SUTURED FLOER HOMOLOGY, FIBRATIONS, AND TAUT DEPTH ONE FOLIATIONS

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ABSTRACT. For an oriented irreducible 3-manifold M with non-empty toroidal boundary, we describe how sutured Floer homology (SFH) can be used to determine all fibered classes in $H^1(M)$. Furthermore, we show that the SFH of a balanced sutured manifold (M,γ) detects which classes in $H^1(M)$ admit a taut depth one foliation such that the only compact leaves are the components of $R(\gamma)$. The latter had been proved earlier by the first author under the extra assumption that $H_2(M)=0$. The main technical result is that we can obtain an extremal Spin^c-structure $\mathfrak{s}$ via a nice and taut sutured manifold decomposition even when $H_2(M)\neq 0$, assuming that $\mathfrak{s}$ supports a group with non-trivial Euler characteristic.

1. INTRODUCTION

Ghiggini [Gh08], Ni [Ni07, Ni09a], and the third author [Ju08, Ju10] showed that knot Floer homology detects fibered knots. Furthermore Ni [Ni09b] proved that HF^+ detects whether a closed irreducible 3-manifold fibers over S^1 if the genus of the fiber is greater than one. Also, Ai and Yi [AN09] showed that Heegaard Floer homology with twisted coefficients detects torus bundles. It is now natural to ask if Heegaard Floer type invariants can also detect whether an arbitrary 3-manifold with non-empty boundary is fibered (in which case the boundary has to be toroidal). If M is an irreducible 3-manifold such that the map $H_2(\partial M;\mathbb{Q})\to H_2(M;\mathbb{Q})$ is surjective, that is, if M is an irreducible 3-manifold which is the exterior of a link in a rational homology sphere, then, as we shall see in Proposition 5.2, it is straightforward that the methods of [Ju08, Ju10] imply that sutured Floer homology detects whether or not M is fibered. In this paper, we show that this restriction on M can be dropped, in particular, we will prove the following theorem (for a precise formulation, we refer the reader to Section 5).

Theorem 1.1. *Sutured Floer homology detects whether or not an irreducible 3-manifold M with non-empty boundary is fibered, and determines all fibered classes in $H^1(M)$.*

In order to state our second main result we first recall that a *sutured manifold* (M,γ) is loosely speaking a 3-manifold M, together with a decomposition of its boundary

$$\partial M=(-R_-)\cup\gamma\cup R_+,$$

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where γ is a collection of annuli, and R_- and R_+ are oriented subsurfaces of ∂M. In the following, we will, for the most part, restrict ourselves to *taut balanced sutured manifolds*; these are sutured manifolds such that M is irreducible, and $R_\pm$ have minimal complexity and no closed components. We refer to Section 2.1 for details.

In this paper, we will also study the existence of taut foliations on a sutured manifold. By a *foliation on a sutured manifold* (M, γ), we mean a codimension 1 transversely oriented foliation on the 3-manifold M such that all the leaves of $\mathcal{F}$ are transverse to γ and tangential to R_- and R_+. We say that a foliation $\mathcal{F}$ is *taut* if there exists a curve or properly embedded arc in M that is transverse to the leaves of $\mathcal{F}$ and that intersects every leaf of $\mathcal{F}$ at least once. Also, there exists the notion of depth for foliations. For example, a foliation is said to be of depth 0 if all leaves are compact. In this paper, we say that a foliation of depth greater than 0 is *indecomposable* if the components of $R_\pm$ are the only compact leaves. We refer to Section 4.1 for details.

The third author [Ju06] introduced the notion of a relative Spin^c-structure on a balanced sutured manifold (M, γ), and associated to a Spin^c-structure $\mathfrak{s}$, an abelian group $SFH(M, \gamma, \mathfrak{s})$ called the *sutured Floer homology* of (M, γ) at $\mathfrak{s}$. Let $\operatorname{Supp}(M, \gamma)$ denote the set of those Spin^c-structures on (M, γ) for which $SFH(M, \gamma, \mathfrak{s}) \neq 0$. We write

$$SFH(M, \gamma) := \bigoplus_{\mathfrak{s} \in \operatorname{Supp}(M,\gamma)} SFH(M, \gamma, \mathfrak{s});$$

this is a relatively $\mathbb{Z}_2$-graded abelian group. Furthermore, the relative grading restricts to a relative grading on each term $SFH(M, \gamma, \mathfrak{s})$.

It follows from work of the third author [Ju06, Ju08], see also Corollary 4.5, that an irreducible balanced sutured manifold admits a taut depth 0 foliation (or equivalently, it is diffeomorphic to a product $(R \times I, \partial R \times I)$ for a compact oriented surface R) if and only if $SFH(M, \gamma) \cong \mathbb{Z}$. In particular, this shows that sutured Floer homology detects whether or not an irreducible balanced sutured manifold admits a taut depth 0 foliation.

Our second main theorem states that sutured Floer homology also detects whether or not an indecomposable balanced sutured manifold (M, γ) admits an indecomposable taut depth one foliation. In order to state the theorem, we need to introduce a few more definitions. First, we can associate to an indecomposable taut depth one foliation $\mathcal{F}$ on (M, γ) a cohomology class $\lambda(\mathcal{F}) \in H^1(M)$ in a natural way, see Section 4.1 for details. Second, given a cohomology class $\alpha \in H^1(M)$, we define

$$SFH_\alpha(M, \gamma) := \bigoplus SFH(M, \gamma, \mathfrak{s}),$$

where we take the direct sum over all Spin^c-structures that pair minimally with α; refer to Section 2.3 for details. We can now formulate our second main theorem.

Theorem 1.2. *Suppose (M, γ) is a connected, indecomposable balanced sutured manifold, and let $\alpha \in H^1(M)$. Then $SFH_\alpha(M, \gamma) \cong \mathbb{Z}$ if and only if there exists an indecomposable taut depth one foliation $\mathcal{F}$ with $\lambda(\mathcal{F}) = \alpha$.*

The well-versed reader might be excused for having a sense of déja vu. In fact, the theorem was proved by the first author [Al13] for strongly balanced sutured manifolds under the extra assumption that $H_2(M) = 0$. This assumption was introduced for technical reasons to ensure that in the course of the proof one only works with balanced sutured manifolds. We will resolve this technical issue using the Alexander polynomial of a sutured manifold (see [FJR11] and [BP01]), and show that one can drop the condition on $H_2(M)$. We also note that the proof of the 'if' direction of this theorem is verbatim the same as in [Al13]. We remove the strongly balanced condition on (M, γ) with the aid of Theorem 4.2. This includes a characterization of the set of outer Spin^c-structures for a nice taut decomposing surface as the ones that pair minimally with $[S]$. So in the decomposition formula for SFH, we do not have to refer to relative Chern classes, which are only defined when (M, γ) is strongly balanced.

In Section 2.3, we introduce the notion of an extremal Spin^c-structure for a given balanced sutured manifold. Loosely speaking, a Spin^c-structure is extremal if it defines a vertex of the convex hull of the support of sutured Floer homology. With this notion, we then immediately obtain the following corollary to Theorem 1.2.

Corollary 1.3. *A connected, taut balanced sutured manifold (M, γ) admits an indecomposable taut depth one foliation if and only if there exists an extremal* Spin^c*-structure with $SFH(M, \gamma, \mathfrak{s}) \cong \mathbb{Z}$.*

The key new tool in the proof of Theorems 1.1 and 1.2 is Theorem 3.1 which allows us to study fibrations and taut depth one foliations even in the presence of non-trivial second homology. It states that, given a taut balanced sutured manifold (M, γ) and a class $\alpha \in H^1(M)$ such that $\chi(SFH_\alpha(M, \gamma)) \neq 0$, if a properly embedded surface S is dual to α, then the surface obtained by removing the closed components of S is also dual to α. In particular, for every such $\alpha \in H^1(M)$, there exists a decomposing surface S dual to α which *has no closed components*, is well-groomed in the sense of Gabai, and gives a taut decomposition. This ensures that the result (M', γ') of decomposing (M, γ) along S is balanced (most importantly, $R_\pm$ has no closed components), in which case we shall see in Theorem 4.2 that $SFH(M', \gamma') \cong SFH_\alpha(M, \gamma)$. The proof of Theorem 3.1 relies on a careful study of the relationship of the surface S to an Alexander invariant of the pair (M, R_-) defined in [FJR11], which is the Euler characteristic of sutured Floer homology.

The paper is organized as follows. In Section 2, we recall the definition of a sutured manifold, the definition of a sutured manifold decomposition, and basic properties of sutured Floer homology. In Section 3, we prove Theorem 3.1, which is our main technical result. In Section 4 we first recall some basic properties and definitions for foliations on sutured manifolds and we then give a proof of Theorem 1.2. Finally, in Section 5, we state and prove our results that imply that sutured Floer homology detects fibered classes on 3-manifolds with non-empty boundary.

Conventions and notations. We assume that all 3-manifolds are compact and oriented, and the foliations we refer to are smooth and transversely oriented. Given a manifold W and a submanifold $S \subset W$, we denote by $N(S)$ a tubular neighbourhood of S in W. When referring to homology or cohomology groups with integer coefficients, we suppress the coefficient ring $\mathbb{Z}$ in our notation.

2. Sutured manifolds

In this Section we recall the definition of a sutured manifold, the definition of a sutured manifold decomposition, and basic properties of sutured Floer homology.

2.1. Definition of sutured manifolds.

A sutured manifold (M, R_-, R_+, γ) consists of a 3-manifold M, together with a decomposition of its boundary

$$\partial M = -R_- \cup \gamma \cup R_+$$

into oriented submanifolds such that

1. γ is a disjoint union of annuli,
2. R_- and R_+ are disjoint,
3. if A is a component of γ, then $R_- \cap A$ is connected and is a boundary component of both A and R_-, and similarly for $R_+ \cap A$. Furthermore,
$$[R_+ \cap A] = [R_- \cap A] \in H_1(A),$$
where we endow $R_\pm \cap A$ with the orientation coming from the boundary of $R_\pm$.

Following the standard convention, we usually write (M, γ) instead of (M, R_-, R_+, γ), and might denote $R_\pm$ by $R_\pm(\gamma)$ when it is not clear from the context which sutured manifold we are referring to. Note that the notion of sutured manifolds is due to Gabai [Ga83], but our definition is less general in so far as we do not allow "toroidal sutures."

Following Gabai, we say that a sutured manifold (M, γ) is *taut* if M is irreducible, and if $R_\pm$ are incompressible and Thurston-norm minimising in $H_2(M, \gamma)$. We are mostly interested in the study of taut foliations on sutured manifolds. By the work of Gabai [Ga83], if a sutured manifold carries a taut foliation, then it is taut. It is therefore reasonable to restrict ourselves henceforth to the study of taut sutured manifolds.

We now recall the following definitions from [Ju06] and [Ju08], respectively.

1. The sutured manifold (M, γ) is called *balanced* if $\chi(R_+) = \chi(R_-)$, and if M and $R_\pm$ have no closed components.
2. The sutured manifold (M, γ) is said to be *strongly balanced* if (M, γ) is balanced, and if for every component F of ∂M the equality $\chi(F \cap R_-) = \chi(F \cap R_+)$ holds.

An example of a strongly balanced, taut, sutured manifold is given as follows. Let R be a compact oriented surface with no closed components. Then

$$(R \times [-1,1], \partial R \times [-1,1])$$

forms a sutured manifold, which we refer to as a *product sutured manifold.* Note that $R_+ = R \times \{1\}$ and $R_- = R \times \{-1\}$. It is clear that this sutured manifold is strongly balanced and taut.

If a sutured manifold (M, γ) carries a codimension one, transversely oriented foliation $\mathcal{F}$, then $\chi(R_+) = \chi(R_-)$, cf. [Ju10, Proposition 3.6]. So if (M, γ) carries a foliation, has at least one suture on each boundary component, and M has no closed components, then it is balanced.

2.2. **Definition of sutured manifold decompositions.** We now define a key operation on a sutured manifold (M, γ), called a *sutured manifold decomposition.* Intuitively, this involves cutting the manifold M along a properly embedded oriented surface S, and adding one side of S to R_+ and the other side to R_-. The new γ is where the new R_+ and R_- meet.

Definition. Let (M, γ) be a sutured manifold.

(1) We say that a properly embedded surface $S \subset M$ is a *decomposing surface* if S has no closed components, and if given any component A of γ, each component λ of $S \cap A$ is either a properly embedded non-separating arc in A, or it is a simple closed curve in the same homology class as $A \cap R_-$.

(2) A decomposing surface S for (M, γ) defines a *sutured manifold decomposition*

$$(M, R_-, R_+, \gamma) \rightsquigarrow^S (M', R'_-, R'_+, \gamma'),$$

where $M' := M \setminus \operatorname{int}(N(S))$ and

$$\begin{aligned} \gamma' &:= (\gamma \cap M') \cup N(S'_+ \cap R_-) \cup N(S'_- \cap R_+), \\ R'_+ &:= ((R_+ \cap M') \cup S'_+) \setminus \operatorname{int}(\gamma'), \\ R'_- &:= ((R_- \cap M') \cup S'_-) \setminus \operatorname{int}(\gamma'), \end{aligned}$$

where S'_+ (respectively S'_-) are the components of $\partial N(S) \cap M'$ whose normal vectors points out of (respectively into) M.

If both (M, γ) and (M', γ') are taut, then (M', γ') is simpler than (M, γ) with respect to some notion of complexity defined by Gabai [Ga83]. Furthermore, as shown by Gabai [Ga83], a sutured manifold (M, γ) is taut if and only if there is a sequence of sutured manifold decompositions starting with (M, γ) and ending with a product sutured manifold.

2.3. **Sutured Floer homology.** As we mentioned in the introduction, the third author [Ju06] introduced the notion of a relative Spin^c-structure on a balanced sutured manifold (M,γ), and showed that the set $\mathrm{Spin}^c(M,\gamma)$ of relative Spin^c-structures on (M,γ) admits a canonical free and transitive $H_1(M)$-action, which makes the set $\mathrm{Spin}^c(M,\gamma)$ into a $H_1(M)$-torsor.

Furthermore, the third author [Ju06] associated to a Spin^c-structure $\mathfrak{s}$ an abelian group $SFH(M,\gamma,\mathfrak{s})$ called the *sutured Floer homology* of (M,γ) at $\mathfrak{s}$. It extends the hat version of Heegaard Floer homology of closed 3-manifolds defined by Ozsváth and Szabó [OS04a, OS04b] to 3-manifolds with boundary. The group

$$SFH(M,\gamma) := \bigoplus SFH(M,\gamma,\mathfrak{s})$$

is finitely generated. It thus follows in particular that the *support* of $SFH(M,\gamma)$, which is defined to be

$$\mathrm{Supp}(M,\gamma) := \{\mathfrak{s} \in \mathrm{Spin}(M,\gamma) :\ SFH(M,\gamma,\mathfrak{s}) \neq 0\},$$

is a finite set. Furthermore, $SFH(M,\gamma)$ carries a relative $\mathbb{Z}_2$-grading, which restricts to a $\mathbb{Z}_2$-grading on each summand $SFH(M,\gamma)$. Given an orientation of the vector space $H_*(M,R_-;\mathbb{R})$, we obtain a lift of this relative $\mathbb{Z}_2$-grading to an absolute one.

We pick an identification of $\mathrm{Spin}^c(M)$ with $H_1(M)$. We say that $\alpha \in H^1(M;\mathbb{R})$ *pairs minimally* with a Spin^c-structure $\mathfrak{s} \in \mathrm{Spin}^c(M,\gamma)$ if $\alpha(\mathfrak{s}) \leq \alpha(\mathfrak{t})$ for all $\mathfrak{t} \in \mathrm{Supp}(M,\gamma)$ and $\alpha(\mathfrak{s}) = \alpha(\mathfrak{t})$ for some $\mathfrak{t} \in \mathrm{Supp}(M,\gamma)$. We then define

$$SFH_\alpha(M,\gamma) := \bigoplus_{\{\mathfrak{s}\in\mathrm{Spin}^c(M,\gamma):\ \alpha \text{ pairs minimally with } \mathfrak{s}\}} SFH(M,\gamma,\mathfrak{s}).$$

Note that we can and will view $SFH_\alpha(M,\gamma)$ as an abelian group with a relative $\mathbb{Z}_2$-grading. Also note that this definition of $SFH_\alpha(M,\gamma)$ does not depend on the choice of identification of $\mathrm{Spin}^c(M,\gamma)$ with $H_1(M)$. Furthermore, this definition of $SFH_\alpha(M,\gamma)$ is clearly equivalent to [Ju10, Definition 4.12] when (M,γ) is strongly balanced.

Finally, we say that a Spin^c-structure $\mathfrak{s} \in \mathrm{Supp}(M,\gamma)$ is *extremal* if there exists an $\alpha \in H^1(M;\mathbb{R})$ such that α pairs minimally with $\mathfrak{s}$, and such that α does not pair minimally with any other $\mathfrak{t} \in \mathrm{Supp}(M,\gamma)$.

If $H_1(M)$ is torsion-free, then it is straightforward to see that $\mathfrak{s}$ is extremal if and only if it is a vertex of the convex hull of $\mathrm{Supp}(M,\gamma)$ viewed as a subset of $H_1(M;\mathbb{R}) = H_1(M) \otimes \mathbb{R}$. Again, this does not depend on the choice of the identification of $\mathrm{Spin}^c(M,\gamma)$ with $H_1(M)$.

3. The main technical theorem

In this section we state and prove our main technical theorem. This theorem is needed for the proofs of both of our main results, namely Theorems 1.1 and 1.2.

In order to state our main technical theorem we need a few more definitions. Let (M,γ) be a balanced sutured manifold.

(1) We say that a surface $S \subset M$ *touches* R_- if $S \cap R_- \neq \emptyset$.
(2) We say a properly embedded surface S in M and a cohomology class $\alpha \in H^1(M)$ are *dual* if α is the Poincaré dual of $[S] \in H_2(M, \partial M)$.
(3) The group $SFH(M,\gamma)$ carries a relative $\mathbb{Z}_2$-grading, which restricts to a grading on each $SFH(M,\gamma,\mathfrak{s})$, and thus also to each $SFH_\alpha(M,\gamma)$. For a $\mathbb{Z}_2$-graded finitely generated abelian group G, we denote by $\chi(G)$ the difference of the ranks of the two gradings. Note that $\chi(G)$ is well-defined up to sign.

The goal of this section is to prove the following proposition.

Theorem 3.1. *Let (M,γ) be a connected balanced sutured manifold, and let $\alpha \in H^1(M)$ be a class such that*

$$\chi(SFH_\alpha(M,\gamma)) \neq 0.$$

If α is dual to a properly embedded surface S, then α is also dual to the union of the components of S that touch R_-.

The proof of Theorem 3.1 will occupy the remainder of this section, and is the technical heart of this paper. The key idea is to translate the information on the sutured Floer homology into information on a certain one-variable Alexander polynomial of (M, R_-), and to then study the corresponding Alexander module of (M, R_-) and its relation to the surface S.

3.1. **Alexander polynomials of pairs of spaces.** Before we can introduce Alexander polynomials of sutured manifolds, we first introduce Alexander polynomials of pairs of spaces in general.

Let F be a free abelian group, and let M be a finitely generated $\mathbb{Z}[F]$-module. Since $\mathbb{Z}[F]$ is Noetherian, there exists a finite resolution

$$\mathbb{Z}[F]^r \xrightarrow{A} \mathbb{Z}[F]^s \to M \to 0.$$

By adding columns of zeros, we can assume that $r \geq s$. The *order of M* is now defined as the greatest common divisor of all $s \times s$-minors of A. It is well-known that up to multiplication by a unit in $\mathbb{Z}[F]$, that is, up to multiplication by an element of the form $\pm f$ for $f \in F$, this definition does not depend on the choice of the resolution. We refer to [Tu01, Section I.4.2] for details on orders.

Now let (X, Y) be a pair of topological spaces. Let $\psi\colon H_1(X) \to F$ be a homomorphism to a free abelian group. We denote by $p\colon \tilde{X} \to X$ the universal abelian cover of X, and we write $\tilde{Y} := p^{-1}(Y)$. Note that $C_*(\tilde{X}, \tilde{Y}; \mathbb{Z})$ is a $\mathbb{Z}[H_1(X)]$-module. We can thus consider the following chain complex of $\mathbb{Z}[F]$-modules:

$$C_*^\psi(X, Y; \mathbb{Z}[F]) := C_*(\tilde{X}, \tilde{Y}; \mathbb{Z}) \otimes_{\mathbb{Z}[H_1(X)]} \mathbb{Z}[F].$$

We then denote the corresponding homology modules by

$$H_*^\psi(X, Y; \mathbb{Z}[F]).$$

(We will drop the superscript ψ if it is understood from the context.) Furthermore, we denote by

$$\Delta^{\psi}_{X,Y} \in \mathbb{Z}[F]$$

the order of the $\mathbb{Z}[F]$-module $H^{\psi}_*(X,Y;\mathbb{Z}[F])$. Note that $\Delta^{\psi}_{X,Y}$ is well-defined up to multiplication by a unit in $\mathbb{Z}[F]$, that is, up to multiplication by an element of $\mathbb{Z}[F]$ of the form $\pm f$ for $f \in F$. Several times, we will be interested in homomorphisms of $H_1(X)$ to $F = \mathbb{Z}$. In this case, we identify the group ring of F with $\mathbb{Z}[x^{\pm 1}]$.

We denote by $\mathbb{Q}(F)$ the quotient field of $\mathbb{Z}[F]$, and define $H^{\psi}_*(X,Y;\mathbb{Q}(F))$ to be the homology of the chain complex

$$C^{\psi}_*(X,Y;\mathbb{Q}(F)) := C_*(\tilde{X},\tilde{Y};\mathbb{Z}) \otimes_{\mathbb{Z}[H_1(X)]} \mathbb{Q}(F).$$

The following elementary lemma is an immediate consequence of the fact that $\mathbb{Q}(F)$ is flat over $\mathbb{Z}[F]$.

Lemma 3.2. *The Alexander polynomial $\Delta^{\psi}_{X,Y}$ is non-zero if and only if*

$$H^{\psi}_*(X,Y;\mathbb{Q}(F)) = 0.$$

We also have the following basic lemma.

Lemma 3.3. *Let (X,Y) be a pair of compact topological spaces with X connected. Let $\psi\colon H_1(X) \to F$ be a homomorphism to a free abelian group. Then*

$$H^{\psi}_0(X,Y;\mathbb{Q}(F)) \cong \begin{cases} \mathbb{Q}(F) & \text{if } Y = \emptyset \text{ and } \psi \text{ is trivial,} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We first consider the case when Y is the empty set. Since X is connected, it follows from the standard expression for the 0-th twisted homology (see [HS97, Section VI]) that we have an isomorphism

$$H^{\psi}_0(X;\mathbb{Q}(F)) \xrightarrow{\cong} \mathbb{Q}(F)/\{v - \psi(g)v :\ v \in \mathbb{Q}(F) \text{ and } g \in H_1(X)\}.$$

Since $\mathbb{Q}(F)$ is a one-dimensional vector space, it follows that $H^{\psi}_0(X;\mathbb{Q}(F)) = 0$ if we quotient out by a non-zero subspace. But this in turn happens if and only if $\psi\colon H_1(X) \to F$ is non-trivial.

Now suppose that Y is non-empty. We denote the components of Y by Y_i for $i \in I$. Let $\phi_i\colon H_1(Y_i;\mathbb{Z}) \to \mathbb{F}$ be the composition of the inclusion map $H_1(Y_i) \to H_1(X)$ with ψ, and we define $\phi\colon H_1(Y;\mathbb{Z}) \to F$ analogously. We then get a commutative diagram

$$\begin{tikzcd} H^{\phi}_0(Y;\mathbb{Q}(F)) \arrow[r, "\cong"] \arrow[d] & \bigoplus\limits_{i\in I} \mathbb{Q}(F)/\{v - \phi_i(g)v :\ v \in \mathbb{Q}(F) \text{ and } g \in H_1(Y_i)\} \arrow[d] \\ H^{\psi}_0(X;\mathbb{Q}(F)) \arrow[r, "\cong"] & \mathbb{Q}(F)/\{v - \psi(g)v :\ v \in \mathbb{Q}(F) \text{ and } g \in H_1(X)\}, \end{tikzcd}$$

where the left vertical map is the map induced by the inclusion, and the right vertical map is the sum of the canonical maps. Since each map $\phi_i\colon H_1(Y_i) \to F$ factors through $\psi\colon H_1(X) \to F$, it follows that for each $i \in I$ we have

$$\operatorname{Im}(\phi_i\colon H_1(Y_i) \to F) \subset \operatorname{Im}(\psi\colon H_1(X) \to F),$$

which in turn implies that the vertical map on the right in the above commutative diagram is an epimorphism for each summand. Since we have at least one summand, the vertical map on the right is an epimorphism. Now it follows from the short exact sequence

$$H_0^{\phi}(Y;\mathbb{Q}(F)) \to H_0^{\psi}(X;\mathbb{Q}(F)) \to H_0^{\psi}(X,Y;\mathbb{Q}(F)) \to 0$$

that $H_0^{\psi}(X,Y;\mathbb{Q}(F)) = 0$. □

3.2. **The Alexander polynomial of a sutured manifold.** Given a sutured manifold (M,γ) and a homomorphism $\psi\colon H_1(M) \to F$, where F is a free abelian group, we refer to Δ^{ψ}_{M,R_-} as *the Alexander polynomial of the triple* (M,γ,ψ). We will now relate the Alexander polynomials of (M,γ) to the sutured Floer homology of (M,R_-). In order to do this, we need one more piece of notation: if H is an abelian group, then given p, $q \in \mathbb{Z}[H]$, we write $p \doteq q$ if p and q are equal up to multiplication by an element of $\mathbb{Z}[H]$ of the form $\pm h$ for $h \in H$.

Recall that, given a balanced sutured manifold (M,γ) and an orientation of the vector space $H_*(M,R_-;\mathbb{R})$, the group $SFH(M,\gamma)$ is an absolutely $\mathbb{Z}_2$-graded, finitely generated abelian group. Since this $\mathbb{Z}_2$-grading restricts to a $\mathbb{Z}_2$-grading on each summand $SFH(M,\gamma,\mathfrak{s})$, it thus makes sense to consider the Euler characteristic $\chi(SFH(M,\gamma,\mathfrak{s})) \in \mathbb{Z}$ for any Spin^c-structure $\mathfrak{s} \in \operatorname{Spin}^c(M,\gamma)$. Using the results and techniques of [FJR11], we will now prove the following proposition.

Proposition 3.4. *Let (M,γ) be a connected balanced sutured manifold. Pick an orientation of $H_*(M,R_-;\mathbb{R})$ and an identification of* $\operatorname{Spin}^c(M,\gamma)$ *with $H = H_1(M)$. Let $\psi\colon H \to F$ be a homomorphism to a free abelian group. Then we have*

$$\Delta^{\psi}_{M,R_-} \doteq \sum_{\mathfrak{s}\in\operatorname{Spin}^c(M,\gamma)} \chi(SFH(M,\gamma,\mathfrak{s}))\cdot\psi(\mathfrak{s}) \in \mathbb{Z}[F].$$

Proof. First of all, given a finitely generated abelian group H, we denote by $\mathbb{Q}(H)$ the ring given by inverting all elements in $\mathbb{Z}[H]$ which are not zero divisors. Given a pair of finite CW-complexes (X,Y), we then denote by $\tau(X,Y) \in \mathbb{Q}(H_1(X))$ Turaev's maximal abelian torsion of (X,Y). We refer to [FJR11, Section 3.4] and [Tu01, Tu02] for details.

We equip the pair (M,R_-) with a CW-structure (X',Y'), and identify $H_1(X')$ with H. By [FJR11, Theorem 1.1], we know that $\tau(X',Y')$ lies in $\mathbb{Z}[H] \subset \mathbb{Q}(H)$, and that in fact

$$\tau(X',Y') \doteq \sum_{\mathfrak{s}\in\operatorname{Spin}^c(M,\gamma)} \chi(SFH(M,\gamma,\mathfrak{s}))\cdot\mathfrak{s} \in \mathbb{Z}[H].$$

Since (M,γ) is balanced, $R_- \neq \emptyset$. It now follows from standard arguments that the pair of CW-complexes (X',Y') is simple homotopy equivalent to a pair of CW-complexes (X,Y), where X is a 2-complex such that all 0-simplices of X lie in Y. Since Reidemeister torsion is invariant under simple homotopy equivalence, it follows that

$$\tau(X,Y) \doteq \tau(X',Y') \in \mathbb{Z}[H],$$

where we also identify $H_1(X)$ with H.

Since (M,γ) is a balanced sutured manifold, it follows from a standard Poincaré duality argument that $\chi(M) = \chi(R_-)$. Hence

$$\chi(X,Y) = \chi(M,R_-) = \chi(M) - \chi(R_-) = 0.$$

We denote by r the number of 1-cells in $X \setminus Y$. Since X is a 2-complex, and since all 0-cells of X lie in Y, it follows from $\chi(X,Y) = 0$ that r also equals the number of 2-cells in $X \setminus Y$.

Let $p\colon \tilde{X} \to X$ be the universal abelian cover of X, and we write $\tilde{Y} := p^{-1}(Y)$. By picking lifts of the cells in $X \setminus Y$ to $\tilde{X}$, we can identify $C_1(\tilde{X},\tilde{Y})$ with $\mathbb{Z}[H]^r$ and $C_2(\tilde{X},\tilde{Y})$ with $\mathbb{Z}[H]^r$. We denote by A the matrix over $\mathbb{Z}[H]$ representing the boundary map $C_2(\tilde{X},\tilde{Y}) \to C_1(\tilde{X},\tilde{Y})$ with respect to the chosen bases. It then follows from [FJR11, Lemma 3.6] that

$$\tau(X,Y) \doteq \det(A) \in \mathbb{Z}[H].$$

We also denote by ψ the extension of $\psi\colon H \to F$ to a ring homomorphism $\mathbb{Z}[H] \to \mathbb{Z}[F]$. Since the Alexander polynomial of a pair only depends on the homotopy type of the pair, we see that $\Delta^{\psi}_{M,R_-} = \Delta^{\psi}_{X,Y}$. Furthermore, note that

$$C_2(\tilde{X},\tilde{Y}) \otimes_{\mathbb{Z}[H]} \mathbb{Z}[F] \xrightarrow{\partial_2} C_1(\tilde{X},\tilde{Y}) \otimes_{\mathbb{Z}[H]} \mathbb{Z}[F] \to H_1^{\psi}(X,Y;\mathbb{Z}[F]) \to 0$$

is a free resolution for $H_1^{\psi}(X,Y;\mathbb{Z}[F])$. This resolution is isomorphic to the resolution

$$\mathbb{Z}[F]^r \xrightarrow{\psi(A)} \mathbb{Z}[F]^r \to H_1^{\psi}(X,Y;\mathbb{Z}[F]) \to 0.$$

It follows from the definition of the order of a $\mathbb{Z}[F]$-module that $\Delta^{\psi}_{X,Y} \doteq \det(\psi(A))$, where $\psi(A)$ is the matrix given by applying $\psi\colon \mathbb{Z}[H] \to \mathbb{Z}[F]$ to all entries of A. Combining the above equalities we then see that

$$\begin{array}{rcl}\Delta^{\psi}_{M,R_-} \doteq \Delta^{\psi}_{X,Y} \doteq \det(\psi(A)) & = & \psi(\det(A)) \\ & \doteq & \psi(\tau(X,Y)) \\ & \doteq & \psi(\tau(X',Y')) \\ & \doteq & \psi\left(\sum_{\mathfrak{s}\in\mathrm{Spin}^c(M,\gamma)} \chi(SFH(M,\gamma,\mathfrak{s}))\cdot\mathfrak{s}\right) \\ & \doteq & \sum_{\mathfrak{s}\in\mathrm{Spin}^c(M,\gamma)} \chi(SFH(M,\gamma,\mathfrak{s}))\cdot\psi(\mathfrak{s}).\end{array}$$

□

3.3. **Proof of Theorem 3.1.** In this section, we prove the following proposition.

Proposition 3.5. *Let (M,γ) be a connected balanced sutured manifold, let*
$$\alpha \in H^1(M) \cong \operatorname{Hom}(H_1(M),\mathbb{Z}),$$
and suppose that S is a properly embedded surface in M that is Poincaré dual to α. If $\Delta^{\alpha}_{M,R_-} \neq 0$, then the class α is also dual to the union of the components of S that touch R_-.

Before we give the proof of Proposition 3.5, we will first show that Theorem 3.1 follows from Proposition 3.5.

Proof of Theorem 3.1. Let (M,γ) be a connected balanced sutured manifold, and suppose that $\alpha \in H^1(M)$ is a class such that
$$\chi(SFH_\alpha(M,\gamma)) \neq 0.$$
Furthermore, let S be a properly embedded surface dual to α. We have to show that α is also dual to the union of the components of S that touch R_-.

First, we pick an identification of $\operatorname{Spin}^c(M,\gamma)$ with $H_1(M)$. By Proposition 3.4,
$$\Delta^{\alpha}_{M,R_-} \doteq \sum_{\mathfrak{s}\in \operatorname{Spin}^c(M,\gamma)} \chi(SFH(M,\gamma,\mathfrak{s}))\cdot x^{\alpha(\mathfrak{s})} \in \mathbb{Z}[x^{\pm 1}].$$
We write
$$d := \min\{\alpha(\mathfrak{s}) : \mathfrak{s} \in \operatorname{Supp}(M,\gamma)\}.$$
It follows from the definition of $SFH_\alpha(M,\gamma)$ that Δ^{α}_{M,R_-} is of the form
$$\Delta^{\alpha}_{M,R_-} = \chi(SFH_\alpha(M,\gamma))\cdot x^d + \text{higher order terms}.$$
Hence, the assumption $\chi(SFH_\alpha(M,\gamma)) \neq 0$ implies that Δ^{α}_{M,R_-} is non-zero. The result now follows from Proposition 3.5. □

We now turn to the proof of Proposition 3.5.

Proof of Proposition 3.5. We denote the components of S by S_e for $e \in E$. We pick disjoint open tubular neighbourhoods $S_e \times [-1,1]$ for $e \in E$ such that

- $S_e \times \{0\} = S_e$,
- $S_e \times [-1,1]$ has the same orientation as M, and
- $(S_e \times [-1,1]) \cap \partial M = \partial S_e \times [-1,1]$.

We denote by W_v for $v \in V$ the components of $M \setminus \bigcup_{e\in E}(S_e \times (-1,1))$. Given an element $e \in E$, we denote by $i(e)$ the element in V with $S_e \times \{-1\} \subset W_{i(e)}$, and similarly, we denote by $t(e)$ the element in V with $S_e \times \{1\} \subset W_{t(e)}$. The pair (V,E), together with the maps $i\colon E \to V$ and $t\colon E \to V$, defines a directed graph that we denote by Γ. We equip Γ with the usual topology. Note that vertices are closed sets, and we view edges without the boundary points, i.e. we view edges as open sets.

We now refer to a vertex or an edge of Γ as "green" if the corresponding subset of M touches R_-, otherwise we refer to a vertex or an edge as "black." We denote by

G the union of the green vertices and edges, and we denote by B the union of the black vertices and edges. Note that the fact that $R_- \neq \emptyset$ implies that there exists at least one vertex which is green. If the whole graph is green, then there is nothing to prove. So assume that this is not the case.

Note that if a component W_v does not touch R_-, then neither does any of its boundary components. In other words, if a vertex of Γ is black, then all the edges adjacent to it are also black. In particular, B is an open subset of Γ. In the next lemma, we will see that B is also "almost" closed.

In order to state the next lemma, we need the notion of a path in Γ. We view any interval $[0, n]$ with $n \in \mathbb{N}$ as a graph in the canonical way with $n+1$ vertices and n edges. We define a *path* in Γ to be a continuous map $P\colon [0, n] \to \Gamma$ that sends vertices to vertices and edges to edges, and such that P restricted to $(0, n)$ is injective. We can now formulate the following lemma.

Lemma 3.6. *If $P\colon [0, n] \to \Gamma$ is a path in Γ such that $P((0, n))$ is black and $P(0)$ is green, then $P(n)$ is black.*

We postpone the proof of the lemma, and first show how to deduce Proposition 3.5 from Lemma 3.6.

Recall that before the lemma we showed that B is open. Since Γ is connected and since B is not all of Γ, it follows that B is not closed. This means that there exists a black edge f with at least one green boundary vertex. Without loss of generality, we can assume that $i(f)$ is green. Assuming Lemma 3.6, we can now prove the following claim.

Claim. The edge f is separating.

In order to show that f is separating, it suffices to prove that for each path $P\colon [0, k] \to \Gamma$ with $P(0) = i(f)$ and $P((0, 1)) = f$, we have $P(k) \neq P(0)$. So let P be such a path. Then there exists a maximal $n \in \{1, \dots, k\}$ such that $P((0, n)) \subset B$. It follows from Lemma 3.6 that $P(n)$ is also black. By the openness of B, we furthermore know that $P((0, n+1) \cap (0, k])$ is also black. It follows from the maximality of k that $k = n$. We thus see that $P(k)$ is black, whereas $P(0) = i(f)$ is green; i.e., we have shown that $P(k) \neq P(0)$. This concludes the proof of the claim.

We denote by Γ' the component of $\Gamma \setminus f$ that contains the vertex $i(f)$. We then denote by X the union of the vertex and edge spaces that correspond to Γ', that is

$$X = \bigcup_{v \in V(\Gamma')} W_v \cup \bigcup_{e \in E(\Gamma')} S_e \times [-1, 1].$$

It then follows easily from the definitions that

$$\partial X = S_f \times \{-1\} \;\cup\; (\partial X \cap \partial M).$$

Put differently, S_f represents the trivial element in $H_2(M, \partial M)$. We thus see that the surfaces S and $\cup_{e \neq f} S_e$ represent the same homology class in $H_2(M, \partial M)$. Since

S_f does not touch R_-, the proposition now follows by induction on the number of components of S.

We now turn to the proof of Lemma 3.6.

Proof of Lemma 3.6. In the following, given subsets $X \subset M$ and $Y \subset X$, we denote by $H_*(X, Y; \mathbb{Q}(x))$ the twisted homology corresponding to the homomorphism $H_1(X) \to H_1(M) \xrightarrow{\alpha} \mathbb{Z}$. Furthermore, given an edge $e \in E$, we denote the map

$$\begin{array}{rcl} H_0(S_e, S_e \cap R_-; \mathbb{Q}(x)) & \to & H_0(W_{i(e)}, W_{i(e)} \cap R_-; \mathbb{Q}(x)) \\ [c] & \mapsto & [c \times \{-1\}] \end{array}$$

by i_e. Similarly, we denote the map

$$\begin{array}{rcl} H_0(S_e, S_e \cap R_-; \mathbb{Q}(x)) & \to & H_0(W_{t(e)}, W_{t(e)} \cap R_-; \mathbb{Q}(x)) \\ [c] & \mapsto & [c \times \{1\}] \end{array}$$

by t_e. These maps now give rise to the following Mayer–Vietoris sequence:

$$\begin{array}{l} \cdots \qquad\qquad \to H_1(M, R_-; \mathbb{Q}(x)) \to \\ \bigoplus_{e\in E} H_0(S_e, S_e \cap R_-; \mathbb{Q}(x)) \xrightarrow{\bigoplus_{e\in E} i_e - t_e} \bigoplus_{v\in V} H_0(W_v, W_v \cap R_-; \mathbb{Q}(x)) \to H_0(M, R_-; \mathbb{Q}(x)) \to \end{array}$$

Since (M, γ) is a balanced sutured manifold, we have $R_- \neq \emptyset$. So by Lemma 3.3, we have that $H_0(M, R_-; \mathbb{Q}(x)) = 0$. Next, since $\Delta^{\alpha}_{M,R_-} \neq 0$, Lemma 3.2 tells us that $H_1(M, R_-; \mathbb{Q}(x)) = 0$. Therefore the map

$$\bigoplus_{e\in E} H_0(S_e, S_e \cap R_-; \mathbb{Q}(x)) \xrightarrow{\bigoplus_{e\in E} i_e - t_e} \bigoplus_{v\in V} H_0(W_v, W_v \cap R_-; \mathbb{Q}(x))$$

in the above long exact sequence is in fact an isomorphism. Note that α restricted to each W_v is zero, since S is Poincaré dual to α. Furthermore, any curve c in S_e can be pushed off S, hence the restriction of α to any S_e is also zero. It thus follows from Lemma 3.3 that for $e \in E$ we have

$$H_0(S_e, S_e \cap R_-; \mathbb{Q}(x)) \cong \begin{cases} 0 & \text{if } e \text{ is green,} \\ \mathbb{Q}(x) & \text{if } e \text{ is black.} \end{cases} \tag{1}$$

Similarly, we see that for $v \in V$, we have

$$H_0(W_v, W_v \cap R_-; \mathbb{Q}(x)) \cong \begin{cases} 0 & \text{if } v \text{ is green,} \\ \mathbb{Q}(x) & \text{if } v \text{ is black.} \end{cases} \tag{2}$$

Now let $P\colon [0,n] \to G$ be a path in G such that $P((0,n))$ is black and $P(0)$ is green. We then consider the diagram

$$\begin{tikzcd}
\bigoplus\limits_{\substack{e \text{ edge}\\ \text{of } [0,n]}} H_0(S_{P(e)}, S_{P(e)} \cap R_-; \mathbb{Q}(x)) \arrow[r, "\bigoplus i_{P(e)} - t_{P(e)}"] \arrow[d] & \bigoplus\limits_{\substack{v \text{ vertex}\\ \text{of } [0,n]}} H_0(W_{P(v)}, W_{P(v)} \cap R_-; \mathbb{Q}(x)) \arrow[d] \\
\bigoplus\limits_{e\in E} H_0(S_e, S_e \cap R_-; \mathbb{Q}(x)) \arrow[r, "\bigoplus i_e - t_e", "\cong"'] & \bigoplus\limits_{v\in V} H_0(W_v, W_v \cap R_-; \mathbb{Q}(x)),
\end{tikzcd}$$

where the vertical maps are for each summand just given by the canonical inclusion. Now recall that the restriction of P to $(0,n)$ is injective, it thus follows that the left vertical map in the above commutative diagram is a monomorphism. Furthermore, note that this diagram is in fact commutative by the choice of our horizontal maps.

The hypothesis that the image of each edge in P is black implies by (1) that in the above commutative diagram, the top-left $\mathbb{Q}(x)$-module is isomorphic to $\mathbb{Q}(x)^n$. Furthermore, the hypothesis that $P(0)$ is green and that $P((0,n))$ is black implies by (2) that the top-right $\mathbb{Q}(x)$-module of the above commutative diagram is isomorphic to $\mathbb{Q}(x)^{n-1} \oplus H_0(W_{P(n)}, W_{P(n)} \cap R_-; \mathbb{Q}(x))$. We can thus rewrite the above commutative diagram as

$$\begin{tikzcd}
\mathbb{Q}(x)^n \arrow[r] \arrow[d] & \mathbb{Q}(x)^{n-1} \oplus H_0(W_{P(n)}, W_{P(n)} \cap R_-; \mathbb{Q}(x)) \arrow[d] \\
\bigoplus\limits_{e\in E} H_0(S_e, S_e \cap R_-; \mathbb{Q}(x)) \arrow[r, "\cong"] & \bigoplus\limits_{v\in V} H_0(W_v, W_v \cap R_-; \mathbb{Q}(x)).
\end{tikzcd}$$

Since the left vertical map is a monomorphism, and since the bottom horizontal map is an isomorphism, it follows from the commutativity that the top horizontal map is a monomorphism. But this is only possible if $H_0(W_{P(n)}, W_{P(n)} \cap R_-; \mathbb{Q}(x)) \neq 0$, which means that $P(n)$ is black. □

□

4. The proof of Theorem 1.2

The proof of Theorem 1.2 is closely modeled on the proof of [Al13, Theorem A]. Proposition 3.5 is the key new ingredient in the proof that allows us do deal with the case when $H_2(M) \neq 0$, is

4.1. **Foliations on sutured manifolds.** In this section, we recall several basic definitions and results about foliations that we need in the statement of Theorem 1.2.

Let (M, γ) be a sutured manifold. As we explained in the introduction, a *foliation on a sutured manifold* (M, γ) is a foliation on the 3-manifold M such that all the leaves of $\mathcal{F}$ are transverse to γ and tangential to R_- and R_+. Furthermore, a foliation $\mathcal{F}$ is *taut* if there exists a curve or properly embedded arc in M that is transverse to the leaves of $\mathcal{F}$ and that intersects every leaf of $\mathcal{F}$ at least once.

A leaf L of a foliation $\mathcal{F}$ on (M, γ) is said to be *depth 0* if it is compact. Recursively, we now define a leaf to be of *depth* k if it is not of depth less than k, and if $\overline{L} \setminus L$ is a collection of leaves of depth less than k. Otherwise, we say that L is of depth ∞. A depth k foliation $\mathcal{F}$ of (M, γ) is a foliation such that all leaves have depth at most k, and $\mathcal{F}$ contains at least one leaf of depth k.

If $\mathcal{F}$ is a depth zero foliation on (M, γ), then the global Reeb stability theorem implies that (M, γ) is a product sutured manifold $(R \times [-1, 1], \partial R \times [-1, 1])$, and $\mathcal{F}$ is equivalent to the product foliation whose leaves are the components of $R \times \{t\}$ for $t \in [-1, 1]$.

We will now study taut depth one foliations a little further. As we mentioned in the introduction, in this paper, we say that a foliation of depth greater than 0 is *indecomposable* if the components of $R_\pm$ are the only compact leaves. If $\mathcal{F}$ is an indecomposable taut depth one foliation on (M, γ), then it follows from Reeb stability (cf. [CC03, Lemma 11.4.4]) that there exists a fiber bundle $p \colon M \setminus (R_- \cup R_+) \to S^1$ such that the fibers are precisely the leaves of $\mathcal{F}$. We denote by

$$\lambda(\mathcal{F}) \colon \pi_1(M) \xrightarrow{\cong} \pi_1(M \setminus (R_- \cup R_+)) \xrightarrow{p_*} \mathbb{Z}$$

the induced map, and we denote by $\lambda(\mathcal{F}) \in H^1(M) \cong \operatorname{Hom}(\pi_1(M), \mathbb{Z})$ also the corresponding cohomology class.

4.2. **Summary of results on sutured Floer homology.** Given a decomposing surface S in a sutured manifold, Gabai [Ga87, Definition 0.2] and the third author [Ju10, Definition 3.20] introduced the notions of S being *well-groomed* and *nice*, respectively. Even though these notions are not very difficult, we treat them as black boxes, and only make use of the following lemma.

Lemma 4.1. *Let $(M, \gamma) \rightsquigarrow^S (M', \gamma')$ be a sutured manifold decomposition. Then the following hold:*

(1) *If S is well-groomed, then (M, γ) and (M', γ') are both taut.*
(2) *If S is nice and (M, γ) is balanced, then (M', γ') is also balanced.*
(3) *If (M, γ) is taut, then any non-zero class in $H^1(M)$ is Poincaré dual to a properly embedded well-groomed decomposing surface.*
(4) *Any union of components of a well-groomed surface is again well-groomed.*
(5) *A well-groomed decomposing surface* with no closed components *can be perturbed into a nice one.*

Here the first and second statement follow immediately from the definitions of well-groomed and nice decomposing surfaces, respectively. The third statement is precisely [Ga87, Lemma 0.7]. The fourth is a straightforward consequence of the definitions, together with [Ga83, Lemma 3.5]. Finally, the last statement is [Ju10, Remark 3.21] (any groomed surface with no closed components can be made into a nice surface by a small perturbation that places its boundary into general position).

We can now state the two results of the third author which we need in the proof. The first result, which is an extension of [Ju10, Proposition 4.13] to balanced sutured manifolds that are not necessarily strongly balanced, describes how sutured Floer homology behaves under decomposition along a nice decomposing surface.

Theorem 4.2. *Let (M, γ) be a taut balanced sutured manifold, and let $\alpha \in H^1(M)$. Suppose that S is a nice decomposing surface that is dual to α. If the result of the sutured manifold decomposition $(M, \gamma) \rightsquigarrow^S (M', \gamma')$ is taut, then*

$$SFH(M', \gamma') \cong SFH_\alpha(M, \gamma).$$

Proof. Let S be a decomposing surface in a balanced sutured manifold (M, γ). Following [Ju08, Definition 1.1] we say that $\mathfrak{s} \in \mathrm{Spin}^c(M, \gamma)$ is *outer with respect to S* if there is a unit vector field v on M whose homology class is $\mathfrak{s}$, and such that $v_p \neq -(\nu_S)_p$ for every $p \in S$. Here ν_S is the unit normal vector field of S with respect to some Riemannian metric on M. We denote by $O_S \subset \mathrm{Spin}^c(M, \gamma)$ the set of all outer Spin^c-structures with respect to S.

According to [Ju08, Theorem 1.3], if (M, γ) is a balanced sutured manifold and $(M, \gamma) \rightsquigarrow^S (M', \gamma')$ is a sutured manifold decomposition along a nice decomposing surface S, then

$$SFH(M', \gamma') \cong \bigoplus_{\mathfrak{s} \in O_S} SFH(M, \gamma, \mathfrak{s}). \tag{3}$$

So the result follows from the following lemma.

Lemma 4.3. *Let (M, γ) be a taut balanced sutured manifold, and let $\alpha \in H^1(M)$. Suppose that S is a nice decomposing surface that is dual to α. If the result of the sutured manifold decomposition $(M, \gamma) \rightsquigarrow^S (M', \gamma')$ is taut, then $\mathfrak{s} \in \mathrm{Spin}^c(M, \gamma)$ pairs minimally with α if and only if $\mathfrak{s} \in O_S$.*

Proof. Let $\mathfrak{s} \in O_S$. First, we show that for $\mathfrak{s}' \in \mathrm{Spin}^c(M, \gamma)$, we have $\langle\, \mathfrak{s} - \mathfrak{s}', \alpha \,\rangle = 0$ if and only if $\mathfrak{s}' \in O_S$. Pick unit vector fields v and v' representing $\mathfrak{s}$ and $\mathfrak{s}'$, respectively, such that v does not coincide with $-\nu_S$ at any point of S, and they satisfy the required boundary condition (in particular, $v = v'$ along ∂M). If $\mathfrak{s}' \in O_S$, then we can also assume that v' does not coincide with ν_S at any point of S. We can homotope v' relative to ∂M such that $v|_S = v'|_S$. After perturbing v' away from S, the set of points p where $v_p = -v'_p$ is a 1-manifold c. We orient c by taking the intersection of the submanifolds $v(M)$ and $-v'(M)$ of the total space of the unit sphere bundle STM, and projecting it to M. Then the homology class of c is Poincaré dual to $\mathfrak{s}-\mathfrak{s}'$. Since $c \cap S = \emptyset$, we obtain that

$$\langle\, \mathfrak{s}-\mathfrak{s}', \alpha\,\rangle = \#(c \cap S) = 0.$$

In the opposite direction, suppose that $\langle\, \mathfrak{s}-\mathfrak{s}', \alpha\,\rangle = 0$. Note that $\mathfrak{s}-\mathfrak{s}'$ is the obstruction to homotoping v to v' on the 2-skeleton of M through unit vector fields relative to ∂M. Hence we can homotope v' through unit vector fields relative to ∂M until $v|_S = v'|_S$. This v' is then a representative of $\mathfrak{s}'$ that never agrees with $-\nu_S$, and so $\mathfrak{s}' \in O_S$.

Next, we prove that if for $\mathfrak{s} \in O_S$ and $\mathfrak{s}' \in \mathrm{Spin}^c(M,\gamma)$, then we have

$$\langle\, \mathfrak{s}-\mathfrak{s}', \alpha\,\rangle = k > 0,$$

then $SFH(M,\gamma,\mathfrak{s}') = 0$ (in other words, $\mathfrak{s}' \not\in \mathrm{Supp}(M,\gamma)$). Represent $\mathfrak{s}$ by a unit vector field v, as above. After homotoping v, we can assume that there is a thin regular neighborhood N of ∂M such that $v_p = \nu_S$ for every $p \in S \setminus N$. Using the flow of v, we obtain a product neighborhood $P = (S \setminus N) \times [-\varepsilon, \varepsilon]$ of $S \setminus N$ such that $v = \partial/\partial t$ on P, where t is the coordinate on $[-\varepsilon, \varepsilon]$. Next, we construct a decomposing surface S' by attaching k small compressible tubes to S, each of which lies in P, and such that each has a single hyperbolic and a single elliptic tangency with the foliation of P by the surfaces $(S \setminus N) \times \{t\}$ for $t \in [-\varepsilon, \varepsilon]$. At the hyperbolic tangency, the orientation of the tube is opposite to that of the leaf, while at the elliptic tangency the orientations agree.

Let $\mathfrak{t} \in O_{S'}$, and choose a unit vector field w representing $\mathfrak{t}$ such that $w_p = (\nu_{S'})_p$ for every $p \in S' \setminus N$, and such that $v_p = w_p$ for every $p \in S \cap N = S' \cap N$. Furthermore, we can assume that w is chosen such that the set $\{\, p \in M \colon v_p = -w_p \,\}$ is a 1-manifold c, oriented as above, representing $\mathfrak{s}-\mathfrak{t}$. Then $|c \cap S'| = k$, as there is exactly one intersection point in each tube in $S' \setminus S$ at the hyperbolic tangency with the horizontal foliation on P. Since each such tangency p is hyperbolic, v_p is positively normal to the foliation, and $w_p = \nu_{S'}$, the intersection sign is positive. Hence

$$\langle\, \mathfrak{s}-\mathfrak{t}, [S']\,\rangle = \#(c \cap S') = k.$$

Of course, $[S] = [S']$, hence $\langle\, \mathfrak{s}-\mathfrak{t}, [S']\,\rangle = \langle\, \mathfrak{s}-\mathfrak{s}', [S']\,\rangle$. So $\langle \mathfrak{t}-\mathfrak{s}', [S']\,\rangle = 0$, and by the previous part, $\mathfrak{s}' \in O_{S'}$ as well.

Let (V,ν) be the result of decomposing (M,γ) along S'. Since S' is compressible, the sutured manifold (V,ν) is not taut, and $SFH(V,\nu) = 0$ (note that (M,γ) is taut, so M is irreducible, and hence so is V). But, according to (3),

$$SFH(V,\nu) \cong \bigoplus_{\mathfrak{s}' \in O_{S'}} SFH(M,\gamma,\mathfrak{s}').$$

This implies that $SFH(M,\gamma,\mathfrak{s}') = 0$ for every $\mathfrak{s}' \in O_{S'}$, which means that $SFH(M,\gamma,\mathfrak{s}') = 0$ for every $\mathfrak{s}' \in \mathrm{Spin}^c(M,\gamma)$ for which $\langle\, \mathfrak{s}-\mathfrak{s}', \alpha\,\rangle = k$.

Since (M,γ) is taut, $SFH(M,\gamma) \neq 0$. Notice that by (3),

$$O_S \cap \mathrm{Supp}(M,\gamma) \neq \emptyset;$$

let $\mathfrak{s}$ be an arbitrary element. Pick an identification of $\mathrm{Spin}^c(M,\gamma)$ with $H_1(M)$. By the above, if $\mathfrak{s}' \in \mathrm{Supp}(M,\gamma)$, then $\alpha(\mathfrak{s}) \le \alpha(\mathfrak{s}')$. Furthermore, we have equality if and only if $\mathfrak{s} \in O_S$. So $\mathfrak{s}' \in \mathrm{Spin}^c(M,\gamma)$ pairs minimally with α if and only if $\mathfrak{s}' \in O_S$. □

□

In the following theorem, we put together [Ju06, Proposition 9.4] and [Ju08, Theorem 9.7].

Theorem 4.4. *An irreducible balanced sutured manifold (M,γ) is a product if and only if $SFH(M,\gamma) \cong \mathbb{Z}$.*

A sutured manifold admits a depth 0 foliation if and only if it is a product. Indeed, then the holonomy group of each leaf is trivial, so, by Reeb stability, M is a fiber bundle over I. We thus obtain the following corollary to Theorem 4.4 that we alluded to in the introduction.

Corollary 4.5. *An irreducible balanced sutured manifold (M,γ) admits a taut depth 0 foliation if and only if $SFH(M,\gamma) \cong \mathbb{Z}$.*

Combining the above two theorems, we now obtain the following lemma.

Lemma 4.6. *Let (M,γ) be a balanced sutured manifold and $\alpha \in H^1(M)$. If*

$$SFH_\alpha(M,\gamma) \cong \mathbb{Z},$$

then α is dual to a nice decomposing surface S such that the decomposition $(M,\gamma) \rightsquigarrow^S (M',\gamma')$ yields a product sutured manifold (M',γ').

Proof. By part (3) of Lemma 4.1, the class α is dual to a well-groomed decomposing surface S'. We denote by S the union of the components of S' that touch R_-. Note that S has no closed components. It follows from Theorem 3.1 that S is also dual to α. By part (4) of Lemma 4.1, the surface S is also well-groomed. Since S has no closed components, it follows from part (5) of Lemma 4.1 that we can perturb S into a nice surface that, by a slight abuse of notation, we also denote by S.

We now perform the decomposition of (M,γ) along S, and call the result (M',γ'). By part (1) of Lemma 4.1, we know that (M',γ') is taut. It now follows from Theorem 4.2 and our assumption on α that

$$SFH(M',\gamma') \cong SFH_\alpha(M,\gamma) \cong \mathbb{Z}.$$

Therefore, by Theorem 4.4, this means that (M',γ') is a product. □

4.3. **Summary of results on foliations.** We now cite two results from the first author [Al13]. We start out with the following lemma that is the content of [Al13, Lemma 4.4], together with [Al13, Remark 7].

Lemma 4.7. *Let (M, γ) be a connected sutured manifold. Suppose that $\mathcal{F}$ is an indecomposable depth one foliation on (M, γ). Then there exists a decomposing surface S with*

$$PD([S]) = \lambda(\mathcal{F}) \in H^1(M)$$

such that the surface decomposition $(M, \gamma) \rightsquigarrow^S (M', \gamma')$ gives rise to a product sutured manifold.

For the reader's convenience, we give a quick outline of the proof. Let $\mathcal{F}$ be an indecomposable depth one foliation. The idea is that by "truncating" an arbitrary noncompact leaf of $\mathcal{F}$ near $R_- \cup R_+$, we obtain a surface S giving a product decomposition. In a bit more detail: as we mentioned already in Section 4.1, it was shown by Cantwell and Conlon [CC03, Lemma 11.4.4] that there exists a fiber bundle $M \setminus (R_- \cup R_+) \to S^1$ with fibers being the leaves of $\mathcal{F}$. So we can take a leaf and remove its ends (which is equivalent to removing a neighbourhood of $R(\gamma)$). This leaves us with a fibration over S^1 whose fibers are compact surfaces, and by removing one of these surfaces, we obtain a product. Finally, note that if c is a loop in M, and $\langle \lambda(\mathcal{F}), [c] \rangle$ denotes the signed intersection number of c with a noncompact leaf L, then $\langle PD([S]), [c] \rangle = \langle \lambda(\mathcal{F}), [c] \rangle$. It thus follows that

$$PD([S]) = \lambda(\mathcal{F}) \in H^1(M).$$

This concludes the sketch of the proof of Lemma 4.7.

Lemma 4.8. [Al13, Lemma C]. *Suppose (M, γ) is a connected sutured manifold. Let $(M, \gamma) \rightsquigarrow^S (M', \gamma')$ be a decomposition along a decomposing surface S such that (M', γ') is a product. Then there exists a depth one foliation $\mathcal{F}$ on (M, γ) such that the components of $R_\pm$ are the only compact leaves, and such that $[S]$ is dual to $\lambda(\mathcal{F}) \in \operatorname{Hom}(\pi_1(M), \mathbb{Z}) \cong H^1(M)$.*

The lemma is proved in [Al13, Section 3.2] by generalising Gabai's construction of depth one foliations to surface decompositions that are not necessarily well-groomed. For more details on Gabai's original construction, see [Ga83, Theorem 5.1].

We will not attempt to sketch the proof of the "only if" direction, but we want to at least give an idea of where the foliation comes from. Given the surface S, we can spin it along $R(\gamma)$ so that S becomes a leaf of an indecomposable depth one foliation $\mathcal{F}$ of (M, γ). We now sketch what we mean by "spinning."

Consider an annulus A as a bundle $I \times S^1$ with fibers homeomorphic to the unit interval I. Now consider how one could create the Reeb foliation of the annulus from this product fibration: essentially by taking the end of the unit intervals and "spinning" them around the two boundary circles $\{0\} \times S^1$ and $\{1\} \times S^1$. This may not be an ideal example in the context of taut foliations, since the Reeb foliation is not taut (there is no properly embedded arc that intersects all leaves transversely). However, if we take the ends of the unit interval fiber in the annulus and spin them in "opposite" directions, then we get a taut foliation.

Our example of spinning in two dimensions can easily be generalised to three dimensions: for an equivalent non-taut example start from a fiber bundle $D^2 \times S^1 \to S^1$, and construct the Reeb foliation of the solid torus in a similar fashion. For the taut example, start with the same fiber bundle, divide the boundary of each D^2 fiber into four intervals and spin the intervals along S^1 such that no two touching ones are spun in the same direction. This is often referred to as the "stacking of chairs" example, and it gives a taut, depth one foliation of the solid torus with four parallel longitudinal sutures, see [Ga84, Example 5.1].

4.4. **Proof of Theorem 1.2.** Suppose (M, γ) is a connected, irreducible balanced sutured manifold, and let $\alpha \in H^1(M)$. We need to show that

$$SFH_\alpha(M, \gamma) \cong \mathbb{Z}$$

if and only if there exists an indecomposable taut depth one foliation $\mathcal{F}$ with $\lambda(\mathcal{F}) = \alpha$.

First, suppose that $SFH_\alpha(M, \gamma) \cong \mathbb{Z}$. Then, in particular, $SFH(M, \gamma) \neq 0$, which implies by [Ju06, Proposition 9.18] that (M, γ) is taut. Then, by Lemma 4.6, it follows that α is dual to a nice decomposing surface S such that the decomposition

$$(M, \gamma) \rightsquigarrow^S (M', \gamma')$$

yields a product sutured manifold (M', γ'). It follows from Lemma 4.8 that we can construct an indecomposable taut depth one foliation $\mathcal{F}$ with

$$\lambda(\mathcal{F}) = PD([S]) \in H^1(M).$$

Conversely, suppose that there exists an indecomposable taut depth one foliation $\mathcal{F}$ of (M, γ) such that $\lambda(\mathcal{F}) = \alpha$. It follows from work of Gabai [Ga83, Theorem 2.12] that (M, γ) is taut. By Lemma 4.7, there exists a decomposing surface S with

$$PD([S]) = \lambda(\mathcal{F}) \in H^1(M)$$

such that the surface decomposition $(M, \gamma) \rightsquigarrow^S (M', \gamma')$ gives rise to a product sutured manifold. It now follows from Theorems 4.2 and 4.4 that

$$SFH_\alpha(M, \gamma) \cong SFH(M', \gamma') \cong \mathbb{Z}.$$

This concludes the proof of Theorem 1.2.

4.5. **Remark.** Sutured Floer homology of a sutured manifold is defined only if each component of the boundary contains at least one suture. The requirement $H_2(M) = 0$ in [Al13, Theorem A] is a direct consequence of this fact: it ensures that for any properly embedded surface in M, its homology class in $H_2(M, \partial M)$ is unchanged if we discard all of its closed components.

If there is a decomposition $(M,\gamma) \rightsquigarrow^S (M',\gamma')$ such that (M,γ) is balanced and (M',γ') is a product, then it is clear that S cannot have closed components. This means that if we know that there is such a decomposition, we can identify an extremal Spin^c-structure such that $SFH(M,\gamma,\mathfrak{s}) \cong \mathbb{Z}$ without the assumption $H_2(M)=0$.

However, conversely, if there is such and extremal Spin^c-structure, we know there exists a surface decomposition giving a product according to the decomposition formulas [Ju08, Theorem 1.3] and [Ju10, Corollary 4.15], but only if we also know that the relevant homology class can be represented by a well-groomed decomposing surface *without closed components* (which we then perturb to a nice decomposing surface). Such a representative always exists when $H_2(M)=0$: take the surface provided by part (3) of Lemma 4.1, and discard its closed components.

The purpose of this paper, contained in Proposition 3.5, is to show that any cohomology class $\alpha \in H^1(M)$ that pairs strictly minimally with an extremal Spin^c-structure, and for which $SFH_\alpha(M,\gamma,\mathfrak{s}) \cong \mathbb{Z}$, can be represented by a nice decomposing surface without closed components.

5. Fibered 3-manifolds with non-empty toroidal boundary

In this final section, we provide a method for deciding whether or not an oriented 3-manifold with non-empty boundary fibers over S^1. We furthermore show how to determine all the fibered classes in $H^1(M)$. Recall that a class

$$\alpha \in H^1(M) \cong \mathrm{Hom}(\pi_1(M),\mathbb{Z})$$

is *fibered* if there exists a fibration, i.e. a locally trivial fiber bundle, $p\colon M \to S^1$ such that

$$\alpha = p_*\colon \pi_1(M) \to \pi_1(S^1) = \mathbb{Z}.$$

Note that for any boundary component of M, the map p also restricts to a fibration over S^1. It follows that all boundary components are tori, and that the restriction of any fibered class to any boundary component is non-zero.

We will now show that sutured Floer homology detects whether a given first cohomology class on an irreducible 3-manifolds with non-empty toroidal boundary is fibered. In order to state our result, we need one more definition. Let M be a 3-manifold such that all the boundary components are tori. A *decoration* of M is a set of non-separating unoriented curves $\{c_1,\dots,c_k\}$ on ∂M such that each component of ∂M contains precisely one such curve. To a decoration $c=\{c_1,\dots,c_k\}$ of M, we associate a set of sutures $\gamma(c)$ by taking a regular neighborhood of $\partial N(c_i)$ in ∂M, where $N(c_i)$ is a regular neighborhood of c_i in ∂M. Note that $(M,\gamma(c))$ is a strongly balanced sutured manifold. For each Spin^c-structure $\mathfrak{s}$, we write

$$SFH(M,c,\mathfrak{s}) = SFH(M,\gamma(c),\mathfrak{s}),$$

and for each $\alpha \in H^1(M)$, we write

$$SFH_\alpha(M,c) = SFH_\alpha(M,\gamma(c)).$$

We can now formulate the following theorem.

Theorem 5.1. *Let M be a connected, irreducible 3-manifold with non-empty toroidal boundary. Let $\alpha \in H^1(M)$ be a class such that the restriction of α to any boundary component is non-zero. We then pick a decoration $c = \{c_1, \dots, c_k\}$ of M such that $\alpha(c_i) \neq 0$ for every $i \in \{1, \dots, k\}$. Then the following statements are equivalent:*

1. *$SFH_\alpha(M, c) \cong \mathbb{Z}$.*
2. *There is an indecomposable taut depth one foliation $\mathcal{F}$ on $(M, \gamma(c))$ with $\lambda(\mathcal{F}) = \alpha$.*
3. *The class α is fibered.*

Proof. The equivalence of (1) and (2) is a special case of Theorem 1.2.

Now we prove the equivalence of (2) and (3). First, suppose that the map $p\colon M \to S^1$ is a fibration in the class α; we denote by $\mathcal{F}_p$ the corresponding depth 0 cooriented foliation of M. Since $\alpha(c_i) \neq 0$, we can arrange that c_i is transverse to $\mathcal{F}_p$ for every $i \in \{1, \dots, k\}$. We view M as a manifold with right angle corners along $\partial\gamma(c)$, and the leaves of $\mathcal{F}_p$ also have corners at $\partial\gamma(c)$. We denote by $A_i^\pm$ the part of $R_\pm$ lying in the component of ∂M containing c_i. Then $A_i^\pm$ is an annulus, and $\mathcal{F}_p$ restricts to $A_i^\pm$ as a fibration with fiber I. We glue the products $A_i^\pm \times I$ to M via identifying $A_i^\pm \times \{0\}$ with $A_i^\pm$ for every $i \in \{1, \dots, k\}$, and call the resulting 3-manifold M'. The orientation of M induces an orientation of M'. Furthermore, let

$$\gamma' = \gamma(c) \cup \bigcup_{i=1}^{k} (\partial A_i^\pm) \times I.$$

We extend $\mathcal{F}_p$ to a foliation $\mathcal{F}$ of (M', γ') such that $A_i^\pm \times \{1\}$ is a compact leaf oriented as $\pm\partial M'$ for every $i \in \{1, \dots, k\}$. Furthermore, each leaf of the restriction of $\mathcal{F}$ to $A_i^\pm \times [0, 1)$ is diffeomorphic to $I \times \mathbb{R}_+$, the interval $I \times \{0\}$ matches with a fiber of $\mathcal{F}_p|_{A_i^\pm}$, and the arcs $I \times \{t\}$ spiral to $A_i^\pm \times \{1\}$ as $t \to \infty$. We can either spiral clockwise or counterclockwise, exactly one of these will make the resulting foliation cooriented. In particular, the coorientability of the foliation implies that $\mathcal{F}|_{\gamma'}$ has no Reeb components. There is a diffeomorphism $d\colon (M', \gamma') \to (M, \gamma(c))$ close to the obvious retraction from M' to M, and $d(\mathcal{F})$ is an indecomposable taut depth one foliation of $(M, \gamma(c))$ with $\lambda(\mathcal{F}) = \alpha$.

In the opposite direction, assume that $\mathcal{F}$ is an indecomposable taut depth one foliation of $(M, \gamma(c))$ such that $\lambda(\mathcal{F}) = \alpha$. Again, we view M as a manifold with corners along $\partial\gamma(c)$. We claim that, in this case, $\mathcal{F}$ is of the form described in the previous paragraph. More precisely, we will show that there is a collar neighborhood N of $R = R_+ \cup R_-$ such that the restriction of $\mathcal{F}$ to $\overline{M \setminus N}$ is a fibration. Indeed, since $\mathcal{F}$ is indecomposable, the union of the compact leaves of $\mathcal{F}$ is R, and hence $M \setminus R$ is the union of the depth one leaves. The holonomy of each depth one leaf is trivial, otherwise there would be leaves limiting on depth one leaves, and hence global Reeb stability implies that $\mathcal{F}$ restricts to a fibration of $M \setminus R$ with leaf space S^1 as M is connected. In particular, in each component γ_0 of $\gamma(c) \setminus R$, the foliation $\mathcal{F}$ restricts to a cooriented fibration over S^1. This determines the foliation $\mathcal{F}|_{\gamma_0}$ uniquely up to isotopy in the annulus γ_0: each leaf is diffeomorphic to $\mathbb{R}$, and spirals clockwise to $R_+ \cap \gamma_0$ and counterclockwise to $R_- \cap \gamma_0$.

Let A be one of the two components of R that meets γ_0. Since M has toroidal boundary, A is an annulus. We now take an identification of A with $S^1 \times I$. Furthermore, let $A \times I$ be a collar neighborhood of A in M such that for every point $(x,t) \in A \approx S^1 \times I$, the arc $\{(x,t)\} \times I$ is transverse to $\mathcal{F}$. Furthermore, we can assume that there is an $\epsilon > 0$ such that

$$\nu = S^1 \times \{0\} \times [0,\epsilon]$$

is a collar neighborhood of $A \cap \gamma_0$ in γ_0 such that $\partial\nu \setminus A = S^1 \times \{0\} \times \{\epsilon\}$ is a curve transverse to $\mathcal{F}$. Then, for every $(x,t,s) \in S^1 \times I \times [0,\epsilon] \approx A \times [0,\epsilon]$, let

$$\psi(x,t,s) = (x,t,\eta(x,t,s)) \in S^1 \times I \times I,$$

where η denotes the third component of ψ, and $t \mapsto \psi(x,t,s)$ is the unique curve of the above form tangent to $\mathcal{F}$ such that $\psi(x,0,s) = (x,0,s)$ (the lift of the curve $t \mapsto (x,t) \in A$ starting from $(x,0,s)$). The map ψ is unique and well-defined if ϵ is sufficiently small. Then ψ defines a collar neighborhood N_A of A on which $\mathcal{F}$ is the product $\mathcal{F}|_\nu \times I$, hence consists of leaves $\mathbb{R} \times I$ spiraling to A, exactly as in the proof of (3) $\Rightarrow$ (2). This way, we obtain a collar neighborhood of each component of R, and the union of these we denote by N. On $\partial(\overline{M \setminus N})$, the foliation $\mathcal{F}$ restricts to a fibration by circles. Finally, every leaf L of $\mathcal{F}|_{\overline{M\setminus N}}$ is compact, as L is a closed subset of a depth one leaf of $\mathcal{F}$, so if a sequence (p_n) in L had no limit point in L, then it would have a subsequence converging to R (as this is the union of the depth zero leaves), contradicting the fact that each p_n lies outside the neighborhood N of R.

In this section, we are mostly interested in the equivalence of (1) and (3). Note that this can also be proved in a more direct way using some of the methods of this paper, not pertaining to foliations. First, suppose that (1) holds. Then, as in the proof of Theorem 1.2, the class α is dual to a nice decomposing surface S such that the decomposition

$$(M,\gamma(c)) \rightsquigarrow^S (M',\gamma')$$

yields a product sutured manifold (M',γ'). Furthermore, since $\alpha(c_i) \neq 0$ for every $i \in \{1,\dots,k\}$, and because S is dual to α, we can assume that ∂S is transverse to c, and hence intersects each component of $\gamma(c)$ in essential arcs. For each component T of ∂M, the set $S \cap T$ consists of parallel oriented curves that divide T into parallel annuli $A_1,\dots,A_l$. After decomposing $(M,\gamma(c))$ along S, there will be exactly one suture in each A_i, and we can take the A_i themselves to be the resulting components of γ'. So $(M',\gamma') \approx (S \times I, \partial S \times I)$, and we obtain M by gluing $S \times \{0\}$ to $S \times \{1\}$. In particular, M is a fiber bundle over S^1 with the surface S, the dual to α, being one of the fibers.

Finally, we show that (3) implies (1). Assume that M fibers over S^1, and that S is a fiber dual to α. Again, we can suppose that c is transverse to S. Note that S is a nice decomposing surface in $(M, \gamma(c))$. As above, the result of decomposing $(M, \gamma(c))$ along S gives a sutured manifold (M', γ') diffeomorphic to $(S \times I, \partial S \times I)$, which is taut. So Theorem 4.2 implies that

$$SFH_\alpha(M, \gamma(c)) \cong SFH(M', \gamma') \cong \mathbb{Z}.$$

This concludes the proof. $\square$

Theorem 5.1 says, in particular, that given a connected, irreducible 3-manifold with non-trivial toroidal boundary, we can use sutured Floer homology to detect whether or not a given class $\alpha \in H^1(M)$ is fibered. Nonetheless, the result is perhaps slightly unsatisfactory, as it does not say how one can determine all fibered classes from sutured Floer homology at once. Just using Theorem 5.1 it is also not clear how one can decide whether there is a fibered class at all.

The issue is that in general there is no decoration which 'works for all non-zero $\alpha \in H^1(M)$'. More precisely, if we fix a decoration $c = \{c_1, \ldots, c_k\}$ of M, then for some classes $\alpha \in H^1(M)$, we might have $\alpha(c_i) = 0$ for some $i \in \{1, \ldots, k\}$. Now let T be a component of ∂M with decoration c_i such that $\alpha(c_i) = 0$. If S is then a surface dual to α in M, then $S \cap \partial T$ consists of say l parallel curves, all isotopic to c_i. Put S in a position where these l curves are all parallel to c_i. When S is a fiber of a fibration, and we decompose (M, γ) along such an S, we obtain (M', γ'), where $M' = S \times I$, but on the component of $\partial S \times I$ containing c_i, there will be 3 parallel components of γ'. Hence, according to [Ju10, Proposition 9.2], if $S \neq D^2$, then $SFH(M', \gamma') \cong (\mathbb{Z}^2)^{\otimes s(\alpha)}$, where

$$s(\alpha) = |\{\, i \in \{1, \ldots, k\} : \alpha(c_i) = 0 \,\}|.$$

In other words, if we first fix the decoration c, condition (3) does not imply (1), unless $s(\alpha) = 0$. On the other hand, if we know that $SFH_\alpha(M, \gamma(c)) \cong (\mathbb{Z}^2)^{\otimes s(\alpha)}$, we cannot use Theorem 3.1 to obtain a nice decomposing surface dual to α that gives a taut decomposition, as we might be stuck with closed components. However, if we put a restriction on the homology of M, we obtain the following.

Proposition 5.2. *Let M be a connected, irreducible 3-manifold with non-empty toroidal boundary, and such that the map $H_2(\partial M; \mathbb{Q}) \to H_2(M; \mathbb{Q})$ is surjective. Pick a decoration $c = \{c_1, \ldots, c_k\}$ of ∂M such that no component of $\partial M \setminus c$ is compressible, and let $\alpha \in H^1(M)$. Then the following are equivalent:*

(1) *The class α is fibered.*
(2) *$SFH_\alpha(M, \gamma(c)) \cong (\mathbb{Z}^2)^{s(\alpha)}$.*

Note that the assumption that $H_2(\partial M; \mathbb{Q}) \to H_2(M; \mathbb{Q})$ is surjective is equivalent to saying that M is the exterior of a link in a rational homology sphere.

Proof. First, suppose that the class α is fibered. Then let S be a fiber of a fibration such that $[S]$ is dual to α. Isotope S such that it is transverse to c, and parallel to c_i whenever $\alpha(c_i) = 0$. We saw above that if we decompose $(M, \gamma(c))$ along S, then we obtain a sutured manifold (M', γ') such that $M \approx S \times I$, and γ' consists of $\partial S \times I$, except that there are three parallel sutures for each component of $\partial S \times I$ that contains a curve c_i for which $\alpha(c_i) = 0$. Since no component of $\partial M \setminus \gamma(c)$ is compressible, (M', γ') is not $D^2 \times I$ with three parallel sutures, which is not taut and hence has $SFH(M', \gamma') = 0$. So [Ju10, Proposition 9.2] and Theorem 4.2 imply that

$$SFH_\alpha(M, \gamma) \cong SFH(M', \gamma') \cong \left(\mathbb{Z}^2\right)^{s(\alpha)}.$$

This shows that (1) implies (2).

Now suppose that (2) holds. Since no component of $\partial M \setminus c$ is compressible, the sutured manifold $(M, \gamma(c))$ is tight. Hence, according to Gabai [Ga83], there exists a well-groomed decomposing surface S such that $PD[S] = \alpha$ (in particular, the result of the decomposition is taut). As usual, we isotope S such that ∂S is transverse to c, and that it is parallel to each curve c_i for which $\alpha(c_i) = 0$. Let S_0 be the union of the closed components of S. Since the map $H_2(\partial M; \mathbb{Q}) \to H_2(M; \mathbb{Q})$ is surjective, the map $j_* \colon H_2(M; \mathbb{Q}) \to H_2(M, \partial M; \mathbb{Q})$ is zero, and hence S_0 is 0-homologous in $H_2(M, \partial M; \mathbb{Q})$, and so also in $H_2(M, \partial M)$ (as the latter is torsion-free). So $S' = S \setminus S_0$ is also dual to α. Furthermore, S' will also produce a taut sutured manifold that we denote (M', γ'). By Theorem 4.2 and our assumption (2),

$$SFH(M', \gamma') \cong \left(\mathbb{Z}^2\right)^{s(\alpha)}.$$

If $\alpha(c_i) = 0$, then the component A_i of $\partial M \setminus N(S)$ containing c_i will have three parallel sutures in (M', γ'). According to [Ju10, Proposition 9.2], removing two of the three parallel sutures from γ' for each of the $s(\alpha)$ curves c_i for which $\alpha(c_i) = 0$ will result in a sutured manifold (M', ν) with $SFH(M', \nu) \cong \mathbb{Z}$. Hence (M', ν) is a product with $R_\pm(\nu) \approx S'$, and so $M' \approx S' \times I$. Furthermore, M is obtained by gluing $S' \times \{0\}$ and $S' \times \{1\}$ via some diffeomorphism, and so α is a fibered class. $\square$

We will now describe a method for detecting all fibered classes (and whether there is a fibered class at all) even when the map $H_2(\partial M; \mathbb{Q}) \to H_2(M; \mathbb{Q})$ is not surjective, and along the way, we will also consider real fibered classes. We say that a class $\alpha \in H^1(M; \mathbb{R})$ is *fibered* if it can be represented by a nowhere vanishing closed 1-form. It is well-known that, for integral classes, the two notions of fiberedness agree. We now denote by

$$\mathcal{F}(M) := \{\alpha \in H^1(M; \mathbb{R}) \,:\, \alpha \text{ is fibered}\}$$

the set of all fibered classes.

In the following, given a decoration $c = \{c_1, \dots, c_k\}$ for a connected, irreducible 3-manifold with non-empty toroidal boundary, we write

$$\mathcal{C}(M, c) = \{\alpha \in H^1(M; \mathbb{R}) :\ \alpha(c_i) \neq 0 \text{ for } i = 1, \dots, k\}.$$

This set contains all classes for which *SFH* can detect whether they are fibered, according to Theorem 5.1. More precisely, let

$$\mathcal{M}(M,c) = \{\alpha \in \mathcal{C}(M,c) : \ SFH_\alpha(M,c) \cong \mathbb{Z}\};$$

the integral classes in $\mathcal{M}(M,c)$ are exactly the fibered classes in $\mathcal{C}(M,\gamma)$. By definition, $\mathcal{M}(M,c) \subset \mathcal{C}(M,c) \subset H^1(M;\mathbb{R})$.

Before we state the next theorem, we recall that if M is a 3-manifold, then it is a consequence of a standard Poincaré duality argument that for any toroidal boundary component T, we have $\operatorname{rank}(\operatorname{Im}(H_1(T) \to H_1(M))) \geq 1$.

Theorem 5.3. *Let M be a connected, irreducible 3-manifold such that ∂M is a non-empty union of tori $T_1, \dots, T_k$, labeled such that*

$$\operatorname{rank}\left(\operatorname{Im}(H_1(T_i) \to H_1(M))\right) = \begin{cases} 2 & \text{for } i \in \{1,\dots,l\}, \text{ and} \\ 1 & \text{for } i \in \{l+1,\dots,k\}. \end{cases}$$

For $i \in \{1,\dots,l\}$, we pick curves c_i^1, $c_i^2 \subset T_i$ such that they span a rank two subspace of $H_1(M)$, and for $i \in \{l+1,\dots,k\}$, we pick a curve $c_i \subset T_i$ which is non-torsion in $H_1(M)$. Then

$$\mathcal{F}(M) = \bigcup_{\{\epsilon_1,\dots,\epsilon_l\} \in \{1,2\}^l} \mathcal{M}(M, c_1^{\epsilon_1}, \dots, c_l^{\epsilon_l}, c_{l+1}, \dots, c_k).$$

Proof. We write

$$\mathcal{C}(M) = \{\alpha \in H^1(M;\mathbb{R}) : \ \alpha|_{T_i} \neq 0 \text{ for } i = 1,\dots,k\}.$$

For any of the sets $\mathcal{C}(M)$, $\mathcal{C}(M,c)$, $\mathcal{F}(M)$, and $\mathcal{M}(M,c)$, we use the same expression decorated with a superscript $\mathbb{Z}$ to indicate their intersection with $H^1(M)$. With these conventions, we can reformulate the equivalence of (1) and (3) in Theorem 5.1 as

$$\mathcal{F}^{\mathbb{Z}}(M) \cap \mathcal{C}^{\mathbb{Z}}(M,c) = \mathcal{M}^{\mathbb{Z}}(M,c). \tag{4}$$

We denote by D the set of the 2^l decorations $\{c_1^{\epsilon_1}, \dots, c_l^{\epsilon_l}, c_{l+1}, \dots, c_k\}$ with $\{\epsilon_1, \dots, \epsilon_l\}$ in $\{1,2\}^l$.

We first prove the following claim.

Claim.

$$\mathcal{F}^{\mathbb{Z}}(M) = \bigcup_{c \in D} \mathcal{M}^{\mathbb{Z}}(M,c).$$

It follows from the earlier discussion that $\mathcal{F}^{\mathbb{Z}}(M) \subset \mathcal{C}^{\mathbb{Z}}(M)$. It also follows easily from the definitions that

$$\mathcal{C}^{\mathbb{Z}}(M) = \bigcup_{c \in D} \mathcal{C}^{\mathbb{Z}}(M,c).$$

If we combine these observations with (4), we obtain that

$$\begin{aligned}\mathcal{F}^{\mathbb{Z}}(M) &= \mathcal{F}^{\mathbb{Z}}(M) \cap \mathcal{C}^{\mathbb{Z}}(M) \\ &= \mathcal{F}^{\mathbb{Z}}(M) \cap \bigcup_{c\in D} \mathcal{C}^{\mathbb{Z}}(M,c) \\ &= \bigcup_{c\in D} \left(\mathcal{F}^{\mathbb{Z}}(M) \cap \mathcal{C}^{\mathbb{Z}}(M,c)\right) \\ &= \bigcup_{c\in D} \mathcal{M}^{\mathbb{Z}}(M,c).\end{aligned}$$

This concludes the proof of the claim.

By a rational open half-space in $H^1(M;\mathbb{R}) \cong H^1(M) \otimes \mathbb{R}$, we mean a subset which is described by a strict rational linear inequality. By a rational cone in $H^1(M;\mathbb{R})$, we mean the intersection of finitely many rational open half-spaces in $H^1(M;\mathbb{R})$. Thurston [Th86] showed that $\mathcal{F}(M)$ is a (possibly empty) union of rational cones. It is a straightforward consequence of the definitions that each $\mathcal{M}(M,c)$ is a rational cone.

Together with the above claim, we now see that $\mathcal{F}(M)$ and $\bigcup_{c\in D} \mathcal{M}(M,c)$ are unions of rational cones that agree on $H^1(M)$. It follows that the two sets are in fact the same. □

Corollary 5.4. *With the notation of Theorem 5.3, we have $\mathcal{F}(M,\gamma) \neq \emptyset$ if and only if $SFH(M,\gamma(c),\mathfrak{s}) \cong \mathbb{Z}$ for an extremal* Spin^c*-structure $\mathfrak{s} \in$* Spin$^c(M,\gamma(c))$ *for one of the 2^l decorations $c = \{c_1^{\epsilon_1},\dots,c_l^{\epsilon_l},c_{l+1},\dots,c_k\}$ for $\{\epsilon_1,\dots,\epsilon_l\} \in \{1,2\}^l$.*

Proof. We first suppose that $\mathcal{F}(M,\gamma) \neq \emptyset$. It then follows immediately from Theorem 5.3 that, for one of the given 2^l decorations c, there exists an extremal Spin^c-structure $\mathfrak{s} \in \text{Spin}^c(M,\gamma(c))$ such that $SFH(M,c,\mathfrak{s}) \cong \mathbb{Z}$.

Now suppose that, for one of the 2^l-decorations c, there exists an extremal Spin^c-structure $\mathfrak{s}$ such that $SFH(M,c,\mathfrak{s}) \cong \mathbb{Z}$. This implies that there exists an $\alpha \in H^1(M;\mathbb{R})$ such that

$$SFH_\alpha(M,c) \cong \mathbb{Z}.$$

It is straightforward to see that the set

$$\{\beta \in H^1(M;\mathbb{R}) \setminus \{0\} \,:\, SFH_\beta(M,c) \cong \mathbb{Z}\}$$

is open in $H^1(M;\mathbb{R})$. Hence, there also exists a class $\beta \in \mathcal{C}(M,c)$ with $SFH_\beta(M,c) \cong \mathbb{Z}$. It thus follows from Theorem 5.3 that M is fibered. □

We finally note that if M is a 3-manifold such that the boundary consists of a single torus T, then $\text{rank}(\text{Im}(H_1(T) \to H_1(M))) = 1$. We can thus record the following special case of Theorem 5.3.

Corollary 5.5. *Let M be a connected, irreducible* 3*-manifold such that the boundary consists of a single torus T. We pick a curve c on T which is non-torsion in $H_1(M)$. Then*

$$\mathcal{F}(M) = \mathcal{M}(M,c).$$

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