The L^2-Alexander torsions of 3-manifolds

Jérôme Dubois [a], Stefan Friedl [b] Wolfgang Lück [c]

[a] *Institut de Mathématiques de Jussieu - Paris Rive Gauche*
Bâtiment S. Germain Case 7012, 75205 Paris Cedex 13, France
[b] *Fakultät für Mathematik, Universität Regensburg*
93040 Regensburg, Germany
[c] *Mathematisches Institut, Universität Bonn*
Endenicher Allee 60, 53115 Bonn, Germany

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Abstract

The aim of this note is to introduce L^2-Alexander torsions for 3-manifolds (which are generalizations of the usual Alexander polynomial and also of the L^2-Alexander invariant defined by Li and Zhang [9]) and to report on calculations for graph manifolds and fibered 3-manifolds. We further announce that given any irreducible 3-manifold there exists a coefficient system such that the corresponding L^2-Alexander torsion detects the Thurston norm. Finally we also state a symmetry formula. *To cite this article: J. Dubois, S. Friedl, W. Lück, C. R. Acad. Sci. Paris, Ser. I 340 (2014).*

Résumé

Torsions d'Alexander L^2 pour les variétés de dimension trois. Le but de cette note est d'introduire les torsions d'Alexander L^2 (généralisations du polynôme d'Alexander usuel et de l'invariant d'Alexander L^2 défini par Li et Zhang [9]) et d'en donner le calcul pour les variétés graphées et les variétés fibrées de dimension 3. On annonce enfin que les torsions d'Alexander L^2 permettent de détecter la norme de Thurston d'une variété de dimension 3 irréductible et qu'elles sont symétriques. *Pour citer cet article : J. Dubois, S. Friedl, W. Lück, C. R. Acad. Sci. Paris, Ser. I 340 (2014).*

Version française abrégée

Dans cette note, nous étudions une nouvelle famille d'invariants des 3-variétés connues sous le nom de *torsions d'Alexander* L^2 et énonçons des propriétés en relation avec la géométrie et la topologie

Email addresses: jerome.dubois@imj-prg.fr (Jérôme Dubois), sfriedl@gmail.com (Stefan Friedl), wolfgang.lueck@him.uni-bonn.de (Wolfgang Lück).

des variétés. A une 3-variété compacte première N dont le bord est vide ou est une réunion de tores, à un élément $\phi \in H^1(N;\mathbb{Z}) = \mathrm{Hom}(\pi_1(N),\mathbb{Z})$ et à un homomorphisme $\gamma\colon \pi_1(N) \to G$, tels que $\phi\colon \pi_1(N) \to \mathbb{Z}$ se factorise à travers γ, est associée la torsion L^2 $\tau^{(2)}(N,\phi,\gamma)$. Il s'agit d'une fonction $\mathbb{R}_{>0} \to [0,\infty)$ définie de la façon suivante. Soit $t \in \mathbb{R}_{>0}$, l'homomorphisme d'anneaux de groupes $\mathbb{Z}[\pi_1(N)] \to \mathbb{R}[G], \sum_{i=1}^n a_i g_i \mapsto \sum_{i=1}^n a_i t^{\phi(g_i)}\gamma(g_i)$ définit une structure de $\mathbb{Z}[\pi_1(N)]$-module à droite sur $\mathbb{R}[G]$ et donc sur l'algèbre de von Neumann $\mathcal{N}(G)$, de plus l'action de $\pi_1(N)$, comme groupe de révêtement, sur le revêtement universel $\widetilde{N}$ fait de $C_*(\widetilde{N};\mathbb{Z})$ un $\mathbb{Z}[\pi_1(N)]$-module à gauche. La torsion L^2 correspondante est désignée par $\tau^{(2)}(N,\phi,\gamma)(t) = \tau^{(2)}\left(\mathcal{N}(G) \otimes_{\mathbb{Z}[\pi_1(N)]} C_*(\widetilde{N};\mathbb{Z})\right)$ et la fonction $\tau^{(2)}(N,\phi,\gamma)\colon \mathbb{R}_{>0} \to [0,\infty),\ t \mapsto \tau^{(2)}(N,\phi,\gamma)(t)$ est appelée la *torsion d'Alexander L^2 du triplet* (N,ϕ,γ). C'est un invariant bien défini à une puissance entière de t près et si N est une variété hyperbolique alors $\tau^{(2)}(N,\phi,\mathrm{id})(t=1) = \exp\left(\frac{\mathrm{vol}(N)}{6\pi}\right)$ (voir Lück-Schick [12]). Les résultats présentés dans cette note sont les suivants :

(i) *Torsion d'Alexander L^2 pour les variétés fibrées* (Théorème 5.2) : pour tout triplet (N,ϕ,γ) comme ci-dessus tel que $N \neq S^1 \times D^2$ et $N \neq S^1 \times S^2$, si ϕ est fibré (c'est-à-dire, s'il existe une fibration $p\colon N \to S^1$ pour laquelle l'homomorphisme induit $p_*\colon \pi_1(N) \to \pi_1(S^1) = \mathbb{Z}$ est égal à ϕ), alors il existe une constante $T \geq 1$ et un entier $s \in \mathbb{Z}$ tels que

$$t^s \cdot \tau^{(2)}(N,\phi,\gamma)(t) = \begin{cases} 1, & \text{si } t \leqslant T^{-1}, \\ t^{x_N(\phi)}, & \text{si } t \geqslant T. \end{cases} \tag{1}$$

Ici $x_N(\phi)$ désigne la norme de Thurston de ϕ (voir Eq. (2)). Pour être précis, dans l'Equation (1) la constante T peut être choisie égale à $\exp(h)$ où h désigne l'entropie de la monodromie correspondant à la fibration (voir [5] pour la définition de l'entropie).

(ii) *Torsion d'Alexander L^2 pour les variétés graphées* (Théorème 5.1) : si la variété $N \neq S^1 \times D^2$ et $N \neq S^1 \times S^2$ est graphée (c'est-à-dire si sa décomposition JSJ ne possède aucune pièce hyperbolique), alors l'Equation (1) reste valable avec $T = 1$ et pour n'importe quel $\phi \in H^1(N;\mathbb{Z})$.

(iii) *Degré de la torsion d'Alexander L^2 et norme de Thurston* (Théorème 5.3) : soit N une 3-variété non graphée, il existe un épimorphisme $\gamma\colon \pi_1(N) \to G$ sur un groupe virtuellement abélien pour lequel l'homomorphisme $\pi_1(N) \to H_1(N;\mathbb{Z})/\text{torsion}$ se factorise à travers γ et tel que pour tout $\phi \in H^1(N;\mathbb{Z})$, la fonction $\tau = \tau^{(2)}(N,\phi,\gamma) : \mathbb{R}_{>0} \to [0,\infty)$ possède la propriété suivante : il existe $d, D \in \mathbb{N}$ et $c, C \in \mathbb{R}\backslash\{0\}$ tels que $\lim_{t\to 0} \frac{\tau(t)}{t^d} = c$ et $\lim_{t\to\infty} \frac{\tau(t)}{t^D} = C$. De plus, le *degré* $\deg \tau = D - d$ de τ est égal à la norme de Thurston $x_N(\phi)$.

(iv) *Propriété de symétrie* (Théorème 5.4) : pour tout triplet (N,ϕ,γ) comme ci-dessus tel que $N \neq S^1 \times D^2$, il existe un entier $n \in \mathbb{Z}$ tel que $n \equiv x_N(\phi) \bmod 2$, vérifiant $\tau^{(2)}(N,\phi,\gamma)(t^{-1}) = t^n \cdot \tau^{(2)}(N,\phi,\gamma)(t)$. Ce résultat peut s'interpréter comme un analogue de la symétrie des polynômes d'Alexander usuels ou tordus (voir [8] et [6]).

1. Introduction

The aim of this note is to discuss a new family of 3-manifolds invariants that has appeared in the last years: the L^2-Alexander torsions, which are functions $\mathbb{R}_{>0} \to [0,\infty)$, originally defined for knots by Li and Zhang [9]. These new kind of torsions can be considered as a generalization of the usual Alexander polynomial (see Theorem 4.1). Our main ambition is to give the definition of these invariants for a compact, connected, oriented 3-manifold, possibly with toroidal boundary, to explicitly compute them in the case of graph manifolds, to give the behavior at 0 and ∞ for fibered manifolds and finally to explain how they determine the Thurston norm for appropriate coefficient systems. The note ends with the study of a symmetry property of the L^2-Alexander torsions similar to the symmetry of the usual Alexander invariant.

In this note, all groups are finitely generated and all manifolds are smooth, orientable, compact and connected. Throughout the paper we refer to [13] for definitions and properties of L^2-torsions.

2. The Fuglede-Kadison determinant and basic definitions

Let G be a group, $\mathbb{Q}[G]$ and $\mathbb{R}[G]$ denote the group rings and $\mathcal{N}(G)$ denotes the von Neumann algebra of G. For an operator $f\colon U \to V$ between two finitely generated Hilbert $\mathcal{N}(G)$-modules or a matrix A over $\mathbb{R}[G]$ there is the notion of being of *determinant class* and of a 'continuous-determinant', the *Fuglede-Kadison determinant*. For these two notions we refer to [13, § 3.2] and we denote by $\det_{\mathcal{N}(G)}(A)$ the Fuglede-Kadison determinant of A.

It is impossible to recall the definition of the Fuglede-Kadison determinant in a few pages. We will thus just list a few properties to give the flavor of this determinant.

- If A is of determinant class, then $\det_{\mathcal{N}(G)}(A)$ is a positive real number.
- If $G = \langle t \rangle$ is an infinite cyclic group and A is a square matrix over $\mathbb{R}[G] = \mathbb{R}[t^{\pm 1}]$ such that the usual determinant $\det(A) \in \mathbb{R}[t^{\pm 1}]$ is non-zero, then $\det_{\mathcal{N}(G)}(A)$ equals the Mahler measure of $\det(A)$.
- The Fuglede-Kadison determinant has many properties which are at least reminiscent of the usual properties of a determinant. For example, if we swap two rows or columns, then the Fuglede-Kadison determinant does not change. Also, under mild assumptions the Fuglede-Kadison determinant is multiplicative for square matrices.
- If A is a matrix over $\mathbb{R}[\Gamma]$ and if Γ is a subgroup of a group G, then $\det_{\mathcal{N}(\Gamma)}(A) = \det_{\mathcal{N}(G)}(A)$.

Here we recall some useful definitions. Let N be a 3-manifold and let $\phi \in H^1(N;\mathbb{Z}) = \mathrm{Hom}(\pi_1(N),\mathbb{Z})$ be a non trivial element.

(i) *The Thurston norm* of ϕ is defined as

$$x_N(\phi) = \min\{\chi_-(\Sigma) \mid \Sigma \subset N \text{ properly embedded surface dual to } \phi\} \tag{2}$$

where given a surface Σ with connected components $\Sigma_1, \ldots, \Sigma_k$ we define its complexity to be $\chi_-(\Sigma) = \sum_{i=1}^k \max\{-\chi(\Sigma_i), 0\}$.

For example, if N is a knot exterior $N = S^3 \setminus \nu K$, where K is a non-trivial knot in S^3 and νK a tubular neighborhood of K, and if $\phi_K\colon \pi_1(S^3 \setminus \nu K) \to \mathbb{Z}$ denotes the abelianization, then $x_N(\phi_K) = 2 \cdot \mathrm{genus}(K) - 1$.

(ii) We say that ϕ is *fibered* if there exists a surface bundle $p\colon N \to S^1$ such that the induced homomorphism $p_*\colon \pi_1(N) \to \pi_1(S^1) = \mathbb{Z}$ is equal to ϕ.

(iii) An *admissible triple* (N, ϕ, γ) consists of a prime 3-manifold N with empty or toroidal boundary, an element $\phi \in H^1(N;\mathbb{Z}) = \mathrm{Hom}(\pi_1(N),\mathbb{Z})$, and a homomorphism $\gamma\colon \pi_1(N) \to G$, such that $\phi\colon \pi_1(N) \to \mathbb{Z}$ factors through γ.

(iv) A function $f\colon \mathbb{R}_{>0} \to [0,\infty)$ is *piecewise monomial* if there exists a subdivision $0 = t_0 < t_1 < \ldots < t_n < t_{n+1} = \infty$ of $[0,\infty)$, a sequence of integers $d_0, \ldots, d_n \in \mathbb{Z}$ and some real numbers $c_0, \ldots, c_n$ such that $f(s) = c_i s^{d_i}$ for all $s \in [t_i, t_{i+1}) \cap \mathbb{R}_{>0}$.

(v) A function $f\colon \mathbb{R}_{>0} \to [0,\infty)$ is *eventually monomial* if there exist $0 < t < T < \infty$, two integers $d, D \in \mathbb{Z}$ and two non-zero real numbers c, C such that $f(s) = cs^d$ for all $s < t$ and $f(s) = Cs^D$ for all $s > T$.

(vi) A function $f\colon \mathbb{R}_{>0} \to [0,\infty)$ is *monomial in the limit* if there exist $d, D \in \mathbb{N}$ and non-zero real numbers c, C such that

$$\lim_{t\to 0} \frac{f(t)}{t^d} = c \text{ and } \lim_{t\to\infty} \frac{f(t)}{t^D} = C.$$

For such a function we define its *degree* to be $\deg f = D - d$ and we say it is *monic* when $c = C = 1$.

3. The L^2-Alexander torsions for 3-manifolds

If $(N, \phi, \gamma\colon \pi_1(N) \to G)$ is an admissible triple, then we associate to it its *L^2-Alexander torsion* $\tau^{(2)}(N,\phi,\gamma)\colon \mathbb{R}_{>0} \to [0,\infty)$ defined as follows. We first equip N with a CW-structure and choose a lift of the cells of N to its universal cover $\widetilde{N}$. For $t \in \mathbb{R}_{>0}$ we consider the ring homomorphism

$$\rho(\phi,\gamma,t)\colon \mathbb{Z}[\pi_1(N)] \to \mathbb{R}[G],\ \sum_{i=1}^n a_i g_i \mapsto \sum_{i=1}^n a_i t^{\phi(g_i)}\gamma(g_i) \tag{3}$$

which defines a $\mathbb{Z}[\pi_1(N)]$-right module structures on the group ring $\mathbb{R}[G]$ and thus also on the von Neumann algebra $\mathcal{N}(G)$. The deck transformation action induces a $\mathbb{Z}[\pi_1(N)]$-left module structure on the chain complex $C_*(\widetilde{N};\mathbb{Z})$. The lifts of the cells of N to $\widetilde{N}$ turn each $\mathcal{N}(G)\otimes_{\mathbb{Z}[\pi_1(N)]} C_*(\widetilde{N};\mathbb{Z})$ into a finitely generated Hilbert $\mathcal{N}(G)$-module. If the chain complex $\left(\mathcal{N}(G)\otimes_{\mathbb{Z}[\pi_1(N)]} C_*(\widetilde{N};\mathbb{Z}), \mathrm{id}\otimes\partial_*\right)$ is not acyclic or if one of the boundary maps is not of determinant class, then we define $\tau^{(2)}(N,\phi,\gamma)(t) := 0$. Otherwise we define the corresponding L^2-torsion as

$$\tau^{(2)}(N,\phi,\gamma)(t) := \tau^{(2)}\left(\mathcal{N}(G)\otimes_{\mathbb{Z}[\pi_1(N)]} C_*(\widetilde{N};\mathbb{Z})\right) = \prod_{i=0}^{2} \det{}_{\mathcal{N}(G)}(\mathrm{id}\otimes\partial_i)^{(-1)^{i+1}}.$$

The resulting function $\tau^{(2)}(N,\phi,\gamma)\colon \mathbb{R}_{>0}\to[0,\infty),\ t\mapsto \tau^{(2)}(N,\phi,\gamma)(t)$ is called the *L^2-Alexander torsion* of the admissible triple (N,ϕ,γ).

We say that two functions $f,g\colon \mathbb{R}_{>0}\to[0,\infty)$ are *equivalent* if there exists an integer $n\in\mathbb{Z}$ such that $f(t) = t^n g(t)$, for all $t\in\mathbb{R}_{>0}$ and we write $f \doteq g$ if it is the case. The equivalence class of $\tau^{(2)}(N,\phi,\gamma)$ is a well-defined topological invariant of the triple (N,ϕ,γ) which does not depend on the chosen CW-structure for N and the choice of the lift of the cells to the universal cover $\widetilde{N}$.

It follows immediately from the definitions, that the evaluation of $\tau^{(2)}(N,\phi,\mathrm{id})$ at $t=1$ does not depend on ϕ and by [12] it is related to the hyperbolic volume of the hyperbolic pieces in the JSJ decomposition of N. More precisely, one has $\tau^{(2)}(N,\phi,\mathrm{id})(t=1) = \prod_{k=1}^r \exp\left(\frac{\mathrm{vol}(N_k)}{6\pi}\right)$ where $N_1,\dots,N_r$ are the hyperbolic pieces in the JSJ decomposition of N.

4. L^2-Alexander torsions for knots

Let K be an oriented knot in S^3. We denote by $X_K = S^3\setminus \nu K$ its exterior. Recall that X_K is a compact 3-manifold whose boundary consists of a single torus and that $H_1(X_K;\mathbb{Z})\cong\mathbb{Z}$ is generated by the meridian of K. The usual abelianization of K is the unique epimorphism $\phi_K\colon \pi_1(X_K)\to\mathbb{Z}$ which sends a (positively oriented) meridian of K to 1.

Observe that (X_K,ϕ_K,ϕ_K) is an admissible triple, which allows us to define the *abelian L^2-Alexander torsion* $\tau^{(2)}(X_K,\phi_K,\phi_K)\colon \mathbb{R}_{>0}\to[0,\infty)$. As we saw above, the equivalence class of this function is a well-defined invariant of K. The following theorem says that the Alexander polynomial $\Delta_K(t)$ entirely determines the abelian L^2-Alexander torsion $\tau^{(2)}(X_K,\phi_K,\phi_K)$ and conversely the abelian L^2-Alexander torsion contains most of the relevant information of the Alexander polynomial.

Theorem 4.1 *Let K be a knot with Alexander polynomial $\Delta_K(z) \in \mathbb{Z}[z^{\pm 1}]$. If we write $\Delta_K(z) = C \cdot z^m \cdot \prod_{i=1}^k (z - a_i)$ with some $C \in \mathbb{Z} \setminus \{0\}, m \in \mathbb{Z}$ and $a_1, \dots, a_k \in \mathbb{C} \setminus \{0\}$, then*

$$\tau^{(2)}(X_K, \phi_K, \phi_K)(t) \doteq C \cdot \prod_{i=1}^k \max\{|a_i|, t\} \cdot \max\{1, t\}^{-1}.$$

We can draw the following observations from Theorem 4.1.

(i) The function $\tau^{(2)}(X_K, \phi_K, \phi_K)\colon \mathbb{R}_{>0} \to [0, \infty)$ is piecewise monomial.

(ii) The degree of the function $\tau^{(2)}(X_K, \phi_K, \phi_K)\colon \mathbb{R}_{>0} \to [0, \infty)$ equals the degree of the Alexander polynomial $\Delta_K(z)$.

(iii) The function $\tau^{(2)}(X_K, \phi_K, \phi_K)\colon \mathbb{R}_{>0} \to [0, \infty)$ is monic if and only if $\Delta_K(z)$ is monic.

Observe that $(X_K, \phi_K, \mathrm{id})$ is also an admissible triple, and we can consider the *full L^2-Alexander torsion* $\tau^{(2)}(K)(t) = \tau^{(2)}(X_K, \phi_K, \mathrm{id})\colon \mathbb{R}_{>0} \to [0, \infty)$. The equivalence class of this function is again a well-defined invariant of K.

The next result asserts that the full L^2-Alexander torsion $\tau^{(2)}(K)$ is a slight variation on the L^2-Alexander invariant $\Delta_K^{(2)}\colon \mathbb{R}_{>0} \to [0, \infty)$. This invariant was first introduced by Li–Zhang [9,10,11], studied by Wegner and the first author [3] and by Ben Aribi [1,2].

Theorem 4.2 *If K is a knot in S^3, then $\Delta_K^{(2)}(t) \doteq \tau^{(2)}(K)(t) \cdot \max\{1, t\}$.*

5. Properties of the L^2-Alexander torsion

A graph manifold is a 3-manifold whose JSJ decomposition has only Seifert fibered pieces. The following theorem shows that 'most' L^2-Alexander torsions of a graph manifold are entirely determined by the Thurston norm.

Theorem 5.1 *Suppose that (N, ϕ, γ) is an admissible triple with N a graph manifold such that $N \neq S^1 \times D^2$ and $N \neq S^1 \times S^2$. If given any JSJ component of N the restriction of γ to a regular fiber has infinite image (e.g. $\gamma = \mathrm{id}$), then*

$$\tau^{(2)}(N, \phi, \gamma)(t) \doteq \begin{cases} 1, & \text{if } t \leqslant 1, \\ t^{x_N(\phi)}, & \text{if } t \geqslant 1. \end{cases}$$

A consequence of Theorem 5.1 is the computation of the L^2-Alexander torsions for any iterated torus knot K and any epimorphism $\gamma\colon \pi_1(X_K) \to G$ onto a group G such that γ factors through the abelianization $\phi_K\colon \pi_1(X_K) \to \mathbb{Z}$. Indeed, it follows from Theorem 5.1 that

$$\tau^{(2)}(X_K, \phi_K, \gamma) \doteq \left(t \mapsto \max\{1, t^{2 \cdot \mathrm{genus}(K) - 1}\}\right).$$

This formula is a generalization of the result obtained in [3, Proposition 5.2] for torus knots.

Let us now consider the case of fibered manifolds. The following theorem says in particular that the L^2-Alexander torsion of a fibered 3-manifold is eventually monomial, furthermore that it is monic and that its degree is determined by the Thurston norm.

Theorem 5.2 *Suppose that (N, ϕ, γ) is an admissible triple such that $N \neq S^1 \times D^2$ and $N \neq S^1 \times S^2$. If ϕ is fibered, then there exists a constant $T \geq 1$ such that*

$$\tau^{(2)}(N, \phi, \gamma)(t) \doteq \begin{cases} 1, & \text{if } t \leqslant T^{-1}, \\ t^{x_N(\phi)}, & \text{if } t \geqslant T. \end{cases}$$

In fact, the constant T can be taken to be $\exp(h)$ where h is the entropy of the monodromy corresponding to the fibration (see [5] for the definition of the entropy).

For a general prime 3-manifold it is in general not true that the L^2-Alexander torsion detects the Thurston norm, but we now see that for a suitable epimorphism γ the corresponding L^2-Alexander torsion does indeed detect the Thurston norm.

Theorem 5.3 *Let N be a prime 3-manifold which is not a graph manifold. There exists an epimorphism $\gamma\colon \pi_1(N) \to G$ onto a virtually abelian group such that the homomorphism $\pi_1(N) \to H_1(N;\mathbb{Z})/\text{torsion}$ factors through γ and such that for any $\phi \in H^1(N;\mathbb{Z})$, the function $\tau^{(2)}(N,\phi,\gamma)$ is monomial in the limit and $\deg \tau^{(2)}(N,\phi,\gamma) = x_N(\phi)$.*

For any admissible triple (N,ϕ,γ) such that $N \neq S^1 \times D^2$, the L^2-Alexander torsion associated to (N,ϕ,γ) is a symmetric function in the sense given in the following result.

Theorem 5.4 *Suppose that (N,ϕ,γ) is an admissible triple such that $N \neq S^1 \times D^2$. For any representative $\tau\colon \mathbb{R}_{>0} \to [0,\infty)$ of $\tau^{(2)}(N,\phi,\gamma)$ there exists an integer $n \in \mathbb{Z}$ with $n \equiv x_N(\phi)$ such that $\tau(t^{-1}) = t^n \cdot \tau(t)$.*

This result can be viewed as a generalization of the symmetry property of the usual and of the twisted Alexander polynomials (see [8,6]).

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