HYPERBOLIC EXAMPLES OF TOPOLOGICALLY SLICE KNOTS

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Abstract. In [FT05] the author and Peter Teichner proved a new sliceness criterion and gave examples of satellite knots to which this criterion applies. These satellite knots can also seen to be topologically slice by applying [CFT07, Theorem 1.5]. In this note we give hyperbolic examples for the main theorem of [FT05] which are a priori not covered by the sliceness criterion of [CFT07].

1. Statement of the theorem

A knot $K \subset S^3$ is called topologically slice if K bounds a locally flat disk $D \subset D^4$. A knot is called smoothly slice if it bounds a smoothly embedded disk in D^4. Clearly a smoothly slice knot is also topologically slice.

In the early 1980's Freedman showed that any knot with trivial Alexander polynomial is topologically slice (see [FQ90, Theorem 11.7B]). In particular Whitehead doubles of knots are topologically slice.

In [FT05] the author and Peter Teichner proved a generalization of Freedman's theorem. In order to state this result we let

$$SR := \langle a, c \mid aca^{-1} = c^2 \rangle \cong \mathbb{Z} \ltimes \mathbb{Z}[1/2] \quad \text{`solvable ribbon group'}.$$

Here the generator a of $\mathbb{Z}$ acts on the normal subgroup $\mathbb{Z}[1/2]$ via multiplication by 2. The following is then the main theorem of [FT05].

Theorem 1.1. [FT05] *Let K be a knot and denote by M_K the 0-surgery on K. Let $G = \mathbb{Z}$ or $G = SR$ and denote the Ore localization of $\mathbb{Z}[G]$ by $\mathbb{K}(G)$. If there exists an epimorphism $\pi_1(M_K) \twoheadrightarrow G$, such that the Blanchfield pairing*

$$H_1(M_K; \mathbb{Z}[G]) \times H_1(M_K; \mathbb{Z}[G]) \to \mathbb{K}(G)/\mathbb{Z}[G]$$

vanishes, then K is topologically slice.

Note that the case $G = \mathbb{Z}$ is actually just a reformulation of Freedman's theorem since there exists a unique epimorphism $\pi_1(M_K) \to \mathbb{Z}$ up to sign and since the vanishing of the Blanchfield pairing is equivalent to $\Delta_K(t) = 1$.

Before we continue we recall the satellite construction of knots. Let K, C be knots. Let $\eta \subset S^3 \setminus K$ be a curve, unknotted in S^3. Then $S^3 \setminus \nu\eta$ is a solid torus. Let

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$\psi : \partial(\overline{\nu\eta}) \to \partial(\overline{\nu C})$ be a diffeomorphism which sends a meridian of η to a longitude of C, and a longitude of η to a meridian of C. The space

$$(S^3 \setminus \nu\eta) \cup_\psi (S^3 \setminus \nu C)$$

is a 3-sphere and the image of K is denoted by $S = S(K, C, \eta)$. We say is the satellite knot with companion C, orbit K and axis η. Note that we replaced a tubular neighborhood of C by a knot in a solid torus, namely $K \subset S^3 \setminus \nu\eta$.

All the examples for Theorem 1.1 and $G = SR$ given in [FT05] (and in the correction [FT06]) are certain satellite knots of the ribbon knot 6_1 (cf. [FT05, Proposition 7.4]).

In [CFT07, Theorem 1.5] Tim Cochran, the author and Peter Teichner showed that under certain circumstances the 'multi–infection' of a topologically slice knot by a string link is still topologically slice. In the special case of an infection by a string knot we get the following result.

Theorem 1.2. *Let K be a topologically slice knot with slice disk D. Let $\eta \subset S^3 \setminus K$ a curve which is the unknot in S^3 and such that η is homotopically trivial in $D^4 \setminus D$. Then the satellite knot $S(K, C, \eta)$ is topologically slice for any knot C.*

Theorem 1.2 can in particular be used to give an alternative proof that Whitehead doubles and the satellite knots of [FT05, FT06] are topologically slice. We refer to [CFT07, Section 4] for full details. It is easy to give examples of knots to which Freedman's theorem applies but which are not satellite knots, e.g. the Kinoshita–Terasaka knots.

Whereas it is easy to compute the Alexander polynomial of a given knot it is much harder to verify whether the condition of Theorem 1.1 is satisfied for $G = SR$. Our main result of this paper is that there exist infinitely many non–satellite knots (more precisely hyperbolic knots) to which Theorem 1.1 applies. More precisely, relying heavily on results of Kawauchi [Ka89a, Ka89b, Ka89c], we can prove the following theorem.

Theorem 1.3. *Given $G = \mathbb{Z}$ or $G = SR$ and $V \in \mathbb{R}$ there exists a hyperbolic knot $K \subset S^3$ with Vol$(S^3 \setminus K) > V$ which has an epimorphism $\pi_1(M_K) \to G$, such that*

$$H_1(M_K; \mathbb{Z}[G]) \times H_1(M_K; \mathbb{Z}[G]) \to \mathbb{K}(G)/\mathbb{Z}[G]$$

vanishes.

2. Proof of Theorem 1.3

First note that in [FT05] it is shown that the vanishing of the Blanchfield pairing

$$H_1(M_K; \mathbb{Z}[G]) \times H_1(M_K; \mathbb{Z}[G]) \to \mathbb{K}(G)/\mathbb{Z}[G]$$

is equivalent to the condition that

$$\mathrm{Ext}^1_{\mathbb{Z}[G]}(H_1(M_K; \mathbb{Z}[G]), \mathbb{Z}[G]) = 0.$$

It is therefore enough to show that given $G = \mathbb{Z}$ or $G = SR$ and $V \in \mathbb{R}$ there exists a hyperbolic knot $K \subset S^3$ with $\mathrm{Vol}(S^3 \setminus K) > V$ which has an epimorphism $\pi_1(M_K) \to G$, such that

$$\mathrm{Ext}^1_{\mathbb{Z}[G]}(H_1(M_K; \mathbb{Z}[G]), \mathbb{Z}[G]) = 0.$$

Before we continue we recall that the derived series of a group $G^{(n)}, n \in \mathbb{N}$ is defined inductively by $G^{(0)} = G$ and $G^{(n+1)} = [G^{(n)}, G^{(n)}]$. The following result of Kawauchi is the key ingredient for the proof of Theorem 1.3.

Theorem 2.1. *Let $K \subset S^3$ be any knot, then for any $V \in \mathbb{R}$ there exists a hyperbolic knot $\tilde{K} \subset S^3$ together with a map $f : (S^3, \tilde{K}) \to (S^3, K)$ such that the following hold:*

1. *$Vol(S^3 \setminus \tilde{K}) > V$,*
2. *the map $\pi_1(S^3 \setminus \tilde{K}) \to \pi_1(S^3 \setminus K)$ is surjective, and*
3. *the induced map $\pi_1(S^3 \setminus \tilde{K})/\pi_1(S^3 \setminus \tilde{K})^{(n)} \to \pi_1(S^3 \setminus K)/\pi_1(S^3 \setminus K)^{(n)}$ is an isomorphism for any n.*

The theorem follows from the theory of almost identical imitations of Kawauchi. More precisely, the theorem follows from combining [Ka89b, Theorem 1.1] with [Ka89a, Properties I and V, p. 450] (cf. also [Ka89c]). Unfortunately the construction is so complex that it is difficult to draw an explicit example for $\tilde{K}$ given a knot K.

Recall that 6_1 is a ribbon knot which satisfies the conditions of Theorem 1.1 (cf. [FT05]). Theorem 1.3 now follows immediately from Theorem 2.1 applied to $K = 6_1$, and from the following proposition applied to $G = \mathbb{Z}$ and $G = SR$.

Proposition 2.2. *Let $K, \tilde{K} \subset S^3$ be knots and $f : (S^3, \tilde{K}) \to (S^3, K)$ a map such that (2) and (3) of Theorem 2.1 hold. Then we get an induced map $M_{\tilde{K}} \to M_K$ with the following two properties:*

1. *The induced map $\pi_1(M_{\tilde{K}}) \to \pi_1(M_K)$ is surjective and $\pi_1(M_{\tilde{K}})/\pi_1(M_{\tilde{K}})^{(n)} \to \pi_1(M_K)/\pi_1(M_K)^{(n)}$ is an isomorphism for any n.*
2. *For any homomorphism $\varphi : \pi_1(M_K) \to G$ to a solvable group G the induced map*
$$H_1(M_{\tilde{K}}; \mathbb{Z}[G]) \to H_1(M_K; \mathbb{Z}[G])$$
is an isomorphism of $\mathbb{Z}[G]$–modules.

We point out that (1) can also be shown using Kawauchi's methods, since the argument of [Ka89a, Property I] shows that the induced map $M_{\tilde{K}} \to M_K$ is an 'imitation' in the sense of [Ka89b] and we can again apply [Ka89a, Property V].

For the proof of Proposition 2.2 we will need the following basic lemma in order to get information on the relation between the fundamental groups of the 0–surgeries of the knots $K, \tilde{K}$ of the Theorem 2.1.

Lemma 2.3. *Let $f : G_1 \to G_2$ be a surjective homomorphism and let $H_1 \subset G_1$ be a normal subgroup. Then $H_2 = f(H_1)$ is normal and furthermore if the induced map*

$G_1/G_1^{(n)} \to G_2/G_2^{(n)}$ *is an isomorphism, then the induced map* $(G_1/H_1)/(G_1/H_1)^{(n)} \to (G_2/H_2)/(G_2/H_2)^{(n)}$ *is an isomorphism as well.*

Proof. Since $f: G_1 \to G_2$ is surjective it follows immediately that $H_2 = f(H_1)$ is normal. For $i = 1, 2$ let $G_i^{(n)} \cdot H_i := \{gh | g \in G_i^{(n)}, h \in H_i\}$. Since H_i and $G_i^{(n)}$ are normal subgroups it follows easily that $G_i^{(n)} \cdot H_i \subset G_i$ is a normal subgroup as well.

Claim.
$$(G_i^{(n)} \cdot H_i)/G_i^{(n)} = \operatorname{Ker}\{G_i/G_i^{(n)} \to (G_i/H_i)/(G_i/H_i)^{(n)}\}.$$

It is easy to see that $(G_i/H_i)^{(n)} = (G_i^{(n)} \cdot H_i)/H_i$. Now let $a \in G_i$ be in the kernel of $G_i \to (G_i/H_i)/(G_i^{(n)}/H_i)$. Then $aH_i \subset G_i^{(n)} \cdot H_i$. The claim is now immediate.

By the 5–lemma and our assumption the lemma now follows once we show that
$$(G_1^{(n)} \cdot H_1)/G_1^{(n)} \to (G_2^{(n)} \cdot H_2)/G_2^{(n)}$$
is an isomorphism. This map is surjective since $f: G_1 \to G_2$ is surjective, but the following commutative diagram shows that it is also injective:
$$\begin{array}{ccc} (G_1^{(n)} \cdot H_1)/G_1^{(n)} & \to & (G_2^{(n)} \cdot H_2)/G_2^{(n)} \\ \downarrow & & \downarrow \\ G_1/G_1^{(n)} & \to & G_2/G_2^{(n)} \end{array}$$
since the bottom map is an isomorphism and the vertical maps are injective. □

Proof of Proposition 2.2. Denote by λ a longitude of K. Since we have a map of pairs $f: (S^3, \tilde{K}) \to (S^3, K)$ it follows that $\tilde{\lambda} = f(\lambda)$ is a longitude of $\tilde{K}$. We denote the corresponding elements in the respective fundamental groups by λ and $\tilde{\lambda}$ as well. Denote by $\langle\langle\lambda\rangle\rangle$ and $\langle\langle\tilde{\lambda}\rangle\rangle$ the normal closures of the subgroups generated by $\lambda \in \pi_1(S^3 \setminus K)$ respectively $\tilde{\lambda} \in \pi_1(S^3 \setminus \tilde{K})$. Note that $\langle\langle\tilde{\lambda}\rangle\rangle = f(\langle\langle\lambda\rangle\rangle)$ since $f: \pi_1(S^3 \setminus \tilde{K}) \to \pi_1(S^3 \setminus K)$ is surjective.

Clearly $\pi_1(M_K) \cong \pi_1(S^3 \setminus K)/\langle\langle\lambda\rangle\rangle$ and $\pi_1(M_{\tilde{K}}) \cong \pi_1(S^3 \setminus \tilde{K})/\langle\langle\tilde{\lambda}\rangle\rangle$. The first statement now follows from Lemma 2.3

Now we turn to the proof of the second statement. Let $\varphi: \pi_1(M_K) \to G$ be a homomorphism such that G is solvable. Denote the induced map $\pi_1(M_{\tilde{K}}) \to \pi_1(M_K) \to G$ by $\tilde{\varphi}$. First note that the induced map $H_1(M_{\tilde{K}}; \mathbb{Z}[G]) \to H_1(M_K; \mathbb{Z}[G])$ is a $\mathbb{Z}[G]$–module homomorphism.

By assumption φ factors through $\pi_1(M_K)/\pi_1(M_K)^{(n)}$ for some n. In particular we have $\pi_1(M_K)^{(n+1)} \subset \operatorname{Ker}(\phi)^{(1)}$ and
$$\begin{aligned} H_1(M_K; \mathbb{Z}[G]) &\cong \operatorname{Ker}(\varphi)/\operatorname{Ker}(\varphi)^{(1)} \\ &\cong (\operatorname{Ker}(\varphi)/\pi_1(M_K)^{(n+1)})/(\operatorname{Ker}(\varphi)^{(1)}/\pi_1(M_K)^{(n+1)}). \end{aligned}$$

We therefore get a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & \mathrm{Ker}(\tilde{\varphi})^{(1)}/\pi_1(M_{\tilde{K}})^{(n+1)} & \to & \mathrm{Ker}(\tilde{\varphi})/\pi_1(M_{\tilde{K}})^{(n+1)} & \to & H_1(M_{\tilde{K}};\mathbb{Z}[G]) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathrm{Ker}(\varphi)^{(1)}/\pi_1(M_K)^{(n+1)} & \to & \mathrm{Ker}(\varphi)/\pi_1(M_K)^{(n+1)} & \to & H_1(M_K;\mathbb{Z}[G]) & \to & 0. \end{array}$$

Clearly the lemma follows once we show that the two vertical maps on the left are isomorphisms. We therefore consider for $i = 0, 1$

$$\begin{array}{ccc} \mathrm{Ker}(\tilde{\varphi})^{(i)}/\pi_1(M_{\tilde{K}})^{(n+1)} & \hookrightarrow & \pi_1(M_{\tilde{K}})/\pi_1(M_{\tilde{K}})^{(n+1)} \\ \downarrow & & \downarrow \cong \\ \mathrm{Ker}(\tilde{\varphi})^{(i)}/\pi_1(M_K)^{(n+1)} & \hookrightarrow & \pi_1(M_{\tilde{K}})/\pi_1(M_{\tilde{K}})^{(n+1)}. \end{array}$$

The horizontal maps are injections, the vertical map on the right is an isomorphism by (1). The vertical map on the left is therefore injective. Since $\pi_1(M_{\tilde{K}}) \to \pi_1(M_K)$ is surjective, it follows that the map on the left is also surjective. □

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