Eta invariants as sliceness obstructions
and their relation to Casson-Gordon invariants

A Dissertation

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ABSTRACT

Eta invariants as sliceness obstructions and their relation to Casson-Gordon invariants

A dissertation presented to the faculty of
the Graduate School of Arts and Sciences of
Brandeis University, Waltham, Massachusetts

by Stefan Friedl

We classify the metabelian unitary representations of $\pi_1(M_K)$, where M_K is the result of zero-surgery along a knot $K \subset S^3$. We show that certain eta invariants associated to metabelian representations $\pi_1(M_K) \to U(k)$ vanish for slice knots and that even more eta invariants vanish for ribbon knots and doubly slice knots. We show that this result contains the Casson-Gordon sliceness obstruction. It turns out that eta invariants can in many cases be easily computed for satellite knots. We use this to study the relation between the eta invariant sliceness obstruction, eta invariant ribboness obstruction, and the L^2-eta invariant sliceness obstruction recently introduced by Cochran, Orr and Teichner. In particular we give an example of a knot which has zero eta invariant and zero metabelian L^2-eta invariant sliceness obstruction but is not ribbon. It is not known whether this knot is slice or not.

Contents

1. Introduction

We study smooth knots $K \cong S^n \subset S^{n+2}$, these form a semigroup under connected sum. This semigroup can be turned into a group by modding out by the subsemigroup of slice knots; a knot $K \subset S^{n+2}$ is called slice if it bounds a smooth $(n+1)$-disk in D^{n+3}. This group is called the knot concordance group in dimension n. It is a natural goal to attempt to understand this group and find complete invariants for when a knot represents a non-zero (or at least non-torsion) element in this group.

Knot concordance in the high-dimensional case, i.e. the case $n > 1$, is well understood. Kervaire [K65] first showed that all even–dimensional knots are slice. For odd–dimensional knots Levine [L69] showed that an odd–dimensional knot is slice if and only if it is algebraically slice, i.e. if the Seifert form has a subspace of half-rank on which it vanishes. This reduces the task of detecting slice knots to an algebraic problem which is well-understood (cf. [L69b]).

The classical case $n = 1$ turned out to be much more difficult to understand. Casson and Gordon [CG78], [CG86] defined certain sliceness obstructions (cf. section 5.1) and used these to give the first examples of knots in S^3 that are algebraically slice but not geometrically slice. Many more examples have been given since then, the most subtle ones were found recently by Cochran, Orr and Teichner [COT01] (cf. section 9).

We will show how to use eta-invariants to detect non–slice knots. Given a closed smooth three–manifold M and a unitary representation $\alpha : \pi_1(M) \to U(k)$, Atiyah, Patodi and Singer [APS75] defined $\eta_\alpha(M) \in \mathbf{R}$, called the eta invariant of (M, α) which has the following property.

Theorem. [3.1] *(Atiyah-Patodi-Singer index theorem) If $(W^4, \beta : \pi_1(W) \to U(k))$ is such that $\partial(W^4, \beta) = n(M^3, \alpha)$ for some $n \in \mathbb{N}$, then*

$$\eta_\alpha(M) = \frac{1}{n}(sign_\beta(W) - ksign(W))$$

where $sign_\beta(W)$ denotes the twisted signature of W.

For a knot K we study the eta invariants associated to the closed manifold M_K, where M_K denotes the result of zero–framed surgery along $K \subset S^3$. Eta invariants in the context of link theory were first studied by Levine [L94]; Letsche [L00] first applied them in the context of knot concordance (cf. section 8.2).

We restrict ourselves to metabelian representations of $\pi_1(M_K)$, i.e. to representations which factor through $\pi_1(M_K)/\pi_1(M_K)^{(2)}$ where $\pi_1(M_K)^{(2)}$ denotes the second commutator subgroup of $\pi_1(M_K)$. We denote the homology of the universal abelian cover of M_K by $H_1(M_K, \Lambda)$, where $\Lambda = \mathbb{Z}[t, t^{-1}]$. Since $\pi_1(M_K)/\pi_1(M_K)^{(1)} = \mathbb{Z}$ we get isomorphisms

$$\pi_1(M_K)/\pi_1(M_K)^{(2)} \cong \pi_1(M_K)/\pi_1(M_K)^{(1)} \ltimes \pi_1(M_K)^{(1)}/\pi_1(M_K)^{(2)} \cong \mathbb{Z} \ltimes H_1(M_K, \Lambda)$$

where $1 \in \mathbb{Z}$ acts on $H_1(M_K, \Lambda)$ by multiplication by t. The following proposition gives a classification of all metabelian unitary representations of $\pi_1(M_K)$, i.e. of all unitary representations of $\mathbb{Z} \ltimes H_1(M_K, \Lambda)$.

Proposition. [**4.3**] *Let H be a Λ-module, then any irreducible representation $\mathbb{Z} \ltimes H \to U(k)$ is conjugate to a representation of the form*

$$\begin{array}{rcl} \alpha_{(z,\chi)} : \mathbb{Z} \ltimes H & \to & U(k) \\ (n,h) & \mapsto & z^n \begin{pmatrix} 0 & \dots & 0 & 1 \\ 1 & \dots & 0 & 0 \\ \vdots & \ddots & & \vdots \\ 0 & \dots & 1 & 0 \end{pmatrix}^n \begin{pmatrix} \chi(h) & 0 & \dots & 0 \\ 0 & \chi(th) & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & \chi(t^{k-1}h) \end{pmatrix} \end{array}$$

where $z \in S^1$ and $\chi : H \to H/(t^k - 1) \to S^1$ is a character which does not factor through $H/(t^l - 1)$ for some $l < k$.

For K a knot, $H_1(M, \Lambda)/(t^k - 1) = H_1(L_k)$, where L_k denotes the k-fold cover of S^3 branched along K. If K is a slice knot and D a slice disk for K, then M_K bounds $N_D := \overline{D^4 \setminus N(D)}$. If a representation $\alpha : \pi_1(M_K) \to U(k)$ extends to $\beta : \pi_1(N_D) \to U(k)$, then $\eta(M_K, \alpha) = \text{sign}_\beta(N_D) - \text{sign}(N_D)$. But $\text{sign}(N_D) = 0$ since $H_*(N_D) = H_*(S^1)$; furthermore Letsche [L00] showed that if $\chi : H_1(M_K, \Lambda)/(t^k - 1) \to S^1$ is of prime power order and $z \in S^1$ is transcendental, then $\alpha_{(z,\chi)}$ extends to β such that $\text{sign}_\beta(N_D) = 0$. We denote the set of irreducible, metabelian represention of this type by $P_k^{irr,met}(\pi_1(M_K))$. We therefore get the following theorem.

Theorem. [**4.7**] *If K is a slice knot, D a slice disk and if $\alpha \in P_k^{irr,met}(\pi_1(M_K))$ extends over $\pi_1(N_D)$, then $\eta_\alpha(M_K) = 0$.*

Now one has to find criteria when a representation α of $\pi_1(M_K)$ extends over $\pi_1(N_D)$. This problem breaks up into two parts, first α has to vanish on $\text{Ker}(\pi_1(M_K) \to \pi_1(N_D))$ and it has to extend from $\text{Im}\{\pi_1(M_K) \to \pi_1(N_D)\}$ to $\pi_1(N_D)$. This leads to the following theorem.

Theorem. [**4.9**] *Let K be a slice knot, k a prime power. Then there exists $P_k \subset H_1(L_k)$ such that $P_k = P_k^\perp$ with respect to the linking pairing $H_1(L_k) \times H_1(L_k) \to \mathbb{Q}/\mathbb{Z}$, such that for any irreducible representation $\alpha : \pi_1(M_K) \to \mathbb{Z} \ltimes H_1(L_k) \to U(k)$ vanishing on $0 \times P_k$ and lying in $P_k^{irr,met}(\pi_1(M_K))$ the representation α will extend over $\pi_1(N_D)$ for some slice disk D and $\eta_\alpha(M_K) = 0$.*

In section 5.1 we recall the Casson–Gordon sliceness obstruction theorem which till the advent of Cochran–Orr–Teichner's L^2-eta invariants proved to be the most effective obstruction. In theorem 5.8 we show that an algebraically slice knot has zero Casson-Gordon sliceness obstruction if and only if it satisfies the vanishing conclusion of theorem 4.9.

The eta invariant approach has several advantages over the Casson-Gordon approach. For example, one can easily show that if $\alpha_1,\dots,\alpha_l$ are representations as in theorem 4.9, then their tensor product will also extend over $\pi_1(N_D)$, and there exists a simple criterion when $\alpha_1\otimes\dots\otimes\alpha_l \in P_k^{irr,met}(\pi_1(M_K))$, which then guarantees that $\eta_{\alpha_1\otimes\dots\otimes\alpha_l}(M_K)=0$ (cf. theorem 4.11). This gives a potentially stronger sliceness obstruction than the Casson–Gordon obstruction.

We then turn to ribbon knots (for a definition cf. section 6). The only fact we use is that a ribbon knot has a slice disk D such that $\pi_1(M_K)\to\pi_1(N_D)$ is surjective. In particular a representation of $\pi_1(M_K)$ extends over $\pi_1(N_D)$ if it vanishes on $\mathrm{Ker}\{\pi_1(M_K)\to\pi_1(N_D)\}$. This allows us to prove the following theorem.

Theorem. **[6.4]** *Let $K\subset S^3$ be a ribbon knot. Then there exists $P\subset H_1(M,\Lambda)$ such that $P=P^\perp$ with respect to the Blanchfield pairing $H_1(M,\Lambda)\times H_1(M,\Lambda)\to \mathbb{Q}(t)/\mathbb{Z}[t,t^{-1}]$ and such that for any $\alpha\in P_k^{irr,met}(\pi_1(M_K))$ vanishing on $0\times P$ we get $\eta_\alpha(M_K)=0$.*

This is a much stronger obstruction theorem than the sliceness–obstruction theorems and could potentially provide a way to disprove the conjecture that each slice knot is ribbon.

We take a quick look at doubly slice knots. We prove a doubly slice obstruction theorem (theorem 7.2), and point out that doubly slice knots actually satisfy the conclusion of the ribbon obstruction theorem. It seems that doubly slice knots have a 'higher chance' of being ribbon than ordinary slice knots.

In section 8 we discuss Gilmer's [G93] and Letsche's [L00] sliceness theorems. It turns out that in fact both theorems have gaps in their proofs. We show to which degree their results still hold and how these follow from theorem 6.4.

Recently Cochran, Orr and Teichner [COT01], [COT02] defined the notion of (n)-solvability for a knot, $n\in\frac{1}{2}\mathbb{N}$, which has in particular the following properties.

(1) A slice knot is (n)-solvable for all n.
(2) A knot is (0.5)-solvable if and only if it is algebraically slice.
(3) A (1.5)-solvable knot has zero Casson–Gordon obstruction.

Given a homomorphism $\varphi:\pi_1(M)\to G$ to a group G, Cheeger and Gromov defined the L^2-eta invariant $\eta_\varphi^{(2)}(M)$ which satisfies a theorem similar to 3.1 if we replace the twisted signature by Atiyah's L^2-signature (cf. theorem 9.4). Cochran, Orr and Teichner used L^2-eta invariants to find examples of knots which are (2.0)–solvable, which in particular have zero Casson–Gordon–invariants, but which are not (2.5)–solvable. Using similar ideas Kim [K02] found examples of knots which are (1.0)–solvable and have zero Casson–Gordon–invariants, but which are not (1.5)–solvable. A quick summary of this theory will be given in section 9.

In section 10 we give examples that show that L^2-eta invariants are not complete invariants for a knot being (0.5)-solvable and (1.5)-solvable. We also show that there exists a knot which is (1.0)-solvable, has zero Casson–Gordon invariants and zero L^2-eta invariant of level 1, but does not satisfy the conclusion of theorem 6.4, i.e. is not ribbon. It's not known whether this knot is slice or not. Furthermore we give an example of a ribbon knot where the conclusion of theorem 4.9 does not hold for non prime–power characters; this shows that the set $P_k^{irr,met}(\pi_1(M_K))$ is in a sense maximal. In all cases we use a satellite construction to get knots whose eta–invariants can be computed explicitly by methods introduced by Litherland [L84] and extended in this thesis.

We conclude with an appendix which contains several algebraic propositions, which we could not find in the literature.

2. Basic knot theory and linking pairings as sliceness–obstructions

Throughout this thesis we will always work in the smooth category. In the classical dimension the theory of knots in the smooth category is equivalent to the theory in the locally flat category. The same is not true for the notion of sliceness. Since a smooth submanifold is always locally flat smooth slice disks are locally flat, but the converse is not true, i.e. there exist knots which are topologically slice, but not smoothly slice (cf. [G86], [E95]).

2.1. Basic knot theory.

By a knot we understand a smooth oriented submanifold of S^3 diffeomorphic to S^1. A smooth surface $F \subset S^3$ with $\partial(F) = S^1$ will be called a Seifert surface for K. Note that a Seifert surface inherits an orientation from K, in particular the map $H_1(F) \to H_1(S^3 \setminus F), a \mapsto a_+$ induced by pushing into the positive normal direction is well-defined. The pairing

$$\begin{array}{rcl} H_1(F) \times H_1(F) & \to & \mathbb{Z} \\ (a,b) & \mapsto & \operatorname{lk}(a, b_+) \end{array}$$

is called the Seifert pairing of F. Any matrix A representing such a pairing for some Seifert surface F is called Seifert matrix for K. Seifert matrices are unique up to S-equivalence (cf. [M65, p. 393] or [L70]). In particular the Alexander polynomial $\Delta_K(t) := \det(At - A^t) \in \mathbb{Z}[t, t^{-1}]/\{\pm t^l\}$ is well-defined and independent of the choice of A.

Each knot comes with a meridian and a longitude, more precisely, let T be a solid torus neighborhood of K. A meridian of K is a non-separating simple closed curve in $\partial(T)$ that bounds a disc in T. A longitude of K is a simple closed curve in $\partial(T)$ that is homologous to K in T and null-homologous in $S^3 \setminus K$. The notion of meridian and longitude is well-defined up to homotopy in $S^3 \setminus K$.

One way of studying knots is to study the manifold $S^3 \setminus K$. Alexander duality shows that $H_1(S^3 \setminus K) = \mathbb{Z}$. Note that the meridian μ generates $H_1(S^3 \setminus K)$ and that it is in fact the unique element in $H_1(S^3 \setminus K)$ such that $\operatorname{lk}(\mu, [K]) = 1$. Later on we'll use the map $\epsilon : \pi_1(S^3 \setminus K) \to H_1(S^3 \setminus K) \to \mathbb{Z}$ given by sending the meridian to 1.

A useful but complicated invariant for a knot K is $\pi_1(S^3 \setminus K, x_0)$ where x_0 is a base point. We normally suppress x_0 in the notation, since different base points give isomorphic groups. Any element in $\pi_1(S^3 \setminus K)$ which is freely homotopic to a meridian is called meridian as well. Note that the meridian elements in $\pi_1(S^3 \setminus K)$ form a conjugacy class.

In general, for a closed subset $S \subset M$ of some manifold M we denote by $N(S)$ some closed tubular neighborhood of S in M. We take the convention that for an open manifold we'll always implicitly take its closure, for example $S^3 \setminus N(K)$ will stand for $\overline{S^3 \setminus N(K)}$.

We'll mostly study invariants of M_K, the result of zero-framed-surgery along K, i.e.

$$M_K = S^3 \setminus N(K) \cup D^2 \times S^1$$

where $\partial(D^2)$ is a longitude of K. After smoothening the corners we can assume that M_K is a smooth manifold. If it is clear with which knot we're dealing we'll drop the index K.

This manifold has the advantage over $S^3 \setminus K$ that it is a closed manifold associated to K. We have $H_1(M_K) \xleftarrow{\cong} H_1(S^3 \setminus K) = \mathbb{Z}$. Denote the image of the meridian μ by μ again, also denote $\pi_1(M_K) \to \mathbb{Z}$ by ϵ as well.

2.2. **Slice knots.** We say that two knots K_1, K_2 are concordant if there exists a smooth submanifold $V \subset S^3 \times [0,1]$ such that $V \cong S^1 \times [0,1]$ and such that $\partial(V) = K_0 \times 0 \cup -K_1 \times 1$. A knot $K \subset S^3$ is called slice if it is concordant to the unknot. Equivalently, a knot is slice if there exists a smooth disk $D \subset D^4$ such that $\partial(D) = K$.

We say that the Seifert pairing on $H_1(F)$ is metabolic if there exists a subspace of half-rank H such that the Seifert pairing vanishes on H. If a knot K has a Seifert matrix of the form

$$A = \begin{pmatrix} 0 & B \\ C & D \end{pmatrix}$$

where $0, B, C, D$ are square matrices, then we say that A is metabolic. It is clear that the Seifert pairing is metabolic if and only if there exists a basis for $H_1(F)$ such that A is metabolic. From the S-equivalence of Seifert matrices it follows that if the pairing on one Seifert surface is metabolic it is also metabolic on any other Seifert surface. If the Seifert pairing of a knot is metabolic then we say that K is algebraically slice.

We need the following classical theorem (cf. [L97, p. 87ff]).

Theorem 2.1. *If K is a slice knot, F a Seifert surface of genus g and $D \subset D^4$ a slice disk, then there exists a two-sided three-manifold $R^3 \subset D^4$ such that $\partial(R) = F \cup_K D$ and $R \cap S^3 = F$.*

We can find a basis $a_1, \dots, a_{2g}$ for $H_1(F) = H_1(F \cup D)$ such that $\langle ma_1, \dots, ma_g \rangle \subset Ker\{H_1(F) \to H_1(R)\} \subset \langle a_1, \dots, a_g \rangle$ for some m, and such that $a_i \cdot a_{g+i} = 1, i = 1, \dots, g$. This means that the rank of $Ker\{H_1(F) \to H_1(R)\}$ is half the rank of $H_1(F)$.

Finally, $\langle a_1, \dots, a_g \rangle$ is a metabolizer for the Seifert pairing.

Definition. Let C be a complex, hermitian matrix, i.e. $C = \bar{C}^t$, then we define the signature sign(C) to be the number of positive eigenvalues of C minus the number of negative ones.

The following is an easy exercise.

Lemma 2.2. *If C is hermitian, non-singular then sign$(PC\bar{P}^t) =$ sign(C) for any P with $\det(P) \neq 0$. If furthermore C is of the form*

$$C = \begin{pmatrix} 0 & B \\ \bar{B}^t & D \end{pmatrix}$$

where B is a square matrix, then sign$(C) = 0$.

In particular, if A is the Seifert matrix of an algebraically slice knot, then $\mathrm{sign}(A + A^t) = 0$. We'll associate more signatures to a knot K in section 3.4.

Most sliceness obstructions make use of the space $N := N_D := D^4 \setminus N(D)$ where D is a slice disk. We summarize a couple of well-known facts about N_D.

Lemma 2.3. *If K is a slice knot and D a slice disk, then*

1. *$\partial(N_D) = M_K$,*
2. *$H_*(N_D) = H_*(S^1)$,*
3. *there exists a map $H_1(N_D) \to \mathbb{Z}$ extending $\epsilon : H_1(M_K) \to \mathbb{Z}$.*

2.3. **Universal abelian cover of M_K and the Blanchfield pairing.** Let K be a knot. Define $X := X_K := S^3 \setminus N(K)$, then $H_1(X_K) \to H_1(M_K)$ is an isomorphism. Denote the infinite cyclic covers of X_K and M_K corresponding to $\epsilon : H_1(X_K) \to H_1(M_K) \to \mathbb{Z}$ by $\tilde{X}$ and $\tilde{M}$. Then $\mathbb{Z} = \langle t \rangle$ acts on $\tilde{X}$ and $\tilde{M}$, therefore $H_1(\tilde{X})$ and $H_1(\tilde{M})$ carry a $\Lambda := \mathbb{Z}[t, t^{-1}]$-module structure. We'll henceforth write $H_1(X, \Lambda)$ for $H_1(\tilde{X})$ and $H_1(M, \Lambda)$ for $H_1(\tilde{M})$. Note that $\tilde{M} = \tilde{X} \cup D^2 \times \mathbf{R}$, in particular $H_1(X, \Lambda) \to H_1(M, \Lambda)$ is an isomorphism.

Our first goal is to understand the Λ-structure of $H_1(M, \Lambda)$ in terms of the Seifert matrices. Let F be a Seifert surface for K and let $I := [-1, 1]/$ Denote by $F \times_K I$ a tubular neighborhood of F in S^3 pinched at $\partial(F) = K$, i.e.

$$F \times_K I \cong F \times I / \sim$$

where $(x, t) \sim (x, 0)$ for all $t \in I, x \in K = \partial(F)$. Now let $Y := S^3 \setminus F \times_K I$. We say Y is the result of slitting S^3 along F.

Claim. The map

$$\begin{array}{rrcl} \lambda_S : & H_1(Y) \times H_1(F) & \to & \mathbb{Z} \\ & (a, b) & \mapsto & \mathrm{lk}(a, b) \end{array}$$

is well-defined and non-singular.

Proof. Consider the following isomorphisms

$$H^1(Y) \cong H^2(S^3, Y) \cong H^2(F \times_{\partial F} I, \partial(F \times_{\partial F} I)) \cong H_1(F \times_{\partial F} I) \cong H_1(F)$$

given by the coboundary map, excision, Lefschetz duality and a homotopy equivalence. Going through the maps one sees that these isomorphisms define the linking pairing, which is hence non-singular. $\square$

We can view $\tilde{X}$ as the disjoint union of $Y \times i, i \in \mathbb{Z}$ glued together along the corresponding copies of $(F \times_K \times \pm 1) \setminus N(K)$. For a basis $a_1, \dots, a_{2g} \in H_1(F)$ denote henceforth by $\alpha_1, \dots, \alpha_{2g} \in H_1(Y)$ the dual basis with respect to λ_S and by $\tilde{\alpha}_1, \dots, \tilde{\alpha}_{2g} \in H_1(Y \times 0) \subset H_1(\tilde{M})$ the lifts of $\alpha_1, \dots, \alpha_{2g} \in H_1(Y)$. Denote the resulting Seifert matrix by A.

Lemma 2.4. *The elements $\tilde{\alpha}_1, \dots, \tilde{\alpha}_{2g} \in H_1(X, \Lambda) = H_1(M, \Lambda)$ generate $H_1(M, \Lambda)$ over Λ and with respect to this generating set $H_1(M, \Lambda) = \Lambda^{2g}/(At - A^t)$. Furthermore multiplication by $t-1$ is an isomorphism of $H_1(M, \Lambda)$.*

Proof. A Mayer-Vietoris sequence shows the first part (cf. [R90, p. 210]), the second part follows from the long exact homology sequence induced by the exact sequence

$$0 \to C_*(\tilde{M}) \xrightarrow{t-1} C_*(\tilde{M}) \to C_*(M) \to 0$$

□

In the following we'll give Λ an involution induced by $\bar{t} = t^{-1}$. Let $S := \{f \in \Lambda | f(1) = 1\}$. The Λ-module $H_1(M, \Lambda)$ is S-torsion since for example the Alexander polynomial $\Delta_K(t)$ lies in S. We recall the definition of the Blanchfield pairing and its main properties (cf. [B57], [K75], [L77]).

Lemma 2.5. *The pairing*

$$\begin{array}{rcl} \lambda_{Bl} : H_1(M_K, \Lambda) \times H_1(M_K, \Lambda) & \to & S^{-1}\Lambda/\Lambda \\ (a, b) & \mapsto & \frac{1}{p(t)} \sum_{i=-\infty}^{\infty} (a \cdot t^i c) t^{-i} \end{array}$$

where $c \in C_2(M_K, \Lambda)$ such that $\partial(c) = p(t)b$ for some $p(t) \in S$, is well-defined. Let F be a Seifert surface for K, and $a_1, \dots, a_{2g} \in H_1(F)$ a basis. With respect to the generating set $\tilde{\alpha}_1, \dots, \tilde{\alpha}_{2g} \in H_1(M_K, \Lambda)$ the pairing $(H_1(M_K, \Lambda), \lambda_{Bl})$ is given by

$$\begin{array}{rcl} \Lambda^{2g}/(At - A^t) \times \Lambda^{2g}/(At - A^t) & \to & S^{-1}\Lambda/\Lambda \\ (v, w) & \mapsto & \bar{v}^t (t-1)(At - A^t)^{-1} w \end{array}$$

It follows that the Blanchfield pairing is non-singular, hermitian, Λ-linear in the second entry and Λ-anti-linear in the first entry.

For any Λ-submodule $P \subset H_1(M_K, \Lambda)$ define

$$P^\perp := \{v \in H_1(M_K, \Lambda) | \lambda_{Bl}(v, w) = 0 \text{ for all } w \in P\}$$

If $P \subset H_1(M, \Lambda)$ is such that $P = P^\perp$ then we say that P is a metabolizer for λ_{Bl} and that λ_{Bl} is metabolic.

Theorem 2.6. *Let $K \subset S^3$ be a knot, F a Seifert surface for K. If there exists a basis $a_1, \dots, a_{2g} \in H_1(F)$ such that $a_1, \dots, a_g$ generates a metabolizer for the Seifert pairing, then $\tilde{\alpha}_{g+1}, \dots, \tilde{\alpha}_{2g} \in H_1(M_K, \Lambda)$ generate a metabolizer for λ_{Bl} over Λ.*

Conversely, if F is a minimal Seifert surface, $P \subset H_1(M_K, \Lambda)$ a metabolizer for the Blanchfield pairing, then there exists a basis $a_1, \dots, a_{2g} \in H_1(F)$ such that $\langle a_1, \dots, a_g \rangle$ is a metabolizer for the Seifert pairing and such that over Λ, P is generated by $\tilde{a}_{g+1}, \dots, \tilde{a}_{2g}$.

In particular the Blanchfield pairing is metabolic if and only if K is algebraically slice.

Proof. Let $R := \mathbb{Z}[t, t^{-1}, (t-1)^{-1}]$. Then $H_1(M, \Lambda)$ is in fact an R-module, since multiplication by $t-1$ is an isomorphism in $H_1(M, \Lambda)$. We get that $(H_1(M, \Lambda), \lambda_{Bl})$ is given by

$$\begin{array}{rcl} R^{2g}/(t-1)^{-1}(At - A^t) \times R^{2g}/(t-1)^{-1}(At - A^t) & \to & S^{-1}R/R \\ (v, w) & \mapsto & \bar{v}^t (t-1)(At - A^t)^{-1} w \end{array}$$

Note that $(t-1)^{-1}(At - A^t)$ is hermitian. The first part of the theorem now follows immediately from proposition C.1.

The converse has been shown by Kearton [K75]. □

We'll now show a more geometrical way of finding a metabolizer for slice knots. In the following assume that K is slice and $D \subset D^4$ a slice disk. By lemma 2.3 there exists a map $H_1(N_D) \to \mathbb{Z}$ extending $\epsilon : H_1(M_K) \to \mathbb{Z}$. If $\tilde{N}_D$ denotes the corresponding $\mathbb{Z}$-fold cover of N_D, then $H_1(\tilde{N}_D)$ has a Λ-module structure. We therefore denote $H_1(\tilde{N}_D)$ by $H_1(N_D, \Lambda)$.

Definition. If A is an R-module, then $T_R A$ denotes the R-torsion submodule of A and $F_R A := A/T_R A$. If $R = \mathbb{Z}$ then we'll write TA for $T_{\mathbb{Z}} A$.

Proposition 2.7. *If K is slice and D any slice disk, then for $P := Ker\{H_1(M_K, \Lambda) \to H_1(N_D, \Lambda)\}$ we get $P \subset P^{\perp}$ and $P^{\perp} = P^{\perp\perp}$.*

Furthermore for $Q := Ker\{H_1(M_K, \Lambda) \to F_{\mathbb{Z}} H_1(N_D, \Lambda)\}$ we get $Q = P^{\perp}$, in particular $Q = Q^{\perp}$.

Proof. There exists a well-defined pairing (cf. [L00])

$$\begin{array}{rrcl} \lambda_{Bl,N} : & T_\Lambda H_2(N, M, \Lambda) \times T_\Lambda H_1(N, \Lambda) & \to & S^{-1}\Lambda/\Lambda \\ & (a, b) & \mapsto & \frac{1}{p(t)} a \cdot c \end{array}$$

where $c \in C_2(N, \Lambda)$ such that $\partial(c) = p(t)b$. In particular for $a \in T_\Lambda H_2(N, M, \Lambda), b \in H_1(M, \Lambda)$ we have

$$\lambda_{Bl}(\partial(a), b) = \lambda_{Bl,N}(a, i_*(b))$$

Claim. $H_2(N, M, \Lambda)$ is Λ-torsion.

Using that $H_1(M, \Lambda)$ is Λ-torsion we get from the long exact sequence that it is enough to show that $H_2(N, \Lambda)$ is Λ-torsion. Consider the following long exact sequence

$$\dots \to H_3(N) \to H_2(N, \Lambda) \xrightarrow{t-1} H_2(N, \Lambda) \to H_2(N) \to H_1(N, \Lambda) \to \dots$$

Using that $H_2(N) = H_3(N) = 0$ we see that $H_2(N,\Lambda) \xrightarrow{t-1} H_2(N,\Lambda)$ is an isomorphism. Since furthermore $H_2(N,\Lambda)$ is finitely generated over Λ it follows that $H_2(N,\Lambda)$ is Λ-torsion (cf. [L77, cor. 1.3]).

Now let $a, b \in P$, then from the exact sequence $T_\Lambda H_2(N,M,\Lambda) \to H_1(M,\Lambda) \to H_1(N,\Lambda)$ it follows that $a = \partial(c)$ for some $c \in T_\Lambda H_2(N,M,\Lambda)$, we get

$$\lambda_{Bl}(a,b) = \lambda_{Bl,N}(c, i_*(b)) = \lambda_{Bl,N}(c,0) = 0$$

This shows that $P \subset P^\perp$. Letsche [L00] shows that furthermore $P^\perp/P$ is $\mathbb{Z}$-torsion and that $P^\perp = P^{\perp\perp}$.

It remains to show that $Q = P^\perp$. Let $q \in Q$, then $mq \in P$ for some m by definition of P and Q. Let $p \in P$, then $m\lambda_{Bl}(q,p) = \lambda_{Bl}(mq,p) = 0 \in S^{-1}\Lambda/\Lambda$. Since $m \notin S$ this implies that in fact $\lambda_{Bl}(q,p) = 0 \in S^{-1}\Lambda/\Lambda$ for all $q \in Q, p \in P$. This shows that $Q \subset P^\perp$. Since $P^\perp/P$ is torsion we also get $P^\perp \subset Q$ by definition of P and Q, hence $Q = P^\perp$. □

2.4. **Finite cyclic covers and linking pairings.** Let K be a knot, k some number. Denote by $M_k = M_{K,k}$ the k-fold cover of M_K corresponding to $\pi_1(M_K) \xrightarrow{\epsilon} \mathbb{Z} \to \mathbb{Z}/k$, similarly define X_k, and denote by $L_k = L_{K,k}$ the k-fold cover of S^3 branched along $K \subset S^3$. Denote by μ some meridian of K, then

$$\begin{array}{rcll} M_k &=& X_k \cup D^2 \times S^1 & \text{where } S^1 = k\mu \\ L_k &=& X_k \cup S^1 \times D^2 & \text{where } \partial(D^2) = k\mu \end{array}$$

Lemma 2.8. *There exist natural isomorphisms*

$$\begin{array}{rcl} H_1(M_k) &=& H_1(X_k) \\ H_1(M_k) &=& H_1(L_k) \oplus \mathbb{Z} \\ H_1(M_k) &=& H_1(M,\Lambda)/(t^k-1) \oplus \mathbb{Z} \\ H_1(L_k) &=& H_1(M,\Lambda)/(t^k-1) \end{array}$$

We'll henceforth identify these groups.

Proof. The first two statements are clear, the $\mathbb{Z}$-part in the second statement is the lift of $k\mu$. From the long exact homology sequence corresponding to

$$0 \to C_*(\tilde{M}) \xrightarrow{\cdot(t^k-1)} C_*(\tilde{M}) \to C_*(M_k) \to 0$$

we get $H_1(M_k) = \mathbb{Z} \oplus H_1(M,\Lambda)/(t^k-1)$, combining this with $H_1(M_k) = \mathbb{Z} \oplus H_1(L_k)$ and identifying the $\mathbb{Z}$-parts as generated in both cases by the lift of $k\mu$ we get the last statement. □

Let F be a Seifert surface for K. Then we can 'build' X_k from the disjoint union of $Y \times i, i = 0, \dots, k-1$ by identifying appropiate copies of $(F \times_K \times \pm 1) \setminus N(K)$, similarly to the way we constructed $\tilde{X}$.

If $a_1, \dots, a_{2g} \in H_1(F)$ is a basis, denote by $\alpha_1, \dots, \alpha_{2g} \in H_1(Y)$ the dual basis, lift them to $\tilde\alpha_1, \dots, \tilde\alpha_{2g} \in H_1(Y \times 0) \subset H_1(X_k)$. These elements and $k\mu$ generate $H_1(X_k) = H_1(M_k)$ as a $\Lambda_k := \Lambda/(t^k - 1)$-module. Denote by A the Seifert matrix of K with respect to the basis $\{a_1, \dots, a_{2g}\}$, then $H_1(M_k) = \mathbb{Z} \oplus \Lambda_k^{2g}/(At - A^t)$.

Proposition 2.9. (1) *The elements $\tilde\alpha_1, \dots, \tilde\alpha_{2g} \in H_1(Y \times 0) \subset H_1(X_k)$ generate $H_1(L_k)$ in fact as a $\mathbb{Z}$-module, with respect to this generating set $H_1(L_k) \cong \mathbb{Z}^{2g}/(\Gamma_k^t \mathbb{Z}^{2h})$ where $\Gamma_k := \Gamma^k - (\Gamma - I)^k$ and $\Gamma := A(A^t - A)^{-1}$.*

(2)
$$|H_1(L_k)| = |\prod_{j=1}^{k} \Delta_K(e^{2\pi i j/k})|$$
where 0 on the right hand side means that $H_1(L_k)$ is infinite.

(3) *$H_1(L_k)$ is torsion if and only if no k^{th} root of unity is a zero of $\Delta_K(t)$.*

(4) *No root of unity of prime power order can be a zero of the Alexander polynomial $\Delta_K(t)$ of a knot K, therefore $H_1(L_k)$ is torsion if k is a prime power.*

Proof. (1) cf. Rolfsen [R90, p. 215],

(2) cf. Gordon [G77, p. 17],

(3) this follows immediately from the above,

(4) the minimal polynomial $\Phi_n(t)$ of an n^{th} root of unity where $n = p^r$ for some prime number p has the property that $\Phi_n(1) = p$ (cf. lemma A.2), but $\Delta_K(1) = 1$.

□

Lemma 2.10. *Let k be any integer such that $H_1(L_k)$ is finite, then the map*
$$\begin{array}{rcl} \lambda_{L_k} = \lambda_L : TH_1(M_k) \times TH_1(M_k) & \to & \mathbb{Q}/\mathbb{Z} \\ (a,b) & \mapsto & \frac{1}{n} a \cdot c \mod \mathbb{Z} \end{array}$$
where $c \in C_2(M_k)$ such that $\partial(c) = na$, defines a symmetric, non-singular pairing.

The pairing of the lemma is called the linking pairing of $H_1(L_k)$. We give a description of λ_L in terms of a Seifert matrix A. We won't need this later, but there doesn't seem to be a description like this in the literature. We will give only a quick outline of the proof.

Definition. For a Seifert matrix A corresponding to a Seifert surface F define
$$E(A,k) := E(k) := \begin{pmatrix} A + A^t & -A^t & 0 & \dots & 0 & 0 \\ -A & A + A^t & -A^t & 0 & \dots & 0 \\ 0 & -A & A + A^t & -A^t & 0 & \vdots \\ 0 & 0 & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & -A & A + A^t & -A^t \\ 0 & 0 & \dots & 0 & -A & A + A^t \end{pmatrix}$$

where the matrix consists of $k-1$ block rows and block columns.

Proposition 2.11. (1) *$E(k)^t$ is a presentation matrix for $H_1(L_k)$, i.e. $H_1(L_k) \cong \mathbb{Z}^d/E(k)\mathbb{Z}^d$.*

(2) *Let k be such that $H_1(L_k)$ is finite, then the linking pairing on L_k is given by*

$$\begin{array}{rcl} \lambda_{L_k} = \lambda_L : \mathbb{Z}^d/E(k) \times \mathbb{Z}^d/E(k) & \to & \mathbb{Q}/\mathbb{Z} \\ (a,b) & \mapsto & a^t E(k)^{-1} b \end{array}$$

Proof. Let W_k be the k-fold covering of D^4 branched along a push in of the Seifert surface F. Then $\partial(W_k) = L_k$ and $E(k)$ presents the intersection pairing of W_k ([K87, p. 283]). Consider

$$\begin{array}{ccccccc} H_2(W_k) & \xrightarrow{j_*} & H_2(W_k, L_k) & \xrightarrow{\partial} & H_1(L_k) & \to & 0 \\ & & \| \wr & & & & \\ & & H^2(W_k) & & & & \\ & & \| \wr & & & & \\ & & \mathrm{Hom}_{\mathbb{Z}}(H_2(W_k), \mathbb{Z}) & & & & \end{array}$$

where the diagonal map $H_2(W_k) \to \mathrm{Hom}_{\mathbb{Z}}(H_2(W_k), \mathbb{Z})$ is just the intersection pairing. Picking corresponding bases we get the first claim. The intersection pairing on W_k can be used to compute the linking pairing on L_k, which then shows part two. $\square$

For any $\mathbb{Z}$-submodule $P \subset TH_1(M_k)$ define

$$P^\perp := \{v \in TH_1(M_k) | \lambda_L(v,w) = 0 \text{ for all } w \in P\}$$

If $P = P^\perp$ then we say that P is a metabolizer for λ_L and say that λ_L is metabolic, if furthermore P is a Λ-module we say that P is a Λ-metabolizer for the linking pairing.

The following proposition is an immediate consequence of theorem 2.1 and propositions 2.11, C.1.

Proposition 2.12. *If K is algebraically slice, k such that $H_1(L_k)$ is finite, then $\lambda_{L,k}$ is metabolic.*

When we considered the Blanchfield pairing we showed that if K is slice, D a slice disk, then a metabolizer is in fact given by $P := \mathrm{Ker}\{H_1(M_K, \Lambda) \to H_1(N_D, \Lambda)\}$. It is a natural question to ask whether $Q_k := \mathrm{Ker}\{TH_1(M_k) \to TH_1(N_k)\}$ is a metabolizer for the linking pairing. If k is a prime power this is true, as the following section shows.

2.5. Homology of prime-power covers and the linking pairing.

Lemma 2.13. *Let Y be a topological space such that $H_*(Y) = H_*(S^1)$ and let $k = p^s$ where p is a prime number. Then $H_*(Y_k) = H_*(Y) \oplus$ torsion. Furthermore $gcd(|TH_1(Y_k)|, p) = 1$.*

The following proof is modelled after [CG86, p. 184].

Proof. For any $n \in \mathbb{N} \cup \{\infty\}$ we can give $H_*(Y_n)$ a Λ-structure. Using $H_2(Y, \mathbb{Z}/p) = 0$ we get the following exact sequence

$$0 \to H_1(Y_\infty, \mathbb{Z}/p) \xrightarrow{t-1} H_1(Y_\infty, \mathbb{Z}/p) \to H_1(Y, \mathbb{Z}/p) \to H_0(Y_\infty, \mathbb{Z}/p) \to 0$$

Since $H_1(Y, \mathbb{Z}/p) \to H_0(Y_\infty, \mathbb{Z}/p)$ is an isomorphism it follows from the sequence that the map $H_1(Y_\infty, \mathbb{Z}/p) \xrightarrow{t-1} H_1(Y_\infty, \mathbb{Z}/p)$ is an isomorphism. Since $H_i(Y) = 0$ for $i > 1$ we also get that $H_i(Y_\infty, \mathbb{Z}/p) \xrightarrow{t-1} H_i(Y_\infty, \mathbb{Z}/p)$ is an isomorphism for $i > 1$. With $\mathbb{Z}/p$–coefficients we get $(t^k - 1) = (t^{p^s} - 1) = (t-1)^{p^s}$, hence multiplication by $(t^k - 1)$ is an automorphism of $H_1(Y_\infty, \mathbb{Z}/p)$ as well. Consider the long exact sequence

$$\dots \to H_1(Y_\infty, \mathbb{Z}/p) \xrightarrow{t^k-1} H_1(Y_\infty, \mathbb{Z}/p) \to H_1(Y_k, \mathbb{Z}/p) \to H_0(Y_\infty, \mathbb{Z}/p) \to 0$$

It follows that $H_1(Y_k, \mathbb{Z}/p) = H_0(Y_\infty, \mathbb{Z}/p) = \mathbb{Z}/p$. Similarly we can show that $H_i(Y_k, \mathbb{Z}/p) = 0$ for $i > 1$. The lemma now follows from the universal coefficient theorem. □

Corollary 2.14. *If k is a prime power, then*

$$\begin{aligned} H_1(M_k) &= \mathbb{Z} \oplus TH_1(M_k) \\ TH_1(M_k) &= H_1(L_k) = H_1(M, \Lambda)/(t^k - 1) \\ gcd(|TH_1(M_k)|, p) &= 1 \end{aligned}$$

Proof. This follows from lemma 2.8 and applying the above lemma to $S^3 \setminus K$ and using $H_1(M_k) = H_1(X_k)$. □

Proposition 2.15. *Let $K \subset S^3$ be a slice knot with slice disk D, k a prime power. Denote the k-fold cover of N_D by N_k. Then $Q := Ker\{TH_1(M_k) \to TH_1(N_k)\}$ is a Λ–metabolizer for λ_L. Furthermore $|Q|^2 = |TH_1(M_k)|$.*

Proof. [CG86] It is clear that Q is a Λ–submodule since $TH_1(M_k) \to TH_1(N_k)$ is a Λ–homomorphism. We'll first show that $Q \subset Q^\perp$, the proof is very similar to the proof of the corresponding claim in proposition 2.7.

From theorem 4.7 and lemma 2.13 we get $H_1(N_k) = \mathbb{Z} \oplus TH_1(N_k)$ and the map $H_1(M_k) \to H_1(N_k)$ sends $TH_1(M_k) \to TH_1(N_k)$, we therefore get an exact sequence

$$\dots \to H_2(N_k) \to H_2(N_k, M_k) \to TH_1(M_k) \to TH_1(N_k) \to H_1(N_k, M_k) \to \dots$$

From lemma 2.13 it follows that $H_2(N_k)$ is torsion, hence $H_2(N_k, M_k)$ is torsion.

There exists a well-defined pairing

$$\begin{aligned} \lambda_{L,N_k} : \ TH_2(N_k, M_k) \times TH_1(N_k) &\to \mathbb{Q}/\mathbb{Z} \\ (a, b) &\mapsto \tfrac{1}{n} a \cdot c \end{aligned}$$

where $c \in C_2(N_k)$ such that $\partial(c) = nb$. In particular for $a \in TH_2(N_k, M_k), b \in TH_1(M_k)$ we have

$$\lambda_{L,N_k}(a, i_*(b)) = \lambda_L(\partial(a), b)$$

Now let $a, b \in Q$, then from the exact sequence

$$H_2(N_k, M_k) = TH_2(N_k, M_k) \to TH_1(M_k) \to TH_1(N_k)$$

it follows that $a = \partial(c)$ for some $c \in TH_2(N_k, M_k)$, we get

$$\lambda_L(a, b) = \lambda_{L,N_k}(c, i_*(b)) = \lambda_{L,N_k}(c, 0) = 0$$

This shows that $Q \subset Q^\perp$.

Denote by W_k the k-fold cover of D^4 branched along D. Then $H_i(W_k) = TH_i(N_k)$ and $H_i(L_k) = TH_i(M_k)$ for $i \geq 1$ furthermore $Q = \mathrm{Ker}\{H_1(L_k) \to H_1(W_k)\}$. The long exact sequence $\dots \to H_i(L_k) \to H_i(W_k) \to H_i(W_k, L_k) \to \dots$ and Lefschetz-duality shows that $|Q|^2 = |TH_1(M_k)|$ (cf. [CG86]) furthermore $|Q||Q^\perp| = |TH_1(M_k)|$ (cf. lemma A.3), it follows that $Q = Q^\perp$. $\square$

Remark. (1) Using theorem 4.12 one can easily produce examples of knots where the linking pairings for all prime powers are metabolic but which are not algebraically slice.

(2) Comparing propositions 2.12 and 2.15 we see that proposition 2.15 gives a weaker sliceness–obstruction, but we'll need the more geometric way of finding a metabolizer later.

2.6. **Relating the Blanchfield pairing to linking pairings via intermediate Blanchfield type pairings.** Let K be a knot and k any number such that $H_1(L_k) = H_1(M, \Lambda)/(t^k - 1)$ is finite. Note that this implies $TH_1(M_k) = H_1(M, \Lambda)/(t^k - 1)$. By proposition 2.9 this is the case if and only if $\Delta_K(w) \neq 0$ for all k^{th} roots of unity w.

The goal is to show that if P is a metaolizer for the Blanchfield pairing of K then the projections of P are metabolizers for linking pairings $\lambda_L : TH_1(M_k) \times TH_1(M_k) \to \mathbb{Q}/\mathbb{Z}$ if $H_1(L_k)$ is finite. Along the way we'll introduce an 'intermediate Blanchfield pairing' which we'll later also use to prove that the prime power Letsche obstruction is equivalent to the weak–ribbon–eta–obstruction (cf. theorem 8.7). Let $\Lambda_k := \Lambda/(t^k - 1)$ and

$$S_k \quad := \quad \{f(t) \in \Lambda_k | f(t) \text{ is not a zero-divisor}\}$$

this set is closed under multiplication. Denote by π_k the natural projections $\Lambda \to \Lambda/(t^k - 1)$ and $H_1(M, \Lambda) \to H_1(M, \Lambda)/(t^k - 1) = TH_1(M_k)$. The Λ-module $H_1(M, \Lambda)$ is S-torsion, for example $\Delta_K(t) = \det(At - A^t)$ is in the annihilator. We assumed that $H_1(L_k)$ is finite, by proposition 2.9 this means in particular that $\gcd(\Delta_K(t), t^k - 1) = 1$, i.e. $\Delta_K(t) \in S_k$, hence the Λ_k-module $TH_1(M_k) = H_1(M, \Lambda)/(t^k - 1)$ is S_k-torsion. We therefore get a Blanchfield pairing

$$\begin{array}{rcl} \lambda_{Bl,k} : TH_1(M_k) \times TH_1(M_k) & \to & S_k^{-1}\Lambda_k/\Lambda_k \\ (a, b) & \mapsto & \frac{1}{p(t)} \sum_{j=0}^{k-1} (a \cdot t^j c) t^{-j} \end{array}$$

where $c \in C_2(M_k)$ such that $\partial(c) = p(t)b$ for some $p(t) \in S_k$. This pairing is non-singular and hermitian over Λ_k.

Lemma 2.16. *Let $a, b \in TH_1(M_k)$ then*

$$\lambda_{Bl,k}(a,b) \quad = \quad \textstyle\sum_{i=0}^{k-1} \lambda_L(a, bt^i)t^{-i}$$

Proof. Let $n = |TH_1(M_k)|$, then $nb = \partial(c)$ for some $c \in C_2(M_k)$. Note that also $n(t^j b) = \partial(t^j c)$ for any j. Since $n \in S_k$ we get

$$\begin{array}{rcll} \lambda_{Bl,k}(a,b) & = & \frac{1}{n}\sum_{i=0}^{k-1}(a \cdot t^i c)t^{-i} & \mod \Lambda_k \\ \lambda_L(a, bt^i) & = & \frac{1}{n} a \cdot (t^i c) & \mod \mathbb{Z} \end{array}$$

□

Note that for a non prime power k we won't get $\pi_k(S) \subset S_k$, hence we don't get a projection map $\pi_k : S^{-1}\Lambda/\Lambda \to S_k^{-1}\Lambda_k/\Lambda_k$. But, if K is a knot such that $H_1(L_k)$ is finite, then $\pi_k(\Delta_K(t)) \in S_k$, in particular for such a K and $a, b \in H_1(M, \Lambda)$, $\pi_k(\lambda_{Bl}(a,b)) \in S_k^{-1}\Lambda_k/\Lambda_k$ is defined.

Lemma 2.17. *Let $a, b \in TH_1(M_k)$ with lifts $\tilde{a}, \tilde{b} \in H_1(M, \Lambda)$, then*

$$\pi_k(\lambda_{Bl}(\tilde{a}, \tilde{b})) \quad = \quad \lambda_{Bl,k}(a,b)$$

Proof. Take $\tilde{c} \in C_2(M, \Lambda)$ such that $\partial(\tilde{c}) = \Delta_K(t)\tilde{b}$, then the statement follows immediately from naturality. □

Proposition 2.18. *If $P \subset H_1(M_K, \Lambda)$ is a metabolizer for $\lambda_{Bl} : H_1(M_K, \Lambda) \times H_1(M_K, \Lambda) \to S^{-1}\Lambda/\Lambda$, then for any k such that $H_1(L_k)$ is finite, $P_k := \pi_k(P) \subset TH_1(M_k)$ is a metabolizer for $\lambda_L : TH_1(M_k) \times TH_1(M_k) \to \mathbb{Q}/\mathbb{Z}$. In particular $|P_k|^2 = |TH_1(M_k)|$.*

Proof. Let $P \subset H_1(M, \Lambda)$ be a metabolizer for λ_{Bl}, i.e. $P = P^\perp$ with respect to λ_{Bl}. Let $P_k := \pi_k(P) \subset TH_1(M_k)$.

We'll first show that $P_k = P_k^\perp$ with respect to $\lambda_{Bl,k}$. Let $Q_{Bl,k} := P_k^\perp \subset TH_1(M_k)$ with respect to $\lambda_{Bl,k}$. From lemma 2.16 we get $\lambda_{Bl,k}(P_k, P_k) = 0$, hence $P_k \subset Q_{Bl,k}$. Let $a_k \in Q_{Bl,k}$. Lift a_k to an element a in $H_1(M, \Lambda)$, then

$$\pi_k(\lambda_{Bl}(a,b)) = \lambda_{Bl,k}(a_k, \pi_k(b)) = 0 \mod \Lambda_k$$

for all $b \in P$. Let $b_1, \dots, b_r$ be Λ-generators for P. We can write $\lambda_{Bl}(a, b_i) = f_i/g_i \mod \Lambda$ such that $\pi_k(g_i) = 1 \in \Lambda_k$. Let $\tilde{a} := a\prod_{j=1}^r g_i$. Then $\lambda_{Bl}(\tilde{a}, b_j) \in \Lambda$ for all $j = 1, \dots, r$, hence $\tilde{a} \in P^\perp = P$, furthermore

$$\pi_k(\tilde{a}) = \pi_k(a)\prod_{j=1}^r \pi_k(g_i) = \pi_k(a) = a_k$$

This shows that $a_k \in P_k$, hence $Q_{Bl,k} \subset P_k$, hence $P_k = P_k^\perp$ with respect to $\lambda_{Bl,k}$.

Now we show that $P_k = P_k^\perp$ with respect to λ_L. Let $Q_L := P_k^\perp$ with respect to λ_L.

Claim. Let $a, b \in TH_1(M_k)$ then

$$\lambda_{Bl,k}(a,b) = 0 \Rightarrow \lambda_L(a,b) = 0 \in \mathbb{Q}/\mathbb{Z}$$

By lemma 2.16 we have $\lambda_{Bl,k}(a,b) = \frac{1}{n}\sum_{j=0}^{k-1} \lambda_L(a, bt^j)t^{-j}$. Since $\{1, t^{-1}, \ldots, t^{-k+1}\}$ is a $\mathbb{Z}$-basis for Λ_k this implies that $\lambda_L(a, bt^j) = 0$ for all $j = 0, \ldots, k-1$, in particular $\lambda_L(a,b) = 0$.

It follows that $\lambda_L(P_k \times P_k) = 0$, therefore $P_k \subset Q_L$. Now let $a \in Q_L$. Then $\lambda_L(a,b) = 0 \mod \mathbb{Z}$ for all $b \in P_k$. Since P_k is a Λ-module we also get $\lambda_L(a, t^j b) = 0$ for all j and all $b \in P_k$. Let $b \in P_k$, then $nb = \partial(c)$ for some $c \in C_2(M_k), n \in \mathbb{Z}$. Since we also get $\partial(t^j b) = n(t^j c)$ it follows that

$$\lambda_{Bl,k}(a,b) = \frac{1}{n}\sum_{j=0}^{k-1}(a \cdot t^j c)t^{-j} = \sum_{j=0}^{k-1} \lambda_L(a, t^j b)t^{-j} = 0 \mod \Lambda_k$$

This shows that $a \in Q_{Bl,k} = P_k$. Therefore $Q_L \subset P_k$.

The last statement follows from lemma A.3. $\square$

3. Introduction to eta-invariants and first application to knots

3.1. **Eta invariants.** Let M^3 be a closed manifold with Riemannian structure g and $\alpha : \pi_1(M) \to U(k)$ a representation. Atiyah, Patodi, Singer [APS75] associated to (M, α) a number $\eta(M, \alpha, g)$.

Denote the trivial representation $\pi_1(M) \to U(1)$ by 1. The number $\eta_\alpha(M) := \eta(\alpha, M) := \eta(M, \alpha, g) - k\eta(M, 1, g)$ is called the (reduced) eta invariant of (M, α). Atiyah-Patodi-Singer [APS75, p. 406] showed that this number is independent of the choice of g.

To state the main theorem concerning eta invariants we need the following definition.

Definition. Let N^4 be a smooth manifold, possibly with boundary. Denote by $\text{sign}(N)$ the signature of the intersection pairing $H_2(N) \times H_2(N) \to \mathbb{Z}$. Now let $\beta : \pi_1(N) \to U(k)$ be a representation. Denote the universal covering of N by $\tilde{N}$. Then $C_*(\tilde{N})$ has a canonical right $\mathbb{Z}\pi_1(N)$-structure and $\mathbb{C}^k$ has a canonical left $\mathbb{Z}\pi_1(N)$-structure. We can define

$$H_*^\beta(N, \mathbb{C}^k) := H_*(C_*(\tilde{N}) \otimes_{\mathbb{Z}\pi_1(N)} \mathbb{C}^k)$$

We can tensor the equivariant pairing $\lambda_I : C_2(\tilde{N}) \times C_2(\tilde{N}) \to \mathbb{Z}\pi_1(N)$ by β to get a hermitian pairing

$$\begin{array}{rcl} H_2^\beta(N, \mathbb{C}^k) \times H_2^\beta(N, \mathbb{C}^k) & \to & \mathbb{C} \\ (a_1 \otimes v_1, a_2 \otimes v_2) & \mapsto & \bar{v}_1^t \alpha(\lambda_I(a_1, a_2)) v_2 \end{array}$$

Denote its signature by $\text{sign}_\beta(N)$, it is called the twisted signature of N, twisted by β.

The main theorem to compute the eta invariant is the following (cf. [APS75]).

Theorem 3.1. *(Atiyah-Patodi-Singer index theorem) Let (M^3, α) as above. If there exists $(W^4, \beta : \pi_1(W) \to U(k))$ with $\partial(W, \beta) = n(M^3, \alpha)$ for some $n \in \mathbb{N}$, then*

$$\eta_\alpha(M) = \frac{1}{n}(sign_\beta(W) - k sign(W))$$

3.2. **Application of eta invariants to knots.** Let K be a knot, we'll study the eta invariants associated to the closed manifold M_K. Eta invariants in the context of knot theory were first studied by Levine [L94], he used the eta invariant to find links which are not concordant to boundary links. Letsche [L00] showed how to use eta invariants to define ribbon–obstructions for classical knots (cf. section 8.2).

Definition. For a group G the derived series is defined by $G^{(0)} := G$ and inductively $G^{(i+1)} := [G^{(i)}, G^{(i)}]$.

The groups $\pi_1(S^3 \setminus K)$ and $\pi_1(M_K)$ are related as follows. The inclusion map $S^3 \setminus K \to M_K$ defines a homomorphism $\pi_1(S^3 \setminus K) \to \pi_1(M_K)$, the kernel is generated by the longitude of K which lies in $\pi_1(S^3 \setminus K)^{(2)}$. In particular $\pi_1(S^3 \setminus K)/\pi_1(S^3 \setminus K)^{(i)} \to \pi_1(M_K)/\pi_1(M_K)^{(i)}$ is an isomorphism for $i = 0, 1, 2$.

We first compute the eta invariant for the trivial knot (cf. [L00, p. 312]).

Lemma 3.2. *Let M_O be the zero-framed surgery on the trivial knot. Then for any $\alpha : \pi_1(M_O) \to U(k)$ we get $\eta_\alpha(M_O) = 0$.*

Proof. Let $\alpha : \pi_1(M_O) \to U(k)$ be a representation. It is a well known fact that $M_O = S^1 \times S^2$. Let $N = S^1 \times D^3$, then we can find $\beta : \pi_1(N) \to \pi_1(S^1) \to U(k)$ such that $\partial(N, \beta) = (M_O, \alpha)$. Since N is homotopy equivalent to S^1 we get $H_2(N) = H_2^\beta(N, \mathbb{C}^k) = 0$, hence the untwisted and twisted signatures vanish, hence $\eta_\alpha(M_O) = 0$. $\square$

Lemma 3.3. *Let K be a slice knot, then doing zero-framed surgery on the concordance from K to the unknot we get a manifold W^4 with the following properties:*

1. *$\partial(W) = M_K \cup -M_O$,*
2. *$H_*(W, M_K) = H_*(W, M_O) = 0$,*
3. *there exists a map $H_1(W) \to \mathbb{Z}$, extending the maps ϵ on $H_1(M_K)$ and $H_1(M_O)$*

The idea is to study under what circumstances the eta invariants of M_K and M_O are the same for a slice knot K, i.e. which eta invariants vanish for slice knots.

3.3. **Concordance invariance of eta invariants.** We quote some definitions, initially introduced by Levine [L94]. Let G be a group, then a G-manifold is a pair (M, α) where M is a compact oriented 3-manifold with components $\{M_i\}$ and α is a collection of homomorphisms $\alpha_i : \pi_1(M_i) \to G$ where each α_i is defined up to inner automorphism.

We call two G-manifolds $(M_j, \alpha_j), j = 1, 2$, homology G-bordant if there exists a G-manifold (N, β) such that $\partial(N) = M_1 \cup -M_2, H_*(N, M_j) = 0$ for $j = 1, 2$ and, up to inner automorphisms of G, $\beta|\pi_1(M_j) = \alpha_j$.

Let $\hat{R}_k(G) := \{\theta : G \to U(k)\}$, If $X_{i\in I}$ is a generating set, then $R_k(G)$ can be embedded in $U(k)^I$, we'll give $R_k(G)$ the ensuing topology. If G is finitely generated, then $\hat{R}_k(G)$ is a real algebraic variety (cf. [L94, p. 83]). Let $R_k(G)$ be the set of all conjugacy classes of $U(k)$-representations of G. Define

$$\begin{array}{rcl} \rho(M, \alpha) : \hat{R}_k(G) & \to & \mathbf{R} \\ \theta & \mapsto & \eta_{\theta\circ\alpha}(M) \end{array}$$

This function factors through $R_k(G)$. It is in general not continuous, but there exists a subvariety $\Sigma \in \hat{R}_k(G)$ of codimension at least 1, such that $\rho(M, \alpha)$ is continuous on

$\hat{R}_k(G) \setminus \Sigma$ (cf. [L94, p. 92]). Furthermore all 'jumps' are integer-valued in the sense that $\rho(M, \alpha) : \hat{R}_k(G) \to \mathbf{R}/\mathbb{Z}$ is continuous.

The goal is to compare $\rho(M_1, \alpha_1), \rho(M_2, \alpha_2)$ for G-homology bordant manifolds, they turn out to be the same outside of a 'special' variety of $\hat{R}_k(G)$.

Definition. A finitely presented $\mathbb{Z}G$-module A is called perfect if $\mathbb{Z} \otimes_{\mathbb{Z}G} A = 0$ where $\mathbb{Z}$ is considered as a $\mathbb{Z}G$-module via the augmentation map $\epsilon : \mathbb{Z}G \to \mathbb{Z}$ defined by $\epsilon(g) = 1$ for $g \in G$. A special subvariety is defined to be a subset of $\hat{R}_k(G)$ of the form

$$\Sigma_A := \{\theta \in \hat{R}_k(G) | \mathbb{C}^k \otimes_{\mathbb{C}G} A_{\mathbb{C}} \neq 0\}$$

where $A_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{Z}} A$ for some perfect $\mathbb{Z}G$-module A (cf. [L94, p. 94]).

Remark. One can define perfect modules and special subvarieties in terms of presentation matrices. Let A be a finitely presented $\mathbb{Z}G$-module and R_A a presentation matrix of size $m \times n$, i.e. the following sequence is exact:

$$(\mathbb{Z}G)^m \xrightarrow{R_A} (\mathbb{Z}G)^n \to A \to 0$$

Then A is perfect if some integral linear combination of the $(n \times n)$-minors of $\epsilon(R_A)$ equals 1. Furthermore $\theta \in \Sigma_A$ if all $(nk \times nk)$-minors of $\theta(R_A)$ are zero. This interpretation shows that Σ_A is in fact a subvariety of $\hat{R}_k(G)$.

We quote the following proposition of Levine [L94, corollary 3.3].

Proposition 3.4. *For any homology G-bordant manifolds $(M_i, \alpha_i), i = 1, 2$ there exists a special subvariety Σ such that $\rho(M_1, \alpha_1) = \rho(M_2, \alpha_2)$ on $\hat{R}_k(G) \setminus \Sigma$.*

We give an outline of the proof.

Proof. Let (N^4, β) be a homology G-bordism of (M_1, α_1) and (M_2, α_2). Then the intersection pairing

$$H_2(N, \mathbb{Z}) \times H_2(N, \mathbb{Z}) \to \mathbb{Z}$$

is identically zero, since $H_2(N, M_1, \mathbb{Z}) = H_2(N, M_2, \mathbb{Z}) = 0$ and the intersection pairing factors through

$$H_2(N, M_1, \mathbb{Z}) \times H_2(N, M_2, \mathbb{Z}) \to \mathbb{Z}$$

Hence $\mathrm{sign}(N) = 0$. One can show that there exists a special variety Σ such that for all $\theta \in \hat{R}_k(G) \setminus \Sigma$ we get $H_2^{\theta\circ\beta}(N, M_i, \mathbb{C}^k) = 0$ for $i = 1, 2$ and hence $\mathrm{sign}_{\theta\circ\beta}(N) = 0$. $\square$

The main difficulty is to find representations which avoid Σ. A first result is the following proposition by Levine [L94, p. 95].

Proposition 3.5. *A special subvariety contains no point of $\hat{R}_k(G)$ which factors through a group of prime power order.*

3.4. $U(1)$**-representations.** We can now prove the following proposition.

Proposition 3.6. *Let K be a slice knot, μ a meridian and $\alpha : \pi_1(M_K) \xrightarrow{\epsilon} \mathbb{Z} \to U(1) = S^1$ a representation. If $\alpha(\mu)$ is transcendental or if $\alpha(\mu)$ is a prime power root of unity, then $\eta_\alpha(M_K) = 0$.*

Proof. Denote by (N, ϵ) the homology $\mathbb{Z}$-bordism between (M_K, ϵ) and (M_O, ϵ) which exists by lemma 3.3. Denote $\mathbb{Z} \to U(1)$ by θ. If $\theta(1)$ is transcendental, then $\theta(\mathbb{Z}) \subset S^1$ contains no algebraic point, in particular θ is disjoint to all (special) subvarieties of $R_1(S^1) = \mathbb{Z}$. The same is true if $\theta(1)$ is a prime power root of unity, since in this case θ factors through a group of prime power order and we can use proposition 3.5. In both cases we therefore get

$$\eta_\alpha(M_K) = \rho(M_K, \epsilon)(\theta) = \rho(M_O, \epsilon)(\theta) = \eta_{\theta\circ\epsilon}(M_O) = 0$$

by proposition 3.4. □

Let K be a knot, A a Seifert matrix, then we define the signature function $\sigma(K) : S^1 \to \mathbb{Z}$ of K as follows (cf. [L69, p. 242])

$$\sigma_z(K) := \sigma_z(A) := \operatorname{sign}(A(1-z) + A^t(1-\bar{z}))$$

It is easy to see that this is independent of the choice of A.

Proposition 3.7. *Let K be a knot, μ a meridian and let $\alpha : \pi_1(M_K) \to U(1)$. If $z := \alpha(\mu)$, then*

(1)
$$\eta_\alpha(M_K) = \sigma_z(K)$$
In particular if α is trivial, then $\eta_\alpha(M_K) = \sigma_1(A) = 0$.

(2) *The function $\sigma_z(K)$ is locally constant outside of the set of zeros of the Alexander polynomial of K, in particular $\sigma_z(K)$ is continuous at $z = 1$.*

(3) *If K is algebraically slice and $z \in S^1$ such that $\Delta_K(z) \neq 0$, then $\sigma_z(K) = 0$. In particular if z is a prime power root of unity, then $\sigma_z(K) = 0$.*

Note that (3) is a strengthening of proposition 3.6.

Proof. (1) cf. [L84].

(2) The function $z \mapsto \sigma_z(A)$ is continuous outside of the set of z's for which $\sigma_z(A)$ is singular, an easy argument shows that these z's are precisely the zeros of the Alexander polynomial.

(3) If $z \in S^1$ such that $\Delta_K(z) \neq 0$ then $A(1-z)+A^t(1-\bar{z})$ is non–degenerate and the first part follows from lemma 2.2. The second part follows from lemma A.2 and the well-known fact that $\Delta_K(1) = 1$.

□

Remark. If K is a slice knot, then not necessarily $\sigma_z(K) = 0$ for all $z \in S^1$. In fact, let K be a slice knot with Seifert matrix

$$A := \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

which exists by proposition 10.9, then $e^{2\pi i/6}$ is a root of $\Delta_K(t)$ and $\sigma_{e^{2\pi i/6}}(K) = -1$. In fact, this signature turns out to be an obstruction to the knot being doubly slice (cf. section 7).

Following Levine [L69] we denote by C_n the cobordism group of knots $S^{2n-1} \subset S^{2n+1}$ and by $G_\epsilon, \epsilon = \pm 1$ the cobordism group of integral matrices A with $\det(A + \epsilon A^t) = 1$. Levine [L69] showed that associating to a knot a Seifert matrix gives a well-defined map $C_n \to G_{(-1)^n}$ which is an isomorphism for $n \geq 3$ and an isomorphism on a subgroup of index 2 for $n = 2$. Put differently, in higher odd dimensions a knot is slice if and only if it is algebraically slice.

Combining results of Matumuto [M77] and Levine [L69b], [L89] we get the following proposition.

Proposition 3.8. *A knot $K \subset S^{2n+1}$ maps to a torsion element in $G_{(-1)^n}$ if and only if there exists a dense subset $Z \in S^1$ such that for all $\alpha : \pi_1(M) \to S^1$ with $\alpha(\mu) \in Z$ we get $\eta_\alpha(M) = 0$.*

In particular, in the case $n \geq 2$ the $U(1)$-eta invariant detects any knot which is not torsion in C_n. In the classical case $n = 1$ Casson and Gordon [CG86] first found an example of a knot which is algebraically slice but is not torsion in C_1. The goal of this thesis is to study to which degree certain non-abelian eta invariants can detect such examples.

4. Metabelian eta-invariants

A group G is called metabelian if $G^{(2)} = \{e\}$, a representation $\varphi : \pi_1(M) \to U(k)$ is called metabelian if it factors through $\pi_1(M)/\pi_1(M)^{(2)}$. Up to now we considered 1-dimensional eta-invariants, which are precisely the eta-invariants for irreducible abelian unitary representations. Now we'll study irreducible metabelian unitary representations of $\pi_1(M_K)$ and its eta-invariants.

Recall that $\tilde{M}_K$ denotes the infinite cyclic cover of M_K corresponding to $\epsilon : H_1(M_K) \to \mathbb{Z}$, therefore $\pi_1(\tilde{M}_K) = \pi_1(M_K)^{(1)}$ and

$$H_1(M_K, \Lambda) = H_1(\tilde{M}_K) \cong \pi_1(M_K)^{(1)}/\pi_1(M_K)^{(2)}$$

The $\Lambda := \mathbb{Z}[t, t^{-1}]$-module structure is given on the right hand side by $t^k \cdot g := \mu^{-k} g \mu^k$ where μ denotes a meridian of K.

We recall the definition of the semi-direct product of two groups. Let G, H be two groups and $\beta : G \to \mathrm{Aut}(H)$ a group homomorphism, then define $G \times_\beta H$ to be the semi-direct product of G and H twisted by β, i.e. the group with underlying set $G \times H$ and group structure given by

$$(g_1, h_1) \cdot (g_2, h_2) := (g_1 \cdot g_2, \beta(g_2)(h_1) \cdot h_2)$$

For a knot K let $\pi := \pi_1(M_K)$, then consider

$$1 \to \pi^{(1)}/\pi^{(2)} \to \pi/\pi^{(2)} \to \pi/\pi^{(1)} \to 1$$

since $\pi/\pi^{(1)} = H_1(M_K) = \mathbb{Z}$ this sequence splits and we get isomorphisms

$$\begin{array}{ccccc} \pi/\pi^{(2)} & \cong & \pi/\pi^{(1)} \ltimes \pi^{(1)}/\pi^{(2)} & \cong & \mathbb{Z} \ltimes \pi^{(1)}/\pi^{(2)} \cong \mathbb{Z} \ltimes H_1(M_K, \Lambda) \\ g & \mapsto & (g, \mu^{-\epsilon(g)} g) & \mapsto & (\epsilon(g), \mu^{-\epsilon(g)} g) \end{array}$$

where $1 \in \mathbb{Z}$ acts by conjugating with μ respectively by multiplying by t. This shows that studying metabelian representations of $\pi_1(M_K)$ corresponds to studying representations of $\mathbb{Z} \ltimes H_1(M_K, \Lambda)$.

4.1. Metabelian representations of $\pi_1(M_K)$.

For a group G denote by $R_k^{irr}(G)$ (resp. $R_k^{irr,met}(G)$) the set of conjugacy classes of irreducible, k-dimensional, unitary (metabelian) representations of G. Recall that for a knot K we can identify

$$R_k^{irr,met}(\pi_1(M_K)) = R_k^{irr}(\mathbb{Z} \ltimes H_1(M_K, \Lambda))$$

In the following let H be a (not necessarily finitely generated) Λ-module and $n \in \mathbb{Z}$ acts on H by multiplication by t^n. The goal is to give a full description of $R_k^{irr}(\mathbb{Z} \ltimes H)$.

It is easy to check the following lemma.

Lemma 4.1. *Let $z \in S^1$ and $\chi : H \to H/(t^k - 1) \to S^1$ a character. Pick a k^{th} root $z^{1/k}$ of z, then*

$$\begin{aligned}\alpha_{(z,\chi)} : \mathbb{Z} \ltimes H &\to U(k)\\ (n,h) &\mapsto (z^{1/k})^n \begin{pmatrix} 0 & \dots & 0 & 1\\ 1 & \dots & 0 & 0\\ \vdots & \ddots & & \vdots\\ 0 & \dots & 1 & 0\end{pmatrix}^n \begin{pmatrix} \chi(h) & 0 & \dots & 0\\ 0 & \chi(th) & \dots & 0\\ \vdots & & \ddots & \vdots\\ 0 & 0 & \dots & \chi(t^{k-1}h)\end{pmatrix}\end{aligned}$$

defines a representation, and the conjugacy class of the representation is independent of the choice of k^{th} root of z.

Note that the definition of $\alpha_{(z,\chi)}$ differs slightly from the definition given in the introduction, which does not change any of the statements.

Lemma 4.2. *Any irreducible representation $\alpha \in R_k^{irr}(\mathbb{Z} \ltimes H)$ is (unitary) conjugate to $\alpha_{(z,\chi)}$ for some $z \in S^1$ and a character $\chi : H \to H/(t^k - 1) \to S^1$ which does not factor through $H/(t^l - 1)$ for some $l < k$.*

Proof. Let $\alpha \in R_k^{irr}(\mathbb{Z} \ltimes H)$. Denote by $\chi_1, \dots, \chi_l : H \to S^1$ the different weights of $\alpha : 0 \times H \to U(k)$. Since H is an abelian group we can write $\mathbb{C}^k = \oplus_{i=1}^l V_{\chi_i}$ where $V_{\chi_i} := \{v \in \mathbb{C}^k | \alpha(0,h)(v) = \chi_i(h)v \text{ for all } h\}$ is the weight space corresponding to χ_i.

Recall that the group structure of $\mathbb{Z} \ltimes H$ is given by

$$(n,h)(m,k) = (n+m, t^m h + k)$$

In particular for all $v \in H$

$$(j,0)(0,t^j h) = (j, t^j h) = (0,h)(j,0)$$

therefore for $A := \alpha(1,0)$ we get

$$A^j \alpha(0,t^j h) = \alpha(j,0)\alpha(0,t^j h) = \alpha(j,t^j h) = \alpha(0,h)\alpha(j,0) = \alpha(0,h)A^j$$

This shows that $\alpha(0,t^j h) = A^{-j}\alpha(0,h)A^j$. Now let $v \in V_{\chi(h)}$, then

$$\alpha(0,h)Av = A\alpha(0,th)A^{-1}Av = A\alpha(0,th)v = A\chi(th)v = \chi(th)Av$$

i.e. $\alpha(1,0) : V_{\chi_i(h)} \to V_{\chi_i(th)}$. Since α is irreducible it follows that, after reordering, $\chi_i(v) = \chi_1(t^i v)$ for all $i = 1, \dots, l$. Note that A^j induces isomorphisms between the weight spaces V_{χ_i} and that $A^l : V_{\chi_1} \to V_{\chi_1}$ is a unitary transformation. In particular it has an eigenvector v, hence $\mathbb{C}v \oplus \mathbb{C}Av \oplus \dots \mathbb{C}A^{l-1}v$ spans an α-invariant subspace. Since α is irreducible it follows that $l = k$ and that each V_{χ_i} is one-dimensional.

Since α is a unitary representation we can find a unitary matrix P such that $P\mathbb{C}e_i = V_i$, in particular, $\alpha_1 := P^{-1}\alpha P$ has the following properties.

(1) $\alpha(0 \times H) = \mathrm{diag}(\chi(h), \chi(th), \dots, \chi(t^{k-1}h))$,

(2) for some $z_1, \dots, z_k \in S^1$

$$\alpha(1,0) := \begin{pmatrix} 0 & \dots & 0 & z_k \\ z_1 & \dots & 0 & 0 \\ \vdots & \ddots & & \vdots \\ 0 & \dots & z_{k-1} & 0 \end{pmatrix}$$

Here we denote by $\operatorname{diag}(b_1, \dots, b_k)$ the diagonal matrix with entries $b_1, \dots, b_k$.

Pick a k^{th} root $z^{1/k}$ of $z := \prod_{i=1}^k z_i$ and let $Q := \operatorname{diag}(d_1, \dots, d_k)$ where $d_i := \frac{\prod_{j=1}^{i-1} z_j}{(z^{1/k})^{i-1}}$. Then $\alpha_2 := Q^{-1}\alpha_1 Q$ has the required properties. □

We can now prove the following proposition.

Proposition 4.3. *There exist well-defined one-to-one correspondences*

$$\begin{aligned} & R_k^{irr}(\mathbb{Z} \ltimes H) \\ \Leftrightarrow\quad & \{(z,\chi) | z \in S^1, \chi : H/(t^k - 1) \to S^1 \text{ irreducible}\}/\sim \\ \Leftrightarrow\quad & \{(z,\chi) | z \in S^1, \chi : H \to S^1 \text{ such that } \chi(t^k h) = \chi(h) \text{ irreducible}\}/\sim \end{aligned}$$

where we say $(z_1, \chi_1) \sim (z_2, \chi_2)$ if $z_1 = z_2$ and $\chi_1(v) = \chi_2(t^l v)$ for some l and we say that a character $\chi : H/(t^k - 1) \to S^1$ is irreducible if it doesn't factor through $H/(t^l - 1)$ for some $l < k$.

Proof. The second correspondence is obviously well-defined and one–to–one. We'll now turn to the first correspondence.

Given α we can associate to it $z := \det(\alpha(1,0))$ and χ a weight of $\alpha : H \to U(k)$. The proof of lemma 4.2 shows that χ is irreducible and that all the weights of α are of the form $\chi_i(h) = \chi(t^{i-1}h), i = 1, \dots, k$. It follows that (z, χ) gives a well-defined equivalence class. Furthermore $\alpha \cong \alpha_{(z,\chi)}$.

Using lemma 4.1 it remains to show that given $z \in S^1$ and $\chi : H/(t^k - 1) \to S^1$ irreducible, $\alpha_{(z,\chi)}$ is irreducible. But this follows immediately from the observation that $\alpha_{(z,\chi)} : 0 \times H \to U(k)$ has k different weights and $\alpha_{(z,\chi)}(1,0)$ permutes the weight spaces. □

Translating the above results back to knot theory we get the following proposition.

Proposition 4.4. *We have natural one-to-one correspondences*

$$\begin{aligned} & R_k^{irr,met}(\pi_1(M)) \Leftrightarrow R_k^{irr}(\mathbb{Z} \ltimes H_1(M,\Lambda)) \Leftrightarrow \\ \Leftrightarrow\quad & R_k^{irr}(\mathbb{Z} \ltimes H_1(M,\Lambda)/(t^k - 1)) \Leftrightarrow R_k^{irr}(\mathbb{Z} \ltimes H_1(L_k)) \\ \Leftrightarrow\quad & \{(z,\chi) | z \in S^1, \chi : H_1(L_k) \to S^1 \text{ irreducible}\}/\sim \end{aligned}$$

4.2. **Representations and special varieties.** In this section we'll find some criteria when metabelian representations avoid special varieties, i.e. when they give concordance invariants.

Definition. For a Λ-torsion module H define $P_k^{irr}(\mathbb{Z} \ltimes H)$ to be the set of conjugacy classes of representations which are conjugate to $\alpha_{(z,\chi)}$ with $z \in S^1$ transcendental and $\chi : H/(t^k - 1) \to S^1$ factoring through a group of prime power order.

If W is a manifold with $H_1(W) \cong \mathbb{Z}$ then we define $P_k^{irr,met}(\pi_1(W)) := P_k^{irr}(\mathbb{Z} \ltimes H_1(W, \Lambda))$.

We need the following theorem, which is basically a slight reformulation of a theorem by Letsche [L00, cor. 3.10].

Theorem 4.5. *Let H be a Λ-torsion module, $G := \mathbb{Z} \ltimes H$, then $P_k^{irr}(G)$ avoids all special subvarieties. In particular if $(M_1, \alpha_1), (M_2, \alpha_2)$ are homology G-bordant and $\theta \in P_k(G)$, then $\eta_{\theta\circ\alpha_1}(M_1) = \eta_{\theta\circ\alpha_2}(M_2)$.*

Proof. Let $\alpha_{(z,\chi)} \in P_k^{irr}(\mathbb{Z} \ltimes H)$, i.e. $z \in S^1$ transcendental and $\chi : H/(t^k - 1) \to \mathbb{Z}/m \to S^1$ where m is a prime power. The first statement of the theorem follows from a result by Letsche [L00, cor. 3.10] once we show that the set $P_k(\mathbb{Z} \ltimes H)$ as defined above is the same as the set $P_k(\mathbb{Z} \ltimes H)$ defined by Letsche. This in turn follows follows immediately from the observation that

(1) all the eigenvalues of $\alpha_{(z,\chi)}(1, 0)$ are of the form $z^{1/k}$ in particular transcendental,
(2) $\alpha_{(z,\chi)} : \mathbb{Z} \ltimes H \to U(k)$ factors through $\mathbb{Z} \ltimes (\mathbb{Z}/m)^k$ and $(\mathbb{Z}/m)^k$ is a group of prime power order where $\mathbb{Z}$ acts by cyclic permutation, i.e. $1 \cdot (v_1, \ldots, v_k) := (v_k, v_1, \ldots, v_{k-1})$.
(3) $H_1(\mathbb{Z} \ltimes H) = \mathbb{Z}$ and $\mathbb{Z} \ltimes H \to \mathbb{Z} \ltimes (\mathbb{Z}/m)^k$ induces an isomorphism of the first homology groups.

The second statement follows from proposition 3.4. $\square$

Proposition 4.6. *Let K be a slice knot with slice disk D. Let $\alpha_{(z,\chi)} \in R_k^{irr,met}(\pi_1(M_K))$. Then $\alpha_{(z,\chi)}$ extends to $\beta \in R_k^{met}(\pi_1(N_D))$ if and only if χ vanishes on $Ker\{H_1(M_K, \Lambda)/(t^k - 1) \to H_1(N_D, \Lambda)/(t^k - 1)\}$.*

Proof. Consider the following diagram

$$\begin{array}{ccccccc}
\pi_1(M_K) & \to & \mathbb{Z} \ltimes H_1(M_K, \Lambda) & \to & \mathbb{Z} \ltimes H_1(M_K, \Lambda)/(t^k - 1) & \xrightarrow{\alpha_{(z,\chi)}} & U(k) \\
\downarrow & & \downarrow & & \downarrow & & \\
\pi_1(N_D) & \to & \mathbb{Z} \ltimes H_1(N_D, \Lambda) & \to & \mathbb{Z} \ltimes H_1(N_D, \Lambda)/(t^k - 1) & &
\end{array}$$

It's clear that $\alpha_{(z,\chi)}$ extends if and only if χ extends, but since S^1 is divisible this is the case if and only if χ vanishes on $\mathrm{Ker}\{H_1(M_K, \Lambda)/(t^k - 1) \to H_1(N_D, \Lambda)/(t^k - 1)\}$. $\square$

Theorem 4.7. *If K is a slice knot and D a slice disk and if $\alpha_{(z,\chi)}$ extends to $\beta \in R_k^{met}(\pi_1(N_D))$, then $\eta_\alpha(M_K) = 0$.*

Proof. We can decompose N_D as $N_D = W^4 \cup_{M_O} S^1 \times D^3$ where $M_O = S^1 \times S^2$ is the zero-framed surgery along the trivial knot in S^3 and W is a homology $\mathbb{Z}$–bordism between M_K and M_O. We can assume that $\alpha = \alpha_{(z,\chi)}$ where $z \in S^1$ transcendental and $\chi : H_1(M)/(t^k - 1) \to S^1$ factors through a group of prime power order.

If $\alpha_{(z,\chi)}$ extends to a representation in $R_k^{met}(\pi_1(N_D))$ then χ vanishes on

$$\operatorname{Ker}\{H_1(M_K, \Lambda)/(t^k - 1) \to H_1(N_D, \Lambda)/(t^k - 1)\}$$

We can then find an extension χ' which also factors through a p-group, hence $\alpha_{(z,\chi')} \in P_k(\pi_1(W))$. The statement now follows from theorem 4.5 and lemma 3.2 since (W, id) is a homology $\mathbb{Z} \ltimes H_1(W, \Lambda)$–bordism between (M_K, i_*) and (M_O, i_*). $\square$

We need a criterion when representations extend.

Proposition 4.8. *Let $A \to B$ be a homomorphism of Λ-torsion modules such that the induced map $A/(t^k - 1) \to B/(t^k - 1)$ is an injection. Then any irreducible representation $\alpha : \mathbb{Z} \ltimes A \to U(k)$ will extend to a representation $\mathbb{Z} \ltimes B \to U(k)$.*

4.3. **Main sliceness obstruction theorem.**

Theorem 4.9. *Let K be a slice knot, k a prime power, then there exists a Λ-metabolizer $P_k \subset TH_1(M_k)$ for the linking pairing, such that for any irreducible representation $\alpha : \pi_1(M_K) \to \mathbb{Z} \ltimes H_1(M_K, \Lambda)/(t^k - 1) \to U(k)$ vanishing on $0 \times P_k$ and lying in $P_k^{irr,met}(\pi_1(M_K))$ we get $\eta_\alpha(M_K) = 0$.*

Proof. Denote the k-fold cover of N_D by N_k. From corollary 2.14 we get $H_1(M, \Lambda)/(t^k - 1) = TH_1(M_k)$, similarly, using lemma 2.3 one gets $H_1(N, \Lambda)/(t^k - 1) = TH_1(N_k)$. Let

$$P_k := \operatorname{Ker}\{TH_1(M_k) \to TH_1(N_k))\}$$

this is a metabolizer for the linking pairing by proposition 2.15.

Let α be an irreducible representation $\pi_1(M) \to \mathbb{Z} \ltimes H_1(M, \Lambda)/(t^k - 1) \to U(k)$ vanishing on $0 \times P_k$ and lying in $P_k^{irr,met}(\pi_1(M))$. Then $\alpha \cong \alpha_{(z,\chi)}$ where $\chi : TH_1(M_k) \to S^1$ vanishes on P_k. The theorem now follows from proposition 4.6 and theorem 4.7. $\square$

In the following we'll show that some eta-invariants of slice knots vanish for non-prime power dimensional irreducible representations.

4.3.1. *Tensor products of representations.*

Let $\alpha \in R_k(G), \beta \in R_l(G)$, then we can form the tensor product $\alpha \otimes \beta \in R_{kl}(G)$.

For $z \in S^1, \chi : H_1(M, \Lambda) \to H_1(M, \Lambda)/(t^k - 1) \to S^1$ we'll explicitely write $\alpha_{(k,z,\chi)}$ for $\alpha_{(z,\chi)} \in R_k^{met}(\pi_1(M))$.

Proposition 4.10. *Let $l_1, l_2 \in \mathbb{N}$ and set $r := gcd(l_1, l_2), s := lcm(l_1, l_2)$. Then*

$$\alpha_{(l_1,z_1,\chi_1)} \otimes \alpha_{(l_2,z_2,\chi_2)} \cong \oplus_{j=0}^{r-1} \alpha_{(s,z_1z_2,\chi^j)}$$

where $\chi^j(v) := \chi_1(v)\chi_2(t^j v)$. In particular if l_1, l_2 are coprime, then

$$\alpha_{(l_1,z_1,\chi_1)} \otimes \alpha_{(l_2,z_2,\chi_2)} \cong \alpha_{(l_1l_2,z_1z_2,\chi_1\chi_2)}$$

If furthermore $\alpha_{(l_1,z_1,\chi_1)}$ and $\alpha_{(l_2,z_2,\chi_2)}$ are irreducible, then $\alpha_{(l_1,z_1,\chi_1)} \otimes \alpha_{(l_2,z_2,\chi_2)}$ is irreducible as well.

Proof. Set $\alpha_i := \alpha_{(l_i,z_i,\chi_i)}, i = 1, 2$. Denote by $e_{11}, \ldots, e_{l_1 1}$ and $e_{12}, \ldots, e_{l_2 2}$ the canonical bases of $\mathbb{C}^{l_1}$ and $\mathbb{C}^{l_2}$. Set $f_{i,j} := e_{i\, mod\, l_1,1} \otimes e_{(i+j)\, mod\, l_2,2}$ for $i = 0, \ldots, s-1, j = 0, \ldots, r-1$. The $f_{i,j}$'s are distinct, therefore $\{f_{i,j}\}_{i=0,\ldots,s-1,j=0,\ldots,r-1}$ form a basis for $\mathbb{C}^{l_1} \otimes \mathbb{C}^{l_2}$. We'll write the representation $\alpha_1 \otimes \alpha_2 \to U(\mathbb{C}^{l_1} \otimes \mathbb{C}^{l_2})$ in terms of this basis, and see that it decomposes as stated. Denote by $\{f_{i,j}^*\}_{i=0,\ldots,s-1,j=0,\ldots,r-1}$ the dual basis. We'll write $e_{i,j}$ for $e_{i\, mod\, l_j,j}$. We compute

$$\begin{aligned}
f_{i,j}^*((\alpha_1 \otimes \alpha_2)(1,0)(f_{k,l})) &= f_{i,j}^*((\alpha_1 \otimes \alpha_2)(1,0)(e_{k,1} \otimes e_{k+l,2})) = \\
&= f_{i,j}^*((\alpha_1(1,0)(e_{k,1}) \otimes \alpha_1(1,0)(e_{k+l,1})) = \\
&= f_{i,j}^*((z_1 e_{k+1,1} \otimes z_2 e_{k+l+1,2})) = f_{i,j}^*(z_1 z_2 f_{k+1,l}) = \\
&= z_1 z_2 \delta_{(i,j),(k+1,l)} \\
f_{i,j}^*((\alpha_1 \otimes \alpha_2)(0,v)(f_{k,l})) &= f_{i,j}^*((\alpha_1 \otimes \alpha_2)(0,v)(e_{k,1} \otimes e_{k+l,2})) = \\
&= f_{i,j}^*(\alpha_1(0,v)(e_{k,1}) \otimes \alpha_1(0,v)(e_{k+l,1})) = \\
&= f_{i,j}^*((\chi_1(t^k v) e_{k,1} \otimes \chi_2(t^{k+l} v) e_{k+l,2})) = \\
&= f_{i,j}^*(\chi_1(t^k v)\chi_2(t^{k+l} v) e_{k,1} \otimes e_{k+l,2}) = \\
&= f_{i,j}^*(\chi_1(t^k v)\chi_2(t^{k+l} v) f_{k,l}) = \chi^l(t^k v)\delta_{(i,j),(k,l)}
\end{aligned}$$

This shows that $\alpha_1 \otimes \alpha_2$ restricts to representations on $V_j := \mathrm{span}(f_{0,j}, \ldots, f_{s-1,j})$ for $j = 0, \ldots, r-1$, furthermore the representation on V_j is isomorphic to $\alpha_{(s,z_1z_2,\chi^j)}$.

The last statement follows from the observation that if $\chi_1\chi_2 : H/(t^{l_1l_2} - 1)$ factors through some $l < l_1 l_2$, then one of the $\chi_i : H \to H/(t^{l_i} - 1)$ factors through $H/(t^k - 1)$ for some $k < l_i$. □

4.3.2. *Strong version of the main theorem.* For a prime number p and a Λ-module H denote by $P_{k,p}^{irr}(\mathbb{Z} \ltimes H)$ the representations $\alpha_{(z,\chi)}$ in $P_k^{irr}(\mathbb{Z} \ltimes H)$ where χ factors through a p-group. Define $P_{k,p}^{irr,met}(\pi_1(M_K)) := P_{k,p}^{irr}(\mathbb{Z} \ltimes H_1(M, \Lambda))$.

Theorem 4.11. *Let K be a slice knot, $k_1, \ldots, k_r$ pairwise coprime prime powers, then there exist Λ-metabolizers $P_{k_i} \subset TH_1(M_{k_i}), i = 1, \ldots, r$ for the linking pairings, such that for any prime number p and any choice of irreducible representations $\alpha_i : \pi_1(M_K) \to \mathbb{Z} \ltimes H_1(M_K, \Lambda)/(t^{k_i} - 1) \to U(k)$ vanishing on $0 \times P_{k_i}$ and lying in $P_{k_i,p}^{irr,met}(\pi_1(M_K))$ we get $\eta_{\alpha_1 \otimes \cdots \otimes \alpha_r}(M_K) = 0$.*

Proof. By theorem 4.9 all the representations $\alpha_1, \dots, \alpha_r$ extend over N_D, hence $\alpha_1 \otimes \dots \otimes \alpha_r$ also extends over N. Write $\alpha = \alpha_{(z_i,\chi_i)}$, then $\alpha_1 \otimes \dots \otimes \alpha_r = \alpha_{(z_1 \cdot \ldots z_r, \chi_1 \cdot \dots \cdot \chi_r)}$ since the k_i are pairwise coprime. This shows that $\alpha_1 \otimes \dots \otimes \alpha_r \in P^{irr,met}_{k_1 \cdot \dots \cdot k_r, p}(\pi_1(M))$, therefore $\eta_{\alpha_1 \otimes \dots \otimes \alpha_r}(M) = 0$ by theorem 4.7. $\square$

Proposition 4.10 shows that $\alpha_1 \otimes \dots \otimes \alpha_r$ is irreducible if $\gcd(k_1, \dots, k_r) = 1$, i.e. the theorem shows that certain non-prime-power dimensional irreducible eta-invariants vanish for slice knots.

We say that a knot K has zero slice–eta–obstruction (SE–obstruction) if the conclusion of theorem 4.9 holds for all prime powers k, and K has zero slice–tensor–eta–obstruction (STE–obstruction) if the conclusion of theorem 4.11 holds for all pairwise coprime prime powers $k_1, \dots, k_r$. Unfortunately I don't have examples of knots which have zero SE–obstruction but non zero STE–obstruction.

Remark. It's easy to find examples of a knot K and one-dimensional representations α, β such that $\eta_\alpha(M_K) = 0$ and $\eta_\beta(M_K) = 0$ but $\eta_{\alpha \otimes \beta}(M_K) \neq 0$. This shows that in general $\eta_\beta(M_K)$ is not determined by $\eta_\alpha(M_K)$ and $\eta_\beta(M_K)$.

The following example shows that in general not all representations are subsummands of tensor products of prime power representations.

Example. Let K be a knot with Seifert matrix $A = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$, then K is algebraically slice. A computation (cf. appendix 2.4) shows that

$$\begin{aligned} H_1(L_2) = &\quad \mathbb{Z}/3 \oplus \mathbb{Z}/3 \\ H_1(L_5) = &\quad \mathbb{Z}/31 \oplus \mathbb{Z}/31 \\ H_1(L_{10}) = &\quad \mathbb{Z}/1023 \oplus \mathbb{Z}/1023 \end{aligned}$$

Since $1023 = 3 \cdot 11 \cdot 31$ we see that $H_1(L_{10})$ supports characters which are not products of characters living on $H_1(L_2), H_1(L_5)$. This shows that there exist irreducible $U(10)$-representations which are not tensor products of lower dimensional representations.

Note that if $TH_1(M_k) = 0$, then $R^{irr}_k(\pi_1(M)) = \emptyset$, therefore theorem 4.11 only gives a non-trivial sliceness obstruction if $TH_1(M_k) \neq 0$ for some prime power k. This is not always the case, in fact Livingston [L01, thm. 0.5] proved the following theorem.

Theorem 4.12. *Let K be a knot. Then there exists a prime power k with $TH_1(M_k) \neq 0$ if and only if $\Delta_K(t)$ has a non-trivial irreducible factor that is not an n-cyclotomic polynomial with n divisible by three distinct primes.*

In chapter 10.4, we'll use this theorem to show that there exists a knot K with $H_1(L_k) = 0$ for all prime powers k, but $H_1(L_6) \neq 0$. This shows that K has non-trivial irreducible $U(6)$-representations, but no unitary irreducible representations of prime power dimensions.

5. Casson-Gordon obstruction

5.1. **The Casson-Gordon obstruction to a knot being slice.** We first recall the definition of the Casson-Gordon obstructions [CG86]. For m a number denote by $C_m \subset S^1$ the unique cyclic subgroup of order m. Let k be some number, using

$$H_1(M_k) \xleftarrow{\cong} H_1(X_k) \to H_1(L_k)$$

we will embed the set of characters on $H_1(L_k)$ into the set of characters on $H_1(M_k)$. For a surjective character $\chi : H_1(M_k) \to H_1(L_k) \to C_m$, set $F_\chi := \mathbb{Q}(e^{2\pi i/m})$.

Let K be a knot and $\chi : H_1(M_k) \to H_1(L_k) \to C_m$ a surjective character. Since $\Omega_3(\mathbb{Z} \times C_m) = H_3(\mathbb{Z} \times C_m)$ is torsion (cf. appendix A.2) there exists $(V_k^4, \epsilon \times \chi)$ and some $r \in \mathbb{N}$ such that $\partial(V_k, \epsilon \times \chi) = r(M_k, \epsilon \times \chi)$. The (surjective) map $\epsilon \times \chi : \pi_1(V_k) \to \mathbb{Z} \times C_m$ defines a $(\mathbb{Z} \times C_m)$-cover $\tilde{V}_\infty$ of V. Then $H_2(C_*(\tilde{V}_\infty))$ and $F_\chi(t)$ have a canonical $\mathbb{Z}[\mathbb{Z} \times C_m]$-module structure and we can form $H_2(C_*(\tilde{V}_\infty) \otimes_{\mathbb{Z}[\mathbb{Z} \times C_m]} F_\chi(t)) =: H_*(V_k, F_\chi(t))$. Since $F_\chi(t)$ is flat over $\mathbb{Z}[\mathbb{Z} \times C_m]$ (cf. lemma A.1) we get

$$H_*(V_k, F_\chi(t)) = H_*(C_*(\tilde{V}_\infty) \otimes_{\mathbb{Z}[\mathbb{Z} \times C_m]} F_\chi(t)) \cong H_*(\tilde{V}_\infty) \otimes_{\mathbb{Z}[\mathbb{Z} \times C_m]} F_\chi(t)$$

which is a free $F_\chi(t)$-module. If χ is a character of prime power order, then the $F_\chi(t)$-valued intersection pairing on $H_2(V_k, F_\chi(t))$ is non-singular (cf. [CG86, p. 190]) and therefore defines an element $t(V_k) \in L_0(F_\chi(t))$ (for a definition of L-groups see appendix B). Denote the image of the ordinary intersection pairing on $H_2(V_k)$ under the map $L_0(\mathbb{C}) \to L_0(F_\chi(t))$ by $t_0(V_k)$.

Proposition 5.1. *If $\chi : H_1(L_k) \to C_m$ is a character with m a prime power, then*

$$\tau(K, \chi) := \frac{1}{r}(t(V_k) - t_0(V_k)) \in L_0(F_\chi(t)) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is a well-defined invariant of $(M_k, \epsilon \times \chi)$, i.e. independent of the choice of V_k.

Proof. Let $(\hat{V}_k, \epsilon \times \chi)$ be a second manifold with $\partial(\hat{V}_k, \epsilon \times \chi) = s(M_k, \epsilon \times \chi)$. Taking disjoint multiples of $\hat{V}_k$ and V_k we can assume that $r = s$. Then $W := V_k \cup_{rM_k} -\hat{V}_k$ is a closed 4-manifold with $t(W) = t(V_k) - t(\hat{V}_k), t_0(W) = t_0(V_k) - t_0(\hat{V}_k)$. It is therefore enough to show that for $(W^4, \epsilon \times \chi)$, W a closed manifold we get $t(W) = t_0(W)$.

But $\Omega_4(\mathrm{pt}) = \mathbb{Z}$ and $\Omega_4(\mathbb{Z} \times C_m) = \Omega_4(\mathrm{pt}) \oplus H_4(\mathbb{Z} \times C_m) = \mathbb{Z} \oplus$torsion (cf. appendix A.2). The torsion free part is in both cases generated by $\mathbb{C}P^2$ (with trivial character for $\Omega_4(\mathbb{Z} \times C_m)$) which shows that the maps t and t_0 coincide, since $L_0(F_\chi(t)) \otimes \mathbb{Q}$ is torsion free. $\square$

Casson and Gordon prove the following theorem (cf. [CG86, p. 192]).

Theorem 5.2. *Let k be a prime power. If $K \subset S^3$ is slice then there exists a subgroup $Q \subset TH_1(M_k)$ with $Q = Q^\perp$ with respect to the linking pairing λ_L, such that for any $\chi : TH_1(M_k) \to C_m$, m a prime power, with $\chi(Q) \equiv 0$ we get $\tau(K, \chi) = 0$.*

We give just a short outline of the proof.

Proof. Assume K is slice and $D \subset D^4$ a slice disk. Let N_k be the k-fold cover of $N := D^4 \setminus N(D)$, then $\partial(N_k) = M_k$. Let $Q := \operatorname{Ker}\{TH_1(M_k) \to TH_1(N_k)\}$ then $Q = Q^\perp$ by proposition 2.15, if $\chi : TH_1(M_k) \to C_m$ is such that $\chi(Q) \equiv 0$ then there exists a map $\tilde{\chi} : H_1(N_k) \to C_{m^l}$ for some l such that the following diagram commutes

$$\begin{array}{ccc} H_1(M_k) & \xrightarrow{\chi} & C_m \\ \downarrow & & \downarrow \\ H_1(N_k) & \xrightarrow{\tilde{\chi}} & C_{m^l} \end{array}$$

Casson and Gordon show that one can use the m^l-fold cover $\tilde{N}_k$ of N_k to compute $\tau(K, \chi)$. Denote the $\mathbb{Z} \times C_{m^l}$ cover of N_k by $\tilde{N}_\infty$. Since m is a prime power we get that $H_2(\tilde{N}_\infty, \mathbb{Q})$ is finite dimensional over $\mathbb{Q}$ (cf. [CG86, p. 191]), hence

$$H_2(N_k, F_{\tilde{\chi}}(t)) \cong H_2(\tilde{N}_\infty) \otimes_{\mathbb{Z}[\mathbb{Z} \times C_m]} F_{\tilde{\chi}}(t) = 0$$

hence $t(N_k) = 0$. But we also have (since k is a prime power) $H_2(N_k, \mathbb{Q}) = 0$ by lemma 2.13, hence $t_0(V_k) = 0$. $\square$

5.2. **Interpretation of Casson-Gordon invariants as eta invariants of M_k.** Let K be a knot, k any number, m a prime power and $\chi : H_1(L_k) \to C_m$ a character and $(V_k^4, \epsilon \times \chi : \pi_1(V_k) \to \mathbb{Z} \times C_m)$ such that $\partial(V_k, \epsilon \times \chi) = r(M_k, \epsilon \times \chi)$.

Let $F \subset \mathbb{C}$ be some number field. We'll always consider F with complex involution and $F(t)$ with the involution given by complex involution and $\bar{t} = t^{-1}$.

Given a transcendental $z \in S^1$ we can define $\tau(K, \chi)(z) \in L_0(\mathbb{C}) \otimes \mathbb{Q}$ (cf. appendix A.2) and hence

$$\tau_z(K, \chi) := \operatorname{sign}(\tau(K, \chi)(z)) \in \mathbb{Q}$$

For a character $\chi : H_1(L_k) \to C_m$ define characters χ^j by setting $\chi^j(v) := \chi(v)^j$, for $z \in S^1$ define

$$\begin{array}{rcccl} \beta_{(z,\chi^j)} : \pi_1(M_k) & \to & H_1(M_k) = \mathbb{Z} \oplus H_1(L_k) & \to & S^1 = U(1) \\ & & (n, v) & \mapsto & z^n \chi^j(v) \end{array}$$

Proposition 5.3. *Let $z \in S^1$ transcendental, then*

$$\tau_z(K, \chi) = \eta_{\beta_{(z,\chi^1)}}(M_k)$$

Proof. Let $z \in S^1$ transcendental, define

$$\begin{array}{rcl} \theta : \mathbb{Z} \times C_m & \to & S^1 \\ (n, y) & \mapsto & z^n y \end{array}$$

then

$$\partial(V_k, \theta \circ (\epsilon \times \chi)) = r(M_k, \theta \circ (\epsilon \times \chi)) = r(M_k, \beta_{(z,\chi^1)})$$

We view $\mathbb{C}$ as a $\mathbb{Z}[\mathbb{Z} \times C_m]$-module via $\theta \circ (\epsilon \times \chi)$ and $\mathbb{C}$ as an $F_\chi(t)$ module via evaluating t to z. Note that both modules are flat by lemma A.1.

$$\begin{aligned} H_2^\beta(V_k, \mathbb{C}) &= H_2(C_*(\tilde{V}_\infty)) \otimes_{\mathbb{Z}[\mathbb{Z}\times C_m]} \mathbb{C} = (H_2(C_*(\tilde{V}_\infty)) \otimes_{\mathbb{Z}[\mathbb{Z}\times C_m]} F_\chi(t)) \otimes_{F_\chi(t)} \mathbb{C} \\ &= H_2(V_k, F_\chi(t)) \otimes_{F_\chi(t)} \mathbb{C} \end{aligned}$$

This also defines an isometry between the forms, i.e. $\mathrm{sign}_\beta(V_k) = t_z(V_k)$. This shows that

$$r\eta_{(z,\chi^1)}(M_k) = \mathrm{sign}_\beta(V_k) - \mathrm{sign}(V_k) = t_z(V_k) - \mathrm{sign}(t_0(V_k)) = r\tau_z(K, \chi)$$

□

Corollary 5.4. *The function*

$$\begin{aligned} S^1 &\to \mathbb{Z} \\ z &\mapsto \eta_{\beta_{(z,\chi)}}(M_K) \end{aligned}$$

is continuous (i.e. constant) outside of a finite set.

Proof. Let $(V_k^4, \epsilon \times \chi)$ such that $\partial(V_k, \epsilon \times \chi) = r(M_k^3, \epsilon \times \chi)$. Let $A(t)$ be a matrix representing the form $t(V_k)$ and let

$$Z := \{z \in S^1 | A(z) \text{ singular or not defined}\}$$

This set is either finite or all of Z. But $t(V_k)$ is non–singular, hence $A(t)$ is non–singular, hence $Z \neq S^1$. Note that

$$\eta_{\beta_{(z,\chi^1)}}(M_k) = \tau_z(K, \chi) = \frac{1}{r}(\mathrm{sign}(A(z) - \mathrm{sign}(V_k)$$

for all $z \in S^1 \setminus Z$. But $z \mapsto \mathrm{sign}(A(z))$ is constant on $S^1 \setminus Z$ since $z \mapsto \mathrm{sign}(A(z))$ has discontinuities only when $A(z)$ is not defined or singular. □

Remark. The eta invariant carries potentially more information than the function $z \to \tau_z(K, \chi)$, since for non-transcendental $z \in S^1$ the number $\tau_z(K, \chi)$ is not defined, whereas $\eta_{\beta_{(z,\chi_1)}}(M_k)$ is still defined. For example the $U(1)$-signatures for slice knots are zero outside the set of singularities, but the eta invariant at the singularities contains information about knots being doubly slice (cf. section 3.4).

For any transcendental $z \in S^1$ we have a well-defined map $\sigma_z : L_0(F_\chi(t)) \to L_0(\mathbb{C}) \to \mathbb{Z}$ given by evaluating $t \mapsto z$ and taking signatures. This extends to a map $\sigma_z : L_0(F_\chi(t)) \otimes \mathbb{Q} \to L_0(\mathbb{C}) \otimes \mathbb{Q} \to \mathbb{Q}$. Clearly $\sigma_z(\tau(K, \chi)) = \tau_z(K, \chi)$.

Proposition 5.5. *Let K be a knot, k any number, m a prime power and $\chi : H_1(L_k) \to C_m$ a character, then*

$$\begin{aligned} & \tau(K, \chi) = 0 \in L_0(F_\chi(t)) \otimes_{\mathbb{Z}} \mathbb{Q} \\ \Leftrightarrow\ & \sigma_z(\rho(\tau(K, \chi))) = 0 \in \mathbb{Q} \textit{ for all transcendental } z \in S^1, \rho \in Gal(F_\chi, \mathbb{Q}) \\ \Leftrightarrow\ & \eta_{\beta_{(z,\chi^j)}}(M_k) = 0 \in \mathbb{Z} \textit{ for all } (j, m) = 1, \textit{ all transcendental } z \in S^1 \end{aligned}$$

Proof. The first equivalence is a purely algebraic statement, which follows from theorem B.1. The second follows from proposition 5.3 and the observation that if $\rho \in \mathrm{Gal}(F_\chi, \mathbb{Q})$ sends $e^{2\pi i/m}$ to $e^{2\pi ij/m}$ for some $(j,m)=1$, then $\rho(\tau(K,\chi)) = \tau(K,\chi^j)$ and hence $\sigma_z(\rho(\tau(K,\chi))) = \tau_z(K,\chi^j) = \eta_{\beta_{(z,\chi^j)}}(M_k)$. □

5.3. **Interpretation of Casson-Gordon invariants as eta invariants of M_K.** The goal is to prove a version of proposition 5.5 with eta invariants of M_K instead of eta invariants of M_k.

Proposition 5.6. *Let K be a knot, $\chi : H_1(M_k) \to H_1(L_k) \to C_m$ some character, then for $\beta := \beta_{(z,\chi)} : \pi_1(M_k) \to U(1)$ and $\alpha = \alpha_{(z,\chi)} : \pi_1(M_K) \to U(k)$ we get*

$$\eta_\alpha(M_K) - \eta_\beta(M_k) = \mu(M_K, k)$$

where

$$\mu(M_K, k) := \sum_{j=1}^{k} sign(A(1-e^{2\pi ij/k}) + A^t(1-e^{-2\pi ij/k}))$$

If K is algebraically slice and k such that $H_1(L_k)$ is finite, then $\eta_\alpha(M_K) = \eta_\beta(M_k)$.

Proof. In order to work with α we have to fix a k^{th} root $z^{1/k}$ of z. The representation $\alpha = \alpha_{(z,\chi)}$ can easily be seen to factor as follows

$$\alpha : \pi_1(M_K) \to \mathbb{Z} \ltimes H_1(M,\Lambda) \to \mathbb{Z} \ltimes H_1(L_k) \to \mathbb{Z} \ltimes (C_m)^k \to U(k)$$

where $\mathbb{Z}$ acts on $(C_m)^k$ by permutation. Note that $k\mathbb{Z} \ltimes (C_m)^k = k\mathbb{Z} \times (C_m)^k$. The group $\Omega_3(\mathbb{Z} \ltimes (C_m)^k) = \Omega_3(\mathrm{pt}) \oplus H_3(\mathbb{Z} \ltimes (C_m)^k)$ is torsion since $\Omega_3(\mathrm{pt}) = 0$ and $H_3(\mathbb{Z} \ltimes (C_m)^k)$ is torsion (cf. proposition A.6). Therefore there exists a manifold V^4 with $\partial(V) = rM$ for some r such that $\alpha : \pi_1(M_K) \to \mathbb{Z} \ltimes (C_m)^k \to U(k)$ extends to V. In fact we can and will assume, after some additional surgery, that $\pi_1(V) \cong G := \mathrm{Im}\{\pi_1(M_K) \to \mathbb{Z} \ltimes (C_m)^k\}$. Note that $\mathbb{Z} \times 0 \subset G$ is a direct summand.

We'll use V to compute $\eta_{\alpha_{(z,\chi)}}(M)$. Let V_k be the k-fold cover of V corresponding to

$$\pi_1(V) = G \to \mathbb{Z} \to \mathbb{Z}/k$$

Write $G_k = \mathrm{Ker}\{\pi_1(V) \to \mathbb{Z}/k\}$, note that $G_k \cong \mathbb{Z} \times C_m$. Then $\partial(V_k) = M_k$ and $\chi : \pi_1(M_k) \to U(1)$ extends to V_k, therefore V_k can be used to compute $\eta_{\beta_{(z,\chi)}}(M_k)$. Denote by $\tilde{V}$ the universal cover of V_k and hence of V as well. Consider

$$\begin{array}{rcl} \phi : C_*(\tilde{V}) \otimes_{\mathbb{Z}G_k} \mathbb{C} & \to & C_*(\tilde{V}) \otimes_{\mathbb{Z}G} \mathbb{C}^k \\ \sigma \otimes v & \mapsto & \sigma \otimes (v, 0, \dots, 0) \end{array}$$

and

$$\begin{array}{rcl} \psi : C_*(\tilde{V}) \otimes_{\mathbb{Z}G} \mathbb{C}^k & \to & C_*(\tilde{V}) \otimes_{\mathbb{Z}G_k} \mathbb{C} \\ \sigma \otimes (v_0, \dots, v_{k-1}) & \mapsto & \sigma \otimes v_0 + \sigma\mu \otimes z^{-1/k} v_1 \cdots + \sigma\mu^{k-1} \otimes z^{(-k+1)/k} v_{k-1} \end{array}$$

where $\mu = (1,0) \in \mathbb{Z} \times 0 \subset G = \pi_1(V)$ denotes the image of the meridian under the map $\pi_1(M) \to \pi_1(V)$. Recall that $\mu \in \pi_1(M)$ acts by definition of $\alpha_{(z,\chi)}$ as follows on $\mathbb{C}^k$:

$$\mu \cdot (v_0, \dots, v_{k-1}) = z^{1/k}(v_{k-1}, v_0, \dots, v_{k-2})$$

It is easy to see that ϕ is well-defined, i.e. respects the tensor product. Now consider ψ. We have to show that for $g \in G$ we get

$$\psi(\sigma g \otimes_{\mathbb{Z}G} (v_0, \dots, v_{k-1})) = \psi(\sigma \otimes_{\mathbb{Z}G} g(v_0, \dots, v_{k-1}))$$

This is obvious for $g \in G_k$, it is therefore enough to show this for $g = \mu$. Using that μ^k acts by multiplication by $(z^{1/k})^k = z$ we compute

$$\begin{array}{ll} & \psi(\sigma \otimes \mu(v_0, \dots, v_{k-1})) = \\ = & \psi(\sigma \otimes z^{1/k}(v_{k-1}, v_0, \dots, v_{k-2})) = \\ = & \sigma \otimes z^{1/k} v_{k-1} + \sigma\mu \otimes z^{-1/k} z^{1/k} v_0 + \dots + \sigma\mu^{k-1} \otimes z^{(-k+1)/k} z^{1/k} v_{k-2} = \\ = & \sigma\mu \otimes v_0 + \sigma\mu^2 \otimes z^{-1/k} + \dots + \sigma\mu^k \otimes z^{(-k+1)/k} v_{k-1} = \\ = & \psi(\sigma\mu \otimes (v_0, \dots, v_{k-1})) \end{array}$$

The maps ϕ, ψ are obviously inverses and chain maps, they therefore induce an isomorphism of homology groups

$$\phi_* : H_*^{\beta}(V_k, \mathbb{C}) = H_*(C_*(\tilde{V}) \otimes_{\mathbb{Z}G_k} \mathbb{C}) \to H_*(C_*(\tilde{V}) \otimes_{\mathbb{Z}G} \mathbb{C}^k) = H_*^{\alpha}(V, \mathbb{C}^k)$$

Denote the twisted signature of V with respect to α by $\operatorname{sign}_{\alpha}(V)$ and the twisted signature of V_k with respect to β by $\operatorname{sign}_{\beta}(V_k)$. Let $\sigma_i \otimes v_i \in C_*(\tilde{V}) \otimes_{\mathbb{Z}G_k} \mathbb{C}, i = 1, 2,$ then $\psi(\sigma_i \otimes v_i) = \sigma_i \otimes (v_i, 0, \dots, 0), i = 1, 2$. We'll just write $(v_i, 0)$ for $(v_i, 0, \dots, 0)$. We compute the twisted intersection product

$$\begin{array}{rcl} \psi(\sigma_1 \otimes v_1) \cdot \psi(\sigma_2 \otimes v_2) & = & \sum_{g \in G} (\bar{v}_1, 0)\alpha(((\sigma_1 g) \cdot \sigma_2)g^{-1}) \begin{pmatrix} v_2 \\ 0 \end{pmatrix} = \\ & = & \sum_{g \in G_k} (\bar{v}_1, 0)\alpha(((\sigma_1 h) \cdot \sigma_2)g^{-1}) \begin{pmatrix} v_2 \\ 0 \end{pmatrix} = \\ & = & \sum_{g \in G_k} \bar{v}_1 \beta(((\sigma_1 h) \cdot \sigma_2)g^{-1}) v_2 = \\ & = & (\sigma_1 \otimes v_1) \cdot (\sigma_2 \otimes v_2) \end{array}$$

we used that for $g \in G$ we can write $g = \mu^{\epsilon} h$ where $h \in \pi^{(1)}$ acts diagonally on $\mathbb{C}^k$ and

$$\begin{pmatrix} v & 0 \end{pmatrix} \alpha(\mu^l) \begin{pmatrix} w \\ 0 \end{pmatrix} = 0 \text{ for } l \not\equiv 0 \mod k$$

This shows that $\operatorname{sign}_{\alpha}(V) = \operatorname{sign}_{\beta}(V_k)$. Therefore

$$\begin{array}{rcl} \eta_{\alpha}(M) - \eta_{\beta}(M_k) & = & \frac{1}{r}((\operatorname{sign}_{\alpha}(V) - k\operatorname{sign}(V)) - (\operatorname{sign}_{\beta}(V_k) - \operatorname{sign}(V_k)) = \\ & = & \frac{1}{r}(\operatorname{sign}(V_k) - k\operatorname{sign}(V)) \end{array}$$

Denote by $\varphi : \mathbb{Z}/k \to U(k) = U(\mathbb{C}[\mathbb{Z}/k])$ the canonical representation, then $\operatorname{sign}_\varphi(V) = \operatorname{sign}(V_k)$. Note that $\varphi = \varphi_1 \oplus \dots \oplus \varphi_k$ where $\varphi_j : \mathbb{Z}/k \to U(1), n \to e^{2\pi i j/k}$, hence

$$\begin{aligned}\frac{1}{r}(\operatorname{sign}(V_k) - k\operatorname{sign}(V)) &= \frac{1}{r}(\operatorname{sign}_{\varphi_1\oplus\dots\oplus\varphi_k}(V) - k\operatorname{sign}(V)) = \\ &= \frac{1}{r}\sum_{i=1}^k(\operatorname{sign}_{\varphi_i}(V) - \operatorname{sign}(V)) \\ &= \frac{1}{r}\sum_{j=1}^k \sigma_{2\pi i j/k}(K)\end{aligned}$$

Now proposition 3.7 concludes the proof of the first part of the proposition.

The second statement of the proposition follows from proposition 2.9 and the observation that $A(1-w) + A^t(1-\bar{w})$ for some $w \in S^1$ if and only if $\Delta_K(w) = 0$. $\square$

Combining propositions 5.5 and 5.6 we get the following corollary.

Corollary 5.7. *Let K be a knot, k any number, m a prime power and $\chi : H_1(L_k) \to C_m$ a character, then*

$$\begin{aligned}&\tau(K,\chi) = 0 \in L_0(F_\chi(t)) \otimes_{\mathbb{Z}} \mathbb{Q} \\ \Leftrightarrow\ &\eta_{\alpha_{(z,\chi^j)}}(M) - \mu(M,k) = 0 \in \mathbb{Z} \textit{ for all } (j,m) = 1, \textit{ all transcendental } z \in S^1\end{aligned}$$

In particular if K is algebraically slice and k such that $H_1(L_k)$ is torsion, then we get

$$\begin{aligned}&\tau(K,\chi) = 0 \in L_0(F_\chi(t)) \otimes_{\mathbb{Z}} \mathbb{Q} \\ \Leftrightarrow\ &\eta_{\alpha_{(z,\chi^j)}}(M) = 0 \in \mathbb{Z} \textit{ for all } (j,m) = 1, \textit{ all transcendental } z \in S^1\end{aligned}$$

5.4. **Relation of Casson-Gordon's theorem to the main theorem.** We say that a knot $K \subset S^3$ has zero Casson-Gordon obstruction if for any prime power k there exists a subgroup $Q \subset TH_1(M_k)$ such that $Q = Q^\perp$ with respect to λ_L and such that for any prime power m and $\chi : TH_1(M_k) \to C_m$ with $\chi(Q) \equiv 0$ we get $\tau(K,\chi) = 0 \in L_0(F_\chi(t)) \otimes \mathbb{Q}$.

The following is an immediate consequence of corollary 5.7.

Theorem 5.8. *Let K be an algebraically slice knot. Then K has zero SE–obstruction if and only if K has zero Casson-Gordon obstruction.*

6. Obstructions to a knot being ribbon

In the following we'll fix a tubular neighborhood $S^3 \times [0,1] \subset D^4$ such that $S^3 = S^3 \times 0$.

Definition. A knot $K \subset S^3 = S^3 \times \{0\}$ is called ribbon if there exists a smooth disk D in $S^3 \times [0,1] \subset D^4$ bounding K such that the projection map $S^3 \times [0,1] \to [0,1]$ is a Morse map and has no local minima. Such a slice disk is called a ribbon disk.

Proposition 6.1. *If K is ribbon and $D \subset D^4$ a ribbon disk, then the maps*

$$\begin{array}{rcl} i_* : \pi_1(S^3 \setminus K) & \to & \pi_1(D^4 \setminus D) \\ \pi_1(M_K) & \to & \pi_1(N_D) \\ H_1(M_K, \Lambda) & \to & H_1(N_D, \Lambda) \end{array}$$

are surjective.

Proof. The first statement is shown in [G81, lemma 3.1] and [K75b, lemma 2.1]. The other statements follow immediately from the first one. □

The following proposition gives an equivalent definition of ribbon knots.

Proposition 6.2. *A knot K is ribbon if and only if K is the boundary of a singular disk $f : D^2 \to S^3$ which has the property that each component of self-intersection is an arc $A \subset f(D^2)$ for which $f^{-1}(A)$ consists of two arcs in D^2 and one lies in int(D^2).*

Proof. If D is such a singular disk then we can push it into $S^3 \times [0,1] \subset D^4$ to remove the singularities. It is obvious that we can arrange the pushing such that the map $D \to [0,1]$ has no local minima.

The converse is shown in [T69, lemma 1.28]. □

It is a longstanding conjecture of Fox (cf. [F61, problem 25]) that slice knots are ribbon. We'll prove a ribbon-obstruction theorem (theorem 6.4) which ostensibly will be stronger than the corresponding theorem for slice knots (theorem 4.11), and therefore potentially gives a way to show that a given slice knot is not ribbon.

6.1. Main ribbon-obstruction theorem.

Proposition 6.3. *Assume that K is ribbon, D a ribbon disk, then $H_1(N_D, \Lambda)$ is $\mathbb{Z}$-torsion free, in particular $P := Ker\{H_1(M_K, \Lambda) \to H_1(N_D, \Lambda)\}$ is a metabolizer for λ_{Bl}.*

Proof. According to [L77, thm. 2.1 and prop. 2.4] there exists a short exact sequence

$$0 \to \mathrm{Ext}^2_\Lambda(H_1(N, M, \Lambda)) \to \overline{H_1(N, \Lambda)} \to \mathrm{Ext}^1_\Lambda(H_2(N, M, \Lambda)) \to 0$$

Here $\overline{H_1(N, \Lambda)}$ denotes $H_1(N, \Lambda)$ with involuted Λ-module structure, i.e. $t \cdot v := t^{-1}v$. Furthermore $\mathrm{Ext}^1_\Lambda(H_2(N, M, \Lambda))$ is $\mathbb{Z}$-torsion free (cf. [L77, prop. 3.2]). In order to show that $H_1(N,\Lambda)$ is $\mathbb{Z}$-torsion free it is therefore enough to show that $H_1(N,M,\Lambda)=0$. Consider the exact sequence

$$H_1(M,\Lambda)\to H_1(N,\Lambda)\to H_1(N,M,\Lambda)\to H_0(M,\Lambda)\to H_0(N,\Lambda)\to 0$$

The last map is an isomorphism. By proposition 6.1 the first map is surjective. It follows that $H_1(N,M,\Lambda)=0$.

The second part follows immediately from proposition 2.7. □

Theorem 6.4. *Let $K\subset S^3$ be a ribbon knot. Then there exists a Λ-subspace $P\subset H_1(M_K,\Lambda)$ such that $P=P^\perp$ and such that for any $\alpha\in P_k^{irr}(\pi_1(M_K))$ vanishing on $0\times P$ we get $\eta_\alpha(M_K)=0$.*

Proof. Let $P:=\operatorname{Ker}\{H_1(M_K,\Lambda)\to H_1(N_D,\Lambda)\}$ where D is a ribbon disk for K. Then $P=P^\perp$ by proposition 6.3. Let $\alpha\in P_k(\pi_1(M))$ which vanishes on $0\times P$. Then α defines a representation of $\mathbb{Z}\ltimes H_1(M,\Lambda)/P$, but $H_1(M,\Lambda)/P\to H_1(N,\Lambda)$ is an isomorphism by proposition 6.1. Hence α defines a representation of $\mathbb{Z}\ltimes H_1(N,\Lambda)$, hence α extends to $\pi_1(N)$. The theorem now follows from proposition 4.7. □

As a corollary we get the following, somewhat weaker ribbonness obstruction, which we'll use in section 10.4 to show that a certain knot is not ribbon.

Corollary 6.5. *Let K be a ribbon knot and k any number such that $H_1(L_k)$ is finite. There exists $P_k\subset TH_1(M_k)$ such $P_k=P_k^\perp$ with respect to λ_{L_k} and such that for all $\chi:TH_1(M_k)\to S^1$ of prime power order, vanishing on P_k, and for all transcendental $z\in S^1$ we get $\eta_{\alpha_{(z,\chi)}}(M_K)=0$.*

Proof. Let $P:=\operatorname{Ker}\{H_1(M_K,\Lambda)\to H_1(N_D,\Lambda)\}$ where D is a ribbon disk for K. Then $P=P^\perp$ with respect to λ_{Bl}. Let k be such that $H_1(L_k)$ is finite. Let $P_k:=\pi_k(P)\subset TH_1(M_k)=H_1(M,\Lambda)/(t^k-1)$. Then P_k is a metabolizer for $\lambda_{L,k}$ by proposition 2.18. Let $\chi:TH_1(M_k)\to S^1$ be a character of prime power order, vanishing on P_k, and $z\in S^1$ transcendental. Then $\alpha=\alpha_{(z,\chi)}$ vanishes on $0\times P$ since χ vanishes on $\pi_k(P)$, therefore we get $\eta_{\alpha_{(z,\chi)}}(M_K)=0$ by theorem 6.4. □

Remark. If we compare the main slice obstruction theorem 4.11 with corollary 6.5 and theorem 6.4 we see that the ribbon obstruction theorem is stronger in two respects. When K is ribbon

(1) we can find metabolizers P_k which all lift to the same metabolizer of the Blanchfield pairing,
(2) the representations for non-prime power dimensions don't have to be tensor products (cf. the example after theorem 4.11).

Remark. We say that a knot K is (k)-homotopically ribbon if K is slice, and there exists a slice disk D such that

$$\pi_1(S^3\setminus K)/\pi_1(S^3\setminus K)^{(k+1)}\to \pi_1(D^4\setminus D)/\pi_1(D^4\setminus D)^{(k+1)}$$

is surjective. It is clear that theorem 6.4 holds for (1)-homotopically ribbon knots. The concept of homotopically ribbon knots was first introduced by Casson and Gordon [CG83].

6.2. **The Casson-Gordon obstruction to a knot being ribbon.** For completeness sake we quickly recall Casson-Gordon's ribbon obstruction theorem. We won't need it later, but it may be interesting to compare it to theorem 6.4 since it loosens the 'other' prime power condition.

Let $K \subset S^3$ be a knot and k a number. Let $\chi : H_1(L_k) \to C_m \subset S^1$ be a character. Then there exists a finite number r such that $r \cdot L_k$ bounds a 4-manifold W such that χ extends to $\pi_1(W) \to C_m$. Define

$$\sigma(K, \chi) = \frac{1}{r}(\mathrm{sign}_\chi(W) - \mathrm{sign}(W))$$

Casson and Gordon show that this number is well-defined and prove the following theorem [CG86, p. 154]

Theorem 6.6. *Let K be a ribbon knot with ribbon disk D and k a prime power. Denote the k-fold covering of D^4, branched along the ribbon disk, by W_k. Then there exists $P_k \subset H_1(L_k)$ such that $P_k = P_k^\perp$ and such that if $\chi : H_1(L_k) \to S^1$ is a character, vanishing on P_k, and if $\pi_1(W_k)$ is finite, then $\sigma(K, \chi) = \pm 1$.*

Note that χ does not necessarily factor through a group of prime-power order. The condition that $\pi_1(W_k)$ is finite is of course very restrictive, but it is satisfied for example in the case Casson and Gordon consider.

Remark. Ruberman [R88] and Matic [M88] showed independently that the conclusion of the theorem also holds for smoothly slice knots. It does not necessarily hold for locally flat slice knots.

Finally we quote the following theorem.

Theorem 6.7. [CG86] *Let $K \subset S^3$. Let $\chi : H_1(L_k) \to C_m$ be a character inducing a cover $\tilde{L}_k$ of L_k. If $H_1(\tilde{L}_k, \mathbb{Q}) = 0$ then*

$$|\sigma(K, \chi) - \tau_1(K, \chi))| \leq 1$$

7. Obstructions to a knot being doubly slice

A knot $K \subset S^3$ is called doubly slice (or doubly null-concordant) if there exists an unknotted two-sphere $S \subset S^4$ such that $S \cap S^3 = K$. It is clear that a doubly slice knot is in particular slice. Fox [F61] first posed the question which slice knots are doubly slice .

We say that knot K is algebraically doubly slice if K has a Seifert matrix of the form $\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}$ where B, C are square matrices of the same size.

We quote some results about doubly slice knots (cf. [S71], [L89] and [R83]).

Theorem 7.1. (1) *If $K \subset S^3$ is doubly slice then K is algebraically doubly slice.*
(2) *If K is algebraically doubly slice then $\sigma_z(K) = 0$ for all $z \in S^1$.*
(3) *There exist slice knots which are not doubly slice.*
(4) *There are knots in all odd dimensions which are algebraically doubly slice but not doubly slice.*

We prove the following new doubly slice obstruction theorem.

Theorem 7.2. *Let $K \subset S^3$ be a doubly slice knot. Then there exist Λ-subspaces $P_1, P_2 \subset H_1(M_K, \Lambda)$ such that*

(1) *$H_1(M_K, \Lambda) = P_1 \oplus P_2$,*
(2) *for $i = 1, 2$ we have $P_i = P_i^\perp$ and for any $\alpha \in P_k^{irr}(\pi_1(M_K))$ vanishing on $0 \times P_i$ we get $\eta_\alpha(M_K) = 0$.*

Proof. Let $S \subset S^4$ be an unknotted two-sphere such that $S \cap S^3 = K$. Intersecting S with $\{(x_1, \dots, x_5) \in \mathbf{R}^5 | x_5 \geq 0\}$ and $\{(x_1, \dots, x_5) \in \mathbf{R}^5 | x_5 \leq 0\}$ we can write write $S = D_1^2 \cup_K D_2^2$ and $S^4 = D_1^4 \cup_{S^3} D_2^4$. Let $N_i = D_i^4 \setminus D_i^2$, then $N_1 \cap N_2 = S^3 \setminus K$ and $N_1 \cup N_2 = S^4 \setminus S$. From the Mayer-Vietoris sequence we get

$$H_1(M_K, \Lambda) = H_1(N_1, \Lambda) \oplus H_1(N_2, \Lambda)$$

since $H_1(M_K, \Lambda) = H_1(S^3 \setminus K, \Lambda)$ and $H_1(S^4 \setminus S, \Lambda) = 0$ since S is trivial.

Now let $P_i := \mathrm{Ker}\{H_1(M, \Lambda) \to H_1(N_i, \Lambda)\}$, the proof concludes as the proof of the main ribbon obstruction theorem 6.4 since $H_1(M_K, \Lambda) \to H_1(N_i, \Lambda)$ is surjective for $i = 1, 2$. □

Remark. (1) The proof of theorem 7.2 shows in particular that if K is doubly slice we can find a slice disk D such that $H_1(N_D, \Lambda)$ is $\mathbb{Z}$-torsion free.
(2) Comparing theorem 7.2 with theorems 4.11 and 6.4 we see that doubly slice knots immediately have zero (doubly) ribbon obstruction.

Question 7.3. *Using the notation of the proof we get from the van Kampen theorem that for a doubly slice knot*

$$\mathbb{Z} = \pi_1(D_1^4 \setminus D_1^2) *_{\pi_1(S^3 \setminus K)} \pi_1(D_2^4 \setminus D_2^2)$$

Can we conclude that $\pi_1(S^3 \setminus K) \to \pi_1(D_i^4 \setminus D_i^2)$ *is surjective for at least one* i*? If yes, then this would show that a doubly slice knot is in fact homotopically ribbon. One can go further and ask whether doubly slice knots are in fact ribbon or doubly ribbon. The first part of the question is of course a weaker version of the famous 'slice equals ribbon' conjecture.*

This question can be but in purely group theoretic terms. If H, G_1, G_2 *are groups normally generated by one element,* $f_i : H \to G_i$ *maps which send a normal generator to a normal generator, then if* $G_1 *_H G_2 \cong \mathbb{Z}$ *can we say that either* f_1 *or* f_2 *is surjective?*

Remark. Taehee Kim [K02] introduced the notion of (n, m)-solvability $(n, m \in \frac{1}{2}\mathbb{N})$ and introduced L^2-eta invariants to find highly non–trivial examples of non doubly slice knots.

8. Gilmer's and Letsche's obstruction

We say that $K \subset S^3$ has zero weak–ribbon–eta–obstruction (WRE-obstruction) if there exists a Λ-subspace $P \subset H_1(M_K, \Lambda)$ such that $P = P^\perp$ with respect to the Blanchfield pairing and such that for any prime power k and any irreducible representation $\alpha : \pi_1(M_K) \to \mathbb{Z} \ltimes H_1(M_K, \Lambda) \to U(k)$ vanishing on $0 \times P$ and lying in $P_k^{irr,met}(\pi_1(M_K))$ we get $\eta_\alpha(M_K) = 0$.

We say that a knot $K \subset S^3$ has zero weak–tensor–ribbon–eta–obstruction (WTRE–obstruction) if there exists a Λ-subspace $P \subset H_1(M_K, \Lambda)$ such that $P = P^\perp$ with respect to the Blanchfield pairing and such that for any prime number p and any choice of prime powers $k_1, \ldots, k_r$ and irreducible representations $\alpha_i : \pi_1(M_K) \to \mathbb{Z} \ltimes H_1(M, \Lambda) \to U(k_i)$ vanishing on $0 \times P$ and lying in $P_{k_i,p}^{irr,met}(\pi_1(M_K))$ we get $\eta_{\alpha_1 \otimes \cdots \otimes \alpha_r}(M_K) = 0$.

Proposition 8.1. *Let K be a slice knot and D a slice disk such that*

$$P := Ker\{H_1(M_K, \Lambda) \to H_1(N_D, \Lambda)\}$$

is a metabolizer for the Blanchfield pairing, then K has zero WTRE–obstruction.

Proof. Let k be a prime power. Denote the k-fold cover of N_D by N_k. From corollary 2.14 we get $H_1(M, \Lambda)/(t^k - 1) = TH_1(M_k)$, similarly, using lemma 2.3 one gets $H_1(N, \Lambda)/(t^k - 1) = TH_1(N_k)$.

Let $\alpha \in P_k^{irr,met}(\pi_1(M))$ which vanishes on $0 \times P$, then we can find $z \in S^1, \chi : H_1(M, \Lambda) \to TH_1(M_k) \to S^1$ such that $\alpha = \alpha_{(z,\chi)}$. Then χ vanishes on $P_k := P/(t^k - 1) \in TH_1(M_k)$. Let

$$Q_k := \mathrm{Ker}\{TH_1(M_k) \to TH_1(N_k))\}$$

It is clear that $P_k \subset Q_k$. Since k is a prime power we know that $|Q_k|^2 = |TH_1(M_k)|$ by proposition 2.15, and by proposition 2.18 we also know that $|P_k|^2 = |TH_1(M_k)|$. This shows that $P_k = Q_k$.

The proof of theorem 4.9 now shows that $\eta_\alpha(M) = 0$. Thus K has zero WRE–obstruction, the proof of theorem 4.11 shows that K has even zero WTRE–obstruction. $\square$

Let D be a slice disk for K such that

$$P := \mathrm{Ker}\{H_1(M_K, \Lambda) \to H_1(N_D, \Lambda)\} = \mathrm{Ker}\{H_1(M_K, \Lambda) \to F_{\mathbb{Z}} H_1(N_D, \Lambda)\}$$

then $P := P^\perp$ by proposition 2.7. This is in particular the case when $H_1(N_D, \Lambda)$ is $\mathbb{Z}$-torsion free. From the proofs of theorem 7.2 and of proposition 6.3 it follows that any ribbon knot and any doubly–slice knot has zero WTRE–obstruction.

Question 8.2. *Let K be a slice knot, can we always find a slice disk D such that*

$$Ker\{H_1(M_K, \Lambda) \to H_1(N_D, \Lambda)\} = Ker\{H_1(M_K, \Lambda) \to F_{\mathbb{Z}} H_1(N_D, \Lambda)\}$$

Remark. Similarly to the above proposition one can show the following. Let K be a slice knot and D a slice disk. Let $P := \operatorname{Ker}\{H_1(M,\Lambda) \to F_{\mathbb{Z}}H_1(N_D,\Lambda)\}$. Then $P = P^\perp$. Let $\chi : H_1(M,\Lambda) \to S^1$ is a character vanishing on P of prime power order q such that $\gcd(q,g) = 1$ for any $g \in T_{\mathbb{Z}}H_1(N_D,\Lambda)$. Then $\eta_{\alpha_{(z,\chi)}} = 0$ for any transcendental z.

8.1. **Gilmer's obstruction.** Let K be a knot, F a Seifert surface. Pick a basis $a_1,\dots,a_{2g}$ for $H_1(F)$, denote by A the corresponding Seifert matrix. Let $\Gamma := (A^t - A)^{-1}A^t$ and k such that $H_1(L_k)$ is finite. Define $\varphi_k : H_1(F) \to H_1(F)$ to be the endomorphism given by $\Gamma^k - (\Gamma-1)^k$ and define $B^k \subset H_1(F,\mathbb{Q}/\mathbb{Z})$ to be the kernel of $\varphi_k \otimes \mathbb{Q}/\mathbb{Z}$. For a prime number p define B^k_p to be the p-primary part of B^k.

Let Y be S^3 slit along F (cf. section 2.3). Denote by $\alpha_1,\dots,\alpha_{2g} \in H_1(Y)$ the dual basis with respect to Alexander duality, i.e. $\operatorname{lk}(a_i,\alpha_j) = \delta_{ij}$.

Recall that we can construct X_k from glueing together k copies of Y, there are therefore k canonical lifts of Y to $X_k \subset L_k$. Pick one, denote the lifts of α_i by $\tilde{\alpha}_i$. These generate $H_1(L_k)$, in fact (cf. proposition 2.9)

$$H_1(L_k) = \left(\bigoplus \mathbb{Z}\tilde{\alpha}_i\right)/\Gamma_k^t$$

Claim. The map

$$\begin{array}{rcl} B^k & \to & H^1(L_k,\mathbb{Q}/\mathbb{Z}) = \operatorname{Hom}(H_1(L_k),\mathbb{Q}/\mathbb{Z}) \\ \sum r_i a_i & \mapsto & (\tilde{\alpha}_j \mapsto r_j) \text{ where } r_j \in \mathbb{Q}/\mathbb{Z} \end{array}$$

is well-defined and an isomorphism.

This is a special case of the following. Let $H := \mathbb{Z}^h/C^t$ for some matrix C. One can easily see, that we can identify $\operatorname{Hom}(H,\mathbb{Q}/\mathbb{Z})$ with the set of column vectors with $\mathbb{Q}/\mathbb{Z}$-entries which become integral when multiplied by C.

In the following, for H a $\mathbb{Z}$-torsion module, we identify $\operatorname{Hom}(H,\mathbb{Q}/\mathbb{Z})$ and $\operatorname{Hom}(H,S^1)$ via

$$\begin{array}{rcl} \operatorname{Hom}(H,\mathbb{Q}/\mathbb{Z}) & \to & \operatorname{Hom}(H,S^1) \\ f & \mapsto & e^{2\pi i f} \end{array}$$

The map $B^k \to H^1(L_k,\mathbb{Q}/\mathbb{Z})$ is not canonical, since we picked a lift $Y \to L_k$. Let $\chi \in B^k$, the k different lifts give k characters $\chi_i \in H^1(L_k,\mathbb{Q}/\mathbb{Z}), i = 1,\dots,k$. But the $\chi_i, i = 1,\dots,k$ give the same Casson-Gordon invariant, hence we get a well-defined Casson-Gordon invariant $\tau(K,\chi) \in L_0(F_\chi(t)) \otimes \mathbb{Q}$ for $\chi \in B^k$.

We say that a knot K has zero Gilmer obstruction for a Seifert surface F if there exists a metabolizer H for the Seifert pairing on $H_1(F)$ such that for all prime powers k and primes p, $\tau(K, B^k_p \cap (H \otimes \mathbb{Q}/\mathbb{Z})) = 0$. More precisely, for a character $\chi \in B^k_p \cap (H \otimes \mathbb{Q}/\mathbb{Z})$ we get $\tau(K,\chi) = 0 \in L_0(F_\chi(t))$.

Gilmer's theorem (cf. [G93, p. 5]) says that a slice knot has zero Gilmer obstruction for all Seifert surfaces. Unfortunately the proof has a gap. On page 6, the statement that $H \otimes \mathbb{Q}/\mathbb{Z}$ is the kernel of μ_* (Gilmer's notation in the paper) is not necessarily true since tensoring with $\mathbb{Q}/\mathbb{Z}$ is not exact. Furthermore the proof of the cancellation lemma 5 has a gap as well, namely on line 12. Note that the same problem appears in Gilmer's earlier paper [G83]. The following weaker statement is correct.

Theorem 8.3. *Let K be a ribbon knot, then K has zero Gilmer obstruction for at least one F.*

Proof. This follows from the proof of Gilmer's theorem (cf. [G93, p. 5]) if we can avoid the above mentioned problem. In fact, it is enough to show that one can find a Seifert surface F and a manifold R^3 such that $\partial(R) = F \cup_K D$, such that $H_1(R)$ is torsion-free. This can be done for ribbon knots, using the following observation of Gilmer.

If K is ribbon, D a ribbon disk, then we can construct a Seifert surface F for K by resolving the ribbon intersections. We do surgery to this surface in the 4-ball along unknots on F around each ribbon intersection. The trace of this surgery is a 3-manifold R which has boundary $F \cup_K D$ and is a handlebody, in particular $H_1(R)$ is $\mathbb{Z}$-torsion free. □

Theorem 8.4. *A knot K has zero Gilmer obstruction for all Seifert surfaces if and only if K has zero WRE–obstruction.*

Proof. Assume that K has zero WRE–obstruction. Let F be a Seifert surface for K. For all prime powers k fix a map $B^k \to \mathrm{Hom}(H_1(L_k), \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}(H_1(L_k), S^1)$ induced by a lifting $Y \to L_k$.

By assumption there exists a Λ-subspace $P \subset H_1(M_K, \Lambda)$ such that $P = P^\perp$ with respect to the Blanchfield pairing and such that for any prime power k and any irreducible representation $\alpha : \pi_1(M_K) \to \mathbb{Z} \ltimes H_1(M_K, \Lambda) \to U(k)$ vanishing on $0 \times P$ and lying in $P_k(\pi_1(M_K))$, we get $\eta_\alpha(M_K) = 0$.

Let g be the genus of F. Trotter [T73] shows that we can find a basis $b_1, \dots, b_{2g}$ for $H_1(F)$ such that for some $1 \le \tilde{g} \le g$ the following holds for the Seifert matrix A:

(1) the matrix $\tilde{A}$ obtained by restricting A to the first $2\tilde{g}$ rows and columns is a minimal Seifert matrix for K,
(2) the matrix A is obtained from $\tilde{A}$ by elementary row and column enlargements (cf. [T73]).

Schematically A is of the following form

$$A = \begin{pmatrix} \tilde{A} & * & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ * & * & * & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

where $\tilde{A}$ is minimal, in particular invertible (cf. [T73]). Denote by $\beta_1, \dots, \beta_{2g} \in H_1(Y)$ the dual basis. We can lift $\beta_1, \dots, \beta_{2g}$ to elements $\tilde{\beta}_1, \dots, \tilde{\beta}_{2g}$ which generate $H_1(M, \Lambda)$, in fact with respect to this generating see we have $H_1(M, \Lambda) = \Lambda^{2g}/(At - A^t)$. It is easy to see that $\tilde{\beta}_{2\tilde{g}+1} = 0, \dots, \tilde{\beta}_{2g} = 0 \in H_1(M, \Lambda)$, in particular $H_1(M, \Lambda) = \Lambda^{2\tilde{g}}/(\tilde{A}t - \tilde{A}^t)$.

The proof of the main theorem in Kearton [K75] shows that we can find a basis $a_1, \dots, a_{2\tilde{g}}$ for $V := \text{span}\{b_1, \dots, b_{2\tilde{g}}\} \subset H_1(F)$ such that $a_1, \dots, a_{\tilde{g}}$ generate a metabolizer for the Seifert pairing restricted to V and such that the corresponding dual elements $\tilde{\alpha}_{\tilde{g}+1}, \dots, \tilde{\alpha}_{2\tilde{g}} \in H_1(M_K, \Lambda)$ span the given metabolizer $P \subset H_1(M_K, \Lambda)$. Now let $H := \text{span}\{a_1, \dots, a_{\tilde{g}}, b_{2\tilde{g}+2}, b_{2\tilde{g}+4}, \dots, b_{2g}\} \subset H_1(F)$. We'll show that H has the required properties.

It is clear that H is a metabolizer for the Seifert pairing. Now let k be a prime power and $\chi \in B^k \cap (H \otimes \mathbb{Q}/\mathbb{Z})$ a character of prime power order m. From the definition of $B^k \to \text{Hom}(H_1(L_k), \mathbb{Q}/\mathbb{Z})$ it is clear that the $\chi^j, j = 1, \dots, m$ correspond to characters $H_1(L_k) \to S^1$ vanishing on $\text{span}\{\tilde{\alpha}_{\tilde{g}+1}, \dots, \tilde{\alpha}_{2\tilde{g}}\} \subset H_1(L_k)$, which in turn correspond to characters $\chi^j : H_1(M_K, \Lambda) \to S^1$ vanishing on $P = \text{span}\{\tilde{\alpha}_{\tilde{g}+1}, \dots, \tilde{\alpha}_{2\tilde{g}}\} \subset H_1(M_K, \Lambda)$. For all transcendental $z \in S^1$ we get $\alpha_{(z,\chi^j)} \in P_k(\pi_1(M_K))$, therefore $\eta_{\alpha_{(z,\chi^j)}}(M) = 0$. Using corollary 5.7 we see that this implies $\tau(K, \chi) = 0 \in L_0(F_\chi(t))$.

Now assume that the conclusion of Gilmer's theorem holds. Let F be a Seifert surface of genus g. Let $H \subset H_1(F)$ be the metabolizer for the Seifert pairing, then we can find a basis $a_1, \dots, a_{2g}$ for $H_1(F)$ such that $a_1, \dots, a_g$ is a basis for H. From proposition 2.6 it follows that the lifts $\tilde{\alpha}_{g+1}, \dots, \tilde{\alpha}_{2g} \in H_1(M, \Lambda)$ span a metabolizer $P \subset H_1(M_K, \Lambda)$. Let k be a prime power and let $\alpha_{(z,\chi)} \in P_k^{irr,met}(\pi_1(M_K))$ be a representation such that $\chi : H_1(M_K, \Lambda) \to H_1(L_k) \to S^1$ vanishes on P. It is easy to see that under the isomorphism $B^k \cong \{H_1(L_k) \to \mathbb{Q}/\mathbb{Z}\}$ the character χ lies in $B^k \cap (H \otimes \mathbb{Q}/\mathbb{Z})$, hence $\eta_{\alpha_{(z,\chi)}}(M_K) = 0$ for all transcendental z by corollary 5.7. □

Corollary 8.5. *If K has zero Gilmer obstruction for one Seifert surface, then K has zero Gilmer obstruction for all Seifert surfaces.*

This follows immediately from the above theorem, since if the conclusion of Gilmer's theorem holds for one Seifert surface, then K has zero WRE–obstruction, which in turn implies that the conclusion of Gilmer's theorem holds for all Seifert surfaces.

8.2. **Letsche's obstruction.** For $x \in H_1(M_K, \Lambda)$ define

$$\begin{array}{rcl} B_x : \mathbb{Z} \ltimes H_1(M_K, \Lambda) & \to & \mathbb{Z} \ltimes S^{-1}\Lambda/\Lambda \\ (n, y) & \mapsto & (n, \lambda_{Bl}(x, y)) \end{array}$$

Therefore any $x \in H_1(M_K, \Lambda)$ defines a map

$$\alpha_x : \pi_1(M_K) \to \pi_1(M_K)/\pi_1(M_K)^{(2)} \cong \mathbb{Z} \ltimes H_1(M_K, \Lambda) \xrightarrow{B_x} \mathbb{Z} \ltimes S^{-1}\Lambda/\Lambda$$

We say that a knot K has zero (prime power) Letsche obstruction if there exists a Λ-subspace $P \subset H_1(M_K, \Lambda)$ such that $P = P^\perp$ with respect to the Blanchfield pairing and such that for any (prime power) k and any $x \in P$ and $\theta \in R_k(\mathbb{Z} \ltimes S^{-1}\Lambda/\Lambda)$ such that $\theta \circ \alpha_x \in P_k^{irr}(\pi_1(M_K))$ we get $\eta_{\theta\circ\alpha_x}(M_K) = 0$.

Letsche (cf. [L00, p. 313]) proved that every slice knot has zero Letsche obstruction. Unfortunately the statement of the last paragraph of the proof of lemma 2.21 is incorrect. The following weaker statement is correct.

Theorem 8.6. *Let $K \subset S^3$ be a slice knot and D a slice disk. If $Ker\{H_1(M_K, \Lambda) \to H_1(N_D, \Lambda)\}$ is a metabolizer for the Blanchfield pairing, then K has zero Letsche obstruction. In particular ribbon knots and doubly–slice knots have zero Letsche obstruction.*

We give a complete proof, which differs somewhat from Letsche's original proof.

Proof. Let $x \in P$. Considering the long exact sequence we see that $x = \partial(w)$ for some $w \in H_2(N, M, \Lambda)$. The proof of proposition 2.7 shows that in fact $H_2(N, M, \Lambda) = T_\Lambda H_2(N, M, \Lambda)$ and that there exists a Blanchfield pairing

$$\lambda_{Bl,N} : T_\Lambda H_2(N, M, \Lambda) \times T_\Lambda H_1(N, \Lambda) \to S^{-1}\Lambda/\Lambda$$

such that $\lambda_{Bl}(x, y) = \lambda_{Bl,N}(w, i_*(y))$ for $y \in H_1(M, \Lambda)$. We get a commutative diagram (cf. [L00, cor. 2.9])

$$\begin{array}{ccccccc}
\pi_1(M) & \to & \mathbb{Z} \ltimes H_1(M, \Lambda) & \xrightarrow{B_x} & \mathbb{Z} \ltimes S^{-1}\Lambda/\Lambda & \xrightarrow{\theta} & U(k) \\
\downarrow & & \downarrow & & \| & & \| \\
\pi_1(N) & \to & \mathbb{Z} \ltimes T_\Lambda H_1(N, \Lambda) & \xrightarrow{B_w} & \mathbb{Z} \ltimes S^{-1}\Lambda/\Lambda & \xrightarrow{\theta} & U(k)
\end{array}$$

This shows that $\theta \circ \alpha_x$ extends over $\pi_1(N)$. The first part of the theorem now follows from theorem 4.7.

The second part follows from proposition 2.7, proposition 6.3 and the remark after theorem 7.2. $\square$

Theorem 8.7. *Let K be a knot. If K has zero WTRE–obstruction, then K has zero Letsche obstruction. Furthermore K has zero prime power Letsche obstruction iff K has zero WRE–obstruction.*

The proof of this theorem will occupy the remainder of this section.

Remark. Let $k_1, \dots, k_s$ be a set of powers of different prime numbers, and $k = \prod_{i=1}^s k_i$. Proposition 8.10 and the proof of theorem 8.7 show that K has zero Letsche obstructions if and only if $\eta_\beta(M_K) = 0$ for any $\beta \in P_{k,p}^{irr}(\pi_1(M_K))$ of the form $\beta = \theta_1 \circ \alpha_x \otimes \dots \otimes \theta_s \circ \alpha_x, x \in P, \theta_i \in P_{k_i,p}^{irr}(\mathbb{Z} \ltimes S^{-1}\Lambda/\Lambda)$.

On the other hand, the proof of theorem 8.7 shows that K has WTRE–obstruction if and only if $\eta_\beta(M_K) = 0$ for any $\beta \in P_{k,p}^{irr}(\pi_1(M_K))$ of the form $\beta = \theta_1 \circ \alpha_{x_1} \otimes \dots \otimes \theta_s \circ \alpha_{x_s}, x_1, \dots, x_s \in P, \theta_i \in P_{k_i,p}^{irr}(\mathbb{Z} \ltimes S^{-1}\Lambda/\Lambda)$.

Question 8.8. *Are all representations of the form $\theta_1 \circ \alpha_{x_1} \otimes \cdots \otimes \theta_s \circ \alpha_{x_s}$ in fact of the form $\tilde{\theta}_1 \circ \alpha_x \otimes \cdots \otimes \tilde{\theta}_s \circ \alpha_x$ for some $x \in P$ and representations $\tilde{\theta}_i$?*

We first need some auxiliary propositions before we start the proof of theorem 8.7 in earnest. In the following we'll identify the Λ-modules $\Lambda/p(t)$ and $p(t)^{-1}\Lambda/\Lambda$. Note that for $p(t), q(t)$ the injection $p(t)^{-1}\Lambda/\Lambda \to (p(t)q(t))^{-1}\Lambda/\Lambda$ corresponds under this identification to $\Lambda/p(t) \xrightarrow{\cdot q(t)} \Lambda/p(t)q(t)$.

Proposition 8.9. *Let k be a prime power and $p(t) \in \Lambda$ such that $p(1) = 1$ and $\alpha : \mathbb{Z} \ltimes p(t)^{-1}\Lambda/\Lambda \to U(k)$ an irreducible representation. Then α extends to an irreducible representation $\tilde{\alpha} : \mathbb{Z} \ltimes S^{-1}\Lambda/\Lambda \to U(k)$.*

From the proof it is clear that the proposition is in general not true for k a composite number.

Proof. Pick $p_i(t) \in S, i = 0, 1, \ldots$ such that $p_i(1) = 1$ and such that for any $f(t) \in S$ we get $f(t) | \prod_{i=1}^{s} p_i(t)$ for some s. Let $q_i(t) := \prod_{j \leq i} p_j(t)$ and $f_i : \Lambda/q_i(t) \to \Lambda/q_{i+1}(t)$ induced by multiplication by $p_{i+1}(t)$. This defines a directed system $(\Lambda/q_i(t), f_i)_{i\geq 1}$. It is easy to see that

$$S^{-1}\Lambda/\Lambda \cong \lim_{\to}((\Lambda/q_i(t), f_i)_{i\geq 1})$$

This means in particular that

$$R_k(\mathbb{Z} \ltimes S^{-1}\Lambda/\Lambda) \cong R_k(\lim_{\to}((\mathbb{Z} \ltimes \Lambda/q_i(t), \mathrm{id} \times f_i)_{i\geq 1}))$$

The inclusion mapping $f_i : \Lambda/q_i(t) \to \Lambda/q_{i+1}(t)$ descends to a map $\cdot p_{i+1}(t) : \Lambda/q_i(t)/(t^k - 1) \to \Lambda/q_{i+1}(t)/(t^k - 1)$. Since k is the power of a prime p we know that the irreducible factors of $t^k - 1$ are $t - 1$ and $\Phi_{p^r}(t)$, which have the property that $\Phi_{p^r}(1) = p$ (cf. lemma A.2). Since $p_{i+1}(1) = 1$ we get therefore $\gcd(t^k - 1, p_{i+1}(t)) = 1$.

Claim. Let $p(t), q(t) \in \Lambda$ such that $p(1) = q(1) = 1$ and k such that $\gcd(t^k - 1, q(t)) = 1$, then

$$\Lambda/(p(t), t^k - 1) \xrightarrow{\cdot q(t)} \Lambda/(p(t)q(t), t^k - 1)$$

is injective.

Let $g(t)$ represent an element in the kernel. Then $g(t)q(t) = 0$ in $\Lambda/(p(t)q(t), t^k - 1)$, which means that $g(t)q(t) = a(t)p(t)q(t) + b(t)(t^k - 1)$ for some $a(t), b(t) \in \Lambda$. But $\gcd(q(t), t^k - 1) = 1$ implies that $q(t)|b(t)$ therefore $g(t) = a(t)p(t) + c(t)(t^k - 1)$ for some $c(t) \in \Lambda$. This shows that $g(t) = 0 \in \Lambda/(p(t), t^k - 1)$. This concludes the proof of the claim.

Applying this to the above situation we get that $\cdot p_{i+1}(t) : \Lambda/q_i(t)/(t^k - 1) \to \Lambda/q_{i+1}(t)/(t^k - 1)$ is an injection for all i. We can now apply proposition 4.8 to complete the proof. $\square$

Proposition 8.10. *Let $p(t) \in \Lambda$ such that $p(1) = 1$. If $k = \prod_{i=1}^{s} k_i$ such that $k_1, \dots, k_s$ are powers of different prime numbers, then an irreducible representation $\alpha : \mathbb{Z} \ltimes p(t)^{-1}\Lambda/\Lambda \to U(k)$ factors through $\mathbb{Z} \ltimes S^{-1}\Lambda/\Lambda$ if and only if $\alpha = \alpha_1 \otimes \cdots \otimes \alpha_s$ with $\alpha_i \in R^{irr}_{k_i}(\mathbb{Z} \ltimes \Lambda/p(t))$. Furthermore if p is a prime number, then $\alpha \in P^{irr}_{k,p}(\mathbb{Z} \ltimes \Lambda/p(t))$ if and only if $\alpha_i \in P^{irr}_{k_i,p}(\mathbb{Z} \ltimes \Lambda/p(t))$ for all i.*

Proof. One direction is clear, since if $\alpha = \alpha_1 \otimes \cdots \otimes \alpha_s, \alpha_i \in R^{irr}_{k_i}(\mathbb{Z} \ltimes \Lambda/p(t))$, then all the α_i's extend over $\mathbb{Z} \ltimes S^{-1}\Lambda/\Lambda$ by proposition 8.9, but then the tensor product extends as well.

Now we show the converse. For simplicity's sake we'll show the converse only for the case $s = 2$. Let $h(t) := \frac{(t^k-1)(t-1)}{(t^{k_1}-1)(t^{k_2}-1)}$, an easy argument (e.g. using lemma A.2) shows that indeed $h(1) = 1$, in fact $h(t)$ is the largest polynomial dividing $t^k - 1$ with this property.

Claim. Let $q(t) \in \Lambda$ such that $q(1) = 1$, then the kernel of

$$\Lambda/(q(t), t^k - 1) \xrightarrow{\cdot h(t)} \Lambda/(q(t)h(t), t^k - 1)$$

is the Λ-subspace generated by $\frac{(t^{k_1}-1)(t^{k_2}-1)}{t-1} = \frac{t^k-1}{h(t)}$. In particular, if $\chi : \Lambda/(q(t), t^k - 1) \to S^1$ factors through $\Lambda/(q(t)h(t), t^k - 1)$, then it has to vanish on $\frac{t^k-1}{h(t)}\Lambda/(q(t), t^k - 1) \subset \Lambda/(q(t), t^k - 1)$.

It is clear that the subspace generated by $\frac{(t^{k_1}-1)(t^{k_2}-1)}{t-1} = \frac{t^k-1}{h(t)} =: f(t)$ lies in the kernel. Conversely assume that $g(t)$ represents an element in the kernel. Then

$$g(t)h(t) = a(t)q(t)h(t) + b(t)(t^k - 1)$$

for some $a(t), b(t) \in \Lambda$. Therefore $(g(t) - a(t)q(t))h(t) = b(t)(t^k - 1)$. Note that $f(t)|(t^k - 1)$ and $\gcd(f(t), h(t)) = 1$, therefore $f(t)|(g(t) - a(t)q(t))$, i.e. $g(t) - a(t)q(t) = f(t)c(t)$ for some $c(t) \in \Lambda$. But this shows that $g(t) = f(t)c(t) \in \Lambda/(q(t), t^k - 1)$. This completes the proof of the claim.

Recall that for $\alpha_{(z_i,\chi_i)} \in R^{irr}_{k_i}(\mathbb{Z} \ltimes \Lambda/p(t)), i = 1, 2$ we get $\alpha_{(z_1,\chi_1)} \otimes \alpha_{(z_2,\chi_2)} = \alpha_{(z_1z_2,\chi_1\chi_2)} \in R^{irr}_{k_1k_2}(\mathbb{Z} \ltimes \Lambda/p(t))$, and conversely, if for $\alpha = \alpha_{(z,\chi)}$ we have $\chi = \chi_1 \cdot \chi_2$ with $\chi_i : \Lambda/(p(t), t^k - 1) \to \Lambda/(p(t), t^{k_i} - 1) \to S^1$, then α can be written as a tensor product.

Claim. Let $q(t) \in \Lambda$ such that $q(1) = 1$, then the kernel of

$$\Lambda/(q(t), t^k - 1) \to \Lambda/(q(t), t^{k_1} - 1) \times \Lambda/(q(t), t^{k_2} - 1)$$

is the Λ-subspace generated by $\frac{(t^{k_1}-1)(t^{k_2}-1)}{t-1} = \frac{t^k-1}{h(t)}$.

It is clear that the subspace generated by $\frac{(t^{k_1}-1)(t^{k_2}-1)}{t-1}$ lies in the kernel. Conversely, suppose that $g(t)$ represents an element in the kernel. Then, after adding an element in $q(t)\Lambda$ we can assume that $g(t)$ is of the form

$$g(t) = a(t)(t^{k_1}-1) = b(t)(t^{k_2}-1) + c(t)q(t)$$

for some $a(t), b(t), c(t) \in \Lambda$. Since $\gcd(t^{k_1}-1, \frac{t^{k_2}-1}{t-1}) = 1$ we can find $r(t), s(t) \in \Lambda$ such that

$$r(t)(t^{k_1}-1) + s(t)\frac{t^{k_2}-1}{t-1} = 1$$

Combining the two equations we get

$$\begin{aligned} g(t) = a(t)(t^{k_1}-1) &= b(t)(t^{k_2}-1) + c(t)q(t)(r(t)(t^{k_1}-1) + s(t)\tfrac{t^{k_2}-1}{t-1}) = \\ &= (c(t)q(t)s(t) + b(t)(t-1))\tfrac{t^{k_2}-1}{t-1} + c(t)q(t)r(t)(t^{k_1}-1) \end{aligned}$$

Since $\gcd(t^{k_1}-1, \frac{t^{k_2}-1}{t-1}) = 1$ it now follows that $(t^{k_1}-1)|(c(t)q(t)s(t) + b(t)(t-1))$, therefore

$$g(t) = d(t)(t^{k_1}-1)\frac{t^{k_2}-1}{t-1} + c(t)r(t)(t^{k_1}-1)q(t)$$

for some $d(t) \in \Lambda$, which proves the claim.

Now assume that $\alpha = \alpha_{(z,\chi)} \in R_k^{irr}(\mathbb{Z} \ltimes \Lambda/p(t))$ extends to $\mathbb{Z} \ltimes S^{-1}\Lambda/\Lambda$ then $\chi : \Lambda/(p(t), t^k-1)$ factors through $\Lambda/p(t) \to \Lambda/(p(t), t^k-1) \xrightarrow{\cdot h(t)} \Lambda/(p(t)h(t), t^k-1) \to S^1$. In particular by the first claim χ has to vanish on $\frac{t^k-1}{h(t)}\Lambda/(p(t), t^k-1) \subset \Lambda/(p(t), t^k-1)$. From the second claim it follows that χ can be written as a product of characters $\chi_i : \Lambda/(q(t), t^k-1) \to \Lambda/(q(t), t^{k_i}-1) \to S^1, i = 1, 2$. But this means that $\alpha_{(z,\chi,k)} = \alpha_{(z,\chi_1,k_1)} \otimes \alpha_{(1,\chi_2,k_2)}$.

The second part of the proposition follows from the observation that χ factors through a p-group if and only if all the χ_i do. $\square$

Now we can turn to the proof of theorem 8.7. Assume that K has zero WTRE–obstruction. Denote the corresponding metabolizer by P. Let $x \in P, \theta \in R_k(\mathbb{Z} \ltimes S^{-1}\Lambda/\Lambda)$ such that $\theta \circ \alpha_x \in P_{k,p}^{irr}(\pi_1(M_K))$. Since the Blanchfield pairing factors through $\Delta_K(t)^{-1}\Lambda/\Lambda$ we can in fact find $\beta_x : \pi_1(M_K) \to \mathbb{Z} \ltimes \Delta_K(t)^{-1}\Lambda/\Lambda$ and $\gamma \in P_{k,p}^{irr}(\mathbb{Z} \ltimes \Delta_K(t)^{-1}\Lambda/\Lambda)$ such that $\theta \circ \alpha_x = \gamma \circ \beta_x$. Note that γ is irreducible, since $\pi_1(M) \to \mathbb{Z} \ltimes \Delta_K(t)^{-1}\Lambda/\Lambda$ is surjective. Now write $k = \prod_{i=1}^s k_i$ such that $k_1, \dots, k_s$ are powers of different prime numbers, then according to proposition 8.10 we can write $\gamma = \gamma_1 \otimes \dots \otimes \gamma_s$ with $\gamma_i \in P_{k_i,p}^{irr}(\mathbb{Z} \ltimes \Delta_K(t)^{-1}\Lambda/\Lambda)$. Let $\alpha_i := \gamma_i \circ \beta_x$, then $\alpha_i \in P_{k_i,p}^{irr,met}(\pi_1(M_K))$ and $\theta \circ \alpha_x = \alpha_1 \otimes \dots \otimes \alpha_s$, and therefore by assumption

$$\eta_{\theta\circ\alpha_x}(M_K) = \eta_{\alpha_1\otimes\dots\otimes\alpha_s}(M_K) = 0$$

This also shows that if K has zero WRE–obstruction, then it has zero prime power Letsche–obstruction.

Now assume that K has zero prime power Letsche-obstruction, denote the corresponding metabolizer by P. It is enough to show that for any prime power k any representation $\alpha \in P_k^{irr,met}(\pi_1(M))$ vanishing on $0 \times P$, is actually of the form $\theta \circ \alpha_x$ for some $x \in P$, $\theta \in R_k(\mathbb{Z} \ltimes S^{-1}\Lambda/\Lambda)$.

Now let $\alpha = \alpha_{(z,\chi)} \in P_k^{irr,met}(\pi_1(M))$ be a representation vanishing on $0 \times P$, i.e. $\chi(P) \equiv 0$. To show it is of the form $\theta \circ \alpha_x$ we need to use an 'intermediate Blanchfield pairing', connecting the linking pairing with the Blanchfield pairing . We have pairings (cf. section 2.6)

$$\begin{array}{rrcl} \lambda_L: & TH_1(M_k) \times TH_1(M_k) & \to & \mathbb{Q}/\mathbb{Z} \\ \lambda_{Bl,k}: & TH_1(M_k) \times TH_1(M_k) & \to & S_k^{-1}\Lambda_k/\Lambda_k \\ \lambda_{Bl}: & H_1(M,\Lambda) \times H_1(M,\Lambda) & \to & S^{-1}\Lambda/\Lambda \end{array}$$

which are related as follows. For $\tilde{c}, \tilde{d} \in H_1(M,\Lambda)$ and $c := \pi_k(\tilde{c}), d := \pi_k(\tilde{d}) \in TH_1(M_k) = H_1(M,\Lambda)/(t^k - 1)$ we have

$$\begin{array}{rcl} \lambda_{Bl,k}(c,d) & = & \sum_{j=0}^{k-1} \lambda_L(c, t^j d)t^{-j} \\ \lambda_{Bl}(\tilde{c},\tilde{d}) & = & \lambda_{Bl,k}(c,d) \in S^{-1}\Lambda/\Lambda/(t^k-1) = S_k^{-1}\Lambda_k/\Lambda_k \end{array}$$

Furthermore $\lambda_L(tc,td) = \lambda_L(c,d)$. Since λ_L is non-degenerate, and S^1 is divisible, we can find $\tilde{\theta} : \mathbb{Q}/\mathbb{Z} \to S^1$ and $x \in TH_1(M_k)$ such that $\chi(v) = \tilde{\theta} \circ \lambda_L(x,v)$ for all v. Since $\chi : H_1(M,\Lambda) \to TH_1(M_k) \to S^1$ vanishes on P hence on P_k, and since $P_k = P_k^{\perp}$ (cf. proposition 2.18) we see that in fact $x \in P_k$. Using $\chi(t^l v) = \tilde{\theta} \circ \lambda_L(x, t^l v) = \tilde{\theta} \circ \lambda_L(t^{-l}x, v)$ we get that α factors as follows:

$$\pi_1(M) \to \mathbb{Z} \ltimes TH_1(M_k) \xrightarrow{\mathrm{id} \times (\lambda_L(x,-),\lambda_L(t^{-1}x,-),\dots,\lambda_L(t^{-k+1}x,-))} \mathbb{Z} \ltimes (\mathbb{Q}/\mathbb{Z})^k \to U(k)$$

where $\mathbb{Z}$ acts by cyclic permutation on $(\mathbb{Q}/\mathbb{Z})^k$, i.e. $1 \cdot (r_0,\dots,r_{k-1}) = (r_{k-1}, r_0, \dots, r_{k-2})$. Since $\{1, t, \dots, t^{k-1}\}$ forms a $\mathbb{Z}$-basis for Λ_k we get isomorphisms

$$\begin{array}{rcl} \mathbb{Z} \ltimes (\mathbb{Q}/\mathbb{Z})^k & \cong & \mathbb{Z} \ltimes (\mathbb{Q}/\mathbb{Z})[t,t^{\pm 1}]/(t^k-1) \cong \mathbb{Z} \ltimes (\mathbb{Q} \otimes \Lambda_k)/\Lambda_k \\ (n,(r_0,\dots,r_{k-1})) & \mapsto & (n, r_0 + r_1 t + \dots + r_{k-1}t^{k-1}) \end{array}$$

Using the relation between λ_L and $\lambda_{Bl,k}$ we see that the representation α factors in fact as follows:

$$\pi_1(M) \to \mathbb{Z} \ltimes TH_1(M_k) \xrightarrow{\mathrm{id} \times \lambda_{Bl,k}(x,-)} \mathbb{Z} \ltimes (\mathbb{Q} \otimes \Lambda_k)/\Lambda_k \to U(k)$$

Using that $(\mathbb{Q} \otimes \Lambda_k)/\Lambda_k \to S_k^{-1}\Lambda_k/\Lambda_k$ is an embedding and applying proposition 4.8 we see that α factors in fact as follows

$$\pi_1(M) \to \mathbb{Z} \ltimes TH_1(M_k) \xrightarrow{\mathrm{id} \times \lambda_{Bl,k}(x,-)} \mathbb{Z} \ltimes S_k^{-1}\Lambda_k/\Lambda_k \to U(k)$$

Let $\tilde{x} \in P$ be a lift of $x \in P_k = P/(t^k - 1)$. Consider the following commuting diagram:

$$\begin{tikzcd} \pi_1(M) \arrow[r] \arrow[dr] & \mathbb{Z} \ltimes H_1(M,\Lambda) \arrow[r, "\mathrm{id} \times \lambda_{Bl}(\tilde{x},-)"] \arrow[d] & \mathbb{Z} \ltimes S^{-1}\Lambda/\Lambda \arrow[d] & \\ & \mathbb{Z} \ltimes TH_1(M_k) \arrow[r, "\mathrm{id} \times \lambda_{Bl,k}(x,-)"] & \mathbb{Z} \ltimes S_k^{-1}\Lambda_k/\Lambda_k \arrow[r] & U(k) \end{tikzcd}$$

this shows that α is in fact of the form $\alpha = \tilde{\theta} \circ \alpha_x$ for some $x \in P$, $\tilde{\theta} \in R_k(\mathbb{Z} \ltimes S^{-1}\Lambda/\Lambda)$ such that $\tilde{\theta} \circ \alpha_x = \alpha \in P_k(\pi_1(M))$. This completes the proof.

9. The Cochran-Orr-Teichner-sliceness obstruction

9.1. **The Cochran-Orr-Teichner-sliceness filtration.** We give a short introduction to the sliceness filtration introduced by Cochran, Orr and Teichner [COT01]. Recall that for a group G we denote by $G^{(i)}$ the i^{th} derived group of G, defined inductively by $G^{(0)} := G$ and $G^{(i+1)} := [G^{(i)}, G^{(i)}]$. For a CW-complex W denote by $W^{(n)}$ the cover corresponding to $\pi_1(W)^{(n)}$. Denote the equivariant intersection form

$$H_2(W^{(n)}) \times H_2(W^{(n)}) \to \mathbb{Z}[\pi_1(W)/\pi_1(W)^{(n)}]$$

by λ_n, and the self-intersection form by μ_n. An (n)-Lagrangian is a submodule $L \subset H_2(W^{(n)})$ on which λ_n and μ_n vanish and which maps onto a Lagrangian of $\lambda_0 : H_2(W) \times H_2(W) \to \mathbb{Z}$.

Definition. [COT01, p. 6, p. 58] A knot K is called (n)-solvable if M_K bounds a spin 4-manifold W such that $H_1(M_K) \to H_1(W)$ is an isomorphism and such that W admits two dual (n)-Lagrangians. This means that λ_n pairs the two Lagrangians non-singularly and that the projections freely generate $H_2(W)$.

A knot K is called $(n.5)$-solvable if M_K bounds a spin 4-manifold W such that $H_1(M_K) \to H_1(W)$ is an isomorphism and such that W admits an (n)-Lagrangian and a dual $(n+1)$-Lagrangian.

We call W an (n)-solution respectively $(n.5)$-solution for K.

Remark.
1. The size of an (n)-Lagrangian depends only on the size of $H_2(N_D)$, in particular if K is slice, D a slice disk, then N_D is an (n)-solution for K for all n, since $H_2(N_D) = 0$ and $N_D = D^4 \setminus N(D)$ is spin.
2. By the naturality of covering spaces and homology with twisted coefficients it follows that if K is (h)-solvable, then it is (k)-solvable for all $k < h$.

Theorem 9.1.

$$\begin{array}{rcl} K \textit{ is (0)-solvable} & \Leftrightarrow & \textit{Arf}(K) = 0 \\ K \textit{ is (0.5)-solvable} & \Leftrightarrow & K \textit{ is algebraically slice} \\ K \textit{ is (1.5)-solvable} & \Rightarrow & \textit{Casson-Gordon invariants vanish and } K \textit{ algebraically slice} \end{array}$$

The converse of the last statement is not true, i.e. there exist algebraically slice knots which have zero Casson-Gordon invariants but are not (1.5)*-solvable.*

The first part, the third part and the $\Leftarrow$ direction of the second part have been shown by Cochran, Orr and Teichner [COT01, p. 6, p. 72, p. 66, p. 73]. We'll prove the $\Rightarrow$ direction of the second part, since there's no complete proof in the literature. Taehee Kim [K02] showed that there exist (1.0)-solvable knots which have zero Casson-Gordon invariants, but are not (1.5)-solvable (cf. also proposition 10.14).

Theorem 9.2. *Let K be a knot, F a Seifert surface and $H \subset H_1(F)$ a metabolizer for the Seifert pairing. Then there exists a* (0.5)*-solution W and a manifold $R^3 \subset W$ with*

$\partial(R) = F \cup D^2$, $D^2 \subset M_K$ being the core of the surgery, such that $H = Ker\{H_1(F) \to H_1(R)\}$.

Proof. Pick a basis $a_1, \dots, a_{2g} \in H_1(F)$ such that $H = \langle a_1, \dots, a_g \rangle_{\mathbb{Z}}$. Denote the Seifert matrix corresponding to the basis $a_1, \dots, a_{2g} \in H_1(F)$ by A. We can represent a_i by embedded circles which we'll also denote by a_i.

We can find a map $\iota : F \times [0,1] \times [-1,1] \to D^4$ with the following properties:

(1) $\iota|_{(F \times 0 \times 0)}$ is the usual embedding of F, $\iota|_{(K \times s \times t)}$ is constant in the (s,t) direction.
(2) $\iota(\text{int}(F) \times 0 \times [-1,1]) \subset S^3$.
(3) $\iota|_{(int(F) \times [0,1] \times [-1,1])}$ is an embedding.

The map ι should be viewed as a bicollared push-in of F into D^4. Write F_p for $\iota(F \times 1 \times 0)$. Let $N := D^4 \setminus (F_p \times \text{int}(D^2))$. Note that $H_1(N) = \mathbb{Z}$. Let C be a handlebody of genus g. Pick a diffeomorphism $f : F_p \cup_{K \times 0} K \times [0,1] \cup_{K \times 1} D^2 \to C$ such that $\text{Ker}\{f_* : H_1(F_p) \to H_1(C)\} = \langle a_1, \dots, a_g \rangle$. Let $W := N \cup_{f \times id|_{F_p \times S^1}} C \times S^1$. We claim that W has the required properties.

First note that $\partial(W) \cong M_K$, we'll identify $\partial(W)$ and M_K. One easily sees that $H_1(M_K) \to H_1(W)$ is an isomorphism. Denote the $\langle t \rangle = \mathbb{Z}$–fold covers of N, W by $\tilde{N}, \tilde{W}$.

We'll first study the homology of N. We can build $\tilde{N}$ from glueing together $\mathbb{Z}$-copies of $D_F := (D^4 \setminus \iota(F \times [0,1] \times (-1,1))) \cap N$, which is diffeomorphic to D^4, along $\mathbb{Z}$-copies of $F_{tr} := \iota(F \times [0,1] \times 0) \cap N$. Note that $H_1(F) = H_1(F_{tr}) = H_1(F_p)$. Let $a_{\pm 1} := \iota(a_i, \frac{1}{2}, \pm 1)$. Denote by $c_{\pm 1}(a_i)$ the cone on $a_{\pm 1}$ in D_F. Let $\tilde{\beta}_i^N := c_-(a_i) \cup -tc_+(a_i) \in H_2(\tilde{N})$. A Mayer-Vietoris argument shows that $H_2(\tilde{N}) \cong \oplus_{i=1}^{2g} \Lambda \tilde{\beta}_i^N$ and $H_1(\tilde{N}) = 0$. It's easy to see that the projections $\beta_i^N \in H_2(N)$ are dual to $a_{g+i} \in H_1(F_p)$ (where we take the index modulo $2g$) via the linking pairing, in particular $\beta_i^N = a_i \times S^1 \in H_2(N)$.

Ko shows that the intersection pairing on $H_2(\tilde{N})$ with respect to $\tilde{\beta}_1^N, \dots, \tilde{\beta}_{2g}^N$ is given by

$$A(1 - t^{-1}) + A^t(1 - t)$$

and the self-intersection pairing is given by

$$\tilde{\beta}_i^N \mapsto (1 - t^{-1})\text{lk}(a_i, a_{i,+}) \in \Lambda / \{p(t) - p(t^{-1}) | p(t) \in \Lambda\}$$

Since A is metabolic it follows that the Λ-intersection pairing and the self-intersection pairing vanish on $\langle \tilde{\beta}_1^N, \dots, \tilde{\beta}_g^N \rangle_\Lambda$

Now let's turn to the homology of W. Consider the following Mayer-Vietoris sequence

$$\to \underbrace{H_2(C \times \mathbf{R})}_{=0} \oplus H_2(\tilde{N}) \xrightarrow{i_*} H_2(\tilde{W}) \xrightarrow{\partial} H_1(F_p \times \mathbf{R}) \to H_1(C \times \mathbf{R}) \oplus \underbrace{H_1(\tilde{N})}_{=0} \to \dots$$

Ko [K89, p. 539] showed that there exists a Λ-basis $\tilde{\beta}_1, \dots, \tilde{\beta}_{2g} \in H_2(\tilde{W})$ such that i_* with respect to the bases $\tilde{\beta}_i^N$ and $\tilde{\beta}_i$ is given by the matrix

$$J := \operatorname{diag}(\underbrace{1-t, \dots, 1-t}_{g \text{ times}}, \underbrace{1, \dots, 1}_{g \text{ times}})$$

and such that $\partial(\tilde{\beta}_i) = a_i, i = 1, \dots, g$. Since the map i_* preserves intersection numbers and since the intersection forms are are Λ-linear we see that that the Λ-intersection paring and the Λ-self-intersection pairing of W vanish on $\langle \tilde{\beta}_1, \dots, \tilde{\beta}_g \rangle_\Lambda$.

Denote the projections of $\tilde{\beta}_i$ by β_i. The intersection pairing on $H_2(W)$ vanishes by naturality on $\langle \beta_1, \dots, \beta_g \rangle_{\mathbb{Z}}$. We are done once we show that $H_2(W)$ is of rank $2g$ and that $\beta_1, \dots, \beta_g$ span a subspace of rank g. Consider the Mayer-Vietoris sequence

$$\begin{array}{rcccccl} \dots \to & H_2(F_p \times S^1) & \to & H_2(C \times S^1) \oplus H_2(N) & \to & H_2(W) \to \\ \to & H_1(F_p \times S^1) & \to & H_1(C \times S^1) \oplus H_1(N) & \to & \dots \end{array}$$

One easily sees that $H_2(W)$ is of rank $2g$. Consider

$$\begin{array}{ccc} H_2(\tilde{W}) & \to & H_1(F_p \times \mathbf{R}) \\ \downarrow & & \downarrow \\ H_2(W) & \to & H_1(F_p \times S^1) \end{array}$$

From this commutative diagram it follows that $\beta_1, \dots, \beta_g$ span a subspace of rank g since the image of $\tilde{\beta}_1, \dots, \tilde{\beta}_g$ in $H_1(F_p \times S^1)$ spans a subspace of rank g. Thus W is a (0.5)-solution for K. It's clear that $R := C \cup \iota(F \times [0,1] \times 0)$ has the required properties. $\square$

We conclude this section with a side remark on algebraically doubly slice knots. A knot $K \subset S^3$ is called doubly slice if there exists an unknotted two-sphere $S \subset S^4$ such that $S \cap S^3 = K$.

We say that a knot K is algebraically doubly slice if K has a Seifert matrix of the form $\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}$ where B, C are square matrices of the same size. Sumners [S71] showed that any doubly slice knot is also algebraically doubly slice.

We get the following corollary from the proof of theorem 9.2.

Corollary 9.3. *Let K be a knot which is algebraically doubly slice, then K is in fact (1)-solvable.*

Proof. Let F be a Seifert surface and $a_1, \dots, a_{2g} \in H_1(F)$ such that $\langle a_1, \dots, a_g \rangle_{\mathbb{Z}}$ and $\langle a_{g+1}, \dots, a_{2g} \rangle_{\mathbb{Z}}$ are dual metabolizers for the Seifert pairing. We use the notation and results of the above proof. The Λ-intersection pairing and the Λ-self intersection pairing vanish on $\langle \tilde{\beta}_1, \dots, \tilde{\beta}_g \rangle_\Lambda$ and $\langle \tilde{\beta}_{g+1}, \dots, \tilde{\beta}_{2g} \rangle_\Lambda$. It only remains to show that

$\beta_{g+1}, \ldots, \beta_{2g} \in H_2(W)$ span a subspace of dimension g. Consider

$$\begin{array}{ccccc} & & H_2(\tilde{N}) & \to & H_2(\tilde{W}) \\ & & \downarrow & & \downarrow \\ H_2(F_p \times S^1) \to H_2(C \times S^1) & \oplus & H_2(N) & \to & H_2(W) \end{array}$$

Note that

$$(H_2(C \times S^1) \oplus H_2(N))/H_2(F_p \times S^1) \cong H_2(N)/\langle a_1 \times S^1, \ldots, a_g \times S^1 \rangle$$

Since $\beta_{g+i} = \beta^N_{g+i} = a_{g+i} \times S^1$ we get that

$$\langle \tilde{\beta}^N_1, \ldots, \tilde{\beta}^N_g \rangle_{\mathbb{Z}} \quad \to \quad H_2(N)/\langle a_1 \times S^1, \ldots, a_g \times S^1 \rangle$$

is an injection, which completes the proof. □

9.2. L^2**-eta invariants as sliceness-obstructions.** In this section we'll very quickly summarize some L^2-signature and L^2-eta invariant theory.

Let M^3 be a smooth manifold and $\varphi : \pi_1(M) \to G$ a homomorphism, then Cheeger and Gromov [CG85] defined an invariant $\eta^{(2)}_\varphi(M) \in \mathbf{R}$, the (reduced) L^2-eta invariant. When it's clear which homomorphism we mean, we'll write $\eta^{(2)}_G(M, G)$ for $\eta^{(2)}_\varphi(M)$.

It W^4 is a smooth manifold and $\psi : \pi_1(W) \to G$ a homomorphism then Atiyah [A76] defined an L^2-signature $\text{sign}^{(2)}(W, \psi)$ which agrees with the definition of L^2-signature given by Cochran, Orr and Teichner [COT01, lemma 5.9]. The fundamental theorem linking these definitions is the following.

Theorem 9.4. [COT01, Remark 5.10] *If $\partial(W, \psi) = (M^3, \varphi)$, then*

$$\eta^{(2)}_\varphi(M) = sign^{(2)}(W, \psi) - sign(W)$$

Cochran, Orr and Teichner study when L^2-signatures vanish for homomorphisms $\pi_1(M_K) \to G$, where G is a PTFA-group. PTFA stands for poly-torsion-free-abelian, and means that there exists a normal subsequence where each quotient is torsion-free-abelian.

Theorem 9.5. [COT02, p. 5] *Let G be a PTFA-group with $G^{(n)} = 1$. If K is a knot, and $\varphi : \pi_1(M_K) \to G$ a homomorphism which extends over a $(n.5)$-solution of M_K, then $\eta^{(2)}_\varphi(M_K) = 0$. In particular if K is slice and φ extends over N_D for some slice disk D, then $\eta^{(2)}_\varphi(M_K) = 0$.*

It's a crucial ingredient in the proposition that the group G is a PTFA-group, for example it's not true in general that $\eta^{(2)}_{\mathbb{Z}/k}(M_K) = 0$ for a slice knot K. One can show that $\eta^{(2)}_{\mathbb{Z}/k}(M_K) = \sum_{j=1}^k \sigma_{e^{2\pi i j/k}}(K)$, but this can be non-zero for some slice knot K, e.g. take a slice knot with Seifert matrix

$$A\begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Then $\eta^{(2)}_{\mathbb{Z}/6}(M_K) = -2$ (cf. section 10.4).

This theorem gives an easy way to define strong sliceness-obstructions. It should be compared to proposition 4.7. For eta-invariants we needed not only that the representations extend over $\pi_1(N_D)$, but that they are also of a certain type.

Let $\mathbb{Q}\Lambda := \mathbb{Q}[t, t^{-1}]$.

Theorem 9.6. [COT01] *Let K be a slice knot, then*

(1) $\eta^{(2)}_{\mathbb{Z}}(M_K) = 0$.
(2) *There exists a metabolizer $P_{\mathbb{Q}} \subset H_1(M_K, \mathbb{Q}\Lambda)$ for the Blanchfield pairing*

$$\lambda_{Bl,\mathbb{Q}} : H_1(M_K, \mathbb{Q}\Lambda) \times H_1(M_K, \mathbb{Q}\Lambda) \to \mathbb{Q}(t)/\mathbb{Q}[t, t^{-1}]$$

such that for all $x \in P_{\mathbb{Q}}$ we get $\eta^{(2)}(M_K, \beta_x) = 0$ where β_x denotes the map

$$\pi_1(M_K) \to \mathbb{Z} \ltimes H_1(M_K, \Lambda) \to \mathbb{Z} \ltimes H_1(M_K, \mathbb{Q}\Lambda) \xrightarrow{\lambda_{Bl,\mathbb{Q}}(x,-)} \mathbb{Z} \ltimes \mathbb{Q}(t)/\mathbb{Q}[t, t^{-1}]$$

Proof. Let D be a slice disk for K. The first part follows immediately from proposition 9.5 since $H_1(M_K) \to H_1(N_D)$ is an isomorphism. The proof of proposition 2.7 shows that $P_{\mathbb{Q}} := \operatorname{Ker}\{H_1(M_K, \mathbb{Q}\Lambda) \to H_1(N_D, \Lambda\mathbb{Q})\}$ is a metabolizer for $\lambda_{Bl,\mathbb{Q}}$. The proof of theorem 8.6 shows that β_x extends over $\pi_1(N_D)$. The second part now follows from proposition 9.5. □

Lemma 9.7. *The maps*

$$\begin{array}{rcl} \{\textit{metabolizer of } \lambda_{Bl}\} & \leftrightarrow & \{\textit{metabolizer of } \lambda_{Bl,\mathbb{Q}}\} \\ P & \mapsto & P \otimes \mathbb{Q} \\ P_{\mathbb{Q}} \cap H_1(M_K, \Lambda) & \leftarrow & P_{\mathbb{Q}} \end{array}$$

are inverses of each other and therefore define a bijection.

Proof. Let P be a metabolizer for λ_{Bl} and assume that $nv \in P$ for some $v \in H_1(M_K, \Lambda), n \in \mathbb{Z}$, then in fact $v \in P$. This follows from the observation that if $np \in \Lambda \cap S^{-1}\Lambda/\Lambda$ for some $p \in S^{-1}\Lambda/\Lambda, n \in \mathbb{Z}$, then $p \in \Lambda$ since $S \cap \mathbb{Z} = \{\pm 1\}$. The lemma now follows immediately. □

We say that a knot K has zero L^2-eta invariant of level 0 if $\eta^{(2)}_{\mathbb{Z}}(M_K) = 0$ and K has zero L^2-eta invariant of level 1 if there exists a metabolizer $P_{\mathbb{Q}} \subset H_1(M_K, \mathbb{Q}\Lambda)$ for $\lambda_{Bl,\mathbb{Q}}$ such that for all $x \in P_{\mathbb{Q}}$ we get $\eta^{(2)}_{\beta_x}(M_K)) = 0$. Note that if K has zero

L^2-eta invariant of level 1 then by the above lemma and theorem 2.6 K is in particular algebraically slice.

10. Examples

In this section we'll construct

(1) a knot which has zero L^2-eta invariant of level 0, but is not algebraically slice,
(2) a (1)-solvable knot which has zero L^2-eta invariant of level 1, but non-zero SE–obstruction,
(3) a knot which has zero L^2-eta invariant of level 1, zero STE–obstruction, but is not ribbon, and
(4) a knot which has zero STE–obstruction but non-zero L^2–eta invariant of level 1,
(5) a ribbon knot K such that there exists no metabolizer P_5 of $(H_1(L_5), \lambda_L)$ with the property that $\tau(K, \chi) = 0$ for all (including non prime power) characters $\chi : TH_1(M_5) \to S^1$.

The idea is in each case to start out with a slice knot K and make 'slight' changes via a satellite construction, the change in the eta invariants can be computed explicitly.

10.1. **Satellite knots.** We'll give two definitions of satellite knots. The first one is easier to visualize, the second one is more useful for general constructions. Let $K \subset S^1 \times D^2$ be an oriented knot. Let g be the oriented generator of $H_1(S^1 \times D^2) = \mathbb{Z}$. The number $w = w(K, S^1 \times D^2)$ defined by $[K] = wg \in H_1(T)$ is called the winding number of $K \subset S^1 \times D^2$. Now let $C \subset S^3$ be an oriented knot.

Pick an orientation preserving diffeomorphism $\varphi : S^1 \times D^2 \to N(C)$ such that $\varphi_*(g) = [C] \in H_1(N(C))$ and $\varphi(S^1 \times 1)$ is a longitude for C. Denote the image of K by S and the image of $1 \times S^1$ by A. Then $S \subset S^3$ is called the satellite knot with companion C, orbit K, winding number w and axis A, we write $S = S(K, C)$.

We give an alternative definition of satellite knots. Let $K, C \in S^3$ be knots, and $A \subset S^3 \setminus K$ a curve, unknotted in S^3, then $S^3 \setminus N(A)$ is a torus. Let $\varphi : \partial(N(A)) \to \partial(N(C))$ be a diffeomorphism which sends a meridian of A to a longitude of C and a longitude of A to a meridian of C. The space

$$S^3 \setminus N(A) \cup_\varphi S^3 \setminus N(C)$$

is a 3-sphere and the image of K is denoted by $S = S(K, C, A)$.

Pick $\psi : S^3 \setminus N(A) \to S^1 \times D^2$ such that a meridian of A gets send to $S^1 \times z$ for some $z \in S^1$, then it is easy to see that $S(K, C, A) = S(\psi(K), C)$ furthermore these two constructions give the same set of knots.

Proposition 10.1. *If $K \subset S^1 \times D^2 \subset S^3$ and C are slice (ribbon), then any satellite knot with orbit K and companion C is slice (ribbon) as well.*

Proof. Let $K \subset S^1 \times D^2 \subset S^3$ and C be slice knots. Let $\phi : S^1 \times I \to S^3 \times I$ be a null-concordance for C, i.e. $\phi(S^1 \times 0) = C$ and $\phi(S^1 \times 1)$ is the unknot. We can extend this to a map $\phi : S^1 \times D^2 \times I \to S^3 \times I$ such that $\phi : S^1 \times D^2 \times 0$ is the zero-framing for C. Now consider

$$\psi : S^1 \times I \to K \times I \hookrightarrow S^1 \times D^2 \times I \xrightarrow{\phi} S^3 \times I$$

Note that $\phi : S^1 \times D^2 \times 1$ is a zero framing for the unknot, since linking numbers are concordance invariants and $\phi : S^1 \times D^2 \times 0$ is the zero framing for C. This shows that $\psi : S^1 \times 1 \to S^3 \times 1$ gives the satellite knot of the unknot with orbit K, i.e. K itself. Therefore ψ gives a concordance between $S = \psi(S^1 \times 0)$ and $K = \psi(S^1 \times 1)$, but K is null-concordant, therefore S is null-concordant as well.

Now assume that K, C are ribbon. Then we can find a concordance ϕ which has no minima under the projection $S^1 \times [0,1] \to S^3 \times [0,1] \to [0,1]$. It is clear that ψ also has no minima, capping off with a ribbon disk for K we get a disk bounding S with no minima, i.e. S is ribbon. $\square$

Proposition 10.2. [COT02, p. 8] *Let K be an (n)-solvable knot, C any (0)-solvable knot, $A \subset S^3 \setminus K$ such that A is the unknot in S^3 and $[A] \in \pi_1(S^3 \setminus K)^{(n)}$. Then $S = S(K, C, A)$ is (n)-solvable.*

Note that we need that at least C is (0)-solvable, i.e. $Arf(C) = 0$ which is equivalent to $\Delta_C(-1) \equiv \pm 1 \mod 8$.

10.2. **Eta invariants and L^2-eta invariants of satellite knots.** We'll compute the eta invariants of satellite knots with winding number zero, following Litherland [L84, p. 338], but generalizing his results to any S^1-character living on cyclic cover of S^3 branched along S.

Let S be a satellite knot with companion C, orbit K, axis A and winding number 0 and let k be a number, not necessarily a prime power.

The curve $A \subset S^3 \setminus N(K)$ is null-homologous since the winding number is zero, and therefore lifts to curves $\tilde{A}_1, \ldots, \tilde{A}_k \in L_{K,k}$. Let $N(\tilde{A}_1), \ldots, N(\tilde{A}_k)$ be disjoint tubular neighborhoods of the $\tilde{A}_i$ projecting down to some fixed neighborhood $N(A)$. Lift the meridian and longitude of A to $L_{K,k}$ and call the resulting curves meridian and longitude of $\tilde{A}_1, \ldots, \tilde{A}_k$.

Claim.

$$L_{S,k} \cong (L_{K,k} \setminus \cup_{i=1}^k N(\tilde{A}_i)) \cup_{\partial(N(\tilde{A}_i)) = \partial(N(C) \times i)} \bigcup_{i=1}^k (S^3 \setminus N(C)) \times i$$

where $\partial(N(\tilde{A}_i))$ and $\partial(N(C)) \times i$ are identified as in the construction of satellite knots, this means the meridian of $\tilde{A}_i$ gets identified with the longitude of C in $(S^3 \setminus N(C)) \times i$ and the longitude of $\tilde{A}_i$ gets identified with the meridian of C in $(S^3 \setminus N(C)) \times i$.

Proof. Denote the right hand side of the statement by L, consider the following map:

$$\begin{array}{rrcl} p: & L & \to & S^3 \setminus N(A) \cup_{\partial(N(A))=\partial(N(C))} S^3 \setminus N(C) \\ & (x,(y,i)) & \mapsto & (\pi(x), y) \end{array}$$

where $\pi : L_{K,k} \to S^3$ denotes the canonical projection. It is now easy to check that (L,p) is the k-fold cover of $S^3 \setminus N(A) \cup S^3 \setminus N(C)$ branched along K, which is just the k-fold cover of S^3 branched along S. $\square$

Lemma 10.3. *There are canonical isomorphisms*

$$H_1(L_{K,k}) \xleftarrow{\cong} H_1(L_{K,k} \setminus \cup_{i=1}^k N(\tilde{A}_i))/\{\textit{meridians of } \tilde{A}_i\} \xrightarrow{\cong} H_1(L_{S,k})$$

and the isomorphism $H_1(L_{K,k}) \to H_1(L_{S,k})$ preserves the linking pairing.

Proof. It is easy to see that the first map is an isomorphism. A Meyer-Vietoris sequence argument shows that the map

$$H_1(L_{K,k} \setminus \cup_{i=1}^k N(\tilde{A}_i))/\{\text{meridians of } \tilde{A}_i\} \to H_1(L_{S,k})$$

is an isomorphism. It remains to show the statement about the linking pairings. Let $x, y \in H_1(L_{K,k})$, they can be represented by cycles in $L_{K,k} \setminus \cup_{i=1}^k N(\tilde{A}_i)$, call the cycles x, y as well. Let D_K be a 2-chain in $L_{K,k}$ with $\partial(D_K) = ry$ for some $r > 0$, transverse to the $\tilde{A}_i$. Note that $D_K \cap N(\tilde{A}_i) \subset S^3 \setminus N(C)$ represents multiples of the longitude of C, hence is null-homologous and therefore bounds a surface F_i. Now let $D_S := D_K \setminus (D_K \cap \cup_{i=1}^k N(\tilde{A}_i)) \cup \cup_{i=1}^k F_i \subset L_{S,k}$, then $\partial D_S = ry$. We get

$$\lambda_L^K(x,y) = \frac{1}{r} x \cdot D_K = \frac{1}{r} x \cdot D_S = \lambda_L^S(x,y)$$

$\square$

This lemma shows in particular that we can identify the set of characters on $H_1(L_{K,k})$ with the set of characters on $H_1(L_{S,k})$. Recall that for a knot K, a number k, a character $\chi : H_1(L_k) \to S^1$ and $z \in S^1$ we defined

$$\begin{array}{rrcl} \beta^K_{(\chi,z)}: & \pi_1(M_k) & \to & S^1 \\ & g & \mapsto & \chi(g) z^{\epsilon(g)} \end{array}$$

where $\epsilon : \pi_1(M_k) \to \mathbb{Z}$ is the canonical surjection. Note that if $A \subset S^3 \setminus K$ is an axis for $S = S(K, C, A, 0)$, then it is null-homologous and therefore lifts to curves $\tilde{A}_1, \dots, \tilde{A}_k \in L_{K,k}$.

Proposition 10.4. *If $S = S(K,C,A)$ is a satellite knot with winding number zero, then*

$$(H_1(M_K,\Lambda), \lambda_{Bl}) \cong (H_1(M_S,\Lambda), \lambda_{Bl})$$

and S, K have S-equivalent Seifert matrices. In particular K is algebraically slice if and only if S is algebraically slice.

The proof of the first part is similar to the above lemma and will be omitted. The second part follows from a theorem of Trotter's [T73].

Theorem 10.5. *Let $S = S(K, C, A, 0)$ be a satellite knot. Let k be any number, $z \in S^1$ and $\chi : H_1(L_{S,k}) \to S^1$ a character, denote the corresponding character $H_1(L_{K,k}) \to S^1$ by χ as well. Then*

$$\eta(M_{S,k}, \beta^S_{(\chi,z)}) = \eta(M_{K,k}, \beta^K_{(\chi,z)}) + \sum_{i=1}^{k} \eta(M_C, \alpha_i)$$

where α_i denotes the representation $\pi_1(M_C) \to U(1)$ given by $g \mapsto \chi(\tilde{A}_i)^{\epsilon(g)}$.

Remark. This theorem and its proof are basically contained in Litherland [L84]. Litherland proves a general statement how to compute the Casson-Gordon invariant of S in terms of the Casson-Gordon invariant of K and the basic invariants of C. Translating the proof into the language of eta invariants gives the proof of theorem 10.5. The advantage of looking at eta invariants is that they make sense even if k, m are not prime powers, whereas Casson-Gordon invariants are only defined if the corresponding intersection forms are non-singular. This case was not considered by Litherland, but his methods carry through without problems.

Proof. Write β_S for $\beta^S_{(\chi,z)}$ and β_K for $\beta^K_{(\chi,z)}$. The representation β_K factors through a group of the form $\mathbb{Z} \times \mathbb{Z}/m$. Since $\Omega_3(\mathbb{Z} \times \mathbb{Z}/m)$ is torsion (cf. appendix A.2) and $\Omega_3(\mathbb{Z}) = 0$ we can find $r > 0$ and pairs $(W_K, \psi_K), (W_1, \psi_1), \dots, (W_k, \psi_k)$ of 4-manifolds and 1-dimensional characters such that

$$\begin{array}{rcl} \partial(W_K, \psi_K) & = & r(M_{K,k}, \beta_K) \\ \partial(W_i, \psi_i) & = & r(M_C, \alpha_i), \quad i = 1, \dots, k \end{array}$$

Let $U \subset M_C$ be the surgery solid torus and let $V_i \subset M_{K,k}$ be a set of disjoint tubular neighborhoods of $\tilde{A}_i$. For $i = 1, \dots, k, j = 1, \dots, r$ denote by U_{ij} the copy of U in the j^{th} boundary component of W_i, and let V_{ij} be the copy of V_i in the j^{th} boundary component of W_K. We can construct

$$W_S := W_K \cup_{V_{ij} = U_{ij}} \bigcup_{i=1}^{k} W_i$$

where each U_{ij} is glued to V_{ij}, so that $\partial(W_S) = rM_{S,k}$. Note that ψ_i and ψ_K factor through the first homology group and agree on the common boundary by definition of α_i. We therefore get a well-defined map $\psi_S : \pi_1(W_S) \to H_1(W_S) \to U(1)$, so that

$$\partial(W_S, \psi_S) = r(M_{S,k}, \beta_S)$$

We can use (W_S, ψ_S) to compute $\eta(M_{S,k}, \beta_S)$. We are done once we show that the signatures of the ordinary and the twisted intersection pairing of W_S is the sum of the corresponding signatures of W_K and $W_1,\dots,W_k$. We'll show this using Wall additivity. Let

$$\begin{array}{rcl} A_K & := & \operatorname{Ker}\{H_1(\partial(V_{ij}))\to H_1(M_{K,k}\setminus V_{ij})\}=\mathbb{Z}\cdot\{\text{meridian of } C\} \\ A_i & := & \operatorname{Ker}\{H_1(\partial(U_{ij}))\to H_1(M_C\setminus U_{ij})\}=\mathbb{Z}\cdot\{\text{longitude of } \tilde{A}_i\}, \quad i=1,\dots,k \end{array}$$

We see that A_K and A_i agree under the identification maps. We will show that the same statement is true for twisted homology, in fact we'll show that for non-trivial coefficients $H_1(\partial(V_{ij}))=H_1(\partial(U_{ij}))=0$. This follows once we show that for any torus T and non-trivial representation $\beta:\pi_1(T)\to S^1$ of finite order m which fixes a generator v of $H_1(T)$ we get $H_1^\beta(T,\mathbb{C})=0$. Indeed, denote by $\tilde{T}$ the m-fold cover of T corresponding to α, then using lemma A.1 we get

$$H_1^\beta(T,\mathbb{C})=H_1(C_*(\tilde{T},\mathbb{Q})\otimes_{\mathbb{Q}[\mathbb{Z}/m]}\mathbb{C})=H_1(C_*(\tilde{T},\mathbb{Q}))\otimes_{\mathbb{Q}[\mathbb{Z}/m]}\mathbb{C}=H_1(\tilde{T},\mathbb{Q})\otimes_{\mathbb{Q}[\mathbb{Z}/m]}\mathbb{C}$$

Note that $H_1(\tilde{T},\mathbb{Q})$ is a trivial $\mathbb{Q}[\mathbb{Z}/m]$-module since β fixes a generator of $H_1(T)$. But $\mathbb{C}$ is a non-trivial $\mathbb{Q}[\mathbb{Z}/m]$-module, hence $H_1^\beta(T,\mathbb{C})=H_1(\tilde{T},\mathbb{Q})\otimes_{\mathbb{Q}[\mathbb{Z}/m]}\mathbb{C}=0$. The result on signatures now follows from Wall additivity (cf. [W69] and [L84, p. 330]). □

Corollary 10.6. *Let $S=S(K,C,A,0)$ be a satellite knot. Let k be any number such that $H_1(L_{K,k})$ is finite, $z\in S^1$ and $\chi:H_1(L_{S,k})\to S^1$ a character, denote the corresponding character $H_1(L_{K,k})\to S^1$ by χ as well. Then*

$$\eta(M_S,\alpha^S_{(\chi,z)})=\eta(M_K,\alpha^K_{(\chi,z)})+\sum_{i=1}^k\eta(M_C,\alpha_i)$$

where α_i denotes the representation $\pi_1(M_C)\to U(1)$ given by $g\mapsto\chi(\tilde{A}_i)^{\epsilon(g)}$.

Proof. Using the above theorem and proposition 5.6 we see that it is enough to show that $\mu(M_S,k)=\mu(M_K,k)$. By proposition 10.4 the knots K and S have S-equivalent Seifert matrices, in particular the twisted signatures are the same (cf. proposition 3.7), and from proposition 5.6 we get $\mu(M_S,k)=\mu(M_K,k)$. □

We can slightly generalize the theorem as follows (cf. [L01]). Let $K\in S^3$ be a knot and $A_1,\dots,A_s\in S^3\setminus K$ be simple closed curves which form the unlink in S^3 and such that $[A_i]=0\in H_1(S^3\setminus K)$. Let $C_1,\dots,C_s$ be knots. Then we can inductively form satellite knots by setting $S_0:=K$ and S_i the satellite formed with orbit S_{i-1}, companion C_i and axis A_i. Note that the winding number is 0 at each point since $A_0,\dots,A_s$ form the unlink and since $[A_i]=0\in H_1(S^3\setminus K)$. We write

$$S_i=:S(K,C_1,\dots,C_i,A_1,\dots,A_i)$$

Theorem 10.7. *Let $S:=S(K,C_1,\dots,C_s,A_1,\dots,A_s)$ as above. Let k be any number such that $H_1(L_{K,k})$ is finite, $z\in S^1$ and $\chi:H_1(L_{S,k})\to S^1$ a character, denote the* *corresponding character $H_1(L_{K,k}) \to S^1$ by χ as well. Then*

$$\eta(M_S, \alpha^S_{(\chi,z)}) = \eta(M_K, \alpha^K_{(\chi,z)}) + \sum_{j=1}^{s}\sum_{i=1}^{k} \eta(M_{C_j}, \alpha_{ij})$$

where α_{ij} denotes the representation $\pi_1(M_{C_j}) \to U(1)$ given by $g \mapsto \chi((\tilde{A}_j)_i)^{\epsilon(g)}$, $(\tilde{A}_j)_1, \dots, (\tilde{A}_j)_k$ being the lifts of A_j to $L_{S,k}$.

We conclude this section by quoting a theorem by Cochran, Orr and Teichner on the computation of L^2-eta invariants for satellite knots (cf. [COT02, p. 8] and [K02, prop. 5.3]).

Theorem 10.8. *Let $S = S(K, C, A)$ with $A \in \pi_1(S^3 \setminus K)^{(1)}$, in particular A defines an element in $H_1(M_K, \Lambda)$. Let $x \in H_1(M_S, \Lambda) = H_1(M_K, \Lambda)$, then*

$$\eta^{(2)}(M_S, \beta_x) = \begin{cases} \eta^{(2)}(M_K, \beta_x) + \eta^{(2)}(M_C, \mathbb{Z}) & \text{if } A \neq 0 \in H_1(M_K, \Lambda) \\ \eta^{(2)}(M_K, \beta_x) & \text{if } A = 0 \in H_1(M_K, \Lambda) \end{cases}$$

10.3. **Construction of slice and ribbon knots.** We say that an integral matrix A is a Seifert matrix if $\det(A - A^t) = 1$. We define $\Delta_A(t) := \det(At - A^t)$.

Proposition 10.9. (1) *Any Seifert matrix A can be realized as the Seifert matrix of a knot $K \subset S^3$. Any metabolic Seifert matrix can be realized by a ribbon knot.*

(2) *Any polynomial $f(t) \in \mathbb{Z}[t, t^{-1}]$ such that $f(1) = 1$ and $f(t) = t^l f(t^{-1})$ for some l can be realized as the Alexander polynomial of a knot, and if $f(t)$ is a norm, i.e. $f(t) = t^l g(t) g(t^{-1})$ for some $g(t) = a_0 + a_1 t + \dots + a_d t^d \in \mathbb{Z}[t, t^{-1}]$ and some l, then $f(t)$ can be realized as the Alexander polynomial of a ribbon knot.*

Proof. (1) The first part is a very classical statement, first proven by Seifert [S34]. We'll give a quick proof to be able to prove the second statement. Let A be a Seifert matrix of size $2g$, the matrix $A - A^t$ is skew-symmetric, we can therefore find an integral invertible matrix P such that $P(A - A^t)P^t$ is of the form $g \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, i.e. $P(A - A^t)P^t$ is block diagonal, with blocks $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Without loss of generality we can assume that $A - A^t$ is of this form. Write $A = (a_{ij})$. We can find oriented simple closed curves $L_1, \dots, L_{2g} \in \mathbf{R}^3 \subset S^3$ with the following properties:

(a) We can write $L_j = A_j \cup_{C_j} B_j$, where

(i)

$$\begin{aligned} \partial(A_{2j-1}) &= C_{2j-1} = e^{2\pi(4j-1)i/(4g)} \cup -e^{2\pi(4j-3)i/(4g)} \\ \partial(A_{2j}) &= C_{2j} = e^{2\pi 4j/(4g)} \cup -e^{2\pi(4j-2)i/(4g)} \end{aligned}$$

the '–' sign denoting orientation and not coordinates,

(ii) $A_j \setminus C_j \subset S^3 \setminus D^2$,

(iii) B_{2j-1} is the cone of C_j on $\frac{1}{2}e^{2\pi(4j-2)/(4g)}$, B_{2j} is the cone of C_j on $\frac{1}{2}e^{2\pi(4j-1)/(4g)}$,

(iv) $A_1, \dots, A_{2g}$ are disjoint.

(b) $\mathrm{lk}(L_i, L_j^+) = a_{ij}$ for $i \neq j$. Here we denote by L_j^+ the curve L_j pushed of D^2 in the positive direction.

Now attach bands to D^2 along the A_j with twisting number a_{jj} and smoothen the boundary, Denote the resulting surface by F. We can give F the orientation induced by the canonical orientation of D^2, then $\partial(F)$ is an oriented knot with the required properties.

Now assume that A is metabolic. We can assume that $A = (a_{ij})$ is such that $a_{2k-1,2l-1} = 0, k, l = 1, \dots, g$. Then have to arrange the $L_1, L_3, \dots, L_{2g-1}$ such that $\mathrm{lk}(L_{2k-1}, L_{2l-1}^+) = 0$. In fact we can and will arrange $L_1, L_3, \dots, L_{2g-1}$ such that they form the unlink. Form F as above. Pick a 'tubular' neighborhood $L_i \times [-1,1] \subset \mathrm{int}(F)$ for $i = 1, 3, \dots, 2g-1$. We can find disks $D_i^j, i = 1, 3, \dots, 2g-1, j = 1, 2$ such that

(a) $D_i^j \cap D_k^l = \emptyset$ unless $i = k, j = l$,

(b) $\partial(D_i^1) = L_i \times -1, \partial(D_i^2) = L_i \times 1$,

(c) the intersections $\mathrm{int}(D_i^j)$ with F are transversal.

Now form a new surface $\tilde{F}$ from F by replacing $L_i \times [-1,1], i = 1, 3, \dots, 2g-1$ by $D_i^1 \cup D_i^2$.We claim that $\tilde{F}$ is a ribbon disk for K. It is clear that $\tilde{F}$ has only ribbon-type self-intersections. Pushing $\tilde{F}$ into D^4 we get an embedded manifold, it is easy to see that the intersection form is 0, therefore the embedded manifold is a disk (cf. [K87, p. 217]).

(2) The first statement was proven by Seifert [S34]. Terasaka [T59] shows that any polynomial of the form $f(t) = t^l g(t) g(t^{-1})$ can be realized by a slice knot. But this shows that $f(t)$ can be realized by a metabolic Seifert matrix, hence, by part 1, by a ribbon knot.

□

10.4. **Examples.** Let

$$B_1 := \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

this Seifert matrix is obviously metabolic. The Alexander polynomial is $\Delta_{B_1}(t) = (t^2 - t + 1)^2$. The signature function $z \mapsto \sigma_z(B_1)$ is zero outside of the set of zeros of the Alexander polynomial since the form is metabolic. The zeros are $e^{2\pi i/6}, e^{2\pi 5i/6}$

and at both points the signature is -1. Let

$$B_2 := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Then $\Delta_{B_2}(t) = t^2 - t + 1$, and

$$\sigma_{e^{2\pi i t}}(B_3) = \begin{cases} 2 & \text{for } t \in (\frac{1}{6}, \frac{5}{6}) \\ 0 & \text{for } t \in [0, \frac{1}{6}) \cup (\frac{5}{6}, 1] \end{cases}$$

The Alexander polynomial is $\Delta_{B_2}(t) = t^2 - t + 1$. The zeros of the Alexander polynomial are $e^{2\pi i/6}, e^{2\pi 5i/6}$, the signature function is 2 for $z = e^{\varphi i}, \pi/3 < \varphi < 5/3\pi$ and 0 for $z = e^{\varphi i}, \varphi < \pi/3$ or $\varphi > 5/3\pi$.

Finally let

$$B_3 := \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 \end{pmatrix}$$

Then $\Delta_{B_3}(t) = \Phi_{14}(t) = 1 - t + t^2 - t^3 + t^4 - t^5 + t^6$, and

$$\sigma_{e^{2\pi i t}}(B_3) = \begin{cases} 2 & \text{for } t \in (\frac{1}{14}, \frac{3}{14}) \cup (\frac{5}{14}, \frac{9}{14}) \cup (\frac{11}{14}, \frac{13}{14}) \\ 0 & \text{for } t \in [0, \frac{1}{14}) \cup (\frac{3}{14}, \frac{5}{14}) \cup (\frac{9}{14}, \frac{11}{14}) \cup (\frac{13}{14}, 1] \end{cases}$$

Proposition 10.10 (Example 1)**.** *There exists a (0)-solvable knot K with zero L^2-eta invariant of level 0 but that is not algebraically slice.*

Proof. Recall that for a knot $\mathrm{Arf}(K) = 0$ if and only if $\Delta_K(-1) \equiv \pm 1 \mod 8$. Now let K be a knot with Seifert matrix $7B_3 \oplus -6B_2$, then $\mathrm{Arf}(K) = 7\mathrm{Arf}(B_3) - 6\mathrm{Arf}(B_2) = 0$, using the above calculations we get $\sigma_z(K) = 2$ for $z = e^{2\pi i k/5}, k = 1, 2, 3, 4$ and $\int_{S^1} \sigma_z(K) = 0$. This shows that K has all the required properties. $\square$

For the following example we need the following technical lemma to construct satellite knots with the right properties.

Lemma 10.11. (1) *Let F be a Seifert surface for a knot K of genus g. Then there exist simple closed curves $A_1, \ldots, A_{2g} \in S^3 \setminus F$ which form the unlink in S^3 and such that the corresponding homology classes give a basis for $H_1(S^3 \setminus F)$.*

(2) *Let $\tilde{A}_1, \ldots, \tilde{A}_s$ be a collection of homology classes in $H_1(L_{K,k})$. We can find simple closed curves $A_1, \ldots, A_s \subset S^3 \setminus K$, which form the unlink in S^3 and where each component represents the trivial element in $H_1(S^3 \setminus K)$, such that for all $i = 1, \ldots, s$ the homology class $\tilde{A}_i$ is represented by one of the k lifts of A_i to L_k.*

Proof. (1) We can view F as a disk with $2g$ 1-handles attached. The meridians of these handles will, properly chosen, give the required curves $A_1, \dots, A_{2g}$,
(2) cf. Livingston [L01, p. 12].

□

Proposition 10.12 (Example 2)**.** *There exists a* (1)*-solvable knot K with zero L^2-eta invariant of level 1, but non-zero SE-invariants.*

Proof. Let $p(t) = -2t + 5 - 2t^{-1}$. By Kearton [K73] there exists a knot $K \subset S^7$ such that it is Blanchfield pairing is isomorphic to

$$\begin{array}{rcl} \Lambda/p(t)^2 \times \Lambda/p(t)^2 & \to & S^{-1}\Lambda/\Lambda \\ (a,b) & \mapsto & \bar{a}p(t)^{-2}b \end{array}$$

In particular there exists a Seifert matrix A such that

$$\begin{array}{rcl} \Lambda^{2g}/(At - A^t) \times \Lambda^{2g}/(At - A^t) & \to & S^{-1}\Lambda/\Lambda \\ (a,b) & \mapsto & \bar{a}^t(t-1)(At - A^t)^{-1}b \end{array}$$

is isomorphic to the given pairing. Since the Blanchfield pairing is metabolic, A is metabolic, and using proposition 10.9 we can realize A by a slice knot $K \subset S^3$. A computation using proposition 2.9 shows that $|H_1(L_{K,4})| = 225$.

Let

$$N := \max_{P_4 \text{metabolizer for } \lambda_{L,4}} \{ |\eta(M_K, \alpha_{(\chi,z)})| \quad | \chi : H_1(L_{K,4})/P_4 \to S^1, z \in S^1 \}$$

Note that the number of possible χ's is finite since $H_1(L_4)$ is finite, furthermore given χ, the function $z \mapsto \eta(M_K, \alpha_{(\chi,z)})$ is locally constant with finitely many jumps, i.e. assumes only finitely many values (cf. corollary 5.4 and proposition 5.6). This shows that N is in fact a finite number.

Let F be a Seifert surface for K. Let $A_1, \dots, A_{2g} \in S^3 \setminus F$ be as in lemma 10.11, part 1. Denote the knot of the proof of proposition 10.10 by D and let $C = (N+1) \cdot D$, and form the iterated satellite knot

$$S := S(K, C, \dots, C, A_1, \dots, A_{2g})$$

We claim that the satellite knot $S := S(K, C, A)$ satisfies the conditions stated in the proposition. S is (1)-solvable by theorem 10.2 and has zero L^2-eta invariant of level 1 by theorem 10.8 since K is slice, $\int_{S^1} \sigma_z(C) = 0$ by construction of C and since K, and therefore also S, has a unique metabolizer for the Blanchfield pairing.

We have to show that for all $P_4 \subset H_1(L_{S,4})$ with $P_4 = P_4^\perp$ with respect to the linking pairing $\lambda_{S,4}$, we can find a non-zero character $\chi : H_1(L_{S,4}) \to S^1$ of prime power order, vanishing on P_4, such that for one transcendental z we get $\eta_{\alpha_{(z,\chi)}}(M_S) \neq 0$.

Let P be a metabolizer and $\chi : H_1(L_{S,4}) \to S^1$ a non-trivial character of order 5, vanishing on P. Denote the corresponding character on $H_1(L_{K,4})$ by χ as well. For any $z \in S^1$ we get by corollary 10.6

$$\eta(M_S, \alpha^S_{(\chi,z)}) = \eta(M_K, \alpha^K_{(\chi,z)}) + \sum_{j=1}^{2g}\sum_{i=1}^{4} \eta(M_C, \alpha_{ij})$$

where α_{ij} denotes the representation $\pi_1(M_C) \to U(1)$ given by $g \mapsto \chi((\tilde{A}_j)_i)^{\epsilon(g)}$ and $(\tilde{A}_j)_i$ denotes the i^{th} lift of A_j to $L_{K,k}$. By definition of N and by proposition 3.7 we get

$$\begin{array}{rcl} \eta(M_S, \alpha^S_{(\chi,z)}) & \geq & -N + \sum_{j=1}^{2g}\sum_{i=1}^{4} \eta(M_C, \alpha_{ij}) = -N + \sum_{j=1}^{2g}\sum_{i=1}^{4} \sigma_{\chi((\tilde{A}_j)_i)}(C) \\ & = & -N + \sum_{j=1}^{2g}\sum_{i=1}^{4} (N+1)\sigma_{\chi((\tilde{A}_j)_i)}(B_2) \end{array}$$

Note that $\eta(M_C, \alpha_{ij}) \geq 0$ for all i, j since $\sigma_{e^{2\pi i j/5}}(C) \geq 0$ for $j = 0, \ldots, 4$. The lifts $(\tilde{A}_j)_i$ generate $H_1(L_{K,4})$, hence $\chi((\tilde{A}_j)_i) \neq 1$ for at least one (i, j) since χ is non-trivial. But $\sigma_w(C) = 2(N+1)$ for $w = e^{2\pi i j/5}, j = 1, 2, 3, 4$. It follows that $\eta(M_S, \alpha^S_{(\chi,z)}) \geq -N + 2(N+1) > 0$ for all z. □

Proposition 10.13 (Example 3)**.** *There exists a knot S which is algebraically slice, (1)-solvable, has zero STE–obstruction and zero L^2-eta invariant of level 1 but does not satisfy the condition for theorem 6.4, i.e. S is not ribbon.*

Proof. Denote by $\Phi_{30}(t) = 1 + t - t^3 - t^4 - t^5 + t^7 + t^8$ the minimal polynomial of $e^{2\pi i/30}$. As in the proof of proposition 10.12 there exists a ribbon knot K such that the Blanchfield pairing is isomorphic to

$$\begin{array}{rcl} \Lambda/\Phi_{30}(t)^2 \times \Lambda/\Phi_{30}(t)^2 & \to & S^{-1}\Lambda/\Lambda \\ (a, b) & \mapsto & \bar{a}\Phi_{30}(t)^{-2}b \end{array}$$

An explicit example of such a knot is given by Taehee Kim [K02, Section 2]. Note that K has a unique metabolizer P for the Blanchfield pairing. Furthermore $H_1(L_{K,k}) = 0$ for all prime powers k by theorem 4.12, but a computation using proposition 2.9 shows that $|H_1(L_{K,6})| = 625$.

Now let C be the knot of the proof of proposition 10.10. Let A be a curve in $S^3 \setminus K$, unknotted in S^3, which lifts to an element $\tilde{A}$ in the 6-fold cover which presents a non-trivial element of order 5 in $H_1(M_{K,6})/P_6$ where $P_6 := \pi_6(P) \in H_1(M_{K,6})$ is the projection of the unique metabolizer for the Blanchfield pairing.

We claim that the satellite knot $S := S(K, C, A)$ satisfies the conditions stated in the proposition. By proposition 10.4 the knot S is algebraically slice, since K is algebraically slice. Since $H_1(L_{S,k}) = H_1(L_{K,k}) = 0$ for all prime powers k, we get $R_k^{irr,met}(\pi_1(M_K)) = \emptyset$ for all prime powers k, hence S has zero STE–obstruction. K is (1)-solvable by theorem 10.2 and has zero L^2-eta invariants of level 1 by theorem 10.8.

As remarked above, the Blanchfield pairing of S has a unique metabolizer P. Let $\chi : H_1(L_{S,6}) \to S^1$ be a non-trivial character of order 5, vanishing on $P_6 := \pi_6(P) \subset$

$H_1(L_{S,6})$ such that $\chi(\tilde{A}) \neq 1 \in S^1$. By corollary 10.6

$$\eta(M_S, \alpha^S_{(\chi,z)}) = \eta(M_K, \alpha^K_{(\chi,z)}) + \sum_{i=1}^{k} \eta(M_C, \alpha_i)$$

where α_i denotes the representation $\pi_1(M_C) \to U(1)$ given by $g \mapsto \chi(\tilde{A}_i)^{\epsilon(g)}$. The first term is zero since K is ribbon and P is the unique metabolizer of the Blanchfield pairing (cf. theorem 6.4). The second term is non-zero since $\chi(\tilde{A}_i) \neq 1$ for at least one i and by the properties of C. This shows that $\eta(M_S, \alpha^S_{(\chi,z)}) \neq 0$, i.e. S is not ribbon by theorem 6.4. □

For completeness sake we add the following example which has been first found by Taehee Kim [K02].

Proposition 10.14 (Example 4)**.** *There exists a knot S which is algebraically slice, (1)-solvable, has zero STE–obstruction but non-zero L^2-eta invariant of level 1.*

Proof. Let K be as in the proof of proposition 10.13 and C a knot with Seifert matrix B_1 and $A \in \pi_1(S^3 \setminus K)^{(1)}$ unknotted in S^3. Then the proof of proposition 10.13 shows that $S = S(K, C, A)$ has zero STE–obstruction and non-zero L^2-eta invariant of level 1 by theorem 10.8. □

Proposition 10.15 (Example 5)**.** *There exists a ribbon knot S with the following property. There exists a prime power k such that there exists no metabolizer P_k of $(H_1(L_k), \lambda_L)$ with the property that $\tau(K, \chi) = 0$ for all characters $\chi : H_1(L_k) \to S^1$.*

Note that by *all* characters we don't restrict ourselves to prime power characters. The proof is similar to the proof of the above proposition.

Proof. By proposition 10.9 we can find a ribbon knot K with Alexander polynomial

$$\Delta_K(t) = f(t)f(t^{-1}) \text{ where } f(t) = 4 - 3t + 2t^2 + 4t^3 - 7t^4 + t^5 + 2t^6 - 3t^7 + t^8$$

a computation shows that $H_1(L_{K,5}) = 1296 = 36^2$. Let

$$N := \max_{P_5 \text{ metabolizer for } \lambda_{L,5}} \{ |\eta(M_K, \alpha_{(\chi,z)})| \quad | \chi : H_1(L_{K,5})/P_5 \to S^1, z \in S^1 \}$$

As in the proof of proposition 10.12 we see that N is finite.

Let $\tilde{A}_1, \dots, \tilde{A}_s \in H_1(L_{K,5})$ be all elements. Let $A_1, \dots, A_s \in S^3 \setminus K$ be as in lemma 10.11, part 2. Let C be a ribbon knot with Seifert matrix $\oplus_{i=1}^{N+1} B_1$, and form the iterated satellite knot

$$S := S(K, C, \dots, C, A_1, \dots, A_s)$$

Note that S is ribbon by proposition 10.1.

According to corollary 5.7, part 2, it is enough to show that for each metabolizer P_5 for λ_L we can find a character $\chi : H_1(L_{S,5}) \to S^1$ vanishing on P_5 such that for a transcendental $z \in S^1$, $\eta(M_S, \alpha_{(\chi,z)}) \neq 0$. Let P_5 be a metabolizer for λ_5 and $\chi : H_1(L_5) \to S^1$ a non-trivial character of order 6, vanishing on P_5. Then for all $z \in S^1$ we get

$$\begin{aligned} \eta(M_{S,5}, \alpha^S_{(\chi,z)}) &= \eta(M_{K,5}, \alpha^K_{(\chi,z)}) + \textstyle\sum_{j=1}^s \sum_{i=1}^5 \eta(M_{C_j}, \alpha_{ij}) \\ &\leq N + \textstyle\sum_{j=1}^s \sum_{i=1}^5 (N+1)\sigma_{\chi((\tilde{A}_j)_i)}(B_1) \end{aligned}$$

Since χ is of order 6 and by construction of $A_1, \dots, A_s$, we can find (i,j) such that $\chi((\tilde{A}_j)_i) = e^{2\pi i/6}$, but recall that $\sigma_z(B_1) = 0$ for all z except for $z = e^{2\pi i/6}, e^{2\pi 5i/6}$ where $\sigma_z(B_1) = -1$. This shows that $\eta(M_S, \alpha^S_{(\chi,z)}) \leq N + (N+1)(-1) = -1$. $\square$

Remark. (1) The Alexander polynomial of K looks unnecessarily big, but it was the polynomial of lowest degree I could find with $|H_1(L_k)|$ being divisible by 6 for some prime power k.

(2) The proposition shows that theorems 4.9 and 6.4 can't be strenghtened to include all non prime power characters.

(3) The above example shows that the set $P_k^{irr,met}(\pi_1(M_K))$ is in a sense maximal, i.e. that the prime power condition on the the characters is indeed necessary.

(4) When we compare the above proposition with theorem 6.6 and 6.7 we see, that K does not have a ribbon disk D with $\pi_1(W_5)$ finite, where W_5 denotes the 5-fold cover of D^4 branched along D.

Appendix A. Auxiliary propositions

A.1. **Algebra lemmas.**

Lemma A.1. *Let G be a finite group, $\chi : G \to C_m$ a character of order m. Then $\mathbb{C}$ and $\mathbb{Q}(e^{2\pi i/m})$ are flat over $\mathbb{Z}[G]$ and $\mathbb{C}$ and $\mathbb{Q}(t)(e^{2\pi i/m})$ are flat over $\mathbb{Z}[\mathbb{Z} \times G] = \mathbb{Z}[\langle t\rangle \times G]$.*

Proof. First note that for any ring R and any multiplicative subset S, the ring $S^{-1}R$ is flat over R (cf. [L93, p. 613]), furthermore $\mathbb{C}$ is flat over $\mathbb{Q}(e^{2\pi i/m})$. Since flatness is transitive it remains to show that for $K = \mathbb{Q}$ and for $K = \mathbb{Q}(t)$ we get $K(e^{2\pi i/m})$ is flat over $K[G]$. From Maschke's theorem (cf. [L93, p. 666]) we know that the group ring $K[G]$ is semisimple, i.e. any $K[G]$-module is semisimple. This concludes the proof. □

For a number d denote by $\Phi_d(t)$ the minimal polynomial of a primitive d^{th} root of unity. Let p be a prime number, then $\Phi_p(t) = 1 + t + \cdots + t^{p-1}$, hence $\Phi_p(1) = p$. Recall the general equation

$$t^{m-1} + \cdots + t + 1 = \frac{t^m - 1}{t - 1} = \prod_{d|m, d\neq 1} \Phi_d(t)$$

An induction argument shows the following lemma.

Lemma A.2. *Let $d \in \mathbb{N}$, then $\Phi_d(1) = 1$ if d is a composite number and $\Phi_d(1) = p$ if d is a power of a prime p.*

Lemma A.3. *Let $\lambda : A \times A \to \mathbb{Q}/\mathbb{Z}$ be a non-singular form on a finite $\mathbb{Z}$-module A, and $N \subset A$ a submodule, then $|N||N^\perp| = |A|$.*

Proof. Since λ is non-singular we get $N \cong \mathrm{Hom}_{\mathbb{Z}}(A/N^\perp, \mathbb{Q}/\mathbb{Z})$, but since $\mathbb{Z}$ is a PID it is easy to see that for any finite $\mathbb{Z}$-module B we have $|\mathrm{Hom}_{\mathbb{Z}}(B, \mathbb{Q}/\mathbb{Z})| = |B|$. □

A.2. **Cobordism groups and group homology.**

Let G be a group, then a G-manifold is a pair (M, α) where M is a compact oriented manifold with components $\{M_i\}$ and α is a collection of homomorphisms $\alpha_i : \pi_1(M_i) \to G$ where each α_i is defined up to inner automorphism.

We call two G-manifolds $(M_j, \alpha_j), j = 1, 2$, G-cobordant if there exists a G-manifold (N, β) such that $\partial(N) = M_1 \cup -M_2$ and, up to inner automorphisms of G, $\beta|\pi_1(M_j) = \alpha_j$.

Denote by $\Omega_k(G)$ the cobordism group of G-manifolds of dimension k, and let $\Omega_k := \Omega_k(\text{trivial group})$. The following lemma summarizes some well-known facts.

Lemma A.4. [CF64]

(1) $\Omega_3(G) = \Omega_3 \oplus H_3(G)$,

(2) Ω_3 *is the trivial group,*
(3) $\Omega_4 \cong \mathbb{Z}$ *via the signature of the intersection pairing.*

We'll need the following facts about group homology.

Lemma A.5. (1) $H_i(\mathbb{Z}) = \mathbb{Z}$ *for* $i = 0, 1$ *and* $H_i(\mathbb{Z}) = 0$ *for* $i \geq 2$,
(2) $H_i(\mathbb{Z} \times A) = H_i(A) \oplus H_{i-1}(A)$,
(3) *let* G *be a finite group, then* $H_i(G)$ *is finite for* $i \geq 1$.

Proof. The first statement follows from the fact that $S^1 = K(\mathbb{Z}, 1)$, the second follows from the Künneth-theorem. Now let G be a finite group. Consider the canonical G-cover $EG \to BG := K(G, 1)$. There's an equivariant lifting map $H_*(BG) \to H_*(EG)$ such that the composition with the map $H_*(EG) \to H_*(BG)$ induced by projection is just multiplication by $m := |G|$. But $\pi_*(EG) = 0$, hence $\tilde{H}_*(EG) = 0$. This shows that $\tilde{H}_i(BG) = \tilde{H}_i(G)$ is m-torsion. □

Proposition A.6. *Let* A *be a finite group and* $\beta : \mathbb{Z} \to Aut(A)$ *a homomorphism. Then* $H_i(\mathbb{Z} \ltimes A)$ *is torsion for* $i \geq 2$.

Proof. Since A is finite there exists $n \in \mathbb{N}$ such that $\beta(n\mathbb{Z}) = \mathrm{id}$. We therefore get an exact sequence

$$0 \to \mathbb{Z} \times A \xrightarrow{(\cdot n, \mathrm{id})} \mathbb{Z} \ltimes A \to \mathbb{Z}/n \to 0$$

We get maps

$$K(\mathbb{Z} \times A, 1) \to K(\mathbb{Z} \ltimes A, 1) \to K(\mathbb{Z}/n, 1)$$

which form a fibration since any map, in particular $K(\mathbb{Z} \ltimes A, 1) \to K(\mathbb{Z}/n, 1)$ can be made into a fibration up to homotopy, but the corresponding fiber turns out to be $K(\mathbb{Z} \times A, 1)$.

We'll use a spectral sequence argument to show that $H_i(\mathbb{Z} \ltimes A)$ is torsion for $i \geq 2$. From the above lemma it follows that $H_i(\mathbb{Z} \times A)$ is torsion for $i \geq 2$, and that $H_i(\mathbb{Z}/n)$ is torsion for $i \geq 1$.

For any fibration $F \to E \to B$ there exists a spectral sequence with $E^2_{p,q} = H_p(B, H_q(F))$ which converges to $H_{p+q}(E)$ where $H_p(B, H_q(F))$ denotes homology coming from the natural $\mathbb{Z}[\pi_1(B)]$-structure of $H_q(F)$. To show that $H_i(E) = H_i(\mathbb{Z} \ltimes A)$ is torsion it is enough to show that

$$E^2_{p,q} = H_p(B, H_q(F)) = H_p(\mathbb{Z}/n, H_q(\mathbb{Z} \times A))$$

is torsion for all p, q with $p + q = i$. This is clearly true for $p = 0, q = i \geq 2$ since $H_i(\mathbb{Z} \times A)$ is torsion for $i \geq 2$.

For a cyclic group one can explicitely compute the homology (cf. [HS71, p. 200]). Let t be a generator of $\mathbb{Z}/n$. If B is a $\mathbb{Z}/n$-module, then define $\psi, \varphi : B \to B$ via

$$\begin{aligned} \varphi(b) &:= (t-1)b \\ \psi(b) &:= (t^{n-1} + t^{n-2} + \cdots + t + 1)b \end{aligned}$$

Then for $i \geq 1$ we get

$$\begin{aligned} H_{2i-1}(\mathbb{Z}/n, B) &= \operatorname{Ker}(\varphi)/\operatorname{Im}(\psi) \\ H_{2i}(\mathbb{Z}/n, B) &= \operatorname{Ker}(\psi)/\operatorname{Im}(\varphi) \end{aligned}$$

In our case $B = H_q(\mathbb{Z} \times A)$. It follows that $H_p(\mathbb{Z}/n, H_q(\mathbb{Z} \times A))$ is torsion for $q \geq 2$ since in this case $H_q(\mathbb{Z} \times A)$ is torsion.

For $q = 0, 1$ we get $H_q(\mathbb{Z} \times A) = \mathbb{Z} \times A$. Consider the map $\mathbb{Z} \hookrightarrow \mathbb{Z} \times A \xrightarrow{t} \mathbb{Z} \times A \to \mathbb{Z}$, since $t^n = \mathrm{id}$ this is an isomorphism, i.e. multiplication by 1 or -1. Now it is easy to see that $H_i(\mathbb{Z}/n, \mathbb{Z} \times A)$ is torsion for $i \geq 1$. $\square$

Appendix B. L-groups and signatures

For a ring R with involution we denote by $L_0(R,\epsilon), \epsilon = \pm 1$, the Witt group of ϵ–hermitian non–singular forms over finitely generated free R-modules (cf. [R98]). We'll always assume that a form is anti-linear in the first argument and linear in the second argument. We'll abbreviate $L_0(R)$ for $L_0(R,+1)$.

Any hermitian form (V,θ) over R can be represented, after choosing a basis for V, by a hermitian matrix A. This matrix is unique up to conjugation. Similarly an element in $L_0(R,\epsilon)$ can be represented by a matrix, but then the matrix is unique only up to conjugation and matrix cobordism.

In the following, given a field $F \subset \mathbb{C}$, closed under complex conjugation, the rings $F[t,t^{-1}], F(t)$ will always be equipped with the involution given by the complex involution on F and $\bar{t} := t^{-1}$.

Let $R \subset \mathbb{C}$ be such that all positive elements are squares, then by Sylvester's theorem

$$\begin{array}{rcl} L_0(R,\epsilon) & \to & \mathbb{Z} \\ A & \mapsto & \text{sign}(\sqrt{\epsilon}A) \end{array}$$

is an isomorphism. This canonically extends to an isomorphism $\text{sign}L_0(R,\epsilon)\otimes\mathbb{Q} \to \mathbb{Q}$. In particular $L_0(\mathbb{C},\pm 1) = L_0(\bar{\mathbb{Q}},\pm 1) \cong \mathbb{Z}$ via the signature map, where we denote by $\bar{\mathbb{Q}}$ the algebraic closure of $\mathbb{Q}$. Since we are interested in studying to which degree signatures determine forms we'll work in this section with $\tilde{L}_0(R,\epsilon) := L_0(R,\epsilon)\otimes\mathbb{Q}$.

Let F be a Galois extension of $\mathbb{Q}$ with a (possibly trivial) involution. Denote by $G(F)$ the set of all $\mathbb{Q}$-linear embeddings $F \to \bar{\mathbb{Q}}$ preserving the involution. For $\rho \in G(F)$ denote the induced maps $L_0(F,\epsilon) \to L_0(\bar{\mathbb{Q}},\epsilon), L_0(F(t),\epsilon) \to L_0(\bar{\mathbb{Q}}(t),\epsilon), \dots$ by ρ as well. Denote by $G_0(F) \subset G(F)$ any subset such that for each $\rho \in G(F)$ there exists precisely one $\tilde{\rho} \in G_0(F)$ with $\tilde{\rho} = \rho$ or $\tilde{\rho} = \bar{\rho}$.

If $\tau = (V,\theta)\otimes\frac{p}{q} \in \tilde{L}_0(F(t),\pm 1)$ and $z \in S^1$ is transcendental, then we can consider $\mathbb{C}$ as an $F(t)$ module, and $\tau(z) := (V \otimes_{F(t)} \mathbb{C},\theta)\otimes\frac{p}{q}$ is well-defined.

The goal of this section is to prove the following theorem.

Theorem B.1. *Let F be a Galois extension of $\mathbb{Q}$, then for $\tau \in \tilde{L}_0(F(t),\epsilon)$ we get*

$$\begin{array}{rl} & \tau = 0 \in \tilde{L}_0(F(t),\epsilon) \\ \Leftrightarrow & \mathit{sign}(\rho(\tau)(z)) = 0 \text{ for all } \rho \in G_0(F) \text{ and all transcendental } z \in S^1 \end{array}$$

B.1. The groups $\tilde{L}_0(F), \tilde{L}_0(F(t))$.

We quote a result from Ranicki [R98, p. 493].

Proposition B.2. *The following maps are isomorphisms*

$$\begin{array}{rcccl} \tilde{L}_0(F) & \to & \bigoplus_{\rho\in G_0(F)}\tilde{L}_0(\bar{\mathbb{Q}}) & \to & \mathbb{Q}^{|G_0(F)|} \\ A & \mapsto & (\rho(A))_{\rho\in G_0(F)} & \mapsto & (\mathit{sign}(\rho(A)))_{\rho\in G_0(F)} \end{array}$$

Consider the case where $F := \mathbb{Q}[t]/q(t)$, $q(t)$ irreducible and $q(t) = uq(t^{-1})$ for some unit $u \in \mathbb{Q}[t^{-1}, t]$. Then there's an involution given by $\bar{t} = t^{-1}$, which is non-trivial if $q(t) \neq t-1, t+1$. In this case the set $G(F)$ corresponds canonically to the set of all roots of $q(t)$ lying in S^1 and $G_0(F)$ corresponds to all roots $z \in S^1$ of $q(t)$ with $\mathrm{Im}(z) \geq 0$.

Theorem B.3. *Let F be a Galois extension of $\mathbb{Q}$, then for $\tau \in \tilde{L}_0(F(t), \epsilon)$ we get*

$$\tau = 0 \in \tilde{L}_0(F(t), \epsilon) \Leftrightarrow \rho(\tau) = 0 \in \tilde{L}_0(\bar{\mathbb{Q}}(t), \epsilon) \text{ for all } \rho \in G_0(F)$$

This result was stated by Litherland [L84, p. 358], but there doesn't seem to be a proof in the literature. To simplify the notation we'll only prove the case $\epsilon = 1$.

We need more definitions and results from [R98, ch. 39C].

Definition. Let F be a field with a possibly trivial involution. Then define $LAut^0_{fib}(F, \epsilon)$ to be the Witt group of triples (V, θ, f) where V is a vector space over F, θ an ϵ-hermitian form on V and f an isometry of (V, θ) such that $(f-1)$ is an automorphism as well. Let $\tilde{L}Aut^0_{fib}(F, \epsilon) := LAut^0_{fib}(F, \epsilon) \otimes \mathbb{Q}$

Proposition B.4. [R98, p. 533] *Let F be a field with (possibly trivial) involution.*

(1) *There exists a split exact sequence*

$$0 \to \tilde{L}_0(F[t, t^{-1}], \epsilon) \to \tilde{L}_0(F(t), \epsilon) \to \tilde{L}Aut^0_{fib}(F, -\epsilon) \to 0$$

(2) *Denote by $\overline{M}(F)$ the set of irreducible monic polynomials $p(t)$ in $F[t]$ with the added property that $\overline{p(t)} = up(t)$ for some unit $u \in F[t, t^{-1}]$ and $\overline{M}^0(F) := \overline{M}(F) \setminus \{t-1\}$. For $p(t) \in \overline{M}^0(F)$ define*

$$\begin{array}{rrcl} r_{p(t)}: & \tilde{L}Aut^0_{fib}(F, \epsilon) & \to & \tilde{L}_0(F[t, t^{-1}]/p(t), \epsilon) \\ & (V, \theta, f) & \to & (Ker\{p(f): V \to V\}, \tilde{\theta}) \end{array}$$

where $\tilde{\theta}(a, b) = \sum_{i=0}^{deg(p)-1} \theta(a, bt^i)t^{-i}$ and t acts by f. Then

$$\prod_{p(t) \in \overline{M}^0(F)} r_{p(t)}: \tilde{L}Aut^0_{fib}(F, \epsilon) \xrightarrow{\cong} \bigoplus_{p(t) \in \overline{M}^0(F)} \tilde{L}_0(F[t, t^{-1}]/p(t), \epsilon)$$

is an isomorphism and the inverse map is given by

$$\begin{array}{rcl} \tilde{L}_0(F[t, t^{-1}]/p(t), \epsilon) & \to & \tilde{L}Aut^0_{fib}(F, \epsilon) \\ (V, \theta) & \mapsto & (V, tr_{(F[t, t^{-1}]/p(t))/F} \circ \theta, t) \end{array}$$

(3) *The map*

$$\begin{array}{rcl} \tilde{L}_0(F, \epsilon) & \to & \tilde{L}_0(F[t, t^{-1}], \epsilon) \\ (V, \theta) & \mapsto & (V, \theta) \otimes_F F[t, t^{-1}] \end{array}$$

is an isomorphism.

There exists a commuting diagram of exact sequences ($G_0 = G_0(F)$)

$$\begin{array}{ccccccccc} 0 & \to & \tilde{L}_0(F[t,t^{-1}]) & \to & \tilde{L}_0(F(t)) & \to & \tilde{L}Aut^0_{fib}(F,-1) & \to & 0 \\ & & \downarrow \prod_{\rho\in G_0}\rho & & \downarrow \prod_{\rho\in G_0}\rho & & \downarrow \prod_{\rho\in G_0}\rho & & \\ 0 & \to & \bigoplus_{\rho\in G_0}\tilde{L}_0(\bar{\mathbb{Q}}[t,t^{-1}]) & \to & \bigoplus_{\rho\in G_0}\tilde{L}_0(\bar{\mathbb{Q}}(t)) & \to & \bigoplus_{\rho\in G_0}\tilde{L}Aut^0_{fib}(\bar{\mathbb{Q}},-1) & \to & 0 \end{array}$$

From propositions B.2 and B.4 it follows that the first vertical map is an injection. Once we show that the last vertical map is an injection as well it follows that the middle vertical map is an injection, this will prove theorem B.3.

For $p \in F[t,t^{-1}]$ irreducible we'll write $F_p := F[t,t^{-1}]/p(t)$. Note that there exists a canonical correspondence

$$\begin{array}{rcl} \{(\rho,z)|\rho\in G(F) \text{ and } z\in S^1 \text{ such that } \rho(p)(z)=0\} & \leftrightarrow & G(F_p) \\ (\rho,z) & \mapsto & (\rho_z : \sum a_i t^i \to \rho(a_i)z^i) \end{array}$$

since $F/\mathbb{Q}$ is Galois. Consider

$$\begin{array}{ccccc} \tilde{L}Aut^0_{fib}(F,\epsilon) & \overset{\cong}{\to} & \displaystyle\bigoplus_{p\in\overline{M}^0(F)}\tilde{L}_0(F_p,\epsilon) \hookrightarrow \bigoplus_{\rho\in G_0}\bigoplus_{p\in\overline{M}_0(F)}\bigoplus_{\substack{z\in S^1\setminus\{1\}\\ \rho(p)(z)=0}}\tilde{L}_0(\bar{\mathbb{Q}},\epsilon) & \xrightarrow{\mu_{\rho,z}} & \mathbb{Q} \\ \downarrow \prod_{\rho\in G_0}\rho & & & & \\ \displaystyle\bigoplus_{\rho\in G_0}\tilde{L}Aut^0_{fib}(\bar{\mathbb{Q}},\epsilon) & \overset{\cong}{\to} & \bigoplus_{\rho\in G_0}\bigoplus_{z\in S^1\setminus\{1\}}\tilde{L}_0(\bar{\mathbb{Q}},\epsilon) & \xrightarrow{\sigma_{\rho,z}} & \mathbb{Q} \end{array}$$

where $\mu_{\rho,z}$ and $\sigma_{\rho,z}$ denotes the composition of projection maps on the corresponding $\tilde{L}_0(\bar{\mathbb{Q}},\epsilon)$ summand and taking signatures. Note that $\mu_{\rho,z}$ is well-defined, since different $p(t)$'s have disjoint zero sets. Define $\mu_{\rho,z}$ to be the zero map if z is not a root for any $\rho(p(t))$.

Proposition B.5. *Let $p \in \overline{M}_0(F)$. For $(V,\theta) \in \tilde{L}_0(F_p,\epsilon)$ we get*

$$\sigma_{\rho,z}(V,\theta) = \mu_{\rho,z}(V,\theta) \text{ for all } \rho \in G_0(F) \text{ and } z \in S^1\setminus\{1\} \text{ such that } \rho(p)(z)=0$$

Corollary B.6. *The map*

$$\prod_{\rho\in G_0(F)}\rho : \tilde{L}Aut^0_{fib}(F,\epsilon) \to \bigoplus_{\rho\in G_0(F)}\tilde{L}Aut^0_{fib}(\bar{\mathbb{Q}},\epsilon)$$

is an injection.

Note that this corollary concludes the proof of theorem B.3. We'll first prove the corollary.

Proof. The induced map

$$\prod_{\rho,z\in S^1\setminus\{1\}}\mu_{\rho,z} : \tilde{L}_0(F_p,\epsilon) \to \bigoplus_{\rho,z\in S^1\setminus\{1\}}\mathbb{Q}$$

is an injection. From the proposition it also follows that the induced map

$$\prod_{\rho\in G_0(F),z\in S^1\setminus\{1\}} \sigma_{\rho,z} : \tilde{L}_0(F_p,\epsilon) \to \bigoplus_{\rho\in G_0(F),z\in S^1} \mathbb{Q}$$

is an injection. Since different p's have disjoint sets of zeros it follows that

$$\prod_{\rho\in G_0(F),z\in S^1\setminus\{1\}} \sigma_{\rho,z} : \bigoplus_{p\in\overline{M}^0(F)} \tilde{L}_0(F_p,\epsilon) \to \bigoplus_{\rho\in G_0(F),z\in S^1} \mathbb{Q}$$

is an injection as well. But this implies that the intermediate map

$$\prod_{\rho\in G_0(F)} \rho : \tilde{L}Aut^0_{fib}(F,\epsilon) \to \bigoplus_{\rho\in G_0(F)} \tilde{L}Aut^0_{fib}(\bar{\mathbb{Q}},\epsilon)$$

is an injection as well. $\square$

Now we'll prove the proposition.

Proof. Let $p \in \overline{M}_0(F)$. Denote the zeros of $\rho(p)(t)$ by $\alpha_1,\ldots,\alpha_n$. Pick a zero α of $\rho(p)$, we can assume that $\alpha=\alpha_1$. Denote the induced embedding $F_p \to \bar{\mathbb{Q}}$ by ρ_α. Consider

$$\begin{array}{ccccccccc}
\tilde{L}_0(\bar{\mathbb{Q}},\epsilon) & \xleftarrow{r_{t-z}} & \tilde{L}Aut^0_{fib}(\bar{\mathbb{Q}},\epsilon) & \xleftarrow{\rho} & \tilde{L}Aut^0_{fib}(F,\epsilon) & \leftarrow & \tilde{L}_0(F_p,\epsilon) & \xrightarrow{\rho_\alpha} & \tilde{L}_0(\bar{\mathbb{Q}},\epsilon) \\
\theta_l & \leftarrow & \bar{\mathbb{Q}}\otimes_F (V, tr_{F_p/F}\circ\theta, t) & \leftarrow & (V, tr_{F_p/F}\circ\theta, t) & \leftarrow & (V,\theta) & \mapsto & \theta_r
\end{array}$$

We have to show that $\operatorname{sign}(\theta_l) = \operatorname{sign}(\theta_r)$. Note that θ_r denotes the form

$$\begin{array}{rcl}
\theta_r : V\otimes_{F_p}\bar{\mathbb{Q}} \times V\otimes_{F_p}\bar{\mathbb{Q}} & \to & \bar{\mathbb{Q}} \\
(v_1\otimes_{F_p} z_1, v_2\otimes_{F_p} z_2) & \mapsto & \bar{z}_1\rho_\alpha(\theta(v_1,v_2))z_2
\end{array}$$

here F_p acts on $\bar{\mathbb{Q}}$ via ρ_α.

Now we have to understand θ_l. In the following we'll view $\bar{\mathbb{Q}}$ as an F-module via ρ. The form $\bar{\mathbb{Q}}\otimes_F (V, tr_{F_p/F}\circ\theta, t)$ is given by

$$\begin{array}{ccccc}
V\otimes_F\bar{\mathbb{Q}}\times V\otimes_F\bar{\mathbb{Q}} & \xrightarrow{\mathrm{tr}_{F_p/F}\circ\theta} & \bar{\mathbb{Q}}\otimes_F F\otimes_F\bar{\mathbb{Q}} & \to & \bar{\mathbb{Q}} \\
(v_1\otimes_F z_1, v_2\otimes_F z_2) & \mapsto & \bar{z}_1\otimes_F \mathrm{tr}_{F_p/F}(\theta(v_1,v_2))\otimes_F z_2 & \mapsto & \bar{z}_1\rho(\mathrm{tr}_{F_p/F}(\theta(v_1,v_2)))z_2
\end{array}$$

Denote by $\bar{\mathbb{Q}}_p$ the ring $F_p\otimes_F\bar{\mathbb{Q}} = \bar{\mathbb{Q}}[t,t^{-1}]/\rho(p(t))$. It is easy to see that the map

$$\bar{\mathbb{Q}}_{p(t)} = F_p\otimes_F\bar{\mathbb{Q}} \xrightarrow{\mathrm{tr}_{F_p/F\otimes_F}\mathrm{id}} F\otimes_F\bar{\mathbb{Q}} \to \bar{\mathbb{Q}}$$

coincides with $\mathrm{tr}_{\bar{\mathbb{Q}}_p/\bar{\mathbb{Q}}} : \bar{\mathbb{Q}}_p \to \bar{\mathbb{Q}}$. Therefore the form $\bar{\mathbb{Q}}\otimes_F (V, tr_{F_p/F}\circ\theta, t)$ is given by

$$\begin{array}{ccccc}
V\otimes_F\bar{\mathbb{Q}}\times V\otimes_F\bar{\mathbb{Q}} & \to & \bar{\mathbb{Q}}_p & \xrightarrow{\mathrm{tr}_{\bar{\mathbb{Q}}_p/\bar{\mathbb{Q}}}} & \bar{\mathbb{Q}} \\
(v_1\otimes_F z_1, v_2\otimes_F z_2) & \mapsto & \theta(v_1,v_2)\otimes_F \bar{z}_1 z_2 & \mapsto & \mathrm{tr}_{\bar{\mathbb{Q}}_p/\bar{\mathbb{Q}}}(\theta(v_1,v_2)\otimes_F \bar{z}_1 z_2)
\end{array}$$

We can write $V \otimes_F \bar{\mathbb{Q}} = V_1 \oplus \cdots \oplus V_n$ where $V_i := \mathrm{Ker}\{(t-\alpha_i) : V \otimes_F \bar{\mathbb{Q}} \to V \otimes_F \bar{\mathbb{Q}}\}$ since the minimal polynomial of t is $p(t) = \prod_{i=1}^n (t-\alpha_i)$. Then θ_l is given by restricting the above form to V_1.

We can decompose the $\bar{\mathbb{Q}}[t]$-module $\bar{\mathbb{Q}}_p = \bar{\mathbb{Q}}[t]/\rho(p(t))$ as follows

$$\bar{\mathbb{Q}}_p = \bigoplus_{i=1}^n \mathrm{Ker}\{(t-\alpha_i) : \bar{\mathbb{Q}}_p \to \bar{\mathbb{Q}}_p\} = \bigoplus_{i=1}^n \bar{\mathbb{Q}}_i$$

where $\bar{\mathbb{Q}}_i := \mathrm{Ker}\{(t-\alpha_i) : \bar{\mathbb{Q}}_p \to \bar{\mathbb{Q}}_p\}$. Note that $\dim_{\bar{\mathbb{Q}}}(\bar{\mathbb{Q}}_i) = 1$. Consider the following map

$$\begin{array}{rccc} \mu_{\alpha_i} : & \bar{\mathbb{Q}}_i & \to & \bar{\mathbb{Q}} \\ & p(t) & \mapsto & p(\alpha_i) \end{array}$$

This defines an isomorphism of $\bar{\mathbb{Q}}$-algebras. Then the trace function is given by

$$\begin{array}{rccc} \mathrm{tr}_{\bar{\mathbb{Q}}_p/\bar{\mathbb{Q}}} : & \oplus_{i=1}^n \bar{\mathbb{Q}}_i = \bar{\mathbb{Q}}_p & \to & \bar{\mathbb{Q}} \\ & (z_1, \ldots, z_n) & \mapsto & \sum_{i=1}^n \mu_i(z_i) \end{array}$$

since $\mathrm{tr}_{\bar{\mathbb{Q}}_p/\bar{\mathbb{Q}}} = \mathrm{tr}_{(\bigoplus_{i=1}^n \bar{\mathbb{Q}}_i)/\bar{\mathbb{Q}}} = \sum_{i=1}^n \mathrm{tr}_{\bar{\mathbb{Q}}_i/\bar{\mathbb{Q}}}$. The form $\theta \otimes_F \bar{\mathbb{Q}} : V \otimes_F \bar{\mathbb{Q}} \times V \otimes_F \bar{\mathbb{Q}} \to \bar{\mathbb{Q}}_p$ restricts to a form $V_1 \times V_1 \to \bar{\mathbb{Q}}_1$ and θ_l is given by $V_1 \times V_1 \to \bar{\mathbb{Q}}_1 \xrightarrow{\mathrm{tr}} \bar{\mathbb{Q}}$.

We can now compute θ_l. Let $\sum_{j=1}^{s_1} v_{1j} \otimes_F z_{1j}, \sum_{l=1}^{s_2} v_{1l} \otimes_F z_{1l} \in V_1$, then

$$\begin{array}{rcl} \theta_l(\sum_{j=1}^{s_1} v_{1j} \otimes_F z_{1j}, \sum_{l=1}^{s_2} v_{1l} \otimes_F z_{1l}) & = & \mathrm{tr}_{\bar{\mathbb{Q}}_p/\bar{\mathbb{Q}}}(\sum_{j=1}^{s_1} \sum_{l=1}^{s_2} \theta(v_{1j}, v_{2l}) \otimes_F \bar{z}_{1j} z_{2l}) = \\ & = & \mu_\alpha(\sum_{j=1}^{s_1} \sum_{l=1}^{s_2} \rho(\theta(v_{1j}, v_{2l}))\bar{z}_{1j} z_{2l}) = \\ & = & \sum_{j=1}^{s_1} \sum_{l=1}^{s_2} \rho_\alpha(\theta(v_{1j}, v_{2l}))\bar{z}_{1j} z_{2l} \end{array}$$

Consider the following sequence of canonical isomorphisms:

$$V_1 = \mathrm{Ker}\{(t-\alpha) : V \otimes_F \bar{\mathbb{Q}} \to V \otimes_F \bar{\mathbb{Q}}\} \cong (V \otimes_F \bar{\mathbb{Q}}) \otimes_{\bar{\mathbb{Q}}[t]} \bar{\mathbb{Q}} \cong V \otimes_{F[t]} \bar{\mathbb{Q}} \cong V \otimes_{F_p} \bar{\mathbb{Q}}$$

The resulting isomorphism is given by

$$\begin{array}{rcc} V_1 = \mathrm{Ker}\{(t-\alpha) : V \otimes_F \bar{\mathbb{Q}} \to V \otimes_F \bar{\mathbb{Q}}\} & \cong & V \otimes_{F_p} \bar{\mathbb{Q}} \\ \sum_{j=1}^s v_j \otimes_F z_j & \mapsto & \sum_{j=1}^s v_j \otimes_{F_p} z_j \end{array}$$

It now follows immediately that the forms θ_l, θ_r are isomorphic. □

B.2. **The group $\tilde{L}_0(\bar{\mathbb{Q}}(t))$.** We need some more facts.

Proposition B.7. [R98, p. 533] *Let F be a field with (possibly trivial) involution. The splitting $\tilde{L}Aut^0_{fib}(F, -\epsilon) \to \tilde{L}_0(F(t), \epsilon)$ in the exaxt sequence*

$$0 \to \tilde{L}_0(F[t, t^{-1}], \epsilon) \to \tilde{L}_0(F(t), \epsilon) \to \tilde{L}Aut^0_{fib}(F, -\epsilon) \to 0$$

is given by

$$(V, \theta, f) \mapsto (V \otimes_F F(t), (v, w) \to (1-t^{-1})\theta((1-f)^{-1}v, w) + \epsilon(1-t)\overline{\theta((1-f)^{-1}w, v)})$$

Theorem B.8. *Let* $\tau \in \tilde{L}_0(\bar{\mathbb{Q}}(t))$, *then*

$$\tau = 0 \in \tilde{L}_0(\bar{\mathbb{Q}}(t)) \Leftrightarrow \tau(z) = 0 \in \tilde{L}_0(\bar{\mathbb{Q}}) \text{ for all transcendental } z$$

Proof. Proposition B.4, part (1) and (3), shows that there exists an isomorphism

$$\tilde{L}_0(\bar{\mathbb{Q}}) \oplus \tilde{L}Aut^0_{fib}(\bar{\mathbb{Q}}, -1) \to \tilde{L}_0(\bar{\mathbb{Q}}(t))$$

Let $Z := S^1 \setminus \{1\} \cap \bar{\mathbb{Q}}$, then $\overline{M}_0(\bar{\mathbb{Q}}) = \{t - z | z \in Z\}$. Using that $\tilde{L}_0(\bar{\mathbb{Q}}) \cong \mathbb{Q}$ via the signature we get, using part (2) of proposition B.4, isomorphisms

$$\begin{array}{ccccc} \bigoplus_{z\in Z} \mathbb{Z} & \to & \bigoplus_{z\in Z} \tilde{L}_0(\bar{\mathbb{Q}}[t,t^{-1}]/(t-z), -1) & \to & \tilde{L}Aut^0_{fib}(\bar{\mathbb{Q}}, -1) \\ (n_z)_{z\in Z} & \mapsto & \bigoplus n_z(\bar{\mathbb{Q}}[t,t^{-1}]/(t-\alpha), i) & \mapsto & \bigoplus n_z(\bar{\mathbb{Q}}, i, z) \end{array}$$

The isomorphisms above and proposition B.7 show that in $\tilde{L}_0(\bar{\mathbb{Q}}(t))$ the form τ is equivalent to

$$(\bar{\mathbb{Q}}(t), 1) \otimes r_0 \oplus \bigoplus_{j=1}^{s} (\bar{\mathbb{Q}}(t), i(1-t^{-1})(1-\bar{\alpha}_j)^{-1} + i(1-t)(1-\alpha_j)^{-1}) \otimes r_i$$

where $z_j \in S^1 \setminus \{1\}, j = 1, \ldots, s$ are distinct and $r_0, \ldots, r_s \in \mathbb{Q}$. Note that $\tau = 0$ if and only if $r_0 = r_1 = \cdots = r_s = 0$.

We can assume that $r_i \in \mathbb{N}$ for all i, and hence restrict ourselves to forms in $L_0(\bar{\mathbb{Q}}(t))$. Then the matrix

$$A(t) := (1) \otimes r_0 \oplus \bigoplus_{j=1}^{s} (i(1-t^{-1})(1-\bar{\alpha}_j)^{-1} + i(1-t)(1-\alpha_j)^{-1}) \otimes r_i$$

represents τ. The signature function $z \mapsto \operatorname{sign}(A(z))$ is locally constant, its only jumps are when $\det(A(t)) = 0$, i.e. when

$$(1-t^{-1})(1-\bar{z}_j)^{-1} + (1-t)(1-z_j)^{-1} = 0 \text{ for some } j$$

i.e. when $t = \frac{1-\bar{z}_j}{1-z_j} \in S^1$. It is clear that $\operatorname{sign}(A(1)) = n_0$ and that the jump of the signature function at $= \frac{1-\bar{z}_j}{1-z_j} \in S^1$ is $2r_j$. The proposition follows now easily since $(\bar{\mathbb{Q}} \cap S^1) \subset S^1$ is dense. $\square$

Combining theorems B.3 and B.8 we now get a proof for theorem B.1.

Appendix C. Matrices and linking pairings

Let R be a commutative ring with involution which has a quotient field K. For any $d \times d$-matrix A over R with $\det(A) \neq 0$ and $A = \bar{A}^t$ denote by $\lambda(A)$ the form

$$\begin{array}{rcl} \lambda(A) : R^d/AR^d \times R^d/AR^d & \to & K/R \\ (a,b) & \mapsto & a^t A^{-1} b \end{array}$$

This form is easily seen to be non-singular and hermitian.

Proposition C.1. *If A is a matrix of size $2f \times 2f$ with $\det(A) \neq 0$ such that for some matrix P, invertible over R*

$$PAP^t = \begin{pmatrix} 0 & C \\ C^t & D \end{pmatrix}$$

where C, D are $(f \times f)$-matrices. Then $\lambda(A)$ is metabolic, in fact $(P^{-1}(0 \times R^f))/AR^{2f}$ is a metabolizer.

Proof. The map $R^{2f} \to R^{2f}, v \to Pv$ induces an isomorphism $\lambda(A) \cong \lambda(PAP^t)$ of forms. It therefore suffices to show that $\lambda(PAP^t)$ is metabolic. Without loss of generality we can therefore assume that $A = \begin{pmatrix} 0 & C \\ C^t & D \end{pmatrix}$. Then

$$A^{-1} = \begin{pmatrix} \tilde{D} & (C^t)^{-1} \\ C^{-1} & 0 \end{pmatrix}$$

for some matrix $\tilde{D}$. Let $Q := (0 \times R^f)/AR^{2f}$, it is easy to see that $\lambda(A)(Q \times Q) \equiv 0$, i.e. $Q \subset Q^\perp$. We want to show that in fact $Q = Q^\perp$. Let $\begin{pmatrix} x \\ y \end{pmatrix} \in R^{2f}$ where $x, y \in R^f$ such that $\lambda(A)\left(\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} 0 \\ z \end{pmatrix}\right) = 0 \mod R$ for all $z \in R^f$, i.e. $\begin{pmatrix} x \\ y \end{pmatrix} \in Q^\perp$. This implies that for all $e_i = (0, \dots, 0, 1, 0, \dots, 0)^t \in R^f, i = f+1, \dots, 2f$ we get

$$\begin{array}{l} \lambda(A)\left(\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} 0 \\ e_i \end{pmatrix}\right) = x^t (C^t)^{-1} e_i = 0 \mod R \\ \Rightarrow e_i^t C^{-1} x =: v_i \in R \Rightarrow C^{-1}x = v := (v_1, \dots, v_f)^t \in R^f \Rightarrow x = Cv \end{array}$$

But then we get

$$\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 0 & C \\ C^t & D \end{pmatrix} \begin{pmatrix} 0 \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ y - Dv \end{pmatrix}$$

i.e. $(x, y) \in Q$. This shows that indeed $Q = Q^\perp$. The statement about the metabolizer follows immediately. $\square$

Notations:

π_K	section 2.1
$M = M_K$	section 2.1
ϵ	section 2.1
μ	section 2.1
$\tilde{\pi}_K$	section 2.1
$N(A)$	section 2.1
$N = N_D$	section 2.2
$\mathrm{sign}(A)$	section 2.2
$X = X_K, X_k$	section 2.3
$H_1(M, \Lambda)$	section 2.3
$TH_1(M, \Lambda)$	section 2.3
Λ	section 2.3
S	section 2.3
λ_{Bl}	section 2.3
M_k	section 2.4
L_k	section 2.4
X_k	section 2.4
Λ_k	section 2.4
π_k	section 2.6
$G^{(i)}$	section 3.2
$\sigma_z(A), \sigma_z(K)$	section 3.4
$\alpha_{(z,\chi)} \in R_k(\pi_1(M))$	section 4.1
λ	section 4.1
$\beta_{(z,\chi)} \in R_1(\pi_1(M_k))$	section 5.2
F_χ	section 5.1
$\Omega_k(G)$	appendix $A.2$
$L_0(F)$	appendix B

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