Detecting the Thurston norm and fibered classes

Stefan Friedl (joint with Stefano Vidussi)

September 2010

The Thurston norm of a 3-manifold

Let N be a 3-manifold and $\phi \in H^1(N) \cong H_2(N, \partial N)$.

The Thurston norm of a 3-manifold

Let N be a 3-manifold and $\phi \in H^1(N) \cong H_2(N, \partial N)$. We can represent ϕ by an embedded surface and we want to measure its complexity.

The Thurston norm of a 3-manifold

Let N be a 3-manifold and $\phi \in H^1(N) \cong H_2(N, \partial N)$. We can represent ϕ by an embedded surface and we want to measure its complexity. Given a surface S with connected components $S_1, \dots, S_k$ we define its *complexity* to be

$$x(S) = \sum_{S_i \neq S^2, D^2} -\chi(S_i).$$

The Thurston norm of a 3-manifold

Let N be a 3-manifold and $\phi \in H^1(N) \cong H_2(N, \partial N)$. We can represent ϕ by an embedded surface and we want to measure its complexity. Given a surface S with connected components $S_1, \dots, S_k$ we define its *complexity* to be

$$x(S) = \sum_{S_i \neq S^2, D^2} -\chi(S_i).$$

We define

$$x(\phi) = \min\{x(S) \mid S \subset N \text{ represents } \phi\}$$

the *Thurston norm* of ϕ.

The Thurston norm of a 3-manifold

Let N be a 3-manifold and $\phi \in H^1(N) \cong H_2(N, \partial N)$. We can represent ϕ by an embedded surface and we want to measure its complexity. Given a surface S with connected components $S_1, \dots, S_k$ we define its *complexity* to be

$$x(S) = \sum_{S_i \neq S^2, D^2} -\chi(S_i).$$

We define

$$x(\phi) = \min\{x(S) \,|\, S \subset N \text{ represents } \phi\}$$

the *Thurston norm* of ϕ. Properties:

The Thurston norm of a 3-manifold

Let N be a 3-manifold and $\phi \in H^1(N) \cong H_2(N, \partial N)$. We can represent ϕ by an embedded surface and we want to measure its complexity. Given a surface S with connected components $S_1, \dots, S_k$ we define its *complexity* to be

$$x(S) = \sum_{S_i \neq S^2, D^2} -\chi(S_i).$$

We define

$$x(\phi) = \min\{x(S) \mid S \subset N \text{ represents } \phi\}$$

the *Thurston norm* of ϕ. Properties:
(1) $x(n\phi) = |n| \cdot x(\phi)$ for $x \in \mathbb{Z}$,

The Thurston norm of a 3-manifold

Let N be a 3-manifold and $\phi \in H^1(N) \cong H_2(N, \partial N)$. We can represent ϕ by an embedded surface and we want to measure its complexity. Given a surface S with connected components $S_1, \dots, S_k$ we define its *complexity* to be

$$x(S) = \sum_{S_i \neq S^2, D^2} -\chi(S_i).$$

We define

$$x(\phi) = \min\{x(S) \,|\, S \subset N \text{ represents } \phi\}$$

the *Thurston norm* of ϕ. Properties:
(1) $x(n\phi) = |n| \cdot x(\phi)$ for $x \in \mathbb{Z}$,
(2) $x(\phi_1 + \phi_2) \leq x(\phi_1) + x(\phi_2)$.

The Thurston norm of a 3-manifold

Let N be a 3-manifold and $\phi \in H^1(N) \cong H_2(N, \partial N)$. We can represent ϕ by an embedded surface and we want to measure its complexity. Given a surface S with connected components $S_1, \dots, S_k$ we define its *complexity* to be

$$x(S) = \sum_{S_i \neq S^2, D^2} -\chi(S_i).$$

We define

$$x(\phi) = \min\{x(S) \mid S \subset N \text{ represents } \phi\}$$

the *Thurston norm* of ϕ. Properties:
(1) $x(n\phi) = |n| \cdot x(\phi)$ for $x \in \mathbb{Z}$,
(2) $x(\phi_1 + \phi_2) \leq x(\phi_1) + x(\phi_2)$.
Property (1) implies that x extends to a seminorm on $H^1(N; \mathbb{Q})$.

The Thurston norm of a 3-manifold

Let N be a 3-manifold and $\phi \in H^1(N) \cong H_2(N, \partial N)$. We can represent ϕ by an embedded surface and we want to measure its complexity. Given a surface S with connected components $S_1, \dots, S_k$ we define its *complexity* to be

$$x(S) = \sum_{S_i \neq S^2, D^2} -\chi(S_i).$$

We define

$$x(\phi) = \min\{x(S) \,|\, S \subset N \text{ represents } \phi\}$$

the *Thurston norm* of ϕ. Properties:
(1) $x(n\phi) = |n| \cdot x(\phi)$ for $x \in \mathbb{Z}$,
(2) $x(\phi_1 + \phi_2) \leq x(\phi_1) + x(\phi_2)$.
Property (1) implies that x extends to a seminorm on $H^1(N; \mathbb{Q})$.
Properties (1) and (2) imply that x extends to a seminorm on $H^1(N; \mathbb{R})$.

The Thurston norm of a 3-manifold

Let N be a 3-manifold and $\phi \in H^1(N) \cong H_2(N, \partial N)$. We define

$$x(\phi) = \min\{x(S) \mid S \subset N \text{ represents } \phi\}$$

the *Thurston norm* of ϕ. Properties:

(1) $x(n\phi) = |n| \cdot x(\phi)$ for $x \in \mathbb{Z}$,

(2) $x(\phi_1 + \phi_2) \leq x(\phi_1) + x(\phi_2)$.

Properties (1) and (2) imply that x extends to a seminorm on $H^1(N; \mathbb{R})$.

The Thurston norm of a 3-manifold

Let N be a 3-manifold and $\phi \in H^1(N) \cong H_2(N, \partial N)$. We define

$$x(\phi) = \min\{x(S) \mid S \subset N \text{ represents } \phi\}$$

the *Thurston norm* of ϕ. Properties:
(1) $x(n\phi) = |n| \cdot x(\phi)$ for $x \in \mathbb{Z}$,
(2) $x(\phi_1 + \phi_2) \leq x(\phi_1) + x(\phi_2)$.
Properties (1) and (2) imply that x extends to a seminorm on $H^1(N; \mathbb{R})$. The set

$$\{\phi \in H^1(N; \mathbb{R}) \mid x(\phi) \leq 1\}$$

is called the *Thurston norm ball* of N.

The Thurston norm of a 3-manifold

Let N be a 3-manifold and $\phi \in H^1(N) \cong H_2(N, \partial N)$. We define

$$x(\phi) = \min\{x(S) \mid S \subset N \text{ represents } \phi\}$$

the *Thurston norm* of ϕ. Properties:
(1) $x(n\phi) = |n| \cdot x(\phi)$ for $x \in \mathbb{Z}$,
(2) $x(\phi_1 + \phi_2) \leq x(\phi_1) + x(\phi_2)$.
Properties (1) and (2) imply that x extends to a seminorm on $H^1(N; \mathbb{R})$. The set

$$\{\phi \in H^1(N; \mathbb{R}) \mid x(\phi) \leq 1\}$$

is called the *Thurston norm ball* of N. Thurston showed that this is a polytope and the vertices are rational.

The Thurston norm of a 3-manifold

Let N be a 3-manifold and $\phi \in H^1(N) \cong H_2(N, \partial N)$. We define

$$x(\phi) = \min\{x(S) \mid S \subset N \text{ represents } \phi\}$$

the *Thurston norm* of ϕ. Properties:
(1) $x(n\phi) = |n| \cdot x(\phi)$ for $x \in \mathbb{Z}$,
(2) $x(\phi_1 + \phi_2) \leq x(\phi_1) + x(\phi_2)$.
Properties (1) and (2) imply that x extends to a seminorm on $H^1(N; \mathbb{R})$. The set

$$\{\phi \in H^1(N; \mathbb{R}) \mid x(\phi) \leq 1\}$$

is called the *Thurston norm ball* of N. Thurston showed that this is a polytope and the vertices are rational. We say $\phi \in H^1(N; \mathbb{N}) = \mathrm{Hom}(\pi_1(N), \mathbb{Z})$ is *fibered* if there exists a fibration $p : N \to S^1$ such that $\phi = p_* : \pi_1(N) \to \mathbb{Z}$.

The Thurston norm of a 3-manifold

Let N be a 3-manifold and $\phi \in H^1(N) \cong H_2(N, \partial N)$. We define

$$x(\phi) = \min\{x(S) \,|\, S \subset N \text{ represents } \phi\}$$

the *Thurston norm* of ϕ. Properties:
(1) $x(n\phi) = |n| \cdot x(\phi)$ for $x \in \mathbb{Z}$,
(2) $x(\phi_1 + \phi_2) \leq x(\phi_1) + x(\phi_2)$.
Properties (1) and (2) imply that x extends to a seminorm on $H^1(N; \mathbb{R})$. The set

$$\{\phi \in H^1(N; \mathbb{R}) \,|\, x(\phi) \leq 1\}$$

is called the *Thurston norm ball* of N. Thurston showed that this is a polytope and the vertices are rational. We say $\phi \in H^1(N; \mathbb{N}) = \text{Hom}(\pi_1(N), \mathbb{Z})$ is *fibered* if there exists a fibration $p : N \to S^1$ such that $\phi = p_* : \pi_1(N) \to \mathbb{Z}$. We say $\phi \in H^1(N; \mathbb{Q})$ is fibered if an integral multiple is fibered.

The Thurston norm of a 3-manifold

Let N be a 3-manifold and $\phi \in H^1(N) \cong H_2(N, \partial N)$. We define

$$x(\phi) = \min\{x(S) \mid S \subset N \text{ represents } \phi\}$$

the *Thurston norm* of ϕ. Properties:
(1) $x(n\phi) = |n| \cdot x(\phi)$ for $x \in \mathbb{Z}$,
(2) $x(\phi_1 + \phi_2) \leq x(\phi_1) + x(\phi_2)$.
Properties (1) and (2) imply that x extends to a seminorm on $H^1(N; \mathbb{R})$. The set

$$\{\phi \in H^1(N; \mathbb{R}) \mid x(\phi) \leq 1\}$$

is called the *Thurston norm ball* of N. Thurston showed that this is a polytope and the vertices are rational. We say $\phi \in H^1(N; \mathbb{N}) = \text{Hom}(\pi_1(N), \mathbb{Z})$ is *fibered* if there exists a fibration $p : N \to S^1$ such that $\phi = p_* : \pi_1(N) \to \mathbb{Z}$. We say $\phi \in H^1(N; \mathbb{Q})$ is fibered if an integral multiple is fibered. We say $\phi \in H^1(N; \mathbb{R})$ is fibered if we can represent it by a non-degenerate closed 1-form.

The Thurston norm of a 3-manifold

Let N be a 3-manifold and $\phi \in H^1(N) \cong H_2(N, \partial N)$. We define

$$x(\phi) = \min\{x(S) \mid S \subset N \text{ represents } \phi\}$$

the *Thurston norm* of ϕ. Properties:
(1) $x(n\phi) = |n| \cdot x(\phi)$ for $x \in \mathbb{Z}$,
(2) $x(\phi_1 + \phi_2) \leq x(\phi_1) + x(\phi_2)$.
Properties (1) and (2) imply that x extends to a seminorm on $H^1(N; \mathbb{R})$. The set

$$\{\phi \in H^1(N; \mathbb{R}) \mid x(\phi) \leq 1\}$$

is called the *Thurston norm ball* of N. We say $\phi \in H^1(N; \mathbb{R})$ is fibered if we can represent it by a non-degenerate closed 1-form.

The Thurston norm of a 3-manifold

Let N be a 3-manifold and $\phi \in H^1(N) \cong H_2(N, \partial N)$. We define

$$x(\phi) = \min\{x(S) \mid S \subset N \text{ represents } \phi\}$$

the *Thurston norm* of ϕ. Properties:
(1) $x(n\phi) = |n| \cdot x(\phi)$ for $x \in \mathbb{Z}$,
(2) $x(\phi_1 + \phi_2) \leq x(\phi_1) + x(\phi_2)$.
Properties (1) and (2) imply that x extends to a seminorm on $H^1(N; \mathbb{R})$. The set

$$\{\phi \in H^1(N; \mathbb{R}) \mid x(\phi) \leq 1\}$$

is called the *Thurston norm ball* of N. We say $\phi \in H^1(N; \mathbb{R})$ is fibered if we can represent it by a non-degenerate closed 1-form. Thurston showed that if ϕ is fibered, then ϕ sits on the cone of an open top-dimensional face,

The Thurston norm of a 3-manifold

Let N be a 3-manifold and $\phi \in H^1(N) \cong H_2(N, \partial N)$. We define

$$x(\phi) = \min\{x(S) \,|\, S \subset N \text{ represents } \phi\}$$

the *Thurston norm* of ϕ. Properties:
(1) $x(n\phi) = |n| \cdot x(\phi)$ for $x \in \mathbb{Z}$,
(2) $x(\phi_1 + \phi_2) \leq x(\phi_1) + x(\phi_2)$.
Properties (1) and (2) imply that x extends to a seminorm on $H^1(N; \mathbb{R})$. The set

$$\{\phi \in H^1(N; \mathbb{R}) \,|\, x(\phi) \leq 1\}$$

is called the *Thurston norm ball* of N. We say $\phi \in H^1(N; \mathbb{R})$ is fibered if we can represent it by a non-degenerate closed 1-form. Thurston showed that if ϕ is fibered, then ϕ sits on the cone of an open top-dimensional face, furthermore any other ψ in the same cone is also fibered.

The Thurston norm of a 3-manifold

Let N be a 3-manifold and $\phi \in H^1(N) \cong H_2(N, \partial N)$. We define

$$x(\phi) = \min\{x(S) \,|\, S \subset N \text{ represents } \phi\}$$

the *Thurston norm* of ϕ. Properties:
(1) $x(n\phi) = |n| \cdot x(\phi)$ for $x \in \mathbb{Z}$,
(2) $x(\phi_1 + \phi_2) \leq x(\phi_1) + x(\phi_2)$.
Properties (1) and (2) imply that x extends to a seminorm on $H^1(N; \mathbb{R})$. The set

$$\{\phi \in H^1(N; \mathbb{R}) \,|\, x(\phi) \leq 1\}$$

is called the *Thurston norm ball* of N. We say $\phi \in H^1(N; \mathbb{R})$ is fibered if we can represent it by a non-degenerate closed 1-form. Thurston showed that if ϕ is fibered, then ϕ sits on the cone of an open top-dimensional face, furthermore any other ψ in the same cone is also fibered.
'Either all classes in a cone are fibered, or none are.'

Twisted Alexander polynomials

Let N be a 3-manifold, $\pi := \pi_1(N)$, $\alpha : \pi \to GL(R, k)$ a representation and $\phi \in H^1(N; \mathbb{Z})$ non-trivial.

Twisted Alexander polynomials

Let N be a 3-manifold, $\pi := \pi_1(N)$, $\alpha : \pi \to GL(R, k)$ a representation and $\phi \in H^1(N; \mathbb{Z})$ non-trivial. Let $\tilde{N}$ be the universal cover, then consider

$$C_*(\tilde{N}) \otimes_{\mathbb{Z}[\pi]} R[t^{\pm 1}] \otimes R^k,$$

Twisted Alexander polynomials

Let N be a 3-manifold, $\pi := \pi_1(N)$, $\alpha : \pi \to GL(R,k)$ a representation and $\phi \in H^1(N;\mathbb{Z})$ non-trivial. Let $\tilde{N}$ be the universal cover, then consider

$$C_*(\tilde{N}) \otimes_{\mathbb{Z}[\pi]} R[t^{\pm 1}] \otimes R^k,$$

This is a chain complex over $R[t^{\pm 1}]$ and we denote its homology by $H_1^\alpha(N; R^k[t^{\pm 1}])$.

Twisted Alexander polynomials

Let N be a 3-manifold, $\pi := \pi_1(N)$, $\alpha : \pi \to GL(R, k)$ a representation and $\phi \in H^1(N; \mathbb{Z})$ non-trivial. Let $\tilde{N}$ be the universal cover, then consider

$$C_*(\tilde{N}) \otimes_{\mathbb{Z}[\pi]} R[t^{\pm 1}] \otimes R^k,$$

This is a chain complex over $R[t^{\pm 1}]$ and we denote its homology by $H_1^\alpha(N; R^k[t^{\pm 1}])$. We consider a free resolution:

$$R[t^{\pm 1}]^r \xrightarrow{D} R[t^{\pm 1}]^s \to H_1^\alpha(N; R^k[t^{\pm 1}]) \to 0$$

Twisted Alexander polynomials

Let N be a 3-manifold, $\pi := \pi_1(N)$, $\alpha : \pi \to GL(R,k)$ a representation and $\phi \in H^1(N;\mathbb{Z})$ non-trivial. Let $\tilde{N}$ be the universal cover, then consider

$$C_*(\tilde{N}) \otimes_{\mathbb{Z}[\pi]} R[t^{\pm 1}] \otimes R^k,$$

This is a chain complex over $R[t^{\pm 1}]$ and we denote its homology by $H_1^\alpha(N; R^k[t^{\pm 1}])$. We consider a free resolution:

$$R[t^{\pm 1}]^r \xrightarrow{D} R[t^{\pm 1}]^s \to H_1^\alpha(N; R^k[t^{\pm 1}]) \to 0$$

and we define

$$\Delta^\alpha_{N,\phi} = \text{gcd of } s \times s\text{-minors of } D.$$

The twisted Alexander polynomial (TAP) of (N, ϕ, α).

TAP, fibered 3-manifolds and the Thurston norm

Let N be a 3-manifold and $\phi \in H^1(N;\mathbb{Z})$.

Let N be a 3-manifold and $\phi \in H^1(N;\mathbb{Z})$.

Theorem 1. (F-Kim) If $\alpha : \pi_1(N) \to GL(R,k)$ is a representation, then

$$\deg(\Delta^{\alpha}_{N,\phi}) = kx(\phi) + \text{error term}.$$

Let N be a 3-manifold and $\phi \in H^1(N;\mathbb{Z})$.

Theorem 1. (F-Kim) If $\alpha : \pi_1(N) \to GL(R,k)$ is a representation, then

$$\deg(\Delta^{\alpha}_{N,\phi}) = kx(\phi) + \text{error term}.$$

Theorem 2. (Cha, Goda-Kitano-Morifuji and F-Kim) If ϕ is a fibered class and $\alpha : \pi_1(N) \to GL(R,k)$ a representation, then
(1) $\Delta^{\alpha}_{N,\phi}$ is monic,

TAP, fibered 3-manifolds and the Thurston norm

Let N be a 3-manifold and $\phi \in H^1(N;\mathbb{Z})$.

Theorem 1. (F-Kim) If $\alpha : \pi_1(N) \to GL(R,k)$ is a representation, then

$$\deg(\Delta^{\alpha}_{N,\phi}) = kx(\phi) + \text{error term}.$$

Theorem 2. (Cha, Goda-Kitano-Morifuji and F-Kim) If ϕ is a fibered class and $\alpha : \pi_1(N) \to GL(R,k)$ a representation, then
(1) $\Delta^{\alpha}_{N,\phi}$ is monic, (2) $\deg(\Delta^{\alpha}_{N,\phi}) = kx(\phi) + \text{error term}$.

TAP, fibered 3-manifolds and the Thurston norm

Let N be a 3-manifold and $\phi \in H^1(N;\mathbb{Z})$.

Theorem 1. (F-Kim) If $\alpha : \pi_1(N) \to GL(R,k)$ is a representation, then

$$\deg(\Delta^{\alpha}_{N,\phi}) = kx(\phi) + \text{error term}.$$

Theorem 2. (Cha, Goda-Kitano-Morifuji and F-Kim) If ϕ is a fibered class and $\alpha : \pi_1(N) \to GL(R,k)$ a representation, then
(1) $\Delta^{\alpha}_{N,\phi}$ is monic, (2) $\deg(\Delta^{\alpha}_{N,\phi}) = kx(\phi) + \text{error term}$.

Theorem 3. (F-Vidussi) If for any representation $\alpha : \pi_1(N) \to GL(R,k)$ we have
(1) $\Delta^{\alpha}_{N,\phi}$ is monic, (2) $\deg(\Delta^{\alpha}_{N,\phi}) = kx(\phi) + \text{error term}$.

TAP, fibered 3-manifolds and the Thurston norm

Let N be a 3-manifold and $\phi \in H^1(N;\mathbb{Z})$.

Theorem 1. (F-Kim) If $\alpha : \pi_1(N) \to GL(R,k)$ is a representation, then

$$\deg(\Delta^{\alpha}_{N,\phi}) = kx(\phi) + \text{error term}.$$

Theorem 2. (Cha, Goda-Kitano-Morifuji and F-Kim) If ϕ is a fibered class and $\alpha : \pi_1(N) \to GL(R,k)$ a representation, then
(1) $\Delta^{\alpha}_{N,\phi}$ is monic, (2) $\deg(\Delta^{\alpha}_{N,\phi}) = kx(\phi) + \text{error term}$.

Theorem 3. (F-Vidussi) If for any representation $\alpha : \pi_1(N) \to GL(R,k)$ we have
(1) $\Delta^{\alpha}_{N,\phi}$ is monic, (2) $\deg(\Delta^{\alpha}_{N,\phi}) = kx(\phi) + \text{error term}$.
then ϕ is a fibered class.

TAP, fibered 3-manifolds and the Thurston norm

Let N be a 3-manifold and $\phi \in H^1(N;\mathbb{Z})$.

Theorem 1. (F-Kim) If $\alpha : \pi_1(N) \to GL(R,k)$ is a representation, then

$$\deg(\Delta^{\alpha}_{N,\phi}) = kx(\phi) + \text{error term}.$$

Theorem 2. (Cha, Goda-Kitano-Morifuji and F-Kim) If ϕ is a fibered class and $\alpha : \pi_1(N) \to GL(R,k)$ a representation, then
(1) $\Delta^{\alpha}_{N,\phi}$ is monic, (2) $\deg(\Delta^{\alpha}_{N,\phi}) = kx(\phi) + \text{error term}$.

Theorem 3. (F-Vidussi) If for any representation $\alpha : \pi_1(N) \to GL(R,k)$ we have
(1) $\Delta^{\alpha}_{N,\phi}$ is monic, (2) $\deg(\Delta^{\alpha}_{N,\phi}) = kx(\phi) + \text{error term}$.
then ϕ is a fibered class.

So twisted Alexander polynomials detect fibered 3-manifolds.

TAP, fibered 3-manifolds and the Thurston norm

Let N be a 3-manifold and $\phi \in H^1(N;\mathbb{Z})$.

Theorem 1. (F-Kim) If $\alpha : \pi_1(N) \to GL(R,k)$ is a representation, then

$$\deg(\Delta^{\alpha}_{N,\phi}) = kx(\phi) + \text{error term}.$$

Theorem 2. (Cha, Goda-Kitano-Morifuji and F-Kim) If ϕ is a fibered class and $\alpha : \pi_1(N) \to GL(R,k)$ a representation, then
(1) $\Delta^{\alpha}_{N,\phi}$ is monic, (2) $\deg(\Delta^{\alpha}_{N,\phi}) = kx(\phi) + \text{error term}$.

Theorem 3. (F-Vidussi) If for any representation $\alpha : \pi_1(N) \to GL(R,k)$ we have
(1) $\Delta^{\alpha}_{N,\phi}$ is monic, (2) $\deg(\Delta^{\alpha}_{N,\phi}) = kx(\phi) + \text{error term}$.
then ϕ is a fibered class.

So twisted Alexander polynomials detect fibered 3-manifolds.
Instead of proving Theorem 3 I will now show that results announced by Dani Wise give a stronger version of Theorem 3 and a converse to Theorem 1.

The theorem/claim of Dani Wise.

Let G be a graph. We obtain a group as follows:

The theorem/claim of Dani Wise.

Let G be a graph. We obtain a group as follows:
To each vertex of G we associate a generator,

The theorem/claim of Dani Wise.

Let G be a graph. We obtain a group as follows:
To each vertex of G we associate a generator,
and to each edge we associate the relation that the two corresponding generators commute.

The theorem/claim of Dani Wise.

Let G be a graph. We obtain a group as follows:
To each vertex of G we associate a generator,
and to each edge we associate the relation that the two corresponding generators commute.
Such a group is called a *right angled Artin group* (RAAG).

The theorem/claim of Dani Wise.

Let G be a graph. We obtain a group as follows:
To each vertex of G we associate a generator,
and to each edge we associate the relation that the two corresponding generators commute.
Such a group is called a *right angled Artin group* (RAAG).

Example:

The theorem/claim of Dani Wise.

Let G be a graph. We obtain a group as follows:
To each vertex of G we associate a generator,
and to each edge we associate the relation that the two corresponding generators commute.
Such a group is called a *right angled Artin group* (RAAG).

Example: (1) If G has no edges we obtain a free group.

The theorem/claim of Dani Wise.

Let G be a graph. We obtain a group as follows:
To each vertex of G we associate a generator,
and to each edge we associate the relation that the two corresponding generators commute.
Such a group is called a *right angled Artin group* (RAAG).

Example: (1) If G has no edges we obtain a free group.
(2) If G is a complete graph (i.e. any two vertices are connected by an edge)

The theorem/claim of Dani Wise.

Let G be a graph. We obtain a group as follows:
To each vertex of G we associate a generator,
and to each edge we associate the relation that the two corresponding generators commute.
Such a group is called a *right angled Artin group* (RAAG).

Example: (1) If G has no edges we obtain a free group.
(2) If G is a complete graph (i.e. any two vertices are connected by an edge) then we obtain a free abelian group.

The theorem/claim of Dani Wise.

Let G be a graph. We obtain a group as follows:
To each vertex of G we associate a generator,
and to each edge we associate the relation that the two corresponding generators commute.
Such a group is called a *right angled Artin group* (RAAG).

Example: (1) If G has no edges we obtain a free group.
(2) If G is a complete graph (i.e. any two vertices are connected by an edge) then we obtain a free abelian group.
So RAAG lie between free groups and free abelian groups.

The theorem/claim of Dani Wise.

Let G be a graph. We obtain a group as follows:
To each vertex of G we associate a generator,
and to each edge we associate the relation that the two corresponding generators commute.
Such a group is called a *right angled Artin group* (RAAG).

Example: (1) If G has no edges we obtain a free group.
(2) If G is a complete graph (i.e. any two vertices are connected by an edge) then we obtain a free abelian group.
So RAAG lie between free groups and free abelian groups.

We now say G is a SRAAG if G is a subgroup of a RAAG.

The theorem/claim of Dani Wise.

Let G be a graph. We obtain a group as follows:
To each vertex of G we associate a generator,
and to each edge we associate the relation that the two corresponding generators commute.
Such a group is called a *right angled Artin group* (RAAG).

We now say G is a SRAAG if G is a subgroup of a RAAG.

Theorem (Wise) Let N be a hyperbolic 3-manifold with $b_1(N) \geq 2$ (or $b_1(N) = 1$ and N is not fibered), then π admits a finite index subgroup π' which is a SRAAG.

The theorem/claim of Dani Wise.

Let G be a graph. We obtain a group as follows:
To each vertex of G we associate a generator,
and to each edge we associate the relation that the two corresponding generators commute.
Such a group is called a *right angled Artin group* (RAAG).

We now say G is a SRAAG if G is a subgroup of a RAAG.

Theorem (Wise) Let N be a hyperbolic 3-manifold with $b_1(N) \geq 2$ (or $b_1(N) = 1$ and N is not fibered), then π admits a finite index subgroup π' which is a SRAAG.

The theorem/claim of Dani Wise.

Let G be a graph. We obtain a group as follows:
To each vertex of G we associate a generator,
and to each edge we associate the relation that the two corresponding generators commute.
Such a group is called a *right angled Artin group* (RAAG).

We now say G is a SRAAG if G is a subgroup of a RAAG.

Theorem (Wise) Let N be a hyperbolic 3-manifold with $b_1(N) \geq 2$ (or $b_1(N) = 1$ and N is not fibered), then π admits a finite index subgroup π' which is a SRAAG.

This is a fantastic theorem (as I will explain) but the proof has not been verified yet.

The theorem/claim of Dani Wise.

Let G be a graph. We obtain a group as follows:
To each vertex of G we associate a generator,
and to each edge we associate the relation that the two corresponding generators commute.
Such a group is called a *right angled Artin group* (RAAG).

We now say G is a SRAAG if G is a subgroup of a RAAG.

Theorem (Wise) Let N be a hyperbolic 3-manifold with $b_1(N) \geq 2$ (or $b_1(N) = 1$ and N is not fibered), then π admits a finite index subgroup π' which is a SRAAG.

Corollaries (for N as in the theorem):

The theorem/claim of Dani Wise.

Let G be a graph. We obtain a group as follows:
To each vertex of G we associate a generator,
and to each edge we associate the relation that the two corresponding generators commute.
Such a group is called a *right angled Artin group* (RAAG).

We now say G is a SRAAG if G is a subgroup of a RAAG.

Theorem (Wise) Let N be a hyperbolic 3-manifold with $b_1(N) \geq 2$ (or $b_1(N) = 1$ and N is not fibered), then π admits a finite index subgroup π' which is a SRAAG.

Corollaries (for N as in the theorem):
(1) $\pi = \pi_1(N)$ is subgroup separable (LERF)

The theorem/claim of Dani Wise.

Let G be a graph. We obtain a group as follows:
To each vertex of G we associate a generator,
and to each edge we associate the relation that the two corresponding generators commute.
Such a group is called a *right angled Artin group* (RAAG).

We now say G is a SRAAG if G is a subgroup of a RAAG.

Theorem (Wise) Let N be a hyperbolic 3-manifold with $b_1(N) \geq 2$ (or $b_1(N) = 1$ and N is not fibered), then π admits a finite index subgroup π' which is a SRAAG.

Corollaries (for N as in the theorem):
(1) $\pi = \pi_1(N)$ is subgroup separable (LERF)
i.e. for any finitely generated subgroup $A \subset \pi$ and any $g \notin A$

The theorem/claim of Dani Wise.

Let G be a graph. We obtain a group as follows:
To each vertex of G we associate a generator,
and to each edge we associate the relation that the two corresponding generators commute.
Such a group is called a *right angled Artin group* (RAAG).

We now say G is a SRAAG if G is a subgroup of a RAAG.

Theorem (Wise) Let N be a hyperbolic 3-manifold with $b_1(N) \geq 2$ (or $b_1(N) = 1$ and N is not fibered), then π admits a finite index subgroup π' which is a SRAAG.

Corollaries (for N as in the theorem):
(1) $\pi = \pi_1(N)$ is subgroup separable (LERF)
i.e. for any finitely generated subgroup $A \subset \pi$ and any $g \notin A$ there exists a homomorphism $\alpha : \pi \to G$ to a *finite* group G such that $\alpha(g) \notin \alpha(A)$.

The theorem/claim of Dani Wise.

Let G be a graph. We obtain a group as follows:
To each vertex of G we associate a generator,
and to each edge we associate the relation that the two corresponding generators commute.
Such a group is called a *right angled Artin group* (RAAG).

We now say G is a SRAAG if G is a subgroup of a RAAG.

Theorem (Wise) Let N be a hyperbolic 3-manifold with $b_1(N) \geq 2$ (or $b_1(N) = 1$ and N is not fibered), then π admits a finite index subgroup π' which is a SRAAG.

Corollaries (for N as in the theorem):
(1) $\pi = \pi_1(N)$ is subgroup separable (LERF)
i.e. for any finitely generated subgroup $A \subset \pi$ and any $g \notin A$ there exists a homomorphism $\alpha : \pi \to G$ to a *finite* group G such that $\alpha(g) \notin \alpha(A)$.
'We can tell that $g \notin A$ in a finite quotient'

The theorem/claim of Dani Wise.

Let G be a graph. We obtain a group as follows:
To each vertex of G we associate a generator,
and to each edge we associate the relation that the two corresponding generators commute.
Such a group is called a *right angled Artin group* (RAAG).

We now say G is a SRAAG if G is a subgroup of a RAAG.

Theorem (Wise) Let N be a hyperbolic 3-manifold with $b_1(N) \geq 2$ (or $b_1(N) = 1$ and N is not fibered), then π admits a finite index subgroup π' which is a SRAAG.

Corollaries (for N as in the theorem):
(1) $\pi = \pi_1(N)$ is subgroup separable (LERF)
(2) π is virtually residually torsion-free nilpotent.

The theorem/claim of Dani Wise.

Let G be a graph. We obtain a group as follows:
To each vertex of G we associate a generator,
and to each edge we associate the relation that the two corresponding generators commute.
Such a group is called a *right angled Artin group* (RAAG).

We now say G is a SRAAG if G is a subgroup of a RAAG.

Theorem (Wise) Let N be a hyperbolic 3-manifold with $b_1(N) \geq 2$ (or $b_1(N) = 1$ and N is not fibered), then π admits a finite index subgroup π' which is a SRAAG.

Corollaries (for N as in the theorem):
(1) $\pi = \pi_1(N)$ is subgroup separable (LERF)
(2) π is virtually residually torsion-free nilpotent.
(i.e. there exists a finite index subgroup π' such that for any non-trivial $g \in \pi'$ there exists a homomorphism $\alpha : \pi' \to G$ to a torsion-free nilpotent group such that $\alpha(g)$ is non-trivial.)

The theorem/claim of Dani Wise.

Let G be a graph. We obtain a group as follows:
To each vertex of G we associate a generator,
and to each edge we associate the relation that the two corresponding generators commute.
Such a group is called a *right angled Artin group* (RAAG).

We now say G is a SRAAG if G is a subgroup of a RAAG.

Theorem (Wise) Let N be a hyperbolic 3-manifold with $b_1(N) \geq 2$ (or $b_1(N) = 1$ and N is not fibered), then π admits a finite index subgroup π' which is a SRAAG.

Corollaries (for N as in the theorem):
(1) $\pi = \pi_1(N)$ is subgroup separable (LERF)
(2) π is virtually residually torsion-free nilpotent.
(3) π is linear over $\mathbb{Z}$

The theorem/claim of Dani Wise.

Let G be a graph. We obtain a group as follows:
To each vertex of G we associate a generator,
and to each edge we associate the relation that the two corresponding generators commute.
Such a group is called a *right angled Artin group* (RAAG).

We now say G is a SRAAG if G is a subgroup of a RAAG.

Theorem (Wise) Let N be a hyperbolic 3-manifold with $b_1(N) \geq 2$ (or $b_1(N) = 1$ and N is not fibered), then π admits a finite index subgroup π' which is a SRAAG.

Corollaries (for N as in the theorem):
(1) $\pi = \pi_1(N)$ is subgroup separable (LERF)
(2) π is virtually residually torsion-free nilpotent.
(3) π is linear over $\mathbb{Z}$
i.e. π is a subgroup of $GL(n, \mathbb{Z})$ for some n.

The theorem/claim of Dani Wise.

Let G be a graph. We obtain a group as follows:
To each vertex of G we associate a generator,
and to each edge we associate the relation that the two corresponding generators commute.
Such a group is called a *right angled Artin group* (RAAG).

We now say G is a SRAAG if G is a subgroup of a RAAG.

Theorem (Wise) Let N be a hyperbolic 3-manifold with $b_1(N) \geq 2$ (or $b_1(N) = 1$ and N is not fibered), then π admits a finite index subgroup π' which is a SRAAG.

Corollaries (for N as in the theorem):
(1) $\pi = \pi_1(N)$ is subgroup separable (LERF)
(2) π is virtually residually torsion-free nilpotent.
(3) π is linear over $\mathbb{Z}$
(4) for any k there exists a finite cover N' with $b_1(N') \geq k$

Recall: π is LERF if for any f.g. $A \subset \pi$ and $g \notin A$ there is a map $\alpha : \pi \to G$ to a *finite* group G s.t. $\alpha(g) \notin \alpha(A)$.

Twisted Alexander polynomials and fibered 3-manifolds

Recall: π is LERF if for any f.g. $A \subset \pi$ and $g \notin A$ there is a map $\alpha : \pi \to G$ to a *finite* group G s.t. $\alpha(g) \notin \alpha(A)$.

Theorem. Let N a 3-mfd and $\phi \in H^1(N)$. Suppose that for any $\alpha : \pi \to G$ we have $\Delta^{\alpha}_{N,\phi}$ is non-zero

Twisted Alexander polynomials and fibered 3-manifolds

Recall: π is LERF if for any f.g. $A \subset \pi$ and $g \notin A$ there is a map $\alpha : \pi \to G$ to a *finite* group G s.t. $\alpha(g) \notin \alpha(A)$.

Theorem. Let N a 3-mfd and $\phi \in H^1(N)$. Suppose that for any $\alpha : \pi \to G$ we have $\Delta^{\alpha}_{N,\phi}$ is non-zero and $\pi_1(N')$ is LERF for any hyperbolic JSJ component N'

Twisted Alexander polynomials and fibered 3-manifolds

Recall: π is LERF if for any f.g. $A \subset \pi$ and $g \notin A$ there is a map $\alpha : \pi \to G$ to a *finite* group G s.t. $\alpha(g) \notin \alpha(A)$.

Theorem. Let N a 3-mfd and $\phi \in H^1(N)$. Suppose that for any $\alpha : \pi \to G$ we have $\Delta^{\alpha}_{N,\phi}$ is non-zero and $\pi_1(N')$ is LERF for any hyperbolic JSJ component N' then ϕ is a fibered class.

Twisted Alexander polynomials and fibered 3-manifolds

Recall: π is LERF if for any f.g. $A \subset \pi$ and $g \notin A$ there is a map $\alpha : \pi \to G$ to a *finite* group G s.t. $\alpha(g) \notin \alpha(A)$.

Theorem. Let N a 3-mfd and $\phi \in H^1(N)$. Suppose that for any $\alpha : \pi \to G$ we have $\Delta^{\alpha}_{N,\phi}$ is non-zero and $\pi_1(N')$ is LERF for any hyperbolic JSJ component N' then ϕ is a fibered class.

This was proved by F-Vidussi in 2006.

Recall: π is LERF if for any f.g. $A \subset \pi$ and $g \notin A$ there is a map $\alpha : \pi \to G$ to a *finite* group G s.t. $\alpha(g) \notin \alpha(A)$.

Theorem. Let N a 3-mfd and $\phi \in H^1(N)$. Suppose that for any $\alpha : \pi \to G$ we have $\Delta^{\alpha}_{N,\phi}$ is non-zero and $\pi_1(N')$ is LERF for any hyperbolic JSJ component N' then ϕ is a fibered class.

This was proved by F-Vidussi in 2006. We have the following immediate corollary:

Recall: π is LERF if for any f.g. $A \subset \pi$ and $g \notin A$ there is a map $\alpha : \pi \to G$ to a *finite* group G s.t. $\alpha(g) \notin \alpha(A)$.

Theorem. Let N a 3-mfd and $\phi \in H^1(N)$. Suppose that for any $\alpha : \pi \to G$ we have $\Delta^{\alpha}_{N,\phi}$ is non-zero and $\pi_1(N')$ is LERF for any hyperbolic JSJ component N' then ϕ is a fibered class.

This was proved by F-Vidussi in 2006. We have the following immediate corollary:

Corollary (if Wise's theorem is correct) Let N be any 3-manifold and $\phi \in H^1(N; \mathbb{Z})$ non-fibered, then there exists $\alpha : \pi \to G$ with $\Delta^{\alpha}_{N,\phi} = 0$.

Recall: π is LERF if for any f.g. $A \subset \pi$ and $g \notin A$ there is a map $\alpha : \pi \to G$ to a *finite* group G s.t. $\alpha(g) \notin \alpha(A)$.

Theorem. Let N a 3-mfd and $\phi \in H^1(N)$. Suppose that for any $\alpha : \pi \to G$ we have $\Delta^{\alpha}_{N,\phi}$ is non-zero and $\pi_1(N')$ is LERF for any hyperbolic JSJ component N' then ϕ is a fibered class.

This was proved by F-Vidussi in 2006. We have the following immediate corollary:

Corollary (if Wise's theorem is correct) Let N be any 3-manifold and $\phi \in H^1(N;\mathbb{Z})$ non-fibered, then there exists $\alpha : \pi \to G$ with $\Delta^{\alpha}_{N,\phi} = 0$.

We now turn to the proof of the theorem.

Recall: π is LERF if for any f.g. $A \subset \pi$ and $g \notin A$ there is a map $\alpha : \pi \to G$ to a *finite* group G s.t. $\alpha(g) \notin \alpha(A)$.

Theorem. Let N a 3-mfd and $\phi \in H^1(N)$. Suppose that for any $\alpha : \pi \to G$ we have $\Delta^{\alpha}_{N,\phi}$ is non-zero and $\pi_1(N')$ is LERF for any hyperbolic JSJ component N' then ϕ is a fibered class.

Proof (if $\pi_1(N)$ is LERF): Let Σ surface dual to ϕ.

Twisted Alexander polynomials and fibered 3-manifolds

Recall: π is LERF if for any f.g. $A \subset \pi$ and $g \notin A$ there is a map $\alpha : \pi \to G$ to a *finite* group G s.t. $\alpha(g) \notin \alpha(A)$.

Theorem. Let N a 3-mfd and $\phi \in H^1(N)$. Suppose that for any $\alpha : \pi \to G$ we have $\Delta^{\alpha}_{N,\phi}$ is non-zero and $\pi_1(N')$ is LERF for any hyperbolic JSJ component N' then ϕ is a fibered class.

Proof (if $\pi_1(N)$ is LERF): Let Σ surface dual to ϕ. Take $\alpha : \pi_1(N) \to G$ with $\Delta^{\alpha}_{N,\phi} \neq 0$.

Recall: π is LERF if for any f.g. $A \subset \pi$ and $g \notin A$ there is a map $\alpha : \pi \to G$ to a *finite* group G s.t. $\alpha(g) \notin \alpha(A)$.

Theorem. Let N a 3-mfd and $\phi \in H^1(N)$. Suppose that for any $\alpha : \pi \to G$ we have $\Delta^{\alpha}_{N,\phi}$ is non-zero and $\pi_1(N')$ is LERF for any hyperbolic JSJ component N' then ϕ is a fibered class.

Proof (if $\pi_1(N)$ is LERF): Let Σ surface dual to ϕ. Take $\alpha : \pi_1(N) \to G$ with $\Delta^{\alpha}_{N,\phi} \neq 0$. We get

$$H_1(N; \mathbb{Z}[G][t^{\pm 1}]) \to H_0(\Sigma; \mathbb{Z}[G][t^{\pm 1}]) \to H_0(N \setminus \Sigma; \mathbb{Z}[G][t^{\pm 1}]) \to H_0(N; \mathbb{Z}[G][t^{\pm 1}])$$

Twisted Alexander polynomials and fibered 3-manifolds

Recall: π is LERF if for any f.g. $A \subset \pi$ and $g \notin A$ there is a map $\alpha : \pi \to G$ to a *finite* group G s.t. $\alpha(g) \notin \alpha(A)$.

Theorem. Let N a 3-mfd and $\phi \in H^1(N)$. Suppose that for any $\alpha : \pi \to G$ we have $\Delta^{\alpha}_{N,\phi}$ is non-zero and $\pi_1(N')$ is LERF for any hyperbolic JSJ component N' then ϕ is a fibered class.

Proof (if $\pi_1(N)$ is LERF): Let Σ surface dual to ϕ. Take $\alpha : \pi_1(N) \to G$ with $\Delta^{\alpha}_{N,\phi} \neq 0$. We get

$$H_1(N;\mathbb{Z}[G][t^{\pm 1}]) \to H_0(\Sigma;\mathbb{Z}[G][t^{\pm 1}]) \to H_0(N \setminus \Sigma;\mathbb{Z}[G][t^{\pm 1}]) \to H_0(N;\mathbb{Z}[G][t^{\pm 1}])$$

$H_1(N;\mathbb{Z}[G][t^{\pm 1}])$ is $\mathbb{Z}[t^{\pm 1}]$-torsion (since $\Delta^{\alpha}_{N,\phi} \neq 0$!)

Twisted Alexander polynomials and fibered 3-manifolds

Recall: π is LERF if for any f.g. $A \subset \pi$ and $g \notin A$ there is a map $\alpha : \pi \to G$ to a *finite* group G s.t. $\alpha(g) \notin \alpha(A)$.

Theorem. Let N a 3-mfd and $\phi \in H^1(N)$. Suppose that for any $\alpha : \pi \to G$ we have $\Delta^\alpha_{N,\phi}$ is non-zero and $\pi_1(N')$ is LERF for any hyperbolic JSJ component N' then ϕ is a fibered class.

Proof (if $\pi_1(N)$ is LERF): Let Σ surface dual to ϕ. Take $\alpha : \pi_1(N) \to G$ with $\Delta^\alpha_{N,\phi} \neq 0$. We get

$$H_1(N; \mathbb{Z}[G][t^{\pm 1}]) \to H_0(\Sigma; \mathbb{Z}[G][t^{\pm 1}]) \to H_0(N \setminus \Sigma; \mathbb{Z}[G][t^{\pm 1}]) \to H_0(N; \mathbb{Z}[G][t^{\pm 1}])$$

$H_1(N; \mathbb{Z}[G][t^{\pm 1}])$ is $\mathbb{Z}[t^{\pm 1}]$-torsion (since $\Delta^\alpha_{N,\phi} \neq 0$!) and $H_0(N; \mathbb{Z}[G][t^{\pm 1}])$ is $\mathbb{Z}[t^{\pm 1}]$-torsion (calculation).

Recall: π is LERF if for any f.g. $A \subset \pi$ and $g \notin A$ there is a map $\alpha : \pi \to G$ to a *finite* group G s.t. $\alpha(g) \notin \alpha(A)$.

Theorem. Let N a 3-mfd and $\phi \in H^1(N)$. Suppose that for any $\alpha : \pi \to G$ we have $\Delta^\alpha_{N,\phi}$ is non-zero and $\pi_1(N')$ is LERF for any hyperbolic JSJ component N' then ϕ is a fibered class.

Proof (if $\pi_1(N)$ is LERF): Let Σ surface dual to ϕ. Take $\alpha : \pi_1(N) \to G$ with $\Delta^\alpha_{N,\phi} \neq 0$. We get

$$H_1(N; \mathbb{Z}[G][t^{\pm 1}]) \to H_0(\Sigma; \mathbb{Z}[G][t^{\pm 1}]) \to H_0(N \setminus \Sigma; \mathbb{Z}[G][t^{\pm 1}]) \to H_0(N; \mathbb{Z}[G][t^{\pm 1}])$$

$H_1(N; \mathbb{Z}[G][t^{\pm 1}])$ is $\mathbb{Z}[t^{\pm 1}]$-torsion (since $\Delta^\alpha_{N,\phi} \neq 0$!) and $H_0(N; \mathbb{Z}[G][t^{\pm 1}])$ is $\mathbb{Z}[t^{\pm 1}]$-torsion (calculation). We conclude that $\text{rank}_{\mathbb{Z}[t^{\pm 1}]}(H_0(\Sigma; \mathbb{Z}[G][t^{\pm 1}])) = \text{rank}_{\mathbb{Z}[t^{\pm 1}]}(H_0(N \setminus \Sigma; \mathbb{Z}[G][t^{\pm 1}]))$.

Twisted Alexander polynomials and fibered 3-manifolds

Recall: π is LERF if for any f.g. $A \subset \pi$ and $g \notin A$ there is a map $\alpha : \pi \to G$ to a *finite* group G s.t. $\alpha(g) \notin \alpha(A)$.

Theorem. Let N a 3-mfd and $\phi \in H^1(N)$. Suppose that for any $\alpha : \pi \to G$ we have $\Delta^\alpha_{N,\phi}$ is non-zero and $\pi_1(N')$ is LERF for any hyperbolic JSJ component N' then ϕ is a fibered class.

Proof (if $\pi_1(N)$ is LERF): Let Σ surface dual to ϕ. Take $\alpha : \pi_1(N) \to G$ with $\Delta^\alpha_{N,\phi} \neq 0$. We get

$$H_1(N; \mathbb{Z}[G][t^{\pm 1}]) \to H_0(\Sigma; \mathbb{Z}[G][t^{\pm 1}]) \to H_0(N \setminus \Sigma; \mathbb{Z}[G][t^{\pm 1}]) \to H_0(N; \mathbb{Z}[G][t^{\pm 1}])$$

$H_1(N; \mathbb{Z}[G][t^{\pm 1}])$ is $\mathbb{Z}[t^{\pm 1}]$-torsion (since $\Delta^\alpha_{N,\phi} \neq 0$!) and $H_0(N; \mathbb{Z}[G][t^{\pm 1}])$ is $\mathbb{Z}[t^{\pm 1}]$-torsion (calculation). We conclude that $\operatorname{rank}_{\mathbb{Z}[t^{\pm 1}]}(H_0(\Sigma; \mathbb{Z}[G][t^{\pm 1}])) = \operatorname{rank}_{\mathbb{Z}[t^{\pm 1}]}(H_0(N \setminus \Sigma; \mathbb{Z}[G][t^{\pm 1}]))$.

Summarizing we proved, if $\Delta^\alpha_{N,\phi} \neq 0$, then

$$\operatorname{rank}_{\mathbb{Z}[t^{\pm 1}]}(H_0(\Sigma; \mathbb{Z}[G][t^{\pm 1}])) = \operatorname{rank}_{\mathbb{Z}[t^{\pm 1}]}(H_0(N \setminus \Sigma; \mathbb{Z}[G][t^{\pm 1}])).$$

Twisted Alexander polynomials and fibered 3-manifolds

Recall: π is LERF if for any f.g. $A \subset \pi$ and $g \notin A$ there is a map $\alpha : \pi \to G$ to a *finite* group G s.t. $\alpha(g) \notin \alpha(A)$.

Theorem. Let N a 3-mfd and $\phi \in H^1(N)$. Suppose that for any $\alpha : \pi \to G$ we have $\Delta^{\alpha}_{N,\phi}$ is non-zero and $\pi_1(N')$ is LERF for any hyperbolic JSJ component N' then ϕ is a fibered class.

We proved, that given $\alpha : \pi_1(N) \to G$, if $\Delta^{\alpha}_{N,\phi} \neq 0$, then
$\mathrm{rank}_{\mathbb{Z}[t^{\pm 1}]}(H_0(\Sigma; \mathbb{Z}[G][t^{\pm 1}])) = \mathrm{rank}_{\mathbb{Z}[t^{\pm 1}]}(H_0(N \setminus \Sigma; \mathbb{Z}[G][t^{\pm 1}]))$.

Recall: π is LERF if for any f.g. $A \subset \pi$ and $g \notin A$ there is a map $\alpha : \pi \to G$ to a *finite* group G s.t. $\alpha(g) \notin \alpha(A)$.

Theorem. Let N a 3-mfd and $\phi \in H^1(N)$. Suppose that for any $\alpha : \pi \to G$ we have $\Delta^{\alpha}_{N,\phi}$ is non-zero and $\pi_1(N')$ is LERF for any hyperbolic JSJ component N' then ϕ is a fibered class.

We proved, that given $\alpha : \pi_1(N) \to G$, if $\Delta^{\alpha}_{N,\phi} \neq 0$, then

$$\mathrm{rank}_{\mathbb{Z}[t^{\pm 1}]}(H_0(\Sigma; \mathbb{Z}[G][t^{\pm 1}])) = \mathrm{rank}_{\mathbb{Z}[t^{\pm 1}]}(H_0(N \setminus \Sigma; \mathbb{Z}[G][t^{\pm 1}])).$$

$$\|\qquad\qquad\qquad\qquad\|$$

$$\mathrm{rk}_{\mathbb{Z}[t^{\pm 1}]}(H_0(\Sigma; \mathbb{Z}[G]) \otimes \mathbb{Z}[t^{\pm 1}])$$

Twisted Alexander polynomials and fibered 3-manifolds

Recall: π is LERF if for any f.g. $A \subset \pi$ and $g \notin A$ there is a map $\alpha : \pi \to G$ to a *finite* group G s.t. $\alpha(g) \notin \alpha(A)$.

Theorem. Let N a 3-mfd and $\phi \in H^1(N)$. Suppose that for any $\alpha : \pi \to G$ we have $\Delta^{\alpha}_{N,\phi}$ is non-zero and $\pi_1(N')$ is LERF for any hyperbolic JSJ component N' then ϕ is a fibered class.

We proved, that given $\alpha : \pi_1(N) \to G$, if $\Delta^{\alpha}_{N,\phi} \neq 0$, then

$$\begin{array}{ccc} \operatorname{rank}_{\mathbb{Z}[t^{\pm 1}]}(H_0(\Sigma; \mathbb{Z}[G][t^{\pm 1}])) & = & \operatorname{rank}_{\mathbb{Z}[t^{\pm 1}]}(H_0(N \setminus \Sigma; \mathbb{Z}[G][t^{\pm 1}])). \\ \| & & \| \\ \operatorname{rk}_{\mathbb{Z}[t^{\pm 1}]}(H_0(\Sigma; \mathbb{Z}[G]) \otimes \mathbb{Z}[t^{\pm 1}]) & = & \operatorname{rk}_{\mathbb{Z}[t^{\pm 1}]} H_0(N \setminus \Sigma; \mathbb{Z}[G]) \otimes \mathbb{Z}[t^{\pm 1}] \end{array}$$

Recall: π is LERF if for any f.g. $A \subset \pi$ and $g \notin A$ there is a map $\alpha : \pi \to G$ to a *finite* group G s.t. $\alpha(g) \notin \alpha(A)$.

Theorem. Let N a 3-mfd and $\phi \in H^1(N)$. Suppose that for any $\alpha : \pi \to G$ we have $\Delta^{\alpha}_{N,\phi}$ is non-zero and $\pi_1(N')$ is LERF for any hyperbolic JSJ component N' then ϕ is a fibered class.

We proved, that given $\alpha : \pi_1(N) \to G$, if $\Delta^{\alpha}_{N,\phi} \neq 0$, then

$$\begin{array}{ccc} \operatorname{rank}_{\mathbb{Z}[t^{\pm 1}]}(H_0(\Sigma; \mathbb{Z}[G][t^{\pm 1}])) & = & \operatorname{rank}_{\mathbb{Z}[t^{\pm 1}]}(H_0(N \setminus \Sigma; \mathbb{Z}[G][t^{\pm 1}])). \\ \| & & \| \\ \operatorname{rk}_{\mathbb{Z}[t^{\pm 1}]}(H_0(\Sigma; \mathbb{Z}[G]) \otimes \mathbb{Z}[t^{\pm 1}]) & = & \operatorname{rk}_{\mathbb{Z}[t^{\pm 1}]} H_0(N \setminus \Sigma; \mathbb{Z}[G]) \otimes \mathbb{Z}[t^{\pm 1}] \\ \| & & \| \\ \operatorname{rk}_{\mathbb{Z}} H_0(\Sigma; \mathbb{Z}[G]) & & \end{array}$$

Twisted Alexander polynomials and fibered 3-manifolds

Recall: π is LERF if for any f.g. $A \subset \pi$ and $g \notin A$ there is a map $\alpha : \pi \to G$ to a *finite* group G s.t. $\alpha(g) \notin \alpha(A)$.

Theorem. Let N a 3-mfd and $\phi \in H^1(N)$. Suppose that for any $\alpha : \pi \to G$ we have $\Delta^{\alpha}_{N,\phi}$ is non-zero and $\pi_1(N')$ is LERF for any hyperbolic JSJ component N' then ϕ is a fibered class.

We proved, that given $\alpha : \pi_1(N) \to G$, if $\Delta^{\alpha}_{N,\phi} \neq 0$, then

$$\begin{array}{ccc}
\operatorname{rank}_{\mathbb{Z}[t^{\pm 1}]}(H_0(\Sigma; \mathbb{Z}[G][t^{\pm 1}])) & = & \operatorname{rank}_{\mathbb{Z}[t^{\pm 1}]}(H_0(N \setminus \Sigma; \mathbb{Z}[G][t^{\pm 1}])). \\
\| & & \| \\
\operatorname{rk}_{\mathbb{Z}[t^{\pm 1}]}(H_0(\Sigma; \mathbb{Z}[G]) \otimes \mathbb{Z}[t^{\pm 1}]) & = & \operatorname{rk}_{\mathbb{Z}[t^{\pm 1}]} H_0(N \setminus \Sigma; \mathbb{Z}[G]) \otimes \mathbb{Z}[t^{\pm 1}] \\
\| & & \| \\
\operatorname{rk}_{\mathbb{Z}} H_0(\Sigma; \mathbb{Z}[G]) & = & \operatorname{rk}_{\mathbb{Z}} H_0(N \setminus \Sigma; \mathbb{Z}[G]).
\end{array}$$

Recall: π is LERF if for any f.g. $A \subset \pi$ and $g \notin A$ there is a map $\alpha : \pi \to G$ to a *finite* group G s.t. $\alpha(g) \notin \alpha(A)$.

Theorem. Let N a 3-mfd and $\phi \in H^1(N)$. Suppose that for any $\alpha : \pi \to G$ we have $\Delta^{\alpha}_{N,\phi}$ is non-zero and $\pi_1(N')$ is LERF for any hyperbolic JSJ component N' then ϕ is a fibered class.

We proved, that given $\alpha : \pi_1(N) \to G$, if $\Delta^{\alpha}_{N,\phi} \neq 0$, then

$$\begin{array}{ccc} \mathrm{rank}_{\mathbb{Z}[t^{\pm1}]}(H_0(\Sigma;\mathbb{Z}[G][t^{\pm1}])) & = & \mathrm{rank}_{\mathbb{Z}[t^{\pm1}]}(H_0(N\setminus\Sigma;\mathbb{Z}[G][t^{\pm1}])). \\ \| & & \| \\ \mathrm{rk}_{\mathbb{Z}[t^{\pm1}]}(H_0(\Sigma;\mathbb{Z}[G])\otimes\mathbb{Z}[t^{\pm1}]) & = & \mathrm{rk}_{\mathbb{Z}[t^{\pm1}]}H_0(N\setminus\Sigma;\mathbb{Z}[G])\otimes\mathbb{Z}[t^{\pm1}] \\ \| & & \| \\ \mathrm{rk}_{\mathbb{Z}}\, H_0(\Sigma;\mathbb{Z}[G]) & = & \mathrm{rk}_{\mathbb{Z}}\, H_0(N\setminus\Sigma;\mathbb{Z}[G]). \\ \| & & \| \\ \frac{|G|}{\mathrm{Im}(\pi_1(\Sigma)\to G)} & & \end{array}$$

Twisted Alexander polynomials and fibered 3-manifolds

Recall: π is LERF if for any f.g. $A \subset \pi$ and $g \notin A$ there is a map $\alpha : \pi \to G$ to a *finite* group G s.t. $\alpha(g) \notin \alpha(A)$.

Theorem. Let N a 3-mfd and $\phi \in H^1(N)$. Suppose that for any $\alpha : \pi \to G$ we have $\Delta^\alpha_{N,\phi}$ is non-zero and $\pi_1(N')$ is LERF for any hyperbolic JSJ component N' then ϕ is a fibered class.

We proved, that given $\alpha : \pi_1(N) \to G$, if $\Delta^\alpha_{N,\phi} \neq 0$, then

$$\begin{array}{ccc} \mathrm{rank}_{\mathbb{Z}[t^{\pm 1}]}(H_0(\Sigma;\mathbb{Z}[G][t^{\pm 1}])) & = & \mathrm{rank}_{\mathbb{Z}[t^{\pm 1}]}(H_0(N\setminus\Sigma;\mathbb{Z}[G][t^{\pm 1}])). \\ \| & & \| \\ \mathrm{rk}_{\mathbb{Z}[t^{\pm 1}]}(H_0(\Sigma;\mathbb{Z}[G])\otimes\mathbb{Z}[t^{\pm 1}]) & = & \mathrm{rk}_{\mathbb{Z}[t^{\pm 1}]}H_0(N\setminus\Sigma;\mathbb{Z}[G])\otimes\mathbb{Z}[t^{\pm 1}] \\ \| & & \| \\ \mathrm{rk}_{\mathbb{Z}}\, H_0(\Sigma;\mathbb{Z}[G]) & = & \mathrm{rk}_{\mathbb{Z}}\, H_0(N\setminus\Sigma;\mathbb{Z}[G]). \\ \| & & \| \\ \frac{|G|}{\mathrm{Im}(\pi_1(\Sigma)\to G)} & = & \frac{|G|}{\mathrm{Im}(\pi_1(N\setminus\Sigma)\to G)} \end{array}$$

Twisted Alexander polynomials and fibered 3-manifolds

Recall: π is LERF if for any f.g. $A \subset \pi$ and $g \notin A$ there is a map $\alpha : \pi \to G$ to a *finite* group G s.t. $\alpha(g) \notin \alpha(A)$.

Theorem. Let N a 3-mfd and $\phi \in H^1(N)$. Suppose that for any $\alpha : \pi \to G$ we have $\Delta^{\alpha}_{N,\phi}$ is non-zero and $\pi_1(N')$ is LERF for any hyperbolic JSJ component N' then ϕ is a fibered class.

So we proved, given $\alpha : \pi_1(N) \to G$, if $\Delta^{\alpha}_{N,\phi} \neq 0$, then

(∗) $$\frac{|G|}{|\mathrm{Im}(\pi_1(\Sigma)\to G)|} = \frac{|G|}{|\mathrm{Im}(\pi_1(N\setminus\Sigma)\to G)|}$$

Twisted Alexander polynomials and fibered 3-manifolds

Recall: π is LERF if for any f.g. $A \subset \pi$ and $g \notin A$ there is a map $\alpha : \pi \to G$ to a *finite* group G s.t. $\alpha(g) \notin \alpha(A)$.

Theorem. Let N a 3-mfd and $\phi \in H^1(N)$. Suppose that for any $\alpha : \pi \to G$ we have $\Delta^{\alpha}_{N,\phi}$ is non-zero and $\pi_1(N')$ is LERF for any hyperbolic JSJ component N' then ϕ is a fibered class.

So we proved, given $\alpha : \pi_1(N) \to G$, if $\Delta^{\alpha}_{N,\phi} \neq 0$, then

(∗) $$\frac{|G|}{|\mathrm{Im}(\pi_1(\Sigma)\to G)|} = \frac{|G|}{|\mathrm{Im}(\pi_1(N\setminus\Sigma)\to G)|}$$

Now assume Σ is not a fiber, i.e. $\pi_1(\Sigma) \not\subseteq \pi_1(N \setminus \Sigma)$.

Recall: π is LERF if for any f.g. $A \subset \pi$ and $g \notin A$ there is a map $\alpha : \pi \to G$ to a *finite* group G s.t. $\alpha(g) \notin \alpha(A)$.

Theorem. Let N a 3-mfd and $\phi \in H^1(N)$. Suppose that for any $\alpha : \pi \to G$ we have $\Delta^{\alpha}_{N,\phi}$ is non-zero and $\pi_1(N')$ is LERF for any hyperbolic JSJ component N' then ϕ is a fibered class.

So we proved, given $\alpha : \pi_1(N) \to G$, if $\Delta^{\alpha}_{N,\phi} \neq 0$, then

$$(*) \qquad \frac{|G|}{|\mathrm{Im}(\pi_1(\Sigma)\to G)|} = \frac{|G|}{|\mathrm{Im}(\pi_1(N\setminus\Sigma)\to G)|}$$

Now assume Σ is not a fiber, i.e. $\pi_1(\Sigma) \not\subseteq \pi_1(N \setminus \Sigma)$. Pick $g \in \pi_1(N \setminus \Sigma) \setminus \pi_1(\Sigma)$.

Recall: π is LERF if for any f.g. $A \subset \pi$ and $g \notin A$ there is a map $\alpha : \pi \to G$ to a *finite* group G s.t. $\alpha(g) \notin \alpha(A)$.

Theorem. Let N a 3-mfd and $\phi \in H^1(N)$. Suppose that for any $\alpha : \pi \to G$ we have $\Delta^\alpha_{N,\phi}$ is non-zero and $\pi_1(N')$ is LERF for any hyperbolic JSJ component N' then ϕ is a fibered class.

So we proved, given $\alpha : \pi_1(N) \to G$, if $\Delta^\alpha_{N,\phi} \neq 0$, then

$$(*) \qquad \frac{|G|}{|\mathrm{Im}(\pi_1(\Sigma) \to G)|} = \frac{|G|}{|\mathrm{Im}(\pi_1(N \setminus \Sigma) \to G)|}$$

Now assume Σ is not a fiber, i.e. $\pi_1(\Sigma) \not\subseteq \pi_1(N \setminus \Sigma)$. Pick $g \in \pi_1(N \setminus \Sigma) \setminus \pi_1(\Sigma)$. Since $\pi_1(N)$ is LERF we can find $\pi_1(N) \to G$ such that $\alpha(g) \notin \alpha(\pi_1(\Sigma))$.

Recall: π is LERF if for any f.g. $A \subset \pi$ and $g \notin A$ there is a map $\alpha : \pi \to G$ to a *finite* group G s.t. $\alpha(g) \notin \alpha(A)$.

Theorem. Let N a 3-mfd and $\phi \in H^1(N)$. Suppose that for any $\alpha : \pi \to G$ we have $\Delta^{\alpha}_{N,\phi}$ is non-zero and $\pi_1(N')$ is LERF for any hyperbolic JSJ component N' then ϕ is a fibered class.

So we proved, given $\alpha : \pi_1(N) \to G$, if $\Delta^{\alpha}_{N,\phi} \neq 0$, then

(∗) $$\frac{|G|}{|\mathrm{Im}(\pi_1(\Sigma) \to G)|} = \frac{|G|}{|\mathrm{Im}(\pi_1(N \setminus \Sigma) \to G)|}$$

Now assume Σ is not a fiber, i.e. $\pi_1(\Sigma) \not\subseteq \pi_1(N \setminus \Sigma)$. Pick $g \in \pi_1(N \setminus \Sigma) \setminus \pi_1(\Sigma)$. Since $\pi_1(N)$ is LERF we can find $\pi_1(N) \to G$ such that $\alpha(g) \notin \alpha(\pi_1(\Sigma))$. Hence $|\mathrm{Im}(\pi_1(\Sigma) \to G)|$

Recall: π is LERF if for any f.g. $A \subset \pi$ and $g \notin A$ there is a map $\alpha : \pi \to G$ to a *finite* group G s.t. $\alpha(g) \notin \alpha(A)$.

Theorem. Let N a 3-mfd and $\phi \in H^1(N)$. Suppose that for any $\alpha : \pi \to G$ we have $\Delta^\alpha_{N,\phi}$ is non-zero and $\pi_1(N')$ is LERF for any hyperbolic JSJ component N' then ϕ is a fibered class.

So we proved, given $\alpha : \pi_1(N) \to G$, if $\Delta^\alpha_{N,\phi} \neq 0$, then

$$(*) \qquad \frac{|G|}{|\mathrm{Im}(\pi_1(\Sigma)\to G)|} = \frac{|G|}{|\mathrm{Im}(\pi_1(N\setminus\Sigma)\to G)|}$$

Now assume Σ is not a fiber, i.e. $\pi_1(\Sigma) \not\subseteq \pi_1(N \setminus \Sigma)$. Pick $g \in \pi_1(N \setminus \Sigma) \setminus \pi_1(\Sigma)$. Since $\pi_1(N)$ is LERF we can find $\pi_1(N) \to G$ such that $\alpha(g) \notin \alpha(\pi_1(\Sigma))$. Hence

$$|\mathrm{Im}(\pi_1(\Sigma) \to G)| = |\mathrm{Im}(\pi_1(\Sigma) \to \pi_1(N \setminus \Sigma) \to G)|$$

Twisted Alexander polynomials and fibered 3-manifolds

Recall: π is LERF if for any f.g. $A \subset \pi$ and $g \notin A$ there is a map $\alpha : \pi \to G$ to a *finite* group G s.t. $\alpha(g) \notin \alpha(A)$.

Theorem. Let N a 3-mfd and $\phi \in H^1(N)$. Suppose that for any $\alpha : \pi \to G$ we have $\Delta^{\alpha}_{N,\phi}$ is non-zero and $\pi_1(N')$ is LERF for any hyperbolic JSJ component N' then ϕ is a fibered class.

So we proved, given $\alpha : \pi_1(N) \to G$, if $\Delta^{\alpha}_{N,\phi} \neq 0$, then

(∗) $$\frac{|G|}{|\mathrm{Im}(\pi_1(\Sigma)\to G)|} = \frac{|G|}{|\mathrm{Im}(\pi_1(N\setminus\Sigma)\to G)|}$$

Now assume Σ is not a fiber, i.e. $\pi_1(\Sigma) \not\subseteq \pi_1(N \setminus \Sigma)$. Pick $g \in \pi_1(N \setminus \Sigma) \setminus \pi_1(\Sigma)$. Since $\pi_1(N)$ is LERF we can find $\pi_1(N) \to G$ such that $\alpha(g) \notin \alpha(\pi_1(\Sigma))$. Hence

$$|\mathrm{Im}(\pi_1(\Sigma) \to G)| = |\mathrm{Im}(\pi_1(\Sigma) \to \pi_1(N \setminus \Sigma) \to G)|$$
$$< |\mathrm{Im}(\pi_1(N \setminus \Sigma) \to G)|$$

Twisted Alexander polynomials and fibered 3-manifolds

Recall: π is LERF if for any f.g. $A \subset \pi$ and $g \notin A$ there is a map $\alpha : \pi \to G$ to a *finite* group G s.t. $\alpha(g) \notin \alpha(A)$.

Theorem. Let N a 3-mfd and $\phi \in H^1(N)$. Suppose that for any $\alpha : \pi \to G$ we have $\Delta^{\alpha}_{N,\phi}$ is non-zero and $\pi_1(N')$ is LERF for any hyperbolic JSJ component N' then ϕ is a fibered class.

So we proved, given $\alpha : \pi_1(N) \to G$, if $\Delta^{\alpha}_{N,\phi} \neq 0$, then

(∗) $$\frac{|G|}{|\mathrm{Im}(\pi_1(\Sigma) \to G)|} = \frac{|G|}{|\mathrm{Im}(\pi_1(N \setminus \Sigma) \to G)|}$$

Now assume Σ is not a fiber, i.e. $\pi_1(\Sigma) \not\subseteq \pi_1(N \setminus \Sigma)$. Pick $g \in \pi_1(N \setminus \Sigma) \setminus \pi_1(\Sigma)$. Since $\pi_1(N)$ is LERF we can find $\pi_1(N) \to G$ such that $\alpha(g) \notin \alpha(\pi_1(\Sigma))$. Hence

$$|\mathrm{Im}(\pi_1(\Sigma) \to G)| = |\mathrm{Im}(\pi_1(\Sigma) \to \pi_1(N \setminus \Sigma) \to G)| < |\mathrm{Im}(\pi_1(N \setminus \Sigma) \to G)|$$

But this contradicts (*)!

Corollaries

Theorem (if Wise's theorem is correct) Let N be any 3-manifold and $\phi \in H^1(N;\mathbb{Z})$ non-fibered, then there exists $\alpha : \pi \to G$ with $\Delta^{\alpha}_{N,\phi} = 0$.

Corollaries

Theorem (if Wise's theorem is correct) Let N be any 3-manifold and $\phi \in H^1(N;\mathbb{Z})$ non-fibered, then there exists $\alpha : \pi \to G$ with $\Delta^{\alpha}_{N,\phi} = 0$.

Corollary (Silver-Williams) Let N be any 3-manifold and $\phi \in H^1(N;\mathbb{Z}) = \mathrm{Hom}(\pi_1(N),\mathbb{Z})$. If ϕ is non-fibered, then $\mathrm{Ker}(\phi) \in \pi_1(N)$ admits uncountably many finite covers.

Corollaries

Theorem (if Wise's theorem is correct) Let N be any 3-manifold and $\phi \in H^1(N;\mathbb{Z})$ non-fibered, then there exists $\alpha : \pi \to G$ with $\Delta^{\alpha}_{N,\phi} = 0$.

Corollary (Silver-Williams) Let N be any 3-manifold and $\phi \in H^1(N;\mathbb{Z}) = \text{Hom}(\pi_1(N), \mathbb{Z})$. If ϕ is non-fibered, then $\text{Ker}(\phi) \in \pi_1(N)$ admits uncountably many finite covers.

Corollary (F-Vidussi) Let M be an S^1-bundle over a closed 3-manifold N.

Corollaries

Theorem (if Wise's theorem is correct) Let N be any 3-manifold and $\phi \in H^1(N;\mathbb{Z})$ non-fibered, then there exists $\alpha : \pi \to G$ with $\Delta^{\alpha}_{N,\phi} = 0$.

Corollary (Silver-Williams) Let N be any 3-manifold and $\phi \in H^1(N;\mathbb{Z}) = \mathrm{Hom}(\pi_1(N), \mathbb{Z})$. If ϕ is non-fibered, then $\mathrm{Ker}(\phi) \in \pi_1(N)$ admits uncountably many finite covers.

Corollary (F-Vidussi) Let M be an S^1-bundle over a closed 3-manifold N. If M is symplectic, then N fibers over S^1.

Corollaries

Theorem (if Wise's theorem is correct) Let N be any 3-manifold and $\phi \in H^1(N;\mathbb{Z})$ non-fibered, then there exists $\alpha : \pi \to G$ with $\Delta^{\alpha}_{N,\phi} = 0$.

Corollary (Silver-Williams) Let N be any 3-manifold and $\phi \in H^1(N;\mathbb{Z}) = \mathrm{Hom}(\pi_1(N), \mathbb{Z})$. If ϕ is non-fibered, then $\mathrm{Ker}(\phi) \in \pi_1(N)$ admits uncountably many finite covers.

Corollary (F-Vidussi) Let M be an S^1-bundle over a closed 3-manifold N. If M is symplectic, then N fibers over S^1.

Given a knot $K \subset S^3$ denote by $MN(K)$ the Morse-Novikov number (i.e. the minimal number of critical points of an S^1-valued Morse function).

Corollaries

Theorem (if Wise's theorem is correct) Let N be any 3-manifold and $\phi \in H^1(N;\mathbb{Z})$ non-fibered, then there exists $\alpha : \pi \to G$ with $\Delta^{\alpha}_{N,\phi} = 0$.

Corollary (Silver-Williams) Let N be any 3-manifold and $\phi \in H^1(N;\mathbb{Z}) = \mathrm{Hom}(\pi_1(N),\mathbb{Z})$. If ϕ is non-fibered, then $\mathrm{Ker}(\phi) \in \pi_1(N)$ admits uncountably many finite covers.

Corollary (F-Vidussi) Let M be an S^1-bundle over a closed 3-manifold N. If M is symplectic, then N fibers over S^1.

Given a knot $K \subset S^3$ denote by $MN(K)$ the Morse-Novikov number (i.e. the minimal number of critical points of an S^1-valued Morse function). For example, K is fibered if and only if $MN(K) = 0$.

Corollaries

Theorem (if Wise's theorem is correct) Let N be any 3-manifold and $\phi \in H^1(N;\mathbb{Z})$ non-fibered, then there exists $\alpha : \pi \to G$ with $\Delta^{\alpha}_{N,\phi} = 0$.

Corollary (Silver-Williams) Let N be any 3-manifold and $\phi \in H^1(N;\mathbb{Z}) = \mathrm{Hom}(\pi_1(N),\mathbb{Z})$. If ϕ is non-fibered, then $\mathrm{Ker}(\phi) \in \pi_1(N)$ admits uncountably many finite covers.

Corollary (F-Vidussi) Let M be an S^1-bundle over a closed 3-manifold N. If M is symplectic, then N fibers over S^1.

Given a knot $K \subset S^3$ denote by $MN(K)$ the Morse-Novikov number (i.e. the minimal number of critical points of an S^1-valued Morse function). For example, K is fibered if and only if $MN(K) = 0$.

Corollary (?). Let K be a non-fibered knot. Then there exists $\lambda > 0$ such that $MN(nK) > \lambda n$.

Agol's theorem

Recall that a SRAAG is a subgroup of a RAAG, which in turn is defined from a graph.

Agol's theorem

Recall that a SRAAG is a subgroup of a RAAG, which in turn is defined from a graph.

Theorem. (Agol 2006) If N is a 3-manifold such that $\pi_1(N)$ is virtually a SRAAG

Agol's theorem

Recall that a SRAAG is a subgroup of a RAAG, which in turn is defined from a graph.

Theorem. (Agol 2006) If N is a 3-manifold such that $\pi_1(N)$ is virtually a SRAAG (i.e. a finite index subgroup of $\pi_1(N)$ is a SRAAG)

Agol's theorem

Recall that a SRAAG is a subgroup of a RAAG, which in turn is defined from a graph.

Theorem. (Agol 2006) If N is a 3-manifold such that $\pi_1(N)$ is virtually a SRAAG (i.e. a finite index subgroup of $\pi_1(N)$ is a SRAAG) then for any non-fibered $\phi \in H^1(N;\mathbb{Z})$

Agol's theorem

Recall that a SRAAG is a subgroup of a RAAG, which in turn is defined from a graph.

Theorem. (Agol 2006) If N is a 3-manifold such that $\pi_1(N)$ is virtually a SRAAG (i.e. a finite index subgroup of $\pi_1(N)$ is a SRAAG) then for any non-fibered $\phi \in H^1(N;\mathbb{Z})$ there exists a finite cover $p : N' \to N$ such that $p^*(\phi) \in H^1(N';\mathbb{Z})$

Agol's theorem

Recall that a SRAAG is a subgroup of a RAAG, which in turn is defined from a graph.

Theorem. (Agol 2006) If N is a 3-manifold such that $\pi_1(N)$ is virtually a SRAAG (i.e. a finite index subgroup of $\pi_1(N)$ is a SRAAG) then for any non-fibered $\phi \in H^1(N;\mathbb{Z})$ there exists a finite cover $p : N' \to N$ such that $p^*(\phi) \in H^1(N';\mathbb{Z})$ sits on the boundary of a fibered cone.

Agol's theorem

Recall that a SRAAG is a subgroup of a RAAG, which in turn is defined from a graph.

Theorem. (Agol 2006) If N is a 3-manifold such that $\pi_1(N)$ is virtually a SRAAG (i.e. a finite index subgroup of $\pi_1(N)$ is a SRAAG) then for any non-fibered $\phi \in H^1(N;\mathbb{Z})$ there exists a finite cover $p : N' \to N$ such that $p^*(\phi) \in H^1(N';\mathbb{Z})$ sits on the boundary of a fibered cone.

(In particular if $\pi_1(N)$ is virtually a SRAAG, the 3-manifold is virtually fibered.)

Agol's theorem

Recall that a SRAAG is a subgroup of a RAAG, which in turn is defined from a graph.

Theorem. (Agol 2006) If N is a 3-manifold such that $\pi_1(N)$ is virtually a SRAAG (i.e. a finite index subgroup of $\pi_1(N)$ is a SRAAG) then for any non-fibered $\phi \in H^1(N;\mathbb{Z})$ there exists a finite cover $p : N' \to N$ such that $p^*(\phi) \in H^1(N';\mathbb{Z})$ sits on the boundary of a fibered cone.

(In particular if $\pi_1(N)$ is virtually a SRAAG, the 3-manifold is virtually fibered.)

The combination of Wise's theorem (unverified) and Agol's theorem (verified) implies that if N is a Haken hyperbolic 3-manifold, then a finite cover of N fibers over S^1.

Agol's theorem

Recall that a SRAAG is a subgroup of a RAAG, which in turn is defined from a graph.

Theorem. (Agol 2006) If N is a 3-manifold such that $\pi_1(N)$ is virtually a SRAAG (i.e. a finite index subgroup of $\pi_1(N)$ is a SRAAG) then for any non-fibered $\phi \in H^1(N;\mathbb{Z})$ there exists a finite cover $p : N' \to N$ such that $p^*(\phi) \in H^1(N';\mathbb{Z})$ sits on the boundary of a fibered cone.

(In particular if $\pi_1(N)$ is virtually a SRAAG, the 3-manifold is virtually fibered.)

The combination of Wise's theorem (unverified) and Agol's theorem (verified) implies that if N is a Haken hyperbolic 3-manifold, then a finite cover of N fibers over S^1. This a most stunning result.

Agol's theorem

Recall that a SRAAG is a subgroup of a RAAG, which in turn is defined from a graph.

Theorem. (Agol 2006) If N is a 3-manifold such that $\pi_1(N)$ is virtually a SRAAG (i.e. a finite index subgroup of $\pi_1(N)$ is a SRAAG) then for any non-fibered $\phi \in H^1(N;\mathbb{Z})$ there exists a finite cover $p : N' \to N$ such that $p^*(\phi) \in H^1(N';\mathbb{Z})$ sits on the boundary of a fibered cone.

Agol's theorem

Recall that a SRAAG is a subgroup of a RAAG, which in turn is defined from a graph.

Theorem. (Agol 2006) If N is a 3-manifold such that $\pi_1(N)$ is virtually a SRAAG (i.e. a finite index subgroup of $\pi_1(N)$ is a SRAAG) then for any non-fibered $\phi \in H^1(N;\mathbb{Z})$ there exists a finite cover $p : N' \to N$ such that $p^*(\phi) \in H^1(N';\mathbb{Z})$ sits on the boundary of a fibered cone.

Theorem (F-Vidussi 2010) Let N be a hyperbolic 3-manifold and $\phi \in H^1(N;\mathbb{Z})$. Then there exists $\alpha : \pi_1(N) \to U(k)$ such that

$$\deg(\Delta^{\alpha}_{N,\phi}) = kx(\phi) + \text{error term}.$$

Agol's theorem

Recall that a SRAAG is a subgroup of a RAAG, which in turn is defined from a graph.

Theorem. (Agol 2006) If N is a 3-manifold such that $\pi_1(N)$ is virtually a SRAAG (i.e. a finite index subgroup of $\pi_1(N)$ is a SRAAG) then for any non-fibered $\phi \in H^1(N;\mathbb{Z})$ there exists a finite cover $p : N' \to N$ such that $p^*(\phi) \in H^1(N';\mathbb{Z})$ sits on the boundary of a fibered cone.

Theorem (F-Vidussi 2010) Let N be a hyperbolic 3-manifold and $\phi \in H^1(N;\mathbb{Z})$. Then there exists $\alpha : \pi_1(N) \to U(k)$ such that

$$\deg(\Delta^{\alpha}_{N,\phi}) = kx(\phi) + \text{error term.}$$

(In particular TAP detect the knot genus for hyperbolic knots.)

Agol's theorem

Theorem. If $\pi_1(N)$ is virtually a SRAAG then for a non-fibered $\phi \in H^1(N)$ there is $p : N' \to N$ such that $p^*(\phi) \in H^1(N'; \mathbb{Z})$ sits on the bdy of a fibered cone.

Theorem (F-Vidussi 2010) Let N be a hyperbolic 3-manifold and $\phi \in H^1(N; \mathbb{Z})$. Then there exists $\alpha : \pi_1(N) \to U(k)$ such that

$$\deg(\Delta^{\alpha}_{N,\phi}) = kx(\phi) + \text{error term.}$$

Agol's theorem

Theorem. If $\pi_1(N)$ is virtually a SRAAG then for a non-fibered $\phi \in H^1(N)$ there is $p : N' \to N$ such that $p^*(\phi) \in H^1(N';\mathbb{Z})$ sits on the bdy of a fibered cone.

Theorem (F-Vidussi 2010) Let N be a hyperbolic 3-manifold and $\phi \in H^1(N;\mathbb{Z})$. Then there exists $\alpha : \pi_1(N) \to U(k)$ such that

$$\deg(\Delta^{\alpha}_{N,\phi}) = kx(\phi) + \text{error term.}$$

Proof.

Agol's theorem

Theorem. If $\pi_1(N)$ is virtually a SRAAG then for a non-fibered $\phi \in H^1(N)$ there is $p : N' \to N$ such that $p^*(\phi) \in H^1(N'; \mathbb{Z})$ sits on the bdy of a fibered cone.

Theorem (F-Vidussi 2010) Let N be a hyperbolic 3-manifold and $\phi \in H^1(N; \mathbb{Z})$. Then there exists $\alpha : \pi_1(N) \to U(k)$ such that

$$\deg(\Delta^{\alpha}_{N,\phi}) = kx(\phi) + \text{error term}.$$

Proof. Let $\phi \in H^1(N; \mathbb{Z})$ non-fibered.

Agol's theorem

Theorem. If $\pi_1(N)$ is virtually a SRAAG then for a non-fibered $\phi \in H^1(N)$ there is $p : N' \to N$ such that $p^*(\phi) \in H^1(N';\mathbb{Z})$ sits on the bdy of a fibered cone.

Theorem (F-Vidussi 2010) Let N be a hyperbolic 3-manifold and $\phi \in H^1(N;\mathbb{Z})$. Then there exists $\alpha : \pi_1(N) \to U(k)$ such that

$$\deg(\Delta^{\alpha}_{N,\phi}) = kx(\phi) + \text{error term.}$$

Proof. Let $\phi \in H^1(N;\mathbb{Z})$ non-fibered. By Wise and Agol we can find k-fold cover $p : N' \to N$ s.t. $\phi' = p^*(\phi) \in H^1(N';\mathbb{Z})$ sits on the boundary of a fibered cone of N'.

Agol's theorem

Theorem. If $\pi_1(N)$ is virtually a SRAAG then for a non-fibered $\phi \in H^1(N)$ there is $p : N' \to N$ such that $p^*(\phi) \in H^1(N'; \mathbb{Z})$ sits on the bdy of a fibered cone.

Theorem (F-Vidussi 2010) Let N be a hyperbolic 3-manifold and $\phi \in H^1(N; \mathbb{Z})$. Then there exists $\alpha : \pi_1(N) \to U(k)$ such that

$$\deg(\Delta^{\alpha}_{N,\phi}) = kx(\phi) + \text{error term.}$$

Proof. Let $\phi \in H^1(N; \mathbb{Z})$ non-fibered. By Wise and Agol we can find k-fold cover $p : N' \to N$ s.t. $\phi' = p^*(\phi) \in H^1(N'; \mathbb{Z})$ sits on the boundary of a fibered cone of N'. The multivariable Alexander polynomial of N' (Alexander norm) picks up $x(\phi')$.

Agol's theorem

Theorem. If $\pi_1(N)$ is virtually a SRAAG then for a non-fibered $\phi \in H^1(N)$ there is $p : N' \to N$ such that $p^*(\phi) \in H^1(N';\mathbb{Z})$ sits on the bdy of a fibered cone.

Theorem (F-Vidussi 2010) Let N be a hyperbolic 3-manifold and $\phi \in H^1(N;\mathbb{Z})$. Then there exists $\alpha : \pi_1(N) \to U(k)$ such that

$$\deg(\Delta^{\alpha}_{N,\phi}) = kx(\phi) + \text{error term}.$$

Proof. Let $\phi \in H^1(N;\mathbb{Z})$ non-fibered. By Wise and Agol we can find k-fold cover $p : N' \to N$ s.t. $\phi' = p^*(\phi) \in H^1(N';\mathbb{Z})$ sits on the boundary of a fibered cone of N'. The multivariable Alexander polynomial of N' (Alexander norm) picks up $x(\phi')$. Hence $\deg(\Delta^{\alpha'}_{N',\phi'}) = x(\phi') + \ldots$ for some $\alpha' : \pi_1(N') \to U(1)$.

Agol's theorem

Theorem. If $\pi_1(N)$ is virtually a SRAAG then for a non-fibered $\phi \in H^1(N)$ there is $p : N' \to N$ such that $p^*(\phi) \in H^1(N'; \mathbb{Z})$ sits on the bdy of a fibered cone.

Theorem (F-Vidussi 2010) Let N be a hyperbolic 3-manifold and $\phi \in H^1(N; \mathbb{Z})$. Then there exists $\alpha : \pi_1(N) \to U(k)$ such that

$$\deg(\Delta^{\alpha}_{N,\phi}) = kx(\phi) + \text{error term.}$$

Proof. Let $\phi \in H^1(N; \mathbb{Z})$ non-fibered. By Wise and Agol we can find k-fold cover $p : N' \to N$ s.t. $\phi' = p^*(\phi) \in H^1(N'; \mathbb{Z})$ sits on the boundary of a fibered cone of N'. The multivariable Alexander polynomial of N' (Alexander norm) picks up $x(\phi')$. Hence $\deg(\Delta^{\alpha'}_{N',\phi'}) = x(\phi') + \ldots$ for some $\alpha' : \pi_1(N') \to U(1)$. Let $\alpha : \pi_1(N) \to U(k)$ the induced representation,

Agol's theorem

Theorem. If $\pi_1(N)$ is virtually a SRAAG then for a non-fibered $\phi \in H^1(N)$ there is $p : N' \to N$ such that $p^*(\phi) \in H^1(N'; \mathbb{Z})$ sits on the bdy of a fibered cone.

Theorem (F-Vidussi 2010) Let N be a hyperbolic 3-manifold and $\phi \in H^1(N; \mathbb{Z})$. Then there exists $\alpha : \pi_1(N) \to U(k)$ such that

$$\deg(\Delta^{\alpha}_{N,\phi}) = kx(\phi) + \text{error term.}$$

Proof. Let $\phi \in H^1(N; \mathbb{Z})$ non-fibered. By Wise and Agol we can find k-fold cover $p : N' \to N$ s.t. $\phi' = p^*(\phi) \in H^1(N'; \mathbb{Z})$ sits on the boundary of a fibered cone of N'. The multivariable Alexander polynomial of N' (Alexander norm) picks up $x(\phi')$. Hence $\deg(\Delta^{\alpha'}_{N',\phi'}) = x(\phi') + \ldots$ for some $\alpha' : \pi_1(N') \to U(1)$. Let $\alpha : \pi_1(N) \to U(k)$ the induced representation, then $\deg(\Delta^{\alpha}_{N,\phi}) =$

Agol's theorem

Theorem. If $\pi_1(N)$ is virtually a SRAAG then for a non-fibered $\phi \in H^1(N)$ there is $p : N' \to N$ such that $p^*(\phi) \in H^1(N'; \mathbb{Z})$ sits on the bdy of a fibered cone.

Theorem (F-Vidussi 2010) Let N be a hyperbolic 3-manifold and $\phi \in H^1(N; \mathbb{Z})$. Then there exists $\alpha : \pi_1(N) \to U(k)$ such that

$$\deg(\Delta^{\alpha}_{N,\phi}) = kx(\phi) + \text{error term.}$$

Proof. Let $\phi \in H^1(N; \mathbb{Z})$ non-fibered. By Wise and Agol we can find k-fold cover $p : N' \to N$ s.t. $\phi' = p^*(\phi) \in H^1(N'; \mathbb{Z})$ sits on the boundary of a fibered cone of N'. The multivariable Alexander polynomial of N' (Alexander norm) picks up $x(\phi')$. Hence $\deg(\Delta^{\alpha'}_{N',\phi'}) = x(\phi') + \dots$ for some $\alpha' : \pi_1(N') \to U(1)$. Let $\alpha : \pi_1(N) \to U(k)$ the induced representation, then
$\deg(\Delta^{\alpha}_{N,\phi}) = \deg(\Delta^{\alpha'}_{N',\phi'}) =$

Agol's theorem

Theorem. If $\pi_1(N)$ is virtually a SRAAG then for a non-fibered $\phi \in H^1(N)$ there is $p : N' \to N$ such that $p^*(\phi) \in H^1(N'; \mathbb{Z})$ sits on the bdy of a fibered cone.

Theorem (F-Vidussi 2010) Let N be a hyperbolic 3-manifold and $\phi \in H^1(N; \mathbb{Z})$. Then there exists $\alpha : \pi_1(N) \to U(k)$ such that

$$\deg(\Delta^{\alpha}_{N,\phi}) = kx(\phi) + \text{error term}.$$

Proof. Let $\phi \in H^1(N; \mathbb{Z})$ non-fibered. By Wise and Agol we can find k-fold cover $p : N' \to N$ s.t. $\phi' = p^*(\phi) \in H^1(N'; \mathbb{Z})$ sits on the boundary of a fibered cone of N'. The multivariable Alexander polynomial of N' (Alexander norm) picks up $x(\phi')$. Hence $\deg(\Delta^{\alpha'}_{N',\phi'}) = x(\phi') + \ldots$ for some $\alpha' : \pi_1(N') \to U(1)$. Let $\alpha : \pi_1(N) \to U(k)$ the induced representation, then $\deg(\Delta^{\alpha}_{N,\phi}) = \deg(\Delta^{\alpha'}_{N',\phi'}) = x(\phi') + \ldots =$

Agol's theorem

Theorem. If $\pi_1(N)$ is virtually a SRAAG then for a non-fibered $\phi \in H^1(N)$ there is $p : N' \to N$ such that $p^*(\phi) \in H^1(N'; \mathbb{Z})$ sits on the bdy of a fibered cone.

Theorem (F-Vidussi 2010) Let N be a hyperbolic 3-manifold and $\phi \in H^1(N; \mathbb{Z})$. Then there exists $\alpha : \pi_1(N) \to U(k)$ such that

$$\deg(\Delta^{\alpha}_{N,\phi}) = kx(\phi) + \text{error term}.$$

Proof. Let $\phi \in H^1(N; \mathbb{Z})$ non-fibered. By Wise and Agol we can find k-fold cover $p : N' \to N$ s.t. $\phi' = p^*(\phi) \in H^1(N'; \mathbb{Z})$ sits on the boundary of a fibered cone of N'. The multivariable Alexander polynomial of N' (Alexander norm) picks up $x(\phi')$. Hence $\deg(\Delta^{\alpha'}_{N',\phi'}) = x(\phi') + \ldots$ for some $\alpha' : \pi_1(N') \to U(1)$. Let $\alpha : \pi_1(N) \to U(k)$ the induced representation, then $\deg(\Delta^{\alpha}_{N,\phi}) = \deg(\Delta^{\alpha'}_{N',\phi'}) = x(\phi') + \ldots = kx(\phi) + \ldots.$