CONCORDANCE OF LINKS AND THEIR COMPONENTS

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1. Introduction

The following theorem was first proved by Cochran [Co91] (building on techniques introduced in [Co85]) and Cochran–Orr [CO90, CO93]. This result was also recently reproved by Cha and Ruberman [CR11].

Theorem 1.1. *There exist links that have components which are smoothly concordant to the unknot but that are not topologically concordant to any link with components which have trivial Alexander polynomials.*

In this note we will give an alternative quick proof of the theorem.

2. Proof of Theorem 1.1

Given an oriented two component link $L = K_1 \cup K_2$ we can consider the corresponding Alexander polynomial $\Delta_L(x_1, x_2) \in \mathbb{Z}[x_1^{\pm 1}, x_2^{\pm 1}]$. The Alexander polynomial is well–defined up to multiplication by a unit in $\mathbb{Z}[x_1^{\pm 1}, x_2^{\pm 1}]$. In the following given $f, g \in \mathbb{Z}[x_1^{\pm 1}, x_2^{\pm 1}]$ we write $f \doteq g$ if f and g differ by multiplication by a unit in $\mathbb{Z}[x_1^{\pm 1}, x_2^{\pm 1}]$.

The following theorem was proved by Kawauchi [Ka78].

Theorem 2.1. *Let L_1 and L_2 be topologically concordant links, then there exist non–zero $f(x_1, x_2), g(x_1, x_2) \in \mathbb{Z}[x_1^{\pm 1}, x_2^{\pm 1}]$ such that*

$$\Delta_{L_1}(x_1, x_2) \cdot f(x_1, x_2) \cdot f(x_1^{-1}, x_2^{-1}) \doteq \Delta_{L_2}(x_1, x_2) \cdot g(x_1, x_2) \cdot g(x_1^{-1}, x_2^{-1}).$$

The following theorem is an immediate consequence of the Torres condition on the Alexander polynomial of a link. We refer to [Ka96, Theorem 7.4.1] for details.

Theorem 2.2. *Let $L = K_1 \cup K_2$ is 2–component link with linking number $\mathrm{lk}(K_1, K_2) = 1$, then $\Delta_L(t, 1)$ equals the Alexander polynomial of K_1 and $\Delta_L(1, t)$ equals the Alexander polynomial of K_2.*

We now say that a polynomial $p(x_1, x_2) \in \mathbb{Z}[x_1^{\pm 1}, x_2^{\pm 1}]$ is *norm–free* if there is no non–trivial $g \in \mathbb{Z}[x_1^{\pm 1}, x_2^{\pm 1}]$ such that $g(x_1, x_2) \cdot g(x_1^{-1}, x_2^{-1})$ divides $p(x_1, x_2)$. (Here by non–trivial we mean that g is not a monomial.) We now obtain the following corollary:

Corollary 2.3. *Let L be a 2–component link with linking number equal to 1 such that $\Delta_L(x_1, x_2)$ is norm–free. If L' is a link concordant to L, then $\Delta_L(t, 1)$ divides the Alexander polynomial of the first component of L' and $\Delta_L(1, t)$ divides the Alexander polynomial of the second component of L'.*

Proof. Let L' be a link which is concordant to L. By Theorem 2.1 there exist non–zero $f(x_1, x_2), g(x_1, x_2) \in \mathbb{Z}[x_1^{\pm 1}, x_2^{\pm 1}]$ such that

$$\Delta_{L_1}(x_1, x_2) \cdot f(x_1, x_2) \cdot f(x_1^{-1}, x_2^{-1}) \doteq \Delta_{L_2}(x_1, x_2) \cdot g(x_1, x_2) \cdot g(x_1^{-1}, x_2^{-1}).$$

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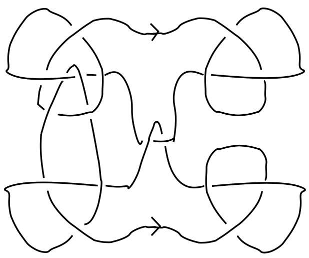
Since $\Delta_L(x_1, x_2)$ is norm–free and since $\mathbb{Z}[x_1^{\pm 1}, x_2^{\pm 1}]$ is a unique factorization domain, it follows easily that $\Delta_L(x_1, x_2)$ divides $\Delta_{L'}(x_1, x_2)$. The corollary now immediately follows from Theorem 2.2. □

We now let $K = T\# - T$, where T denotes the trefoil. It is well–known that K is concordant to the unknot. We now consider the following link: The components



are both copies of K, and the linking number is equal to 1. But a direct calculation shows that $\Delta_L(x_1, x_2)$ can be written as $\sum_{i,j} a_{ij} x_1^i x_2^j$ where the non–zero coefficients a_{ij} are given in the following table:

$i \backslash j$	0	1	2	3	4
2	2	−5	7	−5	2
1	−5	13	−18	13	−5
0	7	−18	25	−18	7
−1	−5	13	−18	13	−5
−2	2	−5	7	−5	2

Note that $\Delta_L(x_1, x_2)$ factors as follows:

$$(1 - x_1 + x_1^2) \cdot (1 - x_2 + x_2^2) \cdot (2 - 3x_1 + 2x_1^2 + x_2(-3 + 5x_1 - 3x_1^2) + x_2^2(2 - 3x_1 + 2x_1^2)).$$

It is easy to see that each factor is irreducible, and it now follows that $\Delta_L(x_1, x_2)$ is norm–free. If L' is any link concordant to L, then by Corollary 2.3 the polynomial

$$\Delta_L(t, 1) = \Delta_L(1, t) = \Delta_K(t) = (t^{-1} - 1 + t)^2$$

divides the Alexander polynomial of either component of L'.

Remark. One can in fact produce infinitely many such links by taking the above link and doing a band sum for one component with a slice knot.

References

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