THE ALGEBRAIC UNKNOTTING NUMBER

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on joint work with Maciej Borodzik.

Caveat: As is customary in talks some of the technical details in the note below might be imprecise or slightly incorrect. For precise statements see the references.

1. Definitions and the statement of the main result

1.1. **The Alexander polynomial.** Let $K \subset S^3$ be a knot. It follows from Alexander duality that $H_1(S^3 \setminus \nu K) \cong \mathbb{Z} = \langle t \rangle$. We can thus consider

$$H_1(\mathbb{Z}\text{–fold cover of } S^3 \setminus \nu K)$$

which is a module over the group ring of $\mathbb{Z} = \langle t \rangle$, i.e. it is a module over $\Lambda := \mathbb{Z}[t^{\pm 1}]$. This module is called the *Alexander module* of K and we will denote it by $H_1(S^3 \setminus \nu K; \Lambda)$.

The Alexander module is finitely presented over Λ, in fact there exists a square $r \times r$–matrix A over Λ such that

$$H_1(S^3 \setminus \nu K; \Lambda) \cong \Lambda^r / A\Lambda^r,$$

and we now define the *Alexander polynomial* of K to be

$$\Delta_K(t) := \det(A).$$

This definition (up to multiplication by a unit in Λ) can be shown to be independent of the presentation matrix A. One can furthermore show that $\Delta_K(t)$ is always non–zero, in fact $\Delta_K(1) = \pm 1$ for any knot K.

Example 1.1. (1) *Let K be the unknot, then $S^3 \setminus \nu K$ is a solid torus, hence its Alexander module is zero, and hence the Alexander polynomial is one.*

(2) *If K is the trefoil, then $\Delta_K(t) = t^2 - t + 1$, more generally, if K is the (p, q)–torus knot, then*

$$\Delta_K(t) = \frac{t^{pq} - 1}{(t^p - 1)(t^q - 1)}.$$

(3) *'Most' knots have non–trivial Alexander polynomial (i.e. not equal to* 1*), but there are non–trivial knots with trivial Alexander polynomial.*

1.2. **The (algebraic) unknotting number.** The *unknotting number* of a knot K is defined as

$$u(K) := \begin{array}{c}\text{the minimal number of crossing changes necessary}\\ \text{to turn } K \text{ into the unknot.}\end{array}$$

The unknotting number is one of the most elementary invariants of a knot, but also one of the most intractable. Whereas upper bounds can be found readily using diagrams, it is much harder to find non–trivial lower bounds.

In this talk we will for the most part study a closely related invariant, namely the *algebraic unknotting number*

$$a(K) := \begin{array}{c}\text{the minimal number of crossing changes necessary}\\ \text{to turn } K \text{ into a knot with trivial Alexander polynomial.}\end{array}$$

By [Fo93] and [Sae99] this is equivalent to the original definition by Murakami [Muk90] given in terms of 'algebraic unknotting moves' on Seifert matrices. In particular upper bounds on $a(K)$ can be obtained from a Seifert matrix alone.

It is clear that $u(K) \geq a(K)$, and that in general this is not an equality. For example for any non–trivial knot K with trivial Alexander polynomial we have $u(K) \geq 1$ and $a(K) = 0$.

1.3. **The Blanchfield pairing.** Let $K \subset S^3$ be a knot. We write $X = X(K) = S^3 \setminus \nu K$, $\Lambda = \mathbb{Z}[t^{\pm 1}]$ and $\Omega = \mathbb{Q}(t)$. We consider

$$\Phi\colon H_1(X;\Lambda) \to H_1(X,\partial X;\Lambda) \to \overline{H^2(X;\Lambda)} \xleftarrow{\cong} \overline{H^1(X;\Omega/\Lambda)} \to \overline{\mathrm{Hom}_\Lambda(H_1(X;\Lambda),\Omega/\Lambda)}.$$

Here the first map is the inclusion induced map, the second map is Poincaré duality, the third map comes from the long exact sequence in cohomology corresponding to the coefficients $0 \to \Lambda \to \Omega \to \Omega/\Lambda \to 0$, and the last map is the evaluation map. All maps are isomorphisms and we thus obtain a non-singular hermitian pairing

$$\begin{array}{rcl}\lambda(K)\colon H_1(X(K);\Lambda) \times H_1(X(K);\Lambda) & \to & \Omega/\Lambda \\ (a,b) & \mapsto & \Phi(a)(b),\end{array}$$

called the *Blanchfield pairing of* K (see also [Bl57]).

Given a hermitian $n \times n$-matrix A over Λ with $\det(A) \neq 0$ we consider the non–singular hermitian pairing

$$\begin{array}{rcl}\lambda(A)\colon \Lambda^n/A\Lambda^n \times \Lambda^n/A\Lambda^n & \to & \Omega/\Lambda \\ (a,b) & \mapsto & \overline{a}^T A^{-1} b.\end{array}$$

We define

$$n(K) := \begin{array}{l}\text{the minimal size of a hermitian matrix } A = A(t) \text{ over } \Lambda \text{ such that}\\ \quad (1) \quad \lambda(A) \cong \lambda(K), \text{ and}\\ \quad (2) \quad A(1) \text{ is congruent over } \mathbb{Z} \text{ to a diagonal matrix.}\end{array}$$

It remains to verify that this definition makes sense, i.e. that such a matrix always exists. Let V be any matrix of size $2k$ which is S–equivalent to a Seifert matrix for K. Note that $V - V^t$ is antisymmetric and it satisfies $\det(V - V^t) = (-1)^k$. After a base change we can thus arrange that

$$V - V^t = \begin{pmatrix} 0 & \mathrm{id}_k \\ -\mathrm{id}_k & 0 \end{pmatrix}.$$

Now consider $A_K(t)$ which is defined as

$$\begin{pmatrix} (1-t^{-1})^{-1}\mathrm{id}_k & 0 \\ 0 & \mathrm{id}_k \end{pmatrix} V \begin{pmatrix} \mathrm{id}_k & 0 \\ 0 & (1-t)\mathrm{id}_k \end{pmatrix} + \begin{pmatrix} \mathrm{id}_k & 0 \\ 0 & (1-t^{-1})\mathrm{id}_k \end{pmatrix} V^t \begin{pmatrix} (1-t)^{-1}\mathrm{id}_k & 0 \\ 0 & \mathrm{id}_k \end{pmatrix}.$$

Then using work of Kearton [Ke75] we can show that $\lambda(A_K(t)) \cong \lambda(K)$.

We consider $A(t) = A_K(t) \oplus (1)$, then $A(1)$ represents an indefinite, odd symmetric bilinear pairing over $\mathbb{Z}$, hence it is diagonalizable. We thus showed that $n(K)$ is defined and that

$$n(K) \leq \deg \Delta_K(t) + 1.$$

1.4. **The main results.** Our main result in [BF12b] states that $n(K)$ gives a lower bound on the algebraic unknotting number.

Theorem 1.2. (F–Borodzik) *For any knot K we have*

$$n(K) \leq a(K).$$

Furthermore $n(K)$ is to the best of our knowledge the optimal 'classical' lower bound on the unknotting number. (Here by a 'classical' invariant we mean an invariant determined by the Seifert matrix.) More precisely in [BF12b] we will show that $n(K)$ subsumes the following classical lower bounds on the unknotting number:

(1) the Levine–Tristram signatures [Tr69, Lev69, Mus65],
(2) the Nakanishi index [Na81] (i.e. the minimal number of generators of the Alexander module),
(3) the Lickorish obstruction [Lic85, CL86] to $u(K) = 1$ in terms of the linking pairing on the 2–fold branched cover,
(4) the Jabuka obstruction [Ja09] to $u(K) = 1$,
(5) the Livingston [Liv11] invariant which gives a lower bound on the topological 4–genus and hence on the algebraic unknotting number,
(6) the Stoimenow obstruction [St04] to $u(K) = 2$.

We conclude this section with a few remarks:

(1) Note that for some of the above it was only known that they give lower bounds on the ordinary unknotting number, our result shows that they all give lower bounds on the algebraic unknotting number.
(2) There is no algorithm for computing $n(K)$, in particular the above invariants can be seen as approaches to calculating $n(K)$.

(3) We conjecture that in general $n(K) = a(K)$. For $n(K) = 1$ this was shown by Fogel [Fo93, Fo94].
(4) In order to detect the difference between the unknotting number and the algebraic unknotting number one needs deeper invariants, e.g. gauge theory [CL86, KM93], Khovanov homology [Ras10] and Heegaard-Floer homology [OS03, OS05, Ow08]. Note though that these invariants are often very hard to calculate in practice.

1.5. **The linking pairing and examples.** If $A(t)$ is a matrix over Λ which represents the Blanchfield pairing, then the integral matrix $A(-1)$ represents the linking pairing

$$l(K) : H_1(\Sigma_2(K)) \times H_1(\Sigma_2(K)) \to \mathbb{Q}/\mathbb{Z}$$

where $\Sigma_2(K)$ denotes the 2–fold branched cover of K. The main theorem above then implies the following:

Theorem 1.3. *If $n(K) = n$, then there exists a symmetric $n \times n$–matrix A over $\mathbb{Z}$ which has the following three properties:*

(1) $|\det(A)| = |\Delta_K(-1)|$,
(2) $l(A) \cong 2l(K)$,
(3) *A modulo two equals the identity matrix.*

If sign$(K) = 2n \cdot \epsilon$ with $\epsilon \in \{-1, 1\}$, then we can furthermore arrange that A has the following two properties:

(4) *A is ϵ–definite,*
(5) *the diagonal entries of A modulo four are equal to $-\epsilon$.*

Remark. Theorem 1.3 is closely related to [Ow08, Theorem 3].

This result gives computable obstructions to $n(K)$ having a given value. The resulting $n(K) = 1$ obstruction is precisely the Lickorish obstruction, but the obstructions to $n(K) = 2$, $n(K) = 3$ etc. are new. We wrote a computer program 'knotorious' and we found that among all knots with up to 12 crossings there exist 21 knots for which we can use this method to show that $n(K) \geq 3$, where all previous classical invariants were inconclusive.

In particular we can now determine the algebraic unknotting number for all knots up to 11 crossings and we can determine the algebraic unknotting number for all but 19 knots with 12 crossings.

An example of a knot we can not deal with is the knot $K = 12a_{50}$. We know that either $n(K) = 1$ or $n(K) = 2$. Its Blanchfield pairing is isometric to

$$\begin{array}{rcl} \Lambda/p \times \Lambda/p & \to & \Omega/\Lambda \\ (v, w) & \mapsto & \frac{1}{p}\overline{v}qw, \end{array}$$

where

$$\begin{array}{rcl} p & = & \Delta_K(t) = -8t^3 + 20t^2 - 30t + 3 - 30t^{-1} + 20t^{-2} - 8t^{-3}, \\ q & = & -t^3 + 7t^2 - 13t + 17 - 13t^{-1} + 7t^{-2} - t^{-3}. \end{array}$$

The knot has $n(K) = 1$ if and only if there exists an automorphism of Λ/p (as a Λ-module), which transforms this pairing into a pairing of the form $(v, w) \mapsto \pm\overline{v}w/p$. Put differently, we have $n(K) = 1$ if and only if there exists an $f \in \Lambda$ such that $qf\overline{f} = \pm 1 \pmod{p}$.

A full discussion of the examples is given on Maciej Borodzik's webpage [BF12a].

2. Proof that $n(K) \leq u(K)$

In order to simplify the discussion we will show that given a knot K we have

$$n(K) \leq u(K).$$

We start out with a preliminary discussion of the effect of a crossing change on 0–framed surgeries.

Given a knot K we denote by $N(K)$ its zero–framed surgery. If K' is obtained from K by a crossing change, then we can obtain $N(K')$ from $N(K)$ through ± 1–surgery on one of the two curves 'circling' the crossing change. Put differently, adding a handle to $N(K)$ along such a curve with framing ± 1, we obtain a cobordism W between $N(K)$ and $N(K')$.

Note that one of the above two curves is null–homologous, if we use this curve, then $H_1(N(K)) \to H_1(W)$ and $H_1(N(J)) \to H_1(W)$ are isomorphisms.

Now let K be a knot which can be turned into the trivial knot J using n crossing changes. We can then also turn the unknot J into K using n crossing changes. By applying the above discussion to the n crossing changes we obtain a 4–manifold W with the following properties:

(1) $\partial W = N(K) \cup N(J) = S^1 \times S^2$,
(2) $\mathbb{Z} = \pi_1(N(J)) \to \pi_1(W)$ is surjective and it induces an isomorphism on first homology, hence $\pi_1(W) = \mathbb{Z}$,
(3) $H_1(N(K)) \to H_1(W)$ is an isomorphism,
(4) $b_2(W) = n$,
(5) the intersection pairing on $H_2(W)$ is diagonalizable.

One can now show that

$$H_2(W; \Lambda) \cong \Lambda^n,$$

and that any matrix $A(t)$ over Λ representing the equivariant intersection form

$$H_2(W; \Lambda) \times H_2(W; \Lambda) \to \Lambda$$

is a presentation matrix for the Blanchfield pairing. The matrix $A(1)$ represents the ordinary intersection pairing on $H_2(W)$, which by the above is diagonalizable. This concludes the proof of the inequality $n(K) \leq u(K)$.

3. Questions

We conclude with a few questions and problems related to $n(K)$.

(1) We showed that $n(K) \le \deg \Delta_K(t)+1$. However, we do not know of a single knot, where this is an equality. So one can ask, whether in general $n(K) \le \deg \Delta_K(t)$ holds.
(2) Given any knot K, do we have the following equality

$$n(K) = \begin{array}{c}\text{minimal size of a hermitian}\\ \text{matrix } A \text{ over } \Lambda \text{ with } \lambda(A) \cong \lambda(K)\end{array} \quad ?$$

Put differently, is the condition in the definition of $n(K)$ that $A(1)$ be diagonal over $\mathbb{Z}$ necessary?
(3) How can we show that $K = 12a_{50}$ satisfies $n(K) > 1$? The Levine–Tristram signatures and the obstruction from the linking form on the 2–fold branched cover are inconclusive. We looked at Blanchfield/linking forms on higher cyclic covers, but to our chagrin these did not give us extra information for any knot so far.
(4) Is $n(K)$ invariant under mutation? It is an open question whether the unknotting number is preserved under mutation. The S–equivalence class of a Seifert matrix (and thus the isometry type of the Blanchfield pairing) is preserved under positive mutation. On the other hand the S–equivalence class (in fact the isomorphism class of the Alexander module) is in general not preserved under negative mutation (see [Ke89]).
(5) The lower bounds on the unknotting number coming from Heegaard–Floer homology have so far been extracted only from the 2–fold branched cyclic cover of K (see [OS05, Ow08]). Can higher cyclic covers give further information?

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