CENTRALIZERS IN 3-MANIFOLD GROUPS

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Abstract. Using the Geometrization Theorem we prove a result on centralizers in fundamental groups of 3-manifolds. This result had been obtained by Jaco and Shalen and by Johannson using different techniques.

1. Introduction

In this paper we will study centralizers in fundamental groups of 3-manifolds. By a 3–manifold we will always mean a compact, orientable, connected, irreducible 3-manifold with empty or toroidal boundary.

Let π be a group. The *centralizer* of an element $g \in \pi$ is defined to be the subgroup

$$C_\pi(g) := \{h \in \pi \,|\, gh = hg\}.$$

Determining centralizers is an important step towards understanding a group. The goal of this note is to give a new proof of the following theorem.

Theorem 1.1. *Let N be a 3-manifold. We write $\pi = \pi_1(N)$. Let $g \in \pi$. If $C_\pi(g)$ is non-cyclic, then one of the following holds:*

(1) *there exists a JSJ torus or a boundary torus T and $h \in \pi$ such that $g \in h\pi_1(T)h^{-1}$ and such that*

$$C_\pi(g) = h\pi_1(T)h^{-1},$$

(2) *there exists a Seifert fibered component M and $h \in \pi$ such that $g \in h\pi_1(M)h^{-1}$ and such that*

$$C_\pi(g) = hC_{\pi_1(M)}(h^{-1}gh)h^{-1}.$$

If N is Seifert fibered, then the theorem holds trivially, and if N is hyperbolic, then it follows from well-known properties of hyperbolic 3-manifold groups (we refer to Section 3.1 for details). If N is neither Seifert fibered nor hyperbolic, then by the Geometrization Theorem N has a non-trivial JSJ decomposition, in particular N is Haken, and in that case the theorem was proved by Jaco and Shalen [6, Theorem VI.1.6] and independently by Johannson [7, Proposition 32.9].

In this note we will give an alternative proof of Theorem 1.1 for 3-manifolds with non-trivial JSJ decomposition using the Geometrization Theorem proved by Perelman. Our proof involves basic facts about fundamental groups of Seifert fibered spaces and hyperbolic 3-manifolds and it consists of a careful study of the fundamental group of the graph of groups corresponding to the JSJ decomposition.

In order to determine centralizers of 3-manifolds it thus suffices to understand centralizers of Seifert fibered spaces. For the reader's convenience we recall the results of Jaco–Shalen and Johannson. Let N be a Seifert fibered 3-manifold with a given Seifert fiber structure. Then there exists a projection map $p\colon N \to B$ where B is the base orbifold. We denote by $B' \to B$ the orientation cover, note that this is either the identity or a 2-fold cover. Following [6] we refer to $p_*^{-1}(\pi_1(B'))$ as the *canonical subgroup* of $\pi_1(N)$. If f is a regular fiber of the Seifert fibration, then we refer to the subgroup of $\pi_1(N)$ generated by f as the *fiber subgroup*. Recall that if N is non-spherical, then the fiber subgroup is infinite cyclic and normal. (Note that the fact that the fiber subgroup is normal implies in particular that it is well-defined, and not just up to conjugacy.)

Remark. Note that the definition of the canonical subgroup and of the fiber subgroup depend on the Seifert fiber structure. By [10, Theorem 3.8] (see also [9] and [6, II.4.11]) a Seifert fibered 3-manifold N admits a unique Seifert fiber structure unless N is either covered by S^3, $S^2 \times \mathbb{R}$, or the 3-torus, or $N = S^1 \times D^2$ or if N is an I-bundle over the torus or the Klein bottle.

The following theorem, together with Theorem 1.1, now classifies centralizers of non-spherical 3-manifolds.

Theorem 1.2. *Let N be a non-spherical Seifert fibered 3-manifold with a given Seifert fiber structure. Let $g \in \pi = \pi_1(N)$ be a non-trivial element. Then the following hold:*

1. *if g lies in the fiber group, then $C_\pi(g)$ equals the canonical subgroup,*
2. *if g does not lie in the fiber group, then the intersection of $C_\pi(g)$ with the canonical subgroup is abelian, in particular $C_\pi(g)$ admits an abelian subgroup of index at most two,*
3. *if g does not lie in the canonical subgroup, then $C_\pi(g)$ is infinite cyclic.*

The first statement is [6, Proposition II.4.5]. The second and the third statement follow from [6, Proposition II.4.7]. Using Theorems 1.1 and 1.2 one can now immediately obtain results on root structures and the divisibility of elements in 3-manifold groups. We refer to [1, Section 4] for details.

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2. Graphs of groups

In this section we summarize some basic definitions and facts concerning graphs of groups and their fundamental groups. We refer to [2, 3, 11] for missing details.

2.1. **Graphs.** A *graph* $\mathcal{Y}$ consists of a set $V = V(\mathcal{Y})$ of *vertices* and a set $E = E(\mathcal{Y})$ of *edges*, and two maps $E \to V \times V : e \mapsto (o(e), t(e))$ and $E \to E : e \mapsto \overline{e}$, subject to the following condition: for each $e \in E$ we have $\overline{\overline{e}} = e$, $\overline{e} \neq e$, and $o(e) = t(\overline{e})$. We sometimes also denote $\overline{e}$ by e^{-1}. Throughout this paper, all graphs are understood to be connected and finite (i.e., their vertex sets and edge sets are finite).

2.2. **The fundamental group of a graph of groups.** Let $\mathcal{Y}$ be a graph. A *graph* $\mathcal{G}$ *of groups based on* $\mathcal{Y}$ consists of families $\{G_v\}_{v\in V(\mathcal{Y})}$ and $\{G_e\}_{e\in E(\mathcal{Y})}$ of groups satisfying $G_e = G_{\overline{e}}$ for every $e \in E(\mathcal{Y})$, together with a family $\{\varphi_e\}_{e\in E(\mathcal{Y})}$ of monomorphisms $\varphi_e : G_e \to G_{t(e)}$ $(e \in E(\mathcal{Y}))$. We will refer to $\mathcal{Y}$ as the *underlying graph* of $\mathcal{G}$.

Let $\mathcal{G}$ be a graph of groups based on a graph $\mathcal{Y}$. We recall the construction of the fundamental group $G = \pi_1(\mathcal{G})$ of $\mathcal{G}$ from [11, I.5.1]. First, consider the *path group* $\pi(\mathcal{G})$ which is generated by the groups G_v $(v \in V(\mathcal{Y}))$ and the elements $e \in E(\mathcal{Y})$ subject to the relations

$$e\varphi_e(g)\overline{e} = \varphi_{\overline{e}}(g) \qquad (e \in E(\mathcal{Y}), g \in G_e).$$

By a *path in* $\mathcal{Y}$ from a vertex v to a vertex w we mean a sequence $(e_1, e_2, \dots, e_n)$ where $o(e_1) = v, t(e_i) = o(e_{i+1}), i = 1, \dots, n-1$ and $t(e_n) = w$.

By a *path in* $\mathcal{G}$ from a vertex v to a vertex w we mean a sequence

$$(g_0, e_1, g_1, e_2, \dots, e_n, g_n),$$

of elements in E where $(e_1, \dots, e_n)$ is a path of length n in $\mathcal{Y}$ from v to w and where $g_0 \in G_v$ and where $g_i \in G_{t(e_i)}$ for $i = 1, \dots, n$. We write $l(\gamma) = n$ and call it the length of γ. We say that the path γ *represents* the element

$$g = g_0 e_1 g_1 e_2 \cdots e_n g_n$$

of $\pi(\mathcal{G})$.

Let now w be a fixed vertex of $\mathcal{Y}$. We will refer to a path from w to w as a *loop based at* w. The fundamental group $\pi_1(\mathcal{G}, w)$ of $\mathcal{G}$ (with base point w) is defined to be the subgroup of $\pi(\mathcal{G})$ consisting of elements represented by loops based at w. If $w' \in V(\mathcal{Y})$ is another base point, and g is an element of $\pi(\mathcal{G})$ represented by a path from w' to w, then $\pi_1(\mathcal{G}, w') \to \pi_1(\mathcal{G}, w) \colon t \mapsto g^{-1}tg$ is an isomorphism. By abuse of notation we write $\pi_1(\mathcal{G})$ to denote $\pi_1(\mathcal{G}, w)$ if the particular choice of base point is irrelevant.

Now let $v \in V$. Pick a path g from v to w. Then the map $G_v \to \pi_1(\mathcal{G}, w)$ given by $t \mapsto g^{-1}tg$ defines a group morphism which is injective (see again [11, I.5.2, Corollary 1]). In particular the vertex groups define subgroups of $\pi_1(\mathcal{G}, w)$ which are well-defined up to conjugation. Given a graph of groups $\mathcal{G}$ and a base vertex w it is always understood that for each vertex v we picked once and for all a path from v to w.

We will later on make use of the following operations on paths. Given a path p in $\mathcal{G}$ from v_1 to v_2 we write $o(p) = v_1$ and $t(p) = v_2$. Given two paths

$$\begin{aligned} p &:= (g_0, e_1, g_1, e_2, \ldots, e_n, g_n), \text{ and} \\ q &:= (h_0, f_1, h_1, f_2, \ldots, f_m, h_m), \end{aligned}$$

with $t(p) = o(q)$ we define

$$p * q := (g_0, e_1, g_1, e_2, \ldots, e_n, g_n \cdot h_0, f_1, h_1, f_2, \ldots, f_m, h_m)$$

which is a path from $o(p)$ to $t(q)$. Furthermore, given a path

$$p := (g_0, e_1, g_1, e_2, \ldots, e_n, g_n)$$

we define the inverse path to be

$$p^{-1} := (g_n^{-1}, \overline{e_n}, \ldots, g_1^{-1}, \overline{e_1}, g_0^{-1}).$$

Note that p^{-1} is a path from $t(p)$ to $o(p)$.

2.3. **Reduced paths.** A path $(g_0, e_1, g_1, e_2, \ldots, e_n, g_n)$ in $\mathcal{G}$ is *reduced* if it satisfies one of the following conditions:

(1) $n = 0$, or
(2) $n > 0$ and $g_i \notin \varphi_{e_i}(G_{e_i})$ for each index i such that $e_{i+1} = \overline{e_i}$.

Given $g \in \pi(\mathcal{G})$ we define its length $l(g)$ to be the length of a reduced path representing it. Note that this is well-defined (see [11, p. 4]), i.e. any g is represented by a reduced path and the definition is independent of the choice of the reduced path. Also note that

$$l(g) = \min\{l(p) \,|\, p \text{ a path which represents } g\}.$$

Note that $l(g) = 0$ if and only if g lies in G_v for some $v \in V$.

We say that $s = (g_0, e_1, g_1, e_2, \ldots, e_n, g_n)$ is *cyclically reduced* if s is reduced and if one of the following holds:

(1) $n = 0$, or
(2) $e_1 \neq \overline{e_n}$, or
(3) $e_1 = \overline{e_n}$ but $g_n g_0$ is not conjugate to an element in $\operatorname{Im}(\varphi_{e_n})$.

Note that a reduced loop $s = (g_0, e_1, g_1, e_2, \dots, e_n, g_n)$ is cyclically reduced if and only if the element it represents has minimal length in its conjugacy class in the path group $\pi(\mathcal{G})$.

We say that $g \in \pi_1(\mathcal{G}, w)$ is *cyclically reduced* if there exists a cyclically reduced loop which represents it. It is straightforward to see that g is cyclically reduced if and only if any reduced loop representing it is cyclically reduced. Also note that if g is cyclically reduced, then $l(g^n) = n \cdot l(g)$.

Any element g of the path groups $\pi(\mathcal{G})$ is conjugate in $\pi(\mathcal{G})$ to a cyclically reduced element s, we can thus define $cl(g) = l(s)$. Note that this is independent of the choice of s. Note that if g is cyclically reduced, then a straightforward argument shows that $l(g^n) = n \cdot l(g)$. In particular given any g we have $cl(g^n) = n \cdot cl(g)$.

3. Fundamental groups of 3-manifolds

In the next two sections we cover properties of fundamental groups of hyperbolic 3-manifold groups and of Seifert fibered spaces, before we return to the study of 3-manifold groups in general.

3.1. **Fundamental groups of hyperbolic 3-manifolds.** Let N be a 3-manifold. We say that N is hyperbolic if the interior admits a complete metric of finite volume and constant sectional curvature equal to -1.

Throughout this section we write

$$U := \left\{ \begin{pmatrix} \varepsilon & a \\ 0 & \varepsilon \end{pmatrix} \text{ with } \varepsilon \in \{-1, 1\} \text{ and } a \in \mathbb{C} \right\} \subset \mathrm{SL}(2, \mathbb{C}).$$

Note that U is an abelian subgroup of $\mathrm{SL}(2, \mathbb{C})$. Recall that $A \in \mathrm{SL}(2, \mathbb{C})$ is called *parabolic* if it is conjugate to an element in U. We say that A is *loxodromic* if A is diagonalizable with eigenvalues λ, λ^{-1} such that $|\lambda| > 1$. We recall the following well known proposition.

Proposition 3.1. *Let N be a hyperbolic 3-manifold. Then the following hold:*

(1) *There exists a faithful discrete representation $\rho\colon \pi_1(N) \to \mathrm{SL}(2, \mathbb{C})$.*
(2) *Let $g \in \pi_1(N)$, then $\rho(g)$ is either parabolic or loxodromic.*
(3) *An element $g \in \pi_1(N)$ is conjugate to an element in a boundary component if and only if $\rho(g)$ is parabolic.*

(4) *Let T be a boundary torus, then there exists a matrix $P \in \mathrm{SL}(2,\mathbb{C})$ such that $P\rho(\pi_1(T))P^{-1} \subset U$.*
(5) *Let $g \in \pi_1(N)$. Then $C_g(\pi_1(N))$ is either infinite cyclic or a free abelian group of rank two. The latter case occurs precisely when g is conjugate to an element in a boundary component T and in that case $C_g(\pi_1(N))$ is a conjugate of $\pi_1(T)$.*

We include the proof of the proposition for completeness' sake.

Proof. (1) A hyperbolic 3-manifold N admits a faithful discrete representation $\pi_1(N) \to \mathrm{Isom}(\mathbb{H}^3) = \mathrm{PSL}(2,\mathbb{C})$. Thurston (see [12, Section 1.6]) showed that this representation lifts to a faithful discrete representation $\pi_1(N) \to \mathrm{SL}(2,\mathbb{C})$.
(2) This follows immediately from considering the Jordan transform of $\rho(g)$ and from the fact that the infinite cyclic group generated by $\rho(g)$ is discrete in $\mathrm{SL}(2,\mathbb{C})$.
(3) This is well-known.
(4) This statement follows easily from the fact that $\pi_1(T) \subset \mathrm{SL}(2,\mathbb{C})$ is a discrete subgroup isomorphic to $\mathbb{Z}^2$.
(5) By (1) we can view $\pi = \pi_1(N)$ as a discrete, torsion-free subgroup of $\mathrm{SL}(2,\mathbb{C})$. Note that the centralizer of any non-trivial matrix in $\mathrm{SL}(2,\mathbb{C})$ is abelian (this can be seen easily using the Jordan normal form of such a matrix). Now let $g \in \pi \subset \mathrm{SL}(2,\mathbb{C})$ be non-trivial. Since π is torsion-free and discrete in $\mathrm{SL}(2,\mathbb{C})$ it follows easily that $C_\pi(g)$ is in fact either infinite cyclic or a free abelian group of rank two. It now follows from [13, Proposition 5.4.4] (see also [10, Corollary 4.6] for the closed case) that there exists a boundary component S and $h \in \pi_1(N)$ such that
$$C_\pi(g) = h\pi_1(S)h^{-1}.$$
□

Given a group π we say that an element g is *divisible by an integer* n if there exists an $h \in \pi$ with $g = h^n$. We say g is *infinitely divisible* if g is divisible by infinitely many integers. The following lemma is an immediate consequences of Proposition 3.1 (5).

Lemma 3.2. *Let $\pi \subset \mathrm{SL}(2,\mathbb{C})$ be a discrete torsion-free group. Then π does not contain any non-trivial elements which are infinitely divisible.*

Let π be a group. We say that a subgroup $H \subset \pi$ is *division closed* if for any $g \in \pi$ and $n > 0$ with $g^n \in H$ the element g already lies in H. The following lemma is an immediate consequence of Proposition 3.1 (2) and (5) and from the observation that $A \subset \mathrm{SL}(2,\mathbb{C})$ is parabolic (respectively loxodromic) if and only if a non-trivial power of A is parabolic (respectively loxodromic).

Lemma 3.3. *Let N be a 3-manifold such that the interior of N is a hyperbolic 3-manifold of finite volume. Let T be a boundary component of N. Then $\pi_1(T) \subset \pi_1(N)$ is division closed.*

Let π be a group. We say that a subgroup H is *malnormal* if $gHg^{-1} \cap H$ is trivial for any $g \notin H$. The following lemma is well-known.

Lemma 3.4. *Let N be a hyperbolic 3-manifold.*

(1) *Let T be a boundary torus. Then $\pi_1(T) \subset \pi_1(N)$ is malnormal.*
(2) *Let T_1 and T_2 be distinct boundary tori. Then for any $g \in \pi_1(N)$ we have $\pi_1(T_1) \cap g\pi_1(T_2)g^{-1} = \{e\}$.*

3.2. **Fundamental groups of Seifert fibered manifolds.** Let N be a Seifert fibered space with regular fiber c. First note that if T is a boundary torus, then the Seifert fibration restricted to T induces a product structure. It follows that $c \in \pi_1(T)$ and that c is indivisible in $\pi_1(T) \cong \mathbb{Z}^2$.

The following results summarize the key properties of fundamental groups of Seifert fibered spaces which are relevant to our discussion.

Theorem 3.5. *Let N be a Seifert fibered 3-manifold with regular fiber c. Then there exists an $s \in \mathbb{N}$ with the following property: If T is a boundary component, and if $g \notin \pi_1(T)$ but some power of g lies in $\pi_1(T)$, then there exists $d \leq s$ such that $g^d = c$ or $g^d = c^{-1}$.*

Proof. Let N be a Seifert fibered 3-manifold with boundary. Let s be the maximum order of a singular fiber of the fibration. Let T be a boundary component, and let $g \notin \pi_1(T)$ such that some power of g lies in $\pi_1(T)$. We denote by $p : N \to B$ the projection to the base orbifold. We denote by b the boundary curve of B corresponding to T. Note that $p(g) \notin \langle b \rangle$ but a power of $p(g)$ lies in $\langle b \rangle$. It follows easily from [6, Remark II.3.1] that $p(g)$ is of finite order. In particular g corresponds to a singular fiber, and then it follows from the definition of s that there exists a $d \leq s$ such that $g^d = c$ or $g^d = c^{-1}$. □

Lemma 3.6. *Let N be a Seifert fibered 3-manifold with regular fiber c and let T be a boundary component. Let $g \in \pi_1(T)$ which is not a power of c, then $C_g(\pi_1(N)) = \pi_1(T)$.*

Proof. We denote by $p : N \to B$ the projection to the base orbifold. Note that $p(g) \in \pi_1(B)$ is non-trivial. It follows easily from [6, Remark II.3.1] that $C_{p(g)}(\pi_1(B))$ is the group generated by the boundary curve of N corresponding to T. It follows easily that $C_g(\pi_1(N)) = \pi_1(T)$. □

The following lemma is also well-known. It can be proved in a similar fashion as Lemma 3.6 by considering the equivalent problem in the fundamental group of the base manifold.

Lemma 3.7. *Let N be a Seifert fibered 3-manifold. Denote by $c \in \pi_1(N)$ the element represented by a regular fiber.*

1. *Let T be a boundary torus and $g \in \pi_1(N) \setminus \pi_1(T)$, then $\pi_1(T) \cap g\pi_1(T)g^{-1} = \langle c \rangle$.*
2. *Let T_1 and T_2 be distinct boundary tori. Then for any $g \in \pi_1(N)$ we have $\pi_1(T_1) \cap g\pi_1(T_2)g^{-1} = \langle c \rangle$.*

We conclude with the following lemma.

Lemma 3.8. *Let N be a non-spherical Seifert fibered manifold. Then $\pi_1(N)$ does not contain non-trivial elements which are infinitely divisible.*

Proof. Let N be a Seifert fibered manifold. Then there exists a finite cover N' which is an S^1-bundle over a surface S (see e.g. [5, p. 391] for details). We write $\Gamma = \pi_1(S)$, $\pi = \pi_1(N)$ and $\pi' = \pi_1(N')$. If N is non-spherical then the long exact sequence in homotopy implies that there exists a short exact sequence

$$1 \to \mathbb{Z} \to \pi' \to \Gamma \to 1.$$

Since $\mathbb{Z}$ and Γ are well-known not to admit any non-trivial infinitely divisible elements, it follows easily that π' does not admit a non-trivial infinitely divisible element. We write $n = [\pi : \pi']$. Since N is non-spherical we know that π is torsion-free. Note that if $g \in \pi$ is non-trivial, then g^n lies in π' and it is also non-trivial. It is now easy to see that π can not admit a non-trivial infinitely divisible element either. □

3.3. **3-manifolds and graphs of groups.** In this section we recall the well-known interpretation of 3-manifold groups as the fundamental group of a graph of groups. Let N be an irreducible, closed, oriented 3-manifold. Recall that the JSJ tori are a minimal collection $\{T_1, \dots, T_k\}$ of tori such that the complements of the tori are either atoroidal or Seifert fibered.

We denote by $\mathcal{G}(N)$ the corresponding JSJ graph, i.e. the vertex set $V = V(\mathcal{G})$ of $\mathcal{G}$ consists of the set of components of N cut along $T_1, \dots, T_k$ pieces and the set $E = E(\mathcal{G})$ of (unoriented) edges consists of the set of JSJ tori $T_1, \dots, T_k$. We sometimes denote the JSJ tori by $T_e, e \in E$ and we denote the components of N cut along $\cup_{e \in E} T_e$ by $N_v, v \in V$. We equip each T_e with an orientation, we thus obtain two canonical embeddings $i_\pm$ of T_e into N cut along T_e. We then denote by $o(e) \in V$ the unique vertex with $i_-(T_e) \in N_{i(e)}$ and we denote by $t(e) \in V$ the unique vertex with $i_+(T_e) \in N_{f(e)}$.

Suppose that N has a non-trivial JSJ decomposition. Then given a Seifert fibered component N_v of the JSJ decomposition of N we denote by $c_v \in \pi_1(N_v)$ the group element defined by a corresponding regular fiber. Note that c_v is well-defined up to inversion (see [14, Lemma 1] or [4]).

We conclude this section with the following theorem.

Theorem 3.9. *Let N be a closed, oriented 3-manifold. Denote by $\mathcal{G} = \mathcal{G}(N)$ the corresponding JSJ graph. If e is an edge such that $o(e)$ and $t(e)$ correspond to Seifert fibered spaces, then $\varphi_e^{-1}(c_{t(e)}) \neq c_{o(e)}^{\pm 1}$.*

Proof. If $\varphi_e^{-1}(c_{t(e)})$ was equal to $c_{o(e)}^{\pm 1}$, then $N_{o(e)}$ and $N_{t(e)}$ would have Seifert fiber structures which (after an isotopy) match along the edge torus. But this contradicts the minimality of the JSJ decomposition. □

4. Proof of the main results

4.1. **Divisibility in 3-manifold groups.** We will first prove the following theorem.

Theorem 4.1. *Let N be a 3-manifold. If N is not spherical, then $\pi_1(N)$ does not contain any non-trivial elements which are infinitely divisible.*

Proof. Let N be a non-spherical 3-manifold and let $x \in \pi_1(N)$ be a non-trivial element. Since the statement of theorem is independent of the choice of base point and conjugation we can without loss of generality assume that $l(x) = cl(x)$. We write $l = l(x)$.

First suppose that $l > 0$. Suppose we have $y \in \pi_1(N)$ and n such that $y^n = x$. Note that $0 < cl(x) = cl(y^n) = n \cdot cl(y)$. It now follows immediately that $n \leq l = cl(x)$.

Now suppose that $l = 0$. Note that this means that x lies in a vertex group $\pi_1(N_w)$. We now define

$$d := \max\{n \in \mathbb{N} \mid x = y^n \text{ for some } y \in \pi_1(N_w)\}.$$

Note that $d < \infty$ by Lemmas 3.2 and 3.8. Furthermore, given a Seifert fibered component N_v we define

$$s_v := \text{maximum of the orders of the singular fibers of } N_v.$$

Finally we define s to be the maximum over all s_v. If there are no Seifert fibered components, then we set $s = 1$. The following claim now implies the theorem.

Claim. If there exists $y \in \pi_1(N)$ and $n \in \mathbb{N}$ with $y^n = x$, then $n \leq ds$.

Suppose we have $y \in \pi_1(N)$ and n such that $y^n = x$. Note that $0 = l(x) = cl(x) = cl(y^n) = n \cdot cl(y)$. It now follows that $cl(y) = 0$. If $l(y) = 0$, then $y \in \pi_1(N_w)$, hence the conclusion holds trivially by the definition of d. Now suppose that $l(y) > 0$. Then there exists a reduced path $p = (g_0, e_1, g_1, \dots, e_l, g_l)$ from w to a vertex v and $z \in \pi_1(N_v)$ such that y is represented by $p * z * p^{-1}$. Among all such pairs (p, z) we pick a pair which minimizes the length of p.

Since p is minimal and $l(p) > 0$ we see that $g_l z g_l^{-1} \notin \mathrm{Im}(\varphi_{e_l})$. On the other hand $p * z^n * p^{-1}$ represents $y^n = x$, hence this path is reduced, which implies that $g_l z^n g_l^{-1} \in \mathrm{Im}(\varphi_{e_l})$. It follows that $\mathrm{Im}(\varphi_{e_l})$ is not division closed, using Lemma 3.3 we conclude that N_v is Seifert fibered.

We denote by c_v the regular fiber of N_v. Recall that by Theorem 3.5 there exists $r|s_v$ with $g_l z^r g_l^{-1} = c_v$. It also follows from Theorem 3.5 that $g_l z^n g_l^{-1} = c_v^m \in \mathrm{Im}(\varphi_{e_l})$ for some m. Note that $n = mr$.

We can now apply Lemmas 3.4 and 3.7, Theorem 3.9 and the fact that p is reduced to conclude that

$$(g_0, e_1, g_1, \dots, e_{l-1}, g_{l-1}\varphi_{e_l}^{-1}(c_v^m)g_{l-1}^{-1}, e_{l-1}^{-1}, \dots, g_1^{-1}, e_1^{-1}, g_0^{-1})$$

is reduced. It follows that $l = 1$. Note that

$$x = g_0\varphi_{e_1}^{-1}(c_v^m)g_0^{-1} = (g_0\varphi_{e_1}^{-1}(c_v)g_0^{-1})^m.$$

It follows that $m \leq d$. We also have $r \leq s_v \leq s$. We now conclude that $n = mr \leq ds$.

□

4.2. **Commuting elements in 3-manifold groups.**

Theorem 4.2. *Let N be a 3-manifold. Let $x, y \in \pi_1(N)$ with $x = yxy^{-1}$. Then one of the following holds:*

1. *x and y generate a cyclic group in $\pi_1(N)$, or*
2. *there exists a JSJ torus T such that x and y lie in a conjugate of $\pi_1(T) \subset \pi_1(N)$, or*
3. *there exists a Seifert fibered component M of the JSJ decomposition such that x and y lie in a conjugate of $\pi_1(M) \subset \pi_1(N)$.*

Proof. Let N be a 3-manifold. Denote by $\mathcal{G} = \mathcal{G}(N)$ the corresponding JSJ graph with vertex set V and edge set E. We denote by $w \in V$ the vertex which contains the base point of N. We denote the vertex groups by $G_v = \pi_1(N_v), v \in V$.

The theorem holds trivially for Seifert fibered spaces, we can therefore assume that N is not a Seifert fibered space, in particular that N is not spherical. Suppose we have $x, y \in \pi_1(N)$ with $x = yxy^{-1}$. By the symmetry of x and y we can without loss of generality assume that $cl(x) \le cl(y)$. Note that the statement of the theorem does not change under conjugation and change of base point, we can therefore without loss of generality assume that $cl(x) = l(x)$.

We represent y by a reduced loop $p = (h_0, f_1, h_1, \ldots, f_{l-1}, h_{l-1}, f_l, h_l)$ based at w. If $l = 0$, then $l(x) = 0$ as well since $l(x) = cl(x) \le cl(y) \le l(y) = 0$. In that case we are done by Proposition 3.1 (5). We thus henceforth only consider the case that $l \ge 1$.

After conjugating x and y with h_l we can without loss of generality assume that $h_l = 1$. Recall that p being reduced implies that for $i = 2, \ldots, l$ the following holds:

$$f_i \ne \overline{f_{i-1}} \text{ or } f_i = \overline{f_{i-1}} \text{ and } h_{i-1} \notin \operatorname{Im}(\varphi_{f_{i-1}}). \tag{4.1}$$

We first study the case that $l(x) = 0$, i.e. $x \in G_w$. Clearly we can assume that x is non-trivial.

Now consider

$$p * x * p^{-1} = (h_0, f_1, h_1, \ldots, f_l, x, f_l^{-1}, \ldots, h_1^{-1}, f_1^{-1}, h_0^{-1}).$$

This path is not reduced since yxy^{-1} can be represented by a path of length zero. It follows that $x \in \operatorname{Im}(\varphi_{f_l})$. We can now represent $x = yxy^{-1}$ by the following path:

$$(h_0, f_1, h_1, \ldots, f_{l-1}, h_{l-1}\varphi_{f_l}^{-1}(x)h_{l-1}^{-1}, f_{l-1}^{-1}, \ldots, h_1^{-1}, f_1^{-1}, h_0^{-1}). \tag{4.2}$$

Case 1: $l = 1$, i.e. $y = (h_0, f_1, 1)$. In that case $yxy^{-1} = x$ is represented by $h_0\varphi_{f_1}^{-1}(x)h_0^{-1}$. It follows that $x \in \operatorname{Im}(\varphi_{f_1})$ and $x \in h_0 \operatorname{Im}(\varphi_{\overline{f_1}})h_0^{-1}$. But if $t(f_1) = o(f_1)$ is hyperbolic this is not possible by Lemma 3.4 since the two boundary tori of $N_{t(f_1)} = N_{o(f_1)}$ corresponding to the edge f_1 are obviously different. If $t(f_1) = o(f_1)$ is Seifert fibered, then we can similarly exclude this case by appealing to Lemma 3.7 and Theorem 3.9.

Case 2: The vertex $o(f_l)$ is hyperbolic. It follows easily from (4.1) and Lemma 3.4 that the path (4.2) is reduced. Since the path represents x this implies in particular that $l = 1$. We thus reduced Case 2 to Case 1.

Case 3: The vertex $o(f_l)$ is Seifert fibered and $\varphi_{f_l}^{-1}(x) \notin \langle c_{o(f_l)} \rangle$. Note that Lemma 3.7 together with Theorem 3.9 and (4.1) implies that the path (4.2) is reduced, i.e. $l = 1$. We thus also reduced Case 3 to Case 1.

Case 4: The vertex $o(f_l)$ is Seifert fibered, $\varphi_{f_l}^{-1}(x) \in \langle c_{o(f_l)} \rangle$ and $l > 1$. Note that by Theorem 3.5 (2) this implies that $h_{l-1}\varphi_{f_l}^{-1}(x)h_{l-1}^{-1} \in \operatorname{Im}(\varphi_{f_{l-1}})$. We can thus represent x by

$$(h_0, f_1, \dots, f_{l-2},\ h_{l-2} \cdot \varphi_{f_{l-1}}^{-1}(h_{l-1}\varphi_{f_l}^{-1}(x)h_{l-1}^{-1}) \cdot h_{l-2}^{-1},\ f_{l-2}^{-1}, \dots, f_1^{-1}, h_0^{-1}).$$

If $o(f_{l-1})$ is hyperbolic, then the argument of Case 2 immediately shows that $l = 2$. If $o(f_{l-1})$ is Seifert fibered, then it follows from Theorems 3.5 and 3.9 and from Lemma 3.7 (2) that $h_{l-2}\cdot\varphi_{f_{l-1}}^{-1}(h_{l-1}\varphi_{f_l}^{-1}(x)h_{l-1}^{-1})\cdot h_{l-2}^{-1} \notin \langle c_{o(f_{l-1})} \rangle$. The argument of Case 3 immediately shows that again $l = 2$.

We now showed that $l = 2$, we thus see that x equals

$$h_0 \cdot \varphi_{f_{l-1}}^{-1}(h_{l-1}\varphi_{f_l}^{-1}(x)h_{l-1}^{-1}) \cdot h_0^{-1}.$$

If $o(f_1) = t(f_2)$ is hyperbolic, then $x \in \operatorname{Im}(\varphi_{f_2})$ and $x \in h_0 \operatorname{Im}(\varphi_{\overline{f_1}})h_0^{-1}$. It follows from Lemma 3.4 that $f_1 = \overline{f_2}$ and $h_0 \in \operatorname{Im}(\varphi_{\overline{f_1}})$. If we change the base point to $o(f_2) = t(f_1)$ we see that x is represented by $\varphi_{f_2}^{-1}(x) \in G_{o(f_2)}$ and y is represented by $\varphi_{f_1}(h_0)h_1 \in G_{o(f_2)}$. If on the other hand $o(f_1) = t(f_2)$ is Seifert fibered, then it follows from Theorem 3.9 that $x \notin \langle c_{t(f_2)} \rangle$. It now follows easily from Lemma 3.7 that $f_1 = \overline{f_2}$ and $h_0 \in \operatorname{Im}(\varphi_{\overline{f_1}})$. We conclude the argument as above.

We now turn to the case that $l(x) > 0$. We claim that Conclusion (1) holds. By Theorem 4.1 we can find $z \in \pi_1(N)$ which is indivisible and $n > 0$ with $x = z^n$. Without loss of generality assume that z is cyclically reduced. We claim that y is a power of z as well. We represent z by a reduced loop $q = (g_0, e_1, g_1, \dots, e_k, g_k)$. We now consider the path $p * q^n * p^{-1}$ which is given by

$$(h_0, f_1, h_1, \dots, f_l, h_l \cdot g_0, e_1, g_1, \dots, e_k, g_k \cdot h_l^{-1}, f_l^{-1}, \dots, h_1^{-1}, f_1^{-1}, h_0^{-1}).$$

This loop has to be reduced since $l > 0$ and therefore the loop is longer than the loop q^n which represents the same element. We conclude that one of the following conditions hold:

(1) $f_l = \overline{e_1}$ and $h_l g_0 \in \operatorname{Im}(\varphi_{f_l})$, or
(2) $e_k = f_l$ and $g_k h_l^{-1} \in \operatorname{Im}(\varphi_{e_k})$.

Note though that not both conclusions can hold, otherwise x would not be cyclically reduced. Now suppose that (1) holds and (2) does not hold. A straightforward induction argument now shows that $p = p' * q^{-1}$ for some reduced path p'. On the other hand, if (2) holds and (1) does not hold, then a straightforward induction argument shows that $p = q^{-1} * p'$ for some reduced path p'.

Claim. If $l(p') = 0$, then p' represents the trivial element.

If $l(p') = 0$, then we denote by y' the element represented by p'. Suppose that y' is non-trivial. In that case we have $y'x^n(y')^{-1} = x^n$ for any n, in particular $x^n y' x^{-n} = y'$. It follows from the discussion of Cases 1, 2, 3 and 4 above that $l(x^n) \leq 2$ for any n. Since x is cyclically reduced and $l(x) > 0$ this case can not occur. This concludes the proof of the claim.

If p' represents the trivial element we are clearly done. If not, then $l(p') > 0$ and we can do an induction argument on the length of p' to show that y is in fact a power of z. $\square$

4.3. **Proof of Theorem 1.1.** For the reader's convenience we recall the statement of Theorem 1.1.

Theorem 4.3. *Let N be a 3-manifold. We write $\pi = \pi_1(N)$. Let $g \in \pi$. If $C_\pi(g)$ is non-cyclic, then one of the following holds:*

(1) *there exists a JSJ torus or a boundary torus T and $h \in \pi$ such that $g \in h\pi_1(T)h^{-1}$ and such that*
$$C_\pi(g) = h\pi_1(T)h^{-1},$$
(2) *there exists a Seifert fibered component M and $h \in \pi$ such that $g \in h\pi_1(M)h^{-1}$ and such that*
$$C_\pi(g) = hC_{\pi_1(M)}(h^{-1}gh)h^{-1}.$$

Proof. Let N be a 3-manifold and let $g \in \pi = \pi_1(N)$. If for any $h \in C_\pi(g)$ the group generated by g and h is cyclic, then either $C_\pi(g)$ is cyclic, or g is infinitely divisible. Since the former case is excluded by Theorem 4.1 the latter case has to hold.

Now suppose that $C_\pi(g)$ is not cyclic and suppose that there exist an $h \in C_\pi(g)$ such that the group generated by g and h is not cyclic. It follows from Theorem 4.2 that one of the following three cases occurs:

(1) there exists a JSJ torus T such that g lies in a conjugate of $\pi_1(T) \subset \pi_1(N)$,
(2) there exists a Seifert fibered component M of the JSJ decomposition such that g lies in a conjugate of $\pi_1(M) \subset \pi_1(N)$,

First suppose there exists a JSJ torus T such that g lies in a conjugate of $\pi_1(T) \subset \pi_1(N)$. Without loss of generality we can assume that $g \in \pi_1(T)$. We first consider the case that the two JSJ components abutting T are different. We denote these two components by M_1 and M_2. By Proposition 3.1 (5) the following claim implies the theorem in this case.

Claim. There exists an $i \in \{1, 2\}$ such that
$$C_\pi(g) = C_{\pi_1(M_i)}(g).$$

Let $h \in C_\pi(g)$. It follows easily from the proof of Theorem 4.2 that either $h \in \pi_1(M_1)$ or $h \in \pi_1(M_2)$. If M_1 is hyperbolic, then it follows from Lemma 3.2 and from Proposition 3.1 (5) that $h \in \pi_1(T)$. It follows that $C_\pi(g) = C_{\pi_1(M_2)}(g)$. Similarly we deal with the case that M_2 is hyperbolic. Finally assume that M_1 and M_2 are Seifert fibered. We denote by c_1 and c_2 the regular fibers of M_1 and M_2. If g is not a power of c_1, then it follows from Lemma 3.6 that $C_\pi(g) = C_{\pi_1(M_2)}(g)$, similarly if g is not a power of c_2. Recall that c_1 and c_2 are indivisible in $\pi_1(T)$ and that by Theorem 3.9 we have $c_1 \neq c_2^{\pm 1}$. It follows that g is either not a power of c_1 or not a power of c_2.

The case that the torus is non-separating can be dealt with similarly. We leave this to the reader. Also, if there exists a Seifert fibered component M of the JSJ decomposition such that g lies in a conjugate of $\pi_1(M) \subset \pi_1(N)$ and such that g does not lie in the image of a boundary torus, then it follows easily from the proof of Theorem 4.2 that

$$C_\pi(g) = C_{\pi_1(M)}(g).$$

□

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