Topology

Stefan Friedl

Contents (detailed)

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CHAPTER 1

Mini-Introduction

A *knot* is a submanifold of S^3 that is diffeomorphic to S^1 and a *link* is a submanifold of S^3 such that each component is diffeomorphic to S^1. Knots and links are a fun and much-loved topic in topology. In these lecture notes we will introduce these concepts in detail and we will use fundamental groups and "Fox calculus" to define the (twisted) Alexander polynomials of knots and links. This puts the lecture notes in the tradition of the classical book by Crowell and Fox [**CF77**]. In contrast to most other books on knot theory we will eschew homology and cohomology.

We will put a particular stress on explaining concepts precisely and on giving precise and rigorous proofs. For statements on geometric topology and differential topology we mostly refer to [**Fri24**].

These lecture notes are organized as follows:

- In Chapter 2 we will introduce knots and links, we will discuss basic examples, concepts and constructions and we will present some basic results. In particular we will introduce the meridian of an oriented knot and we will show that it is essentially unique.
- In Chapter 3 we will study the fundamental group $\pi_1(S^3 \setminus L)$ of a link. In particular we will calculate this group if L is a torus knot.
- In Chapter 4 we will introduce link diagrams and we will show that every link admits a diagram. Furthermore we will formulate the Reidemeister moves that can be used to relate any two diagrams of a given link.
- In Chapter 5 we will present an algorithm, which takes as input any diagram of a link L gives, and which gives as output the so-called Wirtinger presentations of $\pi_1(S^3 \setminus L)$, which is a presentation of deficiency one. This calculation allows us in particular to show that for any oriented m-component link the abelianization of $\pi_1(S^3 \setminus L)$ is naturally isomorphic to $\mathbb{Z}^m$.
- In Chapter 6 we will use the fact that for an oriented knot K we have a natural isomorphism $\pi_1(S^3 \setminus K)_{\mathrm{ab}} \xrightarrow{\cong} \mathbb{Z}$ to introduce the linking number $\mathrm{lk}(K, J)$ with any oriented knot J in the complement of K. Using the Wirtinger presentation we can give a diagrammatic way to calculate the linking number.
- In Chapter 7 we use the linking number to introduce the longitude of an oriented knot and we will show that it is essentially unique. We will use the meridian and the longitude of an oriented link to introduce the fairly simple minded coloring polynomials of an oriented knot, which will turn out to be surprisingly effective in distinguishing knots.
- In Chapter 8 we will introduce the purely algebraic concept of the Alexander function $\Delta_{\pi,\Phi} \in \mathbb{Q}(t_1, \ldots, t_m)$ of a deficiency-one group π and an epimorphism $\Phi\colon \pi \to \mathbb{Z}^m$ onto a free abelian group of rank $m \geq 1$.

- In Chapter 9 we will study the Alexander function of fundamental groups of link complements. A very mild variation on this concept leads us to the Alexander polynomial of oriented knots and links.
- In Chapter 10 we will show that the Alexander polynomial of an oriented link is always symmetric.
- In Chapter 11 we will introduce the concept of a Seifert surface of a knot and we will use Seifert's algorithm to show that every knot admits a Seifert surface.
- In Chapter 12 we will introduce the notion of a fibered knot and we will show that torus knots are fibered.
- In Chapter 13 we will show that degrees of Alexander polynomials give lower bounds on the genus of a knot and we will show that Alexander polynomials of fibered knots have specific properties.
- In Chapter 14 we will introduce twisted Alexander functions of groups. This concept generalizes the above "untwisted" notion. We will outline how some of the "untwisted" results generalize to the "twisted" setup.
- In Chapter 15 we will use the twisted Alexander function to define, not surprisingly, the twisted Alexander polynomials of knots and links and we will see that these again give lower bounds on the knot genus and obstructions to fiberedness. In general these twisted invariants give better information than the original Alexander polynomial.
- In Chapter 16 we will compare many different notions of equivalence of links.

CHAPTER 2

Introduction to knots and links

In this chapter we will define knots and links and we will discuss several examples, basic results and notions.

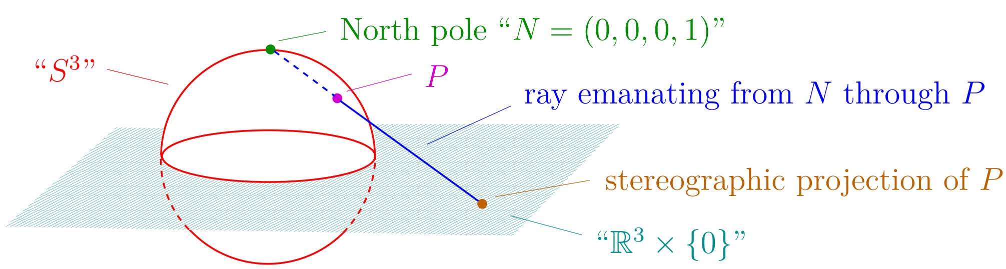
2.1. **Knots and links.** We start out with the following little lemma.

Lemma 2.1. (Stereographic Projection Lemma) We refer to the map[1]

$$\begin{array}{rcl} \Phi\colon S^3 & \to & \mathbb{R}^3 \cup \{\infty\} \\ (x_1,x_2,x_3,x_4) & \mapsto & \begin{cases} \left(\dfrac{x_1}{1-x_4}, \dfrac{x_2}{1-x_4}, \dfrac{x_3}{1-x_4}\right), & \text{if } x_4 < 1, \\ \infty, & \text{if } x_4 = 1. \end{cases} \end{array}$$

as the stereographic projection. This map has the following properties:

(1) The map Φ sends the North Pole $N := (0,0,0,1) \in S^3$ to ∞.
(2) For any $P \in S^3$ that does not equal the North Pole N the point $\Phi(P) \in \mathbb{R}^3$ is the unique point such that the ray emanating from N and that goes through P intersects the plane $\mathbb{R}^3 \times \{0\}$ in $(\Phi(P), 0)$.
(3) For any $(v_1,v_2,v_3,0) \in S^3 \cap (\mathbb{R}^3 \times \{0\})$ we have $\Phi(v_1,v_2,v_3,0) = (v_1,v_2,v_3)$.
(4) The map Φ is a homeomorphism.
(5) The restriction of Φ to a map $S^3 \setminus \{N\} \to \mathbb{R}^3$ is an orientation-preserving diffeomorphism.



Proof. The statements follow easily from the definitions. ∎

Identification. By definition and the Stereographic Projection Lemma 2.1 we have identifications

$$S^3 =_{\mathrm{i}} \left\{ (z,w) \in \mathbb{C}^2 \,\middle|\, |w|^2 + |z|^2 = 1 \right\} =_{\mathrm{i}} \left\{ (w,x,y,z) \,\middle|\, w^2+x^2+y^2+z^2 = 1 \right\} \underset{\uparrow}{=_{\mathrm{i}}} \mathbb{R}^3 \cup \{\infty\}.$$

via the stereographic projection as defined in the Stereographic Projection Lemma 2.1

[1] We equip $\mathbb{R}^3 \cup \{\infty\}$ with the topology where open neighborhoods of ∞ are given by sets of the form $(\mathbb{R}^3 \setminus K) \cup \{\infty\}$ where $K \subset \mathbb{R}^3$ is a compact subset.

We will go back and forth between these models without mentioning these maps. In particular we will use the Stereographic Projection Lemma 2.1 (5) to view $\mathbb{R}^3$ as a submanifold of the smooth manifold $S^3 =_{\mathrm{i}} \mathbb{R}^3 \cup \{\infty\}$.

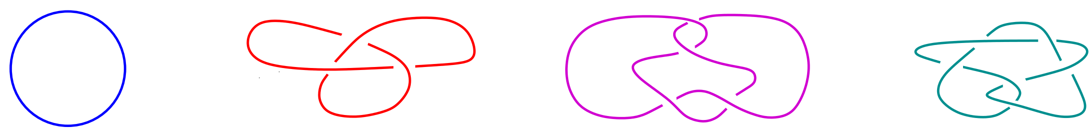
Now we turn to the definition of a knot.

Definition. A knot is a submanifold of $S^3 =_{\mathrm{i}} \mathbb{R}^3 \cup \{\infty\}$ that is diffeomorphic to S^1.

We start out with the most boring but also the most important knot.

Definition. The trivial knot is defined as $\{(x, y, 0) \in \mathbb{R}^3 \,|\, x^2 + y^2 = 1\} \subset \mathbb{R}^3 \cup \{\infty\} =_{\mathrm{i}} S^3$. In the literature the trivial knot is often also called the unknot.

In the figure below we show the trivial knot together with two other examples of knots. More precisely, we show 1-dimensional submanifolds of $\mathbb{R}^3$ that we view as 1-dimensional submanifolds of $\mathbb{R}^3 \cup \{\infty\} =_{\mathrm{i}} S^3$.[2]



trivial knot (left-handed) trefoil figure-8 knot stevedore knot

Remark. The definition of a knot is supposed to model the "physical objects" that we have in mind and that are sketched in the figure above. It is therefore perhaps at first not clear why we consider knots in $S^3 =_{\mathrm{i}} \mathbb{R}^3 \cup \{\infty\}$ instead of knots in $\mathbb{R}^3$. The reason is that topologists prefer, if possible, to work with compact spaces. In particular the compact space $S^3 =_{\mathrm{i}} \mathbb{R}^3 \cup \{\infty\}$ is often strongly preferable to the non-compact space $\mathbb{R}^3$. For peace of mind we will show in the Link-in-$\mathbb{R}^3$-S^3-Lemma 2.5 that the theory of knots in $\mathbb{R}^3$ is essentially the same as the theory of knots in S^3. □

The notion of a knot can be generalized to the notion of a link:

Definition. Let $m \in \mathbb{N}_0$.

(1) An m-component link is a submanifold of S^3 that is diffeomorphic to the disjoint union of m copies of S^1 together with a bijection from $\{1, \ldots, m\}$ to the set of components of L.
(2) Given an m-component link $L \subset S^3$ and given $i \in \{1, \ldots, m\}$ we denote by L_i the i-th component of L.

[2]In principle it is possible to give a precise description of all these four knots. For example we just gave a definition of the trivial knot as $\{(x, y, 0) \,|\, x^2 + y^2 = 1\} \subset S^3 = \mathbb{R}^3 \cup \{\infty\}$. Precise definitions, and 3-dimensional models of the trefoil and the figure-8 knot can also be found here:

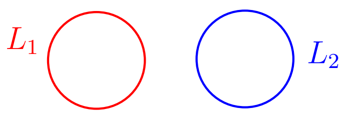
trefoil: `https://www.desmos.com/3d/3c1777edd9` (thanks to Johannes Zander)
figure-8 knot: `https://www.desmos.com/3d/aab54cb3df` (thanks to Filip Misev)

Later, on page 38, we will give an alternative precise definition of the trefoil. It is also clear that one can give a precise description of the stevedore knot, but this description would be painful to write down and it would not add to our understanding. We therefore stick with the picture, with the understanding, that if somebody was challenging us, we could write down a precise description in coordinates. But it is considered very impolite to challenge a topologist to give a rigorous description.

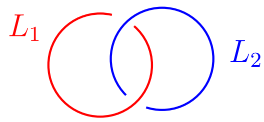
Definition. Let $m \in \mathbb{N}_0$. We define

$$\text{trivial } m\text{-component link} := \sum_{i=1}^{m} \{(x, y-3i, 0) \in \mathbb{R}^3 \,|\, x^2+y^2=1\} \subset \mathbb{R}^3 \cup \{\infty\} =_{\mathrm{i}} S^3$$

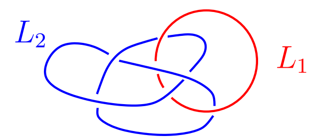
with the obvious numbering of the components of L.



trivial 2-component link



Hopf link



link where one component is a trefoil and one is the trivial knot

Remark. It follows from the classification of 1-dimensional smooth manifolds that a link is the same as a 1-dimensional submanifold of S^3 that is closed as a smooth manifold (together with a numbering of the components). □

Now we want to say that two links are "the same" if one can be "deformed" into the other.

Definition. We say that two m-component links $L = L_1 \sqcup \cdots \sqcup L_m$ and $\widetilde{L} = \widetilde{L}_1 \sqcup \cdots \sqcup \widetilde{L}_m$ are smoothly isotopic if there exists a smooth isotopy from L to $\widetilde{L}$, i.e. if there exists a smooth map

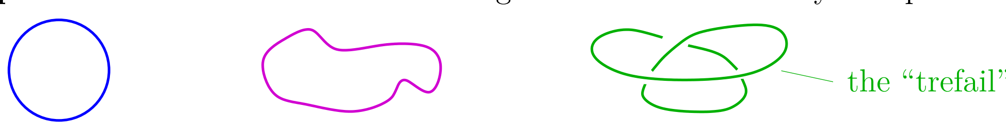
$$\begin{array}{rcl} F\colon L \times [0,1] & \to & S^3 \\ (z,t) & \mapsto & F(z,t) \end{array}$$

such that the following hold:[3]

(1) For each $t \in [0,1]$ the map $F_t\colon L \to S^3$ is a smooth embedding.
(2) We have $F_0 = \mathrm{id}$ and for each $i \in \{1,\dots,m\}$ we have $F_1(L_i) = \widetilde{L}_i$.

Remark. One can easily show that the property of being smoothly isotopic defines an equivalence relation on the set of links. □

Example. The three knots shown in the figure below are smoothly isotopic.



Usually we do not distinguish two knots if they are smoothly isotopic. For example, any knot that is smoothly isotopic to a trivial knot is called *trivial knot*. Sometimes we also say that a knot that is smoothly isotopic to a trivial knot is *trivial*.

□

Playing around with pictures for some time shows that it might be quite difficult to show that the trefoil is smoothly isotopic to the trivial knot. This arouses the suspicion, that the trefoil is in fact not smoothly isotopic to the trivial knot. This raises the following question.

Question 2.2. How can we show that the trefoil is not smoothly isotopic to the trivial knot?

[3] As usual, given a map $F\colon X \times [0,1] \to Y$ and given $t \in [0,1]$ we denote by $F_t\colon X \to Y$ the map that is given by $x \mapsto F(x,t)$.

The following proposition allows us to relate the above question to the question, whether two topological spaces are diffeomorphic.

Proposition 2.3. (Link–Smooth Isotopy Proposition) Let L and $\widetilde{L}$ be two links in S^3. If L and $\widetilde{L}$ are smoothly isotopic, then there exists an orientation-preserving diffeomorphism between the link complements $S^3 \setminus L$ and $S^3 \setminus \widetilde{L}$.

The key input is the following very general theorem which is proved in [**Fri24**].

Theorem 2.4. (Isotopy Extension Theorem) Let M be a smooth manifold with $\partial M = \varnothing$ and let K be a compact smooth manifolds. Given any smooth isotopy $F\colon K \times [0,1] \to M$ there exists a diffeotopy $G\colon M \times [0,1] \to M$ with the following properties:

(1) $G_0 = \mathrm{id}$.
(2) We have $G \circ (F_0 \times \mathrm{id}) = F\colon K \times [0,1] \to M$, in other words, we have
$$G(F_0(x),t) \;=\; F(x,t) \qquad \text{for all } x \in K \text{ and } t \in [0,1].$$

Proof of the Link–Smooth Isotopy Proposition 2.3. Let
$$\begin{array}{rcl} F\colon L \times [0,1] & \to & S^3 \\ (z,t) & \mapsto & F(z,t) \end{array}$$
be a smooth isotopy from a link L to a knot $\widetilde{L}$. By the Isotopy Extension Theorem 2.4 we can extend the smooth isotopy F to a diffeotopy of S^3. This means in particular that there exists a diffeotopy
$$\begin{array}{rcl} G\colon S^3 \times [0,1] & \to & S^3 \\ (z,t) & \mapsto & G(z,t) \end{array}$$
from the identity to a diffeomorphism $\Phi = G_1\colon S^3 \to S^3$ with $\Phi|_K = F_1$, in particular with $\Phi(L) = \widetilde{L}$. The map $\Phi\colon S^3 \to S^3$ restricts to an orientation-preserving diffeomorphism $\Phi\colon S^3 \setminus L \to S^3 \setminus \widetilde{L}$. ■

The following lemma says that the theory of links in S^3 up to smooth isotopy is essentially the same as the theory of links in $\mathbb{R}^3$ up to smooth isotopy.

Lemma 2.5. (Link-in-$\mathbb{R}^3$-S^3-Lemma)

(1) Every link in $S^3 =_{\mathrm{i}} \mathbb{R}^3 \cup \{\infty\}$ is smoothly isotopic to a link in $\mathbb{R}^3$.
(2) Let $L, \widetilde{L}$ be two links in $\mathbb{R}^3$. If L and $\widetilde{L}$ are smoothly isotopic in $S^3 =_{\mathrm{i}} \mathbb{R}^3 \cup \{\infty\}$, then they are also smoothly isotopic in $\mathbb{R}^3$.

Sketch of proof. As usual we make the identification $S^3 =_{\mathrm{i}} \mathbb{R}^3 \cup \{\infty\}$ where $e_4 = N = \infty$.

(a) Let L be a link in $S^3 =_{\mathrm{i}} \mathbb{R}^3 \cup \{\infty\}$. Since $\dim(L) = 1 < 3 = \dim(S^3)$ we know that $L \neq S^3$. Therefore we can pick $P \in S^3 \setminus L$. Note that there exists an $A \in \mathrm{SO}(4)$ with $A \cdot P = N$. Since $\mathrm{SO}(4)$ is connected the matrix A gives rise to a diffeotopy $F\colon S^3 \times [0,1] \to S^3$ with $F_0 = \mathrm{id}$ and such that F_1 is given by multiplication by A. Clearly F defines a smooth isotopy from $L = F_0(L)$ to $F_1(L) \subset S^3 \setminus \{N\} = \mathbb{R}^3$.
(b) Let $L, \widetilde{L}$ be links in $\mathbb{R}^3$ and let $H\colon L \times [0,1] \to S^3$ be a smooth isotopy from L to $\widetilde{L}$. We start out with the following rather technical claim:

Claim. There exists a smooth embedding $\Phi\colon \overline{B}^3 \to \mathrm{SO}(4)$ such that $\Phi(0) = \mathrm{id}$ and such that $\Phi(\overline{B}^3) \cdot e_4 = \{\Phi(x) \cdot e_4 \,|\, x \in \overline{B}^3\}$ is a neighborhood of $e_4 = \infty$ in $S^3 =_{\mathrm{i}} \mathbb{R}^3 \cup \{\infty\}$.

Proof. We leave it to the reader to verify that the following map has all the desired properties:

$$\begin{aligned} \Phi\colon \overline{B}^3 \;&\to\; \mathrm{SO}(4) \\ x \;&\mapsto\; \text{Gram-Schmidt orthogonalization process (starting with the last column)} \\ &\qquad\qquad \text{applied to the matrix } \underbrace{(e_1 \;\; e_2 \;\; e_3 \;\; e_4 + (x,0))}_{\in \mathrm{GL}_+(4,\mathbb{R})}. \end{aligned}$$

⊞

Next note that it follows fairly easily from $\Phi(0) = \mathrm{id}$ and from $L, \widetilde{L} \subset \mathbb{R}^3$ that there exists an $\epsilon \in (0,1)$ such that for every $x \in \overline{B}^3_\epsilon$ we have $\Phi(x) \cdot L \subset \mathbb{R}^3$ and $\Phi(x) \cdot \widetilde{L} \subset \mathbb{R}^3$. Since $\dim(L \times [0,1]) = 2 < 3 = \dim(S^3)$ and since H is smooth we can pick $x \in \overline{B}^3_\epsilon$ with $\Phi(x) \cdot e_4 \notin S^3 \setminus H(L \times [0,1])$. (We refer to [**Fri24**] for details.) Note that this means that $(\Phi(x)^{-1} \circ H)(L \times [0,1]) \subset \mathbb{R}^3 = S^3 \setminus \{e_4\}$.

We now have the following three smooth isotopies

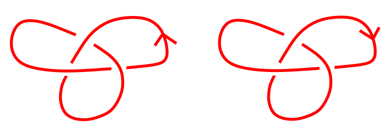
$$\begin{aligned} L \times [0,1] &\to \mathbb{R}^3 \\ (v,t) &\mapsto \Phi(t \cdot x)^{-1}(v) \end{aligned} \qquad \begin{aligned} L \times [0,1] &\to \mathbb{R}^3 \\ (v,t) &\mapsto \Phi(x)^{-1} \cdot H(v,t) \end{aligned} \qquad \begin{aligned} \widetilde{L} \times [0,1] &\to \mathbb{R}^3 \\ (v,t) &\mapsto \Phi((1-t) \cdot x)^{-1}(v). \end{aligned}$$

Combining these three smooth isotopies gives us the desired smooth isotopy from L to $\widetilde{L}$ in $\mathbb{R}^3$. ■

2.2. Orientations.

In many settings it is also natural to equip knots and links with orientations. This leads us to the following definition:

Definition. Let $m \in \mathbb{N}_0$. A link is called oriented if it is oriented as a 1-dimensional smooth manifold.



trefoils with opposite orientation

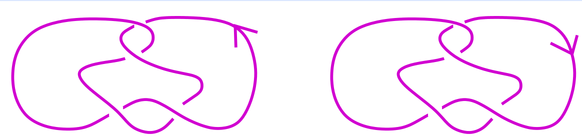
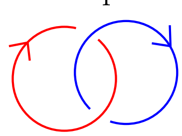


figure-8 knot with opposite orientations

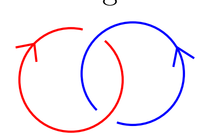
Definition. We say that two *oriented* links $L = L_1 \sqcup \cdots \sqcup L_m$ and $\widetilde{L} = \widetilde{L}_1 \sqcup \cdots \sqcup \widetilde{L}_m$ are smoothly isotopic, if there exists a smooth isotopy $F\colon L \times [0,1] \to S^3$ from L to $\widetilde{L}$ such that for each $i \in \{1,\dots,m\}$ the map $F_1\colon L_i \to \widetilde{L}_i$ is an orientation preserving diffeomorphism.

Example. We consider the two Hopf links H^+ and H^- that are shown in the figure below.

positive Hopf link H^-



negative Hopf link H^+



The difference is that we flipped the orientation of the blue component. Using the fundamental group one can easily show that these two oriented Hopf links are not smoothly isotopic. □

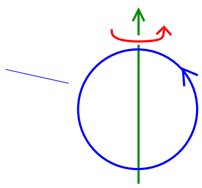
In light of the above example we now restrict ourselves to knots. For knots one can wonder whether the orientation of a knot really makes a difference. This suggests the following definition:

Definition.

(1) Let K be an oriented knot. We define the reverse K^{rev} of K to be the knot K with the opposite orientation.
(2) A knot K is called reversible if the two orientations on K give smoothly isotopic oriented knots.[4]

Example. As we see in the figure below it is basically clear that the trivial knot is reversible.

trivial knot



rotation around the z-axis
by the angle π turns the trivial knot
into itself but flips the orientation

In Exercise 2.1 we will show that the trefoil and the figure-8 knot are also reversible. □

This leads us to the following question:

Question 2.6. Is every knot reversible?

2.3. **The mirror image of a link.** We continue with the following nicely geometric definition.

Definition. Given a link $L \subset S^3$ the reflection of L in any hyperplane of $\mathbb{R}^4$ is called mirror of L and denoted by L^{mir}.

Lemma 2.7. (Link Mirror Lemma)

(1) Let $L \subset S^3$ be a link. The reflections in any two hyperplanes of $\mathbb{R}^4$ give rise to smoothly isotopic links.
(2) Let $L \subset \mathbb{R}^3$ be a link. As usual, using the identification $\mathbb{R}^3 \cup \{\infty\} =_{\text{i}} S^3$, we view L as a link in S^3. The image of L under a reflection of an affine hyperplane in $\mathbb{R}^3$ corresponds under this embedding to a mirror of L.

Sketch of proof.

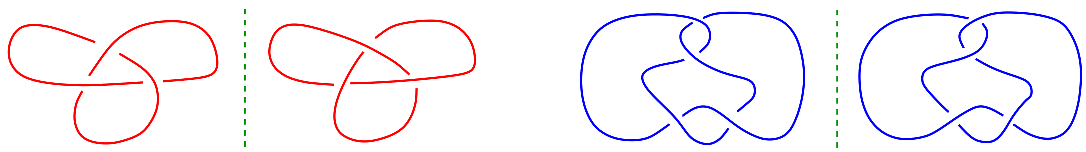
(1) Let $L \subset S^3$ be a link and let H_1 and H_2 be two hyperplanes of $\mathbb{R}^4$. Elementary linear algebra shows that there exists an $A \in \text{SO}(4)$ with $A \cdot H_1 = H_2$. More elementary linear algebra shows that $\text{SO}(4)$ is path-connected. It follows that there exists a smooth path $\gamma\colon [0,1] \to \text{SO}(4)$ with $\gamma(0) = \text{id}$ and $\gamma(1) = A$. We now consider the diffeotopy
$$\begin{array}{rcl} F\colon S^3 \times [0,1] & \to & S^3 \\ (x,t) & \mapsto & \text{reflection in the hyperplane } \gamma(t)\cdot H_1 \text{ applied to } x \end{array}$$
The restriction of $F\colon S^3 \times [0,1] \to S^3$ to $L \times [0,1] \to S^3$ gives us the desired smooth isotopy between the reflections of L in H_1 and H_2.
(2) Let $L \subset \mathbb{R}^3$ be a link. First note that the same argument as in (1) shows that reflection in any two affine hyperplanes of $\mathbb{R}^3$ lead to smoothly isotopic links in $\mathbb{R}^3$. Thus it remains to show that there exists a single hyperplane H of $\mathbb{R}^3$ such that the reflection of L in H corresponds to the mirror image of $L \subset \mathbb{R}^3 \subset S^3$ as defined above. Next

[4] In the literature what we call a reversible knot is often called an invertible knot. We reserve the adjective invertible for a related, but different notion.

note that the stereographic projection, that we used in the Stereographic Projection Lemma 2.1, commutes with the reflection in the $(x_1 = 0)$-hyperplane of $\mathbb{R}^4$ respectively $\mathbb{R}^3$. Thus we see that the hyperplane $(x_1 = 0)$ has the desired property. ■

Convention. Let $L \subset S^3$. Since usually we only care about links up to smooth isotopy we use the Link Mirror Lemma 2.7 to talk of "the mirror image" of L instead of the slightly more correct "a mirror image of L".

Example. Keeping the Link Mirror Lemma 2.7 (2) in mind, we see in the figure below the trefoil and its mirror and we also see the figure-8 link and its mirror.



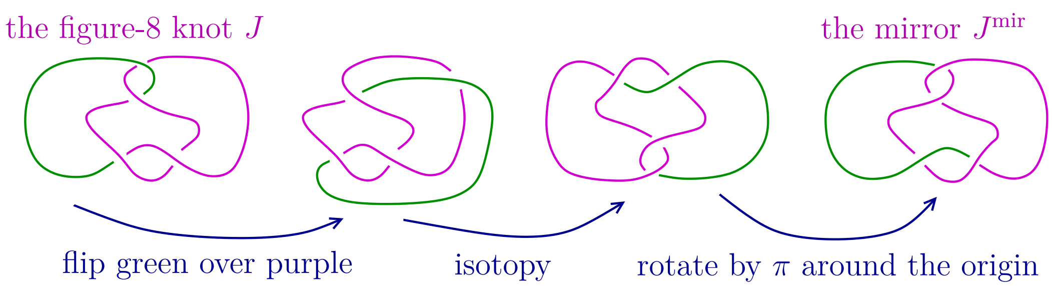
the trefoil K and its mirror K^{mir} the figure-8 knot J and its mirror J^{mir}

As for orientations the question arises, whether the mirror image of a link is "truly different" from the original link. We are thus naturally led to the following definition:

Definition. We say that a link L in $\mathbb{R}^3$ is amphichiral if it is smoothly isotopic to its mirror image, otherwise we call the link L chiral.

Examples.

(1) The trivial knot equals of course its mirror, in particular the trivial knot is amphichiral.
(2) In the figure below we show that the figure-8 knot J is smoothly isotopic to its mirror image J^{mir}, i.e. the figure-8 knot is also amphichiral.



It is now natural to ask whether the trefoil is amphichiral. But endless hours of playing around with the trefoil do not lead to any success. So the suspicion arises that the following question should be answered in the negative.

Question 2.8. Is the trefoil amphichiral?

2.4. **The split union of links.**

In this and the following section we will introduce two ways to construct new links out of given links. In both cases the construction will depend on choices and we will want to argue that up to smooth isotopy the choices make no difference.

We will deal with this issue using the following proposition.

Theorem 2.9. (Smooth Ball Embedding Theorem) Let M be an oriented n-dimensional smooth manifold. Let $\varphi, \psi\colon \overline{B}^n \to M \setminus \partial M$ be orientation-preserving smooth embeddings of $\overline{B}^n$. There exists a diffeotopy

$$G\colon M \times [0,1] \;\to\; M$$

from the identity $G_0 = \mathrm{id}_M$ to a diffeomorphism $G_1\colon M \to M$ such that the following diagram commutes:

$$\begin{tikzcd} & \overline{B}^n \arrow[dl, "\varphi"'] \arrow[dr, "\psi"] & \\ M \arrow[rr, "G_1", "\cong"'] & & M. \end{tikzcd}$$

Finally, if $\varphi(0) = \psi(0)$, then we can find a diffeotopy rel $\{\varphi(0)\}$.

Sketch of proof. We sketch the proof for $M = \mathbb{R}^n$ and if $\varphi(0) = \psi(0) = 0$. In this case the key observation in the proof is the surprising fact that for any smooth embedding $f\colon \overline{B}^n \to \mathbb{R}^n$ with $f(0) = 0$ the map

$$\begin{array}{rcl} F\colon \overline{B}^n \times [0,1] & \to & \mathbb{R}^n \\ (x,t) & \mapsto & \begin{cases} \frac{1}{t}\cdot f(t\cdot x), & \text{if } t \neq 0, \\ \mathrm{D}f_0 \cdot x, & \text{if } t = 0 \end{cases} \end{array}$$

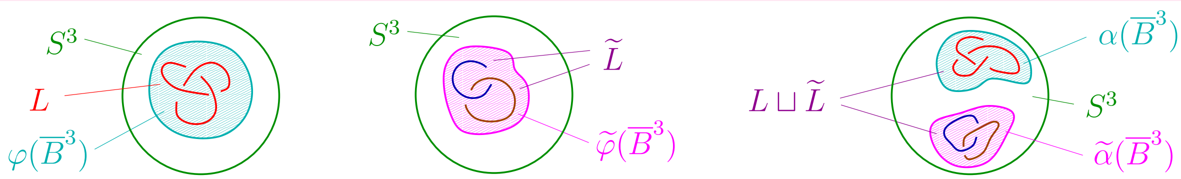
is a smooth isotopy from the map that is given by multiplication by $\mathrm{D}f_0 \cdot x$ to the original map $f\colon \overline{B}^n \to \mathbb{R}^n$. We refer to [**Fri24**] for details and how to deduce the theorem from this observation. ■

Now let us introduce the first way to construct new (oriented) links out of given ones.

Lemma 2.10. (Link–Split Union Lemma) Let L and $\widetilde{L}$ be two oriented links in S^3.

(1) We can pick an orientation-preserving embedding $\varphi\colon \overline{B}^3 \to S^3$ with $L \subset \varphi(B^3)$ and we can pick an orientation-preserving embedding $\widetilde{\varphi}\colon \overline{B}^3 \to S^3$ with $\widetilde{L} \subset \widetilde{\varphi}(B^3)$.

(2) We pick two orientation-preserving embeddings $\alpha, \widetilde{\alpha}\colon \overline{B}^3 \to S^3$ with disjoint images. The smooth isotopy type of $\alpha(\varphi^{-1}(L)) \cup \widetilde{\alpha}(\widetilde{\varphi}^{-1}(\widetilde{L}))$ is well-defined and it only depends on the smooth isotopy type of L and $\widetilde{L}$.



Definition. We continue with the notation from the Link–Split Union Lemma 2.10.

(1) We refer to $\alpha(\varphi^{-1}(L)) \cup \widetilde{\alpha}(\widetilde{\varphi}^{-1}(\widetilde{L}))$ as the split union $L \sqcup \widetilde{L}$ of the links L and $\widetilde{L}$.

(2) We say that a link is splittable if it is the split union of two non-empty links. Otherwise we call the link unsplittable.[5]

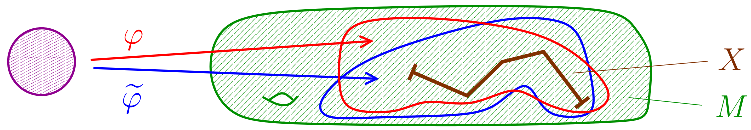
Sketch of proof.

(1) We will prove this statement in Exercise 2.10.

[5]It follows almost immediately from the definitions that a link is splittable if and only if there exist two disjoint smoothly embedded 3-balls in S^3 each of which contains at least one component of L.

(2) We start out with the following statement which might actually be of independent interest.

Claim. Let X be a compact subset of an oriented n-dimensional smooth manifold M and let $\varphi, \widetilde{\varphi}\colon \overline{B}^n \to M$ be two orientation-preserving smooth embeddings with $X \subset \varphi(B^n)$ and $X \subset \widetilde{\varphi}(B^n)$. There exists a smooth isotopy $\Theta\colon \overline{B}^n \times [0,1] \to M$ such that $\Theta_0 = \varphi$, $\Theta_1 = \widetilde{\varphi}$ and such that for each $t \in [0,1]$ we have $X \subset \Theta_t(B^n)$.



Proof. If X is empty, then there is nothing to show. So now assume that X is non-empty. We pick $P \in X$. We can assume without loss of generality that $\varphi(0) = \widetilde{\varphi}(0) = P$. Note that this means that we can view φ and $\widetilde{\varphi}$ as tubular maps for the 0-dimensional submanifold $\{P\} \subset M$. It follows from the Smooth Ball Embedding Theorem 2.9 that there exists a smooth isotopy $\Theta\colon \overline{B}^n \times [0,1] \to M$ such that $\Theta_0 = \varphi$, $\Theta_1 = \widetilde{\varphi}$ and such that for each $t \in [0,1]$ we have $\Theta_t(0) = P$.

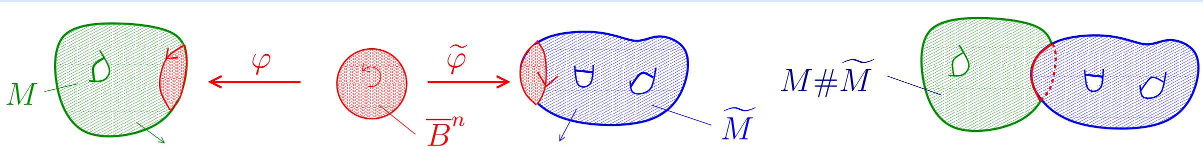
It is straightforward to show that there exists an open neighborhood U of P such that for each $t \in [0,1]$ we have $U \subset \Theta_t(B^n)$. If $X \subset U$, then we are done. Otherwise we use the fact that X is by hypothesis compact to reduce the general case to the case that $X \subset U$. ⊞

Now we go back the actual statement of the lemma. First note that it follows from the claim that the smooth isotopy type of $\alpha(\varphi^{-1}(L)) \cup \widetilde{\alpha}(\widetilde{\varphi}^{-1}(\widetilde{L}))$ does not depend on the choice of φ and $\widetilde{\varphi}$. Next note that it follows easily from the Smooth Ball Embedding Theorem 2.9 that the smooth isotopy type of $\alpha(\varphi^{-1}(L)) \cup \widetilde{\alpha}(\widetilde{\varphi}^{-1}(\widetilde{L}))$ does not depend on the choice of α and $\widetilde{\alpha}$. Finally note that it follows easily from the Isotopy Extension Theorem 2.4 that the smooth isotopy type of $\alpha(\varphi^{-1}(L)) \cup \widetilde{\alpha}(\widetilde{\varphi}^{-1}(\widetilde{L}))$ only depends on the smooth isotopy type of L and $\widetilde{L}$. ■

2.5. **The connected sum of oriented submanifolds and knots.** In this section we will introduce the connected sum $K \# \widetilde{K}$ of two oriented knots $K, \widetilde{K}$. The definition requires some preparation. Let us first recall the definition of the connected sum of smooth manifolds.

Definition. Let $n \in \mathbb{N}$ and let M and $\widetilde{M}$ be two oriented connected non-empty n-dimensional smooth manifolds. We pick an orientation-preserving smooth embedding $\varphi\colon \overline{B}^n \to M \setminus \partial M$. Furthermore we pick an orientation-reversing smooth embedding $\widetilde{\varphi}\colon \overline{B}^n \to \widetilde{M} \setminus \partial \widetilde{M}$. We define the connected sum of M and $\widetilde{M}$ as

$$M \# \widetilde{M} := \big(M \setminus \varphi(B^n)\big) \sqcup \big(\widetilde{M} \setminus \widetilde{\varphi}(B^n)\big)/\sim \quad \text{where } \varphi(P) \sim \widetilde{\varphi}(P) \text{ for all } P \in S^{n-1}.$$



Example. The above figure shows convincingly that the connected sum of a surface of genus g with a surface of genus h is a surface of genus $g+h$. □

The following proposition summarizes some key facts about the connected sum of smooth manifolds:

Proposition 2.11. (Manifold–Smooth Connected Sum-Proposition) Let $n \in \mathbb{N}$ and let M and $\widetilde{M}$ be two oriented connected n-dimensional smooth manifolds. Furthermore let $\varphi\colon \overline{B}^n \to M \setminus \partial M$ be an orientation-preserving smooth embedding and finally let $\widetilde{\varphi}\colon \overline{B}^n \to \widetilde{M} \setminus \partial\widetilde{M}$ be an orientation-reversing smooth embedding. The following statements hold:

(0) The resulting connected sum $M\#\widetilde{M}$ is a smooth manifold with a unique orientation that coincides with the orientations of $M \setminus \varphi(B^n)$ and $\widetilde{M} \setminus \widetilde{\varphi}(B^n)$. Up to an orientation-preserving diffeomorphism the definition does not depend on the choice of φ and $\widetilde{\varphi}$.
(1) There exists an explicit diffeomorphism from $M\#S^n$ to M.
(2) The connected sum is commutative, i.e. $M\#\widetilde{M}$ admits an orientation-preserving diffeomorphism to $\widetilde{M}\#M$.
(3) If $n \geq 2$ or if all the manifolds are closed, then the connected sum is associative, i.e. $(M\#M')\#M''$ admits an orientation-preserving diffeomorphism to $M\#(M'\#M'')$.

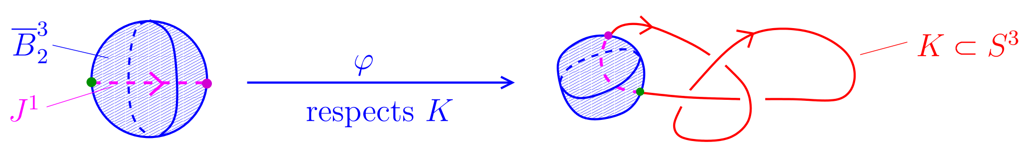
Proof. The fact that $M\#\widetilde{M}$ is well-defined up to an orientation-preserving diffeomorphism is a pretty straightforward consequence of the Smooth Ball Embedding Theorem 2.9 and the Isotopy Extension Theorem 2.4. We refer to [**Fri24**] for details and the proofs of the other statements. ■

Next we want to introduce the connected sum of submanifolds. This requires the following definition:

Definition. Let $n \in \mathbb{N}$ and let $k \in \{1, \ldots, n-1\}$.

(1) We denote by $\overline{B}^n_2$ and $\overline{B}^k_2$ the closed balls of radius 2 in $\mathbb{R}^n$ and $\mathbb{R}^k$.
(2) We set $J^k := \{(x, 0) \in \overline{B}^n_2 \,|\, x \in \overline{B}^k_2\}$ and we view equip J^k with the obvious orientation.
(3) Let M be an oriented n-dimensional smooth manifold and let $K \subset M$ be an oriented k-dimensional submanifold. Furthermore let $\varphi\colon \overline{B}^n_2 \to M \setminus \partial M$ be a smooth embedding.
    (a) We say that φ respects K if $\varphi(J^k) = \varphi(\overline{B}^n_2) \cap K$, if $\varphi\colon \overline{B}^n_2 \to M$ is orientation-preserving and if $\varphi|_{J^k}\colon J^k \to \varphi(\overline{B}^n_2) \cap K$ is orientation-preserving.
    (b) We say that φ anti-respects K if the same conditions as in (a) hold, except that we now demand that both maps are orientation-reversing.

In the figure below we illustrate the definition of a map that respects K in the special case that $M = \mathbb{R}^3 \cup \{\infty\}_i = S^3$ and that K is in fact a knot.



The following proposition says that maps that respect a submanifold K always exist and that in a sense they are unique.

Proposition 2.12. (Respectful Embedding Proposition) Let M be an oriented n-dimensional smooth manifold. Furthermore let $K \subset M$ be an oriented k-dimensional submanifold.

(1) If K is non-empty, then there exists a smooth embedding $\overline{B}^n_2 \to M$ that respects K.
(2) If K is connected, then for any two smooth embeddings $\varphi\colon \overline{B}^n_2 \to M$ and $\psi\colon \overline{B}^n_2 \to M$ that respect K there exists a diffeotopy $F\colon M \times [0,1] \to M$ such that $F_0 = \mathrm{id}$, such that $F_1 \circ \varphi = \psi$ and such that each $F_t \circ \varphi$ respects K.

The obvious analogues of (1) and (2) also hold for "anti-respect" instead of "respect".

Sketch of proof. The first statement follows easily from the definition of a submanifold. Now let us turn to the proof of the second statement. Thus we assume that K is connected. Let $\varphi\colon \overline{B}^n_2 \to M$ and $\psi\colon \overline{B}^n_2 \to M$ be two continuous maps that respect K. It follows from a variation on the Smooth Ball Embedding Theorem 2.9) that there exists a smooth isotopy $I\colon \overline{B}^n \times [0,1] \to M$ such that $I_0 = \varphi$, such that $I_1 = \psi$ and such that each $I_t \circ \varphi$ respects K.

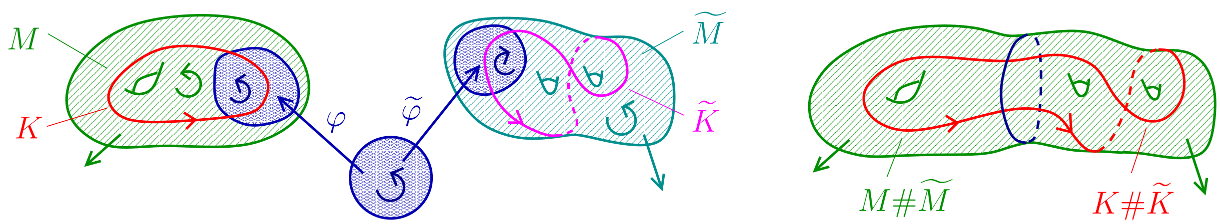
It follows from a slight generalization of the Isotopy Extension Theorem 2.4 that there exists a diffeotopy $F\colon M \times [0,1] \to M$ with the following properties:

(1) $F_0 = \mathrm{id}$.
(2) For all $x \in \overline{B}^n$ and all $t \in [0,1]$ we have $F(I_0(x), t) = F(x, t)$.

This diffeotopy has all the desired properties. ■

Now we can define the connected sum of smooth submanifolds.

Definition. Let M be an oriented n-dimensional smooth manifold and let $K \subset M$ be an oriented k-dimensional submanifold. Furthermore let $\widetilde{M}$ be an oriented n-dimensional smooth manifold and let $\widetilde{K} \subset \widetilde{M}$ be an oriented k-dimensional submanifold. We pick a smooth embedding $\varphi\colon \overline{B}^n_2 \to M$ that respects K and we pick a smooth embedding $\widetilde{\varphi}\colon \overline{B}^n_2 \to \widetilde{M}$ that anti-respects $\widetilde{K}$. We use $\varphi\colon \overline{B}^n \to M$ and $\widetilde{\varphi}\colon \overline{B}^n \to \widetilde{M}$ to define $M \# \widetilde{M}$. Furthermore we use[6] $\overline{B}^k = \overline{B}^k_1 \xrightarrow{x \mapsto (x,0)} J^n_k \xrightarrow{\varphi} K$ and $\overline{B}^k = \overline{B}^k_1 \xrightarrow{x \mapsto (x,0)} J^n_k \xrightarrow{\widetilde{\varphi}} \widetilde{K}$ to define the connected sum $K \# \widetilde{K}$. We refer to the resulting pair $(M \# \widetilde{M}, K \# \widetilde{K})$ as the connected sum of (M, K) and $(\widetilde{M}, \widetilde{K})$.



The next proposition can be viewed as an analogue of the Manifold–Smooth Connected Sum-Proposition 2.11.

[6]Note that here we deliberately restrict ourselves from balls of radius 2 to balls of radius 1.

Proposition 2.13. (Submanifold–Connected Sum Proposition) Let M be an oriented n-dimensional smooth manifold and let K be a closed oriented k-dimensional submanifold of M. Furthermore let $\widetilde{M}$ be an oriented n-dimensional smooth manifold and let $\widetilde{K}$ be a closed oriented k-dimensional submanifold of $\widetilde{M}$.

(1) The subset $K\#\widetilde{K}$ is a closed k-dimensional smooth submanifold of $M\#\widetilde{M}$.
(2) If M and $\widetilde{M}$ are both connected, then the oriented diffeomorphism type of the pair $(M\#\widetilde{M}, K\#\widetilde{K})$ does not depend on the choice of the map that respects K and the map that anti-respects $\widetilde{K}$.

Sketch of proof.

(1) We leave it to the reader to verify this statement. Note that here one needs to make use of the fact that the maps that (anti-) respect the submanifold give us control on $\overline{B}{}^n_2$ and not just on $\overline{B}{}^n$.
(2) This statement follows quite easily from the Respectful Embedding Proposition 2.12. ■

Now we can finally specialize to our beloved setting of knots in S^3.

Definition.

(1) Let K and $\widetilde{K}$ be two oriented knots in S^3. We pick a smooth embedding $\varphi\colon \overline{B}{}^3_2 \to S^3$ that respects K and we pick a smooth embedding $\widetilde{\varphi}\colon \overline{B}{}^3_2 \to S^3$ that anti-respects $\widetilde{K}$. We perform the connected sum $(S^3\#S^3, K\#\widetilde{K})$ and we use the semi-obvious orientation-preserving diffeomorphism $S^3\#S^3 \to S^3$ to view $K\#\widetilde{K}$ as an oriented knot in S^3.[7]
(2) We say that a knot K is prime[8] if it is not smoothly isotopic to the connected sum of two non-trivial knots.

Examples.

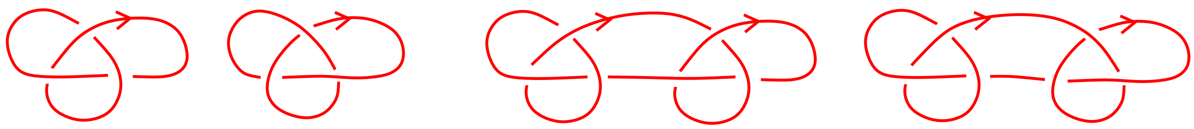
(1) In the figure below we show the connected sum of the trefoil K and the figure-8 knot $\widetilde{K}$. Admittedly it can take a minute to connect the picture to the actual definition.

trefoil K figure-8 knot $\widetilde{K}$ connected sum $K\#\widetilde{K}$

In the next figure we show the connected sum of trefoil K and of the reverse of the figure-8 knot $\widetilde{K}^{\text{rev}}$. The figure shows that the orientation of the knots plays a role.
(2) In the figure below we show the trefoil and its "mirror image" K^{mir}, i.e. the reflection of K in the xy-hyperplane.
The connected sum $K\#K$ is called the *granny knot* and the connected sum $K\#K^{\text{mir}}$ is called the *square knot*. It is notoriously tricky to distinguish the square knot and the

[7] It requires some thought why this diffeomorphism $S^3\#S^3 \to S^3$ is unique up to isotopy. An easy way out is to appeal to the Cerf Theorem 16.2.
[8] In principle it would be more suitable to call such a knot "irreducible". The name "prime" is justified by the Knot–Prime Decomposition Theorem 11.13.

trefoil K $\qquad$ $\widetilde{K}^{\text{rev}}$ $\qquad$ connected sum $K\#\widetilde{K}^{\text{rev}}$



trefoil K $\qquad$ mirror image K^{mir} $\qquad$ granny knot $K\#K$ $\qquad$ square knot $K\#K^{\text{mir}}$

granny knot, even up to taking "mirror images". We will see later in Exercise 7.8 that the granny knot and the square knot are indeed not smoothly isotopic. □

The following proposition is now an analogue of the Manifold–Smooth Connected Sum-Proposition 2.11.

Proposition 2.14. (Knot Connected Sum Proposition) On the set of smooth isotopy classes of oriented knots the connected sum operation is well-defined and it has the following properties:

(1) The trivial knot is a neutral element.
(2) The connected sum operation is commutative.
(3) The connected sum operation is associative.

In other words, the connected sum operation defines a commutative monoid structure on the set of smooth isotopy classes of oriented knots.

Sketch of proof. First note that it follows almost immediately from the Isotopy Extension Theorem 2.4 together with the Submanifold–Connected Sum Proposition 2.13 that the connected sum operation is indeed well-defined on the set of isotopy classes of oriented knots.[9]

The proof of Statements (1)–(3) is a slight variation on the proof of the corresponding statements of the Manifold–Smooth Connected Sum-Proposition 2.11. We leave it to the reader to fill in the details. ■

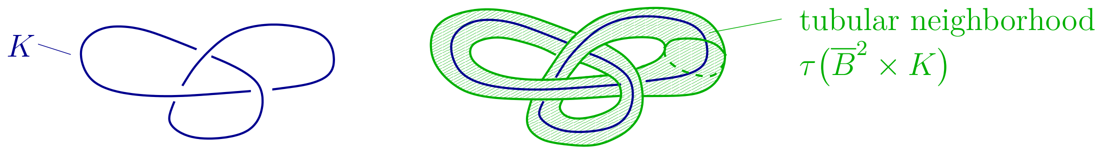
2.6. Tubular maps, exteriors and meridians.

We continue our discussion of knots and links with the following fairly general definition.

Definition. Let M be a 3-dimensional smooth manifold with $\partial M = \varnothing$ and let $L \subset M$ be a closed 1-dimensional smooth submanifold.

(1) A tubular map is a smooth embedding $\tau\colon \overline{B}^2 \times L \to M$ such that for any $P \in L$ we have $\tau(0, P) = P$.
(2) The image of a tubular map is called a tubular neighborhood of L.

[9]Note though that the statement does not follow just from the Submanifold–Connected Sum Proposition 2.13 since we claim that $K\#\widetilde{K}$ is well-defined up to isotopy in S^3. This is a priori not the same statement as saying that the oriented diffeomorphism type of $(S^3\#S^3, K\#\widetilde{K})$ is well-defined.



The following theorem says that tubular maps always exist and that they are unique in a suitable sense.

Theorem 2.15. (Link Tubular Map Theorem) Let M be an oriented 3-dimensional smooth manifold with $\partial M = \varnothing$ and let $L \subset M$ be a closed oriented 1-dimensional smooth submanifold.

(1) There exists an orientation-preserving[10] tubular map $\tau\colon \overline{B}^2 \times L \to M$.
(2) Let $\sigma, \tau\colon \overline{B}^2 \times L \to M$ be two orientation-preserving tubular maps for L. There exists a diffeotopy $F\colon M \times [0,1] \to M$ and a smooth map $g\colon L \to \mathrm{SO}(2)$ with the following properties:
    (a) $F_0 = \mathrm{id}$.
    (b) F is a homotopy rel L.
    (c) The diffeomorphism $F_1\colon M \to M$ restricts to the map

$$\begin{array}{rcl} \sigma(\overline{B}^2 \times L) & \to & \tau(\overline{B}^2 \times L) \\ \sigma(v, x) & \mapsto & \tau(g(x) \cdot v, x). \end{array}$$

Proof.

(1) This statement is a special case of the Product Tubular Neighborhood Theorem together with the Tubular Map Lemma from [**Fri24**].
(2) This statement is a special case of the the Tubular Map–Uniqueness Theorem which is formulated and proved in [**Fri24**]. ■

Definition. Let L be an link in S^3. By the Link Tubular Map Theorem 2.15 (1) there exists a tubular map $\tau\colon \overline{B}^2 \times L \to S^3$. We refer to $X_L := S^3 \setminus \tau(B^2 \times L)$ as the exterior of L.

A priori the definition of the exterior depends on the choice of the tubular map. Fortunately the following lemma tells us that this is not a real issue.

Lemma 2.16. (Link Exterior Lemma) Let L be a link and let $\tau\colon \overline{B}^2 \times L \to S^3$ be a tubular map.

(1) The exterior $X_L := S^3 \setminus \tau(B^2 \times L)$ of L is a compact smooth submanifold of S^3 with $\partial X_L = \tau(S^1 \times L)$. In particular X_L inherits an orientation from S^3.
(2) The exterior X_L is well-defined up to an orientation-preserving diffeomorphism.
(3) If L and L' are smoothly isotopic links, then there exists an orientation-preserving diffeomorphism between the link exteriors X_L and $X_{L'}$.
(4) The inclusion $X_L \to S^3 \setminus L$ is a homotopy equivalence.

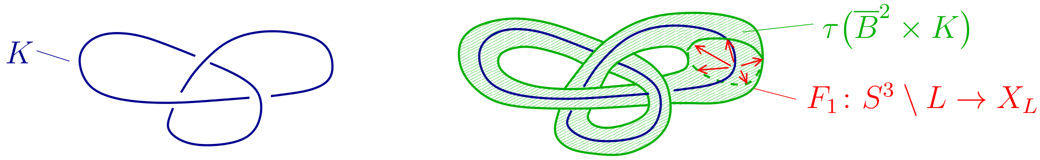
Proof.

(1) This statement follows from standard facts about smooth embeddings, see [**Fri24**].

[10]We equip $\overline{B}^2$ with the standard orientation and we equip $\overline{B}^2 \times L$ with the product orientation.

(2) This statement is an easy consequence of the Link Tubular Map Theorem 2.15 (2)
(3) This statement follows from (2) together with the Isotopy Extension Theorem 2.4.
(4) Let $\tau\colon \overline{B}^2 \times L \to S^3$ be a tubular map. We consider the map that retracts $S^3 \setminus L$ radially to X_L, i.e. we consider the map

$$\begin{aligned} F\colon (S^3 \setminus L) \times [0,1] \;&\to\; S^3 \setminus L \\ (x,t) \;&\mapsto \begin{cases} \tau(v\cdot((1-t) + \frac{t}{\|v\|}), y), & \text{if } x = \tau(v,y) \text{ for } (v,y) \in (\overline{B}^2 \setminus \{0\}) \times L, \\ x, & \text{if } (x,t) \notin \tau(B^2 \times L). \end{cases} \end{aligned}$$



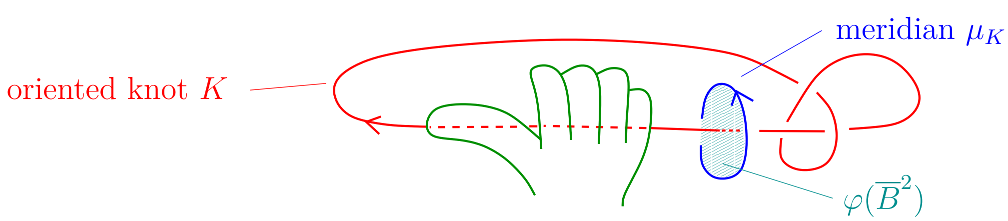
Since the map is continuous on two closed subsets that cover $(S^3 \setminus L) \times [0,1]$ we see that F is continuous. It is now clear that F is a deformation retraction from $S^3 \setminus L$ to X_L. It follows that the inclusion $X_L \to S^3 \setminus L$ is a homotopy equivalence. ■

We move on to a particularly important definition:

Definition. Let K be an oriented knot in S^3. Let $\mu_K \subset S^3 \setminus K$ be an oriented smoothly embedded loop. We say that μ_K is a meridian of K if there exists a smooth embedding $\varphi\colon \overline{B}^2 \subset S^3$ with the following three properties:

(1) The map $\varphi|_{S^1}$ defines an orientation-preserving diffeomorphism $S^1 \to \mu_K$.
(2) The image $\varphi(\overline{B}^2)$ intersects K transversally in a single point P[11].
(3) A positive basis for $\mathrm{T}_P(\varphi(\overline{B}^2))$ followed by a positive basis for $\mathrm{T}_P K$ gives a positive basis for $\mathrm{T}_P S^3$.

Remark. More casually speaking, the Meridian Proposition 2.17 says that a meridian of an oriented knot K is any knot μ which "circles once around the knot K" where the orientation is given by the right-hand-rule, i.e. if the thumb points into the direction of the knot, then the fingers point into the direction of the meridian.

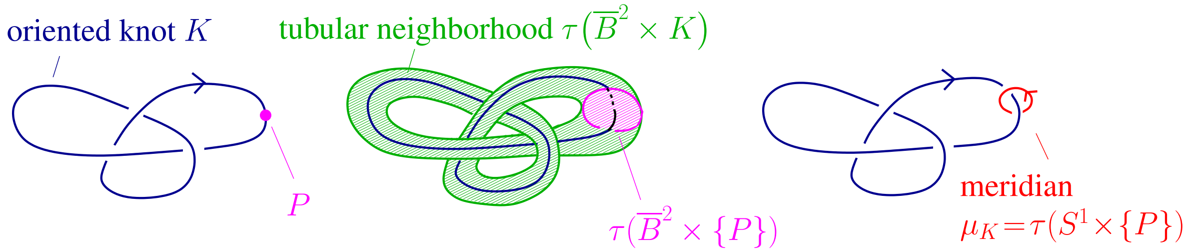


Proposition 2.17. (Meridian Proposition) Let L be an oriented link in S^3 and let K be a component of L.

(1) K admits a meridian.
(2) Let $\tau\colon \overline{B}^2 \times L \to S^3$ be an orientation-preserving tubular map. For any $P \in K$ the image $\tau(S^1 \times \{P\})$ is a meridian of K.

[11] Here "intersects transversally in a single point" means that $\varphi(\overline{B}^2) \cap K$ consists of a single point P and that $\mathrm{T}_P\varphi(\overline{B}^2) + \mathrm{T}_P K = \mathrm{T}_P S^3$.

(3) Any two meridians for K in $S^3 \setminus L$ are smoothly isotopic in $S^3 \setminus L$.
(4) Given any meridian μ of the component K there exists an orientation-preserving tubular map $\tau\colon \overline{B}^2 \times L \to S^3$ and a $P \in K$ with $\mu = \tau(S^1 \times \{P\})$.



Sketch of proof of the Meridian Proposition 2.17.

(1) This statement follows easily from taking any submanifold chart for a point on K.
(2) This statement follows almost immediately from the definitions.
(3) Since being smoothly isotopic is an equivalence relation it suffices to show that any meridian is smoothly isotopic to a meridian that arises from an orientation-preserving tubular map.

Let $\mu \subset S^3 \setminus K$ be a meridian. We pick a corresponding smooth embedding $\varphi\colon \overline{B}^2 \to S^3$ as in the definition of the meridian. It follows immediately from the Link Tubular Map Theorem 2.15 that there exists an orientation-preserving smooth embedding $\tau\colon \overline{B}^2_2 \times K \to S^3$ such that $\tau(0, P) = P$ for all $P \in K$.[12] It remains to prove the following claim:

Claim. There exists a smooth isotopy $H\colon S^1 \times [0,1] \to S^3 \setminus K$ with $H_0 = \varphi$ and such that for all $x \in S^1$ we have $H_1(x) = (x, P)$.

Proof. We prove the claim in four steps:

(a) We pick a point $Q \neq P$ on K. Since $\tau(B^2_2 \times (K \setminus \{Q\}))$ is an open neighborhood of $P = \tau(0, P)$ there exists an $\epsilon > 0$ such that $\varphi(\overline{B}^2_\epsilon) \subset \tau(B^2_2 \times (K \setminus \{Q\}))$. After applying the smooth isotopy

$$\begin{array}{rcl} F\colon \overline{B}^2 \times [0,1] & \to & S^3 \\ (x,t) & \mapsto & \varphi(x \cdot ((1-t) + \epsilon \cdot t)) \end{array}$$

we can assume that $\varphi(\overline{B}^2)$ is contained in $\tau(B^2_2 \times (K \setminus \{Q\}))$.

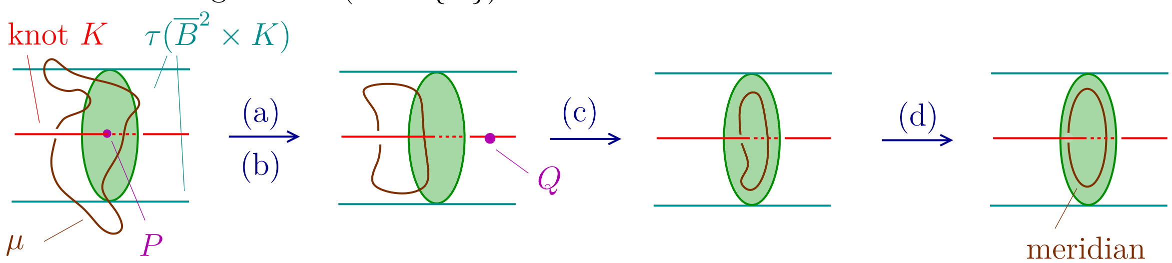
(b) By (a) we can consider the map

$$\Theta\colon \overline{B}^2 \xrightarrow{\varphi} \tau(\overline{B}^2_2 \times (K \setminus \{Q\})) \xrightarrow{\tau(v,z) \mapsto \tau(v,P)} \tau(\overline{B}^2 \times \{P\}).$$

It follows easily from hypothesis (1) that the differential of Θ at 0 is an isomorphism. It follows from the Inverse Mapping Theorem [**Fri24**] that there exists an $\epsilon > 0$ such that the restriction of Θ to $\overline{B}^2_\epsilon$ is a smooth embedding. As in (a) we can assume, after a smooth isotopy that is given by radial shrinking, that Θ is in fact a smooth embedding. Note that it follows from the fact that τ is orientation-preserving and hypothesis (2) that this smooth embedding $\Theta\colon \overline{B}^2 \to \tau(\overline{B}^2_2 \times \{P\})$ is actually orientation-preserving.

[12] Here $\overline{B}^n_r$ denotes the closed n-ball around the origin of radius r.

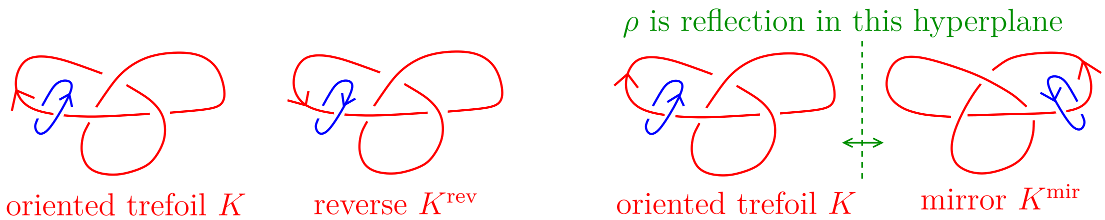
(c) Since $K \setminus \{Q\}$ is diffeomorphic to an open interval we can find a smooth deformation retraction from $K \setminus \{Q\}$ to $\{P\}$. If we take the "product" of this deformation retraction with the identity on $\overline{B}^2$ we see that our smooth embedding $\varphi\colon \overline{B}^2 \to S^3$ is actually smoothly isotopic, rel $(0, P)$ to a smooth embedding $\overline{B}^2 \to \tau(B_2^2 \times \{P\})$.
(d) It follows from the Smooth Ball Embedding Theorem 2.9 that the smooth embedding $\overline{B}^2 \to \tau(B_2^2 \times \{P\})$ from (c) is smoothly isotopic, rel 0, to the obvious smooth embedding $\overline{B}^2 \to \tau(\overline{B}^2 \times \{P\})$. ⊞



(4) Let μ be a meridian of K. As above we obtain from the Link Tubular Map Theorem 2.15 an orientation-preserving smooth embedding $\tau\colon \overline{B}^2 \times K \to S^3$ such that $\tau(0, P) = P$ for all $P \in K$. We fix $P \in K$. By (2) we know that $\nu := \tau(S^1 \times \{P\})$ is a meridian of K. By (3) we know that there exists a smooth isotopy $F\colon \mu \times [0,1] \to S^3 \setminus L$ with $F_0 = \mathrm{id}$ and $F_1(\mu) = \nu$. We use the constant isotopy $L \times [0,1] \to L$ to extend F to a smooth isotopy of $\mu \cup L$. Finally we use the Isotopy Extension Theorem 2.4 to extend this smooth isotopy of $\mu \cup L$ to a diffeotopy $H\colon S^3 \times [0,1] \to S^3$ with $H_0 = \mathrm{id}$. It follows that $H_1 \circ \tau\colon \overline{B}^2 \times L \to S^3$ is an orientation-preserving tubular map such that $(H)1 \circ \tau)(S^1 \times \{P\}) = F(\mu) = \nu$. ■

We conclude this chapter with the following lemma which tells us what effect reversing the orientation and taking mirror images have on meridians.

Lemma 2.18. (Meridian–Symmetries Lemma) Let $K \subset S^3$ be an oriented knot with meridian μ_K.
(1) A meridian for K^{rev} is given by $(\mu_K)^{\mathrm{rev}}$.
(2) Let $\rho\colon S^3 \to S^3$ be a reflection in a hyperplane. Then a meridian for $K^{\mathrm{mir}} = \rho(K)$ is given by $\rho(\mu_K)^{\mathrm{rev}}$.



Proof. We recall the definition from page 27 of a meridian. Let J be an oriented knot. We say that an oriented knot $\nu \subset S^3 \setminus J$ is a *meridian of* J if there exists a smooth embedding $\varphi\colon \overline{B}^2 \subset S^3$ with the following three properties:

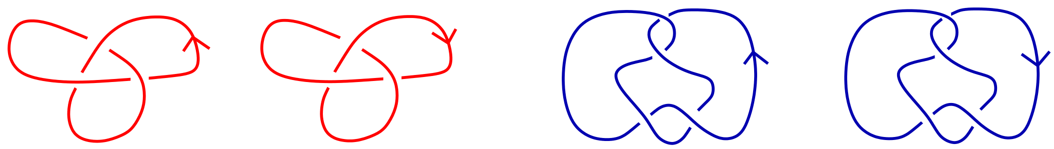
(a) The map $\varphi|_{S^1}$ defines an orientation-preserving diffeomorphism $S^1 \to \nu$.
(b) The image $\varphi(\overline{B}^2)$ intersects J transversally in a single point P.

(c) A positive basis for $T_P(\varphi(\overline{B}^2))$ together with a positive basis for $T_P J$ gives a positive basis for $T_P S^3$.

In both cases of the lemma it is clear that the first two conditions of the definition of a meridian are satisfied. The third condition follows from some elementary linear algebra. We leave these considerations to the reader. ■

Exercises for Chapter 2.

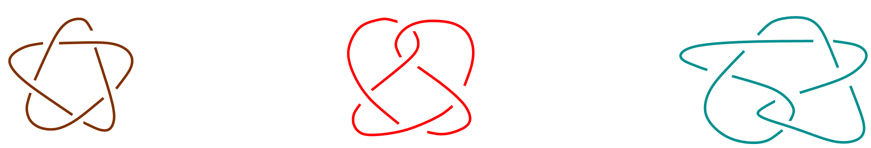
Exercise 2.1. Show that the trefoil and the figure-8 knot are reversible. In other words, show that the two oriented knots shown in the figure to the left, respectively to the right, are smoothly isotopic.



trefoil with the two orientations figure-8 knot with the two orientations

Exercise 2.2. Show that the two Hopf links H_+ and H_-, that are shown in the figure below, are not smoothly isotopic.
Hint. Consider the fundamental group of the complement of the red component.

Exercise 2.3. In `https://www.desmos.com/3d/92d2884a72`
an explicit description of the figure-8 knot is given. Give an explicit description for the cinquefoil, the three-twist knot and the stevedore knot that are shown below.



cinquefoil three-twist knot stevedore knot

Exercise 2.4.

(a) Find a 3-component link that "seems to be linked" but such that each 2-component sublink is unlinked.
(b) For any $n \in \mathbb{N}$ find an n-component link that "seems to be linked" but such that each $(n-1)$-component sublink is unlinked.

Remark. Such links were first studied by Hermann Brunn [**Bru92**] and are often called *Brunnian links*.

Exercise 2.5. We consider the Hopf map

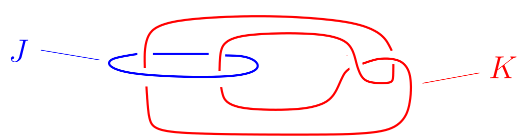
$$\begin{array}{rcl} H\colon S^3=\{(z_1,z_2)\in\mathbb{C}^2\,|\,|z_1|^2+|z_2|^2=1\} & \to & S^2 =_{\mathrm{i}} \mathbb{C}\mathrm{P}^1 \\ (z_1,z_2) & \mapsto & [z_1:z_2]. \end{array}$$

(a) Let $P\in S^2$. What does $H^{-1}(\{P\})\subset S^3$ look like?
(b) Let $P,Q\in S^2$ be two distinct points. What does $H^{-1}(\{P,Q\})\subset S^3$ look like?
(c) Let $P,Q,R\in S^2$ be three distinct points. What does $H^{-1}(\{P,Q,R\})\subset S^3$ look like?

Remark. You can make use of the fact that the answers do not depend on the choice of the points.

Exercise 2.6. We consider the two links shown in the figure below. In precisely one of the two cases the red component bounds a smoothly embedded disk that is disjoint from the other components. Which case is it?

Exercise 2.7. We consider the two disjoint knots J and K shown below. Is the link $L=J\sqcup K$ smoothly isotopic to the link $L=K\sqcup J$ (with the opposite order)?



Exercise 2.8. Let $K,J\subset S^3$ be oriented knots. We assume that there exists an orientation-preserving diffeomorphism $f\colon X_K\to X_J$ with $f(\mu_K)=\mu_J$. Show that there exists an orientation-preserving diffeomorphism $f\colon S^3\to S^3$ with $f(K)=J$.
Hint. Use the Handle Attachment Proposition from [**Fri24**] to attach a 2-handle to X_K and X_J along $\mu_K\subset\partial X_K$ and $\mu_J\subset\partial X_J$.

Exercise 2.9. Let $K\subset\mathbb{R}^3\subset\mathbb{R}^3\cup\{\infty\}=_{\mathrm{i}} S^3$ be a knot.

(a) Show that there exists a smooth embedding $f\colon S^2\to\mathbb{R}^3\setminus K$ such that $f(S^2)$ is a retract of $\mathbb{R}^3\setminus K$.
(b) Show that there exists a continuous map $g\colon\mathbb{R}^3\setminus K\to S^3\setminus K$ such that the induced map $g_*\colon\pi_1(\mathbb{R}^3\setminus K)\to\pi_1(S^3\setminus K)$ is an isomorphism.
(c) Show that $(S^3\setminus K)\vee S^2$ is homotopy equivalent to $\mathbb{R}^3\setminus K$.

Exercise 2.10. Let $K\subset S^3$ be a compact subspace. Show that there exists a smooth embedding $\varphi\colon\overline{B}^3\to S^3$ with $K\subset\varphi(B^3)$.

Exercise 2.11. Show that if a link $L\subset S^3$ is splittable, then there exists an embedding $f(S^2)\to S^3\setminus L$ such that $f(S^2)$ is a retract of $S^3\setminus L$.
Remark. If the reader already knows some facts about homotopy groups, then the reader will notice that the exercise implies that $\pi_2(S^3\setminus L)$ is non-zero.

Exercise 2.12.

(a) Let $X \sqcup Y$ be a subspace of S^3. We assume that there exists smooth embeddings $\alpha, \beta\colon \overline{B}^3 \to S^3$ with disjoint images and such that $X \subset \alpha(B^3)$ and $Y \subset \beta(B^3)$. Show that $S^3 \setminus (X \sqcup Y)$ is homotopy equivalent to $(S^3 \setminus X) \vee S^2 \vee (S^3 \setminus Y)$.

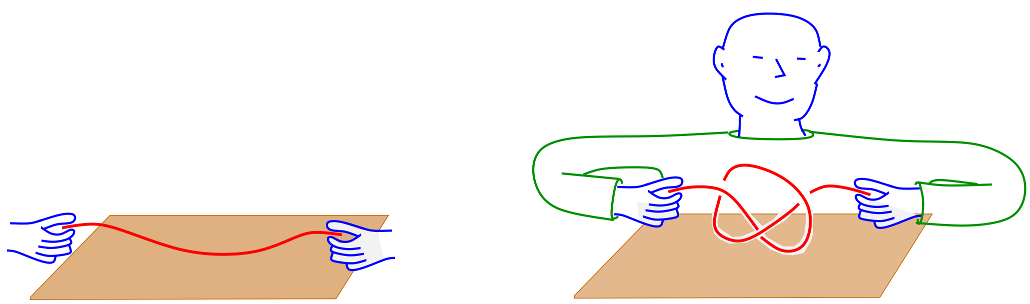
(b) Show that (a) can be used to give a solution to Exercise 2.9.

Remark. If the reader already knows some facts about homotopy groups, then the reader will notice that the exercise implies that $\pi_2(S^3 \setminus L)$ is non-zero.

Exercise 2.13. Show that for any link $L \subset S^3$ the complement $S^3 \setminus L$ is path-connected.
Hint. Make use of the Link Tubular Map Theorem 2.15.

Exercise 2.14. Show that there exists a closed 3-dimensional smooth manifold and a closed 1-dimensional smooth manifold $L \subset M$ such that L does *not* admit a tubular map.
Remark. This shows in the Link Tubular Map Theorem 2.15 we cannot drop the hypothesis that M is orientable.

Exercise 2.15. Can you pick up a loose unknotted string with both hands and turn it into a knotted string as shown in the figure below, while holding on to the ends of the string? Either way, which smooth isotopy types of knotted strings can you produce this way?



CHAPTER 3

Fundamental groups of link complements

The main idea of the following chapters is to study knots and links using the most powerful tool that currently is at our disposal, namely the fundamental group. More precisely, given a link $L \subset S^3$ we will study the fundamental group $\pi_1(S^3 \setminus L)$. In this chapter we will calculate this group for all torus knots. We will use this result to show in particular that the trefoil is not smoothly isotopic to the trivial knot.

3.1. The fundamental group of the complement of a link.

Our idea now is to study the fundamental group of link complements. The following lemma says that we do not really have to worry about base points.

Lemma 3.1. (Link Complement–Connected Lemma) Given any link $L \subset S^3$ the complement $S^3 \setminus L$ is path-connected.

Proof. The lemma can be proved using the Link Tubular Map Theorem 2.15. Cleverly we already filled in the details in Exercise 2.13. ■

The following lemma shows that fundamental groups can (at least in principle) be used to distinguish knots and links up to smooth isotopy:

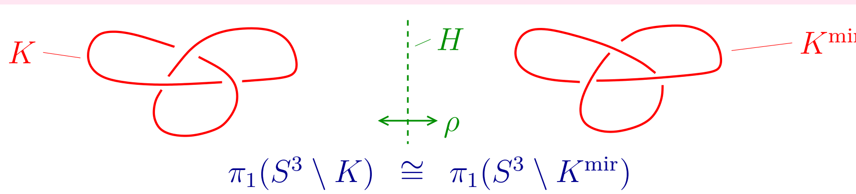
Lemma 3.2. (Isotopic Link-π_1-Lemma) Let $L, \widetilde{L}$ be links in S^3. If L and $\widetilde{L}$ are smoothly isotopic, then $\pi_1(S^3 \setminus L) \cong \pi_1(S^3 \setminus \widetilde{L})$.

Proof. We suppose that L and $\widetilde{L}$ are smoothly isotopic. It follows from the Link–Smooth Isotopy Proposition 2.3 that the link complements $S^3 \setminus L$ and $S^3 \setminus \widetilde{L}$ are diffeomorphic. Evidently this implies that the groups $\pi_1(S^3 \setminus L)$ and $\pi_1(S^3 \setminus \widetilde{L})$ are isomorphic. ■

Remark. For a link $L \subset \mathbb{R}^3 \subset \mathbb{R}^3 \cup \{\infty\}_i = S^3$ one could also consider the fundamental group $\pi_1(\mathbb{R}^3 \setminus L)$. In Exercise 3.1 we will see that $\pi_1(\mathbb{R}^3 \setminus L) \cong \pi_1(S^3 \setminus L)$. As we will see shortly, somewhat surprisingly, technically it is often easier to work with $S^3 \setminus L$ instead of $\mathbb{R}^3 \setminus L$. □

Before we head towards calculations, let us first state the following sobering lemma, which implies that just looking at the fundamental $\pi_1(S^3 \setminus L)$ is useless for distinguishing a link from its mirror:

Lemma 3.3. (Mirror Link-π_1-Lemma) Given any link L in S^3 there exists an isomorphism $\pi_1(S^3 \setminus L) \cong \pi_1(S^3 \setminus L^{\mathrm{mir}})$.



Proof. By definition of the mirror we have $L^{\mathrm{mir}} = \rho(L)$ where $\rho\colon S^3 \to S^3$ is the reflection in some hyperplane $H \subset \mathbb{R}^4$. Note that ρ restricts to a diffeomorphism $S^3 \setminus L \to S^3 \setminus L^{\mathrm{mir}}$. It follows that the induced map $\rho_*\colon \pi_1(S^3 \setminus L) \cong \pi_1(S^3 \setminus L^{\mathrm{mir}})$ is an isomorphism. ■

The name of the game is now to determine $\pi_1(S^3 \setminus L)$ for interesting knots and links, e.g. for the trivial knot and the trefoil. The trick is to find an elegant description of the complements of the trivial knot and the trefoil. For this purpose it is useful to introduce one more way to think about spheres. We will do so in the next section.

3.2. One more avatar of spheres.

As we just mentioned, we want to get one more way to think about spheres. Even though in this course we only really care about S^3, let us formulate the following lemma in a more general context:

Lemma 3.4. (Sphere–Solid Tori-Decomposition Lemma) Let $m, n \in \mathbb{N}_0$. We consider the following two maps:

$$\begin{array}{rcl} \Phi\colon S^{m-1} \times \overline{B}^n & \to & A := \left\{(x,y) \in \mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n} \,\middle|\, \|x\|^2 + \|y\|^2 = 1 \text{ and } \|y\|^2 \geq \frac{1}{2}\right\} \\ (a,b) & \mapsto & \left(\frac{a}{\sqrt{\|a\|^2+\|b\|^2}}, \frac{b}{\sqrt{\|a\|^2+\|b\|^2}}\right) \end{array}$$

and

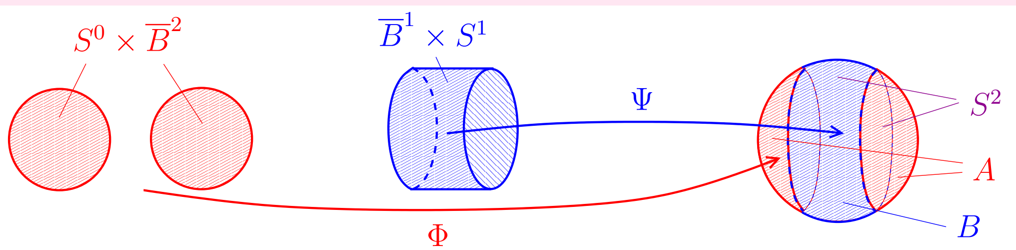
$$\begin{array}{rcl} \Psi\colon \overline{B}^m \times S^{n-1} & \to & B := \left\{(x,y) \in \mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n} \,\middle|\, \|x\|^2 + \|y\|^2 = 1 \text{ and } \|x\|^2 \geq \frac{1}{2}\right\} \\ (a,b) & \mapsto & \left(\frac{a}{\sqrt{\|a\|^2+\|b\|^2}}, \frac{b}{\sqrt{\|a\|^2+\|b\|^2}}\right). \end{array}$$

These maps have the following properties:

(1) The map $\Phi\colon S^{m-1} \times \overline{B}^n \to A$ is an orientation-preserving diffeomorphism.
(2) The map $\Psi\colon \overline{B}^m \times S^{n-1} \to B$ is a diffeomorphism, it is orientation-preserving if and only if m is even.
(3) The map

$$\begin{array}{rcl} \Theta\colon (S^{m-1} \times \overline{B}^n) \cup_{S^{m-1} \times S^{n-1}} (\overline{B}^m \times S^{n-1}) & \to & S^{m+n-1} \\ [P] & \mapsto & \begin{cases} \Phi(P), & \text{if } P \in S^{m-1} \times \overline{B}^n, \\ \Psi(P), & \text{if } P \in \overline{B}^m \times S^{n-1} \end{cases} \end{array}$$

is well-defined and it is a diffeomorphism.



Convention. We use the diffeomorphism Θ from the Sphere–Solid Tori-Decomposition Lemma 3.4 (2) to add the smooth manifold $(S^{m-1} \times \overline{B}^n) \cup_{S^{m-1} \times S^{n-1}} (\overline{B}^m \times S^{n-1})$ to our list of avatars of S^{m+n-1}.

Sketch of proof.

(1) We consider the map

$$\begin{array}{rcl}\widetilde{\Phi}\colon B=\big\{(x,y)\in\mathbb{R}^m\times\mathbb{R}^n=\mathbb{R}^{m+n}\,\big|\,\|x\|^2+\|y\|^2=1\ \&\ \|y\|^2\geq\tfrac{1}{2}\big\} & \to & S^{m-1}\times\overline{B}^n\\ (x,y) & \mapsto & \Big(\frac{z}{\|y\|},\frac{w}{\|y\|}\Big).\end{array}$$

One can easily verify that $\widetilde{\Phi}$ and Φ are inverses of one another and that Φ and $\widetilde{\Phi}$ are smooth. This observation implies that $\Phi\colon\overline{B}^m\times S^{n-1}\to B$ is a diffeomorphism. We leave the proof that Φ is orientation-preserving to the reader or alternatively to [**Fri24**].

(2) This proof is almost identical to the proof of (1).

(3) It is clear that Φ and Ψ agree on $S^{m-1}\times S^{n-1}$. It follows that the given map

$$\Theta\colon\,(S^{m-1}\times\overline{B}^n)\cup_{S^{m-1}\times S^{n-1}}(\overline{B}^m\times S^{n-1})\ \to\ S^{m+n-1}$$

is well-defined and continuous. Using (1) one easily verifies that the map is a bijection. It follows from the Compact-Hausdorff Proposition that the map Θ is a homeomorphism. Going through the definition of the smooth structure of the smooth manifold on the left shows that the map is actually a diffeomorphism. Finally it follows from (1) and (2) that the diffeomorphism is actually orientation-preserving. ■

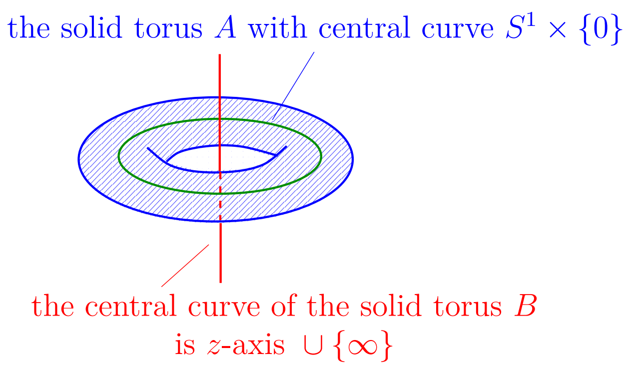
We are mostly interested in the case $m=n=2$. As we mentioned on page 14 we have an identification $S^3=_{\mathrm{i}}\mathbb{R}^3\cup\{\infty\}$ and we view $\mathbb{R}^3$ as a submanifold of the smooth manifold $S^3=_{\mathrm{i}}\mathbb{R}^3\cup\{\infty\}$. In particular we can view A and B as submanifolds of $S^3=_{\mathrm{i}}\mathbb{R}^3\cup\{\infty\}$. In this setting we have

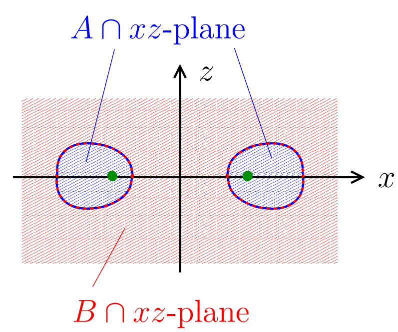
$$\text{“central curve of } A\text{”}\ =\ \Phi(S^1\times\{0\})\ =\ S^1\times\{0\}$$

and

$$\text{“central curve of } B\text{”}\ =\ \Psi(\{0\}\times S^1)\ =\ \text{the } z\text{-axis}\cup\{\infty\}.$$

We visualize the solid torus A and the central curve of the solid torus B in the figure below. The solid torus B is more difficult to illustrate. It consists of a closed disk attached to each point on the central curve. For example the closed disk attached to the origin is just the “obvious” closed disk in the xy-plane that touches the torus $\Phi(S^1\times S^1)$.[13]



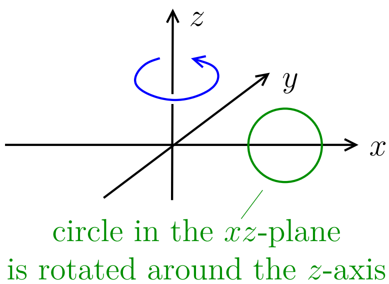


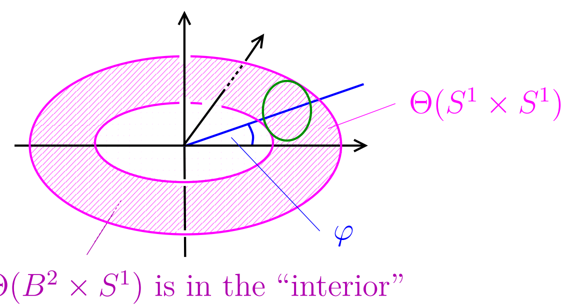
[13] What is the disk attached to the point ∞?

Next note that the map

$$\begin{array}{rcl} \Theta\colon \overline{B}^2 \times S^1 & \to & \mathbb{R}^3 \\ ((x,y),\exp(\mathrm{i}\varphi)) & \mapsto & \underbrace{\begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{rotation around } z\text{-axis}} \cdot \underbrace{\begin{pmatrix} 1+\frac{1}{3}x \\ 0 \\ -\frac{1}{3}y \end{pmatrix}}_{\substack{\text{describes circle} \\ \text{in } xz\text{-plane}}} \end{array}$$

is a smooth embedding. This (or obvious variations thereof) is usually viewed as the standard smooth embedding of the solid torus into $\mathbb{R}^3$.





The above embedding and the Sphere–Solid Tori-Decomposition Lemma 3.4 give us two smooth embeddings of the solid torus $\overline{B}^2 \times S^1$ into $\mathbb{R}^3$. The following lemma says that these two smooth embeddings are essentially the same.

Lemma 3.5. (Torus Embeddings–Isotopic Lemma) Let $\Omega\colon S^3 \setminus \{N\} \to \mathbb{R}^3$ be the stereographic projection as defined in the Stereographic Projection Lemma 2.1. We consider the two smooth embeddings

$$\begin{array}{rl} \overline{B}^2 \times S^1 \xrightarrow{\Theta} \mathbb{R}^3 & \\ ((x,y),e^{\mathrm{i}\varphi}) \mapsto \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1+\frac{x}{2} \\ 0 \\ -\frac{y}{2} \end{pmatrix} \end{array} \quad \text{and} \quad \begin{array}{c} \overline{B}^2 \times S^1 \xrightarrow{\Xi} S^3 \setminus \{N\} \xrightarrow{\Omega} \mathbb{R}^3 \\ (z,w) \mapsto \left(\frac{w}{\sqrt{|z|^2+|w|^2}}, \frac{z}{\sqrt{|z|^2+|w|^2}} \right) \end{array}$$

provided by the above and provided by the Sphere–Solid Tori-Decomposition Lemma 3.4. There exists a diffeotopy G of S^3 rel $\{(0,0)\} \times S^1$ with $G_0 = \mathrm{id}$ and $G_1 \circ \Theta = \Omega \circ \Xi$.

Convention. In all the future examples we will not distinguish between the two smooth embeddings of the solid torus $\overline{B}^2 \times S^1 \to \mathbb{R}^3$ that are given in the Torus Embeddings–Isotopic Lemma 3.5.

Proof. The lemma follows immediately from the Link Tubular Map Theorem 2.15 since both maps are orientation-preserving tubular maps of the trivial knot. ■

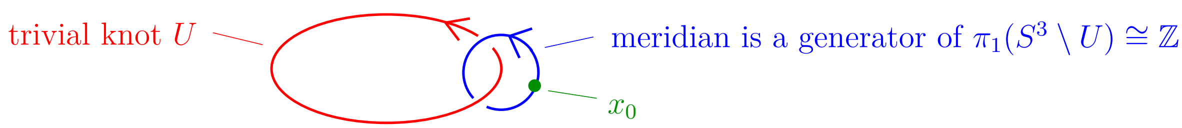
3.3. **Fundamental groups of complements of the trivial knot.** In this and the following sections we will use the description of S^3 from the previous section to give precise descriptions of the complements of the trivial knot and of torus knots (which contains the trefoil as a special case). These descriptions will allow us to determine the fundamental groups of complements of the trivial knot and of torus knots.

First let us recall the definition from page 14:

Definition. The trivial knot is defined as $\{(x, y, 0) \in \mathbb{R}^3 \,|\, x^2 + y^2 = 1\} \subset \mathbb{R}^3 \cup \{\infty\} =_{\mathrm{i}} S^3$.

Now we turn to the calculation of the fundamental groups of the complement of the trivial knot:

Proposition 3.6. (Trivial Knot-π_1-Proposition) Let U be the trivial knot.
(1) Under the identification $\mathbb{R}^3 \cup \{\infty\} =_{\mathrm{i}} S^3$ from page 14 the trivial knot corresponds to $U = \{(z, 0) \in \mathbb{C}^2 \,|\, |z|^2 = 1\}$.
(2) $S^3 \setminus U$ is diffeomorphic to $S^1 \times \mathbb{C}$.
(3) We have an isomorphism $\pi_1(S^3 \setminus U) \cong \mathbb{Z}$. In fact a generator of $\pi_1(S^3 \setminus U)$ is given by a meridian (see page 27 for the definition of a meridian).



Remark. Let $U \subset \mathbb{R}^3 \subset \mathbb{R}^3 \cup \{\infty\} =_{\mathrm{i}} S^3$ be the trivial knot. In the Trivial Knot-π_1-Proposition 3.6 we just showed that $S^3 \setminus U$ is diffeomorphic to $S^1 \times \mathbb{C}$. On the other hand, in Exercise 3.3 we will see that $\mathbb{R}^3 \setminus U$ is homotopy equivalent to the wedge $S^1 \vee S^2$. Thus we see that it is much easier to understand the complement of the trivial knot in S^3 than the complement of the trivial knot in $\mathbb{R}^3$. This is one instance where it pays off to study knots in S^3 instead of $\mathbb{R}^3$. □

Proof.
(1) This statement follows immediately from the Stereographic Projection Lemma 2.1.
(2) Note that
$$\begin{aligned} S^3 \setminus U &= \{(z, w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1\} \setminus \{(z, 0) \mid |z| = 1\} \\ &= \{(z, w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1 \text{ and } |w|^2 > 0\}. \end{aligned}$$
Now basically the same argument as in the proof of the Sphere–Solid Tori-Decomposition Lemma 3.4 shows that the map
$$\begin{aligned} \Phi\colon S^1 \times \mathbb{C} &\to S^3 \setminus U = \left\{(z, w) \in \mathbb{C}^2 \,\middle|\, |w|^2 + |z|^2 = 1 \text{ and } |w|^2 > 0\right\} \\ (a, b) &\mapsto \left(\frac{b}{\sqrt{|a|^2+|b|^2}}, \frac{a}{\sqrt{|a|^2+|b|^2}}\right) \end{aligned}$$
is a diffeomorphism. Let $f\colon S^1 \to S^1 \times \mathbb{C}$ be the embedding given by $z \mapsto (z, 0)$. We now consider the following isomorphisms:
$$\pi_1(S^3 \setminus U) \xleftarrow{\Phi_*} \pi_1(S^1 \times \mathbb{C}) \xleftarrow{f_*} \pi_1(S^1) \xleftarrow{\cong} \mathbb{Z}.$$
isomorphism since Φ is a homeomorphism; isomorphism since f is a homotopy equivalence; explicit isomorphism from [**Fri24**]
(3) Note that a generator of $\pi_1(S^1)$ is represented by the loop $[0, 1] \to S^1$ that is given by $t \mapsto \exp(2\pi i t)$. One can easily verify that, using the notation from (2), the image $\Phi(f(S^1))$ is a meridian of K. ■

3.4. **Torus knots.** In this section we introduce the family of torus knot, which as a special case contains the trefoil knot.

Definition.

(1) Let $p, q \in \mathbb{Z}$ be coprime. We refer to the image of the map

$$\begin{aligned} \varphi_{p,q}\colon \mathbb{R} &\to S^3 \\ t &\mapsto \left(\tfrac{1}{\sqrt{2}}\exp(\mathrm{i}\cdot p\cdot t), \tfrac{1}{\sqrt{2}}\exp(\mathrm{i}\cdot q\cdot t)\right). \end{aligned}$$

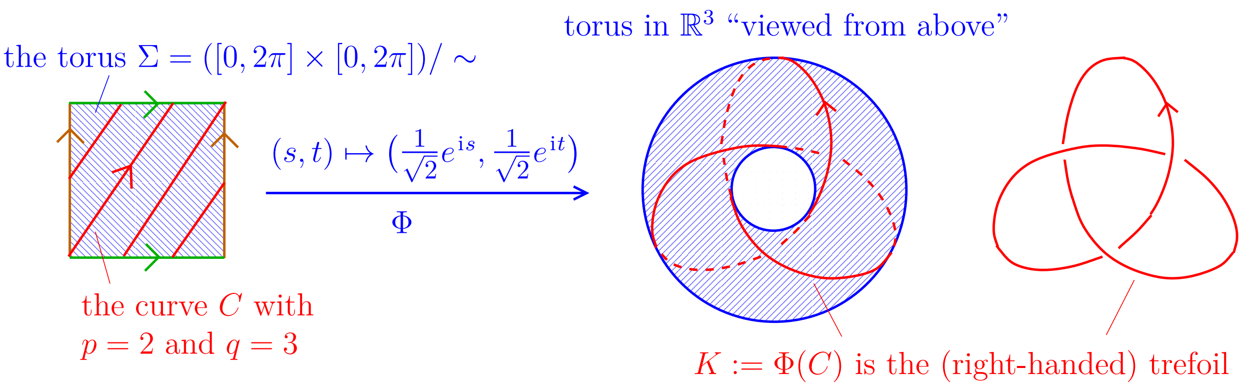
as the (p,q)-torus knot[20] $T(p,q)$. We equip the torus knot with the orientation which turns the map $\varphi_{p,q}\colon \mathbb{R} \to T(p,q)$ into an orientation-preserving local diffeomorphism.

(2) We define the right-handed trefoil to be the $(2,3)$-torus knot and we define the left-handed trefoil to be the $(-2,3)$-torus knot.

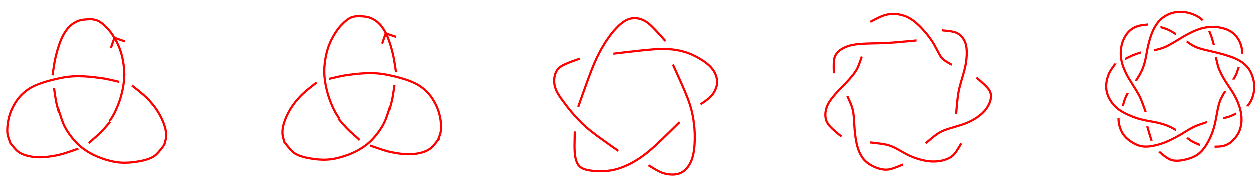
Visualisation.

(1) We give an alternative different description of torus knots: Let $p, q \in \mathbb{Z}$ be coprime. We consider the curve $C = \{(p\cdot t, q\cdot t) \mid t \in \mathbb{R}\}$ on the torus $\Sigma = ([0,2\pi]\times[0,2\pi])/\sim$. By definition the (p,q)-torus knot $T(p,q)$ is the image of C under the map

$$\begin{aligned} \Phi\colon \Sigma = ([0,2\pi]\times[0,2\pi])/\sim &\to S^3 \\ (s,t) &\mapsto \left(\tfrac{1}{\sqrt{2}}\exp(\mathrm{i}s), \tfrac{1}{\sqrt{2}}\exp(\mathrm{i}t)\right). \end{aligned}$$



(2) In the following figure we illustrate several examples of torus knots:



right-handed trefoil $T(2,3)$ | left-handed trefoil $T(-2,3)$ | $T(5,2)$ | $T(7,2)$ | $T(3,-8)$

(3) A visualization of the (p,q)-torus knots is also given by Johannes Zander in

`https://www.desmos.com/3d/4ba6cb90d2`.

Lemma 3.7. (Torus Knot Lemma)

(0) Each torus knot is indeed a knot.

(1) All torus knots $T(n,1)$ and $T(1,n)$ are smoothly isotopic to the trivial knot.

[20] We will show in the Torus Knot Lemma 3.7 that $T(p,q)$ is indeed a knot.

(2) For any coprime $p, q \in \mathbb{N}_0$ we have:
  (a) $T(-p,-q) = T(p,q)^{\text{rev}}$.
  (b) $T(-p,q)$ and $T(p,-q)$ are mirrors of $T(p,q)$.
  (c) $T(p,q)$ is reversible, i.e. it is smoothly isotopic to $T(p,q)^{\text{rev}}$.
  (d) $T(p,q)$ is smoothly isotopic to $T(q,p)$.

Proof.

(0) Let $p, q \in \mathbb{Z}$ be coprime. We consider again the map

$$\begin{array}{rcl} \varphi_{p,q}\colon \mathbb{R} & \to & S^3 \\ t & \mapsto & \left(\frac{1}{\sqrt{2}}\exp(\mathrm{i}\cdot p\cdot t), \frac{1}{\sqrt{2}}\exp(\mathrm{i}\cdot q\cdot t)\right). \end{array}$$

One can easily verify that this map is an immersion and that it descends to a smooth embedding $\mathbb{R}/2\pi\mathrm{i}\mathbb{Z} \to S^3$, whose image is precisely the (p,q)-torus knot $T(p,q)$. It follows that the image $\varphi_{p,q}(\mathbb{R}^3) = T(p,q)$ is indeed a smooth submanifold of S^3 that is diffeomorphic to S^1. In other words, $T(p,q)$ is indeed a knot.

Using the Orientation-Action Proposition **??** (1) one can easily show we can indeed equip the torus knot with a unique orientation which turns the map $\varphi_{p,q}\colon \mathbb{R} \to T(p,q)$ into an orientation-preserving local diffeomorphism.

(1) We will prove this statement in Exercise 3.5.

(2) (a) It is clear that as unoriented submanifolds we have $T(-p,-q) = T(p,q)$. But the two maps from $\mathbb{R}$ to the torus knots differ by a minus sign. This shows that as oriented knots we have $T(-p,-q) = T(p,q)^{\text{rev}}$.

  (b) First we consider the map
$$\begin{array}{rcl} g\colon S^3 & \to & S^3 \\ (z,w) & \mapsto & (\overline{z}, w). \end{array}$$

  First note that we have $T(-p,q) = g(T(p,q))$ as oriented knots. Next note that, with respect to real coordinates, g is just the reflection in the $(x_2 = 0)$-hyperplane. It follows that $T(-p,q) = g(T(p,q)) = T(p,q)^{\text{mir}}$ as oriented knots. Similarly we see that $T(p,-q)$ is obtained from $T(p,q)$ by reflection in the $(x_4 = 0)$-hyperplane.

  (c) For the purpose of this proof we denote by $\rho_i\colon S^3 \to S^3$ the reflection in the $(x_i = 0)$-hyperplane. Note that as oriented knots we have

$$\begin{array}{rcl} T(p,q)^{\text{rev}} \overset{\text{by (a)}}{=} T(-p,-q) \overset{\text{by (d)}}{=} \rho_4(T(p,-q)) & \overset{\text{by (d)}}{=} & \rho_2(\rho_4(T(p,q))) \\ & \underset{\substack{\text{smoothly isotopic by the} \\ \text{Link Mirror Lemma 2.7}}}{\cong} & \rho_4(\rho_4(T(p,q))) \underset{\text{since } \rho_4\circ\rho_4 = \mathrm{id}}{=} T(p,q). \end{array}$$

  (d) We consider the map

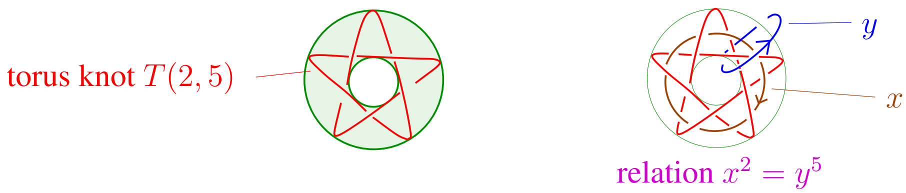
$$\begin{array}{rcl} f\colon S^3 & \to & S^3 \\ (z,w) & \mapsto & (w,z) \end{array} \quad \text{which with real coordinates we can write as} \quad \begin{array}{rcl} f\colon S^3 & \to & S^3 \\ v & \mapsto & \underbrace{\begin{pmatrix} 0 & \mathrm{id}_2 \\ \mathrm{id}_2 & 0 \end{pmatrix}}_{=:A} \cdot v. \end{array}$$

Evidently $T(q,p) = f(T(p,q))$ as oriented knots. Note that $A \in \mathrm{SO}(4)$. Since $\mathrm{SO}(4)$ is path-connected we see that f is diffeotopic to the identity. It follows that $T(q,p)$ is smoothly isotopic to $T(p,q)$. ■

3.5. **Fundamental groups of complements of torus knots.** The following proposition shows that fundamental groups of torus knot complements have very elegant and concise presentations:

Proposition 3.8. (Torus Knot-π_1-Proposition) For any coprime $p, q \in \mathbb{N}$ we have

$$\pi_1(S^3 \setminus T(p,q)) \;\cong\; \langle x, y \mid x^p = y^q \rangle.$$



Example.

(1) As we pointed out above, the torus knots $T(b,1)$ and $T(1,b)$ are just the trivial knot in disguise. Evidently the groups $\langle x, y \mid x^b = y^1 \rangle$ and $\langle x, y \mid x^1 = y^b \rangle$ are both isomorphic to $\mathbb{Z}$. Thus the Torus Knot-π_1-Proposition 3.8 (1) actually contains most of the results of the Trivial Knot-π_1-Proposition 3.6.

(2) Let $K = T(2,3)$ be the trefoil. It follows from the Torus Knot-π_1-Proposition 3.8 that $\pi_1(S^3 \setminus K) \cong \langle x, y \mid x^2 = y^3 \rangle$. □

In the proof of the Torus Knot-π_1-Proposition 3.8, and most other calculations of fundamental groups later in this part of the lecture notes we need the following theorem from [**Fri24**].

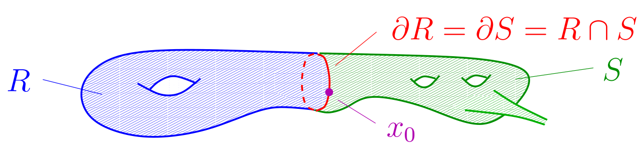
Theorem 3.9. (Seifert-van Kampen Theorem) Let M be an m-dimensional smooth manifold and let $R, S \subset M$ be two codimension zero submanifolds such that the following hold:

(1) $M = R \cup S$.
(2) R and S are closed subsets of M.
(3) $R \cap S$ is a single component of ∂R and it is a single component of ∂S.[21]

For every base point $x_0 \in R \cap S$ the inclusion-induced map

$$\pi_1(R, x_0) *_{\pi_1(R \cap S, x_0)} \pi_1(S, x_0) \;\to\; \pi_1(M, x_0)$$

is an isomorphism.



In our applications we will often perform amalgamations along infinite cyclic groups. For these discussions the following notation comes in handy:

[21]Here we mean by ∂R and ∂S the boundary of R and S viewed as smooth manifolds in their own right.

Notation. Let G and $\widetilde{G}$ be two groups and let $\mu \in G$ and $\widetilde{\mu} \in \widetilde{G}$ be elements. We denote by $G *_{\mu=\widetilde{\mu}} \widetilde{G}$ the amalgamated product corresponding to the group homomorphisms $\mathbb{Z} \to G$ and $\mathbb{Z} \to \widetilde{G}$ that are given by $1 \mapsto \mu$ and $1 \mapsto \widetilde{\mu}$.

Proof of the Torus Knot-π_1-Proposition 3.8. By the Sphere–Solid Tori-Decomposition Lemma 3.4 there exists a diffeomorphism

$$\Phi\colon S^3 \xrightarrow{\cong} \underbrace{(S^1 \times \overline{B}^2)}_{=:A} \cup_{S^1\times S^1 = S^1 \times S^1} \underbrace{(\overline{B}^2 \times S^1)}_{=:B}$$

which is, up to scaling by a factor of $\sqrt{2}$, the identity on $\Sigma := S^1 \times S^1$. Thus it remains to prove the following claim:

Claim. For $K := \{(\exp(ps\,\mathrm{i}), \exp(qs\,\mathrm{i})) \mid s \in \mathbb{R}\} \subset \Sigma \subset A \cup B$ there exists an isomorphism $\pi_1((A \cup B) \setminus K) \cong \langle x, y \,|\, x^p = y^q \rangle$.

Proof. We start out with the following simple observation:

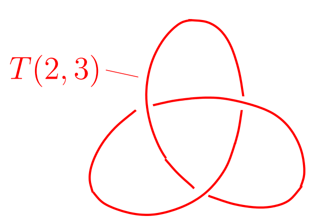
(a) A and B are codimension-zero submanifolds of $A \cup B$.
(b) A and B are closed subsets of $A \cup B$.
(c) We have $\partial A = \partial B = A \cap B = \Sigma$.

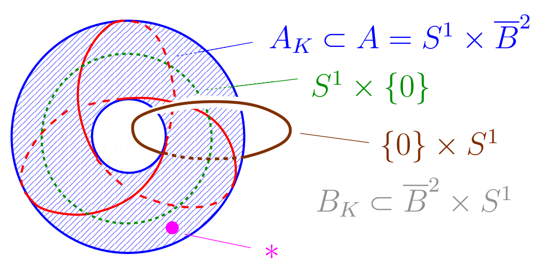
We write $A_K := A \setminus K, B_K := B \setminus K$ and $\Sigma_K := \Sigma \setminus K$. Since K is a closed subset of $A \cup B$ we obtain easily from the above that the following statements hold:

(a′) A_K and B_K are codimension-zero submanifolds of $X_K := (A \cup B) \setminus K$.
(b′) A_K and B_K are closed subsets of X_K.
(c′) $\partial A_K = \partial B_K = A_K \cap B_K = \Sigma_K$.

Next note that $x := S^1 \times \{0\}$ is a deformation retract of A_K and that $y := \{0\} \times S^1$ is a deformation retract of B_K. Finally note that Σ_K admits a deformation retraction to a parallel copy of K. More precisely, for $\epsilon > 0$ sufficiently small, there exists a deformation retraction from Σ_K to the loop

$$z \;=\; \{(\exp(ps\,\mathrm{i} + \epsilon), \exp(qs\,\mathrm{i})) \mid s \in \mathbb{R}\} \;\subset\; \Sigma_K.$$





We pick a base point $* \in \Sigma_K$. By an abuse of notation we denote by x, y and z also the loops corresponding to the circles x, y and z with the obvious orientation. Then the inclusion-induced homomorphisms

$$\begin{array}{ccccc} \pi_1(A_K, *) & \leftarrow & \pi_1(\Sigma_K, *) & \to & \pi_1(B_K, *) \\ i_*(g) & \leftarrow\!\shortmid & g & \mapsto & j_*(g) \end{array} \qquad \text{become} \qquad \begin{array}{ccccc} \langle x \rangle & \leftarrow & \langle z \rangle & \to & \langle y \rangle \\ x^p & \leftarrow\!\shortmid & z & \mapsto & y^q. \end{array}$$

Thus we see that

$$\pi_1(X_K,*) \underset{\substack{\uparrow\\ \text{by the Seifert–van Kampen}\\ \text{Theorem 3.9, which we can apply by}\\ \text{the above statements (a′), (b′) and (c′)}}}{=} \pi_1(A_K,*) *_{\pi_1(\Sigma_K,*)} \pi_1(B_K,*) \underset{\substack{\uparrow\\ \text{by the above}\\ \text{discussion}}}{\cong} \langle x\rangle *_{x^p=y^q} \langle y\rangle = \langle x,y \mid x^p = y^q\rangle.$$

■

Let again U be the trivial knot and let K be the trefoil. The obvious question is whether or not $\pi_1(S^3 \setminus K) = \langle x, y \mid x^2 = y^3\rangle$ is isomorphic to $\pi_1(S^3 \setminus U) = \mathbb{Z}$. As usual we first consider the abelianization of $\pi_1(S^3 \setminus K)$. For better or worse can easily show that the abelianization of $\pi_1(S^3 \setminus K)$ is isomorphic to $\mathbb{Z}$. Despite this slight setback the following lemma, which answers Question 2.2, does fortunately show that the group $\pi_1(S^3 \setminus K) = \langle x, y \mid x^2 = y^3\rangle$ is not isomorphic to $\pi_1(S^3 \setminus U) = \mathbb{Z}$.

Proposition 3.10. (Trefoil–Non-Trivial Proposition)
(1) The group $\langle x, y \mid x^2 = y^3\rangle$ admits an epimorphism onto the permutation group S_3, in particular the group $\langle x, y \mid x^2 = y^3\rangle$ is non-abelian and it is not isomorphic to $\mathbb{Z}$.
(2) The trefoil is not smoothly isotopic to the trivial knot.

Proof.
(1) One can easily verify there exists a unique homomorphism

$$\Phi\colon \langle x, y \mid x^2 = y^3\rangle \;\to\; S_3 = \text{permutation group on three elements}$$

with

$$\Phi(x) = \sigma := (1\,2) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

and

$$\Phi(y) = \tau := (1\,2\,3) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}.$$

The permutations σ and τ do not commute, hence the image of Φ is a non-abelian subgroup of S_3.[22] On the other hand the abelian group $\mathbb{Z}$ cannot admit an epimorphism onto a non-abelian group.
(2) It follows from (1), the Trivial Knot-π_1-Proposition 3.6 and the Isotopic Link-π_1-Lemma 3.2 that the trefoil is not smoothly isotopic to the trivial knot. ■

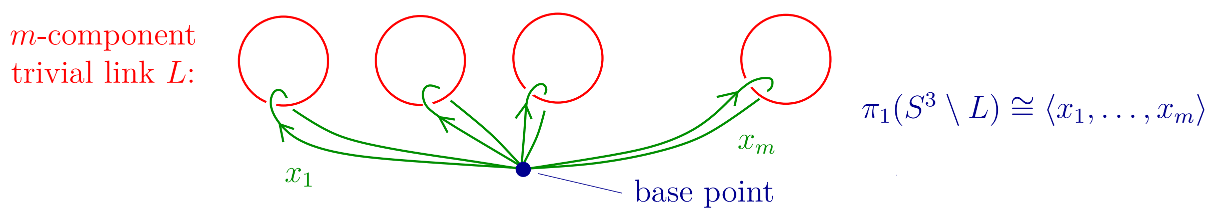
3.6. Splits unions and fundamental groups.
In this section we discuss the split union of links and the effect this construction has on fundamental groups of link complements:

Proposition 3.11. (Split Union-π_1-Proposition) Let L and $\widetilde{L}$ be two oriented links in S^3. We consider the split union $L \sqcup \widetilde{L}$ as defined on page 20. There exists an isomorphism

$$\pi_1(S^3 \setminus (L \sqcup \widetilde{L})) \;\cong\; \pi_1(S^3 \setminus L) * \pi_1(S^3 \setminus \widetilde{L}).$$

Example. Let L be the trivial m-component link. Evidently L is the split union of m copies of the trivial knot.
It follows from the Split Union-π_1-Proposition 3.11 and the Trivial Knot-π_1-Proposition 3.6 that $\pi_1(S^3 \setminus L)$ is a free group on m generators. In the figure above we show generators. These can be obtained easily from the proof. □

[22] In fact it is easy to see that the only non-abelian subgroup of S_3 is S_3 itself, i.e. Φ is an epimorphism onto S_3.



Proof. We start out with two basic claims:

Claim 1. Given any link $J \subset S^3$ there exists a smooth isotopy from J to a link in $S^3_{>0}$.

Proof. There are many ways to prove this claim. One approach would be to modify the proof of the Link-in-$\mathbb{R}^3$-S^3-Lemma 2.5. We leave it to the reader to find their preferred proof. ⊞

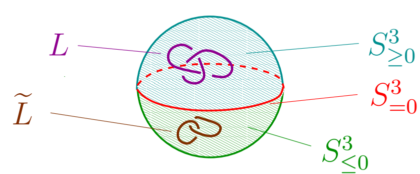
Claim 2. Given any compact $X \subset S^3_{>0}$ and given any $x_0 \in S^3_{=0}$ the inclusion induced map $\pi_1(S^3_{\geq 0} \setminus X, x_0) \to \pi_1(S^3 \setminus X, x_0)$ is an isomorphism.

Proof. By the Seifert-van Kampen Theorem 3.9 we have the following pushout diagram:

$$\begin{tikzcd} \pi_1(S^3_{=0}, x_0) \arrow[r] \arrow[d] & \pi_1(S^3_{\geq 0} \setminus X, x_0) \arrow[d] \\ \pi_1(S^3_{\leq 0}, x_0) \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \pi_1(S^3 \setminus K). \end{tikzcd}$$

The fundamental groups to the left are trivial. This implies of course that the left vertical map is an isomorphism. It follows from that the right vertical map is also an isomorphism. ⊞

Now we turn to the actual proof of the proposition. Let L and $\widetilde{L}$ be two oriented links in S^3. It follows from Claim 1 and the Link–Split Union Lemma 2.10 (2) that we can assume that $L \subset S^3_{>0}$ and similarly we can arrange that $\widetilde{L} \subset S^3_{<0}$.



Let $\varphi\colon \overline{B}^3 \to S^3_{\geq 0} \subset S^3$ and $\widetilde{\varphi}\colon \overline{B}^3 \to S^3_{\leq 0} \subset S^3$ be the obvious smooth embeddings. Cleverly we pick $\alpha = \varphi$ and $\widetilde{\alpha} = \widetilde{\varphi}$. It follows from the definition on page 20 that $L \sqcup \widetilde{L} = L \cup \widetilde{L}$. Finally we see that

$$\pi_1(S^3 \setminus (L \sqcup \widetilde{L})) \;=\; \pi_1((S^3_{\geq 0} \setminus L) \cup_{S^3_{=0}} (S^3_{\leq 0} \setminus \widetilde{L})) \;\underset{\substack{\uparrow \\ \text{follows from the Seifert-van Kampen Theorem 3.9} \\ \text{and the fact that } \pi_1(S^2) = 0}}{\cong}\; \pi_1(S^3_{\geq 0} \setminus L) * \pi_1(S^3_{\leq 0} \setminus \widetilde{L}).$$

■

For the record we state the following theorem which can be viewed as a converse to the Split Union-π_1-Proposition 3.11.

Theorem 3.12. (Link Kneser Theorem) Let L be an oriented link in S^3. If there exists an isomorphism $\pi_1(S^3 \setminus L) \cong G_1 * G_2$, then we can write $L = L_1 \sqcup L_2$, where L_1 and L_2 are unions of components of L, such that $\pi_1(S^3 \setminus L) \cong G_i$ for $i = 1, 2$.

Proof. The theorem is a relative straightforward consequence of the more general Kneser Theorem in 3-manifold topology which is stated and proved in [**Hem76**, Section 7] and [**Cal14**, Theorem 3.9]. ■

We conclude this section with one more explicit example of a link:

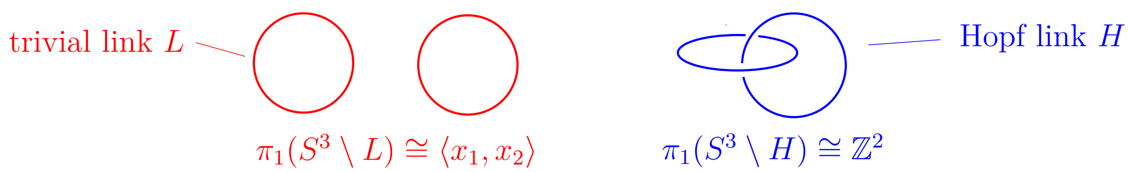
Definition. We refer to

$$H \ := \ (\{0\} \times S^1) \sqcup (S^1 \times \{0\}) \ \subset \ (\overline{B}^2 \times S^1) \cup_{S^1 \times S^1} (S^1 \times \overline{B}^2) =_{\mathrm{i}} S^3$$

as the Hopf link.

Lemma 3.13. (Hopf Link–Lemma)

(1) The Hopf link H is smoothly isotopic to the link
$$\{(x,y,0) \in \mathbb{R}^3 \,|\, x^2+y^2=1\} \cup \{(x,0,z) \in \mathbb{R}^3 \,|\, (x-1)^2+z^2=1\} \ \subset \ \mathbb{R}^3 \subset S^3.$$
(2) $S^3 \setminus H$ is diffeomorphic to $S^1 \times S^1 \times (-1,1)$.
(3) $\pi_1(S^3 \setminus H) \cong \mathbb{Z}^2$.
(4) The Hopf link is not smoothly isotopic to the trivial 2-component link.



Sketch of proof.

(1) This follows fairly easily from the identifications given by the Sphere–Solid Tori-Decomposition Lemma 3.4 and the Stereographic Projection Lemma 2.1.
(2) As in the proof of the Trivial Knot-π_1-Proposition 3.6 one can write down an explicit diffeomorphism. We will fill in the details in Exercise 3.4.
(3) This statement follows immediately from (2).
(4) Let L be 2-component trivial link. On page 42 we showed that $\pi_1(S^3 \setminus L)$ is the free group on two generators. This group is non-abelian, so by the calculation in (3) it follows that $\pi_1(S^3 \setminus L)$ is not isomorphic to $\pi_1(S^3 \setminus H)$. It follows from the Isotopic Link-π_1-Lemma 3.2 that L is not smoothly isotopic to H. ■

3.7. **Connected sum of knots and fundamental groups.** In this section we will consider the effect of the connected sum operation of knots on fundamental groups. Before we can state our main result we need to formulate a fairly general lemma:

Lemma 3.14. (Loop-π_1-Lemma) Let M be a path-connected smooth manifold and let $x_0 \in M$.

(1) Let $C \subset M$ be an oriented submanifold of M that is diffeomorphic to S^1. We make two choices:
 - We pick an orientation-preserving diffeomorphism $f\colon S^1 \to C$.
 - Since M is path-connected we can pick a path $\gamma\colon [0,1] \to M$ from x_0 to $f(1) \in C$.

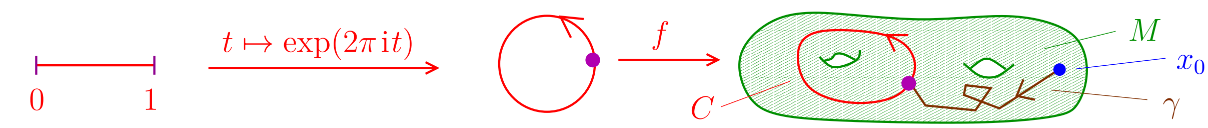
 The element

$$\Big[\ \underbrace{\gamma * (t \mapsto f(\exp(2\pi\,\mathrm{i}t))) * \overline{\gamma}}_{\substack{\text{the loop in } (M,x_0) \text{ that is given} \\ \text{by the concatenation of} \\ (1)\ \text{the path } \gamma \text{ from } x_0 \text{ to } \gamma(1) \\ (2)\ \text{the loop } t \mapsto f(\exp(2\pi\,\mathrm{i}t)) \text{ in } \gamma(1) \\ (3)\ \text{the ``inverse'' path } \overline{\gamma} \text{ from } \gamma(1) \text{ to } x_0}}\ \Big] \ \in\ \pi_1(M,x_0)$$

is well-defined up to conjugation.

(2) Let $C, \widetilde{C} \subset M$ be two oriented submanifolds of M that are diffeomorphic to S^1. If C and $\widetilde{C}$ are smoothly isotopic, then C and $\widetilde{C}$ define the same conjugacy class in $\pi_1(M, x_0)$.

(3) Let $C \subset M$ be an oriented submanifold of M that is diffeomorphic to S^1. If we denote by C^{rev} the same submanifold with opposite orientation, then C and C^{rev} define elements of $\pi_1(M, x_0)$ that are inverses of one another.



Sketch of proof. We leave it to the reader to provide the proof. ■

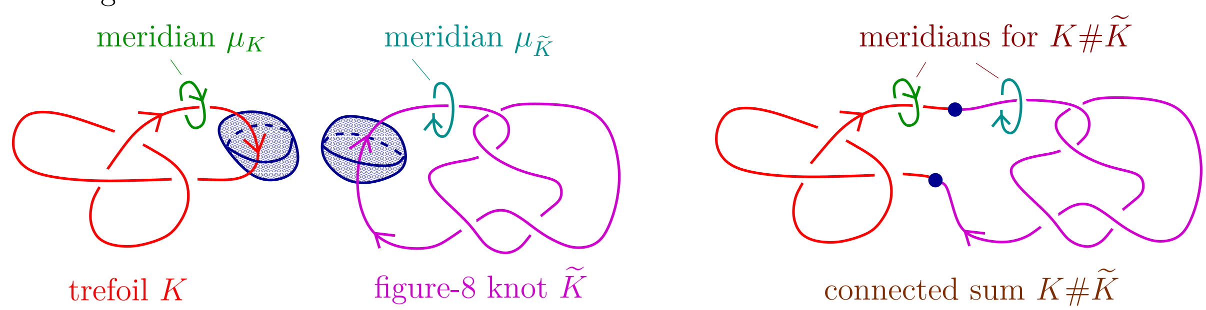
The Loop-π_1-Lemma 3.14 allows us to introduce the following slightly dangerous notation:

Notation.

(1) Let M be a path-connected smooth manifold, let $x_0 \in M$ and let $C \subset M$ be an oriented submanifold of M that is diffeomorphic to S^1. We denote by $[C] \in \pi_1(M, x_0)$ the element, well-defined up to conjugation, that we introduced in the Loop-π_1-Lemma 3.14.

(2) Let $L \subset S^3$ be a link and let K be an oriented component of L. It follows from the Meridian Proposition 2.17 and the Loop-π_1-Lemma 3.14 that for any $x_0 \in S^3 \setminus L$ the meridian μ_K defines an element in $\pi_1(S^3 \setminus L, x_0)$ that is well-defined up to conjugation. If there is no danger of confusion, then we denote this element by $\mu_K \in \pi_1(S^3 \setminus L, x_0)$ as well.

Next recall that on page 24 we introduced the connected sum $K \# \widetilde{K}$ of two oriented knots K and $\widetilde{K}$. For the reader's convenience we recall the definition (together with the meridians) in the figure below:



Proposition 3.15. (Knot Connected Sum-π_1-Proposition) Let K and $\widetilde{K}$ be two oriented knots in S^3. For any base points $x_0 \in S^3 \setminus K$ and $\widetilde{x}_0 \in S^3 \setminus \widetilde{K}$ and for any

meridians μ_K and $\mu_{\widetilde{K}}$ there exists an isomorphism

$$\pi_1(S^3 \setminus (K\#\widetilde{K})) \;\cong\; \pi_1(S^3 \setminus K, x_0) *_{\mu_K=\mu_{\widetilde{K}}} \pi_1(S^3 \setminus \widetilde{K}, \widetilde{x}_0).$$

Example. Let K be the trefoil $K = T(2,3)$. Using the Torus Knot-π_1-Proposition 3.8 and the Knot Connected Sum-π_1-Proposition 3.15 one can show, with enough patience, that the sets $\mathrm{Hom}(\pi_1(S^3 \setminus K), S_3)$ and $\mathrm{Hom}(\pi_1(S^3 \setminus (K\#K)), S_3)$ have different cardinalities. This implies that the trefoil K and the connected sum $K\#K$ are not smoothly isotopic. □

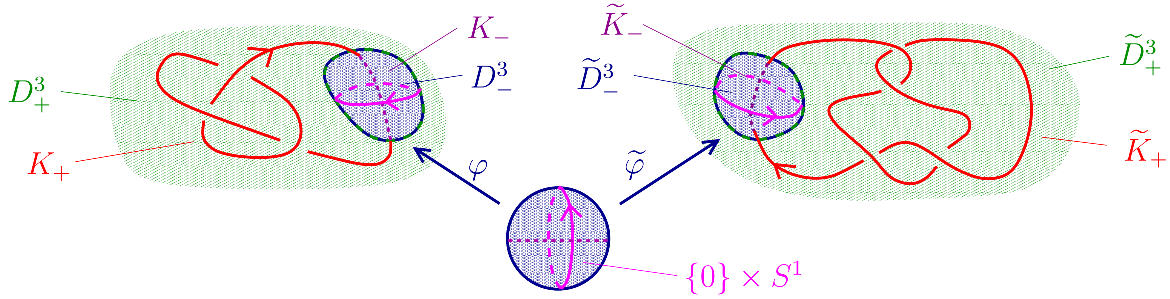
Sketch of proof. We start out with the following observation:

Claim 1. The statement of the proposition is independent of the choice of base points, choice of meridians and choice of paths connecting the meridians to the base points.

Proof. The claim is an almost immediate consequence of the Loop-π_1-Lemma 3.14 and the following group theoretic result: If π, $\widetilde{\pi}$ are groups, if $\nu \in \pi$ and $\widetilde{\nu} \in \widetilde{\pi}$ and if $f\colon \pi \to \pi'$ is an isomorphism, then $\pi *_{\nu=\widetilde{\nu}} \widetilde{\pi}$ is isomorphic to $\pi' *_{f(\nu)=\widetilde{\nu}} \widetilde{\pi}$. ⊞

Now we turn to the actual proof of the proposition.

- As in the definition of $K\#\widetilde{K}$ we pick a smooth embedding $\varphi\colon \overline{B}^3_2 \to S^3$ that respects K and we pick a smooth embedding $\widetilde{\varphi}\colon \overline{B}^3_2 \to S^3$ that anti-respects $\widetilde{K}$.[23]
- We write $D^3_+ := S^3 \setminus \varphi(B^3)$, $D^3_- := \varphi(\overline{B}^3)$, $D^3_0 := \varphi(S^2)$ and $K^* := D^3_* \cap K$.
- We write $\widetilde{D}^3_+ := S^3 \setminus \widetilde{\varphi}(B^3)$, $\widetilde{D}^3_- := \widetilde{\varphi}(\overline{B}^3)$, $\widetilde{D}^3_0 := \widetilde{\varphi}(S^2)$ and $\widetilde{K}^* := \widetilde{D}^3_* \cap \widetilde{K}$.
- It follows easily from the definitions that $\mu_K = \varphi(\{0\} \times S^1)$ and $\mu_{\widetilde{K}} = \widetilde{\varphi}(\{0\} \times S^1)$ are meridians for the oriented knots K and $\widetilde{K}$.[24]



Claim 2. The two inclusion-induced homomorphisms $\pi_1(D^3_+ \setminus K_+) \to \pi_1(S^3 \setminus K)$ and $\pi_1(\widetilde{D}^3_+ \setminus \widetilde{K}_+) \to \pi_1(S^3 \setminus \widetilde{K})$ are isomorphisms.

Proof. By the Seifert-van Kampen Theorem 3.9 we have the following pushout diagram:

$$\begin{tikzcd}
\pi_1(D^3_0 \setminus K_0) \arrow[r] \arrow[d] & \pi_1(D^3_+ \setminus K_+) \arrow[d] \\
\pi_1(D^3_- \setminus K_-) \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \pi_1(S^3 \setminus K).
\end{tikzcd}$$

Note that it follows easily from the fact that $\varphi\colon \overline{B}^3 \to S^3$ respects K that the inclusion $D^3_0 \setminus K_0 \to D^3_- \setminus K_-$ is a homotopy equivalence. This implies that in the above pushout diagram the left vertical map is an isomorphism. It follows that the right vertical map is also an isomorphism. The argument for $\widetilde{K}$ is of course basically the same. ⊞

[23]We refer to page 22 for the definitions of such maps.

[24]For $\widetilde{K}$ one needs to think a little bit about the orientations.

Next we consider the following maps:

isomorphism by the Seifert-van Kampen Theorem 3.9

$$\begin{array}{ccccc}
\pi_1(\underbrace{(S^3\#S^3)\setminus(K\#\widetilde{K})}_{=S^3\setminus(K\#\widetilde{K})}) & \cong & \pi_1(D^3_+\setminus K_+) & *_{\pi_1(D^3_0\setminus K_0)=\pi_1(\widetilde{D}^3_0\setminus\widetilde{K}_0)} & \pi_1(\widetilde{D}^3_+\setminus\widetilde{K}_+) \\
 & & (1)\ \downarrow\cong & & (1)\ \downarrow\cong \\
 & & \pi_1(D^3_+\setminus K_+) & *_{\mu_K=\mu_{\widetilde{K}}} & \pi_1(\widetilde{D}^3_+\setminus\widetilde{K}_+) \\
 & & (2)\ \downarrow\cong & & (2)\ \downarrow\cong \\
 & & \pi_1(S^3\setminus K) & *_{\mu_K=\mu_{\widetilde{K}}} & \pi_1(S^3\setminus\widetilde{K}).
\end{array}$$

It remains to show that the vertical maps are isomorphisms:

(1) Regarding the top vertical maps we make three observations:
   (a) As we mentioned above, $\mu_K = \varphi(\{0\}\times S^1)$ and $\mu_{\widetilde{K}} = \widetilde{\varphi}(\{0\}\times S^1)$ are meridians for the oriented knots K and $\widetilde{K}$.
   (b) The identification $D^3_0 = \widetilde{D}^3_0$ identifies μ_K with $\widetilde{\mu}_K$ as oriented submanifolds.
   (c) μ_K is a deformation retract $D^3_0\setminus K_0$ and $\mu_{\widetilde{K}}$ is a deformation retract $D^3_0\setminus\widetilde{K}_0$.

   It follows from this discussion that the top vertical maps are isomorphisms.

(2) It follows easily from Claim 2 that the bottom vertical maps are isomorphisms. ■

Exercises for Chapter 3.

Exercise 3.1.

(a) Let $n\geq 3$ and furthermore let $K\subset\mathbb{R}^n$ be a compact subset. We view K also as a subset of $S^n =_{\text{i}} \mathbb{R}^n\cup\{\infty\}$. Let $x_0\in\mathbb{R}^n\setminus K$. Show that the inclusion-induced map $\pi_1(\mathbb{R}^n\setminus K, x_0)\to\pi_1(S^n\setminus K, x_0)$ is an isomorphism.
(b) Does the conclusion of (a) also hold for $n=2$?
(c) Does the conclusion of (c) also hold for non-compact subsets of $\mathbb{R}^n$?

Exercise 3.2. Given $k\in\mathbb{Z}$ we consider the map

$$\begin{array}{rcl} \varphi_k\colon S^1\times S^1 & \to & S^1\times S^1 \\ (w,z) & \mapsto & (w\cdot z^k, z). \end{array}$$

Show that $(S^1\times\overline{B}^2)\cup_{\varphi_k}(S^1\times\overline{B}^2)$ is homeomorphic to S^3 if and only if $k\in\{-1,1\}$.
Hint. It might be helpful to compute the fundamental group or the first homology group of this topological space.

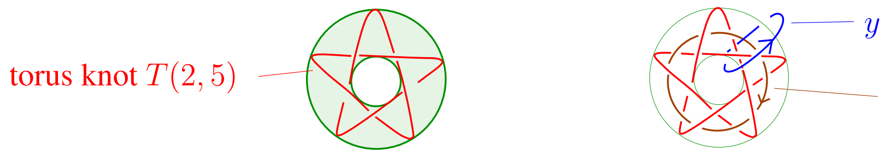
Exercise 3.3.

(a) Let $K\subset\mathbb{R}^3$ be the trivial knot. Show that $\mathbb{R}^3\setminus K$ is homotopy equivalent to $S^1\vee S^2$.
(b) Let $L\subset\mathbb{R}^3$ be the m-component unlink. Show that $\mathbb{R}^3\setminus L$ is homotopy equivalent to a wedge of spheres of suitable dimensions.

Exercise 3.4. Let $H\subset S^3$ be the Hopf link. Show that $S^3\setminus H$ is diffeomorphic to $S^1\times S^1\times(-1,1)$.

Exercise 3.5. Show that for every $n\in\mathbb{Z}$ the torus knots $T(n,1)$ and $T(1,n)$ are smoothly isotopic to the trivial knot.

Exercise 3.6. In the figure below we show the torus knot $T(2,5)$ and two loops x and y. Show directly (without worrying about base points) that $x^2 = y^5 \in \pi_1(S^3 \setminus T(2,5))$.



Exercise 3.7. Give an *explicit* diffeotopy from the map

$$\begin{array}{rcl} f\colon S^3 & \to & S^3 \\ v & \mapsto & \begin{pmatrix} 0 & \mathrm{id}_2 \\ \mathrm{id}_2 & 0 \end{pmatrix} \cdot v \end{array}$$

to the identity.

Exercise 3.8. As usual we consider the torus $S^1 \times S^1 \subset S^3 =_{\mathrm{i}} (\overline{B}^2 \times S^1) \cup_{S^1 \times S^1} (S^1 \times \overline{B}^2)$. Given $m, n \in \mathbb{Z}$ we consider $L(m,n) := \{(y,z) \in S^1 \times S^1 \,|\, y^m = z^n\}$.

(a) What is the number of components of $L(m,n)$?
(b) For which $m, n \in \mathbb{Z}$ is $L_{m,n}$ a torus knot?
(c) Draw $L(2,2)$ and $L(4,6)$.

Remark. The link $L(m,n)$ is called the *torus link* $L(m,n)$.

Exercise 3.9. Let $p, q \in \mathbb{N}$ coprime. Recall that in the proof of the Torus Knot-π_1-Proposition 3.8 we gave an explicit isomorphism $\langle x, y \mid x^p = y^q \rangle \to \pi_1(S^3 \setminus T(p,q))$.

(a) Show that $\pi_1(S^3 \setminus T(p,q))_{\mathrm{ab}} \cong \mathbb{Z}$.
(b) Find a word in x and y that represents a generators of $\pi_1(S^3 \setminus T(p,q))_{\mathrm{ab}}$.
(c) We consider the trefoil $T(2,3)$. Find a word in x and y that represents a meridian of the trefoil.
(d) Do (c) for all torus knots.
Remark. This might be a bit tricky.

Exercise 3.10. Let $n \in \mathbb{N}_0$. We consider the dihedral group

$$D_n \;=\; \mathbb{Z}_n \rtimes_{\varphi} \mathbb{Z}_2 \;\cong\; \langle x, t \,|\, x^n, t^2, t \cdot x \cdot t^{-1} = x^{-1} \rangle.$$

semidirect product where $\varphi\colon \mathbb{Z}_n \to \mathbb{Z}_n$ is given by multiplication by -1

Distinguish the torus knots $T(2,p)$ by looking at epimorphisms onto dihedral group.

Exercise 3.11. Let $n \in \mathbb{N}_0$. As in Exercise 3.10 we consider the dihedral group

$$D_n \;=\; \mathbb{Z}_n \rtimes_{\varphi} \mathbb{Z}_2 \;\cong\; \langle x, t \,|\, x^n, t^2, t \cdot x \cdot t^{-1} = x^{-1} \rangle.$$

Let K_1 and K_2 be two oriented knots and let p be a prime. Use the Knot Connected Sum-π_1-Proposition 3.15 to prove that the following two statements are equivalent:

(1) There exists an epimorphism $\pi_1(S^3 \setminus (K_1 \# K_2)) \to D_p$.
(2) There exists an $i \in \{1,2\}$ such that $\pi_1(S^3 \setminus K_i)$ admits an epimorphism onto D_p.

Exercise 3.12. Let $p, q \in \mathbb{N}$ be coprime. We set $\Gamma(p,q) := \langle x, y \mid x^p = y^q \rangle$ and we consider the subgroup $C(p,q)$ of $\Gamma(p,q)$ that is generated by x^p and y^q. We denote by $T(p,q)$ the

(p, q)-torus knot. In the Torus Knot-π_1-Proposition 3.8 we showed that there exists an isomorphism $\pi_1(S^3 \setminus T(p, q)) \cong \Gamma(p, q)$.

(a) Show that $\Gamma(p, q)/C(p, q) \cong \mathbb{Z}_p * \mathbb{Z}_q$.

(b) The *center* of a group π is defined as $C(\pi) := \{g \in \pi \,|\, gh = hg \text{ for all } h \in \pi\}$.
  (i) We assume that $p, q \neq 1$. Show that $C(\mathbb{Z}_p * \mathbb{Z}_q) = \{e\}$.
  (ii) Show that $C(\Gamma(p, q)) = C(p, q)$.

(c) Let (p, q) and (p', q') be two pairs of coprime natural numbers with $1 < p < q$ and $1 < p' < q'$. Show that if $(p, q) \neq (p', q')$, then the torus knots $T(p, q)$ and $T(p', q')$ are not smoothly isotopic.

Exercise 3.13. Let $K \subset S^3$ be an oriented knot with exterior X_K and meridian $\mu_K \subset \partial X_K$. We pick a base point $x_0 \in \mu_K$. Show that $\pi_1(X_K, x_0)$ is normally generated by μ_K.
Hint. Attach a 2-handle to X_K along the meridian. What can you say about the fundamental group of the resulting topological space?

Exercise 3.14. Let K and $\widetilde{K}$ be oriented knots. Show that the fundamental groups of $S^3 \setminus (K \# \widetilde{K})$ and $S^3 \setminus (K \# (\widetilde{K}^{\text{mir}})^{\text{rev}})$ are isomorphic.

Exercise 3.15. We consider the two links L and $\widetilde{L}$ that are shown in the figure below.

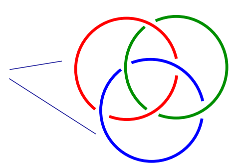
(a) Show that there exists an orientation-preserving diffeomorphism $X_L \to X_{\widetilde{L}}$ which preserves the ordering of the boundary components.

(b) Show that the two links L and $\widetilde{L}$ are not smoothly isotopic.

the two links are not smoothly isotopic but
the complements are diffeomorphic

Exercise 3.16. Show that the Borromean rings (which are shown in the figure below) are non-trivial by considering one component as an element in π_1 of the complement of the other two components.

Borromean rings



Exercise 3.17. Let $K \subset S^3$ be a knot. We consider the "physical cone"

$$C \;:=\; \{r \cdot P \,|\, P \in K \text{ and } r \in [0, 1]\} \;\subset\; \overline{B}^4.$$

(a) Show that C is homeomorphic to the disk $\overline{B}^2$.

(b) We suppose that K is smoothly isotopic to the trivial knot. Show that C is a topological submanifold of the topological manifold $\overline{B}^4$.

Exercise 3.18. Let $K \subset S^3$ be a knot such that $\pi_1(S^3 \setminus K) \not\cong \mathbb{Z}$. As in Exercise 3.17 we consider the "physical cone"

$$C \;:=\; \{r \cdot P \,|\, P \in K \text{ and } r \in [0, 1]\} \;\subset\; \overline{B}^4.$$

In Exercise 3.17 we showed that C is homeomorphic to the disk $\overline{B}^2$. In the following we will show that C is not a submanifold of the topological manifold $\overline{B}^4$.

(a) Let $\alpha\colon \pi_1(S^3 \setminus K) \to \mathbb{Z}$ and $\beta\colon \mathbb{Z} \to \pi_1(S^3 \setminus K)$ be two homomorphisms. Show that the composition $\beta \circ \alpha\colon \pi_1(S^3 \setminus K) \to \pi_1(S^3 \setminus K)$ is not an isomorphism.

(b) Suppose that F is a proper 2-dimensional topological submanifold of $\overline{B}^4$. Show that given any point $P \in F \cap B^4$ there exists a continuous map $\Phi\colon \overline{B}^4 \to \overline{B}^4$ that is an embedding such that $\Phi(0) = P$ and such that $\Phi(\overline{B}^2 \times \{0\}) = \Phi(\overline{B}^4) \cap C$.

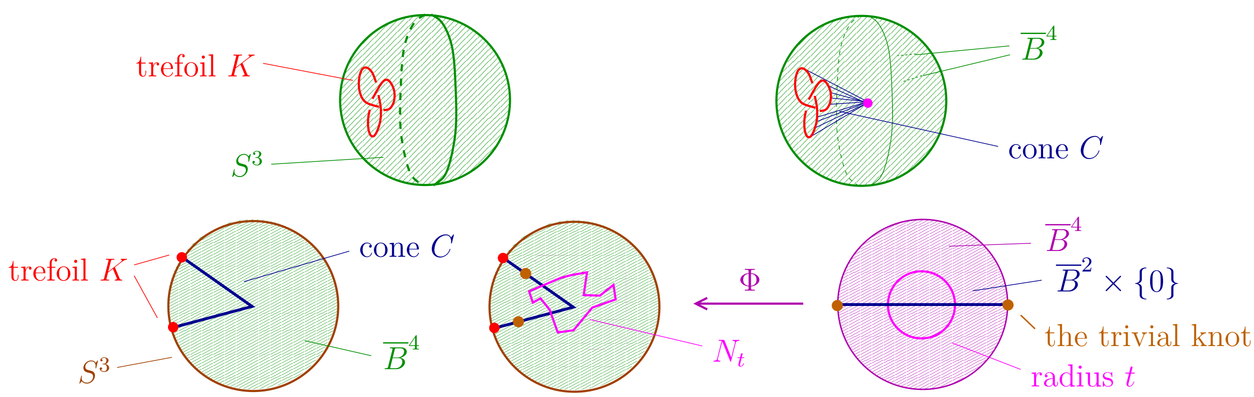
(c) Show that C is not a *topological submanifold* of $\overline{B}^4$.

Hint. Evidently the idea is to apply (c) to the point $P = 0 \in C$. The difficulty is that the map Φ from (c) is not just a rescaling of the identity but it is potentially a completely different chart. To deal with the problem we introduce some notation.

(i) Given $I \subset [0,1]$ we write $D_I := \{v \in \overline{B}^4 \mid \|v\| \in I\}$.
(ii) Given $I \subset [0,1]$ we write $N_I := \Phi(D_I)$.
(iii) Given $t \in (0,1]$ we write $D_t := D_{\{t\}}$ and $N_t := N_{\{t\}}$.

Show that the following statements hold:

(1) Given any $t \in (0,1]$ there exists an $s \in (0,1)$ with $D_{[0,s]} \subset N_{[0,t)}$.
(2) Given any $s \in (0,1]$ there exists a $t \in (0,1)$ with $N_{[0,t]} \subset D_{[0,s)}$.
(3) Given any choice of $r < s$ in $(0,1]$ the two inclusion maps $D_r \setminus C \to D_{[r,s]} \setminus C$ and $D_s \setminus C \to D_{[r,s]} \setminus C$ are homotopy equivalences, in particular they induce isomorphism of fundamental groups. Each of the fundamental groups is isomorphic to $\pi_1(S^3 \setminus K)$.
(4) Given any choice of $r < s$ in $(0,1]$ the two inclusion maps $N_r \setminus C \to N_{[r,s]} \setminus C$ and $N_s \setminus C \to N_{[r,s]} \setminus C$ are homotopy equivalences, in particular they induce isomorphism of fundamental groups. Each of the fundamental groups is isomorphic to $\mathbb{Z}$.
(5) Use the above discussion together with (b) to show that such Φ cannot exist.



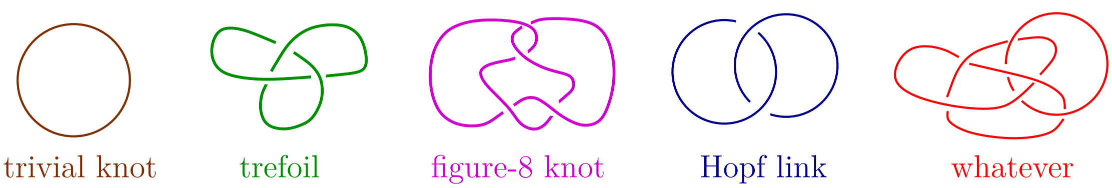
CHAPTER 4

Link diagrams

In the previous chapter we gave a completely rigorous calculation of the isomorphism type of the fundamental group of the complement of torus links and we used this to show that the trefoil is not smoothly isotopic to the trivial knot.

Our next goal is to calculate fundamental groups of complements of arbitrary links. But to calculate such fundamental groups we first need to find a way to describe all links in a reasonable way. In this chapter we will introduce the notion of a diagram of a link. We will see that diagrams are convenient ways to describe links. In the next chapter we will explain the Wirtinger algorithm which, given a link diagram for a link $L \subset S^3$, lets us determine a presentation of $\pi_1(S^3 \setminus L)$ from a diagram.

4.1. **Definition and existence of link diagrams.** In Chapter 2 we showed lots of 2-dimensional figures which present knots and links in $\mathbb{R}^3$. Some of these are shown in the figure below:

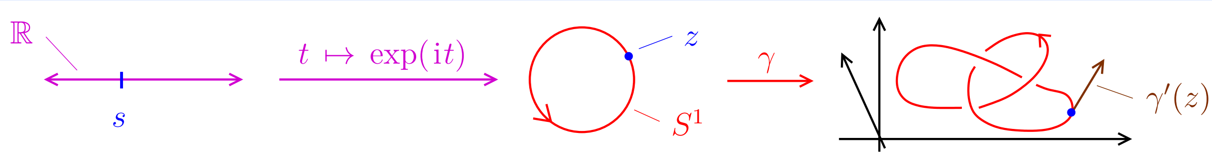


Such 2-dimensional pictures, which clearly indicate a link, are usually called “diagrams”. It takes some thought to say precisely what a “diagram” is supposed to be.[25] In this chapter we will give a formal definition of a link diagram and we will see that every link admits such a diagram. In the course we will see on many occasions that link diagrams are a convenient tool for describing links.

Before we start with link theory we introduce one bit of notation

Notation. Given a smooth map $\gamma\colon S^1 \to \mathbb{R}^n$ and $z = \exp(is) \in S^1$ we write

$$\gamma'(z) \;:=\; \text{derivative at the point } s \text{ of the function } \mathbb{R} \to \mathbb{R}^n \text{ given by } t \mapsto \gamma(\exp(it)).$$



[25]Unless one applies the popular all-purpose “I know it when I see it” -definition.

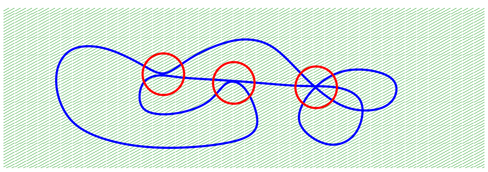
Definition. A map $\gamma\colon \bigsqcup_{i=1}^{m} S^1_i \to \mathbb{R}^2$ is called diagrammatic if the following conditions are satisfied:

(a) The map is an immersion, in our case this just means that γ is smooth and that for each $z \in \bigsqcup_{i=1}^{m} S^1_i$ we have $\gamma'(z) \neq 0$.

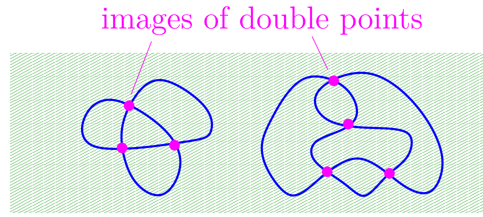
(b) If $z \neq w \in \bigsqcup_{i=1}^{m} S^1_i$ satisfy $\gamma(z) = \gamma(w)$, then $\gamma'(z)$ and $\gamma'(w)$ are linearly independent.

(c) For every $P \in \gamma\big(\bigsqcup_{i=1}^{m} S^1_i\big)$ the preimage $\gamma^{-1}(P)$ consists of either one or two points.

Any $z \in \bigsqcup_{i=1}^{m} S^1_i$ for which there exists a $w \neq z \in \bigsqcup_{i=1}^{m} S^1_i$ with $\gamma(z) = \gamma(w)$ is called a double point of γ.



not allowed in a diagrammatic map



images of diagrammatic maps $S^1 \to \mathbb{R}^2$

We start out with the following lemma which we will use subconsciously on many occasions.

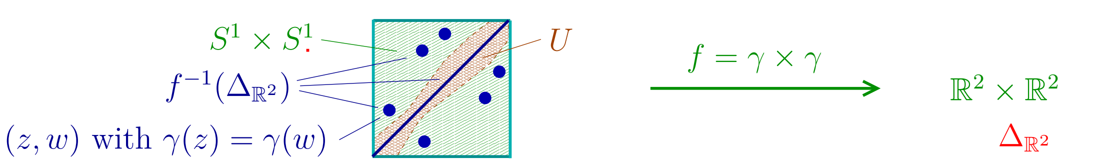
Lemma 4.1. (Diagram–Double Point Lemma) Every diagrammatic map has only finitely many double points.

Sketch of proof. To simplify the notation a little bit we only deal with the case $m = 1$, i.e. we consider a diagrammatic map $\gamma\colon S^1 \to \mathbb{R}^2$. Recall that this means that γ has the following properties:

(a) For each $z \in S^1$ we have $\gamma'(z) \neq 0$.
(b) If $z \neq w \in S^1$ satisfy $\gamma(z) = \gamma(w)$, then $\gamma'(z)$ and $\gamma'(w)$ are linearly independent.
(c) For every $P \in \gamma(S^1)$ the preimage $\gamma^{-1}(P)$ consists of either one or two points.

Next we consider the map
$$\begin{array}{rcl} f\colon S^1 \times S^1 & \to & \mathbb{R}^2 \times \mathbb{R}^2 \\ (z,w) & \mapsto & (\gamma(z), \gamma(w)). \end{array}$$

Given a set X we denote by $\Delta_X = \{(x,x) \mid x \in X\}$ the diagonal in $X \times X$. It follows easily from the definitions that
$$\#\text{double points of } \gamma \;=\; \tfrac{1}{2} \cdot \#(f^{-1}(\Delta_{\mathbb{R}^2}) \setminus \Delta_{S^1}).$$



Thus it suffices to prove the following claim:

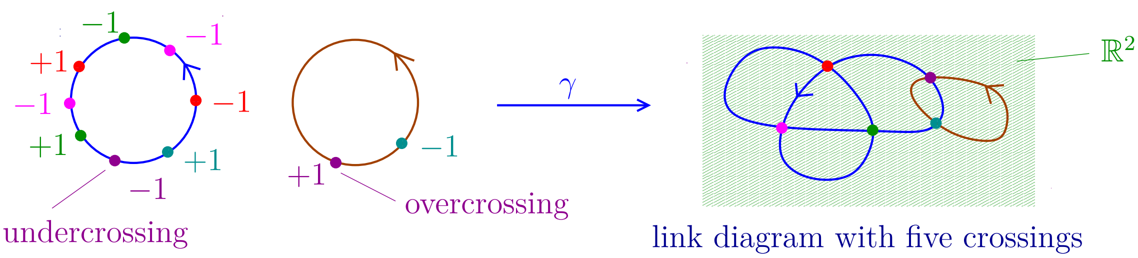
Claim. The set $f^{-1}(\Delta_{\mathbb{R}^2}) \setminus \Delta_{S^1}$ is finite.

Proof. We make the following observations:

(1) Since $\Delta_{\mathbb{R}^2} \subset \mathbb{R}^2 \times \mathbb{R}^2$ is closed we see that $f^{-1}(\Delta_{\mathbb{R}^2}) \subset S^1 \times S^1$ is closed.
(2) It follows from (a) that there exists an open neighborhood U of Δ_{S^1} such that $U \cap f^{-1}(\Delta_{\mathbb{R}^2}) = \Delta_{S^1}$.
(3) It follows from (b) that each $(z, w) \in (f^{-1}(\Delta_{\mathbb{R}^2}) \setminus \Delta_{S^1})$ is an isolated point in $f^{-1}(\Delta_{\mathbb{R}^2})$.

The above observations imply that $f^{-1}(\Delta_{\mathbb{R}^2}) \setminus \Delta_{S^1}$ is a discrete subset of the compact topological space $(S^1 \times S^1) \setminus U$. It follows that $(f^{-1}(\Delta_{\mathbb{R}^2}) \setminus \Delta_{S^1})$ is finite. ■

Definition.
(1) A link diagram is a diagrammatic map $\gamma\colon \bigsqcup_{i=1}^{m} S_i^1 \to \mathbb{R}^2$ together with a map
$$c\colon \{\text{double points of } \gamma\} \;\to\; \{\pm 1\}$$
which has the property that for $z \neq w \in \bigsqcup_{i=1}^{m} S_i^1$ with $\gamma(z) = \gamma(w)$ we have $c(z) \neq c(w)$.
(2) A knot diagram is a link diagram with $m = 1$.
(3) Let (γ, c) be a link diagram. Let z be a double point of γ. If $c(z) = +1$, then we say that the double point is an overcrossing, otherwise we call it an undercrossing.
(4) We refer to the images of the double points as the crossings of the link diagram.



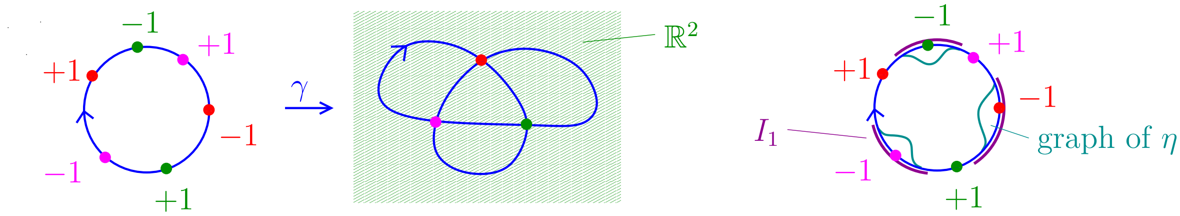
The next lemma shows that link diagrams give rise to links.

Lemma 4.2. (Diagram-to-Link Lemma) Let $\left(\gamma\colon \bigsqcup_{i=1}^{m} S_i^1 \to \mathbb{R}^2, c\right)$ be a link diagram. We can pick disjoint compact segments $I_1, \dots, I_k$ of $\bigsqcup_{i=1}^{m} S_i^1$ and we can pick a smooth map $\eta\colon \bigsqcup_{i=1}^{m} S_i^1 \to [-1, 0]$ with the following properties:

(a) Each segment contains a unique undercrossing and each undercrossing is contained in some segment.
(b) No segment contains an overcrossing.
(c) The value of η outside the segments is equal to 0.
(d) On each segment the function η has a unique local minimum, namely at the corresponding undercrossing where the value of η is equal to -1.

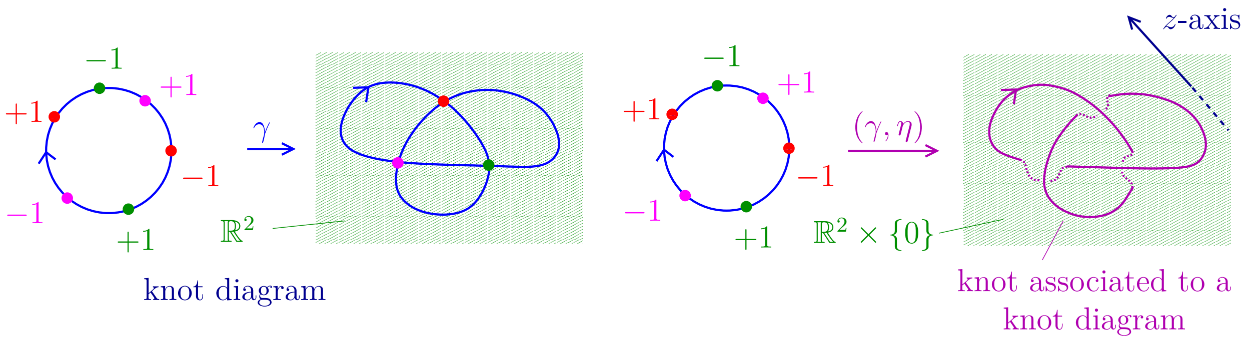
The following two statements hold:
(1) The image of $\bigsqcup_{i=1}^{m} S_i^1$ under the map $\bigsqcup_{i=1}^{m} S_i^1 \to \mathbb{R}^3$ given by $z \mapsto (\gamma(z), \eta(z))$ (together with the obvious ordering of the components) is an m-component link.
(2) Any two choices of segments and η as above give rise to smoothly isotopic links.

If we equip $\bigsqcup_{i=1}^{m} S_i^1$ with the standard orientation, then in the above statements we can also replace "link" by "oriented link".



Definition. The Diagram-to-Link Lemma 4.2 shows that a link diagram gives rise to an essentially unique (oriented) link. We refer to this link as the link associated to the link diagram. At times we will rather blur the difference between a link diagram and the associated link.

Example. In the figure below we show to the left a knot diagram. To the right we show the associated knot.



It is pretty clear that the resulting knot is smoothly isotopic to the trefoil. □

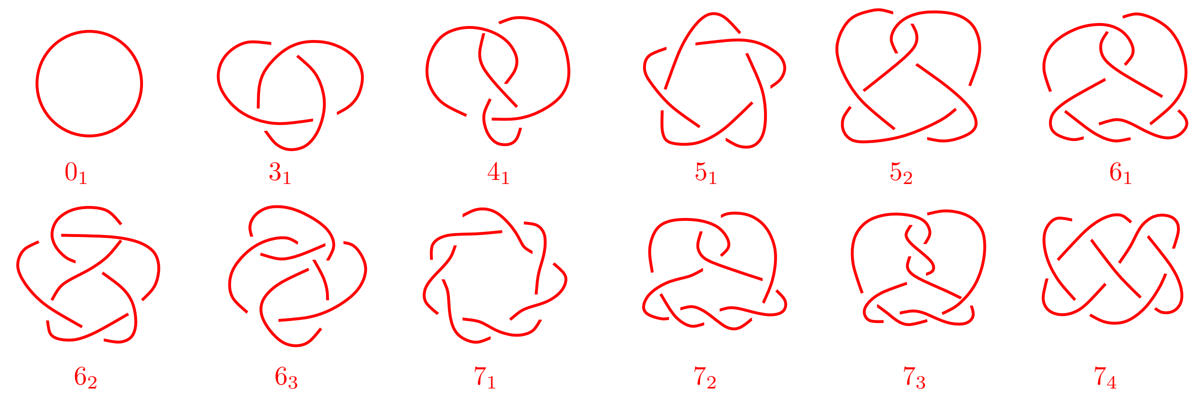
Proof. It is elementary to show that such segments $I_1, \dots, I_k$ and that such a smooth function $\eta\colon \bigsqcup_{i=1}^{m} S^1 \to [-1,0]$ exist. Given such a function it is clear that the map $z \mapsto (\gamma(z), \eta(z))$ is injective. Since $\bigsqcup_{i=1}^{m} S_i^1$ is compact and since $\mathbb{R}^3$ is Hausdorff it follows that the map is actually an embedding. Since γ is an immersion it follows that the map $z \mapsto (\gamma(z), \eta(z))$ is in fact even a smooth embedding. In summary we see that the image is a one-dimensional submanifold of $\mathbb{R}^3 \cup \{\infty\} =_{\text{i}} S^3$ that is diffeomorphic to $\bigsqcup_{i=1}^{m} S_i^1$. In other words, the image is a link.

Now suppose that we are given two functions η and η' with the properties stated as above. We consider the map

$$\begin{aligned} H\colon \big(\bigsqcup_{i=1}^{m} S_i^1\big) \times [0,1] \;&\to\; \mathbb{R}^3 \cup \{\infty\} =_{\text{i}} S^3 \\ (z,t) \;&\mapsto\; (\gamma(z), \eta(z)\cdot(1-t) + \eta'(z)\cdot t). \end{aligned}$$

This map is smooth. By construction we know that for any two distinct points $z, w \in \bigsqcup_{i=1}^{m} S_i^1$ with $\gamma(z) = \gamma(w)$ we have $\eta(z) < \eta(w)$ if and only if $c(z) = -1$ if and only if $\eta'(z) < \eta'(w)$. Using this observation one can easily verify that each H_t is a smooth embedding. In other words, the map H_t is a smooth isotopy between the two (oriented) links corresponding to η and η'. ■

In practice, i.e. in future examples, we do what every other sane topologist does, namely we draw a suggestive picture and we interpret it as a link diagram and a corresponding link in the obvious way. For example in the figure below we show the first prime knots (ordered by crossing number) in the standard table of knots (where we do not distinguish between knots and their mirror image).



We refer to the standard textbooks on knot theory [**Rol90, BZH14**] for a continuation of the table. But it should already be pretty clear from this table that knots provide a straightforward way to produce an almost endless list of interesting topological problems. In the Diagram-to-Link Lemma 4.2 we saw that any link diagram gives rise to a link. The following proposition gives us the converse:

Proposition 4.3. (Link Diagram Existence Proposition)
(1) Given any (oriented) link $L \subset S^3$ there exists a link diagram such that L is smoothly isotopic to the link associated to the link diagram.
(2) Given any (oriented) link $L \subset \mathbb{R}^3$ there exists an $A \in \mathrm{SO}(3)$ such that the projection of $A \cdot L$ onto the xy-plane defines a link diagram for L.

We postpone the rather lengthy proof of the Link Diagram Existence Proposition 4.3 to the next section.

The Link Diagram Existence Proposition 4.3 allows us to introduce the following definition:

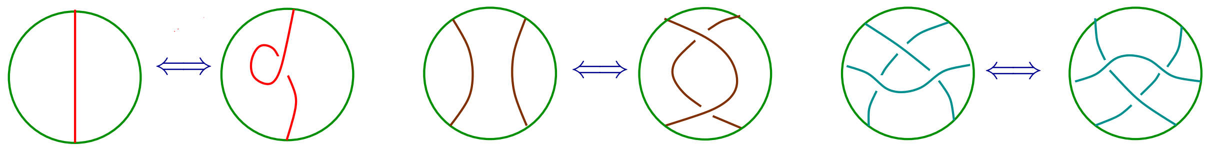
Definition. Let L be a link. We define the crossing number $c(L)$ of L as the minimal number of crossings of any diagram of L.

As an aside let us formulate the following open question:

Question 4.4. Is the crossing number additive under the connected sum operation? More precisely, if K and $\widetilde{K}$ are oriented knots, is $c(K \# \widetilde{K}) = c(K) + c(\widetilde{K})$?

We have now seen that every link, up to smooth isotopy, arises from a link diagram. The question arises, when do two link diagrams give rise to smoothly isotopic links? The following theorem gives a complete answer. We formulate the theorem in a slightly informal way since we will not really make use of it.

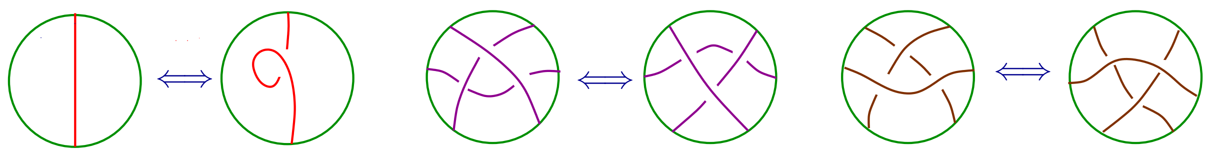
Theorem 4.5. (Reidemeister Moves Theorem) Two link diagrams give rise to smoothly isotopic links if and only if the two diagrams are related by a finite sequence of smooth isotopies of $\mathbb{R}^2$ and Reidemeister moves. The three Reidemeister moves are illustrated in the figure below.



Reidemeister move 1 Reidemeister move 2 Reidemeister move 3

Proof. The theorem goes back to the work of Kurt Reidemeister [**Rei26**, p. 26], [**Rei32**, Chapter 3] in 1932 and independently James Alexander and Garland Briggs [**AB27**, p. 563]. The formulations and the proofs in [**Rei26, Rei32, AB27**] are in terms of "polygonal links". The statement of the theorem is well known in our setting, where links are defined as submanifolds of S^3 and it appears in any textbook on link theory. A satisfactory proof can be found in [**OSS15**, Theorem B.1.1]. ■

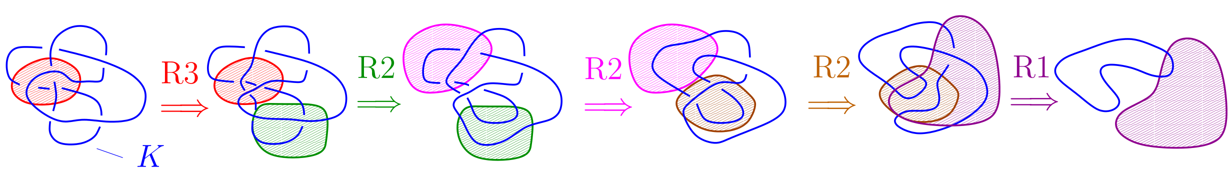
Example. In Exercise 4.8 we will see that the following variations on the above Reidemeister moves are actually consequences of the original Reidemeister moves.



Reidemeister move 1′ Reidemeister move 3′ Reidemeister move 3″

To preserve our sanity we will not distinguish in our language between the original Reidemeister moves and the various variations. □

Example. In the figure below we show to the left a diagram of some knot K and we show how, using a sequence of Reidemeister moves, one can turn the diagram into the "trivial diagram".



Remark. Throughout these lecture notes we will see on several occasions that the tools of algebraic topology, e.g. fundamental groups, linking pairings and Reidemeister torsion can be used fruitfully to study knots and links. The Link Diagram Existence Proposition 4.3 and the Reidemeister Moves Theorem 4.5 open up an alternative route to studying links. More precisely, these two results show that there exists a bijection

$$\begin{array}{c}\text{(oriented) link diagrams up to smooth isotopy}\\ \text{and Reidemeister moves}\end{array} \xrightarrow{\cong} \text{(oriented) links up to smooth isotopy.}$$

This bijection lies at the heart of the definition of several link invariants, like the Jones polynomial [**Jon85**] that was introduced in 1984 and the HOMFLY-PT polynomial [**FYH$^+$85, PT87**] that was discovered in 1985. These polynomial invariants are quite different from any of the invariants that are obtained through algebraic-topological methods. We refer to [**Lic97**, Chapters 3 and 15] and [**BZH14**, Chapter 17] for more information on these invariants. □

4.2. Proof of the Link Diagram Existence Proposition 4.3.

The proof of the Link Diagram Existence Proposition 4.3 mostly rests on the next lemma. The formulation of this lemma requires the following notation:

Notation. Given $v \in S^2$ we denote by

$$\begin{array}{rcl} \pi_v\colon \mathbb{R}^3 & \to & v^\perp := \{w \in \mathbb{R}^3 \,|\, \langle v, w\rangle = 0\} \\ w & \mapsto & w - \langle v, w\rangle \cdot v \end{array}$$

the projection onto the orthogonal complement of v.

Now we can formulate the lemma.

Lemma 4.6. (Link Generic Projection Lemma) Let $L \subset \mathbb{R}^3$ be a link. We pick a diffeomorphism $f\colon \bigsqcup_{i=1}^m S^1_i \to L$. We consider the following three properties of a vector $v \in S^2$:

(a) The map $\pi_v \circ f\colon \bigsqcup_{i=1}^m S^1_i \to v^\perp = \{w \in \mathbb{R}^3 \,|\, \langle v, w\rangle = 0\}$ is an immersion.

(b) Whenever $z \neq w \in \bigsqcup_{i=1}^m S^1_i$ satisfy $(\pi_v \circ f)(z) = (\pi_v \circ f)(w)$, then $(\pi_v \circ f)'(z)$ and $(\pi_v \circ f)'(w)$ are linearly independent.

(c) For every $P \in (\pi_v \circ f)\big(\bigsqcup_{i=1}^m S^1_i\big)$ the preimage $(\pi_v \circ f)^{-1}(P)$ consists of either one or two points.

The set

$$\{v \in S^2 \,|\, v \text{ satisfies (a), (b) and (c)}\}$$

has full measure in S^2.[26]

Proof of the Link Generic Projection Lemma 4.6. To simplify the notation we only consider the case of a knot. The proof for links is essentially the same.

Let $K \subset \mathbb{R}^3$ be a knot. We pick a diffeomorphism $f\colon S^1 \to K$. We say that a subset of a smooth manifold W is *large* if it is open and if it has full measure in W. Note that the intersection of two large subsets is again large. Furthermorenote that any self-diffeomorphism of S^2 sends large sets to large sets.

It is convenient to consider an extra property of vectors in S^2 which is somewhat weaker than the above property (c):

(c′) given any $P \in (\pi_v \circ f)(S^1)$ the preimage $(\pi_v \circ f)^{-1}(P)$ consists of finitely many points.

We write

$$V(a) := \{v \in S^2 \,|\, v \text{ satisfies (4)}\}.$$

Similarly we define $V(b), V(c), V(a,b), V(a,b,c')$ and so on. Note that with this language we need to show that $V(a,b,c)$ has full measure.

Let $v \in S^2$. We start out with the following three elementary observations regarding the projections π_v.

(1) Given $u \in \mathbb{R}^3 \setminus \{0\}$ we have

$$\pi_v(u) = 0 \iff u \in \mathbb{R} \cdot v \iff \frac{u}{\|u\|} \in \{v, -v\}.$$

(2) The projection π_v is linear, hence for any $P \in \mathbb{R}^3$ we have $(\mathrm{D}\pi_v)_P = \pi_v$.

[26] We refer to [**Fri24**] for the definition of a subset of full measure.

(3) It follows from (2) and the chain rule that given any $z \in S^1$ we have the equality $(\pi_v \circ f)'(z) = \pi_v(f'(z))$.

Next we consider the three maps[27] [28]

$$\varphi\colon S^1 \to S^2 \qquad \psi\colon \overbrace{\{(z,w) \in S^1 \times S^1 \mid z \neq w\}}^{=:M} \to S^2 \qquad \& \qquad \rho\colon S^2 \to S^2$$
$$z \mapsto \frac{f'(z)}{\|f'(z)\|} \qquad (z,w) \mapsto \frac{f(z)-f(w)}{\|f(z)-f(w)\|} \qquad v \mapsto -v.$$

We make the following observations:

(4) If X is a subset of full measure (respectively large subset) of S^2, then it follows from the above discussion that $X \cap \rho(X)$ is also a subset of full measure (respectively large).
(5) Since f is smooth we see that the map φ is also smooth.
(6) Given $v \in S^2$ it follows easily from (1) and (3) that $\pi_v \circ f\colon S^1 \to \mathbb{R}^2$ is an immersion if and only if neither v nor $-v$ lies in $\varphi(S^1)$. In other words, we have the equality $V(a) = (S^2 \setminus \varphi(S^1)) \cap \rho(S^2 \setminus \varphi(S^1))$.
(7) Note that $M = \{(z,w) \in S^1 \times S^1 \mid z \neq w\}$ is an open subset of the smooth manifold $S^1 \times S^1$, thus it is a smooth manifold in a natural way. With this smooth manifold structure the map $\psi\colon M \to S^2$ is easily seen to be smooth.
(8) For each $(x,y) \in S^1$ we make the identification $\mathrm{T}_{(x,y)}S^1 = \mathbb{R}$ via the basis vector $(-y,x)$. Given $(z,w) \in M$ we use the above identification to make the identification $\mathrm{T}_{(z,w)}M = \mathrm{T}_{(z,w)}(S^1 \times S^1) = \mathbb{R}^2$. Furthermore recall that given $v \in S^2$ we know that we have the equality $\mathrm{T}_v S^2 = v^\perp = \{w \in \mathbb{R}^3 \mid \langle v,w\rangle = 0\} \subset \mathbb{R}^3$.
(9) Using the identifications from (8) a rather elementary calculation[29] shows that for a point $(z,w) \in M$ we can write the differential $D\psi_{(z,w)}$, viewed as a homomorphism $\mathrm{T}_{(z,w)}M = \mathbb{R}^2 \to \mathrm{T}_{\psi(z,w)}S^2 \subset \mathbb{R}^3$, in the following way as a (3×2)–matrix:
$$D\psi_{(z,w)} \;=\; \frac{1}{\|f(z)-f(w)\|} \cdot \underbrace{\begin{pmatrix}\pi_{\psi(z,w)}(f'(z)) & -\pi_{\psi(z,w)}(f'(w))\end{pmatrix}}_{(3\times 2)\text{–matrix, in particular two columns}}.$$
(10) It follows from (9) that $(z,w) \in M$ is a regular point of ψ if and only if the vectors $\pi_{\psi(z,w)}(f'(z))$ and $\pi_{\psi(z,w)}(f'(w))$ are linearly independent.
(11) Let $v \in S^2$. It follows immediately from (1) and the linearity of π_v that we have $\pi_v(f(z)) = \pi_v(f(w))$ if and only if $\psi(z,w) = v$ or $\psi(z,w) = -v$.
(12) We denote by $\sigma\colon S^1 \times S^1$ the diffeomorphism given by $(z,w) \mapsto (w,z)$. It follows immediately from the definitions that $\psi \circ \sigma = \rho \circ \psi$.

[27] Since f is an immersion we see that the map φ is well-defined, i.e. we do not divide by zero. Similarly, since f is in particular injective we see that the map ψ is well-defined, i.e. once again we do not divide by zero.

[28] The astute reader will notice that throughout the argument it might be more reasonable to work with maps φ and ψ that take values in $\mathbb{RP}^2 = S^2/x \sim -x$ instead of taking values in S^2. We stick with the maps to S^2 since S^2 has the advantage that its tangent spaces can be described easily as vector subspaces of $\mathbb{R}^3$, which makes it easy to write down differentials. Also, the approach of working with maps to $\mathbb{RP}^2$ instead creates an extra layer of notation which is as annoying as our notation which requires the extra map ρ.

[29] This calculation can be performed easily using the chain rule and by writing ψ as the composition of the map $(z,w) \mapsto f(z) - f(w)$ followed by the map $P \mapsto \frac{P}{\|P\|}$.

(13) It follows easily from (3), (10), (11) and (12) that
$$V(b) \;=\; \{\text{regular values of } \psi\}.$$
After these initial remarks we turn to our first claim.

Claim 1.
(α) The sets $V(a)$ and $V(a,b)$ are large.
(β) We have $V(a)=V(a,c')$ and $V(a,b)=V(a,b,c')$.

Proof. We prove the two statements of the claim in several steps.

(i) Since S^1 is compact we obtain that $S^2\setminus\varphi(S^1)$ is an open subset of S^2. Furthermore, since φ is smooth and since $\dim(S^1)<\dim(S^2)$ we obtain that $S^2\setminus\varphi(S^1)$ is a subset of full measure. In summary, we have shown that $S^2\setminus\varphi(S^1)$ is large. It follows from (5) and (6) that $V(a)=(S^2\setminus\varphi(S^1))\cap\rho(S^2\setminus(\varphi(S^1)))$ is large.

(ii) It follows from Sard's Theorem (see [**Fri24**]) that the set of regular values of ψ is a subset of full measure. It follows from (i) and (13) that $V(a,b)=V(a)\cap V(b)$ has full measure.

(iii) Let $v\in V(a)$. It follows from (11)and from the fact that S^1 is compact that $\psi^{-1}(v)$ is finite. In particular we see that $V(a,c')=V(a)$ and thus also $V(a,b,c')=V(a,b)$.

(iv) It remains to show that $V(a,b)$ is open. Let $v\in V(a,b)=V(a)\cap V(b)$. From (13) and (iii) we obtain an open neighborhood U of $v\in S^2$ that is still contained in $V(b)$. Since $V(a)$ is open we see that $U\cap V(a)$ is an open neighborhood of $v\in S^2$ that is still contained in $V(a,b)$. It follows that $V(a,b)$ is indeed open. ⊞

Before we continue it is perhaps helpful to summarize what we have shown so far: In the claim we have seen that $V(a,b,c')=V(a,b)$ is large, in particular that it has full measure. But our actual goal is to show that $V(a,b,c)$ has full measure.

To do so we consider the map
$$\begin{array}{rrcl}\Xi\colon & \overbrace{\{(x,y,z)\in S^1\times S^1\times S^1\,|\,x\neq y \text{ and } x\neq z\}}^{=:N} & \to & S^2\times S^2\\ & (x,y,z) & \mapsto & (\psi(x,y),\psi(x,z)).\end{array}$$
We make the following simple observations:

(14) We view N as a smooth manifold in an obvious way. Evidently the map Ξ is smooth.

(15) It follows easily from (1) that we have $v\in V(c)$ if and only if none of the four vectors $(\pm v,\pm v)$ lies in $\Xi(N)$.

(16) If one of the four vectors $(\pm v,\pm v)$ lies in $\Xi(N)$, then so does (v,v). This can be seen as follows: If $\Xi(x,y,z)\in\{(\pm v,\pm v)\}$, then $f(x),f(y),f(z)$ lie on the line $\mathbb{R}\cdot v$. After possibly permuting x,y,z we can assume that $f(x)=f(y)+r\cdot v$ and that $f(x)=f(z)+s\cdot v$ with $r,s>0$. But with this permutation we have $\Xi(x,y,z)=(v,v)$.

Claim 2. We consider the "partial diagonal"
$$\Delta \;:=\; \{(v,v)\in S^2\times S^2\,|\,v\in V(a,b)\}.$$
The complement of $\Xi(N)\cap\Delta$ has full measure in Δ.

Proof. In the previous claim we saw that $V(a,b)$ is an open subset of S^2. Using this fact it is straightforward to see that Δ is a 2-dimensional submanifold of $S^2\times S^2$. Thus it suffices to prove that if $(x,y,z)\in N$ satisfies $\Xi(x,y,z)=(v,v)$ for some $v\in V(a,b)$, then

Ξ intersects Δ transversally, i.e. we have the equality

$$(\mathrm{D}\Xi_{(x,y,z)})(\mathrm{T}_{(x,y,z)}N) + \mathrm{T}_{(v,v)}\Delta \;=\; \mathrm{T}_{(v,v)}(S^2 \times S^2) \;\subset\; \mathbb{R}^3 \times \mathbb{R}^3 = \mathbb{R}^6.$$

We write $p = \pi_v(f'(x))$, $q = \pi_v(f'(y))$ and $r = \pi_v(f'(z))$ and we write $\mu = \frac{1}{\|f(x)-f(y)\|}$ and $\nu = \frac{1}{\|f(x)-f(z)\|}$. By (10) and (13), and by the fact that $v \in V(b)$, we know that p, q and r are pairwise linearly independent. In particular q and r form a basis for $\mathrm{T}_v S^2$. Note that this implies that $\begin{pmatrix} q \\ q \end{pmatrix}$ and $\begin{pmatrix} r \\ r \end{pmatrix}$ form a basis for $\mathrm{T}_{(v,v)}(S^2 \times S^2)$. Thus, using (9) we see that

$$(\mathrm{D}\Xi_{(x,y,z)})(\mathrm{T}_{(x,y,z)}N) + \mathrm{T}_{(v,v)}\Delta \;=\; \text{span of the columns of } \begin{pmatrix} \mu\cdot p & -\mu\cdot q & 0 & q & r \\ \nu\cdot p & 0 & -\nu\cdot r & q & r \end{pmatrix}.$$

Using that p, q and r are pairwise linearly independent it is not hard to see that the five columns of the matrix on the right span a 4-dimensional subspace of $\mathbb{R}^6$. Since this subspace is contained in the 4-dimensional subspace $\mathrm{T}_{(v,v)}(S^2 \times S^2)$ we see that it equals $\mathrm{T}_{(v,v)}(S^2 \times S^2)$. ⊞

We consider the "diagonal map" $d\colon V(a,b) \to \Delta \subset S^2 \times S^2$ that is given by $d(v) = (v,v)$. One can easily show that $d\colon V(a,b) \to \Delta$ is a diffeomorphism. It follows from (16) that $V(a,b,c) = d^{-1}(\Delta \setminus \Xi(N))$. By Claim 2 we know $\Delta \setminus \Xi(N)$ is a subset of full measure of Δ. Since d is a diffeomorphism we obtain that $V(a,b,c) = d^{-1}(\Delta \setminus \Xi(N))$ has full measure in $V(a,b)$. We saw above that $V(a,b)$ has full measure in S^2. It follows that $\Delta \setminus \Xi(N)$ has full measure in S^2. ■

Proof of the Link Diagram Existence Proposition 4.3. Recall that in the Link-in-$\mathbb{R}^3$-S^3-Lemma 2.5 we showed that every link in S^3 is smoothly isotopic to a link in the subspace $\mathbb{R}^3 \subset \mathbb{R}^3 \cup \{\infty\} =_{\mathrm{i}} S^3$. It follows that it remains to prove the following claim:

Claim. Given any (oriented) link $L \subset \mathbb{R}^3$ there exists an $A \in \mathrm{SO}(3)$ such that the projection of $A \cdot L$ onto the xy-plane defines a link diagram for L.

Proof. Let $L \subset \mathbb{R}^3$ be a link. We pick $v \in S^2$ as in the Link Generic Projection Lemma 4.6. Note that, possibly after applying a matrix in SO(3) to L, we can assume that $v = e_3$. We define $\gamma := \pi_{e_3} \circ f\colon S^1 \to \mathbb{R}^2 \times \{0\} = \mathbb{R}^2$. It follows immediately from the fact that $v = e_3$ has the three properties (a), (b) and (c) stated in the Link Generic Projection Lemma 4.6 that γ is diagrammatic. Next we consider the function

$$\begin{aligned} \epsilon\colon S^1 &\to \mathbb{R} \\ w &\mapsto z\text{-coordinate of } f(w). \end{aligned}$$

Furthermore, given $z_1 \neq z_2 \in S^1$ with $\gamma(z_1) = \gamma(z_2)$ we define

$$c(z_i) \;:=\; \begin{cases} +1, & \text{if } \epsilon(z_i) > \epsilon(z_{3-i}), \\ -1, & \text{if } \epsilon(z_i) < \epsilon(z_{3-i}). \end{cases}$$

It is clear that (γ, c) is a link diagram. We pick a function $\eta\colon S^1 \to [-1,0]$ as in the Diagram-to-Link Lemma 4.2. We consider the map

$$\begin{aligned} H\colon S^1 \times [0,1] &\to \mathbb{R}^3 \\ (z,t) &\mapsto (\gamma(z), (1-t)\cdot\epsilon(t) + t\cdot\eta(t)). \end{aligned}$$

Clearly this map is smooth. Given any $z, w \in S^1$ with $\gamma(z) = \gamma(w)$ we have $\eta(z) < \eta(w)$ if and only if $c(z) = -1$ if and only if $\eta(z) < \eta(w)$. From this observation we deduce that each H_t is actually a smooth embedding. Thus we see that H is a smooth isotopy from L to the link associated to the link diagram (γ, η). ■

Exercises for Chapter 4.

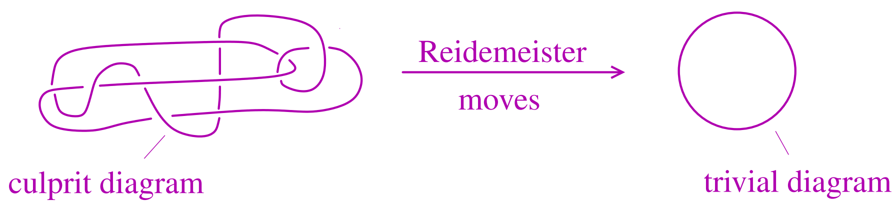
Exercise 4.1. Let $\gamma\colon S^1 \to \mathbb{R}^2$ be a smooth map such that the following condition is satisfied:

$(*)$ If $z \neq w \in S^1$ satisfy $\gamma(z) = \gamma(w)$, then $\gamma'(z)$ and $\gamma'(w)$ are linearly independent.

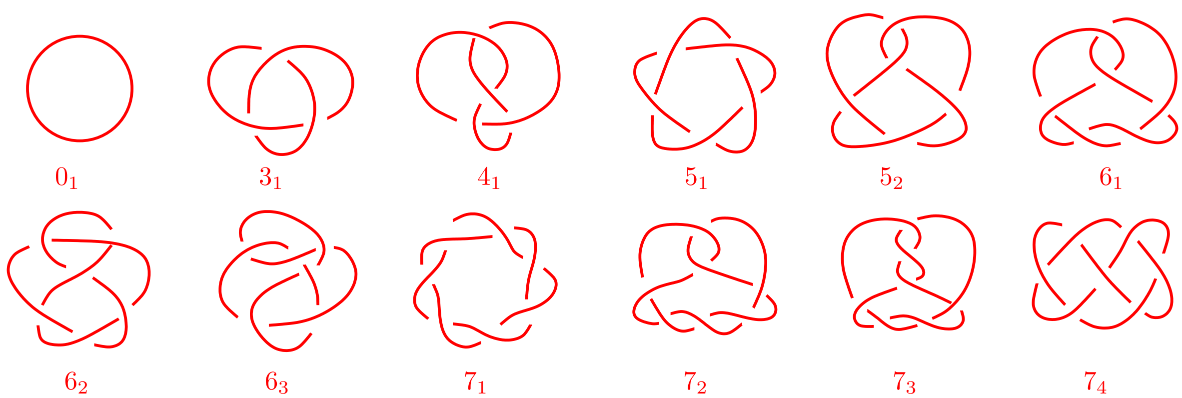
A point $z \in S^1$ for which there exists a $w \neq z \in S^1$ with $\gamma(z) = \gamma(w)$ is called a double point of γ. Does γ necessarily have finitely many double points?

Exercise 4.2. We consider the two knot diagrams shown in the figure below. Use Reidemeister moves to show that the associated knots are smoothly isotopic.

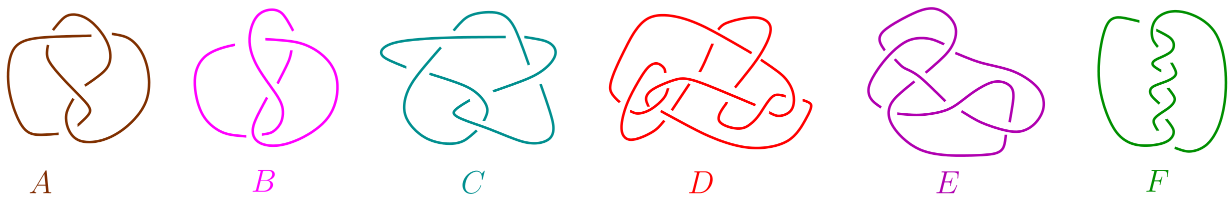
Exercise 4.3. Use Reidemeister moves to turn the diagram to the left into the trivial diagram on the right.
Remark. This diagram is sometimes called the *culprit diagram*, since one first needs to *increase* the number of crossings.



Exercise 4.4. We show again the list of knots from page 55.



In the following we show six different knots. Each knot is smoothly isotopic (up to mirror image) to precisely one of the knots from the above table. Match the knots to the corresponding knot in the table.
Solution. A: Figure-8, B: trefoil, C: stevedore 6_1, D: unknot, E: 6_3, F: 5_1



Exercise 4.5. Let L be a link. Recall that the crossing number $c(L)$ is defined as the smallest number of crossings of a link diagram for L.

(a) What are the knots with crossing number ≤ 4?
(b) What are the 2-component links with crossing number ≤ 4?
(c) Let $p, q \in \mathbb{N}$ be coprime. We consider the torus knot $T(p,q)$, as defined on page 38. What is the best *upper* bound on the crossing number of $T(p,q)$ that you can find?
Remark. It is usually quite hard to give a *lower* bound on the crossing number of a given knot or link.

Exercise 4.6. Let L be an oriented m-component link. Furthermore let $\gamma\colon \bigsqcup_{i=1}^{m} S^1_i \to \mathbb{R}^2$ and $c\colon \{\text{double points of } \gamma\} \to \{\pm 1\}$ be two maps that form a diagram for L.

(a) What is a diagram for L^{mir}?
(b) What is a diagram for L^{rev}?

Exercise 4.7. A *coloring of a link diagram* is a coloring of each strand (i.e. each segment between two undercrossings) by one of three colors, say red, green blue, such that the following two conditions hold:

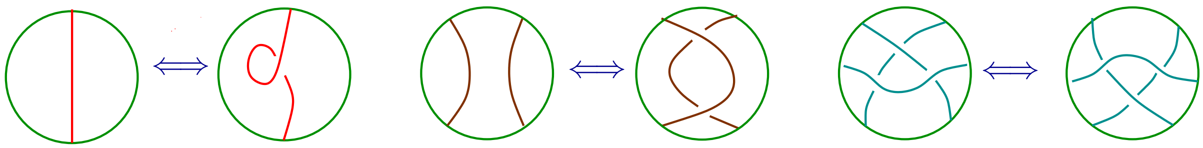
(1) At each crossing either all colors are the same or all three colors appear.
(2) At least two colors get used in the diagram.

We say that the diagram is *colorable* if a coloring exists. In the figure we show colorings of two diagrams of the trefoil.

two diagrams for the right-handed trefoil

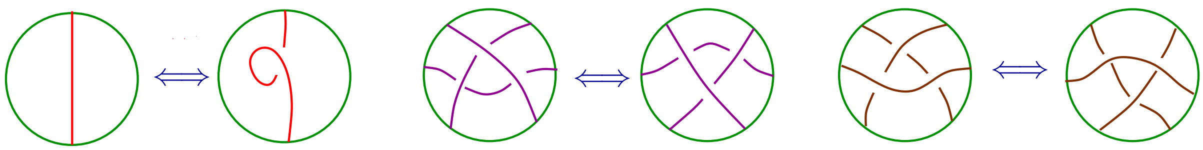
(a) Show that if two diagrams represent smoothly isotopic links, then either both are colorable or neither is colorable. In other words show that colorability is a link invariant and it makes sense to say that a link is colorable.
Hint. Use the Reidemeister Moves Theorem 4.5.
(b) Is the unknot colorable?
(c) Is the figure-8 knot colorable?
(d) Let K and $\widetilde{K}$ be two oriented knots. What is the connection between the colorability of K, $\widetilde{K}$ and the connected sum $K \# \widetilde{K}$?

Exercise 4.8. In the following we show, for the reader's convenience, the original Reidemeister moves:



Reidemeister move 1 Reidemeister move 2 Reidemeister move 3

Show that the following alternative Reidemeister moves are a consequence of our original Reidemeister moves.



Reidemeister move 1′ Reidemeister move 3′ Reidemeister move 3″

Exercise 4.9. We introduce the following language:

- An *arc* in 3-manifold M is a proper smooth submanifold that is diffeomorphic to $[0, 1]$.
- We say that a disjoint collection $A_1, \ldots, A_k \subset \overline{B}^3$ is *trivial* if it is properly smoothly isotopic to the collection of arcs $\overline{B}^3 \cap (\{(\frac{i}{2k}, 0)\} \times \mathbb{R})$, $i = 1, \ldots, k$.
- We say that a link $L \subset S^3 = S^3_{\geq 0} \cup_{S^3_{=0}} S^3_{\leq 0} =_{\mathrm{i}} \overline{B}^3_+ \cup_{S^2} \cup \overline{B}^3_-$ is in *bridge position* if L intersects S^2 transversally and if $L \cup \overline{B}^3_+$ and $L \cap \overline{B}^3_-$ is a trivial collection of arcs.

Now we can formulate the following exercises:

(a) Show that every link is smoothly isotopic to a link in bridge position.
(b) Let L be an m-component link. Show that $\pi_1(S^3 \setminus L)_{\mathrm{ab}} \cong \mathbb{Z}^m$.

We define the *bridge number* $b(L)$ of a link L to be the minimal k such that $b(L)$ is smoothly isotopic to a link in bridge position and such that the number of arcs equals k.

(c) Show that $b(K) = 1$ if and only if K is the trivial knot.
(d) Show that the bridge number of the trefoil and the figure-8 knot equals 2.
(e) The *meridional rank* $m(L)$ of a link L is the minimal $k \in \mathbb{N}_0$ such that $\pi_1(S^3 \setminus L)$ is generated by k meridians. Show that $m(L) \leq b(L)$.
Remark. It is an open question where $m(L) = b(L)$ holds for all links.

non-trivial arcs trivial arcs Hopf link in bridge position

CHAPTER 5

Wirtinger presentations

In Chapter 3 we calculated the isomorphism types of the fundamental group of the complement of any torus knot. But that approach does not generalize to complements of other knots and links. In the previous chapter we learned about link diagrams and we saw that every link can be described by a link diagram.

In this chapter we will explain the Wirtinger algorithm which, given a link diagram for a link L, lets us determine a presentation of $\pi_1(S^3 \setminus L)$. In principle this allows us to calculate the fundamental group for any given link. The problem is that it is quite hard to extract meaningful information from the fundamental group. In this chapter we will develop just enough tools to at least show that the trivial knot, the trefoil and the figure-8 link are pairwise different, i.e. we will show that they are pairwise not smoothly isotopic. In the next chapter we will learn about the Alexander polynomial which will be a powerful way to extract invariants from fundamental groups of link complements.

5.1. **The HNN-Gluing Theorem.** Before we turn to the Wirtinger presentation let us first build up our expertise on calculating fundamental groups.

Definition. Let π and Γ be two groups and let $\alpha, \beta\colon \Gamma \to \pi$ be two homomorphisms. We refer to

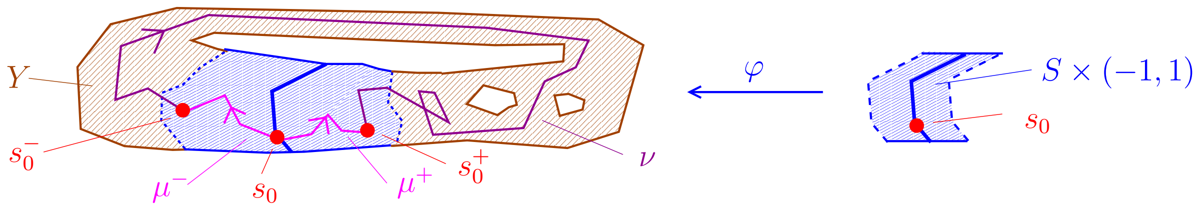
$$\langle \pi, t \mid \alpha(\Gamma) = t \cdot \beta(\Gamma) \cdot t^{-1} \rangle \;:=\; (\pi * \langle t \rangle) \,/\, \langle\langle \{\alpha(g) \cdot t \cdot \beta(g)^{-1} \cdot t^{-1}\}_{g \in \Gamma} \rangle\rangle$$

as the HNN-extension corresponding to (π, α, β).

Example. Let π be a group and let Γ be the trivial group. It follows immediately from the definitions that the corresponding HNN-extension is the free product $\pi * \langle t \rangle$. □

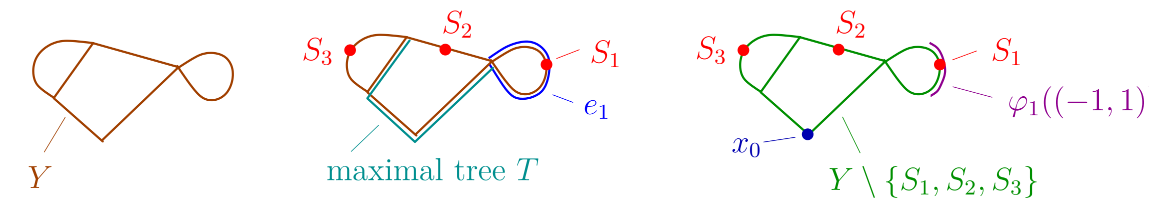
Theorem 5.1. (HNN–Gluing Theorem) Let Y be a topological space and let $S \subset Y$ be a path-closed connected non-empty subset such that $Y \setminus S$ is also path-connected. Furthermore let $\varphi\colon S \times (-1, 1) \to Y$ be an open embedding such that for each $s \in S$ we have $\varphi(s, 0) = s$. We introduce the following objects:
(1) We denote by $\iota_\pm\colon S \to Y \setminus S$ the map given by $x \mapsto \varphi(x, \pm\frac{1}{2})$.
(2) We pick $s_0 \in S$.
(3) We pick $s_0^- \in \varphi(S \times \frac{-1}{2})$ and we pick a path μ^- in $\varphi(S \times [-1, 0])$ from s_0 to s_0^-.
(4) We pick $s_0^+ \in \varphi(S \times \frac{1}{2})$ and we pick a path μ^+ in $\varphi(S \times [0, 1])$ from s_0 to s_0^+.
(5) We pick a path ν in $Y \setminus S$ from s_0^- to s_0^+.
(6) We write $\alpha_- := \iota_{-*}\colon \Gamma = \pi_1(S, s_0) \to \pi_1(Y \setminus S, s_0^-)$.
(7) We write [30] $\alpha_+ := \nu_* \circ \iota_{+*}\colon \pi_1(S, s_0) \to \pi_1(Y \setminus S, s_0^+) \to \pi_1(Y \setminus S, s_0^-)$.
The inclusion $Y \setminus S \to Y$ and the map $t \mapsto [\nu * \overline{\mu^+} * \mu^-] \in \pi_1(Y, s_0^-)$ induce an isomorphism

$$\langle \pi_1(Y \setminus S, s_0^-), t \mid \alpha_-(\pi_1(S, s_0)) = t \cdot \alpha_+(\pi_1(S, s_0)) \cdot t^{-1} \rangle \xrightarrow{\cong} \pi_1(Y, s_0^-).$$



Proof. This theorem follows fairly easily from HNN-Seifert–van Kampen Theorem which is stated and proved in [**Fri24**]. ■

Example. Let Y be a compact connected non-empty topological graph and let $T \subset Y$ be a maximal tree. We pick $x_0 \in T$. Let $e_1, \dots, e_n$ be the edges of Y that are not contained in T. For $i = 1, \dots, n$ we pick a point S_i in the interior of e_i. Note that for each $i \in \{1, \dots, n\}$ we can pick an open embedding $\varphi_i\colon (-1,1) \to e_i$ with $\varphi_i(0) = S_i$.



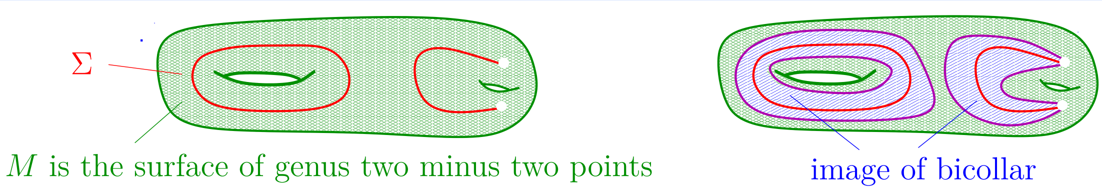
We now see that

$$\pi_1(G, x_0) \underset{\substack{\uparrow \\ \text{we apply the HNN–Gluing Theorem 5.1} \\ \text{altogether } n \text{ times}}}{\cong} \langle \pi_1(G \setminus \{S_1, \dots, S_n\}, x_0), t_1, \dots, t_n \rangle \underset{\substack{\uparrow \\ \text{since } G \setminus \{S_1, \dots, S_n\} \text{ is contractible}}}{\cong} \langle t_1, \dots, t_n \rangle.$$

□

In most applications of the HNN–Gluing Theorem 5.1 we will deal with the case that Y is a smooth manifold and that S is a codimension one submanifold of Y. For such a case it is helpful to introduce the following definition.

Definition. Let M be a smooth manifold and let $\Sigma \subset M$ be a proper codimension-one submanifold of M with $\partial\Sigma = \varnothing$.[31] A bicollar for the submanifold Σ is a smooth embedding $\beta\colon [-1,1] \times \Sigma \to M \setminus \partial M$ such that the following conditions are satisfied:
(1) For all $x \in \Sigma$ we have $\beta(0, x) = x$.
(2) $\beta([-1,1] \times \Sigma)$ is a closed subset of M.

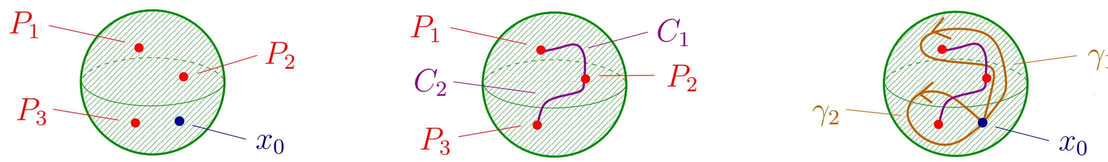


In the following we will use on several occasions the following theorem which is proved in [**Fri24**].

[30] The homomorphisms $\mu_*\colon \pi_1(Y \setminus S, s_0^+) \to \pi_1(Y \setminus S, s_0^-)$ is the isomorphism given by the path μ.
[31] Since we assume that Σ has no boundary the condition that Σ is a proper submanifold just means that it is a closed subset of M.

Theorem 5.2. (Bicollar Neighborhood Theorem) Let M be an oriented smooth manifold and let $\Sigma \subset M$ be a proper oriented codimension-one submanifold of M with $\partial\Sigma = \varnothing$. There exists an orientation-preserving bicollar $[-1,1] \times \Sigma \to M$.

Example. Let $P_1, \dots, P_n$ be pairwise distinct points on S^2. Let $C = C_1 \cup \cdots \cup C_{n-1} \subset S^2$ be a 1-dimensional smooth submanifold such that each C_i is diffeomorphic to $[0,1]$ and such that $\partial C_i = \{P_i, P_{i+1}\}$. Let $x_0 \in S^2 \setminus \{P_1, \dots, P_n\}$ be a base point. Furthermore let $\gamma_1, \dots, \gamma_{n-1} \colon [0,1] \to S^2 \setminus \{P_1, \dots, P_n\}$ be smooth loops in x_0 such that γ_i intersects C_i transversally in a single point and such that $\gamma_i \cap C_j = \varnothing$ for $i \neq j$.



One can easily show that $\pi_1(S^2 \setminus (C_1 \cup \cdots \cup C_{n-1}), x_0)$ is simply connected. It follows from the Bicollar Neighborhood Theorem 5.2, applied to the submanifolds $C_i \cap (S^2 \setminus \{P_1, \dots, P_n\})$, that we can apply the HNN–Gluing Theorem 5.1 altogether $n-1$ times. We see that the loops $\gamma_1, \dots, \gamma_{n-1}$ represent a basis for $\pi_1(S^2 \setminus \{P_1, \dots, P_n\}, x_0)$. $\square$

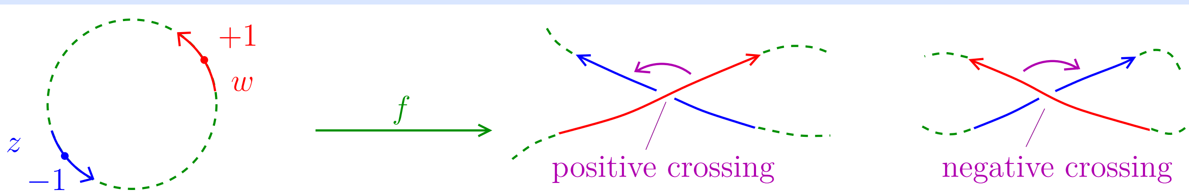
Now we are fully equipped to calculate the fundamental groups of any link that is represented by a diagram. We will do so in next section.

5.2. **The Wirtinger presentation.** In the previous chapter we showed in the Link Diagram Existence Proposition 4.3 that we can describe any link via a link diagram. In this section we will present an explicit algorithm that, given a link diagram of a link L, produces a presentation for the fundamental group of the complement $S^3 \setminus L$.

Before we can state the promised algorithm we need to introduce a few more definitions.

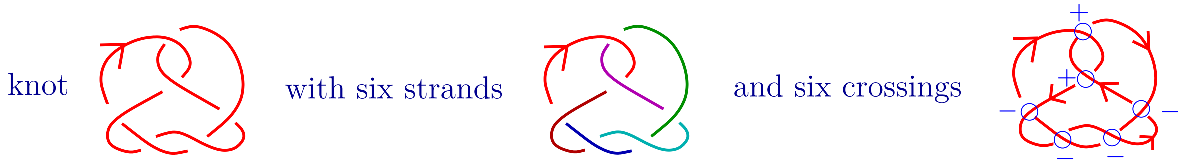
Definition. Let $\left(f \colon \bigsqcup_{i=1}^{m} S^1_i \to \mathbb{R}^2, c\right)$ be a link diagram, in the sense of the definition on page 53.

(1) We refer to the images of the double points as the crossings of the link diagram.
(2) Let $c \in f\big(\bigsqcup_{i=1}^{m} S^1_i\big)$ be a crossing. Let $z, w \in \bigsqcup_{i=1}^{m} S^1_i$ with $f(z) = f(w) = x$ and such that $c(z) = -1$ and $c(w) = 1$, in other words, z is an undercrossing and w is an overcrossing. We say that c is a positive crossing if the ordered basis $(f'(w), f'(z))$ is a positive basis for $\mathbb{R}^2$. Otherwise we say that x is a negative crossing.



Definition. Let $L \subset \mathbb{R}^2 \times [-1,0] \subset \mathbb{R}^3$ be the oriented m-component link that is associated to the link diagram $\left(f \colon \bigsqcup_{i=1}^{m} S^1_i \to \mathbb{R}^2, c\right)$, in the sense of the definition on page 54.

(3) We refer to the components of $L \setminus (\mathbb{R}^2 \times \{-1\})$ as the strands of L.



We start out with two minor lemmas:

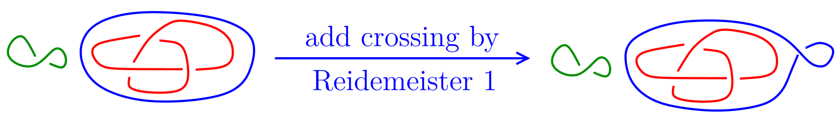
Lemma 5.3. (Crossing–Strand Lemma) If each component of a link diagram has at least one crossing, then the number of crossings equals the number of strands.

Proof. This lemma will be proved in Exercise 5.3. ■

The following lemma says that we can always arrange that the hypothesis of the Crossing–Strand Lemma 5.3 is satisfied:

Lemma 5.4. (One Crossing Exists Lemma) Every link admits a diagram such that each component has at least one crossing.

Proof. Let L be a link. By the Link Diagram Existence Proposition 4.3 we know that L admits a link diagram.



If we have a component without a crossing, then we just apply a Reidemeister 1 move to get a crossing. ■

The following proposition shows how to determine the fundamental group of the complement of a link that is associated to a link diagram. We will sacrifice a little bit of rigor in the formulation in an attempt to make it more readable.

Proposition 5.5. (Wirtinger[32] Presentation Proposition) Let L be a link that is associated to a given link diagram such that each component has at least one crossing. We enumerate the strands by $x_1, \ldots, x_n$ and we enumerate the crossings by $1, \ldots, n$.[33][34] For the i-th crossing we define a relation $r_i \in \langle x_1, \ldots, x_n \rangle$ as shown in the figure below.

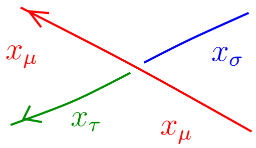
(1) We consider the base point $\diamond = (0, 0, 2)$. We have an explicit isomorphism
$$\langle x_1, \ldots, x_n \,|\, r_1, \ldots, r_n \rangle \xrightarrow{\cong} \pi_1(S^3 \setminus L, \diamond)$$
where each x_i is given by a meridian corresponding to the i-th strand.

(2) In (1) we can drop any one of the relations and we still obtain an isomorphism.

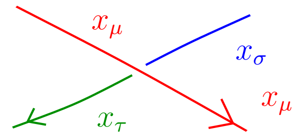
[32] The Wirtinger Presentation Proposition was first proved by the Austrian Mathematician Wilhem Wirtinger in 1905, it seems like this result was first published in [**Art25a**]. We refer to [**Epp99**, p. 253] for more on the history of this result.

[33] It follows from the Crossing–Strand Lemma 5.3 that the number of crossings equals the number of strands.

[34] For a knot it is customary to enumerate the strands cyclically $x_1, \ldots, x_n$, starting at a random strand and to say that the crossing r_i is between strand x_i and x_{i+1}.



the positive crossing gives rise to the relation $r_i = x_\mu \cdot x_\tau \cdot x_\mu^{-1} \cdot x_\sigma^{-1}$



the negative crossing gives rise to the relation $r_i = x_\mu^{-1} \cdot x_\tau \cdot x_\mu \cdot x_\sigma^{-1}$

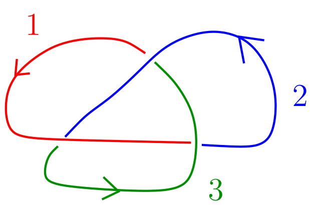
Definition. Given a link L, any presentation with n generators and $n-1$ relators as obtained from the Wirtinger Presentation Proposition 5.5 (2) is called a Wirtinger presentation of $\pi_1(S^3 \setminus L)$.

We now proceed as follows:

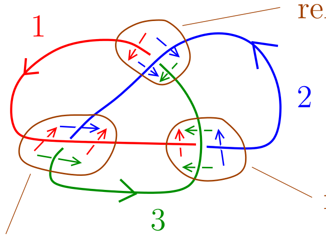
(1) We will use the Wirtinger Presentation Proposition 5.5 to give a new calculation of the fundamental group of the complement of the trefoil and of the Hopf link.
(2) In the next Section 5.3 we will provide the proof of the Wirtinger Presentation Proposition 5.5.
(3) In Section 5.4 we will consider more applications of the Wirtinger Presentation Proposition 5.5.

Examples.

(1) We consider the knot diagram for the trefoil K that is shown in the figure below to the left. We number the strands as shown. To the right we show how the three crossings give rise to three relations.



knot diagram for the trefoil with three negative crossings



relation $x_2^{-1} \cdot x_1 \cdot x_2 \cdot x_3^{-1}$

relation $x_3^{-1} \cdot x_2 \cdot x_3 \cdot x_1^{-1}$

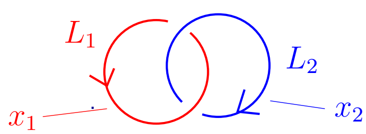
relation $x_1^{-1} \cdot x_3 \cdot x_1 \cdot x_2^{-1}$

It follows from this figure and the Wirtinger Presentation Proposition 5.5 (2) that there exists an isomorphism[35]

$$\langle x_1, x_2, x_3 \,|\, x_2^{-1} \cdot x_1 \cdot x_2 \cdot x_3^{-1}, x_3^{-1} \cdot x_2 \cdot x_3 \cdot x_1^{-1}, \underbrace{\cancel{x_1^{-1} \cdot x_3 \cdot x_1 \cdot x_2^{-1}}}_{\text{not needed}} \rangle \xrightarrow{\cong} \pi_1(S^3 \setminus K).$$

(2) We consider the diagram of the Hopf link $H = L_1 \cup L_2$ that is shown below.

Hopf link



both crossings are positive, we get relations
$x_1 \cdot x_2 \cdot x_1{}^{-1} \cdot x_1{}^{-1}$
$x_2 \cdot x_1 \cdot x_2{}^{-1} \cdot x_1{}^{-1}$.

[35]In the Torus Knot-π_1-Proposition 3.8 we showed that $\pi_1(S^3 \setminus K) \cong \langle x, y \mid x^2 = y^3 \rangle$. In particular we have now shown that the two groups $\langle x_1, x_2, x_3 \,|\, x_3^{-1} x_2 x_3 x_1^{-1}, x_2^{-1} x_1 x_2 x_3^{-1} \rangle$ and $\langle x, y \mid x^2 = y^3 \rangle$ are isomorphisms. In Exercise 5.7 we will give a purely algebraic proof of this statement.

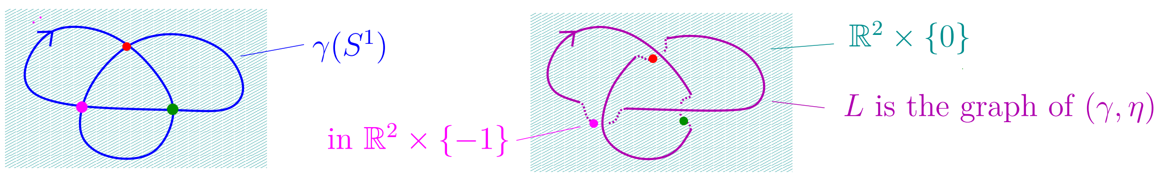
We see that

$$\pi_1(S^3 \setminus H) \underset{\substack{\uparrow \\ \text{by the Wirtinger Presentation}}}{\xleftarrow{\cong}} \langle x_1, x_2 \,|\, x_1 \cdot x_2 \cdot x_1{}^{-1} \cdot x_1{}^{-1} \rangle \;=\; \langle x_1, x_2 \,|\, [x_1, x_2] \rangle \underset{\substack{\uparrow \\ \text{Proposition 5.5 (2)}}}{\cong} \mathbb{Z}^2.$$

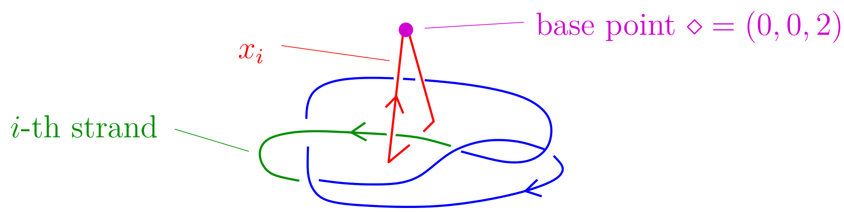
□

5.3. Proof of the Wirtinger Presentation Proposition 5.5. We start out by preparing the scene:

- Let $\big(\gamma\colon \bigsqcup_{i=1}^{m} S^1 \to \mathbb{R}^2, c\big)$ be a link diagram with n crossings such that each component has at least one crossing.
- We pick a smooth function $\eta\colon \bigsqcup_{i=1}^{m} S^1_i \to [-1, 0]$ as in the statement of the Diagram-to-Link Lemma 4.2.
- We denote by $L \subset \mathbb{R}^2 \times [-1, 0]$ the image of the map $\bigsqcup_{i=1}^{m} S^1_i \to \mathbb{R}^3$ that is given by $z \mapsto (\gamma(z), \eta(z))$.



- We enumerate the strands by $x_1, \ldots, x_n$. It follows from the Crossing–Strand Lemma 5.3 that we have n crossings. We enumerate the crossings by $1, \ldots, n$.
- We work with the base point $\diamond = (0, 0, 2) \in \mathbb{R}^3$ which lies "above" the link L.
- For the i-th strand we denote, by abuse of notation, by x_i also the oriented triangle that starts at $\diamond$ and "circles once around the i-th strand" according to the "right-hand rule". Note that this loop is path homotopic to a meridian "around the strand x_i".
- For the i-th crossing we define a relation $r_i \in \langle x_1, \ldots, x_n \rangle$ as on page 68.



We break the proof of the statements of the Wirtinger Presentation Proposition 5.5 into three parts:

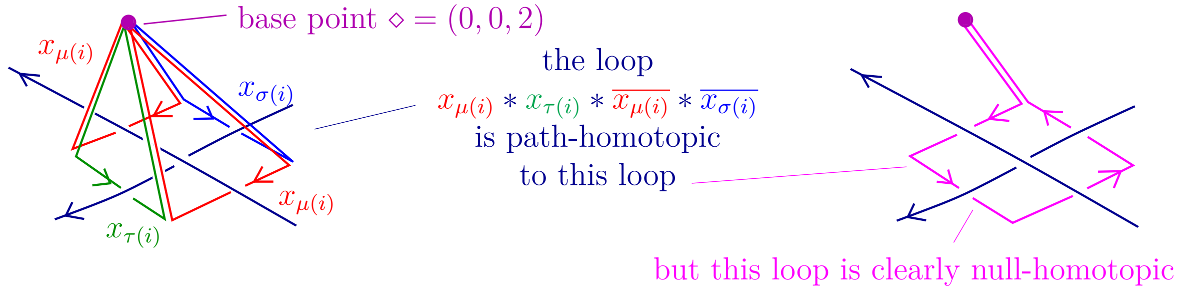
(1) (a) We construct an explicit homomorphism

$$\langle x_1, \ldots, x_n \,|\, r_1, \ldots, r_n \rangle \;\to\; \pi_1(S^3 \setminus L, \diamond)$$

(b) We show that the homomorphism from (a) is an isomorphism.

(2) We show that in (1a) we can drop one relation.

Proof of the Wirtinger Presentation Proposition 5.5 (1a). Let $i \in \{1, \ldots, n\}$. We consider the i-th crossing. First we deal with the case that the crossing is positive. By "sliding" the various loops $x_{\sigma(i)}$ along the strands and by some further path-homotopies we can assume that we are in the situation in the figure below on the left. If one follows the loop $x_{\mu(i)} * x_{\tau(i)} * \overline{x_{\mu(i)}} * \overline{x_{\tau(i)}}$ on its journey one realizes that on three occasions one goes back and forth to the base point. Thus the loop is actually path-homotopic to the loop shown in the figure below on the right. But that loop is clearly null-homotopic. Almost the same logic applies to negative crossings.



This discussion shows that there exists a unique group homomorphism

$$\langle x_1,\dots,x_n \,|\, \text{relations corresponding to crossings}\rangle \;\to\; \pi_1(S^3\setminus L,\diamond)$$

with $x_i \mapsto [x_i]$. ■

Before we can continue we need to state and prove the following lemma:

Lemma 5.6. (Punctured Sphere–Normal Generators Lemma) Let $P_1,\dots,P_n$ be pairwise distinct points on S^2 and let $x_0 \in S^2 \setminus \{P_1,\dots,P_n\}$ be a base point. Let $\varphi_1,\dots,\varphi_n\colon \overline{B}^2 \to S^2\setminus\{x_0\}$ be smooth embeddings with disjoint images such that for each $i\in\{1,\dots,n\}$ we have $\varphi_i(0)=P_i$. For $i=1,\dots,n$ we set $C_i := \varphi_i(S^1)$.

(1) The elements[36] $C_1,\dots,C_n \in \pi_1(S^2\setminus\{P_1,\dots,P_n\},x_0)$ form a *normal*[37] generating set of $\pi_1(S^2\setminus\{P_1,\dots,P_n\},x_0)$.
(2) In fact, for any $i\in\{1,\dots,n\}$ the elements[38] $C_1,\dots,\widehat{C}_i\dots,C_n$ also form a *normal* generating set of $\pi_1(S^2\setminus\{P_1,\dots,P_n\},x_0)$.

Proof of Lemma 5.6. After reordering the points we might as well assume that $i=n$. The lemma follows from the following little calculation:

$$\begin{aligned}\{e\} \;=\; \pi_1(\mathbb{R}^2) \;\underset{\text{since } S^2\setminus\{P_n\}\cong\mathbb{R}^2}{\cong}\; \pi_1(S^2\setminus\{P_n\}) \;&\cong\; \pi_1((S^2\setminus\{P_1,\dots,P_n\})\cup\varphi_1(B^2)\cdots\cup\varphi_{n-1}(\overline{B}^2))\\ &\cong\; \pi_1((S^2\setminus\{P_1,\dots,P_n\}))/\langle\!\langle [C_1],\dots,[C_n]\rangle\!\rangle.\end{aligned}$$

follows from applying the Seifert-van Kampen Theorem 3.9
altogether $n-1$ times and since $\varphi_i(S^1)$ is a deformation retract of $\varphi_i(\overline{B}^2)\setminus\{P_i\}$ ■

[37]Recall that by the Loop-π_1-Lemma 3.14 we know that the submanifolds $C_1,\dots,C_n$ give rise to elements of $\pi_1(S^2\setminus\{P_1,\dots,P_n\},x_0)$ which are well-defined up to conjugation. This indeterminacy has no influence on the statement of the lemma.

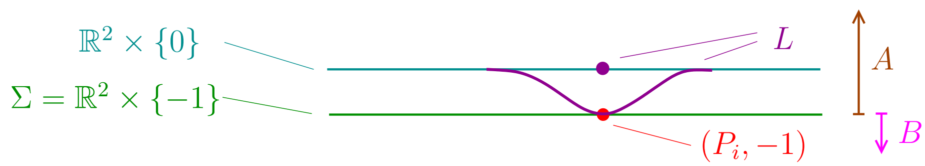
[38]We consider the free group $\langle x,y\rangle$ on two generators. The conjugates x and $yxyx^{-1}y^{-1}$ do *not* form a generating set for the free group (this follows from Whitehead's Algorithm [**Whi36**]). Using this algebraic fact one can easily show that the with the "wrong paths" the $C_1,\dots,C_n$ are not a generating set of the fundamental group.

Proof for the Wirtinger Presentation Proposition 5.5 (1b). We continue with the above notation and we add some notation:

- Somewhat similar to the proof of Torus Knot-π_1-Proposition 3.8 we consider the following subspaces of $\mathbb{R}^3 \cup \{\infty\} =_{\mathrm{i}} S^3$:

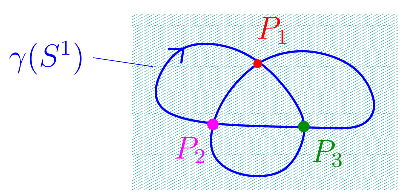
$$\begin{array}{rcl} A & := & \{(x,y,z) \in \mathbb{R}^3 \,|\, z \geq -1\} \cup \{\infty\}, \\ B & := & \{(x,y,z) \in \mathbb{R}^3 \,|\, z \leq -1\} \cup \{\infty\}, \\ \Sigma & := & \{(x,y,z) \in \mathbb{R}^3 \,|\, z = -1\} \cup \{\infty\} \;=\; A \cap B \end{array}$$

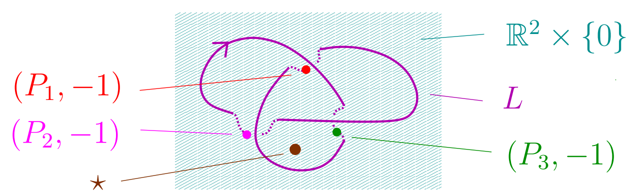
  and we set $A_L := A \setminus L$, $B_L := B \setminus L$ and $\Sigma_L := \Sigma \setminus L$.



- We pick a base point $\star \in \Sigma_L$.
- We denote by $P_1, \ldots, P_n \subset \mathbb{R}^2$ the crossings of γ.
- Note that it follows from property (d) of the Diagram-to-Link Lemma 4.2 that

$$L \cap (\mathbb{R}^2 \times \mathbb{R}_{\leq -1}) \;=\; L \cap (\mathbb{R}^2 \times \{-1\}) \;=\; \{(P_1,-1), \ldots, (P_n,-1)\}.$$





Claim 0. The inclusion maps induce an isomorphism

$$\pi_1(A_L, \star) *_{\pi_1(\Sigma_L, \star)} \pi_1(B_L, \star) \xrightarrow{\cong} \pi_1(S^3 \setminus L, \star).$$

Proof. Using the Stereographic Projection Lemma 2.1 and elementary arguments one can make the following observations:

(a) A and B are codimension-zero submanifolds of $\mathbb{R}^3 \cup \{\infty\} =_{\mathrm{i}} S^3$ with $A \cup B = S^3$.
(b) A and B are closed subsets of $\mathbb{R}^3 \cup \{\infty\} =_{\mathrm{i}} S^3$.
(c) We have $\partial A = \partial B = A \cap B = \Sigma$.

Since L is a closed subset of $A \cup B$ we obtain easily from the above that the following statements hold:

(a′) A_L and B_L are codimension-zero submanifolds of $S^3 \setminus L$ with $A_L \cup B_L = S^3 \setminus L$.
(b′) A_L and B_L are closed subsets of $S^3 \setminus L$.
(c′) $\partial A_L = \partial B_L = A_L \cap B_L = \Sigma_L$.

It follows from these observations that we can apply the Seifert–van Kampen Theorem 3.9, which gives us immediately the desired result. ⊞

Our next goal is to understand the groups $\pi_1(A_L, \star)$, $\pi_1(B_L, \star)$ and $\pi_1(\Sigma_L, \star)$. We handle these three groups consecutively. We start out with the easiest group on our list, namely the group $\pi_1(B_L, \star)$:

Claim 1. The group $\pi_1(B_L,\star)$ is trivial.

Proof. One can easily show that the map

$$\begin{array}{rcl} B\times[0,1] & \to & B \\ (P,t) & \mapsto & \begin{cases} \infty, & \text{if } P=\infty \text{ or } t=1,\\ P-(0,0,\frac{1}{1-t}-1), & \text{otherwise}\end{cases}\end{array}$$

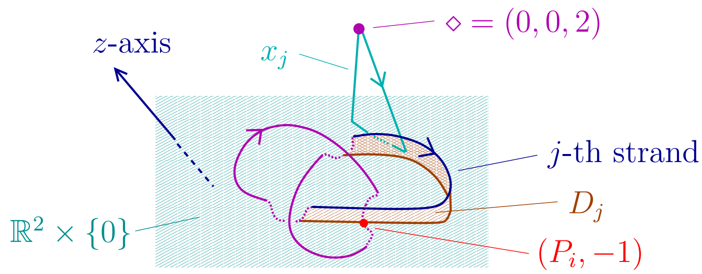
is continuous. This is a deformation retraction from B to $\{\infty\}$. Note that this map restricts to a deformation retraction from B_L to $\{\infty\}$. This shows that B_L is contractible. It follows that $\pi_1(B_L,\star)$ is indeed trivial. ⊞

In the following we will describe the group $\pi_1(A_L,\diamond)$ where, as before, $\diamond=(0,0,2)$.

(1) We write

$$D \;:=\; \Big\{(\gamma(z),w)\,\Big|\, z\in\bigsqcup_{i=1}^m S^1_i \text{ and } w\in[-1,\eta(z))\Big\}\setminus\bigcup_{i=1}^n\{(P_i,-1)\}\;\subset\;\mathbb{R}^2\times[-1,0].$$

(2) We enumerate the strands of the link.
(3) For $j\in\{1,\dots,n\}$ we denote by D_j the unique component of D that contains the j-th strand in its closure.
(4) For $j=1,\dots,n$ we denote by x_j a loop in $\diamond=(0,0,2)$ that is the boundary of a triangle that intersects D_j precisely once in a positive direction.



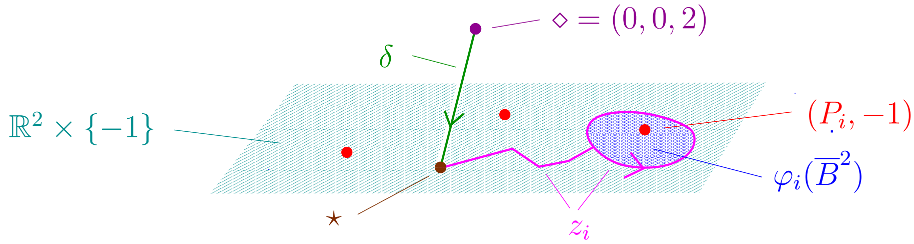
Claim 2. The obvious map $\langle x_1,\dots,x_n\rangle\to\pi_1(A_L,\diamond)$ is an isomorphism.

Proof. We start out with three observations:

(1) One can easily verify that each D_j is a proper smooth submanifold of A_L. It follows from the Bicollar Neighborhood Theorem 5.2 that each D_j has a bicollar $\beta\colon D_j\times[-1,1]\to A_L$.
(2) We set
$$Z \;:=\; A\setminus\Big\{(\gamma(z),w)\,\Big|\, z\in\bigsqcup_{i=1}^m S^1_i \text{ and } w\in[-1,\eta(z)]\Big\}.$$
Note that $Z\cup(D_1\cup\dots\cup D_n)=A_L$.
(3) Similar to Claim 1 we consider the deformation retraction
$$\begin{array}{rcl} A\times[0,1] & \to & A \\ (P,t) & \mapsto & \begin{cases} \infty, & \text{if } P=\infty \text{ or } t=1,\\ P+(0,0,\frac{1}{1-t}-1), & \text{otherwise}\end{cases}\end{array}$$
from A to $\{\infty\}$. It is straightforward to verify that this map restricts to a deformation retraction from Z to $\{\infty\}$.
(4) Each D_j is simply connected.

It follows from (1) that we can apply the HNN–Gluing Theorem 5.1 altogether n times. It follows from (3) and (4) that the obvious map $\langle x_1,\dots,x_n\rangle \to \pi_1(A_L,\star)$ is indeed an isomorphism. ⊞

Finally we turn to the group $\pi_1(\Sigma_L,\star)$. We pick orientation-preserving smooth embeddings $\varphi_i\colon \overline{B_L}^2 \to \mathbb{R}^2\times\{-1\}$, $i=1,\dots,n$ with disjoint images and such that for each i we have $\varphi_i(0)=(P_i,-1)$. As in the Loop-π_1-Lemma 3.14 we pick paths in Σ_L from $\star$ to a point on $\varphi_i(S^1)$ and we consider the loops in $\star$ that are given by the paths and by going "once around" $\varphi_i(S^1)$. We denote the resulting loops by $z_1,\dots,z_n$.



Claim 3. The loops $z_1,\dots,z_n$ form a normal generating set of $\pi_1(\Sigma_L,\star)$.

Proof. This statement follows immediately from the Punctured Sphere–Normal Generators Lemma 5.6 (1). ⊞

Let δ be the straight path in A_L from $\star\in\mathbb{R}^2\times\{-1\}$ to $\diamond=(0,0,2)$. Now we can finally prove the desired statement. Indeed, we have

$$\begin{array}{rll} \pi_1(S^3\setminus L,\star) & \overset{\text{by Claim 0}}{\xleftarrow{\cong}} \ \pi_1(A_L,\star) *_{\pi_1(\Sigma_L,\star)} \pi_1(B_L,\star) & \overset{\text{by Claim 1}}{\xrightarrow{\cong}} \ \pi_1(A_L,\star)/\langle\langle \pi_1(\Sigma_L,\star)\rangle\rangle \\ & \xrightarrow{\delta_*} \ \pi_1(A_L,\diamond)/\delta_*\langle\langle \pi_1(\Sigma_L,\star)\rangle\rangle & \underset{\text{follows from Claims 2 and 3}}{\xleftarrow{\cong}} \ \langle x_1,\dots,x_n\rangle/\langle\langle \delta_*(z_1),\dots,\delta_*(z_n)\rangle\rangle \\ & \underset{\substack{\text{as on page 70 we see that each } \delta_*(z_i) \text{ is conjugate to } r_i\text{, it follows} \\ \langle\langle z_1,\dots,z_n\rangle\rangle = \langle\langle r_1,\dots,r_n\rangle\rangle}}{=} \ \langle x_1,\dots,x_n\rangle/\langle\langle r_1,\dots,r_n\rangle\rangle. & \end{array}$$

■

Proof for the Wirtinger Presentation Proposition 5.5 (2). We want to show that we can drop any one of the relations in the above presentation. Therefore let $j\in\{1,\dots,n\}$. By the Punctured Sphere–Normal Generators Lemma 5.6 (1) that $z_1,\dots,\widehat{z_j},\dots,z_n$ also form a normal generating set of $\pi_1(\Sigma_L,\star)$. Therefore we see that in the above equality we can drop the relation r_j. ■

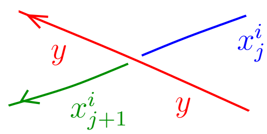
5.4. **More applications of the Wirtinger Presentation Proposition 5.5.** The following corollary to the Wirtinger Presentation Proposition 5.5 gives us the disappointing news that our old trick of using abelianizations of fundamental groups is not particularly useful for distinguishing links.

Corollary 5.7. (Link Group-Abelianization Corollary) Let $L=L_1\cup\cdots\cup L_m$ be an oriented m-component link. We denote by $\mu_1,\dots,\mu_m\subset S^3\setminus L$ the meridians of the components of L. There exists a unique isomorphism

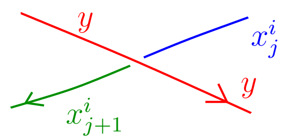
$$\Phi\colon \pi_1(S^3\setminus L)_{\mathrm{ab}} \ \to\ \mathbb{Z}^m$$

such that[39] $\Phi(\mu_i) = e_i$ for $i = 1, \dots, m$.

Proof. Let $L = L_1 \cup \dots \cup L_m$ be an oriented m-component link. By the Link Diagram Existence Proposition 4.3 and the One Crossing Exists Lemma 5.4 such that each component has at least one crossing. We enumerate the strands by $x_1^1, \dots, x_{k_1}^1, x_1^2, \dots, x_{k_2}^2, \dots, x_1^m, \dots, x_{k_m}^m$ where for each $i \in \{1, \dots, m\}$ the strands $x_1^i, \dots, x_{k_i}^i$ are cyclically ordered strands of the i-th component. Furthermore we enumerate the crossings by $c_1^1, \dots, c_{k_1}^1,\ c_1^2, \dots, c_{k_2}^2, \dots, c_1^m, \dots, c_{k_m}^m$ where any crossing c_j^i separates the strands x_j^i and x_{j+1}^i (where we interpret x_{k_i+1} as x_1.) Note that the relation r_i^j corresponding to the crossing c_i^j is of the form $y^\epsilon \cdot x_{j+1}^i \cdot y^{-\epsilon} \cdot (x_j^i)^{-1}$ where y is some generator and $\epsilon \in \{-1, 1\}$.



$$r_j^i = y \cdot x_{j+1}^i \cdot y^{-1} \cdot (x_j^i)^{-1}$$



$$r_j^i = y^{-1} \cdot x_{j+1}^i \cdot y \cdot (x_j^i)^{-1}$$

It follows from this discussion, the Wirtinger Presentation Proposition 5.5 (2) and basic facts about abelianizations of presentations that a presentation matrix for the abelian group $\pi_1(S^3 \setminus L)_{\mathrm{ab}}$ is given by the block matrix

$$\begin{pmatrix} J_1 & 0 & \dots & 0 \\ 0 & J_2 & & 0 \\ \vdots & & \ddots & 0 \\ 0 & 0 & \dots & J_m \end{pmatrix} \qquad \text{where} \qquad J_i := \underbrace{\begin{pmatrix} 1 & 0 & \dots & -1 \\ -1 & 1 & & 0 \\ 0 & \ddots & \ddots & \vdots \\ 0 & \dots & -1 & 1 \end{pmatrix}}_{(k_i \times k_i)\text{-matrix}}.$$

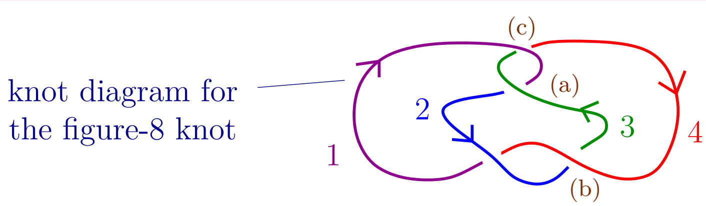
Elementary linear algebra now gives us the desired result. ■

After this sobering corollary we move on to the next example.

Lemma 5.8. (Figure 8 Knot–π_1-Lemma) Let J be the oriented figure-8 knots shown in the figure below. There exists an isomorphism

$$\pi_1(S^3 \setminus J) \ \cong\ \langle x_1, x_2, x_3, x_4 \,|\, \underbrace{x_3 \cdot x_2 \cdot x_3^{-1} \cdot x_1^{-1}}_{\text{crossing (a)}}, \underbrace{x_4^{-1} \cdot x_3 \cdot x_4 \cdot x_2^{-1}}_{\text{crossing (b)}}, \underbrace{x_1 \cdot x_4 \cdot x_1^{-1} \cdot x_3^{-1}}_{\text{crossing (c)}} \rangle$$

where each x_i corresponds to a meridian.



Proof. The lemma follows immediately from the Wirtinger Presentation Proposition 5.5 (2) and staring at the above diagram. ■

[39]As we discussed on page 45, the i-th meridian defines an element in $\pi_1(S^3 \setminus L)$ that is well-defined up to conjugation. It follows that the i-th meridian defines a well-defined element of $\pi_1(S^3 \setminus L)_{\mathrm{ab}}$.

On its own the Figure 8 Knot–π_1-Lemma 5.8 is not particularly enlightening. It is initially not clear whether this group is isomorphic to $\mathbb{Z}$ or to the fundamental group of the complement of the trefoil. By the Link Group-Abelianization Corollary 5.7 we know that there is no hope that the abelianization will tell us something interesting.

So we need a different approach. Recall that in the Trefoil–Non-Trivial Proposition 3.10 we used the (non-) existence of an epimorphism onto the symmetric group S_3 to distinguish the trefoil from the trivial knot. The idea now is to repeat this trick with a different finite group. In fact using dihedral groups we can prove the following corollary:

Corollary 5.9. (Trefoil–Figure 8–Non-Isotopic Corollary) The trivial knot, the trefoil and the figure-8 knot are pairwise not smoothly isotopic.

Proof. Recall for each $n \in \mathbb{N}$ we defined the dihedral group $D_n = \langle x, t \,|\, x^n, t^2, t \cdot x \cdot t^{-1} = x^{-1}\rangle$. It is encouraging that $D_3 \cong S_3$ thus we try our luck with D_5.[40] With enough time on our hands we obtain the following table:

	trivial knot	trefoil	figure-8 knot
epimorphism onto S_3	✗	✓	✗
epimorphism onto the dihedral group D_5	✗	✗	✓

The corollary follows from this table and the Isotopic Link-π_1-Lemma 3.2. ■

This approach of trying to distinguish knots by the (non-) existence of epimorphisms onto finite groups is rather ad hoc, but as we have seen, not without its merits. This motivates the following notation:

Notation. Let π be a group. We denote by $\mathcal{F}(\pi)$ the set of isomorphism classes of finite groups G for which there exists an epimorphism $\pi \to G$.

With this notation we can formulate the following generalization of the Trefoil–Figure 8–Non-Isotopic Corollary 5.9.

Proposition 5.10. (Trefoil–Figure 8–Detection Proposition) Let K be either the trivial knot, the figure-8 knot or a torus knot Furthermore let $\widetilde{K}$ be another knot. If $\mathcal{F}(\pi_1(S^3 \setminus K) = \mathcal{F}(\pi_1(S^3 \setminus \widetilde{K}))$, then $\widetilde{K}$ is smoothly isotopic to K or to the mirror K^{mir}.

Proof. This proposition is proved in [**BF20, BR20**] and [**Wil19**, Theorem B]. ■

We conclude this section with the following question which is still open.

Question 5.11. Let K and $\widetilde{K}$ be two knots in S^3. If $\mathcal{F}(\pi_1(S^3 \setminus K)) = \mathcal{F}(\pi_1(S^3 \setminus \widetilde{K}))$, does it follow that K is smoothly isotopic to $\widetilde{K}$ or to $\widetilde{K}^{\text{mir}}$?

5.5. **The Reidemeister-Schreier process.** Even though the Link Group-Abelianization Corollary 5.7 was quite disappointing, using a clever jiu-jitsu move we will see in this section that we can actually turn the Link Group-Abelianization Corollary 5.7 to our advantage. The key to doing so is contained in the following definition and the subsequent lemma.

Definition. Let π be a group and let $n \in \mathbb{N}$. We say that a subgroup $\Gamma \subset \pi$ is cocyclic of order n if Γ is a normal subgroup such that π/Γ is isomorphic to $\mathbb{Z}_n$.

[40] The reader might ask, why we do not try our luck with D_4. But for reasons we cannot get into right now, given a knot K and an even number $2n$ there cannot be an epimorphism $\pi_1(S^3 \setminus K) \to D_{2n}$.

Lemma 5.12. (Cocyclic Subgroup Lemma) If π is a group whose abelianization is isomorphic to $\mathbb{Z}$, then given any $n \in \mathbb{N}$ there exists a unique subgroup that is cocyclic of order n. It is given by the kernel of any epimorphism $\pi \to \mathbb{Z}_n$.

Proof. Let π be a group whose abelianization is isomorphic to $\mathbb{Z}$ and let $n \in \mathbb{N}$. We denote by $\psi\colon \pi \to \pi_{\text{ab}}$ the natural projection and we pick an isomorphism $\varphi\colon \pi_{\text{ab}} \xrightarrow{\cong} \mathbb{Z}$. It is clear that $(\varphi \circ \psi)^{-1}(n \cdot \mathbb{Z})$ is a subgroup of π that is cocyclic of order n. On the other hand, suppose that Γ is a subgroup of π that is cocyclic of order n. We pick an isomorphism $\theta\colon \pi/\Gamma \cong \mathbb{Z}_n$. It follows that there exists a unique epimorphism $\alpha\colon \mathbb{Z} \to \mathbb{Z}_n$ such that the following diagram commutes:

$$\begin{tikzcd}
\pi \arrow[r, "\psi", "\cong"'] \arrow[d, two heads, "g\mapsto g\Gamma"'] & \pi_{\text{ab}} \arrow[r, "\varphi", "\cong"'] & \mathbb{Z} \arrow[d, two heads, "\alpha"] \\
\pi/\Gamma \arrow[rr, "\theta", "\cong"'] & & \mathbb{Z}_n.
\end{tikzcd}$$

It follows immediately that $\Gamma = (\varphi \circ \psi)^{-1}(n \cdot \mathbb{Z})$. ■

The Link Group-Abelianization Corollary 5.7 and the Cocyclic Subgroup Lemma 5.12 allow us to make the following definition:

Definition. Given a knot K and $n \in \mathbb{N}$ we denote by $\pi(K, n)$ the unique subgroup of $\pi_1(S^3 \setminus K)$ that is cocyclic of order n.

Example. If K is the trivial knot, then we saw in the Trivial Knot-π_1-Proposition 3.6 that we have an isomorphism $\pi_1(S^3 \setminus K) \cong \mathbb{Z}$. It follows easily that for any $n \in \mathbb{N}$ we have $\pi(K, n) \cong n \cdot \mathbb{Z}$, in particular $\pi(K, n)$ is isomorphic to $\mathbb{Z}$. □

We have now a new tool for trying to distinguish knots:

Lemma 5.13. (Cocyclic Subgroup–Invariant Lemma) Let K and J be two knots. If K and J are smoothly isotopic, then given any $n \in \mathbb{N}$ the groups $\pi(K, n)$ and $\pi(J, n)$ are isomorphic. In particular the abelianizations of these groups are isomorphic.

The question arises, given a knot K and $n \in \mathbb{N}$, how can we determine $\pi(K, n)$ from a given presentation for $\pi_1(S^3 \setminus K)$? Fortunately the following proposition gives us an explicit algorithm. To facilitate the uptake we formulate the following proposition in a slightly informal way.

Proposition 5.14. (Reidemeister-Schreier Process) Let $\pi = \langle x_1, \dots, x_m, t \,|\, r_1, \dots, r_l \rangle$ be a group and let $\phi\colon \pi \to \mathbb{Z}_n$ be an epimorphism such that $\phi(x_1) = \dots = \phi(x_m) = 0$ and $\phi(t) = 1$. Then a presentation for the subgroup $\ker(\phi\colon \pi \to \mathbb{Z}_n)$ is given by

$$\langle \underbrace{y_{1,0}, \dots, y_{1,n-1}, \dots, y_{m,0}, \dots, y_{m,n-1}, u}_{m \cdot n + 1 \text{ generators}} \,|\, \underbrace{s_1^0, \dots, s_1^{n-1}, \dots, s_l^0, \dots, s_l^{n-1}}_{l \cdot n \text{ relations}} \rangle$$

where each s_i^j is obtained as follows:

(1) For $i = 1, \dots, l$ and $j = 0, \dots, n-1$ we write $t^j \cdot r_i \cdot t^{-j}$ as an expression in

$$x_1, tx_1t^{-1}, \dots, t^{n-1}x_1t^{-n+1}, \dots, x_m, tx_mt^{-1}, \dots, t^{n-1}x_mt^{-n+1} \text{ and } t^n.$$

(2) We replace each term $t^j x_i t^{-j}$ by $y_{i,j}$ and we replace t^n by u. We obtain a word in the new generators and denote it by s_i^j.

The inclusion map $\ker(\phi\colon \pi \to \mathbb{Z}_n) \to \pi$ is then given by sending each $y_{i,j}$ to $t^j x_i t^{-j}$ and by sending u to t^n.

Proof. This proposition is a special case of a solution to a more general problem: Namely, suppose that we are given a finitely presented group $\pi = \langle X \,|\, R\rangle$ and an epimorphism $\alpha\colon \pi \to G$ onto a finite group. How can we determine a presentation of $\ker(\alpha)$ from the given data?

Such an algorithm is provided from the Reidemeister-Schreier process that is explained in [**LS77**, Chapter II.4], [**MKS04**, Chapter 2.3] or [**Bog08**, Chapter 2.9]. Our proposition is a special case of this more general algorithm. ■

Instead of studying the proof of the Reidemeister-Schreier Proposition 5.14 our time is better spent trying to understand explicit examples.

Example. Let $K \subset S^3$ be the trefoil. We write $\pi = \pi_1(S^3 \setminus K)$. In the Torus Knot-$\pi_1$-Proposition 3.8 we showed that we have an isomorphism $\pi \cong \langle x, t \mid \underbrace{x^3 \cdot t^{-2}}_{=r}\rangle$. Note that there exists a unique epimorphism $\phi\colon \pi \to \mathbb{Z}_2$ with $\phi(x) = 0$ and $\phi(t) = 1$. Evidently $\pi(K,2) = \ker(\phi\colon \pi \to \mathbb{Z}_2)$. Now we will use the Reidemeister-Schreier Proposition 5.14 to find a presentation $\langle y_0, y_1, u \,|\, s_0, s_1\rangle$ for $\ker(\phi\colon \pi \to \mathbb{Z}_2)$. We write

$$\begin{aligned} t^0 \cdot r \cdot t^{-0} &= \underbrace{x^3}_{=y_0^3} \cdot \underbrace{t^{-2}}_{=u^{-1}} \\ t^1 \cdot r \cdot t^{-1} &= t^1 \cdot \underbrace{x^3 \cdot t^{-2}}_{=r} \cdot t^{-1} = t \cdot x^3 \cdot t^{-3} = (txt^{-1})^3 \cdot t^{-2} = (\underbrace{txt^{-1}}_{y_1^3})^3 \cdot \underbrace{t^{-2}}_{=u^{-1}}. \end{aligned}$$

Thus we see that

$$\pi(K,2) \cong \ker(\phi\colon \pi \to \mathbb{Z}_2) \underset{\uparrow}{\cong} \langle y_0, y_1, u \,|\, y_0^3 \cdot u^{-1}, y_1^3 \cdot u^{-1}\rangle \underset{\uparrow}{\cong} \langle y_0, y_1 \,|\, y_0^3 = y_1^3\rangle.$$

by the Reidemeister-Schreier Proposition 5.14 — the last relation implies that $y_1^3 = u$, the isomorphism follows from Tietze transformations

We can now easily compute that the abelianization of the above group $\pi(K,2) = \ker(\phi\colon \pi \to \mathbb{Z}_2)$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}_3$.

This calculation, together with the Cocyclic Subgroup–Invariant Lemma 5.13 and the above discussion of the trivial knot, gives a new proof of the Trefoil–Non-Trivial Proposition 3.10. □

Evidently we would like to continue with the figure-8 knot. But this case is harder than the trefoil. The case of the trefoil was fairly easily to handle, since we started out with a presentation where one generator got sent to one, and the other ones got sent to zero. Unfortunately this is not the case for Wirtinger presentations, so we will need to first "rearrange" the presentation. The key tool for doing so is the following lemma:

Lemma 5.15. (Presentation–Substitution Lemma) Let $X = \{x_1, \ldots, x_k\}$ be a finite set and furthermore let $r = \{r_1, \ldots, r_l\}$ be a finite subset of $\langle X\rangle = \langle x_1, \ldots, x_k\rangle$. Suppose that we can write each $r_i(x_1, \ldots, x_k)$ as a word $s_i(x_1, \ldots, x_{k-1}, x_k \cdot w)$ where w is a word in $x_1,\dots,x_{k-1}$, then the homomorphism

$$\langle x_1,\dots,x_k\,|\,r_1,\dots,r_l\rangle \;\to\; \langle x_1,\dots,x_{k-1},t\,|\,s_1,\dots,s_l\rangle$$

that is induced by $x_i\mapsto x_i$, $i=1,\dots,k-1$, and $x_k\mapsto tw^{-1}$ is an isomorphism.

Definition. We refer to the isomorphism of the Presentation–Substitution Lemma 5.15 as the isomorphism given by the substitution $t=x_k\cdot w$.

Example. A picky reader might decide that the formulation of the Presentation–Substitution Lemma 5.15 is not completely precise. But the following example should make the meaning clear:

$$\langle a,b\,|\,a^{-1}ba^{-1}bab^{-1}\rangle \;=\; \langle a,b\,|\,\underbrace{a^{-1}b\cdot a^{-1}b\cdot a\cdot(a^{-1}b)^{-1}\cdot a^{-1}}_{=a^{-1}ba^{-1}bab^{-1}}\rangle \underset{\substack{\uparrow\\ \text{substitution } c=a^{-1}b}}{=} \langle a,c\,|\,c\cdot c\cdot a\cdot c^{-1}\cdot a^{-1}\rangle.$$

□

Proof. Note that there exists in fact a unique homomorphism

$$\langle x_1,\dots,x_k\,|\,r_1,\dots,r_l\rangle \;\to\; \langle x_1,\dots,x_{k-1},t\,|\,s_1,\dots,s_l\rangle$$

that has the property that $x_i\mapsto x_i$, $i=1,\dots,k-1$ and that $x_k\mapsto t\cdot w^{-1}$. Furthermore, we have a map from the right to left by the reverse substitution $t\mapsto x_k\cdot w$. It is straightforward to verify that these homomorphisms are inverses of one another, hence both are isomorphisms. ■

In the following example we consider the figure-8 knot K and we carry out the first steps needed to calculate the isomorphism type of $\pi(K,2)$.

Example. Let K be the figure-8 knot. In the Figure 8 Knot–π_1-Lemma 5.8 we saw that we have an isomorphism

$$\pi_1(S^3\setminus K)\;\cong\;\langle y_1,y_2,y_3,y_4\,|\,y_1^{-1}y_4y_1y_3^{-1},\,y_3^{-1}y_2y_3y_1^{-1},\,y_2^{-1}y_1y_2y_4^{-1}\rangle.$$

We denote by $\phi\colon \pi_1(S^3\setminus K)\to\mathbb{Z}_2$ the epimorphism that is given by sending each y_i to 1. We set $t:=y_1$. The goal is to replace the generators y_1,y_2,y_3 by generators x_1,x_2,x_3 which satisfy $\phi(x_i)=0$. In fact we have

$$\begin{aligned}\pi_1(S^3\setminus K) &= \langle t,y_2,y_3,y_4\,|\,t^{-1}y_4ty_3^{-1},\,y_3^{-1}y_2y_3t^{-1},\,y_2^{-1}ty_2y_4^{-1}\rangle\\ &= \langle t,y_2,y_3,y_4\,|\,\underbrace{t^{-1}y_4t^{-1}tty_3^{-1}}_{=t^{-1}y_4ty_3^{-1}},\,\underbrace{t^{-1}ty_3^{-1}y_2t^{-1}ty_3t^{-1}}_{=y_3^{-1}y_2y_3t^{-1}},\,\underbrace{t^{-1}ty_2^{-1}ty_2t^{-1}ty_4^{-1}}_{=y_2^{-1}ty_2y_4^{-1}}\rangle\\ &\underset{\uparrow}{=} \langle t,x_2,x_3,x_4\,|\,t^{-1}x_4tx_3^{-1},\,t^{-1}x_3^{-1}x_2tx_3,\,t^{-1}x_2^{-1}tx_2x_4^{-1}\rangle.\end{aligned}$$

substitution $x_i=y_it^{-1}$, $x_i^{-1}=ty_i^{-1}$, see the Presentation–Substitution Lemma 5.15

At this point we have a presentation to which we can apply the Reidemeister-Schreier Proposition 5.14. A heroic calculation shows that $\pi(K,2)_{\mathrm{ab}}\cong\mathbb{Z}\oplus\mathbb{Z}_5$. Thus we obtain a new proof that the figure-8 knot and the trefoil are not smoothly isotopic. □

The above procedure for distinguishing knots works fairly well in practice. For example, with enough time at one's hand, or alternatively a convenient computer implementation, one can use the above methods to show that all the knots shown in the figure on page 55 are in fact pairwise different. In fact this is exactly what was done by James Alexander and Garland Briggs [**AB27**, p. 563] when they first classified "small knots" (without the computer though.)

Nonetheless, the above approach is not overly satisfying. It would be much better to have invariants of knots that are much quicker to calculate. We will come back to this challenge in the next chapter.

5.6. Appendix: High-dimensional knots.

Since the study of knots in S^3 was so much fun, it would be a shame not to consider high-dimensional knots.

Definition. An n-dimensional knot in S^k is a smooth submanifold of S^k that is diffeomorphic to S^n. For $n = 1$ and $k = 3$ we say that such a knot is classical. When we consider the case $n > 1$ and $k > 3$, then we say that the knot is high-dimensional.

For the most part we will now restrict ourselves to the case of codimension two knots. We start out with a boring example:

Example. Let $n, k \in \mathbb{N}$ with $k \geq n$. We refer to $U := \{(x, 0) \in S^{n+2} \subset \mathbb{R}^{k+1} \,|\, x \in S^n\}$ as the *trivial n-dimensional knot in S^k*. □

Definition. We say that an n-dimensional knot in S^k is trivial if it is smoothly isotopic to the trivial n-dimensional knot in S^k.

We continue with two unsurprising lemmas.

Lemma 5.16. (Isotopic Link-π_1-Lemma) Let $n \in \mathbb{N}$. Let K and $\widetilde{K}$ be two n-dimensional knots in S^{n+2}. If K and $\widetilde{K}$ are smoothly isotopic, then $\pi_1(S^{n+2} \setminus K) \cong \pi_1(S^{n+2} \setminus \widetilde{K})$.

Proof. The proof of this statement is verbatim the same as the proof of the Isotopic Link-π_1-Lemma 3.2. ■

Lemma 5.17. (Trivial Knot-π_1-Lemma) Let $n \in \mathbb{N}$. If K is a trivial n-dimensional knot in S^{n+2}, then $\pi_1(S^{n+2} \setminus K) \cong \mathbb{Z}$.

Proof. Let K be a trivial n-dimensional knot in S^{n+2}. We see that

$$\pi_1(S^{n+2} \setminus K) \underset{\substack{\uparrow \\ \text{by the Isotopic Link-}\pi_1\text{-Lemma 5.16}}}{\cong} \pi_1(S^{n+2} \setminus U) \overset{\substack{\text{follows from Exercise 5.20} \\ \downarrow}}{\cong} \pi_1(S^1 \times \mathbb{R}^{n+1}) \cong \pi_1(S^1) \cong \mathbb{Z}.$$

■

Evidently the question now is whether there exist non-trivial high-dimensional knots. Or perhaps better, the question is whether we will manage to explicitly construct such examples. The good news is, we can indeed do so.

In the following we will present a construction, which has its origins in the work of Emil Artin [**Art25b**] and Eric Zeeman [**Zee65**], that turns a codimension-two knot in S^{n+2} into a family of codimension-two knots in S^{n+3}.

Definition. We start out with the following preparations:

(1) We make the identification $S^{n+2} =_{\mathrm{i}} \overline{B}^{n+2}_{+} \cup_{S^{n+1}} \overline{B}^{n+2}_{-}$.
(2) By the Sphere–Solid Tori-Decomposition Lemma 3.4 we can make the identification $S^{n+3} =_{\mathrm{i}} (S^1 \times \overline{B}^{n+2}) \cup_{S^1 \times S^{n+1}} (\overline{B}^2 \times S^{n+1})$.
(3) We make the identification $\mathbb{R}^{n+2} = \mathbb{C} \times \mathbb{R}^n$. In particular we view $\overline{B}^{n+2}$ as a subspace of $\mathbb{C} \times \mathbb{R}^n$.
(4) Given $z \in S^1$ let

$$\begin{array}{rcl} \rho_z \colon \overline{B}^{n+2} & \to & \overline{B}^{n+2} \\ (w,x) & \mapsto & (z\cdot w, x) \end{array}$$

be the rotation of $\overline{B}^{n+2}$ by the "angle" $z \in S^1$ around "the $0 \times \overline{B}^{n}$-axis".

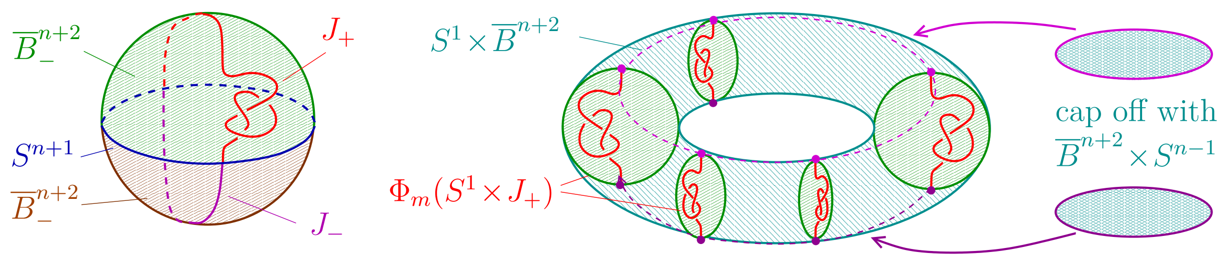
Now let $K \subset S^{n+2}$ be an n-dimensional knot. It follows fairly easily from the Smooth Ball Embedding Theorem 2.9 and the fact that K is a smooth submanifold that, after possibly applying a diffeotopy of S^{n+2}, we can assume that $J_- := K \cap \overline{B}_-^{n+2} = \{(0,w)\,|\,w \in \overline{B}^{n}\}$ is the "trivial disk knot" in $\overline{B}^{n+2}$. We denote by $J_+ := K \cap \overline{B}_+^{n+2}$ the "other disk knot". Next, given $m \in \mathbb{Z}$ we denote by Φ_m the diffeomorphism[41]

$$\begin{array}{rcl} \Phi_m \colon S^1 \times \overline{B}^{n+2} & \to & S^1 \times \overline{B}^{n+2} \\ (z,x) & \mapsto & (z, \rho_{z^m}(x)). \end{array}$$

Note that Φ_0 is the identity[42] and also note that each Φ_m leaves the subset $S^1 \times S^{n-1}$ invariant as a subset. Thus we can define the m-twist spin $S_m(K)$ as follows:

$$S_m(K) \;:=\; \underbrace{\Phi_m(S^1 \times J_+)}_{\subset S^1\times \overline{B}^{n+2}} \cup_{S^1\times S^{n-1}} \underbrace{(\overline{B}^2 \times S^{n-1})}_{\subset \overline{B}^2\times S^{n+1}} \;\subset\; \underbrace{(S^1 \times \overline{B}^{n+2}) \cup_{S^1\times S^{n+1}} (\overline{B}^n \times S^{n+1})}_{=S^{n+3}}.$$

In the figure below we do our best to illustrate this construction for $n = 1$. Again informally speaking, $S_m(K)$ is given by spinning the disk knot J_+ around the S^1-direction, performing m spins around J_+ as you go around S^1, and then capping off the result by $\overline{B}^2 \times S^{n-1}$.



As we will see, it follows easily from the following proposition that, perhaps unsurprisingly, some twist spins are non-trivial. Perhaps more surprisingly the proposition also says that (± 1)-twist spins are always trivial.

Proposition 5.18. (Spun Knot-π_1-Proposition) Let $K \subset S^{n+2}$ be an n-dimensional knot.

(1) For the 0-twist spin there exists an isomorphism $\pi_1(S^{n+3} \setminus S_0(K)) \cong \pi_1(S^{n+2} \setminus K)$.
(2) For every $\epsilon \in \{-1, 1\}$ the ϵ-twist spin knot $S_\epsilon(K)$ is trivial.

[41] Informally speaking Φ_m spins $\overline{B}^{n+2}$ altogether m times as we go around the S^1-direction.
[42] As a start it suffices to consider the trivial case $m = 0$.

Sketch of proof. Let $K \subset S^{n+2}$ be an n-dimensional knot.

(1) We have the following isomorphisms:[43]

it follows from the Seifert–van Kampen Theorem 3.9, that this map is an isomorphism, note that here we use that $\Phi_0(S^1 \times J_+) = S^1 \times J_+$

$$\begin{tikzcd}
\pi_1(S^{n+3} \setminus S_0(K)) & \pi_1(S^1 \times (\overline{B}^{n+2} \setminus J_+)) *_{\pi_1(S^1 \times (S^{n+1} \setminus S^{n-1}))} \pi_1(\overline{B}^2 \times (S^{n+1} \setminus S^{n-1})) \arrow[l, "\cong"'] \arrow[d, "\cong"'] \\
& (\langle t \rangle \times \pi_1(\overline{B}^{n+2} \setminus J_+)) *_{\langle s \rangle \times \langle t \rangle} \langle s \rangle \\
& \pi_1(\overline{B}^{n+2} \setminus J_+) \arrow[u, "\cong"] \arrow[d, "\cong"'] \\
& \pi_1(S^{n+2} \setminus K).
\end{tikzcd}$$

isomorphisms given by the Product-π_1-Proposition

isomorphism by the group theoretic fact that $\Gamma \xrightarrow{\cong} (\langle t \rangle \times \Gamma) *_{\langle s \rangle \times \langle t \rangle} \langle s \rangle$

isomorphism by the Seifert–van Kampen-Theorem 3.9, see the argument on page 46

(2) This statement is proved implicitly in [**Zee65**, p. 486] and explicitly in [**FO15**, Corollary 2.1]. The reason for writing [**FO15**] was that it felt easier to prove the statement, than to try to read the proofs written down by others. So we warmly recommend to try to prove the statement on your own instead of reading the proofs in the literature. ■

Corollary 5.19. (Non-Trivial High-Dimensional Knot Existence Corollary) Given any $n \in \mathbb{N}$ there exists an n-dimensional knot in S^{n+2} that is non-trivial.

Proof. Let $K \subset S^3$ be the trefoil and let $n \in \mathbb{N}$. We see that

$$\pi_1(S^{n+2} \setminus (n-1)\text{-st iterated 0-twist spin of } K) \underset{\substack{\uparrow \\ \text{by the Spun Knot-}\pi_1\text{-Proposition 5.18}}}{\cong} \pi_1(S^3 \setminus K) \overset{\substack{\text{by the Torus Knot-}\pi_1\text{-Proposition 3.8 and} \\ \text{the Trefoil–Non-Trivial Proposition 3.10} \\ \downarrow}}{\not\cong} \mathbb{Z} \underset{\substack{\uparrow \\ \text{by the Trivial Knot-}\pi_1\text{-Lemma 5.17}}}{=} \pi_1(S^{n+2} \setminus U).$$

It follows from the Isotopic Link-π_1-Lemma 5.16 that the $(n-1)$-st iterated 0-twist spin of the trefoil is a non-trivial knot. ■

Exercises for Chapter 5.

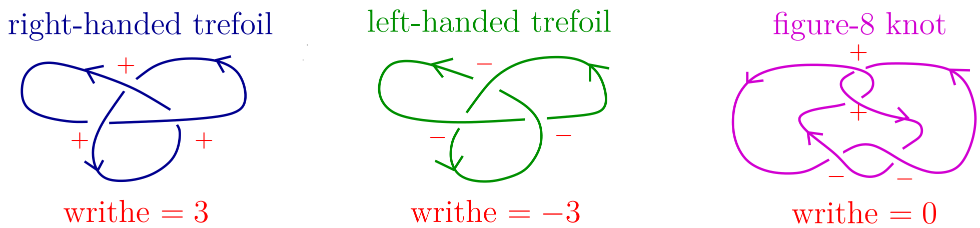
Exercise 5.1. Use the HNN–Gluing Theorem 5.1 to calculate the fundamental groups of the torus and the Klein bottle, just using the fact that $\pi_1(S^1) \cong \mathbb{Z}$.

Exercise 5.2. Let X be a path-connected topological space that admits a non-empty path-connected subspace S such that $X \setminus S$ is also non-empty and path-connected. Does $\pi_1(X)$ admit an epimorphism onto $\mathbb{Z}$?

[43] By now the reader will be fully aware of the fact that it is supremely painful to completely keep track of all identifications, base points, induced maps and so on. Thus we also took some liberties to improve readability.

Exercise 5.3. Show that if a knot diagram has at least one crossing, then the number of crossings equals the number of strands.

Exercise 5.4. Given a diagram D of a knot K we pick an orientation and we define the writhe $w(D)$ of D as the sum of the signs of the crossings.



(a) Show that the writhe of a knot is independent of the orientation.
(b) Let D and $\widetilde{D}$ be two diagrams for the same knot $K \subset S^3$. Show that the following two statements are equivalent:
  (i) D and $\widetilde{D}$ are related by sequence of Reidemeister moves 2 and 3 (and all the obvious variations thereof).
  (ii) The writhes of D and $\widetilde{D}$ agree.

  Remark. Of course you should make use of the Reidemeister Moves Theorem 4.5.

Exercise 5.5. Let π be a group. We say that a subset $S \subset \pi$ *normally generates* π if $\langle\langle S \rangle\rangle = \pi$. We define the *weight of* π as

$$w(\pi) := \min\{\#S \mid S \text{ is a subset of } \pi \text{ that normally generates } \pi\} \in \mathbb{N}_0 \cup \{\infty\}.$$

(a) Determine the weight of $\mathbb{Z}^m$.
(b) Determine the weight of the free group on m generators.
(c) Let $L \subset S^3$ be an m-component link. Show that $w(\pi_1(S^3 \setminus L)) = m$.

Exercise 5.6. Let K_1 and K_2 be two oriented knots in S^3. Show that for $i = 1, 2$ there exists an epimorphism $\pi_1(S^3 \setminus (K_1 \# K_2)) \to \pi_1(S^3 \setminus K_i)$.
Remark. Evidently you should make use of the Knot Connected Sum-π_1-Proposition 3.15. But what result from the current chapter do you also need?

Exercise 5.7. Give a purely algebraic proof that the two groups $\langle x, y \mid x^2 = y^3 \rangle$ and $\langle a, b, c \mid c^{-1}bca^{-1}, b^{-1}abc^{-1} \rangle$ are isomorphic.
Remark. You could use Tietze transformations and you could use the Presentation–Substitution Lemma 5.15.

Exercise 5.8.
(a) Let K be an oriented knot with diagram D and let $\widetilde{K}$ be an oriented knot with diagram $\widetilde{D}$. How can you obtain a diagram for $K \# \widetilde{K}$ from D and $\widetilde{D}$?
(b) Let K and $\widetilde{K}$ be two oriented knots in S^3. Use the Wirtinger[44] Presentation Proposition 5.5 to show that there exists an isomorphism

$$\pi_1(S^3 \setminus (K \# \widetilde{K})) \;\cong\; \pi_1(S^3 \setminus K, x_0) *_{\mu_K = \mu_{\widetilde{K}}} \pi_1(S^3 \setminus \widetilde{K}, \widetilde{x}_0).$$

  Remark. This gives a new proof of the Knot Connected Sum-π_1-Proposition 3.15.

Exercise 5.9. Given $m \in \mathbb{N}$ we denote by $m \cdot K$ the m-fold connected sum of the trefoil (with a fixed orientation). Show that for $m \neq n \in \mathbb{N}$ the knots $m \cdot K$ and $n \cdot K$ are not

smoothly isotopic.
Hint. Use Exercise 5.6 and count the number of epimorphisms from fundamental groups onto the permutation group S_3.

Exercise 5.10. We start out with two definitions:

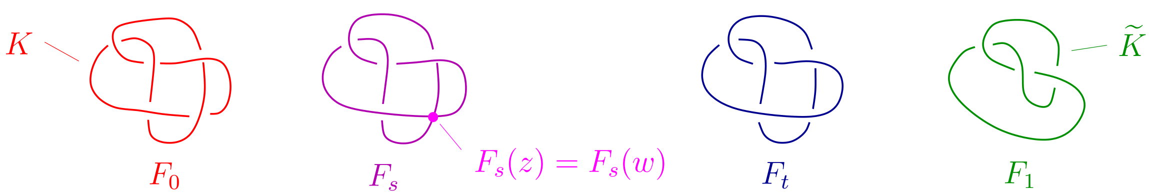
- The *deficiency* of a finite presentation $\langle x_1, \dots, x_k \,|\, r_1, \dots, r_l \rangle$ is defined to be $k - l$.
- Let π be a finitely presented group. We define the *deficiency* $\operatorname{def}(\pi)$ *of* π as the maximal deficiency of a finite presentation for π.

Let $m \in \mathbb{N}$. What are the possible deficiencies of the groups $\pi_1(S^3 \setminus L)$ where L is an m-component link?

Exercise 5.11. We say that two knots K and $\widetilde{K}$ in $\mathbb{R}^3$ are related by a *crossing change* if there exists a smooth map $F\colon S^1 \times [0,1] \to \mathbb{R}^3$ and some $s \in (0,1)$ with the following properties:

(1) We have $F_0(S^1) = K$ and $F_1(S^1) = \widetilde{K}$.
(2) For every $t \neq s$ the map $F_t\colon S^1 \to S^3$ is an embedding.
(3) The map $F_s\colon S^1 \to S^3$ is an immersion and there exist $w, z \in S^1$ with the following properties:
   (i) the restriction of F_s to $S^1 \setminus \{w, z\}$ is an injection,
   (ii) $F_s(w) = F_s(z)$ and the vectors $F_s'(w)$ and $F_s'(z)$ are linearly independent.

In the figure below we illustrate the definition of a crossing change.

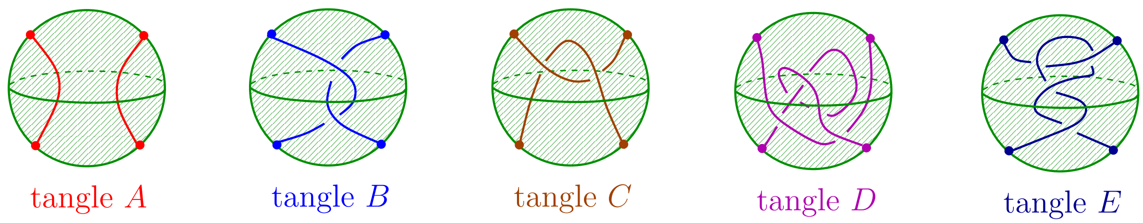


Given two knots K and $\widetilde{K}$ the minimal number of crossing changes needed to turn K into $\widetilde{K}$ is called the *Gordian distance between* K *and* $\widetilde{K}$. The Gordian distance of a knot K to the trivial knot is called the *unknotting number of* K. (In practice these numbers are notoriously difficult to calculate.)

(a) Show that any two knots in $\mathbb{R}^3$ are related by a sequence of finitely many crossing changes. In other words, show that the Gordian distance between two knots is finite.
   Hint. You could use the fact that every knot admits a knot diagram.
(b) What are the best lower and upper bounds you can find on the Gordian distance between the trefoil and the figure-8 knot?

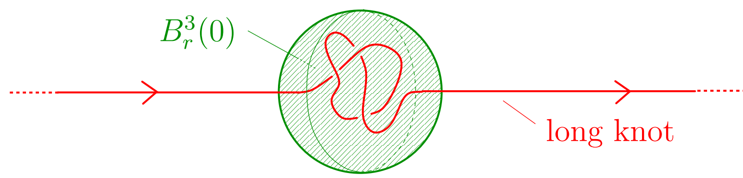
Exercise 5.12. A *tangle* is a proper 1-dimensional submanifold of $\overline{B}^3$. We consider the five tangles shown in the figure below. Note that the boundary of each of the five tangles is the same.

(a) For which pairs S, T of the five tangles does there exist an orientation-preserving diffeomorphism $\varphi\colon \overline{B}^3 \to \overline{B}^3$ with $\varphi(S) = T$?
(b) For which pairs S, T of the five tangles does there exist an orientation-preserving diffeomorphism $\varphi\colon \overline{B}^3 \to \overline{B}^3$ with $\varphi(S) = T$ and with $\varphi|_{S^2} = \mathrm{id}_{S^2}$?



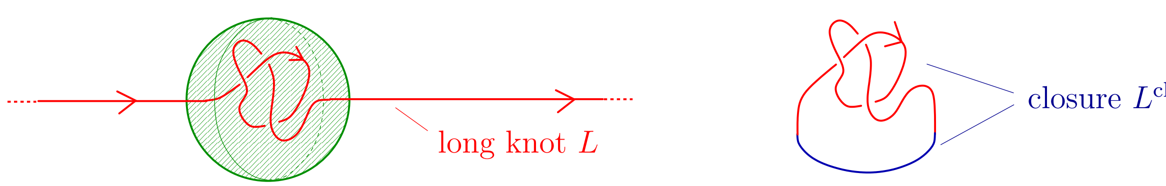
Exercise 5.13. We start out with a few definitions.

(1) A *long knot* is a submanifold K of $\mathbb{R}^3$ that is diffeomorphic to $\mathbb{R}$ and which has the property that there exists an $r > 0$ such that
$$K \cap (\mathbb{R}^3 \setminus B^3_r(0)) \;=\; (\mathbb{R} \times \{(0,0)\}) \cap (\mathbb{R}^3 \setminus B^3_r(0)).$$
A long knot comes with a natural orientation given by "going from $-\infty$ to ∞". We refer to $U := \mathbb{R} \times \{0,0\}$ as the *trivial long knot*.



(2) In this exercise we say that two long knots K and J are *equivalent* if there exists a smooth isotopy $F\colon \mathbb{R} \times [0,1] \to \mathbb{R}^3$ such that $F_0(\mathbb{R}) = K$, $F_1(\mathbb{R}) = J$ and such that there exists a $C > 0$ such that for each $t \in [0,1]$ and each $x \in \mathbb{R}$ with $|x| \geq C$ we have $F_t(x) = (x,0,0)$.

(3) Let L be a long knot. We pick an $r > 0$ such that
$$L \cap B^3_r(0) \;\subset\; \mathbb{R} \times \{(0,0)\}.$$
We denote by
$$\begin{array}{rcl} \Phi\colon \overline{B}^3_{2r}(0) & \to & S^3 \\ x & \mapsto & \left(\frac{x}{2r}, \sqrt{1 - |\frac{x}{2r}|^2}\right) \end{array}$$
the obvious smooth embedding whose image is precisely the upper hemisphere. We refer to
$$L^{\mathrm{cl}} \;:=\; \Phi\big(L \cap \overline{B}^3_{2r}(0)\big) \cup \big\{\big(x,0,0,\sqrt{1-x^2}\,\big)\,|\, x \in [-1,1]\big\}$$
as the *closure* L^{cl} *of the long knot* L. Note that the closure inherits an orientation from L.



Now we turn to the actual exercises:

(a) There is a pretty obvious definition of a composition $L\#L'$ of long knots L and L'. Turn this idea into a proper definition.
(b) Show that $L\#L'$ and $L'\#L$ are equivalent.
(c) Show that the fundamental groups of $\mathbb{R}^3 \setminus L$ and $S^3 \setminus L^{\mathrm{cl}}$ are isomorphic.

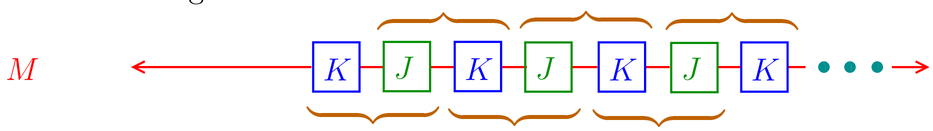
(d) Show that the closure L^{cl} is well-defined up to smooth isotopy and show that the map

$$\begin{array}{rcl} \{\text{long knots up to equivalence}\} & \to & \{\text{oriented knots up to smooth isotopy}\} \\ [L] & \mapsto & [L^{\mathrm{cl}}] \end{array}$$

is a bijection.

Exercise 5.14. We continue with the discussion of long knots from Exercise 5.14.

(a) Let K and J be two long knots such that $K\#J$ is equivalent to the trivial long knot. We consider the 1-dimensional smooth submanifold M of $\mathbb{R}^3$ that is shown in the figure below. We denote by $U = \mathbb{R} \times \{(0,0)\}$ the trivial long knot. Show that there exists a diffeomorphism $\Phi\colon (\mathbb{R}^3, M) \to (\mathbb{R}^3, U)$ and show that there exists a diffeomorphism $\Phi\colon (\mathbb{R}^3, M) \to (\mathbb{R}^3, K)$.
Hint. Make use of the fact that by Exercise 5.13 (b) we know that $J\#K$ is also equivalent to the long trivial knot.



Remark. The braces are supposed to give you an idea for the proof.

(b) Let $K \subset S^3$ be an oriented knot. Suppose that there exists an oriented knot $J \subset S^3$ such that $K\#J$ is the trivial knot. Show that there exists a *homeomorphism* $\Theta\colon S^3 \to S^3$ such that $\Theta(K)$ equals the trivial knot.

Remark. This argument is known as the “Mazur swindle” [**Maz60**].

Exercise 5.15.

(a) We consider the epimorphism

$$\phi\colon \mathbb{Z}_2 * \mathbb{Z}_2 = \langle a, b \,|\, a^2, b^2 \rangle \;\to\; \mathbb{Z}_2$$

that is given by $\phi(a) = \phi(b)$. Show that $\ker(\phi)$ is isomorphic to $\mathbb{Z}$.
Hint. Use the Presentation–Substitution Lemma 5.15 and the Reidemeister-Schreier Process 5.14.

(b) We consider the connected sum $\mathbb{R}\mathrm{P}^3\#\mathbb{R}\mathrm{P}^3$ of two copies of the real projective space $\mathbb{R}\mathrm{P}^3$. Note that $\pi_1(\mathbb{R}\mathrm{P}^3\#\mathbb{R}\mathrm{P}^3) \cong \mathbb{Z}_2 * \mathbb{Z}_2$. What is the covering of $\mathbb{R}\mathrm{P}^3\#\mathbb{R}\mathrm{P}^3$ corresponding to $\ker(\phi)$?

Exercise 5.16. Let K be a knot. Show that $\pi_1(S^3 \setminus K)$ admits an epimorphism onto the permutation group S_3 if and only if K is colorable in the sense of Exercise 4.7

Exercise 5.17. Let A and B be two finitely generated abelian groups with $\mathcal{F}(A) = \mathcal{F}(B)$. In other words, we assume that A and B have the same finite quotients. Are A and B necessarily isomorphic?

Exercise 5.18. Let K be the trefoil. Compute the abelianization of the group $\pi(K, 3)$.

Exercise 5.19.

(a) Let K and $\widetilde{K}$ be two oriented knots. What is the connection between $\pi(K, 2)$, $\pi(\widetilde{K}, 2)$ and $\pi(K\#\widetilde{K}, 2)$?

(b) Show that given any $n \in \mathbb{N}$ there exists a knot such that $\pi_1(S^3 \setminus K)$ has at least n generators.

Exercise 5.20. Let $n \in \mathbb{N}$ and let $k \in \{1, \ldots, n-1\}$. We consider the submanifold $K := \{(x, 0) \in \mathbb{R}^{n+1} \,|\, x \in S^k\}$. Show that $S^n \setminus K$ is diffeomorphic to $S^{n-k-1} \times \mathbb{R}^{k+1}$.

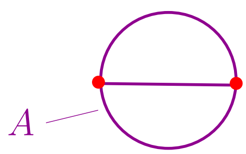
Exercise 5.21. Let $K \subset S^3$ be an oriented knot and let $m \in \mathbb{Z}$. As on page 80 we consider the m-twist spin $S_m(K) \subset S^4$. We view S^3 as a subset of S^4 via the embedding $z \mapsto (z, 0)$. The intersection $S_m(K) \cap S^3$ is a knot. What is the smooth isotopy type of this knot? A possible answer is that this knot is a connected sum of K with some suitable other knot.

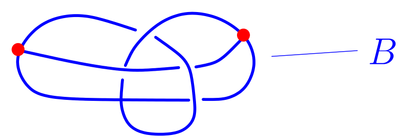
Exercise 5.22. Let $m \in \mathbb{N}$. Show that there exist two 2-component links $L = L_1 \sqcup L_2$ and $\widetilde{L} = \widetilde{L}_1 \sqcup \widetilde{L}_2$ in S^{2m+1} with $\dim(L_1) = \dim(L_2) = \dim(\widetilde{L}_1) = \dim(\widetilde{L}_2) = m$ but such that L and $\widetilde{L}$ are not smoothly isotopic.

Exercise 5.23. We consider the "spatial graphs" $A \subset \mathbb{R}^3$ and $B \subset \mathbb{R}^3$ that are shown in the figure below.

(a) (i) Determine a presentation for $\pi_1(\mathbb{R}^3 \setminus A)$.
(ii) What is the abelianization of $\pi_1(\mathbb{R}^3 \setminus A)$?

(b) Replace A by B and again solve (i) and (ii).





CHAPTER 6

Linking numbers

In this short chapter we introduce the linking number of two disjoint oriented knots. The linking number is one of the most basic invariants of topology, which we will encounter on several occasions and in different guises throughout these notes.

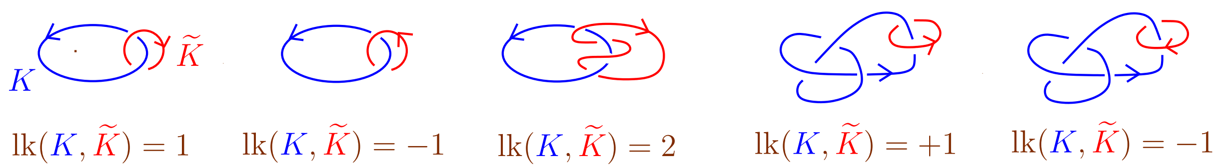
6.1. Definition of Linking numbers and basic properties.

The Link Group-Abelianization Corollary 5.7 allows us to introduce the following basic and rather useful definition:

Definition. Let K and $\widetilde{K}$ be two disjoint oriented knots. By the Link Group-Abelianization Corollary 5.7 we have $\pi_1(S^3 \setminus K)_{\mathrm{ab}} = \mathbb{Z} \cdot \mu_K$. Note that it follows from the Loop-π_1-Lemma 3.14 and the discussion on page 45 that $\widetilde{K}$ gives rise to a well-defined element of $\pi_1(S^3 \setminus K)_{\mathrm{ab}}$ which we denote by $[\widetilde{K}]$. We now define the linking number $\mathrm{lk}(K, \widetilde{K})$ to be the unique integer such that the following equality holds in $\pi_1(S^3 \setminus K)_{\mathrm{ab}} = \mathbb{Z} \cdot \mu_K$:

$$[\widetilde{K}] \;=\; \mathrm{lk}(K, \widetilde{K}) \cdot \mu_K \;\in\; \pi_1(S^3 \setminus K)_{\mathrm{ab}} = \mathbb{Z} \cdot \mu_K.$$

Example. In the figure below we show some basic examples of linking numbers:

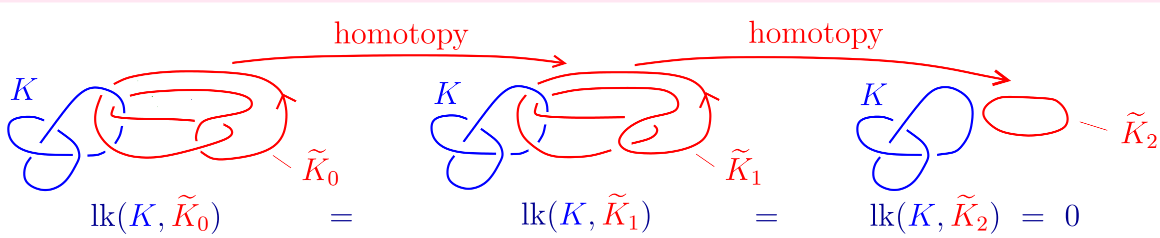


These result from the observation that for the trivial knot K we have $\pi_1(S^3 \setminus K) = \mathbb{Z} \cdot [\mu_K]$ and from the fact that for any oriented knot K we clearly have $\mathrm{lk}(K, \mu_K) = 1$. $\square$

In the following lemmas we collect some basic facts about linking numbers:

Lemma 6.1. (Linking Number-Homotopy-Lemma) Let K be an oriented knot and let $\widetilde{K}$ and $\widetilde{K}'$ be two oriented knots in the complement of K. If $\widetilde{K}$ and $\widetilde{K}'$ are homotopic in the complement of K, i.e. if there exists a continuous map $F\colon \widetilde{K} \times [0,1] \to S^3 \setminus K$ such that $F_0 = \mathrm{id}$ and such that $F_1\colon \widetilde{K} \to \widetilde{K}'$ is an orientation-preserving diffeomorphism, then

$$\mathrm{lk}(K, \widetilde{K}) \;=\; \mathrm{lk}(K, \widetilde{K}').$$



Proof. It follows easily from our hypothesis that $[\widetilde{K}] = [\widetilde{K}'] \in \pi_1(S^3 \setminus K)_{\mathrm{ab}}$. It follows immediately from this observation that $\mathrm{lk}(K, \widetilde{K}) = \mathrm{lk}(K, \widetilde{K}')$. $\blacksquare$

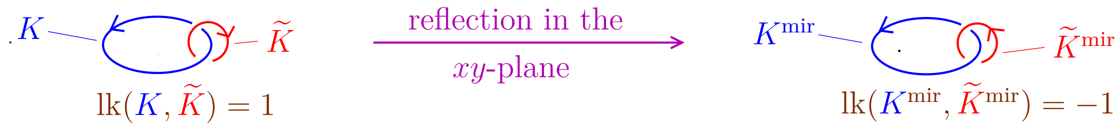
Lemma 6.2. (Linking Number–Orientation Lemma) For any disjoint oriented knots K and $\widetilde{K}$ we have

$$\underbrace{\mathrm{lk}(K^{\mathrm{rev}},\widetilde{K}) \;=\; \mathrm{lk}(K,\widetilde{K}^{\mathrm{rev}})}_{\text{knots with orientation reversed}} \;=\; -\,\mathrm{lk}(K,\widetilde{K}).$$

Proof. The lemma follows easily from the definitions and the following two facts:

(1) It follows from the Meridian–Symmetries Lemma 2.18 together with the Loop-π_1-Lemma 3.14 (3) that $[\mu_{K^{\mathrm{rev}}}] = [(\mu_K)^{\mathrm{rev}}] = -[\mu_K] \in \pi_1(S^3\setminus K)_{\mathrm{ab}}$.
(2) It follows from the Loop-π_1-Lemma 3.14 (3) that $[\widetilde{K}^{\mathrm{rev}}] = -[\widetilde{K}] \in \pi_1(S^3\setminus K)_{\mathrm{ab}}$. ■

Lemma 6.3. (Linking Number–Mirror Image Lemma) For any two disjoint oriented knots K and $\widetilde{K}$ we have
$$\mathrm{lk}(K^{\mathrm{mir}},\widetilde{K}^{\mathrm{mir}}) \;=\; -\mathrm{lk}(K,\widetilde{K}).$$



Proof. Let K and $\widetilde{K}$ be two disjoint oriented knots. Let $\rho\colon S^3\to S^3$ be the reflection in a hyperplane. We have the following equalities in $\pi_1(S^3\setminus K^{\mathrm{mir}})_{\mathrm{ab}}$:

$$\begin{aligned}
\mathrm{lk}(K^{\mathrm{mir}},\widetilde{K}^{\mathrm{mir}})\cdot\mu_{K^{\mathrm{mir}}} &\overset{\substack{\text{definition of } \mathrm{lk}(K^{\mathrm{mir}},\widetilde{K}^{\mathrm{mir}})\\ \downarrow}}{=} [\widetilde{K}^{\mathrm{mir}}] \overset{\substack{\text{definition of } K^{\mathrm{mir}}\\ \downarrow}}{=} [\rho(\widetilde{K})] \overset{\substack{\text{definition of } \mathrm{lk}(K,\widetilde{K})\\ \downarrow}}{=} [\rho(\mathrm{lk}(K,\widetilde{K})\cdot\mu_K)]\\
&= [\mathrm{lk}(K,\widetilde{K})\cdot\rho(\mu_K)] = \mathrm{lk}(K,\widetilde{K})\cdot[\rho(\mu_K)] \underset{\substack{\uparrow\\ \text{by the Meridian–Symmetries Lemma 2.18 (2) we know}\\ \text{that a meridian for } K^{\mathrm{mir}}=\rho(K) \text{ is given by } \rho(\mu_K)^{\mathrm{rev}},\\ \text{the equality now follows from the Loop-}\pi_1\text{-Lemma 3.14 (3)}}}{=} \mathrm{lk}(K,\widetilde{K})\cdot(-\mu_{K^{\mathrm{mir}}}).
\end{aligned}$$

From this calculation it follows that $\mathrm{lk}(K^{\mathrm{mir}},\widetilde{K}^{\mathrm{mir}}) = -\mathrm{lk}(K,\widetilde{K})$. ■

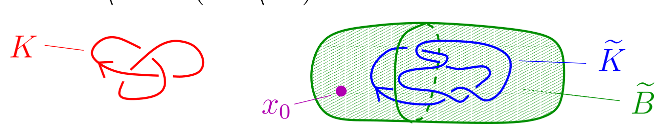
The following lemma implies that if the linking number of two disjoint knots is non-zero, then they form a non-splittable link:

Lemma 6.4. (Linking Number-of-Split Link Lemma) Let K and $\widetilde{K}$ be two disjoint oriented knots. If the link $K\sqcup\widetilde{K}$ is splittable, then $\mathrm{lk}(K,\widetilde{K})=0$.

Proof. Let K and $\widetilde{K}$ be two disjoint oriented knots such that the link $K\sqcup\widetilde{K}$ is splittable. It follows almost immediately from the definition of a splittable link on page 20 that we can write $S^3 = B\cup\widetilde{B}$ where B and $\widetilde{B}$ are smooth submanifolds with the following properties:

(a) B and $\widetilde{B}$ are diffeomorphic to $\overline{B}^3$.
(b) We have $B\cap\widetilde{B} = \partial B = \partial\widetilde{B}$.
(c) $K\subset B\setminus\partial B$ and $\widetilde{K}\subset\widetilde{B}\setminus\partial\widetilde{B}$.

Note that by (c) we have $S^3\setminus K = (B^3\setminus K)\cup\widetilde{B}$.

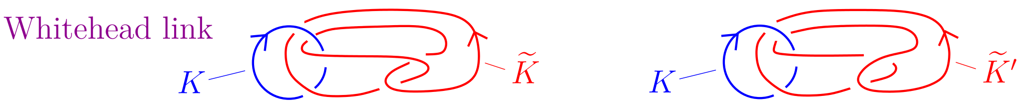


It now follows from (b) and the Seifert-van Kampen Theorem 3.9 that for any $x_0 \in \partial B = \partial\widetilde{B}$ the inclusions $B^3 \setminus K \to S^3 \setminus K$ and $\widetilde{B} \to S^3 \setminus K$ induce an isomorphism

$$\pi_1(B^3 \setminus K, x_0) * \pi_1(\widetilde{B}, x_0) \;\to\; \pi_1(S^3 \setminus K, x_0).$$

Since $\pi_1(\widetilde{K}) \to \pi_1(\widetilde{B}) \cong \{e\}$ is the trivial map we now see that $\pi_1(\widetilde{K}) \to \pi_1(S^3 \setminus K)$ is the trivial map. This implies that $[\widetilde{K}]$ is the trivial element in $\pi_1(S^3 \setminus K, x_0)$. This in turn implies that $\operatorname{lk}(K, \widetilde{K}) = 0$. ■

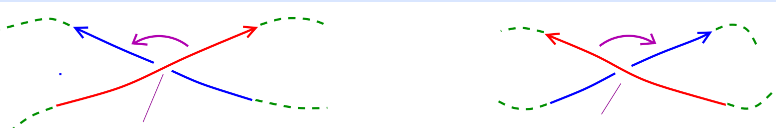
Example. It is not completely unreasonable to ask whether the converse to the Linking Number-of-Split Link Lemma 6.4 holds. In the figure below we show two disjoint oriented knots K and $\widetilde{K}$. Note that in the complement of K the closed curve $\widetilde{K}$ is *homotopic* to the closed curve $\widetilde{K}'$ that is shown in the figure to the right.



As above on page 87 it follows easily from the above figure and the Linking Number-Homotopy-Lemma 6.1 that $\operatorname{lk}(K, \widetilde{K}) = 0$. It seems unlikely that $K \sqcup \widetilde{K}$ is splittable. But as of right now we lack the tools to prove that the link is indeed not splittable. We will deal with this issue in Exercise 9.12.

6.2. **Linking numbers via diagrams.** Our next goal is to find a practical way for calculating linking numbers. Before we can achieve this goal it is convenient to remind ourselves of the definition of positive and negative crossings from page 66:

Definition. A crossing in a link diagram is called positive (respectively negative) if the direction of the upper strand followed by the direction of the lower strand give a positive (respectively negative) basis for $\mathbb{R}^2$.



From the definition it is not clear how to calculate the linking number. To calculate the linking number we need a way to describe elements in a Wirtinger presentation. This leads us to the following lemma, which is of more general interest:

Lemma 6.5. (Element-in-Wirtinger Presentation Lemma) Let L be an oriented link and let $J \subset S^3 \setminus L$ be an oriented knot. We assume that the link $L \sqcup J$ is associated to a link diagram such that each component has at least one crossing. We consider the base point $\diamond = (0, 0, 2)$. Let

$$\langle x_1, \ldots, x_n \,|\, r_1, \ldots, r_{n-1} \rangle \xrightarrow{\cong} \pi_1(S^3 \setminus L, \diamond)$$

be the explicit isomorphism from the Wirtinger Presentation Proposition 5.5. We pick a base point P on J, away from a crossing. We do the following:

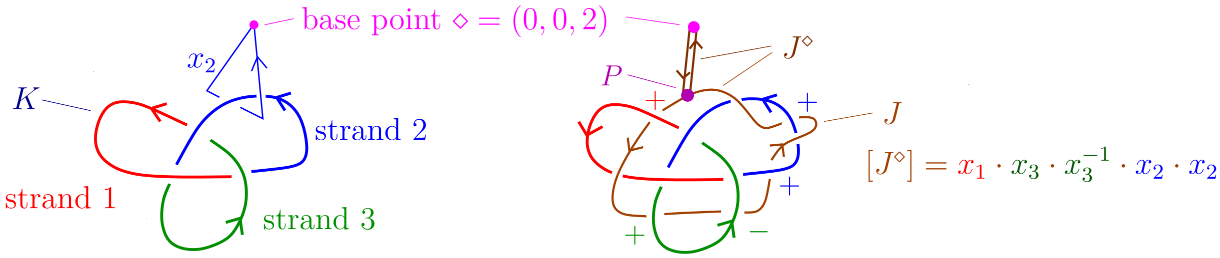
(a) We denote by $J^\diamond$ the loop that goes straight from $\diamond$ to P, goes once around J in the given direction, and that finally goes straight back to $\diamond$.

(b) We start with the empty word. We walk along J in the positive orientation, starting and ending at P. Whenever we go under L (at the strand i of L) we do the following:
- We right-multiply our word with x_i if it is a positive crossing.
- We right-multiply our word with x_i^{-1} if it is a negative crossing.

We denote the resulting word by $w_J \in \langle x_1, \ldots, x_n \rangle$.

Under the above isomorphism the element $[w_J] \in \langle x_1, \ldots, x_n \,|\, r_1, \ldots, r_{n-1} \rangle$ corresponds to $[J^\diamond] \in \pi_1(S^3 \setminus L, \diamond)$.

Example. In the following figure we consider the trefoil K with the usual diagram and we show an oriented knot $J \subset S^3 \setminus K$:



On page 68 we showed that we have an isomorphism

$$\langle x_1, x_2, x_3 \,|\, x_2^{-1} \cdot x_1 \cdot x_2 \cdot x_3^{-1}, x_3^{-1} \cdot x_2 \cdot x_3 \cdot x_1^{-1} \rangle \xrightarrow{\cong} \pi_1(S^3 \setminus K).$$

It follows almost immediately from the Element-in-Wirtinger Presentation Lemma 6.5 that $[J^\diamond] = x_1 \cdot x_3 \cdot x_3^{-1} \cdot x_2 \cdot x_2 = x_1 \cdot {x_2}^2$.

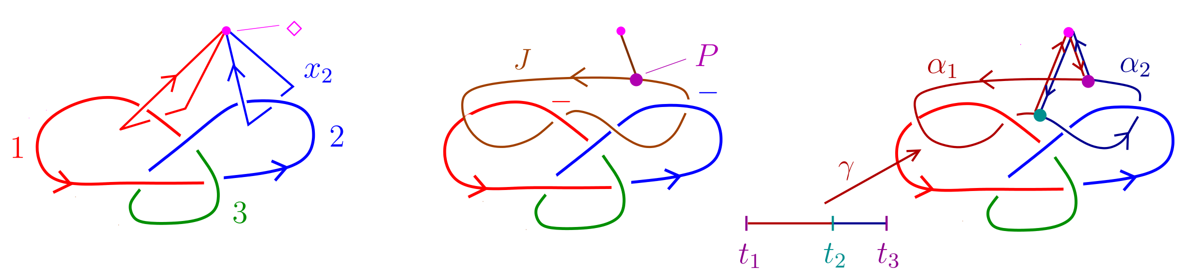
Sketch of proof. We make the following preparations:

(1) For $i \in \{1, \ldots, n\}$ we denote by x_i the loop in $\diamond$ that goes once around the i-th strand in the usual way.

(2) We pick a smooth map $\gamma\colon [0,1] \to S^3$ with the following properties:
  (a) We have $\gamma(0) = \gamma(1) = P$.
  (b) γ descends to a diffeomorphism $\gamma\colon [0,1]/0 \sim 1 \to J$.

(3) We pick $0 = t_1 < t_1 < \cdots < t_{m+1} = 1$ such that for each $j \in \{1, \ldots, m\}$ the image $\gamma([t_j, t_{j+1}]$ contains precisely one point where we go under L. Let $\sigma_j \in \{1, \ldots, n\}$ be the corresponding strand. Furthermore we set $\epsilon_j := +1$ if the crossing is positive and we set $\epsilon_j := -1$ if the crossing is negative.

(4) For $j = 1, \ldots, m$ we denote by α_j the loop that starts in $\diamond$, goes straight to $\gamma(t_j)$, goes along γ to $\gamma(t_{j+1})$ and then goes straight back to $\diamond$.



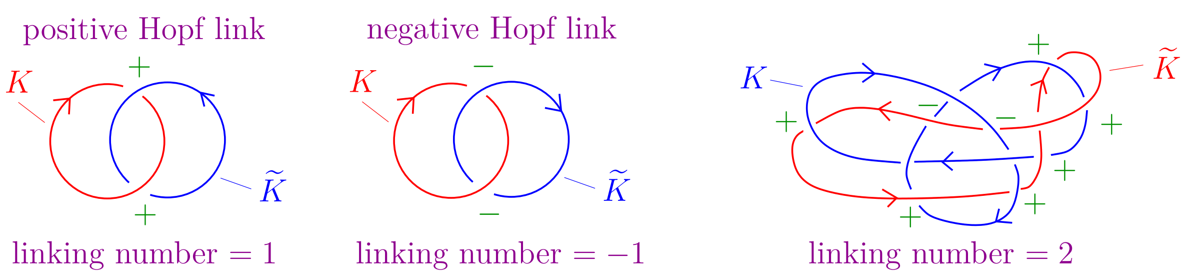
Now we see that in $\pi_1(S^3 \setminus L, \diamond)$ we have the following equalities:

$$[J^\diamond] \underset{\substack{\uparrow \\ \text{since the loops } J^\diamond \text{ and} \\ \alpha_1 * \cdots * \alpha_m \text{ are homotopic}}}{=} [\alpha_1 * \cdots * \alpha_m] \underset{\substack{\uparrow \\ \text{definition of the} \\ \text{product in } \pi_1(S^3\setminus L,\diamond)}}{=} [\alpha_1] \cdot \ldots \cdot [\alpha_m] \underset{\substack{\uparrow \\ \text{since the loop } \alpha_i \text{ is} \\ \text{homotopic to } x_{\sigma_i}^{\epsilon_i}}}{=} [x_{\sigma_1}^{\epsilon_1}] \cdot \ldots \cdot [x_{\sigma_m}^{\epsilon_m}] \underset{\substack{\uparrow \\ \text{by definition we have} \\ x_{\sigma_1}^{\epsilon_1} \cdot \ldots \cdot x_{\sigma_m}^{\epsilon_m} = w_J}}{=} [w_J]. \qquad \blacksquare$$

Using the Element-in-Wirtinger Presentation Lemma 6.5 we can now prove the following popular proposition:

Proposition 6.6. (Linking Number-via-Crossings Proposition) Let K and $\widetilde{K}$ be two disjoint oriented knots. If the link $K \sqcup \widetilde{K} \subset S^3$ is associated to a link diagram, then

$$\operatorname{lk}(K, \widetilde{K}) \;=\; \begin{array}{rl} & \frac{1}{2} \cdot \text{number of positive crossings between } K \text{ and } \widetilde{K} \\ - & \frac{1}{2} \cdot \text{number of negative crossings between } K \text{ and } \widetilde{K}. \end{array}$$



Proof. We introduce the following notation: We denote by $U(\widetilde{K}, K)$ the set of crossings where $\widetilde{K}$ crosses under K and we denote by $O(\widetilde{K}, K)$ the set of crossings where $\widetilde{K}$ crosses over K. For each crossing c of K and $\widetilde{K}$ we denote by $\epsilon_c \in \{-1, 1\}$ the sign of the crossing. We start out with the following claim:

Claim 1. We have
$$\operatorname{lk}(K, \widetilde{K}) \;=\; \sum_{c \in U(\widetilde{K}, K)} \epsilon_c.$$

Proof. We consider the base point $\diamond = (0, 0, 2)$. Let

$$\langle x_1, \ldots, x_n \,|\, r_1, \ldots, r_n \rangle \;\xrightarrow{\cong}\; \pi_1(S^3 \setminus K, \diamond)$$

be the explicit isomorphism from the Wirtinger Presentation Proposition 5.5. Recall that each x_i is a meridian for K. Now we want to apply the Element-in-Wirtinger Presentation Lemma 6.5 to $\widetilde{K}$. Thus we do the following: We pick a base point P on $\widetilde{K}$ and we walk along $\widetilde{K}$ in the positive orientation. At every undercrossing with K (at the strand i of K), we record x_i if it is a positive crossing and we record x_i^{-1} if it is a negative crossing. In summary we get a word of the form $x_{\sigma_1}^{\epsilon_1} \cdot \ldots \cdot x_{\sigma_m}^{\epsilon_m}$ with $\sigma_1, \ldots, \sigma_m \in \{1, \ldots, n\}$ and $\epsilon_1, \ldots, \epsilon_m \in \{-1, 1\}$. We now see that we have the following equalities in the abelian group $\pi_1(S^3 \setminus K; \diamond)_{\text{ab}} = \mathbb{Z} \cdot \mu_K$:

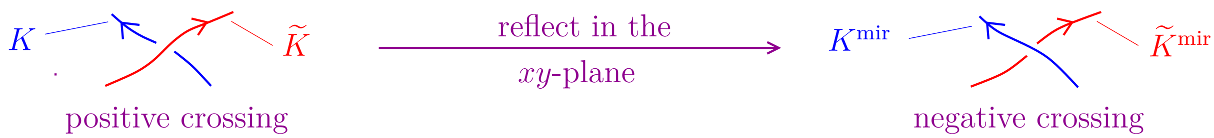
$$\operatorname{lk}(K, \widetilde{K}) \cdot \mu_K \underset{\substack{\uparrow \\ \text{by definition} \\ \text{of } \operatorname{lk}(K,\widetilde{K})}}{=} [\widetilde{K}^\diamond] \underset{\substack{\uparrow \\ \text{by the Element-in-Wirtinger} \\ \text{Presentation Lemma 6.5}}}{=} [x_{\sigma_1}^{\epsilon_1} \cdot \ldots \cdot x_{\sigma_m}^{\epsilon_m}] \overset{\substack{\text{we use additive notation} \\ \downarrow}}{=} \sum_{j=1}^{m} \epsilon_j \cdot [x_{\sigma(j)}] \underset{\substack{\uparrow \\ \text{since each } x_i \\ \text{is a meridian}}}{=} \sum_{j=1}^{m} \epsilon_j \cdot \mu_K \;=\; \Big(\sum_{c \in U(\widetilde{K}, K)} \epsilon_c \Big) \cdot \mu_K.$$

It follows that $\operatorname{lk}(K, \widetilde{K}) = \sum_{c \in U(\widetilde{K}, K)} \epsilon_c$. $\boxplus$

Claim 2. We have
$$\operatorname{lk}(K,\widetilde{K}) \;=\; \sum_{c\in O(\widetilde{K},K)} \epsilon_c.$$

Proof. Let K^{mir} and $\widetilde{K}^{\text{mir}}$ be the reflections of K and $\widetilde{K}$ in the xy-plane. Note that $K^{\text{mir}} \cup \widetilde{K}^{\text{mir}}$ has an obvious diagram with the following properties:

- The points where $\widetilde{K}$ crosses over K become points where $\widetilde{K}^{\text{mir}}$ crosses under K^{mir}.
- The signs of the crossings flip.



We now see that
$$\operatorname{lk}(K,\widetilde{K}) \underset{\uparrow}{=} -\operatorname{lk}(K^{\text{mir}},\widetilde{K}^{\text{mir}}) \underset{\uparrow}{=} -\sum_{c\in U(\widetilde{K}^{\text{mir}},K^{\text{mir}})} \epsilon_c \underset{\uparrow}{=} -\sum_{c\in O(\widetilde{K},K)} (-\epsilon_c) = \sum_{c\in O(\widetilde{K},K)} \epsilon_c.$$
by the Linking Number Mirror Image Lemma 6.3 — by Claim 1 — since reflections turns over- into undercrossings and since reflections flip signs of crossings ⊞

In summary it follows that

$$\begin{aligned}\operatorname{lk}(K,\widetilde{K}) &\overset{\text{by Claim 1 and Claim 2}}{\overset{\downarrow}{=}} \tfrac{1}{2}\cdot\Big(\sum_{c\,\in\, U(\widetilde{K},K)} \epsilon_c + \sum_{c\,\in\, O(\widetilde{K},K)} \epsilon_c\Big) = \tfrac{1}{2}\cdot\Big(\sum_{c\,\in\, U(\widetilde{K},K)\,\cup\, O(\widetilde{K},K)} \epsilon_c\Big)\\ &= \tfrac{1}{2}\cdot\left(\begin{array}{rl} & \text{number of positive crossings between } K \text{ and } \widetilde{K}\\ - & \text{number of negative crossings between } K \text{ and } \widetilde{K}\end{array}\right).\end{aligned}$$
■

Using the Linking Number-via-Crossings Proposition 6.6 we can easily prove the following proposition:

Proposition 6.7. (Linking Number-Symmetry Proposition) For any disjoint oriented knots K and $\widetilde{K}$ we have
$$\operatorname{lk}(\widetilde{K},K) \;=\; \operatorname{lk}(K,\widetilde{K}).$$

Proof. Let K and $\widetilde{K}$ be two disjoint oriented knots. By the Link Diagram Existence Proposition 4.3 and the One Crossing Exists Lemma 5.4 we know that there exists a diagram for the link $K \sqcup \widetilde{K}$ such that each component has at least one crossing. We can thus use the Linking Number-via-Crossings Proposition 6.6 to calculate the linking numbers of K and $\widetilde{K}$. The symmetry of the linking numbers now follows from the observation that the formula in the Linking Number-via-Crossings Proposition 6.6 is symmetric in K and $\widetilde{K}$. ■

Exercises for Chapter 6.

Exercise 6.1. Let K and K' be oriented knots and let J be a knot in the complement of K and K'. Show that if K and K' are homotopic in the complement of J, then
$$\operatorname{lk}(K,J) \;=\; \operatorname{lk}(K',J).$$

Exercise 6.2. Provide an example of an oriented 3-component link $L = L_1 \sqcup L_2 \sqcup L_3$ such that for any $i \neq j$ the following two statements hold:

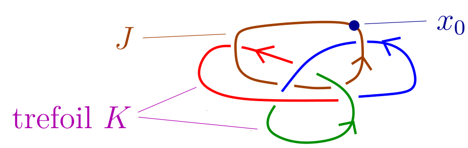
(1) The linking number $\operatorname{lk}(L_i, L_j)$ is zero.

(2) The link $L_i \sqcup L_j$ is non-trivial.

Exercise 6.3. We consider the pairs of knots $K_1, \widetilde{K}_1$ and $K_2, \widetilde{K}_2$ shown in the figure below.
(a) Determine $\operatorname{lk}(K_1, \widetilde{K}_1)$.
(b) Determine $\operatorname{lk}(K_2, \widetilde{K}_2)$.

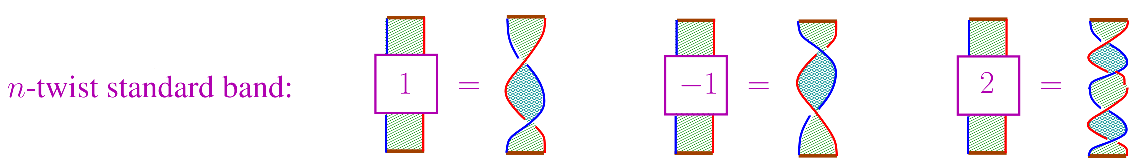
Exercise 6.4. For which matrices $A = (a_{ij}) \in \mathrm{M}(n \times n, \mathbb{Z})$ does there exist an oriented n-component link $L = L_1 \sqcup \cdots \sqcup L_n$ such that for any $i \neq j$ we have $\operatorname{lk}(L_i, L_j) = a_{ij}$?

Exercise 6.5. We consider the trefoil K and the oriented knot $J \subset S^3 \setminus K$ that is shown in the figure below. Let $x_0 \in J$. Show that $[J] \in \pi_1(S^3 \setminus K, x_0)$ is non-trivial.
Hint. You could try to find an epimorphism $\varphi\colon \pi_1(S^3 \setminus K, x_0) \to G$ onto a finite group G such that $\varphi([J])$ is non-trivial.

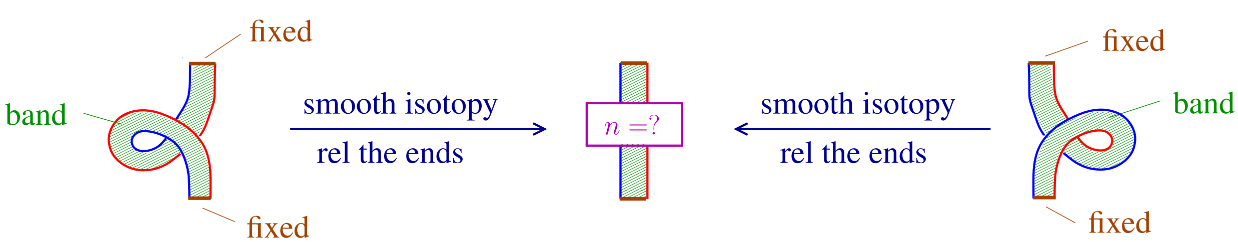


Exercise 6.6. In the literature the Linking Number-via-Crossings Proposition 6.6 often gets used as a definition of the linking number. Use the Reidemeister Moves Theorem 4.5 to make sense of this approach. More precisely, verify by hand that the right hand side of the Linking Number-via-Crossings Proposition 6.6 is unchanged under any of the Reidemeister moves.

In the last two exercises we look at "bands" in $\mathbb{R}^3$ where the two brown ends are fixed. In the figure below we show the definition of the *n-twist standard band*:



Exercise 6.7. We consider the two bands that are shown in the figure below to the left and right. For each of the two bands determine the $n \in \mathbb{Z}$ such that the band is smoothly isotopic (rel the ends) to the n-twist standard band.



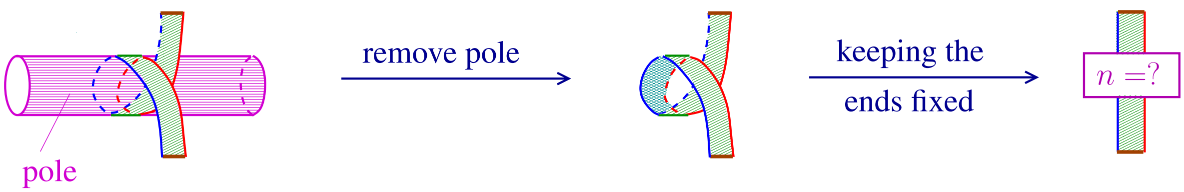
Exercise 6.8. Suppose we are given a "horizontal pole". We want to wrap a band around the pole such that the ends are fixed as before, such that the band is "tight" and such that the band is tangent to the pole. We then remove the pole and we consider the question, whether the resulting band is twisted or not.

(a) For the band shown in the figure below, what is the corresponding number of twists in the standard band?

(b) How do you have to change the wrapping to get the opposite number of twists?

(c) Can you wrap the band around the pole several times in such a way that the band, which we obtain from removing the pole, is untwisted?

Solution. Yes. First wrap couple of times in one way, then go underneath the band, but otherwise wrap the same way.



CHAPTER 7

Longitudes and coloring polynomials

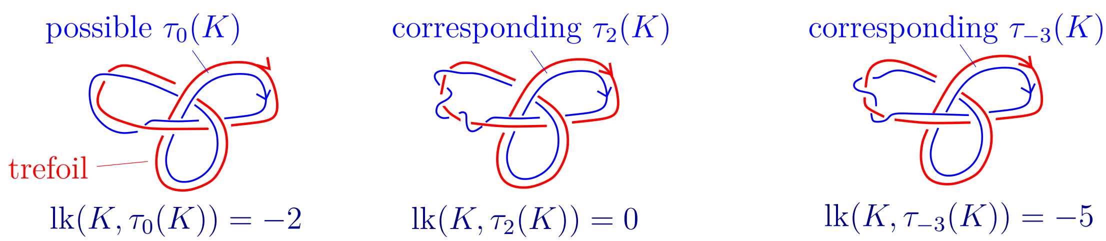
In this chapter we will use the linking number, which we defined in the previous chapter, to introduce the "longitude" of an oriented knot. We will use the pair of a meridian and a longitude to define "coloring polynomials" of oriented knots. These are fairly simply minded but surprisingly powerful invariants of knots.

7.1. **Longitudes.** The following lemma is the key to introducing the longitude of an oriented knot:

Lemma 7.1. (Sputnik Lemma) Let K be an oriented knot, let $\tau\colon \overline{B}^2 \times K \to S^3$ be an orientation-preserving tubular map and let $\gamma\colon S^1 \to K$ be an orientation-preserving diffeomorphism.

(1) Given any $m \in \mathbb{Z}$ we define the knot $\tau_m(K)$ as the image of the smooth embedding
$$\begin{array}{rcl} S^1 & \to & S^3 \\ z & \mapsto & \tau(z^m, \gamma(z)). \end{array}$$
For any $m, n \in \mathbb{Z}$ we have $\operatorname{lk}(K, \tau_n(K)) - \operatorname{lk}(K, \tau_m(K)) = n - m$.

(2) There exists a unique $m \in \mathbb{Z}$ with $\operatorname{lk}(K, \tau_m(K)) = 0$.



Proof. Let $k \in \mathbb{Z}$. We introduce the following loops
$$\begin{array}{rcl} \gamma_k\colon [0,1] & \to & S^1 \times S^1 \\ t & \mapsto & (e^{2\pi i t \cdot k}, e^{2\pi i t}), \end{array} \qquad \begin{array}{rcl} \alpha\colon [0,1] & \to & S^1 \times S^1 \\ t & \mapsto & (e^{2\pi i t}, 1) \end{array} \quad \text{and} \quad \begin{array}{rcl} \beta\colon [0,1] & \to & S^1 \times S^1 \\ t & \mapsto & (1, e^{2\pi i t}). \end{array}$$
Note that these loops define elements in $\pi_1(S^1 \times S^1, (1,1))$. Now let K be an oriented knot, let $\tau\colon \overline{B}^2 \times K \to S^3$ be an orientation-preserving tubular map and let $\gamma\colon S^1 \to K$ be an orientation-preserving diffeomorphism.

(1) Let $m, n \in \mathbb{Z}$. We have the following equalities in $\pi_1(S^3 \setminus K, \tau(1,1))_{\text{ab}}$:
$$\begin{array}{rcl} \operatorname{lk}(K, \tau_n(K)) \cdot \mu_K \overset{\substack{\text{definition of } \operatorname{lk}(K,\tau_n(K)) \\ \downarrow}}{=} [\tau_n(K)] & = & [(\tau \circ \gamma_n)(S^1)] \qquad\qquad = \tau_*([\gamma_n]) \overset{\substack{\text{this equality in } \pi_1(S^1 \times S^1, (1,1)) \text{ follows} \\ \text{the calculation of } \pi_1 \text{ of a torus} \\ \downarrow}}{=} \tau_*([\alpha^n] \cdot [\beta]) \\ & = & \tau_*([\alpha^{n-m}] \cdot [\alpha^m] \cdot [\beta]) \quad = \tau_*([\alpha^{n-m}]) + \tau_*([\gamma_m]) \\ & \underset{\substack{\uparrow \\ \text{definition of } \mu_K}}{=} & \mu_K^{n-m} + [(\tau \circ \gamma_m)(S^1)] \underset{\substack{\uparrow \\ \text{definition of } \operatorname{lk}(K,\tau_m(K))}}{=} (n - m + \operatorname{lk}(K, \tau_m(K))) \cdot \mu_K. \end{array}$$

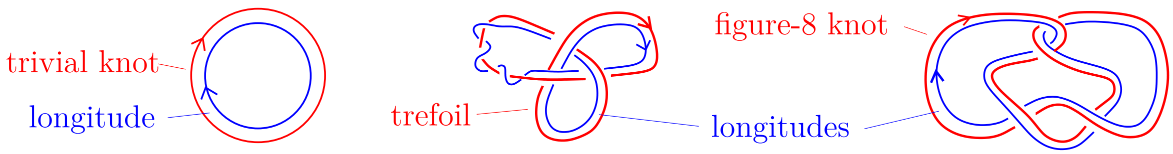
It follows that $\operatorname{lk}(K, \tau_n(K)) = n - m + \operatorname{lk}(K, \tau_m(K))$.

(2) This statement follows immediately from (1). ■

On page 27 we introduced the meridian of an oriented knot. We now introduce its loyal partner, namely the longitude:

Definition. Let K be an oriented knot. Recall that by the Link Tubular Map Theorem 2.15 there exists an orientation-preserving tubular map $\tau\colon \overline{B}^2 \times K \to S^3$. By the Sputnik Lemma 7.1 there exists a unique $m \in \mathbb{Z}$ with $\operatorname{lk}(K, \tau_m(K)) = 0$. We refer to $\tau_m(K)$ as the longitude λ_K of K.

Example. Loosely speaking the longitude runs parallel to the knot K. But to ensure that the linking number of K and its longitude λ_K is zero we might have to introduce some twists around K.



Using the Linking Number-via-Crossings Proposition 6.6 one can easily verify that in the above figure the blue knots are indeed meridians of the red knots. □

Lemma 7.2. (Longitude Lemma) Every oriented knot K admits a longitude and any two longitudes are smoothly isotopic in the complement of K.

Sketch of proof. The existence follows immediately from the Link Tubular Map Theorem 2.15 (1) together with the existence statement of the Sputnik Lemma 7.1. The uniqueness follows easily from the Link Tubular Map Theorem 2.15 (2) and the uniqueness statement of the Sputnik Lemma 7.1. We leave it to the reader to fill in the details. ■

7.2. **Longitudes and symmetries.** We start out with recalling the following definitions from pages 18 and 18:

Definition. Let K be an oriented knot.

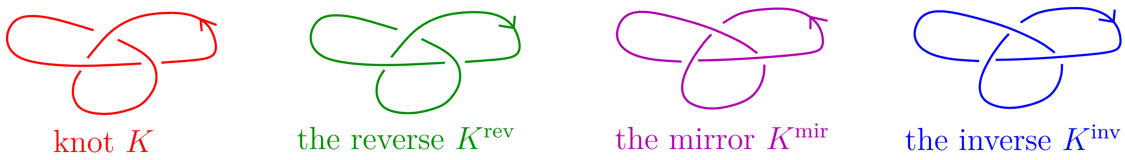
(1) The reverse K^{rev} is defined as the knot K with the reverse orientation.
(2) The mirror K^{mir} is defined as the reflection of K in any hyperplane of $\mathbb{R}^4$.[45] [46] We give the mirror K^{mir} the orientation that turns the reflection into an orientation-preserving diffeomorphism $K \to K^{\text{mir}}$.
(3) The inverse K^{inv} is defined as the mirror of the reverse of K (which of course equals the reverse of the mirror).[47]

The following lemma is an extension of the Meridian–Symmetries Lemma 2.18:

[45] Note that we showed in the Link Mirror Lemma 2.7 that, up to a smooth isotopy, the definition of K^{mir} does not depend on the choice of the hyperplane.

[46] Note that on page 18 we introduced the mirror of an unoriented knot. For the purpose of this section it is better to work with oriented knots.

[47] The names "reverse" and "inverse" at times also get used with different meanings. We follow [**Con70**, p. 336] for our naming convention.



Lemma 7.3. (Meridian-Longitude–Symmetries Lemma) Let $K \subset S^3$ be an oriented knot with knot exterior X_K and meridian $\mu_K \subset \partial X_K$ and longitude $\lambda_K \subset \partial X_K$. Furthermore let $\rho\colon S^3 \to S^3$ be the reflection in a hyperplane. We obtain the following table:

	meridian	longitude
original knot K	μ_K	λ_K
reverse K^{rev}	μ_K^{rev}	λ_K^{rev}
mirror image $K^{\text{mir}} = \rho(K)$	$\rho(\mu_K)^{\text{rev}}$	$\rho(\lambda_K)$
inverse $K^{\text{inv}} = \rho(K)^{\text{rev}}$	$\rho(\mu_K)$	$\rho(\lambda_K)$.

Proof. First note that the statements regarding the inverse follow immediately from the statements regarding the reverse and the mirror. We proved the statements regarding the meridians in the Meridian–Symmetries Lemma 2.18. The statement about the longitude of K^{rev} follows easily from the Linking Number–Orientation Lemma 6.3 and the statement about the longitude of K^{mir} follows equally easily from the Linking Number–Mirror Image Lemma 6.3. ■

Now the question arises whether an oriented knot is smoothly isotopic to any of its doppelgänger. This leads us to the following definitions from pages 18 and 19:

Definition. Let K be an oriented knot.

(1) We say that K is reversible if K is smoothly isotopic to its reverse.
(2) We say that K is amphichiral if K is smoothly isotopic to its mirror image. Otherwise we call the knot K chiral.[48]
(3) We say that K is invertible if K is smoothly isotopic to its inverse.

Example. As we saw in the figure on page 18, it is basically clear that the trivial knot is amphichiral and reversible. Furthermore, in the figure on page 19 we showed that the figure-8 knot J is amphichiral. Also note that in Exercise 2.1 we showed that the trefoil and the figure-8 knot are reversible. □

Theses examples lead us to the following two questions:

Question 2.6. Is every knot reversible?

Question 2.8. Is the trefoil chiral?

Before we show how to answer both questions let us introduce the following simple-minded definition:

Definition.

(1) A group-pair system is a triple (G, g_1, g_2) consisting of a group G together with two elements $g_1, g_2 \in G$.
(2) An isomorphism between group-pair systems (G, g_1, g_2) and (H, h_1, h_2) is a group isomorphism $\varphi\colon G \to H$ with $\varphi(g_1) = h_1$ and $\varphi(g_2) = h_2$.

The following definition gives us the group-pair system we care about most:

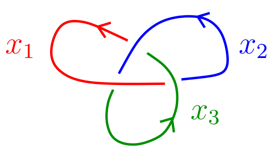
Definition. Let $K \subset S^3$ be an oriented knot. By the Link Tubular Map Theorem 2.15 there exists an orientation-preserving tubular map $\tau\colon \overline{B}^2 \times K \to S^3$.

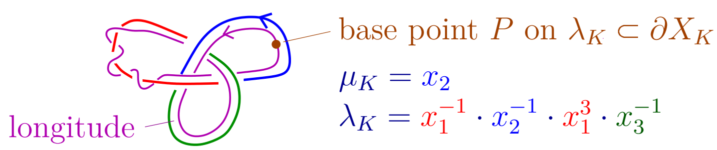
(1) As on page 26 we refer to $X_K := S^3 \setminus \tau(B^2 \times K)$ as the exterior of K.
(2) We use the tubular map to define the meridian $\mu_K \subset \partial X_K$ as on page 27 and the longitude $\lambda_K \subset \partial X_K$ as on page 96.
(3) Let $x \in K$. We refer to the group-pair system

$$(\pi_1(X_K, \tau(1,x)), \underbrace{\mu_K, \lambda_K}_{\text{group elements by the Loop-}\pi_1\text{-Lemma 3.14}})$$

as the group-pair system of the oriented knot K. Note that it follows easily from the Link Tubular Map Theorem 2.15 (2) that the isomorphism type of this group-pair system is an invariant of the oriented knot K.

Example. We consider the following knot diagram for the trefoil K:





We pick a base point $P \in \lambda_K \subset \partial X_K$. As on page 68 we get the presentation

$$\langle x_1, x_2, x_3 \,|\, x_2^{-1} \cdot x_1 \cdot x_2 \cdot x_3^{-1}, x_3^{-1} \cdot x_2 \cdot x_3 \cdot x_1^{-1} \rangle \xrightarrow{\cong} \pi_1(S^3 \setminus K, P) \underset{\text{by the Link Exterior Lemma 2.16 (4)}}{\xleftarrow{\cong}} \pi_1(X_K, P)$$

where x_1, x_2, x_3 are meridians that correspond to the three strands. It follows easily from the discussion of the longitude of the trefoil on page 96 and the Element-in-Wirtinger Presentation Lemma 6.5 that $\mu_K = x_2$ and $\lambda_K = x_1^{-1} \cdot x_2^{-1} \cdot x_1^3 \cdot x_3^{-1}$. In summary, the group-pair system of K is isomorphic to the group-pair system

$$(\underbrace{\langle x_1, x_2, x_3 \,|\, x_2^{-1} \cdot x_1 \cdot x_2 \cdot x_3^{-1}, x_3^{-1} \cdot x_2 \cdot x_3 \cdot x_1^{-1} \rangle}_{\cong \pi_1(S^3 \setminus K, P) \cong \pi_1(X_K, P)}, \underbrace{x_2}_{=\mu_K}, \underbrace{x_1^{-1} \cdot x_2^{-1} \cdot x_1^3 \cdot x_3^{-1}}_{=\lambda_K}).$$

□

The following proposition can be viewed as a refinement of the Isotopic Link-π_1-Lemma 3.2 and the Mirror Link-π_1-Lemma 3.3:

Proposition 7.4. (Knot–Group-Pair System Proposition)

(1) Let K and J be two oriented knots. If K and J are smoothly isotopic, then the corresponding group-pair systems are isomorphic.
(2) Given an oriented knot K with a group-pair system $(\pi_K, \mu_K, \lambda_K)$ the following hold:
    (a) The group-pair system of the reverse K^{rev} is isomorphic to $(\pi_K, \mu_K^{-1}, \lambda_K^{-1})$.
    (b) The group-pair system of the mirror K^{mir} is isomorphic to $(\pi_K, \mu_K^{-1}, \lambda_K)$.
    (c) The group-pair system of the inverse K^{inv} is isomorphic to $(\pi_K, \mu_K, \lambda_K^{-1})$.

Sketch of proof.

(1) This statement follows easily from the Isotopy Extension Theorem 2.4.

(2) This statement follows basically immediately from the Meridian-Longitude–Symmetries Lemma 7.3 together with the Loop-π_1-Lemma 3.14 (3). ■

7.3. Coloring polynomial. The Knot–Group-Pair System Proposition 7.4 raises the following question: How we can show that two group-pair systems are not isomorphic? On page 75 we saw that looking at finite quotients of groups can be a convenient tool for distinguishing groups. We now apply the same idea to group-pair systems. There are various ways of making this idea precise. We will use an approach due to Michael Eisermann [**Eis07**].

We start out with the following definition:

Definition. Let G be a group. We consider the group ring

$$\mathbb{Z}[G] \;:=\; \Big\{ \sum_{i=1}^{m} r_i \cdot g_i \,\Big|\, r_1, \dots, r_m \in \mathbb{Z} \text{ and } g_1, \dots, g_m \in G \Big\}.$$

We equip this abelian group with the involution that is induced by $g \mapsto g^{-1}$, more precisely, we set

$$\overline{\sum_{i=1}^{m} r_i \cdot g_i} \;:=\; \sum_{i=1}^{m} r_i \cdot g_i^{-1}.$$

We can now introduce a new knot invariant:

Definition. Let G be a *finite* group and let $x \in G$.

(1) Given a group-pair system (π, g_1, g_2) where π is a finitely generated group we define

$$\mathrm{P}^x_G(\pi, g_1, g_2) \;:=\; \overset{\text{finite sum by Exercise 7.3}}{\overset{\downarrow}{\sum_{\{\varphi \in \mathrm{Hom}(\pi, G)\,|\,\varphi(g_1) = x\}}}} \varphi(g_2) \;\in\; \mathbb{Z}[G].$$

It is clear that isomorphic group-pair systems give the same element in $\mathbb{Z}[G]$.

(2) Let K be an oriented knot. Following [**Eis07**, Definition 1.2] we refer to

$$\mathrm{P}^x_G(K) \;:=\; \mathrm{P}^x_G(\text{group-pair system } (\pi_1(X_K), \mu_K, \lambda_K) \text{ of } K) \;\in\; \mathbb{Z}[G]$$

as the (G, x)-coloring polynomial of K.

The following lemma collects a few basic facts about coloring polynomials:

Lemma 7.5. (Coloring Polynomial Lemma) Let G be a *finite* group and let $x \in G$.

(1) If K and $\widetilde{K}$ are smoothly isotopic oriented knots, then $\mathrm{P}^x_G(K) = \mathrm{P}^x_G(\widetilde{K})$.

(2) Let K be an oriented knot.

(a) For the reverse K^{rev} we have $\mathrm{P}^x_G(K^{\mathrm{rev}}) = \overline{\mathrm{P}^{x^{-1}}_G(K)}$

(b) For the mirror K^{mir} we have $\mathrm{P}^x_G(K^{\mathrm{mir}}) = \mathrm{P}^{x^{-1}}_G(K)$.

(c) For the inverse K^{inv} we have $\mathrm{P}^x_G(K^{\mathrm{inv}}) = \overline{\mathrm{P}^x_G(K)}$.

Proof.

(1) This statement follows easily from the Knot–Group-Pair System Proposition 7.4 (1).

(2) In the following we perform the calculation for (a). The other statements are proved in a similar fashion. So let K be an oriented knot. We write $\pi_K := \pi_1(X_K)$. We see that

$$\overline{\mathrm{P}_G^{x^{-1}}(K)} = \overline{\sum_{\substack{\varphi\in\mathrm{Hom}(\pi_K,G)\\ \varphi(\mu_K)=x^{-1}}} \varphi(\lambda_K)} \underset{\substack{\uparrow\\ \text{since } \varphi \text{ is a}\\ \text{homomorphism}}}{=} \sum_{\substack{\varphi\in\mathrm{Hom}(\pi_K,G)\\ \varphi(\mu_K^{-1})=x}} \varphi(\lambda_K^{-1}) \underset{\substack{\uparrow\\ \text{by the Knot–Group-Pair System}\\ \text{Proposition 7.4 (2a)}}}{=} \mathrm{P}_G^x(K^{\mathrm{rev}}).$$

■

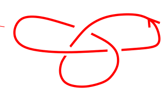
The art now is to find, given a knot, a finite group G and element $x \in G$ such that the (G, x)-coloring polynomial has content. In the following examples and results we will see how this can be done in practice.

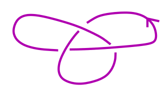
As a first application we can now finally answer Question 2.8 in the affirmative:

Proposition 7.6. (Trefoil-is-Chiral Proposition) The trefoil is chiral.

Proof. We equip the trefoil K with either orientation. We consider the alternating group $G = A_5$ and the element $x = (1\,2\,3\,4\,5) \in A_5$. An elementary (but somewhat lengthy) calculation shows that there exist precisely six epimorphisms $\pi_1(S^3 \setminus K) \to A_5$ which send the meridian to $x \in A_5$.

left handed trefoil K
$\mathrm{P}_{A_5}^x(K) = 1 + 5x$

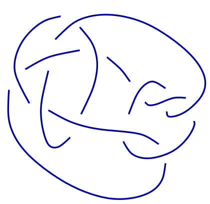




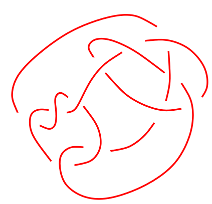
right handed trefoil K^{mir}
$\mathrm{P}_{A_5}^x(K^{\mathrm{mir}}) = 1 + 5x^{-1}$

One can then calculate that $\mathrm{P}_{A_5}^x(K) = 1 + 5x$ and[49] $\mathrm{P}_{A_5}^x(K^{\mathrm{mir}}) = 1 + 5x^{-1}$ (see [**Eis07**, p. 307]). Since $x \neq x^{-1} \in A_5$ we obtain from the Coloring Polynomial Lemma 7.5 (1) that K is chiral. ■

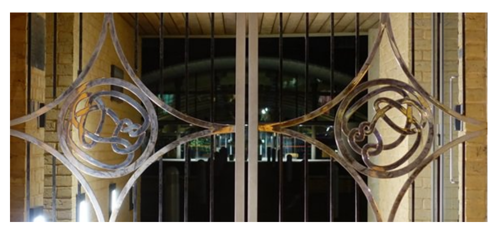
Example. We consider the Conway knot and the Kinoshita-Terasaka knot that are shown in the figure below[50]



Conway knot



Kinoshita-Terasaka knot



Newton institute Cambridge

This pair of knots is notoriously hard to distinguish. Now let G be the Mathieu group M_{11}, this is the unique simple group of order $7920 = 2^4 \cdot 3^2 \cdot 5 \cdot 11$ and the smallest of the sporadic simple groups, see [**CCN+85**] for details. As is explained in [**Eis07**, p. 315], one way of describing $G = M_{11}$ is to note that it is the subgroup of the alternating group A_{11} that is generated by

$$x = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 1 \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 2 & 3 & 5 & 7 & 10 & 1 & 8 & 6 & 11 & 9 & 4 \end{pmatrix}.$$

[49] In fact, since by Exercise 2.1 we know that the trefoil K is reversible we see that the mirror is smoothly isotopic to the inverse, thus we see that $\mathrm{P}_{A_5}^x(K^{\mathrm{mir}}) = \mathrm{P}_{A_5}^x(K^{\mathrm{inv}}) = \overline{\mathrm{P}_{A_5}^x(K)} = \overline{1+5x} = 1 + 5x^{-1}$.

[50] The knots are named after the mathematician John Conway [**Con70**] and after the mathematicians Shinichi Kinoshita and Hidetaka Terasaka [**KT57**].

Note that x has order 11 in A_{11} and thus also in $G = M_{11}$. By [**Eis07**, p. 317] we have the following table:[51]

$$\begin{array}{lcllcl}
\mathrm{P}^x_G(K) & = & 1+11x^3+11x^7, & \qquad \mathrm{P}^x_G(C) & = & 1+11x^3+11x^7, \\
\mathrm{P}^x_G(K^{\mathrm{rev}}) & = & 1+22x^3+11x^7, & \qquad \mathrm{P}^x_G(C^{\mathrm{rev}}) & = & 1+11x^3+11x^5+11x^7, \\
\mathrm{P}^x_G(K^{\mathrm{mir}}) & = & 1+11x^4+22x^8, & \qquad \mathrm{P}^x_G(C^{\mathrm{mir}}) & = & 1+11x^4+11x^6+11x^8, \\
\mathrm{P}^x_G(K^{\mathrm{inv}}) & = & 1+11x^4+11x^8, & \qquad \mathrm{P}^x_G(C^{\mathrm{inv}}) & = & 1+11x^4+11x^8.
\end{array}$$

Since x has order 11 in G we see that the four elements in each of the two columns are different. This shows that the Kinoshita-Terasaka knot and the Conway knot are neither reversible, nor amphichiral nor invertible. Since the collection of four elements in $\mathbb{Z}[G]$ appearing in the two columns are different we also see that no flavor of the Kinoshita-Terasaka knot is smoothly isotopic to any flavor of the Conway knot. $\square$

For the record let us formulate the following proposition, which gives in particular a negative answer to Question 2.6.

Proposition 7.7. (Knot-without-Symmetries Proposition) There exists an oriented knot that is neither reversible, nor amphichiral nor invertible.

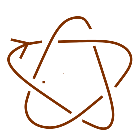
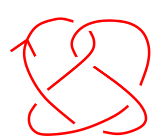
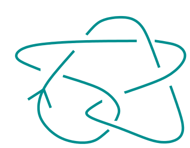
Proof. This proposition follows immediately from the above example. $\blacksquare$

Exercises for Chapter 7.

Exercise 7.1. The *writhe* of a knot diagram is defined as the number of positive crossings minus the number of negative crossings.

(a) Determine the writhes of the standard diagrams of the two trefoils and of the figure-8 knot.
(b) Show that every knot admits a diagram of writhe zero.
(c) Let D be a knot diagram of writhe zero. Show that a longitude is given by taking a "parallel copy" of the diagram in $\mathbb{R}^2$.

Exercise 7.2. Find longitudes for the three oriented knots that are shown in the figure below.

cinquefoil  three-twist knot  stevedore knot 

Exercise 7.3. Let π be a finitely generated group and let G be a finite group. Show that the set $\mathrm{Hom}(\pi, G)$ of homomorphisms from π to G is a finite set.

Exercise 7.4. Let G be a group and $x \in G$. Determine $\mathrm{P}^x_G(\text{trivial knot})$.

Exercise 7.5. Can it happen that the coloring polynomial $\mathrm{P}^x_G(K)$ of a knot is zero?

[51]Note that the Coloring Polynomial Lemma 7.5 implies that the coloring polynomial P^x_G of the inverse is determined by the coloring polynomial of the original knot. The lemma makes no such claim for the coloring polynomial of the mirror, in fact the following table shows that the coloring polynomial P^x_G of the mirror is not determined by the coloring polynomial P^x_G of the original knot.

Exercise 7.6.
(a) Is the knot 5_1 chiral?
(b) Is the knot 5_1 reversible?

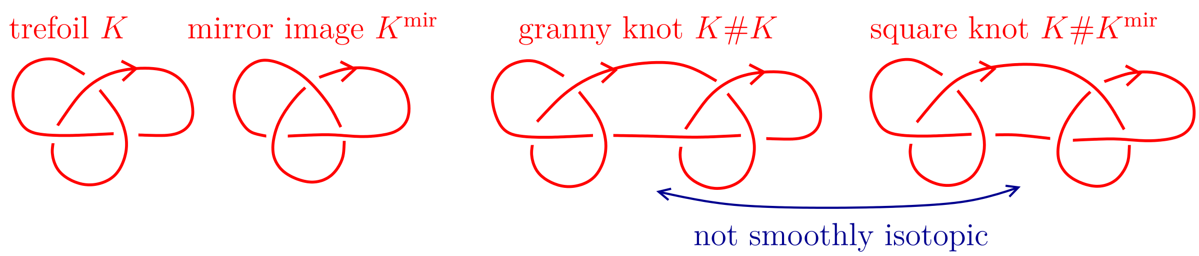
Exercise 7.7. Let G be a group. As we will see on page 103, we can equip $\mathbb{Z}[G]$ with a naive multiplication given by the multiplication in G and distributivity. Now let K and $\widetilde{K}$ be two oriented knots. We consider the connected sum $K\#\widetilde{K}$ as defined on page 24. Let G be a finite group and let $x \in G$. Show that we have the following equality in the group ring $\mathbb{Z}[G]$:

$$\mathrm{P}_G^x(K\#\widetilde{K}) \; = \; \mathrm{P}_G^x(K)\cdot \mathrm{P}_G^x(\widetilde{K}) \quad \in \mathbb{Z}[G].$$

Hint. Use the Knot Connected Sum-π_1-Proposition 3.15.

Exercise 7.8. Let K be the trefoil.
(a) Show that the granny knot $K\#K$ is not smoothly isotopic to its mirror image.
(b) Show that the square knot $K\#K^{\mathrm{mir}}$ is neither smoothly isotopic to the granny knot $K\#K$ nor to its mirror image.

Hint. Use Exercise 7.7.



Exercise 7.9. Let M be a closed 3-dimensional smooth manifold and let $\Phi\colon \overline{B}^3 \to M$ be a fixed smooth embedding. This smooth embedding gives us a map

$$\begin{array}{ccc} \begin{array}{c}\text{set of smooth isotopy}\\ \text{classes of knots in } \overline{B}^3\end{array} & \to & \begin{array}{c}\text{set of smooth isotopy}\\ \text{classes of knots in } M\end{array} \\ K & \mapsto & \Phi(K). \end{array}$$

Is this map always injective?
Solution. Not if M is not orientable.

CHAPTER 8

The Alexander function of groups

In this chapter we will introduce, following work of Ralph Fox [**Fox53**] and Masaaki Wada [**Wad94**], the Alexander function of groups that admit a presentation of deficiency one. In the next chapter we will use the group theoretic work of the current chapter to define the Alexander polynomial of knots and links.

8.1. Group rings.

Before we can introduce the Alexander functions of groups we need to make a few preparations.

Definition. Let G be a group. We consider the abelian group

$$\mathbb{Z}[G] \;:=\; \Big\{ \sum_{i=1}^{m} r_i \cdot g_i \,\Big|\, r_1,\dots,r_m \in \mathbb{Z} \text{ and } g_1,\dots,g_m \in G \Big\}$$

which we equip with the "obvious" multiplication that is given by

$$\Big(\sum_{i=1}^{m} r_i \cdot g_i\Big) \cdot \Big(\sum_{j=1}^{n} s_j \cdot h_j\Big) \;:=\; \sum_{i=1}^{m}\sum_{j=1}^{n} \underbrace{r_i \cdot_{\mathbb{Z}} s_j}_{\in \mathbb{Z}} \cdot \underbrace{(g_i \cdot_G h_j)}_{\in G}.$$

We refer to $\mathbb{Z}[G]$ as the group ring of G. One can easily show that $\mathbb{Z}[G]$ is a ring with multiplicatively neutral element $1 := 1_{\mathbb{Z}} \cdot e_G$.

Example. If H is a free abelian group of rank m, then any choice of a basis $t_1,\dots,t_m$ for H gives rise to a natural isomorphism from the multivariable Laurent polynomial ring $\mathbb{Z}[t_1^{\pm 1},\dots,t_m^{\pm 1}]$ to the group ring $\mathbb{Z}[H]$. □

Notation. Let $\varphi\colon G \to H$ be a group homomorphism. We consider the ring homomorphism

$$\begin{aligned} \varphi_*\colon \mathbb{Z}[G] \;&\to\; \mathbb{Z}[H] \\ \sum_{i=1}^{m} r_i \cdot g_i \;&\mapsto\; \sum_{i=1}^{m} r_i \cdot \varphi(g_i). \end{aligned}$$

8.2. Fox calculus.

In this section we study the purely algebraic notion of Fox derivatives, which was first introduced by Ralph Fox [**Fox53**].

Proposition 8.1. (Fox Derivative Proposition) Let $\langle x_1,\dots,x_k\rangle$ be the free group on generators $x_1,\dots,x_k$. For $i \in \{1,\dots,k\}$ there exists a unique map

$$\frac{\partial}{\partial x_i}\colon \mathbb{Z}[\langle x_1,\dots,x_k\rangle] \;\to\; \mathbb{Z}[\langle x_1,\dots,x_k\rangle]$$

with the following properties:

(1) For any $i,j \in \{1,\dots,k\}$ we have $\frac{\partial}{\partial x_i} x_j = \delta_{ij}$.

(2) For any $u,v \in \mathbb{Z}[\langle x_1,\dots,x_k\rangle]$ we have

$$\frac{\partial}{\partial x_i}(u \cdot v) \;=\; \frac{\partial}{\partial x_i}(u) + u \cdot \frac{\partial}{\partial x_i}(v) \qquad \text{(Leibniz Rule)}.$$

(3) The map $\frac{\partial}{\partial x_i}$ is a homomorphism of abelian groups.

Proof. We delay the proof of the Fox Derivative Proposition 8.1 to the next section. ■

Definition.
(1) We refer to the map
$$\frac{\partial}{\partial x_i}\colon \mathbb{Z}[\langle x_1,\dots,x_k\rangle] \;\to\; \mathbb{Z}[\langle x_1,\dots,x_k\rangle]$$
from the Fox Derivative Proposition 8.1 as the *i-th Fox derivative*.
(2) At times we write $\frac{\partial r}{\partial x_i} := \frac{\partial}{\partial x_i}r$.

As we just mentioned, we postpone the proof of the Fox Derivative Proposition 8.1 to the next section. In the following we will first state and prove two lemma on Fox derivatives which allow us to get acquainted with the weird properties of Fox derivatives. The familiarity with the axioms will also guide us in the proof of the Fox Derivative Proposition 8.1.

The following lemma should be read as an addendum to the Fox Derivative Proposition 8.1.

Lemma 8.2. (Fox Derivative Properties Lemma) Let $\langle x_1,\dots,x_k\rangle$ be the free group on generators $x_1,\dots,x_k$ and let $i\in\{1,\dots,k\}$.
(4) If $e\in\langle x_1,\dots,x_k\rangle$ is the trivial element, then
$$\frac{\partial}{\partial x_i}e \;=\; 0.$$
(5) Let $u\in\langle x_1,\dots,x_k\rangle$.
  (a) For $n\in\mathbb{N}$ we have $$\frac{\partial}{\partial x_i}\,u^n \;=\; (1+u+\dots+u^{n-1})\cdot\frac{\partial u}{\partial x_i}.$$
  (b) $$\frac{\partial}{\partial x_i}u^{-1} \;=\; -u^{-1}\cdot\frac{\partial u}{\partial x_i}.$$
(6) Let $n\in\mathbb{Z}$.
  (a) For $j\ne i$ we have $$\frac{\partial}{\partial x_i}\,x_j^n \;=\; 0.$$
  (b) We have
$$\frac{\partial}{\partial x_i}\,x_i^n \;=\; \begin{cases} \sum\limits_{k=0}^{n-1} x_i^k, & \text{if } n>0,\\ 0, & \text{if } n=0,\\ -\sum\limits_{k=n}^{-1} x_i^k, & \text{if } n<0.\end{cases}$$
  In particular we have $\frac{\partial}{\partial x_i}x_i^{-1} = -x_i^{-1}$.

Proof.
(4) Note that
$$\frac{\partial}{\partial x_i}\,e \;=\; \frac{\partial}{\partial x_i}\,(e\cdot e) \;\underset{\substack{\uparrow\\ \text{by the Leibniz rule}}}{=}\; \frac{\partial}{\partial x_i}\,e + e\cdot\frac{\partial}{\partial x_i}\,e \;\underset{\substack{\uparrow\\ \text{since } 1=1\cdot e=e}}{=}\; 2\cdot\frac{\partial}{\partial x_i}\,e.$$
It follows that $\frac{\partial}{\partial x_i}\,e = 0$.
(5) Let $u\in\langle x_1,\dots,x_k\rangle$.
  (a) Given $n\in\mathbb{N}$ we see that
$$\frac{\partial}{\partial x_i}\,u^n \;=\; \frac{\partial}{\partial x_i}\,(u\cdot u^{n-1}) \;\underset{\substack{\uparrow\\ \text{by the Leibniz rule}}}{=}\; \frac{\partial}{\partial x_i}\,u + u\cdot\frac{\partial}{\partial x_i}\,u^{n-1} \;\underset{\substack{\uparrow\\ \text{iterating the argument}}}{=}\; \frac{\partial u}{\partial x_i}\cdot\sum_{k=0}^{n-1}u^k.$$

(b) Note that
$$0 \underset{\substack{\uparrow \\ \text{by (4)}}}{=} \frac{\partial}{\partial x_i}\, e \;=\; \frac{\partial}{\partial x_i}\,(u\cdot u^{-1}) \underset{\substack{\uparrow \\ \text{by the Leibniz rule}}}{=} \frac{\partial}{\partial x_i}\, u + u\cdot \frac{\partial}{\partial x_i}\, u^{-1}.$$
It follows that $\frac{\partial}{\partial x_i}\, u^{-1} = -u^{-1}\cdot \frac{\partial}{\partial x_i} u$.

(6) (a) This statement follows immediately from (4), (5a), (5b) and the fact that for $j \neq i$ we have $\frac{\partial}{\partial x_i} x_j = 0$.

(b) For $n \in \mathbb{N}$ this statement follows from (5a). For $n = 0$ this statement follows from (4). Finally note that for $n < 0$ we have
$$\frac{\partial}{\partial x_i}\, x_i^n \;=\; \frac{\partial}{\partial x_i}\,(x_i^{-1})^{-n} \underset{\substack{\uparrow \\ \text{by (5a)}}}{=} \Bigg(\sum_{k=0}^{-n-1} (x_i^{-1})^k\Bigg)\cdot \frac{\partial x_i^{-1}}{\partial x_i} \underset{\substack{\uparrow \\ \text{by (5b)}}}{=} \Bigg(\sum_{k=n+1}^{0} x_i^k\Bigg)\cdot(-x_i^{-1}) \;=\; -\sum_{k=n}^{-1} x_i^k.$$
■

Before we can formulate the next lemma we need to introduce the following harmless concept.

Definition. Let G be a group. We refer to the map
$$\begin{array}{rcl} \epsilon := \epsilon_G\colon \mathbb{Z}[G] & \to & \mathbb{Z} \\ \sum\limits_{i=1}^{m} r_i\cdot g_i & \mapsto & \sum\limits_{i=1}^{m} r_i \end{array}$$
as the augmentation homomorphism. One can easily verify that the augmentation homomorphism is in fact a ring homomorphism.[52]

The following lemma gives us a mysterious identity which will be crucial later on.

Lemma 8.3. (Fox Derivative Identity) Given any $u \in \mathbb{Z}[\langle x_1,\dots,x_k\rangle]$ we have
$$u - \underset{\substack{\uparrow \\ \text{augmentation } \mathbb{Z}[\langle x_1,\dots,x_k\rangle]\to\mathbb{Z}}}{\epsilon(u)} \;=\; \sum_{i=1}^{k} \frac{\partial u}{\partial x_i}\cdot(x_i - 1).$$

Proof. Since both sides are $\mathbb{Z}$-linear it suffices to prove the equality for $u \in \langle x_1,\dots,x_k\rangle$. Note that in this case we has $\epsilon(u) = 1$. In other words, it suffices to prove the following claim:

Claim. For every $u \in \langle x_1,\dots,x_k\rangle$ we have
$$\sum_{i=1}^{k} \frac{\partial u}{\partial x_i}\cdot(x_i - 1) \;=\; u - 1.$$

Proof. First note that if u is the trivial element, then it follows from the definitions and the Fox Derivative Properties Lemma 8.2 (4) that both sides are zero. In particular the equality holds.

[52] In fact the augmentation homomorphism is essentially the same as the ring homomorphism corresponding to the group homomorphisms $G \to \{e\}$.

It now remains to show that if the equality holds for some $u \in \langle x_1, \dots, x_k \rangle$, then it also holds for $u \cdot x_s$ and $u \cdot x_s^{-1}$, where $s \in \{1, \dots, k\}$ is arbitrary. We calculate

$$\begin{aligned}\sum_{i=1}^{k} \frac{\partial}{\partial x_i}(u \cdot x_s) \cdot (x_i - 1) &\overset{\substack{\text{Leibniz rule}\\ \downarrow}}{=} \sum_{i=1}^{k} \Big(\frac{\partial}{\partial x_i} u + u \cdot \overbrace{\frac{\partial}{\partial x_i} x_s}^{=\delta_{is}} \Big) \cdot (x_i - 1) \\ &= \underbrace{\sum_{i=1}^{k} \frac{\partial}{\partial x_i} u \cdot (x_i - 1)}_{\substack{= u-1, \text{ by the induction}\\ \text{hypothesis}}} + u \cdot (x_s - 1) \quad = \quad u - 1 + u \cdot x_s - u \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad = \quad u \cdot x_s - 1.\end{aligned}$$

Since this is so much fun we also calculate

$$\begin{aligned}\sum_{i=1}^{k} \frac{\partial}{\partial x_i}(u \cdot x_s^{-1}) \cdot (x_i - 1) &\overset{\substack{\text{Leibniz rule}\\ \downarrow}}{=} \sum_{i=1}^{k} \Big(\frac{\partial}{\partial x_i} u + u \cdot \overbrace{\frac{\partial}{\partial x_i} x_s^{-1}}^{=-\delta_{is} \cdot x_i^{-1}} \Big) \cdot (x_i - 1) \\ &= \underbrace{\sum_{i=1}^{k} \frac{\partial}{\partial x_i} u \cdot (x_i - 1)}_{\substack{= u-1, \text{ by the induction}\\ \text{hypothesis}}} - u \cdot x_s^{-1} \cdot (x_s - 1) \quad = \quad u - 1 - u + u \cdot x_s^{-1} \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad = \quad u \cdot x_s^{-1} - 1.\end{aligned}$$

■

8.3. Proof of the Fox Derivative Proposition 8.1.

This section is solely dedicated to the following task:

Proof of the Fox Derivative Proposition 8.1. We start out with some preparations:

(1) A *word in* $x_1, \dots, x_k$ is a finite sequence $(x_{s_1}^{\epsilon_1}, \dots, x_{s_t}^{\epsilon_t})$ where $s_1, \dots, s_t \in \{1, \dots, k\}$ and $\epsilon_1, \dots, \epsilon_t \in \mathbb{Z}$. We refer to $\ell(w) := t$ as the *length of the word* w.

(2) We denote by W the set of all words in $x_1, \dots, x_k$. We define a multiplication on W by "juxtaposition":
$$(x_{s_1}^{\mu_1}, \dots, x_{s_p}^{\mu_p}) \cdot (x_{t_1}^{\nu_1}, \dots, x_{t_q}^{\nu_q}) \;=\; (x_{s_1}^{\mu_1}, \dots, x_{s_p}^{\mu_p}, x_{t_1}^{\nu_1}, \dots, x_{t_q}^{\nu_q}).$$
Clearly this product turns W into a monoid.

(3) We say that a word $(x_{s_1}^{\epsilon_1}, \dots, x_{s_t}^{\epsilon_t})$ is *unreduced* if one of the following occurs:
   (i) There exists an $i \in \{1, \dots, t\}$ with $\epsilon_i = 0$.
   (ii) There exists a $j \in \{1, \dots, t-1\}$ with $s_j = s_{j+1}$.
   Otherwise we say that the word is *reduced*.

(4) The free group $\langle x_1, \dots, x_k \rangle$ is defined as the set of reduced words in $x_1, \dots, x_k$. In [**Fri24**] it is shown that there exists a unique group structure on $\langle x_1, \dots, x_k \rangle$ such that the map $\Theta\colon W \to \langle x_1, \dots, x_k \rangle$ given by $\Theta(x_{s_1}^{\epsilon_1}, \dots, x_{s_t}^{\epsilon_t}) := x_{s_1}^{\epsilon_1} \cdot \ldots \cdot x_{s_t}^{\epsilon_t}$ is a surjective monoid homomorphism.

After these preparations we start out with the actual proof. For the remainder of the proof we fix $i \in \{1, \dots, k\}$.

Claim 1. There exists a unique map[53]
$$D_i\colon W \;\to\; \mathbb{Z}[\langle x_1, \dots, x_k \rangle]$$

with the following properties:
(0) We have $D_i() = 0$.
(1) (a) For $j \neq i$ and any $n \in \mathbb{Z}$ we have $D_i(x_j^n) = 0$.
(b) For $n \in \mathbb{Z}$ we have $D_i(x_i^n) = \begin{cases} 1 + x_i \cdots + x_i^{n-1}, & \text{if } n > 0, \\ 0, & \text{if } n = 0, \\ -x_i^{-1} - \cdots - x_i^n, & \text{if } n < 0. \end{cases}$
(2) For any $u, v \in W$ we have the following equality in $\mathbb{Z}[\langle x_1, \dots, x_k \rangle]$:

$$D_i(u, v) \; = \; D_i(u) + \Theta(u) \cdot D_i(v) \qquad \text{(Leibniz Rule)}.$$

Proof. The three rules tell us iteratively in a unique way how to define D_i of a given word $(x_{s_1}^{\epsilon_1}, \dots, x_{s_t}^{\epsilon_t})$ "from left to right". More precisely, given $(x_{s_1}^{\epsilon_1}, \dots, x_{s_t}^{\epsilon_t}) \in W$ we set

$$D_i(x_{s_1}^{\epsilon_1}, \dots, x_{s_t}^{\epsilon_t}) \; = \; \sum_{j=1}^{t} x_{s_1}^{\epsilon_1} \cdot \ldots \cdot x_{s_{j-1}}^{\epsilon_{j-1}} \cdot \begin{cases} 1 + x_{s_j} \cdots + x_{s_j}^{\epsilon_j - 1}, & \text{if } \epsilon_j > 0, \\ 0, & \text{if } \epsilon_j = 0, \\ -x_{s_j}^{-1} - \cdots - x_{s_j}^{\epsilon_j}, & \text{if } \epsilon_j < 0. \end{cases}$$

By definition (0) and (1) hold. Furthermore one can easily verify that (2) holds. We leave it to the reader to fill in the details. ⊞

Claim 2. For any $m, n \in \mathbb{Z} \setminus \{0\}$ we have

$$D_i(x_i^m, x_i^n) \; = \; D_i(x_i^{m+n}).$$

Proof. First note that this is a statement which holds in $\mathbb{Z}[x_i^{\pm 1}] \subset \mathbb{Z}[\langle x_1, \dots, x_k \rangle]$. Next note that $\mathbb{Z}[x_i^{\pm 1}]$ can also be viewed as a subring of $\mathbb{Q}(x_i)$. Furthermore note that it follows easily from the definitions that in $\mathbb{Q}(x_i)$ we have for any $d \in \mathbb{Z}$ the equality

$$D_i(x_i^d) \; = \; \frac{1 - x_i^d}{1 - x_i}.$$

Note that in $\mathbb{Q}(x_i)$ we have the following equalities:

$$D_i(x_i^m, x_i^n) \underset{\substack{\uparrow \\ \text{by Claim 1 (2)}}}{=} D_i(x_i^m) + x_i^m \cdot D_i(x_i^n) \underset{\substack{\uparrow \\ \text{by the above trick}}}{=} \frac{1 - x_i^m}{1 - x_i} + x_i^m \cdot \frac{1 - x_i^n}{1 - x_i} = \frac{1 - x_i^{m+n}}{1 - x_i} \underset{\substack{\uparrow \\ \text{again the above trick}}}{=} D_i(x_i^{m+n}).$$

Since $\mathbb{Z}[x_i^{\pm 1}]$ is a subring of $\mathbb{Q}(x_i)$ we see that the desired equality actually also holds in $\mathbb{Z}[x_i^{\pm 1}] \subset \mathbb{Z}[\langle x_1, \dots, x_k \rangle]$. ⊞

Claim 3. If we have $w, w' \in W$ with $\Theta(w) = \Theta(w') \in \langle x_1, \dots, x_k \rangle$, then $D_i(w) = D_i(w')$.

Proof. It follows by the above definition of the free group $\langle x_1, \dots, x_k \rangle$ that any element in $\langle x_1, \dots, x_k \rangle$ is represented by a *unique* reduced word. In particular, by reflexivity we can assume that w' is reduced. We now need to prove that for any unreduced word w in $x_1, \dots, x_k$ with $\Theta(w) = \Theta(w')$ we have $D_i(w) = D_i(w')$. We prove this statement by induction on the length of w. If $\ell(w) = 0$, then $\Theta(w) = e$ and thus $w' = ()$. It follows from Claim 1 (0) that $D_i(w) = e = D_i(w')$.

[53]This map is of course motivated by the Fox Derivative Properties Lemma 8.2.

Now suppose that we have proved the statement for all w of length $< t$. Next let $w = (x_{s_1}^{\epsilon_1}, \dots, x_{s_t}^{\epsilon_t})$ be an unreduced word of length t. We have to distinguish two cases:

(i) There exists a $j \in \{1, \dots, t\}$ with $\epsilon_j = 0$. This means that we can write $w = (u, x_{s_j}^0, v) \cdot v$ with $u, v \in W$. It follows that

$$D_i(w) \;=\; D_i(u, x_{s_j}^0, v) \underset{\substack{\uparrow \\ \text{by the Leibniz Rule}}}{=} D_i(u) + u \cdot \underbrace{D_i(x_{s_j}^0)}_{=0,\text{ by (1)}} + \underbrace{u \cdot (x_{s_j}^0)}_{=u} \cdot D_i(v) \underset{\substack{\uparrow \\ \text{by the Leibniz Rule}}}{=} D_i(u, v).$$

(ii) There exists a $j \in \{1, \dots, t-1\}$ with $s_j = s_{j+1}$. Note that this means that we can write $w = (u, x_s^\mu, x_s^\nu, v)$. We set[54] $\widetilde{w} = (u, x_s^{\mu+\nu}, v)$. Evidently $\Theta(w) = \Theta(\widetilde{w})$. Since $\ell(\widetilde{w}) < \ell(w) = t$ it suffices, by induction, to prove that $D_i(w) = D_i(\widetilde{w})$. First we consider the case that $i = s$. In this case we have the following equalities in $\mathbb{Z}[\langle x_1, \dots, x_k \rangle]$:

$$\begin{aligned} D_i(w) \;&=\; D_i(u, x_i^\mu, x_i^\nu, v) \overset{\substack{\text{by the Leibniz Rule} \\ \downarrow}}{=} D_i(u) + u \cdot (D_i(x_i^\mu, x_i^\nu) \cdot v + x_i^{\mu+\nu} \cdot D_i(v)) \\ &\underset{\substack{\uparrow \\ \text{by Claim 2}}}{=} D_i(u) + u \cdot (D_i(x_i^{\mu+\nu}) \cdot v + x_i^{\mu+\nu} \cdot D_i(v)) \underset{\substack{\uparrow \\ \text{by the Leibniz rule}}}{=} D_i(u, x_i^{\mu+\nu}, v). \end{aligned}$$

The case $i \neq s$ is done in a similar fashion, but it is even simpler since it follows immediately from the definition of D_i that for $i \neq s$ we have $D_i(x_s^\mu, x_s^\nu) = 0$ and $D_i(x_s^{\mu+\nu}) = 0$. ⊞

Claim 4. The map

$$\begin{aligned} \frac{\partial}{\partial x_i} \colon \mathbb{Z}[\langle x_1, \dots, x_k \rangle] \;&\to\; \mathbb{Z}[\langle x_1, \dots, x_k \rangle] \\ \sum_{j=1}^m a_j \cdot w_j \;&\mapsto\; \sum_{j=1}^m a_j \cdot \underset{\substack{\uparrow \\ \text{where } \Theta(\widetilde{w}_i) = w_i}}{D_i(\widetilde{w}_j)} \end{aligned}$$

is well-defined, it has all the desired properties and it is the unique such homomorphism.

Proof. First note that it follows from Claim 3 that the given map is well-defined.

(1) It follows from Claim 1 that $\frac{\partial}{\partial x_i} x_j = D_i(x_j) = \delta_{ij}$.
(3) The map is by definition a homomorphism of abelian groups.
(2) Let $u, v \in \langle x_1, \dots, x_k \rangle$. We pick $\widetilde{u}, \widetilde{v} \in W$ with $\Theta(\widetilde{u}) = u$ and $\Theta(\widetilde{v}) = v$. Note that

$$\frac{\partial}{\partial x_i}(u \cdot v) \underset{\substack{\uparrow \\ \text{since } \Theta \text{ is a homomorphism we} \\ \text{have } \Theta(\widetilde{u}, \widetilde{v}) = \Theta(\widetilde{u}) \cdot \Theta(\widetilde{v}) = u \cdot v, \\ \text{the equality thus follows from} \\ \text{from the definition of } \frac{\partial}{\partial x_i}}}{=} D_i(\widetilde{u}, \widetilde{v}) \underset{\substack{\uparrow \\ \text{by Claim 1}}}{=} D_i(\widetilde{u}) + \Theta(\widetilde{u}) \cdot D_i(\widetilde{v}) \underset{\substack{\uparrow \\ \text{by definition of } \frac{\partial}{\partial x_i}}}{=} \frac{\partial}{\partial x_i}(u) + u \cdot \frac{\partial}{\partial x_i}(v).$$

It follows easily from (3) that the desired equality also holds for all $u, v \in \mathbb{Z}[\langle x_1, \dots, x_k \rangle]$. Finally the uniqueness of $\frac{\partial}{\partial x_i}$ follows from the uniqueness statement of Claim 1. ■

8.4. **The Alexander function.** The reader might be wondering, where is this going? In this section we will give an answer to this pertinent question.

[54] Note that it is possible that $\mu + \nu = 0$, which means that we could end up with a word that is unreduced in the sense of (i).

Definition.

(1) Given any finite presentation $P = \langle x_1, \dots, x_k \,|\, r_1, \dots, r_l \rangle$ we denote by
$$\mu = \mu_P \colon \langle x_1, \dots, x_k \rangle \;\to\; P$$
the natural projection.

(2) Let π be a group. A finite presentation for the group π consists of a finite presentation $\langle x_1, \dots, x_k \,|\, r_1, \dots, r_l \rangle$ together with an isomorphism
$$\gamma \colon \langle x_1, \dots, x_k \,|\, r_1, \dots, r_l \rangle \xrightarrow{\cong} \pi.$$

(3) The deficiency of a finite presentation $\langle x_1, \dots, x_k \,|\, r_1, \dots, r_l \rangle$ is defined to be $k - l$.

We continue with a weird definition:

Definition. Let $P = \langle x_1, \dots, x_k \,|\, r_1, \dots, r_l \rangle$ be a finite presentation.

(1) We define the Jacobi matrix $J(P)$ to be the $(l \times k)$-matrix
$$J(P) \;:=\; \Big\{ \mu_{P*}\Big(\underbrace{\frac{\partial r_i}{\partial x_j}}_{\in \mathbb{Z}[\langle x_1, \dots, x_k \rangle]} \Big) \Big\}_{\substack{i=1,\dots,l \\ j=1,\dots,k}} \;\in\; \mathrm{M}(l \times k, \mathbb{Z}[P]).$$

(2) Given a group homomorphism $\Phi \colon P \to H$ we write
$$J^{\Phi}(P) \;:=\; \Phi_*(J(P)) \;\in\; \mathrm{M}(l \times k, \mathbb{Z}[H]).$$
where Φ_* is the induced ring homomorphism $\mathbb{Z}[P] \to \mathbb{Z}[H]$

Before we can state the first result of the section we need to introduce just a little bit of extra notation:

Notation.

(1) Let H be a torsion-free abelian group. We denote by $\mathbb{Q}(H)$ the quotient field of the domain $\mathbb{Z}[H]$.

(2) Let R be a ring and let $M \in \mathrm{M}(l \times k, R)$ be a matrix. Given $i \in \{1, \dots, k\}$ we denote by M_i the $l \times (k-1)$-matrix which we obtain by deleting the i-th column.

In the Wirtinger Presentation Proposition 5.5 (2) we saw that fundamental groups of link complements admit presentations of deficiency one. Since these are the groups we are mostly interested in, we now restrict ourselves to such groups.

Proposition 8.4. (Presentation Function Quotient Proposition) Suppose we are given a presentation $P = \langle x_1, \dots, x_k \,|\, r_1, \dots, r_{k-1} \rangle$ of *deficiency one*. Let $\Phi \colon P \to H$ be a non-trivial homomorphism to a free abelian group H.

(1) There exists an $i \in \{1, \dots, k\}$ such that $\Phi(x_i)$ is non-trivial.

(2) Let $i \in \{1, \dots, k\}$ with $\Phi(x_i) \neq 0$. The term[55]
$$\Delta_{P,\Phi} \;:=\; \frac{\det(J^{\Phi}(P)_i)}{\Phi_*(x_i - 1)} \;\in\; \mathbb{Q}(H)$$
is well-defined (i.e. independent of the choice of i) up to multiplication by a sign.

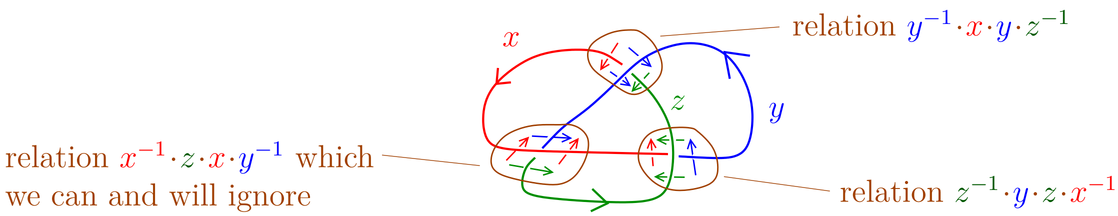
[55] The fact that $\Phi(x_i)$ is non-trivial and the hypothesis that H is a free abelian group implies that $\Phi_*(x_i - 1)$ is invertible in $\mathbb{Q}(H)$.

Proof. In an attempt to keep the conversation flowing, without getting bogged down by technicalities, we postpone the proof of this proposition to Section 8.5. ■

Example. Let $K \subset S^3$ be the trefoil. We consider the presentation

$$P := \langle x, y, z \,|\, \underbrace{y^{-1}\cdot x\cdot y\cdot z^{-1}}_{=r_1},\ \underbrace{z^{-1}\cdot y\cdot z\cdot x^{-1}}_{=r_2} \rangle \xrightarrow{\cong} \pi_1(S^3 \setminus K)$$

from the discussion on page 68.



One can easily verify that there exists a unique epimorphism $\Phi\colon P \to \langle t\rangle$ such that $\Phi(x) = \Phi(y) = \Phi(z) = t$. We now calculate that the Jacobi matrix $J(P) \in \mathrm{M}(2\times 3, \mathbb{Z}[P])$ is given by

$$J(P) = \begin{pmatrix} \frac{\partial r_1}{\partial x} & \frac{\partial r_1}{\partial y} & \frac{\partial r_1}{\partial z} \\ \frac{\partial r_2}{\partial x} & \frac{\partial r_2}{\partial y} & \frac{\partial r_2}{\partial z} \end{pmatrix} \underset{\uparrow}{=} \begin{pmatrix} y^{-1} & y^{-1}\cdot x - y^{-1} & -y^{-1}\cdot x\cdot y\cdot z^{-1} \\ -z^{-1}\cdot y\cdot z\cdot x^{-1} & z^{-1} & z^{-1}\cdot y - z^{-1} \end{pmatrix} \underset{\uparrow}{=} \begin{pmatrix} y^{-1} & y^{-1}\cdot x - y^{-1} & -1 \\ -1 & z^{-1} & z^{-1}\cdot y - z^{-1} \end{pmatrix}$$

this follows from the Fox Derivative Proposition 8.1 and the Fox Derivative Properties Lemma 8.2 applied to $r_1 = y^{-1}\cdot x\cdot y\cdot z^{-1}$ and $r_2 = z^{-1}\cdot y\cdot z\cdot x^{-1}$

since $y^{-1}\cdot x\cdot y\cdot z^{-1} = e$ and $z^{-1}\cdot y\cdot z\cdot x^{-1} = e$

We obtain the following equalities in $\mathbb{Q}(t)$:

$$\frac{\det(J^{\Phi}(P)_1)}{\Phi_*(x-1)} = \frac{\det\left(\Phi_*\begin{pmatrix} \cancel{y^{-1}} & y^{-1}\cdot x - y^{-1} & -1 \\ \cancel{-1} & z^{-1} & z^{-1}\cdot y - z^{-1} \end{pmatrix}\right)}{\Phi_*(x-1)} = \frac{\det\begin{pmatrix} \cancel{t^{-1}} & 1-t^{-1} & -1 \\ \cancel{-1} & t^{-1} & 1-t^{-1} \end{pmatrix}}{t-1} = \frac{t^{-2}-t^{-1}+1}{t-1}$$

$$\frac{\det(J^{\Phi}(P)_2)}{\Phi_*(y-1)} = \frac{\det\left(\Phi_*\begin{pmatrix} y^{-1} & \cancel{y^{-1}\cdot x - y^{-1}} & -1 \\ -1 & \cancel{z^{-1}} & z^{-1}\cdot y - z^{-1} \end{pmatrix}\right)}{\Phi_*(y-1)} = \frac{\det\begin{pmatrix} t^{-1} & \cancel{1-t^{-1}} & -1 \\ -1 & \cancel{t^{-1}} & 1-t^{-1} \end{pmatrix}}{t-1} = \frac{-t^{-2}+t^{-1}-1}{t-1}$$

$$\frac{\det(J^{\Phi}(P)_3)}{\Phi_*(z-1)} = \frac{\det\left(\Phi_*\begin{pmatrix} y^{-1} & y^{-1}\cdot x - y^{-1} & \cancel{-1} \\ -1 & z^{-1} & \cancel{z^{-1}\cdot y - z^{-1}} \end{pmatrix}\right)}{\Phi_*(z-1)} = \frac{\det\begin{pmatrix} t^{-1} & 1-t^{-1} & \cancel{-1} \\ -1 & t^{-1} & \cancel{1-t^{-1}} \end{pmatrix}}{t-1} = \frac{t^{-2}-t^{-1}+1}{t-1}.$$

So all three terms do indeed agree up to a sign. □

Example. Since the previous example was so much fun, let us consider again the trefoil $K = T(2,3)$, but this time with a very different presentation. By the Torus Knot-π_1-Proposition 3.8 we know that we have a presentation $Q := \langle x, y \mid x^2\cdot y^{-3}\rangle \xrightarrow{\cong} \pi_1(S^3 \setminus K)$. Note that

$$J(Q) = \begin{pmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \end{pmatrix} \underset{\uparrow}{=} \begin{pmatrix} 1+x & x^2\cdot(-y^{-1}-y^{-2}-y^{-3}) \end{pmatrix}.$$

this follows from the Fox Derivative Proposition 8.1 and the Fox Derivative Properties Lemma 8.2 (5) applied to $r = x^2\cdot y^{-3}$

Up to sign the map $\Phi\colon Q \to \langle t\rangle$ given by $\Phi(x) = t^3$ and $\Phi(y) = t^2$ is the unique epimorphism onto $\langle t\rangle$. We calculate

$$\frac{\det(J^{\Phi}(Q)_{1})}{\Phi_*(x-1)} = \frac{\det(\Phi_* \left(\cancel{1+x} \;\; x^2\cdot(-y^{-1}-y^{-2}-y^{-3})\right))}{\Phi_*(x-1)} = \frac{\left(\cancel{1+t^3} \;\; t^6\cdot(-t^{-2}-t^{-4}-t^{-6})\right)}{t^3-1} = \frac{t^2-t+1}{t-1},$$

$$\frac{\det(J^{\Phi}(Q)_{2})}{\Phi_*(y-1)} = \frac{\det(\Phi_* \left(1+x \;\; \cancel{x^2\cdot(-y^{-1}-y^{-2}-y^{-3})}\right))}{\Phi_*(y-1)} = \frac{\left(1+t^3 \;\; \cancel{t^6\cdot(-t^{-2}-t^{-4}-t^{-6})}\right)}{t^2-1} = -\frac{t^2-t+1}{t-1}.$$

So we get (up to a sign), the same result as with the Wirtinger presentation which we used above. Magic! □

The Presentation Function Quotient Proposition 8.4 is a little surprising and cute, but on its own not particularly useful. We want to obtain invariants of groups, not of presentations. Fortunately the above two examples give us the hope that we have actually a group invariant at our hands.

Theorem 8.5. (Alexander Function Theorem) Let π be a group and let $\Phi\colon \pi \to H$ be a non-trivial homomorphism to a free abelian group H. If π admits a presentation $\gamma\colon P = \langle x_1,\dots,x_k \,|\, r_1,\dots,r_{k-1}\rangle \xrightarrow{\cong} \pi$ of deficiency one, then

$$\Delta_{\pi,\Phi} \;:=\; \Delta_{P,\Phi\circ\gamma} \;=\; \underbrace{\frac{\det(J^{\Phi}(P)_{i})}{\Phi_*(x_{i}-1)}}_{\substack{\text{where } \Phi(x_i) \text{ is}\\ \text{non-trivial}}} \;\in\; \mathbb{Q}(H)$$

is well-defined, i.e. independent of the choice of the presentation, up to multiplication by an element of the form $\pm h$ with $h \in H$.

Proof. We will provide the proof of the theorem in Section 8.6. ■

The indeterminacy statement in the Alexander Function Theorem 8.5 motivates the following notation:

Notation. Let H be a torsion-free abelian group. Given $p, q \in \mathbb{Q}(H)$ we write $p \doteq q$ if there exists an $\epsilon \in \{-1, 1\}$ and an $h \in H$ with $p = \epsilon \cdot h \cdot q$.

The Alexander Function Theorem 8.5 allows us to make the following definition:

Definition. Let π be a group and let $\Phi\colon \pi \to H$ be a non-trivial homomorphism to a free abelian group H. If π admits a presentation $\gamma\colon P = \langle x_1,\dots,x_k \,|\, r_1,\dots,r_{k-1}\rangle \to \pi$ of deficiency one, then we refer to

$$\Delta_{\pi,\Phi} \;:=\; \Delta_{P,\Phi\circ\gamma} \;:=\; \Delta_{P,\Phi\circ\gamma} \;=\; \underbrace{\frac{\det(J^{\Phi\circ\gamma}(P)_{i})}{\Phi_*(x_{i}-1)}}_{\substack{\text{where } \Phi(x_i) \text{ is}\\ \text{non-trivial}}} \;\in\; \mathbb{Q}(H)$$

as the Alexander function[62] of (π, Φ). By the Alexander Function Theorem 8.5 the Alexander function is is well-defined up to "$\doteq$".

Before we consider examples let us get the following basic lemma out of the way:

[62]We call $\Delta_{\pi,\Phi}$ Alexander *function* since $\Delta_{\pi,\Phi} \in \mathbb{Q}(H)$ can be viewed as a rational function, i.e. the quotient of two polynomials. Later on we will introduce the Alexander *polynomial* of an oriented link, which is a Laurent polynomial, i.e. an element of $\mathbb{Z}[t_1^{\pm 1},\dots,t_m^{\pm 1}]$.

Lemma 8.6. (Alexander Function–Functorial Lemma)

(1) Let π and $\widetilde{\pi}$ be two groups which admit presentations of deficiency one, let $f\colon \widetilde{\pi} \to \pi$ be an isomorphism and let $\Phi\colon \pi \to H$ be a non-trivial homomorphisms to a free abelian group H. Then
$$\Delta_{\widetilde{\pi},\Phi\circ f} \;=\; \Delta_{\pi,\Phi} \quad \in \mathbb{Q}(H).$$

(2) Let π be a group which admits a presentations of deficiency one. Let $\Phi\colon \pi \to G$ and $f\colon G \to H$ be two homomorphisms to free abelian groups such that $f \circ \Phi\colon \pi \to H$ is non-trivial. Then[63]
$$f_*(\Delta_{\pi,\Phi}) \;=\; \Delta_{\pi,f\circ\Phi} \quad \in \mathbb{Q}(H).$$

Proof.

(1) This statement is an immediate consequence of the Alexander Function Theorem 8.5.

(2) By (1) we can assume that $\pi = \langle x_1,\dots,x_k \,|\, r_1,\dots,r_{k-1}\rangle$. We pick $i \in \{1,\dots,k\}$ such that $(f\circ\Phi)(x_i) \neq 0$. We see that
$$\begin{aligned} f_*(\Delta_{\pi,\Phi}) \;&=\; f_*\Big(\frac{\det(\Phi_*(J(\pi)_i))}{\Phi_*(x_i-1)}\Big) \;=\; \frac{f_*(\det(\Phi_*(J(\pi)_i)))}{f_*(\Phi_*(x_i-1))} \\ &\underset{\substack{\uparrow \\ \text{since determinants commute with ring homomorphisms}}}{=} \frac{\det((f\circ\Phi)_*(J(\pi)_i)))}{(f\circ\Phi)_*(x_i-1))} \;=\; \Delta_{\pi,f\circ\Phi}. \end{aligned}$$
■

Example.

(1) Let $\pi = \pi_1(S^3 \setminus T_{2,3})$ be the fundamental group of the complement of the trefoil $T_{2,3}$. For at least one of the two epimorphisms $\pi \to \langle t\rangle$ we obtain from the above calculations that $\Delta_{\pi,\Phi} \doteq \frac{t^2-t+1}{t-1}$.

(2) Let $\pi = \mathbb{Z}$ and let $\Phi\colon \mathbb{Z} \to \langle t\rangle$ be the epimorphism with $\Phi(1) = t$. We consider the deficiency-one presentation that is given by the obvious map $\gamma\colon \langle s\,|\,\rangle \to \mathbb{Z}$. We see that
$$\Delta_{\mathbb{Z},\Phi} \;=\; \Delta_{\langle s\,|\,\rangle,\Phi\circ\gamma} \;=\; \frac{\det(J^{\Phi\circ\gamma}(\langle s\,|\,\rangle)_1)}{(\Phi\circ\gamma)(s)-1} \;=\; \frac{\det((0\times 0)\text{-matrix})}{\Phi(1)-1} \;=\; \frac{1}{t-1}.$$
This calculation, together with the calculation in (1) and the Alexander Function–Functorial Lemma 8.6 gives us a new, and much more systematic, proof that the fundamental group of the complement of the trefoil is not isomorphic to $\mathbb{Z}$.

(3) Let $\pi = \mathbb{Z}^2$ and let $\Phi\colon \mathbb{Z}^2 \to H$ be a non-trivial homomorphism to a free abelian group H. We consider the presentation that is given by the obvious homomorphism $\gamma\colon P := \langle x,y\,|\,xyx^{-1}y^{-1}\rangle \to \mathbb{Z}^2$. Note that[64]
$$J^\Phi(P) \;=\; \Phi_*\big(\tfrac{\partial r}{\partial x} \;\; \tfrac{\partial r}{\partial y}\big) \;=\; \Phi_*(1-\underbrace{xyx^{-1}}_{=y} \;\; x-\underbrace{xyx^{-1}y^{-1}}_{=1}) \;=\; (1-\Phi(y) \;\; \Phi(x)-1).$$
If $\Phi(x)$ is non-trivial, then
$$\Delta_{\mathbb{Z}^2,\Phi} \;=\; \Delta_{P,\Phi\circ\gamma} \;=\; \frac{\det(J^\Phi(P)_1)}{\Phi(x)-1} \;=\; \frac{\det(\cancel{1-\Phi(y)} \;\; \Phi(x)-1)}{\Phi(x)-1} \;\doteq\; 1 \;\in\; \mathbb{Q}(H).$$
Basically the same calculation also gives us the same conclusion if $\Phi(y)$ is non-trivial.

[63]Recall that $f\colon G \to H$ induces a ring homomorphism $f\colon \mathbb{Z}[G] \to \mathbb{Z}[H]$. It is clear that it also induces a ring homomorphism $f_*\colon \{\frac{r}{s} \in \mathbb{Q}(G)\,|\,r,s \in \mathbb{Z}[G], f_*(s) \neq 0\} \to \mathbb{Q}(H)$.

[64]To simplify the notation we will ignore γ in our notation.

(4) Let $\pi = \langle x, y\rangle$ be the free group on two generators and let $\Phi\colon \langle x, y\rangle \to H$ be a non-trivial homomorphism to a free abelian group H. We consider the deficiency-one presentation that is given by the equality $\langle x, y \,|\, e\rangle \to \langle x, y\rangle$. If $\Phi(x)$ is non-trivial, then we see that

$$\Delta_{\langle x,y\rangle,\Phi} \;=\; \frac{\det(J^{\Phi}(\langle x, y \,|\, e\rangle)_1)}{\Phi(x)-1} \;=\; \frac{\det(\Phi(0\;\;0)_1)}{\Phi(x)-1} \;=\; 0 \;\in\; \mathbb{Q}(H).$$

The same calculation also gives us the same conclusion if $\Phi(y)$ is non-trivial. Note that we get the same result for any free group of rank ≥ 2. □

The remainder of the chapter is occupied with the proofs of the Presentation Function Quotient Proposition 8.4 and the Alexander Function Theorem 8.5. In the next chapter we will consider (a slight variation of) the Alexander function of fundamental groups of link complements. At that point we will consider examples ad nausea.

8.5. **Proof of the Presentation Function Quotient Proposition 8.4.** In this section we will provide the proof of the Presentation Function Quotient Proposition 8.4. The proof is mostly a consequence of the following two lemmas:

Lemma 8.7. (Jacobi Matrix–Vector Lemma) Let $P = \langle x_1, \dots, x_k \,|\, r_1, \dots, r_l\rangle$ be a finite presentation. As before we denote by $J(P)$ the Jacobi matrix of P, which is a $(l \times k)$-matrix over $\mathbb{Z}[P]$. We have

$$J(P)\cdot \begin{pmatrix} 1-\mu_P(x_1) \\ \vdots \\ 1-\mu_P(x_k)\end{pmatrix} \;=\; 0.$$

Proof. We write $\mu := \mu_{P*}\colon \mathbb{Z}[\langle x_1, \dots, x_k\rangle] \to \mathbb{Z}[P]$. Note that for any $j \in \{1, \dots, l\}$ we have

$$\begin{aligned} j\text{-th row of } J(P)\cdot \begin{pmatrix} 1-\mu(x_1) \\ \vdots \\ 1-\mu(x_k)\end{pmatrix} \;&=\; \Big(\mu\big(\tfrac{\partial r_1}{\partial x_j}\big)\;\dots\;\mu\big(\tfrac{\partial r_k}{\partial x_j}\big)\Big)\cdot \begin{pmatrix} 1-\mu(x_1) \\ \vdots \\ 1-\mu(x_k)\end{pmatrix} \\ &=\; \sum_{i=1}^{k} \mu\big(\tfrac{\partial r_i}{\partial x_j}\big)\cdot(1-\mu(x_i)) \\ &\underset{\substack{\uparrow \\ \text{since } \mu \text{ is a ring} \\ \text{homomorphism}}}{=}\; \mu\Big(\sum_{i=1}^{k} \frac{\partial r_i}{\partial x_j}\cdot(1-x_i)\Big) \;\underset{\substack{\uparrow \\ \text{by the Fox Derivative} \\ \text{Identity 8.3}}}{=}\; \mu(r_i-1) \;=\; \underbrace{\mu(r_i)}_{=e=1}-1 \;=\; 0. \end{aligned}$$

■

Lemma 8.8. (Matrix–Quotient Lemma) Let $\mathbb{F}$ be a field, let $M \in \mathrm{M}((k-1)\times k, \mathbb{F})$ be a matrix and let $w = \begin{pmatrix} w_1 \\ \vdots \\ w_k\end{pmatrix} \in \mathbb{F}^k$ such that $M \cdot w = 0$. For any $i, j \in \{1, \dots, k\}$ with $w_i \neq 0$ and $w_j \neq 0$ we have[65]

$$\frac{\det(M_i)}{w_i} = \pm\,\frac{\det(M_j)}{w_j}.$$

Proof. To simplify the notation we now assume that $i = 1$ and $j = 2$. (Note though that here we pick up a sign indeterminacy.) We denote the k columns of M by $c_1, \dots, c_k$.

[65] Recall that M_i and M_j are the matrices that we obtain from M by deleting the i-th respectively j-th column.

Note that with this notation our hypothesis that $M \cdot w = 0$ turns into the statement that $w_1 \cdot c_1 + w_2 \cdot c_2 + w_3 \cdot c_3 + \cdots + w_k \cdot w_k = 0$. Since we assume that $w_1 \neq 0$ and $w_2 \neq 0$ we can divide this equality by $w_1 \cdot w_2$ and we obtain the following equality:

$$(*) \qquad \frac{1}{w_2} \cdot c_1 + \frac{1}{w_1} \cdot c_2 + \frac{w_3}{w_1 \cdot w_2} \cdot c_3 + \cdots + \frac{w_k}{w_1 \cdot w_2} \cdot c_k \;=\; 0.$$

Finally we perform the following calculation:

$$\begin{aligned} \frac{\det(M_1)}{w_1} &= \frac{\det(c_2 \;\; c_3 \; \ldots \; c_k)}{w_1} = \det\left(\frac{c_2}{w_1} \;\; c_3 \; \ldots \; c_k \right) \overset{\substack{\text{multilinearity of the determinant} \\ \downarrow}}{=} \det\Big(\underbrace{\frac{c_2}{w_1} + \sum_{i=3}^{k} \frac{w_i}{w_1 \cdot w_2} \cdot c_i}_{-\frac{c_1}{w_2},\ \text{by } (*)} \;\; c_3 \; \ldots \; c_k \Big) \\ &\underset{\substack{\uparrow \\ \text{follows from } (*)}}{=} \det\left(\frac{-c_1}{w_2} \;\; c_3 \; \ldots \; c_k \right) = -\frac{\det(c_1 \;\; c_3 \; \ldots \; c_k)}{w_2} = -\frac{\det(M_2)}{w_2}. \end{aligned}$$

■

Proof. To simplify the notation we now assume that $i = 1$ and $j = 2$. (Note though that here we pick up a sign indeterminacy.) We denote the k columns of M by $c_1, \ldots, c_k$. Note that with this notation our hypothesis that $M \cdot w = 0$ turns into the statement that $w_1 \cdot c_1 + w_2 \cdot c_2 + w_3 \cdot c_3 + \cdots + w_k \cdot w_k = 0$. Since we assume that $w_1 \neq 0$ and $w_2 \neq 0$ we can divide this equality by $w_1 \cdot w_2$ and we obtain the following equality:

$$(*) \qquad c_1 \cdot w_1 + c_2 \cdot w_2 + c_3 \cdot w_3 + \cdots + c_k \cdot w_k \;=\; 0.$$

The lemma follows from the following calculation:

$$\begin{aligned} \det(M_1) \cdot w_2 &= \det(c_2 \;\; c_3 \; \ldots \; c_k) \cdot w_2 \overset{\substack{\text{multilinearity of the determinant} \\ \downarrow}}{=} \det\left(c_2 \cdot w_2 \;\; c_3 \; \ldots \; c_k \right) \\ &\underset{\substack{\uparrow \\ \text{follows from } (*)}}{=} \det\big((-c_1 \cdot w_1 - c_3 \cdot w_3 - \cdots - c_k \cdot w_k) \;\; c_3 \; \ldots \; c_k \big) \\ &\underset{\substack{\uparrow \\ \text{follows from the multilinearity of the determinant}}}{=} \det\left(-c_1 \cdot w_1 \;\; c_3 \; \ldots \; c_k \right) \underset{\uparrow}{=} -\det(c_1 \;\; c_3 \; \ldots \; c_k) \cdot w_1 = -\det(M_2) \cdot w_1. \end{aligned}$$

■

Proof of the Presentation Function Quotient Proposition 8.4. As before we consider a presentation $P = \langle x_1, \ldots, x_k \,|\, r_1, \ldots, r_{k-1} \rangle$ of deficiency one and a non-trivial homomorphism $\Phi \colon P \to H$ to a free abelian group H.

(1) Since $x_1, \ldots, x_k$ is a generating set of P and since $\Phi \colon P \to H$ is non-trivial homomorphism we see that at least one $\Phi(x_i)$ is non-zero.

(2) Let $i \neq j \in \{1, \ldots, k\}$ with $\Phi(x_i) \neq 0$ and with $\Phi(x_j) \neq 0$. It follows easily from the Jacobi Matrix–Vector Lemma 8.7, the fact that $\Phi_* \colon \mathbb{Z}[P] \to \mathbb{Z}[H] \subset \mathbb{Q}(H)$ is a ring homomorphism and the Matrix–Quotient Lemma 8.8 that

$$\frac{\det(J^\Phi(P)_i)}{\Phi_*(x_i - 1)} \;=\; \pm \frac{\det(J^\Phi(P)_j)}{\Phi_*(x_j - 1)} \;\in\; \mathbb{Q}(H).$$

■

8.6. **Proof of the Alexander Function Theorem 8.5.** We need to show that two deficiency-one presentations of a given group lead to essentially the same invariant. The idea is to go between any two presentations using Tietze transformations, and to see that Tietze transformations do not change the invariant. Unfortunately this will require us to generalize our definition of the Alexander function to arbitrary finite presentations. This program is carried out in the following three subsections.

8.6.1. **Proof of Theorem 8.5: Tietze Transformations.** We start out with the following definition, which might be familiar from an earlier course in group theory:

Definition. Let
$$\phi\colon \langle x_1,\dots,x_k \,|\, r_1,\dots,r_l\rangle \xrightarrow{\cong} \pi$$
be a presentation of a group π. We obtain new presentations of π as follows:

(T1) For any element s of $\langle\!\langle r_1,\dots,r_l\rangle\!\rangle \subset \langle X\rangle$ we obtain the presentation
$$\langle x_1,\dots,x_k \,|\, r_1,\dots,r_l,s\rangle \underset{\substack{\uparrow\\ \text{since } s\in\langle\!\langle r_1,\dots,r_l\rangle\!\rangle \text{ the groups are identical}}}{=} \langle x_1,\dots,x_k \,|\, r_1,\dots,r_l\rangle \xrightarrow{\phi} \pi.$$

(T2) For any $x\notin\{x_1,\dots,x_k\}$ and any $s,t\in\langle x_1,\dots,x_k\rangle$ we obtain the presentation
$$\begin{array}{rcl} \langle x_1,\dots,x_k,x \,|\, r_1,\dots,r_l,s\cdot x\cdot t\rangle & \to & \langle x_1,\dots,x_k \,|\, r_1,\dots,r_l\rangle \xrightarrow{\phi} \pi. \\ x_i & \mapsto & x_i \\ x & \mapsto & s^{-1}\cdot t^{-1} \end{array}$$

The above two methods, and their inverses, of obtaining new presentations out of a given one are called Tietze transformations.

Example. As an illustration let us show three tricks:

(a) If we have a presentation, then we can replace any relator by a conjugate. For example, if $\langle x,y\,|\,r(x,y),s(x,y)\rangle$ is a presentation, then we have
$$\langle x,y\,|\,r(x,y)\rangle \overset{(T1)}{=} \langle x,y\,|\,r(x,y),x^{-1}\cdot r(x,y)\cdot x\rangle \overset{(T1)}{=} \langle x,y\,|\,x^{-1}\cdot r(x,y)\cdot x\rangle.$$

(b) One can "eliminate" a generator in suitable settings. For example, suppose we have a presentation of the form $P=\langle x,y,z\,|\,x\cdot v(y,z)^{-1},w(x,y,z)\rangle$ where $v(y,z)\in\langle y,z\rangle$ and $w(x,y,z)\in\langle x,y,z\rangle$. We now see that
$$\begin{aligned}\langle x,y,z\,|\,x\cdot v(y,z)^{-1},w(x,y,z)\rangle &\overset{(T1)}{=} \langle x,y,z\,|\,x\cdot v(y,z)^{-1},w(x,y,z),w(v(y,z),y,z)\rangle\\ &\overset{(T1)}{=} \langle x,y,z\,|\,x\cdot v(y,z)^{-1},w(v(y,z),y,z)\rangle\\ &\overset{(T2)}{=} \langle y,z\,|\,w(v(y,z),y,z)\rangle.\end{aligned}$$

(c) If $\langle x,y\,|\,r(x,y)\rangle$ is a presentation, then we can "substitute y by $\widetilde{y}=y^{-1}$" as follows:
$$\begin{aligned}\langle x,y\,|\,r(x,y)\rangle &\overset{(T2)}{=} \langle x,y,\widetilde{y}\,|\,r(x,y),\widetilde{y}\cdot y\rangle \overset{(T1)}{=} \langle x,y,\widetilde{y}\,|\,r(x,y),\widetilde{y}\cdot y,r(x,\widetilde{y}^{-1})\rangle\\ &\overset{(T1)}{=} \langle x,y,\widetilde{y}\,|\,\widetilde{y}\cdot y,r(x,\widetilde{y}^{-1})\rangle \overset{(T2)}{=} \langle x,\widetilde{y}\,|\,r(x,\widetilde{y}^{-1})\rangle.\end{aligned}$$

For example let $K\subset S^3$ be the trefoil. We now see that
$$\begin{aligned}\pi_1(S^3\setminus K) &\overset{\substack{\text{see page 68}\\ \downarrow}}{=} \langle x,y,z\,|\,y^{-1}\cdot x\cdot y\cdot z^{-1},\ z^{-1}\cdot y\cdot z\cdot x^{-1}\rangle \overset{\substack{\text{apply (a)}\\ \downarrow}}{=} \langle x,y,z\,|\,x\cdot y\cdot z^{-1}\cdot y^{-1},\ z^{-1}\cdot y\cdot z\cdot x^{-1}\rangle\\ &\underset{\substack{\uparrow\\ \text{apply (b) to } x\cdot y\cdot z^{-1}\cdot y^{-1}}}{=} \langle y,z\,|\,z^{-1}\cdot y\cdot z\cdot y\cdot z^{-1}\cdot y^{-1}\rangle \underset{\substack{\uparrow\\ \text{apply (a)}}}{=} \langle y,z\,|\,y\cdot z\cdot y\cdot z^{-1}\cdot y^{-1}\cdot z^{-1}\rangle\\ &= \langle y,z\,|\,y\cdot z\cdot y = z\cdot y\cdot z\rangle.\end{aligned}$$
□

Now we can state the following absolutely essential theorem:

Theorem 8.9. (Tietze Theorem) Any two finite presentations for a given group are related by a finite sequence of Tietze transformations.

Proof. The proof is actually quite short and elementary. We refer to [**Fri24**] for details. ■

The Tietze Theorem 8.9 will be the key ingredient in the proof of the Alexander Function Theorem 8.5, which we provide in the next section.

8.6.2. **Proof of Theorem 8.5: Arbitrary presentations.** The catch with the idea of using the Tietze Theorem 8.9 to relate two deficiency-one presentations is that inbetween we might leave the cozy world of deficiency-one presentations. We thus need to generalize the notion of an Alexander function to arbitrary finite presentations.

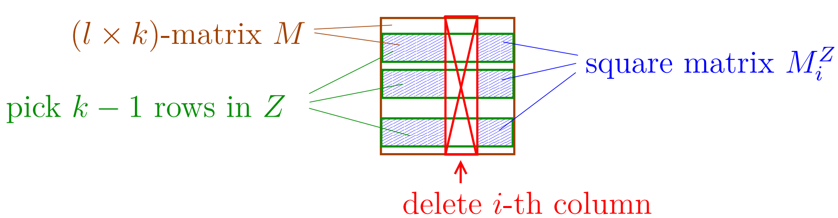
The idea, given a presentation $\langle x_1,\dots,x_k\,|\,r_1,\dots,r_l\rangle$, is as follows:

(1) We again look at the Jacobi matrix $J(P)$.
(2) Again we delete a column and we obtain a $(l\times(k-1))$-matrix.
(3) To obtain a square matrix we now consider just $k-1$ rows at any given time.
(4) To get a well-defined invariant we need to consider all possibilities to consider $k-1$ rows and we use the greatest common divisor to get an invariant.

To put this idea into practice we need to introduce the following notation:

Notation. Let R be a ring and let $M\in \mathrm{M}(l\times k,R)$ be a matrix.

(1) Given $i\in\{1,\dots,k\}$ we denote by M_i the $l\times(k-1)$-matrix which is the result of deleting the i-th column.
(2) Given $Z\subset\{1,\dots,l\}$ with $\#Z=k-1$ we denote by M^Z the $(k-1)\times k$-matrix that is given by the rows[66] in Z.
(3) Let $i\in\{1,\dots,k\}$ and let $Z\subset\{1,\dots,l\}$ with $\#Z=k-1$. We consider the corresponding $(k-1)\times(k-1)$-matrix $M_i^Z:=(M^Z)_i$.



The following definition is surely familiar from an earlier algebra course:

Definition. Let R be a commutative ring.

(1) Given $r,s\in R$ we write $r|s$ if there exists a $t\in R$ with $s=t\cdot r$.
(2) Given a subset $M\subset R$ we say that $r\in R$ is a divisor of M if for every $m\in M$ we have $r|m$.
(3) Let R be a unique factorization domain (UFD). Let $M\subset R$ be a subset. There exists an element $\gcd_R(M)\in R$ such that $\gcd_R(M)$ is a divisor of M and such that if t is another divisor of $\gcd_R(M)$, then $t|\gcd_R(M)$. Note that $\gcd_R(M)$ is well-defined up to multiplication by a unit in R.

Example. Let R be a UFD. We have $\gcd_R(R)=1$ and $\gcd_R(\emptyset)=0$. □

[66]The mnemonic device here is that "Z" stands for "Zeilen". Unfortunately R for "rows" is already taken.

The following lemma is a generalization of the Presentation Function Quotient Proposition 8.4.

Lemma 8.10. (Independence-of-Generator Lemma) Let $P = \langle x_1, \dots, x_k \,|\, r_1, \dots, r_l \rangle$ be a presentation and let $\Phi\colon P \to H$ be a non-trivial homomorphism to a free abelian group H.

(1) There exists an $i \in \{1, \dots, k\}$ with $\Phi(x_i) \neq 0$.
(2) Let $i \in \{1, \dots, k\}$ with $\Phi(x_i) \neq 0$. The term[67] [68]

$$\Delta_{P,\Phi} := \frac{\gcd_{\mathbb{Z}[H]}\{\det(J^\Phi(P)_i^Z)\}_{\substack{Z \subset \{1,\dots,l\} \\ \#Z = k-1}}}{\Phi_*(x_i - 1)} \in \mathbb{Q}(H)$$

is well-defined (i.e. independent of the choice of i) up to multiplication by a sign.

Proof. Let $P = \langle x_1, \dots, x_k \,|\, r_1, \dots, r_l \rangle$ be a presentation and let $\Phi\colon P \to H$ be a non-trivial homomorphism to a free abelian group H. The proof of Statement 1 is identical to the proof of the Presentation Function Quotient Proposition 8.4 (1). Thus let us turn to the proof of Statement 2. Let $i, j \in \{1, \dots, k\}$ with $\Phi(x_i) \neq 0$ and $\Phi(x_j) \neq 0$. Furthermore let $Z \subset \{1, \dots, l\}$ with $\#Z = k - 1$. It follows easily from the Jacobi Matrix–Vector Lemma 8.7, the fact that $\Phi_*\colon \mathbb{Z}[P] \to \mathbb{Z}[H] \subset \mathbb{Q}(H)$ is a ring homomorphism and the Matrix–Quotient Lemma 8.8 that[69]

$$\frac{\det(J^\Phi(P)_i^Z)}{\Phi_*(x_i - 1)} \doteq \frac{\det(J^\Phi(P)_j^Z)}{\Phi_*(x_j - 1)} \in \mathbb{Q}(H).$$

It follows that

$$\det(J^\Phi(P)_i^Z) \cdot \Phi_*(x_j - 1) \doteq \det(J^\Phi(P)_j^Z) \cdot \Phi_*(x_i - 1) \in \mathbb{Z}[H].$$

Taking greatest common divisors in $\mathbb{Z}[H]$ we obtain the following equality in $\mathbb{Z}[H]$:

$$\gcd_{\mathbb{Z}[H]}\{\det(J^\Phi(P)_i^Z)\cdot\Phi_*(x_j - 1)\}_{\substack{Z \subset \{1,\dots,l\} \\ \#Z = k-1}} \doteq \gcd_{\mathbb{Z}[H]}\{\det(J^\Phi(P)_j^Z)\cdot\Phi_*(x_i - 1)\}_{\substack{Z \subset \{1,\dots,l\} \\ \#Z = k-1}}.$$

When we take greatest common divisors we can pull out common factors. Thus we obtain that

$$\gcd_{\mathbb{Z}[H]}\{\det(J^\Phi(P)_i^Z)\}_{\substack{Z \subset \{1,\dots,l\} \\ \#Z = k-1}} \cdot\Phi_*(x_j - 1) \doteq \gcd_{\mathbb{Z}[H]}\{\det(J^\Phi(P)_j^Z)\}_{\substack{Z \subset \{1,\dots,l\} \\ \#Z = k-1}} \cdot\Phi_*(x_i - 1).$$

From this we finally obtain the desired equality

$$\frac{\gcd_{\mathbb{Z}[H]}\{\det(J^\Phi(P)_i^Z)\}_{\substack{Z \subset \{1,\dots,l\} \\ \#Z = k-1}}}{\Phi_*(x_i - 1)} \doteq \frac{\gcd_{\mathbb{Z}[H]}\{\det(J^\Phi(P)_j^Z)\}_{\substack{Z \subset \{1,\dots,l\} \\ \#Z = k-1}}}{\Phi_*(x_j - 1)} \in \mathbb{Q}(H). \blacksquare$$

8.6.3. **Proof of Theorem 8.5: Conclusion.** The following lemma is generalization of the Alexander Function Theorem 8.5.

[67] Note that for a presentation of deficiency one we have to take $Z = \{1, \dots, k-1\}$ and we see that the definition agrees with the definition from the Presentation Function Quotient Proposition 8.4.
[68] By [**Lan93**, Theorem IV.23] we know that $\mathbb{Z}[H]$ is a unique factorization domain.
[69] One might think that we are done, but since $\mathbb{Q}(H)$ is a field it does not make much sense to consider greatest common divisors at this point.

Lemma 8.11. (Independence-of-Presentation Lemma) Let π be a group and let $\Phi\colon \pi \to H$ be a non-trivial homomorphism to a free abelian group H. If $\gamma\colon P \to \pi$ and $\delta\colon Q \to \pi$ are finite presentations, then

$$\Delta_{P,\Phi} \doteq \Delta_{Q,\Phi\circ\delta} \in \mathbb{Q}(H).$$

Proof. By the Tietze Theorem 8.9 we know that the two presentations $\gamma\colon \pi \to P$ and $\delta\colon \pi \to Q$ are related by a finite sequence of Tietze transformations, as defined on page 115. So it suffices to deal with the case that the presentations P and Q are related by a single Tietze transformation.

(T1) We consider the following situation: We have

$$\underbrace{\langle x_1,\dots,x_k \mid r_1,\dots,r_l,s\rangle}_{=Q} = \underbrace{\langle x_1,\dots,x_k \mid r_1,\dots,r_l\rangle}_{=P} \xrightarrow{\gamma} \pi$$

where $s \in \langle\langle r_1,\dots,r_l\rangle\rangle \subset \langle X\rangle$. We see that the Jacobi matrix $J(Q)$ equals the Jacobi matrix $J(P)$, except that we are adding the row $\frac{\partial s}{\partial x_i}$, $i=1,\dots,k$. But since $s \in \langle\langle r_1,\dots,r_l\rangle\rangle$ one obtains easily from the Fox Derivative Properties Lemma 8.2 (5) that there exist $\lambda_1,\dots,\lambda_l \in \mathbb{Z}[P]$ such that for every $i \in \{1,\dots,k\}$ we have

$$\frac{\partial s}{\partial x_i} = \sum_{j=1}^{l} \lambda_j \cdot \frac{\partial r_j}{\partial x_i}.$$

In other words the extra row of $J(Q)$ is just a linear combination of the previous rows.

Claim. Let $Z \subset \{1,\dots,l\}$ with $\#Z = k-2$. For any $i \in \{1,\dots,k\}$ we have

$$\det\left(J^{\Phi}(Q)_i^{Z\cup\{l+1\}}\right) = \sum_{\substack{j=1\\ j\notin Z}}^{l} \Phi_*(\lambda_j)\cdot \det\left(J^{\Phi}(Q)_i^{\{Z\cup\{j\}}\right).$$

Proof. First note that:

$$\begin{aligned}(l+1)\text{-st row of } J^{\Phi}(Q) &= \Phi_*((l+1)\text{-st row of } Q) \overset{\text{by the above discussion}}{\underset{\downarrow}{=}} \Phi_*\Big(\sum_{j=1}^{l}\lambda_j\cdot j\text{-th row of } Q\Big)\\ &\underset{\substack{\uparrow\\ \text{since } \Phi_* \text{ is a ring homomorphism}}}{=} \sum_{j=1}^{l}\Phi_*(\lambda_j)\cdot\Phi_*(j\text{-th row of } Q).\end{aligned}$$

The claim follows easily from this calculation and the multilinearity of the determinant. ⊞

Finally note that for any $i \in \{1,\dots,k\}$ we have

$$\begin{aligned}\gcd_{\mathbb{Z}[H]}\{\det(J^{\Phi}(Q)_i^{Z})\}_{\substack{Z\subset\{1,\dots,l+1\}\\ \#Z=k-1}} &\overset{\substack{\text{follows easily from the above claim}\\ \downarrow}}{\doteq} \gcd_{\mathbb{Z}[H]}\{\det(J^{\Phi}(Q)_i^{Z})\}_{\substack{Z\subset\{1,\dots,l\}\\ \#Z=k-1}}\\ &\underset{\substack{\uparrow\\ \text{since the first } l \text{ rows of } J(Q) \text{ and } J(P) \text{ agree}}}{\doteq} \gcd_{\mathbb{Z}[H]}\{\det(J^{\Phi}(P)_i^{Z})\}_{\substack{Z\subset\{1,\dots,l\}\\ \#Z=k-1}},\end{aligned}$$

which then implies that $\Delta_{Q,\Phi\circ\gamma} \doteq \Delta_{P,\Phi\circ\gamma} \in \mathbb{Q}(H)$.

(T2) We consider the following situation: We have $x \notin \{x_1,\dots,x_k\}$, $s,t \in \langle x_1,\dots,x_k\rangle$ and the presentations

$$\overbrace{\langle x_1,\dots,x_k,x \mid r_1,\dots,r_l, s\cdot x\cdot t\rangle}^{=Q} \xrightarrow{\Theta} \overbrace{\langle x_1,\dots,x_k \mid r_1,\dots,r_l\rangle}^{=P} \xrightarrow{\gamma} \pi.$$
$$\begin{aligned} x_i &\mapsto x_i \\ x &\mapsto s^{-1}\cdot t^{-1}\end{aligned}$$

We see that
$$J(Q) \;=\; \begin{pmatrix} J(P) & 0 \\ * & s\end{pmatrix}.$$

Finally note that for any $i \in \{1,\dots,k\}$ we have

$$\begin{aligned}\gcd_{\mathbb{Z}[H]}\{\det(J^\Phi(Q)_i^Z)\}_{\substack{Z\subset\{1,\dots,l+1\}\\ \#Z=k}} &\overset{\substack{\text{the determinant is zero unless } l+1\in Z\\ \downarrow}}{\doteq} \gcd_{\mathbb{Z}[H]}\{\det(J^\Phi(Q)_i^Z)\}_{\substack{Z\subset\{1,\dots,l+1\}\\ \#Z=k,\ l+1\in Z}} \\ &\underset{\substack{\uparrow\\ \text{since } \Phi(s) \text{ is a unit}}}{\doteq} \gcd_{\mathbb{Z}[H]}\{\det(J^\Phi(P)_i^Z)\}_{\substack{Z\subset\{1,\dots,l\}\\ \#Z=k-1}},\end{aligned}$$

which then implies that $\Delta_{Q,\Phi\circ\gamma\circ\Theta} \doteq \Delta_{P,\Phi\circ\gamma} \in \mathbb{Q}(H)$. ■

Proof of the Alexander Function Theorem 8.5. We just proved the Independence-of-Presentation Lemma 8.11, which is a generalization of the Alexander Function Theorem 8.5. So we are done! ■

Exercises for Chapter 8.

Exercise 8.1. Does there exist a variation on the Fox derivative which satisfies the "real" product rule. More precisely: Let $\langle x_1,\dots,x_k\rangle$ be the free group on generators $x_1,\dots,x_k$ and let $i\in\{1,\dots,k\}$. Does there exist a map
$$D_i\colon \mathbb{Z}[\langle x_1,\dots,x_k\rangle] \;\to\; \mathbb{Z}[\langle x_1,\dots,x_k\rangle]$$
with the following properties:
(1) We have $D_i(x_j)=\delta_{ij}$.
(2) For any $u,v\in\mathbb{Z}[\langle x_1,\dots,x_k\rangle]$ we have $D_i(u\cdot v)=D_i(u)\cdot v+u\cdot D_i(v)$.
(3) D_i is a homomorphism of abelian groups.

Exercise 8.2. Let $m,n\in\mathbb{Z}$. We consider the Baumslag-Solitar group
$$\mathrm{BS}(m,n) \;=\; \langle x,y \,|\, x^{-1}\cdot y^m\cdot x \;=\; y^n\rangle.$$
We consider the epimorphism $\Phi\colon \mathrm{BS}(m,n)\to\langle t\rangle$ that is given by $\Phi(x)=t$ and $\Phi(y)=e$. Determine the Alexander function $\Delta_{\mathrm{BS}(m,n),\Phi}\in\mathbb{Q}(t)$.

Exercise 8.3. Show that given any polynomial $f(t)\in\mathbb{Z}[t^{\pm1}]$ there exists a group π of deficiency one and an epimorphism $\Phi\colon\pi\to\langle t\rangle$ such that $\Delta_{\pi,\Phi}\doteq\frac{f(t)}{t-1}$.

Exercise 8.4. Let K be the trefoil. Show that given any $m\in\mathbb{Z}$ there exists a presentation P for $\pi_1(S^3\setminus K)$ such that
$$\Delta_{P,\Phi} \;=\; \pm t^m\cdot\frac{t^2-t+1}{t-1}$$
where Φ denotes one of the two epimorphisms $\pi_1(S^3\setminus K)\to\langle t\rangle$.

Exercise 8.5. Let $T_{2,3}$ be the trefoil. Show that the presentation $\langle a, b \,|\, aba = bab \rangle$ can be turned into the presentation $\langle x, y \,|\, x^2 \cdot y^{-3} \rangle$ using Tietze transformations.
Hint. Consider $x = bab$ and $y = ab$.

Exercise 8.6. We consider the presentation $P = \langle a, b \,|\, aba = bab \rangle$. Show that there exist two epimorphisms $\Phi\colon P \to \mathbb{Z}$ and compute $\Delta_{P,\Phi}$.
Remark. The discussion on page 115 shows P is also a presentation for the fundamental group of the trefoil complement. But it is fun to go through the exercise without making use of this fact.

Exercise 8.7. Let R be a UFD. Convince yourself that $\gcd_R(\varnothing) = 0$.

Exercise 8.8. Let π be a group that admits a finite presentation and let $\Phi\colon \pi \to H$ be a non-trivial homomorphism to a free abelian group. Let $\gamma\colon P = \langle x_1, \ldots, x_k \,|\, r_1, \ldots, r_l \rangle \to \pi$ be a finite presentation. Given $m \in \mathbb{N}_0$ we define $E_m(P)$ to be the ideal of $\mathbb{Z}[H]$ generated by the $(k-m) \times (k-m)$-minors of the Jacobi matrix $J^{\Phi\circ\gamma}(P) \in \mathrm{M}(l \times k, \mathbb{Z}[H])$.

(a) Show that the ideal $E_m(P)$ does not depend on the choice of the presentation.

The greatest common divisor of $E_m(P)$ is sometimes called the m-th Alexander polynomial of (π, Φ).

(b) What is the relationship between the first Alexander polynomial and the Alexander function?
(c) Compute the Alexander polynomials for finitely generated free groups, for free abelian groups and for the groups $\pi_1(\Sigma_g)$ where Σ_g is the surface of genus g.
(d) Show that for $g \geq 2$ the surface group $\pi_1(\Sigma_g)$ is neither free nor free abelian.

Remark. This invariants are discussed in greater detail in [**CF77**, Chapter VII].

Exercise 8.9. Let π be a group that admits a presentation $P = \langle x_1, \ldots, x_k \,|\, r_1, \ldots, r_{k-1} \rangle$ of deficiency one and let $\Phi\colon \pi \to H$ be a non-trivial homomorphism to a free abelian group. Let $i \in \{1, \ldots, k\}$ such that $\Phi(x_i)$ is non-trivial. We introduce the following objects:

- Let $A \in \mathrm{M}(r \times r, \mathbb{Z}[H])$ be a square matrix. A *codimension m minor of A* is the determinant of any square matrix that one obtains from A by crossing out m rows and m columns.
- We define the *k-th order Alexander function of* (π, Φ) as

$$\Delta^k_{\pi,\Phi} \;:=\; \frac{\gcd_{\mathbb{Z}[H]}\{\text{all codimension } k \text{ minors of } J^\Phi(P)_i\}}{\Phi(x_i) - 1}.$$

  Note that by definition $\Delta^0_{\pi,\Phi} = \Delta_{\pi,\Phi}$.

Now we can formulate your tasks:

(a) Show that $\Delta^k_{\pi,\Phi}$ is well-defined up to "$\doteq$".
(b) Let $\pi = \mathbb{Z}^2 * \mathbb{Z}^2$ and let $\Phi\colon \pi \to \mathbb{Z}^2$ be the obvious homomorphism. Show that $\Delta_{\pi,\Phi} = 0$ and show $\Delta^1_{\pi,\Phi} \neq 0$.
(c) Let $g \in \mathbb{N}$. Use the above higher order Alexander functions to show that the genus g surface group $\langle x_1, \ldots, x_{2g} \,|\, [x_1, x_2] \cdot \cdots \cdot [x_{2g-1}, x_{2g}] \rangle$ is not a free group.

CHAPTER 9

The Alexander polynomial of knots and links

In this chapter we will use the results from the last chapter to finally introduce the Alexander polynomial of knots and links. The Alexander polynomial of a knot can be introduced in many different ways, a long list of different definitions is for example provided in [**Ari21**]. In this part of the lecture notes we use an approach due to Crowell–Fox [**CF77**] and Masaaki Wada [**Wad94**]. This approach has the great advantage that it requires only fundamental groups and it is completely self-contained.

9.1. **Alexander polynomial of oriented knots.** Before we turn to the definition of the Alexander polynomial of oriented knots and link let us recall, for the reader's convenience, the group theoretic definition from page 111 we rely on:

Definition. Let π be a group, let $\Phi\colon \pi \to H$ be a non-trivial homomorphism to a torsion-free abelian group and let $\gamma\colon \langle x_1,\dots,x_n \,|\, r_1,\dots,r_{n-1}\rangle \to \pi$ be a presentation of deficiency 1. We pick $i \in \{1,\dots,n\}$ such that $(\Phi\circ\gamma)(x_i)$ is non-trivial. We set

$$\Delta_{\pi,\Phi} \;\coloneqq\; \frac{\det\left((\Phi\circ\gamma)_*\begin{pmatrix} \frac{\partial r_1}{\partial x_1} & \dots & \cancel{\frac{\partial r_1}{\partial x_i}} & \dots & \frac{\partial r_1}{\partial x_n} \\ \vdots & & \vdots & & \vdots \\ \frac{\partial r_{n-1}}{\partial x_1} & \dots & \cancel{\frac{\partial r_{n-1}}{\partial x_i}} & \dots & \frac{\partial r_{n-1}}{\partial x_n}\end{pmatrix}\right)}{(\Phi\circ\gamma)_*(x_i-1)} \;\in\; \mathbb{Q}(H).$$

In the Presentation Function Quotient Proposition 8.4 and the Alexander Function Theorem 8.5 we showed that $\Delta_{\pi,\Phi}$ is well-defined, i.e. independent of the choice of the presentation and the choice of i, up to multiplication by an element of the form $\pm h$ with $h \in H$.

After this reminder we turn to the definition of the Alexander polynomial of oriented knots:

Definition. Let $K \subset S^3$ be an oriented knot. As usual we denote by μ_K a meridian of K. By the Link Group-Abelianization Corollary 5.7 there exists a unique epimorphism

$$\Phi_K\colon \pi_1(S^3\setminus K) \;\to\; \langle t\rangle$$

such that $\Phi_K(\mu_K) = t$. By the Wirtinger Presentation Proposition 5.5 we know that $\pi_1(S^3\setminus K)$ admits a presentation of deficiency one. Thus it makes sense to define[70]

$$\Delta_K \;=\; \Delta_K(t) \;\coloneqq\; (t-1)\cdot \underbrace{\Delta_{\pi_1(S^3\setminus K),\Phi_K}}_{\text{as defined above}} \;\in\; \mathbb{Q}(t).$$

We refer to $\Delta_K(t)$ as the Alexander polynomial of K.

Example.

[70]Here we use the obvious identification of the group ring $\mathbb{Z}[\langle t\rangle]$ with the Laurent polynomial ring $\mathbb{Z}[t^{\pm 1}]$. This way we end up with the quotient field $\mathbb{Q}(t)$.

(1) Let K be the trivial knot. We calculate

$$\Delta_K(t) \underset{\substack{\uparrow \\ \text{by the Trivial Knot-}\pi_1\text{-Proposition 3.6}}}{\doteq} (t-1)\cdot\Delta_{\mathbb{Z},\Phi_K} \underset{\substack{\uparrow \\ \text{see page 112}}}{\doteq} (t-1)\cdot\frac{1}{t-1} = 1.$$

(2) Let $K \subset S^3$ be the trefoil. We calculate

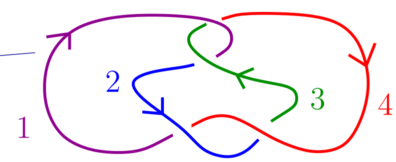
$$\Delta_K(t) \underset{\substack{\uparrow \\ \text{by definition}}}{\doteq} (t-1)\cdot\Delta_{\pi_1(S^3\setminus K),\Phi_K} \underset{\substack{\uparrow \\ \text{see page 110}}}{\doteq} (t-1)\cdot\frac{t^2-t+1}{t-1} = t^2-t+1.$$

(3) Let K be the oriented figure-8 knot as in the figure below. By the Figure 8 Knot–π_1-Lemma 5.8 there exists an isomorphism

$$\gamma\colon P := \langle x_1, x_2, x_3, x_4 \,|\, \underbrace{x_3\cdot x_2\cdot x_3^{-1}\cdot x_1^{-1}}_{=r_1}, \underbrace{x_4^{-1}\cdot x_3\cdot x_4\cdot x_2^{-1}}_{=r_2}, \underbrace{x_1\cdot x_4\cdot x_1^{-1}\cdot x_3^{-1}}_{=r_3}\rangle \xrightarrow{\cong} \pi_1(S^3\setminus K)$$

such that the x_i correspond to meridians.

figure-8 knot



Note that

$$\begin{aligned} J^{\Phi_K\circ\gamma}(P) &= (\Phi_K\circ\gamma)_*\left(\frac{\partial r_i}{\partial x_j}\right)_{\substack{i=1,2,3\\ j=1,2,3,4}} \\ &\underset{\substack{\uparrow \\ \text{uses that for a relation } aba^{-1}d^{-1} \text{ we} \\ \text{have } aba^{-1}d^{-1}=e \text{ and } aba^{-1}=d}}{=} (\Phi_K\circ\gamma)_*\begin{pmatrix} -1 & x_3 & 1-x_1 & 0 \\ 0 & -1 & x_4^{-1} & x_4^{-1}x_3-x_4^{-1} \\ 1-x_3 & 0 & -1 & x_1 \end{pmatrix} \underset{\substack{\uparrow \\ \text{since the } x_i \text{ are meridians we} \\ \text{have } (\Phi_K\circ\gamma)(x_i)=t}}{=} \begin{pmatrix} -1 & t & 1-t & 0 \\ 0 & -1 & t^{-1} & 1-t^{-1} \\ 1-t & 0 & -1 & t \end{pmatrix}. \end{aligned}$$

It follows that

$$\begin{aligned} \Delta_K(t) &\doteq (t-1)\cdot\frac{\det(J^{\Phi_K\circ\gamma}(P)_1)}{(\Phi_K\circ\gamma)_*(x_i-1)} \doteq (t-1)\cdot\frac{\det\begin{pmatrix} t & 1-t & 0 \\ -1 & t^{-1} & 1-t^{-1} \\ 0 & -1 & t \end{pmatrix}}{t-1} \\ &\doteq -1+3t-t^2. \end{aligned}$$

□

The following proposition explains in particular why we called the above objects "polynomials".

Proposition 9.1. (Knot–Alexander Polynomial Proposition)

(1) The Alexander polynomial $\Delta_K(t)$ of an oriented knot K is well-defined up to "$\doteq$", i.e. it is well-defined up to multiplication by $\pm t^k$ for some $k \in \mathbb{Z}$.
(2) The Alexander polynomial Δ_K is an element of $\mathbb{Z}[t^{\pm 1}]$.
(3) If K and $\widetilde{K}$ are two oriented knots that are smoothly isotopic, then $\Delta_K \doteq \Delta_{\widetilde{K}}$.

Proof.

(1) This statement follows immediately from the Alexander Function Theorem 8.5.[71]

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[71]There is an alternative approach to showing that we have a well-defined knot invariant: By the Reidemeister Moves Theorem 4.5 we know that any two diagrams for a link are related by a sequence of

(2) Let $K \subset S^3$ be an oriented knot and let $\Phi_K\colon \pi_1(S^3 \setminus K)_{\text{ab}} \to \langle t \rangle$ be the unique epimorphism with $\Phi_K(\mu_K) = t$. By the Wirtinger Presentation Proposition 5.5 we know that there exists a presentation $\gamma\colon P := \langle x_1, \dots, x_n \,|\, r_1, \dots, r_{n-1} \rangle \xrightarrow{\cong} \pi_1(S^3 \setminus K)$ where each x_i corresponds to a meridian. We see that

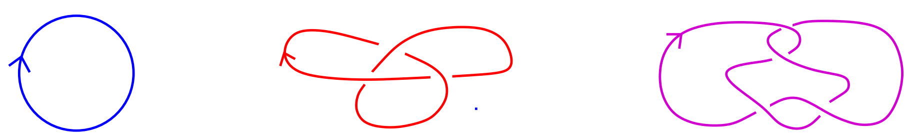
$$\begin{aligned}\Delta_K(t) &\overset{\substack{\text{by definition}\\ \downarrow}}{\doteq} (t-1)\cdot\Delta_{\pi_1(S^3\setminus K),\Phi_K} \overset{\substack{\text{by definition}\\ \downarrow}}{\doteq} (t-1)\cdot\frac{\det(J^{\Phi_K\circ\gamma}(P)_1)}{(\Phi_K\circ\gamma)(x_1)-1}\\ &\qquad\qquad\qquad\qquad\qquad \underset{\substack{\uparrow\\ \text{since } x_1 \text{ is a meridian}}}{=} (t-1)\cdot\frac{\det(J^{\Phi_K\circ\gamma}(P)_1)}{t-1} \;=\; \underbrace{\det(J^{\Phi_K\circ\gamma}(P)_1)}_{\substack{\text{determinant of a matrix}\\ \text{with entries in } \mathbb{Z}[t^{\pm 1}]}} \in \mathbb{Z}[t^{\pm 1}].\end{aligned}$$

(3) Let K and $\widetilde{K}$ be two oriented knots that are smoothly isotopic. As in the Isotopic Link-π_1-Lemma 3.2 we note that it follows from the Link–Smooth Isotopy Proposition 2.3 that there exists an orientation-preserving diffeomorphism $f\colon S^3 \setminus K \to S^3 \setminus \widetilde{K}$. It follows easily from the Meridian Proposition 2.17 that the image of a meridian of K is a meridian of $\widetilde{K}$. This shows that there exists an isomorphism $f_*\colon \pi_1(S^3\setminus K) \to \pi_1(S^3\setminus\widetilde{K})$ such that the following diagram commutes:

$$\begin{tikzcd} \pi_1(S^3 \setminus K) \arrow[rr, "f_*", "\cong"'] \arrow[dr, "\Phi_K"'] & & \pi_1(S^3 \setminus \widetilde{K}) \arrow[dl, "\Phi_{\widetilde{K}}"] \\ & \langle t \rangle. & \end{tikzcd}$$

The desired statement now follows from the Alexander Function–Functorial Lemma 8.6. ■

Example. It follows from the above calculations and the Knot–Alexander Polynomial Proposition 9.1 that the trivial knot, the trefoil and the figure-8 knot are pairwise not smoothly isotopic.



$\Delta_K \doteq 1$ $\qquad$ $\Delta_K \doteq 1 - t + t^2$ $\qquad$ $\Delta_K \doteq 1 - 3t + t^2$

Thus we recover the Trefoil–Figure 8–Non-Isotopic Corollary 5.9, but this time using a much more systematic approach. □

It is now natural to try to calculate the Alexander polynomial of more oriented knots, e.g. of torus knots. For the torus knots $T(p,q)$ we showed in the Torus Knot-π_1-Proposition 3.8 that $\pi_1(S^3 \setminus T(p,q)) \cong \langle x, y \,|\, x^p \cdot y^{-q} \rangle$. The slight catch with this calculation is that the

Reidemeister moves. This theorem can be used to show that any two Wirtinger presentations are related by a sequence of Wirtinger presentations, in particular they are related by a sequence of presentations of deficiency one. This way one can show that the Alexander polynomial (defined via Wirtinger presentations) is well-defined without going through presentations of uncertain deficiency. This approach is carried out in [**Wad94**].

role of the orientation of $T(p,q)$ is not entirely clear and we do not know what element in the presentation corresponds to a meridian. Therefore we will first develop the theory of Alexander polynomials a little further, before we return to torus knots.

9.2. Alexander polynomial of oriented links. We move to the definition of the Alexander polynomial of an oriented link with ≥ 2 components:

Definition. Let $m \geq 2$ and let $L \subset S^3$ be an oriented m-component link. By the Link Group-Abelianization Corollary 5.7 there exists a unique epimorphism

$$\Phi_L\colon \pi_1(S^3 \setminus L) \;\to\; \langle t_1, \dots, t_m \rangle_{\mathrm{ab}}$$

such that for $i = 1, \dots, m$ we have $\Phi_L(\mu_{L_i}) = t_i$. By the Wirtinger Presentation Proposition 5.5 we know that $\pi_1(S^3 \setminus L)$ admits a presentation of deficiency one. Thus it makes sense to define[72]

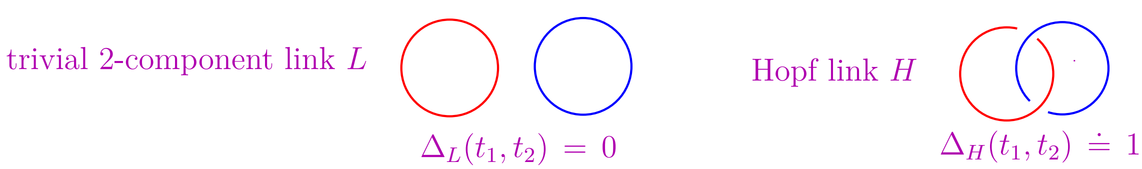
$$\Delta_L \;=\; \Delta_L(t_1, \dots, t_m) \;:=\; \underbrace{\Delta_{\pi_1(S^3\setminus L),\Phi_L}}_{\substack{\text{as defined on}\\ \text{page 111}}} \;\in\; \mathbb{Q}(t_1, \dots, t_m)$$

as the Alexander polynomial of L.

Example.

(1) We consider the m-component trivial link L with $m \geq 2$ with any choice of orientation. We calculate
$$\Delta_L(t_1, \dots, t_m) \;\underset{\substack{\uparrow\\ \text{see page 42}}}{\doteq}\; \Delta_{\langle x_1,\dots,x_m\rangle,\Phi_L} \;\underset{\substack{\uparrow\\ \text{follows from the calculation on}\\ \text{page 112 and the fact that } m\geq 2}}{\doteq}\; 0.$$

(2) Let H be the Hopf link with any choice of orientation. We calculate
$$\Delta_H(t_1, t_2) \;\underset{\substack{\uparrow\\ \text{by the Hopf Link–Lemma 3.13}}}{\doteq}\; \Delta_{\mathbb{Z}^2,\Phi_L} \;\underset{\substack{\uparrow\\ \text{see page 112}}}{\doteq}\; 1.$$



We continue with the analogue of the Knot–Alexander Polynomial Proposition 9.1.[73]

Proposition 9.2. (Link–Alexander Polynomial Proposition) Let $m \in \mathbb{N}_{\geq 2}$.

(1) The Alexander polynomial $\Delta_L(t_1, \dots, t_m)$ of an oriented m-component link L is well-defined up to "$\doteq$", i.e. it is well-defined up to multiplication by a monomial in $t_1, \dots, t_m$.
(2) The Alexander polynomial Δ_L is an element of $\mathbb{Z}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$.

---

[72]Here we use the obvious identification of the group ring $\mathbb{Z}[\langle t_1, \dots, t_m \rangle_{\mathrm{ab}}]$ with the Laurent polynomial ring $\mathbb{Z}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$. This way we end up with the quotient field $\mathbb{Q}(t_1, \dots, t_m)$.

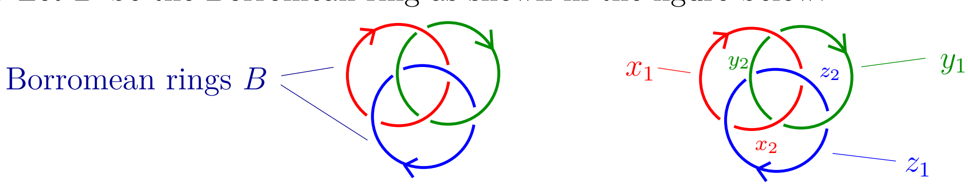
[73]If one compares the proofs of the Knot–Alexander Polynomial Proposition 9.1 (2) and the Link–Alexander Polynomial Proposition 9.2 (2) one can see where the slight asymmetry between knots and links in the definition of the Alexander polynomial comes from.

(3) If L and $\widetilde{L}$ are two oriented m-component links that are smoothly isotopic, then $\Delta_L \doteq \Delta_{\widetilde{L}}$.

Proof.

(1) This statement follows again immediately from the Alexander Function Theorem 8.5.
(2) Let $m \geq 2$ and let $L = L_1 \cup \cdots \cup L_m \subset S^3$ be an oriented m-component link. We denote by $\Phi_L \colon \pi_1(S^3 \setminus L)_{\text{ab}} \to \langle t_1, \ldots, t_m \rangle_{\text{ab}}$ the unique epimorphism with $\Phi_L(\mu_{L_i}) = t_i$. By the Wirtinger Presentation Proposition 5.5 we know that there exists a presentation $\gamma \colon P := \langle x_1, \ldots, x_n \,|\, r_1, \ldots, r_{n-1} \rangle \xrightarrow{\cong} \pi_1(S^3 \setminus L)$ where each x_i is a meridian. Since $m \geq 2$ we can pick $i_1, i_2 \in \{1, \ldots, n\}$ with $(\Phi_L \circ \gamma)(x_{i_1}) = t_1$ and with $(\Phi_L \circ \gamma)(x_{i_2}) = t_2$. Note that by definition we have the following equality in $\mathbb{Q}(t_1, \ldots, t_m)$:
$$\Delta_L(t_1, \ldots, t_m) \;\doteq\; \frac{\det(J^{\Phi_L \circ \gamma}(P)_{i_1})}{t_1 - 1} \;\doteq\; \frac{\det(J^{\Phi_L \circ \gamma}(P)_{i_2})}{t_2 - 1}.$$
This implies that we have the following equality in $\mathbb{Q}(t_1, \ldots, t_m)$:
$$\underbrace{\det(J^{\Phi_L \circ \gamma}(P)_{i_1})}_{\in \mathbb{Z}[t_1^{\pm 1}, \ldots, t_m^{\pm 1}]} \cdot (t_2 - 1) \;\doteq\; \underbrace{\det(J^{\Phi_L \circ \gamma}(P)_{i_2})}_{\in \mathbb{Z}[t_1^{\pm 1}, \ldots, t_m^{\pm 1}]} \cdot (t_1 - 1).$$
We make the following observations:
(a) Note that $J^{\Phi_L \circ \gamma}(P)_{i_1}$ and $J^{\Phi_L \circ \gamma}(P)_{i_2}$ are matrices with entries in $\mathbb{Z}[t_1^{\pm 1}, \ldots, t_m^{\pm 1}]$. It follows that $\det(J^{\Phi_L \circ \gamma}(P)_{i_1})$ and $\det(J^{\Phi_L \circ \gamma}(P)_{i_2})$ both lie in $\mathbb{Z}[t_1^{\pm 1}, \ldots, t_m^{\pm 1}]$.
(b) We consider again the above equality
$$\det(J^{\Phi_L \circ \gamma}(P)_{i_1}) \cdot (t_2 - 1) \;\doteq\; \det(J^{\Phi_L \circ \gamma}(P)_{i_2}) \cdot (t_1 - 1).$$
By (a) we know that all factors lie in $\mathbb{Z}[t_1^{\pm 1}, \ldots, t_m^{\pm 1}]$. Since $\mathbb{Z}[t_1^{\pm 1}, \ldots, t_m^{\pm 1}]$ is a UFD and since $\gcd(t_1 - 1, t_2 - 1) = 1$ we see that $\det(J^{\Phi_L \circ \gamma}(P)_{i_1}) = (t_1 - 1) \cdot f$ for some $f \in \mathbb{Z}[t_1^{\pm 1}, \ldots, t_m^{\pm 1}]$.
(c) It follows that
$$\Delta_L(t_1, \ldots, t_m) \;\doteq\; \frac{\det(J^{\Phi_L \circ \gamma}(P)_{i_1})}{t_1 - 1} \;=\; \frac{(t_1 - 1) \cdot f}{t_1 - 1} \;=\; f \;\in\; \mathbb{Z}[t_1^{\pm 1}, \ldots, t_m^{\pm 1}].$$
(3) The proof of this statement is basically identical to the proof of the Knot–Alexander Polynomial Proposition 9.1 (3). ■

Example. Let B be the Borromean ring as shown in the figure below.



From the diagram and the Wirtinger Presentation Proposition 5.5 we obtain a presentation with generators $x_1, x_2, y_1, y_2, z_1, z_2$ and relations $p_1, p_2, q_1, q_2, r_1, r_2$ where
$$\begin{aligned} p_1 &= x_1 \cdot y_1 \cdot x_1^{-1} \cdot y_2^{-1} \\ p_2 &= z_2^{-1} \cdot x_2 \cdot z_2 \cdot x_1^{-1} \end{aligned}$$
and where the other relations are given by cyclically permuting the letters x, y, z. We consider the obvious epimorphism $\Phi_B \colon \pi_1(S^3 \setminus B) \to \langle x, y, z \rangle_{\text{ab}}$. Let P be the presentation

we obtain by removing the last relation. Note that

$$\det(J^{\Phi_B}(P)_6) \;=\; \det\begin{pmatrix} 1-y & 0 & x & -1 & 0 & \cancel{0} \\ -1 & z^{-1} & 0 & 0 & 0 & \cancel{-z^{-1}+z^{-1}\cdot x} \\ 0 & 0 & 1-z & 0 & y & \cancel{-1} \\ 0 & -x^{-1}+x^{-1}\cdot y & -1 & x^{-1} & 0 & \cancel{0} \\ z & -1 & 0 & 0 & 1-x & \cancel{0} \end{pmatrix}.$$

It follows that

$$\Delta_B \underset{\uparrow}{=} \frac{\det(J^{\Phi_B}(P))_6)}{\Phi_B(z_2)-1} \;\doteq\; \frac{(1-z)\cdot(1-x)\cdot(1-y)\cdot(1-z)}{z-1} \;\doteq\; (1-x)\cdot(1-y)\cdot(1-z).$$

the author would like to point that he calculated $\det(J^{\Phi_B}(P)_6)$ by hand

$\square$

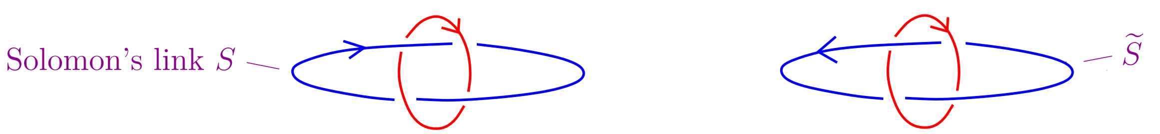
9.3. **The role of orientations.**

A priori the Alexander polynomial of a knot and a link depends on a choice of an orientation for each component. In this section we will study to what degree the Alexander polynomial depends on this choice. We start out with the following lemma:

Lemma 9.3. (Alexander Polynomial–Orientation Lemma) Let L be an oriented m-component link and let $\epsilon_1, \ldots, \epsilon_m \in \{-1, 1\}$. We denote by $L^{(\epsilon_1,\ldots,\epsilon_m)}$ the link L but where we flipped the orientation of the i-th component precisely if $\epsilon_i = -1$. Then

$$\Delta_{L^{(\epsilon_1,\ldots,\epsilon_m)}}(t_1, \ldots, t_m) \;\doteq\; \Delta_L(t_1^{\epsilon_1}, \ldots, t_m^{\epsilon_m}).$$

Example. In the figure we show Solomon's link S.



$\Delta_S(t_1, t_2) \;\doteq\; 1 + t_1 \cdot t_2$ $\qquad$ $\Delta_{\widetilde{S}}(t_1, t_2) \;\doteq\; 1 + t_1^{-1} \cdot t_2$

We see that if we flip the orientation of the blue component, then the Alexander polynomials are not equivalent, so the oriented links are not smoothly isotopic. $\square$

Proof. Let L be an oriented m-component link and let $\epsilon_1, \ldots, \epsilon_m \in \{-1, 1\}$. We make a few preparations:

(1) We write $L' := L^{(\epsilon_1,\ldots,\epsilon_m)}$.
(2) We write $\pi := \pi_1(S^3 \setminus L) = \pi_1(S^3 \setminus L')$.
(3) (a) Let $\mu_1, \ldots, \mu_m$ be meridians for the oriented link L.
  (b) Let $\mu'_1, \ldots, \mu'_m$ be the corresponding meridians for the oriented link L'.
(4) It follows from the Meridian–Symmetries Lemma 2.18 that $\mu'_i = \mu_i$ if $\epsilon_i = 1$ and that $\mu'_i = \mu_i^{\mathrm{rev}}$ if $\epsilon_i = -1$.
(5) As on page 124 we introduce the following notation:
  (a) We denote by $\Phi_L \colon \pi \to \langle t_1, \ldots, t_m\rangle_{\mathrm{ab}}$ the epimorphism given by $\Phi_L(\mu_i) = t_i$.
  (b) We denote by $\Phi_{L'} \colon \pi \to \langle t_1, \ldots, t_m\rangle_{\mathrm{ab}}$ the epimorphism given by $\Phi_{L'}(\mu'_i) = t_i$.
(6) We denote by $\Theta \colon \langle t_1, \ldots, t_m\rangle_{\mathrm{ab}} \to \langle t_1, \ldots, t_m\rangle_{\mathrm{ab}}$ the homomorphism that is given by $t_i \mapsto t_i^{\epsilon_i}$. Note that it follows from (4) and the Loop-π_1-Lemma 3.14 (3) that the following diagram commutes:

$$\begin{tikzcd} & \pi \arrow[dl, "\Phi_L"'] \arrow[dr, "\Phi_{L'}"] & \\ \langle t_1,\dots,t_m\rangle_{\mathrm{ab}} \arrow[rr, "\Theta"] & & \langle t_1,\dots,t_m\rangle_{\mathrm{ab}}. \end{tikzcd}$$

Now we turn to the actual proof of the promised equality. In the following we first consider the case $m \geq 2$. Sharp eyes will notice that the desired result is actually an immediate consequence of the Alexander Function–Functorial Lemma 8.6 (2). Alternatively one can also do a direct calculation:

- We pick a presentation $P = \langle x_1,\dots,x_n \,|\, r_1,\dots,r_{n-1}\rangle$ of deficiency one for π.
- We pick an $i \in \{1,\dots,n\}$ such that $\Phi_L(x_i)$ is non-trivial.

We then see that

$$\begin{aligned} \Delta_{L'}(t_1,\dots,t_m) &\overset{\substack{\text{by definition of } \Delta_{L'}\\ \downarrow}}{\doteq} \frac{\det(\Phi_{L'*}(J(P)_i))}{\Phi_{L'*}(x_i-1)} \overset{\substack{\text{since } \Phi_{L'} = \Theta\circ\Phi_L\\ \downarrow}}{\doteq} \frac{\det((\Theta\circ\Phi_L)_*(J(P)_i))}{(\Theta\circ\Phi_L)_*(x_i-1)} \overset{\substack{\text{since determinants commute}\\ \text{with ring homomorphisms}\\ \downarrow}}{\doteq} \Theta_*\Big(\frac{\det(\Phi_{L*}(J(P)_i))}{\Phi_{L*}(x_i-1)}\Big) \\ &\underset{\substack{\uparrow\\ \text{by definition of } \Delta_L}}{\doteq} \Theta_*(\Delta_L(t_1,\dots,t_m)) \underset{\substack{\uparrow\\ \text{by definition of } \Theta}}{\doteq} \Delta_L(t_1^{\epsilon_1},\dots,t_m^{\epsilon_m}). \end{aligned}$$

The case $m=1$ is basically identical, we just need to multiply the intermediate terms by $t-1$ and we need to notice that $t^{-1}-1 = (-t^{-1})\cdot(t-1) \doteq t-1$. ∎

A perspicacious reader might have noticed that all the Alexander polynomials that we have calculated so far are symmetric, up to "$\doteq$". It turns out that this is not a coincidence, this is in fact a general phenomenon:

Theorem 9.4. (Alexander Polynomial–Symmetry Theorem) Given any oriented m-component link L we have

$$\Delta_L(t_1,\dots,t_m) \;\doteq\; \Delta_L(t_1^{-1},\dots,t_m^{-1}).$$

Proof. The proof of this theorem is much harder than all the other results on Alexander polynomials. We outsource it to the next chapter. ∎

Next we recall the following definition from page 18:

Definition. Let $L \subset S^3$ be an oriented m-component link. The reverse L^{rev} is defined as the link L with the reverse orientation. In other words, $L^{\mathrm{rev}} = L^{(-1,\dots,-1)}$.

The following lemma, which builds on the Alexander Polynomial–Symmetry Theorem 9.4, unfortunately shows that the Alexander polynomial cannot be used to distinguish a link from its reverse.

Lemma 9.5. (Alexander Polynomial–Reverse Lemma) For any oriented m-component link $L \subset S^3$ we have

$$\Delta_{L^{\mathrm{rev}}}(t_1,\dots,t_m) \;\doteq\; \Delta_L(t_1,\dots,t_m).$$

Proof. We note that

$$\Delta_{L^{\mathrm{rev}}}(t_1,\dots,t_m) \underset{\substack{\uparrow\\ \text{follows from the Alexander}\\ \text{Polynomial–Orientation Lemma 9.3}}}{\doteq} \Delta_L(t_1^{-1},\dots,t_m^{-1}) \underset{\substack{\uparrow\\ \text{by the Alexander Polynomial–Symmetry}\\ \text{Theorem 9.4}}}{\doteq} \Delta_L(t_1,\dots,t_m).$$

∎

9.4. **Alexander polynomials of unoriented knots.** The Alexander Polynomial–Reverse Lemma 9.5 implies that for knots the Alexander polynomial does not depend on the choice of an orientation. This is a double edge sword: On the negative side this has the consequence that we cannot use Alexander polynomial to distinguish an oriented knot from its reverse. On the positive side it has the advantage that we get an invariant of unoriented knots and when calculating the invariant, we do not have to worry about orientations:

Definition. Let $K \subset S^3$ be a knot. We pick an epimorphism

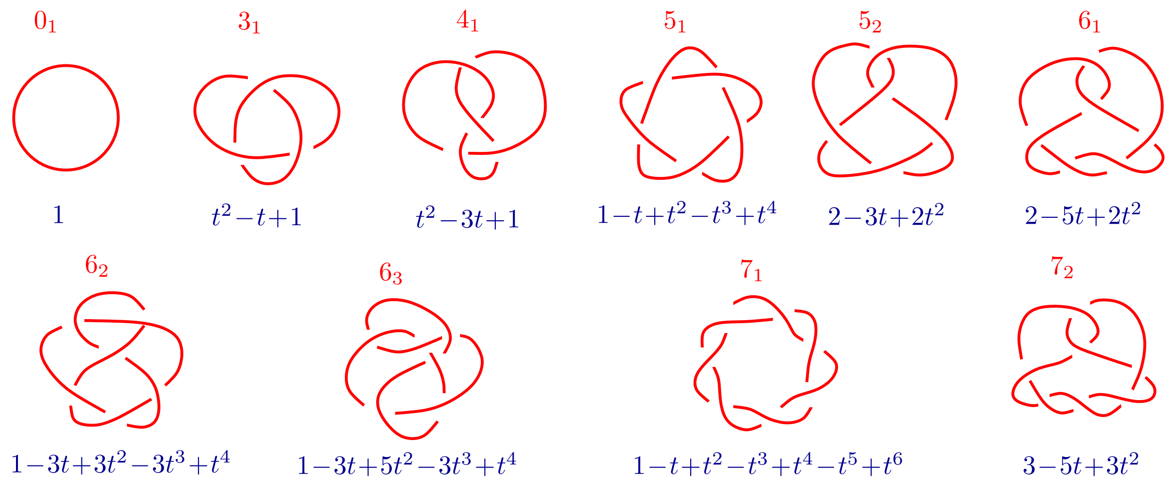
$$\Phi\colon \pi_1(S^3 \setminus K)_{\mathrm{ab}} \ \to\ \langle t\rangle.$$

We refer to

$$\Delta_K(t) \ := \ (t-1)\cdot \Delta_{\pi_1(S^3\setminus K),\Phi} \ \in\ \mathbb{Z}[t^{\pm 1}]$$

as the Alexander polynomial of K. It follows from the Alexander Polynomial–Reverse Lemma 9.5 that this invariant is well-defined, i.e. it is independent of the choice of Φ.

Example. It is clear that with enough brain or machine power one can calculate the Alexander polynomial of any knot that is given by a diagram. For example we already calculated the Alexander polynomials of the trivial knot, the trefoil 3_1 and the figure-8 knot 4_1. On page 55 we showed a list of small crossing knots. Using heroic calculations one ends up with the corresponding list of Alexander polynomials:



The following proposition gives us the Alexander polynomials of an infinite family of knots, namely of all torus knots:

Proposition 9.6. (Torus Knot–Alexander Polynomial Proposition) Let $p, q \in \mathbb{N}$ be coprime. For the torus knot $T(p,q)$ we have[74]

$$\Delta_{T(p,q)}(t) \ \doteq\ \frac{(t-1)\cdot(t^{p\cdot q}-1)}{(t^p-1)\cdot(t^q-1)}.$$

Example. For the trivial knot $U = T_{(1,1)}$ we obtain from the Torus Knot–Alexander Polynomial Proposition 9.6 that $\Delta_U(t) \doteq 1$ and for the trefoil $K = T(2,3)$ we obtain that $\Delta_K(t) \doteq \dfrac{(t-1)\cdot(t^6-1)}{(t^2-1)\cdot(t^3-1)} \doteq t^2 - t + 1$. This recovers the results from page 122.

[74]It is a nice algebra exercise to show that the right hand side does indeed lie in $\mathbb{Z}[t^{\pm 1}]$.

Proof. Let $p, q \in \mathbb{N}$ be coprime. Recall that by the Torus Knot-π_1-Proposition 3.8 we know that $\pi_1(S^3 \setminus K) \cong P := \langle x, y \mid x^p \cdot y^{-q} \rangle$. Note that

$$J(P) \;=\; \begin{pmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \end{pmatrix} \underset{\substack{\uparrow \\ \text{by the Fox Derivative Properties Lemma 8.2 (5)}}}{=} \begin{pmatrix} \sum\limits_{i=0}^{p-1} x^i & -x^p \cdot \sum\limits_{i=1}^{q} y^{-i} \end{pmatrix}.$$

Up to sign the homomorphism $\Phi\colon P \to \langle t \rangle$ that is given by $\Phi(x) = t^q$ and $\Phi(y) = t^p$ is the unique epimorphism onto $\langle t \rangle$. We calculate:

$$\begin{aligned} \Delta_{T(p,q)}(t) &= (t-1) \cdot \frac{\det(J^{\Phi}(P)_2)}{\Phi_*(y-1)} \qquad\qquad = (t-1) \cdot \frac{\det\left(\Phi_* \begin{pmatrix} \sum\limits_{i=0}^{p-1} x^i & \cancel{-x^p \cdot \sum\limits_{i=1}^{q} y^{-i}} \end{pmatrix} \right)}{\Phi_*(y-1)} \\ &= (t-1) \cdot \frac{\det \begin{pmatrix} \sum\limits_{i=0}^{p-1} t^{qi} & \cancel{-t^{pq} \cdot \sum\limits_{i=1}^{q} t^{-ip}} \end{pmatrix}}{t^p - 1} = \frac{t-1}{t^p-1} \cdot \sum_{i=0}^{p-1} t^{qi} \underset{\substack{\uparrow \\ \text{elementary algebra}}}{=} \frac{t-1}{t^p-1} \cdot \frac{t^{pq}-1}{t^q-1}. \end{aligned}$$

■

Rather impressively the Torus Knot–Alexander Polynomial Proposition 9.6 allows us to completely classify the torus knots $T(p,q)$ with $p, q \in \mathbb{N}$:

Corollary 9.7. (Torus Knot–Classification Corollary) Let $p_1, q_1 \in \mathbb{N}$ be coprime and let $p_2, q_2 \in \mathbb{N}$ be coprime. We assume that $p_1 \neq 1$ and $q_1 \neq 1$. Then

$$T_{(p_1,q_1)} \text{ and } T_{(p_2,q_2)} \text{ are smoothly isotopic} \quad \Longleftrightarrow \quad \{p_1, q_1\} = \{p_2, q_2\}.$$

Proof. First note that the "$\Leftarrow$"-direction was shown in the Torus Knot Lemma 3.7 (2d).

We turn to the proof of the "$\Rightarrow$"-direction. So suppose that we have coprime $p_1, q_1 \in \mathbb{N}$ and coprime $p_2, q_2 \in \mathbb{N}$ such that $T_{(p_1,q_1)}$ and $T_{(p_2,q_2)}$ are smoothly isotopic. It follows from the Knot–Alexander Polynomial Proposition 9.1 and the Torus Knot–Alexander Polynomial Proposition 9.6 that

$$\frac{(t-1)\cdot(t^{p_1 \cdot q_1}-1)}{(t^{p_1}-1)\cdot(t^{q_1}-1)} \;\doteq\; \frac{(t-1)\cdot(t^{p_2 \cdot q_2}-1)}{(t^{p_2}-1)\cdot(t^{q_2}-1)}.$$

A purely algebraic argument now shows that $\{p_1, q_1\} = \{p_2, q_2\}$. We outsource this argument to Exercise 9.4. ■

The following proposition shows that the Alexander polynomial is multiplicative under the connected sum operation.

Proposition 9.8. (Connected Sum–Alexander Polynomial Proposition) Let J and K be two oriented knots. On page 24 we introduced the connected sum $J \# K$. The following equality holds:

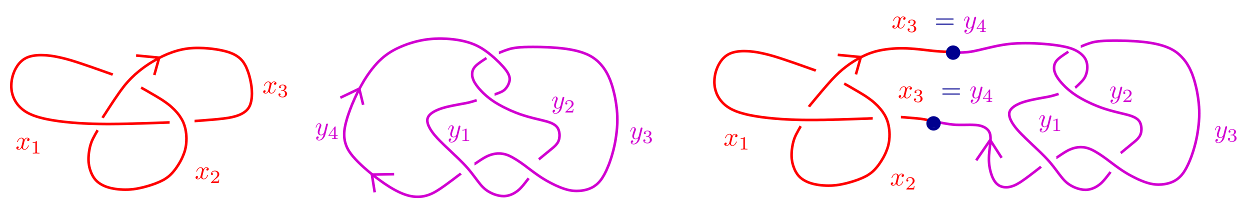
$$\Delta_{J\#K} \;\doteq\; \Delta_J \cdot \Delta_K \;\in\; \mathbb{Z}[t^{\pm 1}].$$

Proof. Let $J, K \subset S^3$ be two oriented knots. By the Wirtinger Presentation Proposition 5.5 we can find presentations

$$\begin{array}{rcrcl} \pi_1(S^3 \setminus J) & = & P & := & \langle x_1, \dots, x_j \,|\, r_1, \dots, r_{j-1} \rangle \\ \pi_1(S^3 \setminus K) & = & Q & := & \langle y_1, \dots, y_k \,|\, s_1, \dots, s_{k-1} \rangle \end{array}$$

where each x_a is a meridian of J and where each y_b is a meridian of K. We now see that

$$\begin{array}{rcl} \pi_1(S^3 \setminus (J \# K)) & \overset{\text{by the Knot Connected Sum-}\pi_1\text{-Proposition 3.15}}{\underset{\downarrow}{\cong}} & \pi_1(S^3 \setminus J) *_{x_j = y_k} \pi_1(S^3 \setminus K) \\ & = & \langle x_1, \dots, x_j, y_1, \dots, y_k \,|\, r_1, \dots, r_{j-1}, s_1, \dots, s_{k-1}, x_j \cdot y_k^{-1} \rangle \;=:\; S. \end{array}$$



trefoil J figure-8 knot K connected sum $J \# K$

It follows that

$$\begin{array}{rcl} \Delta_{J\#K} & = & (t-1) \cdot \dfrac{\det(J^{\Phi_{J\#K}}(S)_{j+k})}{\det(\Phi_{J\#K}(y_k) - 1)} \overset{\text{since } y_k \text{ corresponds to a meridian of } J\#K}{\underset{\downarrow}{=}} (t-1) \cdot \dfrac{\det(J^{\Phi_{J\#K}}(S)_{j+k})}{t-1} = \det(J^{\Phi_{J\#K}}(S)_{j+k}) \\ & = & \det \Phi_{J\#K} \begin{array}{c} \begin{pmatrix} \left(\frac{\partial r_a}{\partial x_b}\right)_{\substack{a=1,\dots,j-1 \\ b=1,\dots,j-1}} & \left(\frac{\partial r_a}{\partial x_j}\right)_{a=1,\dots,j-1} & 0 & \cancel{0} \\ 0 & 0 & \left(\frac{\partial s_a}{\partial y_b}\right)_{\substack{a=1,\dots,k-1 \\ b=1,\dots,k-1}} & \cancel{\left(\frac{\partial r_a}{\partial y_k}\right)_{a=1,\dots,k-1}} \\ 0 & x_j & 0 & \cancel{-y_k^{-1}} \end{pmatrix} \\ \begin{array}{cccc} & \uparrow \; j\text{-th column} & & \uparrow \; (j+k)\text{-th column} \end{array} \end{array} \\ & = & \det \begin{pmatrix} \Phi_J\left(\frac{\partial r_a}{\partial x_b}\right)_{\substack{a=1,\dots,j-1 \\ b=1,\dots,j-1}} & \Phi_J\left(\frac{\partial r_a}{\partial x_j}\right)_{a=1,\dots,j-1} & 0 \\ 0 & 0 & \Phi_K\left(\frac{\partial s_a}{\partial y_b}\right)_{\substack{a=1,\dots,k-1 \\ b=1,\dots,k-1}} \\ 0 & \Phi_J(x_j) & 0 \end{pmatrix} \\ & = & (-1)^{k-1} \cdot \Phi_J(x_j) \cdot \underbrace{\det\left(\Phi_J\left(\frac{\partial r_a}{\partial x_b}\right)_{\substack{a=1,\dots,j-1 \\ b=1,\dots,j-1}}\right)}_{=\Delta_J} \cdot \underbrace{\det\left(\Phi_K\left(\frac{\partial s_a}{\partial y_b}\right)_{\substack{a=1,\dots,k-1 \\ b=1,\dots,k-1}}\right)}_{=\Delta_K} \\ & \doteq & \Delta_J \cdot \Delta_K. \end{array}$$

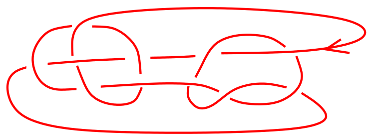
■

The above calculations indicate that the Alexander polynomial is pretty effective at distinguishing knows. The Alexander polynomial is not perfect though. For example in the following figure we show the Conway knot and the Kinoshita-Terasaka knot which we already encountered on page 100.

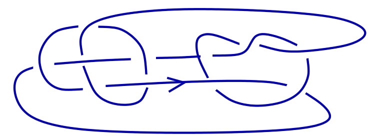
With enough effort one can show that in both case the Alexander polynomials are $\doteq 1$. But on page 100 we already noticed that both knots are rather non-trivial.

9.5. **Alexander polynomials and mirrors.**

In this last section we return to the general study of links. In the Alexander Polynomial–Reverse Lemma 9.5 we showed that the



Kinoshita-Terasaka knot

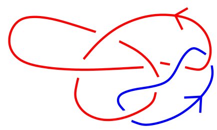
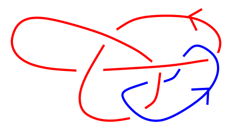


Conway knot

Alexander polynomial of a link and its reverse coincide. In this section we will prove an analogous result for mirrors.

First let us recall the following definitions from page 18:

Definition. Let $L \subset S^3$ be an oriented link. The mirror L^{mir} is defined as the reflection of $L \subset S^3$ in any hyperplane of $\mathbb{R}^4$.[75][76] We give the mirror L^{mir} the orientation that turns the reflection into an orientation-preserving diffeomorphism $L \to L^{\text{mir}}$.

link L   the mirror L^{mir}

The following lemma shows that the Alexander polynomial also cannot be used to distinguish a link from its mirror.

Lemma 9.9. (Alexander Polynomial–Mirror Lemma) For any oriented m-component link $L \subset S^3$ we have

$$\Delta_{L^{\text{mir}}}(t_1, \dots, t_m) \;\doteq\; \Delta_L(t_1^{-1}, \dots, t_m^{-1}) \;\underset{\substack{\uparrow \\ \text{by the Alexander Polynomial} \\ \text{Symmetry Theorem 9.4}}}{\doteq}\; \Delta_L(t_1, \dots, t_m).$$

Proof. Let L be an oriented m-component link. We make a few preparations:

(1) We denote by $\rho\colon \mathbb{R}^4 \to \mathbb{R}^4$ the reflection in some hyperplane of $\mathbb{R}^4$.
(2) We write $L' := L^{\text{mir}} := \rho(L)$.
(3) We write $\pi := \pi_1(S^3 \setminus L)$ and we write $\pi' = \pi_1(S^3 \setminus L')$. Note that ρ induces an isomorphism $\rho_*\colon \pi \to \pi'$.
(4) (a) Let $\mu_1, \dots, \mu_m$ be meridians for the oriented link L.
 (b) Let $\mu_1', \dots, \mu_m'$ be the corresponding meridians of the oriented link L'.
(5) The Meridian–Symmetries Lemma 2.18 implies that for any i we have $\mu_i' = \rho(\mu_i)^{\text{rev}}$.
(6) As on page 124 we introduce the following notation:
 (a) We denote by $\Phi_L\colon \pi \to \langle t_1, \dots, t_m\rangle_{\text{ab}}$ the epimorphism given by $\Phi_L(\mu_i) = t_i$.
 (b) We denote by $\Phi_{L'}\colon \pi \to \langle t_1, \dots, t_m\rangle_{\text{ab}}$ the epimorphism given by $\Phi_{L'}(\mu_i') = t_i$.
(7) We denote by $\Theta\colon \langle t_1, \dots, t_m\rangle_{\text{ab}} \to \langle t_1, \dots, t_m\rangle_{\text{ab}}$ the homomorphism that is given by $t_i \mapsto t_i^{-1}$.
(8) It follows easily from (5) and the Loop-π_1-Lemma 3.14 (3) that the following diagram commutes:

[75]Note that we showed in the Link Mirror Lemma 2.7 that, up to a smooth isotopy, the definition of L^{mir} does not depend on the choice of the hyperplane.
[76]Note that on page 18 we introduced the mirror of an unoriented link. For the purpose of this section it is better to work with oriented links.

$$\begin{tikzcd}
\pi \arrow[rr, "\rho_*"] \arrow[d, "\Phi_L"'] & & \pi' \arrow[d, "\Phi_{L'}"] \\
\langle t_1, \dots, t_m \rangle_{\text{ab}} \arrow[rr, "\Theta"] & & \langle t_1, \dots, t_m \rangle_{\text{ab}}.
\end{tikzcd}$$

Now we turn to the actual proof of the promised equality. In the following we first consider the case $m \geq 2$. Sharp eyes will once again notice that the desired result is actually an immediate consequence of the Alexander Function–Functorial Lemma 8.6 (1) together with the Alexander Polynomial Symmetry Theorem 9.4. Alternatively one can also do a direct calculation:

- We pick a presentation $P = \langle x_1, \dots, x_n \,|\, r_1, \dots, r_{n-1} \rangle$ of deficiency one for π.
- We pick an $i \in \{1, \dots, n\}$ such that $\Phi_L(x_i)$ is non-trivial.

We see that

$$\begin{aligned}
\Delta_{L^{\text{mir}}}(t_1, \dots, t_m) \; &\overset{\substack{\text{by definition of } \Delta_{L^{\text{mir}}} \\ \downarrow}}{\doteq} \; \frac{\det(J^{\Phi_{L'} \circ \rho_*}(P)_1)}{(\Phi_{L'} \circ \rho)_*(x_i - 1)} \; \overset{\substack{\text{by (8)} \\ \downarrow}}{\doteq} \; \frac{\det((\Theta \circ \Phi_L)_*(J(P)_1))}{(\Theta \circ \Phi_L)_*(x_i - 1)} \; \overset{\substack{\text{since determinants commute} \\ \text{with ring homomorphisms} \\ \downarrow}}{\doteq} \; \Theta_*\Big(\frac{\det(\Phi_{L*}(J(P)_1))}{\Phi_{L*}(x_i - 1)} \Big) \\
&\underset{\substack{\uparrow \\ \text{by definition of } \Delta_L}}{\doteq} \; \Theta_*(\Delta_L(t_1, \dots, t_m)) \; \underset{\substack{\uparrow \\ \text{by definition of } \Theta}}{\doteq} \; \Delta_L(t_1^{-1}, \dots, t_m^{-1}) \; \underset{\substack{\uparrow \\ \text{by the Alexander Polynomial} \\ \text{Symmetry Theorem 9.4}}}{=} \; \Delta_L(t_1, \dots, t_m).
\end{aligned}$$

The case $m = 1$ is basically identical, we just need to multiply the intermediate terms by $t - 1$ and we need to notice that $t^{-1} - 1 = (-t^{-1}) \cdot (t - 1) \doteq t - 1$. ■

Exercises for Chapter 9.

Exercise 9.1. Given a non-zero Laurent polynomial $p(t) = \sum_{i=r}^{s} a_i \cdot t^i \in \mathbb{Z}[t^{\pm 1}]$ with $a_r \neq 0$ and $a_s \neq 0$ we define its *degree* as

$$\deg(p(t)) \;=\; \deg\Big(\sum_{i=r}^{s} a_i \cdot t^i \Big) \;=\; s - r.$$

Show that for every knot K we have

$$\deg(\Delta_K(t)) - 1 \;\leq\; \text{crossing number of } K.$$

Exercise 9.2. Let D and D' be two m-component link diagrams and let L and L' be the oriented m-component links that are associated to D and D'. We use the corresponding Wirtinger presentations P and P' to define Alexander functions. Show, using the "'obvious" oriented version of the Reidemeister Moves Theorem 4.5, that if L and L' are smoothly isotopic as oriented links, then the Alexander functions of P and P' in $\mathbb{Q}(t_1, \dots, t_m)$ are the same up to "$\doteq$".

Remark. In other words, the exercise is to show that one can get a link invariant without leaving the world of deficiency-one presentations.

Exercise 9.3.

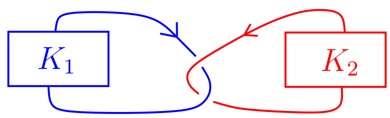
(a) Let G_1, G_2 be groups that admit a presentation of deficiency one. Let $\Phi\colon G_2 \to H$ be a non-trivial homomorphism to a free abelian group H. Show that for any epimorphism $\gamma\colon G_1 \to G_2$ there exists a $p \in \mathbb{Z}[H]$ such that $\Delta_{G_1, \Phi \circ \gamma} = p \cdot \Delta_{G_2, \Phi}$.
Hint. You should make use of the flexibility provided by the Independence-of-Generator Lemma 8.10.

(b) We consider the following variation on (a): We now assume that there exists a monomorphism $\gamma\colon G_1 \to G_2$ such that $\Phi\circ\gamma\colon G_1 \to H$ is non-trivial. Can you find a relationship between $\Delta_{G_1,\Phi\circ\gamma}$ and $\Delta_{G_2,\Phi}$?
(c) Let K_1 and K_2 be two knots. We now suppose that there exists an epimorphism $\pi_1(S^3\setminus K_1) \to \pi_1(S^3\setminus K_2)$. Show that there exists a $p \in \mathbb{Z}[t^{\pm 1}]$ such that $\Delta_{K_1} = p\cdot\Delta_{K_2}$.

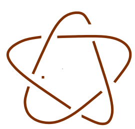
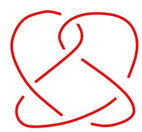
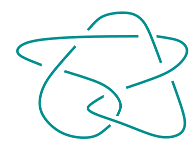
Exercise 9.4. Let $p_1, q_1 \in \mathbb{N}$ be coprime and let $p_2, q_2 \in \mathbb{N}$ be coprime. Show that if $T_{(p_1,q_1)}$ and $T_{(p_2,q_2)}$ are smoothly isotopic, then $\{p_1, q_1\} = \{p_2, q_2\}$.
Remark. By the discussion in the proof of the Torus Knot–Classification Corollary 9.7 it remains to prove the following statement:

$$\frac{(t-1)\cdot(t^{p_1\cdot q_1}-1)}{(t^{p_1}-1)\cdot(t^{q_1}-1)} \doteq \frac{(t-1)\cdot(t^{p_2\cdot q_2}-1)}{(t^{p_2}-1)\cdot(t^{q_2}-1)} \quad\Longrightarrow\quad \{p_1, q_1\} = \{p_2, q_2\}.$$

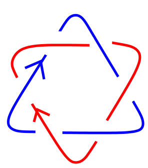
Exercise 9.5. We consider the oriented 2-component link L which is built out of oriented knots K_1 and K_2 as shown in the figure below. Determine $\Delta_L(t_1, t_2)$ in terms of the Alexander polynomials of K_1 and K_2.

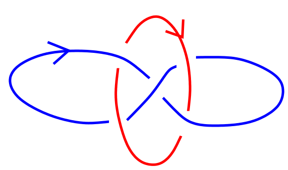


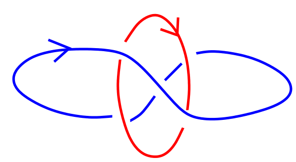
Exercise 9.6. Compute the Alexander polynomials of the cinquefoil, the three-twist knot and the stevedore knot that are shown below.

cinquefoil  three-twist knot  stevedore knot 

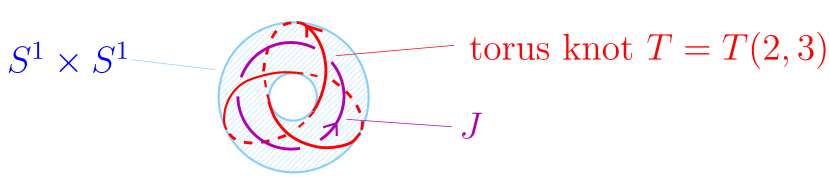
Exercise 9.7. Compute the Alexander polynomial of the Star of David link, the Whitehead link and the mystery link shown in the figure below.


Star of David link


Whitehead link

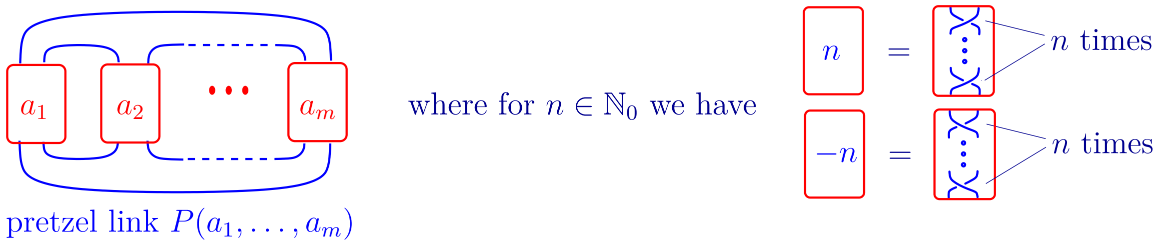

mystery link

Exercise 9.8. Let $p, q \in \mathbb{N}$ be coprime. Let $L = T \cup J$ be the link that is given by the torus knot $T = T(p,q) \subset \overline{B}^2 \times S^1$ and the core curve $J = \{0\} \times S^1$ of the solid torus.
(a) Show that $\pi_1(S^3 \setminus L) \cong \langle x, y, z \,|\, [x,z], x^p \cdot y^{-q} \cdot z^q \rangle$.
(b) Calculate the Alexander polynomial of L.

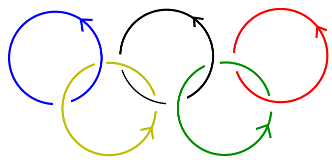


Exercise 9.9. Let $a_1, \ldots, a_m \in \mathbb{Z}$. We refer to the figure below for the definition of the *pretzel link* $P(a_1, \ldots, a_m)$:

(a) For which $a_1, \ldots, a_m$ is $P(a_1, \ldots, a_m)$ a knot?
(b) Show that the Alexander polynomial of $P(-3, 5, 7)$ equals $\doteq 1$.
Hint. Try to find a presentation with few generators.



Exercise 9.10. Compute the Alexander polynomial of the Olympic rings.
Hint. Try to find a presentation with a small number of generators.

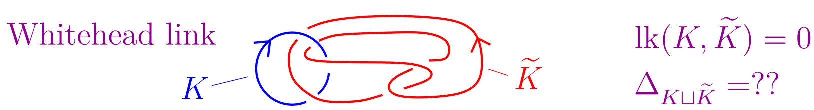


Exercise 9.11. As usual we consider the torus $S^1 \times S^1 \subset S^3 =_{\mathrm{i}} (\overline{B}^2 \times S^1) \cup_{S^1 \times S^1} (S^1 \times \overline{B}^2)$. Let $m, n \in \mathbb{Z}$. We set $g := \gcd(m, n)$. As in Exercise 3.8 we consider the g-component torus link $L(m, n) := \{(y, z) \in S^1 \times S^1 \,|\, y^m = z^n\}$.

(a) Determine $\pi_1(S^3 \setminus L(m, n))$.
(b) Determine the Alexander polynomial of $L(m, n)$.

Exercise 9.12.

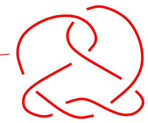
(a) Let L be a link that is splittable. Show that $\Delta_L = 0$.
(b) Show that the Whitehead link, which is shown in the figure below, is not splittable.
Remark. This confirms our suspicion from page 89.

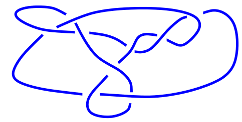


Exercise 9.13. In Exercise 8.9 we introduced higher order Alexander functions $\Delta^n_{\pi,\Phi}$ where π is a group that admits a presentation $P = \langle x_1, \ldots, x_k \,|\, r_1, \ldots, r_{k-1} \rangle$ of deficiency one, $\Phi\colon \pi \to H$ is a non-trivial homomorphism to a free abelian group H and $n \in \mathbb{N}_0$. With the obvious modifications to the definition on page 121 we can now introduce the higher order Alexander polynomials $\Delta^n_K(t)$ of an oriented knot. In the following we consider the two knots 6_1 and 9_{46}.

(a) Show that the 0-th Alexander polynomials equal $2 - 5t + 2t^2$.
(b) Show that the first order Alexander polynomials of 6_1 and 9_{46} are different.

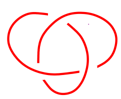
Exercise 9.14. In Exercise 8.9 we introduced higher order Alexander functions $\Delta^n_{\pi,\Phi}$ where π is a group that admits a presentation $P = \langle x_1, \ldots, x_k \,|\, r_1, \ldots, r_{k-1} \rangle$ of deficiency one,

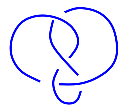
6_1



9_{46}

$\Phi\colon \pi \to H$ is a non-trivial homomorphism to a free abelian group and $n \in \mathbb{N}_0$. With the obvious modifications to the definition on page 124 we can now introduce the higher order Alexander polynomials $\Delta_L^n(t_1, \ldots, t_m)$ of an oriented link with $m \geq 2$ components. Let $L = J \sqcup K$ be the split union of two oriented knots J and K.

(a) If you did not solve Exercise 9.12, then show that $\Delta_L(s,t) = 0$.
(b) Show that $\Delta_L^1(s,t) = \Delta_J(s) \cdot \Delta_K(t)$.

$J = 3_1$



$K = 4_1$

$$\begin{aligned} \Delta_{J \sqcup K}(s,t) &= 0 \\ \Delta_{J \sqcup K}(s,t) &= \Delta_J(s) \cdot \Delta_K(t) \end{aligned}$$

CHAPTER 10

The symmetry theorem for Alexander polynomials

In this chapter we will do the following:

(1) Given a diagram for a link L we introduce the "over-presentation" of $\pi_1(S^3 \setminus L)$ which is somewhat similar to the Wirtinger presentation, but in general it has fewer generators. It thus leads to more efficient calculations of Alexander polynomials.

(2) In the Alexander Polynomial–Symmetry Theorem 9.4 we prove the result, which we already mentioned in the previous chapter, that given any oriented m-component link L we have

$$\Delta_L(t_1, \dots, t_m) \doteq \Delta_L(t_1^{-1}, \dots, t_m^{-1}).$$

The key idea is to calculate the Alexander polynomial twice, namely once using the "over-presentation" and once using the "under-presentation". We will see that these two presentations are "dual" in a suitable sense, which allows us to obtain the symmetry result.

(3) In the Alexander Polynomial-at-1 Proposition 10.7 we will show that for any knot K we have $\Delta_K(1) = \pm 1$. Together with the Alexander Polynomial–Symmetry Theorem 9.4 this characterizes the Laurent polynomials that can arise as Alexander polynomials of knots.

(4) We conclude with two appendices on the Torres conditions for Alexander polynomials of links and on Alexander polynomials of alternating knots. These appendices do not contains proofs, but they contain results and open questions which are worth knowing.

10.1. **The over presentation.** In this section we use a diagram of a link L to introduce the "over-presentation" for $\pi_1(S^3 \setminus L)$. In general this presentation has fewer generators than the Wirtinger presentation from the Wirtinger Presentation Proposition 5.5. This is useful since the effort to calculate Alexander polynomials of a link from a given deficiency one presentation grows rapidly with the number of generators.

In the Diagram-to-Link Lemma 4.2 we showed that any link diagram gives rise to a link. In the following lemma we give a slightly different construction:

Lemma 10.1. (Alternative Diagram-to-Link Lemma) Let $\big(\gamma\colon \bigsqcup_{i=1}^{m} S_i^1 \to \mathbb{R}^2, c\big)$ be a link diagram such that each component has at least one crossing. We can pick a smooth map $\xi\colon \bigsqcup_{i=1}^{m} S_i^1 \to [-1,1]$ with the following properties:

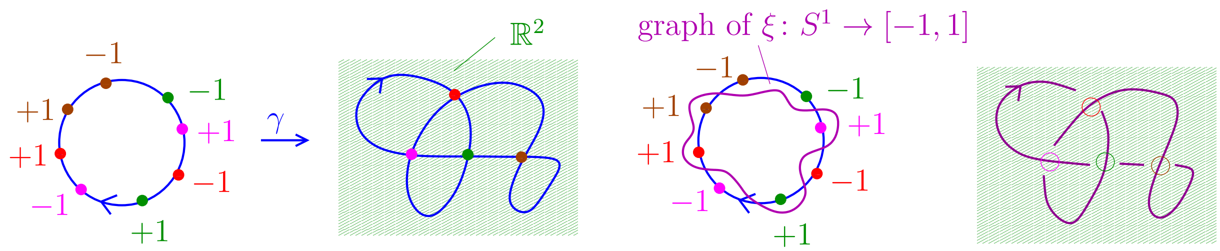
(a) $\xi^{-1}(1)$ is the set of overcrossings (as defined on page 53).
(b) $\xi^{-1}(-1)$ is the set of undercrossings.
(c) Each component of $\xi^{-1}([0,1])$ contains an overcrossing.
(d) Each component of $\xi^{-1}([-1,0])$ contains an undercrossing.

(e) For each $z \in \bigsqcup_{i=1}^{m} S^1_i$ with $\xi(z) = 0$ we have $\xi'(z) \neq 0$.

The following three statements hold:

(1) The image of $\bigsqcup_{i=1}^{m} S^1_i$ under the map $\bigsqcup_{i=1}^{m} S^1_i \to \mathbb{R}^3$ given by $z \mapsto (\gamma(z), \xi(z))$ (together with the obvious ordering of the components) is an m-component link.
(2) Any two choices of ξ give rise to smoothly isotopic links,
(3) The resulting links are smoothly isotopic to the links obtained from the original Diagram-to-Link Lemma 4.2.

If we equip $\bigsqcup_{i=1}^{m} S^1_i$ with the standard orientation, then in the above statements we can also replace "link" by "oriented link".



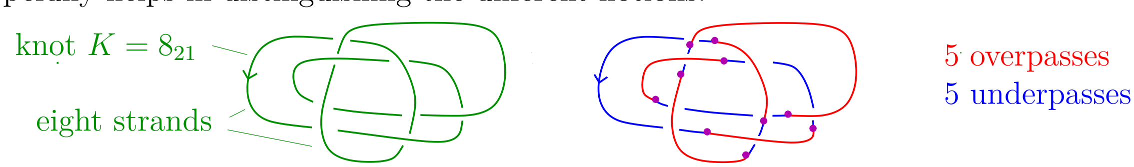
Proof. The proof of the lemma is very similar to the proof of the Diagram-to-Link Lemma 4.2. We leave it to the reader to make the necessary modifications. ■

Definition. Let $L \subset \mathbb{R}^2 \times [-1,1] \subset \mathbb{R}^3$ be the oriented m-component link that is associated to the link diagram $\left(\gamma\colon \bigsqcup_{i=1}^{m} S^1_i \to \mathbb{R}^2, c\right)$ via the Alternative Diagram-to-Link Lemma 10.1.

(1) We refer to each component of $L \cap (\mathbb{R}^2 \times [0,1])$ as an overpass.
(2) We refer to each component of $L \cap (\mathbb{R}^2 \times [-1,0])$ as an underpass.

We will often use the following fact: If each component of a link diagram has at least one crossing, then the same argument as in the Crossing–Strand Lemma 5.3 shows that the number of overpasses equals the number of underpasses.

The definition of an overpass and underpass is similar, but nonetheless different, from the definition of the definition of strand which we gave on page 66. The following figure hopefully helps in distinguishing the different notions:



The following proposition is an analogue of the Wirtinger Presentation Proposition 5.5:

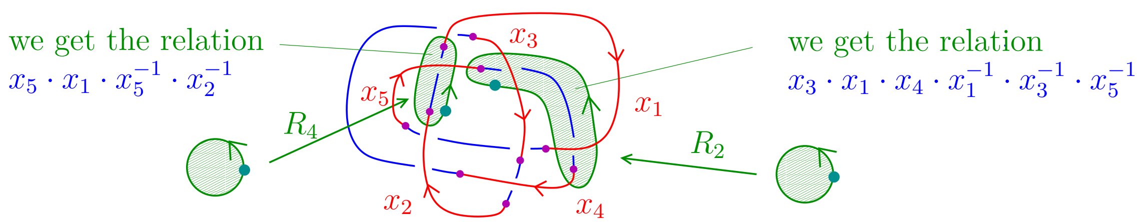
Proposition 10.2. (Over-Presentation Proposition) Let $\left(\gamma\colon \bigsqcup_{i=1}^{m} S^1_i \to \mathbb{R}^2, c\right)$ be a link diagram such that each component has at least one crossing. Let L be the link that we associated in the Alternative Diagram-to-Link Lemma 10.1 to the link diagram. We enumerate the overpasses by $x_1, \dots, x_n$ and we enumerate the underpasses by $1, \dots, n$.

- We pick **orientation-preserving** smooth embeddings $R_i \colon \overline{B}^2 \to \mathbb{R}^2$, $i = 1, \dots, n$ with disjoint images such that the projection of the i-th underpass to $\mathbb{R}^2$ is contained in $R_i(B^2)$ and such that $R_i(S^1)$ intersects $\gamma\big(\bigsqcup_{i=1}^{m} S_i^1\big)$ transversally.
- For the i-th underpass we define a relation $r_i \in \langle x_1, \dots, x_n\rangle$ as follows: We consider the path $[0,1] \to \mathbb{R}^2$ that is given by $t \mapsto R_i(\exp(2\pi i t))$. We start at $t = 0$ and going along $[0,1]$ we record a generator x_j (or its inverse x_j^{-1}) if the path hits the image of the j-th overpass with a positive (negative) sign.

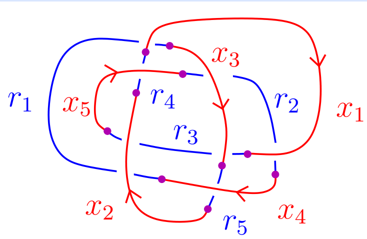
With this notation the following statements hold:

(1) We have an explicit isomorphism
$$\langle \underbrace{x_1, \dots, x_n}_{\substack{\text{correspond to}\\ \text{overpasses}}} \mid \underbrace{r_1, \dots, r_n}_{\substack{\text{correspond to}\\ \text{underpasses}}} \rangle \xrightarrow{\cong} \pi_1(S^3 \setminus L)$$
where each x_i is given by a meridian corresponding to the i-th overpass.

(2) In the presentation (1) we can drop any one of the relations and we still obtain an isomorphism.



Definition. Let $\big(\gamma \colon \bigsqcup_{i=1}^{m} S_i^1 \to \mathbb{R}^2, c\big)$ be a link diagram such that each component has at least one crossing. Let L be the link that we associated in the Alternative Diagram-to-Link Lemma 10.1 to the link diagram. We refer to the presentation of $\pi_1(S^3 \setminus L)$ from the Over-Presentation Proposition 10.2 as the over-presentation of $\pi_1(S^3 \setminus L)$.



we get generators x_1, x_2, x_3, x_4, x_5
and relations

$$\begin{aligned}
r_1 &= x_3 \cdot x_1 \cdot x_2^{-1} \cdot x_4^{-1} \cdot x_2 \cdot x_1^{-1}\\
r_2 &= x_3 \cdot x_1 \cdot x_4 \cdot x_1^{-1} \cdot x_3^{-1} \cdot x_5^{-1}\\
r_3 &= x_2 \cdot x_5 \cdot x_2^{-1} \cdot x_3 \cdot x_1^{-1} : x_3^{-1}\\
r_4 &= x_5 \cdot x_1 \cdot x_5^{-1} \cdot x_2^{-1}\\
r_5 &= x_4^{-1} \cdot x_3^{-1} \cdot x_4 \cdot x_2
\end{aligned}$$

over-presentation for $\pi_1(S^3 \setminus 8_{21})$

Sketch of proof. First recall that in the Alternative Diagram-to-Link Lemma 10.1 we picked a smooth function $\xi \colon \bigsqcup_{i=1}^{m} S_i^1 \to [-1,1]$ with the following properties:

(a) $\xi^{-1}(1)$ is the set of overcrossings.
(b) $\xi^{-1}(-1)$ is the set of undercrossings.
(c) Each component of $\xi^{-1}([0,1])$ contains an overcrossing.

(d) Each component of $\xi^{-1}([-1,0])$ contains an undercrossing.
(e) For each $z \in \bigsqcup_{i=1}^{m} S^1_i$ with $\xi(z) = 0$ we have $\xi'(z) \neq 0$.

Let L be the oriented m-component link that is associated to the link diagram via this function ξ. As before we refer to each component of $L \cap (\mathbb{R}^2 \times [-1,0])$ as an underpass.

It is elementary to see that we can modify the above map ξ to obtain a smooth function $\widetilde{\xi}\colon \bigsqcup_{i=1}^{m} S^1_i \to [-1,1]$ with the following modified properties

(a) $\widetilde{\xi}^{-1}(1) = \xi^{-1}(1)$ is the set of overcrossings.
(b) $\widetilde{\xi}^{-1}(-1)$ is the set of underpasses.
(c) Each component of $\widetilde{\xi}^{-1}([0,1])$ contains an overcrossing.
(d) Each component of $\widetilde{\xi}^{-1}([-1,0])$ contains an undercrossing.
(e) For each $z \in \bigsqcup_{i=1}^{m} S^1_i$ with $\xi(z) = 0$ we have $\xi'(z) \neq 0$.

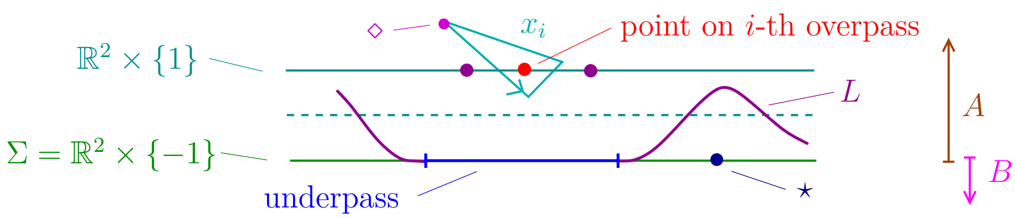
It is easy to verify that

$$\begin{array}{rcl} \Big(\bigsqcup_{i=1}^{m} S^1_i\Big) \times [0,1] & \to & \mathbb{R}^3 \\ (z,t) & \mapsto & (z, \xi(z)\cdot(1-t) + \widetilde{\xi}(z)\cdot t) \end{array}$$

is a smooth isotopy from L to $\widetilde{L}$.

At that point the proof is almost identical to the proof of the Wirtinger Presentation Proposition 5.5 which we gave in Section 5.3. We introduce the following familiar notation:

- We consider the following subspaces of $\mathbb{R}^3 \cup \{\infty\} =_{\mathrm{i}} S^3$:
$$\begin{array}{rcl} A & := & \{(x,y,z) \in \mathbb{R}^3 \,|\, z \geq -1\} \cup \{\infty\}, \\ B & := & \{(x,y,z) \in \mathbb{R}^3 \,|\, z \leq -1\} \cup \{\infty\}, \\ S & := & \{(x,y,z) \in \mathbb{R}^3 \,|\, z = -1\} \cup \{\infty\} \;=\; A \cap B \end{array}$$
and we set $A_L := A \setminus L$, $B_L := B \setminus L$ and $S_L := S \setminus L$.
- We work with the base point $\diamond = (0,0,2) \in \mathbb{R}^3$ which lies "above" the link L.
- For the i-th overpass we denote by x_i an oriented triangle that starts at $\diamond$ and "circles once around the i-th overpass at any overcrossing" according to the "right-hand rule".
- We pick a base point $\star \in S_L$.



Similar to the arguments in Section 5.3 the following claims hold:

Claim 0. The inclusion maps induce an isomorphism
$$\pi_1(A_L,\star) *_{\pi_1(S_L,\star)} \pi_1(B_L,\star) \xrightarrow{\cong} \pi_1(S^3 \setminus L, \star).$$

Claim 1. The group $\pi_1(B_L,\star)$ is trivial.

Claim 2. The obvious map $\langle x_1,\ldots,x_n\rangle \to \pi_1(A_L,\diamond)$ is an isomorphism.

Claim 3. The loops $R_1(S^1), \ldots, R_n(S^1)$ form a normal generating set of $\pi_1(S_L, \star)$.

As on page 73 the proposition is an easy consequence of the above claims. We leave it to the reader to fill in all the details. ■

10.2. **The symmetry of the Alexander polynomial.** In this section we give the long overdue proof of the following theorem:

Theorem 9.4. (Alexander Polynomial–Symmetry Theorem) Given any oriented m-component link L we have

$$\Delta_L(t_1, \ldots, t_m) \doteq \Delta_L(t_1^{-1}, \ldots, t_m^{-1}).$$

The logic of the proof is as follows:

(1) In Subsection 10.2.1 we will introduce the notion of two presentations being "dual". We will see that if a group admits a pair of dual presentations, then the Alexander function is symmetric.
(2) In Subsection 10.2.2 we will introduce the under-presentation of $\pi_1(S^3 \setminus L)$.
(3) In Subsection 10.2.3 we will prove the Dual Presentation Theorem 10.6 which says that, for a given link diagram, the corresponding over-presentation and under-presentation are dual.
(4) In Subsection 10.2.4 we will wrap up the proof of the Alexander Polynomial–Symmetry Theorem 9.4.

10.2.1. **Dual presentations.** We start out with the following standard notation:

Definition.

(1) Let G a group. We equip the corresponding group ring $\mathbb{Z}[G]$ with the involution that is induced by $\overline{g} := g^{-1}$ for $g \in G$. More precisely, we write

$$\overline{\sum_{i=1}^{n} a_i \cdot g_i} \;:=\; \sum_{i=1}^{n} a_i \cdot g_i^{-1}.$$

This involution is natural in the sense that for any group homomorphism $\varphi\colon G \to H$ and any $p \in \mathbb{Z}[G]$ we have $\varphi_*(\overline{p}) = \overline{\varphi_*(p)} \in \mathbb{Z}[H]$.
(2) If G is a free abelian group, then the involution on $\mathbb{Z}[H]$ defines also an involution on the quotient field $\mathbb{Q}(H)$ in an obvious way.

The following is the key definition in the proof of the Alexander Polynomial–Symmetry Theorem 9.4.

Definition. Let π be a group. We denote by $\Phi\colon \pi \to \pi_{\text{ab}}$ the natural epimorphism. We say that two presentations[77] $P = \langle x_1, \ldots, x_n \,|\, r_1, \ldots, r_n \rangle$ and $P^\dagger = \langle y_1, \ldots, y_n \,|\, s_1, \ldots, s_n \rangle$ for π are dual to one another if the following two conditions are satisfied:

(D1) For $i = 1, \ldots, n$ we have $\Phi(x_i) = \Phi(\overline{y_i})$.
(D2) There exist $g_1, \ldots, g_n, h_1, \ldots, h_n \in \pi$ such that for any $i, j = 1, \ldots, n$ we have[78]

$$\Phi\Big(g_i \cdot \frac{\partial r_i}{\partial x_j} \cdot (x_j - 1)\Big) \;=\; \Phi\Big(\overline{h_j \cdot \frac{\partial s_j}{\partial y_i} \cdot (y_i - 1)}\Big).$$

The following proposition makes it clear why we are interested in dual presentations:

Proposition 10.3. (Dual Presentation–Symmetry Proposition) Let π be a group and let $\Phi\colon \pi \to H$ be a homomorphism a free abelian group H. We assume that π admits a pair of dual presentations $P = \langle x_1, \dots, x_n \,|\, r_1, \dots, r_n\rangle$ and $P^\dagger = \langle y_1, \dots, y_n \,|\, s_1, \dots, s_n\rangle$ with the following two properties:

- In each presentation we can drop any one of the relations.
- For all $i \in \{1, \dots, n\}$ the images $\Phi(x_i) \in H$ and $\Phi(y_i) \in H$ are non-trivial.

Then the following equality holds:

$$\Delta_{\pi,\Phi} \;\doteq\; \overline{\Delta_{\pi,\Phi}} \;\in\; \mathbb{Q}(H).$$

Proof. Let π be a group and let $\Phi\colon \pi \to H$ be a non-trivial homomorphism to a free abelian group H. We assume that there exist two presentations $P = \langle x_1, \dots, x_n \,|\, r_1, \dots, r_n\rangle$ and $P^\dagger = \langle y_1, \dots, y_n \,|\, s_1, \dots, s_n\rangle$ for π with the following three properties:

- The presentations are dual to one another.
- In both presentations we can drop any one of the relations.
- For all $i \in \{1, \dots, n\}$ the images $\Phi(x_i) \in H$ and $\Phi(y_i) \in H$ are non-trivial.

Evidently we now want to use these presentations to calculate the Alexander function.

Since Φ is a non-trivial homomorphism there exists an $i \in \{1, \dots, n\}$ such that $\Phi(x_i)$ is non-trivial. To simplify the notation a bit we will assume that $\Phi(x_n)$ is non-trivial. Note that (D1) implies that $\Phi(y_n)$ is also non-trivial.

By our hypothesis we can drop in both presentations the last relation. We denote the resulting presentations by $\widetilde{P}$ and $\widetilde{P}^\dagger$.

Claim. We have
$$\det\Big(\underbrace{\Phi\big(\tfrac{\partial r_i}{\partial x_j}\big)_{i,j=1,\dots,n-1}}_{=J^\Phi(\widetilde{P})_n}\Big) \;\doteq\; \overline{\det\Big(\underbrace{\Phi\big(\tfrac{\partial s_i}{\partial y_j}\big)_{i,j=1,\dots,n-1}}_{=J^\Phi(\widetilde{P}^\dagger)_n}\Big)}.$$

Proof. We have the following equalities in $\mathbb{Z}[H]$:

since Φ is a ring homomorphisms, since determinants commute with ring homomorphisms and since determinants are multilinear

$$\begin{aligned}
\det\Big(\Phi\big(\tfrac{\partial r_i}{\partial x_j}\big)_{i,j=1,\dots,n-1}\Big)\cdot\prod_{j=1}^{n-1}(\Phi(x_j)-1) \;&=\; \det\Big(\Phi\big(\tfrac{\partial r_i}{\partial x_j}\cdot(x_j-1)\big)_{i,j=1,\dots,n-1}\Big)\\
\doteq\; \det\Big(\Phi\Big(\overline{\tfrac{\partial s_j}{\partial y_i}\cdot(y_i-1)}\Big)_{i,j=1,\dots,n-1}\Big) \;&=\; \det\Big(\Phi\Big(\overline{\tfrac{\partial s_i}{\partial y_j}\cdot(y_j-1)}\Big)_{i,j=1,\dots,n-1}\Big)
\end{aligned}$$

by (D2), note that the $g_1, \dots, g_n$ and $h_1, \dots, h_n$ get swallowed by the "$\doteq$"

since $\det(A) = \det(A^T)$

$$=\; \overline{\det\Big(\Phi\big(\tfrac{\partial s_i}{\partial y_j}\big)_{i,j=1,\dots,n-1}\Big)}\cdot\prod_{j=1}^{n-1}(\Phi(x_j)-1).$$

by the same argument as above, we also use that $f \mapsto \overline{f}$ is a ring homomorphism and we use (D1)

[77]To simplify the notation we drop the isomorphism between π and its presentations from the notation.

[78]The notion of dual presentations was introduced by Ralph Fox and Guillermo Torres [**TF54**], see also [**CF77**]. Our definition differs slightly from their definitions since we allow for the $g_1, \dots, g_n$ and $h_1, \dots, h_n \in \pi$.

To obtain the desired equality we need to dispose of the common factor $\prod_{j=1}^{n-1}(\Phi(x_j)-1)$. To achieve this we make two observations:

- By hypothesis we know that for all $j \in \{1,\dots,n\}$ the image $\Phi(x_j) \in H$ is non-trivial. This implies that each $\Phi(x_j)-1 \in \mathbb{Z}[H]$ is non-zero.
- Since H is a free abelian group we know that $\mathbb{Z}[H]$ is a domain.

The desired equality follows from the above calculation and the above two observations. ⊞

We now see that

$$\Delta_{\pi,\Phi} \underset{\substack{\uparrow \\ \text{by the definition on page 111}}}{\doteq} \frac{\det(J^{\Phi}(\widetilde{P})_n)}{\Phi(x_n)-1} \underset{\substack{\uparrow \\ \text{by the definition on page 109}}}{\doteq} \frac{\det\left(\Phi\left(\frac{\partial r_i}{\partial x_j}\right)_{i,j=1,\dots,n-1}\right)}{\Phi(x_n)-1} \underset{\substack{\uparrow \\ \text{by the claim and by (D2)}}}{\doteq} \frac{\overline{\det\left(\Phi\left(\frac{\partial s_i}{\partial y_j}\right)_{i,j=1,\dots,n-1}\right)}}{\overline{\Phi(y_n)-1}}$$

$$\underset{\substack{\uparrow \\ \text{as above, this follows from the various definitions}}}{\doteq} \frac{\overline{\det(J^{\Phi}(\widetilde{P}^{\dagger})_n)}}{\overline{\Phi(y_n)-1}} \doteq \overline{\Delta_{\pi,\Phi}}.$$

■

10.2.2. **The under presentation.** The Dual Presentation–Symmetry Proposition 10.3 shows that one approach to proving the Alexander Polynomial–Symmetry Theorem 9.4 is to prove that $\pi_1(S^3 \setminus L)$ admits a pair of dual presentations.

In this subsection we introduce, given a diagram of a link L, the under-presentation of $\pi_1(S^3 \setminus L)$, which will later see is dual to the corresponding over-presentation of $\pi_1(S^3 \setminus L)$.

Proposition 10.4. (Under-Presentation Proposition) Let $\left(\gamma\colon \bigsqcup_{i=1}^{m} S^1_i \to \mathbb{R}^2, c\right)$ be a link diagram such that each component has at least one crossing. Let L be the link that we associated in the Alternative Diagram-to-Link Lemma 10.1 to the link diagram. We enumerate the underpasses by $y_1,\dots,y_n$ and we enumerate the overpasses by $1,\dots,n$.

- We pick **orientation-reversing**[79] smooth embeddings $S_i\colon \overline{B}^2 \to \mathbb{R}^2$, $i=1,\dots,n$ with disjoint images and such that the projection of the i-th overpass to $\mathbb{R}^2$ is contained in $S_i(B^2)$ and such that $S_i(S^1)$ intersects $\gamma\left(\bigsqcup_{i=1}^{m} S^1_i\right)$ transversally.
- For the i-th overpass we define a relation $s_i \in \langle y_1,\dots,y_n\rangle$ as follows: We consider the path $[0,1] \to \mathbb{R}^2$ that is given by $t \mapsto S_i(\exp(2\pi it))$. We start at $t=0$ and going along $[0,1]$ we record a generator y_j (or its inverse y_j^{-1}) if the path hits the image of the j-th overpass with a negative (positive) sign.

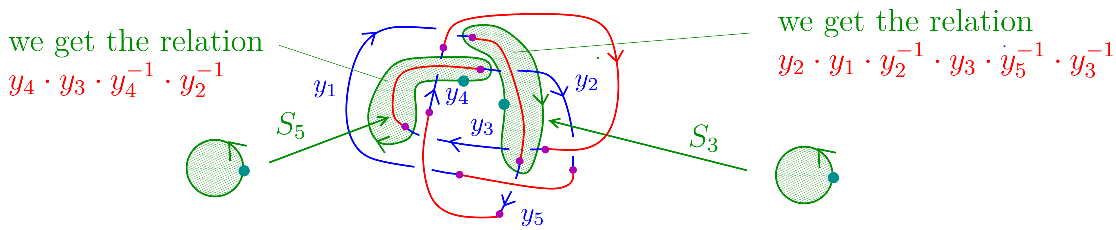
With this notation the following statements hold:

(1) We have an explicit isomorphism

$$\langle \underbrace{y_1,\dots,y_n}_{\substack{\text{correspond to}\\ \text{underpasses}}} \mid \underbrace{s_1,\dots,s_n}_{\substack{\text{correspond to}\\ \text{overpasses}}} \rangle \xrightarrow{\cong} \pi_1(S^3 \setminus L)$$

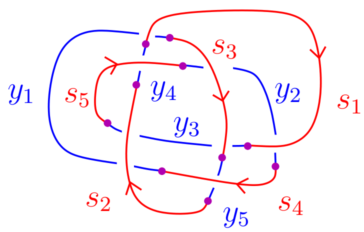
where each y_i is given by the inverse of a meridian corresponding to the i-th underpass.

(2) In the presentation (1) we can drop any one of the relations and we still obtain an isomorphism.



Proof. The proof of this proposition is almost identical to the proof of the Over-Presentation Proposition 10.2 which in turn is almost the same as the proof of the Wirtinger Presentation Proposition 5.5. We leave it to the reader to make the necessary modifications. ∎

Definition. Let $\big(\gamma\colon \bigsqcup_{i=1}^{m} S^1_i \to \mathbb{R}^2, c\big)$ be a link diagram such that each component has at least one crossing. Let L be the link that we associated in the Alternative Diagram-to-Link Lemma 10.1 to the link diagram. We refer to the presentation of $\pi_1(S^3 \setminus L)$ from the Under-Presentation Proposition 10.4 as the under-presentation of $\pi_1(S^3 \setminus L)$.



we get generators and relations y_1, y_2, y_3, y_4, y_5

$$\begin{aligned}
s_1 &= y_4 \cdot y_1 \cdot y_2^{-1} \cdot y_3^{-1} \cdot y_2 \cdot y_1^{-1}\\
s_2 &= y_3^{-1} \cdot y_1^{-1} \cdot y_5^{-1} \cdot y_1 \cdot y_2 \cdot y_4\\
s_3 &= y_2 \cdot y_1 \cdot y_2^{-1} \cdot y_3 \cdot y_5^{-1} \cdot y_3^{-1}\\
s_4 &= y_5 \cdot y_2 \cdot y_5^{-1} \cdot y_1^{-1}\\
s_5 &= y_4 \cdot y_3 \cdot y_4^{-1} \cdot y_2^{-1}
\end{aligned}$$

under-presentation for $\pi_1(S^3 \setminus 8_{21})$

10.2.3. The Dual Presentation Theorem.

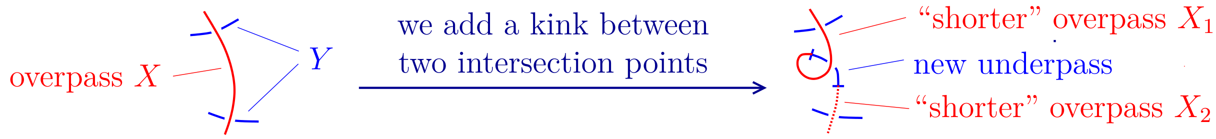
In this subsection we want to show that over- and under-presentations are duals. To simplify our life we restrict ourselves to presentations arising from “simple” diagrams:

Definition. A link diagram is called simple if for each overpass X and each underpass Y intersect in at most one point, i.e. we have $\#(X \cap Y) \leq 1$.

Lemma 10.5. (Simple Diagram Lemma) Every link admits a simple diagram such that each component has at least one crossing.

Sketch of proof. Let L be a link. By the Link Diagram Existence Proposition 4.3 and the One Crossing Exists Lemma 5.4 we know that L admits a diagram such that each component has at least one crossing. For each pair of overpass X and underpass Y with more than one intersection point we add a kink between the intersection points, as shown in the figure below:
This move splits the overpass into two overpasses X_1, X_2, each of which has at least one intersection point with the underpass Y less than the initial overpass. Furthermore it creates an underpass with a single crossing. It is clear that iterating this procedure (and doing the same for underpasses) creates a simple link diagram for L. ∎

[79]Note that throughout we always use the opposite convention to the one used in the Over-Presentation Proposition 10.2. This is a little bit confusing, but it is necessary to obtain a presentation which is dual to the over-presentation.



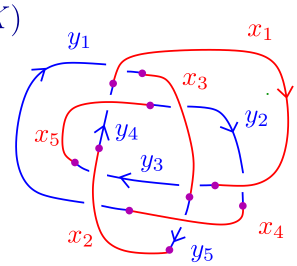
Theorem 10.6. (Dual Presentation Theorem) Let $\left(\gamma\colon \bigsqcup_{i=1}^{m} S_i^1 \to \mathbb{R}^2, c\right)$ be a simple[80] link diagram such that each component has at least one crossing. Let L be the link that we associated in the Alternative Diagram-to-Link Lemma 10.1 to the link diagram. We enumerate the underpasses by $1,\dots,n$ and we enumerate the overpasses by $1,\dots,n$. Since each component has at least one crossing we can and will do this in such a way that the i-th underpass and the i-th overpass correspond to the same component of L. The corresponding over-presentation and under-presentation of $\pi_1(S^3\setminus L)$ are dual to one another.

Example. We consider again $K = 8_{21}$ and in the figure below we consider again our favorite over- and under-presentations:

over-presentation for $\pi_1(S^3\setminus K)$

x_1, x_2, x_3, x_4, x_5

$$\begin{aligned}
r_1 &= x_3\cdot x_1\cdot x_2^{-1}\cdot x_4^{-1}\cdot x_2\cdot x_1^{-1}\\
r_2 &= x_3\cdot x_1\cdot x_4\cdot x_1^{-1}\cdot x_3^{-1}\cdot x_5^{-1}\\
r_3 &= x_2\cdot x_5\cdot x_2^{-1}\cdot x_3\cdot x_1^{-1}\cdot x_3^{-1}\\
r_4 &= x_5\cdot x_1\cdot x_5^{-1}\cdot x_2^{-1}\\
r_5 &= x_4^{-1}\cdot x_3^{-1}\cdot x_4\cdot x_2
\end{aligned}$$



under-presentation for $\pi_1(S^3\setminus K)$

y_1, y_2, y_3, y_4, y_5

$$\begin{aligned}
s_1 &= y_4\cdot y_1\cdot y_2^{-1}\cdot y_3^{-1}\cdot y_2\cdot y_1^{-1}\\
s_2 &= y_3^{-1}\cdot y_1^{-1}\cdot y_5^{-1}\cdot y_1\cdot y_2\cdot y_4\\
s_3 &= y_2\cdot y_1\cdot y_2^{-1}\cdot y_3\cdot y_5^{-1}\cdot y_3^{-1}\\
s_4 &= y_5\cdot y_2\cdot y_5^{-1}\cdot y_1^{-1}\\
s_5 &= y_4\cdot y_3\cdot y_4^{-1}\cdot y_2^{-1}
\end{aligned}$$

For $i=1$ and $j=2$ we see that

$$\begin{aligned}
\Phi_K\big(\tfrac{\partial r_1}{\partial x_2}\cdot(x_2-1)\big) &= \Phi_K\big((-x_3\cdot x_1\cdot x_2^{-1} + x_3\cdot x_1\cdot x_2^{-1}\cdot x_4^{-1})\cdot(x_2-1)\big) \overset{\text{since } \Phi_K(x_i)=t}{\overset{\downarrow}{=}} (-t+1)\cdot(t-1)\\
\Phi_K\big(\overline{\tfrac{\partial s_2}{\partial y_1}\cdot(y_1-1)}\,\big) &= \Phi_K\big(\overline{(-y_3^{-1}\cdot y_1^{-1} + y_3^{-1}\cdot y_1^{-1}\cdot y_5^{-1})\cdot(y_1-1)}\big) \underset{\text{since } \Phi_K(y_i)=t^{-1}}{\underset{\uparrow}{=}} (-t^{-2}+t^{-3})\cdot(t-1).
\end{aligned}$$

Thus we see that the two terms do indeed agree up to a power of t. $\square$

Proof. [81] Let $\left(\gamma\colon \bigsqcup_{i=1}^{m} S_i^1 \to \mathbb{R}^2, c\right)$ be a link diagram such that each component has at least one crossing. Let L be the link that we associated in the Alternative Diagram-to-Link Lemma 10.1 to the link diagram. We adopt the following notation:

(1) Since each component of the diagram has at least one crossing we see, as in the Crossing–Strand Lemma 5.3, that each component has the same number of under- and overpasses.

[80]The hypothesis that the diagram is simple simplifies the proof, but it is not really necessary. We leave it to the reader to modify the proof so that it can handle any diagram such that each component has at least one crossing.

[81]We will take a few minor liberties and omit a few details to keep the proof readable. A completely rigorous proof would certainly require more details.

This implies that we can enumerate the underpasses by $1, \ldots, n$ and we can enumerate the overpasses by $1, \ldots, n$ in such a way that the i-th underpass and the i-th overpass correspond to the same component of L.

(2) We denote by $\Phi_L \colon \pi_1(S^3 \setminus L) \to \langle t_1, \ldots, t_m\rangle_{\mathrm{ab}}$ the epimorphism that sends the i-th meridian to t_i.

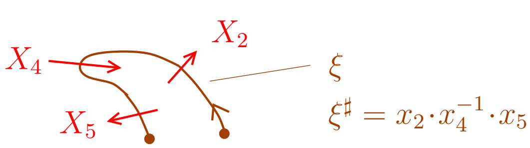
(3) (a) We denote by $X_1, \ldots, X_n$ the projections of the overpasses to $\mathbb{R}^2$.
 (b) We denote by $Y_1, \ldots, Y_n$ the projections of the underpasses to $\mathbb{R}^2$.

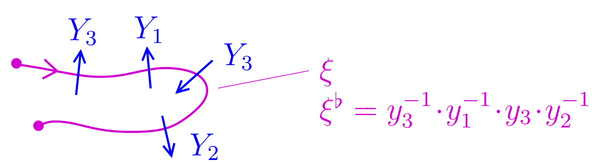
(4) (a) As in the Over-Presentation Proposition 10.2 we pick orientation-preserving smooth embeddings $R_1, \ldots, R_n \colon \overline{B}^2 \to \mathbb{R}^2$ "around the Y_i". We obtain the corresponding over-presentation $\langle x_1, \ldots, x_n \,|\, r_1, \ldots, r_n\rangle$.
 (b) As in the Under-Presentation Proposition 10.4 we pick orientation-reversing smooth embeddings $S_1, \ldots, S_n \colon \overline{B}^2 \to \mathbb{R}^2$ "around the X_i". We obtain the corresponding under-presentation $\langle y_1, \ldots, y_n \,|\, s_1, \ldots, s_n\rangle$.

Next we introduce two new objects:

(a) Given a smooth embedding $\xi \colon [0,1] \to \mathbb{R}^2$ that intersects the overpasses $X_1, \ldots, X_n$ transversally we denote by $\xi^\sharp \in \langle x_1, \ldots, x_n\rangle$ the following word: We start out at 0, going along $[0,1]$ we record a generator x_j (or its inverse x_j^{-1}) if the image of ξ hits the image of the j-th underpass with a positive (negative) sign.

(b) Given a smooth embedding $\xi \colon [0,1] \to \mathbb{R}^2$ that intersects the underpasses $Y_1, \ldots, Y_n$ transversally we denote by $\xi^\flat \in \langle y_1, \ldots, y_n\rangle$ the following word: We start out at 0, going along $[0,1]$ we record a generator y_j (or its inverse y_j^{-1}) if the image of ξ hits the image of the j-th underpass with a negative (positive) sign.





We make the following observations regarding these definitions:

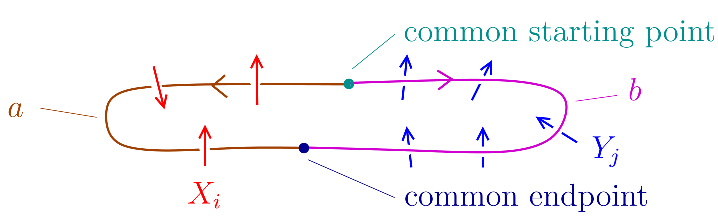
(a) For two suitable smooth embeddings $\xi, \zeta \colon [0,1] \to \mathbb{R}^2$ with $\xi(1) = \zeta(0)$ we have, basically by definition, $(\xi * \zeta)^\sharp = \xi^\sharp \cdot \zeta^\sharp$. Furthermore, we have $R_i(t \mapsto \exp(2\pi i t))^\sharp = r_i$.

(b) The analogues for "$\flat$" also hold.

Claim 1. Suppose we are given the following:

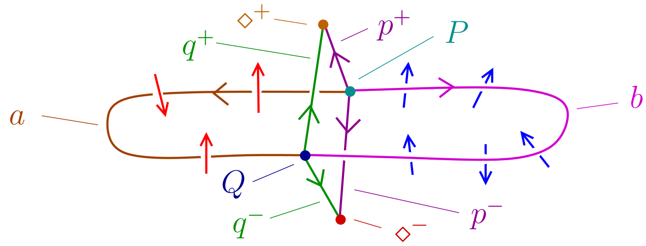
- A smooth embedding $[0,1] \xrightarrow{a} \mathbb{R}^2 \setminus (Y_1 \cup \ldots \cup Y_n)$ that intersects $X_1, \ldots, X_n$ transversally.
- A smooth embedding $[0,1] \xrightarrow{b} \mathbb{R}^2 \setminus (X_1 \cup \ldots \cup X_n)$ that intersects $Y_1, \ldots, Y_n$ transversally.

If $a(0) = b(0)$ and $a(1) = b(1)$, then $\Phi_L(a^\sharp) = \Phi_L(b^\flat)^{-1}$.



Proof. We introduce some extra notation:

- We set $P := a(0) = b(0)$ and $Q := a(1) = b(1)$.
- We write $\diamond^{\pm} := (0, 0, \pm 2)$.
- We denote by $p^{\pm}$ the straight path from P to $\diamond^{\pm}$. Similarly we define $q^{\pm}$.



We make the following observations:

(a) $a^\sharp \in \langle x_1, \dots, x_n \rangle$ describes the loop $\overline{p^+} * a * q^+$ in $\pi_1((\mathbb{R}^2 \times [0,\infty)) \setminus L, \diamond^+) = \langle x_1, \dots, x_n \rangle$.
(b) $b^\flat \in \langle y_1, \dots, y_n \rangle$ describes the loop $\overline{p^-} * b * q^-$ in $\pi_1((\mathbb{R}^2 \times (-\infty, 0]) \setminus L, \diamond^-) = \langle y_1, \dots, y_n \rangle$.
(c) Since the image of a lies in the complement of $Y_1, \dots, Y_n$ we see that a is path-homotopic in $\mathbb{R}^2 \times [-2, 0]$ to the path $p^- * \overline{q^-}$.
(d) Since the image of b lies in the complement of $X_1, \dots, X_n$ we see that b is path-homotopic in $\mathbb{R}^2 \times [0, 2]$ to the path $p^+ * \overline{q^+}$.

We now consider the following diagram:

$$\begin{tikzcd}
\overset{a^\sharp}{\langle x_1, \dots, x_n \rangle} \arrow[r, two heads] \arrow[d, "\mathrm{id}"] & \langle x_1, \dots, x_n \,|\, r_1, \dots, r_{n-1} \rangle \arrow[d, "\mathrm{id}"] & \\
\underset{[\overline{p^+} * a * q^+]}{\pi_1((\mathbb{R}^2 \times [0,\infty)) \setminus L, \diamond^+)} \arrow[r, two heads, "\text{inclusion}", "\text{induced}"'] & \underset{[\overline{p^+} * p^- * \overline{q^-} * q^+]}{\pi_1(S^3 \setminus L, \diamond^+)} \arrow[r, "\Phi_L"] \arrow[d, "(\overline{p^-} * p^+)_*"] & \langle t_1, \dots, t_m \rangle_{\mathrm{ab}} \arrow[d, "\mathrm{id}"] \\
\overset{[\overline{q^-} * \overline{b} * p^-]}{\pi_1((\mathbb{R}^2 \times (-\infty, 0]) \setminus L, \diamond^-)} \arrow[r, two heads, "\text{inclusion}", "\text{induced}"'] & \overset{[\overline{q^-} * q^+ * \overline{p^+} * p^-]}{\pi_1(S^3 \setminus L, \diamond^-)} \arrow[r, "\Phi_L"] & \langle t_1, \dots, t_m \rangle_{\mathrm{ab}} \\
\underset{(b^\flat)^{-1}}{\langle y_1, \dots, y_n \rangle} \arrow[r, two heads] \arrow[u, "\mathrm{id}"] & \langle y_1, \dots, y_n \,|\, s_1, \dots, s_{n-1} \rangle . \arrow[u, "\mathrm{id}"'] &
\end{tikzcd}$$

Using (a)–(d) one sees that the maps send the given elements to the given elements. It follows that $\Phi_L(a^\sharp) = \Phi_L((b^\flat)^{-1})$, which corresponds to the desired result. ⊞

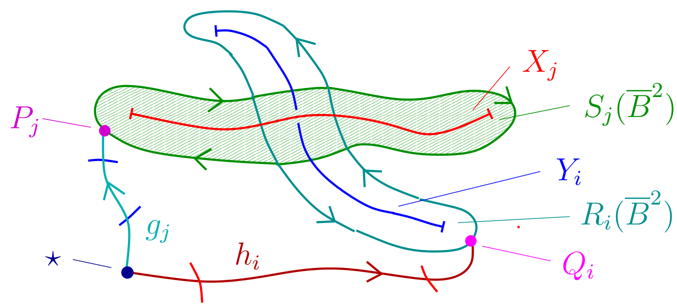
Now we turn to the actual proof that the over- and under-presentation are dual. We have to show that these two presentations satisfy the conditions (D1) and (D2) from page 140. First we consider (D1). Let $i \in \{1, \dots, n\}$. Recall that we arranged that there exists a $k \in \{1, \dots, m\}$ such that the i-th underpass and the i-th overpass both correspond to the k-th component of L. We now see that

$$\Phi_L(x_i) \underset{\uparrow}{=} \Phi_L(\mu_{L_m}) \underset{\uparrow}{=} \Phi_L(y_i^{-1}) = \Phi_L(\overline{y_i}).$$

by construction of the over- and under-presentation

So it remains to prove (D2). This will require a few more pieces of notation:

- We pick a base point $\star \in \mathbb{R}^2 \times \{0\}$.
- For $i = 1, \dots, n$ we pick a path g_i in $\mathbb{R}^2 \setminus (X_1 \cup \dots \cup X_n)$ from $\star$ to $S_i(1)$.
- For $i = 1, \dots, n$ we pick a path h_i in $\mathbb{R}^2 \setminus (Y_1 \cup \dots \cup Y_n)$ from $\star$ to $R_i(1)$.

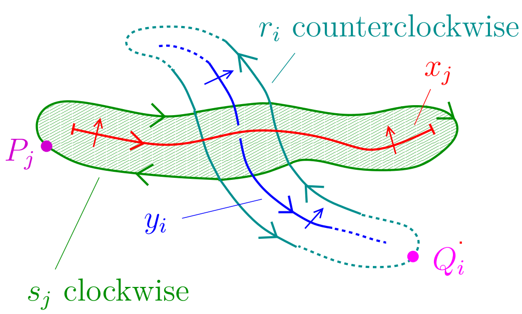


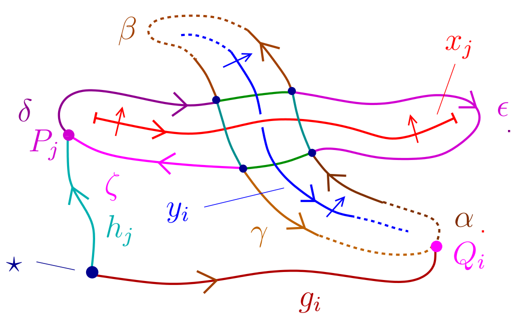
It remains to prove the following claim:

Claim 2. For any $i, j \in \{1, \dots, n\}$ we have
$$\Phi_L\Big(g_i^\sharp \cdot \frac{\partial r_i}{\partial x_j} \cdot (x_j - 1)\Big) \;=\; \Phi_L\Big(\overline{h_j^\flat \cdot \frac{\partial s_j}{\partial y_i} \cdot (y_i - 1)}\Big).$$

Proof. Since we assume that the diagram is simple there exists at most one crossing between the i-th underpass and the j-th overpass. If there is no crossing, then one can easily see that x_j does not appear in the word r_i and y_i does not appear in the word s_j. This implies that both Fox derivatives $\frac{\partial r_i}{\partial x_j}$ and $\frac{\partial s_j}{\partial y_i}$ are zero. Thus the promised equality holds trivially.

Now we consider the much more interesting case, that there exists a crossing. We use the figure below to introduce some further notation.





In the setting of the figure we have
$$\begin{aligned} r_i &= \alpha^\sharp \cdot x_j^\mu \cdot \beta^\sharp \cdot x_j^{-\mu} \cdot \gamma^\sharp &&\in \langle x_1, \dots, x_n\rangle \\ s_j &= \delta^\flat \cdot y_i^\nu \cdot \epsilon^\flat \cdot y_i^{-\nu} \cdot \zeta^\flat &&\in \langle y_1, \dots, y_n\rangle. \end{aligned}$$

(To be more precise, in the figure we have $\mu = 1$ and $\nu = 1$. But we want to take care of all possible orientations.) We calculate

$$\frac{\partial r_i}{\partial x_j} \underset{\uparrow}{=} \frac{\partial}{\partial x_j}(\alpha^\sharp\cdot x_j^\mu\cdot\beta^\sharp\cdot x_j^{-\mu}\cdot\gamma^\sharp) \underset{\uparrow}{=} \frac{\partial}{\partial x_j}(\alpha^\sharp\cdot x_j^\mu\cdot\beta^\sharp\cdot x_j^{-\mu}) = (\alpha^\sharp - \alpha^\sharp\cdot x_j^\mu\cdot\beta^\sharp\cdot x_j^{-\mu})\cdot\frac{x_j^\mu-1}{x_j-1}$$

by the discussion on page 145 since X_i and Y_j have only one intersection point we se that γ and ζ play no role

$$\frac{\partial s_j}{\partial y_i} \overset{\downarrow}{=} \frac{\partial}{\partial y_i}(\delta^\flat\cdot y_i^\nu\cdot\epsilon^\flat\cdot y_i^\nu\cdot\zeta^\flat) \overset{\downarrow}{=} \frac{\partial}{\partial y_i}(\delta^\flat\cdot y_i^\nu\cdot\epsilon^\flat\cdot y_i^\nu) = (\delta^\flat - \delta^\flat\cdot y_i^\nu\cdot\epsilon^\flat\cdot y_i^{-\nu})\cdot\frac{y_i^\nu-1}{y_i-1}.$$

It follows that

$$\begin{aligned} g_i^\sharp \cdot \tfrac{\partial r_i}{\partial x_j} \cdot (x_j - 1) &= (g_i \cdot \alpha)^\sharp \cdot (1 - x_j^\mu \cdot \beta^\sharp \cdot x_j^{-\mu}) \cdot (x_j^\mu - 1) \\ h_j^\flat \cdot \tfrac{\partial s_j}{\partial y_i} \cdot (y_i - 1) &= (h_j \cdot \delta)^\flat \cdot (1 - y_i^\nu \cdot \epsilon^\flat \cdot y_i^{-\nu}) \cdot (y_i^\nu - 1). \end{aligned}$$

This implies that

$$\begin{aligned} g_i^\sharp \cdot \tfrac{\partial r_i}{\partial x_j} \cdot (x_j - 1) &= (g_i * \alpha * x_j^\mu)^\sharp \cdot x_j^{-\mu} \cdot (1 - x_j^\mu \cdot \beta^\sharp \cdot x_j^{-\mu}) \cdot x_j^\mu \cdot \overbrace{x_j^{-\mu} \cdot (x_j^\mu - 1)}^{=1-x_j^{-\mu}} \\ h_j^\flat \cdot \tfrac{\partial s_j}{\partial y_i} \cdot (y_i - 1) &= (h_j * \delta * y_i^\nu)^\flat \cdot y_i^{-\nu} \cdot (1 - y_i^\nu \cdot \epsilon^\flat \cdot y_i^{-\nu}) \cdot y_i^\nu \cdot \underbrace{y_i^{-\nu} \cdot (y_i^\nu - 1)}_{=1-y_i^{-\nu}}. \end{aligned}$$

Since Φ_L is a homomorphism and since the target of Φ_L is abelian we obtain that

$$\begin{aligned} \Phi_L(g_i^\sharp \cdot \tfrac{\partial r_i}{\partial x_j} \cdot (x_j - 1)) &= \Phi_L((g_i * \alpha * x_j^\mu)^\sharp) \cdot \Phi_L(1 - \beta^\sharp) \cdot \Phi_L(1 - x_j^{-\mu}) \\ \Phi_L(h_j^\flat \cdot \tfrac{\partial s_j}{\partial y_i} \cdot (y_i - 1)) &= \Phi_L((h_j * \delta * y_i^\nu)^\flat) \cdot \Phi_L(1 - \epsilon^\flat) \cdot \Phi_L(1 - y_i^{-\nu}). \end{aligned}$$

Claim 2 follows from the above calculations and the observation that Claim 1 give us the following equalities:

$$\begin{aligned} \Phi_L((g_i * \alpha * x_j^\mu)^\sharp) &= \Phi_L((h_j * \delta * y_i^\nu)^\flat)^{-1}, \\ \Phi_L(\beta^\sharp) &= \Phi_L(y_i^{-\nu})^{-1}, \\ \Phi_L(\epsilon^\flat) &= \Phi_L(x_j^\nu)^{-1}. \end{aligned}$$

■

10.2.4. **Proof of the Alexander Polynomial–Symmetry Theorem 9.4.** For the reader's convenience we recall the theorem we need to prove:

Theorem 9.4. (Alexander Polynomial–Symmetry Theorem) Given any oriented m-component link L we have

$$\Delta_L(t_1, \ldots, t_m) \doteq \Delta_L(t_1^{-1}, \ldots, t_m^{-1}).$$

Proof. Let L be an oriented m-component link. We adopt the following standard notation:

(1) We set $\pi := \pi_1(S^3 \setminus L)$ and we denote by $\Phi_L \colon \pi \to \langle t_1, \ldots, t_m \rangle_{\mathrm{ab}}$ the epimorphism given by $\mu_{L_i} \mapsto t_i$.
(2) As usual we identify the group ring of $\langle t_1, \ldots, t_m \rangle_{\mathrm{ab}}$ with the Laurent polynomial ring $\mathbb{Z}[t_1^{\pm 1}, \ldots, t_m^{\pm 1}]$.
(3) Note that the above involution on group rings corresponds to the obvious involution on $\mathbb{Z}[t_1^{\pm 1}, \ldots, t_m^{\pm 1}]$ and $\mathbb{Q}(t_1, \ldots, t_m)$ that is given by $t_i \mapsto t_i^{-1}$.

With the above notation our task can now be reformulated: We need to show that

$$\Delta_L = \overline{\Delta_L} \in \mathbb{Z}[t_1^{\pm 1}, \ldots, t_m^{\pm 1}].$$

Next, recall that by the Simple Diagram Lemma 10.5 we know that L admits a simple diagram such that each component has a crossing. We enumerate the underpasses by $1, \ldots, n$ and we enumerate the overpasses by $1, \ldots, n$. Since each component has at least one crossing we can and will do this in such a way that the i-th underpass and the i-th overpass correspond to the same component of L. Next we apply the Over-Presentation Proposition 10.2 and the Under-Presentation Proposition 10.2 and we introduce the following notation:

- Let $P := \langle x_1, \ldots, x_n \,|\, r_1, \ldots, r_n \rangle$ be the corresponding over-presentation.
- Let $P^\dagger := \langle y_1, \ldots, y_n \,|\, s_1, \ldots, s_n \rangle$ be the corresponding under-presentation.

By the Dual Presentation Theorem 10.6 we know that these two presentations are duals of one another. We make two more observations:

- By the last sentence of the Over-Presentation Proposition 10.2 and of the Under-Presentation Proposition 10.4 we can drop in both cases the n-th relation. We denote the resulting presentations by $\widetilde{P}$ and $\widetilde{P}^\dagger$.
- In both cases we argued that all $\Phi_L(x_i)$ and all $\Phi_L(y_i)$ are non-trivial.

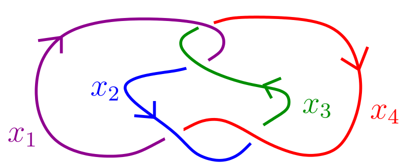
The above observation implies that we can apply the Dual Presentation–Symmetry Proposition 10.3, and we see that
$$\Delta_{\pi,\Phi_L} \doteq \overline{\Delta_{\pi,\Phi_L}}.$$
For links this gives, by definition, immediately the desired statement that $\Delta_L \doteq \overline{\Delta_L}$. For knots basically the same argument works, the only extra ingredient we need is the little observation that $t-1 \doteq t^{-1}-1$. ■

10.3. **The Alexander polynomial evaluated at $t=1$.** We just showed that Alexander polynomials of links are symmetric. For knots we will now see that Alexander polynomials have one more property:

Proposition 10.7. (Alexander Polynomial-at-1 Proposition) For any knot $K \subset S^3$ we have $\Delta_K(1) = \pm 1$.

Proof. Let $K \subset S^3$ be a knot. By the Link Diagram Existence Proposition 4.3 and the One Crossing Exists Lemma 5.4 we know that K admits a diagram with at least one crossing. We order the strands cyclically. By the Wirtinger Presentation Proposition 5.5 we obtain an $n \in \mathbb{N}$, a map $\sigma\colon \{1,\dots,n\} \to \{1,\dots,n\}$ and a Wirtinger presentation
$$P := \langle x_1,\dots,x_n \,|\, r_1,\dots,r_{n-1}\rangle \xrightarrow{\cong} \pi_1(S^3 \setminus K)$$
where each r_j is of the form

(a) $r_j = x_{\sigma(j)} \cdot x_{j+1} \cdot x_{\sigma(j)}^{-1} \cdot x_j^{-1}$ (positive crossing)
(b) $r_j = x_{\sigma(j)}^{-1} \cdot x_{j+1} \cdot x_{\sigma(j)} \cdot x_j^{-1}$ (negative crossing).



$$\begin{array}{rl} \text{the generators are} & x_1, x_2, x_3, x_4 \\ \text{the relations are} & r_1 = x_3 \cdot x_2 \cdot x_3^{-1} \cdot x_1^{-1} \\ & r_2 = x_4^{-1} \cdot x_3 \cdot x_4 \cdot x_2^{-1} \\ & r_3 = x_1 \cdot x_4 \cdot x_1^{-1} \cdot x_3^{-1} \end{array}$$

Let $\Phi\colon P \to \langle t \rangle$ be the unique homomorphism given by $\Phi(x_i) = t$. As before we denote by $\epsilon\colon \mathbb{Z}[t^{\pm 1}] \to \mathbb{Z}$ the augmentation, i.e. the unique ring homomorphism with $\epsilon(t^k) = 1$.

We see that
$$\begin{aligned} \Delta_K(1) \;=\; \epsilon(\Delta_K(t)) &\overset{\substack{\text{definition of } \Delta_K(t)\\ \downarrow}}{=} \epsilon\Big((t-1)\cdot \frac{\det(\Phi_*(J(P))_n)}{\Phi_*(x_n-1)}\Big) \overset{\substack{\text{since } x_n \text{ is a meridian}\\ \downarrow}}{=} \epsilon\Big((t-1)\cdot \frac{\det(\Phi_*(J(P))_n)}{t-1}\Big) \\ &= \epsilon(\det(\Phi_*(J(P))_n)) \underset{\substack{\uparrow \\ \text{since determinants commute}\\ \text{with ring homomorphisms}}}{=} \det((\epsilon \circ \Phi_*)(J(P))_n). \end{aligned}$$

It follows from this discussion that it remains to prove the following claim:

Claim. We have

$$(\epsilon\circ\Phi_*)(J(P)) = \underbrace{\begin{pmatrix} -1 & 1 & 0 & \dots & 0 \\ 0 & -1 & 1 & & 0 \; 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & -1 & 1 \end{pmatrix}}_{(n-1)\times n\text{-matrix}} \quad \text{and thus} \quad (\epsilon\circ\Phi_*)(J(P))_n = \underbrace{\begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & \ddots & \ddots \\ 0 & \dots & 0 & -1 \end{pmatrix}}_{\text{determinant } = (-1)^{n-1}}.$$

Proof. We just need to show that the j-th row of $(\epsilon\circ\Phi_*)(J(P))$ has an entry -1 in column j, an entry 1 in column $j+1$ and zeros otherwise. We look at the two types of relations:

(a) First we suppose that $r_j = x_{\sigma(j)}\cdot x_{j+1}\cdot x_{\sigma(j)}^{-1}\cdot x_j^{-1}$. If $\sigma(j)\neq\{j,j+1\}$, then

$$\begin{array}{rcll} (\epsilon\circ\Phi_*)\big(\frac{\partial r_j}{\partial x_j}\big) & = & (\epsilon\circ\Phi_*)(-x_{\sigma(j)}\cdot x_{j+1}\cdot x_{\sigma(j)}^{-1}\cdot x_j^{-1}) & = \; -1, \\ (\epsilon\circ\Phi_*)\big(\frac{\partial r_j}{\partial x_{j+1}}\big) & = & (\epsilon\circ\Phi_*)(x_{\sigma(j)}) & = \; 1, \\ (\epsilon\circ\Phi_*)\big(\frac{\partial r_j}{\partial x_{\sigma(j)}}\big) & \underset{\substack{\uparrow \\ \text{apply the axioms from the Fox Derivative Proposition 8.1}}}{=} & (\epsilon\circ\Phi_*)(1-x_{\sigma(j)}\cdot x_{j+1}\cdot x_{\sigma(j)}^{-1}) & = \; 0. \end{array}$$

All other Fox derivatives are evidently zero. The calculation for the special case that $\sigma(j)\in\{j,j+1\}$ is almost the same.

(b) We suppose that $r_j = x_{\sigma(j)}^{-1}\cdot x_{j+1}\cdot x_{\sigma(j)}\cdot x_j^{-1}$. This case is treated almost entirely the same way as the previous case. ■

Let $K\subset S^3$ be a knot. By the Alexander Polynomial–Symmetry Theorem 9.4 and the Alexander Polynomial-at-1 Proposition 10.7 we now know that $\Delta_K(t)\doteq\Delta_K(t^{-1})$ and that $\Delta_K(1)=\pm1$. The following proposition shows that these are the only two general properties of Alexander polynomials of knots:

Proposition 10.8. (Alexander Polynomial–Realization Proposition) Given any Laurent polynomial $p(t)\in\mathbb{Z}[t^{\pm1}]$ with $p(t)\doteq p(t^{-1})$ and with $p(1)=\pm1$ there exists a knot $K\subset S^3$ with $\Delta_K(t)=p(t)$.

Proof. The proposition was first proved by Herbert Seifert [**Sei34**, Satz 6]. An alternative proof was given by Jerome Levine [**Lev65**, p. 136]. ■

10.4. Appendix I: The Torres condition.

It is natural to ask whether the Alexander Polynomial-at-1 Proposition 10.7 generalizes to links. To formulate this extension we need to recall the notion of the linking number of two oriented knots which we introduced on page 87

Definition. Let K and $\widetilde{K}$ be two disjoint oriented knots. By the Link Group-Abelianization Corollary 5.7 we have $\pi_1(S^3\setminus K)_{\text{ab}}=\mathbb{Z}\cdot\mu_K$. Note that it follows from the Loop-π_1-Lemma 3.14 that $\widetilde{K}$ gives rise to a well-defined element of $\pi_1(S^3\setminus K)_{\text{ab}}$ which we denote by $[\widetilde{K}]$. We now define the linking number $\operatorname{lk}(K,\widetilde{K})$ to be the unique integer such that the following equality holds in $\pi_1(S^3\setminus K)_{\text{ab}}=\mathbb{Z}\cdot\mu_K$:

$$[\widetilde{K}] \;=\; \operatorname{lk}(K,\widetilde{K})\cdot\mu_K \;\in\; \pi_1(S^3\setminus K)_{\text{ab}}=\mathbb{Z}\cdot\mu_K.$$

Using the notion of a linking number we can now formulate the following theorem:

Theorem 10.9. (Torres Condition Theorem) Let $K = K_1 \cup \cdots \cup K_m$ be an oriented m-component link with $m \geq 2$. For $i = 1, \ldots, m-1$ we set $\lambda_i := \operatorname{lk}(K_i, K_m)$. Then[82]

$$\Delta_K(t_1, \ldots, t_{m-1}, 1) \doteq \begin{cases} (t_1^{\lambda_1} - 1) \cdot \frac{\Delta_{K_1}(t_1)}{t_1 - 1}, & \text{if } m = 2, \\ (t_1^{\lambda_1} \cdot \ldots \cdot t_{m-1}^{\lambda_{m-1}} - 1) \cdot \Delta_{K_1 \sqcup \cdots \sqcup K_{m-1}}(t_1, \ldots, t_{m-1}), & \text{if } m \geq 3. \end{cases}$$

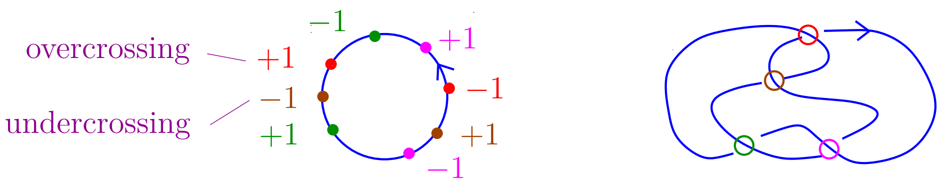
Proof. The theorem was first proved by Guillermo Torres [**Tor53**]. Alternative proofs can be found in [**Kaw96**, Theorem 7.4.1], [**Hil12**, Chapter 5] and [**Tur02**, p. 86]. For the time being we do not attempt to prove this theorem. ■

In the Alexander Polynomial–Realization Proposition 10.8 we saw that we know which Laurent polynomials are realized as the Alexander polynomials of knots. For links this characterization problem is still open:

Problem 10.10. Which multivariable Laurent polynomials in $\mathbb{Z}[t_1^{\pm 1}, \ldots, t_m^{\pm 1}]$ are realized as the Alexander polynomial of an m-component link?

10.5. **Appendix II: Alexander polynomials of alternating knots.** We now introduce a particularly interesting class of knots:

Definition.
(1) A knot diagram $(\gamma \colon S^1 \to \mathbb{R}^2, c)$ is called alternating if on S^1 the overcrossings[83] and undercrossings alternate.
(2) A knot is called alternating if it admits an alternating diagram.



alternating knot diagram — corresponding knot

Example.
(1) All diagrams shown in the list on page 55 are alternating.
(2) One can easily verify that the $(2, q)$-torus knots are alternating.
(3) The "standard diagram" for the knot 8_{19} which is shown below is not alternating. But in principle there could be a different diagram for this knot which is alternating.

trivial knot — trefoil — figure 8 knot — $(7, 2)$-torus knot — 8_{19}

alternating — alternating?

[82]As we will see in Exercise 10.7 the formulation of this theorem would be neater if we had stuck to the Alexander function of fundamental groups.
[83]We refer to page 53 for the definition of an overcrossing and an undercrossing.

Theorem 10.11. (Alexander Polynomial–Alternating Theorem) Let K be a knot. We write

$$\Delta_K(t) \;=\; \sum_{i=m}^{n} a_i \cdot t^i$$

with $a_m \neq 0$ and $a_n \neq 0$. If K is alternating, then the coefficients are alternating in sign, more precisely, there exists an $\epsilon \in \{-1, 1\}$ such that for all $i = m, \ldots, n$ we have $a_i \cdot \epsilon \cdot (-1)^i \geq 0$.

Proof. This theorem was first proved by Kunio Murasugi [**Mur58**, Theorem 4.4]. See also [**Mur60**, Theorem 3.12] for a related result. ■

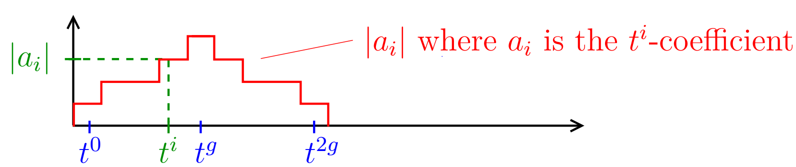
Example. In the figure below we consider the same knots as above, but this time with their Alexander polynomials.

trivial knot	trefoil	figure 8 knot	$(7,2)$-torus knot	8_{19}
1	$1-t+t^2$	$1-3t+t^2$	$1-t+t^2-t^3+t^4-t^5+t^6$	$1-t+t^3-t^5+t^6$

For the first four knots we see that the signs of the coefficients of the Alexander polynomials are indeed alternating. For the knot $K = 8_{19}$ we have $\Delta_K(t) = 1-t+0t^2+t^3+0t^4-t^5+t^6$. We observe that the coefficients are *not* alternating. It follows from the Alexander Polynomial–Alternating Theorem 10.11 that $K = 8_{19}$ does not admit an alternating diagram. □

The Alexander polynomials of knots still holds some mysteries. For example the following conjecture, which goes back to a question of Ralph Fox from 1961 [**Fox62**, p. 170], is still open:

Conjecture 10.12. (Trapezoidal Conjecture) If K is an alternating knot, then we can write $\Delta_K(t) = a_0 + a_1 \cdot t + \cdots + a_{2g} \cdot t^{2g}$ such that for $i = 0, \ldots, g-1$ we have $|a_i| \leq |a_{i+1}|$. (The figure below explains perhaps the name "trapezoidal".)



This conjecture was proved by Kunio Murasugi [**Mur85**] and Peter Ozsváth and Zoltan Szabó [**SO03**, p. 233] for large classes of alternating knots.

Remark. We defined knots as submanifolds of S^3. The definition of an alternating knot is in terms of diagrams, which is arguably somewhat awkward. A diagram-free characterization of alternating knots was given by Josh Greene [**Gre17**] and independently by Josh Howie [**How17**].

Exercises for Chapter 10.

Exercise 10.1. For which diagrams does the over presentation have the same number of generators as the Wirtinger presentation?

Exercise 10.2. Let $m, n \in \mathbb{Z}$. We consider the Baumslag-Solitar group

$$\mathrm{BS}(m,n) \;=\; \langle x, y \,|\, x^{-1} \cdot y^m \cdot x \;=\; y^n \rangle.$$

For which $m, n \in \mathbb{Z}$ does there exist an oriented link $L \subset S^3$ such that $\pi_1(S^3 \setminus L)$ is isomorphic to $\mathrm{BS}(m,n)$?

Exercise 10.3. Let $K \subset S^3$ be a knot.

(a) Show that the number $\Delta_K(-1) \in \mathbb{Z}$ is odd.
(b) Show that $\deg(\Delta_K(t))$ is even.

Exercise 10.4. Let K be a knot let $\xi \in S^1$ such that there exists a prime p and a $k \in \mathbb{N}$ such that $\xi^{p^k} = 1$. Show that $\Delta_K(\xi) \neq 0$.
Hint. Consider the minimal polynomial of ξ and use the Alexander Polynomial-at-1 Proposition 10.7.

Exercise 10.5.

(a) Let $L = L_1 \cup L_2$ be an oriented 2-component link. Show that $\Delta_L(1,1) = \mathrm{lk}(L_1, L_2)$.
(b) Let L be an oriented link with ≥ 3 components. Show that $\Delta_L(1, \dots, 1) = 0$.

Hint. In both cases make use of the Torres Condition Theorem 10.9.

Exercise 10.6. Let $K \subset S^3$ be a knot. By the Link Diagram Existence Proposition 4.3 and the One Crossing Exists Lemma 5.4 we know that K admits a diagram with at least one crossing. We order the strands cyclically. By the Wirtinger Presentation Proposition 5.5 we obtain an $n \in \mathbb{N}$, a map $\sigma\colon \{1, \dots, n\} \to \{1, \dots, n\}$ and a Wirtinger presentation

$$P \;\coloneqq\; \langle x_1, \dots, x_n \,|\, r_1, \dots, r_{n-1} \rangle \;\xrightarrow{\cong}\; \pi_1(S^3 \setminus K)$$

where each r_j is of the form $r_j = x_{\sigma(j)}^{\epsilon_j} \cdot x_{j+1} \cdot x_{\sigma(j)}^{-\epsilon_j} \cdot x_j^{-1}$ for some $\epsilon_j \in \{-1, 1\}$. Show that the diagram is alternating if and only if $\sigma\colon \{1, \dots, n\} \to \{1, \dots, n\}$ is a bijection.

Exercise 10.7. Let $K = K_1 \cup \dots \cup K_m$ be an oriented m-component link with $m \geq 2$. We set $\widehat{K} \coloneqq K_1 \cup \dots \cup K_{m-1}$.

(a) Reformulate the above Torres Condition Theorem 10.9 in terms of the Alexander functions of the two groups $\pi_1(S^3 \setminus K)$ and $\pi_1(S^3 \setminus \widehat{K})$ together with the homomorphisms $\pi_1(S^3 \setminus K) \to \pi_1(S^3 \setminus \widehat{K}) \to \pi_1(S^3 \setminus \widehat{K})_{\mathrm{ab}}$. and $\pi_1(S^3 \setminus \widehat{K}) \to \pi_1(S^3 \setminus \widehat{K})_{\mathrm{ab}}$.
Remark. If done correctly, this should lead to a more unified formulation of the Torres Condition Theorem 10.9.
(b) Prove the Torres Condition Theorem 10.9.
Hint. Start out with a diagram of K and use the Wirtinger presentation of K and $\widehat{K}$.

CHAPTER 11

Seifert surfaces

A Seifert surface for a knot K is defined as a compact orientable connected 2-dimensional submanifold F of S^3 such that $\partial F = K$. In this chapter we will give many examples of Seifert surfaces and we will see that every knot admits a Seifert surface. We will also introduce the "genus of a knot" and we will see that it is additive under the connected sum operation.

In the next Chapter 12 we will introduce the related notion of a "fibered knot". Afterwards, in Chapter 13 we will see that the Alexander polynomial of a knot contains information about the "genus" and the "fiberedness" of a knot.

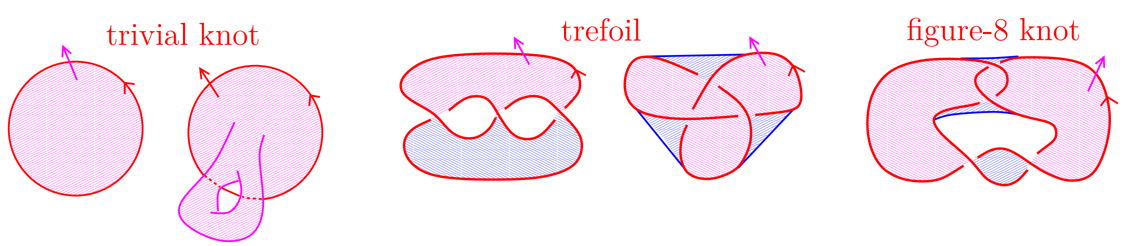
11.1. Seifert surfaces.

The following is one of the two key notions of this chapter.

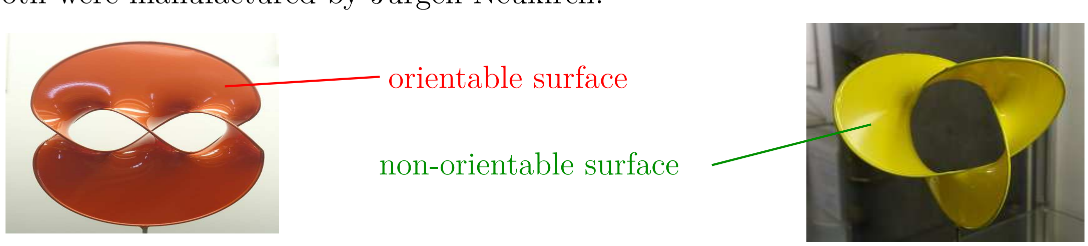
Definition. Let $K \subset S^3$ be a knot. A Seifert surface[84] for K is a compact orientable connected 2-dimensional submanifold F of S^3 such that $\partial F = K$. If K itself is oriented, then we equip a Seifert surface F with the orientation such that $\partial F = K$ as oriented smooth manifolds.

Examples.

(1) In the figure below we show Seifert surfaces for the trivial knot, the trefoil and the figure-8 knot.



(2) In the figure below we see two surfaces of minimal surface area that bound the trefoil, both were manufactured by Jürgen Neukirch.



[84]Seifert surfaces are named after Herbert Seifert (1907-1996) who we had already encountered at the Seifert-van Kampen-Theorem 3.9.

The one on the left is orientable, thus it is a Seifert surface. The one on the right is diffeomorphic to the Möbius band, thus it is not a Seifert surface. □

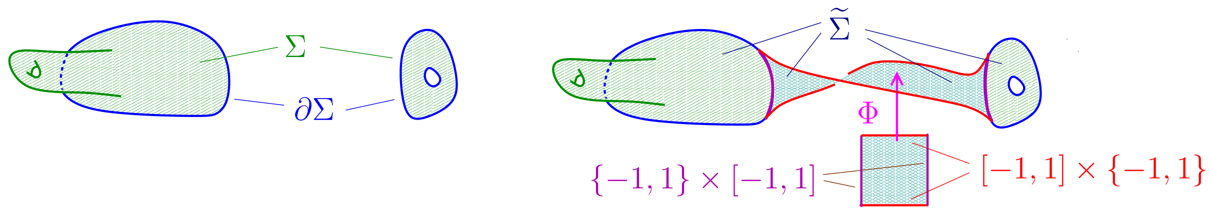
The following proposition is arguably one of the most consequential results in knot theory:

Proposition 11.1. (Seifert Algorithm) Every knot $K \subset S^3$ admits a Seifert surface.

Remark. The above proposition was first proved in 1930 by F. Frankl and Lev Pontryagin [**FP30**]. Shortly afterwards Herbert Seifert [**Sei34**] gave a different proof which is the one we present below. The proof below is very visual, explicit and practical. It is arguably though not the most rigorous proof. We give a more "high-tech" proof in [**Fri24**]. □

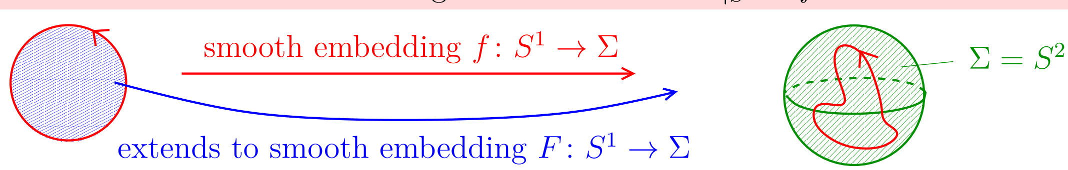
The proof of the Seifert Algorithm 11.1 requires some preparations. First of all we need to introduce the following definition:

Definition. Let Σ be a 2-dimensional smooth submanifold of some smooth manifold M. (In most of our applications we work with $M = \mathbb{R}^3$ or $M = S^3$.) We say that a smooth submanifold $\widetilde{\Sigma}$ of M is obtained from Σ by a band attachment if there exists an embedding $\Phi\colon [-1,1] \times [-1,1] \to \widetilde{\Sigma}$ such that the following statements hold:
(1) $\widetilde{\Sigma} = \Sigma \cup \Phi([-1,1] \times [-1,1])$.
(2) $\Phi([-1,1] \times [-1,1]) \cap \Sigma = \Phi([-1,1] \times [-1,1]) \cap \partial\Sigma = \Phi(\{-1,1\} \times [-1,1])$.
(3) $\partial\widetilde{\Sigma} = (\partial\Sigma \setminus \Phi(\{-1,1\} \times (-1,1)) \cup \Phi([-1,1] \times \{-1,1\})$.



We will also need the following proposition.

Proposition 11.2. (Fill-in-Disk Proposition) Let Σ be a compact orientable 2-dimensional smooth manifold and let $f\colon S^1 \to \Sigma$ be a smooth embedding. If f is null-homotopic, then there exists a smooth embedding $F\colon \overline{B}^2 \to \Sigma$ with $F|_{S^1} = f$.



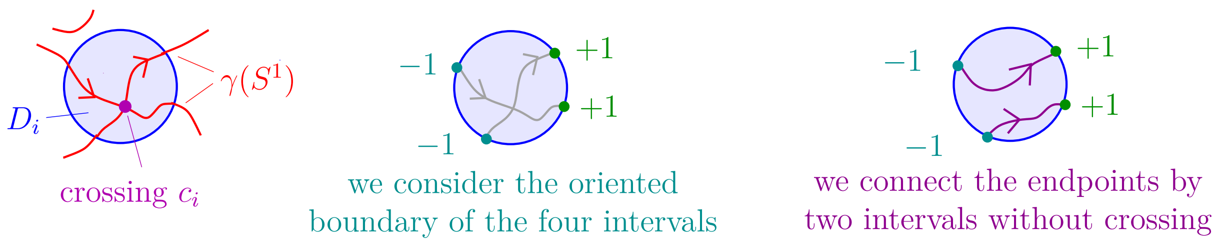
Proof. A proof is provided in [**Fri24**]. ■

Sketch of proof of the Seifert Algorithm 11.1. Let $K \subset S^3$ be a knot. It follows from the Knot Diagram Existence Proposition 4.3 and the Isotopy Extension Theorem 2.4 that we only have to deal with the case that K lies in $\mathbb{R}^3$ and that it arises from a knot diagram $(\gamma\colon S^1 \to \mathbb{R}^2, c)$. We equip K with an orientation and we perform the following three steps:

(1) First step we "resolve the crossings" in the diagram. More precisely we do the following:
 (a) We enumerate the crossings by $c_1, \dots, c_n$.

(b) We pick disjointly smoothly embedded closed disks $D_1, \dots, D_n \subset \mathbb{R}^2$ with the following properties:
   - Each disk D_i contains precisely one crossing, namely c_i.
   - Each ∂D_i intersects $\gamma\colon S^1 \to \mathbb{R}^2$ transversally.
   - Each $D_i \cap \gamma(S^1)$ is the union of two oriented properly smoothly embedded intervals.

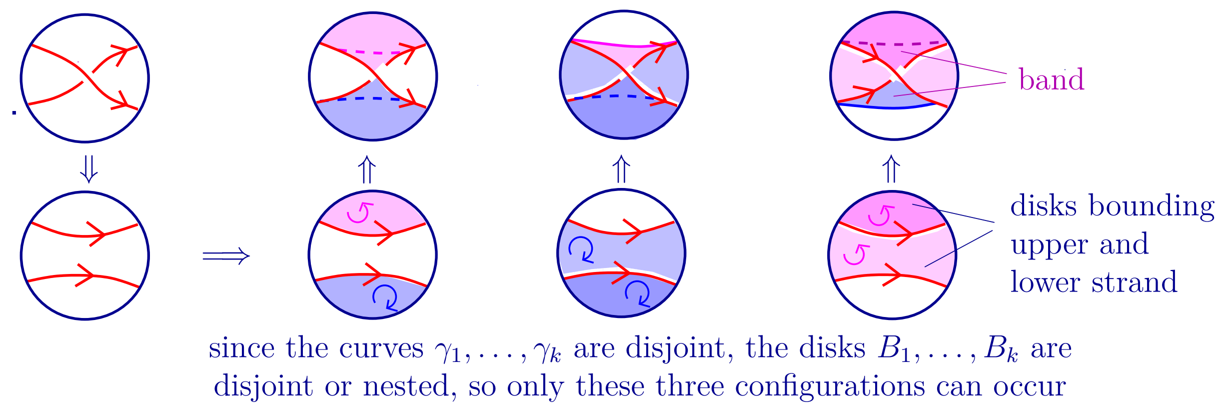
(c) For each disk D_i we do the following: We consider the four *oriented* boundary points of the two oriented intervals. In the disk we pick two *disjoint* oriented properly smoothly embedded intervals such that the boundary is precisely given by the four oriented points. We can arrange that in a small neighborhood of ∂D_i the new intervals agree with the original intervals.



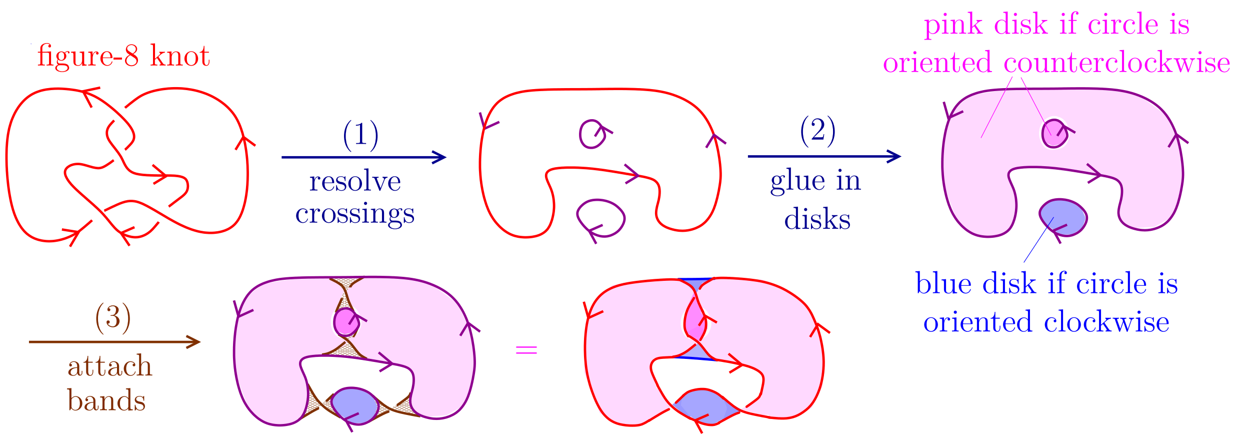
(2) (a) From the operation in (1) we obtain a collection of disjoint simple closed oriented curves $\gamma_1, \dots, \gamma_k$ in $\mathbb{R}^2$. It follows from the Fill-in-Disk Proposition 11.2 that each of these simple closed curves bounds a smoothly embedded disk $B_1, \dots, B_k \subset \mathbb{R}^2$. We equip these disks with the orientation coming from the oriented curves $\gamma_1, \dots, \gamma_k$.
(b) We pick $t_1, \dots, t_k \in \mathbb{R}$ such that $t_i < t_j$ if $B_j \subset B_i$.
(c) We consider the "wedding cake" $W := (B_1 \times \{t_1\}) \cup \dots \cup (B_k \times \{t_k\}) \subset \mathbb{R}^3$.

(3) For each crossing we attach a band to W according to the type of crossing we started out with. In the figure below we show this procedure for one type of crossing. We make the following clarifications and observations:
(a) After a smooth isotopy we can assume that we are in the situation on the left (except for possibly the opposite type of crossing).
(b) The upper and lower interval each sit on the boundary of one of the oriented disks $B_1, \dots, B_k$. These disks can be "above" or "below" the intervals.
(c) Since the curves are disjoint the disks are disjoint or nested (here "nested" means that one is contained in the other).
(d) By (b) there are a priori four combinatorial possibilities for the disks. But it follows from (c) that it is not possible that for the lower strand the disk is above and that at the same time for the upper strand the disk is below. Thus there are only three possibilities.
(e) For each of the three possible cases we show in the figure below how to attach a band to the collection of disks. This band, if we attach it to disks B_i and B_j, can be attached in the cylinder where the z-coordinate lies between t_i and t_j. It follows from (2b) that this cylinder is disjoint from all the other $B_k \times \{t_k\}$.

By construction the resulting submanifold of $\mathbb{R}^3$ is compact and orientable. We leave it to the zealous reader to show that the boundary of the resulting submanifold is smoothly isotopic to K. Since the submanifold we constructed has no closed component we see that the submanifold is a Seifert surface for K. ■



Example. In the figure below we apply the Seifert Algorithm 11.1 to the usual diagram of the figure-8 knot.



We obtain a Seifert disk that is built out of three disks and four bands. □

11.2. **The genus of a surface.** The following definition might be familiar from an earlier course in topology:

Definition.
(1) Given $g, n \in \mathbb{N}_0$ we write

$$\Sigma_{g,n} \;:=\; \text{the surface of genus } g \text{ minus } n \text{ open disks.}$$

Note that n is precisely the number of boundary components.
(2) Let M be a compact orientable connected non-empty 2-dimensional smooth manifold with n boundary components. By the classification of compact 2-dimensional smooth manifolds there exists a unique $g \in \mathbb{N}_0$ such that M is diffeomorphic to some $\Sigma_{g,n}$. We refer to g as the genus(M) of M.

The question is, given a surface, how can we read off its genus in practice? As a first step towards answering this question, we prove following the lemma, which gives us an easy way to read off the genus from the fundamental group:

Lemma 11.3. (Surface-π_1-Genus Lemma) Let F be a compact orientable connected non-empty 2-dimensional smooth manifold with $n \in \mathbb{N}$ boundary components.

(1) The genus of F is the unique $g \in \mathbb{N}_0$ such that $\pi_1(F)$ is a free group on $2g+n-1$ generators.
(2) The genus of F is the unique $g \in \mathbb{N}_0$ such that the abelianization $\pi_1(F)_{\mathrm{ab}}$ is a free abelian group on $2g+n-1$ generators.

Proof.

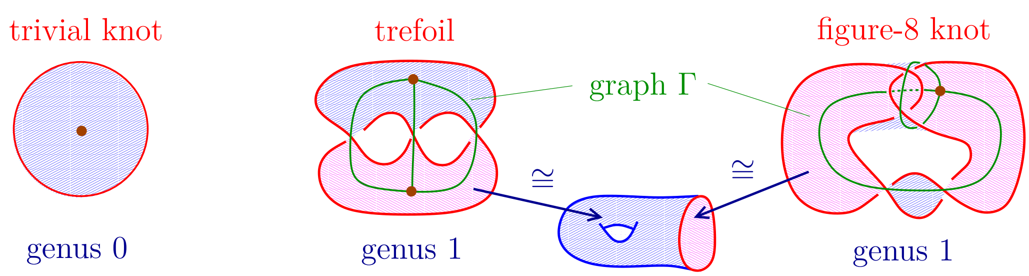
(1) This statement lemma follows from the following two facts:
   (a) This follows from the calculation of fundamental groups of surfaces. For example one could use that M is homotopy equivalent to a wedge of $2g+n-1$ circles.
   (b) The uniqueness of g follows from the fact that two free groups are isomorphic if and only if the bases have the same cardinality.
(2) This statement follows immediately from (1) and calculation of the abelianization of a free group. ■

The following lemma can be useful for determining the genus of a given surface:

Lemma 11.4. (Genus-via-Graph Lemma) Let F be a compact orientable connected 2-dimensional submanifold with one boundary component. If $\Gamma \subset F$ is a topological graph that is a deformation retract of F, then

$$\text{genus of } F \;=\; \tfrac{1}{2}\cdot(1-\chi(F)) \;=\; \tfrac{1}{2}\cdot(1-\#\text{vertices of } \Gamma + \#\text{edges of } \Gamma).$$

Example. In the figure below we use the Genus-via-Graph Lemma 11.4 to determine the genus of some of the Seifert surfaces that we showed on page 154.



Proof. We denote by k the number of vertices and we denote by l the number of vertices. We calculate that

$$\pi_1(\Sigma) \underset{\substack{\uparrow \\ \text{since fundamental groups are a homotopy invariant}}}{\cong} \pi_1(|G|) \overset{\substack{\text{since } |G| \text{ is homotopy equivalent to the wedge of } l-k+1 \text{ circles} \\ \downarrow}}{\cong} \text{free group on } l-k+1 \text{ generators.}$$

The promised result now follows from the Surface-π_1-Genus Lemma 11.3. ■

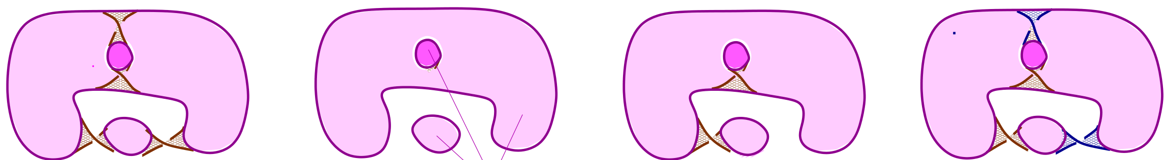
The next lemma is particularly suited for determining the genus of surfaces obtained through the Seifert Algorithm 11.1.

Lemma 11.5. (Genus-via-Bands Lemma) Let Σ be a Seifert surface. If Σ is obtained from k smoothly disjointly embedded disks and the attachment of l bands, then

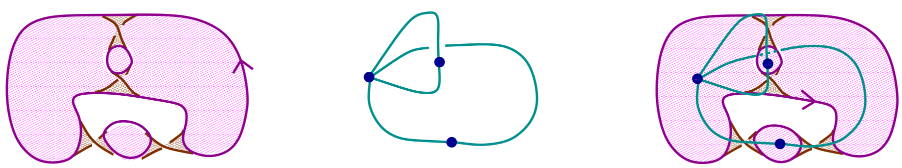
$$\text{genus}(\Sigma) \;=\; \tfrac{1}{2}\cdot(l-k+1).$$

Sketch of proof. We present two different ways to prove the lemma:

(1) Note that a Seifert surface is connected. Since we start out out with k smoothly disjointly embedded disks it takes $k-1$ bands to obtain a connected smooth submanifold M. It follows from a slight generalization of the Manifold–Seifert-van Kampen Theorem 3.9 that M is simply connected. It follows from a slight variation on the HNN–Gluing Theorem 5.1 that each of the remaining $l-(k-1)$ bands increases the rank of the fundamental group by 1, so that at the end we see that $\pi_1(\Sigma)$ is a free group on $l-k+1$ generators. It follows from the Surface-π_1-Genus Lemma 11.3 that genus$(\Sigma) = \frac{1}{2} \cdot (l-k+1)$.



(2) Let $\Gamma = (V, E, \varphi)$ be the abstract graph where V is given by the set of vertices, E is given by the set of bands and $\varphi\colon E \to \mathcal{P}(V)$ is the map which assigns to a band $b \in E$ the disks in V to which the band gets attached. It is not particularly difficult to show that 8 the topological realization $|\Gamma|$ admits an embedding $|\Gamma| \to \Sigma$ such that $|G|$ is a deformation retract of Σ.



Since a Seifert surface is compact, orientable, connected and non-empty we obtain the promised result from the above discussion together with the Genus-via-Graph Lemma 11.4.

In both approaches we leave it to the reader to fill in the technical details. ■

11.3. **The genus of a knot.**

If a knot K bounds a Seifert surface of genus g, then the construction sketched in the figure below shows pretty convincingly that K also bounds Seifert surfaces of any genus $h \geq g$.

So the interesting question is, what is the minimal genus of a Seifert surface for a given knot? This leads us to the following definition:

Definition. Given a knot $K \subset S^3$ we define

$$\text{genus}(K) := \text{minimal genus of a Seifert surface of } K.$$

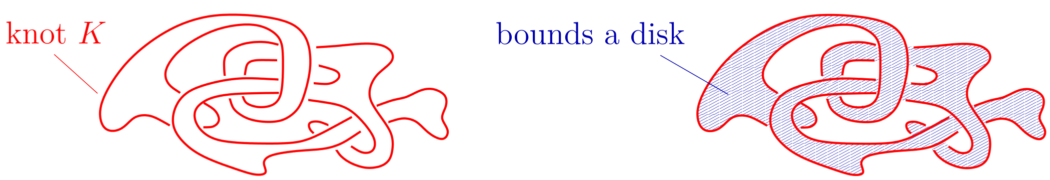
The following lemma gets used subconsciously all the time:

Lemma 11.6. (Genus–Smooth Isotopy Invariant Lemma) Let K and $\widetilde{K}$ be two oriented knots. If K and $\widetilde{K}$ are smoothly isotopic, then $\text{genus}(K) = \text{genus}(\widetilde{K})$.

Proof. This statement is an almost immediate consequence of the Isotopy Extension Theorem 2.4. ■

Note that, almost by definition, a Seifert surface has genus zero if and only if it is diffeomorphic to a disk. The following proposition now shows that trivial knots are precisely the knots that admit a Seifert surface of genus zero:

Proposition 11.7. (Genus-Zero Knots Proposition) Let K be a knot. The following two statements are equivalent:
(1) The knot K has genus zero, i.e. K bounds a smoothly embedded disk.
(2) The knot K is smoothly isotopic to the trivial knot.



Proof. Let K be a knot. We need to prove the equivalence of (1) and (2).

First we prove the almost trivial "(2)⇒(1)"-direction. It is clear that the trivial knot $\{(x,y,0) \in \mathbb{R}^3 \,|\, x^2+y^2=1\} \subset \mathbb{R}^3 \cup \{\infty\} =_{\text{i}} S^3$ bounds a disk. It follows from the Genus–Smooth Isotopy Invariant Lemma 11.6 that any knot that is smoothly isotopic to the trivial knot also bounds a smoothly embedded disk.

Now we turn to the proof of the much more interesting "(1)⇒(2)"-direction.

Thus let $\varphi\colon \overline{B}^2 \to S^3 =_{\text{i}} \mathbb{R}^3 \cup \{\infty\}$ be a smooth embedding with $\varphi(S^1) = K$. It suffices to show that there exists a smooth isotopy from φ to a smooth embedding $\psi\colon \overline{B}^2 \to \mathbb{R}^3$ such that $\psi(S^1) = S^1 \times \{0\} \subset \mathbb{R}^2 \times \{0\}$. We perform the following steps:[85]

(0) Using almost the same argument as in the Link-in-$\mathbb{R}^3$-S^3-Lemma 2.5 one can arrange that, after a smooth isotopy, we have $K \subset \mathbb{R}^3$ and $\varphi(\overline{B}^2) \subset \mathbb{R}^3$.

(1) Note that it follows from elementary linear algebra that there exists an $A \in \text{SL}(3,\mathbb{R})$ such that $A \cdot (\text{D}\varphi_0)(\text{T}_0\overline{B}^2) = \mathbb{R}^2 \times \{0\}$. By the Matrix Group-$\pi_0$-Proposition **??** (2) we know that $\text{SL}(3,\mathbb{R})$ is path-connected. Using the Smooth Path-Connectivity Proposition **??** we can find a smooth path γ in $\text{SL}(3,\mathbb{R})$ from id to A. This path gives rise to the smooth isotopy
$$\begin{array}{rcl} \overline{B}^2 \times [0,1] & \to & \mathbb{R}^3 \\ (x,t) & \mapsto & \gamma(t)\cdot\varphi(x). \end{array}$$
This shows that we might as well assume that $(\text{D}\varphi_0)(\text{T}_0\overline{B}^2) = \mathbb{R}^2 \times \{0\}$.

(2) (a) Let $p\colon \mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ be the projection. It follows from (1) and the chain rule that the map $\text{D}(p\circ\varphi)_0\colon \text{T}_0\overline{B}^2 \to \text{T}_0\mathbb{R}^2$ is an epimorphism. Since both vector

[85]The following steps are similar to the steps in the proof of the Meridian Proposition 2.17.

spaces are 2-dimensional we see that the map is in fact an isomorphism. It follows from the Inverse Mapping Theorem **??** that there exists an open neighborhood U of $0 \in \overline{B}^2$ such that the restriction of $p \circ \varphi \colon \overline{B}^2 \to \mathbb{R}^2$ to $p \circ \varphi \colon U \to \mathbb{R}^2$ is a smooth embedding.

(b) We pick $\epsilon > 0$ such that $\overline{B}^2_\epsilon(0) \subset U$. We consider the smooth isotopy

$$\begin{array}{rcl} \overline{B}^2 \times [0,1] & \to & \mathbb{R}^3 \\ (x,t) & \mapsto & \underbrace{\varphi(((1-t)+t\cdot\epsilon)\cdot x).}_{\substack{\text{for } t=0 \text{ this is the original map } \varphi, \\ \text{for } t=1, \text{ the composition with } p \text{ is} \\ \text{still a smooth embedding}}} \end{array}$$

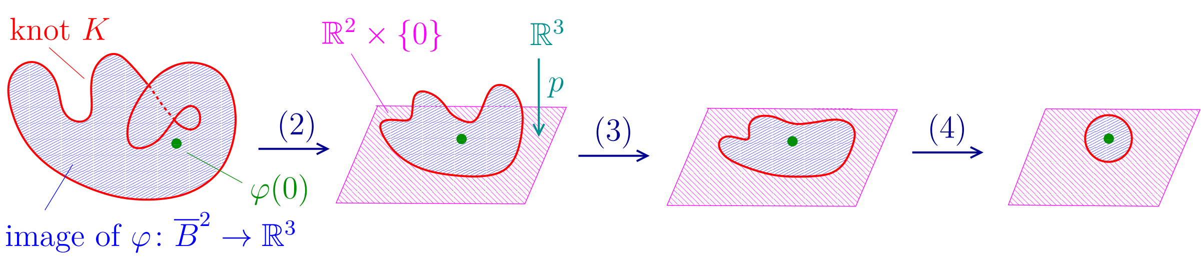
This discussion shows that we can now assume that we have a smooth embedding $\varphi \colon \overline{B}^2 \to \mathbb{R}^3$ such that the map $p \circ \varphi \colon \overline{B}^2 \to \mathbb{R}^2$ is a smooth embedding.

(3) We write $\varphi = \varphi^2_{\mathbb{R}} \times \varphi_{\mathbb{R}} \colon \overline{B}^2 \to \mathbb{R}^2 \times \mathbb{R}$. It follows from (2) that the map

$$\begin{array}{rcl} \overline{B}^2 \times [0,1] & \to & \mathbb{R}^2 \times \mathbb{R} \\ (x,t) & \mapsto & (\varphi_{\mathbb{R}^2}(x), \varphi_{\mathbb{R}}(x)\cdot t) \end{array}$$

is a smooth isotopy. This shows that we only need to deal with a smooth embedding $\varphi \colon \overline{B}^2 \to \mathbb{R}^2 \times \{0\}$.

(4) Now we can finally apply the Smooth Ball Embedding Theorem 2.9 to see that our smooth embedding $\varphi \colon \overline{B}^2 \to \mathbb{R}^2 \times \{0\}$ is smoothly isotopic to the standard smooth embedding $\overline{B}^2 \to \mathbb{R}^2 \times \{0\}$.

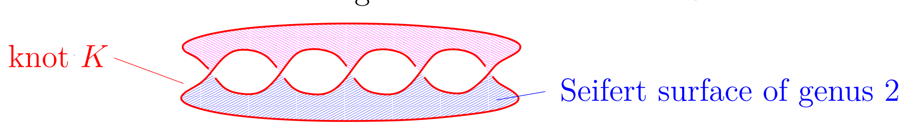


We leave it to the reader to fill in all the technical details. ∎

Example.

(1) We consider again the trefoil. On page 158 we showed that the trefoil admits a Seifert surface of genus 1. Furthermore in the Trefoil–Non-Trivial Proposition 3.10 (2) we saw that the trefoil is not smoothly isotopic to the trivial knot. It follows from the Genus-Zero Knots Proposition 11.7 that genus(trefoil) = 1.

(2) We consider the knot K in the figure below. It admits a Seifert surface of genus two.



But does K also admit a Seifert surface of genus one? This does not seem to be the case, but right now we lack the tools to exclude the possibility that such a Seifert surface exists. □

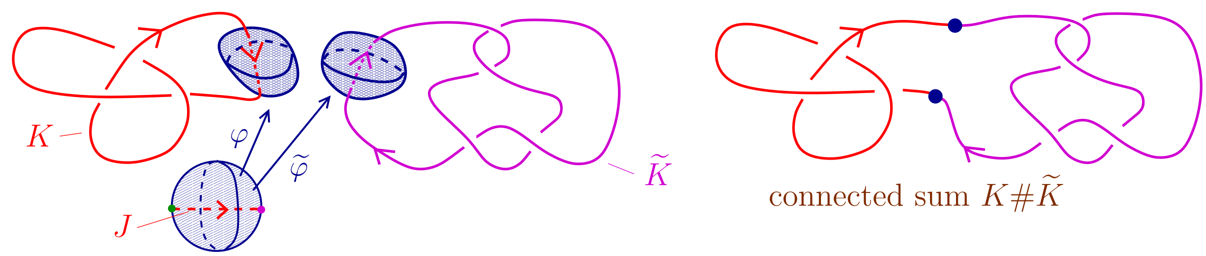
11.4. **Genus and the connected sum operation.** In this section we will see that the genus of a knot is additive under the connected sum operation. It is worthwhile recalling the definition of the connected sum of knots from page 24:

Definition.

(1)
- We set $J := \{(x,0) \in \overline{B}^3_2 \,|\, x \in [-2,2]\}$.
- Let K be an oriented knot. We say that a smooth embedding $\varphi\colon \overline{B}^3_2 \to S^3$ (anti-) respects K if $\varphi(J) = \varphi(\overline{B}^3_2) \cap K$, if $\varphi\colon \overline{B}^3_2 \to S^3$ is orientation-preserving (orientation-reversing) and if $\varphi|_J\colon J \to \varphi(\overline{B}^3_2) \cap K$ is orientation-preserving (orientation-reversing).

(2) Let K and $\widetilde{K}$ be two oriented knots in S^3. We pick a smooth embedding $\varphi\colon \overline{B}^3_2 \to S^3$ that respects K and we pick a smooth embedding $\widetilde{\varphi}\colon \overline{B}^3_2 \to S^3$ that anti-respects $\widetilde{K}$. We define

$$\text{connected sum } K\#\widetilde{K} \;:=\; \underbrace{(K \setminus \varphi(\mathring{J})) \cup (\widetilde{K} \setminus \widetilde{\varphi}(\mathring{J}))}_{\subset (S^3\setminus\varphi(B^3)) \cup (S^3\setminus\widetilde{\varphi}(B^3)) \cong S^3}.$$



Proposition 11.8. (Genus–Connected Sum Proposition) Given any two oriented knots K and $\widetilde{K}$ we have

$$\text{genus}(K\#\widetilde{K}) \;=\; \text{genus}(K) + \text{genus}(\widetilde{K}).$$

The remainder of this section is concerned with a (sketch of) the proof of the Genus–Connected Sum Proposition 11.8. More precisely, in the following two subsections we prove the "$\leq$"-inequality and the "$\geq$"-inequality. Together these two inequalities gives us the desired equality. Afterwards, in Section 11.5 we will use the Genus–Connected Sum Proposition 11.8 to study "prime decompositions" of knot.

In the following two subsections we will take a few liberties. The key ideas of the proofs are very pretty. Carrying out the technical details rigorously leads to a technical mess, involving smooth (sub-) manifolds with corner, which we do not want to get into.

11.4.1. **Proof of the "$\leq$"-inequality of Proposition 11.8.** The proof of the Genus–Connected Sum Proposition 11.8 rests on the shoulders of the following lemma:

Lemma 11.9. (Seifert Surface–Connected Sum Lemma) Let K and $\widetilde{K}$ be two oriented knots and let F and $\widetilde{F}$ be Seifert surfaces for K and $\widetilde{K}$. We introduce some objects and some notation:

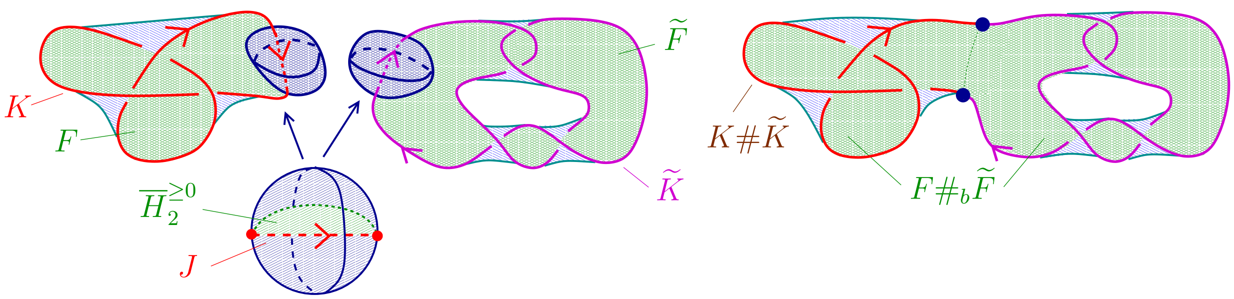
- For $r \in \mathbb{R}_{\geq 0}$ we write $\overline{H}^{\geq 0}_r := \{(x,y,z) \in \overline{B}^3 \,|\, y \geq 0 \text{ and } z = 0\}$ and $H^{\geq 0}_r := \overline{H}^{\geq 0}_r \cap B^3_r$.
- Since F is a submanifold of S^3 we can pick a smooth embedding $\varphi\colon \overline{B}^3_2 \to S^3$ that respects K, such that φ restricts to a smooth embedding $\overline{H}^{\geq 0}_2 \to F$ and such that $\varphi(\overline{B}^3_2) \cap F = \varphi(\overline{H}^{\geq 0}_2) \cap F$.

- Since $\widetilde{F}$ is a submanifold of S^3 we can pick a smooth embedding $\widetilde{\varphi}\colon \overline{B}^3_2 \to S^3$ that anti-respects $\widetilde{K}$, such that $\widetilde{\varphi}$ restricts to a smooth embedding $\overline{H}^{\geq 0}_2 \to \widetilde{F}$ and such that $\widetilde{\varphi}(\overline{B}^3_2) \cap \widetilde{F} = \widetilde{\varphi}(\overline{H}^{\geq 0}_2) \cap \widetilde{F}$.

We now consider the boundary connected sum

$$F \#_{\mathrm{b}} \widetilde{F} \;:=\; \underbrace{(F \setminus \varphi(H^{\geq 0}_1)) \cup (\widetilde{F} \setminus \widetilde{\varphi}(H^{\geq 0}_1))}_{\subset (S^3 \setminus \varphi(B^3)) \cup (S^3 \setminus \widetilde{\varphi}(B^3)) \cong S^3}.$$

The following statements hold:
(1) $F \#_{\mathrm{b}} \widetilde{F}$ is a Seifert surface for $K \# \widetilde{K}$.
(2) We have $\operatorname{genus}(F \#_{\mathrm{b}} \widetilde{F}) = \operatorname{genus}(F) + \operatorname{genus}(\widetilde{F})$.



Sketch of proof of Lemma 11.9.

(1) This statement follows easily from the definitions.[86] We leave it to the reader to fill in the details.
(2) We set $g := \operatorname{genus}(K)$ and $\widetilde{g} := \operatorname{genus}(\widetilde{K})$. We have

$$\begin{aligned} \pi_1(F \#_{\mathrm{b}} \widetilde{F}) \;\; & \overset{\cong}{\leftarrow} \;\; \pi_1(F \setminus \varphi(H^{\geq 0}_1)) * \pi_1(\widetilde{F} \setminus \widetilde{\varphi}(H^{\geq 0}_1)) \;\; \overset{\cong}{\rightarrow} \;\; \pi_1(F) * \pi_1(\widetilde{F}) \\ & \cong \;\; \{\text{free group of rank } 2g\} * \{\text{free group of rank } 2\widetilde{g}\} \\ & = \;\; \text{free group of rank } 2 \cdot (g + \widetilde{g}). \end{aligned}$$

isomorphisms by a mild generalization of the Manifold–Seifert-van Kampen Theorem 3.9

It follows from the Surface-π_1-Genus Lemma 11.3 that $\operatorname{genus}(F \#_{\mathrm{b}} \widetilde{F}) = g + \widetilde{g}$. ■

We can now already prove the "$\leq$"-inequality of the Genus–Connected Sum Proposition 11.8:

Proof of the "$\leq$"-inequality of Proposition 11.8. Let K and $\widetilde{K}$ be oriented knots. We want to prove that

$$\operatorname{genus}(K \# \widetilde{K}) \;\leq\; \operatorname{genus}(K) + \operatorname{genus}(\widetilde{K}).$$

Let F be a Seifert surface for K of minimal genus and let $\widetilde{F}$ be a Seifert surface for $\widetilde{K}$ of minimal genus. As in the Seifert Surface–Connected Sum Lemma 11.9 we form $F \#_{\mathrm{b}} \widetilde{F}$. We now see that

$$\operatorname{genus}(K \# \widetilde{K}) \underset{\substack{\uparrow \\ \text{since } F \#_{\mathrm{b}} \widetilde{F} \text{ is a Seifert} \\ \text{surface for } K \# \widetilde{K}}}{\leq} \operatorname{genus}(F \#_{\mathrm{b}} \widetilde{F}) \underset{\substack{\uparrow \\ \text{by the Seifert Surface–Connected} \\ \text{Sum Lemma 11.9 (2)}}}{=} \operatorname{genus}(F) + \operatorname{genus}(\widetilde{F}) \underset{\substack{\uparrow \\ \text{by choice of } F \text{ and } \widetilde{F}}}{=} \operatorname{genus}(K) + \operatorname{genus}(\widetilde{K}).$$

■

[86] Note that here our careful choice of radii plays an essential role.

11.4.2. **Proof of the "≥"-inequality of Proposition 11.8.** Our proof of the "≥"-inequality of Proposition 11.8 requires two preparations. First of all we need the following proposition.

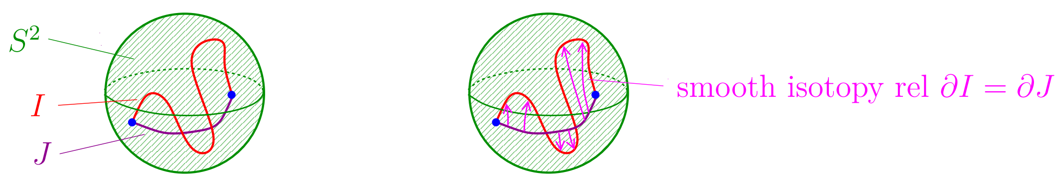
Proposition 11.10. (Fill-in-Disks Proposition) Let $\gamma\colon S^1 \to S^2$ be a smooth embedding. There exist two smooth embeddings $\varphi_1, \varphi_2\colon \overline{B}^2 \to S^2$ with $\varphi_1(\overline{B}^2) \cup \varphi_2(\overline{B}^2) = S^2$, with $\varphi_1(\overline{B}^2) \cap \varphi_2(\overline{B}^2) = \gamma(S^1)$ and such that for $i = 1, 2$ we have $\varphi_i|_{S^1} = \gamma$.

Sketch of proof. We recall the statement of the Fill-in-Disk Proposition 11.2: If Σ is a compact orientable 2-dimensional smooth manifold and if $f\colon S^1 \to \Sigma$ is a smooth embedding which is null-homotopic, then there exists a smooth embedding $F\colon \overline{B}^2 \to \Sigma$ with $F|_{S^1} = f$.

Now let $\gamma\colon S^1 \to S^2$ be a smooth embedding. Note that $\pi_1(S^2) = 0$. We can thus apply the Fill-in-Disk Proposition 11.2 and we obtain a smooth embedding $\varphi_1\colon \overline{B}^2 \to S^2$ with $\varphi_1|_{S^1} = \gamma$. Next we consider $\Sigma := S^2 \setminus \varphi_1(0)$. It follows from the 2-dimensional analogue of the Stereographic Projection Lemma 2.1 that Σ is diffeomorphic to $\mathbb{R}^2$. This implies that Σ is simply connected. We can thus apply the Fill-in-Disk Proposition 11.2 to $\gamma\colon S^1 \to \Sigma$ and we obtain a smooth embedding $\varphi_2\colon \overline{B}^2 \to \Sigma$ with $\varphi_2|_{S^1} = \gamma$. ■

We will also need the following lemma, which might be of independent interest.

Lemma 11.11. (Interval–Isotopy Lemma) Let I and J be two connected 1-dimensional smooth submanifolds of S^2 that are diffeomorphic to the interval $[0, 1]$. If I and J have the same boundary, then I and J are smoothly isotopic rel $\partial I = \partial J$.



Proof of Interval–Isotopy Lemma 11.11. The lemma can be deduced, with some effort, from the Fill-in-Disks Proposition 11.10. To keep the discussion flowing we outsource the proof of the lemma to Appendix 11.6. ■

Now we are finally ready to prove the "≥"-inequality of Proposition 11.8.

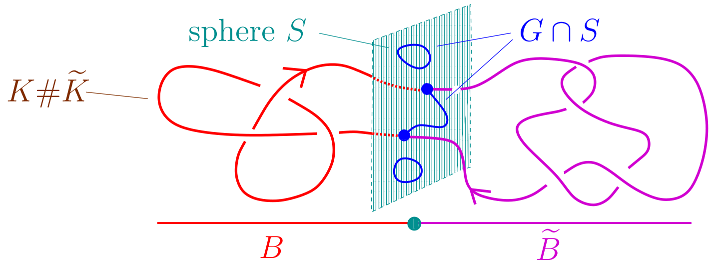
Sketch of proof of the "≥"-inequality of Proposition 11.8. Let K and $\widetilde{K}$ be two oriented knots. We need to prove that

$$\operatorname{genus}(K \# \widetilde{K}) \;\geq\; \operatorname{genus}(K) + \operatorname{genus}(\widetilde{K}).$$

In other words, we need to show that if G is a Seifert surface of the connected sum $K \# \widetilde{K}$, then $\operatorname{genus}(G) \geq \operatorname{genus}(K) + \operatorname{genus}(\widetilde{K})$. Thus let G be a Seifert surface of $K \# \widetilde{K}$. We introduce the following notation:

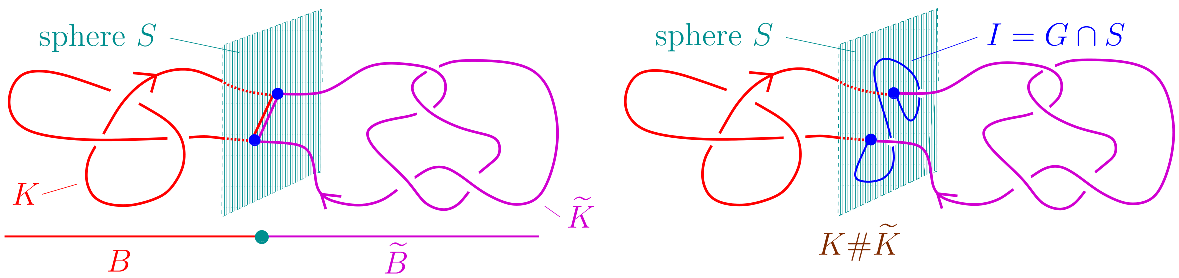
- Let S be the sphere that comes with the definition of the connected sum of K and $\widetilde{K}$.
- Let B and $\widetilde{B}$ be the closed 3-balls that come with the definition of the connected sum of K and $\widetilde{K}$.

Note that it follows from a suitable generalization of the Transversality Theorem of [**Fri24**] that, after a smooth isotopy, we can assume that G intersects S transversally. This implies that $G \cap S$ is a compact 1-dimensional submanifold of S with boundary $\partial G \cap S = (K \# \widetilde{K}) \cap S$.



Next note that it follows from the classification of compact 1-dimensional smooth manifolds that $G \cap S$ has precisely one component that is diffeomorphic to an interval and that all other components are diffeomorphic to S^1. We now prove the desired conclusion by induction on the number of closed components of $G \cap S$.

If $G \cap S$ has no closed component, then $G \cap S$ consists of a single component I which is diffeomorphic to an interval.



We set $F := G \cap B$ and $\widetilde{F} := G \cap \widetilde{B}$. Note that $G = F \#_{\mathrm{b}} \widetilde{F}$. Also note that using the Interval–Isotopy Lemma 11.11 one can now show reasonably easily that (up to an isotopy) $F = G \cap B$ is a Seifert surface for K and that (up to an isotopy) $\widetilde{F} = G \cap \widetilde{B}$ is a Seifert surface for $\widetilde{K}$. It follows that

$$\operatorname{genus}(G) \;=\; \operatorname{genus}(F \#_{\mathrm{b}} \widetilde{F}) \underset{\substack{\uparrow \\ \text{by the Seifert Surface–Connected Sum Lemma 11.9}}}{=} \operatorname{genus}(F) + \operatorname{genus}(\widetilde{F}) \;\geq\; \operatorname{genus}(K) + \operatorname{genus}(\widetilde{K}).$$

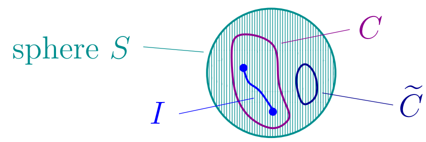
Now suppose that $G \cap S$ has at least one closed component. We start out with the following claim:

Claim 1. There exists a closed component C of $G \cap S$ and a closed disk $D \subset S$ such that $\partial D = C$ and such that $G \cap D = C$.

Proof. This can be seen by an "innermost curve" argument:

(a) We start with a closed component C of $G \cap S$. Note that it follows easily from the Fill-in-Disk Proposition 11.2 that there exist two closed disks $D, D' \subset S$ such that $D \cap D' = \partial D = \partial D' = C$. Let D be the disk that does not contain I.

(b) If $G \cap D = C$, then we are done. If not, then there exists a component $\widetilde{C}$ of $G \cap S$ that is contained in D. We now replace C by $\widetilde{C}$ and start again with (a).

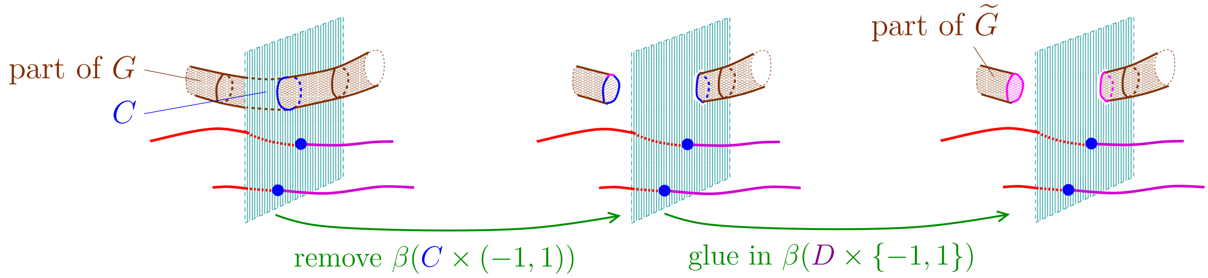


Since $G \cap S$ has only finitely many components the above process will come to an end, and we have found a closed component C of $G \cap S$ that bounds a closed disk $D \subset S$ such that $\partial D = C$ and such that $G \cap D = C$. ⊞

We now cut G along $G \cap D$ and glue in two copies of D. More precisely, by the Bicollar Neighborhood Theorem 5.2 there exists a bicollar for S, i.e. there exists a smooth embedding $\beta\colon S \times [-1,1] \to S^3$ such that $\beta(s,0) = s$ for all $s \in S$. Since G is compact we can arrange that the restriction of β to $C \times [-1,1] \to S^3$ is in fact a bicollar $\beta\colon C \times [-1,1] \to G$ and that $\beta(S \times [-1,1]) \cap G = \beta(C \times [-1,1])$. We now consider

$$\widetilde{G} \;:=\; \underbrace{(G \setminus \beta(C \times (-1,1))) \cup \beta(D \times \{-1,1\})}_{\substack{\text{to be precise, } \widetilde{G} \text{ is this submanifold with the} \\ \text{corner at } \beta(\partial D \times \{-1,1\} \text{ rounded}}}.$$

One can now easily verify that $\widetilde{G}$ is a compact oriented smooth submanifold of S^3 with $\partial \widetilde{G} := K \# K_b$ and with $\widetilde{G} \cap S = (G \cap S) \setminus C$.



The idea now is to replace G by $\widetilde{G}$. Note though that there are a few issues: The surface $\widetilde{G}$ might be disconnected and since it is not entirely clear what the genus of $\widetilde{G}$ is. This leads us to the following claim:

Claim 2. The component $\widetilde{G}_0$ of $\widetilde{G}$ that contains $K \# \widetilde{K}$ is a Seifert surface of $K \# \widetilde{K}$ with $\operatorname{genus}(\widetilde{G}_0) \le \operatorname{genus}(G)$.

Proof. It is basically clear that the component $\widetilde{G}_0$ of $\widetilde{G}$ that contains $K \# \widetilde{K}$ is a Seifert surface of $K \# \widetilde{K}$. It remains to show that $\operatorname{genus}(\widetilde{G}_0) \le \operatorname{genus}(G)$. We start out with two preparations:

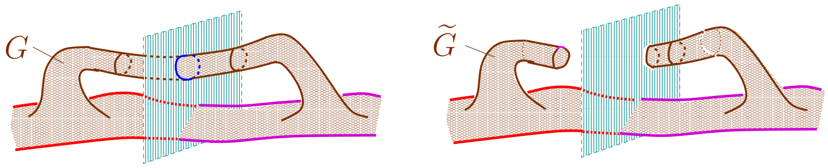
(i) For any compact orientable connected 2-dimensional smooth manifold M with $n \in \mathbb{N}_0$ boundary components we know that the Euler characteristic $\chi(M)$ is given by $\chi(M) = 2 - \operatorname{genus}(M) - n$.

(ii) We consider the Euler characteristic of $\widetilde{G}$. We see that

$$\chi(\widetilde{G}) \underset{\substack{\uparrow \\ \text{follows from } \chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B) \text{ and the} \\ \text{fact that } \chi(C \times \{-1,1\}) = 0 \text{ and } \chi(\partial D \times \{-1,1\}) = 0}}{=} \chi(G) - \chi(C \times [-1,1]) + \chi(D \times \{-1,1\}) \underset{\substack{\uparrow \\ \text{since } \chi(C \times [-1,1]) = 0 \text{ and} \\ \text{since } \chi(D) = 1}}{=} \chi(G) + 2.$$

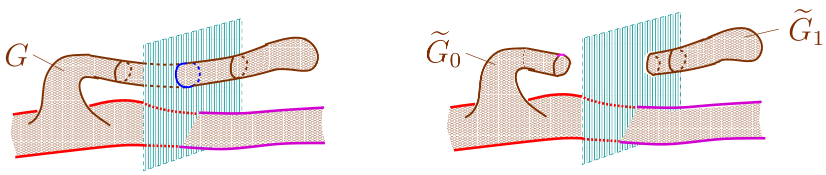
Now we consider two separate cases:

(1) If $\widetilde{G}$ is already connected, then it follows from (i) and (ii) that $\operatorname{genus}(\widetilde{G}) = \operatorname{genus}(G)-1$.



$\operatorname{genus}(\widetilde{G}) = \operatorname{genus}(G) - 1$

(2) If $\widetilde{G}$ is disconnected, then $\widetilde{G} = \widetilde{G}_0 \sqcup \widetilde{G}_1$. Note that $\widetilde{G}_1$ is closed, thus it follows from (ii) that $\chi(\widetilde{G}_1) \leq 2$. It follows that $\chi(\widetilde{G}_0) = \chi(\widetilde{G}) - \chi(\widetilde{G}_1) = \chi(G) + 2 - \underbrace{\chi(\widetilde{G}_0)}_{\leq 2} \geq \chi(\widetilde{G})$.



$\operatorname{genus}(\widetilde{G}_0) \leq \operatorname{genus}(G)$

By (i) this implies that $\operatorname{genus}(\widetilde{G}) \leq \operatorname{genus}(G)$. ⊞

Iterating this procedure we can dispose of the closed components of $G \cap S$ one by one, till we end up with the initial case, that $G \cap S$ has no closed component. ■

11.5. **The prime decomposition theorem.** We start out by recalling the following definition from page 24:

Definition. We say that a knot K is prime if it is not smoothly isotopic to the connected sum of two non-trivial knots.

In principle it would be more suitable to call a knot, which does not admit a non-trivial connected sum decomposition, "irreducible". The name "prime" is justified by the Knot–Prime Decomposition Theorem 11.13 which we will formulate in a minute. But first let us state the following proposition:

Proposition 11.12. (Prime Knot–Genus Proposition) For every non-prime knot K we have $\operatorname{genus}(K) \geq 2$.

Proof. Let K be a knot which is not prime. By definition this means that K is smoothly isotopic to the connected sum of two oriented non-trivial knots J and $\widetilde{J}$. We see that

$$\operatorname{genus}(K) \underset{\uparrow}{=} \operatorname{genus}(J\#\widetilde{J}) \underset{\uparrow}{=} \operatorname{genus}(J) + \operatorname{genus}(\widetilde{J}) \underset{\uparrow}{\geq} 1+1 = 2.$$

by the Genus–Smooth Isotopy Invariant Lemma 11.6

by the Genus–Connected Sum Proposition 11.8

by the Genus-Zero Knots Proposition 11.7, since the knots J and $\widetilde{J}$ are non-trivial ■

The Prime Knot–Genus Proposition 11.12 allows us to prove the first part of the following theorem:

Theorem 11.13. (Knot–Prime Decomposition Theorem) Let $K \subset S^3$ be a knot.

(1) There exist oriented prime knots $K_1, \ldots, K_m$ such that K is smoothly isotopic to the connected sum $K_1\#\ldots\#K_m$.

(2) The oriented knots $K_1, \ldots, K_m$ in (1) are unique (as oriented knots) up to smooth isotopy and reordering.

Proof.

(1) This statement can be proved easily using induction on the genus of K. Indeed, the base case genus$(K) \leq 1$ is an immediate consequence of the Prime Knot–Genus Proposition 11.12. The induction step follows immediately from the Genus–Connected Sum Proposition 11.8 and the associativity of the connected sum operation (which we proved in the Knot Connected Sum Proposition 2.14).

(2) This statement was first proved by Horst Schubert in [**Sch49**]. Alternatively see [**BZH14**, Chapter 7] or [**Lic97**, Theorem 2.12] or [**Cro04**, Theorem 4.5.4] for textbook proofs of this statement. These proofs go well beyond what we can hope to achieve in these notes. ■

11.6. Appendix: Proof of the Interval–Isotopy Lemma 11.11.

In this appendix we will sketch the long overdue proof of the following lemma:

Lemma 11.11. (Interval–Isotopy Lemma) Let I and J be two connected 1-dimensional smooth submanifolds of S^2 that are diffeomorphic to the interval $[0,1]$. If I and J have the same boundary, then I and J are smoothly isotopic rel $\partial I = \partial J$.

Sketch of proof. Let P and Q be the two boundary points of I and J.

Claim. There exists a smooth isotopy rel ∂I from I to a smooth submanifold I' such that I' and J only intersect at the boundary.

Proof. We prove the claim with the following steps:

(0) We equip I and J with the orientation "which goes from P to Q". More formally, we equip them with the orientation such that $\partial I = \partial J = -\{P\} \cup \{Q\}$.

(1) We keep J fixed. By "spiraling around P and Q" we can modify I such that $\widetilde{\mathrm{T}}_P I \neq \widetilde{\mathrm{T}}_P J$ and $\widetilde{\mathrm{T}}_Q I \neq \widetilde{\mathrm{T}}_Q J$ and such that the algebraic intersection number of I and J is zero.We refer to [**Fri24**] for the definition of the algebraic intersection number.

(2) Next we use the Transversality Theorem from [**Fri24**] to make I and J transverse, while keeping J fixed and while keeping (1). Note that this implies that $I \cap J$ is now finite.

(3) We now want to arrange that I and J intersect only at P and Q. If $I \cap J = \{P, Q\}$, then there is nothing to show. Now assume that there are more intersection points. For this situation we need the following subclaim:

   Subclaim. If $I \cap J$ contains more intersection points than P and Q, then there exists a smoothly embedded disk $D \subset S^2 \setminus \{P, Q\}$ such that ∂D is the union of two intervals $\alpha \subset I$ and $\beta \subset J$.

   Proof of subclaim.

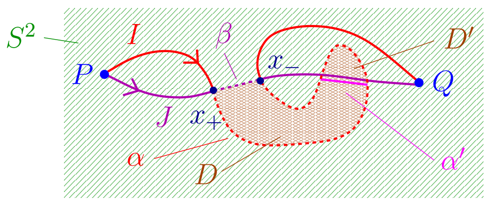
   (a) If there are more intersection points than P and Q, then it follows from the fact that the algebraic intersection number is zero that there exist two adjacent intersection points x_- and x_+ on I with opposite signs.

   (b) Let α and β be the intervals on I and J with endpoints x_-, x_+.

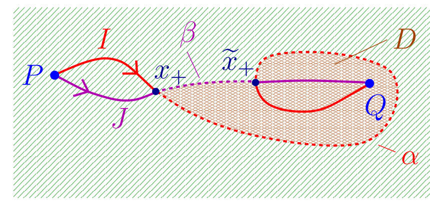
   (c) By the Fill-in-Disk Proposition 11.2 we know that $\alpha \cup \beta$ is the boundary of two smoothly embedded disks. Since the signs of the intersection points are different we see that one of the two disks, we call it D, has the property that in a neighborhood of x_- and x_+ the intersection of D with I and J comes from α and β.

(d) It follows from (c) that $J \cap \partial D = J \cap (\alpha \setminus \{x_-, x_+\}) = \varnothing$. (Here we use that x_- and x_+ are adjacent on I.) It follows easily that P, Q do not lie in D.
(e) It follows from (c) that $I \cap \partial D = I \cap (\beta \setminus \{x_-, x_+\})$. If $I \cap \partial D \neq \alpha$, then let α' be another component of $I \cap D$ and let $D' \subset D$ be the disk that cobounds α' and that does not contain α in the boundary. We then restart the process at (d).

This concludes the proof of the subclaim.

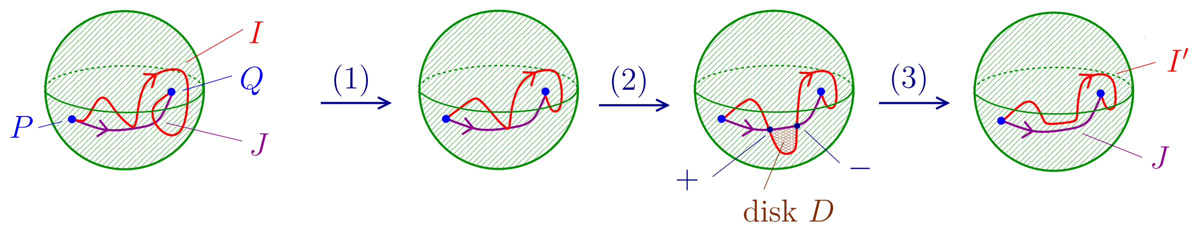


intersection points with opposite sign



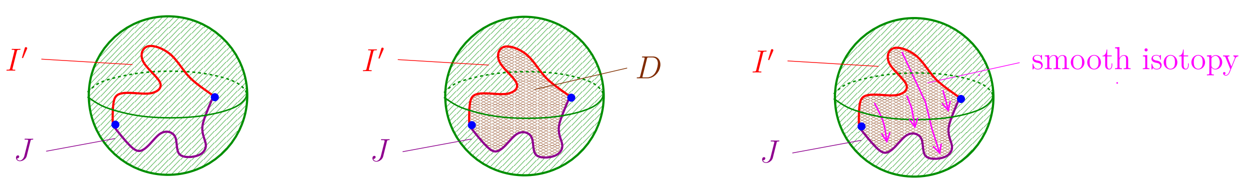
intersection points with same sign, we do not need to consider this case, but the figure shows why the opposite signs are essential for the argument

Let D be a disk as provided by the subclaim. Since the signs of the intersection points are opposite we can use the disk to push I across the disk to remove two intersection points with J.



We iterate this process till I and J only meet at the endpoints. We leave the challenging task of turning the above sketch into a proper proof to the reader. ⊞

In the claim we just showed that there exists a smooth isotopy rel ∂I from I to a smooth submanifold I' such that I' and J only intersect at the boundary. We apply once again the Fill-in-Disk Proposition 11.2, this time to $I' \cup J$, and we see that there exists a disk $D \subset S^2$ with $\partial D = I' \cup J$.



We use this disk to define an isotopy from I' to J. ■

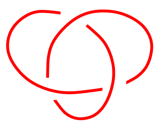
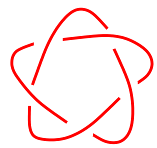
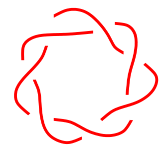
Exercises for Chapter 11.

Exercise 11.1.
(a) Let D and D' be two disks in $\mathbb{R}^2$ such that $\partial D \cap \partial D' = \varnothing$. Show that $D' \subset \mathring{D}$ or $D \subset \mathring{D}'$ or $D \cap D' = \varnothing$.
(b) Can we replace $\mathbb{R}^2$ by any 2-dimensional smooth manifold?

(c) Can we replace $\mathbb{R}^2$ by any topological space?

Exercise 11.2.

(a) Apply Seifert's Algorithm 11.1 to the diagram of the torus knot $T(2,n)$ that is illustrated below. What is the genus of the resulting Seifert surface?
(b) Let $p, q \in \mathbb{N}$ be coprime. Find a suitable diagram for the torus knot $T(p,q)$. Then apply Seifert's Algorithm 11.1. What is the smallest genus that you can achieve?

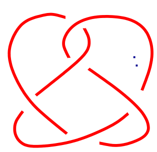
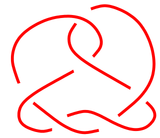
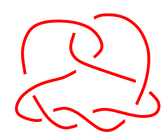
$T(2,3)$  $T(2,5)$  $T(2,7)$ 

Exercise 11.3. Let D be a knot diagram with c crossings. Show that if we apply Seifert's Algorithm 11.1 to this diagram, then we obtain a surface F with $\text{genus}(F) \leq \frac{c}{2}$.

Exercise 11.4. Give an example of a diagram of a knot K where Seifert's Algorithm 11.1 does not give a Seifert surface of minimal genus.
Hint. You could even take K to be the trivial knot.

Exercise 11.5. Does the Seifert surface we obtain from Seifert's Algorithm 11.1 depend on the choice of the orientation of the knot?

Exercise 11.6. We consider the knots 5_2, 6_1 and 7_2 which are shown below (together with their Alexander polynomials.) Show that in each case the genus equals 1.

5_2  6_1  7_2 

$2 - 3t + 2t^2$ $2 - 5t + 2t^2$ $3 - 5t + 3t^2$

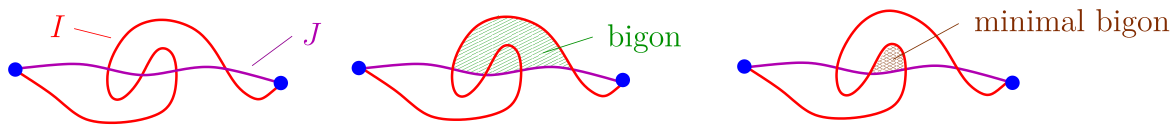
Exercise 11.7. Show that there exists infinitely many pairwise non-smoothly isotopic prime knots.
Remark. There are many approaches to showing this statement. One approach is to provide a non-constructive proof using torus knots and by arguing that their Alexander polynomials have infinitely many different irreducible factors.

Exercise 11.8. We say that an m-component link $L = L_1 \cup \cdots \cup L_m$ is a *boundary link* if there exists a collection of m disjoint Seifert surfaces $F_1, \ldots, F_m$ such that for $i = 1, \ldots, m$ we have $\partial F_i = L_i$.

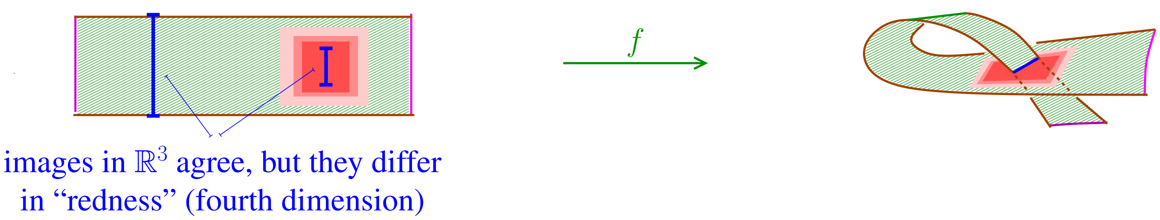
(a) Give an example of a boundary link that is not split.
Hint. You can use the higher order Alexander polynomials of Exercise 9.14.
(b) Show that if L is an m-component boundary link, then there exists an epimorphism from $\pi_1(S^3 \setminus L)$ onto a free group on m generators.
Hint. Use the Seifert surfaces and bicollars to construct an interesting continuous map $S^3 \setminus L \to \bigvee_{i=1}^{m} S^1$.
(c) Show that if $m \geq 2$ and if L is a boundary link, then $\Delta_L = 0$.

Exercise 11.9. Let I and J be two connected 1-dimensional smooth submanifolds of S^2 that are diffeomorphic to the interval $[0,1]$. We assume that I and J have the same boundary and that I and J intersect transversally. A *bigon for I and J* is a disk $D \subset S^2$ such that ∂D is the union of an interval on I and an interval on J. We order bigons by inclusion.

(a) Does a minimal bigon always exist?
(b) Let D be a minimal bigon. Is it true that $\mathring{D}$ is disjoint from I and J?
Solution. No. Problems arise if one of the endpoints lies in the bigon.
(c) Can we use a minimal bigon to reduce the number of intersection points of I and J?



Exercise 11.10. Using a specific color, e.g. "red" we can draw 4-dimensional pictures, where "redness" indicates the fourth dimension. For example in the figure below we show a map $f\colon [0,4] \times [0,1] \to \mathbb{R}^3$ which is not injective, but if we use the "redness" function $r\colon [0,4] \times [0,1] \to \mathbb{R}_{\geq 0}$, then the map $(f,r)\colon [0,4] \times [0,1] \to \mathbb{R}^3 \times \mathbb{R}_{\geq 0}$ is injective.



A knot K is called *smoothly slice* if there exists a proper smooth embedding $\varphi\colon \overline{B}^2 \to \overline{B}^4$ such that $K = \varphi(S^1)$.

(a) In the figure below we show two knots. Precisely two of these knots are smoothly slice. Which ones?

(b) Show that for every oriented knot K the connected sum $K \# K^{\mathrm{inv}} = K\#(K^{\mathrm{mir}})^{\mathrm{rev}}$ is smoothly slice.

CHAPTER 12

Fibered knots

In the last chapter we introduced the notion of a Seifert surface of a knot. In this chapter we introduce the notion of a "fibered knot". These two notions are closely related, since fibered knot have, basically by definition, particularly pleasing Seifert surfaces. As our main example we will see that all torus knots are "fibered".

In the next chapter we will then see that the Alexander polynomial of a fibered knot is of a specific form. This will then allow us, among many other things, to show that there do exist knots that are not "fibered".

12.1. **Smooth bundle maps.** Before we discussion the specific notion of a fibered knot, let us first discuss a few aspects of the more general theory of fiber bundles. :

Definition.

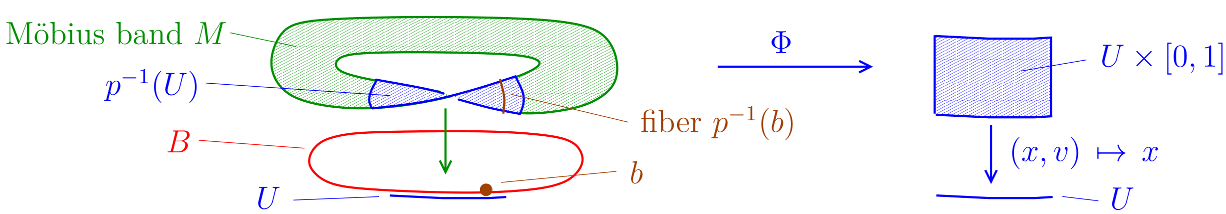
(1) Let M and B be smooth manifolds. We say that a smooth map $p\colon M \to B$ is a smooth bundle map if there exists a smooth manifold F such that for every $b \in B$ there exists a smooth trivialization of p around b, i.e. there exists an open neighborhood U of $b \in B$ and a diffeomorphism $\Phi\colon p^{-1}(U) \to U \times F$ such that the following diagram commutes:

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "\Phi", "\cong"'] \arrow[dr, "p"'] & & U \times F \arrow[dl, "(x,v)\mapsto x"] \\ & U. & \end{tikzcd}$$

For each $b \in B$ we refer to $p^{-1}(b)$ as a fiber of p. Note that each fiber is diffeomorphic to F.

(2) Let M and B be smooth manifolds. We say that M fibers smoothly over B if there exists a smooth bundle map $p\colon M \to B$.

Example. In the following figure we show a map $p\colon M \to B$ from the Möbius band to a circle B.



We also illustrate that this map is a smooth bundle map and that each fiber is diffeomorphic to the interval $[0,1]$. □

In the following we will mostly be interested in fiber bundles over S^1. For such bundles there is a useful alternative description:

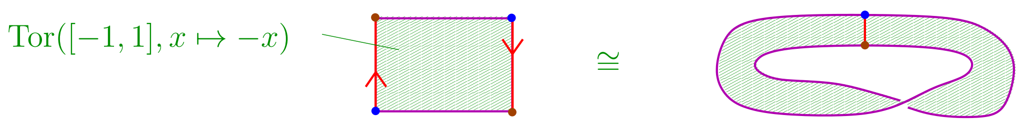
Definition. Let F be a topological space and let $\mu\colon F \to F$ be a continuous map.

(1) We call
$$\mathrm{Tor}(F,\mu) \;:=\; (F\times[0,1])\,/\sim \qquad \text{where } (x,0)\sim(\mu(x),1) \text{ for all } x\in F,$$
the mapping torus of (F,μ).

(2) We refer to the map
$$\begin{array}{rcl} q\colon \mathrm{Tor}(F,\mu) = (F\times[0,1])/(x,0)\sim(\mu(x),1) & \to & S^1 \\ {[(x,t)]} & \mapsto & \exp(2\pi\mathrm{i}t)\end{array}$$
as the natural projection onto S^1

Note that if F is a k-dimensional smooth manifold F and if $\mu\colon F\to F$ is a diffeomorphism, then the mapping torus is naturally a $(k+1)$-dimensional smooth manifold with $\partial(\mathrm{Tor}(F,\mu)) = \mathrm{Tor}(\partial F, \mu|_{\partial F})$.

Example. Let $F=[-1,1]$ and let $\mu\colon[-1,1]\to[-1,1]$ be the diffeomorphism that is given by $x\mapsto -x$.



The corresponding mapping torus $\mathrm{Tor}(F,\mu)$ is, more or less by definition, diffeomorphic to the Möbius band. $\square$

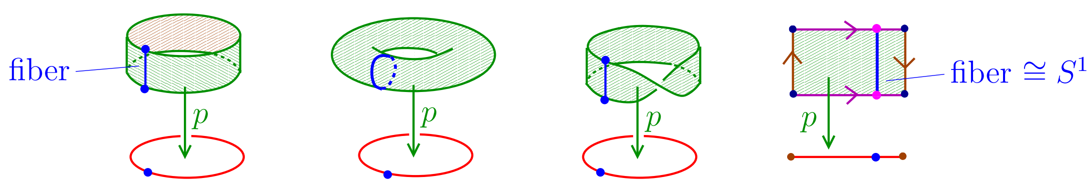
The following lemma says that smooth fiber bundles over S^1 are essentially the same as mapping tori of diffeomorphisms.

Lemma 12.1. (Smooth Mapping Torus–Bundle Lemma)

(1) Let F be a smooth manifold F and let $\mu\colon F\to F$ be a self-diffeomorphism. The natural projection
$$\begin{array}{rcl} q\colon \mathrm{Tor}(F,\mu) = (F\times[0,1])/(x,0)\sim(\mu(x),1) & \to & S^1 \\ {[(x,t)]} & \mapsto & \exp(2\pi\mathrm{i}t)\end{array}$$
is a smooth bundle map with fiber F.

(2) Let M be a smooth manifold. A smooth map $p\colon M\to S^1$ is a smooth fiber map if and only if there exists a smooth manifold F, a diffeomorphism $\mu\colon F\to F$ and a diffeomorphism $\Theta\colon \mathrm{Tor}(F,\mu)\to M$ such that the following diagram commutes:
$$\begin{tikzcd} \mathrm{Tor}(F,\mu) \arrow[rr, "\Theta"] \arrow[dr, "q"', "{[x,t]\mapsto\exp(2\pi\mathrm{i}t)}"] & & M \arrow[dl, "p"] \\ & S^1. & \end{tikzcd}$$

Example. In the figure below we illustrate that the annulus, the Möbius band, the torus and the Klein bottle fiber smoothly over S^1.



In Exercise 12.1 we will use the Smooth Mapping Torus–Bundle Lemma 12.1 to show that all the remaining compact connected 2-dimensional smooth manifolds do *not* fiber smoothly over S^1. $\square$

Proof. Let $\theta\colon [0,1] \to S^1$ be the map that is given by $\theta(t) := \exp(2\pi i t)$.

(1) We denote by $\pi\colon [0,1] \to [0,1]/0 \sim 1$ the natural projection. We consider the two open subsets $U_1 := \pi^{-1}((\frac{1}{4}, \frac{3}{4}))$ and $U_2 := \pi^{-1}([0, \frac{1}{3}) \cup (\frac{2}{3}, 1])$ in $[0,1]/0 \sim 1$. Furthermore we consider the maps

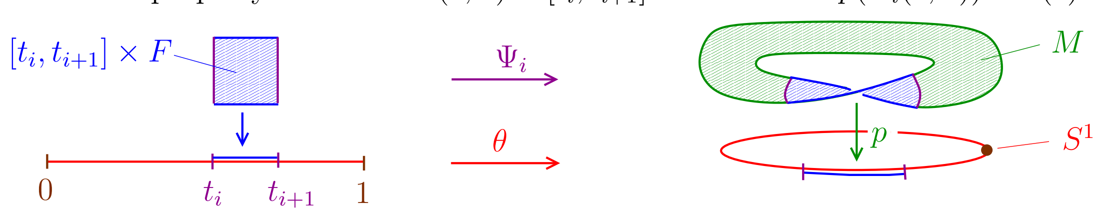
$$\begin{array}{rcl} p^{-1}(U_1) & \xrightarrow{\Phi_1} & U_1 \times F \\ [(t,x)] & \mapsto & (t,x) \end{array} \quad \text{and} \quad \begin{array}{rcl} p^{-1}(U_2) & \xrightarrow{\Phi_2} & U_2 \times F \\ [(t,x)] & \mapsto & \begin{cases} (t, \mu(x)), & \text{if } t \in [0, \frac{1}{3}), \\ (t,x) & \text{if } t \in (\frac{2}{3}, 1]. \end{cases} \end{array}$$

Note that the map on the right is indeed well-defined. One can show quite easily that both maps are homeomorphisms. It is also straightforward to verify that these maps are even diffeomorphisms. Furthermore it is basically clear that Φ_1 and Φ_2 respect the projections onto U_1 and U_2. Therefore, since the two open subsets U_1 and U_2 cover all of $[0,1]/0 \sim 1$. It follows easily from this discussion that the natural map $p\colon \operatorname{Tor}(F,\mu) \to [0,1]/0 \sim 1 \xrightarrow{[t] \mapsto \theta(t)} S^1$ is indeed a smooth bundle map.

(2) Let M be a smooth manifold and let $p\colon M \to S^1$ be a smooth bundle map with fiber F. It follows easily from the definition of a smooth bundle map and the Lebesgue Lemma that there exists an $n \geq 2$ and $0 = t_0 < t_1 < \cdots < t_n = 1$ such that for all $i = 0, \ldots, n-1$ there exists a diffeomorphism

$$\Psi_i\colon [t_i, t_{i+1}] \times F \ \to p^{-1}(\theta([t_i, t_{i+1}]))$$

that has the property that for all $(s,x) \in [t_i, t_{i+1}] \times F$ we have $p(\Psi_i(s,x)) = \theta(s)$.



Given $s \in [t_i, t_{i+1}]$ we denote by $\Psi_i^s\colon F \to p^{-1}(\theta(s))$ the diffeomorphism that is given by the composition of the obvious diffeomorphism $F \to \{s\} \times F$ and the restriction of Ψ_i to $\{s\} \times F$. Furthermore given $i = 0, \ldots, k-1$ we consider the diffeomorphism $\nu_i := (\Psi_{i+1}^{t_i})^{-1} \circ \Psi_i^{t_i}\colon F \to F$ where we set $\Psi_k = \Psi_0$. We set $\mu := \nu_{k-1} \circ \cdots \circ \nu_0\colon F \to F$. Finally we consider the map

$$\begin{array}{rrcl} \Theta\colon & \operatorname{Tor}(F,\mu) & \to & M \\ & [(x, \underset{\substack{\uparrow \\ \text{where } s \in [t_i, t_{i+1}]}}{s})] & \mapsto & \Psi_i(s, (\nu_{i-1} \circ \cdots \circ \nu_0)(x)) \end{array}$$

It follows easily from the definitions that this map is well-defined and that it is a diffeomorphism. $\blacksquare$

12.2. **The Ehresmann Fibration Theorem.** In this section we want to state two theoremsfrom [**Fri24**] which are instrumental in showing that a given smooth map is actually a smooth bundle map. First we need to recall the following definition.

Definition. Let M and B be smooth manifolds and let $f\colon M \to B$ be a smooth map.

(1) Let $P \in M$.
 (a) If $P \in M \setminus \partial M$, then we say that P is a regular point if $\mathrm{D}f_P\colon \mathrm{T}_P M \to \mathrm{T}_{f(P)}B$ is an epimorphism.
 (b) If $P \in \partial M$, then we say that P is a regular point if $\mathrm{D}f_P\colon \mathrm{T}_P(\partial M) \to \mathrm{T}_{f(P)}B$ is an epimorphism.

(2) We say that $Q \in B$ is a regular value if all points in the preimage $f^{-1}(Q)$ are regular.

After this remainder we can now state the following theorem:

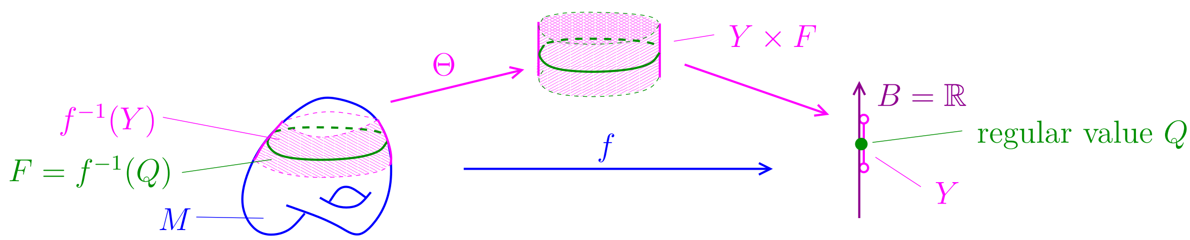
Theorem 12.2. (Neighborhood-Regular Value Theorem) Let $f\colon M \to B$ be a smooth map between smooth manifolds M and B with $\partial B = \emptyset$. Let $Q \in B$ be a regular value of f.

(1) $F := f^{-1}(Q)$ is a proper submanifold of M of dimension $\dim(M) - \dim(B)$.
(2) If M is orientable, then F is also orientable.
(3) If M is compact, then F is also compact.

For the last statement we assume that M is compact.

(4) There exists an open neighborhood Y of $Q \in B$ together with a diffeomorphism $\Theta\colon f^{-1}(Y) \to Y \times F$ such that the following diagram commutes:

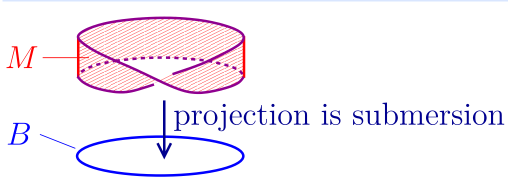
$$\begin{tikzcd} f^{-1}(Y) \arrow[rr, "\Theta"] \arrow[dr, "f"'] & & Y \times F \arrow[dl, "(y,k)\mapsto y"] \\ & Y. & \end{tikzcd}$$

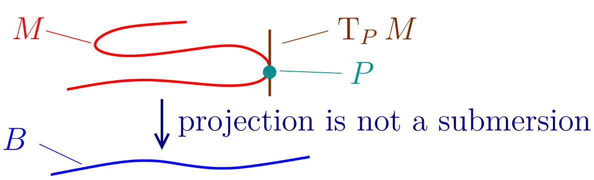


Proof. The theorem is proved in [**Fri24**]. ■

The following definition arises naturally in this context:

Definition. Let M and B be smooth manifold where $\partial B = \emptyset$. A smooth map $\varphi\colon M \to B$ is called a submersion if every $Q \in B$ is a regular value of φ.





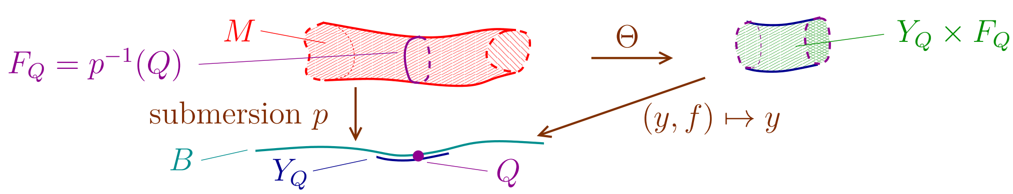
Any smooth bundle map $p\colon M \to B$ to a smooth manifold with $\partial B = \emptyset$ is evidently a submersion. The following theorem, which was first proved by Charles Ehresmann [**Ehr51**] in 1951, says in particular that under some mild hypotheses the converse also holds.

Theorem 12.3. (Ehresmann Fibration Theorem) Let M and B be smooth manifold where $\partial B = \varnothing$. Let $p\colon M \to B$ be a submersion. If M is compact and B is connected, then the map $p\colon M \to B$ is a smooth bundle map.

Proof. Let M and B be smooth manifolds where $\partial B = \varnothing$. Let $p\colon M \to B$ be a submersion. We assume that M is compact and that B is connected. Given $Q \in B$ we set $F_Q := p^{-1}(Q)$. The Neighborhood-Regular Value Theorem 12.2[87] gives us the following two facts:

(a) Each F_Q is naturally a compact smooth manifold of dimension $\dim(M) - \dim(B)$.
(b) For each $Q \in B$ there exists an open neighborhood Y_Q of $Q \in B$ and a diffeomorphism $\Theta\colon p^{-1}(Y_Q) \to Y_Q \times F_Q$ such that the following diagram commutes:

$$\begin{tikzcd} p^{-1}(Y_Q) \arrow[rr, "\Theta"] \arrow[dr, "p"'] & & Y_Q \times F_Q \arrow[dl, "(y,f)\mapsto y"] \\ & Y_Q. & \end{tikzcd}$$

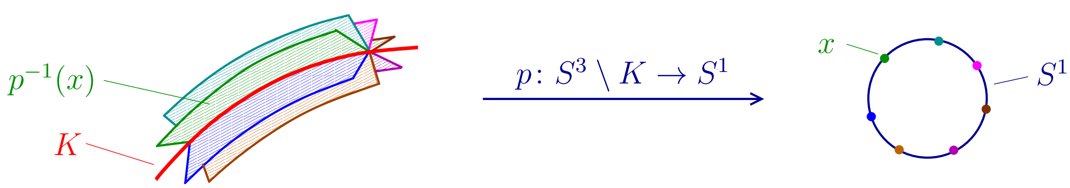


It follows easily from these statements and the hypothesis that B is connected that all F_Q are diffeomorphic. It is now clear that $p\colon M \to B$ is a smooth bundle map. ■

12.3. **Fibered knots.** With the following definition we return to the setting of knots:

Definition. Let $K \subset S^3$ be a knot.

(1) We say that K is fibered if there exists a smooth bundle map $p\colon S^3 \setminus K \to S^1$ such that each fiber $p^{-1}(x)$ together with K forms a Seifert surface for K.
(2) We say that a Seifert surface F is a fiber Seifert surface if $F \setminus K = F \setminus \partial F$ is the fiber of a smooth bundle map $S^3 \setminus K \to S^1$.

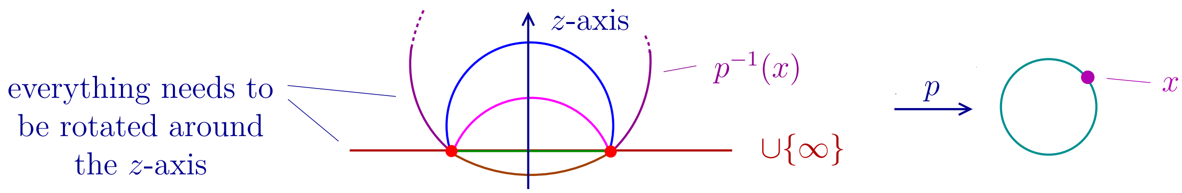


Example. We consider the trivial knot $K = \{(z, 0) \in S^3 \subset \mathbb{C}^2 \,|\, z \in S^1\}$. It is straightforward to verify that the maps

$$\begin{array}{rcl} f\colon S^3 \setminus K & \to & \mathbb{C} \times S^1 \\ (z, w) & \mapsto & \frac{1}{|w|} \cdot (z, w) \end{array} \qquad \text{and} \qquad \begin{array}{rcl} g\colon \mathbb{C} \times S^1 & \to & S^3 \setminus K \\ (z, w) & \mapsto & \frac{(z,w)}{\|(z,w)\|} \end{array}$$

are diffeomorphisms that are inverses of one another. It follows easily from this observation that the map $p\colon S^3 \setminus K \to S^1$ that is given by $p(z, w) = \frac{w}{|w|}$ is a smooth bundle map.

[87] Note that to apply this theorem we use that M is compact.



Next note that for any $x \in S^1$ the map $\varphi\colon \overline{B}^2 \to S^3$ that is given by $a \mapsto (a, x \cdot (1 - |a|^2))$ is a smooth embedding such that $\varphi(\overline{B}^2) = g(\mathbb{C} \times \{x\}) \cup K = p^{-1}(x) \cup K$. This shows that each fiber $p^{-1}(x)$ together with K does indeed form a Seifert surface for K. $\square$

Our main example of fibered knots will be torus knots. It is worth recalling the definition from page 38:

Definition. Let $p, q \in \mathbb{Z}$ be coprime. We refer to

$$T(p,q) \;:=\; \Big\{ \Big(\tfrac{1}{\sqrt{2}} \exp(p \cdot it), \tfrac{1}{\sqrt{2}} \exp(q \cdot it) \Big) \,\Big|\, t \in \mathbb{R} \Big\}$$

as the (p,q)-torus knot $T(p,q)$.

Now we are ready to state the most interesting proposition of this chapter:

Proposition 12.4. (Torus Knot–Fibered Proposition) Let $p, q \in \mathbb{N}$ be coprime.[88]
(1) The corresponding torus knot $T(p,q)$ is fibered.
(2) The fiber has genus $\frac{1}{2}(pq + 1 - p - q)$.

Remark. Alternative expositions and illustrations of parts of the argument and further discussions and generalizations can also be found in [**Mil68**, Lemmas 4.8 and 6.1], [**Sav12**, Chapter 8], [**EN85**, p. 125] and [**Rol90**, Chapter 10.I].

Example. It follows from the Torus Knot–Fibered Proposition 12.4 that the trefoil $T(2,3)$ is fibered. The fibers of a smooth bundle map $p\colon S^3 \setminus T(2,3) \to S^1$ are illustrated in the following video:

`https://www.youtube.com/watch?v=T1So8OCDQ3g`

In the proof of the Torus Knot–Fibered Proposition 12.4 (1) we will use at some point the following little lemma:

Lemma 12.5. (Connected Fiber Lemma) Let M be a connected smooth manifold and let $p\colon M \to B$ be a smooth bundle map. If for some $x \in B$ the fiber $p^{-1}(x)$ has a single boundary component, then $p^{-1}(x)$ is connected.

Sketch of proof of the Connected Fiber Lemma 12.5. Let M be a connected smooth manifold and let $p\colon M \to B$ be a smooth bundle map. Let $x \in B$ such that the fiber $p^{-1}(x)$ has a single boundary component. We set

$$M' \;:=\; \left\{ y \in M \;\middle|\; \begin{array}{l} p(y) \text{ is contained in the unique component} \\ \text{of } p^{-1}(p(y)) \text{ which contains } \partial(p^{-1}(p(y))) \end{array} \right\}.$$

[88]It follows from the Torus Knot Lemma 3.7 (2c) that any torus knot is smoothly isotopic to a (mirror image of a) torus knot $T(p,q)$ with $p, q \in \mathbb{N}$. Since the mirror of a fibered knot is evidently again fibered we see that every torus knot is fibered.

Using the local condition of a smooth bundle map one can easily verify that M' is an open and closed subset of M. Since M is connected and M' is non-empty we see that $M = M'$. It follows that $p^{-1}(x)$ is connected. We leave it to the reader to fill in the details. ■

Proof of the Torus Knot–Fibered Proposition 12.4 (1). Let $p, q \in \mathbb{N}$ be coprime. Note that it follows from the Torus Knot Lemma 3.7 (1) that it suffices to deal with the case that $p \neq q$. Since the factors $\frac{1}{\sqrt{2}}$ in the definition of the (p,q)-torus knot are an eyesore we work throughout this proof with

$$\widetilde{T}(p,q) \;:=\; \{(\exp(p\cdot \mathrm{i}t), \exp(q\cdot \mathrm{i}t)) \,|\, t\in\mathbb{R}\} \;\subset\; \widetilde{S}^3 \;:=\; \{((z,w)\in\mathbb{C}^2 \,|\, |z|^2+|w|^2=2\}$$

instead of $T(p,q)$ and S^3.

Claim 1. We consider the map

$$\begin{array}{rcl} f\colon \widetilde{S}^3 = \{(z,w)\in\mathbb{C}^2 \,|\, |z|^2+|w|^2=2\} & \to & \mathbb{C} \\ (z,w) & \mapsto & z^q - w^p. \end{array}$$

(1) We have $f^{-1}(0) = \widetilde{T}(p,q)$.
(2) 0 is a regular value of f.

Proof.
(1) This statement is elementary and left to the reader.
(2) We start out with two observations:

- Note that for any $(z,w)\in\mathbb{C}^2$ we have
$$\mathrm{D}f_{(z,w)} \;=\; (q\cdot z^{q-1} \quad -p\cdot w^{p-1}) \;\in\; \mathrm{M}(1\times 2,\mathbb{C}) \subset \mathrm{M}(2\times 4,\mathbb{R}).$$
- Given $(z,w)\in S^3$ we have
$$\mathrm{T}_{(z,w)}S^3 \;=\; \{(c,d)\in\mathbb{C}^2 \,|\, \underbrace{\langle (z,w),(c,d)\rangle}_{\substack{\text{scalar product}\\ \text{in } \mathbb{R}^4=\mathbb{C}^2}} = 0\} \;\overset{\substack{\text{since for } a,b\in\mathbb{C}=_{\mathrm{i}}=\mathbb{R}^2 \text{ we have } \langle a,b\rangle = \mathrm{Im}(a \underset{\mathbb{C}}{\cdot} b)\\ \downarrow}}{=}\; \Big\{(c,d)\in\mathbb{C}^2 \,\Big|\, \mathrm{Im}\,\big(\underbrace{z\underset{\mathbb{C}}{\cdot}c + w\underset{\mathbb{C}}{\cdot}d}_{\text{calculation in } \mathbb{C}}\big) = 0\Big\}.$$

  It follows that $\begin{pmatrix} w \\ -z\end{pmatrix}$ and $\begin{pmatrix} \mathrm{i}\cdot w \\ -\mathrm{i}\cdot z\end{pmatrix}$ are vectors in $\mathrm{T}_{(z,w)}S^3$.

Now let $(z,w)\in f^{-1}(0) = \widetilde{T}(p,q)$. We need to show that (z,w) is a regular point of f, i.e. we need to show that the differential $\mathrm{D}f_{(z,w)}\colon \mathrm{T}_{(z,w)}S^3 \to \mathrm{T}_{f(z,w)}\mathbb{C} = \mathbb{C}$ has rank two. Note that

$$\mathrm{D}f_{(z,w)}\cdot\begin{pmatrix} w \\ -z\end{pmatrix} \;\underset{\substack{\uparrow\\ \text{by the above calculation of } \mathrm{D}f_{(z,w)}}}{=}\; (q\cdot z^{q-1} \quad -p\cdot w^{p-1})\cdot\begin{pmatrix} w \\ -z\end{pmatrix} \;=\; \underbrace{q\cdot w\cdot z^{q-1} + p\cdot w^{p-1}\cdot z}_{\substack{\text{non-zero, since } p\neq q \text{ and}\\ \text{since } |w|=|z|=1}}$$

$$\mathrm{D}f_{(z,w)}\cdot\begin{pmatrix} \mathrm{i}\cdot w \\ -\mathrm{i}\cdot z\end{pmatrix} \;\overset{\downarrow}{=}\; (q\cdot z^{q-1} \quad -p\cdot w^{p-1})\cdot\begin{pmatrix} \mathrm{i}\cdot w \\ -\mathrm{i}\cdot z\end{pmatrix} \;=\; \mathrm{i}\cdot(q\cdot w\cdot z^{q-1} + p\cdot w^{p-1}\cdot z).$$

It follows from these calculations that the differential $\mathrm{D}f_{(z,w)}$ has rank two. This concludes the proof that (z,w) is a regular point. ⊞

Claim 2. We consider the map

$$\begin{array}{rcl} \widetilde{f}\colon \widetilde{S}^3 \setminus \widetilde{T}(p,q) \;=\; \widetilde{S}^3 \setminus f(0) & \to & S^1 \\ (z,w) & \mapsto & \frac{z^q-w^p}{|z^q-w^p|} \;=\; \frac{f(z,w)}{|f(z,w)|}. \end{array}$$

Every $(z,w) \in \widetilde{S}^3 \setminus \widetilde{T}(p,q)$ is a regular point.

Proof. We need to show that for every $(z,w) \in \widetilde{S}^3 \setminus \widetilde{T}(p,q)$ the differential $\mathrm{D}\widetilde{f}_{(z,w)}$ is non-zero. We make a few preparations:

(a) We denote by $\pi\colon \mathbb{C} \setminus \{0\} \to S^1$ the map that is given by $z \mapsto \frac{z}{|z|}$.
(b) By definition we have $\widetilde{f} = \pi \circ f\colon \widetilde{S}^3 \setminus \widetilde{T}(p,q) \to S^1$.
(c) A straightforward calculation shows that for every $z \in \mathbb{C} \setminus \{0\}$ and every $v \in \mathbb{C}$ we have $\mathrm{D}\pi_z \cdot v = \mathrm{im}(v \cdot z^{-1}) \cdot \mathrm{i} \cdot \pi(z) \in \mathrm{T}_{\pi(z)}S^1 = \mathbb{R} \cdot \mathrm{i} \cdot \pi(z)$.
(d) It follows from (c) that for $z \in \mathbb{C} \setminus \{0\}$ and $v \in \mathbb{C}$ we have

$$\mathrm{D}\pi_z \cdot v = 0 \iff \mathrm{im}(v \cdot z^{-1}) = 0 \iff v \in \mathbb{R} \cdot z.$$

(e) It follows from the description of $\mathrm{T}_{(z,w)}\widetilde{S}^3$ in Claim 1 that $\begin{pmatrix} \mathrm{i}\cdot z \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ \mathrm{i}\cdot w \end{pmatrix}$ are tangent vectors. In particular any real linear combination of these two vectors is a tangent vector.

Now let $(z,w) \in \widetilde{S}^3 \setminus T(p,q)$. We calculate that

$$\begin{aligned} \mathrm{D}\widetilde{f}_{(z,w)} \cdot \overbrace{\begin{pmatrix} \mathrm{i}\cdot p\cdot z \\ \mathrm{i}\cdot q\cdot w \end{pmatrix}}^{\in \mathrm{T}_{(z,w)}\widetilde{S}^3 \text{ by (e)}} &\underset{\substack{\uparrow \\ \text{by (b) and the chain rule}}}{=} \mathrm{D}\pi_{f(z,w)} \cdot \mathrm{D}f_{(z,w)} \cdot \begin{pmatrix} \mathrm{i}\cdot p\cdot z \\ \mathrm{i}\cdot q\cdot w \end{pmatrix} \underset{\substack{\uparrow \\ \text{calculation of } \mathrm{D}f_{(z,w)} \text{ in Claim 1}}}{=} \mathrm{D}\pi_{f(z,w)} \cdot (q\cdot z^{q-1} \quad p\cdot w^{p-1}) \cdot \begin{pmatrix} \mathrm{i}\cdot p\cdot z \\ \mathrm{i}\cdot q\cdot w \end{pmatrix} \\ &= \underbrace{\mathrm{D}\pi_{z^q-w^p} \cdot (\mathrm{i} \cdot p \cdot q \cdot (z^q - w^p)).}_{\neq 0 \text{ by the discussion in (d)}} \end{aligned}$$

We have thus shown that the differential $\mathrm{D}\widetilde{f}_{(z,w)}$ is non-zero. ⊞

After these preliminary claims we turn to the proof of the actual statement:

Claim 3. The map

$$\begin{array}{rcl} \widetilde{f}\colon \widetilde{S}^3 \setminus \widetilde{T}(p,q) & \to & S^1 \\ (z,w) & \mapsto & \frac{z^q-w^p}{|z^q-w^p|} \;=\; \frac{f(z,w)}{|f(z,w)|} \end{array}$$

is a smooth bundle map.

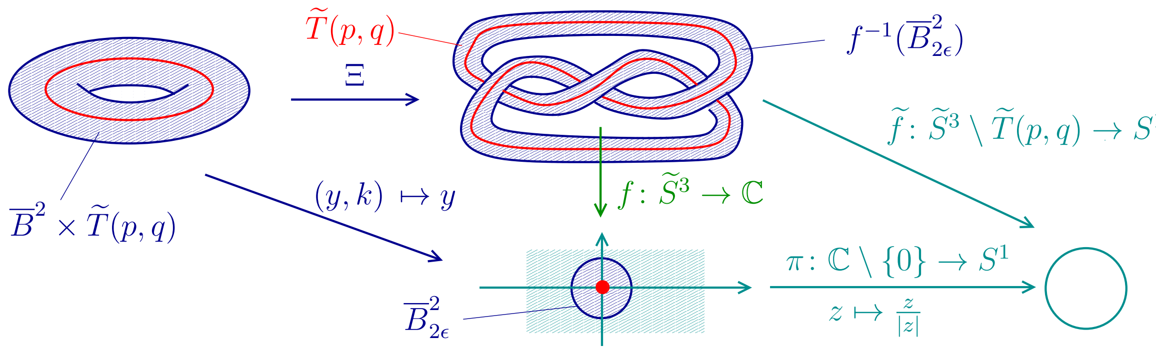
Proof. We first collect what we obtain from Claim 1 and Claim 2:

(1) It follows from Claim 1 and the Neighborhood-Regular Value Theorem 12.2 (4) that there exists an $\epsilon > 0$ and a diffeomorphism $\Xi\colon \overline{B}^2_{2\epsilon} \times \widetilde{T}(p,q) \to f^{-1}(\overline{B}^2_{2\epsilon})$ such that the following diagram commutes:

$$\begin{tikzcd} \overline{B}^2_{2\epsilon} \times \widetilde{T}(p,q) \arrow[rr, "\Xi", "\cong"'] \arrow[dr, "(y,k)\mapsto y"'] & & f^{-1}(\overline{B}^2_{2\epsilon}) \arrow[dl, "f"] \\ & \overline{B}^2_{2\epsilon}. & \end{tikzcd}$$

We set $X := S^3 \setminus f^{-1}(B^2_\epsilon)$.

(2) It follows from Claim 2 that the map $\widetilde{f}\colon \widetilde{S}^3 \setminus \widetilde{T}(p,q) \to S^1$ is a submersion.



Recall that $\pi\colon \mathbb{C}\setminus\{0\} \to S^1$ is the map that is given by $z \mapsto \frac{z}{|z|}$. It follows easily from (1) and (2) that the map $\widetilde{f}\colon X \to S^1$ is a submersion. (Note that the control on ∂X is provided by the commutative diagram in (1) since Ξ restricts to a diffeomorphism $S^1_\epsilon \times \widetilde{T}(p,q) \to \partial X$.) This observation implies that we can apply the Ehresmann Fibration Theorem 12.3 and we deduce that $\widetilde{f}\colon X \to S^1$ is a smooth bundle map. Using (1) one can easily verify that $\pi \circ f = \widetilde{f}\colon f^{-1}(\overline{B}^2_{2\epsilon}) \setminus \widetilde{T}(p,q) \to S^1$ is also a smooth bundle map. It is now pretty straightforward to combine these two observations and to deduce that $\widetilde{f}\colon S^3 \setminus \widetilde{T}(p,q) \to S^1$ is a smooth bundle map. ⊞

Claim 4. For each $x \in S^1$ the fiber $\widetilde{f}^{-1}(x)$ together with $\widetilde{T}(p,q)$ forms a Seifert surface for $\widetilde{T}(p,q)$.

Proof. Let $x \in S^1$. We need to show that the union $\Sigma := \widetilde{f}^{-1}(x) \cup \widetilde{T}(p,q)$ is a Seifert surface for $\widetilde{T}(p,q)$. Note that $\widetilde{S}^3$ is the union of the two open subsets $\widetilde{S}^3 \setminus \widetilde{T}(p,q))$ and $\Xi(B^2_\epsilon \times \widetilde{T}(p,q))$. We first consider the intersection of Σ with these two open subsets separately:

(1) Note that $\Sigma \cap (\widetilde{S}^3 \setminus \widetilde{T}(p,q)) = \widetilde{f}^{-1}(x)$. It follows from the Neighborhood-Regular Value Theorem 12.2 that $\Sigma \cap (\widetilde{S}^3 \setminus \widetilde{T}(p,q)) = \widetilde{f}^{-1}(x)$ is an orientable submanifold with empty boundary of the open subset $S^3 \setminus \widetilde{T}(p,q)$.

(2) Note that $\Sigma \cap \Xi(B^2_\epsilon \times \widetilde{T}(p,q)) = \Xi(\{r \cdot x \,|\, r \in [0,\epsilon)\} \times \widetilde{T}(p,q))$. Since the above map $\Xi\colon \overline{B}^2_\epsilon \times \widetilde{T}(p,q) \to S^3$ is a smooth embedding we see that $\Sigma \cap \Xi(B^2_\epsilon \times \widetilde{T}(p,q))$ is a smooth submanifold with boundary $\Xi(\{0\} \times \widetilde{T}(p,q)) = \widetilde{T}(p,q)$.

It follows easily from these two observations that Σ is a compact oriented submanifold of S^3 with boundary given by $\widetilde{T}(p,q)$.

Finally, eagle eyed readers will have noticed that we still need to argue why Σ is connected. First note that it follows easily from the Connected Fiber Lemma 12.5 that $\Sigma \cap X$ is connected. Furthermore note that $\Sigma \cap \Xi(\overline{B}^2_{2\epsilon} \times \widetilde{T}(p,q)) \cong [0,2\epsilon] \times \widetilde{T}(p,q)$ is clearly connected. It thus follows that Σ is connected. ■

Proof of the Torus Knot–Fibered Proposition 12.4 (2). This statement follows from [**Mil68**, Theorem 7.2]. We will give an alternative proof a little later on page 193. ■

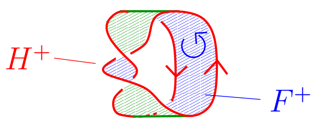
12.4. **Appendix: More fibered knots.** In this appendix we formulate a result which can be used frequently to show that knots are fibered. To state the result it is convenient to generalize some concepts from knots to links.

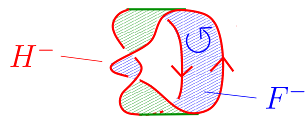
First of all we have the following generalizations of the concepts that we first introduced on pages 154 and 176:

Definition. Let $L \subset S^3$ be an oriented link.

(1) A Seifert surface for L is a compact oriented connected submanifold F of S^3 with $\partial F = L$.[89]
(2) We say that L is fibered if there exists a smooth bundle map $p\colon S^3 \setminus L \to S^1$ such that each fiber $p^{-1}(x)$ together with L forms a Seifert surface for L.
(3) We say that a Seifert surface F is a fiber Seifert surface if $F \setminus L = F \setminus \partial F$ is the fiber of a smooth bundle map $S^3 \setminus L \to S^1$.

Example. In the figure below we show the positive and negative Hopf link $H^\pm$ with Seifert surfaces $F^\pm$.





By the Hopf Link–Lemma 3.13 we already know that the complements $S^3 \setminus H^\pm$ are diffeomorphic to $S^1 \times S^1 \times (-1, 1)$. In Exercise 12.6 we will strengthen this result to conclude that $F^\pm$ are fiber Seifert surfaces for the Hopf links $H^\pm$. We refer to $F^\pm$ as Hopf bands.

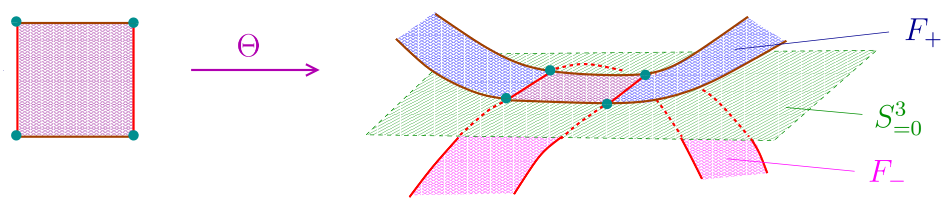
Definition. Let $F_- \subset S^3_{\le 0}$ and $F_+ \subset S^3_{\ge 0}$ be two Seifert surfaces such that there exists a diffeomorphism

$$\Theta\colon [-1,1] \times [-1,1] \;\to\; F_- \cap F_+$$

with the following properties:

(1) $\Theta(\{\pm 1\} \times [-1,1]) = \partial F_- \cap S^3_{=0}$.
(2) $\Theta([-1,1] \times \{\pm 1\}) = \partial F_+ \cap S^3_{=0}$.

We call $F_- \cup F_+$ the plumbing of F_- and F_+. After rounding corners this is a smooth submanifold of S^3.



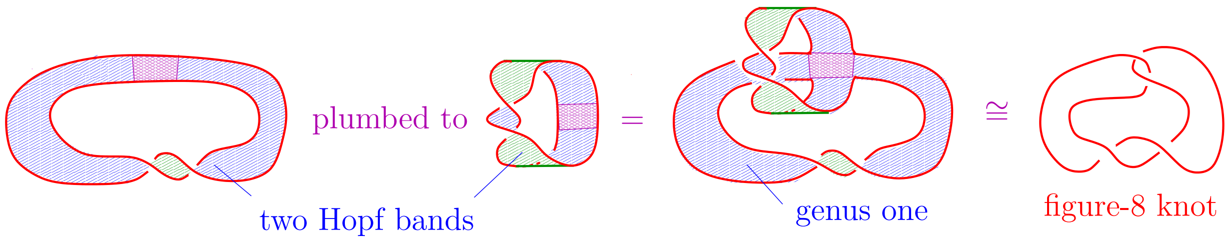
With this definition we can formulate the following surprising theorem.

Theorem 12.6. (Plumbing Fiber Theorem) Let $F \cup F'$ be the plumbing of two Seifert surfaces $F \subset S^3_{\le 0}$ and $F' \subset S^3_{\ge 0}$. If F and F' are fiber Seifert surfaces, then $F \cup F'$ is also a fiber Seifert surface. In particular $\partial(F \cup F')$ is a fibered link.

Proof. This theorem is first stated in [**Sta78**, p. 56]. A proof is given in [**Gab83**, Theorem 3]. The proof goes well beyond our current technical abilities. ■

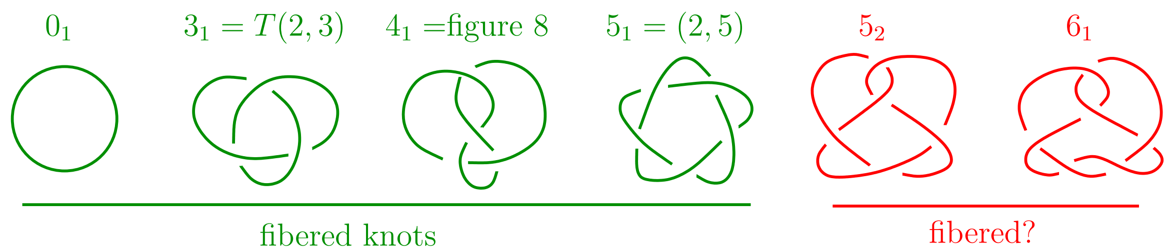
[89]For the record we explain why such Seifert surfaces always exist. The Seifert Algorithm 11.1 can also be applied to the diagram of an oriented link L. The output is a compact oriented smooth submanifold F such that the boundary (with the induced orientation) equals L. The slight catch is that F does not need to be connected. One obtains a oriented connected submanifold with the same boundary by "internal connected sum". Since we will not make use of Seifert surfaces for links we will not elaborate further.

Example. In the figure below we show the plumbing of two Hopf bands. Note that the result is a surface of genus one with connected boundary.



By the Plumbing Fiber Theorem 12.6 the resulting knot is fibered. Since the knot is clearly isotopic to the figure-8 knot this implies that the figure-8 knot is fibered and that the figure-8 knot admits a fiber Seifert surface of genus one. □

We consider again some of the knots that were illustrated on page 55.



It follows from the Torus Knot–Fibered Proposition 12.4 and the above discussion that the first four knots are fibered. But even with the powerful Plumbing Fiber Theorem 12.6 it seems hard to show that 5_2 and 6_1 are fibered. This raises the following question:

Question 12.7. Are the knots 5_2 and 6_1 fibered?

We will return to this question on page 194.

We conclude this chapter with two theorems which we will not prove, but which are worth knowing. First of all we have the following classification of fibered knots that admit a fiber Seifert surface of genus one:

Theorem 12.8. (Fibered Genus-One Knots Theorem)

(1) (a) The trefoil is fibered such that the fiber Seifert surface has genus one.
 (b) The figure 8 knot is fibered such that the fiber Seifert surface has genus one.
(2) If a knot K is fibered such that the fiber Seifert surface has genus one, then K is smoothly isotopic to a trefoil or to the figure 8 knot.

Proof.

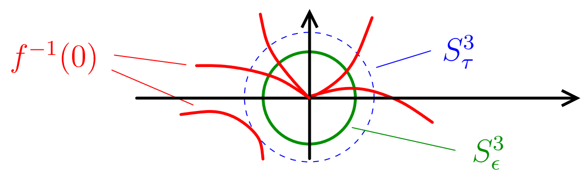
(1) (a) This statement follows from the Torus Knot–Fibered Proposition 12.4 applied to the trefoil $T(2,3)$. A more hands-on proof is given in [**Zee65**, p. 475], [**Rol90**, p. 327-333] and [**BZH14**, p. 77].
 (b) As we just discussed this statement is a consequence of the Plumbing Fiber Theorem 12.6. An alternative proof is are given in [**BZH14**, p. 77].
(2) This statement is proved in [**BZH14**, Theorem 5.15, 6.1 and 15.7]. ■

Finally let us state the following theorem which relates knot theory to algebraic geometry. The theorem goes back to the work of John Milnor [**Mil68**].

Theorem 12.9. (Milnor Fibering Theorem) Let $f(z,w) \in \mathbb{C}[z,w]$ be a complex polynomial in two variables. We assume that $f(z,w)$ is square-free, i.e. we assume that $f(z,w)$ is the product of distinct irreducible polynomials. We assume that $f(0,0)=0$. Given $\epsilon > 0$ we consider the sphere $S^3_\epsilon = \{(z,w) \in \mathbb{C}^2 \,|\, |z|^2+|w|^2=\epsilon^2\}$ and we set $L_\epsilon := f^{-1}(0) \cap S^3_\epsilon$. There exists a $\tau \in \mathbb{R}_{>0} \cup \{\infty\}$ such that for any $\epsilon \in (0,\tau)$ the subset $L_\epsilon \subset S^3_\epsilon$ is a closed 1-dimensional smooth submanifold (in other words, it is a link) and such that the smooth function

$$\begin{array}{rcl} p\colon S^3_\epsilon \setminus L_\epsilon & \to & S^1 \\ (z,w) & \mapsto & \frac{f(z,w)}{|f(z,w)|} \end{array}$$

is a smooth bundle map which has the property that for each $\xi \in S^1$ the fiber $p^{-1}(\xi)$ together with L_ϵ forms a Seifert surface for L_ϵ.



Example. Let $p,q \in \mathbb{N}$ be coprime. We consider the polynomial $f(z,w) = z^p - w^q$. In this case one can in fact take $\tau = \infty$ and one can easily see, as on page 178, that the resulting link $L_\epsilon \subset S^3_\epsilon$ is precisely the (p,q)-torus knot. The links corresponding to more complicated polynomials are studied in detail in [**Wal04**, Chapters 5.3, 5.4] and [**EN85**, Appendix to Chapter 1].

Proof. The theorem follows from [**Mil68**, Theorem 4.8] (with the hypothesis written in [**Mil68**, p. 35]), [**Mil68**, Lemma 6.1] and [**Mil68**, p. 81]. ■

Exercises for Chapter 12.

Exercise 12.1. Recall that that every self-diffeomorphism of S^1 is diffeotopic to the identity or to the reflection in a hyperplane.

(a) Show that every self-diffeomorphism of $[-1,1]$ is diffeotopic to the identity or to the map that is given by $x \mapsto -x$.
(b) Let N be a compact 1-dimensional smooth manifold and let $\mu\colon N \to N$ be a diffeomorphism. Show that the mapping torus $\operatorname{Tor}(N,\mu)$ is diffeomorphic to the annulus, or the torus, or the Möbius band or the Klein bottle.

Exercise 12.2. Let W be a compact 1-dimensional smooth manifold and let $\mu\colon W \to W$ be a diffeomorphism. We consider the mapping torus $\operatorname{Tor}(W,\mu)$.

(a) Show that $\pi_1(\operatorname{Tor}(W,\mu))$ cannot be isomorphic to the fundamental group of a surface Σ_g of genus $g \geq 2$.
(b) Show that $\pi_1(\operatorname{Tor}(W,\mu))$ cannot be isomorphic to the fundamental group of a non-orientable surface N_k of genus $k \geq 3$.

Hint. Consider abelianizations.
Remark. The goal of this exercise is to give a proof of Exercise 12.1 using the fundamental group instead of using the classification of self-diffeomorphisms up to diffeotopy.

Exercise 12.3. Let M be a path-connected smooth manifold and let $p\colon M \to S^1$ be a smooth bundle map.

(a) Show that $p_*\colon \pi_1(M) \to \pi_1(S^1)$ has non-trivial image.
(b) Show that if $p_*\colon \pi_1(M) \to \pi_1(S^1)$ is an epimorphism, then each fiber $p^{-1}(x)$ is connected.
(c) Show that if $p_*\colon \pi_1(M) \to \pi_1(S^1)$ is not an epimorphism, then there exists a smooth bundle map $q\colon M \to S^1$ such that $q_*\colon \pi_1(M) \to \pi_1(S^1)$ is an epimorphism.

Exercise 12.4. Let K be a knot. Show that there exists a continuous map $f\colon S^3 \setminus K \to S^1$ such that for every $x_0 \in M$ the induced map $f_*\colon \pi_1(S^3 \setminus K, x_0) \to \pi_1(S^1, p(x_0))$ is an epimorphism.
Hint. Use the fact that K admits a Seifert surface and use the Bicollar Neighborhood Theorem 5.2.

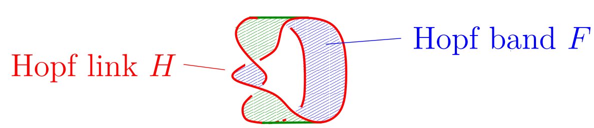
Exercise 12.5. Let $p\colon M \to S^1$ be a smooth bundle map. We suppose that for some $x \in S^1$ the fiber F has a non-compact component and a compact component. Show that M is disconnected.

Exercise 12.6. Let $H \subset S^3$ be the Hopf link as defined on page 44.

(a) Show that there exists a smooth map

$$\Theta\colon S^1 \times S^1 \times [-1,1] \;\to\; S^3$$

with the following two properties:
  (i) The map Θ restricts to a diffeomorphism $S^1 \times S^1 \times (-1,1) \to S^3 \setminus H$.
  (ii) For each $z \in S^1$ the restriction of Φ to $\{z\} \times S^1 \times [-1,1]$ is a smooth embedding with $\Phi(\{z\} \times S^1 \times \{-1,1\}) = H$.
(b) Convince yourself that the Hopf band that is shown below is (after a smooth isotopy) given by $\Phi(\{z\} \times S^1 \times [-1,1])$.



Exercise 12.7. Let K and $\widetilde{K}$ be fibered knots. Is the connected sum $K\#\widetilde{K}$ also fibered?

Exercise 12.8. Show that the trefoil can be written as the boundary of a surface that is obtained from plumbing several Hopf bands.

CHAPTER 13

Alexander polynomials, genus and fiberedness

In this chapter we will explore the geometric content of the Alexander polynomial. More precisely, we will see that the Alexander polynomial gives us a lower bound on the genus of a knot. Furthermore we will see that the Alexander polynomial of a fibered knot is bimonic, which implies of course, that knots with non-bimonic Alexander polynomial are not fibered.

13.1. Genus and the Alexander polynomial.

In this section, given a knot K, we will relate the degree of the Alexander polynomial to the genus of the knot K. First we have to define what we mean by the "degree" of a Laurent polynomial:

Definition. Given a Laurent polynomial $p(t) = \sum\limits_{i=r}^{s} a_i \cdot t^i \in \mathbb{Z}[t^{\pm 1}]$ with $a_r \neq 0$ and $a_s \neq 0$ we define its degree as[90]

$$\deg(p(t)) \;=\; \deg\left(\sum_{i=r}^{s} a_i \cdot t^i\right) \;=\; s - r.$$

We extend this definition to $\deg(0) = -\infty$. Note that if we have $p(t), q(t) \in \mathbb{Z}[t^{\pm 1}]$ with $p(t) \doteq q(t)$, then $\deg(p(t)) = \deg(q(t))$.

Example. We have $\det(-5t^{-2} + 3t^{-1} + 7t^4 - t^5) = 7$ and $\deg(1) = 0$. □

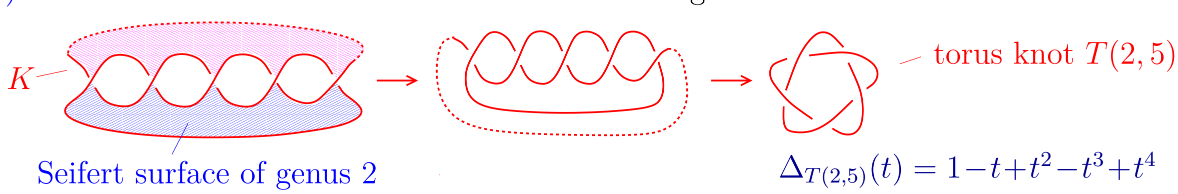
The following is the first of the two main results of this chapter:

Proposition 13.1. (Genus–Alexander Polynomial Proposition) For every knot K we have

$$\deg(\Delta_K(t)) \;\leq\; 2 \cdot \operatorname{genus}(K).$$

Example.

(1) We consider the knot K which is shown in the figure below.



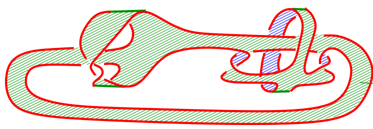
We see a Seifert surface of genus two and on page 161 we were wondering whether K also admits a Seifert surface of genus one. In the figure we show a smooth isotopy from

[90] The name "degree" for this concept is perhaps a little unfortunate, a name like "width" or "breadth" might be more suitable. But since "degree" usually gets used in our context, we stick to this name.

K to the torus knot $T(2,5)$. Thus we see that

$$2 \underset{\substack{\uparrow\\ \text{by the above}\\ \text{example}}}{\geq} \operatorname{genus}(K) \underset{\substack{\uparrow\\ \text{by the Genus–Smooth}\\ \text{Isotopy Invariant Lemma 11.6}}}{=} \operatorname{genus}(T(2,5)) \overset{\substack{\text{by the Genus–Alexander Polynomial Proposition 13.1}\\ \downarrow}}{\geq} \tfrac{1}{2}\cdot\deg(\Delta_{T(2,5)}(t)) \underset{\substack{\uparrow\\ \text{by the Torus Knot–Alexander}\\ \text{Polynomial Proposition 9.6}}}{=} \tfrac{1}{2}\cdot\deg\Bigg(\overbrace{\frac{(t^1-1)\cdot(t^{10}-1)}{(t^2-1)\cdot(t^5-1)}}^{=1-t+t^2-t^3+t^4}\Bigg) = 2.$$

(2) Unfortunately in general the inequality of the Genus–Alexander Polynomial Proposition 13.1 is not an equality. For example on page 100 we introduced the Kinoshita-Terasaka knot, which we show again in the figure below with a different diagram.



We already mentioned that K is a non-trivial knot with $\Delta_K(t) \doteq 1$. Thus it follows that $\deg(\Delta_K(t)) = 0$, but since K is non-trivial we know by the Genus-Zero Knots Proposition 11.7 that $\operatorname{genus}(K) \geq 1$. In fact we will see on page 220 that the genus of K is two.[91] $\square$

To prepare ourselves for the proof of the Genus–Alexander Polynomial Proposition 13.1 we need to digest two lemmas:

Lemma 13.2. (Complement–Surface-π_1-Lemma) If $F \subset S^3$ is a Seifert surface, then $\pi_1(S^3 \setminus F)$ is finitely presented.

Proof of Lemma 13.2. This lemma is a special case of the following more general statement: If M is a compact smooth manifold and F is a compact smooth submanifold of M, then $\pi_1(M \setminus F)$ is finitely presented. This statement is proved in [**Fri24**]. ■

Lemma 13.3. (Laurent Polynomial-Determinant-Degree Lemma) We suppose that we are given matrices $A_{(s-r)\times s} \in \mathrm{M}((s-r)\times s, \mathbb{Z})$ and $B_{r\times s}, C_{r\times s} \in \mathrm{M}(r \times s, \mathbb{Z})$. Then

$$\deg\left(\det\begin{pmatrix} A_{(s-r)\times s} \\ B_{r\times s}\cdot t + C_{r\times s}\end{pmatrix}\right) \leq r.$$

Proof of Lemma 13.3. The statement follows almost immediately from the Leibniz expansion of the determinant, since any summand that appears is the product of $(s-r)$ integers and r linear terms of the form $b + t\cdot c$ with $b, c \in \mathbb{Z}$. ■

Proof of Proposition 13.1. Let K be a knot. We need to prove that for every Seifert surface F of K we have
$$\deg(\Delta_K(t)) \leq 2\cdot\operatorname{genus}(F).$$

So let F be a Seifert surface for K. We make a few preparations:

- We set $g := \operatorname{genus}(F)$.
- We pick $b_0 \in S^3 \setminus F$.
- We pick an orientation for K and we equip F with the corresponding orientation.
- We denote as usual by $\Phi_K\colon \pi_1(S^3 \setminus K, b_0) \to \langle t\rangle$ the unique epimorphism which sends a meridian to t.

[91] The first proof that the genus is two was given in [**Gab86b**, Theorem 5.7].

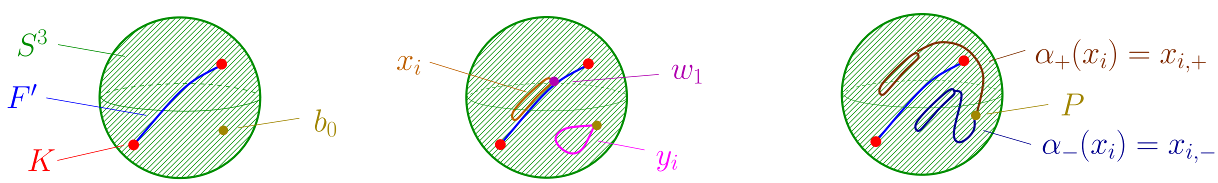
Claim 1. There exists a presentation P of $\pi_1(S^3\setminus K, b_0)$ of the form

$$\pi_1(S^3\setminus K,b_0) \;=\; \left\langle y_1,\dots,y_k,t \;\middle|\; \begin{array}{l} r_1,\dots,r_l \\ t\cdot x_{i,+}\cdot t^{-1}\cdot x_{i,-}^{-1}, i=1,\dots,2g \end{array}\right\rangle$$

where $r_1,\dots,r_l,x_{1,\pm},\dots,x_{2g,\pm}\in\langle y_1,\dots,y_k\rangle$ and where $\Phi_K(y_i)=0$ and $\Phi_K(t)=t$.

Proof. We make a few more preparations:

- We write $F' := F\setminus K$. One can easily verify that F' is a proper submanifold of $S^3\setminus K$. We pick $b_1\in F'$.
- It follows from the Bicollar Neighborhood Theorem 5.2 that there exists an orientation-preserving smooth embedding $\beta\colon [-1,1]\times F'\to S^3\setminus K$ such that $\beta(0,f)=f$ for all $f\in F'$.



Next note that it follows from the above discussion and the HNN–Gluing Theorem 5.1 that there exist two homomorphisms $\alpha_\pm\colon \pi_1(F',b_1)\to\pi_1(S^3\setminus F,b_0)$ and an isomorphism

$$\langle \pi_1(S^3\setminus F,b_0),t\,|\,\alpha_-(\pi_1(F',b_1))=t\cdot\alpha_+(\pi_1(F',b_1))\cdot t^{-1}\rangle \xrightarrow{\cong} \pi_1(S^3\setminus K,b_0).$$

We now need to describe the various groups and homomorphisms involved:

- By the Complement–Surface-π_1-Lemma 13.2 we know that $\pi_1(S^3\setminus F,b_0)$ is finitely presented. Thus we can pick a finite presentation
$$\pi_1(S^3\setminus F,b_0) \;=\; \langle y_1,\dots,y_k\,|\,r_1,\dots,r_l\rangle.$$
- Note that
$$\pi_1(F') \xrightarrow{\cong} \pi_1(F) \;=\; \pi_1\left(\begin{array}{c}\text{surface of genus } g \text{ with one}\\ \text{boundary component}\end{array}\right) \cong \text{free group on } 2g \text{ generators.}$$

It follows that we get a presentation of $\pi_1(S^3\setminus K,b_0)$ of the form

$$P \;=\; \left\langle y_1,\dots,y_k,t \;\middle|\; \begin{array}{l} r_1,\dots,r_l, \\ t\cdot\alpha_+(x_i)\cdot t^{-1}\cdot\alpha_-(x_i)^{-1}, i=1,\dots,2g \end{array}\right\rangle.$$

Note that $\Phi_K(y_i)=0$ and $\Phi_K(t)=t$. For $i=1,\dots,2g$ we now set $x_{i,\pm}:=\alpha_\pm(x_i)$ and we obtain the promised presentation. ⊞

Recall that on page 109 we introduced the Jacobi matrix $J(P)$ of a finite presentation. Note that in our case the Jacobi matrix is a $(l+2g)\times(k+1)$-matrix. The following claim now describes $J^{\Phi_K}(P):=\Phi_{K*}(P)\in \mathrm{M}((l+2g)\times(k+1),\mathbb{Z}[t^{\pm1}])$.

Claim 2. We have

$$J^{\Phi_K}(P) \;=\; \begin{array}{c} \\ l \\ 2g \end{array} \begin{array}{c} \begin{array}{cc} \quad k \quad\quad & \quad 1 \end{array} \\ \begin{pmatrix} A_{l\times k} & 0 \\ B_{2g\times k}\cdot t + C_{2g\times k} & * \end{pmatrix}\end{array}$$

where $A_{l\times k},B_{2g\times k},C_{2g\times k}$ are matrices with entries in $\mathbb{Z}$.

Proof. It follows easily from the defining properties of Fox derivatives and the Fox Derivative Properties Lemma 8.2 that

$$J(P) \;=\; \begin{pmatrix} \left(\frac{\partial r_i}{\partial y_j}\right)_{\substack{i=1,\dots,l,\\ j=1,\dots,k}} & 0 \\ \left(t\cdot\frac{\partial x_{i,+}}{\partial y_j} - t\cdot x_{i,+}\cdot t^{-1}\cdot\frac{\partial x_{i,-}}{\partial y_j}\right)_{\substack{i=1,\dots,2g,\\ j=1,\dots,k}} & * \end{pmatrix}$$

Since $x_1^{\pm},\dots,x_{2g}^{\pm}\in\langle y_1,\dots,y_k\rangle$, since $\Phi_K(y_j)=0$ and since $\Phi_K(t)=t$ we see that the entries of $J^{\Phi_K}(P)$ are precisely of the form we promised. ⊞

We now see that

$$\begin{aligned}\Delta_K(t) &\overset{\text{by the definition on page 121}}{=} (t-1)\cdot\Delta_{\pi_1(S^3\setminus K),\Phi_K}(t)\\ &\underset{\substack{\text{by the Independence-of-Presentation Lemma 8.11, note}\\ \text{that it follows from } \Phi_K(t)=t \text{ that we can use the } t\text{-generator}\\ \text{to calculate the Alexander function}}}{=} \frac{t-1}{\Phi_{K*}(t-1)}\cdot\gcd_{\mathbb{Z}[t^{\pm1}]}\{\det(J^{\Phi_K}(\pi)^Z_{k+1})\}_{\substack{Z\subset\{1,\dots,l+2g\}\\ \#Z=k}} \underset{\substack{\text{follows from } \Phi_K(t)=t\\ \text{and the claim}}}{=}\\ &= \gcd_{\mathbb{Z}[t^{\pm1}]}\Bigg\{\underbrace{\det\begin{pmatrix} A_{l\times k} & \cancel{0}\\ B_{2g\times k}\cdot t + C_{2g\times k} & \cancel{*}\end{pmatrix}^Z}_{\substack{\text{it follows from the Laurent}\\ \text{Polynomial-Determinant-Degree}\\ \text{Lemma 13.3 that degree } \le 2g}}\Bigg\}_{\substack{Z\subset\{1,\dots,l+2g\}\\ \#Z=k}}.\end{aligned}$$

Thus we see that $\Delta_K(t)$ is the greatest divisor of a set of Laurent polynomials of degree $\le 2g$. It follows that $\deg(\Delta_K(t))\le 2g$. But that is exactly what we had set out to show. ■

13.2. Alexander polynomials and fibered knots.

Our next goal is to study Alexander polynomials of fibered knots. We will do so in the following three subsections:

(1) We will show that if K is fibered, then $\pi_1(S^3\setminus K)$ is a semidirect product $F\rtimes_\varphi\langle t\rangle$ where F is a free group.
(2) We will study Alexander functions of semidirect products $F\rtimes_\varphi\langle t\rangle$ where F is a free group.
(3) We will combine (1) and (2) to show that Alexander polynomials of fibered knots are "bimonic". This result will allow us to show that certain knots are not fibered.

13.2.1. Fundamental groups and fibered knots.

By the Smooth Mapping Torus–Bundle Lemma 12.1 we know that if $K\subset S^3$ is fibered, then $S^3\setminus K$ is diffeomorphic to a mapping torus $\mathrm{Tor}(F,\mu)$. Since the Alexander polynomial is determined by the fundamental group it is a good idea to remind ourselves of our earlier results on fundamental groups of mapping tori.

To state the relevant result we need to introduce the following notion from group theory:

Definition. Let N be a group and let $\varphi\colon N\to N$ be an isomorphism.

(1) We define the semidirect product of N and $\langle t\rangle$ with respect to φ as the group $N\rtimes_\varphi\langle t\rangle$ where the underlying set is given by the direct product $N\times\langle t\rangle$, but where the group

multiplication is given by

$$(h, t^m) \cdot (\widetilde{h}, t^{\widetilde{m}}) \;:=\; (h \cdot \varphi^m(\widetilde{h}), t^{m+\widetilde{m}}).$$

We view N and $\langle t\rangle$ as subgroups of $N \rtimes_\varphi \langle t\rangle$ via the two natural monomorphisms $N \to N \rtimes_\varphi \langle t\rangle$ and $\langle t\rangle \to N \rtimes_\varphi \langle t\rangle$.

(2) We refer to the epimorphism $N \rtimes_\varphi \langle t\rangle \to \langle t\rangle$ that is given by $g \mapsto t^0$ for $g \in N$ and $t \mapsto t$ as the natural epimorphism.

Example. Note that, with the above notation, we have for any $x \in N$ that

$$t \cdot x \cdot t^{-1} = ((e,t)\cdot(x,t^0))\cdot(e,t^{-1}) \;=\; (\varphi(x),t)\cdot(e,t^{-1}) \;=\; (\varphi(x),e) \;=\; \varphi(x).$$

Note that this shows that in particular that N is a normal subgroup of $N \rtimes_\varphi \langle t\rangle$.

The definition of the semidirect product allows us to formulate the following proposition:

Proposition 13.4. (Mapping Torus-π_1-Proposition) Let (X, x_0) be a pointed path-connected topological space and let $\mu\colon X \to X$ be a homeomorphism. There exists an isomorphism $\varphi\colon \pi_1(X, x_0) \to \pi_1(X, x_0)$ and an isomorphism

$$\Theta\colon \pi_1(\mathrm{Tor}(X,\mu), [(x_0, 0)]) \xrightarrow{\cong} \pi_1(X, x_0) \rtimes_\varphi \langle t\rangle.$$

Proof. We leave it to the reader to deduce this proposition from the HNN–Gluing Theorem 5.1. ∎

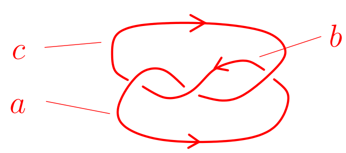
Using the Mapping Torus-π_1-Proposition 13.4 we can easily proof the following proposition:

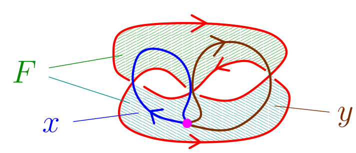
Proposition 13.5. (Fibered Knot-π_1-Proposition) Let $K \subset S^3$ be a fibered knot and let F be a corresponding fiber Seifert surface.[92] We denote by g the genus of F. There exists an isomorphism $\varphi\colon \langle y_1, \dots, y_{2g}\rangle \to \langle y_1, \dots, y_{2g}\rangle$ and an isomorphism

$$\pi_1(S^3 \setminus K) \;\cong\; \langle y_1, \dots, y_{2g}\rangle \rtimes_\varphi \langle t\rangle.$$

If K is oriented, then we can arrange that under this isomorphism the epimorphism $\Phi_K\colon \pi_1(S^3 \setminus K) \to \langle t\rangle$ agrees with the natural epimorphism from the semidirect product to $\langle t\rangle$.

Example. We consider the trefoil K with the diagram and the Seifert surface F that are shown below.





Similar to the discussion on page 68 we see that a presentation for $\pi_1(S^3 \setminus K)$ is given by

$$\pi_1(S^3 \setminus K) \;=\; \langle a, b, c \,|\, \underbrace{c\cdot a\cdot c^{-1}\cdot b^{-1}}_{\text{i.e. } c\cdot a = b\cdot c}, a\cdot b\cdot a^{-1}\cdot c^{-1},\; \underbrace{b\cdot c\cdot b^{-1}\cdot a^{-1}}_{\text{i.e. } a^{-1}\cdot b = b\cdot c^{-1}} \rangle.$$

With $x = a^{-1}\cdot b$ and $y = b^{-1}\cdot c$ and $t = a$ we see that in $\pi_1(S^3 \setminus K)$ we have

$$\begin{aligned} t^{-1}\cdot x\cdot t &= a^{-1}\cdot a^{-1}\cdot b\cdot a &= a^{-1}\cdot b\cdot c^{-1}\cdot a &= a^{-1}\cdot b\cdot c^{-1}\cdot b\cdot b^{-1}\cdot a &= x\cdot y^{-1}\cdot x^{-1}\\ t^{-1}\cdot y\cdot t &= a^{-1}\cdot b^{-1}\cdot c\cdot a &= a^{-1}\cdot b^{-1}\cdot b\cdot c &= a^{-1}\cdot b\cdot b^{-1}\cdot c &= x\cdot y. \end{aligned}$$

[92] We refer to page 176 for the definition of a fiber Seifert surface. Note that such a surface exists precisely if K is fibered.

Let $\varphi\colon \langle x,y\rangle \to \langle x,y\rangle$ be the homomorphism[93] that is given by $\varphi(x) := x\cdot y^{-1}\cdot x^{-1}$ and by $\varphi(y) := x\cdot y$. One can now easily verify, using the above considerations, that the assignment $x \mapsto a^{-1}\cdot b$, $y \mapsto b^{-1}\cdot c$ and $t \mapsto a$ defines an isomorphism $\langle x,y\rangle \rtimes_\varphi \langle t\rangle \to \pi_1(S^3 \setminus K)$.

Proof. Let $K \subset S^3$ be a fibered knot and let F be a fiber Seifert surface of K. By definition this means that there exists a smooth bundle map $p\colon S^3 \setminus K \to S^1$ such that each fiber $p^{-1}(x)$ together with K forms a Seifert surface for K. We set $F' := F \setminus K$. We make a few preparations:

(a) We pick a base point $x_0 \in F'$.
(b) It follows immediately from the Smooth Mapping Torus–Bundle Lemma 12.1 that there exists a self-diffeomorphism $\mu\colon F' \to F'$ such that $S^3 \setminus K$ is diffeomorphic to the mapping torus $\mathrm{Tor}(F',\mu) := (F' \times [0,1])/(x,0) \sim (\mu(x),1)$.

The proposition now follows from the following isomorphisms:

$$\begin{array}{ccccc} \pi_1(S^3 \setminus K) & \underset{\substack{\uparrow \\ \text{by (b)}}}{\cong} & \pi_1(\mathrm{Tor}(F',\mu)) & \underset{\substack{\uparrow \\ \text{by the Mapping Torus-}\pi_1\text{-Proposition 13.4} \\ \text{there exists such an automorphism } \varphi \\ \text{of } \pi_1(F',x_0) \text{ and such that an isomorphism}}}{\cong} & \pi_1(F',x_0) \rtimes_\varphi \langle t\rangle \\ & \cong & \pi_1(F,x_0) \rtimes_\varphi \langle t\rangle & \cong & \langle y_1,\dots,y_{2g}\rangle \rtimes_\varphi \langle t\rangle. \end{array}$$

Finally we assume that K is oriented. If the two epimorphisms to $\langle t\rangle$ match, then we are done. Otherwise we use the isomorphism

$$\begin{array}{rcl} \langle y_1,\dots,y_{2g}\rangle \rtimes_\varphi \langle t\rangle & \to & ,\langle y_1,\dots,y_{2g}\rangle \rtimes_{\varphi^{-1}} \langle \widetilde{t}\rangle \\ (g,t^m) & \mapsto & (g,\widetilde{t}^m) \end{array}$$

to replace the semidirect product by a semidirect product with the "opposite" natural epimorphism to $\langle t\rangle$. ■

For the record we also state the following theorem which gives us the converse to the Fibered Knot-π_1-Proposition 13.5:

Theorem 13.6. (Stallings–Fibered Knot Characterization Theorem) Let $K \subset S^3$ be a knot and let $\Phi\colon \pi_1(S^3 \setminus K) \to \langle t\rangle$ be one of the two epimorphisms. The following statements are equivalent:

(1) K is fibered.
(2) $\pi_1(S^3 \setminus K)$ is isomorphic to a semidirect product $\langle y_1,\dots,y_k\rangle \rtimes \langle t\rangle$.
(3) $\ker(\Phi)$ is a finitely generated free group.
(4) $\ker(\Phi)$ is finitely generated.

Proof.
(1)⇒(2). This statement is the content of the Fibered Knot-π_1-Proposition 13.5.
(2)⇒(3). It follows basically immediately from the definitions that the kernel of the natural homomorphism $\langle y_1,\dots,y_k\rangle \rtimes \langle t\rangle \to \langle t\rangle$ is the free group $\langle y_1,\dots,y_k\rangle$.
(3)⇒(4). This statement is trivial.
(4)⇒(1). This direction was first proved by John Stallings [**Sta62**, Theorem 2]. A textbook proof is given in [**BZH14**, Theorem 5.1]. ■

[93] In Exercise 13.5 we will see that φ is actually an isomorphism.

13.2.2. **Alexander polynomials of semidirect products.** The Fibered Knot-π_1-Proposition 13.5 raises the following question: What can we say about Alexander functions of semidirect products $F \rtimes \langle\varphi\rangle$ where F is a free group? To state the main result of this subsection we need to introduce the following definition:

Definition. We say that a Laurent polynomial $p(t) \in \mathbb{Z}[t^{\pm 1}]$ is monic if the highest coefficient is ± 1. We say a Laurent polynomial $p(t) \in \mathbb{Z}[t^{\pm 1}]$ is bimonic if the lowest and the highest coefficient are ± 1.

Example. The Laurent polynomial $5t - 2t^2 - t^4$ is monic but it is not bimonic. □

Proposition 13.7. (Alexander Polynomial-of-Semidirect Product Proposition) Let $\pi = \langle y_1, \dots, y_k\rangle$ be a finitely generated free group and let $\varphi\colon \pi \to \pi$ be an isomorphism. We consider the corresponding semidirect product $\pi \rtimes_\varphi \langle t\rangle$ and the natural epimorphism $\Phi\colon \pi \rtimes_\varphi \langle t\rangle \to \langle t\rangle$. The following statements hold:
(0) $(t-1)\cdot \Delta_{\pi\rtimes_\varphi\mathbb{Z},\Phi}$ is a Laurent polynomial, i.e. it is contained in $\mathbb{Z}[t^{\pm 1}] \subset \mathbb{Q}(t)$.
(1) $(t-1)\cdot \Delta_{\pi\rtimes_\varphi\mathbb{Z},\Phi}$ is bimonic.
(2) We have $\deg((t-1)\cdot \Delta_{\pi\rtimes_\varphi\mathbb{Z},\Phi}) = k$.

The proof of the Alexander Polynomial-of-Semidirect Product Proposition 13.7 will require the remainder of this subsection. In the next Subsection 13.2.3 we will then return to the study of Alexander polynomials of (fibered) knots.

Lemma 13.8. (Abelianization-of-Jacobi Matrix Lemma) Let π be a finitely generated free group, let $\varphi\colon \pi \to \pi$ be a homomorphism and let $x_1, \dots, x_k$ be a basis for π. We introduce the following notation:

- We denote by $\alpha\colon \pi \to \pi_{\mathrm{ab}}$ the natural projection onto the abelianization.
- For $i = 1, \dots, k$ we set $\overline{x}_i := \alpha(x_i)$. Note that π_{ab} is a free abelian group with basis $\overline{x}_1, \dots, \overline{x}_k$.
- Let $\epsilon\colon \mathbb{Z}[\pi] \to \mathbb{Z}$ be the augmentation homomorphism from page 105, i.e. ϵ is the ring homomorphism which is given by $g \mapsto 1$ for all $g \in \pi$.

Then the following two statements hold:
(1) We have the following equality in $\mathrm{M}(k\times k, \mathbb{Z})$:
$$\epsilon_*\Big(\underbrace{\Big(\frac{\partial\varphi(x_i)}{\partial x_j}\Big)_{i,j=1,\dots,k}}_{(k\times k)\text{-matrix over } \mathbb{Z}[\pi]} \Big) \;=\; \begin{array}{c}\text{matrix that represents } \varphi_*\colon \pi_{\mathrm{ab}} \to \pi_{\mathrm{ab}} \text{ with}\\ \text{respect to the basis } \overline{x}_1, \dots, \overline{x}_k\end{array}$$
(2) If φ is an isomorphism, then $\epsilon_*\big(\big(\frac{\partial\varphi(x_i)}{\partial x_j}\big)_{i,j=1,\dots,k}\big) \in \mathrm{M}(k\times k,\mathbb{Z})$ is invertible.

Proof of Lemma 13.8.
(1) The statement follows easily from the definitions and from applying the following claim to $r = \varphi(x_i)$ for $i = 1, \dots, k$:

Claim. For any $r \in \langle x_1, \dots, x_k\rangle$ and any $i \in \{1, \dots, k\}$ we have
$$\sum_{i=1}^{k} \epsilon\Big(\frac{\partial r}{\partial x_i}\Big)\cdot \overline{x}_i \;=\; \alpha(r) \;\in\; \pi_{\mathrm{ab}}.$$

Proof. The claim follows from the following three observations:

- Basically by definition the claim holds for $r = x_j$ for any $j \in \{1, \dots, k\}$.
- Let $j \in \{1, \dots, k\}$. Recall that it follows from the Fox Derivative Properties Lemma 8.2 (5) that $\frac{\partial x_j^{-1}}{\partial x_i} = -\delta_{ij} \cdot x_i^{-1}$. It follows easily from this observation and from the observation that $\epsilon(x_j^{-1}) = 1$ that the claim holds for $r = x_j^{-1}$.
- We suppose that the equality holds for $u, v \in \langle x_1, \dots, x_k \rangle$. We claim that it then also holds for $u \cdot v$. Indeed, this follows from the following little calculation in π_{ab}:

$$\begin{aligned} \sum_{i=1}^{k} \epsilon\Big(\frac{\partial(u \cdot v)}{\partial x_i}\Big) \cdot \overline{x}_i \ &\overset{\text{by the Leibniz rule}}{=} \ \sum_{i=1}^{k} \epsilon\Big(\frac{\partial u}{\partial x_i} + u \cdot \frac{\partial v}{\partial x_i}\Big) \cdot \overline{x}_i \ \overset{\text{since } \epsilon \text{ is a ring homomorphism and since } \epsilon(u)=1}{=} \ \sum_{i=1}^{k} \epsilon\Big(\frac{\partial u}{\partial x_i}\Big) \cdot \overline{x}_i + \sum_{i=1}^{k} \epsilon\Big(\frac{\partial v}{\partial x_i}\Big) \cdot \overline{x}_i \\ &\underset{\text{since we assume that the equality holds for } u \text{ and } v}{=} \ \alpha(u) \cdot \alpha(v) \ = \ \alpha(u \cdot v). \end{aligned}$$

$\boxplus$

(2) If $\varphi\colon \pi \to \pi$ is an isomorphism, then the induced homomorphism $\varphi_*\colon \pi_{\text{ab}} \to \pi_{\text{ab}}$ is also an isomorphism. This implies that the matrix on the right hand of (1) is an isomorphism. By (1) this also implies that $\epsilon_*\big(\big(\frac{\partial \varphi(x_i)}{\partial x_j}\big)_{i,j=1,\dots,k}\big)$ is an invertible matrix. ■

Lemma 13.9. (Polynomial Matrix–Determinant Lemma) Let $k \in \mathbb{N}_0$ and let R be a commutative ring. For any two matrices $A, B \in \mathrm{M}(k \times k, R)$ we have

$$\det(A \cdot t + B) \ = \ \det(A) \cdot t^n + \text{intermediate terms} + \det(B) \cdot t^0.$$

Proof. This statement follows immediately from the Leibniz formula for determinants. ■

Now we can finally provide the proof of the Alexander Polynomial-of-Semidirect Product Proposition 13.7:

Proof of Proposition 13.7. Let $\pi = \langle y_1, \dots, y_k \rangle$ be a finitely generated free group and let $\varphi\colon \pi \to \pi$ be an isomorphism. We consider the corresponding semidirect product $\pi \rtimes_\varphi \langle t \rangle$ and the natural epimorphism $\Phi\colon \pi \rtimes_\varphi \langle t \rangle \to \langle t \rangle$. Recall that we need to prove the following statements:

(0) $(t-1) \cdot \Delta_{\pi \rtimes_\varphi \mathbb{Z}, \Phi}$ is a Laurent polynomial.
(1) $(t-1) \cdot \Delta_{\pi \rtimes_\varphi \mathbb{Z}, \Phi}$ is bimonic.
(2) We have $\deg((t-1) \cdot \Delta_{\pi \rtimes_\varphi \mathbb{Z}, \Phi}) = k$.

First note that we have the following isomorphisms:

$$\begin{aligned} \langle y_1, \dots, y_k \rangle \rtimes_\varphi \mathbb{Z} \ &\overset{\text{follows easily from the example on page 189}}{\cong} \ \langle y_1, \dots, y_k, t \,|\, t \cdot y_1 \cdot t^{-1} \cdot \varphi(y_1)^{-1}, \dots, y_k \cdot t^{-1} \cdot \varphi(y_k)^{-1} \rangle \\ &= \ \langle y_1, \dots, y_k, t \,|\, t \cdot y_1 \cdot t^{-1} \cdot \varphi(y_1^{-1}), \dots, y_k \cdot t^{-1} \cdot \varphi(y_k^{-1}) \rangle \\ &\underset{\text{substitution } y_i \mapsto x_i^{-1}}{=} \ \langle x_1, \dots, x_k, t \,|\, t \cdot x_1^{-1} \cdot t^{-1} \cdot \varphi(x_1), \dots, x_k^{-1} \cdot t^{-1} \cdot \varphi(x_k) \rangle. \end{aligned}$$

Note that $\Phi(t) = t$ and $\Phi(x_1) = \cdots = \Phi(x_k) = 0$. We now see that

$$(t-1)\cdot\Delta_{\pi\rtimes_\varphi\mathbb{Z},\Phi} \underset{\substack{\text{we calculate the}\\ \text{Alexander function}\\ \text{using the generator ``}t\text{''}}}{=} (t-1)\cdot\frac{\det\left(\Phi\left(\frac{\partial(t\cdot x_i^{-1}\cdot t^{-1}\cdot\varphi(x_i))}{\partial x_j}\right)_{i,j}\right)}{\Phi(t-1)} = \det\left(\Phi\left(\frac{\partial(t\cdot x_i^{-1}\cdot t^{-1}\cdot\varphi(x_i))}{\partial x_j}\right)_{i,j}\right)$$

$$\underset{\substack{\text{by the properties of Fox derivatives}\\ \text{and by definition of }\Phi}}{=} \det\left(\left(-t\cdot\mathrm{id}_k + \Phi\left(\frac{\partial\varphi(x_i)^{-1}}{\partial x_j}\right)\right)_{i,j}\right) \underset{\substack{\text{since }\Phi\text{ is trivial}\\ \text{on }\langle x_1,\dots,x_k\rangle}}{=} \det\left(-t\cdot\mathrm{id}_k + \underbrace{\epsilon\left(\frac{\partial\varphi(x_i)^{-1}}{\partial x_j}\right)_{i,j}}_{\substack{\text{by the Abelianization-of}\\ \text{Jacobi Matrix Lemma 13.8}\\ \text{this matrix is invertible over }\mathbb{Z}}}\right)$$

The desired statement follows from this calculation together with the claim in the beginning of the proof. ■

13.2.3. **Alexander polynomials of fibered knots.** As the reader will surely have noticed, the Fibered Knot-π_1-Proposition 13.5 together with the Alexander Polynomial-of-Semidirect Product Proposition 13.7 gives us an interesting proposition on Alexander polynomials of fibered knots:

Proposition 13.10. (Fibered Knot–Alexander Polynomial Proposition) If $K \subset S^3$ is a fibered knot, then the following two statements hold:

(1) The Alexander polynomial is bimonic.
(2) We have $\deg(\Delta_K(t)) = 2 \cdot$ genus of a fiber Seifert surface.

Example. Let $p, q \in \mathbb{N}$ be coprime. By the Torus Knot–Fibered Proposition 12.4 we know that the torus knot $T(p,q)$ is fibered. Let F be the corresponding fiber Seifert surface. We now calculate that

$$\mathrm{genus}(F) \underset{\substack{\text{by the Fibered Knot–Alexander}\\ \text{Polynomial Proposition 15.7}}}{=} \tfrac{1}{2}\cdot\deg(\Delta_K(t)) \underset{\substack{\text{by the Torus Knot–Alexander}\\ \text{Polynomial Proposition 9.6}}}{=} \tfrac{1}{2}\cdot\deg\left(\frac{(t-1)\cdot(t^{p\cdot q}-1)}{(t^p-1)\cdot(t^q-1)}\right) = \tfrac{1}{2}\cdot(pq+1-p-q).$$

Note that this discussion also provides a proof for Statement (2) of the Torus Knot–Fibered Proposition 12.4. □

Proof. Let $K \subset S^3$ be a fibered knot with corresponding Seifert fiber surface F. We denote by g the genus of F. By the Fibered Knot-π_1-Proposition 13.5 there exists an isomorphism $\varphi\colon \langle y_1,\dots,y_{2g}\rangle \to \langle y_1,\dots,y_{2g}\rangle$ and an isomorphism

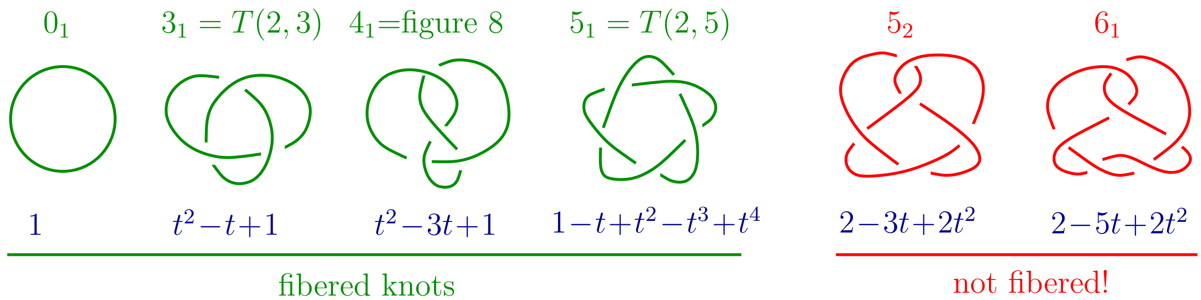
$$\Theta\colon \pi_1(S^3\setminus K) \xrightarrow{\cong} \langle y_1,\dots,y_{2g}\rangle \rtimes_\varphi \langle t\rangle.$$

We denote by $\Phi\colon \langle y_1,\dots,y_{2g}\rangle \rtimes_\varphi \langle t\rangle \to \langle t\rangle$ the natural epimorphism. As we discussed on page 128, we can calculate the Alexander polynomial of K using any epimorphism $\pi_1(S^3\setminus K) \to \langle t\rangle$. Note that

$$\Delta_K(t) \underset{\substack{\text{by the discussion on page 128, we can}\\ \text{calculate the Alexander polynomial of }K\\ \text{using any epimorphism }\pi_1(S^3\setminus K)\to\langle t\rangle}}{=} (t-1)\cdot\Delta_{\pi_K,\Phi\circ\Theta} \underset{\substack{\text{by the above isomorphism and the}\\ \text{Alexander Function Lemma 8.6}}}{=} (t-1)\cdot\Delta_{\langle y_1,\dots,y_{2g}\rangle\rtimes_\varphi\langle t\rangle,\Phi}.$$

The proposition now follows from the Alexander Polynomial-of-Semidirect Product Proposition 13.7. ■

Example. We return to the list of knots that were shown on page 182.



On page 182 we discussed that the first four knots are fibered. In Question 12.7 we were wondering out load whether the knots 5_2 and 6_1 are fibered. If we now consider the list of Alexander polynomials from page 128 we see that the Alexander polynomials of 5_1 and 6_2 are not binomic. It thus follows from the Fibered Knot–Alexander Polynomial Proposition 15.7 that these two knots are not fibered.

Corollary 13.11. (Fiber–Minimal Corollary) Let K be a knot. If K is fibered, then for the corresponding fiber Seifert surface F we have the following equalities:

$$\operatorname{genus}(K) \;=\; \operatorname{genus}(F) \;=\; \tfrac{1}{2}\cdot \deg(\Delta_K(t)),$$

in particular F is a Seifert surface of minimal genus.

Proof. Let K be a fibered knot with corresponding fiber Seifert surface F. We see that

$$\operatorname{genus}(K) \underset{\substack{\uparrow\\ \text{by definition}}}{\leq} \operatorname{genus}(F) \underset{\substack{\uparrow\\ \text{by the Fibered Knot–Alexander}\\ \text{Polynomial Proposition 15.7}}}{=} \tfrac{1}{2}\cdot \deg(\Delta_K(t)) \underset{\substack{\uparrow\\ \text{by the Genus–Alexander}\\ \text{Polynomial Proposition 13.1}}}{\leq} \operatorname{genus}(K).$$

We see that all inequalities have to be equalities and we get the promised result. ■

Remark. Let K be a fibered knot with fiber Seifert surface F. By the Fiber–Minimal Corollary 13.11 we know that F is a Seifert surface of minimal genus. In fact, by [**EL83**, Lemma 5.1] we know that every Seifert surface of minimal genus is smoothly isotopic, rel K, to F (we will also provide a proof in Exercise 16.3). In general though, for non fibered knots, minimal genus Seifert surfaces are not unique, examples are for instance provided in [**Alf70**] and [**Alt12**].

13.3. **Appendix: Alternating knots.** We have now shown that the Alexander polynomial gives lower bounds on the genus and that the Alexander polynomial of a fibered knot is bimonic. The Alexander polynomial is an imperfect invariant though, for example we saw on page 186 that in general it does not determine the genus of a knot.

In this appendix we formulate two theorems which show that in contrast the Alexander polynomial of an *alternating* knot is a rather powerful invariant:

Theorem 13.12. (Alternating-Δ-Genus Theorem) Let K be an alternating knot. The following two statements hold:

(1) $\deg(\Delta_K(t)) = 2\cdot \operatorname{genus}(K)$.
(2) If we apply Seifert's algorithm to an alternating diagram, then we obtain a Seifert surface of minimal genus.

Proof. This theorem was first proved independently by Kunio Murasugi, see [**Mur58**, p. 235 and Theorem 4.1], and Richard Crowell [**Cro59**, Theorem 3.5]. Alternative proofs are given in [**Gab86a**, Theorem 4] and [**Gre17**]. ■

Theorem 13.13. (Alternating-Δ-Fibered Theorem) For any alternating knot K we have

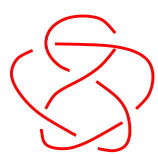
$$\Delta_K(t) \text{ is bimonic} \quad \Longleftrightarrow \quad K \text{ is fibered.}$$

Proof. This theorem was first proved by Kunio Murasugi [**Mur63**, Theorem 2]. The theorem is also a consequence of [**Ni07**, Theorem 1.1]. ■

Exercises for Chapter 13.

Exercise 13.1. We consider the knots 6_2, 6_3 which are shown below (together with their Alexander polynomials.) Show that in each case the genus equals 2.
Remark. This exercise, together with the discussion on pages 157 and 193 (and the observation that 5_1 and 7_1 are torus knots), and Exercise 11.6 gives us the genera for all knots that are shown in the figure on page 128.

6_2



$1 - 3t + 3t^2 - 3t^3 + t^4$

6_3

$1 - 3t + 5t^2 - 3t^3 + t^4$

Exercise 13.2. Given a finitely generated group π we denote, for the purpose of this exercise, by $d(\pi)$ the minimal number of generators of π. Given a knot K we define

$$\widetilde{g}(K) \;:=\; \min\left\{ d(\pi_1(\Sigma)) \;\middle|\; \begin{array}{l} \Sigma \text{ is a compact (possibly non-orientable)} \\ \text{connected submanifold of } S^3 \text{ with } \partial\Sigma = K \end{array} \right\}.$$

(a) Determine $\widetilde{g}$ of the trefoil.
(b) Show that given any $m \in \mathbb{N}$ there exists a knot K with $\operatorname{genus}(K) \geq m$ and $\widetilde{g}(K) = 1$.

Exercise 13.3. Let π and Γ be two groups and let $\alpha, \beta\colon \Gamma \to \pi$ be two homomorphisms. Recall that on page 64 we defined the corresponding HNN-extension:

$$\langle \pi, t \mid \alpha(\Gamma) = t \cdot \beta(\Gamma) \cdot t^{-1} \rangle \;:=\; (\pi * \langle t \rangle) \,/\, \langle\langle \{\alpha(g) \cdot t \cdot \beta(g)^{-1} \cdot t^{-1}\}_{g \in \Gamma} \rangle\rangle.$$

We introduce the following language:

- We say that an HNN-extension is *ascending* if α is an isomorphism.
- We refer to the homomorphism $\Phi\colon \langle \pi, t \mid \alpha(\Gamma) = t \cdot \beta(\Gamma) \cdot t^{-1} \rangle \to \langle t \rangle$ that is given by $g \mapsto t^0$ for $g \in \pi$ and $t \mapsto t$ as the *natural epimorphism*.

Show that if the HNN-extension is ascending, then the Alexander function corresponding to the natural homomorphism is monic and show that in general it is not bimonic.

Exercise 13.4. We say that an m-component link $L = L_1 \cup \cdots \cup L_m$ is a *boundary link* if there exists a collection of m disjoint Seifert surfaces $F_1, \dots, F_m$ such that for $i = 1, \dots, m$ we have $\partial F_i = L_i$.

(a) Give an example of a boundary link that is not split.
Remark. You could use the higher order Alexander polynomials of Exercise 9.13 to show that the boundary link is not split.

(b) Show that if L is an m-component boundary link, then there exists an epimorphism from $\pi_1(S^3 \setminus L)$ onto a free group on m generators.
Hint. Use the Seifert surfaces and bicollars to construct an interesting continuous map $S^3 \setminus L \to \bigvee_{i=1}^{m} S^1$.

(c) Show that if $m \geq 2$ and if L is a boundary link, then $\Delta_L = 0$.

Exercise 13.5. Let $\varphi\colon \langle x, y\rangle \to \langle x, y\rangle$ be the homomorphism that is uniquely determined by $\varphi(x) = x \cdot y^{-1} \cdot x^{-1}$ and $\varphi(y) = x \cdot y$. Show that φ is an isomorphism.

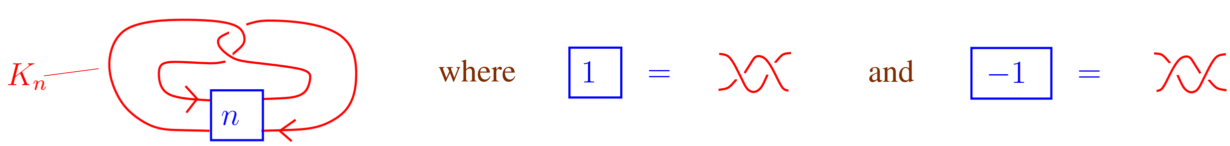
Exercise 13.6. Let K and $\widetilde{K}$ be two disjoint oriented links. Show that if there exists a Seifert surface for K that is disjoint from $\widetilde{K}$, then $\operatorname{lk}(K, \widetilde{K}) = 0$.
Hint. Consider the description of $\pi_1(S^3 \setminus K)$ in the proof of the Genus–Alexander Polynomial Proposition 13.1.

Exercise 13.7. Given $n \in \mathbb{Z}$ we consider the n-twist knot K_n that is shown in the figure below.

(a) Determine the Alexander polynomial $\Delta_{K_n}(t)$.

(b) What is the genus of K_n?

(c) Which n-twist knots are fibered?



CHAPTER 14

The twisted Alexander function: Definition and basic properties

Let π be a group that admits a presentation of deficiency one and let $\Phi\colon \pi \to H$ be a non-trivial homomorphism to a free abelian group. On page 111 we introduced the corresponding Alexander function $\Delta_{\pi,\Phi} \in \mathbb{Q}(H)$, where $\mathbb{Q}(H)$ denotes the quotient field of $\mathbb{Z}[H]$.

In this chapter we will generalize this concept. More precisely if on top of π and Φ we are also given a unique factorization domain (UFD) R together with a homomorphism $\alpha\colon \pi \to \mathrm{GL}(n, R)$, then we will see in this chapter that we can define the twisted Alexander function $\Delta^{\alpha}_{\pi,\Phi} \in Q_R(H)$, where $Q_R(H)$ denotes the quotient field of $R[H]$. The distribution of labor between this and the following chapter is as follows:

(1) In this chapter we will outline the definition and main properties of the twisted Alexander function.
(2) In the following chapter we will use the twisted Alexander function to introduce the twisted Alexander polynomial of oriented knots and links. Furthermore we will study many applications to the theory of knots and links.

Many of the proofs in this and the following chapter are slight generalizations of the earlier "untwisted" statements. To keep this and the next chapter at reasonable lengths, we will often just give outlines of the proofs. We leave it to the reader to fill in the details.

14.1. Definition of the twisted Alexander function.

In this section we want to define twisted Alexander functions corresponding to homomorphisms $\pi \to \mathrm{GL}(n, R)$. We start out with some notation and three really elementary lemmas:

Notation. Given a ring R and given $k, l \in \mathbb{N}_0$ we denote by $\mathrm{M}(k \times l, R)$ the set of $k \times l$-matrices with entries in R. Furthermore, for $k, l, m \in \mathbb{N}_0$ we denote by

$$\mu\colon \mathrm{M}(k \times l, R) \times \mathrm{M}(l \times m, R) \;\to\; \mathrm{M}(k \times m, R)$$

the map that is given by the usual matrix multiplication. Note this map turns $\mathrm{M}(k\times k, R)$ into a ring.

Lemma 14.1. (Group Ring-to-Matrix Ring Lemma) Let π be a group, let S be a commutative ring and let $\gamma\colon \pi \to \mathrm{GL}(n, S)$ be a homomorphism. The map

$$\begin{array}{rcl} \gamma_*\colon \mathbb{Z}[\pi] & \to & \mathrm{M}(n \times n, S) \\ \sum\limits_{i=1}^{k} \underbrace{a_i \cdot g_i}_{\text{where } a_i \in \mathbb{Z} \text{ and } g_i \in \pi} & \mapsto & \sum\limits_{i=1}^{k} a_i \cdot \gamma(g_i) \end{array}$$

is a ring homomorphism.

Proof. The lemma follows almost immediately from the definitions. ∎

Lemma 14.2. (Matrix Multiplication–Change-of-Ring Lemma) Let $\varphi\colon R \to S$ be a ring homomorphism. The maps

$$\begin{array}{rcl} \varphi_*\colon \mathrm{M}(k\times l, R) & \to & \mathrm{M}(k\times l, S) \\ (a_{ij}) & \mapsto & (\varphi(a_{ij})) \end{array}$$

respect matrix multiplication in the sense that for any $k,l,m \in \mathbb{N}_0$ the following diagram commutes:

$$\begin{tikzcd} \mathrm{M}(k\times l, R) \times \mathrm{M}(l\times m, R) \arrow[r, "\mu"] \arrow[d, "\varphi_*\downarrow\ \ \downarrow\varphi_*"'] & \mathrm{M}(k\times m, R) \arrow[d, "\varphi_*"] \\ \mathrm{M}(k\times l, S) \times \mathrm{M}(l\times m, S) \arrow[r, "\mu"] & \mathrm{M}(k\times m, S) \end{tikzcd}$$

commutes.

Proof. The lemma follows again almost immediately from the definitions. ■

Lemma 14.3. (Matrix-of-Matrices Lemma) Let S be a ring and let $n \in \mathbb{N}_0$. The maps[94]

$$\begin{array}{rcl} \Xi\colon \mathrm{M}(k\times l, \mathrm{M}(n\times n, S)) & \to & \mathrm{M}((k\cdot n)\times(l\cdot n), S) \\ (\mathrm{A}_{ij})_{i,j} & \mapsto & \text{matrix where the } (i\cdot n+i')\times(j\cdot n+j') \text{ is} \\ & & \text{given by the } (i',j')\text{-entry of the matrix } A_{ij} \end{array}$$

are isomorphisms of abelian groups that respect matrix multiplication, in the sense that for any $k,l,m \in \mathbb{N}_0$ the following diagram commutes:

$$\begin{tikzcd} \mathrm{M}(k\times l, \mathrm{M}(n,\times n, S)) \times \mathrm{M}(l\times m, \mathrm{M}(n,\times n, S)) \arrow[r, "\mu"] \arrow[d, "\Xi\downarrow\ \ \downarrow\Xi"'] & \mathrm{M}(k\times m, \mathrm{M}(n,\times n, S)) \arrow[d, "\Xi"] \\ \mathrm{M}((k\cdot n)\times(l\cdot n), S) \times \mathrm{M}((l\cdot n)\times(m\cdot n), S) \arrow[r, "\mu"] & \mathrm{M}((k\cdot n)\times(m\cdot n), S) \end{tikzcd}$$

commutes.

$$\Xi\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \Xi\begin{pmatrix} \boxed{\begin{matrix} (A_{11})_{11} & (A_{11})_{12} \\ (A_{11})_{21} & (A_{11})_{22} \end{matrix}} & \boxed{\begin{matrix} (A_{12})_{11} & (A_{12})_{12} \\ (A_{12})_{21} & (A_{12})_{22} \end{matrix}} \\ \boxed{\begin{matrix} (A_{21})_{11} & (A_{21})_{12} \\ (A_{21})_{21} & (A_{21})_{22} \end{matrix}} & \boxed{\begin{matrix} (A_{22})_{11} & (A_{22})_{12} \\ (A_{22})_{21} & (A_{22})_{22} \end{matrix}} \end{pmatrix} = \begin{pmatrix} (A_{11})_{11} & (A_{11})_{12} & (A_{12})_{11} & (A_{12})_{12} \\ (A_{11})_{21} & (A_{11})_{22} & (A_{12})_{21} & (A_{12})_{22} \\ (A_{21})_{11} & (A_{21})_{12} & (A_{22})_{11} & (A_{22})_{12} \\ (A_{21})_{21} & (A_{21})_{22} & (A_{22})_{21} & (A_{22})_{22} \end{pmatrix}$$

Proof. The lemma is elementary and follows quite easily from the definitions. We leave it to the reader to fill in the details. ■

Shortly we will also need the following definition:

Definition. Let π be a group, let S be a commutative ring and let $\gamma\colon \pi \to \mathrm{GL}(n,S)$ be a homomorphism. Given $k,l \in \mathbb{N}$ we denote by γ_* also the map

$$\begin{array}{rcl} \gamma_*\colon\ \mathrm{M}(k\times l, \mathbb{Z}[\pi]) & \to & \mathrm{M}((k\cdot n)\times(l\cdot n), S) \\ (a_{ij}) & \mapsto & \Xi(\underbrace{\gamma_*(a_{ij})}_{\in \mathrm{M}(k\times l, \mathrm{M}(n\times n, S))}). \end{array}$$

[94]In plain English the map Ξ does the following: We view an element of $\mathrm{M}(k\times l, \mathrm{M}(n\times n,S))$ as a $(k\times l)$-matrix of $(n\times n)$-matrices. We now view such a matrix as an $(k\cdot n)\times(l\cdot n)$-matrix in the obvious way. In practice we will often just apply Ξ without specifically pointing out, that we are applying Ξ.

We point out that it follows from the Group Ring-to-Matrix Ring Lemma 14.1, the Matrix Multiplication–Change-of-Ring Lemma 14.2 and the Matrix-of-Matrices Lemma 14.3 that the above maps $\gamma_*\colon \mathrm{M}(k \times l, \mathbb{Z}[\pi]) \to \mathrm{M}((k \cdot n) \times (l \cdot n), S)$ again respect matrix multiplications.

After this tiring list of trivialities, let us continue with a definition, which recalls some concepts introduced on page 109:

Definition. Let $P = \langle x_1, \dots, x_k \,|\, r_1, \dots, r_l \rangle$ be a finite presentation.

(1) As on page 109 we denote by $\mu_P\colon \langle x_1, \dots, x_k \rangle \to P$ the natural projection.
(2) As on page 109 we define the Jacobi matrix $J(P)$ to be the $(l \times k)$-matrix

$$J(P) \;:=\; \Big\{ \mu_{P*}\Big(\underbrace{\frac{\partial r_i}{\partial x_j}}_{\in \mathbb{Z}[\langle x_1, \dots, x_k \rangle]} \Big) \Big\}_{\substack{i=1,\dots,l \\ j=1,\dots,k}} \;\in\; \mathrm{M}(l \times k, \mathbb{Z}[P]).$$

We move onto a new concept:

Definition. Let π be a group, let $\Phi\colon \pi \to H$ be a homomorphism to a free abelian group H, let R be a commutative ring and let $\alpha\colon \pi \to \mathrm{GL}(n, R)$ be a homomorphism. We consider the map

$$\begin{array}{rcl} \Phi \otimes \alpha\colon \pi & \to & \mathrm{GL}(n, R[H]) \\ g & \mapsto & \alpha(g) \cdot \Phi(g). \end{array}$$

It follows easily from the definitions and the hypothesis that R is commutative and that H is abelian that the map $\Phi \otimes \alpha\colon \pi \to \mathrm{GL}(n, R[H])$ is in fact a homomorphism.

Example. Let π be a group, let $\Phi\colon \pi \to H$ be a homomorphism to a free abelian group H, let R be a commutative ring and let $\alpha\colon \pi \to \mathrm{GL}(n, R)$ be a homomorphism. Finally let $x \in \pi$ such that $y := \Phi(x)$ is non-trivial. We can perform the following calculation in the Laurent polynomial ring $R[y^{\pm 1}] \subset R[H]$:

$$\det((\Phi \otimes \alpha)_*(x - 1)) \underset{\substack{\uparrow \\ \text{by the above} \\ \text{definition}}}{=} \det(\alpha(x) \cdot y - \mathrm{id}_n) \underset{\substack{\uparrow \\ \text{by the Polynomial Matrix} \\ \text{Determinant Lemma 13.9}}}{=} \overbrace{\det(\alpha(x)) \cdot y^n + \dots + \det(-\mathrm{id}_n)}^{\text{polynomial in the variable } y \text{ of degree } n} \neq 0$$

$\square$

The following notation is a slight variation on the definition from page 111:

Notation. Let R be a commutative domain.

(1) We denote by Q_R the quotient ring of R.

Let H be a torsion-free abelian group. (Note that this implies that $R[H]$ is a domain.)

(2) We denote by $Q_R(H)$ the quotient ring of $R[H]$.
(3) We denote by R^* the group of multiplicative units of R.
(4) Given $p, q \in Q_R(H)$ we write[95] $p \doteq_R q$ if there exists a unit $\epsilon \in R^*$ and an $h \in H$ with $p = \epsilon \cdot h \cdot q$.

Finally we recall the following notation from page 109:

[95] Since $-1, 1$ are the only units of $\mathbb{Z}$ we see that "$\doteq_{\mathbb{Z}}$" is equivalent to the symbol "$\doteq$" that we defined on page 111.

Notation. Let R be a ring and let $M \in \mathrm{M}(l \times k, R)$ be a matrix. Given $i \in \{1, \dots, k\}$ we denote by M_i the $l \times (k-1)$-matrix which we obtain by deleting the i-th column.

We can now formulate the following analogue of the Presentation Function Quotient Proposition 8.4 and the Alexander Function Theorem 8.5:

Theorem 14.4. (Twisted Alexander Function Theorem) Let π be a group, let H be a free abelian group, let $\Phi\colon \pi \to H$ be a non-trivial homomorphism, let R be a domain and let $\alpha\colon \pi \to \mathrm{GL}(n, R)$ be a homomorphism. Finally assume that we are given a presentation[96]

$$P = \langle x_1, \dots, x_k \,|\, r_1, \dots, r_{k-1} \rangle \xrightarrow{\cong} \pi$$

of deficiency one.

(1) There exists an $i \in \{1, \dots, k\}$ such that $\Phi(x_i)$ is non-trivial.
(2) If $\Phi(x_i)$ is non-trivial, then $\det((\Phi \otimes \alpha)_*(x_i - 1)) \neq 0$.

We pick an $i \in \{1, \dots, k\}$ such that $\Phi(x_i)$ is non-trivial. We consider

$$\Delta^{\alpha}_{\pi,\Phi} \;:=\; \frac{\det((\Phi \otimes \alpha)_*(J(P)_i))}{\det((\Phi \otimes \alpha)_*(x_i - 1))} \;\in\; Q_R(H).$$

The following statements hold:

(3) Up to a sign $\Delta^{\alpha}_{\pi,\Phi}$ does not depend on the choice of i.
(4) If R is a unique factorization domain (UFD), then up to "$\doteq_R$" $\Delta^{\alpha}_{\pi,\Phi}$ does not depend on the choice of the deficiency one presentation.

Proof. As we will see, the proof of the theorem is very similar to the earlier proofs of the Presentation Function Quotient Proposition 8.4 and the Alexander Function Theorem 8.5. We will provide the details in Section 14.3. ■

The Alexander Function Theorem 14.4 allows us to make the following definition:

Definition. Let π be a group, let $\Phi\colon \pi \to H$ be a non-trivial homomorphism to a free abelian group H, let R be a UFD and let $\alpha\colon \pi \to \mathrm{GL}(n, R)$ be a homomorphism. If π admits a presentation $P = \langle x_1, \dots, x_k \,|\, r_1, \dots, r_{k-1} \rangle$ of deficiency one, then we refer to

$$\Delta^{\alpha}_{\pi,\Phi} \;:=\; \underbrace{\frac{\det((\Phi \otimes \alpha)_*(J(P)_i))}{\det((\Phi \otimes \alpha)_*(x_i - 1))}}_{\substack{\text{where } \Phi(x_i) \text{ is} \\ \text{non-trivial}}} \;\in\; Q_R(H)$$

as the twisted Alexander function of (π, Φ, α). By the Twisted Alexander Function Theorem 14.4 the twisted Alexander function is well-defined up to "$\doteq_R$".

Example. Suppose that in the previous definition we have $R = \mathbb{Z}$, $n = 1$ and that we are considering the trivial representation $\tau\colon \pi \to \mathrm{GL}(1, \mathbb{Z})$ that is given by $g \mapsto (1)$. It then follows immediately from the definition that the twisted Alexander function $\Delta^{\tau}_{\pi,\Phi}$ equals the Alexander function $\Delta_{\pi,\Phi}$. □

[96]To simplify the notation, which is already heavy enough, we suppress the isomorphism from the presentation to the group from the notation.

The remainder of this chapter and the next chapter are organized as follows:

(1) In the remainder of this section we will state two very elementary lemmas about twisted Alexander functions.
(2) In Section 14.2 we will do an explicit calculation of a twisted Alexander function, just to get a sense of what is happening.
(3) In Section 14.3 we will finally provide a proof of the Twisted Alexander Function Theorem 14.4.
(4) In Section 14.4 we will slightly generalize our setup, namely we will define twisted Alexander functions for any pair $(\pi, \Phi\colon \pi \to H)$ together with any representation $\alpha\colon \pi \to \operatorname{Aut}_R(M)$.
(5) In Chapter 15 we will apply the theory of twisted Alexander functions to knots and links.

As mentioned above, we now provide two elementary and formal lemmas about twisted Alexander functions. First of all we have the following lemma, which is a generalization of the Alexander Function–Functorial Lemma 8.6:

Lemma 14.5. (Twisted Alexander Function–Functorial Lemma)
(1) Let π and $\widetilde{\pi}$ be two groups which admit presentations of deficiency one, let $f\colon \widetilde{\pi} \to \pi$ be an isomorphism, let $\Phi\colon \pi \to H$ be a non-trivial homomorphisms to a free abelian group H, let R be a UFD and let $\alpha\colon \pi \to \mathrm{GL}(n, R)$ be a homomorphism. Then
$$\Delta^{\alpha\circ f}_{\widetilde{\pi},\Phi\circ f} \;=\; \Delta^{\alpha}_{\pi,\Phi} \quad \in Q_R(H).$$
(2) Let π be a group which admits a presentation of deficiency one. Let $\Phi\colon \pi \to G$ be a homomorphism to a free abelian group, let R be a UFD and let $\alpha\colon \pi \to \mathrm{GL}(n, R)$ be a homomorphism.
   (a) Let H be a free abelian group and let $f\colon G \to H$ be a homomorphism such that $f \circ \Phi\colon \pi \to H$ is non-trivial. Then
$$f_*(\Delta^{\alpha}_{\pi,\Phi}) \;=\; \Delta^{\alpha}_{\pi,f\circ\Phi} \quad \in Q_R(H).$$
   (b) Let S be a UFD. Furthermore let $\varphi\colon R \to S$ be a ring homomorphism. As in the Matrix Multiplication–Change-of-Ring Lemma 14.2 we now denote by $\varphi_*\colon \mathrm{GL}(n, R) \to \mathrm{GL}(n, S)$ the induced homomorphism. Then
$$\varphi_*(\Delta^{\alpha}_{\pi,\Phi}) \;=\; \Delta^{\varphi_*\circ\alpha}_{\pi,\Phi} \quad \in Q_S(G).$$

Proof.

(1) This statement is an immediate consequence of the Twisted Alexander Function Theorem 14.4.

(2) By (1) we can assume that $\pi = \langle x_1,\dots,x_k \,|\, r_1,\dots,r_{k-1}\rangle$. We pick $i \in \{1,\dots,k\}$ such that $(f\circ\Phi)(x_i) \neq 0$.

(a) We see that

$$\begin{array}{rcl} f_*(\Delta^\alpha_{\pi,\Phi}) & = & f_*\Big(\dfrac{\det((\Phi\otimes\alpha)_*(J(\pi)_i))}{\det((\Phi\otimes\alpha)_*(x_i-1))}\Big) \;=\; \dfrac{f_*(\det((\Phi\otimes\alpha)_*(J(\pi)_i)))}{f_*(\det((\Phi\otimes\alpha)_*(x_i-1)))} \\ & \underset{\substack{\uparrow\\ \text{since determinants commute}\\ \text{with ring homomorphisms}}}{=} & \dfrac{\det(f_*((\Phi\otimes\alpha)_*(J(\pi)_i)))}{\det(f_*((\Phi\otimes\alpha)_*(x_i-1)))} \underset{\substack{\uparrow\\ \text{it follows easily from the definitions}\\ \text{that for any matrix } A \text{ over } \mathbb{Z}[\pi] \text{ we}\\ f_*((\Phi\otimes\alpha)_*(A)) = ((f\circ\Phi)\otimes\alpha)(A)}}{=} \dfrac{\det(((f\circ\Phi)\otimes\alpha)_*(J(\pi)_i)}{\det(((f\circ\Phi)\otimes\alpha)_*(x_i-1))} \;=\; \Delta^\alpha_{\pi,f\circ\Phi}. \end{array}$$

(b) This statement also follows easily from the definitions. We leave it to the reader to fill in the details. ■

Before we can formulate the next lemma, we need to introduce a harmless piece of notation:

Notation. Let π be a group, let R be a commutative ring and let $\alpha\colon \pi \to \mathrm{GL}(n,R)$ be a homomorphism. Finally let $A \in \mathrm{GL}(n,R)$. We introduce the conjugated representation

$$\begin{array}{rcl} \alpha^A\colon \pi & \to & \mathrm{GL}(n,R) \\ g & \mapsto & A^{-1}\circ\alpha(g)\circ A. \end{array}$$

Lemma 14.6. (Conjugation Invariance Lemma) Let π be a group, let $\Phi\colon \pi \to H$ be a non-trivial homomorphism to a free abelian group H, let R be a UFD and let $\alpha\colon \pi \to \mathrm{GL}(n,R)$ be a homomorphism. Finally let $A \in \mathrm{GL}(n,R)$. For the conjugated representation $\alpha^A\colon \pi \to \mathrm{GL}(n,R)$ we have

$$\Delta^\alpha_{\pi,\Phi} \;=\; \Delta^{\alpha_A}_{\pi,\Phi} \;\in\; Q_R(H).$$

Proof. One can easily verify that for any $(r\times s)$-matrix M over $\mathbb{Z}[\pi]$ we have

$$(\Phi\otimes\alpha^A)_*(M) \;=\; \operatorname{diag}\underbrace{(A^{-1},\dots,A^{-1})}_{r \text{ copies}}\cdot(\Phi\otimes\alpha)_*(M)\cdot\operatorname{diag}\underbrace{(A,\dots,A)}_{s \text{ copies}}.$$

In particular, if $r = s$, then we see that $\det((\Phi\otimes\alpha^A)_*(M)) = \det((\Phi\otimes\alpha)_*(M))$. The desired equality of twisted Alexander functions follows immediately from this observation and the definition of the twisted Alexander function. ■

14.2. **Example of a twisted Alexander function.** In this section we want to calculate the twisted Alexander function in a particular case, just to get a sense for the definition and for what is happening. The group π under consideration will be $\pi := \pi_1(S^3\setminus T(2,3))$, where $T(2,3)$ is the trefoil. We make two observations:

(i) We already know, e.g. by considering one of the presentations below, that there exists (up to sign) a unique epimorphism $\Phi\colon \pi \to \langle t\rangle$. It follows from the Twisted Alexander Function–Functorial Lemma 14.5 (1) that this indeterminacy has only a negligible effect on any twisted Alexander function.

(ii) We consider the permutation group S_3.
- Using the Presentation–Homomorphism Lemma **??** and say the presentation provided by the Torus Knot-π_1-Proposition 3.8 one can easily verify that, up to conjugation, there exists a unique epimorphism $\varphi\colon \pi_1(S^3\setminus K) \to S_3$.

- We consider the following two special elements of S_3:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \quad \text{(cyclic permutation)}$$
$$\tau = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \quad \text{(swapping of second and third number).}$$

  One can easily show, say using Exercise **??** and using the Presentation–Homomorphism Lemma **??**, that there exists a unique homomorphism $\theta\colon S_3 \to \mathrm{GL}(2,\mathbb{Z})$ with

$$\theta(\sigma) = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \qquad \text{and} \qquad \theta(\tau) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

  In combination we get a homomorphism $\alpha := \theta \circ \varphi\colon \pi \to \mathrm{GL}(2,\mathbb{Z})$.

In the following, for practice, we will perform the calculation of the twisted Alexander function $\Delta^{\alpha}_{\pi,\Phi} \in \mathbb{Q}(t)$ using two different presentations:

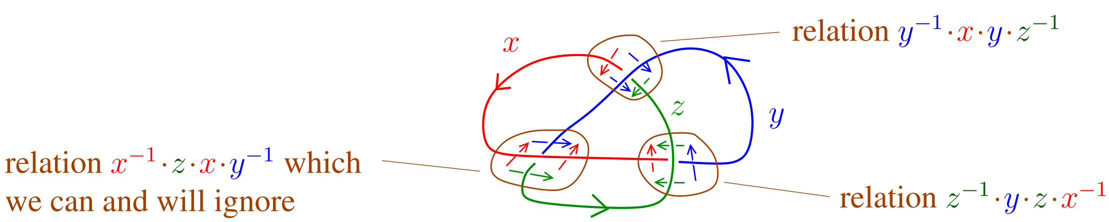
(1) We will use the presentation from the Wirtinger Presentation Proposition 5.5 (2).
(2) We will use the presentation from the Torus Knot-π_1-Proposition 3.8.

These calculations will happen in the following two subsections. The calculations show that calculating twisted Alexander functions is doable, but unfortunately also a pain, which is best left to computers[97] In this particular example it is also not entirely clear, what the resulting twisted Alexander function tries to tell us.

14.2.1. Calculation using the Wirtinger presentation.

Recall that on page 68 we used the Wirtinger Presentation Proposition 5.5 (2) to show that there exists an isomorphism

$$P := \langle x, y, z \,|\, y^{-1}\cdot x\cdot y\cdot z^{-1}, z^{-1}\cdot y\cdot z\cdot x^{-1}, \underbrace{\cancel{x^{-1}\cdot z\cdot x\cdot y^{-1}}}_{\text{not needed}} \rangle \xrightarrow{\cong} \pi_1(S^3 \setminus K).$$



Using the above presentation P and the Presentation–Homomorphism Lemma **??** we obtain the following two homomorphisms:

(i) We obtain an epimorphism $\Phi\colon \pi \to \langle t \rangle$ that is given by $\Phi(x) = \Phi(y) = \Phi(z) = t$.
(ii) We consider again the above permutations $\sigma, \tau \in S_3$. Note that $\sigma^3 = \mathrm{id}$, $\tau^2 = \mathrm{id}$ and $\sigma \circ \tau \circ \sigma = \tau$. We can now easily verify that we have a well-defined epimorphism $\varphi\colon \pi \to S_3$ that is given by
$$\varphi(x) = \tau = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \quad \varphi(y) = \sigma\circ\tau\circ\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \quad \varphi(z) = \sigma^2\circ\tau\circ\sigma^{-2} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}.$$

[97] Or for older people: Better left to students with programming skills.

Note that with $\alpha = \theta \circ \varphi \colon \pi \to \mathrm{GL}(2,\mathbb{Z})$ we now have

$$\begin{aligned}
(\Phi\otimes\alpha)(x) &= \Phi(x)\cdot\alpha(x) &&= t\cdot\theta(\tau) &&= t\cdot\begin{pmatrix}0&1\\1&0\end{pmatrix} &&= \begin{pmatrix}0&t\\t&0\end{pmatrix},\\
(\Phi\otimes\alpha)(y) &= \Phi(y)\cdot\alpha(y) &&= t\cdot\theta(\sigma\circ\tau\circ\sigma^{-1}) &&= t\cdot\begin{pmatrix}-1&-1\\0&1\end{pmatrix} &&= \begin{pmatrix}-t&-t\\0&t\end{pmatrix},\\
(\Phi\otimes\alpha)(z) &= \Phi(z)\cdot\alpha(z) &&= t\cdot\theta(\sigma^2\circ\tau\circ\sigma^{-2}) &&= t\cdot\begin{pmatrix}1&0\\-1&-1\end{pmatrix} &&= \begin{pmatrix}t&0\\-t&-t\end{pmatrix}.
\end{aligned}$$

Next recall that on page 110 we saw that the Jacobi matrix $J(P) \in \mathrm{M}(2\times 3, \mathbb{Z}[P])$ is given by

$$J(P) = \begin{pmatrix} y^{-1} & y^{-1}\cdot x - y^{-1} & -1 \\ -1 & z^{-1} & z^{-1}\cdot y - z^{-1}\end{pmatrix}.$$

We obtain the following equalities in $\mathbb{Q}(t)$:

$$\begin{aligned}
\Delta^{\alpha}_{\pi,\Phi} &= \frac{\det((\Phi\otimes\alpha)_*(J(P)_1)}{\det((\Phi\otimes\alpha)_*(x-1))} = \frac{\det\left((\Phi\otimes\alpha)_*\begin{pmatrix}\cancel{y^{-1}} & y^{-1}\cdot x - y^{-1} & -1\\ \cancel{-1} & z^{-1} & z^{-1}\cdot y - z^{-1}\end{pmatrix}\right)}{\det((\Phi\otimes\alpha)_*(x-1))}\\
&= \frac{\det\left(\begin{array}{c|c} \begin{pmatrix}-t^{-1}&-t^{-1}\\0&t^{-1}\end{pmatrix}\cdot\begin{pmatrix}0&t\\t&0\end{pmatrix}-\begin{pmatrix}-t^{-1}&-t^{-1}\\0&t^{-1}\end{pmatrix} & \begin{pmatrix}-1&0\\0&-1\end{pmatrix}\\ \hline \begin{pmatrix}t^{-1}&0\\-t^{-1}&-t^{-1}\end{pmatrix} & \begin{pmatrix}t^{-1}&0\\-t^{-1}&-t^{-1}\end{pmatrix}\cdot\begin{pmatrix}-t&-t\\0&t\end{pmatrix}-\begin{pmatrix}t^{-1}&0\\-t^{-1}&-t^{-1}\end{pmatrix}\end{array}\right)}{\det\left(\begin{pmatrix}0&t\\t&0\end{pmatrix}-\begin{pmatrix}-1&0\\0&-1\end{pmatrix}\right)}\\
&= \frac{\det\left(\begin{array}{c|c}\begin{pmatrix}-1+t^{-1}&-1+t^{-1}\\1&-t^{-1}\end{pmatrix} & \begin{pmatrix}1&0\\0&1\end{pmatrix}\\ \hline \begin{pmatrix}t^{-1}&0\\-t^{-1}&-t^{-1}\end{pmatrix} & \begin{pmatrix}-1-t^{-1}&-1\\1+t^{-1}&t^{-1}\end{pmatrix}\end{array}\right)}{\det\begin{pmatrix}-1&t\\t&-1\end{pmatrix}} = \frac{\det\left(\begin{array}{cc|cc}-1+t^{-1}&-1+t^{-1}&-1&0\\1&-t^{-1}&0&-1\\ \hline t^{-1}&0&-1-t^{-1}&-1\\-t^{-1}&-t^{-1}&1+t^{-1}&t^{-1}\end{array}\right)}{\det\begin{pmatrix}-1&t\\t&-1\end{pmatrix}}\\
&= \frac{(t^{-2}-1)^2}{1-t^2} = t^{-2}-1.
\end{aligned}$$

14.2.2. **Calculation using the torus knot presentation.** For practice, and as a sanity check, we also calculate the twisted Alexander function of $\pi_1(S^3\setminus T(p,q))$ using a different presentation. Namely, for coprime $p,q\in\mathbb{N}$ we consider the presentation

$$Q := \langle x,y\,|\,x^p\cdot y^{-q}\rangle \xrightarrow{\cong} \pi_1(S^3\setminus T(p,q))$$

from the Torus Knot-π_1-Proposition 3.8, where for the trefoil we have $p=2$ and $q=3$. Using the above presentation Q and the Presentation–Homomorphism Lemma **??** we obtain the following two homomorphisms:

(i) We obtain an epimorphism $\Phi\colon Q\to\langle t\rangle$ that is given by $\Phi(x)=t^q$ and $\Phi(y)=t^p$.
(ii) We obtain an epimorphism $\varphi\colon\langle x,y\,|\,x^2\cdot y^{-3}\rangle\to S_3$ that is given by $\varphi(x)=\tau$ and $\varphi(y)=\sigma$.

Note that with $\alpha = \theta \circ \varphi \colon \pi \to \mathrm{GL}(2, \mathbb{Z})$ we now have

$$\begin{array}{rclclclcl}
(\Phi \otimes \alpha)(x) & = & \Phi(x) \cdot \alpha(x) & = & t \cdot \theta(\tau) & = & t^3 \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & = & \begin{pmatrix} 0 & t^3 \\ t^3 & 0 \end{pmatrix}, \\
(\Phi \otimes \alpha)(y) & = & \Phi(y) \cdot \alpha(y) & = & t^2 \cdot \theta(\sigma) & = & t^2 \cdot \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} & = & \begin{pmatrix} 0 & t^2 \\ -t^2 & -t^2 \end{pmatrix}.
\end{array}$$

Next recall that in the proof of the Torus Knot–Alexander Polynomial Proposition 9.6 we saw that

$$J(Q) = \begin{pmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \end{pmatrix} \underset{\substack{\uparrow \\ \text{by the Fox Derivative Properties} \\ \text{Lemma 8.2 (5) applied to } r = x^p \cdot y^{-q}}}{=} \begin{pmatrix} \sum\limits_{i=0}^{p-1} x^i & -x^p \cdot \sum\limits_{i=1}^{q} y^{-i} \end{pmatrix} \underset{\substack{\uparrow \\ \text{specialized to } p=2 \text{ and } q=3}}{=} \begin{pmatrix} 1+x & -x^2 \cdot (y^{-1}+y^{-2}+y^{-3}) \end{pmatrix}.$$

We obtain the following equalities in $\mathbb{Q}(t)$:

$$\begin{array}{rclcl}
\Delta^{\alpha}_{T(2,3)}(t) & = & \dfrac{\det((\Phi \otimes \alpha)_*(J(Q)_2)}{\det((\Phi \otimes \alpha)_*(y-1))} & = & \dfrac{\det((\Phi \otimes \alpha)_*((1+x \;\; -x^2 \cdot (y^{-1}+y^{-2}+y^{-3})))}{\det((\Phi \otimes \alpha)_*(y-1))} \\[3ex]
& = & \dfrac{\det\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & t^3 \\ t^3 & 0 \end{pmatrix}\right)}{\det\left(\begin{pmatrix} 0 & t^2 \\ -t^2 & -t^2 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)} & = & \dfrac{\det\begin{pmatrix} 1 & t^3 \\ t^3 & 1 \end{pmatrix}}{\det\begin{pmatrix} -1 & t^2 \\ -t^2 & -t^2-1 \end{pmatrix}} \\[3ex]
& = & \dfrac{1-t^6}{1+t^2+t^4} & = & 1-t^2.
\end{array}$$

With a big sigh of relief the author can now proudly point to the fact that the two calculations of $\Delta^{\alpha}_{T(2,3)}(t)$, namely $t^{-2}-1$ and $1-t^2$, do indeed just differ by multiplication by a power of t.[98]

14.3. Proof of the Twisted Alexander Function Theorem 14.4.

In this section we will prove the Twisted Alexander Function Theorem 14.4. As we will see, the proof is an adaptation of the arguments of Sections 8.5 and 8.6, where we dealt with the "untwisted" Alexander function.

In the following arguments we will on several occasions make use of the following elementary lemma:

Lemma 14.7. (Determinant-after-Homomorphism Lemma) Let π be a group, let S be a commutative ring and let $\varphi \colon \pi \to \mathrm{GL}(n, S)$ be a group homomorphism. Furthermore let $A, B \in \mathrm{M}(k \times k, \mathbb{Z}[\pi])$ be two square matrices and let $\lambda \in \mathbb{Z}[\pi]$. The following statements hold:

(1) (a) If we obtain B from A by right multiplication of a column by λ, then
$$\det(\varphi_*(B)) = \det(\varphi_*(A)) \cdot \det(\varphi_*(\lambda)).$$

(b) If we obtain B from A by adding a right-multiple of column to another column, then
$$\det(\varphi_*(B)) = \det(\varphi_*(A)).$$

[98] Here we actually do not need a sign.

(2) (a) If we obtain B from A by left multiplication of a row by λ, then

$$\det(\varphi_*(B)) \;=\; \det(\varphi_*(\lambda))\cdot\det(\varphi_*(A)).$$

(b) If we obtain B from A by adding a left-multiple of a row to another row, then

$$\det(\varphi_*(B)) \;=\; \det(\varphi_*(A)).$$

Proof. In the following we provide the proof of (1a). The proofs of all other statements are basically the same. So suppose that we obtain B from A by right multiplication of a column by λ, then

$$\begin{aligned}\det(\varphi_*(B)) &\overset{\substack{\text{by hypothesis}\\ \downarrow}}{=} \det\left(\varphi_*\left(A\cdot\begin{pmatrix}\mathrm{id}_r & 0 & 0\\ 0 & \lambda & 0\\ 0 & 0 & \mathrm{id}_s\end{pmatrix}\right)\right) &&\overset{\substack{\text{follows from the Matrix Multiplication–Change-of-Ring Lemma 14.2}\\ \text{together with the Matrix-of-Matrices Lemma 14.3}\\ \downarrow}}{=} \det\left(\varphi_*(A)\cdot\varphi_*\begin{pmatrix}\mathrm{id}_r & 0 & 0\\ 0 & \lambda & 0\\ 0 & 0 & \mathrm{id}_s\end{pmatrix}\right)\\ &= \det(\varphi_*(A))\cdot\det\left(\varphi_*\begin{pmatrix}\mathrm{id}_r & 0 & 0\\ 0 & \lambda & 0\\ 0 & 0 & \mathrm{id}_s\end{pmatrix}\right) &&= \det(\varphi_*(A))\cdot\det\begin{pmatrix}\mathrm{id}_{r\cdot n} & 0 & 0\\ 0 & \varphi_*(\lambda) & 0\\ 0 & 0 & \mathrm{id}_{s\cdot n}\end{pmatrix}\\ &= \det(\varphi_*(A))\cdot\det(\varphi_*(\lambda)).\end{aligned}$$

■

Next we recall the following notation from page 116:

Notation. Let S be a ring and let $M \in \mathrm{M}(l\times k, S)$ be a matrix.

(1) Given $i \in \{1,\dots,k\}$ we denote by M_i the $l\times(k-1)$-matrix which is the result of deleting the i-th column.
(2) Given $Z \subset \{1,\dots,l\}$ with $\#Z = k-1$ we denote by M^Z the $(k-1)\times k$-matrix that is given by the rows in Z.
(3) Let $i\in\{1,\dots,k\}$ and let $Z\subset\{1,\dots,l\}$ with $\#Z = k-1$. We consider the corresponding $(k-1)\times(k-1)$-matrix $M_i^Z := (M^Z)_i$.

Using the Determinant-after-Homomorphism Lemma 14.7 we can easily prove the following generalization of the Matrix–Quotient Lemma 8.8:

Lemma 14.8. (Generalized Matrix–Quotient Lemma) Let π be a group, let $k\in\mathbb{N}$, let $M\in\mathrm{M}((k-1)\times k,\mathbb{Z}[\pi])$ and and let $w = \begin{pmatrix} w_1\\ \vdots\\ w_k\end{pmatrix} \in \mathbb{Z}[\pi]^k = \mathrm{M}(k\times 1,\mathbb{Z}[\pi])$ be such that $M\cdot w = 0$. Furthermore let S be a commutative ring and let $\varphi\colon \pi\to \mathrm{GL}(n,S)$ be a homomorphism. For any $i,j\in\{1,\dots,k\}$ with $\det(\varphi_*(w_i))\neq 0$ and $\det(\varphi_*(w_j))\neq 0$ we have the equality

$$\frac{\det(\varphi_*(M_i))}{\det(\varphi_*(w_i))} \;=\pm\; \frac{\det(\varphi_*(M_j))}{\det(\varphi_*(w_j))}.$$

Proof of Lemma 14.8. To simplify the notation we now assume that $i=1$ and $j=2$. (Note though that here we pick up a sign indeterminacy.) We denote the k columns of M by $c_1,\dots,c_k$. Note that with this notation our hypothesis becomes

$$(*) \qquad c_1\cdot w_1 + c_2\cdot w_2 + \dots + c_k\cdot w_k \;=\; 0.$$

We now perform the following calculation:

$$\begin{aligned}
\det(\varphi_*(M_1))\cdot\det(\varphi_*(w_2)) &\overset{\substack{\text{by the Determinant-after-Homomorphism Lemma 14.7 (1)}\\ \downarrow}}{=} \det(\varphi_*(c_2\cdot w_2\ c_3\ \dots\ c_k))\\
&\underset{\substack{\uparrow\\ \text{follows from } (*)}}{=} \det(\varphi_*((-c_1\cdot w_1 - c_3\cdot w_3 - \dots - c_k\cdot w_k)\ c_3\ \dots\ c_k))\\
&\underset{\uparrow}{=} \det(\varphi_*(-c_1\cdot w_1\, c_3\ \dots\ c_k)) \underset{\uparrow}{=} (-1)^n\cdot\det(\varphi_*(M_2))\cdot\det(\varphi_*(w_1)).
\end{aligned}$$

this follows from the Determinant-after-Homomorphism Lemma 14.7 (1)

This of course gives us the desired result. ■

Proof of Theorem 14.4. Let π be a group, let $\Phi\colon \pi\to H$ be a non-trivial homomorphism to a free abelian group H, let R be a domain and let $\alpha\colon \pi\to \mathrm{GL}(n,R)$ be a homomorphism. Finally assume that we are given a presentation

$$P=\langle x_1,\dots,x_k\,|\,r_1,\dots,r_{k-1}\rangle \xrightarrow{\cong} \pi$$

of deficiency one.

(1) Since $x_1,\dots,x_k$ is a generating set of P and since $\Phi\colon P\to H$ is a non-trivial homomorphism we see that at least one $\Phi(x_i)$ is non-zero.
(2) Let $i\in\{1,\dots,k\}$ such that $\Phi(x_i)$ is non-trivial. It follows immediately from the example on page 199 that $\det((\Phi\otimes\alpha)_*(x_i-1))\neq 0$.
(3) We need to prove the following claim:

Claim. Let $i,j\in\{1,\dots,n\}$ such that $\Phi(x_i)$ and $\Phi(x_j)$ are non-trivial. Then
$$\frac{\det((\Phi\otimes\alpha)_*(J(P)_i))}{\det((\Phi\otimes\alpha)_*(x_i-1))} \;=\; \pm\frac{\det((\Phi\otimes\alpha)_*(J(P)_j))}{\det((\Phi\otimes\alpha)_*(x_j-1))}\;\in\; Q_R(H)$$

Proof. First recall that by the Jacobi Matrix–Vector Lemma 8.7 we know that
$$J(P)\cdot\begin{pmatrix}1-\mu_P(x_1)\\ \vdots\\ 1-\mu_P(x_k)\end{pmatrix} \;=\; 0.$$
The claim now follows from the Generalized Matrix–Quotient Lemma 14.8.

(4) We now assume that R is a UFD. We need to show that the definition of $\Delta^{\alpha}_{\pi,\Phi}$ (up to "$\doteq_R$") does not depend on the choice of the presentation. As we will see, the proof of this statement is very similar to the proof of the Alexander Function Theorem 8.5.

First we need to extend our definition of a twisted Alexander function to presentations that are not necessarily of deficiency one. Thus let $P=\langle x_1,\dots,x_k\,|\,r_1,\dots,r_l\rangle$ be any finite presentation for π and let $i\in\{1,\dots,k\}$ such that $\Phi(x_i)$ is non-trivial. Similar to the Independence-of-Generator Lemma 8.10 we consider[99]

$$\Delta^{\alpha}_{P,\Phi} \;:=\; \frac{\gcd_{R[H]}\{\det((\Phi\otimes\alpha)_*(J(P)^Z_i))\}_{\substack{Z\subset\{1,\dots,l\}\\ \#Z=k-1}}}{\det((\Phi\otimes\alpha)_*(x_i-1))}\;\in\; Q_R(H)$$

[99]Here we use that R is a UFD. More precisely, since R is a UFD and since H is a free abelian group it follows from [**Lan93**, Theorem IV.23] that $R[H]$ is a unique factorization domain. Thus it makes sense to talk of the greatest common divisor in this context.

and we see that it is well-defined (i.e. independent of the choice of i) up to multiplication by a sign.

Now let P and Q be two deficiency-one presentations for π. Recall that by the Tietze Theorem 8.9 we know that the two presentations $\gamma\colon \pi \to P$ and $\delta\colon \pi \to Q$ are related by a finite sequence of Tietze transformations, as defined on page 115. So it suffices to deal with the case that the presentations P and Q are related by a single Tietze transformation.

(T1) We consider the following situation: We have

$$\underbrace{\langle x_1,\dots,x_k \mid r_1,\dots,r_l,s\rangle}_{=Q} \;=\; \underbrace{\langle x_1,\dots,x_k \mid r_1,\dots,r_l\rangle}_{=P} \xrightarrow{\gamma} \pi$$

where $s \in \langle\langle r_1,\dots,r_l\rangle\rangle \subset \langle X\rangle$. We see that the Jacobi matrix $J(Q)$ equals the Jacobi matrix $J(P)$, except that we are adding the row $\frac{\partial s}{\partial x_i}$, $i=1,\dots,k$. But since $s \in \langle\langle r_1,\dots,r_l\rangle\rangle$ one obtains easily from the Fox Derivative Properties Lemma 8.2 (5) that there exist $\lambda_1,\dots,\lambda_l \in \mathbb{Z}[P]$ such that for every $i \in \{1,\dots,k\}$ we have

$$\frac{\partial s}{\partial x_i} \;=\; \sum_{j=1}^{l} \lambda_j \cdot \frac{\partial r_j}{\partial x_i}.$$

In other words the extra row of $J(Q)$ is just a "left-linear" combination of the previous rows.

Claim. Let $Z \subset \{1,\dots,l\}$ with $\#Z = k-2$. For any $i \in \{1,\dots,k\}$ we have

$$\det\big((\Phi\otimes\alpha)_*\big(J(Q)_i^{Z\cup\{l+1\}}\big)\big) \;=\; \sum_{\substack{j=1\\ j\notin Z}}^{l} \det(\Phi_*(\lambda_j))\cdot\det\big((\Phi\otimes\alpha)_*\big(J(Q)_i^{\{Z\cup\{j\}}\big)\big).$$

Proof. First note that by the above discussion we have

$$(l+1)\text{-st row of } J(Q) \;=\; \sum_{j=1}^{l} \lambda_j \cdot j\text{-th row of } J(Q).$$

The claim follows easily from this observation and the Determinant-after-Homomorphism Lemma 14.7 (2). ⊞

Finally note that for any $i \in \{1,\dots,k\}$ we have

$$\begin{aligned}
&\gcd_{R[H]}\{\det(J((\Phi\otimes\alpha)_*(Q_i^Z)))\}_{\substack{Z\subset\{1,\dots,l+1\}\\ \#Z=k-1}} \doteq_R\\
&\underset{\substack{\uparrow\\ \text{follows easily from}\\ \text{the above claim}}}{\doteq_R} \gcd_{R[H]}\{\det(J((\Phi\otimes\alpha)_*(Q_i^Z)))\}_{\substack{Z\subset\{1,\dots,l\}\\ \#Z=k-1}}\\
&\underset{\substack{\uparrow\\ \text{since the first } l \text{ rows}\\ \text{of } J(Q) \text{ and } J(P) \text{ agree}}}{\doteq_R} \gcd_{R[H]}\{\det(J((\Phi\otimes\alpha)_*((P_i^Z))\}_{\substack{Z\subset\{1,\dots,l\}\\ \#Z=k-1}}.
\end{aligned}$$

This calculation implies that $\Delta^{\alpha}_{Q,\Phi\circ\gamma} \doteq_R \Delta^{\alpha}_{P,\Phi\circ\gamma} \in Q_R(H)$.

(T2) We consider the following situation: We have $x \notin \{x_1,\dots,x_k\}$, $s,t \in \langle x_1,\dots,x_k\rangle$ and the presentations

$$\begin{array}{rcl} \overbrace{\langle x_1,\dots,x_k,x \mid r_1,\dots,r_l,s\cdot x\cdot t\rangle}^{=Q} & \xrightarrow{\Theta} & \overbrace{\langle x_1,\dots,x_k \mid r_1,\dots,r_l\rangle}^{=P} \xrightarrow{\gamma} \pi. \\ x_i & \mapsto & x_i \\ x & \mapsto & s^{-1}\cdot t^{-1} \end{array}$$

We see that $$J(Q) = \begin{pmatrix} J(P) & 0 \\ * & s \end{pmatrix}.$$

Finally note that for any $i \in \{1,\dots,k\}$ we have

$$\begin{array}{rl} \gcd_{R[H]}\{\det((\Phi\otimes\alpha)_*(J(Q)_i^Z))\}_{\substack{Z\subset\{1,\dots,l+1\}\\ \#Z=k}} & \doteq_R \\ \doteq_R & \gcd_{R[H]}\{\det((\Phi\otimes\alpha)_*(J(Q)_i^Z))\}_{\substack{Z\subset\{1,\dots,l+1\}\\ \#Z=k,\ l+1\in Z}} \\ \uparrow & \\ \text{by the above description of } J(Q) \text{ the} & \\ \text{determinant is zero, unless } l+1\in Z & \\ \doteq_R & \gcd_{R[H]}\{\det((\Phi\otimes\alpha)_*(J(P)_i^Z))\}_{\substack{Z\subset\{1,\dots,l\}\\ \#Z=k-1}}. \\ \uparrow & \\ \text{since } \det((\Phi\otimes\alpha)(s)) = \Phi(s)^n\cdot\det(\alpha(s)) & \\ \text{is a unit in } R[H] & \end{array}$$

This calculation implies that $\Delta^\alpha_{Q,\Phi\circ\gamma\circ\Theta} \doteq_R \Delta^\alpha_{P,\Phi\circ\gamma} \in Q_R(H)$. ■

14.4. Twisted Alexander functions corresponding to representations.

In this final section of this chapter we will generalize our framework a little bit. We start out with a very general definition, which appears in many guises throughout large parts of mathematics:

Definition. Let π be a group and let R be a commutative ring.

(1) An R-representation of π consists of a finitely generated free R-module M and a homomorphism $\pi \to \operatorname{Aut}_R(M)$.
(2) We say that two representations $\varphi\colon \pi \to \operatorname{Aut}_R(M)$ and $\psi\colon \pi\to\operatorname{Aut}_R(N)$ are isomorphic if there exists an isomorphism $\Theta\colon M\to N$ of R-modules such that for each $g\in\pi$ we have $\varphi(g) = \Theta\circ\psi(g)\circ\Theta^{-1}\colon M\to M$.

Example. Let R be a commutative ring.

(1) Let G be any group. The map
$$\begin{array}{rcl} \tau_n = \tau_n^R\colon G & \to & \operatorname{Aut}_R(R^n) \\ g & \mapsto & \mathrm{id}_{R^n} \end{array}$$
is called the trivial n-dimensional R-representation of G. For $R=\mathbb{Z}$ we write $\tau_n := \tau_n^{\mathbb{Z}}$.
(2) Let G be a finite group. We consider the group ring $R[G]$. It follows easily from the definitions that the map
$$\begin{array}{rcl} \rho_G^R\colon \pi & \to & \operatorname{Aut}_R(R[G]) \\ g & \mapsto & \begin{pmatrix} R[G] & \to & R[G] \\ p & \mapsto & g\cdot p \end{pmatrix} \end{array}$$
defines an R-representation of π. We call it the regular R-representation of G. For $R=\mathbb{Z}$ we write $\rho_G := \rho_G^{\mathbb{Z}}$.

(3) Let $n \in \mathbb{N}$.

- The map
$$\begin{array}{rcl} \sigma_n^R\colon S_n & \to & \operatorname{Aut}_R(R^n) \\ \sigma & \mapsto & \left(\begin{array}{rcl} R^n & \to & R^n \\ (v_1,\dots,v_n) & \mapsto & (v_{\sigma(1)},\dots,v_{\sigma(n)}) \end{array} \right) \end{array}$$
is clearly an R-representation of the permutation group S_n. We refer to it as the standard R-representation of the permutation group S_n. For $R = \mathbb{Z}$ we write $\sigma_n := \sigma_n^{\mathbb{Z}}$.
- We set $(R^n)_0 := \{(v_1,\dots,v_n) \in R^n \mid v_1 + \dots + v_n = 0\}$. It is clear that $(R^n)_0$ is preserved under permutation of the coordinates. Thus we obtain a well-defined representation
$$\begin{array}{rcl} \sigma_{n,0}^R\colon S_n & \to & \operatorname{Aut}_R((R^n)_0) \\ \sigma & \mapsto & \left(\begin{array}{rcl} (R^n)_0 & \to & (R^n)_0 \\ (v_1,\dots,v_n) & \mapsto & (v_{\sigma(1)},\dots,v_{\sigma(n)}) \end{array} \right). \end{array}$$
We refer to this representation as the reduced standard R-representation of the permutation group. For $R = \mathbb{Z}$ we write $\sigma_{n,0} := \sigma_{n,0}^{\mathbb{Z}}$. □

The following lemma shows that representations are essentially the same as homomorphisms to matrix groups:

Lemma 14.9. (Representation Classification Lemma) Let π be a group, let R be a commutative ring and let $n \in \mathbb{N}_0$.

(1) Let M be a free R-module of rank n and let $v_1,\dots,v_n$ be a basis of M. The map
$$\begin{array}{rcl} \operatorname{Aut}_R(M) & \to & \operatorname{GL}(n,R) \\ \varphi & \mapsto & \begin{array}{l} \text{matrix representing}^{100}\ \varphi\colon M \to M \text{ with} \\ \text{respect to the ordered basis } (v_1,\dots,v_n) \end{array} \end{array}$$
is an isomorphism.

(2) The isomorphism from (1) defines a natural bijection between the set of isomorphism classes of representations $\pi \to \operatorname{Aut}_R(M)$, where M is any free R-module of rank n, and conjugacy classes of homomorphisms $\pi \to \operatorname{GL}(n,R)$.

Proof. Both statements follow easily from the definitions. We refer to the reader's preferred linear algebra book for details. ■

The Conjugation Invariance Lemma 14.6 allows us to extend our concept of the twisted Alexander function:

Definition. Let π be a group which admits a presentation of deficiency one, furthermore let $\Phi\colon \pi \to H$ be a non-trivial homomorphism to a free abelian group H, let R be a UFD and let $\alpha\colon \pi \to \operatorname{Aut}_R(M)$ be an R-representation. We pick a basis $v_1,\dots,v_n$ of M and we consider the corresponding homomorphism $\widetilde{\alpha}\colon \pi \to \operatorname{GL}(R,n)$. Note that it follows from the Representation Classification Lemma 14.9 and the Conjugation Invariance Lemma 14.6 that
$$\Delta_{\pi,\Phi}^{\alpha} := \Delta_{\pi,\Phi}^{\widetilde{\alpha}} \in Q_R(H)$$
does not depend on the choice of the basis $v_1,\dots,v_n$ and that in fact it does not even depend on the isomorphism type of the representation. We refer to $\Delta_{\pi,\Phi}^{\alpha}$ as the twisted

[100]We refer to page ?? for the definition of this matrix.

Alexander function of (π, Φ, α). By the Twisted Alexander Function Theorem 14.4 the twisted Alexander function is well-defined up to "$\doteq_R$".

This formulate our next lemma we will need the following definition:

Definition. Let π be a group, let R be a commutative ring and let $\alpha\colon \pi \to \operatorname{Aut}_R(M)$ and $\beta\colon \pi \to \operatorname{Aut}_R(N)$ be two R-representations. We define the direct sum representation

$$\begin{array}{rcl} \alpha \oplus \beta\colon \pi & \to & \operatorname{Aut}_R(M \oplus N) \\ g & \mapsto & \left(\begin{array}{rcl} M \oplus N & \to & M \oplus N \\ (m,n) & \mapsto & (\alpha(g) \cdot m, \beta(g) \cdot n). \end{array}\right) \end{array}$$

Example. Let $n \in \mathbb{N}_0$. In Exercise 14.2 we will see that the direct sum of the reduced standard $\mathbb{Q}$-representation $\sigma_{n,0}^{\mathbb{Q}}$ of S_n and the trivial one-dimensional $\mathbb{Q}$-representation $\tau_1^{\mathbb{Q}}$ of S_n is isomorphic to the standard representation $\sigma_n^{\mathbb{Q}}$ of S_n. □

Lemma 14.10. (Direct Sum–Twisted Alexander Function Lemma) Let π be a group which admits a presentation of deficiency one, let $\Phi\colon \pi \to H$ be a non-trivial homomorphism to a free abelian group H, let R be a UFD and let $\alpha\colon \pi \to \operatorname{Aut}_R(M)$ and $\beta\colon \pi \to \operatorname{Aut}_R(N)$ be two R-representations. Then

$$\Delta_{\pi,\Phi}^{\alpha\oplus\beta} \doteq \Delta_{\pi,\Phi}^{\alpha} \cdot \Delta_{\pi,\Phi}^{\beta}.$$

Sketch of proof. The lemma follows easily from the definitions and the observation that the determinant of a diagonal block matrix $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ is just the product of the determinants of A and B. We leave it to the reader to fill in the details. ■

Notation. Let R be a commutative ring.

(1) We denote by R^* the group of multiplicative units of R.

Let H be an abelian group and let $\rho\colon H \to R^*$ be a homomorphism.

(2) We consider the map
$$\begin{array}{rcl} \rho_*\colon R[H] & \to & R[H] \\ \sum\limits_{i=1}^{k} r_i \cdot h_i & \mapsto & \sum\limits_{i=1}^{k} (\rho(h_i) \cdot r_i) \cdot h_i. \end{array}$$
One can easily verify that this map is a ring homomorphism.

(3) Given a group π and a homomorphism $\alpha\colon \pi \to \operatorname{Aut}_R(M)$ we consider the map
$$\begin{array}{rcl} \rho \cdot \alpha\colon \pi & \to & \operatorname{Aut}_R(M) \\ g & \mapsto & (m \mapsto \rho(g) \cdot \alpha(g)). \end{array}$$
One can easily verify that this map is a group homomorphism.

Lemma 14.11. (Modified Twisted Alexander Polynomial Lemma) Let π be a group which admits a presentation of deficiency one, let $\Phi\colon \pi \to H$ be a non-trivial homomorphism to a free abelian group H, let R be a UFD and let $\alpha\colon \pi \to \operatorname{Aut}_R(M)$ be an R-representation. Furthermore let $\rho\colon H \to R^*$ be a homomorphism. With the above notation we have
$$\Delta_{\pi,\Phi}^{(\rho\circ\Phi)\cdot\alpha} \doteq \rho_*(\Delta_{\pi,\Phi}^{\alpha}).$$

We conclude this chapter with a short discussion when the twisted Alexander function is actually a polynomial. To formulate the corresponding result we will need the following definition from representation theory.

Definition. Let π be a group, let $\mathbb{F}$ be a field and let $\alpha\colon \pi \to \operatorname{Aut}_{\mathbb{F}}(M)$ be a representation. We say that the representation is reducible if there exists a subspace $V \subset M$ with $0 \subsetneq V \subsetneq M$ such that for every $g \in \pi$ we have $\alpha(g)(V) = V$. Otherwise we say that the representation is irreducible.

Now we can formulate the last result of this chapter:

Proposition 14.12. (Twisted Alexander Function–Polynomial Proposition) Let π be a group which admits a presentation of deficiency one, let $\Phi\colon \pi \to H$ be a non-trivial homomorphism to a free abelian group H, let $\mathbb{F}$ be a UFD and let $\alpha\colon \pi \to \operatorname{Aut}_R(M)$ be a representations. We assume that one of the following conditions is satisfied:

(i) The image of $\Phi\colon \pi \to H$ is a free abelian group of rank ≥ 2.
(ii) The ring R is a field, the representation α is irreducible and the restriction of α to $\ker(\Phi)$ is non-trivial.

Then $\Delta^{\alpha}_{\pi,\Phi} \in R[H]$.

Sketch of proof. If (i) holds, then it follows by the same argument as in proof of the Link–Alexander Polynomial Proposition 9.2 (2) that $\Delta^{\alpha}_{\pi,\Phi} \in R[H]$. We will not really make use of the case (ii). We will thus not provide a proof. The statement that under the hypothesis (ii) we have $\Delta^{\alpha}_{\pi,\Phi} \in R[H]$ follows from [**FKK12**, Proposition 9.3] and the discussion in [**FV11a**, Chapter 3.3]. ∎

Exercises for Chapter 14.

Exercise 14.1. As in the Baumslag-Solitar Group Proposition **??** we consider the Baumslag-Solitar group

$$\operatorname{BS}(1,2) \;=\; \langle x, y \,|\, x^{-1}\cdot y \cdot x \;=\; y^2\rangle.$$

(a) Show that $\operatorname{BS}(1,2)$ admits (up to conjugation) a unique epimorphism $\varphi\colon\ \operatorname{BS}(1,2) \to S_3$.
(b) Let $\Phi\colon\ \operatorname{BS}(1,2) \to \langle t\rangle$ be the epimorphism that is given by $\Phi(x) = t$ and $\Phi(y) = 1$. Furthermore let $\theta\colon S_3 \to \operatorname{GL}(2,\mathbb{Z})$ be the representation from page 203. Compute the twisted Alexander function $\Delta^{\theta\circ\varphi}_{\operatorname{BS}(1,2),\Phi}$.

Exercise 14.2. Let $n \in \mathbb{N}_0$.

(a) Show that the direct sum of the reduced standard $\mathbb{Q}$-representation $\sigma^{\mathbb{Q}}_{n,0}$ of S_n and the trivial one-dimensional $\mathbb{Q}$-representation $\tau^{\mathbb{Q}}_1$ of S_n is isomorphic to the standard $\mathbb{Q}$-representation $\sigma^{\mathbb{Q}}_n$ of S_n.
(b) Does the conclusion of (a) also hold if we replace $\mathbb{Q}$ by $\mathbb{Z}$?
(c) Does the conclusion of (a) also hold if we replace $\mathbb{Q}$ by $\mathbb{F}_p$ for some prime p?

Exercise 14.3. Let G be a finite group. We denote by $\sigma_{G,\mathbb{Q}}\colon G \to \operatorname{Aut}_{\mathbb{Q}}(\mathbb{Q}[G])$ the regular $\mathbb{Q}$-representation. Show that there exists a representation $\alpha\colon G \to \operatorname{Aut}_{\mathbb{Q}}(\mathbb{Q}[G])$ such that $\sigma_{G,\mathbb{Q}} \cong \tau_{G,\mathbb{Q}} \oplus \alpha$.

Exercise 14.4. Let π be a group that admits a presentation of deficiency one, let $\Phi\colon \pi \to H$ be an epimorphism onto a free abelian group H with $\operatorname{rank}(H) \geq 2$, let $\alpha\colon \pi \to G$ be a homomorphism to a finite group G and let $\rho^{\mathbb{Q}}_G\colon \pi \to \operatorname{Aut}_{\mathbb{Q}}(\mathbb{Q}[G])$ be the corresponding regular representation. Show that there exists an $f \in R[H]$ such that

$$\Delta^{\rho^{\mathbb{Q}}_G}_{\pi,\Phi} \;=\; \Delta_{\pi,\Phi}\cdot f.$$

Hint. Make use of Exercise 14.3.

Exercise 14.5. Let π be a group, let $\mathbb{F}$ be a field and let $\alpha\colon \pi \to \mathrm{GL}(n,\mathbb{F})$ be a representation. Show that the representation is reducible if and only if there exist $r, s \in \{1,\dots,n-1\}$ with $r+s=n$, maps $A\colon \pi \to \mathrm{GL}(r,\mathbb{F})$, $B\colon \pi \to \mathrm{M}(r\times s,\mathbb{F})$ and $C\colon \pi \to \mathrm{GL}(s,\mathbb{F})$ and a matrix $P \in \mathrm{GL}(n,\mathbb{F})$ such that for every $g \in \pi$ we have

$$P \cdot \alpha(g) \cdot P^{-1} \;=\; \begin{pmatrix} A(g) & B(g) \\ 0 & C(g) \end{pmatrix}.$$

CHAPTER 15

The twisted Alexander polynomial of knots and links

Let R be a UFD. In Chapter 14 we associated to a group π (assuming it admits a presentation of deficiency one) together with a non-trivial homomorphism $\pi \to H$ to a free abelian group and a representation $\pi \to \operatorname{Aut}_R(M)$ the twisted Alexander function $\Delta^{\alpha}_{\pi,\Phi} \in Q_R(H)$. Our favorite groups of deficiency one are the fundamental groups $\pi_1(S^3 \setminus L)$ of link complements. In this section we will thus use the general theory of twisted Alexander functions to introduce the twisted Alexander polynomials $\Delta^{\alpha}_{L} \in Q_R(t_1,\dots,t_m)$ of any oriented m-component link together with a representation $\pi_1(S^3 \setminus L) \to \operatorname{Aut}_R(M)$.

This chapter is organized as follows:

(1) In Section 15.1 we will introduce twisted Alexander polynomials of oriented links and we will study basic properties, e.g. we will study the effect of reversing orientations and of taking the mirror.
(2) In Section 15.2 we will see that twisted Alexander polynomials can be used to distinguish knots, in particular they can be used to distinguish the Kinoshita-Terasaka knot from the Conway knot.
(3) In Section 15.3 we will see that twisted Alexander polynomials can be used to give lower bounds on the knot genus. We will use this method to determine the rather elusive genera of the Kinoshita-Terasaka knot and of the Conway knot.
(4) In Section 15.4 we will see that twisted Alexander polynomials can be used to give new obstructions to a knot being fibered.
(5) In the Appendix 15.5 we will discuss the question, for which representations twisted Alexander polynomials of links are symmetric. We will only state a symmetry result from the literature, without proving it.

15.1. **Twisted Alexander polynomials of oriented knots and links.** On page 124 we used Alexander functions to introduce the Alexander polynomial of an oriented link. In a very similar fashion we now use twisted Alexander functions to define the twisted Alexander polynomial of an oriented link:

Definition. Let $m \in \mathbb{N}$, let $L \subset S^3$ be an oriented m-component link, let R be a UFD, let M be a free R-module and let $\pi_1(S^3 \setminus L) \to \operatorname{Aut}_R(M)$ be a representation. Recall that by the Link Group-Abelianization Corollary 5.7 there exists a unique epimorphism

$$\Phi_L\colon \pi_1(S^3 \setminus L) \;\to\; \langle t_1,\dots,t_m\rangle_{\mathrm{ab}}$$

such that for $i = 1,\dots,m$ we have $\Phi_L(\mu_{L_i}) = t_i$. By the Wirtinger Presentation Proposition 5.5 we know that $\pi_1(S^3 \setminus L)$ admits a presentation of deficiency one. Thus it makes sense to define[101]

$$\Delta_L^\alpha \;=\; \Delta_L^\alpha(t_1,\dots,t_m) \;:=\; \underbrace{\Delta^\alpha_{\pi_1(S^3\setminus L),\Phi_L}}_{\substack{\text{as defined on}\\ \text{page 211}}} \;\in\; Q_R(t_1,\dots,t_m)$$

as the twisted Alexander polynomial of L.

Example. Let $m \in \mathbb{N}$ and let $L \subset S^3$ be an oriented m-component link. We denote by $\tau\colon \pi_1(S^3 \setminus L) \to \mathrm{GL}(1,\mathbb{Z})$ the trivial one-dimensional representation. It follows from the discussion on page 200 and the definition on page 121 of the Alexander polynomial of a knot that

$$\Delta_L^\tau(t_1,\dots,t_m) \;=\; \begin{cases} \frac{1}{t-1}\cdot \Delta_K(t), & \text{if } m=1,\\ \Delta_L(t_1,\dots,t_m), & \text{if } m \geq 2.\end{cases}$$

In particular we see that the twisted Alexander polynomial of a knot is - alas - not necessarily a polynomial. Fortunately the Link–Twisted Alexander Polynomial Proposition 15.1 (2) shows that in almost all other cases the twisted Alexander polynomial of a link is in fact a polynomial. □

The following proposition can be viewed as an analogue of the Knot–Alexander Polynomial Proposition 9.1 and the Link–Alexander Polynomial Proposition 9.2:

Proposition 15.1. (Link–Twisted Alexander Polynomial Proposition) Let $m \in \mathbb{N}$, let L be oriented m-component link L, let R be a UFD and let $\alpha\colon \pi_1(S^3 \setminus L) \to \mathrm{Aut}_R(M)$ be a representation.

(1) (a) The twisted Alexander polynomial $\Delta_L^\alpha(t_1,\dots,t_m)$ is well-defined up to "$\doteq_R$", i.e. up to multiplication by a unit in R and up to multiplication by a monomial $t_1^{d_1}\cdot\dots\cdot t_m^{d_m}$.
 (b) If $\beta\colon \pi_1(S^3 \setminus L) \to \mathrm{Aut}_R(N)$ is a representation that is isomorphic to α, in the sense of the definition on page 209, then $\Delta_L^\alpha(t_1,\dots,t_m) \doteq_R \Delta_L^\beta(t_1,\dots,t_m)$.

(2) We assume that we are in one of the following settings:
 (i) We have $m \in \mathbb{N}_{\geq 2}$.
 (ii) The ring R is a field, the representation is irreducible and the image of α is a non-abelian subgroup of $\mathrm{Aut}_R(M)$.

 Then the twisted Alexander polynomial Δ_L^α is an element of $R[t_1^{\pm 1},\dots,t_m^{\pm 1}]$.

(3) If $\widetilde{L}$ is an oriented m-component link that is smoothly isotopic to L, then there exists a representation $\widetilde{\alpha}\colon \pi_1(S^3 \setminus L) \to \mathrm{Aut}_R(M)$ with $\Delta_L^\alpha \doteq_R \Delta_{\widetilde{L}}^{\widetilde{\alpha}}$.

Sketch of proof.

(1) (a) This statement is just a special case of the Twisted Alexander Function Theorem 14.4 (4).
 (b) This statement follows immediately from the Representation Classification Lemma 14.9 together with the Conjugation Invariance Lemma 14.6.

(2) This statement follows immediately from the Twisted Alexander Function–Polynomial Proposition 14.12.

(3) Let $\widetilde{L}$ be an oriented m-component link that is smoothly isotopic to L. As in the Isotopic Link-π_1-Lemma 3.2 we note that it follows from the Link–Smooth Isotopy Proposition 2.3 that there exists an orientation-preserving diffeomorphism $f\colon S^3\setminus L \to S^3\setminus \widetilde{L}$. It follows easily from the Meridian Proposition 2.17 that for each $i \in \{1,\dots,m\}$ the image

[101] It follows from the Conjugation Invariance Lemma 14.6 that we can ignore base points.

of a meridian of L_i is a meridian of $\widetilde{L}_i$. This shows that there exists an isomorphism $f_*\colon \pi_1(S^3\setminus L)\to\pi_1(S^3\setminus\widetilde{L})$ such that the following diagram commutes:

$$\begin{tikzcd} \pi_1(S^3\setminus L) \arrow[rr, "f_*", "\cong"'] \arrow[dr, "\Phi_L"'] & & \pi_1(S^3\setminus \widetilde{L}) \arrow[dl, "\Phi_{\widetilde{L}}"] \\ & \langle t_1,\dots,t_m\rangle_{\mathrm{ab}}. & \end{tikzcd}$$

Given a homomorphism $\alpha\colon \pi_1(S^3\setminus L)\to \mathrm{Aut}_R(M)$ we set $\widetilde{\alpha}:=\alpha\circ(f_*)^{-1}$. The desired statement now follows from the Twisted Alexander Function–Functorial Lemma 14.5 (1). ■

The following elementary lemma is a generalization of the Alexander Polynomial–Orientation Lemma 9.3:

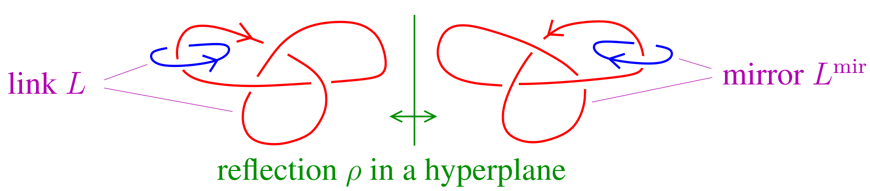
Lemma 15.2. (Alexander Polynomial–Orientation Lemma) Let $m\in\mathbb{N}$ and let L be an oriented m-component link L. Given $\epsilon_1,\dots,\epsilon_m\in\{-1,1\}$ we denote by $L^{(\epsilon_1,\dots,\epsilon_m)}$ the link L but where we flipped the orientation of the i-th component precisely if $\epsilon_i=-1$. For any UFD R and any representation $\alpha\colon\pi_1(S^3\setminus L)\to\mathrm{Aut}_R(M)$ we have

$$\Delta^{\alpha}_{L^{(\epsilon_1,\dots,\epsilon_m)}}(t_1,\dots,t_m)\ \doteq_R\ \Delta^{\alpha}_L(t_1^{\epsilon_1},\dots,t_m^{\epsilon_m}).$$

Sketch of proof. The proof of the lemma is basically the same as the proof of the Alexander Polynomial–Orientation Lemma 9.3. The only real difference is that we need to replace the Alexander Function–Functorial Lemma 8.6 (2) by the Twisted Alexander Function–Functorial Lemma 14.5 (2). We leave it to the reader to fill in the details. ■

Now let L be an oriented m-component link. As on page 18 we consider the mirror L^{mir}. For the untwisted Alexander polynomial we know that

$$\Delta_{L^{\mathrm{mir}}}(t_1,\dots,t_m)\ \underset{\substack{\uparrow\\ \text{by the Alexander Polynomial}\\ \text{Mirror Lemma 9.9}}}{\doteq}\ \Delta_L(t_1^{-1},\dots,t_m^{-1})\ \underset{\substack{\uparrow\\ \text{by the Alexander Polynomial}\\ \text{Symmetry Theorem 9.4}}}{\doteq}\ \Delta_L(t_1,\dots,t_m).$$



We have the following analogue of the Alexander Polynomial–Mirror Lemma 9.9:

Lemma 15.3. (Twisted Alexander Polynomial–Mirror Lemma) Let L be an oriented m-component link, let R be a UFD and let $\alpha\colon\pi_1(S^3\setminus L\to\mathrm{Aut}_R(M)$ be a representation. Let $\rho\colon S^3\to S^3$ be a reflection in a hyperplane. We consider the corresponding mirror $L^{\mathrm{mir}}=\rho(L)$. Then

$$\Delta^{\alpha\circ\rho_*}_{\rho(L)}(t_1,\dots,t_m)\ \doteq\ \Delta^{\alpha}_L(t_1^{-1},\dots,t_m^{-1}).$$

Sketch of proof. The proof of the lemma is basically the same as the proof of the Alexander Polynomial–Mirror Lemma 9.9. The only real difference is that we need to replace the Alexander Function–Functorial Lemma 8.6 (1) by the Twisted Alexander Function–Functorial Lemma 14.5 (1). We leave it to the reader to fill in the details. ■

15.2. **Distinguishing knots and links.** Since the twisted Alexander polynomial is an invariant of an m-component link *together* with a representation, it is a priori not clear how one could use twisted Alexander polynomials to distinguish smooth isotopy classes of oriented links.

In this section we will see that there are workarounds for this issue. In particular the following slight refinement of the Link–Twisted Alexander Polynomial Proposition 15.1 (3) will turn out to be helpful:

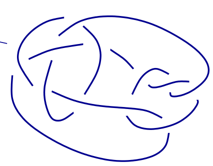
Proposition 15.1. (Link–Twisted Alexander Polynomial Proposition) Let $m \in \mathbb{N}$, let L be oriented m-component link L, let G be a group and let $\varphi\colon \pi_1(S^3 \setminus L) \to G$ be a homomorphism.

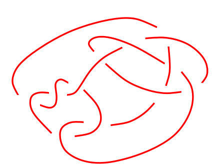
(3′) If $\widetilde{L}$ is an oriented m-component link that is smoothly isotopic to L, then there exists a homomorphism $\widetilde{\varphi}\colon \pi_1(S^3 \setminus L) \to \widetilde{G}$ with $\varphi(\pi_1(S^3 \setminus L)) = \widetilde{\varphi}(\pi_1(S^3 \setminus \widetilde{L}))$ such that for any UFD R and any representation $\alpha\colon G \to \operatorname{Aut}_R(M)$ we have
$$\Delta_L^{\alpha\circ\varphi} \;\doteq_R\; \Delta_{\widetilde{L}}^{\alpha\circ\widetilde{\varphi}}.$$

Proof. The proof of this statement is almost identical to the proof of the Link–Twisted Alexander Polynomial Proposition 15.1 (3). ■

Example. Let C be the Conway knot and let K be the Kinoshita-Terasaka knot, that we introduced on page 100.

Conway knot C





Kinoshita-Terasaka knot K

On page 100 we managed to distinguish these two knots using the somewhat naive but powerful knot coloring polynomial. On the other hand we saw on page 130 that the Alexander polynomial is in both cases $= 1$. Thus the Alexander polynomial cannot distinguish these knots, and it cannot say anything interesting about these knots.

In both cases one can verify that, up to conjugation, there exists a unique epimorphism $\varphi_C\colon \pi_1(S^3 \setminus C) \to A_5 \subset S_5$ and a unique epimorphism $\varphi_K\colon \pi_1(S^3 \setminus K) \to A_5 \subset S_5$. Next let $\sigma_5\colon A_5 \to \mathrm{GL}(5,\mathbb{Z})$ be the permutation representation which we introduced on page 210. One calculates that
$$\begin{aligned} \Delta_C^{\sigma_5\circ\varphi_C}(t) &= ????\\ \Delta_K^{\sigma_5\circ\varphi_K}(t) &= ??\end{aligned}$$
Since calculations over the integers can quickly get out of hand, it is often more efficient to work with a finite field instead of $\mathbb{Z}$. For example, if we replace $\mathbb{Z}$ by $\mathbb{F}_7$, i.e. if we use the representation $\sigma_5^{\mathbb{F}_7}\colon A_5 \to \mathrm{GL}(5,\mathbb{F}_7)$, then we obtain
$$\begin{aligned} \Delta_C^{\sigma_5^{\mathbb{F}_7}\circ\varphi_C}(t) &= 1+t+4t^2+6t^3+t^5+2t^6+5t^7+2t^8+t^9+6t^{11}+4t^{12}+t^{13}+t^{14},\\ \Delta_K^{\sigma_5^{\mathbb{F}_7}\circ\varphi_K}(t) &= 1+5t+6t^2+5t^3+t^4+2t^5+2t^6+2t^7+t^8+5t^9+6t^{10}+5t^{11}+t^{12}.\end{aligned}$$

It follows from these calculations, the Link–Twisted Alexander Polynomial Proposition 15.1 (3′) and the Link–Twisted Alexander Polynomial Proposition 15.1 (1b) that the two knots are *not* smoothly isotopic. $\square$

15.3. **Twisted Alexander polynomials and the knot genus.** In the Genus–Alexander Polynomial Proposition 13.1 we showed that the degree of the Alexander polynomial of a knot gives a lower bound on $2 \cdot \operatorname{genus}(K)$. In this section we will generalize this statement to twisted Alexander polynomials.

To do so we first of all need to slightly generalize the notion of the degree of a Laurent polynomial (which we first introduced on page 185):

Definition. Let R be a domain.

(1) Given a Laurent polynomial $p(t) = \sum\limits_{i=r}^{s} a_i \cdot t^i \in R[t^{\pm 1}]$ with $a_r \neq 0$ and $a_s \neq 0$ we define its degree as[102]
$$\deg(p(t)) \;=\; \deg\left(\sum_{i=r}^{s} a_i \cdot t^i\right) \;=\; s - r.$$
We extend this definition to $\deg(0) = -\infty$.

(2) Given $f(t) \in Q_R(t)$ we write $f(t) = \frac{p(t)}{q(t)}$ with $p(t), q(t) \in R[t^{\pm 1}]$ and we set
$$\deg(f(t)) \;=\; \deg(p(t)) - \deg(q(t)).$$
One can easily verify that this degree is well-defined. It is also elementary to see that if we are given $f(t), g(t) \in Q_R(t)$ with $f(t) \doteq_R g(t)$, then $\deg(f(t)) = \deg(g(t))$.

Example. Let K be an oriented knot and let $\tau\colon \pi_1(S^3 \setminus K) \to \mathrm{GL}(1, \mathbb{Z})$ be the trivial 1-dimensional representation. Then
$$\deg(\Delta_K^\tau(t)) \underset{\substack{\uparrow \\ \text{by the discussion on page 215}}}{=} \deg\left(\frac{\Delta_K(t)}{t-1}\right) \;=\; \deg(\Delta_K(t)) - \deg(t-1) \;=\; \deg(\Delta_K(t)) - 1.$$
$\square$

In light of the above example the following proposition can be viewed as a generalization of the Genus–Alexander Polynomial Proposition 13.1:

Proposition 15.4. (Genus–Twisted Alexander Polynomial Proposition) For every oriented knot K, ever UFD R and every representation $\alpha\colon \pi_1(S^3 \setminus K) \to \mathrm{Aut}_R(M)$ we have[103]
$$\deg(\Delta_K^\alpha(t)) \;\leq\; \operatorname{rank}(M) \cdot (2 \cdot \operatorname{genus}(K) - 1).$$
In other words, we have
$$\operatorname{genus}(K) \;\geq\; \frac{1}{2 \cdot \operatorname{rank}(M)} \cdot \deg(\Delta_K^\alpha(t)) + \frac{1}{2}.$$

Example. Let $K = T(2,3)$ be the trefoil. As we pointed out one page 161, the trefoil has genus one. Recall that in Section 14.2 we gave an explicit example of a representation $\alpha\colon \pi_1(S^3 \setminus K) \to \mathrm{GL}(2, Z)$ such that $\Delta_K^\alpha(t) \doteq 1 - t^2$. Note that in this case the inequality of the Genus–Twisted Alexander Polynomial Proposition 15.4 is in fact an equality. $\square$

Proof. Evidently the proof of this proposition is very similar to the proof of the Genus–Alexander Polynomial Proposition 13.1. In particular we will import some results from the earlier proof.

[103]The expression "$2 \cdot \operatorname{genus}(K) - 1$" is not as random as it looks. For a Seifert surface F of minimal genus we have $2 \cdot \operatorname{genus}(K) - 1 = -\chi(F)$.

Let K be an oriented knot and let R be a UFD. It follows immediately from the definition of the twisted Alexander polynomial on page 211 that it suffices to consider the case that we are given a homomorphism $\alpha\colon \pi_1(S^3\setminus K) \to \mathrm{GL}(n,R)$. We need to prove that for every Seifert surface F of K we have

$$\deg(\Delta_K^\alpha(t)) \;\leq\; n\cdot(2\cdot\mathrm{genus}(F)-1).$$

So let F be a Seifert surface for K. We make a few preparations:

- We set $g := \mathrm{genus}(F)$.
- We pick $b_0 \in S^3\setminus F$.
- We denote as usual by $\Phi_K\colon \pi_1(S^3\setminus K, b_0) \to \langle t\rangle$ the unique epimorphism which sends a meridian to t.

Claim 1. There exists a presentation P of $\pi_1(S^3\setminus K, b_0)$ of the form

$$\pi_1(S^3\setminus K, b_0) \;=\; \left\langle y_1,\dots,y_k,t \;\middle|\; \begin{array}{l} r_1,\dots,r_l \\ t\cdot x_{i,+}\cdot t^{-1}\cdot x_{i,-}^{-1},\, i=1,\dots,2g \end{array} \right\rangle$$

where $r_1,\dots,r_l,x_{1,\pm},\dots,x_{2g,\pm} \in \langle y_1,\dots,y_k\rangle$ and where $\Phi_K(y_i)=0$ and $\Phi_K(t)=t$.

Proof. We proved this claim on page 187. ⊞

Claim 2. We set $\Gamma := \ker(\Phi_K)$. There exist matrices $A_{l\times k}, B_{2g\times k}, C_{2g\times k}$ with entries in $\mathbb{Z}[\Gamma]$ such that

$$J(P) \;=\; \begin{array}{cc} & \begin{array}{cc} k & \qquad\qquad 1 \end{array} \\ \begin{array}{c} l \\ 2g \end{array} & \begin{pmatrix} A_{l\times k} & 0 \\ B_{2g\times k}\cdot t + C_{2g\times k} & * \end{pmatrix} \end{array}$$

Proof. The proof of this claim is very similar to the proof of the corresponding "untwisted" claim on page 187. We leave it to the reader to fill in the details. ⊞

Evidently we now want to use the presentation of Claim 1 and the discussion of Claim 2 to prove the desired inequality. First note that it follows from the proof of the Twisted Alexander Function Theorem 14.4 that we can use the presentation P from Claim 1 to calculate the twisted Alexander polynomial. More precisely, it follows from that proof, applied to the $(k+1)$-st generator (i.e. applied to the generator t), that

$$\Delta_K^\alpha \;=\; \Delta^\alpha_{\pi_1(S^3\setminus K),\Phi_K} \;=\; \frac{\gcd_{R[t^{\pm1}]}\{\det((\Phi_K\otimes\alpha)_*(J(P)^Z_{k+1}))\}_{\substack{Z\subset\{1,\dots,l\}\\ \#Z=k}}}{\det((\Phi_K\otimes\alpha)_*(t-1))} \;\in\; Q_R(t).$$

Next note that it follows from the discussion on page 199 that the degree of the denominator is n. Thus it remains to prove the following claim:

Claim 3. For any $Z\subset\{1,\dots,l\}$ with $\#Z=k$ we have

$$\deg(\det((\Phi_K\otimes\alpha)_*(J(P)^Z_{k+1}))) \;\leq n\cdot 2g.$$

Proof. Let $Z \subset \{1,\dots,l\}$ with $\#Z = k$. We set $Z' := Z \cap \{1,\dots,l\}$ and $Z'' \cap \{k+1,\dots,l+2g\}$. We note that

$$\begin{aligned}\det((\Phi_K\otimes\alpha)_*(J(P)^Z_{k+1})) &\overset{\substack{\text{by claim 2}\\ \downarrow}}{=} \det\left((\Phi_K\otimes\alpha)_*\left(\begin{pmatrix} A_{l\times k} & \cancel{0} \\ B_{2g\times k}\cdot t + C_{2g\times k} & \cancel{*}\end{pmatrix}^Z\right)\right)\\ &= \det\left((\Phi_K\otimes\alpha)_*\begin{pmatrix} A^{Z'}_{l\times k} \\ B^{Z''}_{2g\times k}\cdot t + C^{Z''}_{2g\times k}\end{pmatrix}\right) \underset{\substack{\uparrow\\ \text{since } A, B \text{ and } C \text{ are matrices}\\ \text{over } \mathbb{Z}[\ker(\Phi_K)] \text{ and since } \Phi_K(t)=t}}{=} \det\begin{pmatrix}\alpha_*(A^{Z'}_{l\times k}) \\ \alpha_*(B^{Z''}_{2g\times k})\cdot t + \alpha_*(C^{Z''}_{2g\times k})\end{pmatrix}.\end{aligned}$$

Note that $\alpha_*(B^{Z''}_{2g\times k})$ has $n\cdot 2g$ columns. Thus it follows from from the Laurent Polynomial-Determinant-Degree Lemma 13.3 that the degree of the above determinant is indeed at most $n \cdot 2g$. ■

In the following two subsections we consider two of our favorite challenge examples, namely the Kinoshita-Terasaka knot K and the Conway knot C.

But before we do so, let us, for completeness' sake, state the following theorem, which complements the Genus–Twisted Alexander Polynomial Proposition 15.4:

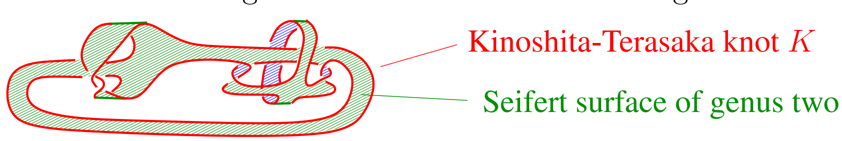
Theorem 15.5. (Twisted Alexander Polynomial–Genus Detection Theorem) Given any oriented knot $K \subset S^3$ there exists a unitary representation $\pi_1(S^3 \setminus K) \to \mathrm{U}(n)$ such that
$$\deg(\Delta^\alpha_K(t)) \;=\; n\cdot(2\cdot \mathrm{genus}(K) - 1).$$

Remark. Unfortunately the Twisted Alexander Polynomial–Genus Detection Theorem 15.5 is not as useful as one might hope, since it does not give us any indications how one could find these elusive unitary representations. □

Proof. This theorem is proved in [**FV15**, Theorem 1.2]. ■

15.3.1. The genus of the Kinoshita-Terasaka knot.

In the figure below we show the Kinoshita-Terasaka knot[104] together with a Seifert surface of genus 2.



Recall that on page 217 we saw that (up to conjugation) there exists a unique epimorphism $\varphi_K\colon \pi_1(S^3\setminus K)\to A_5$. As before we denote by $\sigma_5^{\mathbb{F}_7}\colon A_5 \to \mathrm{Aut}_{\mathbb{F}_7}(\mathbb{F}_7^5)$ the standard representation. On page 217 we saw that
$$\Delta_K^{\sigma_5^{\mathbb{F}_7}\circ\varphi_K}(t) \;=\; 1+5t+6t^2+5t^3+t^4+2t^5+2t^6+2t^7+t^8+5t^9+6t^{10}+5t^{11}+t^{12}.$$

[104]We leave it to the reader to verify that this diagram does indeed represent the Kinoshita-Terasaka knot as defined on page 100.

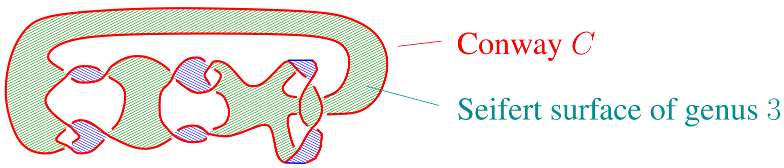
If we plug this twisted Alexander polynomial into the Genus–Twisted Alexander Polynomial Proposition 15.4 we obtain that

$$\operatorname{genus}(K) \underset{\substack{\uparrow \\ \text{by the Genus–Twisted Alexander} \\ \text{Polynomial Proposition 15.4}}}{\geq} \frac{1}{2\cdot\dim_{\mathbb{F}_7}(\mathbb{F}_7^5)}\cdot\deg(\Delta_K^{\sigma_5^{\mathbb{F}_7}\circ\varphi_K}(t))+\frac{1}{2} \underset{\substack{\uparrow \\ \text{by the above} \\ \text{calculation}}}{=} \frac{1}{2\cdot 5}\cdot 12+\frac{1}{2} = \frac{17}{10}.$$

Since the genus is a natural number, we see that $\operatorname{genus}(K)\geq 2$. Thus the upper and lower bounds match, and we have shown that $\operatorname{genus}(K)=2$.

15.3.2. **The genus of the Conway knot.**

In the figure below we show the Conway knot C together with a Seifert surface of genus 3.



Evidently we want to repeat the trick from the previous subsection. On page 217 we saw that (up to conjugation) there exists a unique epimorphism $\varphi_C\colon \pi_1(S^3\setminus C)\to A_5$. As before we denote by $\sigma_5^{\mathbb{F}_7}\colon A_5\to \operatorname{Aut}_{\mathbb{F}_7}(\mathbb{F}_7^5)$ the standard representation. On page 217 we saw that

$$\Delta_C^{\sigma_5^{\mathbb{F}_7}\circ\varphi_C}(t) = 1+t+4t^2+6t^3+t^5+2t^6+5t^7+2t^8+t^9+6t^{11}+4t^{12}+t^{13}+t^{14}.$$

This time we obtain that

$$\operatorname{genus}(C) \underset{\substack{\uparrow \\ \text{by the Genus–Twisted Alexander} \\ \text{Polynomial Proposition 15.4}}}{\geq} \frac{1}{2\cdot\operatorname{rank}_{\mathbb{F}_7}(\mathbb{F}_7^5)}\cdot\deg(\Delta_C^{\sigma_5^{\mathbb{F}_7}\circ\varphi_C}(t))+\frac{1}{2} \underset{\substack{\uparrow \\ \text{by the above} \\ \text{calculation}}}{=} \frac{1}{2\cdot 5}\cdot 14+\frac{1}{2} = \frac{19}{10}.$$

Since the genus is a natural number we see that $\operatorname{genus}(C)\geq 2$. Thus the upper and lower bounds do not match. As we will see shortly, we will get around this nuisance by replacing the standard representation of S_n by the reduced standard representation of S_n.

We will make use of the following lemma, which explains the relationship between the standard representation of S_n by the reduced standard representation of S_n:

Lemma 15.6. (Standard Representation–Decomposition Lemma) Let $n\in\mathbb{N}$ and let $\mathbb{F}$ be some field such that the characteristic of $\mathbb{F}$ is coprime to n. As on page 210 we write $(\mathbb{F}^n)_0 := \{(v_1,\dots,v_n)\in\mathbb{F}^n \,|\, v_1+\dots+v_n=0\}$ and as before we denote by $\sigma_{n,0}^{\mathbb{F}}\colon S_n\to\operatorname{Aut}((\mathbb{F}^n)_0)$ the reduced standard representation. Then we have the following isomorphism of representations of S_n:

$$\sigma_{n,0}^{\mathbb{F}}\oplus\tau_1^{\mathbb{F}} \cong \sigma_n^{\mathbb{F}}.$$

Proof. By hypothesis n is coprime to the characteristic of $\mathbb{F}$. This implies that there exists an $r\in\mathbb{F}$ with $n\cdot r=1$. Now we consider the following homomorphisms of $\mathbb{F}$-vector spaces:

$$\begin{array}{rcl} \Phi\colon (\mathbb{F}^n)_0\oplus\mathbb{F} & \to & \mathbb{F}^n \\ ((v_1,\dots,v_n),f) & \mapsto & (v_1,\dots,v_n)+f\cdot(1,\dots,1) \end{array}$$

and

$$\begin{array}{rcl}\Psi\colon \mathbb{F}^n & \to & (\mathbb{F}^n)_0 \oplus \mathbb{F} \\ (v_1,\dots,v_n) & \mapsto & \big(\underbrace{(v_1,\dots,v_n) - r\cdot(v_1+\dots+v_n)\cdot(1,\dots,1)}_{\text{sum of the coefficients is zero}}, r\cdot(v_1+\dots+v_n)\big).\end{array}$$

It follows easily from the definitions that the two maps are inverses of one another. In particular both are isomorphisms of $\mathbb{F}$-vector spaces. Finally note that it follows easily from the definitions that Φ preserves the S_n-action, thus it defines an isomorphism of S_n-representations. ■

We now return to the example of the Conway knot. The idea now is to replace the reduced standard representation $\sigma_5^{\mathbb{F}_7}$ by the reduced standard representation $\sigma_{5,0}^{\mathbb{F}_7}$. We could calculate $\Delta_C^{\sigma_{5,0}^{\mathbb{F}_7}}(t)$ again by a computer. But it is much more insightful to relate it to the twisted Alexander polynomial $\Delta_C^{\sigma_5^{\mathbb{F}_7}}(t)$ which we already calculated. In the following calculation we will use the following notation:

- We denote by $\rho\colon \mathbb{Z} \to \mathbb{F}_7$ the unique ring homomorphism.
- Given a representation α of S_5 we write $\Delta_C^{\alpha}(t) := \Delta_C^{\alpha\circ\varphi_C}(t)$.

Now we see that

$$\begin{aligned}\Delta_C^{\sigma_5^{\mathbb{F}_7}}(t) &\underset{\substack{\uparrow\\ \text{by the Standard Representation}\\ \text{Decomposition Lemma 15.6}}}{\doteq} \Delta_C^{\sigma_{5,0}^{\mathbb{F}_7}\oplus\tau_1^{\mathbb{F}_7}}(t) \underset{\substack{\uparrow\\ \text{by the Direct Sum}\\ \text{Twisted Alexander}\\ \text{Function Lemma 14.10}}}{\doteq} \Delta_C^{\sigma_{5,0}^{\mathbb{F}_7}}(t)\cdot\Delta_C^{\tau_1^{\mathbb{F}_7}}(t) \underset{\substack{\uparrow\\ \text{since } \tau_1^{\mathbb{F}_7} = \rho_*\circ\tau_1^{\mathbb{Z}} \text{ as}\\ \text{maps } \pi_1(S^3\setminus C)\to \mathrm{GL}(1,\mathbb{F}_7)}}{\doteq} \Delta_C^{\sigma_{5,0}^{\mathbb{F}_7}}(t)\cdot\Delta_C^{\rho_*\circ\tau_1^{\mathbb{Z}}}(t)\\ &\underset{\substack{\uparrow\\ \text{by the Twisted Alexander Function}\\ \text{Functorial Lemma 14.5}}}{\doteq} \Delta_C^{\sigma_{5,0}^{\mathbb{F}_7}}(t)\cdot\rho(\Delta_C^{\tau_1^{\mathbb{Z}}}(t)) \underset{\substack{\uparrow\\ \text{by the discussion}\\ \text{on page 215}}}{\doteq} \Delta_C^{\sigma_{5,0}^{\mathbb{F}_7}}(t)\cdot\rho\Big(\frac{\Delta_C(t)}{t-1}\Big) \underset{\substack{\uparrow\\ \text{on page 130 we saw}\\ \text{that } \Delta_C(t)=1}}{\doteq} \Delta_C^{\sigma_{5,0}^{\mathbb{F}_7}}(t)\cdot\frac{1}{t-1}.\end{aligned}$$

Thus we see that the two twisted Alexander polynomials corresponding to the standard representation and corresponding to the reduced standard representation essentially agree. But since the rank of the reduced representation is lower, we get a better bound on the genus. More precisely, we now obtain that

$$\mathrm{genus}(C) \underset{\substack{\uparrow\\ \text{by the Genus–Twisted Alexander}\\ \text{Polynomial Proposition 15.4}}}{\geq} \frac{1}{2\cdot\mathrm{rank}_{\mathbb{F}_7}((\mathbb{F}_7^5)_0)}\cdot\deg(\Delta_C^{\sigma_{5,0}^{\mathbb{F}_7}\circ\varphi_C}(t)) + \frac12 \underset{\substack{\uparrow\\ \text{by the above}\\ \text{calculation}}}{=} \frac{1}{2\cdot 4}\cdot(14-1)+\frac12 = \frac{17}{8}.$$

Since the genus is a natural number we see that $\mathrm{genus}(C) \geq 3$. Thus the upper and lower bounds now do match. Thus we have finally shown that $\mathrm{genus}(C) = 3$.

15.4. **Twisted Alexander polynomials and fiberedness.** In this section we will generalize the Fibered Knot–Alexander Polynomial Proposition 15.7 to the twisted context. To formulate the main result of this section we need the following generalization of a definition from page 191:

Definition. Let R be a domain.

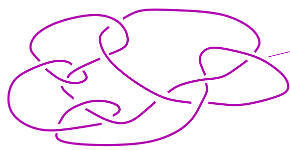
(1) We say that a Laurent polynomial $p(t) \in R[t^{\pm 1}]$ is bimonic if the highest and lowest coefficients both are a unit in R, i.e. they both lie in R^*.
(2) We say $f(t) \in Q_R(t)$ is bimonic if it is the quotient of two bimonic Laurent polynomials.

In light of the discussion on page 215, the following proposition can be viewed as a generalization of the Fibered Knot–Alexander Polynomial Proposition 15.7:

Proposition 15.7. (Fibered Knot–Twisted Alexander Polynomial Proposition) Let K be an oriented knot, let R be a UFD and let $\alpha\colon \pi_1(S^3 \setminus K) \to \operatorname{Aut}_R(M)$ be a representation. If K is fibered, then the following two statements hold:

(1) The twisted Alexander polynomial $\Delta_K^\alpha(t)$ is bimonic.
(2) We have[105] $$\deg(\Delta_K^\alpha(t)) \;=\; \operatorname{rank}(M) \cdot (2 \cdot \operatorname{genus}(K) - 1).$$

Example. We consider the knot $K = 12_{1345} = 12_{n57}$. One can show that its genus equals two and that its Alexander polynomial equals $1 - 2t + 3t^2 - 2t^3 + t^4$.



So the Fibered Knot–Alexander Polynomial Proposition 15.7 has no objections to K being fibered. But one can show that there exists a homomorphism $\varphi\colon \pi_1(S^3 \setminus K) \to S_5$ such that
$$\Delta_K^{\sigma_5^{\mathbb{F}_2} \circ \varphi}(t) \;=\; 1 + t + t^3 + t^5 + t^7 + t^8 \;\in\; \mathbb{F}_2[t^{\pm 1}].$$
We see that
$$\deg(\Delta_K^{\sigma_5^{\mathbb{F}_2} \circ \varphi}(t)) \;=\; 8 \;\neq\; 12 \;=\; 4 \cdot (2 \cdot 2 - 1) \;=\; \dim_{\mathbb{F}_2}((\mathbb{F}_2^5)_0) \cdot (2 \cdot \operatorname{genus}(K) - 1).$$
Thus it follows from the Fibered Knot–Twisted Alexander Polynomial Proposition 15.7 that K is not fibered. □

Before we head to the proof of the Fibered Knot–Twisted Alexander Polynomial Proposition 15.7, we state for completeness' sake the following theorem, which can be viewed as a converse to the Fibered Knot–Twisted Alexander Polynomial Proposition 15.7:

Theorem 15.8. (Twisted Alexander Polynomial–Fiberedness Detection Theorem) For any non-fibered knot $K \subset S^3$ there exists a representation $\pi_1(S^3 \setminus K) \to \mathrm{U}(n)$ such that
$$\deg(\Delta_K^\alpha(t)) \;=\; 0.$$

Proof. This theorem is proved in [**FV13**, Theorem 1.2]. (A somewhat weaker fiberedness detection theorem was first proved in [**FV11b**, Theorem 1.2].) ■

The proof of the Fibered Knot–Twisted Alexander Polynomial Proposition 15.7 will occupy the remainder of this chapter. We start out with a few preparations:

[105] Recall that by the Fiber–Minimal Corollary 13.11 we know that genus(K) equals the genus of a Seifert fiber surface.

Definition. Let $\varphi\colon F \to F$ be a homomorphism between two finitely generated free groups. Given a basis $x_1,\dots,x_k$ of F and given a basis $y_1,\dots,y_l$ of G we refer to

$$J^{y_1,\dots,y_l}_{x_1,\dots,x_k}(\varphi) \;:=\; \left(\frac{\partial\varphi(x_i)}{\partial y_j}\right)_{\substack{i=1,\dots,k,\\ j=1,\dots,l}} \;\in\; \mathrm{M}(k\times l,\mathbb{Z}[\pi])$$

as the Jacobi matrix of φ with respect to the bases $x_1,\dots,x_k$ and $y_1,\dots,y_l$. If we are given an endomorphism $\varphi\colon F\to F$ of a free group, then given any basis $x_1,\dots,x_k$ of F we write $J^{x_1,\dots,x_k}(\varphi) := J^{x_1,\dots,x_k}_{x_1,\dots,x_k}(\varphi)$.

Example.

(1) Let F be a free group with basis $x_1,\dots,x_k$. It follows immediately from the definitions that $J^{x_1,\dots,x_k}(\mathrm{id}) = \mathrm{id}_F \in \mathrm{M}(k\times k,\mathbb{Z}[F])$.
(2) Let $\pi = \langle y_1,\dots,y_l \,|\, r_1,\dots,r_k\rangle$ be a finite presentation. Let $\varphi\colon \langle x_1,\dots,x_k\rangle \to \langle y_1,\dots,y_l\rangle$ be the homomorphism given by $\varphi(x_i) = r_i$ for $i=1,\dots,k$. It follows immediately from the definitions that

$$J^{y_1,\dots,y_l}_{x_1,\dots,x_k}(\varphi) \;=\; \underbrace{J(\text{finite presentation } \langle y_1,\dots,y_l \,|\, r_1,\dots,r_k\rangle)}_{\text{as defined on page 109}}.$$

□

In a second we will need the following "chain rule":

Lemma 15.9. (Fox Derivative–Chain Rule) Let X be a finitely generated free group with basis $x_1,\dots,x_m$ and let Y be a finitely generated free group with basis $y_1,\dots,y_n$. Furthermore let $\lambda\colon X\to Y$ be any homomorphism. Given any $g\in\pi$ and given any $j\in\{1,\dots,m\}$ we have the following equality in $\mathbb{Z}[Y]$:

$$\frac{\partial\lambda(g)}{\partial y_j} \;=\; \sum_{r=1}^{m}\lambda\left(\frac{\partial g}{\partial x_r}\right)\cdot\frac{\partial\lambda(x_r)}{\partial y_j}.$$

Proof. We need to prove the desired equality for any $g\in\pi$. Since $x_1,\dots,x_m$ is a basis for π it suffices to prove the following claim:

Claim.

(a) The desired equality holds for any $g = x_i$.
(b) The desired equality holds for any $g = x_i^{-1}$.
(c) If the desired equality holds for $u,v\in X$, then it also holds for $g = u\cdot v$.

Proof.

(a) It follows easily from $\frac{\partial x_j}{\partial x_k} = \delta_{ik}$ that the statement holds for any x_i.
(b) For $g = x_i^{-1}$ we have

$$\frac{\partial\lambda(x_i^{-1})}{\partial x_j} \;=\; \frac{\partial(\lambda(x_i)^{-1})}{\partial x_j} \;\underset{\substack{\uparrow\\ \text{by the Fox Derivative Properties Lemma 8.2 (5)}}}{=}\; -\lambda(x_i)^{-1}\cdot\frac{\partial\lambda(x_i)}{\partial x_j} \;\underset{\substack{\uparrow\\ \text{by (a)}}}{=}\; -\lambda(x_i)^{-1}\cdot\sum_{r=1}^{m}\lambda\left(\frac{\partial x_i}{\partial x_r}\right)\cdot\frac{\partial\lambda(x_r)}{\partial y_j}$$

$$= \sum_{r=1}^{m}\lambda\left(-x_i^{-1}\cdot\frac{\partial x_i}{\partial x_r}\right)\cdot\frac{\partial\lambda(x_r)}{\partial y_j} \;\underset{\substack{\uparrow\\ \text{by the Fox Derivative Properties Lemma 8.2 (5)}}}{=}\; \sum_{r=1}^{m}\lambda\left(\frac{\partial x_i^{-1}}{\partial x_r}\right)\cdot\frac{\partial\lambda(x_r)}{\partial y_j}.$$

(c) Suppose the desired equality holds for $u, v \in X$. Then the following calculation shows that the desired equality also holds for $u \cdot v$:

$$\begin{aligned}\frac{\partial\lambda(u\cdot v)}{\partial x_j} &\overset{\text{by the Leibniz rule}}{=} \frac{\partial\lambda(u)}{\partial x_j} + \lambda(u)\cdot\frac{\partial\lambda(v)}{\partial y_j} \overset{\text{since } u \text{ and } v \text{ satisfy the desired equality}}{=} \sum_{r=1}^{m} \lambda\Big(\frac{\partial u}{\partial x_r}\Big)\cdot\frac{\partial\lambda(x_r)}{\partial y_j} + \lambda(u)\cdot\sum_{r=1}^{m} \lambda\Big(\frac{\partial v}{\partial x_r}\Big)\cdot\frac{\partial\lambda(x_r)}{\partial y_j} \\ &= \sum_{r=1}^{m} \lambda\Big(\frac{\partial u}{\partial x_r} + u\cdot\frac{\partial v}{\partial x_r}\Big)\cdot\frac{\partial\lambda(x_r)}{\partial y_j} \underset{\text{by the Leibniz rule}}{=} \sum_{r=1}^{m} \lambda\Big(\frac{\partial(u\cdot v)}{\partial x_r}\Big)\cdot\frac{\partial\lambda(x_r)}{\partial y_j}.\end{aligned}$$

■

Lemma 15.10. (Jacobi Matrix-Isomorphism Lemma) Let π be a finitely generated free group. If $\varphi\colon \pi \to \pi$ is an isomorphism, then for any basis $x_1, \dots, x_n$ of π the Jacobi matrix $J^{x_1,\dots,x_n}(\varphi) = (\frac{\partial\varphi(x_i)}{\partial x_j})_{i,j=1,\dots,n}$ has a right inverse in $\mathrm{M}(n \times n, \mathbb{Z}[\pi])$.

Proof. We apply the Fox Derivative–Chain Rule 15.9 in the setting where $X = Y = \pi$ and $\lambda = \mathrm{id}$. Since φ is an isomorphism we know that $y_i = \varphi(x_i)$, $i = 1, \dots, n$ also form a basis for π. Given any $i, j \in \{1, \dots, n\}$ we set $g := \varphi(x_i)$ and we obtain from the Fox Derivative–Chain Rule 15.9 that

$$\frac{\partial\varphi(x_i)}{\partial y_j} = \sum_{r=1}^{n} \frac{\partial\varphi(x_i)}{\partial x_r}\cdot\frac{\partial x_r}{\partial y_j}.$$

But this implies that

$$\mathrm{id}_n \underset{\substack{\text{since } \varphi(x_i) = y_i \text{ and} \\ \text{since } \mathrm{id}_n = (\delta_{ij})_{i,j=1,\dots,n}}}{=} \Big(\frac{\partial\varphi(x_i)}{\partial y_j}\Big)_{i,j=1,\dots,n} \underset{\text{by the above}}{=} \Big(\sum_{r=1}^{n} \frac{\partial\varphi(x_i)}{\partial x_r}\cdot\frac{\partial x_r}{\partial y_j}\Big)_{i,j=1,\dots,n} \underset{\text{definition of matrix multiplication}}{=} \overbrace{\Big(\frac{\partial\varphi(x_i)}{\partial x_r}\Big)_{i,r=1,\dots,n}}^{=J^{x_1,\dots,x_n}(\varphi)}\cdot\overbrace{\Big(\frac{\partial x_r}{\partial y_j}\Big)_{j,j=1,\dots,n}}^{\text{matrix over } \mathbb{Z}[\pi]}.$$

We have thus shown that $J^{x_1,\dots,x_n}(\varphi)$ has a right inverse in $\mathrm{M}(n \times n, \mathbb{Z}[\pi])$. ■

Now we can finally provide the proof of the Fibered Knot–Twisted Alexander Polynomial Proposition 15.7.

Proof of Proposition 15.7. Let $K \subset S^3$ be a fibered oriented knot with corresponding Seifert fiber surface F. We denote by g the genus of F. Furthermore let R be a UFD. It follows immediately from the definition of the twisted Alexander polynomial on page 211 and the Fiber–Minimal Corollary 13.11 that it suffices to show that for any homomorphism $\alpha\colon \pi_1(S^3 \setminus K) \to \mathrm{GL}(n, R)$ the following two statements hold:

(1) The twisted Alexander polynomial $\Delta_K^{\alpha}(t)$ is bimonic.
(2) We have $\deg(\Delta_K^{\alpha}(t)) \;=\; n \cdot (2 \cdot g - 1)$.

First recall that by the Fibered Knot-π_1-Proposition 13.5 we know that there exists an isomorphism $\varphi\colon \langle y_1, \dots, y_{2g}\rangle \to \langle y_1, \dots, y_{2g}\rangle$ and an isomorphism

$$\Theta\colon \pi_1(S^3 \setminus K) \xrightarrow{\cong} \langle y_1, \dots, y_{2g}\rangle \rtimes_{\varphi} \langle t\rangle$$

such that under this isomorphism the epimorphism $\Phi_K\colon \pi_1(S^3 \setminus K) \to \langle t\rangle$ agrees with the natural epimorphism $\Phi\colon \langle y_1, \dots, y_{2g}\rangle \rtimes_{\varphi} \langle t\rangle \to \langle t\rangle$ that is given by $\Phi(y_i) = 1$ and $\Phi(t) = t$.

Next note that, as in the proof of the Alexander Polynomial-of-Semidirect Product Proposition 13.7, we have the following isomorphisms:

$$\begin{array}{rcl}\langle y_1,\dots,y_{2g}\rangle \rtimes_\varphi \mathbb{Z} & \overset{\text{follows easily from the example on page 189}}{\cong} & \langle y_1,\dots,y_{2g},t\,|\,t\cdot y_1\cdot t^{-1}\cdot\varphi(y_1)^{-1},\dots,y_{2g}\cdot t^{-1}\cdot\varphi(y_{2g})^{-1}\rangle\\ & = & \langle y_1,\dots,y_{2g},t\,|\,t\cdot y_1\cdot t^{-1}\cdot\varphi(y_1^{-1}),\dots,y_{2g}\cdot t^{-1}\cdot\varphi(y_{2g}^{-1})\rangle\\ & \underset{\text{substitution } y_i\mapsto x_i^{-1}}{=} & \langle x_1,\dots,x_{2g},t\,|\,t\cdot x_1^{-1}\cdot t^{-1}\cdot\varphi(x_1),\dots,x_{2g}^{-1}\cdot t^{-1}\cdot\varphi(x_{2g})\rangle \;=:\; P.\end{array}$$

We will now work with the presentation P that we just found. Using the $(2g+1)$-st generator t we see that

$$\Delta_K^\alpha(t) \;=\; \frac{\det((\Phi\otimes\alpha)_*(J(P)_{2g+1}))}{\det((\Phi\otimes\alpha)_*(t-1))}.$$

By the discussion on page 199 we know that the denominator of the above fraction is bimonic of degree n. Thus it suffices to prove the following claim about the numerator:

Claim. The determinant $\det((\Phi\otimes\alpha)_*(J(P)_{2g+1}))$ is bimonic of degree $n\cdot 2g$.

Proof. We perform the following calculation:

$$\begin{array}{l}\det((\Phi\otimes\alpha)_*(J(P)_{2g+1})) \;=\; \det\Big((\Phi\otimes\alpha)\Big(\dfrac{\partial(t\cdot x_i^{-1}\cdot t^{-1}\cdot\varphi(x_i))}{\partial x_j}\Big)_{i,j=1,\dots,2g}\Big)\\[2ex] \underset{\substack{\text{follows from the Fox Derivative Proposition 8.1 and}\\ \text{the Fox Derivative Properties Lemma 8.2}}}{=} \det\Big((\Phi\otimes\alpha)_*\Big(t\cdot\operatorname{diag}(-x_1,\dots,-x_{2g})+\operatorname{diag}(tx_1^{-1}t^{-1},\dots,tx_{2g}^{-1}t^{-1})\cdot\underbrace{\Big(\dfrac{\partial\varphi(x_i)}{\partial x_j}\Big)_{i,j}}_{=J^{x_1,\dots,x_{2g}}(\varphi)}\Big)\Big)\\[2ex] \underset{\text{by the definition of }\Phi}{=} \det(t\cdot\operatorname{diag}(\alpha(-x_1),\dots,\alpha(-x_{2g}))+\operatorname{diag}(\alpha(tx_1^{-1}t^{-1}),\dots,\alpha(tx_{2g}^{-1}t^{-1}))\cdot\alpha_*(J^{x_1,\dots,x_{2g}}(\varphi)))\\[2ex] \underset{\substack{\text{by the Polynomial Matrix}\\ \text{Determinant Lemma 13.9}}}{=} t^{2g\cdot n}\cdot\prod_{i=1}^{2g}\underbrace{\det(-\alpha(x_i))}_{\in R^*}+t^0\cdot\Big(\prod_{i=1}^{2g}\underbrace{\det(\alpha(tx_i^{-1}t^{-1}))}_{\in R^*}\Big)\cdot\underbrace{\det(\alpha_*(J^{x_1,\dots,x_{2g}}(\varphi)))}_{\in R^*\text{ by the discussion below}}.\end{array}$$

By the Jacobi Matrix-Isomorphism Lemma 15.10 we know that $J^{x_1,\dots,x_{2g}}(\varphi)$ has a right inverse in $M(2g\times 2g,\mathbb{Z}[\pi])$.[106] It follows from the Group Ring-to-Matrix Ring Lemma 14.1, the Matrix Multiplication–Change-of-Ring Lemma 14.2 and the Matrix-of-Matrices Lemma 14.3 that $\alpha_*(J^{x_1,\dots,x_{2g}}(\varphi))$ has a right inverse in $M(2g\cdot n\times 2g\cdot n,R)$. But this implies that $\det(\alpha_*(J^{x_1,\dots,x_{2g}}(\varphi)))$ is a unit in R. ∎

15.5. Appendix: Symmetries of twisted Alexander polynomials.

For ordinary Alexander polynomials we proved the Alexander Polynomial Symmetry Theorem 9.4 which says that for any oriented m-component link L we have

$$\Delta_L(t_1,\dots,t_m) \;\doteq\; \Delta_L(t_1^{-1},\dots,t_m^{-1}).$$

The reader might also remember that the proof was highly non-trivial. It is natural to ask whether a similar type of symmetry statement also holds for twisted Alexander polynomials. It turns out that this is a tricky question and, as we will see, even just to formulate the

[106] Note that at this point the proof starts to differ quite drastically from the original argument in the untwisted case.

corresponding symmetry theorem requires some thought and preparations. Since we will not really make use of the symmetry statement, and since we will not prove it anyway, we are hiding this discussion in an appendix.

First of all we need the following definitions:

Definition. Let R be a commutative ring.

(1) An involution on R is a map $R \to R$, usually written $z \mapsto \overline{z}$, with the following properties:
   (a) $\overline{z+w} = \overline{z} + \overline{w}$ for all $z, w \in R$,
   (b) $\overline{w \cdot z} = \overline{w} \cdot \overline{z}$ for all $z, w \in R$,
   (c) $\overline{\overline{z}} = z$ for all $z \in R$.

(2) Given an R-module M we denote by $\overline{M}$ the R-module where for $m \in \overline{M}$ we have $r \cdot_{\overline{M}} m := \overline{r} \cdot_M m$.

Example.

(1) For any commutative ring the identity map is evidently an involution.
(2) Complex conjugation turns $\mathbb{C}$ into a ring with involution. $\square$

Definition. Let R be a commutative ring with (possibly trivial) involution and let M be a free R-module.

(1) A R-sesquilinear pairing on M is a map $\langle\ ,\ \rangle \colon M \times M \to R$ such that the following holds:
   - For all $v, v' \in M$ and all $w, w' \in M$ we have $\langle v + v', w\rangle = \langle v, w\rangle + \langle v', w\rangle$ and $\langle v, w + w'\rangle = \langle v, w\rangle + \langle v, w'\rangle$.
   - For all $v, w \in M$ and $r \in R$ we have $\langle r \cdot v, w\rangle = r \cdot \langle v, w\rangle$ and $\langle v, r \cdot w\rangle = \overline{r} \cdot \langle v, w\rangle$.

(2) We say that an R-sesquilinear pairing $\langle\ ,\ \rangle \colon M \times M \to R$ is non-singular if the induced maps

$$\begin{array}{rcl} M & \to & \operatorname{Hom}_R(\overline{M}, R) \\ v & \mapsto & \left(\begin{array}{rcl} \overline{M} & \to & R \\ w & \mapsto & \langle v, w\rangle \end{array}\right) \end{array} \quad \text{and} \quad \begin{array}{rcl} M & \to & \overline{\operatorname{Hom}_R(M, R)} \\ w & \mapsto & \left(\begin{array}{rcl} M & \to & R \\ v & \mapsto & \langle v, w\rangle \end{array}\right) \end{array}$$

are isomorphisms of R-modules.[107]

Lemma 15.11. (Dual Representation Lemma) Let R be a commutative ring with (possibly trivial) involution, let $\alpha \colon \pi \to \operatorname{Aut}_R(M)$ be a representation and let

$$\langle -, - \rangle \colon M \times M \ \to\ R$$

be a non-singular R-sesquilinear pairing. In this setting there exists a unique representation $\alpha^\dagger \colon \pi \to \operatorname{Aut}_R(M)$ such that for all $m, m' \in M$ and all $g \in \pi$ we have[108]

$$\langle \alpha(g)(m), m'\rangle \ =\ \langle m, \alpha^\dagger(g^{-1})(m')\rangle.$$

We refer to this representation as the Hermitian adjoint of α.

Example.

[107] It is worth verifying that these maps are homomorphisms of R-modules.

[108] In other words: For any $m, m' \in M$ and any $g \in \pi$ we have $\langle \alpha(g)(m), \alpha^\dagger(g)(m)\rangle = \langle m, m'\rangle$.

(1) Let R be a commutative ring with involution. We consider $M = R^n$ and the non-singular R-sesquilinear pairing

$$\begin{array}{rcl} \langle\ ,\ \rangle\colon R^n \times R^n & \to & R \\ ((v_1,\dots,v_n),(w_1,\dots,w_n)) & \mapsto & \sum\limits_{i=1}^{n} v_i \cdot \overline{w}_i. \end{array}$$

It follows easily from the definitions that given any representation $\alpha\colon \pi \to \mathrm{GL}(n,R)$ and given any $g \in \pi$ we have $\alpha^\dagger(g) = (\overline{\alpha(g)^T})^{-1}$. For $R = \mathbb{R}$ it follows from this discussion that for any orthogonal representation $\alpha\colon \pi \to \mathrm{O}(n)$ we have $\alpha = \alpha^\dagger$ and for $R = \mathbb{C}$ it follows from this discussion that for any unitary representation $\alpha\colon \pi \to \mathrm{U}(n)$ we have $\alpha = \alpha^\dagger$.

(2) Let R be a commutative ring. We consider the pairing

$$\begin{array}{rcl} \langle -,-\rangle\colon R^2 \times R^2 & \to & R \\ \left(\begin{pmatrix} a \\ b\end{pmatrix}, \begin{pmatrix} c \\ d\end{pmatrix}\right) & \mapsto & \det\begin{pmatrix} a & c \\ b & d\end{pmatrix}. \end{array}$$

In Exercise 15.2 we will show that this R-bilinear pairing is non-singular and we will show that for any representation $\alpha\colon \pi \to \mathrm{GL}(2,R)$ we have $\alpha^\dagger(g) = \det(\alpha(g))^{-1}\cdot\alpha(g)$. In particular, if we are given a representation $\alpha\colon \pi \to \mathrm{SL}(2,R)$, then $\alpha^\dagger = \alpha$. □

Proof. Let $g \in \pi$ and let $m' \in M$. Since the pairing is sesquilinear we see that the map

$$\begin{array}{rcl} M & \to & R \\ m & \mapsto & \langle \alpha(g)(m), m'\rangle \end{array}$$

defines an R-linear homomorphism $M \to R$. Since the pairing is non-singular we see that there exists a unique $\widetilde{m} \in M$ such that $\langle m, \widetilde{m}\rangle = \langle \alpha(g)(m), m'\rangle$ for all $m' \in M$. We now set $(\alpha^\dagger(g^{-1}))(m') := \widetilde{m}$. We leave it to the reader that this way we defined a homomorphism $\alpha^\dagger(g)\colon M \to M$, that this defines a homomorphism $\alpha^\dagger\colon \pi \to \mathrm{Aut}_R(M)$ and that this homomorphism has the desired properties. ■

After all these preparations we can now finally formulate the following theorem which, by the discussion on page 215, can be viewed as a generalization of the Alexander Polynomial–Symmetry Theorem 9.4:

Theorem 15.12. (Twisted Alexander Polynomial–Symmetry Theorem) Let L be an oriented m-component link. Furthermore let R be a UFD with involution and let $\alpha\colon \pi_1(S^3 \setminus L \to \mathrm{Aut}_R(M)$ be a representation. For any Hermitian adjoint representation $\alpha^\dagger$ as above we have

$$\Delta_L^\alpha(t_1,\dots,t_m) \ \doteq_R\ \Delta_L^{\alpha^\dagger}(t_1^{-1},\dots,t_m^{-1}).$$

Proof. This theorem is proved in [**HSW10**, Theorem 3.2], building on [**Mil62**, Lemma 2]. We also refer to [**Kit96**, Theorem B], [**Hil12**, Theorem 6.7], and [**FKK12**, Theorem 1.1] for related results.

The courageous reader can also try to prove the theorem, using the approach taken in our proof of the Alexander Polynomial–Symmetry Theorem 9.4. ■

The following summarizes the special cases one mostly cares about:

Corollary 15.13. (Twisted Alexander Polynomial–Symmetry Corollary) Let L be an oriented m-component link L. We suppose that we are given one of the following types of presentations:

(1) (a) An orthogonal representation $\alpha\colon \pi_1(S^3 \setminus L \to \mathrm{O}(n)$, where $R = \mathbb{R}$.
  (b) A unitary representation $\alpha\colon \pi_1(S^3 \setminus L \to \mathrm{U}(n)$, where $R = \mathbb{C}$.
(2) A representation $\alpha\colon \pi_1(S^3 \setminus L) \to \mathrm{SL}(2, R)$ for some UFD R.

Then
$$\Delta_L^\alpha(t_1, \dots, t_m) \ \doteq_R \ \Delta_L^\alpha(t_1^{-1}, \dots, t_m^{-1}).$$

Proof. The corollary follows immediately from the Twisted Alexander Polynomial–Symmetry Theorem 15.12 together with the examples on page 228. ■

Example. We consider again the trefoil T and as on page 18 we denote by T^{rev} the trefoil with the opposite orientation. Next recall that in Section 14.2 we introduced a representation $\alpha\colon \pi_1(S^3 \setminus T) \to \mathrm{GL}(2, \mathbb{Z})$ and we saw that $\Delta_T^\alpha(t) = 1 - t^2$. We now see that

$$\begin{array}{ccccccccc}
& \begin{array}{c}\text{by the Twisted Alexander}\\ \text{Polynomial Orientation}\\ \text{Lemma 15.2}\end{array} & & \begin{array}{c}\text{by the Twisted Alexander}\\ \text{Polynomial Symmetry}\\ \text{Theorem 15.12}\end{array} & & \begin{array}{l}\text{by the example on page 228}\\ \text{where } \rho\colon \pi_1(S^3 \setminus K) \to \{\pm 1\}\\ \text{is given by } \rho(t) = -1\end{array} & & & \\
& \downarrow & & \downarrow & & \downarrow & & & \\
\Delta_{T^{\mathrm{rev}}}^\alpha(t) & \doteq & \Delta_T^\alpha(t^{-1}) & \doteq & \Delta_T^{\alpha^\dagger}(t) & \doteq & \Delta_T^{\rho\cdot\alpha}(t) & & \\
& \doteq & \rho_*(\Delta_T^\alpha(t)) & = & \Delta_T^\alpha(-t) & = & 1 - (-t)^2 & = & 1 - t^2. \\
& \uparrow & & \uparrow & & & & & \\
& \begin{array}{l}\text{by the Modified Twisted}\\ \text{Alexander Polynomial}\\ \text{Lemma 14.11}\end{array} & & \begin{array}{l}\text{follows easily from the}\\ \text{definitions and } \rho(t) = -1\end{array} & & & & &
\end{array}$$

We thus see that $\Delta_{T^{\mathrm{rev}}}^\alpha(t) \doteq \Delta_T^\alpha(t)$. But that is perfectly reasonable since in Exercise 2.1 we showed that the trefoil is reversible, i.e. it is smoothly isotopic to its reverse. □

Exercises for Chapter 15.

Exercise 15.1. Let H be the Hopf link. Show that every twisted Alexander polynomial equals, up to the indeterminacy, 1.

Exercise 15.2. Let R be a commutative ring. We consider the form
$$\begin{array}{rcl} \langle -, - \rangle\colon R^2 \times R^2 & \to & R \\ \left(\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix} \right) & \mapsto & \det \begin{pmatrix} a & c \\ b & d \end{pmatrix}. \end{array}$$

(a) Show that this pairing is non-singular.
(b) Show that for any representation $\alpha\colon \pi \to \mathrm{GL}(2, R)$ and any $g \in \pi$ we have the equality $\alpha^\dagger(g) = \det(\alpha(g))^{-1} \cdot \alpha(g)$.

Exercise 15.3. Let L be an oriented m-component link and let R be a UFD. Show that if L is splittable (in the sense of the definition on page 20), then for any representation $\alpha\colon \pi_1(S^3 \setminus L) \to \mathrm{Aut}_R(M)$ we have $\Delta_L^\alpha = 0$.
Remark. In [**FV15**] the following converse is shown: If L is not splittable then there exists a representation $\pi_1(S^3 \setminus L) \to \mathrm{GL}(k, \mathbb{Q})$ such that $\Delta_L^\alpha \neq 0$.

Exercise 15.4. Let $p, q \in \mathbb{N}$ be coprime. We consider the torus knot $T(p,q)$. Recall that by the Torus Knot-π_1-Proposition 3.8 we have an isomorphism

$$\pi_1(S^3 \setminus T(p,q)) \;\cong\; \underbrace{\langle x, y \mid x^p \cdot y^{-q} \rangle}_{=:\pi}.$$

(a) Show that there exists a unique representation

$$\alpha\colon \langle x, y \mid x^p \cdot y^{-q} \rangle \;\to\; \mathrm{GL}(n, \mathbb{C})$$

such that

$$\alpha(x) = \begin{pmatrix} 0 & 1 & 0 \dots \end{pmatrix} \qquad \text{and} \qquad \alpha(y) = \begin{pmatrix} \exp(2\pi\,\mathrm{i}/q) & 0 & \dots & 0 \\ 0 & \ddots & 0 & \vdots \\ \vdots & & \ddots & 0 \\ 0 & & & \exp(2\pi\,\mathrm{i}/q) \end{pmatrix}.$$

(b) Compute $\Delta^{\alpha}_{T(p,q)}$.
Hint. Modify the proof of the Torus Knot–Alexander Polynomial Proposition 9.6.

(c) Show that for $p = 2$ and $q = 3$ the representation from (a) is conjugate (over $\mathbb{C}$) to the representation $\pi_1(S^3 \setminus K) \to \mathrm{GL}(2, \mathbb{Z})$ that is given in Section 14.2.

Exercise 15.5. Let K_1 and K_2 be two oriented knots, let R be a commutative ring and let $\pi_1(S^3 \setminus (K_1 \# K_2)) \to \mathrm{GL}(n, R)$ be a representation. In the proof of the Knot Connected Sum-π_1-Proposition 3.15 we gave an explicit isomorphism

$$\pi_1(S^3 \setminus K_1) *_{\mu_{K_1} = \mu_{K_2}} \pi_1(S^3 \setminus K_2) \;\xrightarrow{\cong}\; \pi_1(S^3 \setminus (K_1 \# K_2)).$$

We can use this isomorphism to obtain homomorphisms $\rho_i\colon \pi_1(S^3 \setminus K_i) \to \pi_1(S^3 \setminus (K_1 \# K_2))$. Determine the relationship between the twisted Alexander polynomials $\Delta^{\alpha\circ\rho_1}_{K_1}(t)$, $\Delta^{\alpha\circ\rho_2}_{K_2}(t)$ and $\Delta^{\alpha}_{K_1 \# K_2}(t)$.

Remark. This exercise is supposed to generalize the Connected Sum–Alexander Polynomial Proposition 9.8. Note though that we do not just have a completely straightforward product formula.

CHAPTER 16

Different notions of equivalence of links

Let L and $\widetilde{L}$ be two links. If L and $\widetilde{L}$ are smoothly isotopic, then we know by the Link–Smooth Isotopy Proposition 2.3, the Link Exterior Lemma 2.16 and the Isotopic Link-π_1-Lemma 3.2 that the following statements hold:

(1) There exists an orientation-preserving diffeomorphism $S^3 \setminus L \to S^3 \setminus \widetilde{L}$ between the link complements.

(2) There exists an orientation-preserving diffeomorphism $X_L \to X_{\widetilde{L}}$ between the link exteriors.

(3) There exists isomorphisms $\pi_1(S^3 \setminus L) \cong \pi_1(S^3 \setminus \widetilde{L})$ and $\pi_1(X_L) \cong \pi_1(X_{\widetilde{L}})$.

It is natural to ask to what degree the converses of these three conclusions results hold. We will discuss this question in great detail in this chapter.

16.1. **Comparing different notions of equivalence of links.** Before we can state our main result we need to introduce a few rather dry notions. We start out with the following definition which extends the notion of a group-pair system from page 97:

Definition.

(1) A group-pair system is a tuple $(G, P_1, \dots, P_m)$ where G is a group and $P_1, \dots, P_m$ are ordered pairs of elements of G.

(2) We say that two group-pair systems $(G, P_1, \dots, P_m)$ and $(H, Q_1, \dots, Q_m)$ are isomorphic if there exists an isomorphism $\varphi\colon G \to H$ such that for each $i \in \{1, \dots, m\}$ there exists an $h_i \in H$ with $h_i \cdot \varphi(P_i) \cdot h_i^{-1} = Q_i$ as an ordered pair of elements of H. In this case we write $(G, P_1, \dots, P_m) \cong (H, Q_1, \dots, Q_m)$.

Next we introduce a closely related concept:

Definition.

(1) A subgroup system is a tuple $(G, S_1, \dots, S_m)$ where G is a group and $S_1, \dots, S_m$ are subgroups of G.

(2) We say that two subgroup systems $(G, S_1, \dots, S_m)$ and $(H, T_1, \dots, T_m)$ are isomorphic if there exists an isomorphism $\varphi\colon G \to H$ such that for each $i \in \{1, \dots, m\}$ the group $\varphi(S_i)$ is conjugate to T_i. In this case we write $(G, S_1, \dots, S_m) \cong (H, T_1, \dots, T_m)$.

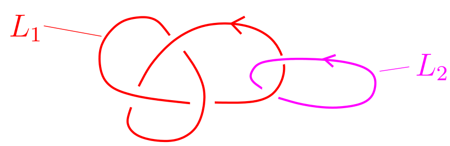
We now return to the study of links:

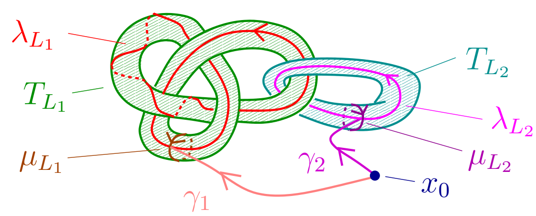
Definition. Let L be an oriented m-component link. We pick an orientation-preserving tubular map $\tau\colon \overline{B}^2 \times L \to S^3$.

(1) We define the exterior $X_L := S^3 \setminus \tau(B^2 \times L)$.

(2) For $i = 1, \ldots, m$ we do the following:
  (a) We denote by T_{L_i} the boundary torus of X_L corresponding to the i-th component of L, i.e. $T_{L_i} = \tau(S^1 \times L_i)$. We denote by $\iota_i \colon T_{L_i} \to X_L$ the inclusion map.
  (b) We pick $y_i \in L_i$ and we define the meridian $\mu_{L_i} := \tau(S^1 \times \{y_i\}) \subset T_{L_i}$.
  (c) We define the longitude $\lambda_{L_i} \subset \partial T_{L_i}$ as on page 96.

(3) Let $x_0 \in X_L$.
  (a) For each boundary component we pick a path γ_i from x_0 to $\tau(1, y_i)$.
  (b) We use this common path to view μ_{L_i} and λ_{L_i} as elements of $\pi_1(X_L, x_0)$. Note that as a pair $(\mu_{L_i}, \lambda_{L_i}) \subset \pi_1(X_L, x_0)$ is well-defined up to a *common* conjugation. The group-pair system $(\pi_1(X_L), \mu_L, \lambda_L)$ of the link L is defined as the group-pair system $(\pi_1(X_L, x_0), (\mu_{L_1}, \lambda_{L_1}), \ldots, (\mu_{L_m}, \lambda_{L_m}))$.[109] [110]
  (c) The peripheral subgroup system $(\pi_1(X_L), \pi_1(T_L))$ of the link L is defined as the fundamental group $\pi_1(X_L, x_0)$ together with the (ordered) family of subgroups $\gamma_{i*}(\iota_{i*}(\pi_1(T_{L_i}), \tau(1, y_i)))$ with $i = 1, \ldots, m$.[111]





Throughout this chapter we adopt the following convention:

Convention. If we talk about the exterior, meridians and longitudes of an oriented link it is understood that we used the same orientation-preserving tubular map to define them.

Now we can formulate the main result of this chapter:

Theorem 16.1. (Link Equivalence Theorem) Let L and $\widetilde{L}$ be two oriented m-component links. We pick orientation-preserving tubular maps for L and $\widetilde{L}$ to define the exteriors X_L and $X_{\widetilde{L}}$ and for $i = 1, \ldots, m$ to define the boundary tori T_{L_i}, $T_{\widetilde{L}_i}$, the meridians μ_{L_i}, $\mu_{\widetilde{L}_i}$ and the longitudes λ_{L_i}, $\lambda_{\widetilde{L}_i}$. We consider the following statements:

[109] Note that this definition generalizes the notion of a group-pair system from page 98 from oriented knots to links.

[110] It follows easily from the Link Tubular Map Theorem 2.15 (2) that the isomorphism type of the group-pair system of an oriented link L is well-defined up to isomorphism.

[111] It follows easily from the Link Tubular Map Theorem 2.15 (2) that the isomorphism type of the peripheral subgroup system of an oriented link L is well-defined up to isomorphism.

$L \cong \widetilde{L}$	L and $\widetilde{L}$ are smoothly isotopic.
$(S^3, L) \cong (S^3, \widetilde{L})$	There exists a diffeomorphism $f\colon S^3 \to S^3$ such that for every $i \in \{1, \ldots, m\}$ we have $f(L_i) = \widetilde{L}_i$ as oriented smooth submanifolds.
$X_L \cong X_{\widetilde{L}}$	There exists a diffeomorphism $f\colon X_L \to X_{\widetilde{L}}$.
$(X_L, \mu_L) \cong (X_{\widetilde{L}}, \mu_{\widetilde{L}})$	There exists a diffeomorphism $f\colon X_L \to X_{\widetilde{L}}$ such that for every $i \in \{1, \ldots, m\}$ we have $f(\mu_{L_i}) = \mu_{\widetilde{L}_i}$ as oriented smooth submanifolds.
$(X_L, \mu_L) \cong (X_{\widetilde{L}}, \pm\mu_{\widetilde{L}})$	There exists a diffeomorphism $f\colon X_L \to X_{\widetilde{L}}$ such that for every $i \in \{1, \ldots, m\}$ we have $f(\mu_{L_i}) = \mu_{\widetilde{L}_i}$ as *unoriented* smooth submanifolds.
$(X_L, \mu_L, \lambda_L) \cong (X_{\widetilde{L}}, \mu_{\widetilde{L}}, \lambda_{\widetilde{L}})$	There exists a diffeomorphism $f\colon X_L \to X_{\widetilde{L}}$ such that for every $i \in \{1, \ldots, m\}$ we have $f(\mu_{L_i}) = \mu_{\widetilde{L}_i}$ and we have $f(\lambda_{L_i}) = \lambda_{\widetilde{L}_i}$ as oriented smooth submanifolds.
$(X_L, T_{L_i}) \cong (X_{\widetilde{L}}, T_{\widetilde{L}_i})$	There exists a diffeomorphism $f\colon X_L \to X_{\widetilde{L}}$ such that for every $i \in \{1, \ldots, m\}$ we have $f(T_{L_i}) = T_{\widetilde{L}_i}$.
$X_L \simeq X_{\widetilde{L}}$	X_L and $X_{\widetilde{L}}$ are homotopy equivalent.
$\pi_1(X_L) \cong \pi_1(X_{\widetilde{L}})$	The groups $\pi_1(X_L)$ and $\pi_1(X_{\widetilde{L}})$ are isomorphic.
$(\pi_1(X_L), \mu_L, \lambda_L) \cong (\pi_1(X_{\widetilde{L}}), \mu_{\widetilde{L}}, \lambda_{\widetilde{L}})$	The group-pair systems are isomorphic.
$(\pi_1(X_L), \pi_1(T_L)) \cong (\pi_1(X_{\widetilde{L}}), \pi_1(T_{\widetilde{L}}))$	The subgroup systems are isomorphic.

We will also use the following variations on the above:

(1) If we write $\cong$, then we mean an orientation-preserving diffeomorphism.
(2) If we write $\overset{\text{or}}{\underset{\text{homeo}}{\cong}}$, then we mean homeomorphism instead of diffeomorphism.
(3) In some of the above statements we replace the link exteriors by the link complements.

With this notation we can formulate the following implications:

$$
\begin{tikzcd}
 & & L \cong \widetilde{L} \arrow[d, Leftrightarrow, "(1)"] & \\
 & & (S^3,L) \underset{\text{or.}}{\cong} (S^3,\widetilde{L}) \arrow[dll, Rightarrow, "(2a)"'] & \\
(X_L,\mu_L,\lambda_L) \cong (X_{\widetilde{L}},\mu_{\widetilde{L}},\lambda_{\widetilde{L}}) \arrow[rr, Rightarrow, "(2b)"'] \arrow[dd, Rightarrow, "(12a)"] & & (X_L,\mu_L) \underset{\text{or.}}{\cong} (X_{\widetilde{L}},\mu_{\widetilde{L}}) \arrow[u, Rightarrow, "(2c)"'] \arrow[d, Rightarrow, "(3)"] & S^3 \setminus L \underset{\text{or.}}{\cong} S^3 \setminus \widetilde{L} \arrow[d, Leftrightarrow, "(5)"] \\
 & & (X_L,\mu_L) \underset{\text{or.}}{\cong} (X_{\widetilde{L}},\pm\mu_{\widetilde{L}}) \arrow[r, Rightarrow, "(4a)"'] \arrow[d, Rightarrow, "\text{(10a) loose control over the orientations of the components}"] & X_L \underset{\text{or.}}{\cong} X_{\widetilde{L}}. \arrow[l, Rightarrow, bend right, "\text{(4b) for knots}"'] \\
(\pi_1(X_L),\mu_L,\lambda_L) \cong (\pi_1(X_{\widetilde{L}}),\mu_{\widetilde{L}},\lambda_{\widetilde{L}}) \arrow[uu, Rightarrow, bend left, "\text{(12b) if } L,\widetilde{L} \text{ are not splittable}^{112}"] & & (X_L,T_L) \underset{\text{or.}}{\cong} (X_{\widetilde{L}},T_{\widetilde{L}}) \arrow[u, Rightarrow, bend left, "\text{(10b) not for links}"] \arrow[ur, Rightarrow, "\text{(11), loose control over the order of the components}"'] &
\end{tikzcd}
$$

If we do not care about the difference between links and their mirror images, then we also have the following implications:

$$
\begin{tikzcd}
L \cong \widetilde{L} \text{ or } L \cong \widetilde{L}^{\text{mir}} \arrow[d, Leftrightarrow, "(1)"] & & \\
(S^3,L) \cong (S^3,\widetilde{L}) \arrow[d, Leftrightarrow, "(2)"] & & \\
(X_L,\mu_L) \cong (X_{\widetilde{L}},\mu_{\widetilde{L}}) \arrow[d, Rightarrow, "(3)"] & & \\
(X_L,\mu_L) \cong (X_{\widetilde{L}},\pm\mu_{\widetilde{L}}) \arrow[r, Rightarrow, "(4a)"'] \arrow[d, Rightarrow, "\text{(10a) loose control over the orientations of the components}"] & X_L \cong X_{\widetilde{L}} \arrow[l, Rightarrow, bend right, "\text{(4b) for knots}"'] \arrow[r, Leftrightarrow, "(5)"] \arrow[d, Leftrightarrow, "(6')"] & S^3 \setminus L \cong S^3 \setminus \widetilde{L} \arrow[d, Leftrightarrow, "(6'')"'] \\
(X_L,T_L) \cong (X_{\widetilde{L}},T_{\widetilde{L}}) \arrow[u, Rightarrow, bend left, "\text{(10b) not for links}"] \arrow[ur, Rightarrow, "(11)"'] \arrow[dd, Rightarrow, "(12a)"] & X_L \underset{\text{homeo.}}{\cong} X_{\widetilde{L}} \arrow[d, Rightarrow, "(9a)"] & S^3 \setminus L \underset{\text{homeo.}}{\cong} S^3 \setminus \widetilde{L} \\
 & X_L \simeq X_{\widetilde{L}} \arrow[u, Rightarrow, bend left, "\text{(9b) if } L,\widetilde{L} \text{ are prime knots}"] \arrow[r, Leftrightarrow, "(8')"] \arrow[d, Rightarrow, "(7a)"] & S^3 \setminus L \simeq S^3 \setminus \widetilde{L} \\
(\pi_1(X_L),\pi_1(T_L)) \cong (\pi_1(X_{\widetilde{L}}),\pi_1(T_{\widetilde{L}})) \arrow[uu, Rightarrow, bend left, "\text{(12b) if } L,\widetilde{L} \text{ are not splittable}^{112}"] & \pi_1(X_L) \cong \pi_1(X_{\widetilde{L}}) \arrow[u, Rightarrow, bend left, "\text{(7b) if } L,\widetilde{L} \text{ are not splittable}^{112}"] \arrow[r, Leftrightarrow, "(8'')"] & \pi_1(S^3 \setminus L) \cong \pi_1(S^3 \setminus \widetilde{L}).
\end{tikzcd}
$$

The proof of the Link Equivalence Theorem 16.1 will require the remainder of this chapter. It is unfortunately unavoidable that we will have to refer to many results which we will prove and discuss in later chapters. Furthermore we will also have to cite lots of results which we will not prove at all.

Even when in the following we provide proofs of statements our arguments will be more sketchy than in the other chapters.

16.2. Proof of the Link Equivalence Theorem 16.1.

In the following twelve subsections we consider all the implications of the Link Equivalence Theorem 16.1.

16.2.1. Proof of (1).

The proof of the Link Equivalence Theorem 16.1 (1) rests on some mundane observations and one very hard result. The latter is worth stating separately:

Theorem 16.2. (Cerf Theorem) Every orientation-preserving diffeomorphism of S^3 is diffeotopic to the identity.

Proof. The theorem was proved by Jean Cerf [**Cer68**], a generalization of the theorem was proved by Allan Hatcher [**Hat83**, p. 553]. ■

Proof of the Link Equivalence Theorem 16.1 (1). Let L and $\widetilde{L}$ be two oriented m-component links. We need to prove the following two statements:

$$L \cong \widetilde{L} \iff (S^3, L) \underset{\text{or.}}{\cong} (S^3, \widetilde{L}) \quad \text{and} \quad L \cong \widetilde{L} \text{ or } L \cong L^{\text{mir}} \iff (S^3, L) \cong (S^3, \widetilde{L}).$$

"$\Rightarrow$". We first suppose that $L \cong \widetilde{L}$, i.e. we assume that L and $\widetilde{L}$ are smoothly isotopic. As we discussed in the proof of the Link–Smooth Isotopy Proposition 2.3, it follows from the Isotopy Extension Theorem 2.4 that there exists an orientation-preserving diffeomorphism $\Phi\colon S^3 \to S^3$ such that for $i = 1, \dots, m$ we have $\Phi(L_i) = \widetilde{L}_i$ as oriented smooth manifolds. We have thus shown that $(S^3, L) \underset{\text{or.}}{\cong} (S^3, \widetilde{L})$.

If L and $\widetilde{L}^{\text{mir}}$ are smoothly isotopic, then we obtain from the above discussion that there exists an orientation-preserving diffeomorphism $(S^3, L) \to (S^3, \widetilde{L}^{\text{mir}})$. If we compose this diffeomorphism with the reflection in a suitable hyperplane we obtain an an orientation-reversing diffeomorphism $(S^3, L) \to (S^3, \widetilde{L})$.

"$\Leftarrow$". First we assume that $(S^3, L) \underset{\text{or.}}{\cong} (S^3, \widetilde{L})$. Thus let $\Phi\colon S^3 \to S^3$ be an *orientation-preserving* diffeomorphism such that for $i = 1, \dots, m$ we have $\Phi(L_i) = \widetilde{L}_i$ as oriented smooth submanifolds. Note that by the above Cerf Theorem 16.2 there exists a diffeotopy $F\colon S^3 \times [0,1] \to S^3$ with $F_0 = \text{id}$ and $F_1 = \Phi$. The restriction of F to $L \times [0,1]$ is the desired smooth isotopy from L to $\widetilde{L}$. We have thus shown that $L \cong \widetilde{L}$.

Finally we suppose that there exists an orientation-reversing diffeomorphism $\Phi\colon S^3 \to S^3$ such that for $i = 1, \dots, m$ we have $\Phi(L_i) = \widetilde{L}_i$ as oriented smooth submanifolds. If we compose this diffeomorphism with a reflection in a hyperplane, then we obtain an orientation-preserving diffeomorphism $\Phi\colon S^3 \to S^3$ such that for $i = 1, \dots, m$ we have $\Phi(L_i) = \widetilde{L}_i^{\text{mir}}$ as oriented smooth submanifolds. By the above this implies that $L \cong \widetilde{L}^{\text{mir}}$. ■

[112] It follows from the Link Kneser Theorem 3.12 that if L and $\widetilde{L}$ are two links with $\pi_1(X_L) \cong \pi_1(\widetilde{X}_L)$, then L is splittable if and only if $\widetilde{L}$ is splittable.

16.2.2. **Proof of (2).** In this subsection we need to show the following implications:

$$\begin{tikzcd}[column sep=small] & & (S^3,L) \underset{\text{or.}}{\cong} (S^3,\widetilde{L}) \arrow[dll, Rightarrow, "\text{(2a)}"'] \\ (X_L,\mu_L,\lambda_L) \cong (X_{\widetilde{L}},\mu_{\widetilde{L}},\lambda_{\widetilde{L}}) \arrow[rr, Rightarrow, "\text{(2b)}"'] & & (X_L,\mu_L) \underset{\text{or.}}{\cong} (X_{\widetilde{L}},\mu_{\widetilde{L}}) \arrow[u, Rightarrow, "\text{(2c)}"'] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} (S^3,L) \cong (S^3,\widetilde{L}) \arrow[d, Leftrightarrow, "\text{(2)}"] \\ (X_L,\mu_L) \cong (X_{\widetilde{L}},\mu_{\widetilde{L}}). \end{tikzcd}$$

In the proof we will need the following theorem on self-diffeomorphisms of the 2-dimensional torus:

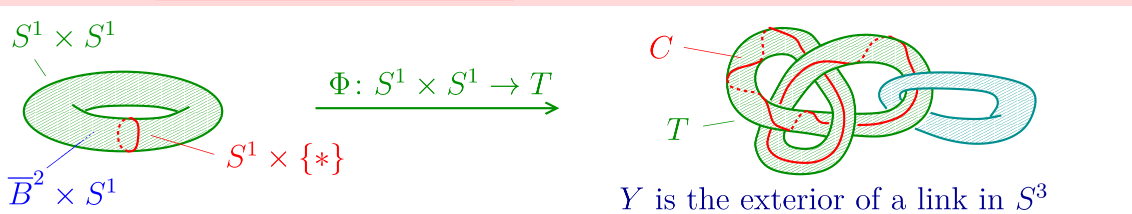
Theorem 16.3. (Torus–Mapping Class Group Theorem)
(1) Given any automorphism $\varphi \in \mathrm{Aut}(\pi_1(S^1 \times S^1, (1,1)))$ there exists a diffeomorphism $f\colon S^1 \times S^1 \to S^1 \times S^1$ with $f_* = \varphi$.
(2) If $\varphi\colon S^1 \times S^1 \to S^1 \times S^1$ is a diffeomorphism such that $\varphi_* = \mathrm{id}$ on $\pi_1(S^1 \times S^1, (1,1))$, then φ is diffeotopic to the identity.

Proof.

(1) For the proof it is best to replace $S^1 \times S^1$ with $T = \mathbb{R}^2/\mathbb{Z}^2$. We make the usual identification $\pi_1(T, [(0,0)]) = \mathbb{Z}^2$. Now let $A \in \mathrm{Aut}(\pi_1(T,[(0,0)])) = \mathrm{Aut}(\mathbb{Z}^2) = \mathrm{GL}(2,\mathbb{Z})$. The map
$$\begin{array}{rcl} f\colon \mathbb{R}^2/\mathbb{Z}^2 & \mapsto & \mathbb{R}^2/\mathbb{Z}^2 \\ [v] & \mapsto & [A \cdot v] \end{array}$$
is clearly a diffeomorphism. Note that f_* is given by multiplication by A.
(2) This statement is proved in [**FM11**, Theorem 2.5] or alternatively in [**Rol90**, Chapter 2.D]. ■

We use the Torus–Self Diffeomorphism Theorem 16.3 to introduce a construction which is very popular in low-dimensional topology:

Proposition 16.4. (Dehn Filling Proposition) Let Y be a 3-dimensional smooth manifold and let T be a component of the boundary ∂Y such that T is diffeomorphic to the 2-dimensional torus. Furthermore let $C \subset T$ be a submanifold such that $[C] \in \pi_1(T)$ can be extended to a basis of $\pi_1(T) \cong \mathbb{Z}^2$.
(1) There exists a diffeomorphism $\Phi\colon S^1 \times S^1 \to \partial Y$ with $\Phi(S^1 \times \{*\}) = C$.
(2) The diffeomorphism type of
$$Y(C) := Y \cup_{\partial Y \leftarrow S^1 \times S^1 : \Phi} (\overline{B}^2 \times S^1) = (Y \sqcup (\overline{B}^2 \times S^1))/\underset{\substack{\uparrow \\ \text{where } \Phi(z,w) \sim (z,w) \text{ for } (z,w) \in S^1 \times S^1}}{\sim}$$
does not depend on the choice of Φ. We refer to the 3-dimensional smooth manifold $Y(C)$ as the Dehn filling of Y along C.



Sketch of proof.

(1) By hypothesis we can extend $[C] \in \pi_1(T)$ to a basis of $\pi_1(T)$. The statement now follows quite easily from the Torus–Self Diffeomorphism Theorem 16.3 (1) and (2).

(2) Let $\Phi, \Psi \colon S^1 \times S^1 \to \partial Y$ be two diffeomorphisms with $\Phi(S^1 \times \{*\}) = C$ and with $\Psi(S^1 \times \{*\}) = C$. We make the following two observations:
  (a) Note that diffeotopic diffeomorphisms $S^1 \times S^1 \to T$ lead to diffeomorphic results.
  (b) It follows easily from the Torus–Self Diffeomorphism Theorem 16.3 (2) that, up to smooth isotopy, $\Psi^{-1} \circ \Phi \colon S^1 \times S^1 \to S^1 \times S^1$ is of the form
$$\begin{array}{rcl} \Theta \colon S^1 \times S^1 & \to & S^1 \times S^1 \\ (z, w) & \mapsto & (z \cdot w^k, w^\epsilon) \end{array}$$
  for some $k \in \mathbb{Z}$ and some $\epsilon \in \{-1, 1\}$. Note that this diffeomorphism extends to a diffeomorphism
$$\begin{array}{rcl} \Theta \colon \overline{B}^2 \times S^1 & \to & \overline{B}^2 \times S^1 \\ (z, w) & \mapsto & (z \cdot w^k, w^\epsilon) \end{array}$$

Using these two observations one can easily show that Φ and Ψ lead to diffeomorphic results. ■

Example. Let $L \subset S^3$ be an m-component link with link exterior X_L, let L_i be a component of L and let $\mu_i \subset \partial X_L$ be a meridian of L_i. We consider the Dehn filling $X(\mu_i)$ as defined in the Dehn Filling Proposition 16.4. In Exercise 16.1 we will see that there exists an orientation-preserving $X_L(\mu_i) \to X_{L \setminus L_i}$ which is the identity on X_L and which sends $\{0\} \times S^1$ to L_i as oriented smooth submanifolds. □

Proof of the Link Equivalence Theorem 16.1 (2). Let L and $\widetilde{L}$ be two oriented m-component links. We need to show the following implications:

$$\begin{tikzcd} & (S^3, L) \underset{\text{or.}}{\cong} (S^3, \widetilde{L}) \arrow[dl, Rightarrow, "(2a)"'] \\ (X_L, \mu_L, \lambda_L) \cong (X_{\widetilde{L}}, \mu_{\widetilde{L}}, \lambda_{\widetilde{L}}) \arrow[r, Rightarrow, "(2b)"'] & (X_L, \mu_L) \underset{\text{or.}}{\cong} (X_{\widetilde{L}}, \mu_{\widetilde{L}}) \arrow[u, Rightarrow, "(2c)"'] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} (S^3, L) \cong (S^3, \widetilde{L}) \arrow[d, Leftrightarrow, "(2)"] \\ (X_L, \mu_L) \cong (X_{\widetilde{L}}, \mu_{\widetilde{L}}). \end{tikzcd}$$

In the following we will prove the oriented statements to the left. We leave it to the reader to modify the argument to deal with the unoriented statement to the right.

(2a). In the following we suppose that $(S^3, L) \underset{\text{or.}}{\cong} (S^3, \widetilde{L})$, i.e. we suppose that there exists an orientation-preserving diffeomorphism $f \colon S^3 \to S^3$ such that for every $i \in \{1, \dots, m\}$ we have $f(L_i) = \widetilde{L}_i$ as oriented smooth submanifolds. We pick an orientation-preserving tubular map $\tau \colon \overline{B}^2 \times L \to S^3$ for L. It is clear that $(f \circ \tau) \circ (\mathrm{id} \circ (f|_L)^{-1}) \colon \overline{B}^2 \times \widetilde{L} \to S^3$ is an orientation-preserving tubular map for $\widetilde{L}$. If we use these tubular maps to define the exteriors, the meridians and the longitudes we see that $f \colon S^3 \to S^3$ restricts to an orientation-preserving diffeomorphism $f \colon X_L \to X_{\widetilde{L}}$ such that for every $i \in \{1, \dots, m\}$ we have $f(\mu_{L_i}) = \mu_{\widetilde{L}_i}$ and we have $f(\lambda_{L_i}) = \lambda_{\widetilde{L}_i}$ as oriented smooth submanifolds. Thus we have shown that $(X_L, \mu_L, \lambda_L) \cong (X_{\widetilde{L}}, \mu_{\widetilde{L}}, \lambda_{\widetilde{L}})$.

(2b). We now suppose that $(X_L, \mu_L, \lambda_L) \cong (X_{\widetilde{L}}, \mu_{\widetilde{L}}, \lambda_{\widetilde{L}})$, i.e. we suppose that there exists a diffeomorphism $f \colon X_L \to X_{\widetilde{L}}$ such that for $i = 1, \dots, m$ we have $f(\mu_{L_i}) = \mu_{\widetilde{L}_i}$ and $f(\lambda_{L_i}) = \lambda_{\widetilde{L}_i}$ as oriented smooth submanifolds. It follows easily that the restriction of f to

$\partial X_L \to \partial X_{\widetilde{L}}$ is orientation-preserving. This implies that the original map $f\colon X_L \to X_{\widetilde{L}}$ is also orientation-preserving. Thus we have shown that $(X_L,\mu_L) \underset{\text{or.}}{\cong} (X_{\widetilde{L}},\mu_{\widetilde{L}})$.
(2c). We suppose that $(X_L,\mu_L) \underset{\text{or.}}{\cong} (X_{\widetilde{L}},\mu_{\widetilde{L}})$, i.e. we suppose that there exists an orientation-preserving diffeomorphism $f\colon X_L \to X_{\widetilde{L}}$ such that for every $i \in \{1,\dots,m\}$ we have $f(\mu_{L_i}) = \mu_{\widetilde{L}_i}$ as oriented smooth manifolds.

Iterating the example preceding the proof we see that $f\colon X_L \to X_{\widetilde{L}}$ extends to an orientation-preserving diffeomorphism $f\colon S^3 \to S^3$ such that for every $i \in \{1,\dots,m\}$ the map f restricts to an orientation-preserving diffeomorphism $f\colon L_i \to \widetilde{L}_i$. Thus we have shown that $(S^3,L) \underset{\text{or.}}{\cong} (S^3,\widetilde{L})$. ■

16.2.3. **Proof of (3).** We need to show that

$$(X_L,\mu_L) \quad\Longrightarrow\quad (X_{\widetilde{L}},\pm\mu_{\widetilde{L}}).$$

In fact, this is of course trivial. It is perhaps more important to say what this step does: In the process we loose control over the orientation of the components of the link. In general the step is not reversible since we saw that on page 101 there exist non-reversible knots, e.g. the Conway knot. For this knot K and its reverse K^{rev} we know, by the Meridian–Symmetries Lemma 2.18, that $(X_K,\mu_K) = (X_K,-(\mu_K)^{\text{rev}}) \cong (X_{K^{\text{rev}}},-\mu_{K^{\text{rev}}})$, but we have $K \not\cong K^{\text{rev}}$, thus it follows from (1) and (2) that we do *not* have $(X_K,\mu_K) \cong (X_{K^{\text{rev}}},\mu_{K^{\text{rev}}})$.

16.2.4. **Proof of (4).** In this subsection we want to show

$$\begin{tikzcd}(X_L,\mu_L) \cong (X_{\widetilde{L}},\pm\mu_{\widetilde{L}}) \arrow[r, Rightarrow, shift right, "\text{(4a)}"'] & X_L \cong X_{\widetilde{L}} \arrow[l, Rightarrow, shift right, bend right, "\text{(4b for knots}"']\end{tikzcd}$$

and we want to prove the corresponding oriented version. The statement (4a) is of course trivial. So it remains to deal with (4b). The proof of the Link Equivalence Theorem 16.1 (4b) rests squarely on the following theorem:

Theorem 16.5. (Gordon-Luecke Theorem) Let K be a knot and let $C \subset \partial X_K$ be an oriented submanifold that is diffeomorphic to S^1 such that $[C] \in \pi_1(\partial X_K)$ can be extended to a basis of $\pi_1(\partial X_K) \cong \mathbb{Z}^2$. If the Dehn filling $Y(C)$, which we introduced in the Dehn Filling Proposition 16.4, is diffeomorphic to S^3, then C is smoothly isotopic in ∂X_K to a meridian of K (as unoriented smooth submanifolds).

Proof. This theorem was proved in the 1980s by Cameron Gordon and John Luecke [**GL89**, Theorem 1]. The proof builds in particular on the work of William Thurston [**Thu82**] for which William Thurston was awarded a Fields medal in 1982. ■

Proof of the Link Equivalence Theorem 16.1 (4b). We need to show the following claim:

Claim. Let K and $\widetilde{K}$ be two oriented knots. If there exists an (orientation-preserving) diffeomorphism $f\colon X_K \xrightarrow{\cong} X_{\widetilde{K}}$, then there exists also an (orientation-preserving) diffeomorphism $g\colon X_K \xrightarrow{\cong} X_{\widetilde{K}}$ such that $g(\mu_K) = \mu_{\widetilde{K}}$ as *unoriented* smooth manifolds.[113]

[113]It is clear that we cannot hope to recover the orientation of μ_K.

Proof. First note that we have the following (orientation-preserving) diffeomorphisms:

$$S^3 \underset{\substack{\uparrow \\ \text{by the example} \\ \text{on page 237}}}{\cong} X_K(\mu_K) \underset{\substack{\uparrow \\ \text{this is clear}}}{\cong} f(X_K)(f(\mu_K)) \underset{\substack{\uparrow \\ \text{since } f(X_K)=X_{\widetilde{K}}}}{=} X_{\widetilde{K}}(f(\mu_K)).$$

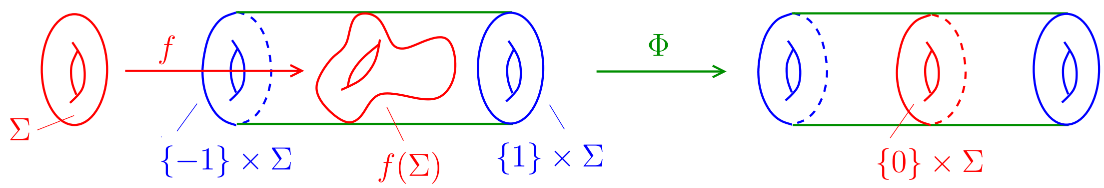
It follows from this discussion and the Gordon-Luecke Theorem 16.5 that $f(\mu_K)$ is smoothly isotopic in $\partial X_{\widetilde{K}}$ to $\mu_{\widetilde{K}}$. Using the Isotopy Extension Theorem 2.4 one can now easily find a diffeotopy from f to an (orientation-preserving) diffeomorphism $g\colon X_K \to X_{\widetilde{K}}$ such that $g(\mu_K) = \mu_{\widetilde{K}}$ (as unoriented smooth submanifolds). ■

16.2.5. **Proof of (5).** We need to show that

$$X_L \cong X_{\widetilde{L}} \iff S^3 \setminus L \cong S^3 \setminus \widetilde{L}$$

and we need to show the corresponding oriented analogue. In the proof of the "$\Leftarrow$"-direction we will need the following non-trivial proposition:

Proposition 16.6. (3D-Product Splitting Proposition) Let Σ be a closed orientable connected 2-dimensional smooth manifold. If $f\colon \Sigma \to (-1,1)\times\Sigma$ is a smooth embedding such that $([-1,1]\times\Sigma)\setminus f(\Sigma)$ is disconnected, then there exists an orientation-preserving diffeomorphism $\Phi\colon [-1,1]\times\Sigma \to [-1,1]\times\Sigma$ which is the identity on a neighborhood of $\{\pm1\}\times\Sigma$ and with $\Phi(f(\Sigma)) = \{0\}\times\Sigma$.



Proof. This proposition can be deduced from [**Hem72**, Theorem 10.5]. Note that [**Hem72**, Theorem 10.5] builds on the "Loop Theorem" (see [**Hem72**, Theorem 4.2]) which is one of the key theorems in the study of 3-dimensional smooth manifolds. ■

Proof of the Link Equivalence Theorem 16.1 (5). Let L and $\widetilde{L}$ be two oriented m-component links. We need to show

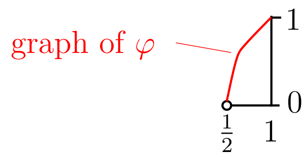
$$X_L \cong X_{\widetilde{L}} \iff S^3 \setminus L \cong S^3 \setminus \widetilde{L}$$

and we need to show the corresponding oriented analogue.
"$\Rightarrow$". In the following we assume that there exists an (orientation-preserving) diffeomorphism $f\colon X_L \xrightarrow{\cong} X_{\widetilde{L}}$. In this case we now need to show that there exists an (orientation-preserving) diffeomorphism $S^3 \setminus L \xrightarrow{\cong} S^3 \setminus \widetilde{L}$. Note that f restricts to an (orientation-preserving) diffeomorphism $f\colon X_L \setminus \partial X_L \xrightarrow{\cong} X_{\widetilde{L}} \setminus \partial X_{\widetilde{L}}$. Thus it remains to prove the following claim:

Claim 1. Let $J \subset S^3$ be an oriented link. There exists an orientation-preserving diffeomorphism between the smooth manifolds $X_J \setminus \partial X_J$ and $S^3 \setminus J$.

Proof. Let $\tau\colon \overline{B}^2 \times J \to S^3$ be an orientation-preserving tubular map for the oriented link J. It is clear that the map $\overline{B}^2 \times J \xrightarrow{(x,t)\mapsto(\frac{1}{2}x,t)} \overline{B}^2 \times J \xrightarrow{\tau} S^3$ is also an orientation-preserving tubular map for J. It follows from the Link Exterior Lemma 2.16 that there exists an orientation-preserving diffeomorphism $S^3 \setminus \tau(B^2 \times J) \to S^3 \setminus \tau(B^2_{\frac{1}{2}} \times J)$. Since the boundary of $S^3 \setminus \tau(B^2_{\frac{1}{2}} \times J)$ equals $\tau(S^1_{\frac{1}{2}} \times J)$ we now see that it suffices to provide an orientation-preserving diffeomorphism $S^3 \setminus \tau(\overline{B}^2_{\frac{1}{2}} \times J) \to S^3 \setminus J$.

One can easily find a diffeomorphism $\varphi\colon (\frac{1}{2}, 1] \to (0,1]$ such that $\varphi(t) = t$ for all $t \in [\frac{3}{4}, 1]$.



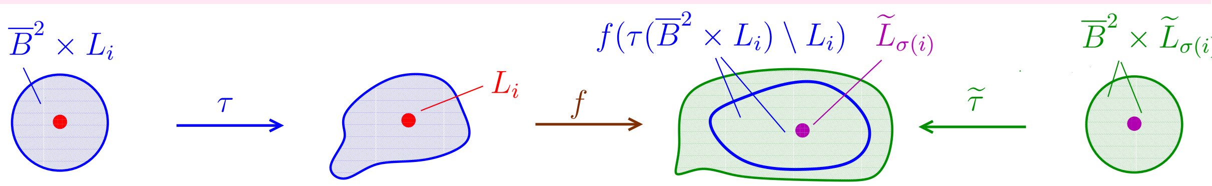
Using the Smooth Pasting Proposition **??** one can now verify that the map

$$\begin{array}{rcl} S^3 \setminus \tau(B^2_{\frac{1}{2}} \times J) & \to & S^3 \setminus J \\ x & \mapsto & \begin{cases} x, & \text{if } x \notin \tau(\overline{B}^2_{\frac{7}{8}} \times J), \\ \tau(\varphi(\|y\|) \cdot y, z) & \text{if } x = \tau(y,z) \text{ with } y \in \overline{B}^2 \setminus \overline{B}^2_{\frac{1}{2}}, z \in J \end{cases} \end{array}$$

is an orientation-preserving diffeomorphism. ⊞

"$\Leftarrow$". In the following we assume that there exists an (orientation-preserving) diffeomorphism $f\colon S^3 \setminus L \xrightarrow{\cong} S^3 \setminus \widetilde{L}$. We need to show that there exists an (orientation-preserving) diffeomorphism $X_L \xrightarrow{\cong} X_{\widetilde{L}}$. We pick an orientation-preserving tubular map $\widetilde{\tau}\colon \overline{B}^2 \times \widetilde{L} \to S^3$ for $\widetilde{L}$.

Claim 2. There exists an orientation-preserving tubular map $\tau\colon \overline{B}^2 \times L \to S^3$ for L and a bijection $\sigma\colon \{1,\dots,m\} \to \{1,\dots,m\}$ such that for every $i \in \{1,\dots,m\}$ we have the inclusion $f(\tau(\overline{B}^2 \times L_i) \setminus L_i) \subset \widetilde{\tau}(\overline{B}^2 \times \widetilde{L}_{\sigma(i)}) \setminus \widetilde{L}_{\sigma(i)}$.



Proof. We pick an orientation-preserving tubular map $\tau\colon \overline{B}^2 \times L \to S^3$ for L. First note that that $\widetilde{\tau}(B^2 \times \widetilde{L}) \subset S^3$ is open. This implies that the complement $S^3 \setminus \widetilde{\tau}(B^2 \times \widetilde{L})$ is compact. We note that it follows from the fact that $f\colon S^3 \setminus L \to S^3 \setminus \widetilde{L}$ is a diffeomorphism that the sets $\{f(S^3 \setminus \tau(B^2_s \times L))\}_{t \in (0,1)}$ form an open cover of $S^3 \setminus \widetilde{\tau}(B^2 \times \widetilde{L})$. It follows from compactness of $S^3 \setminus \widetilde{\tau}(B^2 \times \widetilde{L})$ that there exists an $s \in (0,1)$ such that $S^3 \setminus \widetilde{\tau}(B^2 \times \widetilde{L}) \subset f(S^3 \setminus \tau(B^2_s \times L))$. From this inclusion and the equalities

$$f(S^3 \setminus \tau(B^2_s \times L)) \;=\; f((S^3 \setminus L) \setminus \tau((B^2_s \times L) \setminus L)) \;=\; (S^3 \setminus \widetilde{L}) \setminus f(\tau(B^2_s \times L) \setminus L)$$

it follows that $f(\tau(B^2_s \times L) \setminus L) \subset \widetilde{\tau}(B^2 \times \widetilde{L}) \setminus \widetilde{L}$. We can now "radially rescale" τ to obtain an orientation-preserving tubular map $\tau\colon \overline{B}^2 \times L \to S^3$ for L such that we have the desired inclusion $f(\tau(\overline{B}^2 \times L) \setminus L) \subset \widetilde{\tau}(\overline{B}^2 \times \widetilde{L}) \setminus \widetilde{L}$.

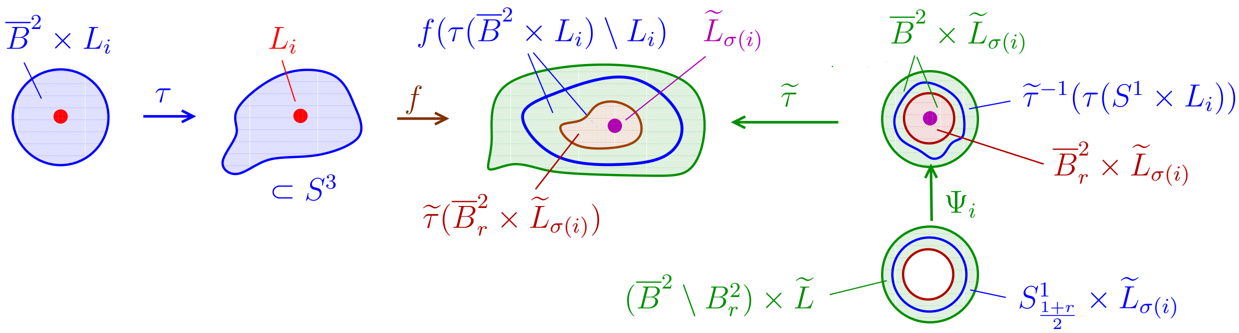
We still need to show the existence of the bijection σ. Let $i \in \{1,\dots,m\}$. Since $f(\tau(\overline{B}^2 \times L_i) \setminus L_i)$ is connected and since $\widetilde{\tau}(\overline{B}^2 \times \widetilde{L}) \setminus \widetilde{L}$ is the disjoint union (as a topological space) of the subspaces $\widetilde{\tau}(\overline{B}^2 \times \widetilde{L}_j) \setminus \widetilde{L}_j$ there exists a unique $\sigma(i) \in \{1,\dots,m\}$ such that $f(\tau(\overline{B}^2 \times L_i) \setminus L_i) \subset \widetilde{\tau}(\overline{B}^2 \times \widetilde{L}_{\sigma(i)}) \setminus \widetilde{L}_{\sigma(i)}$. It remains to argue that the resulting map

$\sigma\colon \{1,\dots,m\} \to \{1,\dots,m\}$ is a bijection. Since we are dealing with a finite set it suffices to show that the map σ is surjective. Thus let $j \in \{1,\dots,m\}$. Since $f(S^3 \setminus \tau(B^2 \times L))$ is compact we obtain, by considering the open cover $\{S^3 \setminus \widetilde{\tau}(\overline{B}^2_s \times \widetilde{L}_j)\}_{t\in(0,1)}$ of $S^3 \setminus \widetilde{L}$ that there exists an $s \in (0,1)$ such that $f(S^3 \setminus \tau(B^2 \times L)) \subset S^3 \setminus \widetilde{\tau}(\overline{B}^2_s \times \widetilde{L}_j)$. This of course implies that $f(S^3 \setminus \tau(B^2 \times L)) \cap \widetilde{\tau}(\overline{B}^2_s \times \widetilde{L}_j) = \varnothing$. Since $f\colon S^3 \setminus L) = S^3 \setminus \widetilde{L}$ is a diffeomorphism (in particular a bijection) we see that there needs to exists an $i \in \{1,\dots,m\}$ with $f(\tau(\overline{B}^2 \times L_i)) \subset \widetilde{\tau}(\overline{B}^2 \times \widetilde{L}_j)$. ⊞

We continue with the notation from claim 2. It remains to prove the following claim:

Claim 3. There exists an orientation-preserving tubular map $\widetilde{\tau}'\colon \overline{B}^2 \times \widetilde{L} \to S^3$ for $\widetilde{L}$ such that f restricts to a diffeomorphism $S^3 \setminus \tau(B^2 \times L) \to S^3 \setminus \widetilde{\tau}'(B^2 \times \widetilde{L})$.

Proof. Note that $f(S^3 \setminus \tau(B^2 \times L))$ is a closed subset of S^3 that is disjoint from $\widetilde{L}$. Since $f(S^3 \setminus L) = S^3 \setminus \widetilde{L}$ we now see that $f(\tau(B^2 \times L) \setminus L) \cup \widetilde{L}$ is an open neighborhood of $\widetilde{L}$. One can easily show that there exists an $r > 0$ such that $\widetilde{\tau}(\overline{B}^2_r \times \widetilde{L}) \subset f(\tau(B^2 \times L) \setminus L) \cup \widetilde{L}$.



Next note that it follows easily from the 3D-Product Splitting Proposition 16.6 that for each $i \in \{1,\dots,m\}$ there exists an orientation-preserving diffeomorphism

$$\Psi_i\colon (\overline{B}^2 \setminus B^2_r) \times \widetilde{L}_{\sigma(i)} \;\to\; (\overline{B}^2 \setminus B^2_r) \times \widetilde{L}_{\sigma(i)},$$

with $\Psi_i(S^1_{\frac{1+r}{2}} \times \widetilde{L}_{\sigma(i)}) = (\widetilde{\tau}^{-1}(\tau(S^1 \times L_i)))$ and which is the identity in a neighborhood of the boundary. It is now straightforward to verify that the desired orientation-preserving tubular map $\widetilde{\tau}'\colon \overline{B}^2 \times \widetilde{L} \to S^3$ for $\widetilde{L}$ is given by the radial rescaling of the following map:

$$\begin{array}{rcl} \overline{B}^2_{\frac{1+r}{2}} \times \widetilde{L} & \to & S^3 \\ (x,y) & \mapsto & \begin{cases} \widetilde{\tau}(x,y), & \text{if } \|x\| \le r, \\ \widetilde{\tau}(\Phi_i^{-1}(x,y)), & \text{if } \|x\| \in [r, \frac{1+r}{2}] \text{ and } y \in \widetilde{L}_{\sigma(i)}. \end{cases} \end{array}$$ ■

16.2.6. **Proof of (6).** We need to show that

$$X_L \cong X_{\widetilde{L}} \;\overset{(6')}{\Longleftrightarrow}\; X_L \underset{\text{homeo.}}{\cong} X_{\widetilde{L}} \quad \text{and} \quad S^3 \setminus L \cong S^3 \setminus \widetilde{L} \;\overset{(6'')}{\Longleftrightarrow}\; S^3 \setminus L \underset{\text{homeo.}}{\cong} S^3 \setminus \widetilde{L}.$$

The "$\Rightarrow$"-direction is of course in both cases trivial. The "$\Leftarrow$"-direction is in both cases a special case of the following deep theorem:

Theorem 16.7. (3D-Homeo-implies-Diffeo) If two 3-dimensional smooth manifolds are homeomorphic, then they are also diffeomorphic.

Proof. This theorem follows from work of Edwin Moise [**Moi52**] [**Moi77**, p. 252 and 253], William Thurston [**Thu97**, Theorem 3.10.8] and James Munkres [**Mun59**, p. 333][**Mun66**][**Mun60**, Theorems 6.2 and 6.3]. We refer to [**Kui99**] and [**Thu97**, Chapter 3.10] for a more detailed discussion. ■

Remark. The statement of the 3D-Homeo-implies-Diffeo 16.7 is rather specific to dimension three. For exampleThere exist closed 4-dimensional topological manifolds that are homeomorphic but not diffeomorphic.

16.2.7. Proof of (7).

We need to show that

$$X_L \simeq X_{\widetilde{L}} \quad \underset{\text{(7b) if } L, \widetilde{L} \text{ are not splittable}}{\overset{\text{(7a)}}{\rightleftarrows}} \quad \pi_1(X_L) \cong \pi_1(X_{\widetilde{L}}).$$

The "⇒"-statement (7a) is just a special case of the well-known fact that homotopy equivalences induces isomorphisms on fundamental groups. The other direction is much more interesting and much harder. It will build on several results which go well beyond the other material of this part of the lecture notes, but which we will cover in later parts of the lecture notes.

In the following discussion it is unavoidable to make use of the notion of higher homotopy groups $\pi_n(X, x_0)$, $n \geq 2$ of a pointed topological space. For the purpose of this section it is enough to know that $\pi_n(X, x_0)$ is defined as the set of homotopy classes of continuous maps $(S^n, *) \to (X, x_0)$. Furthermore we will mention the notion of a CW-complex. But even if the reader is completely unaware of these concepts, it should not be a problem to follow the flow of the conversation.

Definition. Let X be a non-empty topological space. We say that X is aspherical if X is path-connected and if all higher homotopy groups $\pi_n(X)$, $n \geq 2$, vanish.

The following theorem explains why we are interested in these new notions:

Theorem 16.8. (Eilenberg-Maclane Uniqueness Theorem) Let (X, x_0) and (Y, y_0) be two pointed aspherical CW-complexes. For any isomorphism $\varphi\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ of the fundamental groups there exists a homotopy equivalence $f\colon X \to Y$ with $f(x_0) = y_0$ such that $f_* = \varphi\colon \pi_n(X, x_0) \to \pi_n(Y, y_0)$.

Proof. This theorem is proved in [**Fri24**]. ■

The Eilenberg-Maclane Uniqueness Theorem 16.8 sounds like a promising approach to dealing with the "⇐"-statement formulated above. But to apply this theorem we need to figure out which link exteriors are CW-complexes and which link exteriors are aspherical. Fortunately the next theorem says in particular that all link exteriors are CW-complexes:

Theorem 16.9. (Smooth Manifold-CW-Theorem) Every compact smooth manifold M admits a CW-structure.

Proof. A sketch of a proof of this theorem is provided in [**Fri24**]. ■

The question, which link exteriors are aspherical, is more subtle:

Theorem 16.10. (Aspherical Link Exterior Theorem) Let L be a non-empty link. Then

$$X_L \text{ is aspherical} \iff L \text{ is not splittable.}$$

Sketch of proof. The"$\Rightarrow$"-direction is quite straightforward: So let L be a link that is splittable in the sense of the definition on page 20. We need to show that X_L is not aspherical. Basically by definition this implies that there exist smooth embeddings $F_\pm \colon \overline{B}^3 \to S^3$ with $F_-(\overline{B}^3) \cap F_+(\overline{B}^3) = F_-(S^2) = F_+(S^2)$ such that $F_\pm(S^2) \cap L = \varnothing$ and such that $L_- = F_-(\overline{B}^3) \cap L$ and $L_+ = F_+(\overline{B}^3) \cap L$ are both non-empty. This allows us to pick points $P_- \in L_-$ and $P_+ \in L_+$.

It is elementary to see that there exists a retraction from $F_\pm(\overline{B}^3)\setminus\{P_\pm\}$ to $F_\pm(S^2)$. Together these two retractions give us a retraction r from $S^3 \setminus (L_- \cup L_+)$ to the sphere $F_-(S^2)$.[114] It follows from thethe calculation of $\pi_2(S^2)$ that $\pi_2(F_-(S^2)) \cong \mathbb{Z}$. By the functoriality of π_2 we see that $r_* \colon \pi_2(X_L) \to \pi_2(F_-(S^2))$ is an epimorphism. This implies that $\pi_2(X_L)$ is non-trivial. Thus we have shown that X_L is not aspherical.
The"$\Leftarrow$"-direction is much harder. Here we just mention the three main ingredients:

(1) The Sphere Theorem which says that if N is an orientable 3-dimensional smooth manifold N with $\pi_2(N) \neq 0$, then there a smooth embedding $f \colon S^2 \to N$ such that $[f]$ is non-trivial in $\pi_2(N)$. This theorem was proved by Christos Papakyriakopoulos [**Pap57**] and John Whitehead [**Whi58**].
(2) The Generalized Smooth Schoenflies Theorem which says that for any smooth embedding $f \colon S^2 \to S^3$ there exist two smooth embeddings $F_\pm \colon \overline{B}^3 \to S^3$ such that $F_-(\overline{B}^3) \cap F_+(\overline{B}^3) = f(S^2)$.
(3) The Hurewicz Theorem and the calculation of homology groups of topological manifolds which imply that if $L \neq \varnothing$ and if $\pi_n(X_L) = 0$ for some $n \geq 2$, then $\pi_{n+1}(X_L) = 0$.

Now let L be a link that is not splittable. Using (1) and (2) one can easily show that $\pi_2(X_L) = 0$. Iteratively applying (3) we see that $\pi_n(X_L) = 0$ for all $n \geq 2$. Thus we have shown that X_L is aspherical. ■

Proof of the Link Equivalence Theorem 16.1 (7). Let L and $\widetilde{L}$ be two oriented m-component links. We need to show that

$$X_L \simeq X_{\widetilde{L}} \quad \underset{\text{(7b) if } L,\widetilde{L} \text{ are not splittable}}{\overset{\text{(7a)}}{\rightleftarrows}} \quad \pi_1(X_L) \cong \pi_1(X_{\widetilde{L}}).$$

As we already mentioned in the beginning of this subsection, the "$\Rightarrow$"-direction is just a special case of the Homotopy-π_1-Proposition **??**.

We turn to the proof of the "$\Leftarrow$"-direction. We assume that L and $\widetilde{L}$ are not splittable. It follows from the Aspherical Link Exterior Theorem 16.10 that X_L and $X_{\widetilde{L}}$ are both aspherical. The promised statement now follows from the Eilenberg-Maclane Uniqueness Theorem 16.8 together with the Smooth Manifold-CW-Theorem 16.9. ■

[114]This argument is basically the content of Exercise 2.11.

16.2.8. **Proof of (8).** We need to prove

$$X_L \simeq X_{\widetilde{L}} \overset{(8')}{\Longleftrightarrow} S^3 \setminus L \simeq S^3 \setminus \widetilde{L} \quad \text{and} \quad \pi_1(X_L) \cong \pi_1(X_{\widetilde{L}}) \overset{(8'')}{\Longleftrightarrow} \pi_1(S^3 \setminus L) \cong \pi_1(S^3 \setminus \widetilde{L}).$$

Both statements follow from the Link Exterior Lemma 2.16, where we showed that for any link J the inclusion $X_J \to S^3 \setminus J$ is a homotopy equivalence.

16.2.9. **Proof of (9).** We want to prove the following statement:

$$\begin{tikzcd}[column sep=huge] X_L \underset{\text{homeo.}}{\cong} X_{\widetilde{L}} \arrow[r, Rightarrow, "(9a)"] & X_L \simeq X_{\widetilde{L}}. \arrow[l, Rightarrow, bend left, "{(9b)\ \begin{array}{ll}\text{yes,} & \text{if } L \text{ and } \widetilde{L} \text{ are prime knots,}\\ \text{no,} & \text{in general}\end{array}}"] \end{tikzcd}$$

The proof of (9b) for prime knots rests on the following theorem:

Theorem 16.11. (Whitten Theorem) Let K and $\widetilde{K}$ be two prime knots. If $\pi_1(X_K)$ is isomorphic to $\pi_1(X_{\widetilde{K}})$, then X_K is homeomorphic to $X_{\widetilde{K}}$.

Proof. Let K and $\widetilde{K}$ be two prime knots such that the fundamental groups $\pi_1(X_K)$ and $\pi_1(X_{\widetilde{K}})$ are isomorphic. Wilbur Whitten [**Whi87**, Rigidity Theorem][115] showed that X_K is homeomorphic to $X_{\widetilde{K}}$. (In fact, if one reads the proof and the references one sees that the argument actually implies that X_K is diffeomorphic to $X_{\widetilde{K}}$.) ■

Proof of the Link Equivalence Theorem 16.1 (9). Let L and $\widetilde{L}$ be two oriented m-component links. The (9a)-statement is of course clear.

Now we turn to the (9b)-statement. First let L and $\widetilde{L}$ be two prime knots with $X_L \simeq X_{\widetilde{L}}$. By the elementary (7a)-statement this implies that $\pi_1(X_L) \cong \pi_1(X_{\widetilde{L}})$. It thus follows from the Whitten Theorem 16.11 that X_L is homeomorphic to $X_{\widetilde{L}}$.

Now it remains to prove the following claim:

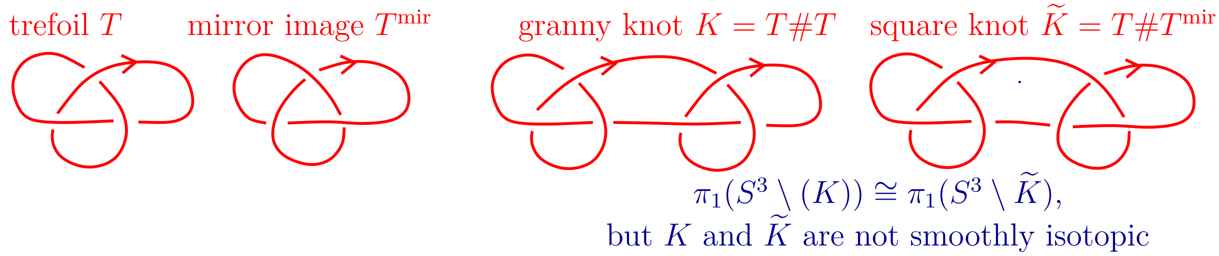
Claim. There exist knots K and $\widetilde{K}$ such that $X_K \simeq X_{\widetilde{K}}$, but such that X_K is not homeomorphic to $X_{\widetilde{K}}$.

Proof. Let T be the trefoil. We equip T with an orientation. As always we denote by T^{rev} the reverse of T, i.e. the trefoil with the opposite orientation. Let $\rho\colon S^3 \to S^3$ be the reflection in a hyperplane of $\mathbb{R}^4$. We set $T^{\text{mir}} := \rho(T)$. Let μ_T be a meridian for T.

[115]Note that [**Whi87**, Rigidity Theorem] is formulated in terms of knot complements, but the proof shows that the statement is really about knot exteriors.

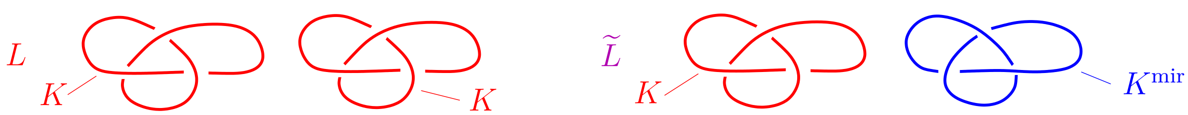
We consider the granny knot $K = T\#T$ and the square knot $\widetilde{K} = T\#T^{\mathrm{mir}}$. Note that

$$\begin{aligned}
\pi_1(S^3 \setminus K) &\overset{\substack{\text{by the Knot Connected Sum-}\pi_1\text{-}\\ \text{Proposition 3.15 applied to } K=T\#T}}{\cong} \pi_1(S^3 \setminus T) *_{\mu_T=\mu_T} \pi_1(S^3 \setminus T) \overset{\text{algebraic isomorphism}}{\cong} \pi_1(S^3 \setminus T) *_{\mu_T=\rho_*(\mu_T)} \rho_*(\pi_1(S^3 \setminus T)) \\
&\underset{\text{since } \rho(S^3\setminus T) = S^3 \setminus \rho(T)}{\cong} \pi_1(S^3 \setminus T) *_{\mu_T=\rho_*(\mu_T)} \pi_1(S^3 \setminus \rho(T)) \underset{\substack{\text{by definition of } T^{\mathrm{mir}}=\rho(T) \text{ and since by the}\\ \text{Meridian–Symmetries Lemma 2.18 we}\\ \text{have } \rho_*(\mu_T)=\mu_{(T^{\mathrm{rev}})^{\mathrm{mir}}} \text{ as oriented smooth submanifolds}}}{\cong} \pi_1(S^3 \setminus T) *_{\mu_T=\mu_{(T^{\mathrm{rev}})^{\mathrm{mir}}}} \pi_1(S^3 \setminus T^{\mathrm{mir}}) \\
&\underset{\text{by Exercise 2.1 we have } T=T^{\mathrm{rev}}}{\cong} \pi_1(S^3 \setminus T) *_{\mu_T=\mu_{T^{\mathrm{mir}}}} \pi_1(S^3 \setminus T^{\mathrm{mir}}) \underset{\substack{\text{by the Knot Connected Sum-}\pi_1\text{-}\\ \text{Proposition 3.15 applied to } \widetilde{K}=T\#T^{\mathrm{mir}}}}{\cong} \pi_1(S^3 \setminus \widetilde{K}).
\end{aligned}$$



We have thus shown that $\pi_1(S^3 \setminus K) \cong \pi_1(S^3 \setminus \widetilde{K})$. From (8″), (7b) we obtain that $X_K \simeq X_{\widetilde{K}}$. On the other hand we saw in Exercise 7.8 that the square knot is neither smoothly isotopic to the granny knot nor to its mirror image. By (1), (2), (3), (4a), (6) this implies that X_K is not homeomorphic to $X_{\widetilde{K}}$. ■

Remark. Let K be the trefoil. We consider the link $L := K \sqcup K$ and the link $\widetilde{L} := K \sqcup K^{\mathrm{mir}}$.



It follows from the Mirror Link-π_1-Lemma 3.3 together with the Split Union-π_1-Proposition 3.11 that $\pi_1(S^3 \setminus L) \cong \pi_1(S^3 \setminus \widetilde{L})$. On the other hand one can show that X_L is not diffeomorphic to $X_{\widetilde{L}}$. So one of the two statements (7b) and (9b) fails. But it is not entirely clear, which one. Put differently, the following question arises: Are $S^3 \setminus L$ and $S^3 \setminus \widetilde{L}$ homotopy equivalent?[116]

16.2.10. **Proof of (10).**

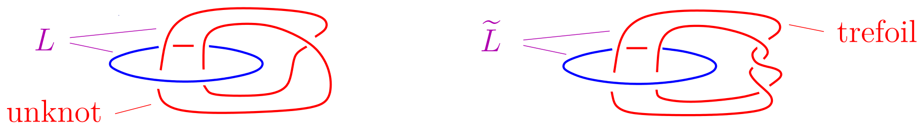
Proof of the Link Equivalence Theorem 16.1 (10). We need to show that

$$\begin{tikzcd}
(X_L, \mu_L) \cong (X_{\widetilde{L}}, \pm\mu_{\widetilde{L}}) \arrow[r, Rightarrow, shift left, "(10a)"] & (X_L, T_L) \cong (X_{\widetilde{L}}, T_{\widetilde{L}}) \arrow[l, Rightarrow, shift left, "(10b)\ \text{no for links}"]
\end{tikzcd}$$

[116]It seems to the author that one can might be able to show that they are not homotopy equivalent by considering π_2 as a module over the group ring $\mathbb{Z}[\pi_1]$, but it seems like a daunting task to turn this into a proper proof.

and we need to show the oriented analogue. First we deal with the (10a)-statement, i.e. we deal with the "$\Rightarrow$"-direction. We assume that there exists an (orientation-preserving) diffeomorphism $f\colon (X_L,\mu_L) \xrightarrow{\cong} (X_{\widetilde{L}},\pm\mu_{\widetilde{L}})$. Recall that this means that there exists an (orientation-preserving) diffeomorphism $f\colon X_L \to X_{\widetilde{L}}$ such that for $i=1,\dots,m$ we have $f(\mu_{L_i}) = \mu_{\widetilde{L}_i}$ as (unoriented) smooth submanifolds. Since f is an diffeomorphism we see that f restricts to a diffeomorphism $\partial X_L \to \partial X_{\widetilde{L}}$. Since for each $i \in \{1,\dots,m\}$ we have $f(\mu_{L_i}) = \mu_{\widetilde{L}_i}$ we see that boundary components of X_L and $X_{\widetilde{L}}$ get matched in the way they are ordered. In other words, for each $i \in \{1,\dots,m\}$ we have $f(T_{L_i}) = T_{\widetilde{L}_i}$.

We turn to the proof of (10b). We need to show that there exist oriented links L and $\widetilde{L}$ with $(X_L,T_L) \cong (X_{\widetilde{L}},T_{\widetilde{L}})$ but such that $(X_L,\mu_L) \ncong (X_{\widetilde{L}},\pm\mu_{\widetilde{L}})$. We consider the two 2-component links L and $\widetilde{L}$ that are shown in the figure below (and which we already considered in Exercise 3.15):



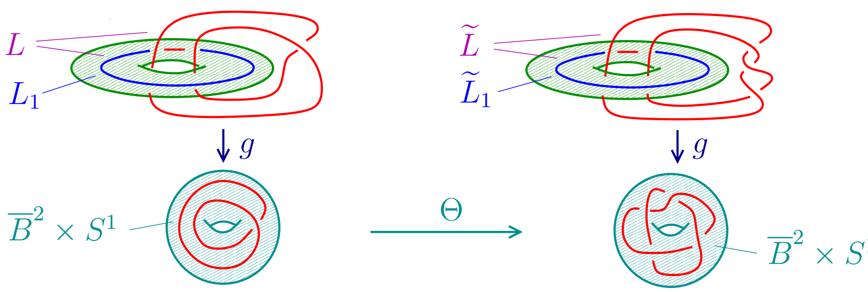
Note that all components of L are the trivial knot, whereas one component of $\widetilde{L}$ is a trefoil. This shows that in no reasonable sense are L and $\widetilde{L}$ smoothly isotopic. It follows from (1), (2) and (3) that there is no reasonable sense in which (X_L,μ_L) and $(X_{\widetilde{L}},\pm\mu_{\widetilde{L}})$ are diffeomorphic. In contrast to this we have the following claim:

Claim. There exists an orientation-preserving diffeomorphism $(X_L,T_L) \to (X_{\widetilde{L}},T_{\widetilde{L}})$.

Proof. Let $L_1 = \widetilde{L}_1$ be the blue component of L and $\widetilde{L}$. Since it is the unknot we know, e.g. by the Sphere–Solid Tori-Decomposition Lemma 3.4, that there exists a diffeomorphism $g\colon X_{L_1} \to \overline{B}^2 \times S^1$. We now consider the diffeomorphism

$$\begin{aligned} \Theta\colon \overline{B}^2 \times S^1 &\to \overline{B}^2 \times S^1 \\ (z,w) &\mapsto (z\cdot w, w). \end{aligned}$$

(Pictorially speaking the diffeomorphism Θ gives the disk $\overline{B}^2$ one full twist as we walk along S^1.)



The figure above should convince the reader that the diffeomorphism $g^{-1}\circ\Theta\circ g\colon X_{L_1} \to X_{\widetilde{L}_1}$ sends L_2 to $\widetilde{L}_2$. It follows that the map $g^{-1}\circ\Theta\circ g$ restricts to an orientation-preserving diffeomorphism $(X_L,T_L) \to (X_{\widetilde{L}},T_{\widetilde{L}})$. ■

16.2.11. **Proof of (11).** We need to show that

$$(X_L,T_L) \cong (X_{\widetilde{L}},T_{\widetilde{L}}) \underset{(11)}{\Longrightarrow} X_L \cong X_{\widetilde{L}}$$

and we need to show the oriented analogue. In fact this statement is trivial. It is perhaps more important to point out what is happening: Going from left to right we loose control over the ordering of the components of the link. Of course in general this step is not reversible.

16.2.12. Proof of (12).

Proof of the Link Equivalence Theorem 16.1 (12). We need to show that

$$\begin{tikzcd} (X_L, T_L) \cong (X_{\widetilde{L}}, T_{\widetilde{L}}) \arrow[d, Rightarrow, shift left, "(12a')"] \\ (\pi_1(X_L), \pi_1(T_L)) \cong (\pi_1(X_{\widetilde{L}}), \pi_1(T_{\widetilde{L}})) \arrow[u, Rightarrow, shift left, "(12b')\ \text{if } L, \widetilde{L} \text{ are not splittable}"] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} (X_L, \mu_L, \lambda_L) \cong (X_{\widetilde{L}}, \mu_{\widetilde{L}}, \lambda_{\widetilde{L}}) \arrow[d, Rightarrow, shift left, "(12a'')"] \\ (\pi_1(X_L), \mu_L, \lambda_L) \cong (\pi_1(X_{\widetilde{L}}), \mu_{\widetilde{L}}, \lambda_{\widetilde{L}}). \arrow[u, Rightarrow, shift left, "(12b'')\ \text{if } L, \widetilde{L} \text{ are not splittable}"] \end{tikzcd}$$

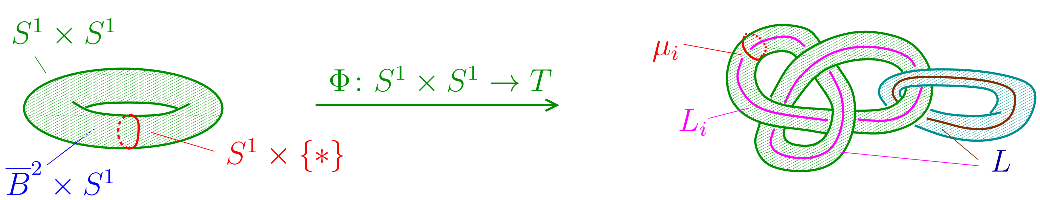
The downward directions are trivial in both cases. We now turn to the proof of the upward directions.

First we consider (12b′)-statement. In the following we assume that there exists an isomorphism $\varphi\colon (\pi_1(X_L), \pi_1(T_L)) \xrightarrow{\cong} (\pi_1(X_{\widetilde{L}}), \pi_1(T_{\widetilde{L}}))$ of subgroup systems. It follows easily from the Waldhausen Theorem [**Wal68**, Corollary 6.5] that there exists a diffeomorphism $f\colon (X_L, T_L) \to (X_{\widetilde{L}}, T_{\widetilde{L}})$ such that $f_* = \varphi$.

Finally we consider (12b′)-statement. In the following we assume that there exists an isomorphism $\varphi\colon (\pi_1(X_L), \mu_L, \lambda_L) \xrightarrow{\cong} (\pi_1(X_{\widetilde{L}}), \mu_{\widetilde{L}}, \lambda_{\widetilde{L}})$ of group-pair systems. Since for each $i \in \{1, \dots, m\}$ the group $\pi_1(T_{L_i})$ is generated by μ_{L_i} and λ_{L_i} we see that φ also defines an isomorphism $\varphi\colon (\pi_1(X_L), \pi_1(T_L)) \xrightarrow{\cong} (\pi_1(X_{\widetilde{L}}), \pi_1(T_{\widetilde{L}}))$. By the above discussion of the (12b′)-statement we see that there exists a diffeomorphism $f\colon (X_L, T_L) \to (X_{\widetilde{L}}, T_{\widetilde{L}})$ such that $f_* = \varphi$. It follows that for each $i \in \{1, \dots, m\}$ we have $f_*(\mu_{L_i}) = \mu_{\widetilde{L}_i} \in \pi_1(T_{L_i})$ and $f_*(\lambda_{L_i}) = \lambda_{\widetilde{L}_i} \in \pi_1(T_{L_i})$. It follows easily from the Torus–Mapping Class Group Theorem 16.3 that f is diffeotopic to a diffeomorphism $f'\colon (X_L, T_L) \to (X_{\widetilde{L}}, T_{\widetilde{L}})$ such that for each $i \in \{1, \dots, m\}$ we have $f(\mu_{L_i}) = \mu_{\widetilde{L}_i}$ and $f(\lambda_{L_i}) = \lambda_{\widetilde{L}_i}$ as oriented smooth submanifolds of T_{L_i}. ∎

Exercises for Chapter 16.

Exercise 16.1. Let $L \subset S^3$ be an m-component link with link exterior X_L, let L_i be a component of L and let $\mu_i \subset \partial X_L$ be a meridian of L_i. We consider the Dehn filling $X(\mu_i)$ as defined in the Dehn Filling Proposition 16.4. Show that there exists an orientation-preserving $X_L(\mu_i) \to X_{L \setminus L_i}$ which is the identity on X_L and which sends $\{0\} \times S^1$ to L_i as oriented smooth manifolds.



Exercise 16.2. Let $L, \widetilde{L}$ be two m-component links. We consider the link exteriors X_L and $X_{\widetilde{L}}$. We denote by $T_1, \dots, T_m$ respectively $\widetilde{T}_1, \dots, \widetilde{T}_m$ the boundary components of X_L

and $X_{\widetilde{L}}$ that correspond to the components of L and $\widetilde{L}$. We consider the following two statements:

(1) There exists a diffeomorphism $f\colon X_L \to X_{\widetilde{L}}$ such that for every $i \in \{1,\dots,m\}$ we have $f(T_i) = \widetilde{T}_i$.
(2) There exists a diffeomorphism $f\colon S^3 \setminus L \to S^3 \setminus \widetilde{L}$ such that for every $i \in \{1,\dots,m\}$ we have ????

Formulate Statement (2) in such a way that the two statements are equivalent and prove the equivalence.
Hint. You might want to look up the notion of ends of a topological space, as introduced on page **??**.

Exercise 16.3. Let K be a fibered knot with fiber Seifert surface F. By the Fiber–Minimal Corollary 13.11 we know that F is a Seifert surface of minimal genus. Show that every Seifert surface G of minimal genus is smoothly isotopic, rel K, to F.
Hint. Consider the covering of $p\colon \widetilde{X}_K \to X_K$ corresponding to the kernel of the abelianization $\pi_1(X_K) \to \pi_1(X_K)_{\mathrm{ab}} \cong \mathbb{Z}$ and make use of the 3D-Product Splitting Proposition 16.6.

Exercise 16.4.
(a) Show that in general the statement of the Link Equivalence Theorem 16.1 (12b′) does not hold for links that are splittable.
(b) Show that in general the statement of the Link Equivalence Theorem 16.1 (12b″) does not hold for links that are splittable.