

TWISTED ALEXANDER POLYNOMIALS OF HYPERBOLIC KNOTS

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ABSTRACT. We study a twisted Alexander polynomial naturally associated to a hyperbolic knot in an integer homology 3-sphere via a lift of the holonomy representation to $SL(2, \mathbb{C})$. It is an unambiguous symmetric Laurent polynomial whose coefficients lie in the trace field of the knot. It contains information about genus, fibering, and chirality, and moreover is powerful enough to sometimes detect mutation.

We calculated this invariant numerically for all 313,209 hyperbolic knots in S^3 with at most 15 crossings, and found that in all cases it gave a sharp bound on the genus of the knot and determined both fibering and chirality.

We also study how such twisted Alexander polynomials vary as one moves around in an irreducible component X_0 of the $SL(2, \mathbb{C})$ -character variety of the knot group. We show how to understand all of these polynomials at once in terms of a polynomial whose coefficients lie in the function field of X_0 . We use this to help explain some of the patterns observed for knots in S^3 , and explore a potential relationship between this universal polynomial and the Culler-Shalen theory of surfaces associated to ideal points.

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1. INTRODUCTION

A fundamental invariant of a knot K in an integral homology 3-sphere Y is its Alexander polynomial Δ_K . While Δ_K contains information about genus and fiberings, it is determined by the maximal metabelian quotient of the fundamental group of the complement $M = Y \setminus K$, and so this topological information has clear limits. In 1990, Lin introduced the *twisted Alexander polynomial* associated to K and a representation $\alpha: \pi_1(M) \rightarrow \mathrm{GL}(n, \mathbb{F})$, where \mathbb{F} is a field. These twisted Alexander polynomials also contain information about genus and fiberings and have been studied by many authors (see the survey [FV3]). Much of this work has focused on those α which factor through a finite quotient of $\pi_1(M)$, which is closely related to studying the ordinary Alexander polynomial in finite covers of M . In contrast, we study here a twisted Alexander polynomial associated to a representation coming from hyperbolic geometry.

Suppose that K is *hyperbolic*, i.e. the complement M has a complete hyperbolic metric of finite volume, and consider the associated holonomy representation $\bar{\alpha}: \pi_1(M) \rightarrow \mathrm{Isom}^+(\mathbb{H}^3)$. Since $\mathrm{Isom}^+(\mathbb{H}^3) \cong \mathrm{PSL}(2, \mathbb{C})$, there are two simple ways to get a linear representation so we can consider the twisted Alexander polynomial: compose $\bar{\alpha}$ with the adjoint representation to get $\pi_1(M) \rightarrow \mathrm{Aut}(\mathfrak{sl}_2\mathbb{C}) \leq \mathrm{GL}(3, \mathbb{C})$, or alternatively lift $\bar{\alpha}$ to a representation $\pi_1(M) \rightarrow \mathrm{SL}(2, \mathbb{C})$. The former approach is the focus of the recent paper of Dubois and Yamaguchi

[DY]; the latter method is what we use here to define an invariant $\mathcal{T}_K(t) \in \mathbb{C}[t^{\pm 1}]$ called the *hyperbolic torsion polynomial*.

The hyperbolic torsion polynomial \mathcal{T}_K is surprisingly little studied. To our knowledge it has only previously been looked at for 2-bridge knots in [Mor, KM1, HM, SW]. Here we show that it contains a great deal of topological information. In fact, we show that \mathcal{T}_K determines genus and fibering for all 313,209 hyperbolic knots in S^3 with at most 15 crossings, and we conjecture this is the case for all knots in S^3 .

1.1. Basic properties. More broadly, we consider here knots in \mathbb{Z}_2 -homology 3-spheres. The ambient manifold Y containing the knot K will always be oriented, not just orientable, and \mathcal{T}_K depends on that orientation. Following Turaev, we formulate \mathcal{T}_K as a Reidemeister-Milnor torsion as this reduces its ambiguity; in that setting, we work with the compact core of M , namely the knot exterior $X := Y \setminus \text{int}(N(K))$ (see Section 2 for details). By fixing certain conventions for lifting the holonomy representation $\bar{\alpha}: \pi_1(X) \rightarrow \text{PSL}(2, \mathbb{C})$ to $\alpha: \pi_1(X) \rightarrow \text{SL}(2, \mathbb{C})$, we construct in Section 4 a well-defined symmetric polynomial $\mathcal{T}_K \in \mathbb{C}[t^{\pm 1}]$. The first theorem summarizes its basic properties:

1.2. Theorem. *Let K be a hyperbolic knot in an oriented \mathbb{Z}_2 -homology 3-sphere. Then \mathcal{T}_K has the following properties:*

- (a) \mathcal{T}_K is an unambiguous element of $\mathbb{C}[t^{\pm 1}]$ which satisfies $\mathcal{T}_K(t^{-1}) = \mathcal{T}_K(t)$. It does not depend on an orientation of K .
- (b) The coefficients of \mathcal{T}_K lie in the trace field of K . If K has integral traces, the coefficients of \mathcal{T}_K are algebraic integers.
- (c) $\mathcal{T}_K(\xi)$ is non-zero for any root of unity ξ . In particular, $\mathcal{T}_K \neq 0$.
- (d) If K^* denotes the mirror image of K , then $\mathcal{T}_{K^*}(t) = \overline{\mathcal{T}_K(t)}$, where the coefficients of the latter polynomial are the complex conjugates of those of \mathcal{T}_K .
- (e) If K is amphichiral then \mathcal{T}_K is a real polynomial.
- (f) The values $\mathcal{T}_K(1)$ and $\mathcal{T}_K(-1)$ are mutation invariant.

Moreover, \mathcal{T}_K both determines and is determined by a sequence of \mathbb{C} -valued torsions of finite cyclic covers of X . Specifically, let X_m be the m -fold cyclic cover coming from the free abelianization of $H_1(X; \mathbb{Z})$. For the restriction α_m of α to $\pi_1(X_m)$, we consider the corresponding \mathbb{C} -valued torsion $\tau(X_m, \alpha_m)$. A standard argument shows that \mathcal{T}_K determines all the $\tau(X_m, \alpha_m)$ (see Theorem 3.1). More interestingly, the converse holds: the torsions $\tau(X_m, \alpha_m)$ determine \mathcal{T}_K (see Theorem 4.5). This latter result follows from work of David Fried [Fri] (see also Hillar [Hil]) and that of Menal-Ferrer and Porti [MFP1].

1.3. Remark. Conjecturally, the torsions $\tau(X_m, \alpha_m)$ can be expressed as analytic torsions and as Ruelle zeta functions defined using the lengths of prime geodesics. See Ray-Singer [RS], Cheeger [Che1, Che2], Müller [Mu1] and Park [Par] for details and background material. We hope that this point of view will be helpful in the further study of \mathcal{T}_K .

The torsions $\tau(X_m, \alpha_m)$ are interesting invariants in their own right. For example, Menal–Ferrer and Porti [MFP1] showed that $\tau(X_m, \alpha_m)$ is non-zero for any m . Furthermore, Porti [Por1] showed that $\tau(X_1, \alpha_1) = \tau(X, \alpha) = \mathcal{T}_K(1)$ is not obviously related to hyperbolic volume. More precisely, using a variation on [Por2, Théorème 4.17] one can show that there exists a sequence of knots K_n whose volumes converge to a positive real number, but the numbers $\mathcal{T}_{K_n}(1)$ converge to zero. See [Por2, MFP1, MFP2] for further results.

1.4. Topological information: genus and fibering. We define $x(K)$ to be the Thurston norm of a generator of $H_2(X, \partial X; \mathbb{Z}) \cong \mathbb{Z}$; if K is null-homologous in Y , then $x(K) = 2 \cdot \text{genus}(K) - 1$, where $\text{genus}(K)$ is the least genus of all Seifert surfaces bounding K . Also, we say that K is *fibred* if X fibers over the circle.

A key property of the ordinary Alexander polynomial Δ_K is that

$$x(K) \geq \deg(\Delta_K) - 1.$$

When K is fibred, this is an equality and moreover the lead coefficient of Δ_K is 1 (here, we normalize Δ_K so that the lead coefficient is positive). As with any twisted Alexander/torsion polynomial, we get similar information out of \mathcal{T}_K :

1.5. Theorem. *Let K be a knot in an oriented \mathbb{Z}_2 -homology sphere. Then*

$$x(K) \geq \frac{1}{2} \deg(\mathcal{T}_K).$$

If K is fibred, this is an equality and \mathcal{T}_K is monic, i.e. has lead coefficient 1.

Theorem 1.5 is an immediate consequence the definitions below and of [FK1, Theorem 1.1] (for the genus bound) and of the work of Goda, Kitano and Morifuji [GKM] (for the fibred case); see also Cha [Cha], Kitano and Morifuji [KM2], Pajitnov [Paj], Kitayama [Kit2], [FK1] and [FV3, Theorem 6.2].

1.6. Experimental results. The calculations in [Cha, GKM, FK1] gave evidence that when one can freely choose the representation α , the twisted torsion polynomial is very successful at detecting both $x(K)$ and non-fibred knots. Moreover [FV1] shows that collectively the twisted torsion polynomials of representations coming from homomorphisms to finite groups determine whether a knot is fibred. However, it is not known if all twisted torsion polynomials together always detect $x(K)$.

Instead of considering many different representations as in the work just discussed, we focus here on a single, albeit canonical, representation. Despite this, we find that \mathcal{T}_K alone is a very powerful invariant. In Section 6, we describe computations showing that the bound on $x(K)$ is sharp for all 313,209 hyperbolic knots with at most 15 crossings; in contrast the bound from Δ_K is not sharp for 2.8% of these knots. Moreover, among such knots \mathcal{T}_K was monic *only* when the knot was fibred, whereas 4.0% of these knots have monic Δ_K but aren't fibred. (Here we computed \mathcal{T}_K numerically to a precision of 250 decimal places, see Section 6.6 for details.)

Given all this data, we are compelled to propose the following, even though on its face it feels quite implausible, given the general squishy nature of Alexander-type polynomials.

1.7. Conjecture. *For a hyperbolic knot K in S^3 , the hyperbolic torsion polynomial \mathcal{T}_K determines $x(K)$, or equivalently its genus. Moreover, the knot K is fibered if and only if \mathcal{T}_K is monic.*

We have not done extensive experiments for knots in manifolds other than S^3 , but so far we have not encountered any examples where \mathcal{T}_K doesn't contain perfect genus and fibering information.

1.8. Topological information: Chirality and mutation. When K is an amphichiral, \mathcal{T}_K is a real polynomial (Theorem 1.2(e)). This turns out to be an excellent way to detect chirality. Indeed, among hyperbolic knots in S^3 with at most 15 crossings, the 353 knots where \mathcal{T}_K is real are *exactly* the amphichiral knots (Section 6.3).

Also, hyperbolic invariants often do not detect mutation, for example the volume [Rub], the invariant trace field [MR, Corollary 5.6.2], and the birationality type of the geometric component of the character variety [CL, Til1, Til2]. The ordinary Alexander polynomial Δ_K is also mutation invariant for knots in S^3 . However, $x(K)$ can change under mutation, and given that $x(K)$ determines the degree of \mathcal{T}_K for all 15 crossing knots, it follows that \mathcal{T}_K can change under mutation; we discuss many such examples in Section 6.4. However, sometimes mutation does preserve \mathcal{T}_K , and we don't know of any examples of two knots with the same \mathcal{T}_K which aren't mutants.

1.9. Character varieties. So far, we have focused on the twisted torsion polynomial of (a lift of) the holonomy representation of the hyperbolic structure on M . However, this representation is always part of a complex curve of representations $\pi_1(M) \rightarrow \mathrm{SL}(2, \mathbb{C})$, and it is natural to study how the torsion polynomial changes as we vary the representation. In Sections 7 and 8, we describe how to understand all of these torsion polynomials at once, and use this to help explain some of the patterns observed in Section 6. For the special case of 2-bridge knots, Kim and Morifuji [Mor, KM1] had previously studied how the torsion polynomial varies with the representation, and our results here extend some of their work to more general knots.

Consider the character variety $X(K) := \mathrm{Hom}(\pi_1(M), \mathrm{SL}(2, \mathbb{C})) // \mathrm{SL}(2, \mathbb{C})$, which is an affine algebraic variety over \mathbb{C} . We show in Section 7 that each $\chi \in X(K)$ has an associated torsion polynomial \mathcal{T}_K^χ . These \mathcal{T}_K^χ vary in an understandable way, in terms of a polynomial with coefficients in the ring of regular functions $\mathbb{C}[X_0]$:

1.10. Theorem. *Let X_0 be an irreducible component of $X(K)$ which contains the character of an irreducible representation. There is a unique $\mathcal{T}_K^{X_0} \in \mathbb{C}[X_0][t^{\pm 1}]$ so that for all $\chi \in X_0$ one has $\mathcal{T}_K^\chi(t) = \mathcal{T}_K^{X_0}(\chi)(t)$. Moreover, $\mathcal{T}_K^{X_0}$ is itself the torsion polynomial of a certain representation $\pi_1(M) \rightarrow \mathrm{SL}(2, \mathbb{F})$, and thus has all the usual properties (symmetry, genus bound, etc.).*

1.11. **Corollary.** *Let K be a knot in an integral homology 3-sphere. Then*

- (a) *The set $\{\chi \in X(K) \mid \deg(\mathcal{T}_K^\chi) = 2x(K)\}$ is Zariski open.*
- (b) *The set $\{\chi \in X(K) \mid \mathcal{T}_K^\chi \text{ is monic}\}$ is Zariski closed.*

It is natural to focus on the component X_0 of $X(K)$ which contains the (lift of) the holonomy representation of the hyperbolic structure, which we call the *distinguished component*. In this case X_0 is an algebraic curve, and we show the following conjecture is implied by Conjecture 1.7.

1.12. **Conjecture.** *Let K be a hyperbolic knot in S^3 , and X_0 be the distinguished component of its character variety. Then $2x(K) = \deg(\mathcal{T}_K^{X_0})$ and $\mathcal{T}_K^{X_0}$ is monic if and only if K is fibered.*

At the very least, Conjecture 1.12 is true for many 2-bridge knots as we discuss in Section 7.6. We also give several explicit examples of $\mathcal{T}_K^{X_0}$ in Section 8 and use these to explore a possible avenue for bringing the Culler-Shalen theory of surfaces associated to ideal points of $X(K)$ to bear on Conjecture 1.12.

1.13. **Other remarks and open problems.** For simplicity, we restricted ourselves here to the study of knots, especially those in S^3 . However, we expect that many of the results and conjectures are valid for more general 3-manifolds. In the broader settings, the appropriate question is whether the twisted torsion polynomial detects the Thurston norm and fibered classes (see [FK1, FK2] and [FV2] for more details).

As we mentioned earlier, there is another natural way to get a torsion polynomial from holonomy representation $\bar{\alpha}: \pi_1(M) \rightarrow \mathrm{PSL}(2, \mathbb{C})$, namely by considering the adjoint representation of $\mathrm{PSL}(2, \mathbb{C})$ on its Lie algebra. The corresponding torsion polynomial was studied by Dubois and Yamaguchi [DY], partly building on work of Porti [Por2]. Here, in Section 5.2 we calculate this alternate invariant for an interesting pair of knots and surprisingly find that its degree is *not* determined by the knot genus.

We conclude this introduction with a few questions and open problems:

- (a) Does \mathcal{T}_K determine the volume of the complement of K ? Some theoretical evidence of a close relationship between \mathcal{T}_K and the volume is given by the recent work of Bergeron and Venkatesh [BV] and the work of Müller [Mu2, Mu3]. Some calculational evidence is also given in [F] and in Section 6.4 in this paper.
- (b) If two knots in S^3 have the same \mathcal{T}_K , are they necessarily mutants? See Section 6.4 for more on this.
- (c) Does the invariant \mathcal{T}_K contain information about symmetries of the knot besides information on chirality?
- (d) Does there exist a hyperbolic knot with $\mathcal{T}_K(1) = 1$?
- (e) If \mathcal{T}_K is a real polynomial, is K necessarily amphichiral?
- (f) For an amphichiral knot, is the top coefficient of \mathcal{T}_K always positive?
- (g) For fibered knots, why is the *second* coefficient of \mathcal{T}_K so often real? This coefficient is the sum of the eigenvalues of the monodromy acting on the twisted homology of the fiber. See Section 6.5 for more.

- (h) Why is $|\mathcal{T}_K(-1)| > |\mathcal{T}_K(1)|$ for 99.99% of the knots considered in Section 6.5?

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2. TWISTED INVARIANTS OF 3-MANIFOLDS

In this section, we review torsions of twisted homology groups and explain how they are used to define the twisted torsion polynomial of a knot together with a representation of its fundamental group to $\mathrm{SL}(2, \mathbb{C})$. We then summarize the basic properties of these torsion polynomials, including how to calculate them.

2.1. Torsion of based chain complexes. Let C_* be a finite chain complex over a field \mathbb{F} . Suppose that each chain group C_i is equipped with an ordered basis c_i and that each homology group $H_i(C_*)$ is also equipped with an ordered basis h_i . Then there is an associated *torsion invariant* $\tau(C_*, c_*, h_*) \in \mathbb{F}^\times := \mathbb{F} \setminus \{0\}$ as described in the various excellent expositions [Mil, Tur3, Tur4, Nic]. We will follow the convention of Turaev, which is the multiplicative inverse of Milnor's invariant. If the complex C_* is acyclic, then we will write $\tau(C_*, c_*) := \tau(C_*, c_*, \emptyset)$.

2.2. Twisted homology. For the rest of this section, fix a finite CW-complex X and set $\pi := \pi_1(X)$. Consider a representation $\alpha: \pi \rightarrow \mathrm{GL}(V)$, where V is a finite-dimensional vector space over \mathbb{F} . We can thus view V as a left $\mathbb{Z}[\pi]$ -module. To define the twisted homology groups $H_*^\alpha(X; V)$, consider the universal cover \tilde{X} of X . Regarding π as the group of deck transformations of \tilde{X} turns the cellular chain complex $C_*(\tilde{X}) := C_*(\tilde{X}; \mathbb{Z})$ into a left $\mathbb{Z}[\pi]$ -module. We then give $C_*(\tilde{X})$ a right $\mathbb{Z}[\pi]$ -module structure via $c \cdot g := g^{-1} \cdot c$ for $c \in C_*(\tilde{X})$ and $g \in \pi$, which allows us to consider the tensor product

$$C_*^\alpha(X; V) := C_*(\tilde{X}) \otimes_{\mathbb{Z}[\pi]} V.$$

Now $C_*^\alpha(X; V)$ is a finite chain complex of vector spaces, and we define $H_*^\alpha(X; V)$ to be its homology.

We call two representations $\alpha: \pi \rightarrow \mathrm{GL}(V)$ and $\beta: \pi \rightarrow \mathrm{GL}(W)$ *conjugate* if there exists an isomorphism $\Psi: V \rightarrow W$ such that $\alpha(g) = \Psi^{-1} \circ \beta(g) \circ \Psi$ for all $g \in \pi$. Note that such a Ψ induces an isomorphism of $H_*^\alpha(X; V)$ with $H_*^\beta(X; W)$.

2.3. Euler structures, homology orientations and twisted torsion of CW complexes. To define the twisted torsion, we first need to introduce certain additional structures on which it depends. (In our final application, most of these will come out in the wash.) First, fix an orientation of each cell of X . Then choose an ordering of the cells of X so we can enumerate them as c_j ; only the relative

order of cells of the same dimension will be relevant, but it is notationally convenient to have only one subscript.

An *Euler lift* for X associates to each cell c_j of X a cell \tilde{c}_j of \tilde{X} which covers it. If \tilde{c}'_j is another Euler lift, then there are unique $g_j \in \pi$ so that $\tilde{c}'_j = g_j \cdot \tilde{c}_j$. We say these two Euler lifts are *equivalent* if

$$\prod_j g_j^{(-1)^{\dim(c_j)}} \text{ represents the trivial element in } H_1(X; \mathbb{Z}).$$

An equivalence class of Euler lifts is called an *Euler structure* on X . The set of Euler structures on X , denoted $\text{Eul}(X)$, admits a canonical free transitive action by $H_1(X; \mathbb{Z})$; see [Tur2, Tur3, Tur4, FKK] for more on these Euler structures. Finally, a *homology orientation* for X is just an orientation of the \mathbb{R} -vector space $H_*(X; \mathbb{R})$.

Now we can define the torsion $\tau(X, \alpha, e, \omega)$ associated to X , a representation α , an Euler structure e , and a homology orientation ω . If $H_*^\alpha(X; V) \neq 0$, we define $\tau(X, \alpha, e, \omega) := 0$, and so now assume $H_*^\alpha(X; V) = 0$. Up to sign, the torsion we seek will be that of the twisted cellular chain complex $C_*^\alpha(X; V)$ with respect to the following ordered basis. Let $\{v_k\}$ be any basis of V , and $\{\tilde{c}_j\}$ any Euler lift representing e . Order the basis $\{\tilde{c}_j \otimes v_k\}$ for $C_*^\alpha(X; V)$ lexicographically, i.e. $\tilde{c}_j \otimes v_k < \tilde{c}_{j'} \otimes v_{k'}$ if either $j < j'$ or both $j = j'$ and $k < k'$. We thus have a based acyclic complex $C_*^\alpha(X; V)$ and we can consider

$$\tau(C_*^\alpha(X; V), c_* \otimes v_*) \in \mathbb{F}^\times.$$

When $\dim(V)$ is even, this torsion is in fact independent of all the choices involved, but when $\dim(V)$ is odd we need to augment it as follows to remove a sign ambiguity.

Pick an ordered basis h_i for $H_*(X; \mathbb{R})$ representing our homology orientation ω . Since we have ordered the cells of X , we can consider the torsion

$$\tau(C_*(X; \mathbb{R}), c_*, h_*) \in \mathbb{R}^\times.$$

We define $\beta_i(X) = \sum_{k=0}^i \dim(H_k(X; \mathbb{R}))$ and $\gamma_i(X) = \sum_{k=0}^i \dim(C_k(X; \mathbb{R}))$, and then set $N(X) = \sum_i \beta_i(X) \cdot \gamma_i(X)$. Following [FKK], which generalizes the ideas of Turaev [Tur1, Tur2], we now define

$$\tau(X, \alpha, e, \omega) := (-1)^{N(X) \cdot \dim(V)} \cdot \tau(C_*^\alpha(X; V), c_* \otimes v_*) \cdot \text{sign}\left(\tau(C_*(X; \mathbb{R}), c_*, h_*)\right)^{\dim(V)}.$$

A straightforward calculation using the basic properties of torsion shows that this invariant does not depend on any of the choices involved, i.e. it is independent of the ordering and orientation of the cells of X , the choice of representatives for e and ω , and the particular basis for V . Similar elementary arguments prove the following lemma. Here $-\omega$ denotes the opposite homology orientation to ω , and note that $(\det \circ \alpha) : \pi \rightarrow \mathbb{F}$ factors through $H_1(X; \mathbb{Z})$.

2.4. Lemma. *If β is conjugate to α , then given $h \in H_1(X; \mathbb{Z})$ and $\epsilon \in \{-1, 1\}$, one has*

$$\tau(X, \beta, h \cdot e, \epsilon \cdot \omega) = \epsilon^{\dim(V)} \cdot ((\det \circ \alpha)(h))^{-1} \cdot \tau(X, \alpha, e, \omega).$$

2.5. Twisted torsion of 3-manifolds. Let N be a 3-manifold whose boundary is empty or consists of tori. We first recall some facts about Spin^c -structures on N ; see [Tur4, Section XI.1] for details. The set $\text{Spin}^c(N)$ of such structures admits a free and transitive action by $H_1(N; \mathbb{Z})$. Moreover, there exists a map $c_1: \text{Spin}^c(N) \rightarrow H_1(N; \mathbb{Z})$ which has the property that $c_1(h \cdot \mathfrak{s}) = 2h + c_1(\mathfrak{s})$ for any $h \in H_1(N; \mathbb{Z})$ and $\mathfrak{s} \in \text{Spin}^c(N)$.

Now consider a triangulation X of N . By [Tur4, Section XI] there exists a canonical bijection $\text{Spin}^c(N) \rightarrow \text{Eul}(X)$ which is equivariant with respect to the actions by $H_1(N; \mathbb{Z}) = H_1(X; \mathbb{Z})$. Given a representation $\alpha: \pi_1(N) \rightarrow \text{GL}(V)$, an element $\mathfrak{s} \in \text{Spin}^c(N)$, and a homology orientation ω for N , we define

$$\tau(N, \alpha, \mathfrak{s}, \omega) := \tau(X, \alpha, e, \omega)$$

where e is the Euler structure on X corresponding to \mathfrak{s} . It follows from the work of Turaev [Tur1, Tur2] that $\tau(N, \alpha, \mathfrak{s}, \omega)$ is independent of the choice of triangulation and hence well-defined. See also [FKK] for more details about $\tau(N, \alpha, \mathfrak{s}, \omega)$.

2.6. Twisted torsion polynomial of a knot. Let K be a knot in a rational homology 3-sphere Y . Throughout, we write $X_K := Y \setminus \text{int}(N(K))$ for the knot exterior, which is a compact manifold with torus boundary. We define an *orientation* of K to be a choice of oriented meridian μ_K ; if Y is oriented, instead of just orientable, this is equivalent to the usual notion. Suppose now that K is oriented. Let $\pi_K := \pi_1(X_K)$ and take $\phi_K: \pi_K \rightarrow \mathbb{Z}$ to be the unique epimorphism where $\phi(\mu_K) > 0$. There is a canonical homology orientation ω_K for X_K as follows: take a point as a basis for $H_0(X_K; \mathbb{R})$ and take $\{\mu_K\}$ as the basis for $H_1(X_K; \mathbb{R})$. We will drop K from these notations if the knot is understood from the context.

For a representation $\alpha: \pi \rightarrow \text{GL}(n, R)$ over a commutative domain R , we define a torsion polynomial as follows. Consider the left $\mathbb{Z}[\pi]$ -module structure on $R^n \otimes_R R[t^{\pm 1}] \cong R[t^{\pm 1}]^n$ given by

$$g \cdot (v \otimes p) := (\alpha(g) \cdot v) \otimes (t^{\phi(g)} p)$$

for $g \in \pi$ and $v \otimes p \in R^n \otimes_R R[t^{\pm 1}]$. Put differently, we get a representation $\alpha \otimes \phi: \pi \rightarrow \text{GL}(n, R[t^{\pm 1}])$. We denote by $Q(t)$ the field of fractions of $R[t^{\pm 1}]$. The representation $\alpha \otimes \phi$ allows us to view $R[t^{\pm 1}]^n$ and $Q(t)^n$ as left $\mathbb{Z}[\pi]$ -modules. Given $\mathfrak{s} \in \text{Spin}^c(X)$ we define

$$\tau(K, \alpha, \mathfrak{s}) := \tau(X_K, \alpha \otimes \phi, \mathfrak{s}, \omega_K) \in Q(t)$$

to be the *twisted torsion polynomial of the oriented knot K* corresponding to the representation α and the Spin^c -structure \mathfrak{s} . Calling $\tau(K, \alpha, \mathfrak{s})$ a polynomial even though it is defined as a rational function is reasonable given Theorem 2.11 below.

2.7. Remark. The study of twisted polynomial invariants of knots was introduced by Lin [Lin]. The invariant $\tau(K, \alpha, \mathfrak{s})$ can be viewed as a refined version of the

twisted Alexander polynomial of a knot and of Wada's invariant. We refer to [Wada, Kit1, FV3] for more on twisted invariants of knots and 3-manifolds.

2.8. The $\mathrm{SL}(2, \mathbb{F})$ torsion polynomial of a knot. Our focus in this paper is on 2-dimensional representations, and we now give a variant of $\tau(K, \alpha, \mathfrak{s})$ which does not depend on \mathfrak{s} or the orientation of K . Specifically, for an *unoriented* knot K in a QHS and a representation $\alpha: \pi \rightarrow \mathrm{SL}(2, \mathbb{F})$ we define

$$(2.9) \quad \mathcal{T}_K^\alpha := t^{\phi(c_1(\mathfrak{s}))} \cdot \tau(K, \alpha, \mathfrak{s})$$

for any $\mathfrak{s} \in \mathrm{Spin}^c(X)$ and choice of oriented meridian μ and show:

2.10. Theorem. *For $\alpha: \pi \rightarrow \mathrm{SL}(2, \mathbb{F})$, the invariant \mathcal{T}_K^α is a well-defined element of $\mathbb{F}(t)$ which is symmetric, i.e. $\mathcal{T}_K^\alpha(t^{-1}) = \mathcal{T}_K^\alpha(t)$.*

We will call $\mathcal{T}_K^\alpha \in \mathbb{F}(t)$ the *twisted torsion polynomial associated to K and α* .

Proof. That the right-hand side of (2.9) is independent of the choice of \mathfrak{s} follows easily from Lemma 2.4 using the observation that $\det((\alpha \otimes \phi)(g)) = t^{2\phi(g)}$ for $g \in \pi$ and the properties of c_1 given in Section 2.5.

The choice of meridian μ affects the right-hand side of (2.9) in two ways: it is used to define the homology orientation ω and the homomorphism $\phi: \pi \rightarrow \mathbb{Z}$. The first doesn't matter by Lemma 2.4, but switching ϕ to $-\phi$ is equivalent to replacing t with t^{-1} . Hence being independent of the choice of meridian is equivalent to the final claim that \mathcal{T}_K^α is symmetric.

Now any $\mathrm{SL}(2, \mathbb{F})$ -representation preserves the bilinear form on \mathbb{F}^2 given by $(v, w) \mapsto \det(vw)$. Using this observation it is shown in [FKK, Theorem 7.3], generalizing [HSW, Corollary 3.4] and building on work of Turaev [Tur1, Tur2], that in our context we have

$$\tau(K, \alpha, \mathfrak{s})(t^{-1}) = t^{2\phi(c_1(\mathfrak{s}))} \cdot \tau(K, \alpha, \mathfrak{s})$$

which establishes the symmetry \mathcal{T}_K^α and hence the theorem. \square

While in general \mathcal{T}_K^α is a rational function, it is frequently a Laurent polynomial or can be computed in terms of the ordinary Alexander polynomial Δ_K .

2.11. Theorem. *Let K be a knot in a QHS and let $\alpha: \pi_K \rightarrow \mathrm{SL}(2, \mathbb{F})$ be a representation.*

- (a) *If α is irreducible, then \mathcal{T}_K^α lies in $\mathbb{F}[t^{\pm 1}]$.*
- (b) *If α is reducible, then $\mathcal{T}_K^\alpha = \mathcal{T}_K^\beta$ where β is the diagonal part of α , i.e. a diagonal representation where $\mathrm{tr}(\beta(g)) = \mathrm{tr}(\alpha(g))$ for all $g \in \pi$.*
- (c) *If α is reducible and factors through $H_1(X_K; \mathbb{Z})/(\text{torsion})$ then*

$$\mathcal{T}_K^\alpha(t) = \frac{\Delta_K(zt) \cdot \Delta_K(z^{-1}t)}{t - (z + z^{-1}) + t^{-1}}$$

where z, z^{-1} are the eigenvalues of $\alpha(\mu_K)$ and Δ_K is the symmetrized Alexander polynomial.

When the ambient manifold Y is an ZHS, then the torsion polynomial of any reducible representation α to $\mathrm{SL}(2, \mathbb{F})$ can be computed by combining (b) and (c); when $H_1(Y; \mathbb{Z})$ is finite but nontrivial, then \mathcal{T}_K^α is the product of the torsion polynomials of two 1-dimensional representations, but may not be directly related to Δ_K .

Proof. Part (a) is due to Kitano and Morifuji [KM2] and is seen as follows. Since α is irreducible, there is a $g \in [\pi, \pi]$ so that $\alpha(g)$ does not have trace 2 (see e.g. Lemma 1.5.1 of [CS] or the first part of the proof of Theorem 1.1 of [KM2]). Then take a presentation of π where g is a generator and apply Proposition 2.13 below with $x_i = g$; since $\phi(g) = 0$ and $\mathrm{tr}(\alpha(g)) \neq 2$ the denominator in (2.14) lies in \mathbb{F}^\times and hence \mathcal{T}_K^α is in $\mathbb{F}[t^{\pm 1}]$.

For Part (b), conjugate α so that it is upper-diagonal

$$\alpha(g) = \begin{pmatrix} a(g) & b(g) \\ 0 & a(g)^{-1} \end{pmatrix} \quad \text{for all } g \in \pi.$$

The diagonal part of α is the representation β given by $g \mapsto \begin{pmatrix} a(g) & 0 \\ 0 & a(g)^{-1} \end{pmatrix}$. It is easy to see, for instance by using (2.14), that $\mathcal{T}_K^\alpha = \mathcal{T}_K^\beta$. Finally, part (c) follows from a straightforward calculation with (2.14), see e.g. [Tur3, Tur4]. \square

2.12. Calculation of torsion polynomials using Fox calculus. Suppose we are given a knot K in a QHS and a representation $\alpha: \pi_1(X_K) \rightarrow \mathrm{SL}(2, \mathbb{F})$. In this section, we give a simple method for computing \mathcal{T}_K^α . As usual, we write $\pi := \pi_1(X_K)$ and $\phi = \phi_K$. We can extend the group homomorphism $\alpha \otimes \phi: \pi \rightarrow \mathrm{GL}(2, \mathbb{F}[t^{\pm 1}])$ to a ring homomorphism $\mathbb{Z}[\pi] \rightarrow M(2, \mathbb{F}[t^{\pm 1}])$ which we also denote by $\alpha \otimes \phi$. Given a $k \times l$ -matrix $A = (a_{ij})$ over $\mathbb{Z}[\pi]$, we denote by $(\alpha \otimes \phi)(A)$ the $2k \times 2l$ -matrix obtained from A by replacing each entry a_{ij} by the 2×2 -matrix $(\alpha \otimes \phi)(a_{ij})$.

Now let $F = \langle x_1, \dots, x_n \rangle$ be the free group on n generators. By Fox (see [Fox1, Fox2, CF] and also [Har, Section 6]) there exists for each x_i a *Fox derivative*

$$\frac{\partial}{\partial x_i} : F \rightarrow \mathbb{Z}[F]$$

with the following two properties:

$$\frac{\partial x_j}{\partial x_i} = \delta_{ij} \quad \text{and} \quad \frac{\partial(uv)}{\partial x_i} = \frac{\partial u}{\partial x_i} + u \frac{\partial v}{\partial x_i} \quad \text{for all } u, v \in F.$$

We also need the involution of $\mathbb{Z}[F]$ which sends $g \in F$ to g^{-1} and respects addition (this is not an algebra automorphism since it induces an anti-homomorphism for multiplication). We denote the image of $a \in \mathbb{Z}[F]$ under this map by \bar{a} , and if A is a matrix over $\mathbb{Z}[F]$ then \bar{A} denotes the result of applying this map to each entry.

The following allows for the efficient calculation of \mathcal{T}_K^α , since π always has such a presentation (e.g. if K is a knot in S^3 one can use a Wirtinger presentation).

2.13. Proposition. *Let K be a knot in a QHS, and $\langle x_1, \dots, x_n \mid r_1, \dots, r_{n-1} \rangle$ be a presentation of π_K of deficiency one. Let A be the $n \times (n-1)$ -matrix with entries $a_{ij} = \frac{\partial r_j}{\partial x_i}$. Fix a generator x_i and consider the matrix A_i obtained from A by deleting the i^{th} row. Then there exists an $l \in \mathbb{Z}$ so that for every even-dimensional representation $\alpha: \pi \rightarrow \text{GL}(V)$ one has*

$$(2.14) \quad \mathcal{T}_K^\alpha(t) = t^l \cdot \frac{\det((\alpha \otimes \phi)(\bar{A}_i))}{\det((\alpha \otimes \phi)(\bar{x}_i - 1))}$$

whenever the denominator is nonzero.

The same formula also holds, up to a sign, when $\dim(V)$ is odd. An easy way to ensure nonzero denominator in (2.14) is to choose an x_i where $\phi(x_i) \neq 0$; then $\det((\alpha \otimes \phi)(\bar{x}_i - 1))$ is essentially the characteristic polynomial of $\alpha(x_i)^{-1}$ and hence nonzero.

2.15. Remark. Wada's invariant (see [Wada]) is defined to be

$$\frac{\det((\alpha \otimes \phi)(A_i^t))}{\det((\alpha \otimes \phi)(x_i - 1))}.$$

In [FV3, p. 53] it is erroneously claimed that, up to multiplication by a power of t , the torsion polynomial \mathcal{T}_K^α agrees with Wada's invariant. Since there seems to be some confusion in the literature regarding the precise relationship between twisted torsion and Wada's invariant, we discuss it in detail in Section 2.20. In that section, we will also see that for representations into $\text{SL}(2, \mathbb{F})$, Wada's invariant does in fact agree with $\mathcal{T}_K^\alpha(t)$. In particular the invariant studied by Kim and Morifuji [KM1] agrees with $\mathcal{T}_K^\alpha(t)$.

Proposition 2.13 is an immediate consequence of:

2.16. Proposition. *Let K, π, A be as above. For each generator x_i , there is an $\mathfrak{s} \in \text{Spin}^c(X_K)$ so that for every even-dimensional representation $\beta: \pi \rightarrow \text{GL}(V)$ one has*

$$(2.17) \quad \tau(X_K, \beta, \mathfrak{s}) = \frac{\det(\beta(\bar{A}_i))}{\det(\beta(\bar{x}_i - 1))} \quad \text{whenever the denominator is nonzero.}$$

The homology orientation ω is suppressed in (2.17) because by Lemma 2.4 it doesn't affect τ as $\dim(V)$ is even.

Proof of Proposition 2.16. Let X be the canonical 2-complex corresponding to the presentation of π , i.e. X has one cell of dimension zero, n cells of dimension one and $n-1$ cells of dimension two. As the Whitehead group of π is trivial [Wal], it follows that X is simple-homotopy equivalent to any other CW-decomposition of X_K ; in particular, it is simple-homotopy equivalent to a triangulation. By standard results (see e.g. [Tur3, Section 8]) we can now use X to calculate the torsion of X_K .

Consider the Euler structure e for X which is given by picking an arbitrary lift of the vertex of X to the universal cover \tilde{X} , and then taking the lift of each

x_i which starts at this basepoint. Reading out the words r_j in x_1, \dots, x_n starting at the basepoint gives a canonical lift for the 2-cells corresponding to the relators. With respect to this basing, the chain complex $C_*(\tilde{X})$ is isomorphic to the following chain complex

$$0 \rightarrow \mathbb{Z}[\pi]^{n-1} \xrightarrow{\partial_2} \mathbb{Z}[\pi]^n \xrightarrow{\partial_1} \mathbb{Z}[\pi] \rightarrow 0.$$

The bases of $C_2(\tilde{X})$ and $C_1(\tilde{X})$ are abusively denoted by $\{r_j\}$ and $\{x_i\}$, and the basis of $C_0(\tilde{X})$ is the lifted basepoint b . Thus

$$\partial_2(r_j) = \sum_i \frac{\partial r_j}{\partial x_i} x_i = \sum_i a_{ij} x_i \quad \text{and} \quad \partial_1(x_i) = (x_i - 1)b.$$

Now fix a basis $\{v_k\}$ for V . If we then view elements $v \in V$ as vertical vectors and $\beta(g)$ as a matrix, the left $\mathbb{Z}[\pi]$ -module structure on V is given by $g \cdot v = \beta(g)v$. Thus in the complex $C_*^\beta(X; V) = C_*(\tilde{X}) \otimes_{\mathbb{Z}[\pi]} V$ we have

$$\begin{aligned} \partial_2(r_j \otimes v_k) &= \sum_i (a_{ij} x_i \otimes v_k) = \sum_i (x_i \cdot \bar{a}_{ij} \otimes v_k) = \sum_i (x_i \otimes \bar{a}_{ij} \cdot v_k) \\ &= \sum_i (x_i \otimes \beta(\bar{a}_{ij})v_k). \end{aligned}$$

Thus with the basis ordering conventions of Section 2.3, the twisted chain complex $C_*^\beta(X; V)$ is given by

$$(2.18) \quad 0 \rightarrow V^{n-1} \xrightarrow{\beta(\bar{A})} V^n \xrightarrow{(\beta(\bar{x}_1-1), \dots, \beta(\bar{x}_n-1))} V \rightarrow 0,$$

where as usual matrices act on the left of vertical vectors.

From now on, we assume that $\det(\beta(\bar{x}_i - 1)) \neq 0$ as otherwise there is nothing to prove. First, consider the case when $\det(\beta(\bar{A}_i)) = 0$. We claim in this case that $C_*^\beta(X; V)$ is not acyclic, and thus (2.17) holds by the definition of τ . Consider any $v \in V^{n-1}$ which is in the kernel of \bar{A}_i ; because $\beta(\bar{x}_i - 1)$ is non-singular, the fact that $\partial^2 = 0$ forces v to be in the kernel of \bar{A} . Hence $H_2^\beta(X; V) \neq 0$ as needed.

When instead $\det(\beta(\bar{A}_i)) \neq 0$, then both boundary maps in (2.18) have full rank and hence the complex is acyclic. Following Section 2.2 of [Tur3] we can use a suitable matrix τ -chain to compute the desired torsion. Specifically [Tur3, Theorem 2.2] gives

$$(2.19) \quad \tau(X, \beta, e) = \frac{\det(\beta(\bar{A}_i))}{\det(\beta(\bar{x}_i - 1))}.$$

Here, we are using that $\dim(V)$ is even, which forces the sign discussed in [Tur3, Remark 2.4] to be positive. Also, the convention of [Tur3] is to record a basis as the rows of a matrix, whereas we use the columns; this is irrelevant since the determinant is transpose invariant. Given (2.19) if we take \mathfrak{s} be the Spin^c -structure corresponding to e , we have established (2.17). \square

2.20. Connection to Wada's invariant. We now explain why the formula (2.14) differs from the one used to define Wada's invariant [Wada], and how Wada's invariant also arises as a torsion of a suitable chain complex. To start, suppose we have a representation $\beta: \pi \rightarrow \mathrm{GL}(d, \mathbb{F})$, where as usual π is the fundamental group of a knot exterior. The representation β makes $V := \mathbb{F}^d$ into both a left and a right $\mathbb{Z}[\pi]$ -module. The left module ${}_{\beta}V$ is defined by $g \cdot v := \beta(g)v$ where $v \in V$ is viewed as a column vector, and the right module V_{β} is defined by $v \cdot g := v\beta(g)$ where now $v \in V$ is viewed as a row vector.

Given a left $\mathbb{Z}[\pi]$ -module W we denote by W^{op} the right $\mathbb{Z}[\pi]$ -module given by $w \cdot f := \overline{f} \cdot w$. Similarly we can define a left module W^{op} for a given right $\mathbb{Z}[\pi]$ -module W . In Section 2, we started with the *left* modules $C_*(\tilde{X})$ and ${}_{\beta}V$ and used the chain complex

$$C_*^{\beta}(\tilde{X}, \mathbb{F}^d) := C_*(\tilde{X})^{op} \otimes_{\mathbb{Z}[\pi]} {}_{\beta}V$$

when defining the torsion.

One could instead consider the chain complex

$$V_{\beta} \otimes_{\mathbb{Z}[\pi]} C_*(\tilde{X}).$$

Here, if $\{v_k\}$ is a basis for V and $\{\tilde{c}_j\}$ is a $\mathbb{Z}[\pi]$ -basis for $C_*(\tilde{X})$, then we endow $V_{\beta} \otimes_{\mathbb{Z}[\pi]} C_*(\tilde{X})$ with the basis $\{v_k \otimes \tilde{c}_j\}$ ordered *reverse* lexicographically, i.e. $v_k \otimes \tilde{c}_j < v_{k'} \otimes \tilde{c}_{j'}$ if either $j < j'$ or both $j = j'$ and $k < k'$.

Suppose now we want to compute the torsion of $V_{\beta} \otimes C_*(\tilde{X})$ using the setup of the proof of Theorem 2.16. Then we have

$$\begin{aligned} \partial_2(v_k \otimes r_j) &= \sum_i (v_k \otimes a_{ij} x_i) = \sum_i (v_k \otimes a_{ij} \cdot x_i) = \sum_i (v_k \cdot a_{ij} \otimes x_i) \\ &= \sum_i (v_k \beta(\alpha_{ij}) \otimes x_i). \end{aligned}$$

Since we are focusing on a right module V_{β} , it is natural to write the matrices for the boundary maps in $V_{\beta} \otimes_{\mathbb{Z}[\pi]} C_*(\tilde{X})$ as matrices which *act on the right of row vectors*. With these conventions one gets the chain complex

$$0 \rightarrow V^{n-1} \xrightarrow{\beta(A^t)} V^n \xrightarrow{(\beta(x_1-1), \dots, \beta(x_n-1))^t} V \rightarrow 0$$

where here A^t denotes the transpose of A , and so A^t is an $(n-1) \times n$ matrix over $\mathbb{Z}[\pi]$. As in the proof of Theorem 2.16, in the generic case [Tur3, Theorem 2.2] gives that

$$\tau(V_{\beta} \otimes_{\mathbb{Z}[\pi]} C_*(\tilde{X})) = \frac{\det(\beta(A_i^t))}{\det(\beta(x_i - 1))}.$$

Up to the sign of the denominator, this is precisely the formula for Wada's invariant given in [Wada].

It's important to note here that $\beta(A^t)$ is not necessarily the same as $(\beta(A))^t$, and hence Wada's invariant may differ from our $\tau(X, \beta)$. However, note that there exists a canonical isomorphism

$$\begin{aligned} V_{\beta} \otimes_{\mathbb{Z}[\pi]} C_*(\tilde{X}) &\rightarrow C_*(\tilde{X})^{op} \otimes_{\mathbb{Z}[\pi]} (V_{\beta})^{op} \\ v \otimes \sigma &\mapsto \sigma \otimes v \end{aligned}$$

which moreover respects the ordered bases. Thus these chain complexes have the same torsion invariant. It's easy to see that the left module $(V_\beta)^{op}$ is isomorphic to ${}_{\beta^*}V$ where $\beta^*: \pi \rightarrow \mathrm{GL}(d, \mathbb{F})$ is the representation given by $\beta^*(g) := (\beta(g)^{-1})^t$. Thus Wada's invariant for β is our torsion $\tau(X, \beta^*)$.

Our focus in this paper is on β of the form $\alpha \otimes \phi$ where $\alpha: \pi \rightarrow \mathrm{SL}(2, \mathbb{F})$ and $\phi: \pi \rightarrow \mathbb{Z}$ is the usual epimorphism. Note that α^* is conjugate to α (see e.g. [HSW]) and hence $(\alpha \otimes \phi)^*$ is conjugate to $\alpha^* \otimes (-\phi)$. Since we argued in Section 2.8 that \mathcal{T}^α is independent of the choice of ϕ , it follows that in this case our \mathcal{T}^α is exactly Wada's invariant for α .

3. TWISTED TORSION OF CYCLIC COVERS

As usual, let K be a knot in a QHS with exterior X and fundamental group π . For an irreducible representation $\alpha: \pi \rightarrow \mathrm{SL}(2, \mathbb{C})$, in this section we relate the torsion polynomial \mathcal{T}_K^α to a sequence of \mathbb{C} -valued torsions of finite cyclic covers of X . We show that the latter determines the former, and will use this connection in Section 4 to prove nonvanishing of the hyperbolic torsion polynomial.

To start, pick an orientation of K to fix the homomorphism $\phi: \pi \rightarrow \mathbb{Z}$. For each $m \in \mathbb{N}$, we denote by X_m the m -fold cyclic cover corresponding to $\pi_m := \phi^{-1}(m\mathbb{Z})$. We denote by α_m the restriction of α to $\pi_m = \pi_1(X_m)$. Since the dimension is even and the image of α_m lies in $\mathrm{SL}(2, \mathbb{C})$, it follows from Lemma 2.4 that the torsion $\tau(X_m, \alpha, \mathfrak{s}, \omega) \in \mathbb{C}$ does not depend on the choice of Spin^c -structure or homology orientation; therefore we denote it by $\tau(X_m, \alpha_m)$. We also let μ_m be the set of all m^{th} roots of unity in \mathbb{C} . The first result of this section is the following (see [DY, Corollary 27] for a related result).

3.1. Theorem. *Let K be a knot in a QHS with exterior X and fundamental group π . Let $\alpha: \pi \rightarrow \mathrm{SL}(2, \mathbb{C})$ be an irreducible representation. Then for every $m \in \mathbb{N}$ we have*

$$\prod_{\zeta \in \mu_m} \mathcal{T}_K^\alpha(\zeta) = \tau(X_m, \alpha_m).$$

Note here since α is irreducible the torsion polynomial \mathcal{T}_K^α is in $\mathbb{C}[t^{\pm 1}]$ by Theorem 2.11(a), and so $\mathcal{T}_K^\alpha(\xi)$ is well-defined for any $\xi \in \mathbb{C}^\times$. Combining Theorem 3.1 with a (generalization of) a result of David Fried [Fri], we will show:

3.2. Theorem. *If $\tau(X_m, \alpha_m)$ is non-zero for every $m \in \mathbb{N}$, then the $\tau(X_m, \alpha_m)$ determine $\mathcal{T}_K^\alpha(t) \in \mathbb{C}(t)$.*

To state the key lemmas, we first need some notation. We denote by γ_m the representation $\pi \rightarrow \mathrm{GL}(\mathbb{C}[\mathbb{Z}_m])$ which is the composite of the epimorphism $\pi \xrightarrow{\phi} \mathbb{Z} \rightarrow \mathbb{Z}_m$ with the regular representation of \mathbb{Z}_m on $\mathbb{C}[\mathbb{Z}_m]$. Given any $\xi \in \mathbb{C}^\times$, we denote by λ_ξ the representation $\pi \rightarrow \mathrm{GL}(1, \mathbb{C})$ which sends $g \in \pi$ to $\xi^{\phi(g)}$. We first prove Theorem 3.1 assuming the following lemmas.

3.3. Lemma. $\tau(X_m, \alpha_m) = \tau(X, \alpha \otimes \gamma_m)$.

3.4. Lemma. *For every $\xi \in \mathbb{C}^\times$ and $\mathfrak{s} \in \mathrm{Spin}^c(X)$ we have*

$$(3.5) \quad \tau(X, \alpha \otimes \phi, \mathfrak{s})(\xi) = \tau(X, \alpha \otimes \lambda_\xi, \mathfrak{s}).$$

Proof of Theorem 3.1. Using Lemma 3.3 and the fact that γ_m and $\bigoplus_{\zeta \in \mu_m} \lambda_\zeta$ are conjugate representations of π , we have

$$\tau(X_m, \alpha_m) = \tau(X, \alpha \otimes \gamma_m) = \prod_{\zeta \in \mu_m} \tau(X, \alpha \otimes \lambda_\zeta, \mathfrak{s}).$$

Note here that while the other terms do not depend on \mathfrak{s} , those in the product at right do since $\alpha \otimes \lambda_\zeta$ is no longer a special linear representation. We now apply Lemma 3.4 to find

$$\begin{aligned} \tau(X_m, \alpha_m) &= \prod_{\zeta \in \mu_m} \tau(X, \alpha \otimes \lambda_\zeta, \mathfrak{s}) = \prod_{\zeta \in \mu_m} \tau(X, \alpha \otimes \phi, \mathfrak{s})(\zeta) \\ &= \prod_{\zeta \in \mu_m} (\zeta)^{-\phi(c_1(\mathfrak{s}))} \mathcal{F}_K^\alpha(\zeta) = \prod_{\zeta \in \mu_m} \mathcal{F}_K^\alpha(\zeta) \end{aligned}$$

where the last two equalities follow from (2.9) and the fact that $\prod \zeta = 1$. \square

Proof of Lemma 3.3. The idea is that for suitable choices one gets an isomorphism

$$C_*^{\alpha_m}(X_m; V) \rightarrow C_*^{\gamma_m \otimes \alpha}(X; \mathbb{C}[\mathbb{Z}_m] \otimes_{\mathbb{C}} V)$$

as *based* chain complexes over \mathbb{C} , and hence their torsions are the same.

Fix a triangulation for X with an ordering c_j of its cells, as well as an Euler lift $c_j \mapsto \tilde{c}_j$ of the cells to the universal cover \tilde{X} . Let $\phi_m: \pi \rightarrow \mathbb{Z}_m$ be the epimorphism whose kernel is $\pi_m = \pi_1(X_m)$, and fix $g \in \pi$ where $\bar{g} = \phi_m(g)$ generates \mathbb{Z}_m .

Consider the triangulation of X_m which is pulled back from that of X , and let c'_j be the cell in X_m which is the image of \tilde{c}_j under $\tilde{X} \rightarrow X_m$. Then each cell of X_m has a unique expression as $g^k \cdot c'_j$ for k in $\{0, \dots, m-1\}$, where here g^k acts on X_m as a deck transformation. We order these cells so that $g^k \cdot c'_j < g^{k'} \cdot c'_{j'}$ if $j < j'$ or both $j = j'$ and $k < k'$. When computing torsion, we'll use the Euler lift $g^k \cdot c'_j \mapsto g^k \cdot \tilde{c}_j$ for X_m .

Let V denote \mathbb{C}^2 with the π -module structure given by α , and let $\{\nu_1, \nu_2\}$ be an ordered basis for V . Consider the map

$$(3.6) \quad f: C_*(\tilde{X}) \otimes_{\mathbb{Z}[\pi_m]} V \rightarrow C_*(\tilde{X}) \otimes_{\mathbb{Z}[\pi]} (\mathbb{C}[\mathbb{Z}_m] \otimes_{\mathbb{C}} V)$$

induced by $\tilde{c} \otimes \nu \mapsto \tilde{c} \otimes (1 \otimes \nu)$; this is well defined since for $h \in \pi_m$ we have

$$\begin{aligned} f((\tilde{c} \cdot h) \otimes \nu) &= (\tilde{c} \cdot h) \otimes (1 \otimes \nu) = \tilde{c} \otimes (h \cdot (1 \otimes \nu)) \\ &= \tilde{c} \otimes ((h \cdot 1) \otimes (h \cdot \nu)) = \tilde{c} \otimes (1 \otimes (h \cdot \nu)) = f(\tilde{c} \otimes (h \cdot \nu)), \end{aligned}$$

where we used $h \in \pi_m$ to see $h \cdot 1 = 1$ in $\mathbb{C}[\mathbb{Z}_m]$. Clearly f is a chain map of complexes of \mathbb{C} -vector spaces, and it is an isomorphism since it sends the elements of the basis $\{(g^k \cdot \tilde{c}_j) \otimes \nu_\ell\}$ to those of the basis $\{\tilde{c}_k \otimes (\bar{g}^{-k} \otimes g^{-k} \cdot \nu_\ell)\}$. Now choose $\{\nu_{k,\ell} = \bar{g}^{-k} \otimes g^{-k} \cdot \nu_\ell\}$ as our basis for $\mathbb{C}[\mathbb{Z}_m] \otimes_{\mathbb{C}} V$ and order them by $\nu_{k,\ell} < \nu_{k',\ell'}$ if $k < k'$ or both $k = k'$ and $\ell < \ell'$. Then with the ordered bases used in Section 2.3, the map f in (3.6) is an isomorphism of *based* chain complexes. In particular, the complexes have the same torsion, which proves the lemma. \square

Proof of Lemma 3.4. Since for any $a \in \mathbb{Z}[\pi]$ we have $(\alpha \otimes \phi)(a)(\xi) = \alpha \otimes \lambda_\xi(a)$ the result should follow by computing both sides of (3.5) with Proposition 2.16. The only issue is that we need to ensure the nonvanishing of the denominators in (2.17) for both $\alpha \otimes \phi$ and $\alpha \otimes \lambda_\xi$. Since α is irreducible, we can choose $g \in [\pi, \pi]$ so that $\text{tr}(\alpha(g)) \neq 2$ (see e.g. [CS, Lemma 1.5.1]). Notice then that $\alpha \otimes \phi(g^{-1} - 1) = \alpha \otimes \lambda_\xi(g^{-1} - 1) = \alpha(g^{-1} - 1)$, and since $\text{tr}(\alpha(g)) \neq 2$ we have $\det(\alpha(g^{-1} - 1)) \neq 0$. Hence if we take a suitable presentation of π where g is a generator, then we can apply Proposition 2.16 with $x_i = g$ to both $\alpha \otimes \phi$ and $\alpha \otimes \lambda_\xi$ and so prove the lemma. \square

We turn now to the proof of Theorem 3.2, which says that typically the torsions $\tau(X_m, \alpha_m)$ collectively determine \mathcal{T}_K^α (by Theorem 3.1, the hypothesis that $\tau(X_m, \alpha_m) \neq 0$ for all m is equivalent to no root of \mathcal{T}_K^α being a root of unity). A polynomial p in $\mathbb{C}[t]$ of degree d is *palindromic* if $p(t) = t^d p(1/t)$, or equivalently if its coefficients satisfy $a_k = a_{d-k}$ for $0 \leq k \leq d$. For any polynomial $p \in \mathbb{C}[t]$ and $m \in \mathbb{N}$, we denote by $r_m(p)$ the resultant of $t^m - 1$ and p , i.e.

$$r_m(p) = \text{Res}(p, t^m - 1) = (-1)^{md} \text{Res}(t^m - 1, p) = (-1)^{md} \prod_{\zeta \in \mu_m} p(\zeta)$$

where here d is the degree of p . The following theorem was proved by Fried [Fri] for $p \in \mathbb{R}[t]$ and generalized by Hillar [Hil] to the case of $\mathbb{C}[t]$.

3.7. Theorem. *Suppose p and q are palindromic polynomials in $\mathbb{C}[t]$. If $r_m(p) = r_m(q) \neq 0$ for all $m \in \mathbb{N}$ then $p = q$.*

Theorem 3.2 now follows easily from Theorems 3.1 and 3.7 and the symmetry of \mathcal{T}_K^α shown in Theorem 2.10.

3.8. Remark. We just saw that, under mild assumptions, the torsions $\tau(X_m, \alpha_m)$ of cyclic covers determine the $\mathbb{C}(t)$ -valued torsion polynomial \mathcal{T}_K^α . It would be very interesting if one could directly read off the degree and the top coefficient of \mathcal{T}_K^α from the $\tau(X_m, \alpha_m)$. See [HL] for some of what's known about recovering a palindromic polynomial p from the sequence $r_m(p)$; in particular, when p is monic and of even degree d , Sturmfels and Zworski conjecture that one only needs to know $r_m(p)$ for $m \leq d/2 + 1$ to recover p .

4. TORSION POLYNOMIALS OF HYPERBOLIC KNOTS

Let K be a hyperbolic knot in an oriented \mathbb{Z}_2 -homology sphere Y . In this section, we define the hyperbolic torsion polynomial \mathcal{T}_K associated to a certain preferred lift to $\text{SL}(2, \mathbb{C})$ of the holonomy representation of its hyperbolic structure.

4.1. The discrete and faithful $\text{SL}(2, \mathbb{C})$ representations. As usual, we write $\pi = \pi_K := \pi_1(X_K)$, and let $\mu \in \pi$ be a meridian for K . The orientation of μ , or equivalently of K , will not matter in this section, but fix one so that $\phi: \pi \rightarrow \mathbb{Z}$ is determined.

From now on assume that $M = Y \setminus K \cong \text{int}(X)$ has a complete hyperbolic structure. The manifold M inherits an orientation from Y , and so its universal cover

\widetilde{M} can be identified with \mathbb{H}^3 by an *orientation preserving* isometry. This identification is unique up to the action of $\text{Isom}^+(\mathbb{H}^3) = \text{PSL}(2, \mathbb{C})$, and the action of π on $\widetilde{M} = \mathbb{H}^3$ gives the *holonomy representation* $\bar{\alpha}: \pi \rightarrow \text{PSL}(2, \mathbb{C})$, which is unique up to conjugation.

4.2. Remark. By Mostow-Prasad rigidity, the complete hyperbolic structure on M is unique. Thus $\bar{\alpha}$ is determined, up to conjugacy, solely by the knot K and the orientation of the ambient manifold Y . A subtle point is that there are actually *two* conjugacy classes of discrete faithful representations $\pi_K \rightarrow \text{PSL}(2, \mathbb{C})$; the other one corresponds to reversing the orientation of Y (not K) or equivalently complex-conjugating the entries of the image matrices.

To define a torsion polynomial, we want a representation into $\text{SL}(2, \mathbb{C})$ rather than $\text{PSL}(2, \mathbb{C})$. Thurston proved that $\bar{\alpha}$ always lifts to a representation $\alpha: \pi \rightarrow \text{SL}(2, \mathbb{C})$; see [Thu] and [Sha, Section 1.6] for details. In fact, there are exactly two such lifts, the other being $g \mapsto (-1)^{\phi(g)} \alpha(g)$; the point is that any other lift has the form $g \mapsto \epsilon(g) \alpha(g)$ for some homomorphism $\epsilon: \pi \rightarrow \{\pm 1\}$, i.e. some element of $H^1(M; \mathbb{Z}_2) = \mathbb{Z}_2$. Now $\bar{\alpha}(\mu)$ is parabolic, and so $\text{tr}(\alpha(\mu)) = \pm 2$. Since Y is a \mathbb{Z}_2 HS, we know $\phi(\mu)$ is odd; hence there is a lift α where $\text{tr}(\alpha(\mu)) = 2$; arbitrarily, we focus on that lift and call it the *distinguished representation*. This representation is determined, up to conjugacy, solely by K (sans orientation). We explain below the simple change that results if we instead required the trace to be -2 .

4.3. The hyperbolic torsion polynomial. For a hyperbolic knot K in an oriented \mathbb{Z}_2 HS, we define the *hyperbolic torsion polynomial* to be

$$\mathcal{T}_K(t) := \mathcal{T}_K^\alpha(t)$$

where $\alpha: \pi \rightarrow \text{SL}(2, \mathbb{C})$ is the distinguished representation. Before proving Theorem 1.2 which summarizes basic properties of $\mathcal{T}_K(t)$, we give a few definitions. The *trace field* \mathbb{F}_K of K is the field obtained by adjoining to \mathbb{Q} the elements $\text{tr}(\alpha(g))$ for all $g \in \pi$; this is a finite extension of \mathbb{Q} and an important number theoretic invariant of the hyperbolic structure on M ; see [MR] for more. We say that K has *integral traces* if every $\text{tr}(\alpha(g))$ is an algebraic integer (this is necessarily the case if M does not contain a closed essential surface, see e.g. [MR, Theorem 5.2.2]). Also, we denote by K^* the result of switching the orientation of the ambient manifold Y ; we call K^* the *mirror image* of K . We call K *amphichiral* if Y has an orientation reversing self-homeomorphism which takes K to itself; equivalently, $K = K^*$ in the category of knots in oriented 3-manifolds.

Theorem 1.2. *Let K be a hyperbolic knot in an oriented \mathbb{Z}_2 -homology 3-sphere. Then \mathcal{T}_K has the following properties:*

- (a) \mathcal{T}_K is an unambiguous element of $\mathbb{C}[t^{\pm 1}]$ which satisfies $\mathcal{T}_K(t^{-1}) = \mathcal{T}_K(t)$. It does not depend on an orientation of K .
- (b) The coefficients of \mathcal{T}_K lie in the trace field of K . If K has integral traces, the coefficients of \mathcal{T}_K are algebraic integers.
- (c) $\mathcal{T}_K(\xi)$ is non-zero for any root of unity ξ .

- (d) If K^* denotes the mirror image of K , then $\mathcal{T}_{K^*}(t) = \overline{\mathcal{T}_K}(t)$, where the coefficients of the latter polynomial are the complex conjugates of those of \mathcal{T}_K .
- (e) If K is amphichiral then \mathcal{T}_K is a real polynomial.
- (f) The values $\mathcal{T}_K(1)$ and $\mathcal{T}_K(-1)$ are mutation invariant.

For the special case of 2-bridge knots and $\xi = \pm 1$, the assertion (c) is also a consequence of the work of Hirasawa-Murasugi [HM] and Silver-Williams [SW].

Proof. Since the distinguished representation α is irreducible, part (a) follows from Theorems 2.10 and 2.11(a).

Next, since M has a cusp, by Lemma 2.6 of [NR] we can conjugate α so that its image lies in $\mathrm{SL}(2, \mathbb{F}_K)$, where \mathbb{F}_K is the trace field; hence $\mathcal{T}_K \in \mathbb{F}_K[t^{\pm 1}]$ proving the first part of (b). To see the other part, first using [MR, Theorem 5.2.4] we can conjugate α so that $\alpha(\pi) \subset \mathrm{SL}(2, \mathcal{O}_{\mathbb{K}})$, where here $\mathcal{O}_{\mathbb{K}}$ is the ring of algebraic integers in some number field \mathbb{K} (which might be a proper extension of \mathbb{F}_K). We now compute \mathcal{T}_K^α by applying Proposition 2.13 to a presentation of π where μ is our preferred generator. Since $\alpha(\mu)$ is parabolic with trace 2, the denominator in (2.14) is $p(t) := \det((\alpha \otimes \phi)(\mu^{-1} - 1)) = (t^k - 1)^2$ where $k = -\phi(\mu) \neq 0$. Thus by (2.14), we know $p(t) \cdot \mathcal{T}_K^\alpha$ is in $\mathcal{O}_{\mathbb{K}}[t^{\pm 1}]$. Then since $p(t) \in \mathbb{Z}[t^{\pm 1}]$ is monic, the lead coefficient of \mathcal{T}_K^α must be integral. An easy inductive argument now shows that all the other coefficients are also integral, proving part (b).

The proof of (c) uses Theorem 3.1, and we handle all m^{th} roots of unity at once. In the notation of Section 3, we have

$$(4.4) \quad \prod_{\zeta \in \mu_m} \mathcal{T}_K(\zeta) = \tau(X_m, \alpha_m).$$

By Menal–Ferrer and Porti [MFP1, Theorem 0.4], which builds on work of Raghunathan [Rag], we have that $H_*^{\alpha_m}(X_m; \mathbb{C}^2) = 0$, or equivalently, $\tau(X_m, \alpha_m)$ is non-zero. Thus by (4.4) we must have $\mathcal{T}_K(\zeta) \neq 0$ for any m^{th} root of unity, establishing part (c).

For (d), the distinguished representation for the mirror knot K^* is $\bar{\alpha}: \pi \rightarrow \mathrm{SL}(2, \mathbb{C})$ where each $\bar{\alpha}(g)$ is the matrix which is the complex conjugate of $\alpha(g)$. Since our choice of orientation for the meridian μ was arbitrary, we can use the same ϕ for when calculating both \mathcal{T}_K and \mathcal{T}_{K^*} . Thus we have

$$\mathcal{T}_{K^*}(t) = t^{\phi(c_1(s))} \tau(X, \bar{\alpha} \otimes \phi) = t^{\phi(c_1(s))} \overline{\tau(X, \alpha \otimes \phi)} = \overline{\mathcal{T}_K}(t)$$

proving (d). Next, claim (e) follows immediately from (d). Finally, claim (f) is a recent result of Menal-Ferrer and Porti [MFP2]. \square

As in Section 3, we now consider the \mathbb{C} -valued torsions $\tau(X_m, \alpha_m)$ of finite cyclic covers of X_K . Somewhat surprisingly, these determine \mathcal{T}_K :

4.5. Theorem. *Let K be a hyperbolic knot in a \mathbb{Z}_2 HS with distinguished representation $\alpha: \pi_K \rightarrow \mathrm{SL}(2, \mathbb{C})$. Then \mathcal{T}_K is determined by the torsions $\tau(X_m, \alpha_m) \in \mathbb{C}$.*

Proof. As discussed in the proof of Theorem 1.2(c), every $\tau(X_m, \alpha_m) \neq 0$, so the result is immediate from Theorem 3.2. \square

4.6. *Remark.* When choosing our distinguished representation, we arbitrarily chose the lift $\alpha: \pi \rightarrow \mathrm{SL}(2, \mathbb{C})$ where $\mathrm{tr}(\alpha(\mu)) = 2$. As discussed, the other lift β is given by $g \mapsto (-1)^{\phi(g)} \alpha(g)$. Note that given $g \in \pi$ we have

$$\begin{aligned} ((\beta \otimes \phi)(g))(t) &= \beta(g) \cdot t^{\phi(g)} = \alpha(g) \cdot (-1)^{\phi(g)} \cdot t^{\phi(g)} \\ &= \alpha(g) \cdot (-t)^{\phi(g)} = ((\alpha \otimes \phi)(g))(-t). \end{aligned}$$

It follows from Proposition 2.13 that $\mathcal{T}_K^\beta(t) = \mathcal{T}_K^\alpha(-t) = \mathcal{T}_K(-t)$. Put differently, using β instead of α would simply replace t by $-t$.

4.7. *Remark.* When Y is not a \mathbb{Z}_2 HS, the choice of lift α of the holonomy representation can have a more dramatic effect on \mathcal{T}^α . For example, consider the manifold $m130$ in the notation of [CHW, CDW]. This manifold is a twice-punctured genus 1 surface bundle over the circle, and as $H_1(M; \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}_8$, there are 4 distinct lifts of the holonomy representation. Two of these lifts give $\mathcal{T}_K^\alpha = (t^2 + t^{-2}) - 2i$ and the other two give $\mathcal{T}_K^\alpha = (t^2 + t^{-2}) + \sqrt{-8 - 8i}(t^1 + t^{-1}) - 6i$ for the two distinct square-roots of $-8 - 8i$. In particular, the fields generated by the coefficients are different; only the latter two give the whole trace field.

5. EXAMPLE: THE CONWAY AND KINOSHITA–TERASAKA KNOTS

The Conway and Kinoshita-Terasaka knots are a famous pair of mutant knots which both have trivial Alexander polynomial. Despite their close relationship, they have different genera. Thus they are a natural place to start our exploration of \mathcal{T}_K , and we devote this section to examining them in detail.

The Conway knot C is the mirror of the knot $11n34$ in the numbering of [HT, HTW]. The program Snap [GN, CGHN] finds that the trace field \mathbb{F} of the hyperbolic structure on the exterior of C is the extension of \mathbb{Q} gotten by adjoining the root $\theta \approx 0.1233737 - 0.5213097i$ of

$$p(x) = x^{11} - x^{10} + 3x^9 - 4x^8 + 5x^7 - 8x^6 + 8x^5 - 5x^4 + 6x^3 - 5x^2 + 2x - 1.$$

Snap also finds the explicit holonomy representation $\pi_C \rightarrow \mathrm{SL}(2, \mathbb{F})$, and one can directly apply Proposition 2.13 to compute \mathcal{T}_C . If we set

$$\eta = \frac{1}{53} (20\theta^{10} + 9\theta^9 + 28\theta^8 + 3\theta^7 + \theta^6 + 19\theta^5 + 10\theta^4 + 47\theta^3 + 6\theta + 1)$$

then $\{\eta, \theta, \theta^2, \dots, \theta^{10}\}$ is an integral basis for $\mathcal{O}_{\mathbb{F}}$, and we find

$$\begin{aligned} \mathcal{T}_C(t) &= (-79\theta^{10} - 35\theta^9 - 111\theta^8 - 11\theta^7 - 4\theta^6 - 71\theta^5 - 38\theta^4 - 187\theta^3 - 2\theta^2 - 24\theta + 206\eta)(t^5 + t^{-5}) \\ &\quad + (257\theta^{10} + 114\theta^9 + 361\theta^8 + 36\theta^7 + 13\theta^6 + 232\theta^5 + 124\theta^4 + 608\theta^3 + 6\theta^2 + 78\theta - 671\eta)(t^4 + t^{-4}) \\ &\quad + (-372\theta^{10} - 165\theta^9 - 523\theta^8 - 51\theta^7 - 21\theta^6 - 334\theta^5 - 183\theta^4 - 877\theta^3 - 11\theta^2 - 111\theta + 972\eta)(t^3 + t^{-3}) \\ &\quad + (373\theta^{10} + 162\theta^9 + 528\theta^8 + 40\theta^7 + 33\theta^6 + 312\theta^5 + 200\theta^4 + 866\theta^3 + 24\theta^2 + 99\theta - 968\eta)(t^2 + t^{-2}) \\ &\quad + (-303\theta^{10} - 115\theta^9 - 445\theta^8 + 14\theta^7 - 75\theta^6 - 152\theta^5 - 227\theta^4 - 649\theta^3 - 73\theta^2 - 29\theta + 749\eta)(t^1 + t^{-1}) \\ &\quad + (116\theta^{10} + 14\theta^9 + 200\theta^8 - 88\theta^7 + 116\theta^6 - 122\theta^5 + 204\theta^4 + 146\theta^3 + 124\theta^2 - 78\theta - 220\eta) \\ &\approx (4.89524 + 0.09920i)(t^5 + t^{-5}) + (-15.68571 - 0.29761i)(t^4 + t^{-4}) + (23.10363 - 0.07842i)(t^3 + t^{-3}) \\ &\quad + (-26.94164 + 4.84509i)(t^2 + t^{-2}) + (38.38349 - 24.49426i)(t^1 + t^{-1}) + (-43.32401 + 44.08061i). \end{aligned}$$

The Kinoshita–Terasaka knot is the mirror of $11n42$. Its trace field is the same as for the Conway knot (since $[\mathbb{F} : \mathbb{Q}]$ is odd, the trace field is also the invariant trace field, which is mutation invariant), and one finds

$$\begin{aligned} \mathcal{T}_{KT}(t) &= (-55\theta^{10} - 24\theta^9 - 78\theta^8 - 6\theta^7 - 5\theta^6 - 45\theta^5 - 29\theta^4 - 128\theta^3 - 5\theta^2 - 15\theta + 142\eta)(t^3 + t^{-3}) \\ &\quad + (293\theta^{10} + 126\theta^9 + 416\theta^8 + 28\theta^7 + 29\theta^6 + 236\theta^5 + 160\theta^4 + 678\theta^3 + 24\theta^2 + 75\theta - 756\eta)(t^2 + t^{-2}) \\ &\quad + (-699\theta^{10} - 291\theta^9 - 1001\theta^8 - 42\theta^7 - 95\theta^6 - 512\theta^5 - 419\theta^4 - 1585\theta^3 - 81\theta^2 - 149\theta + 1785\eta)(t^1 + t^{-1}) \\ &\quad + (790\theta^{10} + 314\theta^9 + 1146\theta^8 + 8\theta^7 + 150\theta^6 + 494\theta^5 + 532\theta^4 + 1738\theta^3 + 136\theta^2 + 126\theta - 1986\eta) \\ &\approx (4.41793 - 0.37603i)(t^3 + t^{-3}) + (-22.94164 + 4.84509i)(t^2 + t^{-2}) + (61.96443 - 24.09744i)(t^1 + t^{-1}) \\ &\quad + (-82.69542 + 43.48539i). \end{aligned}$$

From the above we see that \mathcal{T}_K is not invariant under mutation. Since C and KT have genus 3 and 2 respectively and $\deg(\mathcal{T}_C) = 10$ and $\deg(\mathcal{T}_{KT}) = 6$, we see that Conjecture 1.7 holds for both knots. Also note that the coefficients of these polynomials are not real, certifying the fact that both knots are chiral.

5.1. *Remark.* It was shown in [FK1, Section 5] that twisted Alexander polynomials corresponding to representations over finite fields detect the genus of all knots with at most twelve crossings. For example, for the Conway knot there is a representation $\alpha: \pi_1(X_C) \rightarrow \mathrm{GL}(4, \mathbb{F}_{13})$ such that the corresponding torsion polynomial $\mathcal{T}_C^\alpha \in \mathbb{F}_{13}[t^{\pm 1}]$ has degree 14, and hence

$$x(C) \geq \frac{1}{4} \deg(\mathcal{T}_C^\alpha) = 3.5.$$

In particular this shows that $x(C) = 5$ since $x(C) = 2 \text{genus}(C) - 1$ is an odd integer.

The calculation using the discrete and faithful $\mathrm{SL}(2, \mathbb{C})$ representation is arguably more satisfactory since it gives the equality

$$x(C) = \frac{1}{2} \deg(\mathcal{T}_C)$$

on the nose, and not just after rounding up to odd integers. Interestingly, we have not found an example where this rounding trick applies to \mathcal{T}_K ; at least for knots with at most 15 crossings one always has $x(K) = \deg(\mathcal{T}_K)/2$ (see Section 6).

5.2. The adjoint representation. For an oriented hyperbolic knot K with distinguished representation $\alpha: \pi_1(X_K) \rightarrow \mathrm{SL}(2, \mathbb{C})$, we now consider the adjoint representation

$$\begin{aligned} \alpha_{adj}: \pi_1(X_K) &\rightarrow \mathrm{Aut}(\mathfrak{sl}(2, \mathbb{C})) \\ g &\mapsto A \mapsto \alpha(g) A \alpha(g)^{-1} \end{aligned}$$

associated to α . It is well-known that this representation is also faithful and irreducible. Using sign-refined torsion and the orientation on K , one gets an invariant $\mathcal{F}_K^{adj} \in \mathbb{C}[t^{\pm 1}]$ which is well-defined up to multiplication by an element of the form t^k . We refer to [DY] for details on this construction and for further information on \mathcal{F}_K^{adj} ; one thing they show is that $\mathcal{F}_K^{adj}(t) = -\mathcal{F}_K^{adj}(t^{-1})$ up a power of t , and so \mathcal{F}_K^{adj} has odd degree.

For the Conway knot we calculate that

$$\begin{aligned} \mathcal{F}_C^{adj}(t) \approx & (-0.2788 + 16.4072i)(t^{13} - 1) + (-3.9858 - 20.1706i)(t^{12} - t) + (-4.2204 - 60.5497i)(t^{11} - t^2) \\ & + (52.0953 + 134.5013i)(t^{10} - t^3) + (-147.7856 - 46.07448i)(t^9 - t^4) + (897.2087 + 62.3265i)(t^8 - t^5) \\ & + (-2465.8556 - 1308.0110i)(t^7 - t^6) \end{aligned}$$

and for the Kinoshita-Terasaka knot we found

$$\begin{aligned} \mathcal{F}_{KT}^{adj}(t) \approx & (-0.7378 + 12.4047i)(t^7 - 1) + (29.9408 - 56.5548i)(t^6 - t) \\ & + (-655.7823 - 173.0400i)(t^5 - t^2) + (2056.7509 + 1678.4875i)(t^4 - t^3). \end{aligned}$$

As $\dim(\mathfrak{sl}(2, \mathbb{C})) = 3$, it follows from Theorem [FK1, Theorem 1.1] that

$$(5.3) \quad x(K) \geq \frac{1}{3} \deg(\mathcal{F}_K^{adj}(t)) \quad \text{and hence} \quad x(C) \geq \frac{13}{3} \quad \text{and} \quad x(KT) \geq \frac{7}{3}.$$

Thus using that $x(K)$ is in integer, we get $x(C) \geq 5$ and $x(KT) \geq 3$, which are sharp. Intriguingly, unlike for \mathcal{F}_K one does *not* have equality in (5.3) for these two knots.

6. KNOTS WITH AT MOST 15 CROSSINGS

There are 313,231 prime knots with 15 or fewer crossings [HTW], of which all but 22 are hyperbolic. For each of these hyperbolic knots, we computed a high-precision numerical approximation to \mathcal{F}_K (see Section 6.6 for details), and this section is devoted to describing the various properties and patterns we found.

6.1. Genus. The genus bound from \mathcal{F}_K given in Theorem 1.5 is sharp for all 313,209 hyperbolic knots with 15 or fewer crossings; that is, $x(K) = \deg(\mathcal{F}_K)/2$ for all these knots. In contrast, the ordinary Alexander polynomial fails to detect the genus for 8,834 of these knots, which is 2.8% of the total.

We showed the genus bound from \mathcal{F}_K was sharp using the following techniques to give upper bounds on the genus. First, for the alternating knots (36% of the total), the genus is simply determined by the Alexander polynomial [Mur1, Cro]. For the nonalternating knots, we first did 0-surgery on the knot K to get a closed 3-manifold N ; by Gabai [Gab2], the genus of K is the same as that of the simplest homologically nontrivial surface in N . We then applied the method of Section 6.7 of [DR] to a triangulation of N to quickly find a homologically

nontrivial surface. As this surface need not be minimal genus, when necessary we randomized the triangulation of N until we found a surface whose genus matched the lower bound from \mathcal{T}_K .

6.2. Fiberings. We also found that \mathcal{T}_K gives a sharp obstruction to fibering for all hyperbolic knots with at most 15 crossings. In particular, the 118,252 hyperbolic knots where \mathcal{T}_K is monic are all fibered. In contrast, while the ordinary Alexander polynomial always certifies nonfibering for alternating knots [Mur2, Gab1], among the 201,702 nonalternating knots there are 7,972 or 4.0% whose Alexander polynomials are monic but don't fiber.

To confirm fibering when \mathcal{T}_K was monic, we used a slight generalization of the method of Section 6.11 of [DR]. Again by [Gab2], it is equivalent to show that the 0-surgery N is fibered. Starting with the minimal genus surface S found as above, we split N open along S , and tried to simplify a presentation for the fundamental group of $N \setminus S$ until it was obviously that of a surface group. If it is, then it follows that $N \setminus S = S \times I$ and N is fibered. The difference with [DR] is that we allowed S to be a general normal surface instead of the restricted class of Figure 6.13 of [DR]. We handled this by splitting the manifold open along S and triangulating the result using Regina [Bur2].

6.3. Chirality. For hyperbolic knots with at most 15 crossings, we found that a knot was amphichiral if and only if \mathcal{T}_K had real coefficients. In particular, there are 353 such knots with \mathcal{T}_K real, and SnapPy [CDW] easily confirms that they are all amphichiral. (This matches the count of amphichiral knots from Table A1 of [HTW].)

In contrast, the numbers $\mathcal{T}_K(1)$ and $\mathcal{T}_K(-1)$ do not always detect chirality. For example, the chiral knot $10_{153} = 10n10$ has $\mathcal{T}_K(1) = 4$ and $10_{157} = 10n42$ has $\mathcal{T}_K(-1) = 576$. Moreover, the knot $14a506$ has both $\mathcal{T}_K(1)$ and $\mathcal{T}_K(-1)$ real. (This last claim was checked to the higher precision of 10,000 decimal places.)

6.4. Knots with the same \mathcal{T}_K . While we saw in Section 5 that \mathcal{T}_K is *not* mutation invariant, there are still pairs of knots with the same \mathcal{T}_K . In particular, among knots with at most 15 crossings, there are 2,739 groups of more than one knot that share the same \mathcal{T}_K , namely 2,700 pairs and 39 triples. Here, we do not distinguish between a knot and its mirror image, and having the same \mathcal{T}_K means that the coefficients agree to 5,000 decimal places. Stoimenow found there are 34,349 groups of mutant knots among those with at most 15 crossings, involving some 77,680 distinct knots [Sto]. Thus there are many examples where mutation changes \mathcal{T}_K . However, all of the examples we found of knots with the same \mathcal{T}_K are in fact mutants.

As mentioned, Menal-Ferrer and Porti [MFP2] showed that the evaluations $\mathcal{T}_K(1)$ and $\mathcal{T}_K(-1)$ are mutation invariant. We found 38 pairs of non-mutant knots with the same $\mathcal{T}_K(1)$ and the same $\mathcal{T}_K(-1)$. Suggestively, several of these pairs (including the five pairs shown in [DGST, Figure 3.9], see also [ST, Tables 2 and 3]) are known to be genus-2 mutants. We also found a triple of mutually non-mutant knots $\{10a121, 12a1202, 12n706\}$ where $\mathcal{T}_K(+1) = -4$, and a similar

sextet $\{10n10, 12n881, 13n592, 13n2126, 15n9378, 15n22014\}$ where $\mathcal{T}_K(+1) = 4$; however, within these groups, the value $\mathcal{T}_K(-1)$ did not agree.

6.5. Other patterns. We found two other intriguing patterns which we are unable to explain. The first is that the second highest coefficient of \mathcal{T}_K is often real for fibered knots. In particular, this is the case for 53.1% (62,763 of 118,252) of the fibered knots in this sample. In contrast, the second coefficient is real for only 0.2% (364 of 194,957) of nonfibered knots. (Arguably, the right comparison is with the lead coefficient of \mathcal{T}_K for nonfibered knots; even fewer (0.05%) of these are real.) For fibered knots, the twisted homology of the universal cyclic cover can be identified with that of the fiber; hence the action of a generator of the deck group on this homology of the cover can be thought of as the action of the monodromy of the bundle on the twisted homology of the fiber. The second coefficient of \mathcal{T}_K is then just the sum of the eigenvalues of this monodromy, but it's unclear why this should often be a real number.

The second observation is that $|\mathcal{T}_K(-1)| > |\mathcal{T}_K(1)|$ for all but 22 ($< 0.01\%$) of these knots. The exceptions are nonalternating and all but one (15n151121) is fibered.

6.6. Computational details. The complete software used for these computations, as well as a table of \mathcal{T}_K for all these knots, is available at [DFJ]. The software runs within Sage [Sage], and makes use of SnapPy [CDW] and t3m [CD]. It finds very high-precision solutions to the gluing equations, in the manner of Snap [GN, CGHN], and extracts from this a high-precision approximation to the distinguished representation. Except as noted above, we did all computations with 250 decimal places of precision. However, to save space, only 40 digits were saved in the final table. To guard against error, two of the authors independently wrote programs which computed \mathcal{T}_K , and the output of these programs were then compared for all nonalternating knots with 14 crossings.

7. TWISTED TORSION AND THE CHARACTER VARIETY OF A KNOT

As usual, consider a hyperbolic knot K in a \mathbb{Z}_2 HS, and let $\pi := \pi_1(X_K)$. So far, we have focused on the torsion polynomial of the distinguished representation $\alpha: \pi \rightarrow \mathrm{SL}(2, \mathbb{C})$ coming from the hyperbolic structure. However, this representation is always part of a complex curve of representations $\pi \rightarrow \mathrm{SL}(2, \mathbb{C})$, and it is natural to ask if there is additional topological information in the torsion polynomials of these other representations. In this section, we describe how to understand all of these torsion polynomials at once, and use this to help explain some of the patterns observed in Section 6. For the special case of 2-bridge knots, Kim and Morifuji [Mor, KM1] had previously studied how the torsion polynomial varies with the representation, and we extend here some of their results to more general knots.

To state our results, we must first review some basics about character varieties; throughout, see the classic paper [CS] or the survey [Sha] for details. Consider the *representation variety* $R(K) := \text{Hom}(\pi, \text{SL}(2, \mathbb{C}))$ which is an affine algebraic variety over \mathbb{C} . The group $\text{SL}(2, \mathbb{C})$ acts on $R(K)$ by conjugating each representation; the algebro-geometric quotient $X(K) := R(K) // \text{SL}(2, \mathbb{C})$ is called the *character variety*. More concretely, $X(K)$ is the set of characters of representations $\alpha \in R(K)$, i.e. functions $\chi_\alpha: \pi \rightarrow \mathbb{C}$ of the form $\chi_\alpha(g) = \text{tr}(\alpha(g))$ for $g \in \pi$. When α is irreducible, the preimage of χ_α under the projection $R(K) \rightarrow X(K)$ is just all conjugates of α , but distinct conjugacy classes of reducible representations can sometimes have the same character. Still, it makes sense to call a character irreducible or reducible depending on which kind of representations it comes from.

The character variety $X(K)$ is also an affine algebraic variety over \mathbb{C} ; its coordinate ring $\mathbb{C}[X(K)]$, which consists of all regular functions on $X(K)$, is simply the subring $\mathbb{C}[R(K)]^{\text{SL}(2, \mathbb{C})}$ of regular functions on $R(K)$ which are invariant under conjugation. We start by showing that it makes sense to define a torsion polynomial \mathcal{T}_K^χ for $\chi \in X(K)$ via $\mathcal{T}_K^\chi := \mathcal{T}_K^\alpha$ for any α with $\chi_\alpha = \chi$.

7.1. Lemma. *If $\alpha, \beta \in R(K)$ have the same character, then $\mathcal{T}_K^\alpha = \mathcal{T}_K^\beta$.*

Proof of Lemma 7.1. As discussed, if α is irreducible then β must be conjugate to α ; hence they have the same torsion polynomial. If instead α is reducible, then Theorem 2.11(b) shows that \mathcal{T}_K^α depends only on the diagonal part of α , which can be recovered from its character. Since β must also be reducible and has the same character as α , we again get $\mathcal{T}_K^\alpha = \mathcal{T}_K^\beta$. \square

An irreducible component X_0 of $X(K)$ has $\dim_{\mathbb{C}}(X_0) \geq 1$ since the exterior of K has boundary a torus. There are two possibilities for X_0 : either it consists entirely of reducible characters, or it contains an irreducible character. In the latter case, it turns out that irreducible characters are Zariski open in X_0 , and every character in X_0 is that of a representation with non-abelian image. As the torsion polynomials of reducible representations are boring (see Theorem 2.11 and the discussion immediately following), we focus on those components containing an irreducible character. We denote the union of all such components as $X(K)^{\text{irr}}$; equivalently, $X(K)^{\text{irr}}$ is the Zariski closure of the set of irreducible characters.

It is natural to ask how \mathcal{T}_K^χ varies as a function of χ . We find:

Theorem 1.10. *Let X_0 be an irreducible component of $X(K)^{\text{irr}}$. There is a unique $\mathcal{T}_K^{X_0} \in \mathbb{C}[X_0][t^{\pm 1}]$ so that for all $\chi \in X_0$ one has $\mathcal{T}_K^\chi(t) = \mathcal{T}_K^{X_0}(\chi)(t)$. Moreover, $\mathcal{T}_K^{X_0}$ is itself the torsion polynomial of a certain representation $\pi \rightarrow \text{SL}(2, \mathbb{F})$ and thus has all the usual properties (symmetry, genus bound, etc.).*

We give several explicit examples of $\mathcal{T}_K^{X_0}$ later in Section 8. The following result is immediate from Theorem 1.10.

7.2. Corollary. *Let X_0 be an irreducible component of $X(K)^{\text{irr}}$. Then*

- (a) For all $\chi \in X_0$, we have $\deg(\mathcal{T}_K^\chi) \leq \deg(\mathcal{T}_K^{X_0})$ with equality on a nonempty Zariski open subset.
- (b) If $\mathcal{T}_K^{X_0}$ is monic, then \mathcal{T}_K^χ is monic for all $\chi \in X_0$. Otherwise, \mathcal{T}_K^χ is monic only on a proper Zariski closed subset.

In particular, when X_0 is a curve, the genus bound and fibering obstruction given by \mathcal{T}_K^χ are the same for all $\chi \in X_0$ except on a finite set where \mathcal{T}_K^χ provides weaker information. We can also repackage Corollary 7.2 as a uniform statement on all of $X(K)$.

Corollary 1.11. *Let K be a knot in an integral homology 3-sphere. Then*

- (a) The set $\{\chi \in X(K) \mid \deg(\mathcal{T}_K^\chi) = 2x(K)\}$ is Zariski open.
- (b) The set $\{\chi \in X(K) \mid \mathcal{T}_K^\chi \text{ is monic}\}$ is Zariski closed.

Proof. It suffices to consider the intersections of these sets with each irreducible component X_0 of $X(M)$. If X_0 consists solely of reducible characters, the result is immediate from Theorem 2.11(c). Otherwise, it follows from Corollary 7.2 combined with the fact that $\deg(\mathcal{T}_K^{X_0}) \leq 2x(K)$. \square

We now turn to the proof of Theorem 1.10.

Proof of Theorem 1.10. By Proposition 1.4.4 of [CS], there is an irreducible component R_0 of $R(K)$ where the projection $R_0 \rightarrow X(K)$ surjects onto X_0 . Consider the tautological representation

$$\rho_{\text{taut}}: \pi \rightarrow \text{SL}(2, \mathbb{C}[R_0])$$

which sends $g \in \pi$ to the matrix $\rho_{\text{taut}}(g)$ of regular functions on R_0 so that

$$\rho_{\text{taut}}(g)(\alpha) = \alpha(g) \quad \text{for all } \alpha \in R_0.$$

Since R_0 is irreducible, $\mathbb{C}[R_0]$ is an integral domain. Thus we can consider its field of fractions, i.e. the field of rational functions $\mathbb{C}(R_0)$. Working over $\mathbb{C}(R_0)$ there is an associated torsion polynomial $\mathcal{T}_K^{\text{taut}}$ which is in $\mathbb{C}(R_0)[t^{\pm 1}]$ since ρ_{taut} is irreducible. From Lemma 2.13, it is clear that for every $\alpha \in R_0$ we have $\mathcal{T}_K^\alpha(t) = \mathcal{T}_K^{\text{taut}}(\alpha)(t)$ in $\mathbb{C}[t^{\pm 1}]$. Hence the coefficients of $\mathcal{T}_K^{\text{taut}}$ have well-defined values at every point $\alpha \in R_0$, and so lie in $\mathbb{C}[R_0]$. Now since the torsion polynomial is invariant under conjugation, each coefficient of $\mathcal{T}_K^{\text{taut}}$ lies in $\mathbb{C}[X_0] = \mathbb{C}[R_0]^{\text{SL}(2, \mathbb{C})}$, and hence $\mathcal{T}_K^{\text{taut}}$ descends to an element of $\mathbb{C}[X_0][t^{\pm 1}]$, which is the $\mathcal{T}_K^{X_0}$ we seek. \square

7.3. The distinguished component. It is natural to focus on the component X_0 of $X(M)$ which contains the distinguished representation. In this case X_0 is an algebraic curve, and we refer to it as the *distinguished component*. By Corollary 7.2, the following conjecture that $\mathcal{T}_K^{X_0}$ detects both the genus and fibering of K is implied by Conjecture 1.7.

Conjecture 1.12. *Let K be a hyperbolic knot in S^3 , and X_0 be the distinguished component of its character variety. Then $2x(K) = \deg(\mathcal{T}_K^{X_0})$ and $\mathcal{T}_K^{X_0}$ is monic if and only if K is fibered.*

As we explain in Section 7.6, this conjecture is true for many 2-bridge knots.

One pattern in Section 6 is that \mathcal{T}_K never gave worse topological information than the ordinary Alexander polynomial Δ_K . In certain circumstances, Corollary 7.2 allows us to relate Δ_K to \mathcal{T}_K as we now discuss. First, we can sometimes show that $\mathcal{T}_K^{X_0}$ must contain at least as much topological information as Δ_K .

7.4. Lemma. *Let K be a knot in an \mathbb{Z} HS. Suppose X_0 is a component of $X(K)^{irr}$ which contains a reducible character. Then $\deg(\mathcal{T}_K^{X_0}) \geq 2 \deg(\Delta_K) - 2$ and if Δ_K is nonmonic so is $\mathcal{T}_K^{X_0}$.*

Proof. Let α be a reducible representation whose character lies in X_0 . By Theorem 2.11(c), the torsion polynomial \mathcal{T}_K^α has degree $2 \deg(\Delta_K) - 2$ and its lead coefficient is the square of that of $\Delta_K \in \mathbb{Z}[t^{\pm 1}]$. The result now follows from Corollary 7.2. \square

Now, consider the distinguished representation α and distinguished component $X_0 \subset X(K)$. We say that α is *sufficiently generic* if $\deg(\mathcal{T}_K) = \deg(\mathcal{T}_K^{X_0})$ and \mathcal{T}_K is monic only if $\mathcal{T}_K^{X_0}$ is. Corollary 7.2 suggests that most knots will have sufficiently generic distinguished representations; however, because the distinguished character takes on only algebraic number values, there seems to be no a priori reason why this must always be the case. Regardless, our intuition is that the hypothesis of this next proposition holds quite often:

7.5. Proposition. *Let K be a knot in an \mathbb{Z} HS whose distinguished representation is sufficiently generic, and whose distinguished component of $X(M)$ contains a reducible character. Then $\deg(\mathcal{T}_K) \geq 2 \deg(\Delta_K) - 2$ and if Δ_K is nonmonic so is \mathcal{T}_K .*

7.6. 2-bridge knots. For 2-bridge knots in S^3 , Kim and Morifuji previously studied the torsion polynomial as a function on $X(M)^{irr}$ in [KM1]. As 2-bridge knots are alternating, the ordinary Alexander polynomial Δ_K determines the genus and whether K fibers [Mur2, Cro, Mur2, Gab1]. However, as mentioned, there seems to be no a priori reason that the same must be true for $\mathcal{T}_K^{X_0}$. We now sketch what is known about this special case, starting with two results from [KM1].

7.7. Theorem ([KM1, Theorem 4.2]). *Let K be a hyperbolic 2-bridge knot. Then there exists a component $X_0 \subset X(K)^{irr}$ such that $2x(K) = \deg(\mathcal{T}_K^{X_0})$ and $\mathcal{T}_K^{X_0}$ is monic if and only if K is fibered. In particular, if $X(K)^{irr}$ is irreducible, then Conjecture 1.12 holds for K .*

Proof. Since Δ_K detects the genus, it is nonconstant and so has a nontrivial root. For any knot in an \mathbb{Z} HS, a root of Δ_K gives rise to a reducible representation $\pi_K \rightarrow \mathrm{SL}(2, \mathbb{C})$ with nonabelian image. In the case of 2-bridge knots, the character of any such representation belongs to a component $X_0 \subset X(M)^{irr}$ (see Remark 1.9 and Corollary 2.9 of [HLM], originating in Proposition 2.3 and the comment following it in [Bur1]), and Lemma 7.4 now finishes the proof. \square

7.8. Theorem ([KM1, Lemma 4.8 and Theorem 4.9]). *Let $K = K(p, q)$ be a hyperbolic 2-bridge knot, and let c be the lead coefficient of Δ_K . Suppose there exists a prime divisor ℓ of p so that if $c \neq \pm 1$ then the reduction of $c \bmod \ell$ is not in $\{-1, 0, 1\}$. Then Conjecture 1.12 holds for K .*

Proof sketch. Let X_0 be any component of $X(K)^{irr}$. First, one shows that X_0 contains a character χ where $\text{tr}(\mu_K) = 0$. In Lemma 4.6 of [KM1] this is shown using the particular structure of π_K , and it also follows from the following more general fact:

7.9. Claim. *Let K be a knot in S^3 whose exterior contains no closed essential surface. If X_0 is a component of $X(M)$, then given $a \in \mathbb{C}$ there is a $\chi \in X_0$ where $\text{tr}(\mu_K) = a$.*

Two-bridge knots satisfy the hypothesis of Claim 7.9 by [HT], and the proof of the claim is straightforward from the Culler-Shalen theory of surfaces associated to ideal points of X_0 . Specifically, on the smooth projective model of X_0 the rational function $\text{tr}(\mu_K)$ takes on the value a somewhere, and if this were at an ideal point the associated essential surface would have to be either closed or have meridian boundary; the latter situation also implies the existence of a closed essential surface by [CGLS, Theorem 2.0.3].

The representation corresponding to a χ where $\text{tr}(\mu_K) = 0$ is irreducible but has metabelian image, and in the 2-bridge case one can use this to calculate \mathcal{F}_K^χ explicitly. In particular, in [KM1] they find that, provided there exists a prime ℓ as in the hypothesis, the polynomial \mathcal{F}_K^χ is non-monic and $\deg \mathcal{F}_K^\chi = 2x(K)$. We then apply Corollary 7.2 to see that Conjecture 1.12 holds. \square

Another interesting class of characters in $X(M)^{irr}$ are those of representations where μ_K is parabolic (e.g. the distinguished representation); such parabolic representations must occur on any component X_0 by Claim 7.9. For the 3,830 non-fibered 2-bridge knots with $q < p \leq 287$, we numerically computed \mathcal{F}_K^χ for all such parabolic characters, using a precision of 150 decimal places. In every case, the polynomial \mathcal{F}_K^χ was nonmonic and gave a sharp genus bound. Since 2-bridge knots contain no closed essential surfaces, every component of $X(M)$ is a curve. Thus for all 2-bridge knots with $p \leq 287$ there are only finitely many $\chi \in X(M)$ where \mathcal{F}_K^χ is monic or where $\deg(\mathcal{F}_K^\chi) < 2 \deg(\Delta_K) - 2$, as conjectured in [KM1].

8. CHARACTER VARIETY EXAMPLES

As with many things related to the character variety, while $\mathcal{F}_K^{X_0}$ is a very natural concept, actually computing it can be difficult. Here, we content ourselves with finding $\mathcal{F}_K^{X_0}$ for three of the simplest examples. In each case, there is only one natural choice for X_0 , and moreover it is isomorphic to $\mathbb{C} \setminus \{\text{finite set}\}$. Thus X_0 is rational and $\mathbb{C}(X_0)$ is just rational functions in one variable, which makes it easy to express the answer. We do one fibered example and two that are non-fibered; in all cases the simplest Seifert surface has genus 1.

8.1. Example: m003. We start with the sibling M of the figure-8 complement, which is one of the two orientable cusped hyperbolic 3-manifolds of minimal volume. The manifold is $m003$ in the SnapPea census [CHW, CDW], and is the once-punctured torus bundle over the circle with monodromy $\begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix}$. Its homology is $H_1(M; \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z}$, and it is, for instance, the complement of a null-homologous knot in $L(5, 1)$. After randomizing the triangulation a bit, SnapPy gives the following presentation

$$\pi := \pi_1(M) = \langle a, b \mid bab^3aba^{-2} = 1 \rangle.$$

We will view $X(\pi)$ as a subvariety of $X(\langle a, b \rangle)$, where $\langle a, b \rangle$ is the free group on $\{a, b\}$. Now $X(\langle a, b \rangle) \cong \mathbb{C}^3$ where the coordinates are $(x, y, z) = (\text{tr}(a), \text{tr}(b), \text{tr}(ab))$; this is because the trace of any word $w \in \langle a, b \rangle$ can be expressed in terms of these coordinates using the fundamental relation $\text{tr}(UV) = \text{tr}(U)\text{tr}(V) - \text{tr}(UV^{-1})$ for $U, V \in \text{SL}(2, \mathbb{C})$. Since π is defined by the single relator $R = bab^3aba^{-2}$, the character variety $X(\pi)$ is cut out by the polynomials corresponding to $\text{tr}(R) = 2$, $\text{tr}([a, R]) = 2$, and $\text{tr}([b, R]) = 2$. Using Gröbner bases in [Sage] to decompose $X(\pi)$ into irreducible components over \mathbb{Q} , we found a unique component X_0 which contains an irreducible character, i.e. contains a point where $\text{tr}([a, b]) \neq 2$. Explicitly, the ideal of X_0 is $\langle yz - x - z, xz + 1 \rangle$ and hence X_0 can be bijectively parameterized by

$$f: \mathbb{C} \setminus \{0\} \rightarrow X_0 \quad \text{where} \quad f(u) = (u, 1 - u^2, -1/u).$$

To compute $\mathcal{F}_K^{X_0}$, we consider the curve $R_0 \subset R(\pi)$ lying above X_0 consisting of representations of the form

$$\rho(a) = \begin{pmatrix} u & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \rho(b) = \begin{pmatrix} 0 & v \\ -v^{-1} & 1 - u^2 \end{pmatrix} \quad \text{where} \quad v + v^{-1} = u^{-1}.$$

Such representations are parameterized by $v \in \mathbb{C} \setminus \{0\}$, and hence $\mathbb{C}(R_0) \cong \mathbb{C}(v)$ and we have an explicit $\pi \rightarrow \text{GL}(2, \mathbb{C}(v))$ which is the restriction of the tautological representation. Using Lemma 2.13, we find the torsion polynomial of this representation is

$$t - \frac{2(v^4 + v^2 + 1)}{v^3 + v} + t^{-1}.$$

Substituting in $v = \pm(1 - \sqrt{-4u^2 + 1})/2u$ to eliminate v gives the final answer

$$\mathcal{F}_K^{X_0}(t) = t + \frac{2(u^2 - 1)}{u} + t^{-1}.$$

8.2. Example: m006. The census manifold $M = m006$ can also be described as $5/2$ surgery on one component of the Whitehead link L . (Here our conventions are such that $+1$ surgery on either component of L gives the trefoil knot whereas -1 surgery gives the figure-8 knot). Thus M is, for instance, the complement of a null-homologous knot in the lens space $L(5, 2)$ and again $H_1(M; \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z}$. Using spun-normal surfaces, it is easy to check via [CD] that there is a Seifert surface in M which has genus one with one boundary component.

SnapPy gives the presentation

$$\pi := \pi_1(M) = \langle a, b \mid b^2 abab^2 a^{-2} = 1 \rangle.$$

Changing the generators to $a_{new} = a^{-1}$ and $b_{new} = ab$ rewrites this as

$$\pi = \langle a, b \mid a^3 bab^3 ab = 1 \rangle.$$

Using the same setup as in the last example, we find a single component X_0 containing an irreducible character. The ideal of X_0 is $\langle x - y, y^2 z - z - 1 \rangle$ and hence we can bijectively parameterize X_0 by

$$(8.3) \quad f: \mathbb{C} \setminus \{1, -1\} \rightarrow X_0 \quad \text{where} \quad f(u) = (u, u, (u^2 - 1)^{-1}).$$

Considering the curve of representations given by

$$\rho(a) = \begin{pmatrix} v & 1 \\ 0 & v^{-1} \end{pmatrix} \quad \text{and} \quad \rho(b) = \begin{pmatrix} v^{-1} & 0 \\ \frac{3-2u^2}{u^2-1} & v \end{pmatrix} \quad \text{where} \quad v + v^{-1} = u$$

and again directly applying Lemma 2.13 and eliminating v gives

$$\mathcal{F}_K^{X_0}(t) = \frac{2u^2 - 1}{u^2 - 1} (t + t^{-1}) + \frac{2u^3}{u^2 - 1}.$$

8.4. **m037.** The census manifold $M = m037$ has $H_1(M; \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$, and so is not a knot in a $\mathbb{Z}/2$ -homology sphere. However, this makes no difference in this character variety context. Again, using spun-normal surfaces one easily checks that there is a Seifert surface in M which has genus one with one boundary component. Now $\pi = \langle a, b \mid a^3 ba^2 ba^3 b^{-2} = 1 \rangle$, and this time, there are two components of $X(\pi)$ containing irreducible characters. However, one of these consists entirely of metabelian representations which factor through the epimorphism $\pi \rightarrow \mathbb{Z}/2 * \mathbb{Z}/2 = \langle c, d \mid c^2 = d^2 = 1 \rangle$ given by $a \mapsto c$ and $b \mapsto d$. Focusing on the other component X_0 , it turns out the ideal is $\langle xz - 2y, 4y^2 - z^2 - 4 \rangle$ and so we can parameterize X_0 by

$$f: \mathbb{C} \setminus \{-2, 0, 2\} \rightarrow X_0 \quad \text{where} \quad f(u) = \left(\frac{u^2 + 4}{4u}, \frac{u^2 + 4}{u^2 - 4}, \frac{8u}{u^2 - 4} \right)$$

and then calculate

$$\mathcal{F}_K^{X_0}(t) = \frac{(u+2)^4}{16u^2} (t + t^{-1}) + \frac{(u+2)(u^4 + 4u^3 - 8u^2 + 16u + 16)}{8(u-2)u^2}.$$

8.5. **The role of ideal points.** A key part of the Culler-Shalen theory is the association of an essential surface in the manifold M to each ideal point of a curve $X_0 \subset X(M)$. The details are in e.g. [Sha], but in brief consider the smooth projective model \widehat{X}_0 with its rational map $\widehat{X}_0 \rightarrow X_0$. Now \widehat{X}_0 is a smooth Riemann surface, and the finite number of points where $\widehat{X}_0 \rightarrow X_0$ is undefined are called the *ideal points* of X_0 . To each such point x , there is an associated non-trivial action of $\pi := \pi_1(M)$ on a simplicial tree T_x . One then constructs a surface S in M dual to this action, which can be taken to be essential (i.e. incompressible, boundary incompressible, and not boundary parallel). As minimal complexity Seifert surfaces often arise from an ideal point of some X_0 , a very natural idea is thus to try to use such an ideal point x to say something about $\mathcal{F}_K^{X_0}$. Moreover,

provided that $X_0 \subset X(M)^{irr}$, a surface associated to an ideal point is never a fiber or semifiber, which suggests that one might hope to prove non-monotonicity of $\mathcal{T}_K^{X_0}$ by examining $\mathcal{T}_K^{X_0}(x)$. Thus, we now compute what happens to $\mathcal{T}_K^{X_0}$ at such ideal points in our two non-fibered examples $m006$ and $m037$. (Aside: It is known that even for knots in S^3 not all boundary slopes need arise from ideal points [CT], so it is probably too much to expect that there is always an ideal point which gives a Seifert surface.)

8.6. Ideal points of $m006$. If we view the parameterization (8.3) above as a rational map from $P^1(\mathbb{C}) \rightarrow X_0$, we have ideal points corresponding to $u \in \{-1, 1, \infty\}$. To calculate the boundary slopes of the surfaces associated to each of these, we consider the trace functions of SnapPy's preferred basis μ, λ for $\pi_1(\partial M)$. In our presentation for π , we calculate

$$\begin{aligned}\mathrm{tr}(\mu) &= \mathrm{tr}(a^2 bab) = x(z^2 - z - 1) = -\frac{u(u^4 - u^2 - 1)}{(u-1)^2(u+1)^2} \\ \mathrm{tr}(\lambda) &= \mathrm{tr}(a^3 ba) = -x^3 - xz + 2x = -\frac{u(u^4 - 3u^2 + 3)}{(u-1)(u+1)} \\ \mathrm{tr}(\mu\lambda) &= x^4 - 2x^2 - z + 1 = \frac{(u^2 - 2)(u^4 - u^2 + 1)}{(u-1)(u+1)}.\end{aligned}$$

Now, consider an ideal point x with associated surface S , and pick a simple closed curve γ on $\pi_1(\partial M)$. Then the number of times γ intersects ∂S is twice the order of the pole of $\mathrm{tr}(\gamma)$ at x (here, if $\mathrm{tr}(\gamma)$ has a zero of order m at x , this counts as a pole of order 0, not one of order $-m$). The above formulae thus show that the points $u = 1$ and $u = -1$ give surfaces with boundary slope $\mu\lambda^2$, whereas $u = \infty$ gives one with boundary slope $\mu^3\lambda^{-1}$. The latter is the homological longitude, and as there is only one spun-normal surface with that boundary slope and each choice of spinning direction, it follows that the surface associated to $\xi = \infty$ must be the minimal genus Seifert surface. Thus we're interested in

$$\mathcal{T}_K^{X_0}(u = \infty)(t) = 2(t + t^{-1}) + (\text{simple pole})t^0.$$

8.7. Ideal points of $m037$. This time we have four ideal points corresponding to $u = -2, 2, 0, \infty$. We find

$$\begin{aligned}\mathrm{tr}(\mu) &= \mathrm{tr}(a^2 ba^3) = \frac{u^8 - 48u^6 + 96u^4 - 768u^2 + 256}{64(u-2)(u+2)u^3} \\ \mathrm{tr}(\lambda) &= \mathrm{tr}(a^{-1} ba^3 b) \\ &= -\frac{u^{12} - 72u^{10} + 1264u^8 - 12032u^6 + 20224u^4 - 18432u^2 + 4096}{256(u-2)^2(u+2)^2u^4} \\ \mathrm{tr}(\mu\lambda) &= -\frac{u^8 - 16u^6 + 352u^4 - 256u^2 + 256}{4(u-2)^3(u+2)^3u}.\end{aligned}$$

Hence $\{2, -2\}$ give surfaces with boundary slope $\mu^2\lambda^{-1}$ and $\{0, \infty\}$ give surfaces with boundary slope $\mu^4\lambda^3$. In fact, the homological longitude is $\mu^2\lambda^{-1}$ and again

using spun-normal surfaces one easily checks that surfaces associated to $\{2, -2\}$ are the minimal genus Seifert surface. Thus, we care about

$$\mathcal{T}_K^{X_0}(u=2)(t) = 4(t + t^{-1}) + (\text{simple pole})t^0 \quad \text{and} \quad \mathcal{T}_K^{X_0}(u=-2)(t) = 0.$$

8.8. General picture for ideal points. Based on the preceding examples and a heuristic calculation for tunnel-number one manifolds, we posit:

8.9. Conjecture. *Let K be a knot in a rational homology 3-sphere, and X_0 a component of $X(K)^{irr}$. Suppose x is an ideal point of X_0 which gives a Seifert surface (hence K is nonfibered). Then the lead coefficient of $\mathcal{T}_K^{X_0}$ has a finite value at x .*

Unfortunately, Conjecture 8.9 does not seem particularly promising as an attack on Conjecture 1.12 for distinguishing fibered versus nonfibered cases. Moreover, in terms of looking at such Seifert ideal points to show that $\mathcal{T}_K^{X_0}$ determines the genus, the second ideal point $u = -2$ in Section 8.7 where $\mathcal{T}_K^{X_0}$ vanishes is not a promising sign.

However, when trying to use an ideal point x of X_0 which gives a Seifert surface to understand $\mathcal{T}_K^{X_0}$, it may be wrong to focus on just the value of $\mathcal{T}_K^{X_0}$ at x . After all, there is no representation of π corresponding to x . Rather, as in the construction of the surface associated to x , perhaps one should view x as giving a valuation on $\mathbb{C}(R_0)$, where R_0 is a component of $R(M)$ surjecting onto X_0 . If we unwind the definition of the associated surface and its properties, we are left with the following abstract situation. There is a field \mathbb{F} with an additive valuation $v: \mathbb{F}^\times \rightarrow \mathbb{Z}$ with a representation $\rho: \pi \rightarrow \text{SL}(2, \mathbb{F})$ so that for each $\gamma \in \pi$ we have $v(\text{tr}(\gamma)) \geq |\phi(\gamma)|$ where $\phi: \pi \rightarrow \mathbb{Z}$ is the usual free abelianization homomorphism. This alone is not enough, because even in the fibered case one always has such a setup by looking at an ideal point of a component of $X(M)$ consisting of reducible representations. Thus it seems that the key to such an approach must be to exploit the fact that since X_0 contains an irreducible character there is a $\gamma \in \pi$ with $\phi(\gamma) = 0$ yet $v(\text{tr}(\gamma))$ is arbitrarily large.

REFERENCES

- [BV] N. Bergeron and A. Venkatesh. The asymptotic growth of torsion homology for arithmetic groups. Preprint 2010, 49 pages. [arXiv:1004.1083](https://arxiv.org/abs/1004.1083).
- [Bur1] G. Burde. *SU(2)-representation spaces for two-bridge knot groups*. *Math. Ann.* **288** (1990), 103–119. [MR1070927](https://doi.org/10.1007/BF01200927).
- [Bur2] B. Burton. Regina, a normal surface theory calculator, version 4.6, 2009. <http://regina.sourceforge.net/>
- [CHW] P. J. Callahan, M. V. Hildebrand, and J. R. Weeks. *A census of cusped hyperbolic 3-manifolds*. *Math. Comp.* **68** (1999), 321–332. With microfiche supplement. [MR1620219](https://doi.org/10.1090/S0025-5718-1999-0116202-1).
- [Cha] J. C. Cha. *Fibred knots and twisted Alexander invariants*. *Trans. Amer. Math. Soc.* **355** (2003), 4187–4200. [arXiv:math/0109136](https://arxiv.org/abs/math/0109136), [MR1990582](https://doi.org/10.1090/S0002-9947-003-019905-2).
- [Che1] J. Cheeger. Analytic torsion and Reidemeister torsion. *Proc. Nat. Acad. Sci. U.S.A.* **74** (1977), 2651–2654. [MR0451312](https://doi.org/10.1073/pnas.74.13.2651).
- [Che2] J. Cheeger. *Analytic torsion and the heat equation*. *Ann. of Math. (2)* **109** (1979), 259–322. [MR528965](https://doi.org/10.2307/2373223).
- [CT] E. Chesebro and S. Tillmann. *Not all boundary slopes are strongly detected by the character variety*. *Comm. Anal. Geom.* **15** (2007), 695–723. [arXiv:math/0510418](https://arxiv.org/abs/math/0510418), [MR2395254](https://doi.org/10.1112/S0013-774X(2007)0015-00069-5).
- [CL] D. Cooper and D. D. Long. *Remarks on the A-polynomial of a knot*. *J. Knot Theory Ramifications* **5** (1996), 609–628. [MR1414090](https://doi.org/10.1080/10467179608839409).
- [CGHN] D. Coulson, O. A. Goodman, C. D. Hodgson, and W. D. Neumann. *Computing arithmetic invariants of 3-manifolds*. *Experiment. Math.* **9** (2000), 127–152. [MR1758805](https://doi.org/10.1080/10467170008839409).
- [Cro] R. Crowell. Genus of alternating link types. *Ann. of Math. (2)* **69** (1959), 258–275. [MR0099665](https://doi.org/10.2307/2372665).
- [CF] R. H. Crowell and R. H. Fox. *Introduction to knot theory*. Based upon lectures given at Haverford College under the Philips Lecture Program. Ginn and Co., Boston, Mass., 1963. [MR0146828](https://doi.org/10.1080/10467176308839409).
- [CD] M. Culler and N. M. Dunfield. t3m: a Python 3-manifold library, 2010. <http://t3m.computop.org>
- [CDW] M. Culler, N. M. Dunfield, and J. R. Weeks. SnapPy, a computer program for studying the geometry and topology of 3-manifolds. <http://snappy.computop.org/>
- [CGLS] M. Culler, C. M. Gordon, J. Luecke, and P. B. Shalen. *Dehn surgery on knots*. *Ann. of Math. (2)* **125** (1987), 237–300. [MR881270](https://doi.org/10.2307/2373223).
- [CS] M. Culler and P. B. Shalen. *Varieties of group representations and splittings of 3-manifolds*. *Ann. of Math. (2)* **117** (1983), 109–146. [MR683804](https://doi.org/10.2307/2373223).
- [DY] J. Dubois and Y. Yamaguchi. Multivariable Twisted Alexander Polynomial for hyperbolic three-manifolds with boundary. Preprint 2009, 39 pages. [arXiv:0906.1500](https://arxiv.org/abs/0906.1500).
- [DFJ] N. M. Dunfield, S. Friedl, and N. Jackson. Software and data accompanying this paper. <http://dunfield.info/torsion>
- [DGST] N. M. Dunfield, S. Garoufalidis, A. Shumakovitch, and M. Thistlethwaite. *Behavior of knot invariants under genus 2 mutation*. *New York J. Math.* **16** (2010), 99–123. [arXiv:math/0607258](https://arxiv.org/abs/math/0607258), [MR2657370](https://doi.org/10.1215/10796702-1253770).
- [DR] N. M. Dunfield and D. Ramakrishnan. *Increasing the number of fibered faces of arithmetic hyperbolic 3-manifolds*. *Amer. J. Math.* **132** (2010), 53–97. [arXiv:0712.3243](https://arxiv.org/abs/0712.3243), [MR2597506](https://doi.org/10.2307/2597506).
- [Fox1] R. H. Fox. Free differential calculus. I. Derivation in the free group ring. *Ann. of Math. (2)* **57** (1953), 547–560. [MR0053938](https://doi.org/10.2307/2372665).
- [Fox2] R. H. Fox. Free differential calculus. II. The isomorphism problem of groups. *Ann. of Math. (2)* **59** (1954), 196–210. [MR0062125](https://doi.org/10.2307/2372665).
- [Fri] D. Fried. *Cyclic resultants of reciprocal polynomials*. In *Holomorphic dynamics (Mexico, 1986)*, volume 1345 of *Lecture Notes in Math.*, pages 124–128. Springer, Berlin, 1988. [MR980956](https://doi.org/10.1007/BFb0075056).

- [FJ] S. Friedl and N. Jackson. Approximations to the volume of hyperbolic knots. In T. Morifuji, editor, *Twisted topological invariants and topology of low-dimensional manifolds*, volume 1747 of *RIMS Kôkyûroku*, pages 35–46, 2011. [arXiv:1102.3742](#).
- [FK1] S. Friedl and T. Kim. [The Thurston norm, fibered manifolds and twisted Alexander polynomials](#). *Topology* **45** (2006), 929–953. [arXiv:math/0505594](#), [MR2263219](#).
- [FK2] S. Friedl and T. Kim. [Twisted Alexander norms give lower bounds on the Thurston norm](#). *Trans. Amer. Math. Soc.* **360** (2008), 4597–4618. [arXiv:math/0505682](#), [MR2403698](#).
- [FKK] S. Friedl, T. Kim, and T. Kitayama. Poincaré duality and degrees of twisted Alexander polynomials. Preprint 2011, 42 pages. [arXiv:1107.3002](#).
- [FV1] S. Friedl and S. Vidussi. [Twisted Alexander polynomials detect fibered 3-manifolds](#). *Ann. of Math. (2)* **173** (2001), 1587–1643. [arXiv:0805.1234](#).
- [FV2] S. Friedl and S. Vidussi. [Twisted Alexander polynomials and symplectic structures](#). *Amer. J. Math.* **130** (2008), 455–484. [arXiv:math/0604398](#), [MR2405164](#).
- [FV3] S. Friedl and S. Vidussi. A survey of twisted Alexander polynomials. Preprint 2010, 42 pages. [arXiv:0905.0591](#).
- [Gab1] D. Gabai. [Detecting fibred links in \$S^3\$](#) . *Comment. Math. Helv.* **61** (1986), 519–555. [MR870705](#).
- [Gab2] D. Gabai. [Foliations and the topology of 3-manifolds. III](#). *J. Differential Geom.* **26** (1987), 479–536. [MR910018](#).
- [GKM] H. Goda, T. Kitano, and T. Morifuji. [Reidemeister torsion, twisted Alexander polynomial and fibered knots](#). *Comment. Math. Helv.* **80** (2005), 51–61. [arXiv:math/0311155](#), [MR2130565](#).
- [GN] O. Goodman and W. Neumann. Snap, a computer program for studying arithmetic invariants of hyperbolic 3-manifolds. <http://snap-pari.sourceforge.net/>
- [Har] S. L. Harvey. [Higher-order polynomial invariants of 3-manifolds giving lower bounds for the Thurston norm](#). *Topology* **44** (2005), 895–945. [arXiv:math/0207014](#), [MR2153977](#).
- [HT] A. Hatcher and W. Thurston. [Incompressible surfaces in 2-bridge knot complements](#). *Invent. Math.* **79** (1985), 225–246. [MR778125](#).
- [HLM] H. M. Hilden, M. T. Lozano, and J. M. Montesinos-Amilibia. On the arithmetic 2-bridge knots and link orbifolds and a new knot invariant. *J. Knot Theory Ramifications* **4** (1995), 81–114. [MR96a:57019](#).
- [Hil] C. J. Hillar. [Cyclic resultants](#). *J. Symbolic Comput.* **39** (2005), 653–669. [arXiv:math/0401220](#), [MR2167674](#).
- [HL] C. J. Hillar and L. Levine. [Polynomial recurrences and cyclic resultants](#). *Proc. Amer. Math. Soc.* **135** (2007), 1607–1618. [arXiv:math/0411414](#), [MR2286068](#).
- [HSW] J. A. Hillman, D. S. Silver, and S. G. Williams. [On reciprocity of twisted Alexander invariants](#). *Algebr. Geom. Topol.* **10** (2010), 1017–1026. [arXiv:0905.2574](#), [MR2629774](#).
- [HM] M. Hirasawa and K. Murasugi. [Evaluations of the twisted Alexander polynomials of 2-bridge knots at \$\pm 1\$](#) . *J. Knot Theory Ramifications* **19** (2010), 1355–1400. [arXiv:0808.3058](#), [MR2735813](#).
- [HT] J. Hoste and M. Thistlethwaite. Knotscape, 1999. <http://pzacad.pitzer.edu/~jhoste/HosteWebPages/kntscp.html>
- [HTW] J. Hoste, M. Thistlethwaite, and J. Weeks. [The first 1,701,936 knots](#). *Math. Intelligencer* **20** (1998), 33–48. [MR1646740](#).
- [KM1] T. Kim and T. Morifuji. Twisted Alexander polynomials and character varieties of 2-bridge knot groups. Preprint 2010, 19 pages. [arXiv:1006.4285](#).
- [Kit1] T. Kitano. [Twisted Alexander polynomial and Reidemeister torsion](#). *Pacific J. Math.* **174** (1996), 431–442. [MR1405595](#).
- [KM2] T. Kitano and T. Morifuji. Divisibility of twisted Alexander polynomials and fibered knots. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* **4** (2005), 179–186. [MR2165406](#).

- [Kit2] T. Kitayama. Normalization of twisted Alexander invariants. Preprint 2009, 19 pages. [arXiv:0705.2371](#).
- [Lin] X. S. Lin. Representations of knot groups and twisted Alexander polynomials. *Acta Math. Sin. (Engl. Ser.)* **17** (2001), 361–380. [MR1852950](#).
- [MR] C. Maclachlan and A. W. Reid. *The arithmetic of hyperbolic 3-manifolds*, volume 219 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2003. [MR1937957](#).
- [MFP1] P. Menal-Ferrer and J. Porti. Twisted cohomology for hyperbolic three manifolds, Preprint 2010. 27 pages. [arXiv:1001.2242](#).
- [MFP2] P. Menal-Ferrer and J. Porti. Mutation and $SL(2, \mathbb{C})$ -Reidemeister torsion for hyperbolic knots, Preprint 2011. 19 pages. [arXiv:1108.2460](#).
- [Mil] J. Milnor. Whitehead torsion. *Bull. Amer. Math. Soc.* **72** (1966), 358–426. [MR0196736](#).
- [Mor] T. Morifuji. Twisted Alexander polynomials of twist knots for nonabelian representations. *Bull. Sci. Math.* **132** (2008), 439–453. [MR2426646](#).
- [Mu1] W. Müller. Analytic torsion and R -torsion of Riemannian manifolds. *Adv. in Math.* **28** (1978), 233–305. [MR498252](#).
- [Mu2] W. Müller. Analytic torsion and cohomology of hyperbolic 3-manifolds. Preprint of the Max-Planck Institute, Bonn (2009).
- [Mu3] W. Müller. The asymptotics of the Ray-Singer analytic torsion of hyperbolic 3-manifolds. Preprint 2010, 35 pages. [arXiv:1003.5168](#).
- [Mur1] K. Murasugi. On the genus of the alternating knot. I, II. *J. Math. Soc. Japan* **10** (1958), 94–105, 235–248. [MR0099664](#).
- [Mur2] K. Murasugi. On a certain subgroup of the group of an alternating link. *Amer. J. Math.* **85** (1963), 544–550. [MR0157375](#).
- [NR] W. D. Neumann and A. W. Reid. Arithmetic of hyperbolic manifolds. In *Topology '90 (Columbus, OH, 1990)*, volume 1 of *Ohio State Univ. Math. Res. Inst. Publ.*, pages 273–310. de Gruyter, Berlin, 1992. [MR1184416](#).
- [Nic] L. I. Nicolaescu. *The Reidemeister torsion of 3-manifolds*, volume 30 of *de Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, 2003. [MR1968575](#).
- [Paj] A. V. Pajitnov. Novikov homology, twisted Alexander polynomials, and Thurston cones. *Algebra i Analiz* **18** (2006), 173–209. [MR2301045](#).
- [Par] J. Park. Analytic torsion and Ruelle zeta functions for hyperbolic manifolds with cusps. *J. Funct. Anal.* **257** (2009), 1713–1758. [MR2540990](#).
- [Por1] J. Porti. Private communication, 2009.
- [Por2] J. Porti. Torsion de Reidemeister pour les variétés hyperboliques. *Mem. Amer. Math. Soc.* **128** (1997), x+139. [MR1396960](#).
- [Rag] M. S. Raghunathan. On the first cohomology of discrete subgroups of semisimple Lie groups. *Amer. J. Math.* **87** (1965), 103–139. [MR0173730](#).
- [RS] D. B. Ray and I. M. Singer. R -torsion and the Laplacian on Riemannian manifolds. *Advances in Math.* **7** (1971), 145–210. [MR0295381](#).
- [Rub] D. Ruberman. Mutation and volumes of knots in S^3 . *Invent. Math.* **90** (1987), 189–215. [MR906585](#).
- [Sha] P. B. Shalen. Representations of 3-manifold groups. In *Handbook of geometric topology*, pages 955–1044. North-Holland, Amsterdam, 2002. [MR1886685](#).
- [SW] D. S. Silver and S. G. Williams. Dynamics of twisted Alexander invariants. *Topology Appl.* **156** (2009), 2795–2811. [arXiv:0801.2118](#), [MR2556037](#).
- [Sage] W. Stein et al. *Sage Mathematics Software, Version 4.7*. <http://www.sagemath.org>
- [Sto] A. Stoimenow. Knot data tables. <http://stoimenov.net/stoimeno/homepage/ptab/>
- [ST] A. Stoimenow and T. Tanaka. Mutation and the colored Jones polynomial. *J. Gökova Geom. Topol. GGT* **3** (2009), 44–78. [arXiv:math/0607794](#), [MR2595755](#).
- [Thu] W. P. Thurston. *Three-dimensional geometry and topology. Vol. 1*, volume 35 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1997. Edited by Silvio Levy. [MR1435975](#).

- [Til1] S. Tillmann. [On the Kinoshita-Terasaka knot and generalised Conway mutation](#). *J. Knot Theory Ramifications* **9** (2000), 557–575. [MR1758873](#).
- [Til2] S. Tillmann. [Character varieties of mutative 3-manifolds](#). *Algebr. Geom. Topol.* **4** (2004), 133–149. [math.GT/0306053](#), [MR2059186](#).
- [Tur1] V. G. Turaev. Reidemeister torsion in knot theory. *Uspekhi Mat. Nauk* **41** (1986), 97–147, 240. [MR832411](#).
- [Tur2] V. G. Turaev. Euler structures, nonsingular vector fields, and Reidemeister-type torsions. *Izv. Akad. Nauk SSSR Ser. Mat.* **53** (1989), 607–643, 672. [MR1013714](#).
- [Tur3] V. Turaev. *Introduction to combinatorial torsions*. Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 2001. Notes taken by Felix Schlenk. [MR1809561](#).
- [Tur4] V. Turaev. *Torsions of 3-dimensional manifolds*, volume 208 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 2002. [MR1958479](#).
- [Wada] M. Wada. [Twisted Alexander polynomial for finitely presentable groups](#). *Topology* **33** (1994), 241–256. [MR1273784](#).
- [Wal] F. Waldhausen. Algebraic K -theory of generalized free products. I, II. *Ann. of Math. (2)* **108** (1978), 135–204. [MR0498807](#).

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