Characteristic Classes

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1	24 October		Organizational meeting
2	1 November		All saints day
3	8 November	Gerrit Herrmann	Generalities on Vector Bundles
4	15 November	Gerrit Herrmann	Constructing New Vector Bundles Out of Old
5	22 November	Stefan Wolf	Stiefel-Whitney Classes: Axioms & First Properties
6	29 November	Stefan Wolf	Applications of Stiefel-Whitney classes
7	6 December	Benedikt Löhlein	Grassmann Manifolds
8	13 December	Benedikt Löhlein	The Cohomology of the Grassmannian
9	20 December	Luigi Caputi	The Steenrod Square Operations
10	10 January	Daniel Fauser	The Thom Isomorphism
11	17 January	Benedikt Preis	Existence of Stiefel-Whitney Classes
12	24 January	Georg Lipp	The Euler Class
13	31 January	Enrico Toffoli	Calculations in Manifolds
14	7 February	Enrico Toffoli	Applications to Manifolds

Characteristic classes are cohomological invariants of vector bundles, which have very interesting and different applications. We will focus on the following classes (and their relation):

- Stiefel-Whitney classes, associated to real vector spaces;

- the Euler class, associated to real vector spaces with an orientation;

- Chern Classes, associated to complex vector spaces.

For Stiefel-Whitney and Euler class we will follow a purely algebraic-topological approach, while for the Chern Classes we will opt for a differential-geometric approach due to Chern and Weil.

Detailed plan

For the topics listed without a specific reference, it is to be intended that the material can be found in the main reference next to the title.

1 Organizational meeting.

1. **Generalities on Vector Bundles** ([MS74, Chapter 2]). Recall the definition of a vector bundle and the related basic definitions (e.g. isomorphism and cross-section). Discuss all four (crucial) examples contained in the chapter. Prove Lemma 2.3 and Theorem 2.2.

2. **Constructing New Vector Bundles Out of Old** ([MS74, Chapter 3]). Discuss the five basic construction (a)-(e). Give examples. Explain the clutching construction given in the book of Hatcher ([Ha09, Page 22]) and give examples. Optional: prove [Ha09, Proposition 1.11].

3. **Stiefel-Whitney Classes: Axioms and First Properties** ([MS74, Chapter 4]) Define Stiefel-Whitney classes axiomatically and prove their first immediate properties. Define the total Stiefel-Whitney class. Prove the further properties contained in Lemma 4.1 and Lemma 4.2. Start to discuss the specific properties of the Stiefel-Whitney classes of projective spaces.

4. **Applications of Stiefel-Whitney Classes** ([MS74, Chapter 4]) Go on with the discussion about properties of the Stiefel-Whitney classes of projective spaces. Prove Theorem 4.7, which is a purely algebraic application of the theory. Recall the definition of immersion of manifolds and prove Theorem 4.8. If there is time left, introduce the Stiefel-Whitney numbers and state (with or without proof) Theorem 4.9.

5. **Grassmann Manifolds** ([MS74, Chapter 5]) You have to make sure the audience understands the definition of the Grassmann manifolds and understand Theorem 5.6 and Theorem 5.7.

6. **Cohomology of the Grassmannian**([MS74, Chapters 6 and 7]) Sketch the CW-structure of the infinite Grassmann manifold and state Theorem 6.4 without proof. Use this to derive Theorem 7.1, which describes the structure of the cohomology of these manifolds. To conclude, prove Theorem 7.3, which give uniqueness of the Stiefel-Whitney classes.

7. **The Steenrod Square Operations** ([Ha02, Chapter 4.L]) Construct the cohomological Steenrod square operations and prove their main properties.

8. **Thom Isomorphism** ([MS74, Chapter 10]) Cover the material of Chapter 10 without focusing too much on the technical parts. In particular, describe the important steps in the proof of Theorem 10.4.

9. **Existence of Stiefel-Whitney Classes** ([MS74, Chapter 8]) Use the Thom-Isomorphism and the Steenrod squares to construct Stiefel-Whitney classes and verify the axioms.

10. **Euler Class** ([MS74, Chapter 9]) Recall the class constructed in the talk about Thom Isomorphism. Explain how this relates to an orientation. Define the Euler-class and discuss basic properties of the Euler Class -especially Property 9.7.

11. **Calculations in Manifolds** ([MS74, Chapter 11]) Give a description of Stiefel-Whitney classes in terms of the normal and the tangent bundle of a smoothly embedded manifold. Conclude by defining the slant product, covering the material of the chapter up to page 127.

12. **Applications to Manifolds** ([MS74, Chapter 11,p. 127–137]) Explain pages 127–137. This contains Poincare duality, the connection between Euler class and Euler characteristic and the Wu-formulas.

References

[Ha02] A. Hatcher, *Algebraic Topolology*, Cambridge University Press, 2002

[Ha09] A. Hatcher, *Vector Bundles and K-Theory*, 2009 (Available online)

[MT97] I. Madsen and J. Tornehave, *From calculus to cohomology, de Rham cohomology and characteristic classes*, Cambridge University Press (1997)

[MS74] J. Milnor and J. D. Stasheff, *Characteristic Classes*, Princeton University Press (1974)