

THE L^2 -TORSION FUNCTION AND THE THURSTON NORM OF 3-MANIFOLDS

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ABSTRACT. Let M be an oriented irreducible 3-manifold with infinite fundamental group and empty or toroidal boundary which is not $S^1 \times D^2$. Consider any element ϕ in the first cohomology of M with integer coefficients. Then one can define the ϕ -twisted L^2 -torsion function of the universal covering which is a function from the set of positive real numbers to the set of real numbers. By earlier work of the second author and Schick the evaluation at $t = 1$ determines the volume.

In this paper we show that its degree, which is a number extracted from its asymptotic behavior at 0 and at ∞ , agrees with the Thurston norm of ϕ .

0. INTRODUCTION

Reidemeister torsion is one of the first invariants in algebraic topology which are able to distinguish the diffeomorphism type of closed manifolds which are homotopy equivalent. A prominent example is the complete classification of lens spaces, see for instance [4]. The Alexander polynomial, which is one of the basic invariants for knots and 3-manifolds, can be interpreted as Reidemeister torsion, see for instance [26]. The Reidemeister torsion of a 3-manifold can be generalized in two ways. Either one can twist it with an element in the first cohomology which leads for example to the twisted Alexander polynomial, see for instance [11], or one can consider the L^2 -version of appropriate coverings resulting in L^2 -torsion invariants, see for instance [22, Chapter 3]. Recently there have been attempts to combine these two generalizations and consider twisted L^2 -versions. Such generalizations have been considered under the name of L^2 -Alexander torsion or L^2 -Alexander Conway invariants for knots or 3-manifolds, for instance in [5, 6, 7, 8, 9, 14, 15, 16].

In all of these papers one has to make certain assumptions to ensure that the twisted L^2 -versions are well-defined. They concern L^2 -acyclicity and determinant class. Either these conditions were just assumed to hold, or verified in special cases by a direct computation. A systematic study of these twisted L^2 -invariants under the name L^2 -torsion function has been carried out in [23]. We summarize some of the results of [23] for 3-manifolds. Let M be a 3-manifold. (Here and throughout the paper we assume that all 3-manifolds are compact, connected and oriented with empty or toroidal boundary, unless we say explicitly otherwise.) If M is irreducible and if it has infinite fundamental group then it was shown in [23] that all these necessary conditions are satisfied for the universal covering \widetilde{M} and an element $\phi \in H^1(M; \mathbb{Z})$. The result is an equivalence class of functions

$$\bar{\rho}^{(2)}(M; \phi): (0, \infty) \rightarrow \mathbb{R}$$

where we call two functions $f, g: (0, \infty) \rightarrow \mathbb{R}$ equivalent if for some integer m we have $f(t) - g(t) = m \cdot \ln(t)$. We recall the definition in Section 1.2. Note though that this invariant is *minus the logarithm* of the function defined and studied in

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the aforementioned papers. In those papers the corresponding function was usually referred to as the L^2 -Alexander torsion. The convention of this paper brings us in line with [22].

The evaluation of $\bar{\rho}^{(2)}(M; \phi)$ at $t = 1$ is well-defined and in fact it was shown by the second author and Schick [24, Theorem 0.7] that for any irreducible 3-manifold we have

$$\bar{\rho}^{(2)}(M; \phi)(t = 1) = -\frac{1}{6\pi} \text{vol}(N),$$

where $\text{vol}(N)$ equals the sum of the volumes of the hyperbolic pieces in the JSJ-decomposition of N .

In the sequence of papers [5], [6], [7] and [23] the behavior of $\bar{\rho}^{(2)}(M; \phi)$ as t ‘goes to the extremes’, i.e. as $t \rightarrow 0$ and $t \rightarrow \infty$, was studied. In particular in [23] it was shown that for any representative ρ there exist constants $C \geq 0$ and $D \geq 0$ such that we get for $0 < t \leq 1$

$$C \cdot \ln(t) - D \leq \rho(t) \leq -C \cdot \ln(t) + D,$$

and for $t \geq 1$

$$-C \cdot \ln(t) - D \leq \rho(t) \leq C \cdot \ln(t) + D.$$

Hence the limits $\lim_{t \rightarrow 0} \frac{\rho(t)}{\ln(t)}$ and $\lim_{t \rightarrow \infty} \frac{\rho(t)}{\ln(t)}$ exist and we can define the *degree* of $\bar{\rho}^{(2)}(M; \phi)$ to be

$$\text{deg}(\bar{\rho}^{(2)}(M; \phi)) := \lim_{t \rightarrow \infty} \frac{\rho(t)}{\ln(t)} - \lim_{t \rightarrow 0} \frac{\rho(t)}{\ln(t)}.$$

It is obviously independent of the choice of the representative ρ .

Thurston [29] assigned to ϕ another invariant, its *Thurston norm* $x_M(\phi)$, which we will review in Subsection 1.6.

The main result of our paper says that the functions $\bar{\rho}^{(2)}(M; \phi)$ not only determine the volume of a 3-manifold but that they also determine the Thurston norm. More precisely, we have the following theorem.

Theorem 0.1. *Let M be an irreducible 3-manifold with infinite fundamental group and empty or toroidal boundary which is not homeomorphic to $S^1 \times D^2$. Then we get for any element $\phi \in H^1(M; \mathbb{Z})$.*

$$\text{deg}(\bar{\rho}^{(2)}(M; \phi)) = -x_M(\phi).$$

Actually we get a much more general result, where we can consider not only the universal covering but appropriate G -coverings $G \rightarrow \bar{M} \rightarrow M$ and get estimates for the L^2 -function for all times $t \in (0, \infty)$ which implies the equality of the degree and the Thurston norm, see Theorem 5.1.

The main ingredients for the proof are the estimates for the twisted L^2 -torsion function and approximation techniques presented in [23], the proof of the Virtual Fibration Theorem of Agol [1][2], Wise [35] and Przytycki-Wise(2012) [27] and a careful analysis of Mahler measures.

Added in proof. We just learned that Liu [18] has given a completely independent proof of Theorem 0.1. The techniques used in both papers are at times somewhat similar. Liu [18] goes on to prove several other very interesting results that are not covered in this paper. In particular he proves Theorem 0.1 also for real classes.

Conventions and notations. We view elements in $\mathbb{Z}[G]^k$ always as row vectors. Given a group G and an $m \times n$ -matrix over $\mathbb{Z}[G]$ we denote by the r_A the homomorphism $\mathbb{Z}[G]^m \rightarrow \mathbb{Z}[G]^n$ given by right multiplication by A . Furthermore, given a group homomorphism $\gamma: G \rightarrow H$ we denote by $\gamma(A)$ the matrix over $\mathbb{Z}[H]$ given

by applying γ to all entries. Throughout the paper we assume that all 3-manifolds are compact, connected and oriented, unless we say explicitly otherwise.

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1. REVIEW OF THE ϕ -TWISTED L^2 -TORSION FUNCTION AND THE THURSTON NORM

In this section we recall some basic definitions, notions and results from [5, 6, 23], in order to keep this paper self-contained. For basic information about L^2 -Betti numbers, Fuglede-Kadison determinants and L^2 -torsion we refer to [22].

1.1. Euler structures and Spin^c -structures. Let X be a finite CW complex and let $p: \tilde{X} \rightarrow X$ be the universal covering of X . Following Turaev [30, 31, 33] we define a *fundamental family of cells* to be a choice for each cell in X of precisely one cell in \tilde{X} which projects to the given cell in X .

We write $\pi = \pi_1(X)$ and we denote by $\psi: \pi \rightarrow H_1(\pi; \mathbb{Z}) = H_1(X; \mathbb{Z})$ the abelianization map. Now let $\{e_i\}_{i \in I}$ and $\{\hat{e}_i\}_{i \in I}$ be two fundamental families of cells. After reordering them we can arrange that for each $i \in I$ we have $e_i = g_i \hat{e}_i$ for some $g_i \in G$. We say that two fundamental families of cells are *equivalent* if

$$\sum_{i \in I} (-1)^{\dim(e_i)} \psi(g_i) = 0.$$

The set of equivalence classes of fundamental families of cells on X is called the set $\text{Eul}(X)$ of *Euler structures on X* . Note that the set of Euler structures on X admits a free and transitive action by $H_1(X; \mathbb{Z})$.

We recall some basic facts regarding Spin^c -structures on 3-manifolds, with empty or toroidal boundary. We refer to [33, Chapter XI] for a detailed discussion. Given a 3-manifold M we denote by $\text{Spin}^c(M)$ the set of Spin^c -structures on M . The set $\text{Spin}^c(M)$ comes with a canonical free and transitive action by $H_1(M; \mathbb{Z})$. Given $\mathfrak{s} \in \text{Spin}^c(M)$ we denote by $c_1(\mathfrak{s}) \in H^2(M, \partial M; \mathbb{Z}) = H_1(M; \mathbb{Z})$ its Chern class. The Chern class has the property that for each $\mathfrak{s} \in \text{Spin}^c(M)$ and $h \in H_1(M; \mathbb{Z})$ the following equality holds

$$(1.1) \quad c_1(h\mathfrak{s}) = 2h + c_1(\mathfrak{s}).$$

In [32, 33] Turaev shows that given any CW-structure X for M there exists a canonical $H_1(M; \mathbb{Z}) = H_1(X; \mathbb{Z})$ -equivariant bijection $\text{Spin}^c(M) \rightarrow \text{Eul}(X)$.

1.2. (L^2 -acyclic) admissible pairs and the ϕ -twisted L^2 -torsion function. In [5, 6] the authors and Dubois introduced the ϕ -twisted L^2 -torsion function of a 3-manifold. This definition was later generalized and analyzed in [23, Section 7] for G -coverings of compact connected manifolds in all dimensions.

We start out with the following definitions.

Definition 1.2. (1) In the following, given any abelian group A we will denote

$$A_f = A / \text{tors}(A).$$

- (2) We say that a group homomorphism $\varphi: G \rightarrow H$ is *large*, if the projection map $G \rightarrow H_1(G; \mathbb{Z})_f$ factors through φ .
- (3) An admissible pair (M, μ) consists of an irreducible 3-manifold $M \neq S^1 \times D^2$ with infinite fundamental group and a large group homomorphism $\mu: \pi_1(M) \rightarrow G$ to a residually finite countable group G . Denote by $\overline{M} \rightarrow M$

the G -covering associated to μ . We say that (M, μ) is L^2 -acyclic if the n -th L^2 -Betti number $b_n^{(2)}(\overline{M}; \mathcal{N}(G))$ vanishes for every $n \geq 0$.

Many of the subsequent results will hold in more general situations, e.g. it is not always necessary to assume that G is residually finite or that φ is large. Nonetheless, in an attempt to keep the paper readable we will note state all the results in the maximal generality.

Convention 1.3. If $\varphi: \pi \rightarrow G$ is a large epimorphism, then we can and will identify $\text{Hom}(\pi, \mathbb{R})$ with $\text{Hom}(G, \mathbb{R})$. Furthermore, given any space X we make the usual identifications $H^1(X; \mathbb{R}) = \text{Hom}(H_1(X; \mathbb{Z}), \mathbb{R}) = \text{Hom}(\pi_1(X), \mathbb{R})$. In particular, if $(M, \mu: \pi_1(M) \rightarrow G)$ is an admissible pair, then any $\phi \in H^1(M; \mathbb{R})$ induces a homomorphism $G \rightarrow \mathbb{R}$ that, by a slight abuse of notation, we also denote by ϕ .

The following lemma is an immediate consequence of [13], [19] and the proof of the Geometrization Theorem by Perelman.

Lemma 1.4. *If $M \neq S^1 \times D^2$ is an irreducible 3-manifold with infinite fundamental group, then $(M, \text{id}_{\pi_1(M)})$ is an L^2 -acyclic admissible pair.*

Now consider an L^2 -acyclic admissible pair $(M; \mu: \pi_1(M) \rightarrow G)$ with Spin^c -structure $\mathfrak{s} \in \text{Spin}^c(M)$. Let $\phi \in H^1(M; \mathbb{Q})$. We pick a CW-structure for M , which by abuse of notation we denote again by M . We denote by \widetilde{M} the universal cover of M and we write $\pi = \pi_1(M)$. We pick a fundamental family of cells in \widetilde{M} that corresponds to \mathfrak{s} .

This fundamental family of cells turns $C_*(\widetilde{M})$ into a chain complex of based free $\mathbb{Z}[\pi]$ -left modules. We view $\mathbb{Z}[G]$ as a right $\mathbb{Z}[\pi]$ -module via the homomorphism μ . We obtain the chain complex $\mathbb{Z}[G] \otimes_{\mathbb{Z}[\pi]} C_*(\widetilde{M})$ of based free $\mathbb{Z}[G]$ -left modules.

Now let $t \in (0, \infty)$. We denote by $\phi^* \mathbb{C}_t$ the based 1-dimensional complex G -representation whose underlying complex vector space is \mathbb{C} and on which $g \in G$ acts by multiplication with $t^{\phi(g)}$. Twisting with $\phi^* \mathbb{C}_t$ transforms a $\mathbb{C}G$ -homomorphism $\mathbb{C}G \rightarrow \mathbb{C}G$ given by right multiplication with the element $\sum_{g \in G} \lambda_g \cdot g$ to the $\mathbb{C}G$ -homomorphism $\mathbb{C}G \rightarrow \mathbb{C}G$ given by right multiplication with the element $\sum_{g \in G} \lambda_g \cdot t^{\phi(g)} \cdot g$. It is obvious how this extends to $\mathbb{C}G$ -left linear maps $\mathbb{C}G^m \rightarrow \mathbb{C}G^n$ and then to $\mathbb{C}G \otimes_{\mathbb{Z}\pi} C_*(\widetilde{M})$. Thus twisting $\mathbb{C}G \otimes_{\mathbb{Z}\pi} C_*(\widetilde{M})$ with $\phi^* \mathbb{C}_t$ yields a finite free $\mathbb{C}G$ -chain complex $\eta_{\phi^* \mathbb{C}_t}(\mathbb{C}G \otimes_{\mathbb{Z}\pi} C_*(\widetilde{M}))$ with a $\mathbb{C}G$ -basis.

Given a $\mathbb{C}G$ -linear map $A: \mathbb{C}G^m \rightarrow \mathbb{C}G^n$, we obtain by applying $L^2(G) \otimes_{\mathbb{C}G} -$ a morphism $\Lambda^G(A)$ of finitely generated Hilbert $\mathcal{N}(G)$ -modules $L^2(G)^m \rightarrow L^2(G)^n$. Thus we obtain from $\eta_{\phi^* \mathbb{C}_t}(\mathbb{C}G \otimes_{\mathbb{Z}\pi} C_*(\widetilde{M}))$ by applying $L^2(G) \otimes_{\mathbb{C}G} -$ a finite Hilbert $\mathcal{N}(G)$ -chain complex denoted by $\Lambda^G \circ \eta_{\phi^* \mathbb{C}_t}(\mathbb{C}G \otimes_{\mathbb{Z}\pi} C_*(\widetilde{M}))$. By our hypothesis this chain complex is \det - L^2 -acyclic for $t = 1$. By [23, Theorem 6.7] we know that it is then also \det - L^2 -acyclic for any $t \in (0, \infty)$. In particular the $\mathcal{N}(G)$ -chain complex has well-defined L^2 -torsion for any $t \in (0, \infty)$. Define the ϕ -twisted L^2 -torsion function

$$(1.5) \quad \begin{aligned} \rho^{(2)}(M, \mathfrak{s}; \mu, \phi): (0, \infty) &\rightarrow \mathbb{R} \\ t &\mapsto \rho^{(2)}\left(\Lambda^G \circ \eta_{\phi^* \mathbb{C}_t}(\mathbb{C}G \otimes_{\mathbb{Z}\pi} C_*(\widetilde{M}))\right) \end{aligned}$$

More details of this construction and the proof that it is well-defined can be found in [23, Section 7] and, with slightly different conventions, in [5].

If μ is the identity homomorphism, then we drop it from the notation. Put differently, we write $\rho^{(2)}(M, \mathfrak{s}; \phi) := \rho^{(2)}(M, \mathfrak{s}; \text{id}_{\pi_1(M)}, \phi)$.

1.3. Comparing the ϕ -twisted L^2 -torsion function and the L^2 -Alexander torsion. The ϕ -twisted L^2 -torsion function $\rho^{(2)}(M, \mathfrak{s}; \mu, \phi): (0, \infty) \rightarrow \mathbb{R}$, as considered in this paper and in [23], is designed in an additive setup, as it is the main

convention when dealing with related invariants such as topological L^2 -torsion, analytic L^2 -torsion, analytic Ray-Singer torsion and so on. When dealing with torsion invariants in dimension 3, the multiplicative setting is standard, which is the reason why we defined for instance in [5, 6] the L^2 -Alexander torsion multiplicatively as a function $\tau^{(2)}(M, \mathfrak{s}; \phi, \mu): (0, \infty) \rightarrow [0, \infty)$.

If (M, μ) is L^2 -acyclic, then it follows immediately from the definitions and the conventions used in the various papers and from [23], that these two invariants are related by the formula

$$(1.6) \quad \tau^{(2)}(M, \mathfrak{s}; \phi, \mu) = \exp(-\rho^{(2)}(M, \mathfrak{s}; \mu, \phi)).$$

Notice that (1.6) implies that $\tau^{(2)}(M, \mathfrak{s}; \phi, \mu)$ never takes the value zero. This is a consequence of Theorem 1.8 (1) which was not available when [5] was finished.

Notice the minus sign appearing in the formula (1.6). This has the consequence that the degree $\deg(\tau^{(2)}(M, \mathfrak{s}; \phi, \mu))$ defined in [5] and the degree $\deg(\rho^{(2)}(M, \mathfrak{s}; \mu, \phi))$ defined in the introduction and later again in (1.10) are related by

$$(1.7) \quad \deg(\tau^{(2)}(M, \mathfrak{s}; \phi, \mu)) = -\deg(\rho^{(2)}(M, \mathfrak{s}; \mu, \phi)).$$

In the following we will cite results from [5, 6] about $\tau^{(2)}(M, \mathfrak{s}; \phi, \mu)$, which via (1.7) we reinterpret as results on $\rho^{(2)}(M, \mathfrak{s}; \mu, \phi)$.

1.4. Properties of the ϕ -twisted L^2 -torsion function. The following theorem summarizes some of the key properties of the ϕ -twisted L^2 -torsion function.

Theorem 1.8 (Properties of the twisted L^2 -torsion function). *Let (M, μ) be an L^2 -acyclic admissible pair, let $\phi \in H^1(M; \mathbb{R})$ and let $\mathfrak{s} \in \text{Spin}^c(M)$.*

(1) Pinching estimate

There exist constants C and D such that we get for $0 < t \leq 1$

$$C \cdot \ln(t) - D \leq \rho^{(2)}(M, \mathfrak{s}; \mu, \phi)(t) \leq -C \cdot \ln(t) + D,$$

and for $t \geq 1$

$$-C \cdot \ln(t) - D \leq \rho^{(2)}(M, \mathfrak{s}; \mu, \phi)(t) \leq C \cdot \ln(t) + D;$$

(2) Dependence on the Spin^c -structure

For any $h \in H_1(M; \mathbb{Z})$ we have

$$\rho^{(2)}(M, h\mathfrak{s}; \mu, \phi) = \rho^{(2)}(M, \mathfrak{s}; \mu, \phi) + \ln(t) \cdot \phi(h).$$

(3) Covering formula

Let $p: \widehat{M} \rightarrow M$ be a finite regular covering such that $\ker(\mu) \subset \widehat{\pi} := \pi_1(\widehat{M})$.

We write $\widehat{\phi} := p^\phi$ and we denote by $\widehat{\mu}$ the restriction of μ to $\widehat{\pi}$. Finally we write $\widehat{\mathfrak{s}} := p^*(\mathfrak{s})$. Then for all t we have*

$$\rho^{(2)}(\widehat{M}, \widehat{\mathfrak{s}}; \widehat{\phi}, \widehat{\mu})(t) = [\widehat{N} : N] \cdot \rho^{(2)}(M, \mathfrak{s}, \phi, \mu)(t).$$

(4) Scaling ϕ

Let $r \in \mathbb{R}$. Then we get for all $t \in (0, \infty)$

$$\rho^{(2)}(M, \mathfrak{s}; \mu, r\phi)(t) = \rho^{(2)}(M, \mathfrak{s}; \mu, \phi)(t^r).$$

(5) Symmetry

For any $t \in (0, \infty)$ we have

$$\rho(M, \mathfrak{s}; \mu, \phi)(t^{-1}) = -\phi(c_1(\mathfrak{s})) \ln(t) + \rho(M, \mathfrak{s}; \mu, \phi)(t).$$

Statement (1) is proved in [23, Theorem 7.3], it is one of the main results of that paper. Statement (2) is proved in [5] and [6]. Statement (3) is proved in [23, Theorem 7.3] and [5, Lemma 5.3] without explicitly mentioning Spin^c -structures. Nonetheless, it is straightforward to see that the proofs provided in the literature

also imply the statement about Spin^c -structures. Statement (4) is basically a tautology, see [23, Theorem 7.3] and [5, Lemma 5.2]. Finally Statement (5) is obtained in the proof of Theorem 1.1 of [6].

Define two functions $f_0, f_1: (0, \infty) \rightarrow \mathbb{R}$ to be *equivalent* if there is an $m \in \mathbb{R}$ such that $f_1(t) - f_0(t) = m \cdot \ln(t)$ holds. Because of Theorem 1.8 (2) the equivalence class of the function $\rho^{(2)}(M; \mathfrak{s}; \mu, \phi)$ defined in (1.5) is independent of the choice of the Spin^c -structure, and will be denoted by

$$(1.9) \quad \bar{\rho}^{(2)}(M; \mu, \phi).$$

Theorem 1.8 (1) allows us to define the degree of $\bar{\rho}^{(2)}(M; \mu, \phi)$ by

$$(1.10) \quad \deg(\bar{\rho}^{(2)}(M; \mu, \phi)) = \lim_{t \rightarrow \infty} \frac{\rho(t)}{\ln(t)} - \lim_{t \rightarrow 0} \frac{\rho(t)}{\ln(t)}$$

for any representative $\rho: (0, \infty) \rightarrow \mathbb{R}$ of $\bar{\rho}^{(2)}(M; \mu, \phi)$.

1.5. Approximation. The following is a consequence of one of the main technical results of [23].

Theorem 1.11 (Twisted Approximation inequality). *Let $\phi: G \rightarrow \mathbb{R}$ be a group homomorphism whose image is finitely generated.*

Consider a nested sequence of normal subgroups of G

$$G \supseteq G_0 \supseteq G_1 \supseteq G_2 \supseteq \dots$$

such that G_i is contained in $\ker(\phi)$ and the intersection $\bigcap_{i \geq 0} G_i$ is trivial. Suppose that the index $[\ker(\phi) : G_i]$ is finite for all $i \geq 0$. Put $Q_i := G/G_i$. Let $\phi_i: Q_i \rightarrow \mathbb{R}$ be the homomorphism uniquely determined by $\phi_i \circ \text{pr}_i = \phi$, where $\text{pr}_i: G \rightarrow Q_i$ is the canonical projection.

Fix an (r, s) -matrix $A \in M_{r,s}(\mathbb{Z}G)$. Denote by $A[i]$ the image of A under the map $M_{r,s}(\mathbb{Z}G) \rightarrow M_{r,s}(\mathbb{Z}Q_i)$ induced by the projection pr_i .

Then we get

$$\dim_{\mathcal{N}(G)}(\ker(\Lambda^G \circ \eta_{\phi^* \mathbb{C}_t}(r_A))) = \lim_{i \rightarrow \infty} \dim_{\mathcal{N}(Q_i)}(\ker(\Lambda^{Q_i} \circ \eta_{\phi_i^* \mathbb{C}_t}(r_{A[i]})))$$

and

$$\det_{\mathcal{N}(G)}(\Lambda^G \circ \eta_{\phi^* \mathbb{C}_t}(r_A)) \geq \limsup_{i \rightarrow \infty} \det_{\mathcal{N}(Q_i)}(\Lambda^{Q_i} \circ \eta_{\phi_i^* \mathbb{C}_t}(r_{A[i]})).$$

Proof. Since the image of ϕ is finitely generated, we can choose a monomorphism $j: \mathbb{Z}^d \rightarrow \mathbb{R}$ and an epimorphism $\phi': G \rightarrow \mathbb{Z}^d$ with $\phi = j \circ \phi'$. Now we apply [23, Theorem 6.51] to ϕ' in the special case $V = j^* \mathbb{C}_t$. \square

1.6. The Thurston norm. Recall the definition in [29] of the *Thurston norm* $x_M(\phi)$ of a 3-manifold M and an element $\phi \in H^1(M; \mathbb{Z}) = \text{Hom}(\pi_1(M), \mathbb{Z})$

$$x(\phi) := \min\{\chi_-(F) \mid F \subset N \text{ properly embedded surface dual to } \phi\},$$

where, given a surface F with connected components F_1, F_2, \dots, F_k , we define

$$\chi_-(F) = \sum_{i=1}^k \max\{-\chi(F_i), 0\}.$$

Thurston [29] showed that this defines a seminorm on $H^1(M; \mathbb{Z})$ which can be extended to a seminorm on $H^1(M; \mathbb{R})$ which we also denote by x_M again. In particular we get for $r \in \mathbb{R}$ and $\phi \in H^1(N; \mathbb{R})$

$$(1.12) \quad x_M(r \cdot \phi) = |r| \cdot x_M(\phi).$$

If $p: N \rightarrow M$ is a finite covering with n sheets, then Gabai [12, Corollary 6.13] showed that

$$(1.13) \quad x_N(p^* \phi) = n \cdot x_M(\phi).$$

If $F \rightarrow M \xrightarrow{p} S^1$ is a fiber bundle for a 3-manifold M and compact surface F and $\phi \in H^1(M; \mathbb{Z})$ is given by $H_1(p): H_1(M) \rightarrow H_1(S^1) = \mathbb{Z}$, then by [29, Section 3] we have

$$(1.14) \quad x_M(\phi) = \begin{cases} -\chi(F) & \text{if } \chi(F) \leq 0; \\ 0 & \text{if } \chi(F) \geq 0. \end{cases}$$

2. CALCULATING THE ϕ -TWISTED L^2 -TORSION FUNCTION

The following theorem says that given M and $\psi \in H^1(M; \mathbb{Q})$ the corresponding L^2 -torsion functions can be computed using one fixed square matrix over $\mathbb{Z}[\pi_1(M)]$ together with a well-understood error term.

Theorem 2.1. *Let M be a 3-manifold with $b_1(M) > 0$, let $\mathfrak{s} \in \text{Spin}^c(M)$. We write $\pi = \pi_1(M)$.*

- (1) *Suppose ∂M is non-empty and toroidal. Then there exists an $s \in \pi_1(M)$ and a square matrix A over $\mathbb{Z}[\pi]$ such that the following conditions are satisfied for any large homomorphism $\mu: \pi \rightarrow G$ and any homomorphism $\phi: G \rightarrow \mathbb{R}$:*

(a) $b_n^{(2)}(\overline{M}; \mathcal{N}(G)) = 0$ holds for all $n \geq 0$ if and only if $\dim_{\mathcal{N}(G)}(\ker(\Lambda^G \circ \eta_{\phi^* \mathbb{C}_t}(r_A)))$ vanishes for all $t > 0$.

(b) If (a) is the case, then (M, μ) is L^2 -acyclic and we get

$$\rho^{(2)}(M, \mathfrak{s}; \mu, \phi)(t) = -\ln(\det_{\mathcal{N}(G)}(\Lambda^G \circ \eta_{\phi^* \mathbb{C}_t}(r_A))) + \eta(t)$$

where $\eta(t)$ is given by

$$\eta(t) = \max\{0, |\phi(s)| \cdot \ln(t)\}.$$

- (2) *Suppose M is closed. Then there exist $s, s' \in \pi_1(M)$ and a square matrix A over $\mathbb{Z}[\pi]$ such that the following conditions are satisfied for any large homomorphism $\mu: \pi \rightarrow G$ and any homomorphism $\phi: G \rightarrow \mathbb{R}$:*

(a) $b_n^{(2)}(\overline{M}; \mathcal{N}(G)) = 0$ holds for all $n \geq 0$ if and only if $\dim_{\mathcal{N}(G)}(\ker(\Lambda^G \circ \eta_{\phi^* \mathbb{C}_t}(r_A)))$ vanishes for all $t > 0$.

(b) If (a) is the case, then (M, μ) is L^2 -acyclic and we get

$$\rho^{(2)}(M, \mathfrak{s}; \mu, \phi)(t) = -\ln(\det_{\mathcal{N}(G)}(\Lambda^G \circ \eta_{\phi^* \mathbb{C}_t}(r_A))) + \eta(t)$$

where $\eta(t)$ is given by

$$\eta(t) = \max\{0, |\phi(s)| \cdot \ln(t)\} + \max\{0, |\phi(s')| \cdot \ln(t)\}.$$

Proof. We only treat the case, where ∂M is empty, and leave it to the reader to figure out the details for the case of a non-empty boundary using the proof of [21, Theorem 2.4]. From [25, Proof of Theorem 5.1] we obtain the following:

- (1) a compact 3-dimensional CW -complex X together with a homeomorphism $f: X \rightarrow M$ (in the following we identify $\pi = \pi_1(M) = \pi_1(X)$ using $\pi_1(f)$),
- (2) two sets of generators $\{s_1, \dots, s_a\}$ and $\{s'_1, \dots, s'_a\}$ of π ,
- (3) an $a \times a$ -matrix F over $\mathbb{Z}[\pi]$,

such the cellular $\mathbb{Z}\pi$ -chain complex $C_*(\tilde{X})$ of the universal cover \tilde{X} looks for an appropriate fundamental family of cells like

$$\mathbb{Z}\pi \xrightarrow{\prod_{i=1}^a r_{s'_i-1}} \bigoplus_{i=1}^a \mathbb{Z}\pi \xrightarrow{r_F} \bigoplus_{i=1}^a \mathbb{Z}\pi \xrightarrow{\bigoplus_{i=1}^a r_{s_i-1}} \mathbb{Z}\pi.$$

The based $\mathbb{Z}G$ -chain complex $\mathbb{Z}G \otimes_{\mathbb{Z}\pi} C_*(\tilde{X})$ looks like

$$\mathbb{Z}G \xrightarrow{\prod_{i=1}^a r_{\mu(s'_i)-1}} \bigoplus_{i=1}^a \mathbb{Z}G \xrightarrow{r_{\mu(F)}} \bigoplus_{i=1}^a \mathbb{Z}G \xrightarrow{\bigoplus_{i=1}^a r_{\mu(s_i)-1}} \mathbb{Z}G.$$

Then the Hilbert $\mathcal{N}(G)$ -chain complex $\Lambda^G \circ \eta_{\phi^* \mathbb{C}_t}(C_*(\bar{X}))$ looks like

$$\begin{aligned} L^2(G) \xrightarrow{\prod_{i=1}^a \Lambda^G(r_{\mu(s'_i)-1})} \bigoplus_{i=1}^a L^2(G) \xrightarrow{\Lambda^G \circ \eta_{\phi^* \mathbb{C}_t}(r_{\mu(F)})} \bigoplus_{i=1}^a L^2(G) \\ \xrightarrow{\bigoplus_{i=1}^a \Lambda^G(r_{\mu(s_i)-1})} L^2(G). \end{aligned}$$

Since $b_1(M) > 0$ is non-trivial there exist $i, j \in \{1, \dots, a\}$ such that s_i and s_j represent non-zero elements in $H_1(M; \mathbb{Z})_f$. For later we record, that given any large homomorphism $\mu: \pi \rightarrow G$ the images $\mu(s)$ and $\mu(s')$ have infinite order. We denote by A the matrix that is obtained from F by removing the i -th column and the j -th row.

For $g \in G$ and $t \in (0, \infty)$ let $D(g, t)_*$ be the 1-dimensional Hilbert $\mathcal{N}(G)$ -chain complex which has as first differential $\Lambda^G(r_{t\phi(g).g-1}): L^2(G) \rightarrow L^2(G)$. Provided that $|g| = \infty$ holds, $D(g, t)_*$ is L^2 -det-acyclic and a direct computation using [22, Theorem 3.14 (6) on page 129 and (3.23) on page 136] shows

$$(2.2) \quad \rho^{(2)}(D(g, t)_*) = \ln(\det_{\mathcal{N}(G)}(\Lambda^G(r_{t\phi(g).g-1})) = \max\{|\phi(g)| \cdot \ln(t), 0\}.$$

Now let $\mathfrak{s} \in \text{Spin}^c(M)$ be the Spin^c -structure that corresponds to the above fundamental family of cells. It follows from [5, Lemma 3.2] that the above group elements s, s' and the matrix A have all the desired properties.

If $\mathfrak{t} \in \text{Spin}^c(M)$ is a different Spin^c -structure, then we can write $\mathfrak{t} = h\mathfrak{s}$ for some $h \in H_1(M; \mathbb{Z})$. We pick a representative $g \in \pi$ of h and we multiply one column of A by h to obtain the matrix with the desired properties. \square

3. LOWER BOUNDS

The elementary proof of the next lemma can be found in [23, Lemma 6.9].

Lemma 3.1. *Let $f: L^2(G)^m \rightarrow L^2(G)^n$ be a bounded G -equivariant operator. Then*

$$\det_{\mathcal{N}(G)}(f) \leq \|f\|^{\dim_{\mathcal{N}(G)}(\overline{\text{im}(f)})}.$$

The next result is an improvement of [5, Proposition 9.6].

Lemma 3.2. *Consider bounded G -equivariant operators $f_0, f_1: L^2(G)^m \rightarrow L^2(G)^m$. For $t > 0$ we define*

$$f[t] := f_0 + t \cdot f_1.$$

Suppose that for every $t > 0$ the operator $f[t]: L^2(G)^m \rightarrow L^2(G)^m$ is L^2 -det-acyclic. Put

$$\rho: (0, \infty) \rightarrow (0, \infty), \quad t \mapsto \ln(\det_{\mathcal{N}(G)}(f[t])).$$

Then we get

$$\begin{aligned} \rho(t) &\leq m \cdot \max\{0, \ln(\|f_0\| + \|f_1\|)\} && \text{for } t \leq 1; \\ \rho(t) &\leq \dim_{\mathcal{N}(G)}(\overline{\text{im}(f_1)}) \cdot \ln(t) + m \cdot \max\{0, \ln(2 \cdot \|f_0\| + \|f_1\|)\} && \text{for } t \geq 1. \end{aligned}$$

In particular we get

$$\begin{aligned} \limsup_{t \rightarrow \infty} \frac{\rho(t)}{\ln(t)} &\leq \dim_{\mathcal{N}(G)}(\overline{\text{im}(f_1)}); \\ \liminf_{t \rightarrow 0} \frac{\rho(t)}{\ln(t)} &\geq 0; \\ \limsup_{t \rightarrow \infty} \frac{\rho(t)}{\ln(t)} - \liminf_{t \rightarrow 0} \frac{\rho(t)}{\ln(t)} &\leq \dim_{\mathcal{N}(G)}(\overline{\text{im}(f_1)}). \end{aligned}$$

Proof. It suffices to prove the two inequalities for $\rho(t)$, then the other claims follow. We begin with the case $t \leq 1$. We get from Lemma 3.1

$$\det_{\mathcal{N}(G)}(f[t]) \leq \|f[t]\|^{\dim_{\mathcal{N}(G)}(\overline{\text{im}(f[t])})}.$$

If $\|f[t]\| \leq 1$, this implies $\det_{\mathcal{N}(G)}(f[t]) \leq 1$ and the claim follows. Hence it remains to treat the case $\|f[t]\| > 1$. Then we get because of $\dim_{\mathcal{N}(G)}(\overline{\text{im}(f)}) \leq m$ that

$$\begin{aligned} \det_{\mathcal{N}(G)}(f[t]) &\leq \|f[t]\|^m \\ &= \|f_0 + t \cdot f_1\|^m \\ &\leq (\|f_0\| + t \cdot \|f_1\|)^m \\ &\stackrel{t \leq 1}{\leq} (\|f_0\| + \|f_1\|)^m. \end{aligned}$$

Next we consider the case $t \geq 1$. We have the orthogonal decomposition

$$L^2(G)^m = \overline{\text{im}(f_1)} \oplus \overline{\text{im}(f_1)}^\perp.$$

With respect to this decomposition we get for any bounded G -equivariant operator $g: L^2(G)^m \rightarrow L^2(G)^m$ the decomposition

$$g = \begin{pmatrix} g^{(1,1)} & g^{(1,2)} \\ g^{(2,1)} & g^{(2,2)} \end{pmatrix}.$$

We estimate for $t \geq 1$ using [22, Theorem 3.14 (1) and (2) on page 128]

$$\begin{aligned} \frac{\det_{\mathcal{N}(G)}(f[t])}{t^{\dim_{\mathcal{N}(G)}(\overline{\text{im}(f_1)})}} &= \det_{\mathcal{N}(G)} \left(\begin{pmatrix} t^{-1} \cdot \text{id} & 0 \\ 0 & \text{id} \end{pmatrix} \right) \cdot \det_{\mathcal{N}(G)}(f[t]) \\ &= \det_{\mathcal{N}(G)} \left(\begin{pmatrix} t^{-1} \cdot \text{id} & 0 \\ 0 & \text{id} \end{pmatrix} \circ f[t] \right) \\ &\stackrel{\text{Lemma 3.1}}{\leq} \left\| \begin{pmatrix} t^{-1} \cdot \text{id} & 0 \\ 0 & \text{id} \end{pmatrix} \circ f[t] \right\|^m. \end{aligned}$$

If $\left\| \begin{pmatrix} t^{-1} \cdot \text{id} & 0 \\ 0 & \text{id} \end{pmatrix} \circ f[t] \right\| \leq 1$ the claim is obviously true. Hence it remains to treat the case $\left\| \begin{pmatrix} t^{-1} \cdot \text{id} & 0 \\ 0 & \text{id} \end{pmatrix} \circ f[t] \right\| \geq 1$. Then we get

$$\begin{aligned}
& \frac{\det_{\mathcal{N}(G)}(f[t])}{t^{\dim_{\mathcal{N}(G)}(\overline{\text{im}}(f_1))}} \\
& \leq \left\| \begin{pmatrix} t^{-1} \cdot \text{id} & 0 \\ 0 & \text{id} \end{pmatrix} (f_0 + t \cdot f_1) \right\|^m \\
& = \left\| \begin{pmatrix} t^{-1} f_0^{(1,1)} & t^{-1} f_0^{(1,2)} \\ f_0^{(2,1)} & f_0^{(2,2)} \end{pmatrix} + \begin{pmatrix} f_1^{(1,1)} & f_1^{(1,2)} \\ 0 & 0 \end{pmatrix} \right\|^m \\
& \leq \left(t^{-1} \cdot \left\| \begin{pmatrix} f_0^{(1,1)} & f_0^{(1,2)} \\ 0 & 0 \end{pmatrix} \right\| + \left\| \begin{pmatrix} 0 & 0 \\ f_0^{(2,1)} & f_0^{(2,2)} \end{pmatrix} \right\| + \left\| \begin{pmatrix} f_1^{(1,1)} & f_1^{(1,2)} \\ 0 & 0 \end{pmatrix} \right\| \right)^m \\
& \stackrel{t^{-1} \leq 1}{\leq} \left(\left\| \begin{pmatrix} f_0^{(1,1)} & f_0^{(1,2)} \\ 0 & 0 \end{pmatrix} \right\| + \left\| \begin{pmatrix} 0 & 0 \\ f_0^{(2,1)} & f_0^{(2,2)} \end{pmatrix} \right\| + \left\| \begin{pmatrix} f_1^{(1,1)} & f_1^{(1,2)} \\ 0 & 0 \end{pmatrix} \right\| \right)^m \\
& \leq \left(\left\| \begin{pmatrix} f_0^{(1,1)} & f_0^{(1,2)} \\ f_0^{(2,1)} & f_0^{(2,2)} \end{pmatrix} \right\| + \left\| \begin{pmatrix} f_0^{(1,1)} & f_0^{(1,2)} \\ 0 & 0 \end{pmatrix} \right\| + \left\| \begin{pmatrix} f_1^{(1,1)} & f_1^{(1,2)} \\ 0 & 0 \end{pmatrix} \right\| \right)^m \\
& = (2 \cdot \|f_0\| + \|f_1\|)^m.
\end{aligned}$$

This finishes the proof of Lemma 3.2. \square

For an element $x = \sum_{g \in G} r_g \cdot g$ in $\mathbb{C}G$ define $|x|_1 := \sum_{g \in G} |r_g|$. Given a matrix $A \in M_{r,s}(\mathbb{C}G)$ define

$$(3.3) \quad \|A\|_1 = r \cdot s \cdot \max \{ |a_{j,k}|_1 \mid 1 \leq j \leq r, 1 \leq k \leq s \}.$$

The next theorem can be viewed as saying, that in the acyclic case the degree of the ϕ -twisted L^2 -torsions gives lower bounds on the Thurston norm. This result is thus an analogue of the classical fact, that the degree of the Alexander polynomial gives a lower bound on the knot genus. We refer to [7] for a detailed discussion of various twisted generalizations of the Alexander polynomial of a knot and their relations to the Thurston norm.

Theorem 3.4 (Lower bound). *Let M be an irreducible 3-manifold with infinite fundamental group π . Let $\mathfrak{s} \in \text{Spin}^c(M)$. Then for any $\phi \in H^1(M; \mathbb{Q})$ there exists a constant $D \geq 0$ such that for any large homomorphism $\mu: \pi_1(M) \rightarrow G$ such that (M, μ) is L^2 -acyclic we have*

$$\begin{aligned}
\frac{1}{2}(\phi(c_1(\mathfrak{s})) + x_M(\phi)) \cdot \ln(t) - D & \leq \rho^{(2)}(M, \mathfrak{s}; \mu, \phi)(t) \quad \text{for } t \leq 1; \\
\frac{1}{2}(\phi(c_1(\mathfrak{s})) - x_M(\phi)) \cdot \ln(t) - D & \leq \rho^{(2)}(M, \mathfrak{s}; \mu, \phi)(t) \quad \text{for } t \geq 1.
\end{aligned}$$

In [5, Theorem 9.1] we proved the analogous statement under the extra assumption that $\mu: \pi_1(M) \rightarrow G$ is a homomorphism to a virtually abelian group.

We say that a cohomology class in $H^1(X; \mathbb{Z}) = \text{Hom}(\pi_1(X), \mathbb{Q})$ is *primitive*, if it is a primitive element in $H^1(X; \mathbb{Z})$. Put differently, a class is primitive if the corresponding homomorphism $\pi_1(X) \rightarrow \mathbb{Z}$ is surjective.

In the proof of Theorem 3.4 we will make use of the following elementary lemma.

Lemma 3.5. *Let M be an irreducible 3-manifold with infinite fundamental group and let $\mathfrak{s} \in \text{Spin}^c(M)$. If the conclusion of Theorem 3.4 holds for all primitive $\phi \in H^1(M; \mathbb{Q})$, then it holds for all $\phi \in H^1(M; \mathbb{Q})$.*

Proof. If ϕ is trivial, then clearly there is nothing to prove. So let $\phi \in H^1(M; \mathbb{Q})$ be non-zero. We pick an $r \in \mathbb{Q}_{>0}$ such that $r\phi \in H^1(M; \mathbb{Z})$ is primitive. We denote by D the constant of Theorem 3.4 corresponding to the primitive class $r\phi$.

From Theorem 1.8 (4) and from (1.12) we get for any large homomorphism $\mu: \pi_1(M) \rightarrow G$ such that (M, μ) is L^2 -acyclic that

$$\begin{aligned} \rho^{(2)}(M, \mathfrak{s}; \mu, \phi)(t) &= \rho^{(2)}(M, \mathfrak{s}; \mu, r\phi)(t^{\frac{1}{r}}); \\ x_M(r\phi) &= r \cdot x_M(\phi). \end{aligned}$$

Combining these equalities with the elementary equalities

$$\begin{aligned} \ln(t^{\frac{1}{r}}) &= \frac{1}{r} \ln(t); \\ (r\phi)(c_1(\mathfrak{s})) &= r \cdot \phi(c_1(\mathfrak{s})), \end{aligned}$$

it is straightforward to see that the desired inequalities also hold for μ and ϕ . \square

Proof of Theorem 3.4. By Lemma 3.5 it suffices to prove the statement for every primitive $\phi \in H^1(M; \mathbb{Z})$. We start out with the following claim.

Claim. Given a primitive $\phi \in H^1(M; \mathbb{Z})$ there exists an $\mathfrak{s} \in \text{Spin}^c(M)$ such that for any large homomorphism $\mu: \pi_1(M) \rightarrow G$ such that (M, μ) is L^2 -acyclic the following inequalities hold

$$\begin{aligned} -D &\leq \rho^{(2)}(M, \mathfrak{s}; \mu, \phi)(t) \quad \text{for } t \leq 1; \\ -x_M(\phi) \cdot \ln(t) - D &\leq \rho^{(2)}(M, \mathfrak{s}; \mu, \phi)(t) \quad \text{for } t \geq 1. \end{aligned}$$

In the following we abbreviate

$$\rho(\mu, \phi) = \rho^{(2)}(M, \mathfrak{s}; \mu, \phi).$$

We conclude by inspecting the proof of [5, Proposition 9.2 in Section 9.1] which is based on [10, Section 4], that there exists an $\mathfrak{s} \in \text{Spin}^c(M)$, integers k, l, m with $k, l \geq 0$ and $x_M(\phi) = k - l$, an element $\gamma \in \pi$ with $\phi(\gamma) = 1$, a matrix $A \in M_{k+m, k+m}(\mathbb{Z}K)$ for $K = \ker(\phi)$, such that for any large homomorphism $\mu: \pi_1(M) \rightarrow G$ such that (M, μ) is L^2 -acyclic the following equality holds

$$\rho(\mu, \phi)(t) = -\ln(\max\{1, t\}^{-1} \cdot \det_{\mathcal{N}(G)}(\Lambda^G(r_{\mu(A)}) + t \cdot \mu(\gamma) \cdot \text{id}_{L^2(G)^k} \oplus 0_{L^2(G)^m})).$$

This implies

$$\rho(\mu, \phi)(t) = \begin{cases} -\ln(\det_{\mathcal{N}(G)}(\Lambda^G(r_{\mu(A)}) + t \cdot \mu(\gamma) \cdot \text{id}_{L^2(G)^k} \oplus 0_{L^2(G)^m})) & \text{for } t \leq 1; \\ l \cdot \ln(t) - \ln(\det_{\mathcal{N}(G)}(\Lambda^G(r_{\mu(A)}) + t \cdot \mu(\gamma) \cdot \text{id}_{L^2(G)^k} \oplus 0_{L^2(G)^m})) & \text{for } t \geq 1. \end{cases}$$

Define

$$D = (k + m) \cdot \ln(2 \cdot (\|A\|_1 + 1)).$$

Note that D depends only on ϕ . We conclude from [23, Lemma 6.3] and the monotonicity of \ln that

$$\begin{aligned} D &\geq (k + m) \cdot \ln(2 \cdot \|\Lambda^G(r_{\mu(A)})\| + \|\text{id}_{L^2(G)^k} \oplus 0_{L^2(G)^m}\|) \\ &\geq (k + m) \cdot \ln(\|\Lambda^G(r_{\mu(A)})\| + \|\text{id}_{L^2(G)^k} \oplus 0_{L^2(G)^m}\|). \end{aligned}$$

Therefore we conclude from Lemma 3.2 applied in the case $f_0 = \Lambda^G(r_{\mu(A)})$ and $f_1 = \mu(\gamma) \cdot \text{id}_{L^2(G)^k} \oplus 0_{L^2(G)^m}$ that

$$\ln(\det_{\mathcal{N}(G)}(\Lambda^G(r_{\mu(A)}) + t \cdot \mu(\gamma) \cdot \text{id}_{L^2(G)^k} \oplus 0_{L^2(G)^m})) \leq \begin{cases} D & t \leq 1; \\ k \cdot \ln(t) + D & t \geq 1. \end{cases}$$

This implies

$$\begin{aligned} -D &\leq \rho(\mu, \phi)(t) \quad \text{for } t \leq 1; \\ -(k - l) \cdot \ln(t) - D &\leq \rho(\mu, \phi)(t) \quad \text{for } t \geq 1. \end{aligned}$$

Since $x_M(\phi) = k - l$, this implies the claim.

We now turn to the proof of the desired inequalities in the theorem. Using Theorem 1.8 (2) and equality (1.1) one can easily see that if the desired inequalities hold for one Spin^c -structure of M , then they also hold for all other Spin^c -structures of M . Now let $\mathfrak{s} \in \text{Spin}^c(M)$ be the Euler structure from the claim. Then

$$\begin{aligned} -D &\leq \rho^{(2)}(M, \mathfrak{s}; \mu, \phi)(t) \quad \text{for } t \leq 1; \\ -x_M(\phi) \cdot \ln(t) - D &\leq \rho^{(2)}(M, \mathfrak{s}; \mu, \phi)(t) \quad \text{for } t \geq 1. \end{aligned}$$

By Theorem 1.8 (5) we also know that

$$\rho(M, \mathfrak{s}; \mu, \phi)(t) = \phi(c_1(\mathfrak{s})) \ln(t) + \rho(M, \mathfrak{s}; \mu, \phi)(t^{-1})$$

for all $t \in (0, \infty)$. Combining this equality with the above inequalities we obtain that

$$\begin{aligned} (\phi(c_1(\mathfrak{s})) + x_M(\phi)) \cdot \ln(t) - D &\leq \rho^{(2)}(M, \mathfrak{s}; \mu, \phi)(t) \quad \text{for } t \leq 1; \\ \phi(c_1(\mathfrak{s})) \cdot \ln(t) - D &\leq \rho^{(2)}(M, \mathfrak{s}; \mu, \phi)(t) \quad \text{for } t \geq 1. \end{aligned}$$

Adding the two inequalities for $t \leq 1$ and dividing by two, and doing the same for the inequalities for $t \geq 1$ gives us the desired inequalities

$$\begin{aligned} \frac{1}{2}(\phi(c_1(\mathfrak{s})) + x_M(\phi)) \cdot \ln(t) - D &\leq \rho^{(2)}(M, \mathfrak{s}; \mu, \phi)(t) \quad \text{for } t \leq 1; \\ \frac{1}{2}(\phi(c_1(\mathfrak{s})) - x_M(\phi)) \cdot \ln(t) - D &\leq \rho^{(2)}(M, \mathfrak{s}; \mu, \phi)(t) \quad \text{for } t \geq 1. \end{aligned}$$

□

4. UPPER BOUNDS

Before we can provide upper bounds on the Thurston norm we will need to prove one preliminary result. This lemma will ensure that some information which is only available at 0 and ∞ leads to uniform estimates for all $t > 0$. This will be a key ingredient when we want to apply approximation techniques.

Lemma 4.1. *Let $\phi: G \rightarrow \mathbb{Z}$ be a non-trivial group homomorphism with finite kernel. Let $A \in M_{m,m}(\mathbb{Z}G)$ be a matrix such that $\Lambda^G(r_A): L^2(G)^m \rightarrow L^2(G)^m$ is a weak isomorphism. Then $\Lambda^G \circ \eta_{\phi^* \mathbb{C}_t}(r_A): L^2(G)^m \rightarrow L^2(G)^m$ is L^2 -det-acyclic for any $t > 0$. Put*

$$\rho: (0, \infty) \rightarrow \mathbb{R}, \quad t \mapsto \ln(\det_{\mathcal{N}(G)}(\Lambda^G \circ \eta_{\phi^* \mathbb{C}_t}(r_A))).$$

Suppose that there are real numbers C and D and integers k and l such that

$$\begin{aligned} \lim_{t \rightarrow 0} \rho(t) - k \cdot \ln(t) &= C; \\ \lim_{t \rightarrow \infty} \rho(t) - l \cdot \ln(t) &= D. \end{aligned}$$

Then we get for all $t > 0$

$$\begin{aligned} k \cdot \ln(t) + C &\leq \rho(t); \\ l \cdot \ln(t) + D &\leq \rho(t). \end{aligned}$$

Proof. Choose an integer $n \geq 1$ and an epimorphism $\phi': G \rightarrow \mathbb{Z}$ such that $\phi = n \cdot \text{id}_{\mathbb{Z}} \circ \phi'$. Then we get for the two functions ρ and ρ' associated to ϕ and ϕ' from Theorem 1.8 (4)

$$\rho'(t) = \rho(t^n).$$

Hence we can assume without loss of generality that ρ is surjective, otherwise replace ϕ by ϕ' .

Choose a group homomorphism $s: \mathbb{Z} \rightarrow G$ with $\phi \circ s = \text{id}$. Choose a map of sets $\sigma: \text{im}(s) \setminus G \rightarrow G$ whose composition with the projection $\text{pr}: G \rightarrow \text{im}(s) \setminus G$ is the identity and whose composition with $\phi: G \rightarrow \mathbb{Z}$ is the constant map with value

$0 \in \mathbb{Z}$, see [23, (6.38)]. Let $B \in M_{m \cdot |\ker(\phi)|, m \cdot |\ker(\phi)|}(\mathbb{Z}[\mathbb{Z}])$ be the matrix describing the restriction of $r_A: \mathbb{Z}G^m \rightarrow \mathbb{Z}G^m$ with s , see [23, (6.39) and (6.41)]. Then a direct computation shows for all $t \in (0, \infty)$

$$s^*(\Lambda^G \circ \eta_{\phi^* \mathbb{C}_t}(r_A)) = \Lambda^{\mathbb{Z}} \circ \eta_{(\phi \circ s)^* \mathbb{C}_t}(r_B): L^2(\mathbb{Z})^{m \cdot |\ker(\phi)|} \rightarrow L^2(\mathbb{Z})^{m \cdot |\ker(\phi)|}$$

where s^* denotes restriction with s . We get from [22, Theorem 3.14 (5) on page 128]

$$\ln(\det_{\mathcal{N}(G)}(\Lambda^G \circ \eta_{\phi^* \mathbb{C}_t}(r_A))) = \frac{\ln(\det_{\mathcal{N}(\mathbb{Z})}(s^*(\Lambda^G \circ \eta_{\phi^* \mathbb{C}_t}(r_A))))}{|\ker(\phi)|}.$$

Hence we can assume without loss of generality $G = \mathbb{Z}$ and $\phi = \text{id}_{\mathbb{Z}}$, otherwise replace $\phi: G \rightarrow \mathbb{Z}$ by $\phi \circ s = \text{id}: \mathbb{Z} \rightarrow \mathbb{Z}$ and A by B .

One easily checks

$$r_{\det_{\mathbb{C}[\mathbb{Z}]}(\eta_{\mathbb{C}_t}(r_A))} = \eta_{\mathbb{C}_t}(r_{\det_{\mathbb{C}[\mathbb{Z}]}(A)}): L^2(\mathbb{Z}) \rightarrow L^2(\mathbb{Z}).$$

Because of [23, Lemma 6.25] we can assume without loss of generality $m = 1$, otherwise replace A by the $(1, 1)$ -matrix given by $\det_{\mathbb{C}[\mathbb{Z}]}(A)$.

Let $p(z) \in \mathbb{C}[\mathbb{Z}] = \mathbb{C}[z, z^{-1}]$ be the only entry in the $(1, 1)$ -matrix A . Since $\Lambda^{\mathbb{Z}}(r_A)$ is a weak isomorphism by assumption, p is non-trivial. We can write

$$p(z) = \sum_{n=n_0}^{n_1} c_n \cdot z^n$$

for integers n_0 and n_1 with $n_0 \leq n_1$, complex numbers $c_{n_0}, c_{n_0+1}, \dots, c_{n_1}$ with $c_{n_0} \neq 0$ and $c_{n_1} \neq 0$. We can also write

$$p(z) = c_{n_1} \cdot z^r \cdot \prod_{i=1}^s (z - a_i)$$

for an integer $s \geq 0$, non-zero complex numbers a_1, \dots, a_r and an integer r . We get from [22, (3.23) on page 136]

$$\det_{\mathcal{N}(\mathbb{Z})}(\Lambda^{\mathbb{Z}}(r_p)) = |c_{n_1}| \cdot \prod_{\substack{i=1, \dots, s \\ |a_i| \geq 1}} |a_i|.$$

For $t \in (0, \infty)$ we get

$$p(tz) = c_{n_1} \cdot (tz)^r \cdot \prod_{i=1}^s (tz - a_i) = t^{r+s} \cdot c_{n_1} \cdot z^r \cdot \prod_{i=1}^s \left(z - \frac{a_i}{t}\right),$$

and hence

$$\det_{\mathcal{N}(\mathbb{Z})}(\Lambda^{\mathbb{Z}}(r_{p(tz)})) = t^{r+s} \cdot |c_{n_1}| \cdot \prod_{\substack{i=1, \dots, s \\ |a_i/t| \geq 1}} \left|\frac{a_i}{t}\right| = t^{r+s} \cdot |c_{n_1}| \cdot \prod_{\substack{i=1, \dots, s \\ |a_i| \geq t}} \frac{|a_i|}{t}.$$

This implies for $t \in (0, \infty)$

$$(4.2) \quad \rho(t) = (r+s) \cdot \ln(t) + \ln(|c_{n_1}|) + \sum_{\substack{i=1, \dots, s \\ |a_i| \geq t}} (\ln(|a_i|) - \ln(t)).$$

Define positive real numbers

$$\begin{aligned} T_0 &= \min\{|a_i| \mid i = 1, 2, \dots, s\}; \\ T_\infty &= \max\{|a_i| \mid i = 1, 2, \dots, s\}. \end{aligned}$$

Then we get

$$\rho(t) = \begin{cases} r \cdot \ln(t) + \ln(|c_{n_1}|) + \sum_{i=1}^s \ln(|a_i|) & \text{for } t \leq T_0; \\ (r+s) \cdot \ln(t) + \ln(|c_{n_1}|) & \text{for } t \geq T_\infty. \end{cases}$$

Since by assumption there are real numbers C and D and integers k and l such that

$$\begin{aligned}\lim_{t \rightarrow 0} \rho(t) - k \cdot \ln(t) &= C; \\ \lim_{t \rightarrow \infty} \rho(t) - l \cdot \ln(t) &= D,\end{aligned}$$

we must have $r = k$, $r + s = l$, $C = \ln(|c_{n_1}|) + \sum_{i=1}^s \ln(|a_i|)$ and $D = \ln(|c_{n_1}|)$. Equation (4.2) becomes

$$\rho(t) = l \cdot \ln(t) + D + \sum_{\substack{i=1, \dots, s \\ |a_i| \geq t}} (\ln(|a_i|) - \ln(t)).$$

Since $(\ln(|a_i|) - \ln(t)) \geq 0$ for $|a_i| \geq t$, we get $l \cdot \ln(t) + \ln(D) \leq \rho(t)$ for all $t > 0$. We estimate for $t > 0$

$$\begin{aligned}k \cdot \ln(t) + C &= k \cdot \ln(t) + D + \sum_{i=1}^s \ln(|a_i|) \\ &= k \cdot \ln(t) + D + \sum_{\substack{i=1, \dots, s \\ |a_i| \geq t}} \ln(|a_i|) + \sum_{\substack{i=1, \dots, s \\ |a_i| < t}} \ln(|a_i|) \\ &= r \cdot \ln(t) + D + s \cdot \ln(t) + \sum_{\substack{i=1, \dots, s \\ |a_i| \geq t}} (\ln(|a_i|) - \ln(t)) + \sum_{\substack{i=1, \dots, s \\ |a_i| < t}} (\ln(|a_i|) - \ln(t)) \\ &= l \cdot \ln(t) + D + \sum_{\substack{i=1, \dots, s \\ |a_i| \geq t}} (\ln(|a_i|) - \ln(t)) + \sum_{\substack{i=1, \dots, s \\ |a_i| < t}} (\ln(|a_i|) - \ln(t)) \\ &\leq l \cdot \ln(t) + D + \sum_{\substack{i=1, \dots, s \\ |a_i| \geq t}} (\ln(|a_i|) - \ln(t)) \\ &= \rho(t).\end{aligned}$$

This finishes the proof of Lemma 4.1. \square

Definition 4.3 (Fibered classes). Let M be a 3-manifold and consider an element $\phi \in H^1(M; \mathbb{Q}) = \text{Hom}(\pi_1(M), \mathbb{Q})$. We say that ϕ is *fibered* if there exists a locally trivial fiber bundle $p: M \rightarrow S^1$ and a $k \in \mathbb{Q}$, $k > 0$ such that the induced map $p_*: \pi_1(M) \rightarrow \pi_1(S^1) = \mathbb{Z}$ coincides with $k \cdot \phi$.

Theorem 4.4. *Let $M \neq S^1 \times D^2$ be an irreducible 3-manifold. Then the following two statements hold:*

- (1) *If M is fibered, then for any large homomorphism $\mu: \pi_1(M) \rightarrow G$ to a residually finite group the pair (M, μ) is L^2 -acyclic.*
- (2) *If $\phi \in H^1(M; \mathbb{Z}) = \text{Hom}(\pi_1(M), \mathbb{Z})$ is a primitive fibered class, then there exists a $T \geq 1$ such that for any $\mathfrak{s} \in \text{Spin}^c(M)$ and for any large homomorphism $\mu: \pi_1(M) \rightarrow G$ to a residually finite group the following inequalities hold*

$$\begin{aligned}\rho^{(2)}(M, \mathfrak{s}; \mu, \phi)(t) &= \frac{1}{2}(\phi(c_1(\mathfrak{s})) + x_M(\phi)) \cdot \ln(t) & \text{for } t < \frac{1}{T}; \\ \rho^{(2)}(M, \mathfrak{s}; \mu, \phi)(t) &= \frac{1}{2}(\phi(c_1(\mathfrak{s})) - x_M(\phi)) \cdot \ln(t) & \text{for } t > T.\end{aligned}$$

In fact one can choose T to be the entropy of the monodromy.

Proof. The first statement follows from [20, Theorem 2.1]. Now we denote by T the entropy of the monodromy of the primitive fibered class ϕ . By Theorem 8.2 of [5] there exists an $\mathfrak{s} \in \text{Spin}^c(M)$ such that

$$\begin{aligned} 0 &= \rho^{(2)}(M, \mathfrak{s}; \mu, \phi)(t) \quad \text{for } t < \frac{1}{T}; \\ -x_M(\phi) \cdot \ln(t) &= \rho^{(2)}(M, \mathfrak{s}; \mu, \phi)(t) \quad \text{for } t > T. \end{aligned}$$

The statement of the theorem follows from these inequalities in precisely the same way as we concluded the proof of Theorem 3.4. \square

The next lemma improves on Theorem 4.4 in so far as it gives us some control over $\rho^{(2)}(M, \mathfrak{s}; \mu, \phi)$ for all t . In particular the set of t 's for which we have control does not depend on the choice of ϕ .

Lemma 4.5. *Let $(M, \mu: \pi_1(M) \rightarrow G)$ be an admissible pair and let $\mathfrak{s} \in \text{Spin}^c(M)$. Then for any fibered $\phi \in H^1(M; \mathbb{Q})$ we have*

$$\begin{aligned} \rho^{(2)}(M, \mathfrak{s}; \mu, \phi)(t) &\leq \frac{1}{2}(\phi(c_1(\mathfrak{s})) + x_M(\phi)) \cdot \ln(t) \quad \text{for } t \leq 1; \\ \rho^{(2)}(M, \mathfrak{s}; \mu, \phi)(t) &\leq \frac{1}{2}(\phi(c_1(\mathfrak{s})) - x_M(\phi)) \cdot \ln(t) \quad \text{for } t \geq 1. \end{aligned}$$

Proof. Let $(M, \mu: \pi_1(M) \rightarrow G)$ be an admissible pair such that M admits a fibered class. By Theorem 4.4 the pair (M, μ) is L^2 -acyclic. Let $\mathfrak{s} \in \text{Spin}^c(M)$. The argument of the proof of Lemma 3.5 shows that it suffices to prove the lemma for primitive fibered classes. So let $\phi \in H^1(M; \mathbb{Z}) = \text{Hom}(\pi_1(M), \mathbb{Z})$ be a primitive fibered class.

Consider a nested sequence of in G normal subgroups

$$G \supseteq G_0 \supseteq G_1 \supseteq G_2 \supseteq \dots$$

such that G_i is contained in $\ker(G \rightarrow H_1(G; \mathbb{Z})_f)$, the index $[\ker(G \rightarrow H_1(G; \mathbb{Z})_f) : G_i]$ is finite for $i \geq 0$ and the intersection $\bigcap_{i \geq 0} G_i$ is trivial. Put $Q_i := G/G_i$. Denote by $\text{pr}_i: G \rightarrow Q_i$ the obvious projection. Let $\mu_i: \pi_1(M) \rightarrow Q_i$ be the composition $\text{pr}_i \circ \mu$. The homomorphisms μ_i are again large.

In the following we consider only the case where M is closed, the case with boundary is analogous. We apply Theorem 2.1 (2) to M . We denote the resulting square matrix over $\mathbb{Z}[\pi]$ by A and the resulting elements in the group π by s, s' . We write $A_i = \text{pr}_i(A)$. Define

$$\eta(t) = \max\{0, |\phi(s)| \cdot \ln(t)\} + \max\{0, |\phi(s')| \cdot \ln(t)\}.$$

As above, the pair (M, μ_i) is L^2 -acyclic. Our choice of A and s, s' ensures that

$$\begin{aligned} \rho^{(2)}(M, \mathfrak{s}; \mu, \phi) &= \eta(t) - \ln(\det_{\mathcal{N}(G)}(\Lambda^G \circ \eta_{\phi^* \mathbb{C}_t}(r_A))); \\ \rho^{(2)}(M, \mathfrak{s}; \mu_i, \phi) &= \eta(t) - \ln(\det_{\mathcal{N}(Q_i)}(\Lambda^{Q_i} \circ \eta_{\phi^* \mathbb{C}_t}(r_{A_i}))). \end{aligned}$$

We conclude from Theorem 1.11

$$(4.6) \quad \ln(\det_{\mathcal{N}(G)}(\Lambda^G \circ \eta_{\phi^* \mathbb{C}_t}(r_A))) \geq \limsup_{i \rightarrow \infty} \ln(\det_{\mathcal{N}(Q_i)}(\Lambda^{Q_i} \circ \eta_{\phi^* \mathbb{C}_t}(r_{A_i}))).$$

By Theorem 4.4 there exists a $T \geq 1$ such that for any natural number i we have

$$\begin{aligned} \rho^{(2)}(M, \mathfrak{s}; \mu_i, \phi)(t) &= \frac{1}{2}(\phi(c_1(\mathfrak{s})) + x_M(\phi)) \cdot \ln(t) \quad \text{for } t < \frac{1}{T}; \\ \rho^{(2)}(M, \mathfrak{s}; \mu_i, \phi)(t) &= \frac{1}{2}(\phi(c_1(\mathfrak{s})) - x_M(\phi)) \cdot \ln(t) \quad \text{for } t > T. \end{aligned}$$

This implies

$$\begin{aligned} \ln(\det_{\mathcal{N}(Q_i)}(\Lambda^G \circ \eta_{\phi^* \mathbb{C}_t}(r_{A_i}))) &= \eta(t) - \frac{1}{2}(\phi(c_1(\mathfrak{s})) + x_M(\phi)) \cdot \ln(t) \quad \text{for } t < \frac{1}{T}; \\ \ln(\det_{\mathcal{N}(Q_i)}(\Lambda^G \circ \eta_{\phi^* \mathbb{C}_t}(r_{A_i}))) &= \eta(t) - \frac{1}{2}(\phi(c_1(\mathfrak{s})) - x_M(\phi)) \cdot \ln(t) \quad \text{for } t > T. \end{aligned}$$

Then Lemma 4.1 applied to $\phi: Q_i \rightarrow \mathbb{Z}$ yields

$$\begin{aligned} \ln(\det_{\mathcal{N}(Q_i)}(\Lambda^G \circ \eta_{\phi^* \mathbb{C}_t}(r_{A_i}))) &\geq \eta(t) - \frac{1}{2}(\phi(c_1(\mathfrak{s})) + x_M(\phi)) \cdot \ln(t) \quad \text{for } t \leq 1 \\ \ln(\det_{\mathcal{N}(Q_i)}(\Lambda^G \circ \eta_{\phi^* \mathbb{C}_t}(r_{A_i}))) &\geq \eta(t) - \frac{1}{2}(\phi(c_1(\mathfrak{s})) - x_M(\phi)) \cdot \ln(t) \quad \text{for } t \geq 1. \end{aligned}$$

Since this holds for all $i \geq 0$ and all $t > 0$, we conclude from (4.6)

$$\begin{aligned} \ln(\det_{\mathcal{N}(G)}(\Lambda^G \circ \eta_{\phi^* \mathbb{C}_t}(r_A))) &\geq \eta(t) - \frac{1}{2}(\phi(c_1(\mathfrak{s})) + x_M(\phi)) \cdot \ln(t) \quad \text{for } t \leq 1; \\ \ln(\det_{\mathcal{N}(G)}(\Lambda^G \circ \eta_{\phi^* \mathbb{C}_t}(r_A))) &\geq \eta(t) - \frac{1}{2}(\phi(c_1(\mathfrak{s})) - x_M(\phi)) \cdot \ln(t) \quad \text{for } t \geq 1. \end{aligned}$$

This implies

$$\begin{aligned} \rho^{(2)}(M, \mathfrak{s}; \mu, \phi) &\leq \frac{1}{2}(\phi(c_1(\mathfrak{s})) + x_M(\phi)) \cdot \ln(t) \quad \text{for } t \leq 1; \\ \rho^{(2)}(M, \mathfrak{s}; \mu, \phi) &\leq \frac{1}{2}(\phi(c_1(\mathfrak{s})) - x_M(\phi)) \cdot \ln(t) \quad \text{for } t \geq 1. \end{aligned}$$

This finishes the proof of the above claim.

The proof that the inequalities of the claim imply the desired inequality of the lemma is now word-for-word the same as the argument, using Theorem 1.8 (5), in the proof of Theorem 3.4. We leave the details to the reader. \square

Lemma 4.7. *Let Γ be a group that is virtually finitely generated free abelian. Consider a finite subset $S \subseteq \Gamma$. Then for any natural number n the function*

$$\begin{aligned} \{A \in M_{n,n}(\mathbb{C}\Gamma) \mid \text{supp}_{\Gamma}(A) \subseteq S\} &\rightarrow [0, \infty], \\ A &\mapsto \begin{cases} \det_{\mathcal{N}(\Gamma)}(\Lambda^{\Gamma}(r_A)) & \text{if } \Lambda^{\Gamma}(r_A) \text{ is a weak isomorphism;} \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

is continuous with respect to the standard topology on the source coming from the structure of a finite-dimensional complex vector space.

Proof. Let $i: \mathbb{Z}^d \rightarrow \Gamma$ be an inclusion whose image has finite index in Γ . Fix a map of sets $\sigma: \text{im}(i) \setminus \Gamma \rightarrow \Gamma$ whose composition with the projection $\Gamma \rightarrow \text{im}(i) \setminus \Gamma$ is the identity. Put $m = [\Gamma : \text{im}(i)]$. With this choice the finitely generated free $\mathbb{C}[\mathbb{Z}^d]$ -module $i^* \mathbb{C}\Gamma$ obtained from $\mathbb{C}\Gamma$ by restriction with i inherits a preferred $\mathbb{C}[\mathbb{Z}^d]$ -basis. Hence there is a finite subset $T \subseteq \mathbb{Z}^d$ and a \mathbb{C} -linear (and hence continuous) map

$$i^*: \{A \in M_{n,n}(\mathbb{C}\Gamma) \mid \text{supp}_{\Gamma}(A) \subseteq S\} \rightarrow \{B \in M_{mn,mn}(\mathbb{C}[\mathbb{Z}^d]) \mid \text{supp}_{\mathbb{Z}^d}(B) \subseteq T\}$$

such that $i^* \Lambda^{\Gamma}(r_A) = \Lambda^{\mathbb{Z}^d}(r_{i^* A})$. Since

$$\det_{\mathcal{N}(\mathbb{Z}^d)}(i^* \Lambda^{\Gamma}(r_A)) = m \cdot \det_{\mathcal{N}(\Gamma)}(r_A)$$

holds for any $A \in M_{n,n}(\mathbb{C}\Gamma)$ by [22, Theorem 3.14 (5) on page 128], it suffices to prove the claim in the special case $\Gamma = \mathbb{Z}^d$.

As $\det_{\mathbb{C}[\mathbb{Z}^d]}: M_{n,n}(\mathbb{C}[\mathbb{Z}^d]) \rightarrow M_{1,1}(\mathbb{C}[\mathbb{Z}^d])$ is continuous and for $A \in M_{n,n}(\mathbb{C}[\mathbb{Z}^d])$ with $\text{supp}_{\mathbb{Z}^d}(A) \subseteq S$ we have $\text{supp}_{\mathbb{Z}^d}(\det_{\mathbb{C}[\mathbb{Z}^d]}) \subseteq S^n$ for $S^n = \{g_1 \cdot g_2 \cdot \dots \cdot g_n \mid g_i \in S\}$, we conclude from [23, Lemma 6.25] that it suffices to treat the case $n = 1$. Since the Mahler measure of a non-trivial element $p \in \mathbb{C}[\mathbb{Z}^d]$ is equal to $\det_{\mathbb{C}[\mathbb{Z}^d]}(\Lambda^{\mathbb{Z}^d}(r_p): L^2(\mathbb{Z}^d) \rightarrow L^2(\mathbb{Z}^d))$ and defined to be zero for $p = 0$, Lemma 4.7 follows from a continuity theorem for Mahler measures proved by Boyd [3, p. 127]. \square

Definition 4.8 (Quasi-fibered classes). Let N be a 3-manifold. We call an element $\phi \in H^1(N; \mathbb{R})$ *quasi-fibered*, if there exists a sequence of fibered elements $\phi_n \in H^1(N; \mathbb{Q})$ converging to ϕ in $H^1(N; \mathbb{R})$.

Notice that obviously any fibered ϕ is non-trivial. The next theorem generalizes the inequalities of Lemma 4.5 for fibered classes to quasi-fibered classes. This theorem can be viewed as the key technical result of this paper.

Theorem 4.9 (Upper bound in the quasi-fibered case). *Let (M, μ) be an admissible pair, $\mathfrak{s} \in \text{Spin}^c(M)$ and let $\phi \in H^1(M; \mathbb{R})$ be a quasi-fibered class. Then*

$$\begin{aligned} \rho^{(2)}(M, \mathfrak{s}; \mu, \phi)(t) &\leq \frac{1}{2}(\phi(c_1(\mathfrak{s})) + x_M(\phi)) \cdot \ln(t) && \text{for } t \leq 1; \\ \rho^{(2)}(M, \mathfrak{s}; \mu, \phi)(t) &\leq \frac{1}{2}(\phi(c_1(\mathfrak{s})) - x_M(\phi)) \cdot \ln(t) && \text{for } t \geq 1. \end{aligned}$$

Proof. We only treat the case, where ∂M is empty, the other case is completely analogous: in the proof below one needs to replace Theorem 2.1 (2) by Theorem 2.1 (1). We write $\pi = \pi_1(M)$ and we pick $\mathfrak{s} \in \text{Spin}^c(M)$.

First recall that our assumption that $\mu: \pi \rightarrow G$ is large implies that the projection $\pi \rightarrow H_1(M)_f$ factors through μ and a map $\nu: G \rightarrow H_1(M)_f$. Since G is residually finite we can choose a sequence of normal subgroups of G

$$G \supseteq G_0 \supseteq G_1 \supseteq G_2 \supseteq \cdots$$

such that G_i is contained in $\ker(\nu: G \rightarrow H_1(M)_f)$, the index $[\ker(\nu) : G_i]$ is finite for $i \geq 0$ and the intersection $\bigcap_{i \geq 0} G_i$ is trivial. Put $Q_i := G/G_i$. Denote by $\mu_i: \pi \rightarrow Q_i$ the composition of the projection $\text{pr}_i: G \rightarrow Q_i$ with μ . Note that μ_i is again a large homomorphism. Recall that this implies in particular that we can make the identifications

$$H^1(M; \mathbb{R}) = \text{Hom}(H_1(\pi)_f, \mathbb{R}) = \text{Hom}(\pi, \mathbb{R}) = \text{Hom}(G, \mathbb{R}) = \text{Hom}(G_i, \mathbb{R}).$$

We apply Theorem 2.1 (2) to M and \mathfrak{s} . We denote the resulting square matrix over $\mathbb{Z}[\pi]$ by A and the resulting elements in π by s, s' . For each $i \in \mathbb{N}$ we write $A_i = \text{pr}_i(A)$. Define for any homomorphism $\psi: H_1(M)_f \rightarrow \mathbb{R}$

$$\xi(\psi)(t) = \max \{0, (|\psi \circ \nu \circ \mu(s)| + |\psi \circ \nu \circ \mu(s')|) \cdot \ln(t)\}.$$

We start out with the following claim.

Claim. For each $i \in \mathbb{N}$ we have the inequalities

$$\begin{aligned} \rho^{(2)}(M, \mathfrak{s}; \mu_i, \phi_i)(t) &\leq \frac{1}{2}(\phi(c_1(\mathfrak{s})) + x_M(\phi)) \cdot \ln(t) && \text{for } t \leq 1; \\ \rho^{(2)}(M, \mathfrak{s}; \mu_i, \phi_i)(t) &\leq \frac{1}{2}(\phi(c_1(\mathfrak{s})) - x_M(\phi)) \cdot \ln(t) && \text{for } t \geq 1. \end{aligned}$$

Let $i \in \mathbb{N}$. Since $\phi \in H^1(M; \mathbb{R})$ is quasi-fibered, there exists a sequence of fibered elements $\phi_n \in H^1(M; \mathbb{Q})$ converging to ϕ . By Lemma 4.5 we know that for each i and n we have

$$(4.10) \quad \rho^{(2)}(M, \mathfrak{s}; \mu_i, \phi_n)(t) \leq \frac{1}{2}(\phi_n(c_1(\mathfrak{s})) + x_M(\phi_n)) \cdot \ln(t) \quad \text{for } t \leq 1;$$

$$(4.11) \quad \rho^{(2)}(M, \mathfrak{s}; \mu_i, \phi_n)(t) \leq \frac{1}{2}(\phi_n(c_1(\mathfrak{s})) - x_M(\phi_n)) \cdot \ln(t) \quad \text{for } t \geq 1.$$

By Theorem 2.1 (2) we have

$$(4.12) \quad \rho^{(2)}(M, \mathfrak{s}; \mu_i, \phi_n)(t) = \xi(\phi_n)(t) - \ln(\det_{\mathcal{N}(Q_i)}(\Lambda^{Q_i} \circ \eta_{\phi_n^* C_i}(r_{A_i})));$$

$$(4.13) \quad \rho^{(2)}(M, \mathfrak{s}; \mu_i, \phi)(t) = \xi(\phi)(t) - \ln(\det_{\mathcal{N}(Q_i)}(\Lambda^{Q_i} \circ \eta_{\phi^* C_i}(r_{A_i}))).$$

Since ϕ_n converges to ϕ and the kernel of the projection map $Q_i \rightarrow H_1(M)_f$ is finite, we get from Lemma 4.7 that

$$\lim_{n \rightarrow \infty} \ln(\det_{\mathcal{N}(Q_i)}(\Lambda^{Q_i} \circ \eta_{\phi_n^* C_i}(r_{A_i}))) = \ln(\det_{\mathcal{N}(Q_i)}(\Lambda^{Q_i} \circ \eta_{\phi^* C_i}(r_{A_i}))).$$

This equality, together with Equations (4.12) and (4.13) and the observation that for any $t \in (0, \infty)$ the equality $\lim_{n \rightarrow \infty} \xi(\phi_n)(t) = \xi(\phi)(t)$ holds, implies that

$$\rho^{(2)}(M, \mathfrak{s}; \mu_i, \phi)(t) = \lim_{n \rightarrow \infty} \rho^{(2)}(M, \mathfrak{s}; \mu_i, \phi_n)(t) \quad \text{for all } t \in (0, \infty).$$

The desired inequalities for $\rho^{(2)}(M, \mathfrak{s}; \mu_i, \phi)(t)$ now follow from (4.10) and (4.11). This concludes the proof of the claim.

Now the theorem follows from the claim we just proved and the following claim.

Claim. For each $t \in (0, \infty)$ we have

$$\rho^{(2)}(M, \mathfrak{s}, \mu, \phi)(t) \leq \liminf_{i \rightarrow \infty} \rho^{(2)}(M, \mathfrak{s}, \mu_i, \phi_i)(t).$$

By Theorem 4.4 we know that the pairs (M, μ) and (M, μ_i) are L^2 -acyclic. By Theorem 2.1 (2) we have

$$(4.14) \quad \rho^{(2)}(M, \mathfrak{s}, \mu, \phi)(t) = \xi(\phi)(t) - \ln(\det_{\mathcal{N}(G)}(\Lambda^G \circ \eta_{\phi^* \mathbb{C}_t}(r_A))).$$

Recall that the kernel of $Q_i \rightarrow H_1(M)_f$ is finite und that $Q_i \rightarrow H_1(M)_f$ is surjective. Now we apply Theorem 1.11 to $\phi: G \rightarrow \mathbb{R}$. For all $t \in (0, \infty)$ we obtain

$$\ln(\det_{\mathcal{N}(G)}(\Lambda^G \circ \eta_{\phi^* \mathbb{C}_t}(r_A))) \geq \limsup_{i \rightarrow \infty} \ln(\det_{\mathcal{N}(Q_i)}(\Lambda^{Q_i} \circ \eta_{\phi_i^* \mathbb{C}_t}(r_{A_i}))).$$

The claim is now an immediate consequence of (4.13) and (4.14). \square

For convenience we also state the result which follows from combining Theorem 3.4 with Theorem 4.9.

Theorem 4.15 (Lower and upper bounds combined in the quasi-fibered case). *Let $M \neq S^1 \times D^2$ be an irreducible 3-manifold with infinite fundamental group π . Let $\phi \in H^1(M; \mathbb{R})$ be a quasi-fibered class.*

Then there exists a $D \in \mathbb{R}$ such that for any $\mathfrak{s} \in \text{Spin}^c(M)$ and any large homomorphism $\mu: \pi_1(M) \rightarrow G$, where G is residually finite and countable, the pair (M, μ) is L^2 -acyclic and such that for $t \leq 1$

$$\frac{1}{2}(\phi(c_1(\mathfrak{s})) + x_M(\phi)) \ln t - D \leq \rho^{(2)}(M, \mathfrak{s}; \mu, \phi)(t) \leq \frac{1}{2}(\phi(c_1(\mathfrak{s})) + x_M(\phi)) \ln t$$

and such that for $t \geq 1$

$$\frac{1}{2}(\phi(c_1(\mathfrak{s})) - x_M(\phi)) \ln t - D \leq \rho^{(2)}(M, \mathfrak{s}; \mu, \phi)(t) \leq \frac{1}{2}(\phi(c_1(\mathfrak{s})) - x_M(\phi)) \ln t.$$

In particular we get

$$\deg(\rho(M, \mathfrak{s}; \mu, \phi)) = -x_M(\phi).$$

5. PROOF OF THE MAIN THEOREM

The following is the main theorem of this paper.

Theorem 5.1 (Main theorem). *Let M be an irreducible 3-manifold with infinite fundamental group π which is not a closed graph manifold and not homeomorphic to $S^1 \times D^2$. Let $\mathfrak{s} \in \text{Spin}^c(M)$ and write $\pi = \pi_1(M)$.*

Then there exists a large epimorphism $\alpha: \pi \rightarrow \Gamma$ to a virtually free abelian group such that the following holds: For any $\phi \in H^1(M; \mathbb{Q})$ and any factorization of $\alpha: \pi \rightarrow \Gamma$ into group homomorphisms $\pi \xrightarrow{\mu} G \xrightarrow{\nu} \Gamma$ for a residually finite countable group G , there exists a real number D depending only on ϕ but not on μ such that for $t \leq 1$

$$\frac{1}{2}(\phi(c_1(\mathfrak{s})) + x_M(\phi)) \ln t - D \leq \rho^{(2)}(M, \mathfrak{s}; \mu, \phi)(t) \leq \frac{1}{2}(\phi(c_1(\mathfrak{s})) + x_M(\phi)) \ln t$$

and such that for $t \geq 1$

$$\frac{1}{2}(\phi(c_1(\mathfrak{s})) - x_M(\phi)) \ln t - D \leq \rho^{(2)}(M, \mathfrak{s}; \mu, \phi)(t) \leq \frac{1}{2}(\phi(c_1(\mathfrak{s})) - x_M(\phi)) \ln t.$$

In particular we get

$$\deg(\rho(M, \mathfrak{s}; \mu, \phi)) = -x_M(\phi).$$

Proof. As explained in [5, Section 10], we conclude from combining [1, 2, 17, 27, 28, 34, 35] that there exists a finite regular covering $p: N \rightarrow M$ such that for any $\phi \in H^1(M; \mathbb{R})$ its pullback $p^*\phi \in H^1(N; \mathbb{R})$ is quasi-fibered. Let k be the number of sheets of p . Let $\text{pr}_N: \pi_1(N) \rightarrow H_1(N)_f$ and $\text{pr}_M: \pi_1(M) \rightarrow H_1(M)_f$ be the canonical projections. The kernel of pr_N is a characteristic subgroup of $\pi_1(N)$. The regular finite covering p induces an injection $\pi_1(p): \pi_1(N) \rightarrow \pi_1(M)$ such that the

image of $\pi_1(p)$ is a normal subgroup of $\pi_1(M)$ of finite index. Let Γ be the quotient of $\pi_1(M)$ by the normal subgroup $\pi_1(p)(\ker(\text{pr}_N))$. Let $\alpha: \pi_1(M) \rightarrow \Gamma$ be the projection. Because of $H_1(p; \mathbb{Z})_f \circ \text{pr}_N = \text{pr}_M \circ \pi_1(p)$ we know that $\pi_1(p)(\ker(\text{pr}_N))$ is contained in the kernel of the canonical projection $\text{pr}_M: \pi_1(M) \rightarrow H_1(M)_f$. This implies that $\alpha: \pi_1(M) \rightarrow \Gamma$ is large, which means in particular that there exists precisely one epimorphism $\beta: \Gamma \rightarrow H_1(M)_f$ satisfying $\text{pr}_M = \beta \circ \alpha$. Moreover, $\alpha \circ \pi_1(p)$ factorizes over pr_N into an injective homomorphism $j: H_1(N)_f \rightarrow \Gamma$ with finite cokernel. Hence Γ is virtually a free abelian group.

Consider any factorization of the homomorphism $\alpha: \pi_1(M) \rightarrow \Gamma$ into group homomorphisms $\pi_1(M) \xrightarrow{\mu} G \xrightarrow{\nu} \Gamma$ with residually finite countable G . Let G' be the quotient of $\pi_1(N)$ by the normal subgroup $\pi_1(p)^{-1}(\ker(\mu))$ and $\mu': \pi_1(N) \rightarrow G'$ be the projection. Since the kernel of μ' and of $\mu \circ \pi_1(p)$ agree, there is precisely one injective group homomorphism $i: G' \rightarrow G$ satisfying $\mu \circ \pi_1(p) = i \circ \mu'$. The kernel of μ' is contained in the kernel of $\text{pr}_N: \pi_1(N) \rightarrow H_1(N)_f$ since j is injective and we have $j \circ \text{pr}_N = \nu \circ i \circ \mu'$. Hence there is precisely one group homomorphism $\nu': G' \rightarrow H_1(N)_f$ satisfying $\nu' \circ \mu' = \text{pr}_N$. In particular μ' is a large homomorphism. One easily checks that the following diagram commutes, and all vertical arrows are injective, the indices $[\pi_1(N) : \text{im}(\pi_1(p))]$ and $[\Gamma : H_1(N)_f]$ are finite, and μ', ν' and β are surjective.

$$\begin{array}{ccccc}
 & & \text{pr}_N & & \\
 & & \curvearrowright & & \\
 \pi_1(N) & \xrightarrow{\mu'} & G' & \xrightarrow{\nu'} & H_1(N)_f \\
 \downarrow \pi_1(p) & & \downarrow i & & \downarrow j \\
 \pi_1(M) & \xrightarrow{\mu} & G & \xrightarrow{\nu} & \Gamma \\
 & & & & \downarrow \beta \\
 & & & & H_1(M)_f \\
 & & \text{pr}_M & & \\
 & & \curvearrowleft & &
 \end{array}$$

Since G is residually finite and countable, the group G' is residually finite and countable.

Now let $\mathfrak{s} \in \text{Spin}^c(M)$ and let $\phi \in H^1(M; \mathbb{Q}) = \text{Hom}(H_1(M)_f; \mathbb{Q})$. We write $e' = p^*(e)$ and $\phi' = p^*(\phi)$. Furthermore we write $c = c_1(\mathfrak{s})$ and $c' = c_1(e')$. Since $\phi' \in H^1(N; \mathbb{Q}) = \text{Hom}(H_1(N)_f; \mathbb{Q})$ is quasi-fibered we can appeal to Theorem 4.15. In our context it says that (N, μ') is L^2 -acyclic and that there exists a real number D' depending only on ϕ' but not on μ' such that for $t \leq 1$

$$\frac{1}{2}(\phi'(c') + x_N(\phi')) \ln t - D' \leq \rho^{(2)}(N, e'; \mu', \phi')(t) \leq \frac{1}{2}(\phi'(c') + x_N(\phi')) \ln t$$

and such that for $t \geq 1$

$$\frac{1}{2}(\phi'(c') - x_N(\phi')) \ln t - D' \leq \rho^{(2)}(N, e'; \mu, \phi')(t) \leq \frac{1}{2}(\phi'(c') - x_N(\phi')) \ln t.$$

We now set $D := \frac{1}{k}D'$. The theorem now follows from these inequalities and the following equalities

$$\begin{aligned}
 x_M(\phi) &= \frac{1}{k}x_N(\phi') \\
 \rho^{(2)}(M, \mathfrak{s}; \mu, \phi)(t) &= \frac{1}{k}\rho^{(2)}(N, e'; \mu', \phi')(t) \quad \text{for all } t \\
 \phi(c_1(\mathfrak{s})) &= \frac{1}{k}\phi'(c_1(e')).
 \end{aligned}$$

Here the first equality is (1.13) and the second one Theorem 1.8 (3). The third one follows easily from the definitions. \square

Remark 5.2 (Graph manifolds). The proof of Theorem 4.15 does not cover closed graph manifolds. However, for a graph manifold M together with a large homomorphism $\mu: \pi_1(M) \rightarrow G$ such that (M, μ) is L^2 -acyclic, together with a class

$\phi \in H^1(M; \mathbb{R})$ the L^2 -torsion function has been computed explicitly in [5, Theorem 8.6] to be

$$\bar{\rho}^{(2)}(M; \mu, \phi)(t) \doteq \min\{0, -x_M(\phi) \cdot \ln(t)\},$$

provided that the image of the regular fiber under μ is an element of infinite order and M is neither $S^1 \times D^2$ nor $S^1 \times S^2$. This implies

$$\deg(\bar{\rho}^{(2)}(M; \mu, \phi)) = -x_M(\phi).$$

Remark 5.3 (The role of Γ). In Theorem 5.1 the group Γ is in some sense optimal. Namely, one cannot expect $\Gamma = H_1(M)_f$ and $\beta = \text{id}_\Gamma$ in Theorem 5.1. For instance, let $K \subseteq S^3$ be a non-trivial knot. Take M to be the 3-manifold given by the complement of an open tubular neighborhood of the knot. Then $\deg(\bar{\rho}(M; \mu, \phi))$ for $\mu: \pi_1(M) \rightarrow H_1(M)_f$ the canonical projection and $\phi: H_1(M)_f \xrightarrow{\cong} \mathbb{Z}$ an isomorphism is just the degree of the Alexander polynomial of the knot K which is not the Thurston norm $x_M(\phi)$ in general, see [5, Section 7.3].

Example 5.4 ($S^1 \times D^2$ and $S^1 \times S^2$). Consider a homomorphism $\phi: H_1(S^1 \times D^2) \xrightarrow{\cong} \mathbb{Z}$. Let k be the index $[\mathbb{Z} : \text{im}(\phi)]$ if ϕ is non-trivial, and let $k = 0$ if ϕ is trivial. Then we conclude from Theorem 1.8 (4), (1.14), and [23, Theorem 7.10]

$$\begin{aligned} x_{S^1 \times D^2}(\phi) &= 0; \\ \deg(\widetilde{\bar{\rho}(S^1 \times D^2; \phi)}) &= k. \end{aligned}$$

Hence we have to exclude $S^1 \times D^2$ in Theorem 5.1. Analogously we get

$$\begin{aligned} x_{S^1 \times S^2}(\phi) &= 0; \\ \deg(\widetilde{\bar{\rho}(S^1 \times S^2; \phi)}) &= 2 \cdot k, \end{aligned}$$

so that we cannot replace “irreducible” by “prime” in Theorem 5.1.

We conclude the paper with the proof of Theorem 0.1.

Proof of Theorem 0.1. Let M be an irreducible 3-manifold with infinite fundamental group. If M is a graph manifold, then the statement is proved in Remark 5.2. Now suppose that M is not a graph manifold. In this case the theorem follows from Theorem 5.1, applied in the special case $G = \pi_1(M)$, $\mu = \text{id}_{\pi_1(M)}$ and $\nu = \alpha$. Here we use that by work of Hempel [13] and the proof of the Geometrization Conjecture fundamental groups of 3-manifolds are residually finite. \square

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