DETERMINANTS OF AMPHICHIRAL KNOTS

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Abstract. We give a simple obstruction for a knot to be amphichiral, in terms of its determinant. We work with unoriented knots, and so obstruct both positive and negative amphichirality.

1. Introduction

By a knot we mean a 1-dimensional submanifold of S^3 that is diffeomorphic to S^1. Given a knot K we denote its *mirror image* by mK, the image of K under an orientation reversing homeomorphism $S^3 \to S^3$. We say that a knot K is *amphichiral* if K is (smoothly) isotopic to mK. Note that we consider unoriented knots, so we do not distinguish between positive and negative amphichiral knots.

Arguably the simplest invariant of a knot K is the determinant $\det(K) \in \mathbb{Z}$ which can be introduced in many different ways. The definition that we will work with is that $\det(K)$ is the order of the first homology of the 2-fold cover $\Sigma(K)$ of S^3 branched along K. Alternative definitions are given by $\det(K) = \Delta_K(-1) = J_K(-1)$ where $\Delta_K(t)$ denotes the Alexander polynomial and $J_K(q)$ denotes the Jones polynomial [Li97, Corollary 9.2], [Ka96, Theorem 8.4.2]. We also recall that for a knot K with Seifert matrix A, a presentation matrix for $H_1(\Sigma(K))$ is given by $A + A^T$. In particular one can readily compute the determinant via $\det(K) = \det(A + A^T)$.

The following proposition gives an elementary obstruction for a knot to be amphichiral.

Proposition 1.1. *Suppose K is an amphichiral knot and p is a prime with $p \equiv 3 \mod 4$. Then either p does not divide $\det(K)$ or p^2 divides $\det(K)$.*

The proof uses the linking form on the 2-fold branched cover of K, which will be recalled in Section 2. As we will now discuss, this simple proposition is surprisingly useful at showing that certain knots are not amphichiral.

Example.

(1) For the trefoil 3_1 we have $\det(3_1) = 3$, so Proposition 1.1 immediately implies the very well known fact that the trefoil is not amphichiral. Usually one uses either the signature of a knot or the Jones polynomial to prove that the trefoil is not amphichiral. It came as a surprise to the authors that one can even use the

Date: June 20, 2017.
2010 *Mathematics Subject Classification.* 57M25, 57M27,

determinant to obtain this result. Of course the linking form that is used in the proof of Proposition 1.1 uses similar Poincaré duality information to the signature.

(2) As a reality check, for the figure eight knot 4_1, which is well known to be amphichiral [BZH14, p. 17], we have $\det(4_1) = 5$, which is consistent with Proposition 1.1.

(3) Our condition also provides information on occasions when the signature fails. The chirality of the knots $K = 9_{42}$ and $K = 10_{71}$ is more difficult to detect, since the Tristram-Levine signature function [Le69, Tr69] of K is identically zero and the HOMFLY and Kauffman polynomials of K do not detect chirality. In fact in [RGK94] Chern-Simons invariants were used to show that these knots are not amphichiral. However, a quick look at Knotinfo [CL] shows that $\det(9_{42}) = 7$ and $\det(10_{71}) = 77 = 7 \cdot 11$. Thus both knots fail to satisfy the criterion from Proposition 1.1 and so we see that these two knots are not amphichiral.

(4) On the other hand it is also quite easy to find a knot for which Proposition 1.1, and also Theorem 1.2 below, fail to show that it is not amphichiral. For example, the torus knot $T(5,2) = 5_1$ has determinant $\det(5_1) = 5$, but is not amphichiral since its signature is nonzero.

As we will see, Proposition 1.1 is a straightforward consequence of the slightly more technical Theorem 1.2 below. Before we state that theorem, recall that given an abelian additive group G and a prime p, the *p-primary part of G* is defined as

$$G_p := \{g \in G \,|\, p^k \cdot g = 0 \text{ for some } k \in \mathbb{N}_0\}.$$

The following theorem is the main result of this article.

Theorem 1.2. *Let K be a knot and let p be a prime with $p \equiv 3 \mod 4$. If K is amphichiral, then the p-primary part of $H_1(\Sigma(K))$ is either zero or it is not cyclic.*

Example. Consider the Stevedore's knot 6_1. A Seifert matrix is given by $A = \begin{pmatrix} 1 & 0 \\ 1 & -2 \end{pmatrix}$. By the aforementioned formula a presentation matrix for $H_1(\Sigma(K))$ is given by $A + A^T$. It is straightforward to compute that $H_1(\Sigma(K)) \cong \mathbb{Z}_9$. So it follows from Theorem 1.2, applied with $p = 3$, that the Stevedore's knot is not amphichiral.

Remark.

(1) There are various results in the literature regarding Alexander polynomials of knots that satisfy stronger versions of amphichirality.
   (a) Hartley-Kawauchi [HK79, Theorem 1] have shown that if an oriented knot K is strongly negative amphichiral, that is if there exists an *involution* h of S^3 such that $h(K)$ is the same as mK but with orientation reversed, then $\Delta_K(t^2) = F(t) \cdot F(-t)$ for some integral polynomial $F(t)$ with $F(-t) = t^k F(t^{-1})$.
   (b) If K is an oriented knot that is negative amphichiral, i.e. if there exists an isotopy from K to mK with orientation reversed, then Corey-Michel [CM83, Proposition 1] (see also [Hi12, Theorem 9.3]) gave a condition on the Alexander polynomial.

(c) Hillman [Hi12, Theorem 9.4] states an analogue of [CM83, Proposition 1] for *positive* amphichiral knots.
(d) Finally, a condition on the Alexander polynomial of a positive amphichiral knot was given by Hartley in [Ha80].

The idea of the proof of Proposition 1.1 is similar to the proofs of [CM83, Proposition 1], [Hi12, Theorem 9.3 and 9.4], but these latter results use the Blanchfield form instead of the linking form on the 2-fold branched cover. It does not seem to be straightforward to deduce Proposition 1.1 from the statements of the previous results. We could not find a result in the literature that implies Theorem 1.2, essentially because the previous results on Alexander polynomials mentioned above did not consider the structure of the Alexander module. So to the best of our knowledge, and to our surprise, Theorem 1.2 seems to be new. Notwithstanding, the selling point of our theorem is not that it yields any new information on amphichirality, but that the obstruction is very fast to compute and frequently effective.

(2) As mentioned above, the proof of Theorem 1.2 relies on the study of the linking form on the 2-fold branched cover $\Sigma(K)$. Similarly, as in [Hi12, Theorem 9.3], one can use the Blanchfield form [Bl57] to obtain restrictions on the Alexander polynomial. Moreover one can use twisted Blanchfield forms [Po16] to obtain conditions on twisted Alexander polynomials [Wa94, FV10] of amphichiral knots. Our initial idea had been to use the latter invariant. But we quickly found that even the elementary invariants studied in this paper are fairly successful. To keep the paper short we refrain from discussing these generalisations.

The paper is organised as follows. Linking forms and basic facts about them are recalled in Section 2. The proofs of Theorem 1.2 and Proposition 1.1 are given in Section 3.

Acknowledgments. The authors are grateful to the Hausdorff Institute for Mathematics in Bonn, in whose excellent research atmosphere some of the research of this paper was developed. We are also grateful to Jae Choon Cha and Chuck Livingston [CL] for providing Knotinfo, which is an indispensable tool for studying small crossing knots.

The first author acknowledges the support provided by the SFB 1085 'Higher Invariants' at the University of Regensburg, funded by the DFG. The third author is supported by an NSERC Discovery Grant.

2. Linking forms

Definition.

(i) A *linking form* on a finitely generated abelian group H is a map $\lambda\colon H \times H \to \mathbb{Q}/\mathbb{Z}$ which has the following properties:
  (a) λ is bilinear and symmetric,
  (b) λ is nonsingular, that is the adjoint map $H \to \operatorname{Hom}(H, \mathbb{Q}/\mathbb{Z})$ given by $a \mapsto (b \mapsto \lambda(a, b))$ is an isomorphism.

(ii) Given a linking form $\lambda\colon H \times H \to \mathbb{Q}/\mathbb{Z}$ we denote the linking form on H given by $(-\lambda)(a,b) = -\lambda(a,b)$ by $-\lambda$.

Lemma 2.1. *Let p be a prime and $n \in \mathbb{N}_0$. Every linking form λ on $\mathbb{Z}_{p^n}$ is given by*

$$\begin{array}{rcl} \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^n} & \to & \mathbb{Q}/\mathbb{Z} \\ (a,b) & \mapsto & \lambda(a,b) = \frac{k}{p^n} a \cdot b \in \mathbb{Q}/\mathbb{Z} \end{array}$$

for some $k \in \mathbb{Z}$ that is coprime to p.

Proof. Pick $k \in \mathbb{Z}$ such that $\frac{k}{p^n} = \lambda(1,1) \in \mathbb{Q}/\mathbb{Z}$. By the bilinearity we have $\lambda(a,b) = \frac{k}{p^n} a \cdot b \in \mathbb{Q}/\mathbb{Z}$ for all $a, b \in \mathbb{Z}_{p^n}$. It follows easily from the fact that λ is nonsingular that k needs to be coprime to p. To wit, if $p|k$ then $k = p \cdot k'$, so for any $a \in p^{n-1}\mathbb{Z}_{p^n}$ and $b \in \mathbb{Z}_{p^n}$ we have $\lambda(a,b) = k'apb/p^{n-1} = 0 \in \mathbb{Q}/\mathbb{Z}$. Therefore the non-trivial subgroup $p^{n-1}\mathbb{Z}_{p^n}$ lies in the kernel of the adjoint map, so the adjoint map is not injective. $\square$

We recall the following well known lemma.

Lemma 2.2. *Let $\lambda\colon H \times H \to \mathbb{Q}/\mathbb{Z}$ be a linking form and let p be a prime. The restriction of λ to the p-primary part H_p of H is also nonsingular.*

Proof. It suffices to show that there exists an orthogonal decomposition $H = H_p \oplus H'$. Since H is the direct sum of its p-primary subgroups we only need to show that if p, q are two different primes and if $a \in H_p$ and $b \in H_q$, then $\lambda(a,b) = 0$. So let p and q be two distinct primes. Since p and q are coprime there exist $x, y \in \mathbb{Z}$ with $px + qy = 1$. It follows that $\lambda(a,b) = \lambda((px+qy)a, b) = \lambda(pxa, b) + \lambda(a, qyb) = 0$. $\square$

Let Σ be an oriented rational homology 3-sphere, i.e. Σ is a 3-manifold with $H_*(\Sigma;\mathbb{Q}) \cong H_*(S^3;\mathbb{Q})$. Consider the maps

$$H_1(\Sigma;\mathbb{Z}) \xrightarrow{\mathrm{PD}^{-1}} H^2(\Sigma;\mathbb{Z}) \xleftarrow{\delta} H^1(\Sigma;\mathbb{Q}/\mathbb{Z}) \xrightarrow{\mathrm{ev}} \mathrm{Hom}(H_1(\Sigma;\mathbb{Z}),\mathbb{Q}/\mathbb{Z}),$$

where the maps are given as follows:

(1) the first map is given by the inverse of Poincaré duality, that is the inverse of the map given by capping with the fundamental class of the *oriented* manifold Σ;
(2) the second map is the connecting homomorphism in the long exact sequence in cohomology corresponding to the short exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ of coefficients; and
(3) the third map is the evaluation map.

The first map is an isomorphism by Poincaré duality, the second map is an isomorphism since Σ is a rational homology sphere and so $H^i(\Sigma;\mathbb{Q}) = H_{3-i}(\Sigma;\mathbb{Q}) = 0$ for $i = 1, 2$, and the third map is an isomorphism by the universal coefficient theorem, and the fact that $\mathbb{Q}/\mathbb{Z}$ is an injective $\mathbb{Z}$-module. Denote the corresponding isomorphism by $\Phi_\Sigma\colon H_1(\Sigma;\mathbb{Z}) \to \mathrm{Hom}(H_1(\Sigma;\mathbb{Z}),\mathbb{Q}/\mathbb{Z})$ and define

$$\begin{array}{rcl} \lambda_\Sigma\colon H_1(\Sigma;\mathbb{Z}) \times H_1(\Sigma;\mathbb{Z}) & \to & \mathbb{Q}/\mathbb{Z} \\ (a,b) & \mapsto & (\Phi_\Sigma(a))(b). \end{array}$$

Lemma 2.3. *For every oriented rational homology 3-sphere Σ, the map λ_Σ is a linking form.*

Proof. We already explained why Φ_Σ is an isomorphism, which is equivalent to the statement that λ_Σ is nonsingular. We refer to [Po16] or alternatively to [Fr17, Chapter 48.3] for a proof that λ_Σ is symmetric. □

The following lemma is an immediate consequence of the definitions and the obvious fact that for an oriented manifold M we have $[-M] = -[M]$.

Lemma 2.4. *Let Σ be an oriented rational homology 3-sphere. We denote the same manifold but with the opposite orientation by $-\Sigma$. For any $a, b \in H_1(\Sigma;\mathbb{Z}) = H_1(-\Sigma;\mathbb{Z})$ we have*

$$\lambda_{-\Sigma}(a,b) \;=\; -\lambda_\Sigma(a,b).$$

Let $K \subset S^3$ be a knot and let $\Sigma(K)$ be the 2-fold cover of S^3 branched along K. Note that $\Sigma(K)$ admits a unique orientation such that the projection $p\colon \Sigma(K) \to S^3$ is orientation-preserving outside of the branch locus $p^{-1}(K)$. Henceforth we will always view $\Sigma(K)$ as an oriented manifold.

Lemma 2.5.

(i) Let K and J be two knots. If K and J are (smoothly) isotopic, then there exists an orientation-preserving diffeomorphism between $\Sigma(K)$ and $\Sigma(J)$.

(ii) Let K be a knot. There exists an orientation-reversing diffeomorphism $\Sigma(K) \to \Sigma(mK)$.

Proof. The first statement follows immediately from the isotopy extension theorem [Ko93, Theorem II.5.2]. The second statement is an immediate consequence of the definitions. □

3. Proofs

3.1. **Proof of Theorem 1.2.** Let K be a knot and let p be a prime. By Lemma 2.5 (ii) there exists an orientation-preserving diffeomorphism $f\colon \Sigma(K) \to -\Sigma(mK)$ which means that f induces an isometry from $\lambda_{\Sigma(K)}$ to $\lambda_{-\Sigma(mK)}$. It follows from Lemma 2.4 that f induces an isometry from the linking form $\lambda_{\Sigma(K)}$ to $-\lambda_{\Sigma(mK)}$. In particular f induces an isomorphism $H_1(\Sigma(K);\mathbb{Z})_p \to H_1(\Sigma(mK);\mathbb{Z})_p$ between the p-primary parts of the underlying abelian groups.

Now suppose that K is amphichiral, meaning that mK is isotopic to K. Write $H = H_1(\Sigma(K))$, denote the p-primary part of H by H_p, and let $\lambda_p\colon H_p \times H_p \to \mathbb{Q}/\mathbb{Z}$ be the restriction of the linking form $\lambda_{\Sigma(K)}$ to H_p. It follows from Lemma 2.2 that λ_p is also a linking form. Then by Lemma 2.5 (ii) and the above discussion, there exists an isometry

$$\Phi\colon (H_p, -\lambda_p) \xrightarrow{\cong} (H_p, \lambda_p).$$

Now suppose that H_p is cyclic and nonzero, so that we can make the identification $H_p = \mathbb{Z}_{p^n}$ for some $n \in \mathbb{N}$. By Lemma 2.1, there exists a $k \in \mathbb{Z}$, coprime to p, such that $\lambda_p(a,b) = \frac{k}{p}ab \in \mathbb{Q}/\mathbb{Z}$ for all $a, b \in \mathbb{Z}_{p^n}$.

The isomorphism $\Phi\colon \mathbb{Z}_{p^n} \to \mathbb{Z}_{p^n}$ is given by multiplication by some $r \in \mathbb{Z}$ that is coprime to p. We have that

$$-\frac{k}{p^n} \;=\; (-\lambda_p)(1,1) \;=\; \lambda_p(r\cdot 1, r\cdot 1) \;=\; \frac{k}{p^n}r^2 \;\in\; \mathbb{Q}/\mathbb{Z}.$$

Thus there exists $m \in \mathbb{Z}$ such that $-\frac{k}{p^n} = \frac{k}{p^n}r^2 + m \in \mathbb{Z}$, so $-k = kr^2 + p^n m$. Working modulo p we obtain $-k \equiv kr^2 \mod p$. Since k is coprime to p, it follows that $-1 \equiv r^2 \mod p$.

But it is a well known fact from classical number theory, see e.g. [Co09, p. 133], that for an odd prime p the number -1 is a square mod p if and only if $p \equiv 1 \mod 4$. Thus we have shown that for an amphichiral knot, and a prime p such that H_p is nontrivial and cyclic, we have that $p \equiv 1 \mod 4$. This concludes the proof of (the contrapositive of) Theorem 1.2.

3.2. **The proof of Proposition 1.1.** Let K be an amphichiral knot. Recall that by definition of the determinant we have $\det(K) = |H_1(\Sigma(K))|$. Now let p be a prime with $p \equiv 3 \mod 4$ that divides $\det(K)$. We have to show that p^2 divides $\det(K)$. Denote the p-primary part of $H_1(\Sigma(K))$ by H_p. Since p divides $\det(K)$, we see that H_p is nonzero. By Theorem 1.2, we know that H_p is not cyclic. But this implies that p^2 divides the order of H_p, which in turn implies that p^2 divides $\det(K)$. This concludes the proof of Proposition 1.1.

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