

REPRESENTATION VARIETIES DETECT ESSENTIAL SURFACES

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ABSTRACT. Extending Culler-Shalen theory, Hara and the second author presented a way to construct certain kinds of branched surfaces in a 3-manifold from an ideal point of a curve in the SL_n -character variety. There exists an essential surface in some 3-manifold known to be not detected in the classical SL_2 -theory. We prove that every connected essential surface in a 3-manifold is given by an ideal point of a rational curve in the SL_n -character variety for some n .

1. INTRODUCTION

In this paper we study an extension of Culler-Shalen theory for higher-dimensional representations. In their seminal work [CS] Culler and Shalen established a method to construct essential surfaces in a 3-manifold from an ideal point of a curve in the $SL_2(\mathbb{C})$ -character variety. The method is built on a beautiful combination of the theory of incompressible surfaces in a 3-manifold, the geometry of representation varieties, and Bass-Serre theory [Se1, Se2]. We refer the reader to the exposition [Sh] for literature and related topics on Culler-Shalen theory. Hara and the second author presented an analogous extension of the Culler-Shalen method to the case of higher-dimensional representations [HK]. They showed that certain kinds of branched surfaces (possibly without any branched points) are constructed from an ideal point of a curve in the $SL_n(\mathbb{C})$ -character variety for a general n . Such a branched surface corresponds to a nontrivial splitting of the 3-manifold group as a complex of groups [C, Ha].

The classical theory for 2-dimensional representations is not sufficient to detect all essential surfaces in Haken manifolds. Throughout the paper let M be a compact connected orientable 3-manifold. We denote by $X_n(M)$ the $SL_n(\mathbb{C})$ -character variety of $\pi_1 M$. It was discovered by Boyer and Zhang [BZ], and Motegi [Mo] that there exist infinitely many Haken manifolds M , which are even hyperbolic, such that $X_2(M)$ has no irreducible component of positive dimension. See also [SZ] for further study on the topic. We say that *an essential surface S in M is given by an ideal point χ of a curve in $X_n(M)$* if S is constructed from χ by the Culler-Shalen method or its extension developed in [HK] as described in Subsection 2.3. Hara and the second author formulated and raised the following question [HK, Question 6.1].

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Question 1.1. *Does there exist an essential surface in some 3-manifold M not given by any ideal point of curves in $X_2(M)$ but given by an ideal point of a curve in $X_n(M)$ for some n ?*

The aim of this paper is to show that the extension of Culler-Shalen theory to the case of higher-dimensional representations [HK] detects all essential surfaces in Haken manifolds. The following is the main theorem of this paper, which, in particular, gives an affirmative answer to Question 1.1.

Theorem 1.2. *Every connected essential surface in M is given by an ideal point of a rational curve in $X_n(M)$ for some n .*

The proof of Theorem 1.2 relies on the breakthroughs of Agol [A] and Wise [W], and the subsequent works of Przytycki and Wise [PW1, PW2] on the separability of subgroups in a 3-manifold group. For a given connected essential surface S in M there exists a non-separating lift T of S in some finite cover N of M by the separability of $\pi_1 S$ in $\pi_1 M$. The non-separating surface T defines abelian representations $\pi_1 N \rightarrow \mathrm{SL}_2(\mathbb{C})$ parameterized in \mathbb{C}^\times , which induces an affine curve D_T consisting of representations $\pi_1 M \rightarrow \mathrm{SL}_n(\mathbb{C})$ where n is twice the degree of the cover N . The set of characters of representations in D_T is a desired rational curve in $X_n(M)$ as in the statement of Theorem 1.2, which has a unique ideal point. Then analyzing the structure of the Bruhat-Tits building associated to the function field of D_T , we explicitly construct a PL-map from the universal cover of M to the 1-skeleton of the building. Finally, we show that the inverse image of midpoints of edges by the PL-map is isotopic to parallel copies of S .

The paper is organized as follows. In Section 2 we review the extension of the Culler-Shalen method to the case of $\mathrm{SL}_n(\mathbb{C})$ -representations in [HK]. Here we recall some of the standard facts on $\mathrm{SL}_n(\mathbb{C})$ -character varieties and Bruhat-Tits buildings associated to the special linear group. In Subsection 2.3, we give the precise definition of the sentence ‘*an essential surface is given by an ideal point*’. Section 3 provides a brief exposition on the separability of surface subgroups by Przytycki and Wise [PW2]. Section 4 is devoted to the proof of Theorem 1.2.

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2. SL_n -CULLER SHALEN THEORY

We begin with an overview of the extension of Culler-Shalen theory to the case of higher-dimensional representations in [HK].

2.1. Character varieties. We briefly review the $\mathrm{SL}_n(\mathbb{C})$ -character variety of a finitely generated group. See [LM, Si1, Si2] for more details.

Let π be a finitely generated group. We define the following affine algebraic set

$$R_n(\pi) = \mathrm{Hom}(\pi, \mathrm{SL}_n(\mathbb{C})).$$

The algebraic group $\mathrm{SL}_n(\mathbb{C})$ acts on the affine algebraic set $R_n(\pi)$ by conjugation. We denote by $X_n(\pi)$ the GIT quotient of the action [MFK]:

$$X_n(\pi) = \mathrm{Hom}(\pi, \mathrm{SL}_n(\mathbb{C})) // \mathrm{SL}_n(\mathbb{C}).$$

The affine algebraic set $X_n(\pi)$ is called the $\mathrm{SL}_n(\mathbb{C})$ -character variety of π . By definition the coordinate ring $\mathbb{C}[X_n(\pi)]$ is isomorphic to the subring $\mathbb{C}[R_n(\pi)]^{\mathrm{SL}_n(\mathbb{C})}$ of $\mathbb{C}[R_n(\pi)]$ consisting of $\mathrm{SL}_n(\mathbb{C})$ -invariant functions. Procesi [P, Theorem 1.3] showed that $\mathbb{C}[R_n(\mathbb{C})]^{\mathrm{SL}_n(\mathbb{C})}$ is generated by *trace functions* I_γ for $\gamma \in \pi$ defined by

$$I_\gamma(\rho) = \mathrm{tr} \rho(\gamma)$$

for $\rho \in R_n(\pi)$. Therefore $X_n(\mathbb{C})$ is identified with the set of *characters* χ_ρ for $\rho \in R_n(\pi)$ defined by

$$\chi_\rho(\gamma) = \mathrm{tr} \rho(\gamma)$$

for $\gamma \in \pi$. For a compact connected orientable 3-manifold M , we abbreviate $R_n(\pi_1 M)$ and $X_n(\pi_1 M)$ with $R_n(M)$ and $X_n(M)$ respectively to simplify notation.

Let C be an affine variety, and denote by $\mathbb{C}(C)$ its field of rational functions. We call C an *affine curve* if the transcendence degree of $\mathbb{C}(C)$ over \mathbb{C} equals 1 [F, Section 6.5]. Consider an affine curve C and its projectivisation \overline{C} . The projective curve \overline{C} might not be smooth, but it has a unique smooth model, i.e., there is a smooth projective curve \tilde{C} together with a birational map $\tilde{C} \dashrightarrow \overline{C}$ which is universal [F, Theorem 7.3]. Recall that a birational equivalence induces an isomorphism on the associated fields of rational functions [F, Proposition 6.12]. Thus their fields of rational functions all agree: $\mathbb{C}(C) = \mathbb{C}(\overline{C}) = \mathbb{C}(\tilde{C})$. To a point P of \tilde{C} the local ring $\mathcal{O}_{\tilde{C}, P}$ of \tilde{C} at P is associated. As the point P is a smooth point, the ring $\mathcal{O}_{\tilde{C}, P}$ is a discrete valuation ring, which induces a discrete valuation v_P on $\mathbb{C}(C)$ [F, Section 7.1].

An *ideal point* χ of an affine curve C is a point of its smooth projective model \tilde{C} corresponding to a point of $\overline{C} \setminus C$. We can equip the rational functions $\mathbb{C}(C)$ with the discrete valuation v_χ associated to an ideal point χ as described above.

2.2. Bruhat-Tits buildings. Following the exposition [G], we describe the Bruhat-Tits building [BT1, BT2, IM] associated to the special linear group over a discrete valuation field. See also [AB] for more details on buildings.

Let F be a commutative field equipped with a discrete valuation v which is not necessarily complete. We denote by \mathcal{O}_v the valuation ring associated to v . The *Bruhat-Tits building* associated to $\mathrm{SL}_n(F)$, which is an $(n-1)$ -dimensional simplicial complex B_v , is defined as follows: A vertex of B_v is the homothety class of a lattice in the n -dimensional vector space F^n , where a *lattice* in F^n is a free \mathcal{O}_v -submodule of full rank, and two lattices Λ and Λ' are *homothetic* if $\Lambda = \alpha\Lambda'$ for some $\alpha \in F^\times$. A set of $(m+1)$ vertices s_0, s_1, \dots, s_m forms an m -simplex in B_v if and only if there exist lattices $\Lambda_0, \Lambda_1, \dots, \Lambda_m$ representing s_0, s_1, \dots, s_m respectively such that after relabeling indices we have the flag relation

$$\omega\Lambda_m \subsetneq \Lambda_0 \subsetneq \Lambda_1 \subsetneq \dots \subsetneq \Lambda_m,$$

where ω is an irreducible element of \mathcal{O}_v .

The simplicial complex B_v is known to be an *Euclidean building*, and, in particular, a CAT(0)-space with respect to the standard metric. See for instance [AB, Definition 11.1] for the definition of an Euclidean building. Since $\mathrm{SL}_n(F)$ acts on the set of lattices in F^n so that homothety classes and above flag relations are preserved, $\mathrm{SL}_n(F)$ acts also on B_v . This action is *type-preserving*, i.e., there exists an $\mathrm{SL}_n(F)$ -invariant map $\tau: B_v^{(0)} \rightarrow \mathbb{Z}/n\mathbb{Z}$ such that $\tau|_{\Delta^{(0)}}$ is a bijection for each

$(n - 1)$ -simplex Δ in B_v . Here for a simplicial complex K we denote by $K^{(m)}$ the m -skeleton of K . In particular, for any subgroup G of $\mathrm{SL}_n(F)$ the quotient B_v/G is again an $(n - 1)$ -dimensional simplicial complex.

Remark 2.1. In the case of $n = 2$ the above construction is nothing but the one of the tree associated to $\mathrm{SL}_2(F)$ in [Se1, Se2].

2.3. An ideal point giving an essential surface. We summarize the construction in [HK] of a certain branched surface from an ideal point of a curve in the character variety. Here we restrict our attention to the case where such a branched surface has no branched points, and is an essential surface.

Let C be a curve in $X_n(M)$ and χ an ideal point of C . We denote by $t: R_n(M) \rightarrow X_n(M)$ the quotient map. There exists a curve D in $t^{-1}(C)$ such that $t|_D$ is not a constant map, and a regular map $\tilde{t}|_D: \tilde{D} \rightarrow \tilde{C}$ on the smooth projective models is induced by $t|_D$. We take a lift $\tilde{\chi} \in (\tilde{t}|_D)^{-1}(\{\chi\})$, and denote by $B_{\tilde{\chi}}$ the Bruhat-Tits building associated to $\mathrm{SL}_n(\mathbb{C}(D))$, where the function field $\mathbb{C}(D)$ is equipped with the discrete valuation at $\tilde{\chi}$. The *tautological representation* $\mathcal{P}: \pi_1 M \rightarrow \mathrm{SL}_n(\mathbb{C}(D))$ is defined by

$$\mathcal{P}(\gamma)(\rho) = \rho(\gamma)$$

for $\gamma \in \pi_1 M$ and $\rho \in D$. Pulling back the action of $\mathrm{SL}_n(\mathbb{C}(D))$ on $B_{\tilde{\chi}}$ by \mathcal{P} , we obtain the action of $\pi_1 M$ on $B_{\tilde{\chi}}$. Extending [CS, Theorem 2.2.1] to the case of a general n , Hara and the second author [HK, Corollary 4.5] proved that the action is *nontrivial*, i.e., for every vertex of $B_{\tilde{\chi}}$ its stabilizer subgroup of $\pi_1 M$ is proper.

Recall that a compact orientable properly-embedded surface S in M is called *essential* if for any component S_0 of S the inclusion-induced homomorphism $\pi_1 S_0 \rightarrow \pi_1 M$ is injective, and S_0 is not boundary-parallel nor homeomorphic to the 2-sphere S^2 . We say that *an essential surface S is given by an ideal point χ* if for some lift $\tilde{\chi}$ of χ there exists a PL map $f: M \rightarrow B_{\tilde{\chi}}^{(1)}/\pi_1 M$ whose inverse image of the set of midpoints of the edges in $B_{\tilde{\chi}}$ is isotopic to some number of parallel copies of S .

When $n = 2$, since $\pi_1 M$ nontrivially acts on the tree $B_{\tilde{\chi}}$ without inversions, every ideal point χ gives some essential surface in M [CS, Proposition 2.3.1]. In general, it follows from the proof of [HK, Theorem 4.7] that if $n = 3$ or if ∂M is non-empty, then there exists a PL map $f: M \rightarrow B_{\tilde{\chi}}^{(2)}/\pi_1 M$ such that $f^{-1}(Y)$ is a certain branched surface called *essential tribranched surface* [HK, Definition 2.2], where Y is the union of edges in the first barycentric subdivision of $B_{\tilde{\chi}}^{(2)}/\pi_1 M$ not contained in $B_{\tilde{\chi}}^{(1)}/\pi_1 M$. Note that an essential tribranched surface without any branched points is nothing but an essential surface in the usual sense.

Remark 2.2. The authors [FKN] showed that every closed 3-manifold M with $\mathrm{rank} \pi_1 M \geq 4$ contains an essential tribranched surface.

3. SURFACE SUBGROUP SEPARABILITY

We recall the separability of surface subgroups in a 3-manifold group proved by Przytycki and Wise [PW2], which is a key ingredient of the proof of Theorem 1.2. A subgroup H of a group G is *separable* if H equals the intersection of finite index subgroups of G containing H .

Theorem 3.1. ([PW2, Theorem 1.1]) *Let S be a connected essential surface S in M . Then $\pi_1 S$ is separable in $\pi_1 M$.*

Theorem 3.1 was proved by Przytycki and Wise [PW1] when M is a graph manifold, and by Wise [W] when M is a hyperbolic manifold. In fact, every finitely generated subgroup of $\pi_1 M$ is separable when M is a hyperbolic manifold, by Wise [W] in the case where M has a non-empty boundary and by Agol [A] in the case where M is closed. See also Liu [L] for a refinement of the separability.

The following is a topological interpretation of Theorem 3.1. While it is well-known for experts, nevertheless we give a proof for the sake of completeness. See also [Sc, Lemma 1.4].

Corollary 3.2. *For an essential surface S in M there exists some finite cover of M where the inverse image of S contains a non-separating component.*

Proof. We may assume that S is connected and separating. Let M_+ and M_- be the two components of the complement of S . It follows from [He, Theorem 10.5] that $\pi_1 S$ has index at least two in $\pi_1 M_-$ and in $\pi_1 M_+$. It follows from Theorem 3.1 that there exists an epimorphism $\varphi: \pi_1 M \rightarrow G$ to a finite group such that $\varphi(\pi_1 S) \neq \varphi(\pi_1 M_{\pm})$. In particular, we have

$$[G : \varphi(\pi_1 S)] \geq 2[G : \varphi(\pi_1 M_{\pm})].$$

Let $p: M_{\varphi} \rightarrow M$ be the covering corresponding to $\text{Ker } \varphi$. The numbers of components of $p^{-1}(M_{\pm})$ and $p^{-1}(S)$ are equal to $[G : \varphi(\pi_1 M_{\pm})]$ and $[G : \varphi(\pi_1 S)]$ respectively, and the above inequality implies

$$[G : \varphi(\pi_1 S)] \geq [G : \varphi(\pi_1 M_+)] + [G : \varphi(\pi_1 M_-)].$$

Thus the number of components of $p^{-1}(S)$ is greater than or equal to that of its complement, which shows that some component of $p^{-1}(S)$ is non-separating. \square

4. PROOF OF THE MAIN THEOREM

Now we prove the main theorem. For the readers' convenience we recall the statement.

Theorem 4.1 (Theorem 1.2). *Every connected essential surface in M is given by an the ideal point of a rational curve in $X_n(M)$ for some n .*

Let S be a connected essential surface in M . It follows from Corollary 3.2 that there exists a d -fold covering $p: N \rightarrow M$ for some d such that $p^{-1}(S)$ contains a non-separating component T . Then the proof is divided into two parts: First, we construct a rational curve C_T in $X_{2d}(M)$ with a unique ideal point χ_T , which is determined by T . Second, for a lift $\tilde{\chi}_T$ of χ_T we construct a PL map $f: M \rightarrow B_{\tilde{\chi}_T}^{(1)}$ whose inverse image of the set of midpoints of edges in $B_{\tilde{\chi}_T}$ is isotopic to two parallel copies of S .

4.1. Construction of a curve. We denote by $\psi: \pi_1 N \rightarrow \mathbb{Z}$ the epimorphism induced by the intersection pairing with T . For each $z \in \mathbb{C}^{\times}$ we define the representation $\tilde{\rho}_z: \pi_1 N \rightarrow \text{SL}_2(\mathbb{C})$ to be the composition of ψ and the homomorphism $\mathbb{Z} \rightarrow \text{SL}_2(\mathbb{C})$ which sends an integer k to the matrix

$$\begin{pmatrix} z^k & 0 \\ 0 & z^{-k} \end{pmatrix}.$$

We consider the induced representation $\rho_z : \pi_1 M \rightarrow \text{Aut}(\mathbb{C}[\pi_1 M] \otimes_{\mathbb{C}[\pi_1 N]} \mathbb{C}^2)$ of $\tilde{\rho}_z$. Fixing representatives $\gamma_1, \dots, \gamma_d \in \pi_1 M$ of the elements of $\pi_1 M / p_*(\pi_1 N)$, we have the decomposition

$$\mathbb{C}[\pi_1 M] \otimes_{\mathbb{C}[\pi_1 N]} \mathbb{C}^2 = \bigoplus_{i=1}^d \gamma_i \otimes \mathbb{C}^2,$$

which is naturally identified with \mathbb{C}^{2d} . Thus we regard ρ_z as a representation $\pi_1 M \rightarrow \text{SL}_{2d}(\mathbb{C})$. We now set

$$\begin{aligned} D_T &= \{\rho_z \in R_{2d}(M) : z \in \mathbb{C}^\times\}, \\ C_T &= \{\chi_{\rho_z} \in X_{2d}(M) : z \in \mathbb{C}^\times\}. \end{aligned}$$

Lemma 4.2. (1) *The set D_T is a curve in $R_{2d}(M)$ isomorphic to \mathbb{C}^\times .*
(2) *The set C_T is a rational curve in $X_{2d}(M)$ with a unique ideal point.*

Proof. These sets of representations and characters are constructed along the following commutative diagram:

$$\begin{array}{ccccccc} \mathbb{C}^\times & \longrightarrow & R_2(\mathbb{Z}) & \longrightarrow & R_2(N) & \longrightarrow & R_{2d}(M) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathbb{C} & \longrightarrow & X_2(\mathbb{Z}) & \longrightarrow & X_2(N) & \longrightarrow & X_{2d}(M), \end{array}$$

where the first vertical map sends $z \in \mathbb{C}^\times$ to $z + z^{-1} \in \mathbb{C}$, and the first bottom horizontal map is an isomorphism which sends $w \in \mathbb{C}$ to the character of \mathbb{Z} whose image of $1 \in \mathbb{Z}$ is w . The composition of the top horizontal maps is called Ψ and the composition of the bottom horizontal maps is called Φ . Then the sets D_T and C_T coincide with the images of Ψ and Φ

We may assume $\gamma_1 \in \pi_1 N$, and take $\mu \in \pi_1 N$ with $\psi(\mu) = 1$. Then we have

$$\begin{aligned} \rho_z(\mu) &= \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} \oplus \bigoplus_{i=2}^d \begin{pmatrix} z^{\psi(\gamma_i^{-1}\mu\gamma_i)} & 0 \\ 0 & z^{-\psi(\gamma_i^{-1}\mu\gamma_i)} \end{pmatrix}, \\ \chi_{\rho_z}(\mu) &= z + z^{-1} + \sum_{i=2}^d \left(z^{\psi(\gamma_i^{-1}\mu\gamma_i)} + z^{-\psi(\gamma_i^{-1}\mu\gamma_i)} \right). \end{aligned}$$

Hence the restriction of the map $R_{2d}(M) \rightarrow \mathbb{C}^2$ sending a representation ρ to the vector of the (1, 1)- and (2, 2)-entries of $\rho(\mu)$ gives the inverse regular map $D_T \rightarrow \mathbb{C}^\times$ of Ψ , where \mathbb{C}^\times is identified with the curve $xy - 1$ in \mathbb{C}^2 , and (1) is proved.

We deduce from the second equation above that the map Φ is not constant. Also by fixing an affine space \mathbb{C}^N containing $X_{2d}(M)$, we regard Φ as a map $\mathbb{C} \rightarrow \mathbb{C}^N$. We denote by $\bar{\Phi} : \mathbb{P}^1 \rightarrow \mathbb{P}^N$ the projective extension of Φ . Since $\bar{\Phi}$ is not a constant map, by the completeness of the projective line \mathbb{P}^1 [Mu, Section I.9, Theorem 1] the image \bar{C}_T of $\bar{\Phi}$ is a projective curve, and by the Riemann-Hurwitz formula the curve \bar{C}_T is rational. Therefore the set C_T , which coincides with the intersection of \bar{C}_T and \mathbb{C}^N , is an affine rational curve. Since Φ induces a surjective regular map $\tilde{\Phi} : \mathbb{P}^1 \rightarrow \tilde{C}_T$ on the smooth projective models, the rational curve C_T has a unique ideal point corresponding to the point at infinity of \mathbb{P}^1 , which proves (2). \square

It is a simple matter to check that both the two ideal points of D_T corresponding to 0 and ∞ are lifts of the unique ideal point χ_T of C_T . Let $\tilde{\chi}_T$ be the one corresponding to 0. Then as in Section 2.3 we obtain the nontrivial action $\pi_1 M$ on the Bruhat-Tits building $B_{\tilde{\chi}_T}$ associated to $\text{SL}_{2d}(\mathbb{C}(D_T))$. We identify $\mathbb{C}(D_T)$ with

the standard function field $\mathbb{C}(t)$ and the valuation at $\tilde{\chi}_T$ with the lowest degree of the Laurent expansion of a rational function. Then the vector space $\mathbb{C}(t)^{2d}$ is decomposed into

$$\mathbb{C}[\pi_1 M] \otimes_{\mathbb{C}[\pi_1 N]} \mathbb{C}(t)^2 = \bigoplus_{i=1}^d \gamma_i \otimes \mathbb{C}(t)^2,$$

where $\pi_1 N$ acts on $\mathbb{C}(t)^2$ by the representation $\mathcal{Q}: \pi_1 N \rightarrow \mathrm{SL}_2(\mathbb{C}(t))$ defined by

$$\mathcal{Q}(\gamma) = \begin{pmatrix} t^{\psi(\gamma)} & 0 \\ 0 & t^{-\psi(\gamma)} \end{pmatrix}$$

for $\gamma \in \pi_1 N$, and the tautological representation $\mathcal{P}: \pi_1 M \rightarrow \mathrm{SL}_{2d}(\mathbb{C}(t))$ is given by the left multiplication on $\mathbb{C}[\pi_1 M] \otimes_{\mathbb{C}[\pi_1 N]} \mathbb{C}(t)^2$.

4.2. Construction of a PL-map. We take a triangulation of M containing S as a normal surface. We may assume that the intersection of each tetrahedron with S is connected, if necessary, replacing the triangulation by an appropriate subdivision. The triangulation of M induces ones of N and the universal cover \tilde{M} of M , so that T and its inverse image \tilde{T} by the covering $\tilde{M} \rightarrow N$ are also normal surfaces. Then we take a cellular map $g: N \rightarrow \mathbb{R}/\mathbb{Z}$ such that $g^{-1}([\frac{1}{2}]) = T$, where we consider the cellular structure of \mathbb{R}/\mathbb{Z} with one vertex corresponding to \mathbb{Z} . We define $\tilde{g}: \tilde{M} \rightarrow \mathbb{R}$ to be the $\pi_1 N$ -equivariant lift of g , so that $\tilde{g}^{-1}(\frac{1}{2} + \mathbb{Z}) = \tilde{T}$.

We now define a map $\tilde{f}^{(0)}: \tilde{M}^{(0)} \rightarrow B_{\tilde{\chi}_T}^{(0)}$ as follows. For $s \in \tilde{M}^{(0)}$ we consider the lattice

$$\bigoplus_{i=1}^d \gamma_i \otimes \Lambda_{\tilde{g}(\gamma_i^{-1}s)}$$

in $\mathbb{C}[\pi_1 M] \otimes_{\mathbb{C}[\pi_1 N]} \mathbb{C}(t)^2$, where Λ_n is the lattice in $\mathbb{C}(t)^2$ generated by the vectors

$$\begin{pmatrix} t^n \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ t^{-n} \end{pmatrix}.$$

Note that $\tilde{g}(\gamma_i^{-1}s)$ is an integer for each i by the construction of \tilde{g} . Then we set $\tilde{f}^{(0)}(s)$ to be the homothety class of the above lattice. In the following two lemmas we observe the key properties of $\tilde{f}^{(0)}$.

Lemma 4.3. *The map $\tilde{f}^{(0)}$ is $\pi_1 M$ -equivariant.*

Proof. For $\gamma \in \pi_1 M$ there exist a permutation σ of degree d and $\delta_i \in \pi_1 N$ such that

$$\gamma \gamma_i = \gamma_{\sigma(i)} \delta_i$$

for each i . Then

$$\begin{aligned} \bigoplus_{i=1}^d \gamma \gamma_i \otimes \Lambda_{\tilde{g}(\gamma_i^{-1}s)} &= \bigoplus_{i=1}^d \gamma_{\sigma(i)} \delta_i \otimes \Lambda_{\tilde{g}(\gamma_i^{-1}s)} &= \bigoplus_{i=1}^d \gamma_{\sigma(i)} \otimes \mathcal{Q}(\delta_i) \cdot \Lambda_{\tilde{g}(\gamma_i^{-1}s)} \\ &= \bigoplus_{i=1}^d \gamma_{\sigma(i)} \otimes \Lambda_{\tilde{g}(\gamma_i^{-1}s) + \psi(\delta_i)} &= \bigoplus_{i=1}^d \gamma_{\sigma(i)} \otimes \Lambda_{\tilde{g}(\delta_i \gamma_i^{-1}s)} \\ &= \bigoplus_{i=1}^d \gamma_{\sigma(i)} \otimes \Lambda_{\tilde{g}(\gamma_{\sigma(i)}^{-1} \gamma s)} &= \bigoplus_{i=1}^d \gamma_i \otimes \Lambda_{\tilde{g}(\gamma_i^{-1} \gamma s)} \end{aligned}$$

for $\gamma \in \pi_1 M$ and $s \in \tilde{M}^{(0)}$, which implies that $\tilde{f}^{(0)}$ is a $\pi_1 M$ -equivariant map. \square

Lemma 4.4. *For each tetrahedron Δ in \widetilde{M} the set $\tilde{f}^{(0)}(\Delta^{(0)})$ consists of one vertex if $\gamma_i^{-1} \cdot \Delta$ does not intersect with \widetilde{T} for any i , and two vertices of distance 2 with respect to the graph metric on $B_{\widetilde{\chi}}^{(1)}$ otherwise.*

Proof. If $\gamma_i^{-1} \cdot \Delta$ does not intersect with \widetilde{T} for any i , then it follows from the choice of g that there exists some $n_i \in \mathbb{Z}$ such that

$$\tilde{g}(\gamma_i^{-1} \cdot \Delta^{(0)}) = \{n_i\}$$

for each i , and hence we obtain

$$\tilde{f}^{(0)}(\gamma_i^{-1} \cdot \Delta^{(0)}) = \left\{ \left[\bigoplus_{i=1}^d \gamma_i \otimes \Lambda_{n_i} \right] \right\}.$$

In the following we consider the case where $\gamma_i^{-1} \cdot \Delta$ intersects with \widetilde{T} for $i = i_1, \dots, i_m$. Then the intersections of $\gamma_i^{-1} \cdot \Delta$ with \widetilde{T} are all connected and of same type for $k = 1, \dots, m$, since otherwise $p(T) = S$ implies that the intersection of some tetrahedron in M with S is not a normal disc, which contradicts the choice of the triangulation of M . Thus $\Delta^{(0)}$ is divided into two subsets $\Delta_+^{(0)}$ and $\Delta_-^{(0)}$ satisfying the following:

- (1) there exists some $n_{i_k} \in \mathbb{Z}$ such that

$$\tilde{g}(\gamma_{i_k}^{-1} \cdot \Delta_+^{(0)}) = \{n_{i_k} + 1\} \text{ and } \tilde{g}(\gamma_{i_k}^{-1} \cdot \Delta_-^{(0)}) = \{n_{i_k}\}$$

for $k = 1, \dots, m$;

- (2) there exists some $n_i \in \mathbb{Z}$ such that

$$\tilde{g}(\gamma_i^{-1} \cdot \Delta^{(0)}) = \{n_i\}$$

for $i \neq i_1, \dots, i_m$.

Hence we obtain

$$\tilde{f}^{(0)}(\Delta_+^{(0)}) = \{[\Lambda_+]\} \text{ and } \tilde{f}^{(0)}(\Delta_-^{(0)}) = \{[\Lambda_-]\},$$

where

$$\begin{aligned} \Lambda_+ &= \left(\bigoplus_{k=1}^m \gamma_{i_k} \otimes \Lambda_{n_{i_k}+1} \right) \oplus \left(\bigoplus_{i \neq i_1, \dots, i_m} \gamma_i \otimes \Lambda_{n_i} \right), \\ \Lambda_- &= \left(\bigoplus_{k=1}^m \gamma_{i_k} \otimes \Lambda_{n_{i_k}} \right) \oplus \left(\bigoplus_{i \neq i_1, \dots, i_m} \gamma_i \otimes \Lambda_{n_i} \right). \end{aligned}$$

Since

$$\begin{aligned} t\Lambda'_n &\subsetneq \Lambda_{n+1} \subsetneq \Lambda'_n, \\ t\Lambda'_n &\subsetneq \Lambda_n \subsetneq \Lambda'_n \end{aligned}$$

for $n \in \mathbb{Z}$, where Λ'_n is the lattice in $\mathbb{C}(t)^2$ generated by the vectors

$$\begin{pmatrix} t^n \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ t^{-n-1} \end{pmatrix},$$

we have

$$\begin{aligned} t\Lambda' &\subsetneq \Lambda_+ \subsetneq \Lambda', \\ t\Lambda' &\subsetneq \Lambda_- \subsetneq \Lambda', \end{aligned}$$

where

$$\Lambda' = \left(\bigoplus_{k=1}^m \gamma_{i_k} \otimes \Lambda'_{n_{i_k}} \right) \oplus \left(\bigoplus_{i \neq i_1, \dots, i_m} \gamma_i \otimes \Lambda_{n_i} \right).$$

By the definition of the building $B_{\tilde{\chi}}$ these relations imply that there exist edges in $B_{\tilde{\chi}}$ connecting $[\Lambda_+]$ and $[\Lambda_-]$ with $[\Lambda']$, and hence the distance between $[\Lambda_+]$ and $[\Lambda_-]$ is at most 2 in $B_{\tilde{\chi}}^{(1)}$. We further observe that the matrix

$$\left(\bigoplus_{i=1}^n \gamma_{i_k} \otimes \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \right) \oplus \left(\bigoplus_{i \neq i_1, \dots, i_m} \gamma_i \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right)$$

in $\mathrm{SL}_{2d}(\mathbb{C}(t))$ sends $[\Lambda_-]$ to $[\Lambda_+]$. Since the action of $\mathrm{SL}_{2d}(\mathbb{C}(t))$ on $B_{\tilde{\chi}}$ is type-preserving, the distance between them is exactly equal to 2, and the lemma follows. \square

We are now in position to construct a desired PL map $f: M \rightarrow B_{\tilde{\chi}_T}^{(1)}/\pi_1 M$. It follows from Lemmas 4.3 and 4.4 that $\tilde{f}^{(0)}$ extends to a $\pi_1 M$ -equivariant simplicial map $\tilde{f}: \tilde{N} \rightarrow B_{\tilde{\chi}_T}^{(1)}$ with respect to the 1st barycentric subdivision of the triangulation of \tilde{M} . We define $f: M \rightarrow B_{\tilde{\chi}_T}^{(1)}/\pi_1 M$ to be the quotient of \tilde{f} by $\pi_1 M$. By the construction of \tilde{f} we check at once that the inverse image of the set of midpoints of edges in $B_{\tilde{\chi}_T}$ by \tilde{f} is isotopic to two parallel copies of \tilde{T} . Since \tilde{f} is $\pi_1 M$ -equivariant and since $p(T) = S$, the inverse image of the set of midpoints of edges in $B_{\tilde{\chi}_T}/\pi_1 M$ by f is isotopic to two parallel copies of S . Therefore S is given by χ_T , which completes the proof.

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