &\rightarrow M \\ P &\mapsto H(P, 1) \end{aligned}$$

defines a diffeomorphism from W to W' ,

- (c) for each $t \in [0, 1]$ the map $W \rightarrow M$ given by $P \mapsto H(P, t)$ is an embedding.

If W and W' are two oriented submanifolds, then we also want the following condition to be satisfied:

- (3) the diffeomorphism $W \rightarrow W'$ in (2) is orientation-preserving.
 (2) Let A and B be two submanifolds of M . We say A and B can be made disjoint if there exist disjoint submanifolds A' and B' of M such that A' is isotopic to A and B' is isotopic to B .

Example. We consider the three submanifolds A, B and C of the torus that are illustrated in Figure 372 on the left. As is shown in the same figure on the right one can make A and B disjoint.

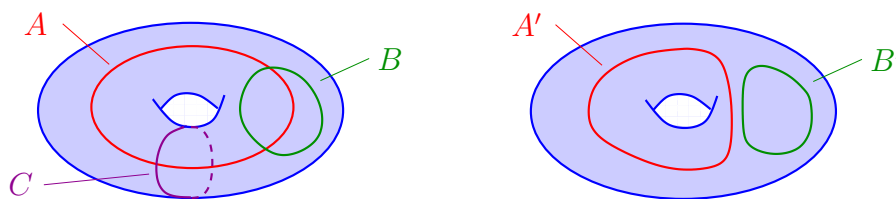


FIGURE 372.

It does not look like we can make the submanifolds A and C disjoint. But how can we prove it? We will record this question for later.

Question 46.10. *Can we make the submanifolds A and C , that are illustrated in Figure 372, disjoint?*

In an attempt to answer Question 46.10 we will introduce the algebraic intersection number of submanifolds. But before we can do so we need to introduce a fairly straightforward.

Definition. Let M be an m -dimensional manifold.⁷⁵¹ Let $A \subset M$ a submanifold of dimension k and let B be a submanifold $V \subset M$ of the complementary dimension $m - k$.

- (1) Let $P \in A \cap B$. We say that A and B *intersect transversely in P* if the equality⁷⁵²

$$T_P A \oplus T_P B = T_P M$$

holds.

- (2) We say that A and B *intersect transversely* if they intersect transversally in each intersection point.

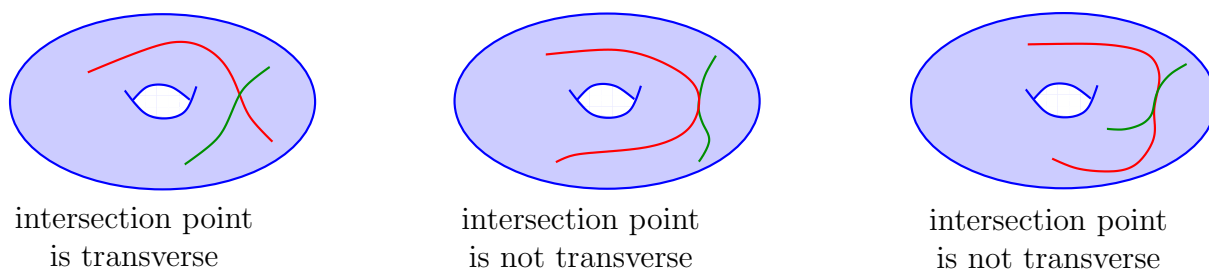


FIGURE 373.

Lemma 46.11. Let M be an m -dimensional manifold, let $A \subset M$ a submanifold of dimension k and let B be a submanifold $V \subset M$ of the complementary dimension $m - k$.

- (1) Let $P \in A \cap B$ be an intersection point. If P is a transverse intersection point, then there exists a chart $\Phi: U \rightarrow B_m(0)$ of M around P such that $\Phi(A) = B_m(0) \cap (\mathbb{R}^k \times 0)$ and $\Phi(B) = B^m(0) \cap (0 \times \mathbb{R}^{n-k})$.
- (2) If A and B intersect transversally and if A, B and M are compact, then the intersection $A \cap B$ consists of finitely many points.

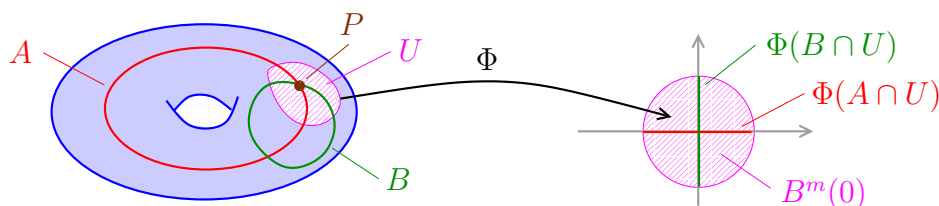


FIGURE 374. Illustration of Lemma 46.11.

Proof. Let M be an m -dimensional manifold, let $A \subset M$ a submanifold of dimension k and let B be a submanifold $V \subset M$ of the complementary dimension $m - k$.

⁷⁵¹Note that in the following we work with manifolds, we will need the smooth atlas in the subsequent discussion.

⁷⁵²Since A is a submanifold of M we can view the tangent space $T_P A$ as a subspace of $T_P M$.

- (1) Let $P \in A \cap B$ be a transverse intersection point. The proof proceeds in two steps:
- (a) After applying a chart of the submanifold A we can assume that M is an open subset of $\mathbb{R}^n$, that $P = 0$ and that $A = (\mathbb{R}^k \times \{0\}) \cap M$.
 - (b) After applying a map of the form

$$\begin{aligned} M &\rightarrow \mathbb{R}^m \\ (x, y) &\mapsto (x + f(y), y) \end{aligned}$$

we can assume that the images of A and B are of the desired form.

We refer to Figure 375 for an illustration of the argument. The details for the argument will be filled in in exercise sheet 4.

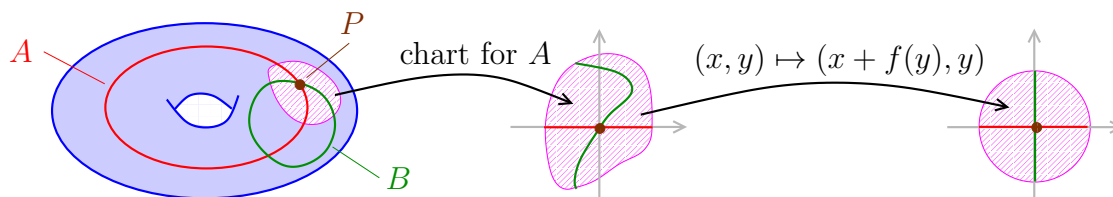


FIGURE 375. Illustration for the proof of Lemma 46.11.

- (2) We now assume that A and B intersect transversally and that A, B and M are compact. The set $A \cap B$ is the intersection of two closed sets, hence it is closed. It follows from (1) that for each $P \in A \cap B$ there exists an open neighborhood U_P of P in M such that U_P contains no other intersection point of A and B . We have an open covering

$$M = (M \setminus A \cap B) \cup \bigcup_{P \in A \cap B} U_P.$$

Since M is compact we can cover M by finitely many of these sets. In particular we only need finitely many sets U_P to cover M . But each U_P contains only one intersection point, hence the number of intersection points is finite. $\square$

Definition. Let M be an m -dimensional oriented manifold, let $A \subset M$ an oriented closed submanifold of dimension k and let B be an oriented closed submanifold $V \subset M$ of the complementary dimension $m - k$. Suppose that A and B intersect transversally.

- (1) To each intersection point P of A and B we assign the sign $\epsilon_P = 1$, if

positive basis of $T_P A$ followed by positive basis of $T_P B$ = positive basis of $T_P M$.

Otherwise we put $\epsilon_P = -1$.⁷⁵³

⁷⁵³It is a straightforward exercise in linear algebra to show that the definition of the sign ϵ_P does not depend on the choice of the positive bases of $T_P U$ and $T_P V$.

(2) We define

$$\text{geometric intersection number of } A \text{ and } B := \#(A \cap B)$$

and

$$\text{algebraic intersection number of } A \text{ and } B := A \cdot B := \sum_{P \in A \cap B} \epsilon_P \in \mathbb{Z}.$$

If A and B do not intersect, then the algebraic intersection number of A and B is of course zero.

Examples.

- (1) In Figure 376 we show pieces of oriented submanifolds of the manifold $\mathbb{R}^2$. On the left the sign of the intersection point is positive, in the other one it is negative.

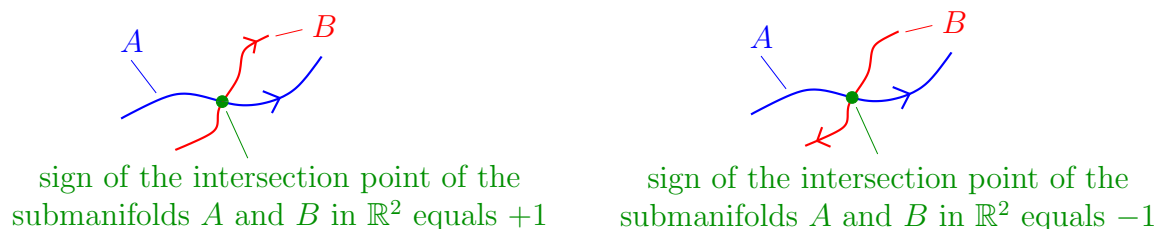


FIGURE 376.

- (2) We consider the surface of genus 2 with the orientation where the normal vector “sticks outward” furthermore we consider the oriented submanifolds A, B, C, D and E that are illustrated in Figure 377. For the submanifolds A and B we have

$$\text{geometric intersection number of } A \text{ and } B = 1,$$

the sign of the intersection point is negative, hence

$$\text{algebraic intersection number of } A \text{ and } B = -1.$$

For the submanifolds A and C we have

$$\text{geometric intersection number of } A \text{ and } C = 2,$$

the intersection points have opposite signs, therefore

$$\text{algebraic intersection number of } A \text{ and } C = -1.$$

finally for the submanifolds D and E we have

$$\text{geometric intersection number of } D \text{ and } E = 2,$$

here both intersection points have negative sign,

$$\text{algebraic intersection number of } D \text{ and } E = -2.$$

We would be able to answer Question 46.10 if we could prove the following statement.

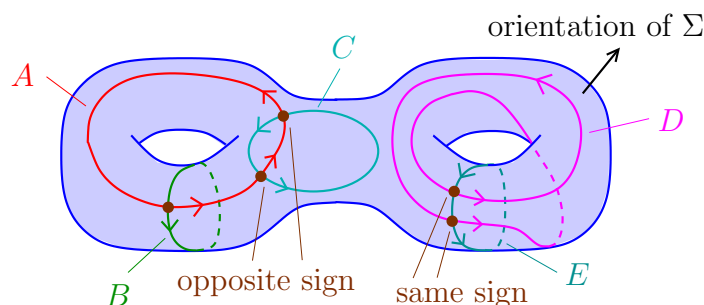


FIGURE 377.

Statement 46.12. *Let M be an oriented compact manifold and let A, A' and B, B' be two pairs of closed oriented submanifolds with $\dim(A) + \dim(B) = \dim(A') + \dim(B') = \dim(M)$. Then the following holds:*

$$A, A' \text{ and } B, B' \text{ are isotopic} \implies A \cdot B = A' \cdot B'.$$

But how can we prove such a statement? In Figure 378 we sketch an idea: We break the isotopies from A to A' and from B to B' into “smaller isotopies” such that each smaller isotopy either does not change the type of intersection points, or two intersection points of opposite sign get created or erased. Even though the idea is fairly convincing, it is very hard to turn this idea into a proper proof.⁷⁵⁴⁷⁵⁵ So for the time being we will not attempt to provide a proof for Statement 46.12.

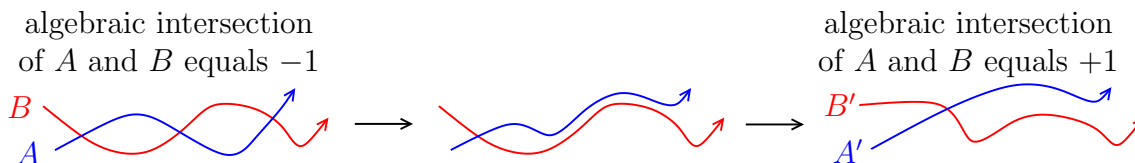


FIGURE 378.

We continue with developing the theory of algebraic intersection numbers a little further. We have the following elementary lemma which follows quite easily from the definitions and some basic linear algebra. The lemma will be proved in exercise sheet 4.

Lemma 46.13. *Let M be an m -dimensional oriented manifold, let A and B be two oriented closed submanifolds of dimensions k and $m - k$ respectively. If A and B intersect transversally then*

$$B \cdot A = (-1)^{\dim(A) \cdot \dim(B)} A \cdot B.$$

⁷⁵⁴You will see it is already hard to prove the statement in dimension 2, let alone in higher dimensions.

⁷⁵⁵Even though we will not follow this approach, it was certainly this type of picture mathematicians had in mind when they developed the concept of algebraic intersection numbers in the early 20th century.

Now suppose that M and N are oriented manifolds of dimensions m respectively n . We had seen in Lemma 19.3 of Analysis IV that $M \times N$ is an $(m + n)$ -dimensional manifold. It is straightforward to show that for each $P \in M$ and $Q \in N$ the obvious map⁷⁵⁶

$$T_P M \oplus T_Q N \rightarrow T_{(P,Q)}(M \times N)$$

is an isomorphism. If $\{v_1, \dots, v_m\}$ is a positive basis for $T_P M$ and $\{w_1, \dots, w_n\}$ is a positive basis for $T_Q N$, then we endow $T_{(P,Q)}(M \times N)$ with the orientation that makes the above isomorphism orientation-preserving. One can easily show that the orientations of M and N induce an orientation of $M \times N$. The following lemma now follows almost immediately from the definitions.

Lemma 46.14. *Let M and N be oriented compact manifolds and let $P \in M$ and $Q \in N$. Then we have*

$$\text{algebraic intersection number of } M \times \{Q\} \text{ and } \{P\} \times N = 1.$$

Example. In Figure 379 we illustrate the statement that in the torus $S^1 \times S^1$ we have

$$\text{algebraic intersection number of } S^1 \times \{1\} \text{ and } \{1\} \times S^1 = 1.$$

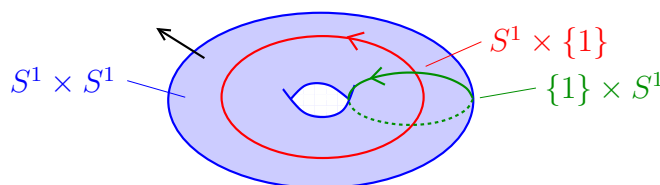


FIGURE 379.

On page 278 we had introduced the notion of a complex manifold, which was the same as a manifold except that we now demanded that all transition maps are biholomorphisms. The same way we can now introduce the notion of a complex submanifold.

Example. We consider the complex projective plane $\mathbb{CP}^n$. On page 278 we had seen that $\mathbb{CP}^n$ is a complex manifold of complex dimension n . It follows easily from the definitions that for any $k \leq n$ the subset

$$A_k := \{[z_0 : \dots : z_k : 0 : \dots : 0] \mid (z_0, \dots, z_k) \in \mathbb{C}^{k+1} \setminus \{0\}\}$$

is a complex submanifold of complex dimension k . Similarly

$$B_k := \{[0 : \dots : 0 : z_{n-k} : \dots : z_n] \mid (z_{n-k}, \dots, z_n) \in \mathbb{C}^{k+1} \setminus \{0\}\}$$

is also a complex submanifold of complex dimension k . Evidently both A_k and B_k are biholomorphic to $\mathbb{CP}^k$.

⁷⁵⁶What is the obvious map?

In the proof of Proposition 17.4 we had seen that every complex manifold comes with a canonical orientation. The following lemma once again follows easily from the definitions and some elementary linear algebra.

Lemma 46.15. *Let M be a compact complex manifold of complex dimension and let A and B be two compact complex submanifolds of M with $\dim(A) + \dim(B) = \dim(M)$. Then the following hold:*

- (1) *The sign at every transverse intersection point of A and B is positive.*
- (2) *If A and B intersect transversally, then the algebraic intersection number equals the geometric intersection number.*

Example. We consider again the complex projective plane $\mathbb{CP}^n$. Let $k \leq n$. As above we consider

$$A_k := \{[z_0 : \dots : z_k : 0 : \dots : 0] \mid (z_0, \dots, z_k) \in \mathbb{C}^{k+1} \setminus \{0\}\}$$

is a complex submanifold of complex dimension k . Similarly

$$B_{n-k} := \{[0 : \dots : 0 : z_k : \dots : z_n] \mid (z_k, \dots, z_n) \in \mathbb{C}^{n-k+1} \setminus \{0\}\}$$

Then

$$A_k \cap B_k = [0 : \dots : 0 : 1 : 0 : \dots : 0]$$

$\uparrow$
 the k -th entry

consists of a single point P . We leave it as an exercise to verify that A_k and B_{n-k} intersect transversally at P . It follows from Lemma 46.15 that $A_k \cdot B_{n-k} = 1$.

46.4. The cup-product and algebraic intersection numbers. Let M be a closed oriented m -dimensional topological manifold. By the Poincaré Duality Theorem 45.1 we know that for any k the map

$$\begin{aligned} H^{m-k}(M) &\rightarrow H_k(M) \\ \varphi &\mapsto \varphi \cap [M] \end{aligned}$$

is an isomorphism. We denote by

$$\text{PD}: H_k(M) \rightarrow H^{m-k}(M)$$

the inverse of this isomorphism.

The following proposition relates the cup-product to algebraic intersection numbers.

Proposition 46.16. *Let M be a closed oriented m -dimensional manifold. Furthermore let $X \subset M$ be an oriented closed k -dimensional submanifold and let Y be an oriented closed $(m-k)$ -dimensional submanifold. We denote by $[X] \in H_k(M)$ and $[Y] \in H_{m-k}(M)$ the images of the fundamental classes of the oriented manifolds A and B under the inclusion maps. If X and Y intersect transversally, then*

$$X \cdot Y = \left\langle \overbrace{\text{PD}([X]) \cup \text{PD}([Y])}^{\in H^m(M)}, [M] \right\rangle.$$

$\in H^{m-k}(M) \quad \in H^{m-k}(M)$

In the following we will only sketch a proof of Proposition 46.16. A full proof is provided in [Br, Theorem VI.11.9] or [DK, Theorem 10.32].

Sketch of proof. Throughout the proof we write $l = m - k$. Put differently, $k + l = m$. We first consider the case that $X \cap Y = \emptyset$ and that X and Y have a trivial tubular neighborhoods $A \times \overline{B}^l$ and $B \times \overline{B}^k$. Since X and Y are disjoint we can assume that the tubular neighborhoods are disjoint.

Evidently we have $X \cdot Y = 0$. So it suffices to show that $\text{PD}([X]) \cup \text{PD}([Y]) = 0$. The following is the key claim.

Claim. We denote by $i^*: H^l(M, M \setminus X \times B^l) \rightarrow H^l(M)$ and $j^*: H^l(M, M \setminus Y \times B^l) \rightarrow H^l(M)$ the obvious maps. Then the following hold:

- (1) there exists an $\alpha \in H^l(M, M \setminus X \times B^l)$ with $\text{PD}([X]) = i^*(\alpha)$, and
- (2) there exists a $\beta \in H^k(M, M \setminus Y \times B^k)$ with $\text{PD}([Y]) = i^*(\beta)$.

We first show that the claim implies the desired statement. By Lemma 44.8 the following diagram commutes

$$\begin{array}{ccc}
 H^k(M, M \setminus X \times B^l) \times H^l(M, M \setminus Y \times B^k) & \xrightarrow{\cup} & H^m(M, \overbrace{M \setminus X \times B^l \cup M \setminus Y \times B^k}^{=M}) \\
 \begin{array}{c} \alpha \downarrow \\ \text{PD}([X]) \end{array} & & \downarrow \\
 \begin{array}{c} \beta \downarrow \\ \text{PD}([Y]) \end{array} & & \\
 H^k(M) \times H^l(M) & \xrightarrow{\cup} & H^m(X).
 \end{array}$$

Since the right upper group is trivial we see that the cup-product of $\text{PD}([X])$ and $\text{PD}([Y])$ has to vanish.

Now we turn to the proof of the claim. Evidently it suffices to provide the proof for the first statement. We consider the following diagram

$$\begin{array}{ccccc}
 H^l(M, M \setminus X \times B^l) & \xrightarrow{\quad} & & & H^l(M) \\
 \downarrow \cap[M] \cong & \swarrow \text{excision} & & & \downarrow \cap[M] \cong \\
 & H^l(X \times \overline{B}^l, X \times S^{l-1}) & \xleftarrow[\cong]{\text{K\"unneth}} & H^0(X) & \\
 & \downarrow \cap[X \times \overline{B}^l] & & \downarrow \cap[X] & \\
 & H_k(X \times \overline{B}^l, X \times S^{l-1}) & \xleftarrow[\cong]{\text{K\"unneth}} & H_k(X) & \\
 \downarrow \cap[M] \cong & \swarrow \text{excision} & & \searrow & \downarrow \cap[M] \cong \\
 H_k(M, M \setminus X \times B^l) & \xleftarrow{\quad} & & & H_k(M).
 \end{array}$$

We claim that the diagram commutes:

- (1) the outer rectangle commutes by Lemma 44.13,

- (2) if follows easily from the definitions that the inner rectangle and the two parallelograms to the left and at the bottom commute.

It is now straightforward to see that the image of $[X]^* \in H^0(X)$ in $H^l(M, M \setminus X \times B^l)$ gets sent via “right-down” to the element in $H_k(M)$ that is given by $[X]$. This concludes the proof of the claim.

- (1) Next we consider the case that $X \cap Y = \emptyset$ where X and Y are not required any more to have a trivial tubular neighborhood. This case is somewhat similar to the previous case, see the aforementioned references for details.
- (2) Next we consider the case that $M = \overline{B}^k \times \overline{B}^{n-k}$ with $X = \overline{B}^k \times \{0\}$ and $Y = \{0\} \times \overline{B}^{n-k}$. In that case we have $X \cdot Y = +1$. On the other hand, as in Proposition 46.2 we can see that the pairing

$$H_k(\overline{B}^k \times \overline{B}^{n-k}; \partial \overline{B}^k \times \overline{B}^{n-k}) \times H_l(\overline{B}^k \times \overline{B}^{n-k}; \overline{B}^k \times \partial \overline{B}^{n-k}) \rightarrow H_k(\overline{B}^k \times \overline{B}^{n-k}; \partial(\overline{B}^k \times \overline{B}^{n-k}))$$

is non-singular. This shows that $\text{PD}([X]) \cup \text{PD}([Y]) = \pm[M]$. A more careful sign analysis shows that the sign is in fact $+1$.

- (3) Finally we have to deal with the general case. For each $P \in X \cap Y$ we can pick a closed neighborhood U_P of P with the following properties:

- (a) The U_P are disjoint.
- (b) For each $P \in X \cap Y$ there exists a chart $\Phi_P: U_P \rightarrow \overline{B}^k \times \overline{B}^{n-k}$ such that $\Phi_P(X \cap U_P) = \overline{B}^k \times \{0\}$ and $\Phi_P(Y \cap U_P) = \{0\} \times \overline{B}^{n-k}$ as oriented submanifolds. Furthermore Φ_P is orientation-preserving if and only if $\epsilon_P = +1$.

Using a Mayer–Vietoris argument we can now break the problem into the cases (1) and (2). We will not execute the details. $\square$

+++++ perhaps should work more with properly embedded submanifolds +++++
 Proposition 46.16 is very versatile. First we show that we can use it to provide a proof of Statement 46.12. More precisely, we have the following proposition.

Proposition 46.17. *Let M be an oriented compact manifold.*

- (1) *If A, A' and B, B' are two pairs of closed oriented submanifolds with $\dim(A) + \dim(B) = \dim(A') + \dim(B') = \dim(M)$, then the following holds:*

$$A, A' \text{ and } B, B' \text{ are isotopic} \implies A \cdot B = A' \cdot B'.$$

- (2) *If A and B are two submanifolds with $\dim(A) + \dim(B) = \dim(M)$ that can be made disjoint, then $A \cdot B = 0$.*

Proof. Let M be an oriented compact manifold and let A, A' and B, B' are two pairs of closed oriented submanifolds with $\dim(A) + \dim(B) = \dim(A') + \dim(B') = \dim(M)$. We have

$$\begin{array}{ccccc} A \cdot B & = & \langle \text{PD}([A]), \text{PD}([B]), [M] \rangle & = & \langle \text{PD}([A']), \text{PD}([B']), [M] \rangle & = & A' \cdot B'. \\ & \uparrow & & \uparrow & & \uparrow & \\ & \text{Proposition 46.16} & & \text{Proposition 25.4} & & \text{Proposition 46.16} & \end{array}$$

This concludes the proof of the first statement. The second statement is an immediate consequence of the first statement. $\square$

Example.

- (1) We can now answer Question 46.10. The submanifolds A and C , that are illustrated in Figure 372, cannot be made disjoint since their algebraic intersection number is non-zero.
- (2) Fibered manifold, show that image of fiber is dual to element in $H^1(M(X, f)) = \text{Hom}(\pi_1(M(X, f)), \mathbb{Z})$ corresponding to the fibration.
- (3) It is tempting to ask whether the converse to Proposition 46.17 (2) holds. Playing around with curves on a surface shows that this looks dubious. More precisely, consider the oriented curves A and B on the surface of genus 2 that are illustrated in Figure 380. Their algebraic intersection number is zero, but it is not clear how one can make them disjoint.⁷⁵⁷

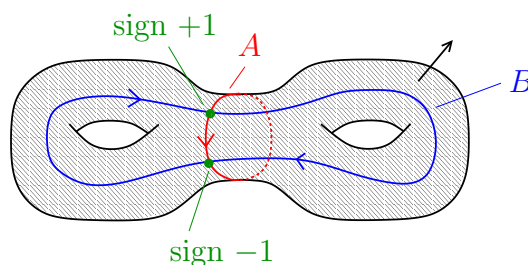


FIGURE 380.

Now we will use Proposition 46.16 “the other way”, i.e. we will use it to determine cup-products using our geometric understanding of submanifolds.

Example. We consider the torus $M = S^1 \times S^1$ together with the submanifolds $A = S^1 \times \{0\}$ and $B = \{0\} \times S^1$ which we endow with the usual orientations. Then

$$\begin{array}{ccccccc} \langle \text{PD}([A]) \cup \text{PD}([B]), [S^1 \times S^1] \rangle & = & A \cdot B & = & +1. \\ & \uparrow & & \uparrow & \\ & \text{Proposition 46.16} & & \text{Figure 381} & \end{array}$$

We also have⁷⁵⁸

$$\begin{array}{ccccccc} \langle \text{PD}([A]) \cup \text{PD}([A']), [S^1 \times S^1] \rangle & = & \langle \text{PD}([A]) \cup \text{PD}([A']), [S^1 \times S^1] \rangle & = & A \cdot A' & = & 0. \\ & \uparrow & & \uparrow & \uparrow & & \\ & \text{since } A \text{ and } A' \text{ are homotopic} & & \text{by Proposition 46.16} & \text{Figure 381} & & \\ & & & \text{since } A \text{ and } A' \text{ are transverse} & & & \end{array}$$

⁷⁵⁷Fun exercise: show that the curves cannot be made disjoint.

⁷⁵⁸There are many other ways to see that $\text{PD}([A]) \cup \text{PD}([A]) = 0$, the easiest is to use the antisymmetry of the cup-product on first cohomology.



FIGURE 381.

In Proposition 46.4 we had determined the cup-product on complex projective space. More precisely, given $n \in \mathbb{N}$ we had shown that the ring $(H^*(\mathbb{CP}^n; \mathbb{Z}), \cup)$ is isomorphic to $\mathbb{Z}[x]/(x^{n+1})$. The following lemma gives a slightly more precise statement regarding the x^i and it also gives a much more geometric proof of Proposition 46.4.

Proposition 46.18. *Let $n \in \mathbb{N}$. For $s \in \{0, \dots, n\}$ we consider the inclusion map*

$$\begin{aligned} i_{s,n}: \mathbb{CP}^s &\rightarrow \mathbb{CP}^n \\ [z_0 : \dots : z_s] &\mapsto [z_0 : \dots : z_s : 0 : \dots : 0]. \end{aligned}$$

We write $x^k = \text{PD}(i_{n-k,n}([\mathbb{CP}^{n-k}]) \in H^{2k}(\mathbb{CP}^n; \mathbb{Z})$. Then for any $k, l \in \{0, \dots, n\}$ we have

$$x^k \cup x^l = \begin{cases} x^{k+l}, & \text{if } k+l \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Let $n \in \mathbb{N}$. For $s \in \{0, \dots, n\}$ we also consider the inclusion map

$$\begin{aligned} t_{s,n}: \mathbb{CP}^s &\rightarrow \mathbb{CP}^n \\ [z_0 : \dots : z_s] &\mapsto [0 : \dots : 0 : z_0 : \dots : z_s]. \end{aligned}$$

Now let $k, l \in \{0, \dots, n\}$. If $k+l > n$, then $x^k \cup x^l \in H^{2(k+l)}(\mathbb{CP}^n; \mathbb{Z}) = 0$. Next we consider the case that $k+l = n$. We consider the map

$$\begin{aligned} \Phi: \mathbb{CP}^l \times [0, 1] &\rightarrow \mathbb{CP}^n \\ ([z], t) &\mapsto [(z \cdot \cos(t \cdot \frac{\pi}{2}), 0) + (0, z \cdot \sin(t \cdot \frac{\pi}{2}))]. \end{aligned}$$

Then

$$\begin{aligned} x^k \cup x^l &= \text{PD}(i_{n-k,n}([\mathbb{CP}^{n-k}])) \cup \text{PD}(i_{n-l,n}([\mathbb{CP}^{n-l}])) \\ &\quad \uparrow \\ &\text{by definition} \\ &= \text{PD}(i_{n-k,n}([\mathbb{CP}^{n-k}])) \cup \text{PD}(t_{n-l,n}([\mathbb{CP}^{n-l}])) \\ &\quad \uparrow \\ &\text{by Proposition 25.4, since } \Phi \text{ is a homotopy from } i_{n-l,l} \text{ to } t_{n-l,n} \\ &= i_{n-k,n}(\mathbb{CP}^{n-k}) \cdot t_{n-l,n}(\mathbb{CP}^{n-l}) = 1. \\ &\quad \uparrow \qquad \qquad \qquad \uparrow \\ &\text{by Proposition 46.16} \qquad \qquad \text{see page 742 for details} \end{aligned}$$

The case that $k+l < n$ is dealt with in a very similar fashion. We will fill in the details in exercise sheet 5. $\square$

We will consider more examples in the next chapter.

46.5. Representing homology classes by manifolds.

Proposition 46.19. *Let M be a compact, orientable n -dimensional manifold. Then every class in $H_{n-1}(M, \partial M; \mathbb{Z})$ can be represented by a submanifold.*

Definition. critical point

Theorem 46.20. (Sard's Theorem)

Theorem 46.21. *Preimage of regular value of a smooth map $p: M \rightarrow S^1$ is an $(n-1)$ -dimensional submanifold of M . If M is orientable, then N is also orientable.⁷⁵⁹*

Now we can provide the proof of Proposition 46.19

Proof of Proposition 46.19. Let M be a compact, orientable n -dimensional manifold and let $\sigma \in H_{n-1}(M, \partial M; \mathbb{Z})$. We write $\phi = \text{PD}(\sigma) \in H^1(M; \mathbb{Z})$. By Proposition 43.7 there exists a smooth map $p: M \rightarrow S^1$ such that $p_*: \pi_1(M) \rightarrow \pi_1(S^1) = \mathbb{Z}$ agrees with ϕ under the identification $H^1(M; \mathbb{Z}) = \text{Hom}(\pi_1(M), \mathbb{Z})$.

By Sard's Theorem 46.20 there exists a regular value $z \in S^1$. We write $W := p^{-1}(z)$. By Theorem 46.21 we know that N is an orientable $(n-1)$ -dimensional manifold. We claim that $[W] = \pm\sigma \in H_{n-1}(M, \partial M; \mathbb{Z})$. Lemma 44.13?? $\square$

For completeness sake we also state the following theorem. It is proved for example in [GS, Proposition 1.2.3]. We will not make use of it.

Theorem 46.22. *Let M be a compact, orientable n -dimensional manifold. Then every class in $H_{n-2}(M, \partial M; \mathbb{Z})$ can be represented by a submanifold.*

⁷⁵⁹In Analysis II we had formulated this theorem for smooth maps $p: U \rightarrow \mathbb{R}$ where U is an open subset of $\mathbb{R}^n$. It is straightforward to generalize this theorem to the case that M is a manifold and to functions that take values in S^1 .

47. THE INTERSECTION FORM OF TOPOLOGICAL MANIFOLDS

47.1. The definition of the intersection form of topological manifolds. Let M be a compact oriented m -dimensional topological manifold. Recall that by the Poincaré Duality Theorem 45.1 we know that for any k the map

$$\begin{aligned} H^{m-k}(M, \partial M) &\rightarrow H_k(M) \\ \varphi &\mapsto \varphi \cap [M] \end{aligned}$$

is an isomorphism. As in the previous chapter we denote by

$$\text{PD}: H_k(M) \rightarrow H^{m-k}(M, \partial M)$$

the inverse of this isomorphism. Given an abelian group A we denote in the following by $FA := A/\text{torsion}$ the maximal torsion-free quotient of A . Note that any form $A \times B \rightarrow \mathbb{Z}$ descends to a form $FA \times FB \rightarrow \mathbb{Z}$.

Now suppose that the manifold M is even-dimensional, i.e. that it is of dimension $m = 2n$ for some n . We refer to the map⁷⁶⁰⁷⁶¹

$$\begin{aligned} Q_M: FH_n(M) \times FH_n(M) &\rightarrow \mathbb{Z} \\ (a, b) &\mapsto \langle \text{PD}(a) \cup \text{PD}(b), [M] \rangle \end{aligned}$$

as the *intersection form of M* .

Remark. Since $\text{PD}: H_n(M) \rightarrow H^n(M)$ is an isomorphism it is clear that the intersection form is basically just a reformulation of the cup-product pairing on cohomology. The big psychological advantage of homology is that very often one can “see homology classes”. For example, in many cases homology classes are represented by submanifolds. In the following we will see that in many situation we can use these submanifolds to easily determine the intersection form of a given manifold.

In the following lemma we summarize some of the key properties of the intersection form.

Proposition 47.1. *Let M be a compact oriented $2n$ -dimensional topological manifold. Then the following hold:*

- (1) *If n is even, then the intersection form Q_M is symmetric, otherwise it is antisymmetric.*⁷⁶²
- (2) *If M is connected and if we make the usual identification $H_0(M; \mathbb{Z}) = \mathbb{Z}$ given by the augmentation map, see Proposition 24.5, then for any $a, b \in FH_n(M; \mathbb{Z})$ we have*

$$\underbrace{Q_M(a, b)}_{\in \mathbb{Z}} = \underbrace{(\text{PD}(a) \cup \text{PD}(b)) \cap [M]}_{\in H_0(M; \mathbb{Z})}.$$

⁷⁶⁰Note that $H_n(M)$ is not necessarily torsion-free. For example $M = S^1 \times \mathbb{RP}^3$ is orientable, but it follows easily from the Künneth Formula 37.5 and the calculation of the homology groups of $\mathbb{RP}^3$ in Lemma 29.19 that $H_2(S^1 \times \mathbb{RP}^3) \cong \mathbb{Z}_2$.

⁷⁶¹Note that $\text{PD}(a)$ and $\text{PD}(b)$ lie in $H^n(M, \partial M)$, hence the cup-product lies in $H^{2n}(M, \partial M)$. The fundamental class $[M]$ is an element in $H_{2n}(M, \partial M)$, hence $\langle \text{PD}(a) \cup \text{PD}(b), [M] \rangle$ is indeed defined.

⁷⁶²Here antisymmetric means that $Q_M(b, a) = -Q_M(a, b)$ for all $a, b \in H_n(M; \mathbb{Z})$.

- (3) If M is closed and if $H_{n-1}(M)$ is torsion-free, then the intersection form is non-singular.

Proof. The first statement is an immediate consequence of Proposition 44.1. The second statement follows from the relationship between the Kronecker-pairing and the cap-product that we had discussed in Lemma 44.11. Finally the third statement is an immediate consequence of (2) and of Proposition 46.2. $\square$

In the next lemma we want to see to what degree the isometry type of the intersection form is an invariant of the homeomorphism type of the manifold.

Lemma 47.2. *Let $f: M \rightarrow N$ be a homeomorphism between two oriented compact even-dimensional topological manifolds. Then⁷⁶³*

$$Q_M \cong \begin{cases} Q_N, & \text{if } f \text{ is orientation-preserving,} \\ -Q_N, & \text{if } f \text{ is orientation-reversing.} \end{cases}$$

Proof. ⁷⁶⁴ Let $f: M \rightarrow N$ be a homeomorphism between two oriented compact $2n$ -dimensional topological manifolds. We write $\varepsilon = +1$ if f is orientation-preserving and $\varepsilon = -1$ if f is orientation-reversing. We start out with the following claim.

Claim. For any $c \in H_n(M; \mathbb{Z})$ we have $\text{PD}_N(f_*(c)) = \varepsilon \cdot f^*(\text{PD}_M(c))$.

So let $c \in H_n(M; \mathbb{Z})$. We have

$$\begin{array}{ccccc} f^*(\text{PD}_M(c)) \cap [M] & = & f^*(\text{PD}_M(c)) \cap (\varepsilon \cdot (f_*)^{-1}([M])) & = & \varepsilon \cdot (f_*(\text{PD}_M(c)) \cap [M]) & = & \varepsilon f_*(c). \\ \uparrow & & & \uparrow & & \uparrow & \\ \text{definition of } \varepsilon & & & \text{Lemma 44.13} & & \text{definition of } \text{PD}_M & \end{array}$$

It follows from the definition of PD_N that we get the desired equality. This concludes the proof of the claim.

Now let $a, b \in FH_n(M; \mathbb{Z})$. Then

$$\begin{array}{ccccc} & \text{Proposition 46.16 (2)} & & & \\ & \downarrow & & & \\ Q_N(f_*(a), f_*(b)) & = & \text{PD}(f_*(a)) \cup \text{PD}(f_*(b)) \cap [N] & & \\ & = & (\varepsilon \cdot f^*(\text{PD}_M(a)) \cup \varepsilon \cdot f^*(\text{PD}_M(b))) \cap (\varepsilon \cdot f_*([M])) & = & \varepsilon \cdot Q_M(a, b). \\ \uparrow & & & \uparrow & \\ \text{by the claim} & & & \text{by Lemma 44.7 and since } \varepsilon^3 = \varepsilon & \end{array}$$

The lemma now follows from the fact that $f_*: FH_n(M; \mathbb{Z}) \rightarrow FH_n(N; \mathbb{Z})$ is an isomorphism. $\square$

⁷⁶³Given a form Q on an abelian group H we evidently denote by $-Q$ the form given by $(-Q)(v, w) := -Q(v, w)$ for all $v, w \in H$.

⁷⁶⁴The proof is perhaps slightly longer than one might expect. In a sense the statement of the lemma is trivial, but we have to pay a price for using the slightly awkward Poincaré Duality isomorphism $H_k(M; \mathbb{Z}) \rightarrow H^{2n-k}(M; \mathbb{Z})$.

Let M be a closed oriented $2n$ -dimensional topological manifold. By Proposition 39.2 we know that $FH_n(M)$ is a finitely generated torsion-free abelian group. By the classification of finitely generated abelian groups it is a free-abelian group. We pick a basis $v_1, \dots, v_k$ and we refer to the matrix

$$(Q_M(v_i, v_j))_{i,j=1,\dots,k}$$

as a *matrix representing the intersection form*.

Examples.

- (1) We first consider the torus $M = S^n \times S^n$. By the Künneth Formula 37.5 we know that a basis for $H_n(S^n \times S^n; \mathbb{Z})$ is given by $[S^n] \times 0, 0 \times [S^n]$. Using the argument of the proof of Lemma 46.3 or alternatively using Proposition 46.16 and an argument as on page 745 we see that with respect to this basis the intersection form of $S^n \times S^n$ is represented by the matrix⁷⁶⁵

$$\begin{pmatrix} 0 & 1 \\ (-1)^n & 0 \end{pmatrix}.$$

- (2) It follows from Proposition 46.18 that the intersection form $H_2(\mathbb{CP}^2; \mathbb{Z}) \times H_2(\mathbb{CP}^2; \mathbb{Z}) \rightarrow \mathbb{Z}$ is represented by the 1×1 -matrix (1).
- (3) Let M be a closed oriented $2n$ -dimensional topological manifold. As usual we denote by $-M$ the same manifold with the opposite orientation. It follows immediately from the definitions that $Q_{-M} = -Q_M$. In particular the intersection form $H_2(\mathbb{CP}^2; \mathbb{Z}) \times H_2(\mathbb{CP}^2; \mathbb{Z}) \rightarrow \mathbb{Z}$ is represented by the 1×1 -matrix (-1) .

47.2. The intersection form and the connected sum operation. Recall that on page 56 we had introduced the connected sum of manifold. In the following we generalize this definition to the connected sum of topological manifolds.

To do so we start out with a definition.

Definition. Let f be a map from an n -dimensional topological manifold X to an n -dimensional topological manifold Y .

- (1) We say f is an *embedding* if $f: X \rightarrow Y$ is injective and if $f(\partial X)$ is a topological submanifold of Y .
- (2) A *closed ball in an n -dimensional topological manifold X* is the image of an embedding $\overline{B}^n \rightarrow X$.

⁷⁶⁵If one is totally honest, then one has to say that the argument of Lemma 46.3 shows that, with respect to the given matrix, the intersection form is represented either by the matrix

$$\begin{pmatrix} 0 & (-1)^n \\ 1 & 0 \end{pmatrix} \quad \text{or by the matrix} \quad \begin{pmatrix} 0 & 1 \\ (-1)^n & 0 \end{pmatrix}.$$

Which of the two matrices it is, is indeed a little trick to ascertain, and Proposition 46.16 can be used to show that it is the latter matrix.

- (3) If X and Y are oriented topological manifolds, then we say that an embedding $f: X \rightarrow Y$ is *orientation-preserving* if the image of $[Y]$ under the map

$$H_n(X, \partial X; \mathbb{Z}) \rightarrow H_n(X, \setminus f(Y)) \xleftarrow{\cong} H_n(X \setminus f(Y \setminus Y), f(\partial Y); \mathbb{Z})$$

agrees with $f_*([Y])$.

We can now define the connected sum of oriented topological manifolds.

Definition. Let M and M' be two oriented n -dimensional topological manifolds and furthermore let $\varphi: \overline{B}^n \rightarrow M \setminus \partial M$ and $\varphi': \overline{B}^n \rightarrow M' \setminus \partial M'$ be two embeddings where φ is orientation-preserving and φ' is orientation-reversing. We define the *connected sum* M and M' as

$$M \# M' := (M \setminus \varphi(B^n)) \sqcup (M' \setminus \varphi'(B^n)) / \{\varphi(P) = \varphi'(P) \mid P \in S^{n-1}\}.$$

We have the following proposition.

Proposition 47.3. *Let M and M' be two connected, oriented n -dimensional topological manifolds.*

- (1) *The homeomorphism type of $M \# M'$ does not depend on the choices.*
- (2) *The connected sum $M \# M'$ is an n -dimensional topological manifold.*
- (3) *If M and M' are oriented, then we can equip $M \# M'$ with an orientation such that the maps $M \setminus \varphi(B^n) \rightarrow M \# M'$ and $M' \setminus \varphi'(B^n) \rightarrow M \# M'$ are orientation-preserving.*

Proof.

- (1) The proof that the homeomorphism type of the connected sum of two n -dimensional topological manifolds does not depend on the choices of the embeddings of the balls is given by combining the argument [Ro, p. 42] together with the “annulus theorem” that was proved in 1969 by Rob Kirby [Ki] for $n \neq 4$ and it was proved in 1982 by Frank Quinn [Qu, Ed] for $n = 4$. Note though that the proof that the connected sum of topological manifolds is well-defined is significantly harder than the proof in the case of manifolds that we had stated in results than the one Proposition 2.7. In particular the proof that the connected sum of topological 4-manifolds was provided only in 1982.
- (2) It is fairly straightforward to show that the connected sum is again a connected orientable topological manifold, see e.g. [Le3, Proposition 6.6] for details.
- (3) The last statement can be proved using a Mayer–Vietoris argument. We leave the details as an exercise to the reader. $\square$

In Proposition 9.9 we had shown that $\pi_1(M \# N)$ is the free product of $\pi_1(M)$ and $\pi_1(N)$ and in Proposition 33.12 we had shown that for each $k \neq 0, n$ we have an isomorphism $H_k(M) \oplus H_k(N) \xrightarrow{H} H_k(M \# N)$. Both propositions were initially stated for manifolds, but verbatim the same proof also works for topological manifolds.

We now have a very similar result for the intersection form of $M \# N$.

Proposition 47.4. *Let M and N be two closed, oriented, connected $2n$ -dimensional topological manifolds. Then there exists an isometry*

$$Q_M \oplus Q_N \cong Q_{M \# N}.$$

Remark. If all homology classes in $H_n(M; \mathbb{Z})$ and $H_n(N; \mathbb{Z})$ can be represented by submanifolds, then the statement of Proposition 47.4 is clear.

Example. We consider the closed surface Σ_3 of genus 3. In Figure 35 we had argued that Σ_3 is the connected sum of g tori. It thus follows from Proposition 47.4 and the discussion on page 750 that the intersection form on $H_1(\Sigma_3; \mathbb{Z}) \cong \mathbb{Z}^6$ is represented by the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}.$$

We obtain the same result in a slightly less rigorous, but arguably more convincing fashion using Proposition 46.16. More precisely, we consider the surface Σ_3 that is illustrated in Figure 382. We give it the usual orientation. We already seen in the discussion of Figure 338 that the oriented curves $a_1, b_1, a_2, b_2, a_3, b_3$ illustrated in Figure 382 represent a basis for $H_1(\Sigma)$. It follows easily from Proposition 46.16 that with respect to this basis the intersection form is represented by the aforementioned 6×6 -matrix.

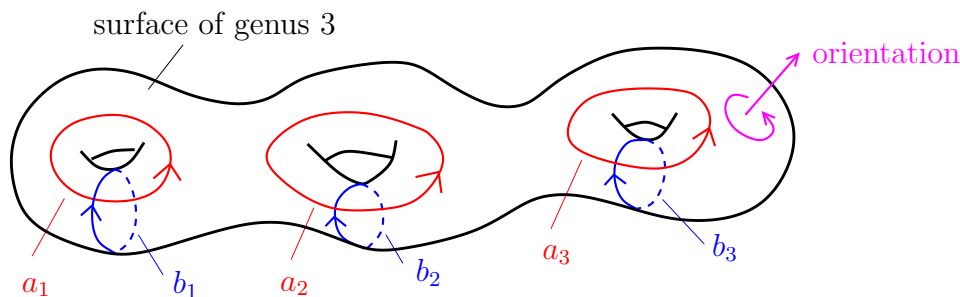


FIGURE 382.

In the proof of Proposition 47.4 we will need the following lemma.

Lemma 47.5. *Let M be an oriented m -dimensional manifold and let W be a compact m -dimensional submanifold. We equip W with the induced orientation from Lemma 39.25. We write $\mathring{W} = W \setminus \partial W$. We fix the following notation:*

- (1) *We denote by i the inclusion map $W \rightarrow M$ and also the inclusion map of the pairs $(W, \partial W) \rightarrow (M, M \setminus \mathring{W})$,*
- (2) *we denote by $p: (M, \partial M) \rightarrow (M, M \setminus \mathring{W})$ the projection map.*

Then the following diagram commutes⁷⁶⁶

$$\begin{array}{ccc} H_k(W; \mathbb{Z}) & \xrightarrow{\text{PD}_W} & H^{m-k}(W, \partial W; \mathbb{Z}) \xrightarrow{\cong \circ (i^*)^{-1}} H^{m-k}(M, M \setminus \overset{\circ}{W}; \mathbb{Z}) \\ \downarrow i_* & & \downarrow p^* \\ H_k(M; \mathbb{Z}) & \xrightarrow{\text{PD}_M} & H^{m-k}(M, \partial M; \mathbb{Z}). \end{array}$$

Proof. By definition of the isomorphisms PD_W and PD_M proving the lemma is equivalent to proving that the following diagram commutes.

$$\begin{array}{ccc} H_k(W; \mathbb{Z}) & \xleftarrow[\cong]{\cap[W]} & H^{m-k}(W, \partial W; \mathbb{Z}) \xleftarrow[\cong]{i^*} H^{m-k}(M, M \setminus \overset{\circ}{W}; \mathbb{Z}) \\ \downarrow i_* & & \downarrow p^* \\ H_k(M; \mathbb{Z}) & \xleftarrow[\cong]{\cap[M]} & H^{m-k}(M, \partial M; \mathbb{Z}). \end{array}$$

So let $\beta \in H_{m-k}(M, M \setminus \overset{\circ}{W}; \mathbb{Z})$. We have

$$\begin{array}{ccccccc} & & \text{by Lemma 39.25} & & & & \\ & & \downarrow & & \xrightarrow{\in H_k(M)} & & \\ i_*(i^*(\beta) \cap [W]) & = & \beta \cap i_*([W]) & = & \beta \cap p_*([M]) & = & p_*(p^*(\beta) \cap [M]) = p^*(\beta) \cap [M]. \\ \uparrow & & & & \uparrow & & \uparrow \\ \text{Lemma 44.12 applied to} & & & & \text{Lemma 44.12 applied to} & & \text{since } p: M \rightarrow M \text{ is id} \\ i: (W, \partial W) \rightarrow (M, M \setminus \overset{\circ}{W}) & & & & p: (M, \partial M) \rightarrow (M, M \setminus \overset{\circ}{W}) & & \text{we have } p_* = \text{id}_* = \text{id} \end{array}$$

We have thus shown that the above diagram does indeed commute. $\square$

Now we can provide the proof of Proposition 47.4.

Proof of Proposition 47.4. Let X and Y be two oriented, connected $2n$ -dimensional topological manifolds with $\partial X \cong S^{n-1}$ and $\partial Y \cong S^{n-1}$. We equip ∂X and ∂Y with the orientation coming from Theorem 39.26. We identify ∂X and ∂Y via an orientation-reversing homeomorphism φ . We write $Z = X \cup_{\varphi} Y$.

We start out with the following claim.

Claim. The inclusions $i: X \rightarrow Z$ and $j: Y \rightarrow Z$ induce an isometry

$$Q_X \oplus Q_Y \cong Q_Z.$$

In the following we view X and Y as submanifolds of Z . By Proposition 33.12 the inclusions induce an isomorphism $H_n(X; \mathbb{Z}) \oplus H_n(Y; \mathbb{Z}) \rightarrow H_n(Z; \mathbb{Z})$. Evidently this descends to an isomorphism $H_n(X; \mathbb{Z}) \oplus H_n(Y; \mathbb{Z}) \rightarrow H_n(Z; \mathbb{Z})$. It suffices to prove the following three statements.

- (1) for $a, b \in H_n(X; \mathbb{Z})$ we have $Q_X(a, b) = Q_Z(i_*(a), i_*(b))$,
- (2) for $a, b \in H_n(Y; \mathbb{Z})$ we have $Q_Y(a, b) = Q_Z(j_*(a), j_*(b))$,
- (3) for $a \in H_n(X; \mathbb{Z})$ and $b \in H_n(Y; \mathbb{Z})$ we have $Q_Z(i_*(a), j_*(b)) = 0$.

⁷⁶⁶Here we use that i induces an isomorphism on cohomology by the cohomological version of the Excision Theorem 39.6 for Topological Manifolds.

We start out with the proof of the first statement. So let $a, b \in H_n(X; \mathbb{Z})$. We denote by $p: (Z, \partial Z) \rightarrow (Z, Z \setminus \mathring{X})$ the projection map, and we denote by $i: (X, \partial X) \rightarrow (Z, Z \setminus \mathring{X})$ the inclusion map. Note that it is a consequence of the Excision Theorem 39.6 for Topological Manifolds that $i_*: H_k(X, \partial X; \mathbb{Z}) \rightarrow H_k(Z, Z \setminus \mathring{X}; \mathbb{Z})$ is an isomorphism for every $k \in \mathbb{N}_0$. Then

$$\begin{aligned}
 Q_X(a, b) &= (\text{PD}_X(a) \cup \text{PD}_X(b)) \cap [X] = ((i^*)^{-1}(\text{PD}_X(a)) \cup (i^*)^{-1}(\text{PD}_X(b))) \cap \underbrace{i_*([X])}_{=p_*([Z])} \\
 &\quad \uparrow \qquad \qquad \qquad \uparrow \\
 &\quad \text{Proposition 46.16 (2)} \qquad \text{Lemma 44.12 applied to } i: (X, \partial X) \rightarrow (Z, Z \setminus \mathring{X}), \\
 &\qquad \qquad \qquad \qquad \qquad \qquad \text{together with Lemma 44.7} \\
 &= (p^*(i^*)^{-1}(\text{PD}_X(a)) \cup p^*(i^*)^{-1}(\text{PD}_X(b))) \cap [Z] \\
 &\quad \uparrow \\
 &\quad \text{Lemma 44.12 applied to } p, \text{ together with Lemma 44.7} \\
 &= (\text{PD}_Z(i_*(a)) \cup \text{PD}_Z(i_*(b))) \cap [Z] = Q_Z(i_*(a), i_*(b)). \\
 &\quad \uparrow \\
 &\quad \text{Lemma 47.5}
 \end{aligned}$$

The second statement is of course proved the same way. We turn to the proof of the third statement. So let $a \in H_n(X; \mathbb{Z})$ and $b \in H_n(Y; \mathbb{Z})$. The argument above shows that we can compute $Q_Z(i_*(a), j_*(b))$ in terms of the cup-product in Z of an element in the image $H^n(Z, Z \setminus \mathring{X}) \rightarrow H^k(Z)$ and an element in the image of $H^n(Z, Z \setminus \mathring{Y}) \rightarrow H^k(Z)$. But $(Z \setminus \mathring{X}) \cup (Z \setminus \mathring{Y}) = Z$. So it follows from Lemma 44.8 that this cup-product is zero. This concludes the proof of the claim. $\square$

Example. On page 750 we had seen that the intersection form of $\mathbb{CP}^2$ is represented by the 1×1 -matrix (1) and we had seen that the intersection form of $-\mathbb{CP}^2$ is represented by the 1×1 -matrix (-1) . It follows from Proposition 47.4 that

$$Q_{\mathbb{CP}^2 \# \mathbb{CP}^2} \cong \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ whereas } Q_{\mathbb{CP}^2 \# -\mathbb{CP}^2} \cong \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Also recall that on page 750 we had shown that

$$Q_{S^2 \times S^2} \cong \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The following question arises.

Question 47.6. *Are the forms defined by the matrices*

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

isometric?

In the next section we will develop the theory of non-singular forms to answer Question 47.6.

47.3. Algebra: Classification of non-singular forms over the real numbers. The following definition is a straightforward generalization of the definition of a form on a real vector space, as given on page 283.

Definition. Let R be a commutative ring.

- (1) A *form over R* is a pair (V, g) consisting of a finitely-generated free R -module V together with a map

$$g: V \times V \rightarrow R$$

that is bilinear, i.e. for all $u, v, w \in V$ and $\mu, \lambda \in R$ we have

$$\begin{aligned} g(\lambda u + \mu v, w) &= \lambda g(u, w) + \mu g(v, w) \\ g(u, \lambda v + \mu w) &= \lambda g(u, v) + \mu g(u, w) \end{aligned}$$

and that is symmetric, i.e. for all $v, w \in V$ we have

$$g(v, w) = g(w, v).$$

Often we just suppress V from the notation and we just speak of a form g .

- (2) Given a form (V, g) we define the *rank* $\text{rank}(g)$ as the rank of the finitely-generated free R -module V .
- (3) Let (V, g) be a form. We consider the homomorphism

$$\begin{aligned} V &\rightarrow V^* = \text{Hom}_R(V, R) \\ v &\mapsto \left(\begin{array}{ccc} W & \rightarrow & R \\ w & \mapsto & g(v, w) \end{array} \right). \end{aligned}$$

This leads us to the following two definitions:

- (a) If this homomorphism is a monomorphism, then we say that g is *non-degenerate*.
- (b) If this homomorphism is an isomorphism, then we say that g is *non-singular*.⁷⁶⁷
- (4) We say that two forms (V, g) and (W, h) are *isometric*, written $(V, g) \cong (W, h)$, if there exists an isometry, i.e. an isomorphism $\varphi: V \rightarrow W$ such that $g(v_1, v_2) = h(\varphi(v_1), \varphi(v_2))$ for all $v_1, v_2 \in V$.
- (5) Given a form (V, g) and a subset $W \subset V$ we write

$$W^\perp := \{v \in V \mid g(v, w) = 0 \text{ for all } w \in W\}.$$

- (6) Given a form (V, g) and a subspace W we denote by $g|_W$ the form that is given by restricting the map $g: V \times V \rightarrow R$ to $W \times W \rightarrow R$.

The proof of the following elementary lemma is left as an exercise to the reader.

Lemma 47.7. *Let R be a commutative ring and let A be a symmetric $n \times n$ -matrix over R . We consider the map*

$$\begin{aligned} \lambda(A): R^n \times R^n &\rightarrow R \\ (v, w) &\mapsto v^T A w. \end{aligned}$$

Then the following hold:

⁷⁶⁷Sometimes non-singular forms are also called *unimodular forms*.

- (1) $\lambda(A)$ is a form over R ,
- (2) every symmetric over R is isometric to $\lambda(A)$ for some symmetric $n \times n$ -matrix over R ,
- (3) the form $\lambda(A)$ is non-degenerate if and only if $\det(A) \neq 0$,
- (4) the form $\lambda(A)$ is non-singular if and only if $\det(A)$ is a unit in R .

In the following, given a symmetric matrix A over a commutative ring we will often just say refer to the form $\lambda(A)$ as the *form* A .

Examples. We now list some symmetric integral matrices with determinant equal to ± 1 . These define in particular non-singular forms over $\mathbb{R}$ but also over $\mathbb{Z}$.

- (1) The 1×1 -matrices $(+1)$ and (-1) .
- (2) The slightly mysterious 8×8 -matrix

$$E_8 = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \end{pmatrix}$$

has determinant equal to $+1$.

- (3) The matrix

$$H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

has determinant -1 .

In the following definition we show how to get new forms out of given forms.

Definition.

- (1) Given forms (V, g) and (W, h) we denote by $(V \oplus W, g \oplus h)$ the form given by the free-module $V \oplus W$ and $(g \oplus h)(v \oplus w, v' \oplus w') := g(v, v') + h(w, w')$.
- (2) Given a form (V, g) we write $-g$ for the form given by $(-g)(v, w) = -g(v, w)$.
- (3) Given a form (V, g) and $n \in \mathbb{Z}$ we define

$$\begin{aligned} n \cdot (V, g) &:= \underbrace{(V, g) \oplus \cdots \oplus (V, g)}_{n\text{-summands}}, & \text{if } n \geq 0, \\ n \cdot (V, g) &:= \underbrace{(V, -g) \oplus \cdots \oplus (V, -g)}_{|n|\text{-summands}}, & \text{if } n \leq 0. \end{aligned}$$

We start out with the classification of forms over the reals.

Definition. Let (V, g) be a form over the real numbers.

- (1) We say g is positive-definite if $g(v, v) > 0$ for all $v \neq 0$.

- (2) We say g is negative-definite if $g(v, v) < 0$ for all $v \neq 0$.
- (3) We say g is indefinite if g is neither positive-definite nor negative-definite.

Examples.

- (1) The form $(+1)$ is positive-definite and the form (-1) is negative-definite.
- (2) We leave it as an exercise to verify that E_8 is positive-definite.
- (3) The form H is easily seen to be indefinite.

Definition. Let (V, g) be a form over the real numbers. We define the *nullity* $\sigma(g)$ as

$$\nu(g) := \max\{\dim(W) \mid W \text{ is a subspace of } V \text{ such that } g|_W \text{ is the zero form}\}.$$

Furthermore we define

$$\begin{aligned} b^+(g) &:= \text{maximal dimension of a subspace } W \text{ of } V \text{ such that } g|_W \text{ is positive-definite,} \\ b^-(g) &:= \text{maximal dimension of a subspace } W \text{ of } V \text{ such that } g|_W \text{ is negative-definite.} \end{aligned}$$

Finally we define the *signature* $\sigma(g)$ as

$$\sigma(g) := b_+(g) - b_-(g).$$

It is clear from the definitions that the nullities and the signatures of two isometric forms over the real numbers are the same.

The following proposition should be familiar from Linear Algebra II.

Proposition 47.8. *Let A be a real symmetric $n \times n$ -matrix..*

- (1) *For every matrix $P \in \text{GL}(n, \mathbb{R})$ the forms $\lambda(A)$ and $\lambda(PAP^t)$ are isometric.*
- (2) *There exists a matrix $P \in \text{GL}(n, \mathbb{R})$ such that PAP^t is a diagonal matrix.*
- (3) *We have*

$$\nu(\lambda(A)) = \text{number of eigenvalues that are equal to zero}$$

and

$$\sigma(\lambda(A)) = \text{number of positive eigenvalues} - \text{number of negative eigenvalues}.$$

*Hereby we count the eigenvalues with multiplicities.*⁷⁶⁸

Example. We write

$$H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ with } P =$$

Proof.

- (1) It is straightforward to verify that the map $v \mapsto P^t v$ defines an isometry from $\lambda(PAP^t)$ to $\lambda(A)$.
- (2) This statement is proved in [Re, Chapter 7].
- (3) This statement is elementary to prove, we leave the details as an exercise to the reader. □

⁷⁶⁸Recall that for a diagonalizable matrix, e.g. a symmetric matrix, the algebraic and geometric multiplicity of an eigenvalue are the same.

The following theorem should also be well-known from Linear Algebra II. Once again we refer to [Re, Theorem 7.2] for a proof.

Theorem 47.9. (Sylvester's Theorem) *Let (V, g) and (W, h) be two forms over the real numbers. Then*

$$(V, g) \text{ and } (W, h) \text{ are isometric} \iff \nu(g) = \nu(h) \text{ and } \sigma(g) = \sigma(h).$$

[Re, Theorem 7.2]

Lemma 47.10. *Let g be a non-degenerate form over a finite-dimensional vector space V . If there exists a subspace W such that $W = W^\perp$, then $\text{sign}(g)$*

signature, Sylvester Theorem (classifies non-singular forms over $\mathbb{R}$), see e.g. Gompf-Stipsicz [GS] p. 9 for a discussion. [MH]

It now follows from Lemma 47.2 that $S^2 \times S^2$, $\mathbb{CP}^2 \# \mathbb{CP}^2$ and $\mathbb{CP}^2 \# -\mathbb{CP}^2$ are pairwise non-homeomorphic.

this shows in particular that in the definition of a connected sum we needed to work with *oriented manifolds*

47.4. Algebra: Non-singular forms over the integers.

Definition. A form (V, g) over the integers is called *even* if for all $v \in V$ we have $(v, v) \equiv 0 \pmod{2}$. Otherwise the intersection form is called *odd*. We refer to the property of g being even or odd as the *parity* of the form.

Examples.

- (1) The forms $(+1)$ and (-1) are odd.
- (2) The form E_8 is even.
- (3) The form H is even
- (4) The direct sum of an odd form with any other form is also odd.

We refer to [GS, Theorem 1.2.14] or alternatively to [MH, Theorem II.5.3] for a proof of the following theorem.

Theorem 47.11. *Two non-singular indefinite forms Q_1 and Q_2 are isometric if and only if they have the same rank, signature and parity.*

As an immediate corollary to Theorem 47.11 we obtain the classification of non-singular indefinite *odd* forms. More precisely, we have the following corollary.

Corollary 47.12. *Let g be a non-singular indefinite odd form. Then*

$$g \cong b^+(g)\langle 1 \rangle \oplus b^-(g)\langle -1 \rangle.$$

Proof. Let g be a non-singular indefinite odd form. The rank, signature and parity agree of the two non-singular indefinite odd forms in the statement of the corollary agree. So these two forms are isometric by Theorem 47.11. $\square$

Now we turn to the study of even forms. We have the following, perhaps somewhat surprising lemma.

Lemma 47.13. *If (V, g) is an even non-singular form over $\mathbb{Z}$, then $\sigma(g) \equiv 0 \pmod{8}$.*

Proof. Given a form (W, h) over $\mathbb{Z}$ we say that a vector $w \in W$ is *characteristic* if for all $u \in W$ we have

$$g(u, w) \equiv g(w, w) \pmod{2}.$$

We make the following observations:

- (1) It follows easily from the definitions that W is even if and only if 0 is a characteristic vector.
- (2) If $W = k \cdot (1) \oplus l \cdot (-1)$, then it is straightforward to show that a vector $u \in \mathbb{Z}^{k+l}$ is characteristic if and only if all its entries are odd.

We will now prove the following slightly stronger claim.

Claim. Let (V, g) be a form over $\mathbb{Z}$ and let $v \in V$ be a characteristic vector. Then

$$g(v, v) \equiv \sigma(g) \pmod{8}.$$

Let $v \in V$ be a characteristic vector of (V, g) . It is straightforward to verify that $v + e_- + e_+$ is a characteristic vector in $(V \oplus \mathbb{Z} \oplus \mathbb{Z}, Q \oplus (1) \oplus (-1))$, where e_- and e_+ are generators of the first respectively the second $\mathbb{Z}$ -summand. By Corollary 47.12 there exists an isometry

$$\Phi: g \oplus (1) \oplus (-1) \cong (b^+(g) + 1) \cdot (1) \oplus (b^-(g) + 1) \cdot (-1).$$

We write $u := \Phi(v)$. Evidently u is a characteristic vector for the form on the right hand side. We have

$$\begin{aligned} & \text{straightforward calculation} \\ & \downarrow \\ g(v, v) &= (g \oplus (1) \oplus (-1))(v + e_- + e_+, v + e_- + e_+) \\ &= ((b^+(g) + 1) \cdot (1) \oplus (b^-(g) + 1) \cdot (-1))(u, u) \\ &= \sum_{i=1}^{b^+(g)+1} \underbrace{(i\text{-th entry of } u)^2}_{\text{odd integer}} + \sum_{i=b^+(g)+2}^{b^+(g)+b^-(g)+2} (-1) \cdot \underbrace{(i\text{-th entry of } u)^2}_{\text{odd integer}} \\ &\equiv \sum_{i=1}^{b^+(g)+1} 1 + \sum_{i=b^+(g)+2}^{b^+(g)+b^-(g)+2} (-1) = (b^+(g) + 1) - (b^-(g) + 1) = \sigma(g) \pmod{8}. \\ & \uparrow \\ & \text{since for each odd number } k \text{ we have } k^2 \equiv 1 \pmod{8} \end{aligned}$$

This concludes the proof of the claim. □

Now we can provide the classification of non-singular indefinite *even* forms.

Corollary 47.14. *Let g be a non-singular indefinite even form. Then*

$$g \cong \frac{\sigma(g)}{8} E_8 \oplus \frac{\text{rank}(g) - |\sigma(g)|}{2} H.$$

Proof. Recall that $\text{rank}(E_8) = \sigma(E_8) = 8$ and that $\text{rank}(H) = 2$ and $\sigma(H) = 0$. Let g be a non-singular indefinite odd form. It follows from the above observation regarding E_8 and H that the rank, signature and parity agree of the two non-singular indefinite even forms in the statement of the corollary agree. So these two forms are isometric by Theorem 47.11. $\square$

We now turn to the study of definite non-singular forms over the integers. We start out with some more examples of such forms. The following lemma is proved in [MH, Lemmas 6.1 and 6.2] and [Se2, p. 51].

Lemma 47.15. *We denote by $e_1, \dots, e_{4m}$ the standard basis for $\mathbb{Z}^{4m}$. We denote by Γ_{4m} the submodule of $\mathbb{Z}^{4m}$ spanned by the vectors $\{e_i + e_j\}_{i,j=1,\dots,4m}$ and $\frac{1}{2}(e_1 + \dots + e_{4m})$. Then the following statements hold:*

- (1) *The restriction of the usual form $\langle, \rangle$ on $\mathbb{Z}^{4m}$ to Γ_{4m} is a positive-definite non-singular form.*
- (2) *The form Γ_{4m} is odd if and only if m is odd.*
- (3) *The form Γ_8 is isometric to E_8 .*
- (4) *The form Γ_{16} is not isometric to $E_8 \oplus E_8$.*

It turns out that there are many more definite forms than indefinite forms and that there is no classification of such forms. For example, by [MH, p. 28] and [Se2, p. 55] we have the following table

dimension	8	16	24	32	40
number of non-isomorphic even positive-definite forms	1	2	24	$\geq 10^7$	$\geq 10^{51}$

Hereby in dimension 8 the unique example is $\Gamma_8 = E_8$ whereas in dimension 16 the unique examples are $\Gamma_8 \oplus \Gamma_8 = E_8 \oplus E_8$ and Γ_{16} .

47.5. Back to topology. We have now introduced lots of invariants, the fundamental group, homotopy groups, homology groups and the cohomology groups with the cup-product. The question now arises, whether we are done at some point, or whether this list of ever new invariants continues forever.

In the different formulations of the Poincaré Conjecture we already seen a case where the above algebraic invariants were enough to identify topological manifolds (namely spheres), up to homeomorphism.

Remark. The intersection form of manifolds is one of the most important invariants of manifolds:

- (1) Mike Freedman proved in 1981 that each form is the intersection form of a topological closed 4-manifold. Furthermore he proved that if M and N are two simply connected closed topological 4-manifolds, then $+++++$ only for even forms!!!!!!
 $+++++$ also note that homotopy equivalence was proved by Whitehead,

see Theorem 1.2.25 in [GS]

M is homeomorphic to $N \iff Q_M$ is isometric to Q_N .

This theorem contains Theorem 23.20 as a special case.⁷⁶⁹

- (2) Simon Donaldson has 1982 proved that the statements of Freedman do not hold for 4-manifolds.⁷⁷⁰ For example he showed that the form which is given by the matrix E_8 according to Freedman the intersection form of a closed topological 4-manifold. But Donaldson showed that it is not the intersection form of a differentiable 4-manifold. Furthermore Donaldson showed that there exists simply-connected 4-dimensional manifolds for which the diffeomorphism type is *not* determined by its intersection form.

+++++++ 10/8 Theorem and 11/8 Conjecture ++++++ Mike Freedman and Simon Donaldson both obtain in 1986 a Fields-medal for these complementary results.

47.6. The signature and finite covers. show that signature zero for 4k-manifolds that are boundary (see Lueck: Seite 159, 153, Scorpan p. 121)

every closed oriented 3-manifold is boundary of a 4-manifold, see [Sv, Corollary 2.5]

Novikov additivity

signature of a manifold multiplicative under finite covers, at least of manifolds, how about topological manifolds?

⁷⁶⁹Indeed, we will later see that if M is a simply-connected n -dimensional topological manifold with $H_2(M) = \cdots = H_n(M) = 0$, then $\pi_i(M) = 0$ for $i = 1, \dots, n-1$. This means, by definition, that M is a homotopy sphere.

⁷⁷⁰The Fields medalist Michael Atiyah described in [At87] the work of Simon Donaldson as follows: “In 1982, when he was a second-year graduate student, Simon Donaldson proved a result that stunned the mathematical world.”

48. TUBULAR NEIGHBORHOODS AND ALEXANDER DUALITY

48.1. Tubular neighborhoods. We start out with the following somewhat technical definition.

Definition. Let M be an m -dimensional manifold and let K be a k -dimensional submanifold.

- (1) A *tubular neighborhood* of K is an m -dimensional submanifold $N(K)$ of N with the following properties:
 - (a) K is contained in the interior of $N(K)$,
 - (b) the boundary of the manifold $N(K)$ equals the boundary of $N(K)$ as a subspace of M ,⁷⁷¹
 - (c) K is a deformation retract of $N(K)$,
 - (d) $\partial N(K)$ is deformation retract of $N(K) \setminus K$.

We refer to $\overset{\circ}{N}(K) := N(K) \setminus K$ as an *open tubular neighborhood* of K .

- (2) A tubular neighborhood $N(K)$ is called *trivial* if there exists a diffeomorphism $\Phi: K \times \overline{B}^{m-k} \rightarrow N(K)$ with $\Phi(P, 0) = P$ for all $P \in K$.

Examples.

- (1) In the Tubular Neighborhood Theorem 2.2 we had shown that if M is an orientable manifold, then every 1-dimensional submanifold admits a trivial tubular neighborhood.
- (2) Let $M = [0, 1] \times [-1, 1] / \sim$ be the Möbius band let $K = [0, 1] \times \{0\} \subset M$ be the submanifold that is given by the central circle. In Figure 383 we will illustrate a tubular neighborhood for K . At least from the picture it is pretty clear that K does not admit a trivial tubular neighborhood.⁷⁷²

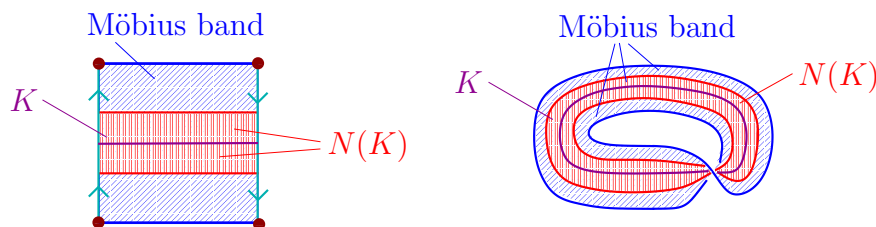


FIGURE 383.

Theorem 48.1. (General Tubular Neighborhood Theorem) *Let M be a manifold and let K be a submanifold. If K is closed,⁷⁷³ then K admits a tubular neighborhood.*

⁷⁷¹Put differently, we have the condition $\partial N(K) = \partial N(K)$ for the two different meanings of “ ∂ ” of a submanifold.

⁷⁷²What would be a rigorous proof for the statement that K does not admit a trivial tubular neighborhood?

⁷⁷³The submanifold K is also a manifold and by “ K is closed” we mean that K is a closed manifold, i.e. K is compact with $\partial K = \emptyset$.

Proof. As for the previous Tubular Neighborhood Theorem 2.2 we refer to [Ko, Section III.2] or alternatively to [Br, Theorem 11.14] and [Le2, Theorem 6.24] for a proof. We leave it as an exercise to bridge the difference in our notation and language to the one provided by the books. $\square$

Remark.

- (1) The tubular neighborhood provided by the General Tubular Neighborhood Theorem 48.1 is unique in an appropriate sense. We refer to [Ko, Section III.3] for details.
- (2) The General Tubular Neighborhood Theorem 48.1 is stated for manifolds instead of the more general context of topological manifolds. The definition of a tubular neighborhood also makes sense for a k -dimensional topological submanifold K of an n -dimensional topological manifold M .
 - (a) If $k = n - 2$, i.e. if K is a codimension 2 submanifold, then some type of tubular neighborhood seems to exist, see e.g. [DV, Theorem 6.8.1] and [FQ, p. 137] for details. When it comes to topological manifolds one should be very careful with references. For example the quoted references do not use the notion of tubular neighborhood as introduced above, it is possible that they only give a neighborhood that satisfies (a), (c) and (d).
 - (b) If $k = n - 3$, i.e. if K is a codimension 3 submanifold, then, according to [DV, p. 347], a tubular neighborhood does not exist in general.

Example. Every tubular neighborhood of $\mathbb{CP}^1$ in $\mathbb{CP}^2$ is non-trivial. Proof: intersection numbers.

We conclude this section with the following lemma.

Lemma 48.2. *Let K be a submanifold of a manifold M . Let $N(K)$ be a tubular neighborhood of K . Then $M \setminus \mathring{N}(K)$ is deformation retract of $M \setminus K$.*

This lemma might appear somewhat technical. But it is quite useful. For example if M and K are compact, then we obtain that the non-compact manifold $M \setminus K$ is homotopy equivalent to the compact manifold $M \setminus \mathring{N}(K)$.

Proof. We denote by r the deformation retraction from $N(K) \setminus K$ to $\partial N(K)$. We consider the map

$$\begin{aligned} f: (M \setminus K) \times [0, 1] &\rightarrow M \setminus K \\ (P, t) &\mapsto \begin{cases} P, & \text{if } P \notin N(K), \\ r(P, t), & \text{if } P \in N(K) \setminus K. \end{cases} \end{aligned}$$

We leave it as an exercise to verify that f is indeed continuous.⁷⁷⁴ Once continuity is verified it is straightforward to show that f is indeed a deformation retraction from $M \setminus K$ to $M \setminus \mathring{N}(K)$. $\square$

⁷⁷⁴This exercise is not entirely straightforward, for example we need the condition (b) of a tubular neighborhood.

48.2. Alexander duality. Let X be a non-empty topological space. In the proof of Proposition 24.5 we had seen that the following is a chain complex

$$\begin{aligned} \dots \xrightarrow{\partial_3} C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\epsilon} \mathbb{Z} \rightarrow 0. \\ \sum_{i=1}^k n_i \sigma_i \mapsto \sum_{i=1}^k n_i. \end{aligned}$$

In a similar fashion to reduced homology we now define the k -th reduced cohomology group $\tilde{H}^k(X)$ of X to be the k -cohomology group of the above chain complex.

The following lemma is the analogue of Lemma 26.1 (1) and Lemma 26.11 (1). We leave the proof as an exercise to the reader.

Lemma 48.3. *Let X be a empty topological space. Then the following statements hold:*

(1) *For every $n \in \mathbb{N}_0$ we have*

$$H^n(X) \cong \begin{cases} \tilde{H}^0(X) \oplus \mathbb{Z}, & \text{if } n = 0, \\ H^n(X), & \text{if } n \neq 0. \end{cases}$$

(2) *For every point P in X and every $n \in \mathbb{N}_0$ we have an isomorphism $\tilde{H}^n(X) \cong H^n(X, P)$.*

Theorem 48.4. (Alexander Duality Theorem) *Let K be a closed submanifold of S^n . Then for any $i \in \mathbb{N}_0$ we have an isomorphism*

$$\tilde{H}_i(S^n \setminus K) \cong \tilde{H}^{n-i-1}(K).$$

Example. Let K be a knot in S^3 , i.e. K is a submanifold of S^3 that is diffeomorphic to S^1 . By the Alexander Duality Theorem 48.4 we have⁷⁷⁵

$$\tilde{H}_i(S^3 \setminus K) \cong \tilde{H}^{2-i}(K) \cong \tilde{H}^{2-i}(S^1) \cong \begin{cases} \mathbb{Z}, & \text{if } i = 1, \\ 0, & \text{otherwise.} \end{cases}$$

More generally, if $L \subset S^3$ is an m -component link, i.e. if L is a 1-dimensional submanifold that is diffeomorphic to the disjoint union of m circles, then the same argument shows that

$$\tilde{H}_i(S^3 \setminus L) \cong \tilde{H}^{2-i}(L) \cong \tilde{H}^{2-i}(\underbrace{S^1 \sqcup \dots \sqcup S^1}_{m\text{-copies}}) \cong \begin{cases} \mathbb{Z}^{m-1}, & \text{if } i = 2, \\ \mathbb{Z}^m, & \text{if } i = 1, \\ 0, & \text{otherwise.} \end{cases}$$

⁷⁷⁵This argument gives a new proof for Proposition 30.1.

Proof of the Alexander Duality Theorem 48.4. Let K be a closed submanifold of S^n . By the General Tubular Neighborhood Theorem 48.1 there exists a closed tubular neighborhood $N(K)$ around K . We first consider the case that $i > 0$. Then

$$\begin{array}{ccccc}
 H_i(S^n \setminus K) & \xleftarrow{\cong} & H_i(S^n \setminus \mathring{N}(K)) & \cong & H^{n-i}(S^n \setminus \mathring{N}(K), \partial N(K)) & \cong & H^{n-i}(S^n, N(K)) \\
 \uparrow & & \uparrow & & \uparrow & & \uparrow \\
 & \text{Lemma 48.2} & & \text{Poincaré Duality Theorem 45.1} & & & \text{Excision Theorem 26.15} \\
 & \cong & \tilde{H}^{n-i-1}(N(K)) & \cong & \tilde{H}^{n-i-1}(K) & & \\
 \uparrow & & \uparrow & & & & \\
 & \text{by the long exact sequence of} & & \text{since } K \text{ is a deformation retract of } N(K) & & & \\
 & \text{the pair } (S^n, N(K)) & & & & &
 \end{array}$$

The case $i = 0$ is proved almost the same way. We leave it as an exercise to verify this case. $\square$

Corollary 48.5. *Let K be a closed submanifold of $\mathbb{R}^n$. Then for any $i \in \mathbb{N}_0$ we have an isomorphism*

$$\tilde{H}_i(\mathbb{R}^n \setminus K) \cong H^{n-i-1}(K).$$

Proof. Let K be a closed submanifold of $\mathbb{R}^n$ and let $i \in \mathbb{N}_0$. Then

$$\begin{array}{ccccc}
 \tilde{H}^{n-i-1}(K) & \cong & \tilde{H}_i(S^n \setminus K) & \cong & \begin{cases} \tilde{H}_i(\mathbb{R}^n \setminus K), & \text{if } i \neq n-1, \\ \tilde{H}_{n-1}(\mathbb{R}^n \setminus K) \oplus \mathbb{Z}, & \text{if } i = n-1. \end{cases} \\
 \uparrow & & \uparrow & & \\
 \text{Alexander Duality Theorem 48.4} & & \text{see exercise sheet 4} & &
 \end{array}$$

The corollary now follows from Lemmas 48.3 and 8.6 (1). $\square$

Corollary 48.6. *Let N be a closed n -dimensional connected manifold. If N is not orientable, then it is not possible to embed it into $\mathbb{R}^{n+1}$.*

Example. It follows from Corollary 48.6 that the real projective space $\mathbb{R}P^2$ and the Klein bottle cannot be embedded into $\mathbb{R}^3$. This answers Question 1.8 in the negative, if the slightly imprecise word “vague” in the formulation of the question is interpreted as “embedded”.

Proof. Let N be a closed n -dimensional connected non-orientable manifold. Suppose that N can be embedded into $\mathbb{R}^{n+1}$. Then

$$\begin{array}{ccccc}
 \mathbb{Z}_2 & \cong & \tilde{H}^{n-1}(N) & \cong & \tilde{H}_0(S^n \setminus N) & \cong & \mathbb{Z}^{\text{number of path-components of } S^n \setminus N} \\
 \uparrow & & \uparrow & & \uparrow & & \\
 \text{Proposition 41.7} & & \text{Alexander Duality} & & \text{Proposition 24.5} & & \\
 & & \text{Theorem 48.4} & & & &
 \end{array}$$

But this is absurd. We have thus obtained the desired contradiction. $\square$

We conclude this chapter with a discussion of several generalizations of the Alexander Duality Theorem 48.4.

First of all, recall that a topological space X is called locally contractible if every point admits an open neighborhood that is contractible, i.e. that is homotopy equivalent to a

point. Examples of locally contractible topological spaces are given by topological manifolds and graphs.

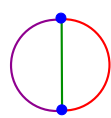
The following theorem is generalization of the Alexander Duality Theorem 48.4 and of Proposition 30.1. It is proved in [Ha, Theorem 3.44].

Theorem 48.7. (Generalized Alexander Duality Theorem) *Let K be a non-empty locally contractible compact subset of S^n . Then for any $i \in \mathbb{N}_0$ we have an isomorphism*

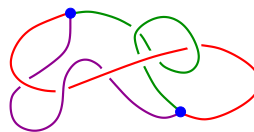
$$\tilde{H}_i(S^n \setminus K) \cong \tilde{H}^{n-i-1}(K).$$

Example. We consider a connected graph G with two vertices and three edges that is illustrated in Figure 384 on the left. On the right of Figure 384 we show it as a subset of S^3 . By the Generalized Alexander Duality Theorem 48.7 we have

$$\begin{array}{ccc} \tilde{H}_i(S^3 \setminus G) & \cong & \tilde{H}^{2-i}(G) \cong \begin{cases} \mathbb{Z}^2, & \text{if } i = 1, \\ 0, & \text{otherwise.} \end{cases} \\ \uparrow & & \uparrow \\ \text{Generalized Alexander} & & \text{similar to Lemma 34.9} \\ \text{Duality Theorem 48.7} & & \end{array}$$



graph G



the graph G as a subset of S^3

FIGURE 384.

Remark. The Generalized Alexander Duality Theorem 48.7 can be generalized even further. More precisely, an analogous statement also holds for any compact subset K of S^n . But hereby one needs to replace the usual singular cohomology on the right hand side by the Čech cohomology of K . We refer to [Ha, p. 256] for more details.

49. THE LINKING FORM ON RATIONAL HOMOLOGY SPHERES

49.1. Rational homology spheres. We now return to our quest to answer our ancient Question 6.7. On page 115 we had seen, using fundamental groups, that if two lens spaces $L(p, q)$ and $L(\tilde{p}, \tilde{q})$ are homeomorphic, then $p = \tilde{p}$. Later on, on page 389 we had seen that higher homotopy groups do not carry any additional information. On page 581 we had determined the homology groups of lens spaces, more precisely we had shown that

$$H_n(L(p, q)) \cong \begin{cases} \mathbb{Z}, & \text{if } n = 0, 3, \\ \mathbb{Z}_p, & \text{if } n = 1, \\ 0, & \text{otherwise.} \end{cases}$$

From the Universal Coefficient Theorem for Cohomology Groups 41.6 it follows that

$$H^n(L(p, q)) \cong \begin{cases} \mathbb{Z}, & \text{if } n = 0, 3, \\ \mathbb{Z}_p, & \text{if } n = 2, \\ 0, & \text{otherwise.} \end{cases}$$

This shows that homology groups and cohomology groups cannot be used to distinguish lens spaces $L(p, q)$ and $L(p, q')$.

In the last chapter we had successfully applied the cup-product to distinguish several manifolds that could not be distinguished just using cohomology groups. It is therefore tempting to try to use cup-products to distinguish lens spaces. But here we immediately encounter the problem, that all cup-products are trivial. For example, the cup-product on the non-trivial group $H^2(L(p, q))$ with itself is zero since it takes values in $H^4(L(p, q)) = 0$.

In this chapter we will develop an analogue of the intersection form, called the *linking form*, for lens spaces, or more generally, for rational homology spheres.

Definition. We say that n -dimensional topological manifold M is a *rational homology sphere*, if for each $k \in \mathbb{N}_0$ we have $H_k(M; \mathbb{Q}) \cong H_k(S^n; \mathbb{Q})$.

We had proved the following lemma in exercise sheet. The proof was a fairly straightforward consequence of the Poincaré Duality isomorphism and the Universal Coefficient Theorem.

Lemma 49.1. *Let M be an n -dimensional topological manifold M . Then the following holds:*⁷⁷⁶

$$M \text{ is a rational homology sphere} \iff \begin{array}{l} M \text{ is connected and orientable, such that} \\ H_i(M; \mathbb{Z}) \text{ is finite for } i = 1, \dots, \lfloor \frac{n-1}{2} \rfloor. \end{array}$$

49.2. More on cohomology groups. Before we can introduce the linking form of rational homology spheres we need to extend some of our definitions and results from cohomology theory. We will do so in this section.

⁷⁷⁶Here recall that given a real number r we denote by $\lfloor r \rfloor = \max\{n \in \mathbb{Z} \mid n \leq r\}$ the result of rounding r down.

Let X be a topological space and let $f: A \rightarrow B$ be a homomorphism between abelian groups. It is straightforward to check that the maps

$$\begin{aligned} f_*: C^n(X; A) = \text{Hom}(C_n(X), A) &\rightarrow C^n(X, B) = \text{Hom}(C_n(X), B) \\ \varphi &\mapsto f \circ \varphi \end{aligned}$$

define a chain map, in particular we get for each n an induced map

$$f_*: H^n(X; A) \rightarrow H^n(X; B)$$

Note that “cohomology is covariant in the coefficients”, i.e. for group homomorphisms $f: A \rightarrow B$ and $g: B \rightarrow C$ we have $(g \circ f)_* = g_* \circ f_*$.

Lemma 49.2. *Let $0 \rightarrow A \xrightarrow{i} B \xrightarrow{p} C \rightarrow 0$ be a short exact sequence of abelian groups and let X be a topological space. Then the following hold:*

- (1) *If $c \in C^n(X; B)$ has the property that $p_*(c) \in C^n(X; C)$ is a cocycle, then there exists a unique cocycle $\beta(c) \in C^{n+1}(X; A)$ with $i_*(\beta(c)) = \delta^n(c) \in C^{n+1}(X; B)$.*
- (2) *For every $n \in \mathbb{N}_0$ there exists a unique homomorphism $H^n(X; C) \xrightarrow{\beta} H^{n+1}(X; A)$ with the following property: if $c \in C^n(X; B)$ has the property that $p_*(c) \in C^n(X; C)$ is a cocycle, then*

$$\beta([p_*(c)]) = [\beta(c)].$$

- (3) *The following sequence of cohomology groups is exact:*

$$\dots \rightarrow H^n(X; A) \xrightarrow{i_*} H^n(X; B) \xrightarrow{p_*} H^n(X; C) \xrightarrow{\beta} H^{n+1}(X; A) \rightarrow \dots$$

Definition. The homomorphism $H^n(X; C) \xrightarrow{\beta} H^{n+1}(X; A)$ from Lemma 49.2 (3) is called the n -th Bockstein homomorphism of the topological space X corresponding to the short exact sequence $0 \rightarrow A \rightarrow B \xrightarrow{p} C \rightarrow 0$.⁷⁷⁷

Proof. Since each $C_n(X)$ is a free abelian group one sees easily that

$$0 \rightarrow C^*(X; A) \xrightarrow{i} C^*(X; B) \xrightarrow{p_*} C^*(X; C) \rightarrow 0$$

is a short exact sequence of cochain complexes. Statements (1) and (2) are proved basically the same way as Lemma 26.8 and Statement (3) is proved the same way as Proposition 26.9. We leave the verification of the details as an exercise to the reader. $\square$

On page 704 we had introduced the cup-product for cohomology groups of a topological space where the coefficients have a ring structure. We will now generalize the notion of the cup-product where we no longer demand that the coefficients have a ring structure.

To do so we recall some notation from 709: first of all, given any $i \in \{0, \dots, n\}$ we write $v_i := (0, \dots, 1, 0, \dots, 0) \in \mathbb{R}^{n+1}$ and secondly, given $a_r, \dots, a_{r+s} \in \Delta^n$ we denote by $[a_r, \dots, a_{r+s}]$ the unique affine map $\Delta^r \rightarrow \Delta^n$ that sends v_i to a_{r+i} for $i = 0, \dots, s$.

⁷⁷⁷The Bockstein homomorphism was introduced by the Russian mathematician Meyer Bockstein (1913-1990).

Definition. Let X be a topological space and let A and B be abelian groups. We define

$$\begin{aligned} C^p(X; A) \times C^q(X; B) &\rightarrow C^{p+q}(X; A \otimes B) \\ \varphi \cup \psi: C_{p+q}(X) &\rightarrow A \otimes B \\ (\varphi, \psi) &\mapsto \sigma \mapsto \underbrace{\varphi(\sigma \circ [v_0, \dots, v_p])}_{\in A} \otimes \underbrace{\psi(\sigma \circ [v_p, \dots, v_{p+q}])}_{\in B}. \end{aligned}$$

We have the following elementary lemma.

Lemma 49.3. *Let X be a topological space and let A and B be abelian groups. Furthermore let $\varphi \in C^p(X; A)$ and $\psi \in C^q(X; B)$. Then*

$$\delta(\varphi \cup \psi) = \delta\varphi \cup \psi + (-1)^p \cdot \varphi \cup \delta\psi \in C^{p+q+1}(X; A \otimes B).$$

Remark. We had proved an analogous statement in Analysis IV for differential forms. More precisely, given a manifold M and $\omega \in \Omega^p(M)$ and $\eta \in \Omega^q(M)$ we had proved that

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^p \cdot \omega \wedge d\eta \in \Omega^{p+q+1}(M).$$

Proof. The lemma follows from an elementary, albeit slightly confusing, calculation. Indeed, let $\varphi \in C^p(X; A)$ and $\psi \in C^q(X; B)$ and let $\sigma: \Delta^{p+q+1} \rightarrow X$ be a singular $(p+q+1)$ -chain. Then we have the following three equalities in $A \otimes B$ ⁷⁷⁸

$$\begin{aligned} \text{(a)} \quad (\delta(\varphi \cup \psi))(\sigma) &= (\varphi \cup \psi) \left(\sum_{j=0}^{p+q+1} (-1)^j \cdot \sigma \circ i_j^{p+q+1} \right) \\ &= (\varphi \cup \psi) \left(\sum_{j=0}^p (-1)^j \cdot \sigma \circ i_j^{p+q+1} \right) + (\varphi \cup \psi) \left(\sum_{j=p+1}^{p+q+1} (-1)^j \cdot \sigma \circ i_j^{p+q+1} \right) \\ &= (\varphi \cup \psi) \left(\sum_{j=0}^p (-1)^j \cdot \sigma \circ i_j^{p+q+1} \right) + (-1)^p \cdot (\varphi \cup \psi) \left(\sum_{k=1}^{q+1} (-1)^k \cdot \sigma \circ i_{k+p}^{p+q+1} \right). \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad (\delta\varphi \cup \psi)(\sigma) &= (\delta\varphi)(\sigma \circ [v_0, \dots, v_{p+1}]) \otimes \psi(\sigma \circ [v_{p+1}, \dots, v_{p+q+1}]) \\ &= \varphi \left(\sum_{j=0}^{p+1} (-1)^j \cdot \sigma \circ [v_0, \dots, v_{p+1}] \circ i_j^{p+1} \right) \otimes \psi(\sigma \circ [v_{p+1}, \dots, v_{p+q+1}]) \end{aligned}$$

and

$$\begin{aligned} \text{(c)} \quad (-1)^p \cdot (\varphi \cup \delta\psi)(\sigma) &= (-1)^p \cdot \varphi(\sigma \circ [v_0, \dots, v_p]) \otimes (\delta\psi)(\sigma \circ [v_p, \dots, v_{p+q+1}]) \\ &= (-1)^p \cdot \varphi(\sigma \circ [v_0, \dots, v_p]) \otimes \psi \left(\sum_{k=0}^{q+1} (-1)^k \cdot \sigma \circ [v_p, \dots, v_{p+q+1}] \circ i_k^{q+1} \right). \end{aligned}$$

We now observe that if we take the sum of (b) and (c), then the $(j = p+1)$ -summand in (b) cancels with the $(k = 0)$ -summand in (c). It is now straightforward to verify that the desired equality (b)+(c)=(a) holds. $\square$

⁷⁷⁸Here in the argument we use on several occasions that for a singular n -cochain ϕ and a singular $(n+1)$ -simplex μ we have by definition

$$(\delta\phi)(\mu) = \phi(\partial\mu) = \phi \left(\sum_{l=0}^{n+1} (-1)^l \cdot \mu \circ i_l^{n+1} \right).$$

Now we can easily prove the following lemma.

Lemma 49.4. *Let X be a topological space and let A and B be abelian groups. Then the map*

$$\begin{aligned} H^p(X; A) \times H^q(X; B) &\rightarrow H^{p+q}(X; A \otimes B) \\ [(\varphi, \psi)] &\mapsto [\varphi \cup \psi] \end{aligned}$$

is well-defined.

Proof. This is an almost immediate consequence of Lemma 49.3. $\square$

We refer to the pairing from Lemma 44.10 again as a cup-product.

The following lemma is an immediate consequence of the definitions and Lemma 44.6.

Lemma 49.5. *Let X be a topological space.*

- (1) *Let $f: A \rightarrow A'$ and $g: B \rightarrow B'$ be group homomorphisms. Then for any $k, l \in \mathbb{N}_0$ the following diagram commutes*

$$\begin{array}{ccc} H^k(X; A) \times H^l(B) & \xrightarrow{\cup} & H^{k+l}(X; A \otimes B) \\ \downarrow f_* \times g_* & & \downarrow (f \otimes g)_* \\ H^k(X; A') \times H^l(B') & \xrightarrow{\cup} & H^{k+l}(X; A' \otimes B'). \end{array}$$

- (2) *Let R be a commutative ring. We denote by $\mu: R \otimes R \rightarrow R$ the map that is given by multiplication in the ring, i.e. it is given by $(r, s) \mapsto r \cdot s$. Then we have the following commutative diagram*

$$\begin{array}{ccc} & H^p(X; R) \times H^q(X; R) & \\ \swarrow \cup & & \searrow \cup \\ H^{p+q}(X; R \otimes R) & \xrightarrow{\mu_*} & H^{p+q}(X; R). \end{array}$$

49.3. The linking form on odd-dimensional rational homology spheres. Now we return to the study of rational homology spheres. We start out with the following elementary lemma.

Lemma 49.6. *For every $(2n + 1)$ -dimensional rational homology sphere the Bockstein homomorphism $H^n(M; \mathbb{Q}/\mathbb{Z}) \rightarrow H^{n+1}(M; \mathbb{Z})$ corresponding to the short exact sequence $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Q} \rightarrow \mathbb{Q}/\mathbb{Z} \rightarrow 0$ is an isomorphism.*

In the following, given an $(2n + 1)$ -dimensional rational homology sphere we denote by $\beta: H^n(M; \mathbb{Q}/\mathbb{Z}) \rightarrow H^{n+1}(M; \mathbb{Z})$ the Bockstein isomorphism corresponding to the short exact sequence $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Q} \rightarrow \mathbb{Q}/\mathbb{Z} \rightarrow 0$ of coefficients.

Proof. It follows from the Poincaré Duality Theorem 45.2 that for the rational homology sphere M we have $H^n(M; \mathbb{Q}) = H^{n+1}(M; \mathbb{Q}) = 0$. Thus it follows from Lemma 49.2 (3) that the Bockstein homomorphism $H^n(M; \mathbb{Q}/\mathbb{Z}) \rightarrow H^{n+1}(M; \mathbb{Z})$ is an isomorphism. $\square$

We continue with the following purely algebraic definition.

Definition.

- (1) A *linking form* on an abelian group H is a bilinear map $\lambda: H \times H \rightarrow \mathbb{Q}/\mathbb{Z}$.
- (2) We say that a linking form $\lambda: H \times H \rightarrow \mathbb{Q}/\mathbb{Z}$ is *symmetric* if $\lambda(a, b) = \lambda(b, a)$ for all $a, b \in H$.
- (3) We say that a linking form $\lambda: H \times H \rightarrow \mathbb{Q}/\mathbb{Z}$ is *nonsingular* if the corresponding map $H \mapsto \text{Hom}(H, \mathbb{Q}/\mathbb{Z})$ that is given by $a \mapsto (b \mapsto \lambda(a, b))$ is an isomorphism.

Example. Let $n \in \mathbb{N}$ and let $k \in \mathbb{Z}$. The map

$$\begin{aligned} \lambda: \mathbb{Z}/n \times \mathbb{Z}/n &\rightarrow \mathbb{Q}/\mathbb{Z} \\ (a, b) &\mapsto \frac{k}{n} a \cdot b \end{aligned}$$

is a symmetric linking form. In exercise sheet 6 we will see that it is non-singular if and only if k is coprime to n .

The following definition can now be viewed as an odd-dimensional analogue of the definition of the intersection form on an oriented compact $4k$ -dimensional manifold that we had given on page 748.

Definition.

- (1) Let S be a commutative ring and let R be a subring of S . We denote the obvious multiplications maps $R \otimes S/R \rightarrow S/R$ and $S/R \otimes R \rightarrow S/R$ by ν .
- (2) Let M be a rational homology sphere of dimension $2n + 1$ with $n \geq 1$.⁷⁷⁹

$$\begin{aligned} \lambda_M: H_n(M; \mathbb{Z}) \times H_n(M; \mathbb{Z}) &\rightarrow \mathbb{Q}/\mathbb{Z} \\ (a, b) &\mapsto \left\langle \underbrace{\nu_* (\beta^{-1}(\text{PD}(a)) \cup \text{PD}(b))}_{\in H^{2n+1}(M; \mathbb{Q}/\mathbb{Z} \otimes \mathbb{Z})}, [M] \right\rangle \end{aligned}$$

as the *linking form* λ_M of M .

Proposition 49.7. *Let M be an $(2n + 1)$ -dimensional rational homology sphere. If n is odd, then the linking form λ_M on $H_n(M; \mathbb{Z})$ is symmetric otherwise it is anti-symmetric.*

We will see that Proposition 49.7 is a straightforward consequence of the next lemma.

Lemma 49.8. *Let S be a commutative ring and let R be a subring of S . We obtain a short exact sequence $0 \rightarrow R \xrightarrow{i} S \xrightarrow{p} S/R \rightarrow 0$. Let X be a topological space. For any $x \in H^k(X; S/R)$ and $y \in H^l(X; S/R)$ we have*

$$\underbrace{\nu_* (\beta(x) \cup y)}_{\in H^{k+l+1}(X; R \otimes S/R)} = (-1)^{k+1} \cdot \underbrace{\nu_* (x \cup \beta(y))}_{\in H^{k+l+1}(X; S/R \otimes R)} \in H^{k+l+1}(X; S/R).$$

⁷⁷⁹+++++ still need to extend definition of cap-product from page 711 to $H^k(X; A) \times H_l(X; B) \rightarrow H_{l-k}(X; A \otimes B)$

Proof of Proposition 49.7 assuming Lemma 49.8. Let M be an $(2n+1)$ -dimensional rational homology sphere and let $a, b \in H_n(M; \mathbb{Z})$. We write $x = \beta^{-1}(\text{PD}(a)) \in H^n(M; \mathbb{Q}/\mathbb{Z})$ and $y = \beta^{-1}(\text{PD}(b)) \in H^n(M; \mathbb{Q}/\mathbb{Z})$. Then

$$\begin{aligned} \lambda_M(a, b) &= \langle \nu_*(x \cup \beta(y)), [M] \rangle \\ &\stackrel{\text{by Lemma 49.8}}{\downarrow} = (-1)^{n+1} \cdot \langle \nu_*(\beta(x) \cup y), [M] \rangle \\ &= (-1)^{n+1} \cdot \langle \nu_*(y \cup \beta(x)), [M] \rangle = \lambda_M(b, a). \\ &\stackrel{\uparrow}{\text{by Proposition 44.1 since } n(n+1) \text{ is even}} \end{aligned}$$

□

Proof of Lemma 49.8. We can and will pick $f \in C^k(X; S)$ and $g \in C^l(X; S)$ such that $[p_*(f)] = x$ and such that $[p_*(g)] = y$. By Lemma 49.2 there exist unique cocycles $\beta(f) \in C^{k+1}(X; R)$ and $\beta(g) \in C^{l+1}(X; R)$ with $i_*(\beta(f)) = \delta^k(f) \in C^{k+1}(X; S)$ and with $i_*(\beta(g)) = \delta^l(g) \in C^{l+1}(X; S)$. We have the following equality in $H^{k+l+1}(X; S \otimes S)$:

$$\begin{aligned} 0 &= [\delta(f \cup g)] = [\delta(f) \cup g + (-1)^k \cdot f \cup \delta(g)] = \underbrace{[i_*(\beta(f)) \cup g]}_{\in C^{k+l+1}(X; R \otimes S)} + (-1)^k \cdot \underbrace{[f \cup i_*(\beta(g))]}_{\in C^{k+l+1}(X; S \otimes R)}. \\ &\quad \uparrow \qquad \qquad \uparrow \\ &\quad \text{by definition} \quad \text{see Lemma 49.3} \\ &\quad \text{of cohomology} \end{aligned}$$

We consider the following commutative diagram of group homomorphisms

$$\begin{array}{ccccc} R \otimes S & \xrightarrow{i \otimes \text{id}} & S \otimes S & \xleftarrow{\text{id} \otimes i} & S \otimes R \\ \text{id} \otimes p \downarrow & & \downarrow \mu & & \downarrow p \otimes \text{id} \\ R \otimes S/R & & S & & S/R \otimes R \\ & \searrow \nu & \downarrow p & \swarrow \nu & \\ & & S/R & & \end{array}$$

where ν and μ stand for the obvious maps that are given by multiplication. From the above equality we get following equality in $H^{k+l+1}(X; S/R)$:

$$\begin{aligned} 0 &= (p \circ \mu)_*([i_*(\beta(f)) \cup g + (-1)^k \cdot f \cup i_*(\beta(g))]) \\ &= [\nu_*(\beta(f) \cup p_*(g)) + (-1)^k \cdot \nu_*(p_*(f) \cup \beta(g))] \\ &\quad \uparrow \\ &\quad \text{by the above commutative diagram of group homomorphisms and Lemma 44.6} \\ &= \nu_*([\beta(f)] \cup [p_*(g)]) + (-1)^k \cdot \nu_*([p_*(f)] \cup [\beta(g)]) \\ &\quad \uparrow \\ &\quad \text{since } \beta(f), p_*(g), p_*(f) \text{ and } \beta(g) \text{ are cocycles} \\ &= \nu_*(\beta(x) \cup y) + (-1)^k \cdot \nu_*(x \cup \beta(y)). \\ &\quad \uparrow \\ &\quad \text{since } [\beta(f)] = \beta([p_*(f)]) := \beta(x) \text{ and } [p_*(g)] = y \text{ and similar for the second summand} \end{aligned}$$

This of course implies the desired statement. □

Definition.

Lemma 49.9. *The linking form λ_M of any $(2n+1)$ -dimensional rational homology sphere is non-singular.*

Proof. Let M be a $(2n+1)$ -dimensional rational homology sphere. It suffices to prove the following claim.

Claim. We consider the map

$$\Phi_M: H_n(M; \mathbb{Z}) \xrightarrow{\text{PD}_M} H^{n+1}(M; \mathbb{Z}) \xleftarrow[\cong]{\beta} H^n(M; \mathbb{Q}/\mathbb{Z}) \xrightarrow{\text{ev}} \text{Hom}_{\mathbb{Z}}(H_n(M; \mathbb{Z}), \mathbb{Q}/\mathbb{Z}).$$

Here the first map is given by Poincaré Duality, the second map is Bockstein the isomorphism given by Lemma 49.6 and the third map is given by the evaluation map. Then the following two statements hold:

- (1) The map Φ_M is an isomorphism.
- (2) For any $a, b \in H_n(M)$ we have $\lambda_M(a, b) = \Phi_M(a)(b)$.

For the first statement it remains to show that the third map in the definition of Φ_M is an isomorphism. But this is an immediate consequence of the Universal Coefficient Theorem 41.6 for Cohomology Groups together with Lemma 41.1 (6).

Now we turn to the proof of the second statement. Let $a, b \in H_n(M; \mathbb{Z})$. Then

by definition and Lemma 44.11

by a slight generalization of Lemma 44.13

$$\begin{array}{ccc} \downarrow & & \downarrow \\ \lambda_M(a, b) = \nu_*(\beta^{-1}(\text{PD}(a)) \cup \text{PD}(b)) \cap [M] & = & \beta^{-1}(\text{PD}(a)) \cap (\text{PD}(b) \cap [M]) = \Phi_M(a)(b). \end{array}$$

This concludes the proof of the claim. $\square$