Topology

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Part I

General topology and basics of category theory

CHAPTER 0

Preliminaries

In this preliminary chapter we introduce some concepts and results from set theory and linear algebra which we will use throughout these notes, but which otherwise have little to do with topology itself. The reader might want to skip this preliminary chapter and directly move to the next chapter, where we actually start with topology.

0.1. **Basic set theory.** Topology builds on top of set theory. We will not go into proper set theory, but whatever we are doing is covered by ZFC, the Zermelo-Frankel set theory with the Axiom of Choice, which was developed by Ernst Zermelo and Adolf Fraenkel [**Zer08, Fra22, Zer30**]. We refer the reader to [**Cie97**] for a detailed introduction. In these notes we mostly employ the language of "naive set theory". In particular, when we describe sets we sometimes use creative notation and language. The hope is that the language is chosen in such a way that any good-willed reader will understand what is meant.

In this section we recall a few basic definitions and results from set theory which will frequently use throughout these notes. Usually we will not refer to the statements explicitly, unless it helps to clarify an argument.

First we introduce the following notation which should mostly be standard.

Notation.

(1) We write $\mathbb{N}_0 \coloneqq \{0, 1, 2, 3, \dots\}$ and $\mathbb{N} \coloneqq \{1, 2, 3, \dots\}$.
(2) We denote by $\varnothing$ the empty set.
(3) Given a set X we write $A \subset X$ if A is a subset of X and we write $A \subsetneq X$ if A is a subset of X with $A \neq X$.
(4) Given a set X and a subset $A \subset X$ we denote by $X \setminus A \coloneqq \{x \in X \mid x \notin A\}$ the complement of A in X.
(5) Given a set X we denote by $\mathcal{P}(X)$ its power set, i.e. $\mathcal{P}(X)$ is the set of all subsets of X.

Definition. A family of subsets of a set X consists of a set I and a map $f \colon I \to \mathcal{P}(X)$.

Remark.

(1) When it comes to families of subsets we often use suggestive notation, e.g. we might write "let $\{U_i\}_{i \in I}$ be a family of subsets of X", which just means that I is a set and the map $I \to \mathcal{P}(X)$ is denoted by $i \mapsto U_i$.
(2) When we write "let $U_1, \dots, U_k$ be subsets" or "let A, B be subsets", then often we will think of these as families of subsets with index sets $\{1, \dots, k\}$ respectively $\{1, 2\}$. □

Convention. Let X be a set. We will view any subset $\mathcal{U} \subset \mathcal{P}(X)$ as a family of subsets of X via the index set $I = \mathcal{U}$ and the map $I = \mathcal{U} \to \mathcal{P}(X)$ given by $U \mapsto U$. This convention might sound silly, but it will be useful on many occasions.

We continue with two of the most important definitions of set theory.

Definition. Let X be a set.

(1) Let $\{U_i\}_{i\in I}$ be a family of subsets of X. We write

$$\begin{aligned} \bigcup_{i\in I} U_i &= \{x \in X \mid \text{ there exists an } i \in I \text{ with } x \in U_i\} \quad \text{(Union)} \\ \bigcap_{i\in I} U_i &= \{x \in X \mid \text{ for every } i \in I \text{ we have } x \in U_i\} \quad \text{(Intersection)}. \end{aligned}$$

In particular, if $I = \varnothing$, then $\bigcup_{i\in I} U_i = \varnothing$ and $\bigcap_{i\in I} U_i = X$.[1]

(2) As we mentioned above, we view subsets $U_1, \dots, U_k$ of X as a family of subsets. We write $U_1 \cup \cdots \cup U_k := \bigcup_{i\in\{1,\dots,k\}} U_i$ and the same way for the intersection.

(3) Let A and B be two subsets of X.

(a) We say that the subsets A and B of X are disjoint if $A \cap B = \varnothing$.

(b) We say A intersects B if $A \cap B \neq \varnothing$.

The very first lemma of these lecture notes says that intersection is distributive with respect to union.

Lemma 0.1. (Intersection–Distributive Lemma) Let X be a set and let $\{U_i\}_{i\in I}$ be a family of subsets of X. Given any subset A of X we have

$$A \cap \left(\bigcup_{i\in I} U_i \right) = \bigcup_{i\in I} (A \cap U_i).$$

Proof. The lemma follows easily from the definitions and elementary logic. ■

Notation. Let $\{U_i\}_{i\in I}$ be a family of subsets of a given set X. If the U_i are pairwise disjoint, i.e. if for any $i \neq j \in I$ we have $U_i \cap U_j = \varnothing$, then at times we write

$$\bigsqcup_{i\in I} U_i := \bigcup_{i\in I} U_i.$$

We move on to the famous de Morgan's Laws from elementary set theory.

Lemma 0.2. (De Morgan's Laws) Let X be a set and let $\{A_i\}_{i\in I}$ be a family of subsets of X. If given a subset $B \subset X$ we denote its complement by $B^c := X \setminus B$, then

$$\left(\bigcap_{i\in I} A_i \right)^c = \bigcup_{i\in I} A_i^c \quad \text{and} \quad \left(\bigcup_{i\in I} A_i \right)^c = \bigcap_{i\in I} A_i^c.$$

Or equivalently, in a more cumbersome notation, we have

$$X \setminus \left(\bigcap_{i\in I} A_i \right) = \bigcup_{i\in I} (X \setminus A_i) \quad \text{and} \quad X \setminus \left(\bigcup_{i\in I} A_i \right) = \bigcap_{i\in I} (X \setminus A_i).$$

Proof. The lemma follows again easily from the definitions and elementary logic. ■

Next we study maps between sets and their effect on subsets.

Notation. Let $f\colon X \to Y$ be a map between sets.

(1) Given a subset $A \subset X$ we write $f(A) := \{f(x) \mid x \in A\}$.

[1] This simple observation will be useful at times.

(2) Given a subset $A \subset Y$ we write $f^{-1}(A) \coloneqq \{x \in X \mid f(x) \in A\}$. Furthermore, given $y \in Y$ we write $f^{-1}(y) \coloneqq f^{-1}(\{y\})$.[2]

The next lemma concerns the interaction between operations on sets and maps between sets.

Lemma 0.3. (Image-Preimage Lemma) Let $f\colon X \to Y$ be a map between two sets X and Y. Then the following statements hold:

(0)	(a)			X	$=$	$f^{-1}(Y)$
	(b)			$\varnothing$	$=$	$f^{-1}(\varnothing)$
(1)	(a)	for every subset $U \subset X$ we have		U	$\subset$	$f^{-1}(f(U))$
	(b)	furthermore, if f is injective[3], then we have		U	$=$	$f^{-1}(f(U))$
(2)	(a)	for every subset $V \subset Y$ we have		V	$\supset$	$f(f^{-1}(V))$
	(b)	furthermore, if f is surjective, then we have		V	$=$	$f(f^{-1}(V))$
(3)		for every family $\{U_i\}_{i\in I}$ of subsets of X we have	$f\left(\bigcup_{i\in I} U_i\right)$		$=$	$\bigcup_{i\in I} f(U_i)$
(4)		for every family $\{U_i\}_{i\in I}$ of subsets of X we have	$f\left(\bigcap_{i\in I} U_i\right)$		$\subset$	$\bigcap_{i\in I} f(U_i)$
(5)	(a)	for every subset $U \subset X$ we have	$f(X \setminus U)$		$\supset$	$f(X) \setminus f(U)$
	(b)	furthermore, if $f^{-1}(f(U)) = U$, then we have	$f(X \setminus U)$		$=$	$f(X) \setminus f(U)$
(6)		for every family $\{V_i\}_{i\in I}$ of subsets of Y we have	$f^{-1}\left(\bigcup_{i\in I} V_i\right)$		$=$	$\bigcup_{i\in I} f^{-1}(V_i)$
(7)		for every family $\{V_i\}_{i\in I}$ of subsets of Y we have	$f^{-1}\left(\bigcap_{i\in I} V_i\right)$		$=$	$\bigcap_{i\in I} f^{-1}(V_i)$
(8)		for every subset $V \subset Y$ we have	$f^{-1}(Y \setminus V)$		$=$	$X \setminus f^{-1}(V)$.

Proof. The lemma follows again from the definitions and elementary logic. ∎

Definition. Let $\{X_i\}_{i\in I}$ be a family of sets. The product of the family $\{X_i\}_{i\in I}$ is defined as the set[4] [5]

$$\prod_{i\in I} X_i \coloneqq \left\{ f\colon I \to \bigcup_{i\in I} X_i \;\middle|\; \text{for every } i \in I \text{ we have } f(i) \in X_i \right\}.$$

Furthermore, given a set X and a set I we write

$$X^I \coloneqq \prod_{i\in I} X.$$

We can now state the Axiom of Choice, which is discussed in greater detail in [**Cie97**, p. 15] and in [**Jec73**].

Axiom of Choice 0.1. For every family of sets $\{X_i\}_{i\in I}$, such that each I is non-empty and such that each X_i is non-empty, the product $\prod_{i\in I} X_i$ is non-empty.

[2] Purists might decry this notation, but it does make notation much more readable.

[3] As a totally irrelevant side remark we would like to point out that the fundamental terms "surjective", "injective" and "bijective" are actually fairly recent additions to the mathematical language. They were first introduced on page 80 of Nicolas Bourbaki's "Théorie des ensembles" first published in 1954, see [**Bou54**]. The reader who has not read the biography of the great mathematician Nicolas Bourbaki is strongly encouraged to do so.

[5] This definition is now possibly beyond the realm of "naive set theory", since it is perhaps a little unclear what the union of arbitrary sets is supposed to mean. But the Zermelo-Fraenkel axioms allow for unions of arbitrary sets.

As we mentioned in the beginning, in these lecture notes we will work with ZFC theory, in particular we will use the Axiom of Choice, even though for the most part it is for our purposes rather irrelevant.

For finitely many sets we adopt the following convenient notation.

Notation.
(1) Given finitely many sets $X_1, \dots, X_n$ we write

$$X_1 \times \cdots \times X_n \ := \ \{(x_1, \dots, x_n) \,|\, \text{ for } i = 1, \dots, n \text{ we have } x_i \in X_i\}.$$

This set has an obvious bijection to the product $\prod_{i=1}^{n} X_i$, and we will freely go back and forth between these two notations.
(2) Given a set X and given $n \in \mathbb{N}$ we write

$$X^n \ := \ \underbrace{X \times \cdots \times X}_{n \text{ times}}.$$

We conclude this section with one last piece of notation.

Notation.
(1) If $\{A_i\}_{i \in I}$ is a family of sets, then we define the disjoint union as the set

$$\bigsqcup_{i \in I} A_i \ := \ \bigcup_{i \in I} (A_i \times \{i\}).$$

(2) If $I = \{1, \dots, k\}$, then we write $A_1 \sqcup \cdots \sqcup A_k := \bigsqcup_{i=1}^{k} A_i$.
(3) We use obvious variations on (1) and (2), for example given sets A and B we refer to

$$A \sqcup B \ := \ (A \times \{1\}) \cup (B \times \{2\})$$

as the disjoint union of A and B

Remark. Let $\{U_i\}_{i \in I}$ be a family of subsets of a given set X such that for any $i \neq j \in I$ we have $U_i \cap U_j = \varnothing$. The expression $\bigsqcup_{i \in I} U_i$ now describes two different sets, namely it describes the subset of X defined on page 35 and it describes the disjoint union as defined above. There is a natural bijection between these two sets and thus we will freely go back and forth between these notations. In practice should not pose any problems. □

0.2. **Relations and equivalence relations on sets.** We start out with recalling the notion of a relation.

Definition. A relation on a set S is a subset of $S \times S$.

The definition of a relation in its full generality is pretty useless. It becomes more interesting once we introduce several adjectives.

Definition. Let S be a set and let $R \subset S \times S$ be a relation on S.
(1) The relation is called reflexive, if for every $s \in S$ we have $(s, s) \in R$.
(2) The relation is called transitive, if for every $(s, t), (t, u) \in R$ we also have $(s, u) \in R$.
(3) The relation is called symmetric, if for every (s, t) in R we also have $(t, s) \in R$.

We adopt the following notation:

Notation. Let S be a set and let $R \subset S \times S$ be a relation on S. Given $x, y \in S$ we write $x \, R \, y$ if and only if $(x, y) \in R$.[6]

Example. We consider the set $\mathbb{R}$ together with the relation

$$\leq \; := \; \{(x, y) \in \mathbb{R}^2 \,|\, y - x \text{ is not negative}\}.$$

One can easily verify that this relation $\leq$ is reflexive and transitive, but that it is not symmetric. Note that given $x, y \in \mathbb{R}$ the above expression "$x \leq y$" has the usual meaning that we are all used to. □

Most of the relations that we will consider are going to be reflexive and transitive. There is a big difference though in the roles of symmetric and (possibly) non-symmetric relations. Let us consider the symmetric ones first. We will turn to the (possibly) non-symmetric relations in the next section.

Definition. An equivalence relation on a set S is a relation that is reflexive, transitive and symmetric.

Convention. Equivalence relations on a set S are almost invariably denoted by "$\sim$" or some variation thereof.

To practice our conventions and notations, the three properties of an equivalence relation "$\sim$" on a set S are precisely the following three statements:

(1) for any $s \in S$ we have $s \sim s$ (reflexivity),
(2) for any $s, t, u \in S$ with $s \sim t$ and $t \sim u$ we have $s \sim u$ (transitivity),
(3) for any $s, t \in S$ with $s \sim t$ we also have $t \sim s$ (symmetry).

We continue with the following definition:

Definition. Let S be a set together with an equivalence relation "$\sim$". An equivalence class of $(S, \sim)$ is a subset C of S with the following three properties:

(a) the subset is non-empty,
(b) given any two $s, t \in C$ we have $s \sim t$, and
(c) given $s \in C$ and $t \in S$ with $s \sim t$ we have $t \in C$.

The following lemma summarizes the key properties of equivalence classes.

Lemma 0.4. (Equivalence Class Lemma) Let S be a set together with an equivalence relation "$\sim$".

(1) Every element of S is contained in a unique equivalence class, namely

$$[s] \; := \; \{t \in S \,|\, t \sim s\}.$$

(2) Two equivalence classes are either disjoint or the same.

Proof. Surely the reader will have seen the proof of the lemma before. Nonetheless we give the proof, in particular to highlight the fact that we do indeed need the symmetry condition.

(1) Let $s \in S$. Since $\sim$ is reflexive we see that $s \in [s]$. Thus it remains to show that $[s] := \{t \in S \,|\, t \sim s\}$ is indeed an equivalence class.
   (a) We already showed that $s \in [s]$, in particular we see that $[s]$ is non-empty.

[6] Most relations that we are working with are denoted by symbols like "$\sim$", "$\leq$", "$\cong$" or "$\simeq$".

(b) Let $t, t' \in [s]$. By definition this means that $t \sim s$ and $t' \sim s$. By symmetry we $s \sim t'$. By transitivity we deduce from $t \sim s$ and $s \sim t'$ that $t \sim t'$.
(c) Let $t \in [s]$ and suppose that $t \sim u$. Since by definition we have $s \sim t$ we obtain from transitivity that $s \sim u$, i.e. we see that $u \in [s]$.

(2) Let C and D be two equivalence classes. We suppose that $C \cap D \neq \varnothing$. Thus there exists an $x \in C \cap D$. We need to show that $C = D$. Thus let $d \in D$. By Condition (2) of an equivalence class applied to D we know that $x \sim d$. By condition (1) of an equivalence class, applied to C, we see that $d \in C$. Thus we have shown that $D \subset C$. Evidently the same argument shows that $C \subset D$. ■

Throughout these notes we adopt the following standard notation.

Notation. Let S be a set together with an equivalence relation "$\sim$".
(1) We denote by $S/\sim$ the set of equivalence classes.
(2) Given $s \in S$ we denote by $[s]$ the unique equivalence class that contains $[s]$.
(3) We refer to the map

$$\begin{array}{rccc} p\colon & X & \to & X/\sim \\ & x & \mapsto & [x] \end{array}$$

as the natural projection.

Example. Let $n \in \mathbb{N}$. We consider the set $\mathbb{Z}$ together with the equivalence relation that is given by

$$x \sim y \quad :\Longleftrightarrow (x-y) \text{ is divisible by } n.$$

There are exactly n equivalence classes, namely

$$\begin{array}{rcl} [0] & = & \{\dots, -2n, -n, 0, n, 2n, 3n, \dots\} \\ [1] & = & \{\dots, 1-2n, 1-n, 1, 1+n, 1+2n, 1+3n, \dots\} \\ & \vdots & \\ [n-1] & = & \{\dots, -2n-1, -n-1, -1, n-1, 2n-1, 3n-1, \dots\}. \end{array}$$

To the despair of many of my cultured friends we write $\mathbb{Z}_n \coloneqq \mathbb{Z}/\sim$. □

Lemma 0.5. (Quotient-Factorization Lemma) Let $\sim$ be an equivalence relation on a set X. We denote by $p\colon X \to X/\sim$ the natural projection. Let $f\colon X \to Y$ be a map to some set Y with the property that $f(x) = f(y)$ whenever $x \sim y$. Then the map

$$\begin{array}{rccc} g\colon & X/\sim & \to & Y \\ & [x] & \mapsto & f(x) \end{array}$$

is well-defined and it is the unique map $g\colon X/\sim \to Y$, such that $f = g \circ p$, i.e. such that the following diagram of maps commutes:

$$\begin{array}{ccc} X & \xrightarrow{p} & X/\sim \\ & \searrow^{f} & \downarrow{\scriptstyle \exists!\, g} \\ & & Y. \end{array}$$

Proof. Let $\sim$ be an equivalence relation on a set X and let $f\colon X \to Y$ be a map with the property that $f(x) = f(y)$ whenever $x \sim y$. Let a be an element of $X/\sim$. Since equivalences classes are by definition non-empty there exists an $x \in X$ with $a = [x]$. We set

$$g(a) \coloneqq f(x).$$

Evidently this definition does not depend on the choice of x. The map g satisfies $f = g \circ p$. Since the projection $X \to X/\sim$ is surjective we see that g is uniquely determined by this property. ■

We will now see that every relation gives rise to an equivalence relation.

Definition. Let S be a set and let R be a relation on S. We say that $x, y \in S$ are equivalent, if there exists a sequence $x_0, \dots, x_k$ of elements in S such that $x = x_0$, $x_k = y$ and such that for each $i \in \{0, \dots, k-1\}$ we have $x_i \, R \, x_{i+1}$ or $x_{i+1} \, R \, x_i$. It is straightforward to verify that "$\sim$" defines an equivalence relation on S. We refer to it as the equivalence relation generated by the relation R.

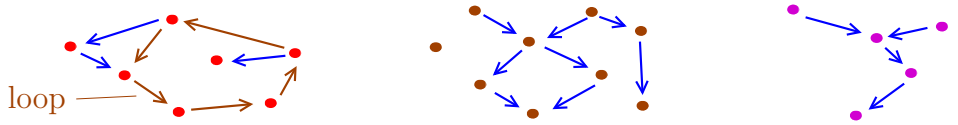
Example. We consider the set $S = \mathbb{R}$ and the relation that is defined by $x \, R \, y$ if and only if $y = x + 1$. The equivalence relation generated by R is the equivalence relation that is given by $x \sim y$ if and only if $x - y \in \mathbb{Z}$. □

Convention. At times we are a little sloppy in our notation and we do not distinguish in our notation between a relation and the equivalence relation it generates.

0.3. **Orders on sets.** In this section we turn our gaze towards relations which are not required to be symmetric. We have the following harmless definition.

Definition.
(1) A relation "$\leq$" on a set S is called antisymmetric if $a \leq b$ and $b \leq a$ implies $a = b$.
(2) A relation that is reflexive, transitive and antisymmetric is called a partial order.
(3) A set together with a partial order is called a partially ordered set.
(4) Let "$\leq$" be a partial order on a set A. Given $a, b \in A$ we write $a < b$ if $a \leq b$ and if $a \neq b$.

Example.
(1) Let X be a subset of $\mathbb{R}$ and denote by $\leq$ the usual "less or equal" relation on the real numbers. Then $(X, \leq)$ is a partially ordered set.
(2) We equip $\mathbb{R} \cup \{\pm\infty\}$ with the relation $\leq$ that is given by the usual relation on $\mathbb{R}$ together with $-\infty \leq a \leq \infty$ for any $a \in \mathbb{R}$. This turns $\mathbb{R} \cup \{\pm\infty\}$ into a partially ordered set.
(3) Let M be a set. Recall that we denote by $\mathcal{P}(M)$ the power set of M, i.e. the set of all subsets of M. Then "being a subset", i.e. the relation given by "$\subset$", is a partial order on $\mathcal{P}(M)$.
(4) Let X be a set. The trivial partial order $\leq_{\text{tr}}$ is the partial order where $x \leq_{\text{tr}} y$ if and only if $x = y$.
(5) We consider the three reflexive transitive relations that are illustrated in the figure below: Here it is understood that $P \leq Q$ if there exists a "directed path" from Q to P.



Example (a) Example (b) Example (c)

(a) The example on the left has a "loop". Given any two points P, Q on the loop we have $P \leq Q$ and $Q \leq P$, hence this example is not a partial order.
(b) This example is a partial order.
(c) This example is also a partial order. □

The following construction lets us build new examples of partially ordered sets out of given ones.

Definition. Let $(A, \leq)$ and $(B, \leq)$ be two partially ordered sets. Given (a, b) and (a', b') in $A \times B$ we define[7]

$$(a, b) < (a', b') \quad :\iff \quad \text{either } (a < a') \text{ or } (a = a' \text{ and } b < b')$$

and we define $(a, b) \leq (a', b')$ if either $(a, b) = (a', b')$ or $(a, b) < (a', b')$. One can easily verify that "$\leq$" is a partial order on $A \times B$. We refer to this partial order on $A \times B$ as the lexicographic order on $A \times B$.

0.4. **Maximal and minimal elements.** We recall the following definition:

Definition. Let $(P, \leq)$ be a partially ordered set and let $Q \subset P$ be a subset.
(1) A maximal element of Q is an element $q \in Q$ such that there is no $r \in Q$ with $q < r$.
(2) A minimal element of Q is an element $q \in Q$ such that there is no $r \in Q$ with $q > r$.

In important cases, maximal elements and minimal elements, if they exist, are unique. This leads us to the following notation:

Notation. If $(P, \leq)$ is a partially ordered set and let $Q \subset P$ be a subset. If the relation is antisymmetric and total, then if a maximum element exists, it is unique, and we denote it by $\max(Q)$. Similarly we define $\min(Q)$, if it exists.

It is clear that any bounded subset $Q \subset \mathbb{Z}$ has a maximal element and a minimal element. For some purposes the following notation is useful:

Notation. Let $Q \subset \mathbb{Z}$ be a subset.
(1) (a) If Q has no upper bound, then we write $\max(Q) := \infty$.
    (b) We write $\max(\varnothing) := -\infty$.
(2) (a) If Q has no lower bound , then we write $\min(Q) := -\infty$.
    (b) We write $\min(\varnothing) := +\infty$.

0.5. **The cardinality of sets.** In this section we introduce the notion of the cardinality of a set.

Definition.
(1) We say that two given sets S and T are equinumerous if there exists a bijection $\varphi \colon S \to T$.[8]
(2) Given a set S we define the cardinality $\#S$ of S as the equinumerosity equivalence class of S in the class of all sets[9]. Put differently we have

$$\#S \ := \ \text{class of all sets } T \text{ that are equinumerous with } S.$$

[7] Note that if $a < a'$, then we do not demand that $b \leq b'$.

(3) We say that a set is finite if there exists an $n \in \mathbb{N}_0$ such that $\#S = \#\{1, \dots, n\}$, otherwise we say that the set is infinite
(4) If a set S is finite, then we write $\#S = n$ for the unique $n \in \mathbb{N}_0$ with $\#S = \#\{1, \dots, n\}$. If S infinite, then sometimes we write $\#S = \infty$[10]. At times we also write $|S| \in \mathbb{N}_0 \cup \{\infty\}$ instead of $\#S \in \mathbb{N}_0 \cup \{\infty\}$.
(5) Let S be a set. If $\#S = n$ for some $n \in \mathbb{N}_0$ or if $\#S = \#\mathbb{N}$, then we say that S is countable, otherwise we say that S is uncountable.

Example. We have $\#(\mathbb{R} \setminus \{0\}) = \#\mathbb{R}$. Indeed a bijection is given by

$$\begin{array}{rcl} \mathbb{R} \setminus \{0\} & \to & \mathbb{R} \\ x & \mapsto & \begin{cases} x, & \text{if } x \in \mathbb{R} \setminus \mathbb{N}, \\ x - 1, & \text{if } x \in \mathbb{N}. \end{cases} \end{array}$$

□

Remark. Another way of putting the definition of cardinality is that given two sets S and T we have $\#S = \#T$ if and only if there exists a bijection from S to T. □

The following lemma gets used on many occasions in these lecture notes.

Lemma 0.6. (Countability Lemma)
(1) Every subset of a countable set is again countable.
(2) Let S be a countable set and let $f\colon S \to T$ be a map. The image $f(S)$ is again countable.
(3) The product of finitely many countable sets is again countable.
(4) The disjoint union of countably many countable sets is again countable.
(5) Let T be a set and let S be a subset such that $T \setminus S$ is countable. Then S is countable if and only if T is countable.
(6) If S is a finite set and if T is a countable set, then the set of maps from S to T is also countable.

Example. We consider the map

$$\begin{array}{rcl} \mathbb{Z} \times \mathbb{N} & \to & \mathbb{Q} \\ (m, n) & \mapsto & \frac{m}{n}. \end{array}$$

Evidently this map is surjective. It follows from the Countability Lemma 0.6 (2) and (3) that $\mathbb{Q}$ is countable. □

Sketch of proof.

(1) Let T be a subset of a countable set S. After excluding the trivial case that S is finite and the case that T is finite one sees that one has to prove the following: Given any infinite subset T of $S = \mathbb{N}$ there exists a bijection $\varphi\colon \mathbb{N} \to T$. Such a bijection is given by defining $\varphi(i)$ as the i-th smallest element of T.

[8]It follows immediately from the definitions that being equinumerous defines an equivalence relation on the class of all sets.
[9]We defined the notion of an equivalence relation on a set, but not on a class. You can ask any of your professors, as long as it is not me, what this now means.
[10]This notation is a little risque. The sets $\mathbb{N}$ and $\mathbb{R}$ are both infinite, but in the $\mathbb{R}$-Uncountable Corollary 0.8 we will see that $\mathbb{N}$ and $\mathbb{R}$ are not equinumerous. Thus we write $\#\mathbb{N} = \infty$ and $\#\mathbb{R} = \infty$, but $\#\mathbb{N} \neq \#\mathbb{R}$.

(2) Let S be a countable set and let $f\colon S \to T$ be a map. We only really need to consider the case that $S = \mathbb{N}$ and that $f(S)$ is infinite. In this case the bijection $\varphi\colon \mathbb{N} \to f(S)$ is given by defining $\varphi(1) = f(1)$ and by defining iteratively

$$\varphi(i+1) \;:=\; \left\{ f(k) \;\middle|\; \begin{array}{c} k \in \mathbb{N} \text{ is the smallest number such} \\ \text{that } f(k) \notin \{\varphi(1), \ldots, \varphi(i)\} \end{array} \right\}.$$

(3) By induction and by ignoring the rather dull case of finite sets it remains to show that $\mathbb{N} \times \mathbb{N}$ is countable. This can be shown by the well-known argument sketched in the figure below.



An alternative, more formal argument would be to consider the map $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$ that is given by $(k, l) \mapsto (2^k \cdot 3^l)$. This map is injective, thus it follows from (1) that $\mathbb{N} \times \mathbb{N}$ is countable.



(4) Let $\{S_i\}_{i \in I}$ be a countable family of countable subsets. It is straightforward to define an injective map $\bigsqcup_{i \in I} S_i \to \mathbb{N} \times \mathbb{N}$. Thus it follows from (1) and (3) that the disjoint union $\bigsqcup_{i \in I} S_i$ is countable.

(5) Let T be a set and let S be a subset such that $T \setminus S$ is countable. If S is countable, then it follows from (4) that $T = (T \setminus S) \cup S$ is countable. On the other hand if T is countable, then it follows from (1) that S is countable.

(6) Let $S = \{v_1, \ldots, v_n\}$ be a finite set and let T be a countable set. We have an obvious bijection between T^n and the set of maps from S to T. Thus it follows from (4) that the set of maps from S to T is countable. ■

We move on to the next definition.

Definition. Let S and T be two sets.

(1) We write $\#S \leq \#T$ if there exists an injection $\varphi\colon S \to T$.

(2) We write $\#S < \#T$ if $\#S \leq \#T$ but $\#S \neq \#T$.

Example. It follows from the Countability Lemma 0.6 (1) that a set S is countable if and only if $\#S \leq \#\mathbb{N}$. □

The following theorem was first proved by Georg Cantor [**Can92**]. The ratio of significance of the statement to length of the proof seems to be unbeatable.

Theorem 0.7. (Cantor's Theorem) Recall that given a set X we denote by $\mathcal{P}(X)$ its power set, i.e. the set of all subsets of X. For every set X we have $\#X < \#\mathcal{P}(X)$.

Proof. First note that the map
$$\begin{array}{rcl} X & \to & \mathcal{P}(X) \\ x & \mapsto & \{x\} \end{array}$$
is evidently injective, hence $\#X \leq \#\mathcal{P}(X)$. Now let $f\colon X \to \mathcal{P}(X)$ be a map. We have to show that it cannot be surjective. We consider the set
$$Y \;:=\; \{x \in X \,|\, x \notin f(x)\}.$$
We claim that Y is not in the image of f. Indeed, if there did exist a $z \in X$ with $f(z) = Y$, then we would have
$$z \in Y \quad \underset{\text{by definition of } Y}{\overset{\uparrow}{\Longleftrightarrow}} \quad z \notin f(z) \quad \underset{\text{since } f(z)=Y}{\overset{\uparrow}{\Longleftrightarrow}} \quad z \notin Y.$$
We have thus obtained a contradiction. ∎

Corollary 0.8. ($\mathbb{R}$-Uncountable Corollary) The set $\mathbb{R}$ of real numbers is uncountable.

Proof. [11] We consider the maps
$$\begin{array}{rcl} f\colon \mathcal{P}(\mathbb{N}) & \to & \{\mathbb{Z}_2\}^{\mathbb{N}} \\ X & \mapsto & \left(\begin{array}{rcl} \mathbb{N} & \to & \mathbb{Z}_2 \\ k & \mapsto & \begin{cases} 1, & \text{if } k \in X \\ 0, & \text{otherwise} \end{cases} \end{array} \right) \end{array} \qquad \text{and} \qquad \begin{array}{rcl} g\colon \{\mathbb{Z}_2\}^{\mathbb{N}} & \to & \mathbb{R} \\ f & \mapsto & \sum_{k=1}^{\infty} \frac{f(k)}{3^k}. \end{array}$$
One can easily verify that $f\colon \mathcal{P}(\mathbb{N}) \to \{\mathbb{Z}_2\}^{\mathbb{N}}$ is a bijection and that $g\colon \{\mathbb{Z}_2\}^{\mathbb{N}} \to \mathbb{R}$ is injective. Thus it follows from Cantor's Theorem 0.7 together with the Countability Lemma 0.6 (1) that $\mathbb{R}$ is uncountable. ∎

Now we can formulate the final result of this section.

Theorem 0.9. (Bernstein-Schröder Theorem)[12] Let S and T be two sets. Then
$$\#S \leq \#T \quad \text{and} \quad \#T \leq \#S \quad \Longrightarrow \quad \#S = \#T.$$

Example. Let U be a non-empty subset of $\mathbb{R}$ which contains an open interval (a, b) with $a < b$. Then $\#U = \#\mathbb{R}$. Indeed, the inclusion $U \to \mathbb{R}$ is of course injective, thus we see that $\#U \leq \#\mathbb{R}$. Furthermore the map $\varphi\colon (a, b) \to \mathbb{R}$ given by $x \mapsto \frac{x}{(x-a)(x-b)}$ defines a bijection between (a, b) and $\mathbb{R}$. Thus $\mathbb{R} \xrightarrow{\varphi^{-1}} (a, b) \to U$ is also an injection. This means that $\#\mathbb{R} \leq \#U$. Therefore it follows from the Bernstein-Schröder-Theorem 0.9 that there exists a bijection from U to $\mathbb{R}$. □

Proof. Let $f\colon A \to B$ and $g\colon B \to A$ be two injective maps between two sets A and B. Without loss of generality we can assume that A and B are disjoint.[13] We denote by $\sim$ the

[11] There are many proofs that the real numbers are uncountable. Perhaps the most beautiful and astounding proof that $\mathbb{Q}$ is countable is given via the amazing Farey sequence, see [**Hat21**, Chapter 3].

[12] The theorem was initially stated by Georg Cantor [**Can87**] in 1887 without a proof and it was first proved by Felix Bernstein [**Ber05**]. Sometimes it is also called the Cantor-Bernstein-Schröder Theorem. The proof we provide is attributed to Julius König [**Kön06**].

[13] If A and B are not disjoint, then we replace A by $A' = A \times \{0\}$ and B by $B' = B \times \{1\}$, then A' and B' are disjoint, since any two elements in A' and B' now have different second coordinate.

equivalence relation on $A \sqcup B$ that is generated by $a \sim f(a)$ for $a \in A$ and by $b \sim g(b)$ for $b \in B$. Note that if $a \in A$ lies in the image of g, then by the injectivity of g there exists a unique element $g^{-1}(a) \in B$ and we have $a \sim g^{-1}(a)$. The same remark works with the roles of f and g reversed.

Claim. For each equivalence class I there exists a bijection $A \cap I \to B \cap I$.

Proof. So let I be an equivalence class and let $a \in A \cap I$.[14] The subset $I \subset A \sqcup B$ consists of all elements equivalent to a. The equivalence class I is then of one of the following three types:[15][16]

		A		B	A	B	A	B	
(1)	a', $\dots$	$f^{-1}(g^{-1}(a))$,		$g^{-1}(a)$,	a,	$f(a)$,	$g(f(a))$,	$f(g(f(a))$, $\dots$	with $a' \in A \setminus g(B)$
(2)	b', $\dots$	$f^{-1}(g^{-1}(a))$,		$g^{-1}(a)$,	a,	$f(a)$,	$g(f(a))$,	$f(g(f(a))$, $\dots$	with $b' \in B \setminus f(A)$
(3)	$\dots$ $\dots$	$f^{-1}(g^{-1}(a))$,		$g^{-1}(a)$,	a,	$f(a)$,	$g(f(a))$,	$f(g(f(a))$, $\dots$	

In case (1) the map f defines a bijection $A \cap I \to B \cap I$, in case (2) the map g^{-1} defines a bijection $A \cap I \to B \cap I$ and in case (3) both f and g^{-1} define a bijection $A \cap I \to B \cap I$.⊞

Recall that by the Equivalence Class Lemma 0.4 we know that equivalence classes always give a disjoint decomposition of a set. In our case, if I_j, $j \in J$ denote the set of equivalence classes of $\sim$ on $A \sqcup B$, then we have

$$A \sqcup B = \bigsqcup_{j \in J} I_j \quad \text{and therefore also} \quad A = \bigsqcup_{j \in J} (A \cap I_j) \text{ and } B = \bigsqcup_{j \in J} (B \cap I_j).$$

Since for each $j \in J$ we have a bijection $A \cap I_j \to B \cap I_j$ we get a bijection $A \to B$. ■

0.6. The Cauchy-Schwarz Inequality.

Definition. Let R be a commutative ring and let V and W be R-modules.

(1) A pairing on $V \times W$ is a bilinear map

$$\varphi \colon V \times W \to R,$$

where by bilinear we mean that for all $v, v' \in V$ and for all $w, w' \in W$ we have the equalities $\varphi(v + v', w) = \varphi(v, w) + \varphi(v', w)$ and $\varphi(v, w + w') = \varphi(v, w) + \varphi(v, w')$.

(2) A pairing is called R-bilinear if for all $\lambda \in R$ and all $v \in V$ and $w \in W$ we have $\varphi(\lambda \cdot v, w) = \lambda \cdot \varphi(v, w)$ and $\varphi(v, \lambda \cdot w) = \varphi(v, w) \cdot \lambda$.

(3) An R-bilinear pairing $\varphi \colon V \times V \to S$ is called symmetric if $\varphi(v, w) = \varphi(w, v)$ for all $v, w \in V$.

We now specialize to the case that the ring R are the real numbers.

Definition. Let V be a real vector space.

(1) We say that a pairing φ is positive-definite if $\varphi(v, v) > 0$ for all $v \in V \setminus \{0\}$.

(2) We say that a pairing φ is positive-semidefinite if $\varphi(v, v) \geq 0$ for all $v \in V \setminus \{0\}$.

---

[14] There always exists an $a \in A \cap I$. Indeed, I is by definition non-empty. If it contains an element in A we are done. Otherwise it contains an element in $b \in B$, but then it also contains $g(b) \in A$, since $b \sim g(b)$.

[15] To clarify the description: In (1) and (2) we look at sets which are described by "half-infinite" sequences starting at a' respectively b'. In (3) we look at a sequence which is infinite in "both directions".

[16] Note that we do not claim that the elements in the given sequences are distinct, for example $f \colon A \to B$ could be a bijection and $g = f^{-1}$ its inverse, then the equivalence class consists only of the two elements a and $g(a)$.

Example. Let $n \in \mathbb{N}_0$. The standard pairing on $\mathbb{R}^n$ is given by

$$\begin{array}{rccl} \langle -,- \rangle \colon \mathbb{R}^n \times \mathbb{R}^n & \to & \mathbb{R} \\ (v,w) & \mapsto & \langle v,w \rangle \coloneqq \sum_{i=1}^{n} v_i \cdot w_i. \end{array}$$

Here we use the standard convention that for $v \in \mathbb{R}^n$ we denote by $v_1, \ldots, v_n$ its coordinates. This standard pairing on $\mathbb{R}^n$ is evidently $\mathbb{R}$-bilinear, symmetric and positive-definite. □

Proposition 0.10. (Cauchy-Schwarz Inequality) Let V be a real vector space and let

$$\begin{array}{rccl} \varphi \colon V \times V & \to & \mathbb{R}_{\geq 0} \\ (v,w) & \mapsto & \varphi(v,w) \end{array}$$

be an $\mathbb{R}$-bilinear form.

(1) If φ is positive semidefinite, then for any $v, w \in V$ we have

$$\varphi(v,w)^2 \;\leq\; \varphi(v,v) \cdot \varphi(w,w).$$

(2) Now suppose that φ is positive-definite. In this equality in (1) holds if and only if v and w are linearly dependent.

Example. Let $v_1, \ldots, v_n \in \mathbb{R}^n$ and let $w_1, \ldots, w_n \in \mathbb{R}$. It follows immediately from the Cauchy-Schwarz Inequality 0.10, applied to the standard pairing on $\mathbb{R}^n$, that

$$(v_1 \cdot w_1 + \cdots + v_n \cdot w_n)^2 \;\leq\; (v_1^2 + \cdots + v_n^2) \cdot (w_1^2 + \cdots + w_n^2).$$ □

Proof. First we just assume that φ is positive semidefinite. Let $v, w \in V$. Given $t \in \mathbb{R}$ we consider

$$f(t) \;\coloneqq\; \varphi(v + t \cdot w, v + t \cdot w) \underset{\substack{\uparrow \\ \text{since } \varphi \text{ is } \mathbb{R}\text{-bilinear and symmetric}}}{=} \varphi(v,v) + 2t \cdot \varphi(v,w) + t^2 \cdot \varphi(w,w).$$

Since φ is positive semidefinite we see that f assumes only non-negative values. Since f is a polynomial function we obtain from elementary analysis that f assumes a minimum, which in turn implies that there exists a $t_0 \in \mathbb{R}$ with $f'(t_0) = 0$. It follows from the second description of $f(t)$ that

$$f'(t) \;=\; 2 \cdot \varphi(v,w) + 2t \cdot \varphi(w,w).$$

This implies that $t_0 \cdot \varphi(w,w) = -\varphi(v,w)$. Now note that

$$\begin{aligned} \overbrace{\varphi(v + t_0 \cdot w, v + t_0 \cdot w)}^{\geq 0} \cdot \overbrace{\varphi(w,w)}^{\geq 0} &= \varphi(w,w) \cdot \varphi(v,v) + 2\overbrace{t_0 \cdot \varphi(w,w)}^{= -\varphi(v,w)} \cdot \varphi(v,w) + \overbrace{t_0^2 \cdot \varphi(w,w) \cdot \varphi(w,w)}^{= \varphi(v,w)^2} \\ &\underset{\substack{\uparrow \\ \text{follows from } t_0 \cdot \varphi(w,w) = -\varphi(v,w)}}{=} \varphi(w,w) \cdot \varphi(v,v) - \varphi(v,w) \cdot \varphi(v,w). \end{aligned}$$

Since φ is positive semidefinite we see that $\varphi(v + t_0 \cdot w, v + t_0 \cdot w) \cdot \varphi(w,w) \geq 0$. Thus follows from the above little calculation above that $\varphi(v,w)^2 \leq \varphi(v,v) \cdot \varphi(w,w)$.

Now suppose that $\varphi(v,w)^2 = \varphi(v,v) \cdot \varphi(w,w)$. This implies, by the above calculation, that $\varphi(v + t_0 \cdot w, v + t_0 \cdot w) \cdot \varphi(w,w) = 0$. Since φ is positive-definite this means that $v + t_0 \cdot w = 0$ or $w = 0$. In either case, v, w are linearly dependent. ■

Finally on some occasions we will need a variation of the Cauchy-Schwarz Inequality 0.10 where we work with pairings over $\mathbb{C}$. This leads us to the following definitions.

Definition. Let V be a complex vector space.
(1) A pairing $\varphi\colon V \times V \to \mathbb{C}$ is called $\mathbb{C}$-sesquilinear if for all $\lambda \in \mathbb{C}$ and all $v, w \in V$ we have $\varphi(\lambda \cdot v, w) = \lambda \cdot \varphi(v, w)$ and $\varphi(v, \lambda \cdot w) = \varphi(v, w) \cdot \overline{\lambda}$.
(2) A $\mathbb{C}$-sesquilinear pairing $\varphi\colon V \times V \to \mathbb{C}$ is called hermitian if $\varphi(v, w) = \overline{\varphi(w, v)}$ for all $v, w \in V$.
(3) We say that a pairing φ is positive-definite if $\varphi(v, v) \in \mathbb{R}_{>0}$ for all $v \in V \setminus \{0\}$.
(4) We say that a pairing φ is positive-semidefinite if $\varphi(v, v) \in \mathbb{R}_{\geq 0}$ for all $v \in V \setminus \{0\}$.

Example. Let $n \in \mathbb{N}_0$. The standard hermitian pairing on $\mathbb{C}^n$ is given by

$$\begin{array}{rccl} \langle -, - \rangle\colon \mathbb{C}^n \times \mathbb{C}^n & \to & \mathbb{R} \\ (v, w) & \mapsto & \langle v, w \rangle := \sum_{i=1}^{n} v_i \cdot \overline{w}_i. \end{array}$$

This standard pairing on $\mathbb{C}^n$ is evidently $\mathbb{C}$-sesquilinear, hermitian and positive-definite. □

Notation. Given $z = x + y\mathrm{i} \in \mathbb{C}$ with $x, y \in \mathbb{R}$ we write $|z| := \sqrt{x^2 + y^2} = \sqrt{z \cdot \overline{z}}$.

Proposition 0.11. (Cauchy-Schwarz Inequality) Let V be a complex vector space and let

$$\begin{array}{rccl} \varphi\colon V \times V & \to & \mathbb{R}_{\geq 0} \\ (v, w) & \mapsto & \varphi(v, w) \end{array}$$

be a hermitian form.
(1) If φ is positive semidefinite, then for any $v, w \in V$ we have

$$|\varphi(v, w)|^2 \leq \varphi(v, v) \cdot \varphi(w, w).$$

(2) Now suppose that φ is positive-definite. In this case equality in (1) holds if and only if v and w are linearly dependent.

Proof. The proof is very similar to the proof of the above Cauchy-Schwarz Inequality 0.10. We leave it to the reader to make the necessary minor modifications. ■

0.7. **The Gram-Schmidt Orthogonalization Process.** The following proposition should be, perhaps in a slightly simplified version, familiar from a course on linear algebra.

Proposition 0.12. (Gram-Schmidt Orthogonalization Process) Let V be a real vector space and let $\langle\ ,\ \rangle\colon V \times V \to \mathbb{R}$ be a positive-definite symmetric $\mathbb{R}$-bilinear[17] form. Given $v \in V$ we write $\|v\| := \sqrt{\langle v, v \rangle}$. Let $v_1, \ldots, v_n$ be a basis of V. We consider the following sequence of n-tuples of vectors in V:[18]

$$\begin{array}{l} u_{0,1}, .., u_{0,n} := v_1, .., v_n \\ u_{1,1}, .., u_{1,n} := \frac{v_1}{\|v_1\|}, \|v_1\| \cdot v_2, v_3, .., v_n \\ \quad \vdots \qquad\qquad \vdots \\ u_{i,1}, .., u_{i,n} := u_{i-1,1}, .., u_{i-1,i-1}, \frac{z}{\|z\|}, \|z\| \cdot v_{i+1}, v_{i+2}, .., v_n \text{ with } z := v_i - \sum_{j=1}^{i-1} \langle v_i, u_{i-1,j} \rangle \cdot u_{i-1,j} \\ \quad \vdots \qquad\qquad \vdots \\ u_{n,1}, .., u_{n,n} := u_{n-1,1}, .., u_{n-1,n-1}, \frac{z}{\|z\|} \qquad \text{with } z := v_n - \sum_{j=1}^{n-1} \langle v_n, u_{n-1,j} \rangle \cdot u_{n-1,j}. \end{array}$$

This sequence has the following properties:

(1) For each $i \in \{1, \ldots, n\}$ the vectors $\{u_{i,1}, \ldots, u_{i,n}\}$ form a basis of V.
(2) For each $i \in \{1, \ldots, n-1\}$ the base change matrix from the basis $v_1, \ldots, v_n$ to the basis $u_{i,1}, \ldots, u_{i,n}$ has determinant 1.
(3) The base change matrix from the basis $v_1, \ldots, v_n$ to the basis $u_{n,1}, \ldots, u_{n,n}$ has positive determinant.
(4) For each $i \in \{1, \ldots, n\}$ the vectors $u_{i,1}, \ldots, u_{i,i}$ are orthonormal.
(5) If $v_1, \ldots, v_n$ are orthonormal, then the sequence of n-tuples of vectors is constant, i.e. for any $i, j \in \{1, \ldots, n\}$ we have $u_{i,j} = v_j$.
(6) If $\varphi\colon V \to V$ is a linear isometry, then the above process commutes with application of φ.

The analogous statement also holds if we are dealing with a positive-definite hermitian $\mathbb{C}$-sesquilinear form over a complex vector space.

Proof. The lemma follows from elementary arguments which we leave to the readers who suffer from linear algebra withdrawal syndrome. ■

0.8. **Group actions.** The following definition might also have appeared in an earlier math course.

Definition. Let X be a set and let G be a group with trivial element e.

(1) An action of G on X is a map

$$\begin{array}{rcl} G \times X & \to & X \\ (g, x) & \mapsto & g \cdot x \end{array}$$

with the following properties:

$$\begin{array}{rcll} e \cdot x & = & x, & \text{for all } x \in X, \\ g \cdot (h \cdot x) & = & (g \cdot_G h) \cdot x, & \text{for all } x \in X \text{ and } g, h \in G. \end{array}$$

($g \cdot (h \cdot x)$: $g \cdot$ group action, $h \cdot$ group action; $g \cdot_G h$: multiplication in the group G)

(2) The action is called free, if $g \cdot x = x$ for some $x \in X$ implies that $g = e$.
(3) We say that G acts transitively, if for every x and y in X there exists a $g \in G$ with $g \cdot x = y$.

Remark. Sometimes what we call an "action" is more precisely referred to as a "left-action". A "right-action" is exactly the same as a left-action except that the second property gets replaced by $g \cdot (h \cdot x) = (h \cdot_G g) \cdot x$. Or, more intuitively, a right-action is a map $X \times G \to X$ such that we have $(x \cdot h) \cdot g = x \cdot (h \cdot_G g)$. Basically all the statements and definitions that we give for left-actions carry over, with minuscule modifications, to the setting of right-actions. □

Examples.

(1) Given any group G the map

[17] Recall that according, to the definition that we gave on page 45, we say that a symmetric $\mathbb{R}$-bilinear form $\langle -, - \rangle\colon V \times V \to \mathbb{R}$ is positive-definite if for every $v \neq 0$ we have $\langle v, v \rangle > 0$.

[18] Our formulation is more complicated than the standard formulation of the Gram-Schmidt Orthogonalization Process. The reason is that we want to have control over the base change matrices. This is important when we show, in the proof of the Matrix Group-π_0-Proposition 2.23 (2), that $\mathrm{SL}(n, \mathbb{R})$ is path-connected.

$$\begin{array}{rcl} G \times G & \to & G \\ (g,h) & \mapsto & ghg^{-1} \end{array}$$

defines an action of G on itself. Let G be a non-trivial group. Considering $h = e$ we see that this action is not free and it is also straightforward to see that the action is not transitive.

(2) Let $\mathbb{F}$ be a field and let V be a k-dimensional vector space over $\mathbb{F}$. We denote by B the set of all ordered bases of V. The map

$$\begin{array}{rcl} \mathrm{GL}(k,\mathbb{F}) \times B & \to & B \\ \big((a_{ij})_{i,j=1,\dots,k}, (v_1,\dots,v_k)\big) & \mapsto & \Big(\sum_{i=1}^{k} a_{i1}v_i, \dots, \sum_{i=1}^{k} a_{ik}v_i\Big) \end{array}$$

is a transitive action. It is an amusing linear algebra exercise to figure out under what circumstances this action is free.

(3) The group $G = \mathbb{Z}^n$ acts on $X = \mathbb{R}^n$ by addition. This action is evidently free but it is not transitive. □

We conclude this section on group actions with the following elementary lemma:

Lemma 0.13. (Group Action-Equivalence Relation Lemma) Let X be a set and let G be a group that acts on X. Then

$$x \sim y \quad :\Longleftrightarrow \quad \text{there exists a } g \in G \text{ such that } g \cdot x = y$$

is an equivalence relation on X.

Proof. The statement follows easily from the definitions. ■

Definition. Let X be a set and let G be a group that acts on X. By the Group Action-Equivalence Relation Lemma 0.13 we know that

$$x \sim_G y \quad :\Longleftrightarrow \quad \text{there exists a } g \in G \text{ such that } g \cdot x = y$$

is an equivalence relation on X. We denote by[19]

$$X/G \;\coloneqq\; X/\sim_G .$$

the set of equivalence classes.

0.9. **Normed vector spaces.** We continue with another definition from linear algebra.

Definition. Let V be a real vector space. A norm on V is a map $\|-\| \colon V \to \mathbb{R}_{\geq 0}$ that has the following three properties:

(1) For every $v \in V$ we have $\|v\| = 0 \iff v = 0.$
(2) For every $v \in V$ and $\lambda \in \mathbb{R}$ we have $\|\lambda \cdot v\| = |\lambda| \cdot \|v\|.$
(3) For every $v, w \in V$ we have $\|v + w\| \leq \|v\| + \|w\|.$

[19] It is arguably more logical to write $G\backslash X$ since we are dealing with a left-action and to reserve the notation X/G for right-actions. We stick to X/G since it is closer to our language "X mod G" and since we will not deal with right-actions at all.

Lemma 0.14. (Pairing–Norm Lemma) Let V be a vector space. If $\langle -,-\rangle \colon V \times V \to \mathbb{R}_{\geq 0}$ is a symmetric positive-definite $\mathbb{R}$-linear pairing, then the map

$$\begin{array}{rcl} V \times V & \to & \mathbb{R}_{\geq 0} \\ v & \mapsto & \|v\| \coloneqq \sqrt{\langle v, v\rangle} \end{array}$$

is a norm.

Proof. It is straightforward to verify that the first two properties of a norm are satisfied. It remains to prove the third property. Given $v, w \in \mathbb{R}^n$ we see that

$$\begin{array}{rcl} \|v+w\|^2 & = & \langle v+w, v+w\rangle \overset{\downarrow}{=} \langle v,v\rangle + \langle v,w\rangle + \langle w,v\rangle + \langle w,w\rangle \overset{\downarrow}{=} \langle v,v\rangle + 2\cdot\langle v,w\rangle + \langle w,w\rangle \\ & \underset{\uparrow}{\leq} & \langle v,v\rangle + 2\cdot\sqrt{\langle v,v\rangle\cdot\langle w,w\rangle} + \langle w,w\rangle \;=\; \left(\sqrt{\langle v,v\rangle} + \sqrt{\langle w,w\rangle}\right)^2 = (\|v\| + \|w\|)^2. \end{array}$$

follows from the $\mathbb{R}$-bilinearity of $\langle\ ,\ \rangle$ — since $\langle w,v\rangle = \langle v,w\rangle$

by the Cauchy-Schwarz Inequality 0.10 we have $\langle v,w\rangle^2 \leq \langle v,v\rangle \cdot \langle w,w\rangle$

By taking square roots on both sides we obtain that $\|v+w\| \leq \|v| + \|w\|$. ■

Definition. The standard pairing on $\mathbb{R}^n$ gives rise to the Euclidean norm

$$\begin{array}{rcl} \|-\|_{\text{Eucl}} \colon \mathbb{R}^n & \to & \mathbb{R}_{\geq 0} \\ v = (v_1, \ldots, v_n) & \mapsto & \sqrt{\langle v, v\rangle} = \sqrt{v_1^2 + \cdots + v_n^2}. \end{array}$$

Usually, when there is no source of confusion, we just write $\|v\|$ instead of $\|v\|_{\text{Eucl}}$.

Example. We will now give more examples of norms $\mathbb{R}^n$. One can easily show that the following maps: are norms and that for $n \geq 2$ they do *not* arise from a positive-definite $\mathbb{R}$-bilinear pairing:

$$\underbrace{\begin{array}{rcl} \|-\|_{\max} \colon \mathbb{R}^n & \to & \mathbb{R}_{\geq 0} \\ (x_1, \ldots, x_n) & \mapsto & \max\{|x_1|, \ldots, |x_n|\} \end{array}}_{\text{maximum norm}} \quad \text{and} \quad \underbrace{\begin{array}{rcl} \|-\|_{\text{Manh}} \colon \mathbb{R}^n & \to & \mathbb{R}_{\geq 0} \\ (x_1, \ldots, x_n) & \mapsto & \sum_{i=1}^{n} |x_i|. \end{array}}_{\text{Manhattan norm}}$$

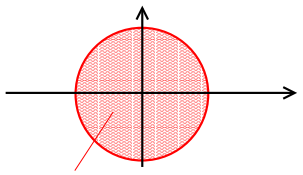
□

Since the human brain is equipped with a great image processor it is usually helpful to turn an abstract definition into an image. In the present context the key to doing so is given by the following definition:

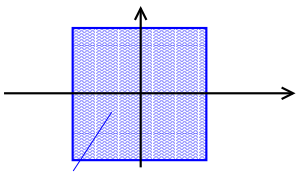
Definition. Given a norm $\|-\| \colon V \to \mathbb{R}_{\geq 0}$ we define

$$\text{norm ball of } \|-\| \;\coloneqq\; \{v \in V \mid \|v\| \leq 1\}.$$

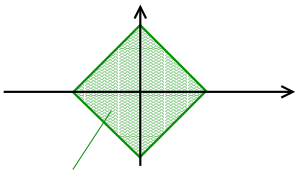
Example. In the figure below we show the norm balls of the Euclidean norm, the maximum norm and the Manhattan norm. □



norm ball of $\|-\|_{\text{Eucl}}$



norm ball of $\|-\|_{\max}$



norm ball of $\|-\|_{\text{Manh}}$

0.10. **Metric spaces.** Finally we recall the notion of a metric space which is most likely familiar from an earlier course on real analysis.

Definition. A metric space is a pair (X, d) consisting of a set X and a metric d on X, i.e. a map

$$\begin{array}{rcl} X \times X & \to & \mathbb{R}_{\geq 0} \coloneqq \{x \in \mathbb{R} \,|\, x \geq 0\} \\ (x, y) & \mapsto & d(x, y) \end{array}$$

with the following properties:

(1) $d(x, y) = 0 \iff x = y$

(2) for all $x, y \in X$ we have $d(x, y) = d(y, x)$ (symmetry)

(3) for all $x, y, z \in X$ we have $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality).

Sometimes we refer to $d(x, y)$ as the distance between x and y.

The following lemma shows that any norm on a vector space V give rise to a metric on V.

Lemma 0.15. (Norm–Metric Lemma) Let V be a real vector space and let $\|-\| \colon V \to \mathbb{R}_{\geq 0}$ be a norm. Then

$$\begin{array}{rcl} V \times V & \to & \mathbb{R}_{\geq 0} \\ (v, w) & \mapsto & d(v, w) \coloneqq \|v - w\| \end{array}$$

is a metric on V.

Proof. The three properties of a metric follow easily from the three properties of a norm. For example, given $v, w \in V$ we see that

$$d(v, w) = \|v - w\| = \|(-1) \cdot (w - v)\| \underset{\substack{\uparrow \\ \text{property (2) of a norm}}}{=} |-1| \cdot \|w - v\| = \|w - v\| = d(w, v).$$

The other two properties of a metric are verified in a similar fashion. ∎

It follows from the Norm–Metric Lemma 0.15 that the Euclidean norm $\|-\|_{\text{Eucl}}$ gives rise to a metric, called the Euclidean metric, which is perhaps the most frequently used example of a metric:

Definition. Given $x, y \in \mathbb{R}^n$ we set

$$d(x, y) \coloneqq \|x - y\|_{\text{Eucl}} \quad \text{where} \quad \|v\|_{\text{Eucl}} = \|(v_1, \ldots, v_n)\|_{\text{Eucl}} \coloneqq \sqrt{v_1^2 + \cdots + v_n^2}\,.$$

We refer to the map

$$\begin{array}{rcl} d \colon \mathbb{R}^n \times \mathbb{R}^n & \to & \mathbb{R}_{\geq 0} \\ (x, y) & \mapsto & d(x, y) \end{array}$$

as the Euclidean metric on $\mathbb{R}^n$.[20] Usually we consider $\mathbb{R}^n$ as a metric space with respect to this Euclidean metric, unless we say something else.

Of course there are many other examples of metrics on $\mathbb{R}^n$. In the following we will see two examples, the first one corresponds to the Manhattan norm and the other one is an example of a metric that does not come from a norm.

Example. Let $n \in \mathbb{R}^n$.

(1) By the Norm–Metric Lemma 0.15 the Manhattan norm from page 50 gives rise to the metric

[20] It follows from the Pairing–Norm Lemma 0.14 together with the Norm–Metric Lemma 0.15 that this is indeed a metric on $\mathbb{R}^n$.

$$\begin{array}{rccl} d_{\text{Manh}}\colon \mathbb{R}^n \times \mathbb{R}^n & \to & \mathbb{R}_{\geq 0} \\ (x,y) & \mapsto & \sum_{i=1}^{n} |x_i - y_i| \end{array}$$

The metric d_{Manh} is called the Manhattan metric since it measures the distance between two points if one can go only vertically (avenues) and horizontally (streets).

(2) We fix a point $P \in \mathbb{R}^n$. We consider the following map:

$$\begin{array}{rccl} d_{\text{SNCF}}\colon \mathbb{R}^n \times \mathbb{R}^n & \to & \mathbb{R}_{\geq 0} & \\ (x,y) & \mapsto & \begin{cases} \|x-y\|, & \text{if } x,y,P \text{ lie on a line,} \\ \|x-P\| + \|y-P\|, & \text{otherwise.} \end{cases} \end{array}$$

One can easily verify that this map is a metric.

Manhattan metric



SNCF-metric



The metric d_{SNCF} is called the SNCF metric since it models the distance in France where every train between two points necessarily has to go through Paris. □

Not surprisingly there are many other examples of metric spaces. The last definition shows that we can define a metric on any set.

Definition. Let X be a set. The map

$$\begin{array}{rccl} X \times X & \to & \mathbb{R}_{\geq 0} & \\ (x,y) & \mapsto & \begin{cases} 0, & \text{if } x = y, \\ 1, & \text{if } x \neq y \end{cases} \end{array}$$

defines a metric on X, it is called the trivial metric.

CHAPTER 1

Topological spaces: basic definitions and properties

In this chapter we give the definition of topological spaces, we provide many examples and we discuss a long list of basic properties. Many of these results might well be familiar to the reader. Nonetheless we attempt to give a fairly complete list of basic statements so that later on we have a firm common foundation to work with.

1.1. Topological spaces.

In this section we give the definition of a topological space and we discuss some very basic definitions and statements. Presumably most readers will have seen this material. Throughout these lecture notes, in the proofs, we strive for giving precise references to earlier results. The main exceptions are the some of the basic lemmas from this and the following chapter, which we will use without referring to them explicitly. The following definition will keep us occupied for the rest of these notes.

Definition.

(1) Let X be a set. A topology on X is a subset $\mathcal{T}$ of the power set $\mathcal{P}(X)$[21] with the following properties:
   (a) the empty set and the entire set X lie in $\mathcal{T}$,
   (b) the intersection of *finitely many* sets in $\mathcal{T}$ is again a set in $\mathcal{T}$,
   (c) the union of *arbitrarily many* sets in $\mathcal{T}$ is again a set in $\mathcal{T}$.
   The sets in $\mathcal{T}$ are called open with respect to $\mathcal{T}$. If $\mathcal{T}$ is understood from the context, then we just say open.

(2) A topological space is a pair $(X, \mathcal{T})$, where X is a set and $\mathcal{T}$ is a topology on X.

Remark.

(1) Usually we suppress the topology from the notation, e.g. usually we just write "let X be a topological space".

(2) If X is a topological space, then the properties of a topology imply immediately the following three statements:
   (a) the empty set and the entire set X are open,
   (b) the intersection of *finitely many* open subsets is again open,
   (c) the union of *arbitrarily many* open subsets is again open.

□

Examples.

(1) Let X be a set, then $\mathcal{T} = \{\varnothing, X\}$ is a topology on X. This topology is sometimes called the trivial topology on X.

(2) Let X be a set. Then $\mathcal{T} = \mathcal{P}(X)$ is also a topology on X. In other words, $\mathcal{T}$ is the topology such that every subset of X is open. This topology is usually referred to as

[21] In more humble words, $\mathcal{T}$ is a set of subsets of X.

the discrete topology on X. Sometimes we say that a topological space is discrete, if the topology is indeed the discrete topology.

(3) Let X be a set. We define $\mathcal{T} \subset \mathcal{P}(X)$ as follows:

$$U \in \mathcal{T} \ :\Longleftrightarrow \text{ either } U = \varnothing \text{ or } U \text{ is the complement of a finite subset of } X.$$

It follows easily from de Morgan's Laws 0.2, that $\mathcal{T}$ is a topology on X. We refer to it as the cofinite topology. Sometimes this topology is also called the finite-complement topology. In a very similar fashion one can define the cocountable topology.[22]

(4) Let X be the empty set. Together with $\mathcal{T} = \{\varnothing\}$ this is a topological space, called the empty topological space. It is a never ending source of counterexamples to carelessly formulated statements.

(5) (a) Let $X = \{*\}$ be a set with a single element $*$. It follows immediately from the definition of a topology there is a unique topology on $X = \{*\}$, namely the topology given by $\mathcal{T} = \{\varnothing, \{*\}\}$. Together with the empty set this is the only case where the topology is already determined by the set. We refer to any topological space consisting of a single point as a singleton.

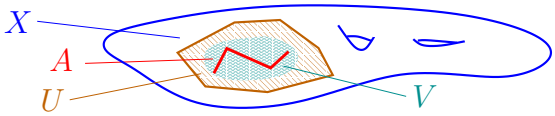
(b) Given a set $X = \{a, b\}$ with two elements it is straightforward to see that there exist four distinct topologies on X. More precisely, there is the trivial and the discrete topology, and there are the two topologies where precisely one of the subsets $\{a\}$ and $\{b\}$ is open.

Not surprisingly the number of topologies on a finite set grows rapidly with the cardinality of the set, we refer to [**KR70, BM02**] for details. □

Definition. Let X be a topological space.

(1) Let $A \subset X$ be a subset. We say that $U \subset X$ is a neighborhood of A if there exists an open subset V such that $A \subset V \subset U$. We say U is an open neighborhood of A, if U is furthermore open. This definition is illustrated in the figure below.

(2) An (open) neighborhood of a point $x \in X$ is an (open) neighborhood of $\{x\}$.



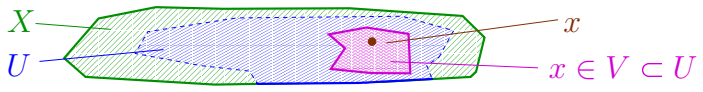
Example. If $X = \mathbb{R}$ and $A = [0, 2)$, then $U = (-1, 3]$ and $V = (-2, \infty)$ are neighborhoods of A in X. □

Using the previous definition we can now state the following lemma which gives a useful criterion for showing that a given subset is open.

Lemma 1.1. (JH-Openness Criterion[23]) Let X be a topological space and let $U \subset X$ be a subset. If for every $x \in U$ there exists a neighborhood V of x that is contained in U, then U is open.

[22] To verify that the axioms are satisfied one will need the Countability Lemma 0.6.

[23] A student, let's call him JH, pointed out to me in an earlier version of these notes, that I was implicitly frequently using the statement of this lemma.

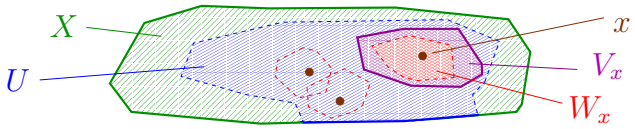


Proof. Let X be a topological space and let $U \subset X$ be a subset. Suppose that for every $x \in U$ there exists a neighborhood V_x of x that is contained in U. By definition of a neighborhood there exists for each $x \in U$ an open subset W_x with $x \in W_x \subset V_x$. We see that

$$U = \bigcup_{x \in U} \{x\} \subset \bigcup_{x \in U} W_x \subset U.$$

(since each W_x is contained in U)

It follows that $U = \bigcup_{x \in U} W_x$, i.e. U is the union of open subsets of X, thus by the third property of a topology, U itself is open. ■



Definition. Let X be a topological space. We say that a subset A of X is closed if the complement $X \setminus A$ is open.

Example. By an *interval* we mean any subset of $\mathbb{R}$ of the form

$$\begin{array}{rcll} (a,b) & := & \{x \in \mathbb{R} \,|\, a < x < b\} & \text{where } a, b \in \mathbb{R} \cup \{\pm\infty\} \\ (a,b] & := & \{x \in \mathbb{R} \,|\, a < x \leq b\} & \text{where } a \in \mathbb{R} \cup \{-\infty\} \text{ and } b \in \mathbb{R} \text{ with } a < b \\ [a,b) & := & \{x \in \mathbb{R} \,|\, a \leq x < b\} & \text{where } a \in \mathbb{R} \text{ and } b \in \mathbb{R} \cup \{\infty\} \text{ with } a < b \\ [a,b] & := & \{x \in \mathbb{R} \,|\, a \leq x \leq b\} & \text{where } a, b \in \mathbb{R} \text{ with } a \leq b. \end{array}$$

One can easily verify that the intervals of the form (a,b), $(-\infty,b)$ and (a,∞) with $a,b \in \mathbb{R}$ are open subsets of $\mathbb{R}$ and that the intervals of the form $[a,b]$, $(-\infty,b]$ and $[a,\infty)$ with $a,b \in \mathbb{R}$ are closed subsets of $\mathbb{R}$. Note that $\varnothing$ and $\mathbb{R}$ are also intervals, they are both open and closed in $\mathbb{R}$. □

The following lemma mirrors the remark on page 53.

Lemma 1.2. (Closed Subset-Lemma) Let X be a topological space.
(1) The empty set and the entire set X are closed.
(2) The intersection of *arbitrarily many* closed subsets is again closed.
(3) The union of *finitely many* closed subsets is again closed.

Proof. The lemma is an immediate consequence of the definition of a topology and de Morgan's Laws 0.2. ■

Remark. Let X be a set. If we specify what we mean by closed subsets such that the three properties of the Closed Subset-Lemma 1.2 are satisfied, then it follows again from de Morgan's Laws 0.2 that the complements of the closed subsets define a topology. □

1.2. **The metric topology.** Recall that on page 51 we introduced the notion of a metric space. We will now see that metric spaces give rise to topological spaces. To do so we will need the following notation:

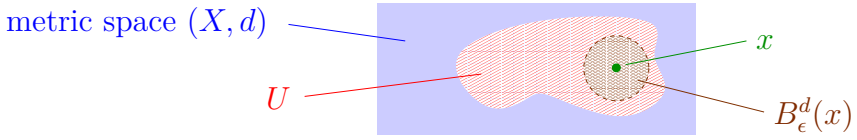
Notation. Let (X, d) be a metric space. Given $x \in X$ and $r \in \mathbb{R}$ we write

$$B_r^d(x) \coloneqq \{y \in X \mid d(x, y) < r\}$$

If the metric d is understood from the context, then sometimes we drop it from the notation.

The following lemma shows that metric spaces give rise to topological spaces.

Lemma 1.3. (Metric-Topology Lemma) Let (X, d) be a metric space. A subset U of X is called open if for every $x \in U$ there exists an $\epsilon > 0$ such that $B_\epsilon^d(x)$ is contained in U. These open subsets define a topology on X.



Proof. We proved this lemma in Analysis IV, alternatively see [**Fri24**]. ■

Definition. Let (X, d) be a metric space.

(1) We refer to the topology introduced in the Metric-Topology Lemma 1.3 as the metric topology.
(2) Unless we say something else we always equip (X, d) with the metric topology.

Example. Let X be a set and let $d\colon X \times X \to \mathbb{R}_{\geq 0}$ be the trivial metric that we introduced on page 52. It is straightforward to see that the corresponding metric topology is the discrete topology. □

Convention. Let $n \in \mathbb{N}$.

(1) Unless we say something else we consider $\mathbb{R}^n$ as a metric space with the Euclidean metric given by

$$d(x, y) \coloneqq \|x - y\| \quad \text{where} \quad \|z\| = \|(z_1, \ldots, z_n)\| \coloneqq \sqrt{z_1^2 + \cdots + z_n^2}.$$

Furthermore, unless we say something else we equip $\mathbb{C}^n$ with the metric given by[24]

$$d(x, y) \coloneqq \|x - y\| \quad \text{where} \quad \|z\| = \|(z_1, \ldots, z_n)\| \coloneqq \sqrt{z_1 \cdot \overline{z_1} + \cdots + z_n \cdot \overline{z}_n}.$$

(2) In particular, unless we say explicitly otherwise, we equip $\mathbb{R}^n$ with the topology arising from the Euclidean metric and we $\mathbb{C}^n$ with the topology arising from the above metric. We call these the Euclidean topologies on $\mathbb{R}^n$ and $\mathbb{C}^n$.

Example. Let $n \in \mathbb{R}^n$ and let $P \in \mathbb{R}^n$. On page 51 we introduced the following metrics:

$$\begin{aligned} d_{\text{Manh}}\colon \mathbb{R}^n &\to \mathbb{R}_{\geq 0} \\ (x, y) &\mapsto \sum_{i=1}^n |x_i - y_i| \end{aligned} \qquad \& \qquad \begin{aligned} d_{\text{SNCF}}\colon \mathbb{R}^n &\to \mathbb{R}_{\geq 0} \\ (x, y) &\mapsto \begin{cases} \|x - y\|, & \text{if } x, y, P \text{ lie on a line,} \\ \|x - P\| + \|y - P\|, & \text{otherwise.} \end{cases} \end{aligned}$$

One can easily show that the Manhattan metric is equivalent to the Euclidean metric and that the SNCF-metric is not equivalent to the Euclidean metric. □

[24] It follows from the observation that $(x + \mathrm{i}y) \cdot \overline{x + \mathrm{i}y} = (x + \mathrm{i}y) \cdot (x - \mathrm{i}y) = x^2 + y^2$ that the metric on $\mathbb{C}^n$ is the Euclidean metric on $\mathbb{C}^n = \mathbb{R}^{2n}$ in slight disguise. In particular it is indeed a metric.

1.3. **Basic definitions.** We proceed with the next definition.
The following proposition gives us important examples of closed and open subsets.

Proposition 1.4. (Ball–Sphere Proposition) Let (X, d) be a metric space, let $y \in X$ and let $r \in \mathbb{R}$.
(1) The set $B_r^d(y) := \{x \in X \mid d(x, y) < r\}$ is open in X.
(2) The set $\overline{B}_r^d(y) := \{x \in X \mid d(x, y) \leq r\}$ is closed in X.
(3) The set $S_r^d(y) := \{x \in X \mid d(x, y) = r\}$ is closed in X.

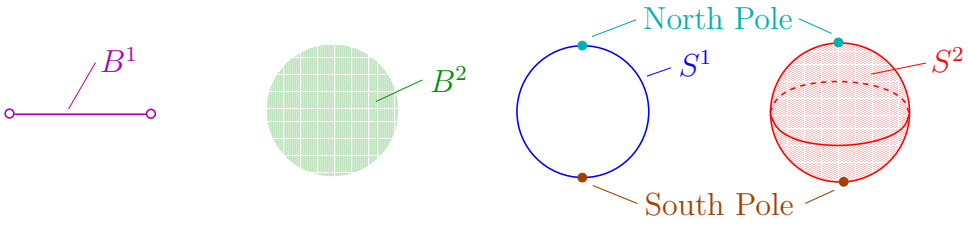
Proof. We proved this proposition in Analysis II. ■

The most frequent application of the Ball–Sphere Proposition 1.4 is to the metric space $\mathbb{R}^n$. Thus we introduce the following notation:

Notation. Let $n \in \mathbb{N}_0$. Given $y \in \mathbb{R}^n$ and $r \in \mathbb{R}$ we write[25] [26]

$$\begin{array}{rcll} B_r^n(y) & := & \{x \in \mathbb{R}^n \mid \|x - y\| < r\} & \text{called open } n\text{-ball} \\ \overline{B}_r^n(y) & := & \{x \in \mathbb{R}^n \mid \|x - y\| \leq r\} & \text{called closed } n\text{-ball} \\ S_r^{n-1}(y) & := & \{x \in \mathbb{R}^n \mid \|x - y\| = r\} & \text{called } (n-1)\text{-sphere.} \end{array}$$

We adopt the following conventions:
(1) A 2-ball is often called a disk and a 1-sphere is often referred to as a circle.
(2) If $y = 0$, then we drop y from the notation and if $r = 1$, then we drop r from the notation.
(3) Sometimes we write rB^n instead of $B_r^n(0)$ and sometimes we write $r\overline{B}^n$ instead of $\overline{B}_r^n(0)$.
(4) We refer to the point $(0, \ldots, 0, 1) \in S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$ as the North Pole of S^n and we refer to the point $(0, \ldots, 0, -1) \in S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$ as the South Pole of S^n.



1.4. **The subspace topology.** The next definition gives us a rich source of new examples of topological spaces.

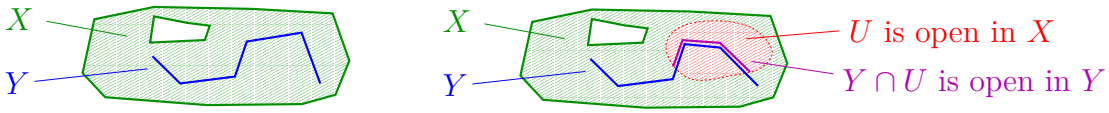
Definition. Let $(X, \mathcal{T})$ be a topological space and let $Y \subset X$ be a subset.
(1) It follows from elementary set theory, see the Intersection–Distributive Lemma 0.1, that

$$\mathcal{S} := \{Y \cap U \mid U \in \mathcal{T}\}$$

is a topology on Y. We refer to $\mathcal{S}$ as the subspace topology on Y induced by Y.

[26] These definitions also make sense for $r < 0$, in that case all of the sets are the empty set. Also note that for $n = 0$ we obtain that $S_r^{n-1}(y) = S_r^{-1}(y) = \varnothing$.

(2) If Y is indeed equipped with the subspace topology, then sometimes we refer to Y as a subspace of X.



Convention. Unless we say something else, we consider each subset Y of $\mathbb{R}^n$ always as a topological space with respect to the subspace topology.

1.5. **Covers of topological spaces.** On many occasions it will be useful to "break" a topological space into smaller pieces. This leads us to the following definition:

Definition. Let X be a topological space.
(1) A cover[27] of a subset $A \subset X$ is a family[28] $\{U_i\}_{i\in I}$ of subsets of X with $A \subset \bigcup_{i\in I} U_i$.
(2) We say that a cover $\{U_i\}_{i\in I}$ is open if each U_i is open. Similarly we define a closed cover.
(3) We say that a cover $\{U_i\}_{i\in I}$ is finite if I is finite. Otherwise we say that the cover is infinite.

The following lemma is crucial for many arguments.

Lemma 1.5. (Cover-Open Subset Lemma) Let X be a topological space and let M be a subset of X.
(1) Let $\{U_i\}_{i\in I}$ be an *arbitrary* open cover of X. The following two statements hold:
  (a) M is open if and only if each intersection $M \cap U_i$ is open in U_i,
  (b) M is closed if and only if each intersection $M \cap U_i$ is closed in U_i.
(2) Let $\{A_i\}_{i\in I}$ be a *finite* closed cover of X. The following two statements hold:
  (a) M is open if and only if each intersection $M \cap A_i$ is open in A_i,
  (b) M is closed if and only if each intersection $M \cap A_i$ is closed in A_i.

Remark. Note that in the Cover-Open Subset Lemma 1.5 we allow an arbitrary family of open subsets but we only allow a finite family of closed subsets. One can easily see that this restriction is necessary. For example, consider $X = \mathbb{R}$ and the infinite family $\{A_i\}_{i\in I}$ of closed subsets of $X = \mathbb{R}$ given by all subsets consisting of a single point. Let $M \subset \mathbb{R}$ be a subset. For each $i \in I$ the intersection $M \cap A_i$ is either the empty set or all of A_i, in particular the intersection $M \cap A_i$ is closed in A_i. But of course there is no reason why M should be a closed subset of $\mathbb{R}$. □

Proof. The proof is rather elementary. The curious reader is referred to [**Fri24**] for details. ■

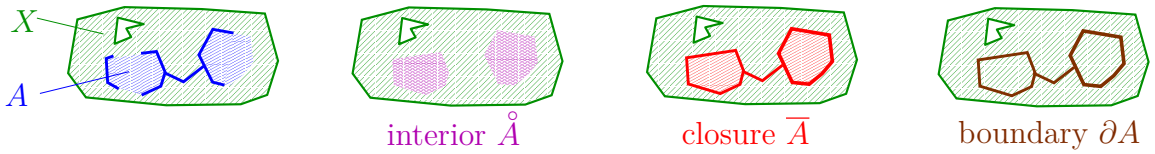
[28] What we call a "cover" is often also called a "covering". We reserve the name "covering" for a completely different concept that we will introduce later.
[28] Recall that according to the definition on page 34 a family of subsets of a given set X consists of a set I and a map $f\colon I \to \mathcal{P}(X)$. We also recall the convention from page 35 which says that we will view any subset $\mathcal{U} \subset \mathcal{P}(X)$ as a family of subsets of X via the index set $I = \mathcal{U}$ and the map $I = \mathcal{U} \to \mathcal{P}(X)$ given by $U \mapsto U$.

1.6. The interior and the closure of a subset.

Definition. Let X be a topological space and let A be a subset of X. We make the following two definitions:[29]

$$\text{interior } \mathring{A} := \text{union of all open subsets of } X \text{ that are contained in } A \qquad \text{closure } \overline{A} := \text{intersection of all closed subsets of } X \text{ that contain } A$$

Sometimes, to avoid confusion, we will also write $\operatorname{int}(A)$ for the interior of a subset A. Furthermore we define the boundary of A in X as $\partial A := \overline{A} \setminus \mathring{A}$.



In the following we will state general properties of the interior, the closure and the boundary of a subset. Evidently the point is not to memorize the following lemmas. Instead they are meant as a resource to turn to, when one is dealing with these objects. Therefore it is warmly recommended to move on to the next section and to return to the present section when the need arises.

Lemma 1.6. (Interior-Closure Lemma) Let X be a topological space and let A be a subset of X.

(1) The interior $\mathring{A}$ is an open subset of X.
(2) The closure $\overline{A}$ of A is a closed subset of X.

Proof.

(1) The union of arbitrarily many open subsets is, by definition of a topology, again an open subset. Thus the interior $\mathring{A}$ is an open subset of X.
(2) In the Closed Subset-Lemma 1.2 we saw that the intersection of arbitrarily many closed subsets is a closed subset. It follows immediately that the closure $\overline{A}$ is a closed subset of X. ■

1.7. The Hausdorff property.

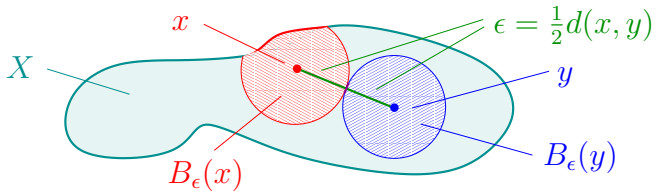
Definition. We say that a topological space X is Hausdorff, if given any two distinct points $x \neq y$ there exist disjoint open neighborhoods U of x and V of y.



The following proposition, which very likely is well-known to most readers, says that topological spaces arising from metric spaces are Hausdorff.

Proposition 1.7. (Metric Space-Hausdorff Proposition) Let (X, d) be a metric space. The corresponding topological space is Hausdorff.

Proof. Let $x, y \in X$ be two different points. We set $\epsilon := \frac{1}{2}d(x, y)$. It follows from the Ball–Sphere Proposition 1.4 that $U := B_\epsilon(x)$ and $V := B_\epsilon(y)$ are neighborhoods of x respectively y. An elementary argument using the triangle inequality shows that U and V are disjoint. ■

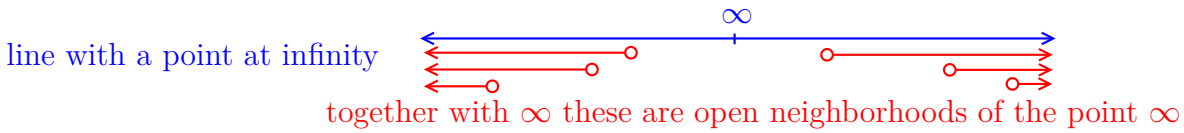


To get some practice for the notion of the Hausdorff property it is convenient to introduce two more examples of topological spaces, which do not come from a metric space. Both new examples are of the type "$\mathbb{R}^n$ together with an extra point".

Definition. Let $n \in \mathbb{N}$. We consider the set $X := \mathbb{R}^n \cup \{\infty\}$, i.e. X consists of $\mathbb{R}^n$ and an extra point ∞. We say that $U \subset X$ is open, if both of the following two conditions are satisfied:

(1) for each point $x \in U \cap \mathbb{R}^n$ there exists an $\epsilon > 0$ such that $B_\epsilon(x) \subset U$,
(2) if $\infty \in U$, then there exists a $C > 0$ such that $\{x \in \mathbb{R}^n \mid \|x\| > C\} \subset U$.

It is straightforward to see that this defines indeed a topology on X. We refer to $\mathbb{R}^n \cup \{\infty\}$ as $\mathbb{R}^n$ with a point at infinity. For $n = 1$ we refer to $\mathbb{R} \cup \{\infty\}$ also as the line with a point at infinity.[30]

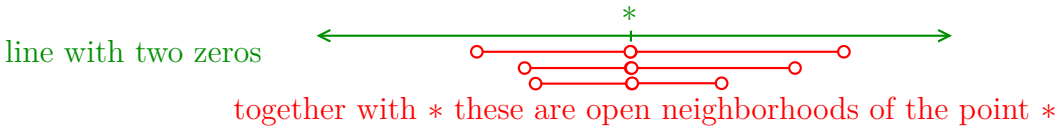


The above example has a mischievous twin brother, which is often a counterexample to carelessly formulated statements.

Definition. We consider the set $X := \mathbb{R} \cup \{*\}$, i.e. X consists of $\mathbb{R}$ and an extra point $*$. We say that $U \subset X$ is open, if the following two conditions are satisfied:

(a) for each point $x \in U \cap \mathbb{R}$ there exists an $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subset U$,
(b) if $* \in U$, then there exists an $\epsilon > 0$ such that $(-\epsilon, 0) \cup (0, \epsilon) \subset U$.

One can easily verify that this is indeed a topology on X. We refer to this topological space as the line with two zeros.[31]



Example. Even though the "line with two zeros" looks at first glance similar to the "line with a point at infinity", they are strikingly different topological spaces. For example it is a fun exercise to show that "$\mathbb{R}^n$ with a point at infinity" is Hausdorff and that "the line with two zeros" is not Hausdorff. □

[30] It is straightforward to see that the subspace topology on $\mathbb{R}^n \subset \mathbb{R}^n \cup \{\infty\}$ is the usual topology on $\mathbb{R}^n$.
[31] It is straightforward to see that the subspace topology on $\mathbb{R}^n \subset \mathbb{R}^n \cup \{*\}$ is the usual topology on $\mathbb{R}^n$.

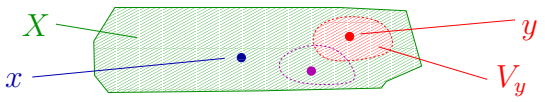
The following lemma gives us a modest but still quite useful consequence of being Hausdorff.

Lemma 1.8. (Points-in-Hausdorff Lemma) Let X be a topological space. If X is Hausdorff, then for every $x \in X$ the corresponding subset $\{x\}$ is closed.

Proof. Let X be a topological space which is Hausdorff. Let $x \in X$. Since X is Hausdorff we know in particular that for every $y \neq x$ there exists an open neighborhoods V_y of y that is disjoint from x. Note that

$$X \setminus \{x\} \;=\; \bigcup_{y \in X \setminus \{x\}} \{y\} \;\underset{\substack{\uparrow \\ \text{since } y \in V_y}}{\subset}\; \bigcup_{y \in X \setminus \{x\}} V_y \;\underset{\substack{\uparrow \\ \text{since } x \notin V_y}}{\subset}\; X \setminus \{x\}.$$

Thus we see that the inclusions are equalities. Since each V_y is open we see that $X \setminus \{x\}$ is the union of open subsets, in particular $X \setminus \{x\}$ is open. In other words, $\{x\}$ is a closed subset of X. ■



1.8. **Compact topological spaces.** In this section we introduce the notion of compactness. Usually it takes a while till one appreciates the brilliance of the definition.

Definition. We say that a topological space X is compact if for each open cover $\{U_i\}_{i \in I}$ of X there exist finitely many indices $i_1, \ldots, i_k \in I$ such that

$$X \;=\; U_{i_1} \cup \cdots \cup U_{i_k}.$$

Remark. The introduction of the notion of compact space by Pavel Alexandrov and Pavel Urysohn [**AU29**] in 1929 shows that often in mathematics the really clever thing is not necessarily the proofs, but coming up with just the right definition. It is difficult to exaggerate the role of the notion of compactness in topology and analysis. □

Example.

(1) Every finite topological space (regardless of the topology) is compact, since there exist only finitely many distinct open subsets. This simple example can sometimes be quite useful.
(2) We consider the topological space $X = \mathbb{R}$. The sets $\{(-n, n)\}_{n \in \mathbb{N}}$ form an open cover of $X = \mathbb{R}$, but evidently we cannot cover $X = \mathbb{R}$ by finitely many of these sets. This shows that $X = \mathbb{R}$ is not compact. □

In the following we will often use the following two naming conventions.

Convention. Let P be a property of a topological space (e.g. being discrete, Hausdorff, compact).

(1) If X is a topological space that satisfies P we might just say that "X is a P-space". For example, a Hausdorff space is a topological space that is Hausdorff.
(2) We say that a subset Y of a topological space X has the property P if Y, equipped with the subspace topology, has the property P.

Before we give some more interesting examples of compact spaces we want to discuss a few properties of compact spaces and compact subsets. The following lemma might be the lemma that gets most often cited in these notes.

Lemma 1.9. (Compact-Closed Lemma) Let X be a topological space.
(1) If X is compact, then any closed subset of X is also compact.
(2) Let $A \subset X$ be a compact subset. If X is Hausdorff, then A is a closed subset of X.

Example.
(1) Let X be a topological space. As we pointed out on page 61, any finite subset of X is compact. Thus if X is Hausdorff, then it follows from the Compact-Closed Lemma 1.9 (2) that any finite subset of X, in particular any point in X, is a closed subset of X. Thus we have found a new proof for the Points-in-Hausdorff Lemma 1.8 (1).
(2) Let X be a topological space, let A be a closed subset and let K be a compact subset of X. It follows from the Compact-Closed Lemma 1.9 (1) that $A \cap K$ is also compact. This simple observation will get used frequently.
(3) We consider the topological space $X = \{0, 1\}$ that is equipped with the trivial topology $\{\varnothing, X\}$. The subset $A = \{0\}$ is evidently compact, but it is not a closed subset of X. This shows that in the formulation of Compact-Closed Lemma 1.9 (2) we cannot drop the hypothesis that X is Hausdorff. □

Proof. We proved this lemma in Analysis II and Analysis IV. Alternatively see [**Fri24**]. ■

Lemma 1.10. (Compact-Discrete Lemma)
(1) Every discrete space that is compact is finite.
(2) Let X be a compact space and let A be a closed subset. If A is discrete, then A is finite.

Remark. Note that the conclusion of the Compact-Discrete Lemma 1.10 (2) does not necessarily hold, if we drop the hypothesis that A is closed. For example, consider the topological space $X = [0, 1]$. The subset $A := \{\frac{1}{n} \mid n \in \mathbb{N}\}$ is a discrete subset, but evidently it is not finite. □

Proof.
(1) Let X be a discrete space. Each point thus defines an open subset. Evidently X is covered by all the open subsets given by the points in X. Since X is compact, we can cover X by finitely many of those open subsets. But this means that X contains only finitely many points.
(2) Let X be a compact space and let A be a closed subset. It follows from the Compact-Closed Lemma 1.9 (1) that A is also compact. So if A is discrete, then it follows from (1) that A is finite. ■

1.9. **The Heine-Borel Theorem.** We start out this section with the following almost self-explanatory definition.

Definition. Let (X, d) be a metric space. A subset A of X is called bounded if there exists a $C \geq 0$ such that for all $x_0, x_1 \in A$ we have $d(x_0, x_1) \leq C$.

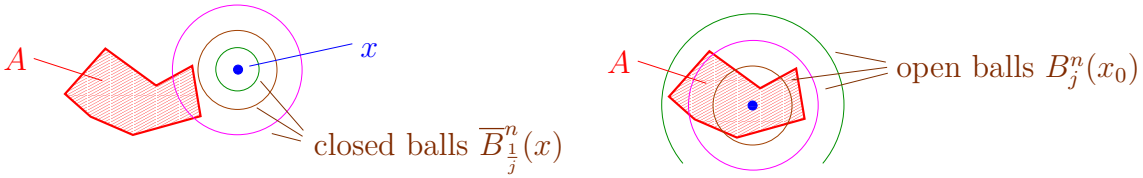
Example. Let (X,d) be a metric space, let $y \in X$ and let $r \geq 0$. It follows easily from the triangle inequality that the balls $B_r^d(y)$, $\overline{B}_r^d(y)$ and the sphere $S_r^d(y)$ are bounded with $C = 2r$. □

Lemma 1.11. (Compact–Closed+Bounded Lemma) Let (X,d) be a metric space and let $A \subset X$ be a subset. If A is compact, then A is closed and it is bounded.

Proof. Let (X,d) be a metric space and let $A \subset X$ be a subset.

(1) We want to show that $X \setminus A$ is open. Let $x \in X \setminus A$. We need to show that there exists an $\epsilon > 0$ with $B_\epsilon^d(x) \subset X \setminus A$. Given $j \in \mathbb{N}$ we consider the subset $U_j = X \setminus \overline{B}_{\frac{1}{j}}^d(x)$. By the Ball–Sphere Proposition 1.4 we know that the U_j are open. Evidently these open subsets cover A. Since A is compact there exist $j_1, \ldots, j_k \in \mathbb{N}$ with $A \subset U_{j_1} \cup \cdots \cup U_{j_k}$. We set $\epsilon := \min\{\frac{1}{j_1}, \ldots, \frac{1}{j_k}\}$. We obtain that $A \subset X \setminus \overline{B}_\epsilon^d(x)$. Thus we have found the desired ϵ.

(2) We need to show that A is bounded. If A is empty, then there is nothing to show. Now suppose that A is non-empty. We pick $x_0 \in A$. It follows from the triangle inequality that it suffices to show that there exists an $R \geq 0$ with $A \subset \overline{B}_R^d(x_0)$. We consider the open subsets $U_j = B_j^d(x_0)$ with $j \in \mathbb{N}$.



Evidently these sets cover A. Since A is compact we obtain that there exist $j_1, \ldots, j_k \in \mathbb{N}$ such that $A \subset U_{j_1} \cup \cdots \cup U_{j_k}$. We set $R := \max\{j_1, \ldots, j_k\}$. We obtain that $A \subset B_R^d(x_0) \subset \overline{B}_R^d(x_0)$. We have thus found the desired R. ■

The ever-popular Heine-Borel Theorem says that for the metric space $X = \mathbb{R}^n$ mercifully the converse to the Compact–Closed+Bounded Lemma 1.11 holds.

Theorem 1.12. (Heine-Borel Theorem) A subset of $\mathbb{R}^n$ is compact if and only if it is bounded and closed.

Proof. We proved this theorem in Analysis II. Alternatively see [**Fri24**]. ■

Example.

(1) Let $y \in \mathbb{R}^n$ and $r \geq 0$. It follows from the Heine-Borel Theorem 1.12 and the Ball–Sphere Proposition 1.4 that the closed ball $\overline{B}_r^n(y)$ and the sphere $S_r^{n-1}(y)$ are compact.

(2) It is an amusing exercise to show, using the Heine-Borel Theorem 1.12, that "$\mathbb{R}^n$ with a point at infinity" is compact.

(3) It is an equally entertaining exercise to show that the "line with two zeros" is not compact.

(4) Since the concepts of being bounded and closed are initially much more intuitive than being compact it is not uncommon that many, especially students, like to replace "compact" by "bounded and closed". But this equality only holds for subsets of $\mathbb{R}^n$, it does not hold for other metric spaces. For example, the set $[0,1] \cap \mathbb{Q}$ is a bounded and closed subset of the metric space $(\mathbb{Q}, d(x,y) = |x-y|)$ it is not compact.

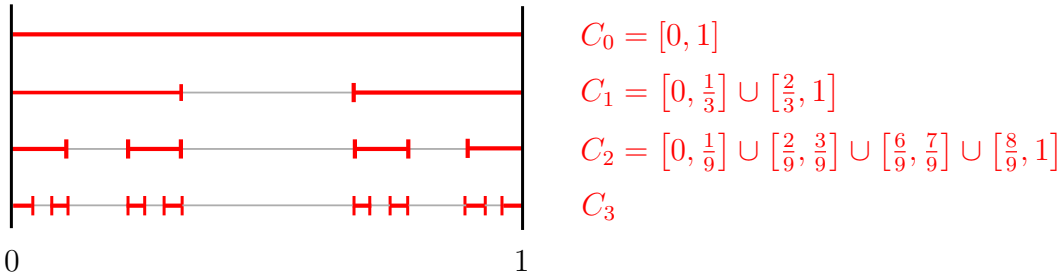
Our final example deserves to be introduced as a proper definition.

Definition. We set $C_0 := [0,1]$. Iteratively we now define sets $C_2, C_3, \dots$ as follows:

$$C_i := \underbrace{\{\tfrac{1}{3}\cdot x \,|\, x \in C_{i-1}\}}_{=\frac{1}{3}\cdot C_{i-1}} \cup \underbrace{\{\tfrac{2}{3}+\tfrac{1}{3}\cdot x \,|\, x \in C_{i-1}\}}_{=\frac{2}{3}+\frac{1}{3}\cdot C_{i-1}}.$$

Note that each C_i consists of 2^i closed intervals of length $\frac{1}{3^i}$ and that we obtain C_i from C_{i-1} by removing the "open middle third" of each interval. We define

$$\text{Cantor set } C := \bigcap_{i\in\mathbb{N}_0} C_i.$$



Lemma 1.13. (Cantor Set–Compact Lemma) The Cantor set is compact.

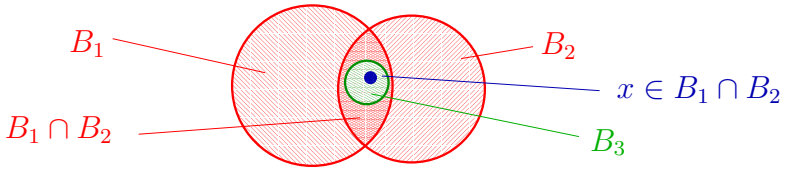
Proof. Each C_i is the union of 2^i closed intervals, hence each C_i is closed. It follows from the Closed Subset-Lemma 1.2 that C itself is a closed subset of $\mathbb{R}$. It is evidently bounded, hence the Heine-Borel Theorem 1.12 implies that the Cantor set is compact. ■

1.10. **Bases of a topology.** In this section we will introduce the notion of a (sub-) basis of a topology. These notions make it possible to define topologies and to describe a given topology. Nonetheless, the topic is somewhat dry and it is best not to spend too much time on this section.

Definition. Let X be a set and let $\mathcal{B} \subset \mathcal{P}(X)$ be a subset of the power set of X. We say that $\mathcal{B}$ has the basis property if the following two conditions are satisfied:

(B1) For every $x \in X$ there exists a $B \in \mathcal{B}$ with $x \in B$.

(B2) Given any $B_1, B_2 \in \mathcal{B}$ and given any $x \in B_1 \cap B_2$ there exists a $B_3 \in \mathcal{B}$ such that $x \in B_3$ and $B_3 \subset B_1 \cap B_2$.



Example. Let (X,d) be a metric space. It follows from an elementary argument, see e.g. the proof of the Ball–Sphere Proposition 1.4 (1), that the set of all open balls

$$\mathcal{B} = \{\text{all balls } B_r(x) \text{ with } r > 0 \text{ and } x \in X\} \subset \mathcal{P}(X)$$

satisfies (B1) and (B2). □

The following lemma justifies our sudden interest in this new definition.

Lemma 1.14. (Topology-from-Basis Lemma) Let X be a set and let $\mathcal{B} \subset \mathcal{P}(X)$. We set

$$\mathcal{T}(\mathcal{B}) \coloneqq \{V \subset X \,|\, \text{for every } x \in V \text{ there exists a } B \in \mathcal{B} \text{ with } x \in B \subset V\}.$$

If $\mathcal{B}$ has the basis property, then $\mathcal{T}(\mathcal{B})$ is a topology on X.

Proof. We proved the lemma in Analysis IV. Alternatively see [**Fri24**]. ■

Definition. Let X be a set.

(1) If $\mathcal{B} \subset \mathcal{P}(X)$ has the basis property, then we refer to $\mathcal{T}(\mathcal{B})$ as the topology on X generated by $\mathcal{B}$.
(2) Conversely, if $\mathcal{T}$ is a given topology on X, if $\mathcal{B} \subset \mathcal{P}(X)$ has the basis property and if $\mathcal{T} = \mathcal{T}(\mathcal{B})$, then we say that $\mathcal{B}$ is a basis of the topology $\mathcal{T}$ of X.

Examples.

(1) If (X, d) is a metric space, then the topology generated by the set of all open balls, is by definition precisely the metric topology we already introduced on page 56.
(2) Let $n \in \mathbb{N}$ and let $\mathbb{R}^n \cup \{\infty\}$ be the topological space defined on page 60. It follows immediately from the definition of the topology that a basis for the topological space $\mathbb{R}^n \cup \{\infty\}$ is given by

$$\mathcal{B} = \left\{B_r^n(x) \,\middle|\, r > 0 \text{ and } x \in \mathbb{R}^n\right\} \cup \left\{\{\infty\} \cup (\mathbb{R}^n \setminus \overline{B}_C^n(0)) \,\middle|\, C \in \mathbb{R}\right\}. \qquad \square$$

CHAPTER 2

Continuity

In the previous chapter we introduced the concept of topological spaces and we discussed various examples and properties. Nonetheless, as it is the theory is rather dull. The theory of topological spaces comes alive with the right notion of suitable maps between topological spaces, namely the notion of continuous maps. The notion of continuity is the main topic of this chapter.

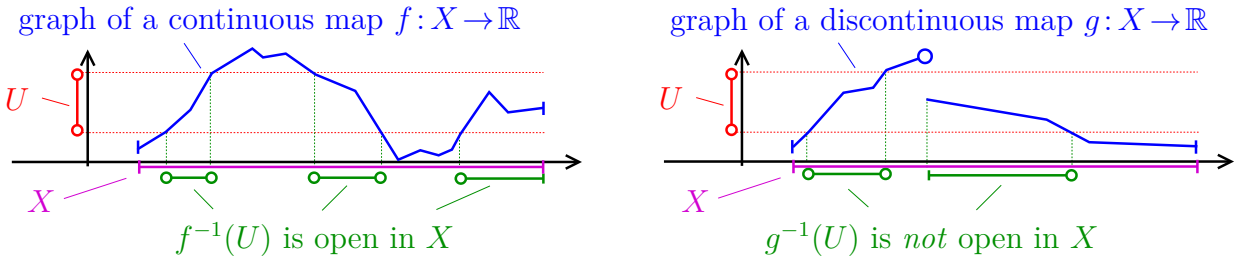
2.1. Continuous maps.

The following is the key definition in the context of topological spaces.

Definition. We say that a map $f\colon X \to Y$ between two topological spaces X and Y is continuous, if for each open subset U in Y the preimage $f^{-1}(U)$ is open in X. Otherwise the map is called discontinuous.

Examples.

(1) For subspaces $X \subset \mathbb{R}$ and maps $\mathbb{R} \to \mathbb{R}$ we see in the figure below that the above definition of continuous matches our intuition.



(2) Let X and Y be two topological spaces. If Y is equipped with the trivial topology or if X is equipped with the discrete topology, then every map $f\colon X \to Y$ is continuous.

(3) We say that a map $f\colon X \to Y$ between two sets is *constant* if for any $x_1, x_2 \in X$ we have $f(x_1) = f(x_2)$. Any constant map $f\colon X \to Y$ between two topological spaces is continuous since the preimage of an open subset of Y is either the empty set or X. This simple example will come in handy on many occasions. □

Lemma 2.1. (Basics-of-Continuity Lemma)

(1) If $f\colon X \to Y$ and $g\colon Y \to Z$ are two continuous maps between topological spaces, then the composition $g \circ f\colon X \to Z$ is also continuous.

(2) Let X be a topological space and let A be a subset of X. If we equip A with the subspace topology, then the inclusion map $i\colon A \to X$ is continuous.

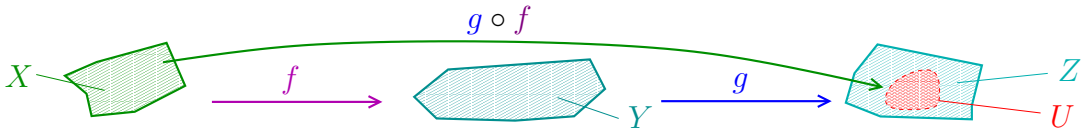
(3) Let $f\colon X \to Y$ be a continuous map between topological spaces and let A be a subset of X. If we equip A with the subspace topology, then the restriction of f to the map $f|_A\colon A \to Y$ is also continuous.
(4) Let $f\colon X \to Y$ be a continuous map. If $Z \subset Y$ is a subset with $f(X) \subset Z$, then the map $f\colon X \to Z$ is also continuous.

Proof.

(1) Let $f\colon X \to Y$ and $g\colon Y \to Z$ be two continuous maps between topological spaces. We need to show that the composition $g \circ f\colon X \to Z$ is also continuous. Let $U \subset Z$ be an open subset. Then

$$(g \circ f)^{-1}(U) \;=\; f^{-1}(g^{-1}(U)) \underset{\uparrow}{=} f^{-1}(\text{the open subset } g^{-1}(U)) \underset{\uparrow}{=} \text{open.}$$

since g is continuous and U is open — since f is continuous



(2) This statement follows immediately from the definitions.
(3) This statement is an immediate consequence of (1) and (2) since the restriction of f to A is precisely the same as $f \circ i$ where $i\colon A \to X$ denotes the inclusion.
(4) For clarity we denote by $g\colon X \to Z$ the map that is given by $x \mapsto f(x)$. Let $U \subset Z$ be an open subset. We need to show that $g^{-1}(U)$ is an open subset of X. By definition of the subspace topology there exists an open subset $V \subset Y$ with $U = Z \cap V$. Thus we see that

$$g^{-1}(U) \;=\; g^{-1}(Z \cap V) \underset{\uparrow}{=} f^{-1}(Z \cap V) \underset{\uparrow}{=} f^{-1}(V) \underset{\uparrow}{=} \text{open subset of } X.$$

definition of g — since $f(X) \subset Z$ — since f is continuous

■

The following proposition makes it much easier to show that a given map is in fact continuous.

Proposition 2.2. (Basis-Continuity Proposition) Let $f\colon X \to Y$ be a map between topological spaces. Let $\mathcal{C}$ be a basis for the topology of Y. Then the following holds

$$f \text{ is continuous} \iff \text{for each } C \in \mathcal{C} \text{ the preimage } f^{-1}(C) \text{ is open in } X.$$

Proof. We proved the proposition in Analysis IV. Alternatively see [**Fri24**]. ■

At times it is also useful to have a characterization of continuous maps in terms of closed subsets.

Lemma 2.3. (Continuity-via-Closed Subsets Lemma) A map $f\colon X \to Y$ between two topological spaces X and Y is continuous if and only if for each closed subset A in Y the preimage $f^{-1}(A)$ is closed in X.

Proof. This lemma is a straightforward consequence of the Image-Preimage Lemma 0.3 (8) and the definitions. We leave it to the reader to fill in the details. ■

By the Basics-of-Continuity Lemma 2.1 (3) we know that the restriction of a continuous map to any subspace is still continuous. The following lemma can be viewed as a converse: If "enough" restrictions of a given map are continuous, then the original map is also continuous.

Proposition 2.4. (Pasting Proposition) Let $f\colon X \to Y$ be a map between topological spaces.

(1) If there exists an open cover $\{W_i\}_{i\in I}$ of X such that each restriction $f|_{W_i}$ is continuous, then f itself is continuous.
(2) If there exists a finite closed cover $\{A_i\}_{i\in I}$ of X such that each restriction $f|_{A_i}$ is continuous, then f itself is continuous.

In practice often the following variation on (2) gets used:

(2′) If X is Hausdorff and if there exists a locally finite cover $\{A_i\}_{i\in I}$ of X such that each A_i is compact and such that each restriction $f|_{A_i}$ is continuous, then f itself is continuous.

Remark.

(1) In practice the Pasting Proposition 2.4 (2) often gets used in the following way: Suppose we are given two topological spaces X and Y and closed subsets $A_1, \ldots, A_m$ of X such that $X = A_1 \cup \cdots \cup A_m$. Furthermore we are given continuous maps $f_i\colon A_i \to Y$, $i = 1, \ldots, m$. If the f_i agree on the overlaps, then the map
$$\begin{array}{ccc} X & \to & Y \\ P & \mapsto & f_i(P) \quad \text{if } P \in A_i \end{array}$$
is well-defined and it follows from the Pasting Proposition 2.4 (2) that it is continuous.
(2) Note that in the Pasting Proposition 2.4 (1) we are allowed to deal with arbitrarily many open subsets whereas in the Pasting Proposition 2.4 (2) we are restricted to finitely many closed subsets. It is clear that the Pasting Proposition 2.4 (2) cannot be generalized to arbitrarily many closed subsets, after all, the restriction of a map $f\colon \mathbb{R} \to \mathbb{R}$ to any one-point subset $\{x\} \subset \mathbb{R}$ is continuous, but clearly not every map $f\colon \mathbb{R} \to \mathbb{R}$ is continuous. □

Proof. Statements (1) and (2) of the Pasting Proposition follow fairly easily from the Cover-Open Subset Lemma 1.5. Statement (2′) follows from Statement (2) and the Compact-Closed Lemma 1.9 (2). We refer to [**Fri24**] for details. ■

The notion of continuity is of course already familiar from earlier analysis courses. The following proposition shows that for maps between metric spaces the earlier "$\epsilon - \delta$"–definition of continuity agrees with the above definition of continuity.

Proposition 2.5. (Metric Continuity Proposition) Let $f\colon X \to Y$ be a map between metric spaces. Then the following holds:
$$f \text{ is continuous} \iff \underset{x\in X}{\forall}\ \underset{\epsilon>0}{\forall}\ \underset{\delta>0}{\exists}\ \underset{x'\in B_\delta(x)}{\forall} d(f(x), f(x')) < \epsilon.$$

Proof. We proved the Metric Continuity Proposition 2.5 in Analysis II. Alternatively see [**Fri24**]. ■

Example. Let $n \in \mathbb{N}$. For each $i \in \{1, \ldots, n\}$ the projection

$$\begin{array}{rcl} \mathbb{R}^n & \to & \mathbb{R} \\ (x_1, \ldots, x_n) & \mapsto & x_i \end{array}$$

is continuous since the "$\epsilon - \delta$"-condition is satisfied with $\delta = \epsilon$. □

The final lemma of this section shows that we can add and multiply continuous real-valued maps.

Lemma 2.6. (Sum-Product-Continuity Lemma) Let X be a topological space. If $f\colon X \to \mathbb{R}$ and $g\colon X \to \mathbb{R}$ are continuous maps, then the maps

$$\begin{array}{rcl} X & \to & \mathbb{R} \\ x & \mapsto & f(x)+g(x) \end{array} \qquad \begin{array}{rcl} X & \to & \mathbb{R} \\ x & \mapsto & f(x)\cdot g(x) \end{array} \qquad \begin{array}{rcl} X & \to & \mathbb{R} \\ x & \mapsto & \max\{f(x), g(x)\} \end{array} \qquad \begin{array}{rcl} X & \to & \mathbb{R} \\ x & \mapsto & \min\{f(x), g(x)\} \end{array}$$

are also continuous. The obvious analogues of the first two statements also hold if we replace $\mathbb{R}$ by $\mathbb{C}$.

Example. Let A be a real $(m \times n)$–matrix. It follows easily from the previous example, the Basics-of-Continuity Lemma 2.1 (1) and the Sum-Product-Continuity Lemma 2.6 that the following map is continuous:

$$\begin{array}{rcl} f\colon \mathbb{R}^n & \to & \mathbb{R}^m \\ v & \mapsto & A \cdot v. \end{array}$$

□

Proof. For brevity's sake we outsource the proof to [**Fri24**]. ■

2.2. Open and closed maps.

We continue with the following straightforward definition.

Definition. Let $f\colon X \to Y$ be a continuous map between topological spaces.
(1) We say that f is open if the image of every open subset of X is open in Y.
(2) We say that f is closed if the image of every closed subset of X is closed in Y.

Examples.

(1) The projection $\mathbb{R}^2 \to \mathbb{R}$ given by $(x, y) \mapsto x$ is easily seen to be an open map, but it is not a closed map.[32]
(2) On the other hand the inclusion $i\colon \mathbb{R} \to \mathbb{R}^2$, $i(x) = (x, 0)$ is evidently not an open map, but it is a closed map. □

Lemma 2.7. (Open-Closed Inclusion Lemma) Let X be a topological space.
(1) If $U \subset X$ is an open subset, then the inclusion map $i\colon U \to X$ is open.
(2) If $A \subset X$ is a closed subset, then the inclusion map $i\colon A \to X$ is closed.

Proof.

(1) Suppose that V is an open subset of U. By definition of the subspace topology there exists an open subset $W \subset X$ with $V = U \cap W$. But then $i(V) = V = U \cap W$ is the intersection of two open subsets of X, thus it is an open subset of X.
(2) The proof of this statement is analogous to the proof of Statement (1). ■

[32] Why is not a closed map?

2.3. **The Compact Image Lemma.** Statement (1) of the following lemma is perhaps the most important feature of compact spaces.

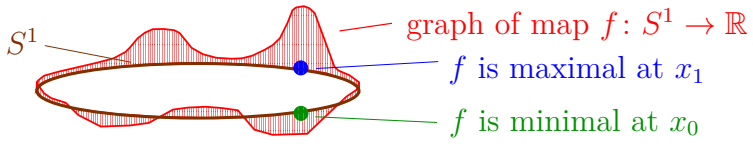
Lemma 2.8. (Compact Image Lemma)

(1) Let $f\colon X \to Y$ be a continuous map. If X is compact, then $f(X)$ is also compact.

(2) Let $f\colon X \to \mathbb{R}$ be a continuous map. If X is compact and non-empty, then f assumes its maximum and its minimum, i.e. there exist $x_0, x_1 \in X$ such that

$$f(x_0) \;\leq\; f(x) \;\leq\; f(x_1) \quad \text{for all } x \in X.$$

(3) Let $f\colon X \to Y$ be a continuous map. If X is compact and if Y is Hausdorff, then $f(X)$ is a closed subset of Y.



Proof.

(1) Let $\{U_i\}_{i\in I}$ be an open cover of the topological space $f(X)$. We need to show that $f(X)$ is contained in the union of finitely many U_i's. We have

$$X \;\underset{\uparrow}{=}\; f^{-1}(f(X)) \;\subset\; f^{-1}\Big(\bigcup_{i\in I} U_i\Big) \;\underset{\uparrow}{=}\; \bigcup_{i\in I} f^{-1}(U_i).$$

by the Image-Preimage Lemma 0.3 (1) — by the Image-Preimage Lemma 0.3 (6)

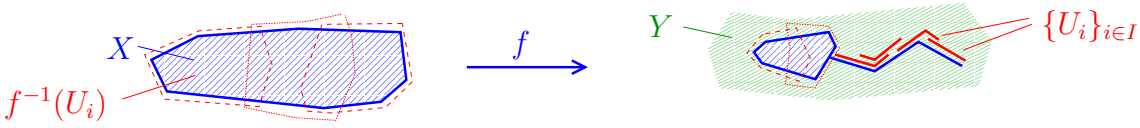
By hypothesis the map $f\colon X \to Y$ is continuous. It follows from the elementary Basics-of-Continuity Lemma 2.1 that the map $f\colon X \to f(X)$ is also continuous, in particular each $f^{-1}(U_i)$ is an open subset of X. This observation together with the above discussion shows that $\{f^{-1}(U_i)\}_{i\in I}$ is an open cover of X. Since X is compact there exist $i_1, \ldots, i_k \in I$ such that

$$X \;=\; f^{-1}(U_{i_1}) \cup \cdots \cup f^{-1}(U_{i_k}).$$

Therefore

$$\begin{aligned} f(X) \;&=\; f\left(f^{-1}(U_{i_1}) \cup \cdots \cup f^{-1}(U_{i_k})\right) \\ &\underset{\uparrow}{=}\; f\left(f^{-1}(U_{i_1})\right) \cup \cdots \cup f\left(f^{-1}(U_{i_k})\right) \;\underset{\uparrow}{\subset}\; U_{i_1} \cup \cdots \cup U_{i_k}. \end{aligned}$$

by the Image-Preimage Lemma 0.3 (3) — by the Image-Preimage Lemma 0.3 (2)



(2) It follows from (1) and the Heine-Borel Theorem 1.12 that $f(X) \subset \mathbb{R}$ is bounded and closed. Since X is non-empty we thus see that $y_0 := \inf(f(X))$ and $y_1 := \sup(f(X))$ exist and they lie in $f(X)$. Any choice of points $x_0 \in f^{-1}(y_0)$ and $x_1 \in f^{-1}(y_1)$ have the desired properties.

(3) This statement is an immediate consequence of Statement (1) and of the Compact-Closed Lemma 1.9 (1). ∎

2.4. **Homeomorphisms.** We introduce the main objects of interest of this section.

Definition.

(1) We say that a map $f\colon X \to Y$ between two topological spaces X and Y is a homeomorphism if the following three properties are satisfied:
   (a) f is continuous,
   (b) f is bijective,
   (c) the inverse map $f^{-1}\colon Y \to X$ is also continuous.
   If there exists a homeomorphism between X and Y we say that X and Y are homeomorphic and sometimes we write $X \cong Y$.

(2) We say that a map $f\colon X \to Y$ between two topological spaces X and Y is an embedding if $f\colon X \to f(X)$, where $f(X)$ is equipped with the subspace topology, is a homeomorphism. We say that the embedding is closed if $f(X)$ is a closed subset of Y and we say that the embedding is open if $f(X)$ is an open subset of Y.

Remark.

(1) If two topological spaces are homeomorphic, then they have the same topological properties, i.e. they share all properties that are defined purely in terms of the topology. For example, if X and Y are homeomorphic, then X is Hausdorff if and only if Y is Hausdorff, X is compact if and only if Y is compact and so on.

(2) It follows from the Open-Closed Inclusion Lemma 2.7 that an open (respectively closed) embedding is also an open (respectively closed) map. □

Before we turn to our first examples we introduce some standard notation.

Notation. Given $n_0 \in \mathbb{N}$ we write

$$\begin{aligned} S^n_{\geq 0} &:= \{(x_1,\dots,x_{n+1}) \in S^n \mid x_{n+1} \geq 0\} && \text{the upper hemisphere} \\ S^n_{\leq 0} &:= \{(x_1,\dots,x_{n+1}) \in S^n \mid x_{n+1} \leq 0\} && \text{the lower hemisphere} \\ S^n_{=0} &:= \{(x_1,\dots,x_{n+1}) \in S^n \mid x_{n+1} = 0\} && \text{the equator.} \end{aligned}$$

Similarly we define

$$\begin{aligned} B^n_{\geq 0} &:= \{(x_1,\dots,x_n) \in B^n \mid x_n \geq 0\} \\ B^n_{\leq 0} &:= \{(x_1,\dots,x_n) \in B^n \mid x_n \leq 0\} \end{aligned}$$

Finally, the same way we also define $\overline{B}^n_{\geq 0}$ and $\overline{B}^n_{\leq 0}$.



Examples.

(1) The maps

$$\begin{aligned} \varphi\colon \overline{B}^n &\to S^n_{\geq 0} \\ x &\mapsto \left(x, \sqrt{1-\|x\|^2}\right) \end{aligned} \qquad \text{and} \qquad \begin{aligned} p\colon S^n_{\geq 0} &\to \overline{B}^n \\ (x_1,\dots,x_n,x_{n+1}) &\mapsto (x_1,\dots,x_n) \end{aligned}$$

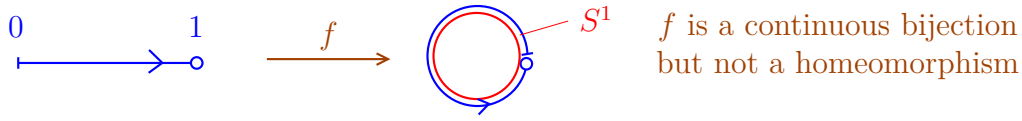
are easily seen to be continuous and inverses to one another. Thus both maps are homeomorphisms. The same way we see that the lower hemisphere is also homeomorphic to the closed ball $\overline{B}^n$.

(2) Let $n \in \mathbb{N}$. It follows easily from the definitions of the standard metrics on $\mathbb{R}^{2n}$ and $\mathbb{C}^n$ that the map

$$\begin{array}{rcl} \mathbb{R}^{2n} & \to & \mathbb{C}^n \\ (x_1, y_1, \ldots, x_n, y_n) & \mapsto & (x_1 + \mathrm{i}y_1, \ldots, x_n + \mathrm{i}y_n) \end{array}$$

is an isometry of metric spaces. Recall that on page 56 we defined the standard topology on $\mathbb{R}^{2n}$ and $\mathbb{C}^n$ to be the metric topology corresponding to the standard metrics. Thus we see that the above map is a homeomorphism. We will use this homeomorphism to make the identification $\mathbb{C}^m =_{\mathrm{i}} \mathbb{R}^{2m}$.[33]

(3) We consider the map
$$\begin{array}{rcl} f\colon [0,1) & \to & S^1 \\ t & \mapsto & \exp(2\pi \mathrm{i} t). \end{array}$$
The map f is a continuous bijection but it is not a homeomorphism. One can see this by either directly showing that f^{-1} is not continuous or by noting that $[0,1)$ is not compact whereas S^1 is compact. The importance of this example cannot be overestimated. It is an unfortunate fact of life that all too often one writes down a continuous map $f\colon X \to Y$ between topological spaces, checks that it is a bijection, and walks away under the warm, but mistaken, impression that one now has found a homeomorphism. □



In many situations one is given an explicit continuous bijection $f\colon X \to Y$ between two topological spaces and one would like to show that f is in fact a homeomorphism. Unfortunately usually it is very painful to write down the explicit inverse map $f^{-1}\colon Y \to X$, let alone to verify that it is continuous. Fortunately we have several results which often help us in circumventing this difficulty.

Lemma 2.9. (Open-Injective Lemma) Let $f\colon X \to Y$ be a continuous map between topological spaces. We assume that f is open or that f is closed.
(1) If f is an injection, then f is an open (respectively closed) embedding.
(2) If f is a bijection, then f is a homeomorphism.

Proof. Let $f\colon X \to Y$ be a continuous map between topological spaces.

(2) We assume that f is a bijection. Furthermore we now assume that f is open. We want to show that f is a homeomorphism. It remains to show that $g := f^{-1}\colon Y \to X$ is continuous. Thus let $U \subset X$ be an open subset. We need to show that $g^{-1}(U)$ is an open subset of Y. But this is obvious, since $g^{-1}(U) = f(U)$ is an open subset since we assume that f is an open map.

Now suppose that the map f is closed. The proof is almost identical to the above argument, except that now we have to use the continuity criterion provided by the Continuity-via-Closed Subsets Lemma 2.3.

(1) We assume that f is an injection and we assume that f is open. We write $Z := f(X)$. Since $f\colon X \to Y$ is open it follows basically immediately from the definition of the subspace topology that $f\colon X \to Z$ is also open. Thus we obtain from (2) that f is an

[33] By an "identification" we mean in these notes that we picked once and for all a homeomorphism. If in the following we write $\mathbb{C}^m =_{\mathrm{i}} \mathbb{R}^{2m}$, then this means that we use the given homeomorphism to go back and forth between these two topological spaces.

embedding. Furthermore, since f is open we see that f is an open embedding. The proof in the case that f is closed is basically identical. ■

The next proposition is certainly one of the most frequently used results in topology.

Proposition 2.10. (Compact-Hausdorff Proposition) Let $f\colon X \to Y$ be a continuous map between topological spaces. If X is compact and if Y is Hausdorff, then the following statements hold:

(1) The map $f\colon X \to Y$ is closed.
(2) If f is an injection, then f is a closed embedding.
(3) If f is a bijection, then f is a homeomorphism.

Remark. On page 72 we gave an example of a continuous bijective map $f\colon [0,1) \to S^1$ that is not a homeomorphism. This shows that we cannot drop the condition that X is compact from the proposition. But what about the "Hausdorff condition"? It is a nice exercise to come up with an example of a bijective continuous map $f\colon X \to Y$ between two topological spaces where X is compact but such that f is nonetheless not a homeomorphism.

Proof. Let $f\colon X \to Y$ be a continuous map between topological spaces. We assume that X is compact and that Y is Hausdorff.

(1) Let A be a closed subset of X. We need to show that $f(A)$ is a closed subset of Y. In fact we see that

$$A \subset X \text{ closed} \underset{\uparrow}{\Longrightarrow} A \text{ is compact} \underset{\uparrow}{\Longrightarrow} f(A) \text{ is compact} \underset{\uparrow}{\Longrightarrow} f(A) \subset Y \text{ is closed.}$$

by the Compact-Closed Lemma 1.9 since X is compact; by the Compact Image Lemma 2.8 (1); by the Compact-Closed Lemma 1.9 (2) since Y is Hausdorff



(2) If f is an injection, then it follows immediately from (1) together with the Open-Injective Lemma 2.9 (1) that f is a closed embedding.
(3) This follows immediately from (2). ■

We formulate our first application of the Compact-Hausdorff Proposition 2.10 (3) as a lemma.

Lemma 2.11. (Stereographic Projection Lemma) Let $n \in \mathbb{N}$. We consider the map

$$\begin{array}{rcl} \Phi\colon S^n & \to & \mathbb{R}^n \cup \{\infty\} \\ (x_1, \ldots, x_{n+1}) & \mapsto & \begin{cases} \left(\dfrac{x_1}{1-x_{n+1}}, \ldots, \dfrac{x_n}{1-x_{n+1}}\right), & \text{if } x_{n+1} < 1, \\ \infty, & \text{if } x_{n+1} = 1. \end{cases} \end{array}$$

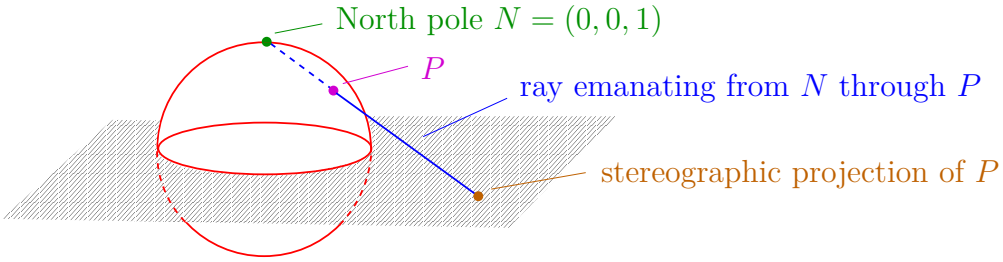
where we equip $\mathbb{R}^n \cup \{\infty\}$ with the topology that we introduced on page 60. We refer to Φ as the stereographic projection. This map has the following properties:

(1) The map Φ sends the North Pole $N = (0, \ldots, 0, 1) \in S^n$ to ∞.
(2) For any $P \in S^n$ that does not equal the North Pole N the point $\Phi(P) \in \mathbb{R}^n$ is the unique point such that the ray emanating from N and that goes through P intersects the plane $\mathbb{R}^n \times \{0\}$ in $(\Phi(P), 0)$.

(3) For any $(v_1, \dots, v_n, 0) \in S^n \cap (\mathbb{R}^n \times \{0\})$ we have $\Phi(v_1, \dots, v_n, 0) = (v_1, \dots, v_n)$.
(4) The map Φ is a homeomorphism.
(5) (a) The restriction of Φ to a map $S^n \setminus \{N\} \to \mathbb{R}^n$ is also a homeomorphism, where the inverse is given by

$$\begin{array}{ccc} \mathbb{R}^n & \to & S^n \setminus \{N\} \\ x & \mapsto & \left(x \cdot \frac{2}{\|x\|^2+1}, \underbrace{\frac{\|x\|^2-1}{\|x\|^2+1}}_{=1-\frac{2}{\|x\|^2+1}} \right). \end{array}$$

(b) The restriction of Φ to a map $S^n_{\leq 0} \to \overline{B}^n$ is a homeomorphism.



Identification. Let $n \in \mathbb{N}$. On many occasions we will use the homeomorphism from the Stereographic Projection Lemma 2.11 to make the identification $S^n =_{\mathrm{i}} \mathbb{R}^n \cup \{\infty\}$. In particular we have the identification $S^2 =_{\mathrm{i}} \mathbb{C} \cup \{\infty\}$.

Proof.

(1) This statement follows immediately from the definition of Φ.
(2) This statement follows from an elementary calculation which we leave to the reader.
(3) This statement follows immediately from the definition of Φ.
(4) Using the Subbasis-Continuity Proposition 2.2 one can easily show that Φ is continuous. We leave it to the reader to verify that Φ is a bijection. Note that S^n is compact by the Heine-Borel Theorem 1.12. Furthermore note that on page 60 we pointed out that $\mathbb{R}^n \cup \{\infty\}$ is Hausdorff. Hence it follows from the Compact-Hausdorff Proposition 2.10 (3) that Φ is a homeomorphism.
(5) It follows almost immediately from (4) that the two maps are homeomorphisms. One can easily verify that the inverse of Φ is indeed given by the map we provided. ■

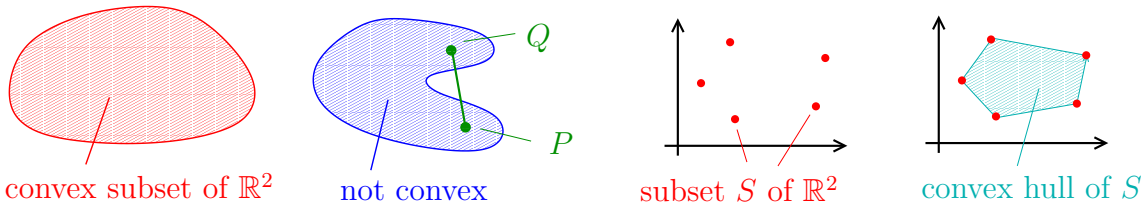
2.5. Convex subsets.

We start out with the definition of a convex subset.

Definition.

(1) We say that a subset $A \subset \mathbb{R}^n$ is convex, if for every two points P and Q in A the segment $\overline{PQ} := \{t \cdot P + (1-t) \cdot Q \mid t \in [0,1]\}$ also lies in A.
(2) Given a subset S of $\mathbb{R}^k$ the convex hull of S is defined as the intersection of all convex subsets of $\mathbb{R}^k$ that contain S. Since the intersection of convex sets is again convex we see that the convex hull of S is a convex subset of $\mathbb{R}^n$.

Examples.



(1) For any point $x \in \mathbb{R}^n$ and any $\epsilon > 0$ the open ball $B^n_\epsilon(x)$ is convex. Indeed, let $P, Q \in B^n_\epsilon(x)$. Then for any $t \in [0,1]$ we have

$$\begin{aligned} \|t\cdot P + (1-t)\cdot Q - x\| &= \|t\cdot(P-x) + (1-t)\cdot(Q-x)\| \\ &\underset{\uparrow}{\leq} t\cdot\|P-x\| + (1-t)\cdot\|Q-x\| \underset{\uparrow}{<} t\cdot\epsilon + (1-t)\cdot\epsilon = \epsilon. \end{aligned}$$

triangle inequality and homogeneity of $\|-\|$ — since $P, Q \in B^n_\epsilon(x)$

We have thus shown that $t\cdot P + (1-t)\cdot Q \in B^n_\epsilon(x)$. Basically the same way one shows that the closed ball $\overline{B}^n_\epsilon(x)$ is also convex.

(2) It is straightforward to verify that any hyperrectangle $[a_1, b_1] \times \cdots \times [a_n, b_n]$ is convex.

(3) We leave it as an exercise to show that the convex hull of points $P_1, \ldots, P_n \in \mathbb{R}^k$ is given by the set

$$\left\{ \sum_{i=1}^n t_i \cdot P_i \,\middle|\, t_1, \ldots, t_n \in \mathbb{R}_{\geq 0} \text{ and } \sum_{i=1}^n t_i = 1 \right\}.$$

(4) One of the most useful properties of convex subsets is that the intersection of arbitrarily many convex subsets is, basically by definition, again convex. □

The following proposition gives a useful criterion for showing that subsets of $\mathbb{R}^n$ are homeomorphic to an open or to a closed ball.

Proposition 2.12. (Convex-to-Ball Proposition)

(1) Let A be a bounded *open* convex subset of $\mathbb{R}^n$ and let $Q \in A$.[34] We consider the map

$$\begin{aligned} \rho_A \colon A \setminus \{Q\} &\to \mathbb{R}_{\geq 0} \\ x &\mapsto \sup\left\{ \|r\cdot(x-Q)\| \,\middle|\, r \in \mathbb{R}_{>0} \text{ and } r\cdot(x-Q)+Q \in A \right\}. \end{aligned}$$

This map is well-defined in the sense that the supremum exists for each $x \in A \setminus \{Q\}$. Furthermore the map ρ_A is continuous and the map

$$\begin{aligned} f_Q \colon A &\to B^n \\ x &\mapsto \begin{cases} (x-Q)\cdot\frac{1}{\rho_A(x)}, & \text{if } x \neq Q, \\ 0, & \text{if } x = Q. \end{cases} \end{aligned}$$

is a homeomorphism with $f_Q(Q) = 0$.

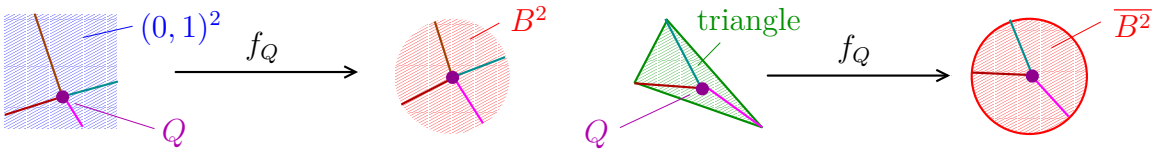
(2) Let A be a bounded *closed*[35] convex subset of $\mathbb{R}^n$. If Q is a point in the interior[36] of A, then the same expression as in (1) defines a homeomorphism[37]

$$f_Q \colon A \to \overline{B}^n = \{x \in \mathbb{R}^n \,|\, \|x\| \leq 1\}$$

with the following properties:

(a) the homeomorphism f_Q sends Q to the origin,

(b) the homeomorphism f_Q restricts to a homeomorphism $\partial A \to S^{n-1}$.



The proof of the Convex-to-Ball Proposition 2.12 relies on the following lemma:

Lemma 2.13. (Distance-to Boundary Continuous Lemma) Let A be a bounded closed convex subset of $\mathbb{R}^n$ such that the origin 0 lies in the interior of A.

(1) The map $\rho_A\colon A\setminus\{0\}\to\mathbb{R}_{\geq 0}$ defined in the Convex-to-Ball Proposition 2.12 is continuous.

(2) The map

$$\begin{array}{rcl} f_0\colon A & \to & \overline{B}^n \\ x & \mapsto & \begin{cases} x\cdot\frac{1}{\rho_A(x)}, & \text{if } x\neq 0, \\ 0, & \text{if } x=0 \end{cases} \end{array}$$

is continuous and it satisfies $f_0^{-1}(S^{n-1})=\partial A$.

Proof of Lemma 2.13. We start out with the following claim:

Claim. The map $\rho_A\colon A\setminus\{0\}\to\mathbb{R}$ is defined, it is continuous, it is bounded from above and it is bounded from below by 1.

Proof. Since A is bounded it follows that ρ_A is defined and that it is bounded from above. It follows from the definition of ρ_A that ρ_A is bounded from below by 1. Therefore it remains to show that ρ_A is continuous. Now let $x\in A\setminus\{0\}$ and let $\epsilon>0$. It suffices to show the following two statements:

(1) there exists a neighborhood U of $x\in A\setminus\{0\}$ with $\rho_A(y)>\rho_A(x)-\epsilon$ for all $y\in U$,
(2) there exists a neighborhood V of $x\in A\setminus\{0\}$ with $\rho_A(y)<\rho_A(x)+\epsilon$ for all $y\in V$.

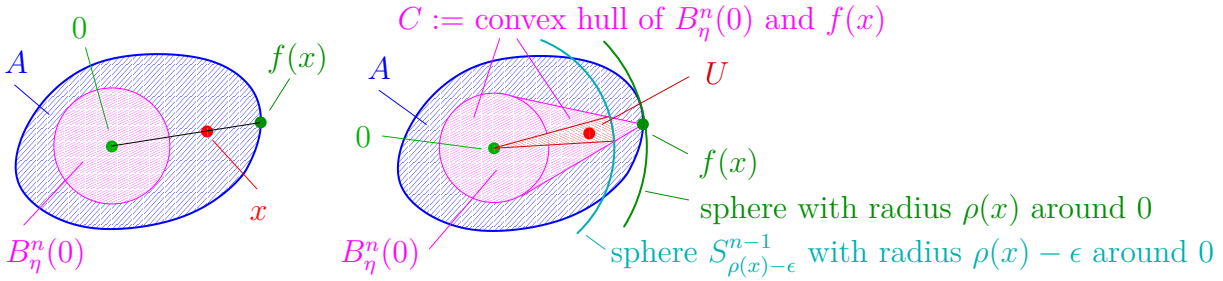
We first show the existence of U. After possibly replacing ϵ by $\min\{\frac{1}{2}\rho_A(x),\epsilon\}$ we can suppose that $\epsilon\in(0,\rho_A(x))$.

Since A is closed we have $f_0(x)\in A$. Since 0 lies in the interior of A there exists an $\eta>0$ such that $B^n_\eta(0)\subset A$. Since A is convex, the convex hull C of $B^n_\eta(0)\cup\{f_0(x)\}$ is still contained[38] in A. Furthermore, since $B^n_\eta(0)$ is open it is straightforward to see that $C':=C\setminus\{f_0(x)\}$ is an open subset of $\mathbb{R}^n$. We denote by $S^{n-1}_{\rho_A(x)-\epsilon}$ the sphere of radius $\rho_A(x)-\epsilon$ around 0. The point $x\cdot\frac{\rho_A(x)-\epsilon}{\|x\|}$ lies on $S^{n-1}_{\rho_A(x)-\epsilon}$ and it lies in C'. Since C' is open there exists an open neighborhood U' on $S^{n-1}_{\rho_A(x)-\epsilon}$ that is contained in C'. We set

$$U := \{r\cdot z\,|\,z\in U' \text{ and } r\in(0,1)\}.$$

[34]The statement becomes slightly more readable if one sets $Q=0$.
[35]Of course "bounded and closed" is in this instance, by the Heine-Borel Theorem 1.12 equivalent to "compact". In this instance we prefer to talk of "bounded closed" to stress the analogy to Statement (1) of the proposition.
[36]Note that the interior might well be empty, i.e. such a point Q might not exist.
[37]This says in particular that if A is a bounded closed convex subset of $\mathbb{R}^n$ with *non-empty interior*, then A is homeomorphic to $\overline{B}^n$.
[38]Why is that the case?

This is an open neighborhood of x and for any $y \in U$ we have $(\rho_A(x)-\epsilon)\cdot y \in A$, i.e. for any $y \in A$ we have $\rho_A(y) \geq \rho_A(x) - \epsilon$. Thus we have shown that the desired neighborhood U exists. The existence of V is proved in a very similar way. We refer to [**Ber09**, Chapter 11.3] for full details. ⊞



From the continuity of ρ_A it follows that f is continuous on $A \setminus \{0\}$. From the fact that ρ_A is bounded it follows easily that f_Q is continuous in 0. Finally one can easily verify, using the convexity of A, that $f_0^{-1}(S^{n-1}) = \partial A$. We leave it to the reader to fill in the details. ■

Proof of the Convex-to-Ball Proposition 2.12.

(2) Let A be a bounded closed convex subset of $\mathbb{R}^n$ and let Q be a point in the interior of A. After a translation we can assume that $Q = 0$ is the origin, in particular we can assume that the origin 0 lies in the interior of A. By the Distance-to Boundary Continuous Lemma 2.13 we know that the map $\rho_A \colon A \setminus \{0\} \to \mathbb{R}_{\geq 0}$ is defined and continuous. Next we consider the map

$$\begin{array}{rccl} f_0 \colon A & \to & \overline{B}^n & \\ x & \mapsto & \begin{cases} x \cdot \frac{1}{\rho_A(x)}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases} \end{array}$$

It is straightforward to verify that this map is bijective[39] and by the Distance-to Boundary Continuous Lemma 2.13 we know that f_0 is continuous and that $f_0^{-1}(S^{n-1}) = \partial A$. Note that our hypothesis on A and the Heine-Borel Theorem 1.12 imply that A is compact. Thus we obtain from the Compact-Hausdorff Proposition 2.10 (3) that f_0 is in fact a homeomorphism. Finally note that it follows from $f_0^{-1}(S^{n-1}) = \partial A$ that f_0 restricts to a homeomorphism $\partial A \to S^{n-1}$.

(1) We denote by $C = \overline{A}$ the closure of A. Since A is open and convex it follows fairly easily that $A = \mathring{C}$. One verifies fairly easily that C is convex. It is clear that C is bounded and it follows from the Interior-Closure Lemma 1.6 that C is closed. One verifies easily that the map f_0 defined for $C = \overline{A}$ restricts to the map f_0 defined for A. Thus it follows from (2) that the map $f_0 \colon C = \overline{A} \to \overline{B}^n$ is a homeomorphism and it restricts to a homeomorphism $f_0 \colon \mathring{C} \to B^n$. Since $\mathring{C} = A$ we get the desired homeomorphism. ■

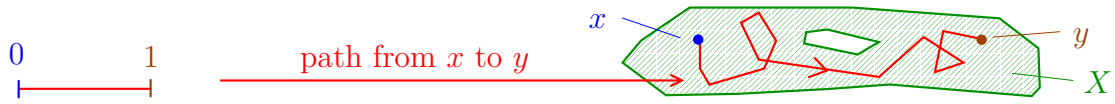
2.6. **The two notions of connected spaces.** In this section we will see that there are two notions of connectedness of a topological space. As we will see, for most "reasonable"

[39]It follows easily from the convexity of A, the hypothesis that $\mathring{A} \neq \varnothing$ and the definition of f_0 that for each $v \in S^{n-1}$ the map f_0 restricts to a bijection on the "ray defined by v", i.e. it restricts to a bijection

$$A \cap \{r \cdot v \,|\, r \in \mathbb{R}_{>0}\} \;\to\; \overline{B}^n \cap \{r \cdot v \,|\, r \in \mathbb{R}_{>0}\}.$$

topological spaces the two notions mercifully coincide. We start out with the more obvious definition.

Definition.
(1) Let X be a topological space. A path is a continuous map $p\colon [0,1] \to X$. Often we say that p is a path from $p(0)$ to $p(1)$.
(2) We say that a topological space X is path-connected if given any two points x and y there exists a path from x to y.



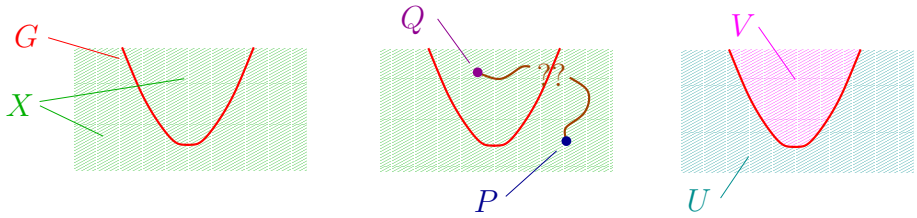
Before we discuss examples let us immediately provide the other, less obvious definition of connectedness.

Definition. We say that a topological space X is connected if the following holds[40]

$$\text{if } X = U \sqcup V \text{ is the disjoint union of open subsets } U, V \quad \Longrightarrow \quad X = U \text{ or } X = V.$$

A topological space which is not connected is called disconnected.

Example.
(1) It follows basically from the definitions that any convex subset of some $\mathbb{R}^n$ is path-connected. But it is much less clear whether say $\mathbb{R}^n$ is connected in the above sense.
(2) Let $G \subset \mathbb{R}^2$ be the graph of the function $f\colon \mathbb{R} \to \mathbb{R}$ given by $x \mapsto x^2$. As in the figure below we consider the complement $X = \mathbb{R}^2 \setminus G$. It is pretty clear that X is not path-connected, surely there is no path from a point P below G to a point above G. But how can one show the non-existence of such a path?[41] But it is clear that X is not connected, since X is the disjoint union of the two open non-empty sets $U := \{(x,y) \in \mathbb{R}^2 \,|\, y < x^2\}$ and $V := \{(x,y) \in \mathbb{R}^2 \,|\, y > x^2\}$.



(3) It follows from the Intermediate Value Theorem that the rational numbers $\mathbb{Q}$ are not path-connected. But $\mathbb{Q}$ is also not connected, for example we can write the topological space $\mathbb{Q}$ as the disjoint union of the two open non-empty subsets $\{x \in \mathbb{Q} \,|\, x^2 < 2\}$ and $\{x \in \mathbb{Q} \,|\, x^2 > 2\}$.

So we see that both notions of connectedness have their advantages and disadvantages. It is often easy to show that a given topological space is path-connected but it can be a pain to show that a given topological space is not path-connected. With connectedness it is precisely the other way round. □

[40] Recall that we write $X = U \sqcup V$ if X is the disjoint union of U and V, i.e. if $X = U \cup V$ and if $U \cap V = \varnothing$.
[41] The fact that such a path does not exist can be shown, with some dexterity, using real analysis. How?

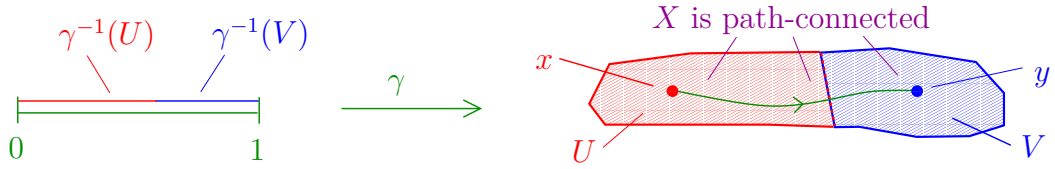
It is basically obvious that any interval is path-connected. But as we will see, it takes some effort to show that the interval $[0,1]$ is connected.

Proposition 2.14. (Interval-Connected Proposition) The interval $[0,1]$ is connected.

Proof. We proved the proposition in Analysis II, alternatively see [**Fri24**]. ■

Corollary 2.15. (Path-Connected-Implies-Connected Corollary) Every path-connected topological space is also connected.

Proof. Let X be a topological space that is path-connected. Suppose that X is not connected. This means that there exist disjoint open non-empty subsets U and V with $X = U \cup V$. Since U and V are non-empty we can find points $x \in U$ and $y \in V$. Since X is path-connected there exists a path $\gamma\colon [0,1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$. Then $\gamma^{-1}(U) \cup \gamma^{-1}(V)$ is a decomposition of $[0,1]$ into two disjoint open non-empty subsets.

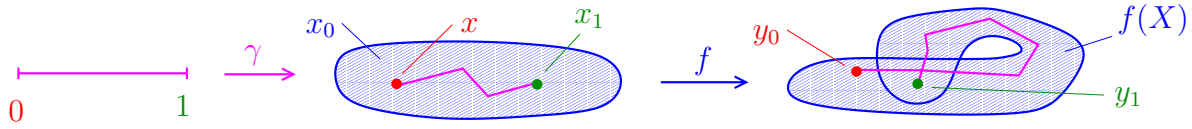


But that is a contradiction to the Interval-Connected Proposition 2.14. ■

Lemma 2.16. (Image-Connected Lemma) Let $f\colon X \to Y$ be a continuous map between topological spaces.
(1) If X is path-connected, then $f(X)$ is also path-connected.
(2) If X is connected, then $f(X)$ is also connected.

Proof.

(1) We suppose that X is path-connected. Let $y_0 = f(x_0)$ and $y_1 = f(x_1)$ be two points in $f(X)$. Since X is path-connected there exists a path $\gamma\colon [0,1] \to X$ from x_0 to x_1. But then $f \circ \gamma\colon [0,1] \to f(X)$ is a path from $f(x_0)$ to $f(x_1)$.



(2) We suppose that $f(Y)$ is disconnected. This means that there exist disjoint open non-empty subsets U and V of $f(Y)$ with $f(Y) = U \cup V$. By the Image-Preimage Lemma 0.3 we know that $f^{-1}(U)$ and $f^{-1}(V)$ are also disjoint and we know that $X = f^{-1}(U) \cup f^{-1}(V)$. By definition of a continuous map $f^{-1}(U)$ and $f^{-1}(V)$ are open. Finally, since $U, V \subset f(X)$ we see that $f^{-1}(U)$ and $f^{-1}(V)$ are non-empty. Thus we have shown that X is also disconnected. ■

In the following lemma we show that most of the topological spaces that we had encountered so far are path-connected and connected. We will use this lemma on numerous occasions without explicitly referring to it.

Lemma 2.17. (Ball-Sphere-Connected Lemma)
(1) Let $n \geq 1$. Every open ball $B_r^n(y)$, every closed ball $\overline{B}_r^n(y)$ and $\mathbb{R}^n$ are path-connected and connected.
(2) Every sphere $S_r^n(y)$ with $n \geq 1$ is path-connected and connected.

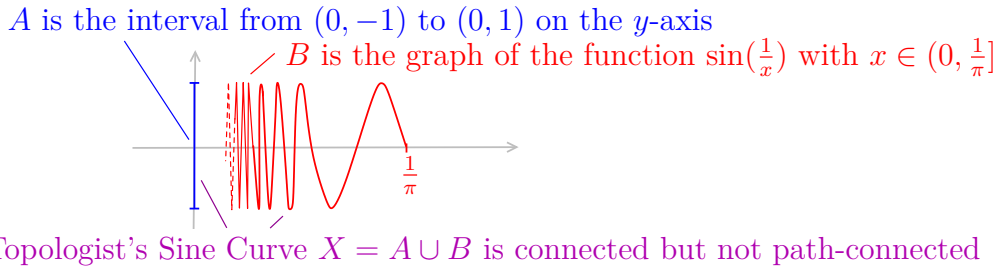
Proof. By the Path-Connected-Implies-Connected Corollary 2.15 it suffices to show that all these topological spaces are path-connected.
(1) It follows from the example on page 75 that all balls (open or closed) and $\mathbb{R}^n$ are convex. But as we already pointed out on page 78, convex subsets are basically by definition path-connected.
(2) We leave it to the reader to show that spheres of dimension ≥ 1 are path-connected. ■

For better or worse, the converse to the Path-Connected-Implies-Connected Corollary 2.15 is not always correct.

Example. The *Topologist's Sine Curve* is defined as the following topological space

$$X \;:=\; \{(0,y)\,|\,y \in [-1,1]\} \cup \{(x,\sin(\tfrac{1}{x}))\,|\,x \in (0,\tfrac{1}{\pi}]\} \;\subset\; \mathbb{R}^2,$$

equipped with the subspace topology. In other words, the Topologist's Sine Curve is the union of the graph of $x \mapsto \sin(\frac{1}{x})$ for $x \in (0,\frac{1}{\pi}]$ and an interval on the y-axis. It is a very instructive exercise to show that X is connected and it is an equally instructive exercise to show that X is *not* path-connected. □



the Topologist's Sine Curve $X = A \cup B$ is connected but not path-connected

Remark. The fact that an example of a topological space that is connected but not path-connected is so "wild" indicates that perhaps for most "reasonable" spaces the two notions of connectedness and path-connectedness agree. In fact in basically all examples of topological spaces that we encounter throughout these notes the two notions of connectedness will agree. □

We continue with the following elementary but useful lemma.

Lemma 2.18. (Connected-to-Discrete Lemma) Every continuous map $f\colon X \to Y$ from a connected space X to a discrete space is constant.

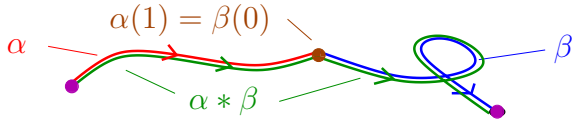
Remark. One could argue that the Connected-to-Discrete Lemma 2.18 is the lemma in topology that is most relevant to real life. More precisely, an immediate consequence of the Connected-to-Discrete Lemma 2.18 is that any *non-constant* map $(a,b) \to A$ from an interval to a finite set is not continuous. This is of huge significance since humanity likes to squeeze a real-parameter (e.g. time, intelligence, performance) into a finite set of values (e.g. historical stages, grades in school and exams, hired-or-fired). Any such function is non-continuous and hence very problematic. □

Proof. We need to show that for every $x \in X$ we have $f(\{x\}) = f(X)$. Since Y is a discrete space we know that $\{f(x)\}$ and $Y \setminus \{f(x)\}$ are open subsets of Y. The preimages $U := f^{-1}(f(x))$ and $V := f^{-1}(Y \setminus \{f(x)\})$ are also open and they are evidently disjoint. Since U is non-empty and since X is connected we see that $V = \varnothing$. But this means that f is constant. ∎

Before we continue we introduce the concatenation of two paths.

Definition. Let X be a topological space and let $\alpha\colon [0,1] \to X$ and $\beta\colon [0,1] \to X$ be two paths with $\alpha(1) = \beta(0)$. We define the concatenation of α and β as the path $\alpha * \beta$ which is given by[42]

$$\begin{array}{rcl} \alpha * \beta\colon [0,1] & \to & X \\ t & \mapsto & \begin{cases} \alpha(2t), & \text{if } t \in \left[0, \frac{1}{2}\right] \\ \beta((2t-1)), & \text{if } t \in \left[\frac{1}{2}, 1\right]. \end{cases} \end{array}$$



The concatenation of paths will play a major role throughout these notes, especially when we introduce the fundamental group of a pointed topological space. In a more humble fashion it also plays a minor role in the proof of the following lemma, which gives a criterion for the union of (path-) connected subsets to be (path-) connected.

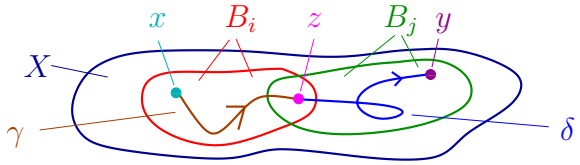
Lemma 2.19. (Union Connected Lemma) Let X be a topological space and let $\{B_i\}_{i\in I}$ be a family of subsets such that the intersection $\bigcap_{i\in I} B_i$ is non-empty.

(1) If each B_i, $i \in I$ is path-connected, then $\bigcup_{i\in I} B_i$ is path-connected.

(2) If each B_i, $i \in I$ is connected, then $\bigcup_{i\in I} B_i$ is connected.

Proof. Let X be a topological space and let $\{B_i\}_{i\in I}$ be a family of subsets such that the intersection $\bigcap_{i\in I} B_i$ contains a point z. We write $Z := \bigcup_{i\in I} B_i$.

(1) We suppose that each B_i, $i \in I$ is path-connected. So let $x, y \in Z$. There exists an $i \in I$ with $x \in B_i$ and there exists a $j \in I$ with $y \in B_j$. Since B_i and B_j are path-connected we can pick a path $\gamma\colon [0,1] \to B_i$ from x to z and we can pick a path $\delta\colon [0,1] \to B_j$ from z to y. Note that the concatenation of the paths γ and δ is a path from x to y.



(2) We suppose that each B_i, $i \in I$ is connected. We write $Z = U \sqcup V$ where U and V are open subsets. Without loss of generality we can assume that $z \in U$. For each $i \in I$ we have a decomposition $B_i = (U \cap B_i) \sqcup (V \cap B_i)$ into disjoint open subsets (here we use the definition of the subspace topology). Since B_i is connected and since $U \cap B_i \neq \varnothing$

[42] It follows from the Pasting Proposition 2.4 (2) and our hypothesis $\alpha(1) = \beta(0)$ that $\alpha * \beta\colon [0,1] \to X$ is a continuous map.

we see that $B_i = U \cap B_i$. Thus we have shown that

$$U = \bigcup_{i \in I} (B_i \cap U) = \bigcup_{i \in I} B_i = Z. \qquad \blacksquare$$

Proof. We leave the proof of the lemma as an exercise to the reader. $\blacksquare$

2.7. **The (path-) components of a topological space.** This section is a follow-up on the previous section. Let us start with the following simple-minded definition.

Definition. Let x and y be two points in a topological space X. We say that the points x and y are path-equivalent, if there exists a continuous map $\gamma\colon [0,1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$.

Lemma 2.20. (Path-Equivalence Lemma) For every topological space X the notion of path-equivalence is indeed an equivalence relation.

Proof. All properties of an equivalence relation are trivial except possibly for transitivity. The fact that path-equivalence is transitive can be shown by concatenating paths. More precisely, if $\alpha\colon [0,1] \to X$ is a path from $P = \alpha(0)$ to $Q = \alpha(1)$ and if $\beta\colon [0,1] \to X$ is a path from $Q = \beta(0)$ to $R = \beta(1)$, then the concatenation $\alpha * \beta\colon [0,1] \to X$ is a path from P to R. $\blacksquare$

The Path-Equivalence Lemma 2.20 allows us to introduce the following definition:

Definition. Let X be a topological space.

(1) We call the path-equivalence classes of points the path-components of X.
(2) We denote by $\pi_0(X)$ the set of path-components of X.[43] Furthermore we write $\pi_0(X) = 0$ if X is non-empty and if it consists of a single path-component.



Example. It is a straightforward consequence of the Intermediate Value Theorem that each point in $\mathbb{Q}$ defines a path-component of $\mathbb{Q}$. $\square$

Lemma 2.21. (Path-Component Lemma) Let X be a topological space.

(1) As a set, X is the disjoint union of its path-components.
(2) Each path-component of X is path-connected.
(3) Each path-component C of X is a maximal path-connected subset, i.e. if $C \subsetneq D$, then D is not path-connected.
(4) Let A be a subset of X.
    (a) If no point in A is path-equivalent to a point in $X \setminus A$, then A is the union of path-components of X.
    (b) If A is path-connected, then A is contained in a path-component of X.
    (c) If A is path-connected and if no point in A is path-equivalent to a point in $X \setminus A$, then A is a path-component of X.

[43] The notation "$\pi_0(X)$" might look rather odd at the moment, but it will become clearer later on, when we introduce the fundamental groups and the higher homotopy groups, why this is a perfectly sensible notation. At that point even the weird definition of "$\pi_0(X) = 0$" will start making more sense.

Proof. The lemma follows immediately from the definitions and basic properties of equivalence relations, see the Equivalence Class Lemma 0.4. We leave it to the reader to fill in the few remaining details. ■

Example. We consider the topological space $\mathbb{R} \setminus \{0\}$. The subsets $\mathbb{R}_{<0}$ and $\mathbb{R}_{>0}$ are evidently path-connected. It follows from the Intermediate Value Theorem that no point in $\mathbb{R}_{<0}$ is path-equivalent to a point in $\mathbb{R}_{>0}$. Thus it follows from the Path-Component Lemma 2.21 (4c) that $\mathbb{R}_{<0}$ and $\mathbb{R}_{>0}$ are the path-components of $\mathbb{R} \setminus \{0\}$. □

2.8. **The path-components of matrix groups.** In this section we will discuss the (path-) connectedness of certain matrix groups over $\mathbb{R}$ and over $\mathbb{C}$. First we introduce the main players.

Notation. Given a commutative ring R we write

$$\begin{aligned} \mathrm{GL}(n,R) &:= \{A \in \mathrm{M}(n\times n,R) \mid \det(A) \text{ is a multiplicative unit in } R\} \\ \mathrm{SL}(n,R) &:= \{A \in \mathrm{M}(n\times n,R) \mid \det(A) = 1\}. \end{aligned}$$

For $\mathbb{K} = \mathbb{R}$ we have the following extra definitions:

$$\begin{aligned} \mathrm{GL}_+(n,\mathbb{R}) &:= \{A \in \mathrm{M}(n\times n,\mathbb{R}) \mid \det(A) > 0\} \\ \mathrm{GL}_-(n,\mathbb{R}) &:= \{A \in \mathrm{M}(n\times n,\mathbb{R}) \mid \det(A) < 0\} \\ \mathrm{O}(n) &:= \{A \in \mathrm{M}(n\times n,\mathbb{R}) \mid A^T\cdot A = \mathrm{id}\} \\ \mathrm{SO}(n) &:= \mathrm{O}(n)\cap \mathrm{SL}(n,\mathbb{R}) = \{A \in \mathrm{M}(n\times n,\mathbb{R}) \mid A^T\cdot A = \mathrm{id} \text{ and } \det(A) = 1\}. \end{aligned}$$

Furthermore for $\mathbb{K} = \mathbb{C}$ we have the following extra definitions:

$$\begin{aligned} \mathrm{U}(n) &:= \{A \in \mathrm{M}(n\times n,\mathbb{C}) \mid \overline{A}^T\cdot A = \mathrm{id}\} \\ \mathrm{SU}(n) &:= \mathrm{U}(n)\cap \mathrm{SL}(n,\mathbb{C}) = \{A \in \mathrm{M}(n\times n,\mathbb{C}) \mid \overline{A}^T\cdot A = \mathrm{id} \text{ and } \det(A) = 1\}. \end{aligned}$$

The following convention explains in what sense these matrix groups are topological spaces.

Identification. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Given $m, n \in \mathbb{N}$ we make the identification

$$\mathrm{M}(m\times n,\mathbb{K}) =_{\mathrm{i}} \mathbb{K}^{m\cdot n}$$

by "juxtaposing" the rows of the matrix. We equip $\mathbb{K}^{m\cdot n}$ and thus also $\mathrm{M}(m\times n,\mathbb{K})$ with the standard topology that we introduced on page 56. We equip each subset of $\mathrm{M}(m\times n,\mathbb{K})$ with the corresponding subspace topology that we introduced on page 57.

Lemma 2.22. (Matrix Group Lemma) Let $n \in \mathbb{N}$.

(1) $\mathrm{GL}(n,\mathbb{R})$, $\mathrm{GL}_+(n,\mathbb{R})$ and $\mathrm{GL}_-(n,\mathbb{R})$, are open subsets of $\mathrm{M}(n\times n,\mathbb{R})$.
(2) $\mathrm{SL}(n,\mathbb{R})$, $\mathrm{O}(n)$ and $\mathrm{SO}(n)$, are closed subsets of $\mathrm{M}(n\times n,\mathbb{R})$.
(3) $\mathrm{O}(n)$ and $\mathrm{SO}(n)$ are compact.
(4) $\mathrm{GL}(n,\mathbb{C})$ is an open subset of $\mathrm{M}(n\times n,\mathbb{C})$.
(5) $\mathrm{SL}(n,\mathbb{C})$, $\mathrm{U}(n)$ and $\mathrm{SU}(n)$, are closed subsets of $\mathrm{M}(n\times n,\mathbb{C})$.
(6) $\mathrm{U}(n)$ and $\mathrm{SU}(n)$ are compact.

Proof. We consider the following two maps

$$\det\colon \begin{array}{ccc} \mathrm{M}(n\times n,\mathbb{R}) & \to & \mathbb{R} \\ A & \mapsto & \det(A) \end{array} \quad \text{and} \quad \varphi\colon \begin{array}{ccc} \mathrm{M}(n\times n,\mathbb{R}) & \to & \mathrm{M}(n\times n,\mathbb{R}) \\ A & \mapsto & A^T\cdot A. \end{array}$$

It follows from the Leibniz Formula **??** and the rather elementary Sum-Product-Continuity Lemma 2.6 that both maps are continuous.

(1) It follows from the above that $\mathrm{GL}(n,\mathbb{R}) = \det^{-1}(\mathbb{R}\setminus\{0\})$, $\mathrm{GL}_+(n,\mathbb{R}) = \det^{-1}((0,\infty))$ and $\mathrm{GL}_-(n,\mathbb{R}) = \det^{-1}((-\infty,0))$ are open subsets of $\mathrm{M}(n\times n,\mathbb{R})$.
(2) Note that it follows from the above and the Continuity-via-Closed Subsets Lemma 2.3 that $\mathrm{SL}(n,\mathbb{R}) = \det^{-1}(\{1\})$, $\mathrm{O}(n) = \varphi^{-1}(\{\mathrm{id}\})$ and $\mathrm{SO}(n) = \mathrm{SL}(n,\mathbb{R})\cap\mathrm{O}(n,\mathbb{R})$ are closed subsets of $\mathrm{M}(n\times n,\mathbb{R})$.
(3) Note that each column of an orthogonal matrix has norm one. This implies that $\mathrm{O}(n)\subset\mathbb{R}^{n^2}$ is a bounded subset of $\mathrm{M}(n\times n,\mathbb{R}) = \mathbb{R}^{n^2}$. It follows from this observation together with (2) and the Heine-Borel Theorem 1.12 that $\mathrm{O}(n)$ and $\mathrm{SO}(n)$ are compact.

The proofs of Statements (4), (5) and (6) are almost identical to the above proofs. ■

The following proposition will hopefully quench the reader's thirst for information on the (path-) connectedness of the matrix groups.

Proposition 2.23. (Matrix Group-π_0-Proposition) Let $n\in\mathbb{N}$.

	topological space	path-components
(1)	$\mathrm{GL}(n,\mathbb{R})$	path-components are $\mathrm{GL}_+(n,\mathbb{R})$ and $\mathrm{GL}_-(n,\mathbb{R})$
(2)	$\mathrm{SL}(n,\mathbb{R})$	path-connected
(3)	$\mathrm{O}(n)$	path-components are $\mathrm{SO}(n)$ and $\mathrm{O}(n)\cap\mathrm{GL}_-(n,\mathbb{R})$
(4)	$\mathrm{GL}(n,\mathbb{C})$	path-connected
(5)	$\mathrm{SL}(n,\mathbb{C})$	path-connected
(6)	$\mathrm{U}(n)$	path-connected
(7)	$\mathrm{SU}(n)$	path-connected.

Proof. The proof is provided in [**Fri24**]. ■

We turn to the proof of Proposition 2.23 for the matrix groups that are defined over $\mathbb{C}$:

Proof of Proposition 2.23 (4), (5), (6) and (7).

Claim 4.

(1) Given any $A\in\mathrm{GL}(n,\mathbb{C})$ there exists a path $\delta\colon[0,1]\to\mathrm{GL}(n,\mathbb{C})$ with $\gamma(0) = A$ and $\gamma(1)\in\mathrm{SL}(n,\mathbb{C})$.
(2) If $A\in\mathrm{U}(n)$, then we can find such a path that takes values in $\mathrm{U}(n)$.

Proof. Let $A\in\mathrm{GL}(n,\mathbb{C})$. We pick $r\in\mathbb{R}_{>0}$ and $\varphi\in\mathbb{R}$ with $\det(A) = r\cdot\exp(\mathrm{i}\varphi)$. The map

$$\begin{array}{rcl}\delta\colon[0,1] & \to & \mathrm{GL}(n,\mathbb{C})\\ t & \mapsto & r^{-t/n}\cdot\exp(-\mathrm{i}\varphi\cdot t/n)\cdot A\end{array}$$

is a path from A to a matrix in $\mathrm{SL}(n,\mathbb{C})$. If $A\in\mathrm{U}(n)$, then $r = 1$ and we see that the path takes values in $\mathrm{U}(n)$. ⊞

Using Claim 4 one sees that it suffices to show that $\mathrm{SL}(n,\mathbb{C})$ and $\mathrm{SU}(n)$ are connected. This is achieved by making straightforward modifications to the Claims 2 and 3 that we considered in the proof of the real case. We leave it to the reader to fill in the details. ■

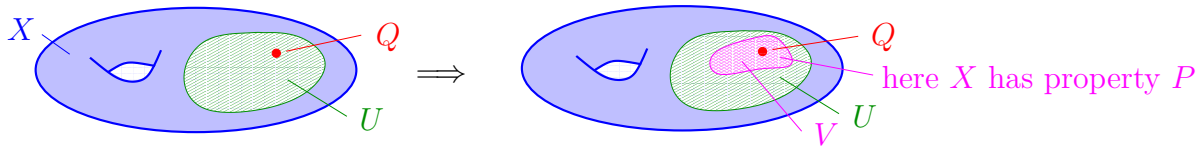
2.9. Local and regional properties.

As so often, let us start out with a definition.

Definition. Let P be a property of topological spaces and let X be a topological space.

(1) We say that X is regionally P if given any $Q \in X$ and any neighborhood U of Q there exists a neighborhood V of Q that is contained in U that has the property P.
(2) We say that X is locally P if given any $Q \in X$ and any neighborhood U of Q there exists an *open* neighborhood V of Q that is contained in U that has the property P.

Note that by definition a topological space that is locally P is also regionally P.



Warning. Our usage of the expression "locally P" is non-standard. In fact most textbooks, see e.g. [**Hat02, Bre93, Rot88**], define "locally P" as what we call "regionally P". (Note though that, as is apparent from [**Mun75**, p. 161], in [**Mun75**] our notion of "locally P" is being used.) The property that we now call "locally P" does not seem to have an established name. For our purpose it is useful to give names to both concepts since both have their uses and justifications. Later on we will also introduce notion of a local property of a map, in this context most books implicitly or explicitly seem to work with open neighborhoods. Thus it seems reasonable to us to use the word "local" whenever we deal with open neighborhoods and to rename the concept when any neighborhood is allowed.

Examples.

(1) Every open subset of $\mathbb{R}^n$ is locally path-connected. Indeed, suppose A is an open subset of $\mathbb{R}^n$. Let $Q \in A$ and let U be a neighborhood of Q in A. Then there exists an $\epsilon > 0$ such that $B^n_\epsilon(x) \subset U$. The ϵ-ball $B^n_\epsilon(x)$ is of course path-connected.
(2) The topological space $X = (-2, 1) \cup [3, 6] \subset \mathbb{R}$ with the usual topology is not path-connected but it is locally path-connected.[44] On the other hand $X = \mathbb{Q}$ is not even locally path-connected.
(3) The line with two zeros, i.e. the topological space $\mathbb{R} \cup \{*\}$ with the topology from page 60, is not Hausdorff, but one can easily show that $\mathbb{R} \cup \{*\}$ is locally Hausdorff. □

The following lemma is the key to showing that for "reasonable" topological spaces the notions of being connected and being path-connected are actually the same.

Lemma 2.24. (Locally Path-Connected Lemma) Let X be a topological space. If X is locally path-connected, then X is connected if and only if X is path-connected.

Proof. First note that if X is path-connected, then we know by the Path-Connected-Implies-Connected Corollary 2.15 that X is connected. This gives us the cheap "if"-direction.

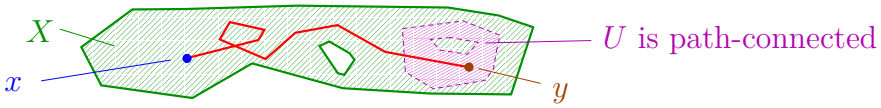
We now assume that X is connected. We start out with the following claim:

Claim 1. Each path-component of X is open.

Proof. Let P be a path-component. We pick $x \in X$. To show that P is open it suffices by the JH-Openness Criterion 1.1 to show that each $y \in P$ admits a neighborhood that is contained in P.

[44] Why is that?

Thus let $y \in P$. We pick a path $p\colon [0,1] \to X$ with $p(0) = x$ and $p(1) = y$. Since X is locally path-connected there exists an open neighborhood U of y that is path-connected. It follows from the Image-Connected Lemma 2.16 (1) together with the Union Connected Lemma 2.19 that $p([0,1]) \cup U$ is path-connected. It follows from the Path-Component Lemma 2.21 (4b) that $p([0,1]) \cup U \subset P$. Finally note that since U is open we know that $p([0,1]) \cup U$ is a neighborhood of y. ⊞



Claim 2. Each path-component of X is closed.

Proof. By Claim 1 we now know that each path-component is open. By the Path-Component Lemma 2.21 (1) we know that the complement of a path-component is the union of path-components. Since the union of open subsets is open we see that each path-component is also closed. ⊞

Since X is connected we obtain from Claim 1 and Claim 2 that each path-component equals X, i.e. X is path-connected. ■

In the following we discuss the notion of being regionally compact.

Examples.

(1) Every open subset of $\mathbb{R}^n$ is regionally compact. Indeed, suppose A is an open subset of $\mathbb{R}^n$. Let $Q \in A$ and let U be a neighborhood of Q in A. Then there exists an $\epsilon > 0$ such that $B^n_\epsilon(x) \subset U$. The closed ϵ-ball $\overline{B}^n_{\frac{\epsilon}{2}}(x)$ is a neighborhood of x and it is compact by the Heine-Borel Theorem 1.12.

(2) Similar to (1) one can easily show that every interval is regionally compact. In fact later on we will mostly need the fact that the interval $[0,1]$ is regionally compact. □

Next we turn from properties of topological spaces to properties of maps.

Definition. Let P be a property of maps between topological spaces and let $f\colon X \to Y$ be a map between topological spaces.

(1) We say that $f\colon X \to Y$ is locally P at $x \in X$ if there exists an open neighborhood U of x and an open neighborhood V of $f(x)$ such that $f\colon U \to V$ has the property P.

(2) We say that f is locally P if it is locally P at any $x \in X$.

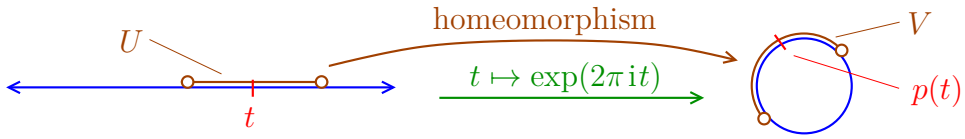
Example.

(1) The map

$$\begin{array}{rcl} p\colon \mathbb{R} & \to & S^1 \\ t & \mapsto & \exp(2\pi i t) \end{array}$$

is a local homeomorphism. Indeed, let $t \in \mathbb{R}$. We set $U = (t - \frac{1}{4}, t + \frac{1}{4})$ and $V = p(U)$. The restriction of p to $p\colon U \to V$ is continuous and a bijection. Furthermore it is not difficult to show that p is an open map. It follows from the Open-Injective Lemma 2.9 (2) that $p\colon U \to V$ is a homeomorphism.

(2) The statement of the Pasting Proposition 2.4 (1) can be reformulated as saying that a map between two topological spaces is continuous, if it is locally continuous. □


homeomorphism
$t \mapsto \exp(2\pi\,\mathrm{i}t)$
U
t
V
$p(t)$

CHAPTER 3

Categories, functors and natural transformations

In the last two chapters we introduced the concept of topological spaces, we studied several natural examples and we discussed various basic concepts. We will now take a short break from discussing topological spaces.

In this chapter we introduce the notion of a category, of a functor and of a natural transformation that many readers will already have encountered in different courses. In the subsequent chapter we will introduce the notions of (co-) products and (co-) limits in categories. These notions might initially appear to be rather abstract, but they play an essential role in topology and they are regularly used in many branches of pure mathematics. These concepts will also guide us for how to proceed with the discussion of topological spaces.

3.1. **Definition and examples of categories.** We start out with the definition of a category. As we will see, the notion of a category is extremely versatile and encompasses many situations.

Definition. A category $\mathcal{C}$ consists of the following data:

(1) A class[45] $\mathrm{Ob}(\mathcal{C})$, the elements of which are called the objects of the category.

(2) For each pair (X, Y) of objects there exists a set[46] [47] $\mathrm{Mor}_{\mathcal{C}}(X, Y)$, the elements of which are called the morphisms from X to Y.

(3) For any three objects X, Y and Z we are provided a map

$$\begin{array}{rcl} \mathrm{Mor}_{\mathcal{C}}(X,Y) \times \mathrm{Mor}_{\mathcal{C}}(Y,Z) & \to & \mathrm{Mor}_{\mathcal{C}}(X,Z) \\ (f,g) & \mapsto & g \circ f, \end{array}$$

called composition map, that satisfies the following axioms:

(A) (associativity): For every $f \in \mathrm{Mor}_{\mathcal{C}}(W,X), g \in \mathrm{Mor}_{\mathcal{C}}(X,Y)$ and $h \in \mathrm{Mor}_{\mathcal{C}}(Y,Z)$ we have

$$(h \circ g) \circ f \;=\; h \circ (g \circ f) \;\in\; \mathrm{Mor}_{\mathcal{C}}(W,Z).$$

(B) (identity): For every object X there exists a morphism $\mathrm{id}_X \in \mathrm{Mor}_{\mathcal{C}}(X,X)$ with the property that

(i) $\mathrm{id}_X \circ f = f$ for all $W \in \mathrm{Ob}(\mathcal{C})$ and all $f \in \mathrm{Mor}_{\mathcal{C}}(W,X)$, and

(ii) $f \circ \mathrm{id}_X = f$ for all $Y \in \mathrm{Ob}(\mathcal{C})$ and all $f \in \mathrm{Mor}_{\mathcal{C}}(X,Y)$.

Before we give a long list of examples let us first state a lemma which will look familiar from a basic algebra course.

Lemma 3.1. (Identity-Uniqueness Lemma) Let $\mathcal{C}$ be a category. For every $X \in \mathrm{Ob}(\mathcal{C})$ there exists a *unique* morphism $\mathrm{id}_X \in \mathrm{Mor}_{\mathcal{C}}(X,X)$ with the property that

(i) $\mathrm{id}_X \circ f = f$ for all $W \in \mathrm{Ob}(\mathcal{C})$ and all $f \in \mathrm{Mor}_{\mathcal{C}}(W,X)$,
(ii) $f \circ \mathrm{id}_X = f$ for all $X \in \mathrm{Ob}(\mathcal{C})$ and all $f \in \mathrm{Mor}_{\mathcal{C}}(X,Y)$.

Proof. The existence of id_X is given by property (3B) of a category. Now suppose that $I \in \mathrm{Mor}_{\mathcal{C}}(X,X)$ has the same property. Then we have

$$\mathrm{id}_X \underset{\substack{\uparrow \\ \text{property (i) of } I \text{ applied to } f=\mathrm{id}_X}}{=} I \circ \mathrm{id}_X \underset{\substack{\uparrow \\ \text{property (ii) of } \mathrm{id}_X \text{ applied to } f=I}}{=} I.$$

■

The Identity-Uniqueness Lemma 3.1 allows us to introduce the following definition:

Definition. Let $\mathcal{C}$ be a category and let $X \in \mathrm{Ob}(\mathcal{C})$. We refer to the morphism that is uniquely defined in the Identity-Uniqueness Lemma 3.1 as the identity morphism id_X.

In the following we introduce all kinds of categories. Many of those will be faithful companions throughout these notes.

Definition.

(a) We refer to the category $\mathcal{Set}$ with

$$\begin{aligned} \mathrm{Ob}(\mathcal{Set}) &:= \text{all sets,} \\ \mathrm{Mor}(X,Y) &:= \text{all maps from } X \text{ to } Y, \end{aligned}$$

and together with the usual composition of maps as the category of sets.

(b) We refer to the category $\mathcal{Top}$ with

$$\begin{aligned} \mathrm{Ob}(\mathcal{Top}) &:= \text{all topological spaces,} \\ \mathrm{Mor}(X,Y) &:= \text{all continuous maps from } X \text{ to } Y, \end{aligned}$$

with the usual composition of maps[48] as the category of topological spaces.

(c) Let $\mathbb{K}$ be a field. We refer to the category $\mathcal{Vec}_{\mathbb{K}}$ with

$$\begin{aligned} \mathrm{Ob}(\mathcal{Vec}_{\mathbb{K}}) &:= \text{all } \mathbb{K}\text{-vector spaces,} \\ \mathrm{Mor}(X,Y) &:= \mathrm{Hom}_{\mathbb{K}}(X,Y) = \text{all } \mathbb{K}\text{-homomorphisms from } X \text{ to } Y, \end{aligned}$$

with the usual composition of homomorphisms as the category of $\mathbb{K}$-vector spaces.

---

[45] As many "working mathematicians" the author has a slightly guilty conscience for not knowing what a "set" and what a "class" is. Therefore the reader is referred to [**Cie97, FP85**] for details. Truth be told, most mathematicians learn from logicians at some point when to say "class" instead of "set", but many mathematicians do not know exactly what they are doing at that point.

[46] One could ask why $\mathrm{Mor}_{\mathcal{C}}(X,Y)$ has to be a set, why not just a class? To the best of my knowledge the axioms on morphisms that we write down also make sense if $\mathrm{Mor}_{\mathcal{C}}(X,Y)$ is only required to be a class. Here's a long discussion which gives a satisfactory answer to people in category theory:

`http://mathoverflow.net/questions/48810/why-need-the-morphisms-to-form-a-set`

I myself do not understand the answer.

[47] If the category is clear from the context, then we just write $\mathrm{Mor}(X,Y)$ instead of $\mathrm{Mor}_{\mathcal{C}}(X,Y)$.

(d) We refer to the category $\mathcal{Group}$ with

$$\begin{aligned} \mathrm{Ob}(\mathcal{Group}) &:= \text{all groups,} \\ \mathrm{Mor}(X,Y) &:= \mathrm{Hom}(X,Y) = \text{all group homomorphisms from } X \text{ to } Y, \end{aligned}$$

with the usual composition of group homomorphisms as the category $\mathcal{Group}$ of groups. Similarly we define the category $\mathcal{AbGroup}$ of abelian groups, the category $\mathcal{Ring}$ of rings and the category $\mathcal{CRing}$ of commutative rings.

(e) A pair of sets is a pair (X, A), where X is a set and A is a subset of X. We refer to the category $\mathcal{PairSet}$ with

$$\begin{aligned} \mathrm{Ob}(\mathcal{PairSet}) &:= \text{all pairs of sets,} \\ \mathrm{Mor}((X,A),(Y,B)) &:= \text{all maps } f \text{ from } X \text{ to } Y \text{ with } f(A) \subset B \end{aligned}$$

with the usual composition of maps as the category of pairs of sets. With the same approach we could also introduce the category of pairs of topological spaces and so on.

The name "morphisms" suggests that morphisms are necessarily maps. But as we will see in the following two examples, this is not necessarily always the case.

Example.

(f) Let G be an arbitrary group. We consider the category $\mathcal{C}$ with

$$\begin{aligned} \mathrm{Ob}(\mathcal{C}) &:= \text{the set with the unique element } *, \\ \mathrm{Mor}(*,*) &:= G. \end{aligned}$$

We define

$$\begin{aligned} \mathrm{Mor}(*,*) \times \mathrm{Mor}(*,*) &\to \mathrm{Mor}(*,*) \\ (f,g) &\mapsto g \circ f := \underbrace{g \cdot f}_{\text{multiplication in } G} \end{aligned}$$

via the group structure on $\mathrm{Mor}(*,*) = G$. The axioms of a category can be deduced easily from the group axioms of G. □

Our last example deserves being a proper definition again.

Definition.

(g) Let $(P, \leq)$ be a partially ordered set, as defined on page 40. We fix a set $\{*\}$ consisting of a single element. We define the order category $\mathcal{Order}_{(P,\leq)}$ to be the category that is given by

$$\begin{aligned} \mathrm{Ob}(\mathcal{Order}_{(P,\leq)}) &= P \\ \mathrm{Mor}(x,y) &= \begin{cases} \{*\}, & \text{if } x \leq y, \\ \varnothing, & \text{otherwise.} \end{cases} \end{aligned}$$

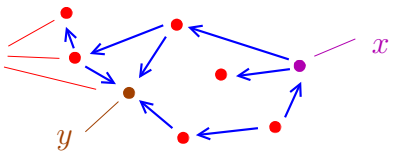
together with the only possible way to compose morphisms.[49]

Given a category $\mathcal{C}$ we now introduce its doppelgänger, namely the corresponding opposite category $\mathcal{C}^{\mathrm{op}}$. The idea is very simple, we keep the objects, but we "invert the direction of the morphisms". This simple concept can be very convenient in future discussions and gives, almost for free, a rich source of new categories.

[48] By the Basics-of-Continuity Lemma 2.1 (1) we know that the composition of continuous maps is indeed continuous.

[49] For example, we have $\mathrm{Mor}(x,y) \neq \varnothing$ and $\mathrm{Mor}(y,z) \neq \varnothing$ precisely if $x \leq y$ and $y \leq z$. But since we are dealing with a partial order we have $x \leq z$, which implies that $\mathrm{Mor}(x,z) = \{*\}$. So the composition of the unique morphism in $\mathrm{Mor}(x,z)$ and the unique morphism in $\mathrm{Mor}(y,z)$ is the unique morphism in $\mathrm{Mor}(x,z)$.

partially ordered set
where $a \leq b$ if there exists a
directed path from a to b



$$\begin{aligned}\mathrm{Mor}(x,y) &= \{*\}\\ \mathrm{Mor}(y,x) &= \varnothing\\ \mathrm{Mor}(x,x) &= \{*\}\\ \mathrm{Mor}(y,y) &= \{*\}\end{aligned}$$

Definition. Let $\mathcal{C}$ be a category. The opposite category $\mathcal{C}^{\mathrm{op}}$ of $\mathcal{C}$ is defined as the category where

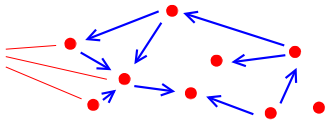
$$\begin{aligned}\mathrm{Ob}(\mathcal{C}^{\mathrm{op}}) &:= \mathrm{Ob}(\mathcal{C})\\ \mathrm{Mor}_{\mathcal{C}^{\mathrm{op}}}(X,Y) &:= \mathrm{Mor}_{\mathcal{C}}(Y,X)\end{aligned}$$

with the composition given by $f \circ_{\mathcal{C}^{\mathrm{op}}} g := g \circ_{\mathcal{C}} f$.

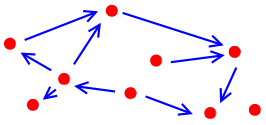
Example.

(1) Let $(P,\leq)$ be a partially ordered set. We consider the corresponding order category $\mathit{Order}_{(P,\leq)}$. The opposite of the order category $\mathit{Order}_{(P,\leq)}$ is the order category corresponding to the partially ordered set $(P,\leq^{\mathrm{op}})$ where $x \leq^{\mathrm{op}} y$ if and only if $y \leq x$.

partially ordered set $(P,\leq)$



$(P,\leq^{\mathrm{op}})$



(2) For any category $\mathcal{C}$ we evidently have $(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} = \mathcal{C}$. □

3.2. **Isomorphisms.** In this section we introduce the notion of an isomorphism, which generalizes the notion that is well-known from linear algebra.

Definition. Let $\mathcal{C}$ be a category and let X, Y be two objects in $\mathcal{C}$.

(1) Let $f \in \mathrm{Mor}_{\mathcal{C}}(X,Y)$ be a morphism and let $g \in \mathrm{Mor}_{\mathcal{C}}(Y,X)$.
  (a) We say that g is left inverse of f if $g \circ f = \mathrm{id}_X$.
  (b) We say that g is right inverse of f if $f \circ g = \mathrm{id}_Y$.
  (c) We say that g is an inverse of f if g is a left and a right inverse of f.

(2) A morphism $f \in \mathrm{Mor}_{\mathcal{C}}(X,Y)$ is called invertible, or an isomorphism, if it admits an inverse.

(3) Two objects $X, Y \in \mathrm{Ob}(\mathcal{C})$ are called isomorphic, if there exists an isomorphism in $\mathrm{Mor}_{\mathcal{C}}(X,Y)$.

(4) Given an object $X \in \mathrm{Ob}(\mathcal{C})$ we refer to the class of all objects that are isomorphic to X as the isomorphism type of X.

Example.

(1) In the category of vector spaces the invertible morphisms are precisely the isomorphisms which are familiar from linear algebra.

(2) In the category of sets the isomorphisms are precisely the bijections.

(3) In the category of topological spaces the isomorphisms are precisely the homeomorphisms.

Lemma 3.2. (Inverse Uniqueness Lemma) Let $\mathcal{C}$ be a category.

(1) Let $X, Y \in \mathrm{Ob}(\mathcal{C})$ and let $f \in \mathrm{Mor}_{\mathcal{C}}(X, Y)$. If f admits a left inverse $\widetilde{g}$ and a right inverse g, then they agree, i.e. $g = \widetilde{g}$. In particular f is invertible.

(2) Let $X, Y \in \mathrm{Ob}(\mathcal{C})$ and let $f \in \mathrm{Mor}(X, Y)$. If f admits an inverse, then the inverse is unique.

Definition. Let $\mathcal{C}$ be a category, let $X, Y \in \mathrm{Ob}(\mathcal{C})$ and let $f \in \mathrm{Mor}_{\mathcal{C}}(X, Y)_{\mathrm{inv}}$ be an invertible morphism. By the Inverse Uniqueness Lemma 3.2 there exists a *unique* morphism $g \in \mathrm{Mor}_{\mathcal{C}}(Y, X)$ with $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$. We refer to this morphism as the inverse of f and usually we denote it by f^{-1}.

Proof of Lemma 3.2.

(1) Let $g \in \mathrm{Mor}_{\mathcal{C}}(Y, X)$ be a morphism with $f \circ g = \mathrm{id}_Y$ and let $\widetilde{g} \in \mathrm{Mor}_{\mathcal{C}}(Y, X)$ be a morphism with $\widetilde{g} \circ f = \mathrm{id}_X$. We have the following equalities of morphisms:

$$g \underset{\substack{\uparrow \\ \text{defining} \\ \text{property of } \mathrm{id}_X}}{=} \mathrm{id}_X \circ g \underset{\substack{\uparrow \\ \text{defining} \\ \text{property of } \widetilde{g}}}{=} (\widetilde{g} \circ f) \circ g \underset{\substack{\uparrow \\ \text{associativity of } \circ}}{=} \widetilde{g} \circ (f \circ g) \underset{\substack{\uparrow \\ \text{defining} \\ \text{property of } g}}{=} \widetilde{g} \circ \mathrm{id}_Y \underset{\substack{\uparrow \\ \text{defining} \\ \text{property of } \mathrm{id}_Y}}{=} \widetilde{g}.$$

(2) Suppose we have $g, \widetilde{g} \in \mathrm{Mor}_{\mathcal{C}}(X, Y)$ with $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$ and with $f \circ \widetilde{g} = \mathrm{id}_Y$ and $\widetilde{g} \circ f = \mathrm{id}_X$. It follows immediately from (1) that $g = \widetilde{g}$. ∎

Lemma 3.3. (Isomorphism Lemma) Let $\mathcal{C}$ be a category.

(1) Let $f \in \mathrm{Mor}_{\mathcal{C}}(X, Y)$ and $g \in \mathrm{Mor}_{\mathcal{C}}(Y, Z)$ be morphisms. If two morphisms out of $f, g, g \circ f$ are invertible, then so is the third and we have $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

(2) "Being isomorphic" is an equivalence relation in the class of objects of $\mathcal{C}$.[50]

Proof. The two statements are basically trivial. We leave it to the nervous reader to fill in the details. ∎

3.3. **Monomorphisms and epimorphisms.** In the context of categories the following definition is very natural.

Definition. Let $\mathcal{C}$ be a category.

(1) A morphism $g \in \mathrm{Mor}_{\mathcal{C}}(X, Y)$ is called a monomorphism if for every two morphisms f_1, f_2 in $\mathrm{Mor}_{\mathcal{C}}(W, X)$ with $g \circ f_1 = g \circ f_2$ we also have $f_1 = f_2$.

(2) A morphism $g \in \mathrm{Mor}_{\mathcal{C}}(X, Y)$ is called an epimorphism if for every two morphisms h_1, h_2 in $\mathrm{Mor}_{\mathcal{C}}(Y, Z)$ with $h_1 \circ g = h_2 \circ g$ we also have $h_1 = h_2$.

Of course the names "monomorphism" and "epimorphism" sound familiar. We will now see that in many, but perhaps not all settings, the new notions coincide with the familiar notions.

[50] The perspicacious reader might notice that here things get logically a little dicey. More precisely, on page 37 we had recalled the notion of an equivalence relation. But we had only defined the notion of an "equivalence relation" on a set and "all topological spaces" do not form a set they "only" form a class. One can also define the notion of an "equivalence relation" on a class, see e.g. [**FP85**, Kapitel 2]. But we do not want to get into the discussion "what is a class?". As a cheap way out we interpret Statement (2) of the lemma as saying that reflexivity, symmetry and transitivity hold for "homotopy equivalence" with the obvious interpretation.

Proposition 3.4. (Monomorphism Proposition) Suppose we are in one of the following categories:

(1) The category *Group* of groups.
(2) The category *AbGroup* of abelian groups.
(3) The category $\mathcal{Vec}_{\mathbb{K}}$ of $\mathbb{K}$-vector spaces, where $\mathbb{K}$ is a fixed field.
(4) The category *Ring* of rings.

If $\varphi\colon G \to H$ is a monomorphism in any of the above categories, then φ, viewed as a map between sets, is injective.

Proof. Let $\varphi\colon G \to H$ be a monomorphism in one of the first three categories. We set $K := \ker(\varphi) := \{g \in G \mid \varphi(g) = e\}$, where e is the trivial element of the group or the zero in the vector space. Note that in the first three categories we listed K is again an object of the category and φ is injective if and only if K is trivial.

We consider the trivial homomorphisms $\alpha\colon K \to G$ and we consider the inclusion homomorphism $\beta\colon K \to G$. Note that $\varphi \circ \alpha = \varphi \circ \beta$. Since φ is a monomorphism we see that $\alpha = \beta$. But this implies that K is trivial.

Finally we turn to the category *Ring* of rings. Note that the argument above does not apply since the multiplicatively neutral element never lies in the kernel, which implies that the kernel of a ring homomorphism is not a ring. Therefore we have to work a little harder. Let $\varphi\colon A \to B$ be a monomorphism in the category *Ring* of rings and let $a_1, a_2 \in A$ with $\varphi(a_1) = \varphi(a_2)$. We consider the polynomial ring $\mathbb{Z}[x]$ and we consider the two homomorphisms $f_1, f_2\colon \mathbb{Z}[x] \to A$ that are defined by $f_1(x) = a_1$ and $f_2(x) = a_2$. Then $(\varphi \circ f_1)(x) = (\varphi \circ f_2)(x)$ and hence $\varphi \circ f_1 = \varphi \circ f_2\colon \mathbb{Z}[x] \to B$. Since φ is a monomorphism we see that $f_1 = f_2$ and thus $a_1 = a_2$. ■

Proposition 3.5. (Epimorphism Proposition) Suppose we are in one of the following categories:

(1) The category *Group* of groups.
(2) The category *AbGroup* of abelian groups.
(3) The category $\mathcal{Vec}_{\mathbb{K}}$ of $\mathbb{K}$-vector spaces, where $\mathbb{K}$ is a fixed field.

If $\varphi\colon G \to H$ is an epimorphism in any of the above categories, then φ, viewed as a map between sets, is surjective.

Example. The perspicacious reader will have noticed that in the Epimorphism Proposition 3.5 we do not say anything about the category of rings. In fact one can easily show that in the category *Ring* of rings the inclusion $\mathbb{Z} \to \mathbb{Q}$ is an epimorphism. □

Proof. The case of the category *AbGroup* of abelian groups and the category of $\mathbb{K}$-vector spaces is straightforward, basically one just needs to consider the "naive cokernel". We leave these cases to the reader.

The case of the category *Group* of groups is trickier, since the cokernel of a group homomorphism is not necessarily a group. We will now deal with this case in detail, following the argument given in [**Lin70**] and [**Par70**, p. 16]. (An alternative proof is given in [**Bur65**, p. 767].) Thus let $\varphi\colon G \to H$ be an epimorphism in the category *Group* of

groups. We consider the set $H/\varphi(G)$ of left cosets of $\varphi(G)$ in H.[51] Let $\{*\}$ be a symbol that is not an element of $H/\varphi(G)$. We set $Y := \{*\} \cup H/\varphi(G)$. We denote by $\mathrm{Bij}(Y)$ the group of bijections $Y \to Y$. We consider the map

$$\begin{array}{rcl} \alpha\colon H & \to & \mathrm{Bij}(Y) \\ h & \mapsto & \left(\begin{array}{ccl} Y & \to & Y \\ y & \mapsto & \begin{cases} (h \cdot k) \cdot \varphi(G), & \text{if } y = k \cdot \varphi(G) \in H/\varphi(G), \\ *, & \text{if } y = *. \end{cases} \end{array} \right) \end{array}$$

One can easily verify that $\alpha\colon H \to \mathrm{Bij}(Y)$ is in fact a homomorphism.

Next let $\sigma \in \mathrm{Bij}(Y)$ be the bijection that swaps $\varphi(G) \in Y$ with $* \in Y$ and that keeps all other elements of Y fixed. Let $\beta\colon H \to \mathrm{Bij}(Y)$ be the homomorphism that is given by $h \mapsto \sigma \circ \alpha(h) \circ \sigma^{-1}$.

Claim. We have $\alpha \circ \varphi = \beta \circ \varphi\colon G \to \mathrm{Bij}(Y)$.

Proof. Let $g \in G$. It follows immediately from the definitions that $\alpha(\varphi(g)) \in \mathrm{Bij}(Y)$ fixes the coset $\varphi(G)$ and that it fixes $*$. It follows that the support[52] of $\alpha(\varphi(g)) \in \mathrm{Bij}(Y)$ and $\sigma \in \mathrm{Bij}(Y)$ are disjoint. This implies that $\alpha(\varphi(g))$ and σ commute. Now we see that

$$\alpha(\varphi(g)) \;=\; \sigma^{-1} \circ (\sigma \circ \alpha(\varphi(g))) \underset{\substack{\uparrow \\ \text{since } \alpha(\varphi(g)) \text{ and } \sigma \text{ commute}}}{=} \sigma^{-1} \circ \alpha(\varphi(g)) \circ \sigma \underset{\substack{\uparrow \\ \text{definition of } \beta}}{=} \beta(\varphi(g)).$$

Thus we have shown that $\alpha \circ \varphi = \beta \circ \varphi\colon G \to \mathrm{Bij}(Y)$. ⊞

Since φ is an epimorphism we obtain from the claim that $\alpha = \beta\colon H \to \mathrm{Bij}(Y)$. We need to show that $\varphi(G) = H$. Let $h \in H$. As we just noticed, we have $\alpha(h) = \beta(h) \in \mathrm{Bij}(Y)$. By definition of β this means that $\alpha(h) = \sigma \circ \alpha(h) \circ \sigma^{-1}$. In other words, we see that $\alpha(h)$ commutes with σ. This in turn implies that $\alpha(h) \in \mathrm{Bij}(Y)$ leaves $\varphi(G) \in Y$ fixed, i.e. we have $h \cdot \varphi(G) = \varphi(G)$. But that implies that $h \in \varphi(G)$. We have thus shown that $\varphi(G) = H$. ■

3.4. Initial and terminal objects.

In this very short section we study the following definition:

Definition. Let $\mathcal{C}$ be a category.

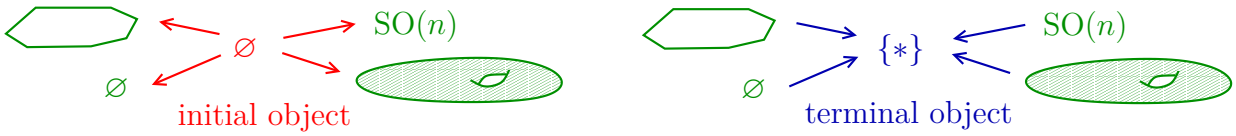
(1) An object $X \in \mathrm{Ob}(\mathcal{C})$ is called initial if for each object $Y \in \mathrm{Ob}(\mathcal{C})$ there exists a unique morphism from X to Y.
(2) An object $X \in \mathrm{Ob}(\mathcal{C})$ is called terminal if for each object $Y \in \mathrm{Ob}(\mathcal{C})$ there exists a unique morphism from Y to X.

Example.

(1) In the category *Set* of sets an initial object is given by the empty set and a terminal object is given by any set consisting of a single element. Similarly in the category *Top* of topological spaces an initial object is given by the empty topological space and a terminal object is given by a topological space consisting of a single element.

[51] In general, if $A \subset H$ is a subgroup, then we obtain an equivalence relation on H by setting $x \sim y$ if there exists an $h \in H$ with $x \cdot h = y$. The set of equivalence classes is denoted by H/A. Note that each equivalence class is of the form $hA = \{h \cdot a \mid a \in A\}$.

[52] Given $\tau \in \mathrm{Bij}(Y)$ we define its support as the subset $\{y \in Y \mid \tau(y) \neq y\}$.



(2) In the category *Group* of groups any group consisting of a single element is an initial and also a terminal object. The same applies to the category *AbGroup* of abelian groups and the analogous statement holds in the category $\mathcal{V}ec_{\mathbb{K}}$ of vector spaces over a field $\mathbb{K}$.
(3) Let X be an initial object of a category $\mathcal{C}$. It follows easily from the definitions that it is also a terminal object of the opposite category $\mathcal{C}^{\mathrm{op}}$. □

Lemma 3.6. (Initial-Terminal Object Uniqueness Lemma) Let $\mathcal{C}$ be a category.
(1) If an initial object exists, then it is unique up to isomorphism.
(2) If a terminal object exists, then it is unique up to isomorphism.

Proof. The statements follow easily from the definition. ■

3.5. **Covariant functors.** In this and the following section we introduce the all-important notions of covariant and contravariant functors.

Definition. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. A covariant functor $F\colon \mathcal{C} \to \mathcal{D}$ consists of a map

$$F\colon \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$$

and for all $X, Y \in \mathrm{Ob}(\mathcal{C})$ of a map

$$F\colon \mathrm{Mor}_{\mathcal{C}}(X,Y) \to \mathrm{Mor}_{\mathcal{D}}(F(X),F(Y)),$$

such that the following axioms are satisfied:
(F1) For all $X \in \mathrm{Ob}(\mathcal{C})$ we have $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$.
(F2) For all $\phi \in \mathrm{Mor}_{\mathcal{C}}(X,Y)$ and all $\psi \in \mathrm{Mor}_{\mathcal{C}}(Y,Z)$ the following equality holds

$$F(\psi \circ \phi) = F(\psi) \circ F(\phi) \quad \in \mathrm{Mor}_{\mathcal{D}}(F(X),F(Z)).$$

Convention. If the covariant functor F is understood from the context, then given a morphism $\phi \in \mathrm{Mor}_{\mathcal{C}}(X,Y)$ we often write ϕ_* instead of $F(\phi)$. The first axiom (F1) then becomes $(\mathrm{id}_X)_* = \mathrm{id}_{F(X)}$ and the second axiom (F2) becomes $(\psi \circ \phi)_* = \psi_* \circ \phi_*$.

Examples.
(1) The map

$$\begin{array}{rcl} F\colon \mathrm{Ob}(\mathcal{V}ec_{\mathbb{R}}) & \to & \mathrm{Ob}(\mathcal{S}et) \\ (V,+,\cdot) & \mapsto & V, \end{array}$$

together with the maps

$$\begin{array}{rcl} \mathrm{Mor}_{\mathcal{V}ec_{\mathbb{R}}}(V,W) & \to & \mathrm{Mor}_{\mathcal{S}et}(V,W) \\ (\varphi\colon V \to W) & \mapsto & (\varphi\colon V \to W) \end{array}$$

define a covariant functor from the category $\mathcal{V}ec_{\mathbb{R}}$ of real vector spaces to the category *Set* of sets. This example can be generalized to many contexts, for example there is a similar covariant functor from the category of topological spaces to the category of sets. Functors of this type are colloquially called *forgetful functor*.
(2) Given any category there is the identity functor with the obvious definition. Funnily enough this functor will soon, namely on page 99, play an interesting role.

(3) Let $\mathcal{Top}$ be the category of topological spaces and let $\mathcal{Set}$ be the category of sets. One can easily show that the map

$$\begin{array}{rcl} F\colon \mathrm{Ob}(\mathcal{Top}) & \to & \mathrm{Ob}(\mathcal{Set}) \\ X & \mapsto & \pi_0(X) := \text{set of path-components of } X, \end{array}$$

and the maps

$$\begin{array}{rcl} \mathrm{Mor}_{\mathcal{Top}}(X,Y) & \to & \mathrm{Mor}_{\mathcal{Set}}(\pi_0(X),\pi_0(Y)) \\ f & \mapsto & \begin{pmatrix} \pi_0(X) & \to & \pi_0(Y) \\ C & \mapsto & \text{unique path-component} \\ & & \text{of } Y \text{ that contains } f(C) \end{pmatrix} \end{array}$$

define a covariant functor from the category $\mathcal{Top}$ of topological spaces to the category $\mathcal{Set}$ of sets.

(4) The maps

$$\begin{array}{rcl} X & \mapsto & (X, \text{discrete topology}) \\ f & \mapsto & f \end{array}$$

define a covariant functor from the category $\mathcal{Set}$ of sets to the category $\mathcal{Top}$ of topological spaces. Evidently we can also replace the discrete topology by the trivial topology. □

We state our next example of a covariant functor as a lemma.

Lemma 3.7. (Fixed Domain-Covariant Functor Lemma) Let $\mathcal{D}$ be a category and let $\mathcal{Set}$ be the category of sets. Furthermore let $W \in \mathrm{Ob}(\mathcal{D})$. The map

$$\begin{array}{rcl} F\colon \mathrm{Ob}(\mathcal{D}) & \to & \mathrm{Ob}(\mathcal{Set}) \\ X & \mapsto & \mathrm{Mor}_{\mathcal{D}}(W,X) \end{array}$$

together with the maps

$$\begin{array}{rcl} \mathrm{Mor}_{\mathcal{D}}(X,Y) & \to & \mathrm{Mor}_{\mathcal{Set}}(\mathrm{Mor}_{\mathcal{D}}(W,X),\mathrm{Mor}_{\mathcal{D}}(W,Y)) \\ \phi & \mapsto & \begin{pmatrix} \mathrm{Mor}_{\mathcal{D}}(W,X) & \to & \mathrm{Mor}_{\mathcal{D}}(W,Y) \\ g & \mapsto & \phi \circ g \end{pmatrix} \end{array}$$

is a covariant functor from the category $\mathcal{D}$ to the category $\mathcal{Set}$ of sets.

Proof. The statement follows easily from the axioms of a category. ■

Example. The example of a covariant functor in the Fixed Domain-Covariant Functor Lemma 3.7 comes with many variations, in particular if $\mathrm{Mor}_{\mathcal{D}}(W,X)$ has more structure than that of a set. For instance let W be a real vector space. Then the above map

$$\begin{array}{rcl} F\colon \mathrm{Ob}(\mathcal{Vec}_{\mathbb{R}}) & \to & \mathrm{Ob}(\mathcal{Vec}_{\mathbb{R}}) \\ V & \mapsto & \underbrace{\mathrm{Hom}_{\mathbb{R}}(W,V)}_{\text{real vector space}}, \end{array}$$

and the maps

$$\begin{array}{rcl} \mathrm{Mor}_{\mathcal{Vec}_{\mathbb{R}}}(U,V) & \to & \mathrm{Mor}_{\mathcal{Vec}_{\mathbb{R}}}(F(U),F(V)) \\ (\phi\colon U \to V) & \mapsto & \begin{pmatrix} \mathrm{Hom}_{\mathbb{R}}(W,U) & \to & \mathrm{Hom}_{\mathbb{R}}(W,V) \\ g & \mapsto & \phi \circ g \end{pmatrix} \end{array}$$

define a covariant functor from the category $\mathcal{Vec}_{\mathbb{R}}$ of real vector spaces to itself. □

3.6. **Contravariant functors.** In this section we introduce the confusing evil cousin of the covariant functor, namely the contravariant functor. We will not make use of this concept

in the course Algebraic Topology I. It will play a role later in Algebraic Topology III, when we consider cohomology groups. We only include this discussion for "general culture".

Definition. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. A contravariant functor $F\colon \mathcal{C} \to \mathcal{D}$ consists of a map

$$F\colon \operatorname{Ob}(\mathcal{C}) \to \operatorname{Ob}(\mathcal{D})$$

and for all $X, Y \in \operatorname{Ob}(\mathcal{C})$ of a map

$$\operatorname{Mor}_{\mathcal{C}}(X, Y) \to \operatorname{Mor}_{\mathcal{D}}(F(Y), F(X))$$

such that the following axioms are satisfied:

(F1) For all $X \in \operatorname{Ob}(\mathcal{C})$ we have $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$.

(F2) For all $\phi \in \operatorname{Mor}_{\mathcal{C}}(X, Y)$ and all $\psi \in \operatorname{Mor}_{\mathcal{C}}(Y, Z)$ we have

$$F(\psi \circ \phi) = F(\phi) \circ F(\psi) \quad \in \operatorname{Mor}_{\mathcal{D}}(F(Z), F(X)).$$

The definition of a contravariant functor looks very much the same as the definition of a covariant functor. The difference is that a contravariant functor "reverses the direction of morphisms" and the orders of $F(\psi)$ and $F(\phi)$ in the second axiom (F2) are swapped. In fact, it follows immediately from the definitions that a contravariant functor $\mathcal{C} \to \mathcal{D}$ is the same as a covariant functor $\mathcal{C} \to \mathcal{D}^{\mathrm{op}}$ to the opposite category.

Convention. If the contravariant functor F is understood from the context, then given a morphism $\phi \in \operatorname{Mor}_{\mathcal{C}}(X, Y)$ we often write ϕ^* instead of $F(\phi)$. The first axiom then becomes $(\mathrm{id}_X)^* = \mathrm{id}_{F(X)}$ and the second axiom becomes $(\psi \circ \phi)^* = \phi^* \circ \psi^*$.

We state our first example as a lemma, namely we have the following analogue of the Fixed Domain-Covariant Functor Lemma 3.7.

Lemma 3.8. (Fixed Target-Contravariant Functor Lemma) Let $\mathcal{D}$ be a category and let $\mathcal{S}et$ be the category of sets. Furthermore let $W \in \operatorname{Ob}(\mathcal{D})$. The map

$$\begin{array}{rcl} F\colon \operatorname{Ob}(\mathcal{D}) & \to & \operatorname{Ob}(\mathcal{S}et) \\ X & \mapsto & \operatorname{Mor}_{\mathcal{D}}(X, W) \end{array}$$

together with the maps

$$\begin{array}{rcl} \operatorname{Mor}_{\mathcal{D}}(X, Y) & \to & \operatorname{Mor}_{\mathcal{S}et}(\operatorname{Mor}_{\mathcal{D}}(Y, W), \operatorname{Mor}_{\mathcal{D}}(X, W)) \\ f & \mapsto & \left(\begin{array}{rcl} \operatorname{Mor}_{\mathcal{D}}(Y, W) & \to & \operatorname{Mor}_{\mathcal{D}}(X, W) \\ g & \mapsto & g \circ f \end{array} \right) \end{array}$$

is a contravariant functor from the category $\mathcal{D}$ to the category $\mathcal{S}et$ of sets.

Proof. Again the statement follows easily from the axioms of a category. ∎

Example. The Fixed Target-Contravariant Functor Lemma 3.8 can be viewed as the mother of all contravariant functors. As in the case of the Fixed Domain-Covariant Functor Lemma 3.7 it spawns many variations if $\operatorname{Mor}_{\mathcal{D}}(X, W)$ has more structure than just being a set.

(1) The map

$$\begin{array}{rcl} F\colon \mathrm{Ob}(\mathcal{Set}) & \to & \mathrm{Ob}(\mathcal{Vec}_{\mathbb{R}}) \\ X & \mapsto & \mathrm{Map}(X,\mathbb{R}) := \underbrace{\text{set of maps } X \to \mathbb{R}}_{\text{vector space in the obvious way}}, \end{array}$$

together with the maps

$$\begin{array}{rcl} \mathrm{Mor}_{\mathcal{Set}}(X,Y) & \to & \mathrm{Mor}_{\mathcal{Vec}_{\mathbb{R}}}(\mathrm{Map}(Y,\mathbb{R}),\mathrm{Map}(X,\mathbb{R})) \\ (\phi\colon X \to Y) & \mapsto & \begin{pmatrix} \mathrm{Map}(Y,\mathbb{R}) & \to & \mathrm{Map}(X,\mathbb{R}) \\ f & \mapsto & f\circ\phi \end{pmatrix} \end{array}$$

is a contravariant functor from the category $\mathcal{Set}$ of sets to the category $\mathcal{Vec}_{\mathbb{R}}$ of real vector spaces.

(2) The map

$$\begin{array}{rcl} F\colon \mathrm{Ob}(\mathcal{Vec}_{\mathbb{R}}) & \to & \mathrm{Ob}(\mathcal{Vec}_{\mathbb{R}}) \\ V & \mapsto & V^* := \mathrm{Hom}_{\mathbb{R}}(V,\mathbb{R}) =: \text{dual vector space of } V, \end{array}$$

together with the maps

$$\begin{array}{rcl} \mathrm{Mor}_{\mathcal{Vec}_{\mathbb{R}}}(U,V) & \to & \mathrm{Mor}_{\mathcal{Vec}_{\mathbb{R}}}(V^*,U^*) \\ (\phi\colon U \to V) & \mapsto & \begin{pmatrix} \mathrm{Hom}_{\mathbb{R}}(V,\mathbb{R}) & \to & \mathrm{Hom}_{\mathbb{R}}(U,\mathbb{R}) \\ f & \mapsto & f\circ\phi \end{pmatrix} \end{array}$$

is a contravariant functor from the category $\mathcal{Vec}_{\mathbb{R}}$ of real vector spaces to itself. □

3.7. **Natural transformations.** In this section we introduce the notion of a natural transformation between two functors. This topic might eventually look even more esoteric, but once one keeps one's eyes open one realizes that the mathematical world is stuffed with natural transformations and natural isomorphisms. They just suffer from the sad fate that often they do not recognized as such.

After this long preamble, let us give the definition of natural transformations.

Definition. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories and let $F, G\colon \mathcal{C} \to \mathcal{D}$ be two covariant functors.

(1) A natural transformation between the functors F and G assigns to each $X \in \mathrm{Ob}(\mathcal{C})$ a morphism $\Phi_X\colon F(X) \to G(X)$ in $\mathcal{D}$ such that for each morphism $\mu\colon X \to Y$ in $\mathcal{C}$ the following diagram of morphisms in $\mathcal{D}$ commutes:

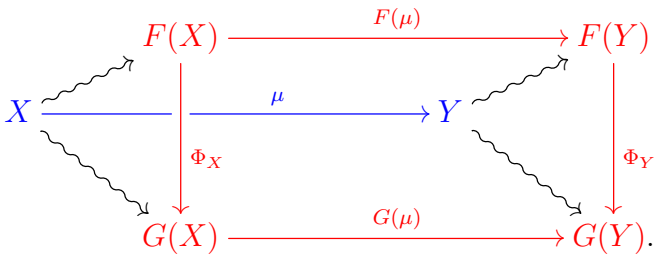
$$\begin{array}{ccc} F(X) & \xrightarrow{F(\mu)} & F(Y) \\ {\scriptstyle \Phi_X}\downarrow & & \downarrow{\scriptstyle \Phi_Y} \\ G(X) & \xrightarrow{G(\mu)} & G(Y). \end{array}$$

(2) If for every $X \in \mathrm{Ob}(\mathcal{C})$ the morphism $\Phi_X\colon F(X) \to G(X)$ is an isomorphism, then we refer to the natural transformation as a natural isomorphism. In this case, given $X \in \mathrm{Ob}(\mathcal{C})$ we often say that $F(X)$ and $G(X)$ are naturally isomorphic.[53]

The same way we can of course also define the notion of a natural transformation and a natural isomorphism between two contravariant functors.

Remark. The notion of a natural transformation can also be summarized in the following diagram:

[53] In the literature the words "natural isomorphism" and "canonical isomorphism" are often used indiscriminately, but whereas the expression "natural isomorphism" has a precise meaning, the expression "canonical



Here the wiggly arrows are not morphisms but they are just meant to remind us which objects a functor associates to the objects X and Y. □

Now we will see that we have encountered natural transformations and natural isomorphisms much more often than one might have initially thought.

Example.

(1) Let $\mathbb{K}$ be a field. We consider the category $\mathcal{V}ec_{\mathbb{K}}$ of vector spaces over $\mathbb{K}$. Given a $\mathbb{K}$-vector space V we write $V^* := \mathrm{Hom}_{\mathbb{K}}(V, \mathbb{K})$ and given $f\colon V \to W$ we denote by $f^*\colon W^* \to V^*$ the induced map. We consider the following two covariant functors from $\mathcal{V}ec_{\mathbb{K}}$ to $\mathcal{V}ec_{\mathbb{K}}$:

$$\begin{array}{rcl} \mathrm{id}\colon V & \mapsto & V \\ (f\colon V \to W) & \mapsto & (f\colon V \to W) \end{array} \quad \text{and} \quad \underbrace{\begin{array}{rcl} F\colon V & \mapsto & (V^*)^* \\ (f\colon V \to W) & \mapsto & ((f^*)^*\colon (V^*)^* \to (W^*)^*). \end{array}}_{\text{this is a covariant functor}}$$

For each $\mathbb{K}$-vector space V we denote by Φ_V the homomorphism

$$\begin{array}{rcl} V & \to & (V^*)^* \\ v & \mapsto & \left(\begin{array}{rcl} V^* & \to & \mathbb{K} \\ (\varphi\colon V \to \mathbb{K}) & \mapsto & \varphi(v) \end{array} \right). \end{array}$$

It is straightforward to verify that these maps define a natural transformation from the identity functor $\mathrm{id}\colon \mathcal{V}ec_{\mathbb{K}} \to \mathcal{V}ec_{\mathbb{K}}$ to the functor $F\colon \mathcal{V}ec_{\mathbb{K}} \to \mathcal{V}ec_{\mathbb{K}}$. If we restrict ourselves to the category $\mathcal{V}ec_{\mathbb{K}}^{\dim<\infty}$ of finite-dimensional $\mathbb{K}$-vector spaces, then the above maps show that any finite-dimensional $\mathbb{K}$-vector space V is naturally isomorphic to its double dual $(V^*)^*$.

(2) We consider the covariant functors

$$\begin{array}{rcl} F\colon \mathcal{R}ing & \to & \mathcal{R}ing \\ R & \mapsto & \mathbb{Z} \\ (f\colon R \to S) & \mapsto & (\mathrm{id}\colon \mathbb{Z} \to \mathbb{Z}) \end{array} \quad \text{and} \quad \begin{array}{rcl} \mathrm{id}\colon \mathcal{R}ing & \to & \mathcal{R}ing \\ R & \mapsto & R \\ (f\colon R \to S) & \mapsto & (f\colon R \to S). \end{array}$$

For each commutative ring R we consider the ring homomorphism

$$\begin{array}{rcl} \Phi_R\colon \mathbb{Z} & \to & R \\ n & \mapsto & n \cdot 1_R. \end{array}$$

By our definition of a ring homomorphism it follows that for each ring homomorphism $f\colon R \to S$ we have $f(1_R) = 1_S$. This implies that the following diagram commutes:

isomorphism" is not a well-defined mathematical concept. However, in the lecture notes we will on occasions write that "some object exists canonically". This means that in the proof we show the existence of the object by giving an explicit construction that does not rely on any choices.

$$\begin{array}{ccc} F(R) = \mathbb{Z} & \xrightarrow{F_*(f)=\mathrm{id}} & \mathbb{Z} = F(S) \\ {\scriptstyle \Phi_R}\downarrow & & \downarrow{\scriptstyle \Phi_S} \\ \mathrm{id}(R) = R & \xrightarrow{\mathrm{id}_*(f)=f} & S = \mathrm{id}(S). \end{array}$$

But this shows that the ring homomorphisms Φ_R define a natural transformation from the covariant functor F to the identity functor id. Thus, given a commutative ring, we refer to this ring homomorphism $\mathbb{Z} \to R$ as the *natural ring homomorphism.* □

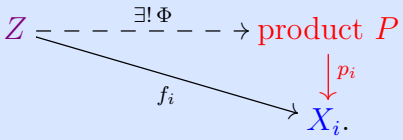
CHAPTER 4

Products and Coproducts – Limits and colimits

In this chapter we will continue with our general discussion of categories. First we will introduce the notion of products and coproducts. Afterwards we will discuss the even more general notions of limits and colimits. These notions will guide us in our exploration of topological spaces, which we will resume in the next chapter.

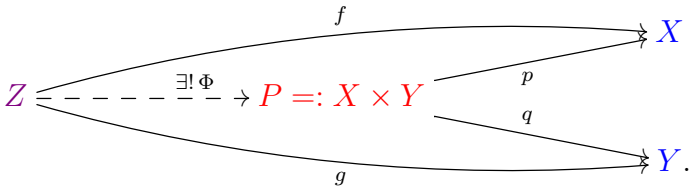
4.1. Products.

Definition. Let $\mathcal{C}$ be a category and let $\{X_i\}_{i\in I}$ be a family of objects in $\mathcal{C}$. A product of $\{X_i\}_{i\in I}$ in the category $\mathcal{C}$ is an object $P \in \mathrm{Ob}(\mathcal{C})$ together with a family of morphisms $\{p_i \colon P \to X_i\}_{i\in I}$ such that the following universal property is satisfied: Whenever we are given a $Z \in \mathrm{Ob}(\mathcal{C})$ and a family of morphisms $\{f_i \colon Z \to X_i\}_{i\in I}$ there exists a unique morphism $\Phi \colon Z \to P$ such that for each $i \in I$ we have $f_i = p_i \circ \Phi$, i.e. such that the following diagram commutes:[54]



Example. Let $\mathcal{C}$ be a category.

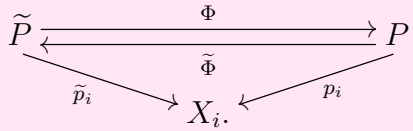
(1) Let $X, Y \in \mathrm{Ob}(\mathcal{C})$ be two objects. A product of X and Y in the category $\mathcal{C}$ is some object $P \in \mathrm{Ob}(\mathcal{C})$ together with two morphisms $p \colon P \to X$ and $q \colon P \to Y$ such that the following universal property is satisfied: Whenever we are given an object $Z \in \mathrm{Ob}(\mathcal{C})$ and two morphisms $f \colon Z \to X$ and $g \colon Z \to Y$ there exists a unique morphism $\Phi \colon Z \to P$ such that $f = p \circ \Phi$ and $g = q \circ \Phi$. The situation is summarized in the following diagram:



(2) If we take the empty family of objects in a category $\mathcal{C}$, then it follows easily from the definitions that a corresponding product is the same as a terminal object in the category $\mathcal{C}$. □

Lemma 4.1. (Product Uniqueness Lemma) Let $\mathcal{C}$ be a category and let $\{X_i\}_{i\in I}$ be a family of objects in $\mathcal{C}$. If we are given products $(P, \{p_i \colon P \to X_i\}_{i\in I})$ and $(\widetilde{P}, \{\widetilde{p}_i \colon \widetilde{P} \to X_i\}_{i\in I})$ then there exist uniquely determined isomorphisms $\Phi \colon \widetilde{P} \to P$ and $\widetilde{\Phi} \colon P \to \widetilde{P}$

which are inverses of one another and such that for each $i \in I$ the following diagram commutes:



Sketch of proof. The two morphisms $\Phi\colon \widetilde{P} \to P$ and $\widetilde{\Phi}\colon P \to \widetilde{P}$ come from the universal properties of P and $\widetilde{P}$. By comparing $\mathrm{id}_P\colon P \to P$ and $\Phi \circ \widetilde{\Phi}\colon P \to P$ one sees, by the uniqueness property, that $\mathrm{id}_P = \Phi \circ \widetilde{\Phi}$ and similarly one sees that $\mathrm{id}_{\widetilde{P}} = \widetilde{\Phi} \circ \Phi$, i.e. Φ and $\widetilde{\Phi}$ are isomorphisms. ■

Notation. Let $\mathcal{C}$ be a category. Let $\{X_i\}_{i\in I}$ be a family of objects in $\mathcal{C}$ such that a product exists.

(1) We usually denote the product by $\prod_{i\in I} X_i$ and we refer to the morphisms $\prod_{i\in I} X_i \to X_i$ as natural projections. Furthermore, given morphisms $f_i\colon Z \to X_i$ we denote the corresponding morphism $Z \to \prod_{i\in I} X_i$ by $\prod_{i\in I} f_i$.

(2) If $I = \{1, \dots, k\}$, then often we denote the product by $X_1 \times \cdots \times X_k$. Furthermore given morphisms $f_i\colon Z \to X_i$, $i = 1, \dots, k$ we denote the corresponding morphism $Z \to X_1 \times \cdots \times X_k$ by $(f_1, \dots, f_k)$.

Lemma 4.2. (Product-of-Sets-and-Groups Lemma)

(1) Let $\mathcal{S}et$ be the category of sets and let $\{X_i\}_{i\in I}$ be a family of sets. The product in $\mathcal{S}et$ is precisely the product

$$\prod_{i\in I} X_i \;:=\; \left\{ f\colon I \to \bigcup_{i\in I} X_i \;\middle|\; \text{for every } i \in I \text{ we have } f(i) \in X_i \right\}$$

as defined on page 36, together with the obvious maps to the X_i that are given by $f \mapsto f(i)$.

(2) Let $\mathcal{G}roup$ be the category of groups and let $\{X_i\}_{i\in I}$ be a family of groups. If I is non-empty, then the product in $\mathcal{G}roup$ is the set

$$\prod_{i\in I} X_i \;=\; \left\{ f\colon I \to \bigcup_{i\in I} X_i \;\middle|\; \text{for every } i \in I \text{ we have } f(i) \in X_i \right\}$$

as in (1), which we now view as a group by the obvious "entry wise" multiplication, together with the obvious homomorphisms to X_i. If I is empty, then the product is defined to be the trivial group.

(3) The same approach as in (2) also applies to the category $\mathcal{A}b\mathcal{G}roup$ of abelian groups and the category $\mathcal{V}ec_{\mathbb{K}}$ of vector spaces over a fixed field $\mathbb{K}$.

Proof.

(1) So let $Z \in \mathrm{Ob}(\mathcal{S}et)$ and let $\{f_i\colon Z \to X_i\}_{i\in I}$ be a family of maps between sets. We define

$$\begin{array}{rcl} \Phi\colon Z & \to & \prod_{i\in I} X_i \\ z & \mapsto & \begin{pmatrix} I & \to & \bigcup_{i\in I} X_i \\ i & \mapsto & f_i(z) \end{pmatrix}. \end{array}$$

It follows immediately from the definition that for each $i \in I$ we have $f_i = p_i \circ \Phi$. It remains to show that Φ is unique. So let $\Psi\colon Z \to \prod_{i\in I} X_i$ be a continuous map such that for each $i \in I$ we have $f_i = p_i \circ \Psi$. We need to show that $\Phi = \Psi$, i.e. we need to show that for each $z \in Z$ and $i \in I$ we have $\Phi(z)(i) = \Psi(z)(i) \in X_i$. But this follows from the observation that $\Phi(z)(i) = p_i(\Phi(z)) = f_i(z) = p_i(\Psi(z)) = \Psi(z)(i) \in X_i$.

(2) If I is non-empty, then the same argument as in (1) also works for the category *Group* of groups. If I is empty, then the trivial group is a group[55], and clearly has the desired properties.

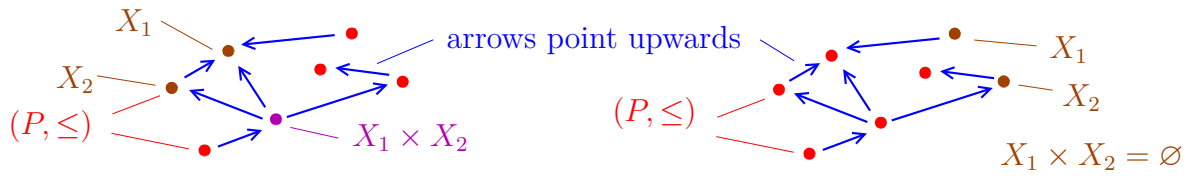
(3) The argument of (2) also works in (3). ■

After these examples one might assume that one gets the hang of the product in categories. The following examples might rob the reader of this illusion.

Example.

(1) Let $(P, \leq)$ be a partially ordered set. We consider the corresponding order category *Order*$_{(P,\leq)}$ as defined on page 90. Given a family $\{X_i\}_{i\in I}$ of objects in *Order*$_{(P,\leq)}$, i.e. given a family $\{X_i\}_{i\in I}$ of elements of P, one can easily show that[56]

$$\prod_{i\in I} X_i := \begin{cases} \inf(\{X_i\}_{i\in I}), & \text{i.e. a maximal element among all lower bounds on } \{X_i\}_{i\in I} \\ \varnothing, & \text{if no such infimum exists.} \end{cases}$$



(2) We introduce a new category. A pointed set is a pair (X, x_0), where X is a set and x_0 is a point in X. We refer to the category $\mathcal{P}\mathit{Set}$ with

$$\begin{aligned} \mathrm{Ob}(\mathcal{P}\mathit{Set}) &:= \text{all pointed sets,} \\ \mathrm{Mor}((X, x_0), (Y, y_0)) &:= \text{all maps } f \text{ from } X \text{ to } Y \text{ with } f(x_0) = y_0 \end{aligned}$$

with the usual composition of maps as the category $\mathcal{P}\mathit{Set}$ of pointed sets. Now let (X, x_0) and (Y, y_0) be two pointed sets. It is an instructive exercise to show that the product of (X, x_0) and (Y, y_0) in the category $\mathcal{P}\mathit{Set}$ is given by the set

$$(X \times Y)/\underbrace{((X \times \{y_0\}) \cup (\{x_0\} \times Y))}_{\substack{\text{the points in this subset get} \\ \text{identified to a single point } *}} \quad \text{together with the point} \quad *$$

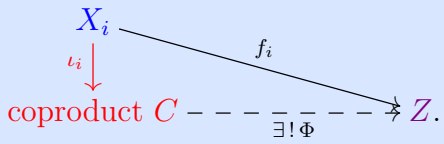
and the obvious morphisms to (X, x_0) and (Y, y_0). □

4.2. **Coproducts.** We now reverse the direction of the morphisms in the definition of a product and we obtain a new interesting definition.

[55]This explains the divergence from the category of sets: For $I = \varnothing$, the product is the empty set, but the empty set is not a group.

[56]In fact it is a very good exercise to provide a proof.

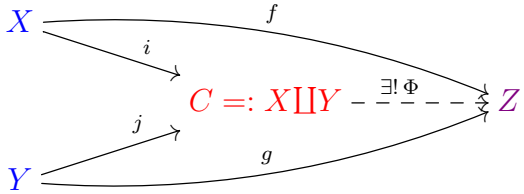
Definition. Let $\mathcal{C}$ be a category and let $\{X_i\}_{i\in I}$ be a family of objects in $\mathcal{C}$. A coproduct of $\{X_i\}_{i\in I}$ in the category $\mathcal{C}$ is an object $C \in \mathrm{Ob}(\mathcal{C})$ together with a family of morphisms $\{\iota_i \colon X_i \to C\}_{i\in I}$ such that the following property is satisfied: If we are given a $Z \in \mathrm{Ob}(\mathcal{C})$ and a family of morphisms $\{f_i \colon X_i \to Z\}_{i\in I}$ there exists a unique morphism $\Phi \colon C \to Z$ such that for each $i \in I$ we have $f_i = \Phi \circ \iota_i$, i.e. such that the following diagram commutes:[57]



Remark. Note that by definition coproducts in $\mathcal{C}$ are products in the opposite category $\mathcal{C}^{\mathrm{op}}$. So we did not introduce a new concept at all. □

Example. Let $\mathcal{C}$ be a category.

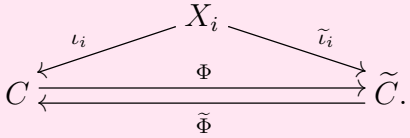
(1) Let $X, Y \in \mathrm{Ob}(\mathcal{C})$ be two objects. A coproduct of X and Y in the category $\mathcal{C}$ is some object $C \in \mathrm{Ob}(\mathcal{C})$ together with two morphisms $i \colon X \to C$ and $j \colon Y \to C$ such that the following universal property is satisfied: Whenever we are given an object $Z \in \mathrm{Ob}(\mathcal{C})$ and two morphisms $f \colon X \to Z$ and $g \colon Y \to Z$ there exists a unique morphism $\Phi \colon C \to Z$ such that $f = \Phi \circ i$ and $g = \Phi \circ j$. The situation is summarized in the following diagram:



(2) If we take the empty family of objects in $\mathcal{C}$, then it follows easily from the definitions that the corresponding coproduct is an initial object in the category $\mathcal{C}$.

(3) Basically by definition coproducts in a category $\mathcal{C}$ correspond to products in the opposite category $\mathcal{C}^{\mathrm{op}}$. □

Lemma 4.3. (Coproduct Uniqueness Lemma) Let $\mathcal{C}$ be a category and let $\{X_i\}_{i\in I}$ be a family of objects in $\mathcal{C}$. If we are given two coproducts $(C, \{\iota_i \colon X_i \to C\}_{i\in I})$ and $(\widetilde{C}, \{\widetilde{\iota}_i \colon X_i \to \widetilde{C}\}_{i\in I})$, then there exist unique isomorphisms $\Phi \colon C \to \widetilde{C}$ and $\widetilde{\Phi} \colon \widetilde{C} \to C$ which are inverses of one another and such that for each $i \in I$ the following diagram commutes:



Proof. The lemma is just a reformulation of the Product Uniqueness Lemma 4.1 since coproducts in $\mathcal{C}$ are products in the opposite category $\mathcal{C}^{\mathrm{op}}$. Alternatively the statement also follows easily from the definitions. ■

Notation. Let $\mathcal{C}$ be a category. Let $\{X_i\}_{i\in I}$ be a family of objects in $\mathcal{C}$ such that a coproduct exists. Often[58] we denote the coproduct by $\coprod_{i\in I} X_i$ and we refer to the morphisms $X_i \to \coprod_{i\in I} X_i$ as natural injections.[59]

Lemma 4.4. (Coproduct-of-Sets-and-Abelian Groups Lemma)

(1) Let $\mathcal{Set}$ be the category of sets and let $\{X_i\}_{i\in I}$ be a family of sets. The coproduct in $\mathcal{Set}$ is the disjoint union

$$\coprod_{i\in I} X_i \coloneqq \bigcup_{i\in I} (X_i \times \{i\}),$$

as defined on page 37, together with the natural injections of the X_i.

(2) Let $\mathcal{AbGroup}$ be the category of *abelian* groups and let $\{X_i\}_{i\in I}$ be a family of groups. The coproduct in $\mathcal{AbGroup}$ is given by

$$\bigoplus_{i\in I} X_i \coloneqq \left\{ (g_i)_{i\in I} \in \prod_{i\in I} X_i \,\middle|\, \text{all but finitely many } g_i \text{ are trivial} \right\}$$

and the homomorphisms

$$\begin{array}{rcl} X_i & \to & \bigoplus_{i\in I} X_i \\ x & \mapsto & \left(\begin{array}{rcl} I & \to & \bigcup_{i\in I} X_i \\ j & \mapsto & \begin{cases} x_i, & \text{if } i = j, \\ 0_{X_j}, & \text{if } i \neq j. \end{cases} \end{array} \right) \end{array}$$

(3) The same approach as in (2) also applies to the category $\mathcal{Vec}_{\mathbb{K}}$ of vector spaces over a fixed field $\mathbb{K}$.

Notation. Let $\{X_i\}_{i\in I}$ be objects in the category $\mathcal{AbGroup}$ of abelian groups. We write

$$\bigoplus_{i\in I} X_i \coloneqq \left\{ (g_i)_{i\in I} \in \prod_{i\in I} X_i \,\middle|\, \text{all but finitely many } g_i \text{ are trivial} \right\}$$

and we refer to it as the direct sum of the X_i. We proceed similarly with objects in the category $\mathcal{Vec}_{\mathbb{K}}$ of vector spaces over a fixed field $\mathbb{K}$.

Proof.

(1) So let $Z \in \mathrm{Ob}(\mathcal{Set})$ and let $\{f_i \colon X_i \to Z\}_{i\in I}$ be a family of maps between sets. We define

$$\begin{array}{rcl} \Phi \colon \coprod_{i\in I} X_i & \to & Z \\ (z_i, i) & \mapsto & f_i(z_i). \end{array}$$

It follows immediately from the definition that for each $i \in I$ we have $f_i = \Phi \circ \iota_i$. It remains to show that Φ is unique. So let $\Psi \colon \coprod_{i\in I} X_i \to Z$ be a continuous map such that for each $i \in I$ we have $f_i = \Psi \circ \iota_i$. We need to show that $\Phi = \Psi$, i.e. we need to show that for each $i \in I$ and each $x_i \in X_i$ we have $\Phi((x_i, i)) = \Psi((x_i, i)) \in Z$. But this follows from the observation that $\Phi((x_i, i)) = \Phi(\iota_i(x_i)) = f_i(z) = \Psi(\iota_i(x_i)) = \Psi((x_i, i)) \in Z$.

[58] In fact, as we will see shortly, in many settings we will see that the coproduct already has an established different notation.

[58] The name *natural injection* is somewhat unfortunate since in general for a morphism it makes no sense to say that it is injective. Even if there is a reasonable notion of injective, the natural injection does not need to be an injection. On the other hand, in all examples we ever care about the natural injection will be an injective map.

(2) Let a $Z \in \mathrm{Ob}(\mathcal{AbGroup})$ and let $\{f_i \colon X_i \to Z\}_{i\in I}$ be a family of homomorphisms. We set[59]

$$\begin{aligned}\Phi \colon \left\{(g_i)_{i\in I} \in \prod_{i\in I} X_i \,\middle|\, \text{all but finitely many } g_i \text{ are trivial}\right\} &\to Z \\ (g_i)_{i\in I} &\mapsto \sum_{\substack{i\in I \text{ with} \\ g_i \neq e}} f_i(g_i).\end{aligned}$$

Note that this map makes sense since we take the product only of finitely elements of Z. Furthermore note that this map is a homomorphism since we work with the category $\mathcal{AbGroup}$ of abelian groups.

It follows immediately from the definition that for each $i \in I$ we have $f_i = \Phi \circ \iota_i$. It remains to show that Φ is unique. So let Ψ be another homomorphism such that for each $i \in I$ we have $f_i = \Psi \circ \iota_i$. We need to show that $\Phi = \Psi$. So suppose that we are given $(g_i)_{i\in I} \in \prod_{i\in I} X_i$ such that there exist only finitely many $i_1, \dots, i_k \in I$ such that the corresponding group element is non-trivial. Then

$$\Phi((g_i)_{i\in I}) = \Phi\Big(\sum_{r=1}^{k} \iota_{i_r}(g_{i_r})\Big) = \sum_{r=1}^{k} \Phi(\iota_{i_r}(g_{i_r})) = \sum_{r=1}^{k} f_{i_r}(g_{i_r}) \underset{\text{same argument backwards}}{\uparrow} = \Psi((g_i)_{i\in I}).$$

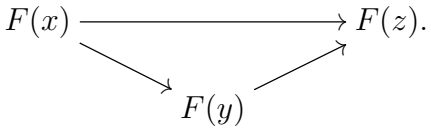
(3) This statement is proved the same way as we proved (2). ■

4.3. **Limits.** In this section and the following section we introduce the very general notion of a limit and of a colimit, which in particular generalize the notion of a product and a coproduct. Before we can introduce these new concepts we need to provide the following definition:

Definition. Let $(P, \leq)$ be a partially ordered set and let $\mathcal{C}$ be a category. A diagram in $\mathcal{C}$ over $(P, \leq)$ is a covariant functor from the order category $\mathcal{Order}_{(P,\leq)}$ to $\mathcal{C}$.

Remark. In plain English a diagram F in a category $\mathcal{C}$ over a partially ordered set $(P, \leq)$ is the following data:

(1) for each $x \in P$ we are given an object $F(x) \in \mathrm{Ob}(\mathcal{C})$,
(2) for each $x < y$ we are given a morphism $F(x) \to F(y)$,[60]
(3) if $x < y < z$, then the following diagram needs to commute:



□

Example.

(1) We consider the partially ordered set $(P, \leq_{\mathrm{tr}})$ that is given by a set P with the trivial partial order $\leq_{\mathrm{tr}}$, i.e. $x \leq_{\mathrm{tr}} y$ if and only if $x = y$. In this setting a diagram over $(P, \leq_{\mathrm{tr}})$ is the same data as a family of objects $\{X_i\}_{i\in P}$.

[59] We use addition notation for the abelian groups.

[60] Note that for $x \in P$ we have $x \leq x$ and the morphism $* \in \mathrm{Mor}(x, x)$ is the identity morphism. Since F is a covariant functor we see that $F(*) = \mathrm{id}_{F(x)}$. This is the reason why we only need to specify morphisms for $x < y$.

(2) We consider the partially ordered set $(\mathbb{N}, \leq)$. In this setting a diagram over $(\mathbb{N}, \leq)$ is the same data as a sequence

$$X_1 \xrightarrow{\varphi_1} X_2 \xrightarrow{\varphi_2} X_3 \xrightarrow{\varphi_3} \dots$$

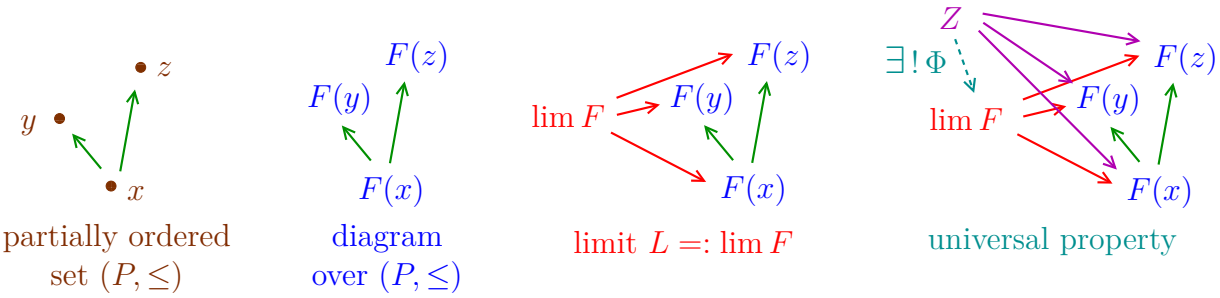
of morphisms $\varphi_i \colon X_i \to X_{i+1}$. For $i < j$ the morphisms $X_i \to X_j$ are understood to be given by $\varphi_{j-1} \circ \dots \circ \varphi_{i+1} \circ \varphi_i$. □

In the following we will introduce the notion of a limit and of a pullback. For the time being we will not make use of these notions. We only include them here for completeness. We will make use of the opposite notions of a colimit and a pushout, that we will introduce in the next section.

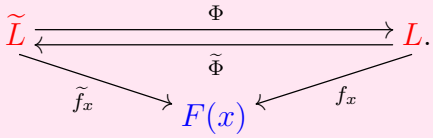
Definition. Let $(P, \leq)$ be a partially ordered set, let $\mathcal{C}$ be a category and let F be a diagram in $\mathcal{C}$ over $(P, \leq)$. A limit of F consists of an object $L \in \mathrm{Ob}(\mathcal{C})$ together with morphisms $f_x \colon L \to F(x)$ for each $x \in P$ such that the following two conditions are satisfied:

(1) For every $x \leq y$ we have $f_y = F(x \to y) \circ f_x \colon L \to F(y)$.
(2) The following universal property holds: If Z is an object in $\mathcal{C}$ and if we are given morphisms $g_x \colon Z \to F(x)$ for each $x \in P$ such that (1) is satisfied, then there exists a unique morphism $\Phi \colon Z \to L$ such that $g_x = f_x \circ \Phi$ for all $x \in P$.

If a limit exists, then often we denote it by $\lim F$. (We will see in the Limit Uniqueness Lemma 4.5 below that limits, if they exist, are essentially unique.)



Lemma 4.5. (Limit Uniqueness Lemma) Let $(P, \leq)$ be a partially ordered set, let $\mathcal{C}$ be a category and let F be a diagram in $\mathcal{C}$ over $(P, \leq)$. Suppose that we are given two limits $(L, \{f_x \colon L \to F(x)\}_{x \in P})$ and $(\widetilde{L}, \{\widetilde{f}_x \colon \widetilde{L} \to F(x)\}_{x \in P})$. Then there exist uniquely determined isomorphisms $\Phi \colon \widetilde{L} \to L$ and $\widetilde{\Phi} \colon L \to \widetilde{L}$ which are inverses of one another and such that for each $x \in P$ the following diagram commutes:



Proof. As in the case of the Product Uniqueness Lemma 4.1 the statement follows easily from the definitions. ■

Example.

(1) If we consider a trivial partially ordered set $(P, \leq_{\mathrm{tr}})$, then the limit is by definition precisely the product in the sense of the definition on page 101.

(2) We consider $(\mathbb{N}, \leq)$ and we consider a diagram in the category $\mathcal{Set}$ of sets where each $\varphi_i \colon X_i \to X_{i+1}$ is an inclusion of subsets of a given set Y. One can easily show that in this case the limit is the first set X_1. □

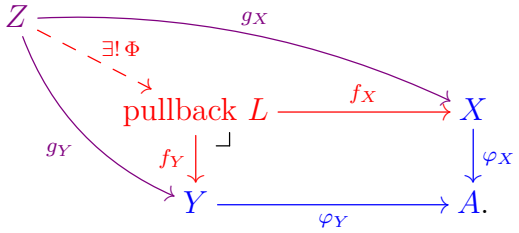
We conclude our discussion of limits with a particularly frequently used special case.

Definition. We consider the set $P = \{0, x, y\}$ with the partial order that is given by the trivial relations $0 \leq 0, x \leq x, y \leq y$ and the non-trivial relations $x \leq 0$ and $y \leq 0$:[61]

the partially ordered set $\begin{array}{ccc} x & & \\ \downarrow & & \\ y & \longrightarrow & 0 \end{array}$ gives rise to the diagram[62] $\begin{array}{ccc} X & & \\ \downarrow{\scriptstyle \varphi_X} & & \\ Y & \xrightarrow{\varphi_Y} & A \end{array}$ the corresponding limit is called a pullback $\begin{array}{ccc} \text{pullback } L & \xrightarrow{f_X} & X \\ {\scriptstyle f_Y}\downarrow & \lrcorner & \downarrow{\scriptstyle \varphi_X} \\ Y & \xrightarrow[\varphi_Y]{} & A. \end{array}$

Here the symbol "⌟" indicates that L is the pullback of the "⌟-shaped" diagram.

Remark. The following diagram summarizes the definition of a pullback:



(1) We start out with a pullback diagram.
(2) The pullback L comes with morphisms $f_X \colon L \to X$ and $f_Y \colon L \to Y$ such that the rectangle commutes.
(3) If we are given an object Z of $\mathcal{C}$ with morphisms $g_X \colon Z \to X$ and $g_Y \colon Z \to Y$ such that the big outer quadrilateral commutes, then there exists a unique morphism $\Phi \colon Z \to L$ that makes the diagram commute. □

Example. We consider the following pullback diagram in the category $\mathcal{Group}$ of groups:

$\begin{array}{ccc} G & & \\ \downarrow{\scriptstyle \varphi} & & \\ \{e\} & \longrightarrow & H \end{array}$ one can easily show that the corresponding pullback is given by $\begin{array}{ccc} \ker(\varphi) & \longrightarrow & G \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \varphi} \\ \{e\} & \longrightarrow & H \end{array}$ □

Lemma 4.6. (Pullback Lemma) Let $\mathcal{C}$ be a category.

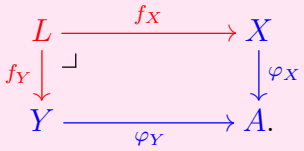
(1) Suppose that $\mathcal{C}$ admits a terminal object $*$. Then the pullback of the diagram

$\begin{array}{ccc} X \times Y & \xrightarrow{p_X} & X \\ {\scriptstyle p_Y}\downarrow & \lrcorner & \downarrow{\scriptstyle \varphi_X} \\ Y & \xrightarrow[\varphi_Y]{} & * \end{array}$

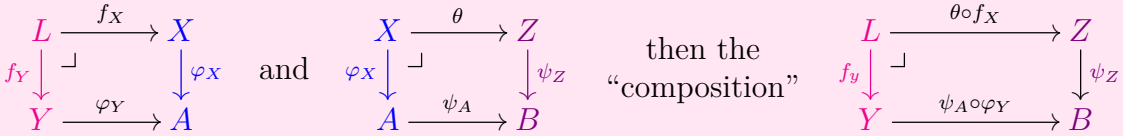
is given by the product $X \times Y$.

[61] We do not need to specify the morphisms from the pullback L to A since it is agrees with $\varphi_X \circ f_X = \varphi_Y \circ f_Y$.
[62] In contrast to the notation above we now write φ_X, φ_Y instead of φ_x, φ_y, since we feel that in this setting this notation is more natural.

(2) Let



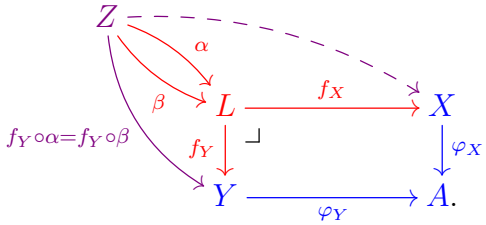
be a pullback.
(a) If φ_X is an isomorphism, then so is the "opposite" morphism f_Y.
(b) If φ_X is a monomorphism, then so is the "opposite" morphism f_Y.
The analogous statements of course also hold with the roles of X and Y swapped.
(3) If we are given pullbacks



is also a pullback.

Proof.

(1) This statement follows easily from the definitions.
(2) (a) This statement also follows easily from the definitions.
(b) We assume that φ_X is a monomorphism. We need to show that f_Y is also a monomorphism. Thus let $\alpha\colon Z \to L$ and $\beta\colon Z \to L$ be two morphisms such that $f_Y \circ \alpha = f_Y \circ \beta$.



We need to show that $\alpha = \beta$. We proceed as follows:
(i) We assume that $f_Y \circ \alpha = f_Y \circ \beta$.
(ii) By (i) we also have $\varphi_Y \circ f_Y \circ \alpha = \varphi_Y \circ f_Y \circ \beta$.
(iii) Since the diagram commutes we get $\varphi_X \circ f_X \circ \alpha = \varphi_X \circ f_X \circ \beta\colon Z \to A$.
(iv) Since φ_X is a monomorphism we see that $f_X \circ \alpha = f_X \circ \beta\colon Z \to X$.
(v) Since we dealing with a pullback there exists a *unique* morphism $\Phi\colon Z \to L$ such that $f_Y \circ \Phi = f_Y \circ \alpha = f_Y \circ \beta$ and such that $f_X \circ \Phi = f_X \circ \alpha = f_X \circ \beta$.
(vi) From (i), (iv) and (v) we obtain that $\alpha = \Phi = \beta$.
(3) This statement follows from purely formal arguments which do not become clearer by writing them down. Thus we leave it to the reader to go through the argument. ■

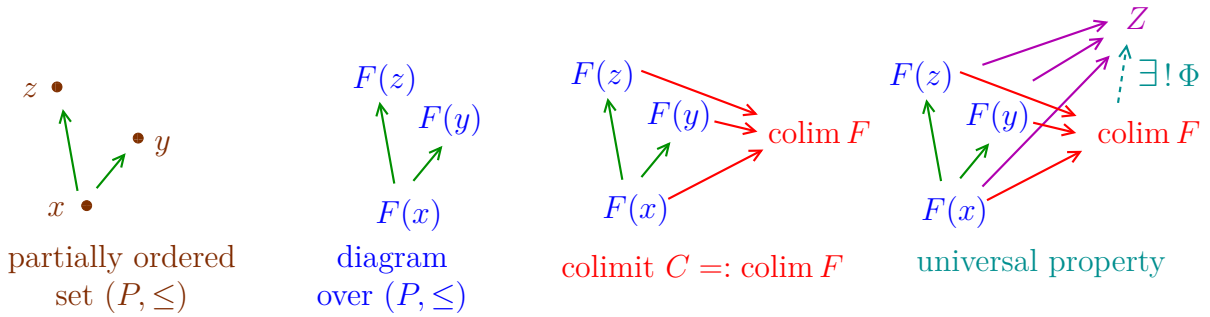
4.4. **Colimits.** In this section we introduce the notion of a colimit, which is very similar to the notion of a limit, except that the directions of the morphisms are reversed.

[62]It is a "meta theorem" that in many categories two morphisms in a pullback diagram, that are opposite of one another, have similar properties.

Definition. Let $(P, \leq)$ be a partially ordered set, let $\mathcal{C}$ be a category and let F be a diagram in $\mathcal{C}$ over $(P, \leq)$. A colimit of F consists of an object $C \in \mathrm{Ob}(\mathcal{C})$ together with morphisms $f_x \colon F(x) \to C$ for each $x \in P$ such that the following two conditions are satisfied:

(1) For every $x \leq y$ we have $f_x = f_y \circ F(x \to y) \colon F(x) \to C$.
(2) The following universal property holds: If Z is an object in $\mathcal{C}$ and if we are given morphisms $g_x \colon F(x) \to Z$ for each $x \in P$ such that (1) is satisfied, then there exists a unique morphism $\Phi \colon C \to Z$ such that $g_x = \Phi \circ f_x$ for all $x \in P$.

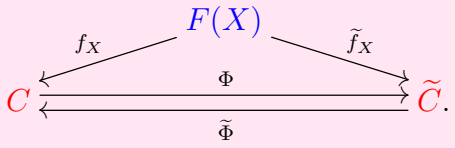
If a colimit exists, then often we denote it by $\operatorname{colim} F$.



Remark. Note that by definition colimits in $\mathcal{C}$ are limits in the opposite category $\mathcal{C}^{\mathrm{op}}$. So we did not introduce a new concept at all. □

Not surprisingly we have the following analogue of the Limit Uniqueness Lemma 4.5.

Lemma 4.7. (Colimit Uniqueness Lemma) Let $(P, \leq)$ be a partially ordered set, let $\mathcal{C}$ be a category and let F be a diagram in $\mathcal{C}$ over $(P, \leq)$. Suppose that we are given two colimits $(C, \{f_x \colon F(x) \to C\}_{x \in P})$ and $(\widetilde{C}, \{\widetilde{f}_x \colon F(X) \to \widetilde{C}\}_{x \in P})$. Then there exist uniquely determined isomorphisms $\Phi \colon C \to \widetilde{C}$ and $\widetilde{\Phi} \colon \widetilde{C} \to C$ which are inverses of one another and such that for each $x \in P$ the following diagram commutes:



Proof. As always the proof follows easily from the definitions. ■

Example.

(1) If we consider a trivial partially ordered set $(P, \leq_{\mathrm{tr}})$, then the colimit is by definition precisely the coproduct, as defined on page 104.
(2) We consider $(\mathbb{N}, \leq)$ and we consider a diagram in the category $\mathcal{Set}$ of sets where each $\varphi_i \colon X_i \to X_{i+1}$ is an inclusion of subsets of a given set Y. We leave it to the reader to show that in this case the colimit is the union of the X_i. □

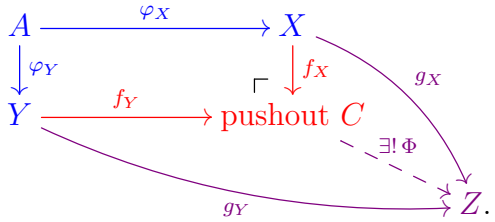
Next we introduce the analogue of the pullback.

Definition. We consider the set $P = \{0, x, y\}$ with the partial order that is given by the trivial relations $0 \leq 0, x \leq x, y \leq y$ and the non-trivial relations $0 \leq x$ and $0 \leq y$:

the partially ordered set $\begin{array}{ccc} 0 & \longrightarrow & x \\ \downarrow & & \\ y & & \end{array}$ gives rise to the diagram $\begin{array}{ccc} A & \xrightarrow{\varphi_X} & X \\ \downarrow{\scriptstyle \varphi_Y} & & \\ Y & & \end{array}$ the corresponding colimit is called a pushout $\begin{array}{ccc} A & \xrightarrow{\varphi_X} & X \\ \downarrow{\scriptstyle \varphi_Y} & \ulcorner & \downarrow{\scriptstyle f_X} \\ Y & \xrightarrow[f_Y]{} & \text{pushout } C. \end{array}$

Here the symbol "$\ulcorner$" indicates that L is the pullback of the "$\ulcorner$-shaped" diagram.

Remark. The following diagram summarizes the definition of a pushout:



(1) We start out with a pushout diagram.
(2) The pushout C comes with morphisms $f_X \colon X \to C$ and $f_Y \colon Y \to C$ such that the rectangle commutes.
(3) If we are given an object Z with morphisms $g_X \colon X \to Z$ and $g_Y \colon Y \to Z$ such that the big outer quadrilateral commutes, then there exists a unique morphism $\Phi \colon C \to Z$ that makes the diagram commute. □

Example. Let G be a group and let A be a *normal* subgroup of G. We consider the following pushout diagram in the category *Group* of groups:

$\begin{array}{ccc} A & \longrightarrow & \{e\} \\ \downarrow & & \\ G & & \end{array}$ one can easily show that the corresponding pushout is given by $\begin{array}{ccc} A & \longrightarrow & \{e\} \\ \downarrow & \ulcorner & \downarrow \\ G & \longrightarrow & G/A. \end{array}$

□

We conclude this section with the following analogue of the Pullback Lemma 4.6.

Lemma 4.8. (Pushout Lemma) Let $\mathcal{C}$ be a category.

(1) If $\mathcal{C}$ has an initial object $*$, then the pushout of

$$\begin{array}{ccc} * & \xrightarrow{\varphi_X} & X \\ \downarrow{\scriptstyle \varphi_Y} & & \downarrow{\scriptstyle \iota_X} \\ Y & \xrightarrow{\iota_Y} & X \coprod Y \end{array}$$

is given by the coproduct $X \coprod Y$.

(2) Let

$$\begin{array}{ccc} A & \xrightarrow{\varphi_X} & X \\ \downarrow{\scriptstyle \varphi_Y} & \ulcorner & \downarrow{\scriptstyle f_X} \\ Y & \xrightarrow{f_Y} & C. \end{array}$$

be a pushout.
(a) If φ_X is an isomorphism, then so is the "opposite" morphism f_Y.
(b) If φ_X is an epimorphism, then so is the "opposite" morphism f_Y.
The analogous statements of course also hold with the roles of X and Y swapped.

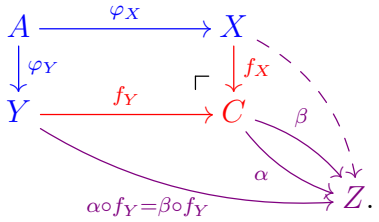
(3) If we are given pushouts

$$\begin{array}{ccc} A & \xrightarrow{\varphi_X} & X \\ {\scriptstyle \varphi_Y}\downarrow & \ulcorner & \downarrow{\scriptstyle f_X} \\ Y & \xrightarrow{f_Y} & L \end{array} \quad \text{and} \quad \begin{array}{ccc} X & \xrightarrow{\theta} & Z \\ {\scriptstyle f_X}\downarrow & \ulcorner & \downarrow{\scriptstyle g_Z} \\ L & \xrightarrow{g_L} & W \end{array} \quad \text{then the "composition"} \quad \begin{array}{ccc} A & \xrightarrow{\theta\circ\varphi_X} & Z \\ {\scriptstyle \varphi_Y}\downarrow & \ulcorner & \downarrow{\scriptstyle g_Z} \\ Y & \xrightarrow{g_L\circ f_Y} & W \end{array}$$

is also a pushout.

Proof.

(1) This statement follows easily from the definitions.
(2) (a) This statement also follows easily from the definitions.
(b) We assume that φ_X is an epimorphism. We need to show that f_Y is also an epimorphism. Let $\alpha\colon C \to Z$ and $\beta\colon C \to Z$ be two morphisms such that $\alpha \circ f_Y = \beta \circ f_Y\colon Y \to Z$.



We need to show that $\alpha = \beta$. We proceed as follows:
(i) We assume that $\alpha \circ f_Y = \beta \circ f_Y\colon Y \to Z$.
(ii) From (i) we obtain that $\alpha \circ f_Y \circ \varphi_Y = \beta \circ f_Y \circ \varphi_Y\colon A \to Z$.
(iii) Since the diagram commutes we get $\alpha \circ f_X \circ \varphi_X = \beta \circ f_X \circ \varphi_X\colon A \to Z$.
(iv) Since φ_X is an epimorphism we see that $\alpha \circ f_X = \beta \circ f_X\colon X \to Z$.
(v) Since we dealing with a pushout there exists a *unique* morphism $\Phi\colon C \to Z$ such that $\Phi \circ f_Y = \alpha \circ f_Y = \beta \circ f_Y$ and such that $\Phi \circ f_X = \alpha \circ f_X = \beta \circ f_X$.
(vi) It follows from (i), (iv) and (v) that $\alpha = \Phi = \beta$.

(3) This statement follows from purely formal arguments which do not become clearer by writing them down. Thus we leave it to the reader to go through the argument. ■

CHAPTER 5

Products and disjoint unions of topological spaces

In Chapter 4 we introduced the concepts of (co-) products and more generally (co-) limits in categories. In this and the following chapter we will see how these concepts play out in the category of topological spaces.

5.1. **The product of topological spaces.** In this section we will see that the category of topological spaces has products. To do so it is convenient to recall the following definition from page 36:

Definition. Let $\{X_i\}_{i\in I}$ be a family of sets. The product of these sets is defined as the set
$$\prod_{i\in I} X_i \coloneqq \left\{ f\colon I \to \bigcup_{i\in I} X_i \;\middle|\; \text{for every } i\in I \text{ we have } f(i)\in X_i \right\}.$$
Sometimes we also denote an element of $\prod_{i\in I} X_i$ by $(x_i)_{i\in I}$.

Proposition 5.1. (Topological-Product Proposition) Let $\{X_i\}_{i\in I}$ be a family of topological spaces. The following statements hold:

(0) The set
$$\mathcal{B} \coloneqq \left\{ \prod_{i\in I} U_i \subset \prod_{i\in I} X_i \;\middle|\; \begin{array}{ll} (1) & \text{each } U_i \text{ is open in } X_i \text{ and} \\ (2) & \text{there exist only finitely many } i\text{'s with } U_i \neq X_i \end{array} \right\}$$
has the basis property that we introduced on page 65. Thus it follows from the Topology-from-Basis Lemma 1.14 that $\mathcal{B}$ defines a topology on $\prod_{i\in I} X_i$.

(1) (a) For any $i\in I$ the projection
$$\begin{array}{rcl} p_i\colon \prod_{i\in I} X_i & \to & X_i \\ f & \mapsto & f(i) \end{array}$$
is continuous with respect to the topology from (0).

(b) Given a topological space Z and given a family $\{f_i\colon Z\to X_i\}_{i\in I}$ of continuous maps the map
$$\begin{array}{rcl} F\colon Z & \to & \prod_{i\in I} X_i \\ z & \mapsto & \begin{pmatrix} I & \to & \bigcup_{i\in I} X_i \\ i & \mapsto & f_i(z) \end{pmatrix}. \end{array}$$
is continuous with respect to the topology from (0).

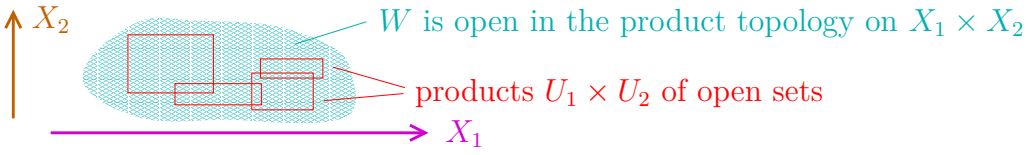
(∗) A map $F\colon Z\to \prod_{i\in I} X_i$ from some topological space Z is continuous if and only if each map $p_i\circ F\colon Z\to X_i$ is continuous.

(2) The topological space $\prod_{i\in I} X_i$ together with the natural projections $p_j\colon \prod_{i\in I} X_i \to X_j$ is a product (in the sense of the definition on page 101) of $\{X_i\}_{i\in I}$ in the category $\mathcal{T}op$ of topological spaces.

Example.

(1) It is worth considering the case that we will use most frequently, namely the case that we are given finitely many topological spaces $X_1, \dots, X_k$. In this case the product topology on $X_1 \times \cdots \times X_k$ is by definition the topology on $X_1 \times \cdots \times X_k$ that is generated by

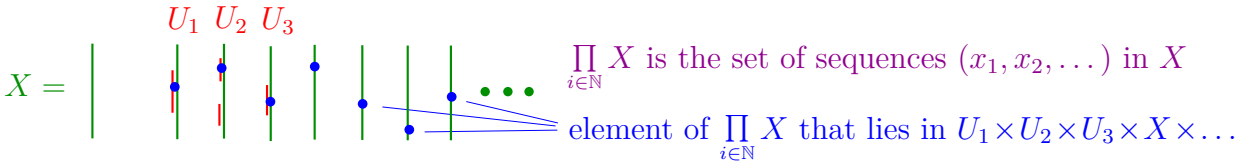
$$\mathcal{B} \;=\; \{U_1 \times \cdots \times U_k \,|\, \text{ each } U_i \text{ is open in } X_i\}.$$



(2) Let X be a topological space. We consider the product

$$X^{\mathbb{N}} \;=\; X \times X \times \ldots \;=\; \mathrm{Maps}(\mathbb{N} \to X) \;=\; \text{set of sequences of points in } X.$$

A basis of the topology is given by the sets of the form $U_1 \times \cdots \times U_k \times X \times \ldots$ where $U_1, \dots, U_k \subset X$ are open. In the figure below we try to illustrate this definition.



Proof of Proposition 5.1. Let $\{X_i\}_{i\in I}$ be a family of topological spaces.

(0) We need to verify that

$$\mathcal{B} \;:=\; \left\{ \prod_{i\in I} U_i \;\subset\; \prod_{i\in I} X_i \;\middle|\; \begin{array}{ll} (1) & \text{each } U_i \text{ is open in } X_i \text{ and} \\ (2) & \text{there exist only finitely many } i\text{'s with } U_i \neq X_i \end{array} \right\}$$

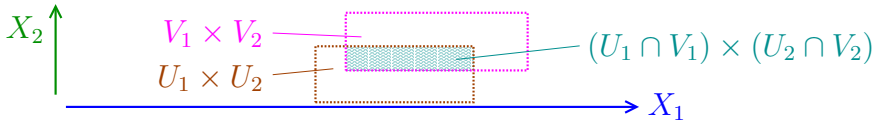
satisfies the conditions (B1) and (B2) formulated on page 65.

(B1) Evidently we have $\prod_{i\in I} X_i \in \mathcal{B}$. It follows immediately that $\mathcal{B}$ satisfies (B1).

(B2) Let $\prod_{i\in I} U_i$ and $\prod_{i\in I} V_i$ be two sets in $\mathcal{B}$. Then

$$\prod_{i\in I} \underbrace{U_i}_{\substack{\text{only finitely} \\ \text{many are} \neq X_i}} \;\cap\; \prod_{i\in I} \underbrace{V_i}_{\substack{\text{only finitely} \\ \text{many are} \neq X_i}} \;=\; \prod_{i\in I} \underbrace{(U_i \cap V_i)}_{\substack{\text{only finitely} \\ \text{many are} \neq X_i}}$$

is again a set in $\mathcal{B}$. Thus $\mathcal{B}$ also satisfies (B2).



(1) (a) It follows immediately from the definitions that the projection maps p_i are continuous with respect to the given topology.

(b) Suppose we are given a topological space Z and continuous maps $f_i\colon Z \to X_i$. We denote by $F\colon Z \to \prod_{i\in I} X_i$ the map that is given by $z \mapsto (i \mapsto f_i(z))$. We claim that F is continuous. By the Subbasis-Continuity Proposition 2.2 it suffices to verify that the preimage of any set in the basis $\mathcal{B}$ is open. So let $\prod_{i\in I} U_i$ be a set in $\mathcal{B}$. By definition there exists a finite set $J \subset I$ such that $U_i = X_i$ for any $i \notin J$. Then

$$F^{-1}\Big(\prod_{i\in I} U_i\Big) = \bigcap_{i\in I} f_i^{-1}(U_i) = \bigcap_{j\in J} \underbrace{f_j^{-1}(U_j)}_{\text{open in } Z} = \begin{array}{c}\text{intersection of } \textit{finitely} \\ \text{many open subsets of } Z.\end{array}$$

This concludes the proof that F is continuous.

(∗) The "only if"-statement follows from (1a) and the fact that the composition of two continuous maps is continuous. The "if"-statement follows easily from (1b).

(2) Let Z be a topological space and let $\{f_i\colon Z \to X_i\}_{i\in I}$ be a family of continuous maps. By the definition of a product on page 101 we need to show that there exists a unique continuous map $\Phi\colon Z \to \prod_{i\in I} X_i$ such that $f_i = p_i \circ \Phi$ for all $i \in I$. To do so, first note that it follows from the Product-of-Sets-and-Groups Lemma 4.2 that the map

$$\begin{array}{rcl} \Phi\colon Z & \to & \prod_{i\in I} X_i \\ z & \mapsto & (f_i(z))_{i\in I} \end{array}$$

is the unique map (of sets) $\Phi\colon Z \to \prod_{i\in I} X_i$ such that for each $i \in I$ we have the equality $f_i = p_i \circ \Phi$. In (1b) we just showed that this map is fortunately continuous. These two observations give us the desired result. ■

Definition. Let $\{X_i\}_{i\in I}$ be a family of topological spaces. We refer to the topology on the set $\prod_{i\in I} X_i$ that we defined in the Topological-Product Proposition 5.1 (0) as the product topology on $\prod_{i\in I} X_i$.

Example.

(1) It follows from the Topological-Product Proposition 5.1 (1a) and (1b) that given any topological space X the

$$\text{diagonal map} \quad \begin{array}{rcl} d\colon X & \to & X \times X \\ x & \mapsto & (x, x) \end{array} \qquad \text{and the projection} \qquad \begin{array}{rcl} p\colon X \times X & \to & X \\ (x, y) & \mapsto & x \end{array}$$

are continuous. Since $p \circ d = \mathrm{id}_X$ we see that $d\colon X \to X \times X$ is an embedding. This fact will come in handy on many occasions.

(2) Let $\{f_i\colon X_i \to Y_i\}_{i\in I}$ be a family of continuous maps between topological spaces. It follows from the Topological-Product Proposition 5.1 (1) together with the Basics-of-Continuity Lemma 2.1 (1) that the map

$$\begin{array}{rcl} \prod_{i\in I} f_i\colon \prod_{i\in I} X_i & \to & \prod_{i\in I} Y_i \\ (x_i)_{i\in I} & \mapsto & (f(x_i))_{i\in I} \end{array}$$

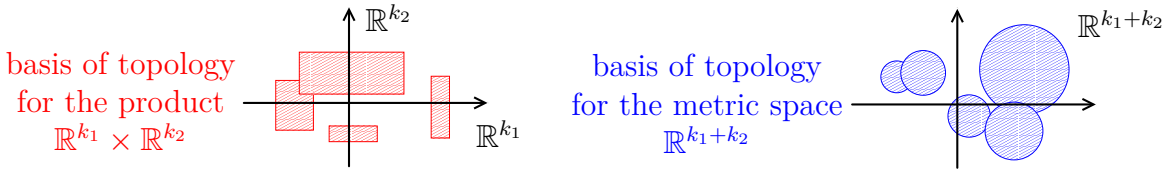
is continuous. □

The following lemma gives us a reinterpretation of the product topology in some of the most frequently studied cases.

Lemma 5.2. (Product Metric Lemma) Let $k_1, \ldots, k_m \in \mathbb{N}_0$. The map

$$\begin{array}{ccc} \overbrace{\mathbb{R}^{k_1+\cdots+k_m}}^{\text{Euclidean topology}} & \to & \overbrace{\mathbb{R}^{k_1} \times \cdots \times \mathbb{R}^{k_m}}^{\text{product of Euclidean topologies}} \\ (x_{1,1}, \ldots, x_{1,k_1}, \ldots, x_{m,1}, \ldots, x_{k,k_m}) & \mapsto & ((x_{1,1}, \ldots, x_{1,k_1}), \ldots, (x_{m,1}, \ldots, x_{m,k_m})) \end{array}$$

is a homeomorphism.



Proof. The proof, which is not very difficult, can be found in [**Fri24**]. ■

Identification. Given $k_1, \ldots, k_m \in \mathbb{N}_0$ we will use the Product Metric Lemma 5.2 to identify the product topological space $\mathbb{R}^{k_1} \times \cdots \times \mathbb{R}^{k_m}$ with the topological space $\mathbb{R}^{k_1+\cdots+k_m}$. We proceed the same way with "$\mathbb{R}$" replaced by "$\mathbb{C}$".

Example. It follows almost immediately from the definition of the topology on the vector space $\mathrm{M}(r \times s, \mathbb{R}) = \mathbb{R}^{r \cdot s}$ on page 83, the example on page 68, the Sum-Product-Continuity Lemma 2.6 and the Product Metric Lemma 5.2 that for any $k, l, m \in \mathbb{N}_0$ the map

$$\begin{array}{ccc} \mathrm{M}(k \times l, \mathbb{R}) \times \mathrm{M}(l \times m, \mathbb{R}) & \to & \mathrm{M}(k \times m, \mathbb{R}) \\ ((a_{ij})_{i,j}, (b_{rs})_{r,s}) & \mapsto & \left(\sum_{j=1}^{l} a_{ij} \cdot b_{js} \right)_{i,s} \end{array}$$

is continuous. The same also applies to matrices over $\mathbb{C}$. □

After these examples let us turn to the study of the more formal properties of the product topology. On several occasions we will use, consciously or unconsciously, the following lemma:

Lemma 5.3. (Product Topology-Basics Lemma)

(1) Let $\{X_i\}_{i \in I}$ be a family of topological spaces.
  (a) For each $i \in I$ let A_i be a subspace of X_i.
    (i) The subspace topology of $\prod_{i \in I} A_i \subset \prod_{i \in I} X_i$ agrees with product topology on $\prod_{i \in I} A_i$.
    (ii) The natural map
    $$\prod_{i \in I} A_i \to \prod_{i \in I} X_i$$
    given by the inclusions $A_i \to X_i$ is an embedding, i.e. it is a homeomorphism onto its image.[63]
  (b) If $J \subset I$ is a subset such that for each $j \in J$ the set X_j consists of a single element x_j, then the map

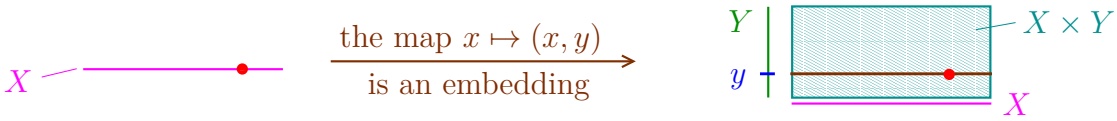
$$\prod_{i\in I\setminus J} X_i \;\to\; \prod_{i\in I} X_i$$
$$(f\colon I\setminus J\to \bigcup_{i\in I\setminus J} X_i) \;\mapsto\; \left(\begin{array}{ccl} I & \to & \bigcup_{i\in I} X_i \\ i & \mapsto & \begin{cases} x_i, & \text{if } i\in J,\\ f(i), & \text{if } i\notin J \end{cases} \end{array}\right).$$

is a homeomorphism.

(2) If $\{X_i\}_{i\in I}$ is a family of topological spaces and if for each $i\in I$ we are given a basis $\mathcal{B}_i$ of the topological space X_i, then

$$\mathcal{C} \;=\; \left\{ \prod_{i\in I} U_i \;\subset\; \prod_{i\in I} X_i \;\middle|\; \begin{array}{ll} (1) & \text{each } U_i\neq X_i \text{ is contained in } \mathcal{B}_i \text{ and} \\ (2) & \text{there exist only finitely many } i\text{'s with } U_i\neq X_i \end{array} \right\}$$

is a basis for $\prod_{i\in I} X_i$.



Proof of Lemma 5.3.

(1) (a) (i) This statement follows easily from the observation that if we take the basis of the topology of the product $\prod_{i\in I} X_i$ from the Topological-Product Proposition 5.1 (0) and if we intersect each basis element with $\prod_{i\in I} A_i$ we obtain precisely the basis of the product $\prod_{i\in I} A_i$ from the Topological-Product Proposition 5.1 (0).

(ii) The given map is clearly an injection. It follows from the Topological-Product Proposition 5.1 (0) that this map is continuous and it follows from (i) that this map is a homeomorphism onto its image.

(b) It follows from the Topological-Product Proposition 5.1 that the given map is continuous. It has an obvious inverse, which is also continuous by the Topological-Product Proposition 5.1.

(2) First note that by definition the sets in $\mathcal{C}$ are open. Next let $W\subset\prod_{i\in I} X_i$ be an open subset and let $(x_i)_{i\in I}\in W$. By definition of the product topology there exist open subsets $V_i\subset X_i$, $i\in I$ such that $X_i=V_i$ for all but finitely many $i\in I$ and such that $(x_i)_{i\in I}\int\prod_{i\in I} V_i\subset W$. Since $\mathcal{B}_i$ is a basis for X_i we see that for each i with $V_i\neq X_i$ there exists a $U_i\in\mathcal{B}_i$ with $x\in U_i\subset V_i$. For all other i we set $U_i:=V_i=X_i$. It follows that

$$(x_i)_{i\in I} \;\in\; \prod_{i\in I} U_i, \subset\; \prod_{i\in I} V_i \;\subset\; W.$$

We deduce that the given set $\mathcal{C}$ is indeed a basis for $\prod_{i\in I} X_i$. ∎

5.2. **Properties of products of topological spaces I.** Throughout these notes we will almost invariably work with the product of only finitely many topological spaces. In this

[63] If I is finite and if each $A_i\subset X_i$ is open (closed), then it follows easily from the definition of the product topology that the natural map $\prod_{i\in I} A_i\to\prod_{i\in I} X_i$ is an open (closed) embedding.

section we will therefore discuss properties of the product of *finitely* many topological spaces. This simplifies some of the arguments and it makes the notation arguably more natural. In the following section we will, for completeness sake, also discuss some of the properties of products of arbitrarily many topological spaces.
The next proposition deals with some of our favorite properties of topological spaces.

Proposition 5.4. (Product Topology-Properties Proposition) Let $X_1, \dots, X_k$ be non-empty[64] topological spaces.
(1) The product $X_1 \times \cdots \times X_k$ is compact if and only if each X_i is compact.
(2) The product $X_1 \times \cdots \times X_k$ is (path)-connected if and only if each X_i is (path)-connected.
(3) The product $X_1 \times \cdots \times X_k$ is Hausdorff if and only if each X_i is Hausdorff.

Proof. We proved the proposition in Analysis IV. Alternatively see [**Fri24**]. ■

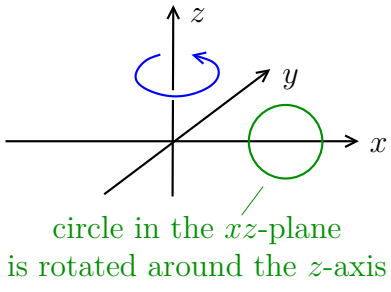
Definition.
(1) We refer to $(S^1)^n$ as the n-dimensional torus. For $n = 2$ we just say the torus.
(2) We refer to $S^1 \times \overline{B}^2$ as the solid torus and we refer to $S^1 \times B^2$ as the open solid torus.
We deduce from the Product Topology-Properties Proposition 5.4 that the n-dimensional torus $(S^1)^n$ and the solid torus $S^1 \times \overline{B}^2$ are compact.

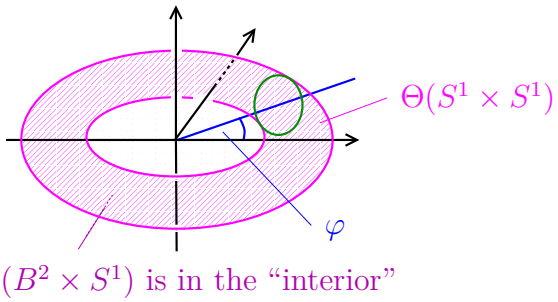
The following lemma allows us to view (solid) tori as subspaces of $\mathbb{R}^3$.

Lemma 5.5. (Torus Embedding Lemma) The map

$$\begin{aligned} \Theta\colon \overline{B}^2 \times S^1 &\to \mathbb{R}^3 \\ ((x,y), \exp(\mathrm{i}\varphi)) &\mapsto \underbrace{\begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{rotation around } z\text{-axis}} \cdot \underbrace{\begin{pmatrix} 1+\frac{1}{3}x \\ 0 \\ -\frac{1}{3}y \end{pmatrix}}_{\substack{\text{describes circle} \\ \text{in } xz\text{-plane}}} \end{aligned}$$

is an embedding. The restrictions of Θ to $S^1 \times S^1$ and $B^2 \times S^1$ are also embeddings.





[64]The "if"-directions also hold if one the X_i is the empty space, since in this case the product is the empty space.

Proof. The map $\Theta\colon \overline{B}^2 \times S^1 \to \mathbb{R}^3$ can be written as the composition of the maps

$$\begin{aligned} \overline{B}^2 \times S^1 \ &\to\ \mathbb{R}^2 \times (\mathbb{R}^2 \setminus \{0\}) \ \xleftarrow{\cong}\ \{(x,y,a,b) \in \mathbb{R}^4 \,|\, a^2+b^2>0\} \ \to\ \mathbb{R}^3 \\ (x,y,a,b) \ &\mapsto \begin{pmatrix} a & -b & 0 \\ b & a & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1+\frac{1}{3}x \\ 0 \\ -\frac{1}{3}y. \end{pmatrix} \end{aligned}$$

The first map is continuous by the Product Topology-Properties Proposition 5.4 (1), the second map is a homeomorphism by the Product Metric Lemma 5.2 and the last map is continuous by the Sum-Product-Continuity Lemma 2.6. This shows that Θ is continuous.

An elementary argument shows that the restriction of Θ to $\overline{B}^2 \times S^1$ is injective. By the Product Topology-Properties Proposition 5.4 (1) we know that the solid torus $\overline{B}^2 \times S^1$ is compact. Thus we obtain from the Compact-Hausdorff Proposition 2.10 (3) that the restriction of Θ to $\Theta\colon \overline{B}^2 \times S^1 \to \Theta(S^1 \times \overline{B}^2)$ is an embedding. But then it follows that the restrictions of Θ to $S^1 \times S^1$ and $B^2 \times S^1$ are embeddings. ∎

5.3. **Properties of products of topological spaces II.** The generalization of the Product Topology-Properties Proposition 5.4 to the product of arbitrarily many topological spaces also holds, see [**Fri24**] for details. We will not really make use of it, but for "general culture" let us formulate the following generalization of the Product Topology-Properties Proposition 5.4 (1).

Proposition 5.6. (Product Topology-Properties Proposition) Let $\{X_i\}_{i\in I}$ be a family of non-empty topological spaces.

(1) The product $\prod_{i\in I} X_i$ is compact if and only if each X_i is compact.

(2) The product $\prod_{i\in I} X_i$ is (path)-connected if and only if each X_i is (path)-connected.

(3) The product $\prod_{i\in I} X_i$ is Hausdorff if and only if each X_i is Hausdorff.

Proof. The proof of all statements of the Product Topology-Properties Proposition 5.6 is basically the same as the proof of the corresponding statements of the Product Topology-Properties Proposition 5.4, *except*, for the "if"-direction of Proposition 5.6 (1) and the "if"-direction of Proposition 5.6 (2) in the connected case. We outsource the proof the "if"-direction of Proposition 5.6 (2) to [**Fri24**]. The proof of the "if"-direction of Proposition 5.6 (1) will be discussed below. ∎

We now deal with the "if"-direction of the Product Topology-Properties Proposition 5.6 (1). This proof is now considerably harder and thus the proposition gets upgraded to a theorem:

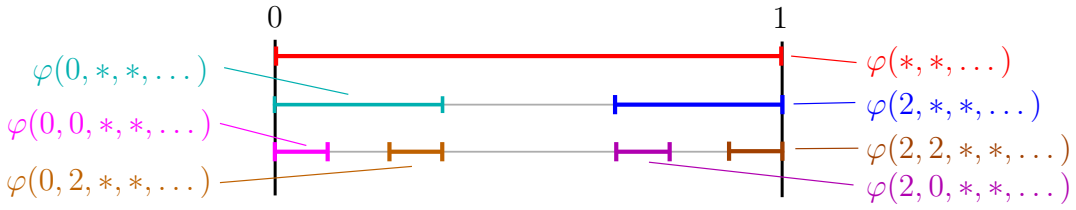
Theorem 5.7. (Tychonoff's Theorem) The product of arbitrarily many compact spaces is again a compact space.

Proof. Let $\{X_i\}_{i\in I}$ be a family of topological space. We will prove Tychonoff's Theorem with $I = \mathbb{N}$ in exercise sheet 2, with copious hints. The proof of the general case builds on the Axiom of Choice. We refer to [**Fri24**] for references to proofs in the literature. ∎

Example. We equip $\{0,2\}$ with the discrete topology. We consider the map

$$\begin{aligned} \varphi\colon \prod_{i\in\mathbb{N}} \{0,2\} &\to C = \text{Cantor set from page 64} \\ (f\colon \mathbb{N}\to\{0,2\}) &\mapsto \sum_{i=1}^{\infty} \frac{f(i)}{3^i}. \end{aligned}$$

It is elementary to see that φ takes values in C.



It is a refreshing exercise, left to the enthusiastic reader, to show that φ is a bijection. The left-hand side is compact by Tychonoff's Theorem 5.7 and the right-hand is evidently Hausdorff. We leave it to the reader to verify, using the Subbasis-Continuity Proposition 2.2 and the JH-Openness Criterion 1.1, that φ is continuous. Thus it follows from the Compact-Hausdorff Proposition 2.10 (3) that φ is a homeomorphism. □

5.4. **Applications of the product topology.** In this short section we see in two instances that the product topology can be quite convenient and that it appears in unexpected settings. First of all we have the following lemma which gives us a surprising characterization of Hausdorff spaces:

Lemma 5.8. (Hausdorff-Diagonal Lemma) For every topological space the following two statements are equivalent:

(i) X is Hausdorff.
(ii) The diagonal $\Delta = \{(x,x) \mid x \in X\}$ is a closed subset of $X \times X$.

Proof. The lemma follows really quite easily from the definitions. We leave it to the reader to fill in the details. ■

Using the Hausdorff-Diagonal Lemma 5.8 we can give an elegant proof of the following lemma:

Lemma 5.9. (Maps-agree-on-Closure Lemma) Let $f, g\colon X \to Y$ be two continuous maps between topological spaces. We assume that Y is Hausdorff.

(1) The set $\{x \in X \mid f(x) = g(x)\}$ is a closed subset of X.
(2) If f and g agree on a subset A, then f and g also agree on the closure $\overline{A}$ of A in X.

Proof.

(1) We consider the diagonal $\Delta = \{(y,y) \in Y \times Y \mid y \in Y\}$. We also consider the map $(f,g)\colon X \to Y \times Y$ that is given by $x \mapsto (f(x), g(x))$. We now see that

$$\{x \in X \mid f(x) = g(x)\} \underset{\text{follows easily from the definitions}}{=} (f,g)^{-1}(\Delta) \underset{\text{since } Y \text{ is Hausdorff we know by the Hausdorff-Diagonal Lemma 5.8 that } \Delta \text{ is a closed subset of } Y \times Y}{=} (f,g)^{-1}(\text{closed subset of } Y \times Y)$$
$$\underset{\text{it follows from the Topological-Product Proposition 5.1 (1b) that } (f,g) \text{ is continuous}}{=} \text{closed subset of } X.$$

(2) This statement follows from (a) and the definition of the closure, see page 59, as the intersection of all closed subsets of X that contain A. ∎

5.5. **The pullback of topological spaces.** On page 108 we introduced the general notion of pullbacks in a category. In this section we show that the category of topological spaces always has pullbacks. It is rather reassuring to hear that this is the case, even though the pullback will only really play a role much later when we study bundles. Thus this section can also be skipped without any major loss.

Proposition 5.10. (Topological-Pullback Proposition) Let $p\colon X \to B$ be a continuous map of topological spaces, let C be a topological space and let $f\colon C \to B$ be a continuous map. We consider the topological space

$$f^*X \coloneqq \{(c,x) \in C \times X \mid f(c) = p(x)\}$$

and the maps

$$\begin{array}{rcl} \varphi\colon f^*X & \to & C \\ (c,x) & \mapsto & c \end{array} \quad \text{and} \quad \begin{array}{rcl} \psi\colon f^*X & \to & X \\ (c,x) & \mapsto & x. \end{array}$$

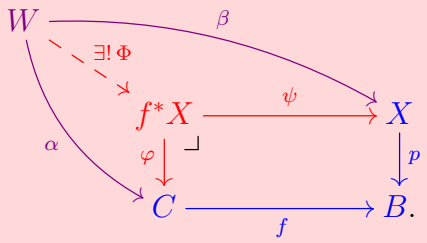
The following statements hold:

(1) (a) The maps $\varphi\colon f^*X \to C$ and $\psi\colon f^*X \to X$ are continuous.

(b) Suppose we are given a topological space W and two continuous maps $\alpha\colon W \to C$ and $\beta\colon W \to X$ such that $f \circ \alpha = p \circ \beta\colon W \to B$. The map

$$\begin{array}{rcl} \Phi\colon W & \to & f^*X \\ w & \mapsto & (\alpha(w), \beta(w)). \end{array}$$

is continuous. In particular we obtain the following commutative diagram:



(∗) A map $\Phi\colon W \to f^*X$ is continuous if and only if the maps $\varphi \circ \Phi\colon W \to C$ and $\psi \circ \Phi\colon W \to X$ are continuous.

(2) The topological space f^*X with the natural maps $\varphi\colon f^*X \to C$ and $\psi\colon f^*X \to X$ is the pullback, in the sense of the definition on page 108, in the category $\mathcal{T}op$ of topological spaces.

Proof.

(1) (1) This statement follows from the Topological-Product Proposition 5.1 (1a) and the Basics-of-Continuity Lemma 2.1 (3).
(2) This statement follows from the Topological-Product Proposition 5.1 (1b) and the Basics-of-Continuity Lemma 2.1 (4).
(∗) The "only if"-statement follows from (1a) and the fact that the composition of two continuous maps is continuous. The "if"-statement follows easily from (1b).

(2) It remains to show uniqueness of Φ. This follows immediately from the definitions and the fact that a point in $C \times X$ is uniquely determined by its projections. ■

Example.

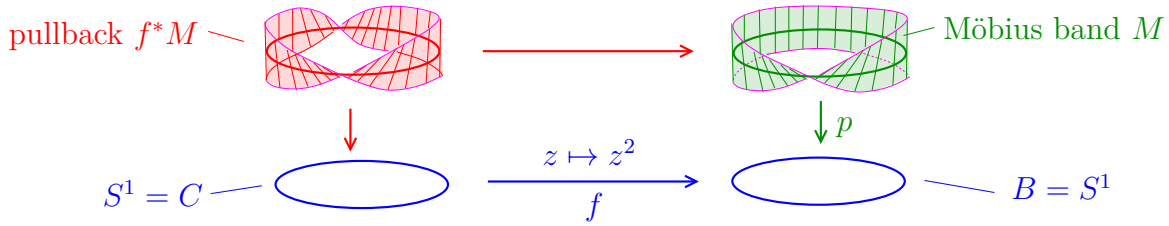
(1) Let $p\colon X \to B$ be a continuous map of topological spaces and let $C \subset B$ be a subspace. We denote by $\iota\colon C \to B$ the inclusion. One can easily verify that the maps

$$\begin{array}{rcl} p^{-1}(C) & \to & \iota^*X \\ x & \mapsto & (p(x), x) \end{array} \qquad \text{and} \qquad \begin{array}{rcl} \iota^*X & \to & p^{-1}(C) \\ (c, x) & \mapsto & x \end{array}$$

are continuous and inverses of one another. Thus we see that ι^*X is naturally homeomorphic to $p^{-1}(C)$.



(2) In the figure below we show the projection $p\colon M \to S^1$ from the Möbius band to S^1 and we see the map $f\colon C = S^1 \to S^1 = B$ that is given by $f(z) = z^2$.



It is a wonderful exercise to convince oneself that the pullback f^*M is a "doubly twisted band" and thus homeomorphic to $S^1 \times [-1, 1]$. □

For completeness we also study our favorite three properties of topological spaces for pullbacks. We will not really make use of this result.

Proposition 5.11. (Pullback-Properties Proposition) Let $p\colon X \to B$ be a continuous map of topological spaces, let C be a topological space and let $f\colon C \to B$ be a continuous map.

(1) If X and C are compact and if X, B, C are Hausdorff, then the pullback f^*X is also compact.
(3) If X and C Hausdorff, then the pullback f^*X is also Hausdorff.

Proof.

(1) We consider the map

$$\begin{array}{rcl} \mu\colon C \times X & \to & B \times B \\ (c, x) & \mapsto & (f(c), p(x)). \end{array}$$

It follows from the Topological-Product Proposition 5.1 (1) that μ is continuous. We denote by $\Delta := \{(b,b) \,|\, b \in B\} \subset B \times B$ the diagonal. Since B is Hausdorff it follows from the Hausdorff-Diagonal Lemma 5.8 that $\Delta \subset B \times B$ is a closed subset. By definition of the pullback we have $f^*X = \mu^{-1}(\Delta) \subset C \times X$. Since X and C are compact and Hausdorff we know by the Product Topology-Properties Proposition 5.4 (1) and (3) that $C \times X$ is compact and Hausdorff. It follows from this discussion and the Compact-Closed Lemma 1.9 (2) that f^*X is indeed compact.

(3) This statement follows easily from the Product Topology-Properties Proposition 5.4 (3). ∎

5.6. **The disjoint union of topological spaces.** In this section we will discuss the coproduct in the category of topological spaces. Before we get to topological spaces it is worth recalling the following definitions from page 37.

Definition. Let $\{A_i\}_{i\in I}$ be a family of sets.

(1) We define the disjoint union as the set

$$\bigsqcup_{i\in I} A_i := \bigcup_{i\in I} (A_i \times \{i\}).$$

As usual we take some liberties with the notation if we have finitely many sets.

(2) Given $j \in I$ we refer to the map $A_j \to \bigsqcup_{i\in I} A_i$ given by $a \mapsto (a,j)$ as the natural injection.

In the Coproduct-of-Sets-and-Abelian Groups Lemma 4.4 we saw that the disjoint union, together with the obvious natural injections, gives us the coproduct in the category of sets. In a second we will see that a very similar statement holds in the context of the category of topological spaces. But before we get to this point, let us introduce the following convention.

Convention. Let $\{A_i\}_{i\in I}$ be a family of sets. For any $i \in I$ we use the natural injection $A_i \to \bigsqcup_{i\in I} A_i$ to think of A_i as a subset of the disjoint union.

The following proposition is the analogue of the Topological-Product Proposition 5.1.

Proposition 5.12. (Topological-Coproduct Proposition) Let $\{X_i\}_{i\in I}$ be a family of topological spaces. We consider the disjoint union $\bigsqcup_{i\in I} X_i$.

(0) (a) We say that $U \subset \bigsqcup_{i\in I} X_i$ is open if and only if for each $i \in I$ the intersection $X_i \cap U$ is an open subset of X_i. This defines a topology on $\bigsqcup_{i\in I} X_i$. We will work with this topology throughout the proposition.

(b) A subset $A \subset \bigsqcup_{i\in I} X_i$ is closed if and only if for each $i \in I$ the intersection $X_i \cap A$ is a closed subset of X_i.

(1) (a) Each natural injection $\iota_j \colon X_j \to \bigsqcup_{i\in I} X_i$ is continuous. In fact it is an embedding that is open and closed.[65]

(b) If Z is a topological space and if $\{f_i \colon X_i \to Z\}_{i\in I}$ is a family of continuous maps, then the map

$$\begin{array}{ccc} \bigsqcup_{i\in I} X_i & \to & Z \\ (x,i) & \mapsto & f_i(x) \end{array}$$

is continuous.

(∗) A map $f\colon \bigsqcup_{i\in I} X_i \to Z$ to a topological space Z is continuous if and only if for each $i \in I$ the map $f \circ \iota_i\colon X_i \to Z$ is continuous.

(2) The topological space $\bigsqcup_{i\in I} X_i$ together with the natural injections ι_i is a coproduct of $\{X_i\}_{i\in I}$ in the category *Top* of topological spaces.

Proof.

(0) (a) This statement follows easily from the definitions.

(b) This statement follows from an elementary argument.

(1) (a) It follows immediately from writing down the definitions that each natural injection $X_j \to \bigsqcup_{i\in I} X_i$ is continuous, and that it is open and closed. It follows from the Open-Injective Lemma 2.9 that the natural injection is an embedding that is open and closed.

(b) Let $\{f_i\colon X_i \to Z\}_{i\in I}$ be a family of continuous maps. We consider the corresponding map $F\colon \bigsqcup_{i\in I} X_i \to Z$. Let $U \subset Z$ be an open subset. For each $i \in I$ we have $F^{-1}(U) \cap X_i = f_i^{-1}(U)$. This shows that $F^{-1}(U)$ is open, i.e. F is continuous.

(∗) The "only if"-statement follows from (1a) and the fact that the composition of two continuous maps is continuous. The "if"-statement follows easily from (1b).

(2) Let Z be a topological space and let $\{f_i\colon X_i \to Z\}_{i\in I}$ be a family of continuous maps. By the definition of the coproduct on page 104 we have to show that there exists a *unique* continuous map $\Phi\colon \bigsqcup_{i\in I} X_i \to Z$ such that for each $i \in I$ we have $f_i = \Phi \circ \iota_i$. In (1b) we defined such a map Φ and we saw that it is continuous. It remains to prove uniqueness. The proof of uniqueness is basically identical to the argument in the proof of the Coproduct-of-Sets-and-Abelian Groups Lemma 4.4 (1) where we dealt with the coproduct in the category of sets. ■

Definition. Let $\{X_i\}_{i\in I}$ be a family of topological spaces. We refer to the topology on $\bigsqcup_{i\in I} X_i$ that we introduced in the Topological-Coproduct Proposition 5.12 as the disjoint union topology. We refer to this topological space as the disjoint union of the topological spaces X_i.[66]

The following lemma plays the role of the Product Topology-Properties Proposition 5.4.

Lemma 5.13. (Disjoint Union Topology-Properties Lemma) Let $\{X_i\}_{i\in I}$ be a family of topological spaces.

(1) The following two statements are equivalent:

(a) The disjoint union $\bigsqcup_{i\in I} X_i$ is compact.

(b) Each X_i is compact and there are only finitely many $i \in I$ such that $X_i \neq \varnothing$.

(2) (a) Each X_i is an open and closed subset of $\bigsqcup_{i\in I} X_i$.

[65] In most cases this statement allows us to safely view each X_j as a closed and open subspace of the disjoint union $\bigsqcup_{i\in I} X_i$.

[66] It would perhaps make more sense to refer to $\bigsqcup_{i\in I} X_i$ as the coproduct of the X_i. But we prefer to stick to the more classical name.

(b) The path-components of the disjoint union $\bigsqcup_{i\in I} X_i$ are precisely the path-components of the X_i.

(3) The disjoint union $\bigsqcup_{i\in I} X_i$ is Hausdorff if and only if each X_i is Hausdorff.

Proof. We prove the statements of the lemma in a different order than they were stated.

(2) (a) Let $i \in I$. Note that for $j \neq i$ we have $X_i \cap X_j = \varnothing$, which is open and closed. Thus it follows from the Topological-Coproduct Proposition 5.12 (0a) and that X_i is an open and closed subset.

(b) This statement follows from (2a) and the Path-Component Lemma 2.21 (4).

(1) We first prove the "(b)⇒(a)"-direction. Thus we assume that each X_i is compact and that there are only finitely many $i_1, \ldots, i_k \in I$ such that the corresponding topological space is non-empty. Let $\{U_j\}_{j\in J}$ be an open cover of $\bigsqcup_{i\in I} X_i$. Since each X_i is compact we can find for each $i \in I$ a finite subset $J_i \subset J$ such that $\bigsqcup_{j\in J_i}(U_j \cap X_i) = X_i$. It is now clear that $X = \bigcup_{r\in\{1,\ldots,k\}} X_i = \bigcup_{r\in\{1,\ldots,k\}} \bigsqcup_{j\in J_{i_r}} U_j$.

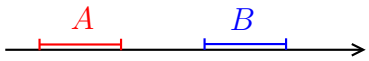
We turn to the proof of the "(a)⇒(b)"-direction. Thus we assume that $\bigsqcup_{i\in I} X_i$ is compact. It follows from (2a) together with the Compact-Closed Lemma 1.9 (1) that each X_i is compact. Next note that by (2a) we also know that the $\{X_i\}_{i\in I}$ form an open cover of the compact topological space $\bigsqcup_{i\in I} X_i$. Thus we see that there exists a finite subset $J \subset I$ with $\bigsqcup_{j\in J} X_j = \bigsqcup_{i\in I} X_i$. In particular $X_i = \varnothing$ for $i \notin J$.

(3) This statement follows easily from the definitions and (2a). ■

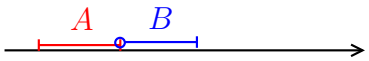
Example. Let X be a topological space and let $A, B \subset X$ be subsets of X. It follows from the Topological-Coproduct Proposition 5.12 (1b) that the obvious map

$$\underbrace{A \sqcup B}_{\text{disjoint union topology}} \quad \to \quad \underbrace{A \cup B}_{\text{subspace topology}}$$

is continuous. If A and B are disjoint, then this map is also a bijection. To be more specific, let $X = \mathbb{R}$. For $A = [0, 1]$ and $B = [2, 3]$ one can show by hand that the map is open, thus a homeomorphism.



$A \sqcup B$ is homeomorphic to $A \cup B \subset \mathbb{R}$



$A \sqcup B$ is *not* homeomorphic to $A \cup B \subset \mathbb{R}$

On the other hand for $A = [0, 1]$ and $(1, 2]$ the above map is certainly not a homeomorphism, for example $A \sqcup B$ is not compact whereas $A \cup B = [0, 2]$ is compact. In [**Fri24**] we formulate a lemma which gives a criterion for showing that $A \sqcup B$ is homeomorphic to $A \cup B$. □

CHAPTER 6

Quotients of topological spaces

In Chapter 5 we introduced the product and disjoint union (vulgo coproduct) of topological spaces. In this chapter we introduce the very flexible notion of a quotient of a topological space by an equivalence relation. This will allow us to introduce some of our favorite topological spaces, namely surfaces and projective spaces, and it will allow us to study the pushout in the category of topological spaces.

6.1. **The quotient topology.** On page 37 we introduced the notion of an equivalence relation. We recall the following notation from page 39.

Notation. Let $\sim$ be an equivalence relation on a set X.

(1) Given $x \in X$ we denote by $[x]$ the unique equivalence class that contains x.

(2) We refer to the map
$$\begin{array}{rccc} p\colon & X & \to & X/\sim \\ & x & \mapsto & [x] \end{array}$$
as the natural projection.

(3) Let $f\colon X \to Y$ be a map between sets such that $f(x) = f(y)$ whenever $x \sim y$. By the Quotient-Factorization Lemma 0.5 the map
$$\begin{array}{rccc} g\colon & X/\sim & \to & Y \\ & [x] & \mapsto & f(x) \end{array}$$
is well-defined and it is the unique map $g\colon X/\sim \to Y$ such that $f = g \circ p$, i.e. such that the following diagram of maps commutes:
$$\begin{array}{ccc} X & \xrightarrow{p} & X/\sim \\ & \searrow^{f} & \downarrow g \\ & & Y. \end{array}$$
We refer to this map g as the induced map.

In this chapter we introduce and study the quotient of topological space by an equivalence relation. This notion is, in contrast to the (co-) product, not a categorical notion. Nonetheless there are structural similarities and the following proposition can be viewed as the analogue of the Topological-Product Proposition 5.1 and of the Topological-Coproduct Proposition 5.12.

Proposition 6.1. (Topological-Quotient Proposition) Let $\sim$ be an equivalence relation on a topological space X. We denote by $p\colon X \to X/\sim$ the natural projection from X onto the set of equivalence classes $X/\sim$. The following statements hold:

(0) (a) The set
$$\mathcal{T} \coloneqq \{U \subset X/\sim \mid p^{-1}(U) \text{ is open in } X\}$$
is a topology on $X/\sim$. We use this topology throughout this proposition.[67]

(b) A subset $A \subset X/\sim$ is closed if and only if the preimage $p^{-1}(A)$ is closed in X.

(1) (a) The natural projection $p\colon X \to X/\sim$ is continuous.

(b) Let Y be a topological space and let $f\colon X \to Y$ be a continuous map with the property that $f(x) = f(y)$ whenever $x \sim y$. The induced map

$$\begin{array}{rccc} g\colon & X/\sim & \to & Y \\ & [x] & \mapsto & f(x) \end{array}$$

is also continuous.

(∗) A map $f\colon X/\sim \to Y$ is continuous if and only if the map $f \circ p\colon X \to Y$ is continuous.

(2) If $\mathcal{S}$ is a topology on $X/\sim$ such that $(X/\sim, \mathcal{S})$ has properties (1a) and (1b), then $\mathcal{S}$ equals the topology from (0).

Proof. Let $\sim$ be an equivalence relation on a topological space X.

(0) (a) This statement follows easily from the Image-Preimage Lemma 0.3 (6) and (7).

(b) This statement follows easily from the Image-Preimage Lemma 0.3 (8).

(1) (a) It follows immediately from the definition of the topology on $X/\sim$ that the natural projection $p\colon X \to X/\sim$ is continuous.

(b) Let $f\colon X \to Y$ be a continuous map such that $f(x) = f(y)$ whenever $x \sim y$. We need to show that the induced map $g\colon X/\sim \to Y$ is continuous as well. Let $U \subset Y$ be open. We need to show that $g^{-1}(U) \subset X/\sim$ is open. By definition of the quotient topology on $X/\sim$ we need to show that $p^{-1}(g^{-1}(U))$ is open in X. We observe that

$$p^{-1}(g^{-1}(U)) \;=\; (g\circ p)^{-1}(U) \;\underset{\text{by the commutative diagram}}{\overset{\uparrow}{=}}\; f^{-1}(U) \;\underset{\text{since } f \text{ continuous}}{\overset{\uparrow}{=}}\; \text{open subset}\,.$$

We have thus shown that g is continuous.

(∗) The "only if"-statement follows from (1a) and the fact that the composition of two continuous maps is continuous. The "if"-statement follows easily from (1b).

(2) Let $\mathcal{S}$ be a topology on $X/\sim$ such that the following conditions are satisfied:

(a′) The natural projection $p\colon X \to (X/\sim, \mathcal{S})$ is continuous.

(b′) Whenever we are given a topological space Y and a continuous map $f\colon X \to Y$ with the property that $f(x) = f(y)$ whenever $x \sim y$, then the corresponding induced map $(X/\sim, \mathcal{S}) \to Y$ is also continuous.

We apply (b) and (a′) to obtain that the identity map $\Phi\colon (X/\sim, \mathcal{T}) \to (X/\sim, \mathcal{S})$ is continuous. This implies that $\mathcal{S} \subset \mathcal{T}$. Similarly we apply (a) and (b′) to obtain that the identity map $\Psi\colon (X/\sim, \mathcal{S}) \to (X/\sim, \mathcal{T})$ is continuous, which tells us that $\mathcal{T} \subset \mathcal{S}$. In summary we see that $\mathcal{S} = \mathcal{T}$. ∎

The following definition will not come as a surprise.

Definition. Let $\sim$ be an equivalence relation on a topological space X.

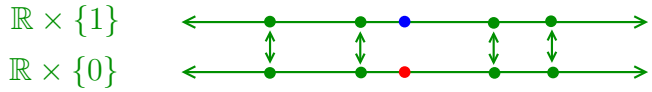
(1) We refer to the topology on $X/\sim$ from the Topological-Quotient Proposition 6.1 (0a) as the quotient topology and we refer to $X/\sim$ as a quotient space of X.

[67] In other words, $U \subset X/\sim$ is open if and only if $p^{-1}(U) \subset X$ is open.

(2) Let Y be a topological space and let $f\colon X \to Y$ be a continuous map with the property that $f(x) = f(y)$ whenever $x \sim y$.
   (a) We refer to the corresponding map $X/\sim \to Y$ as the induced map.
   (b) At times we denote the induced map by $\overline{f}$. Sometimes, when there is no danger of confusion we denote the induced map just by f itself.
   (c) Sometimes we also say that the map $f\colon X \to Y$ descends to a map $X/\sim \to Y$.

As before we care about compactness, connectedness and the Hausdorff property of topological spaces. We will see shortly that compactness and connectedness is preserved by going to quotients. The Hausdorff property is more tricky and we start out with the following cautionary example:

Example. We consider the topological space $X = (\mathbb{R}\times\{0\})\cup(\mathbb{R}\times\{1\})$ and the equivalence relation that is generated by $(x,0) \sim (x,1)$ for all $x \neq 0$.



We denote by $p\colon X \to X/\sim$ the natural projection. One can easily show that $p((0,0))$ and $p((0,1))$ cannot be separated by open neighborhoods. One can also easily show that $X/\sim$ is homeomorphic to the line with two zeros, which is not Hausdorff. This shows that the quotient of a Hausdorff space is not necessarily Hausdorff. □

Definition. Let X be a topological space and let $\sim$ be an equivalence relation on X. Recall that an equivalence relation is, by the definition on page 37, a subset of $X \times X$ with suitable properties. We say that the relation $\sim$ is closed if $\sim$ is a closed subset of $X \times X$.

The following lemma is the analogue of the Product Topology-Properties Proposition 5.4 and the Disjoint Union Topology-Properties Lemma 5.13.

Proposition 6.2. (Quotient Topology-Properties Proposition) Let $\sim$ be an equivalence relation on a topological space X.

(1) If X is compact, then $X/\sim$ is also compact.
(2) If X is (path)-connected, then $X/\sim$ is also (path)-connected.
(3) If X is Hausdorff and compact and if $\sim$ is a closed equivalence relation on X, then $X/\sim$ is Hausdorff.

Proof. By the Topological-Quotient Proposition 6.1 (1a) we know that the natural projection $X \to X/\sim$ is continuous. The first two statements of the lemma thus follow from the Compact Image Lemma 2.8 and the Image-Connected Lemma 2.16. The proof of Statement (3) is non-trivial but dull, we refer to [**Fri24**] for details. ■

A careful reading of these notes shows that the following lemma gets used on many occasions without giving it its due credit.

Lemma 6.3. (Twice Quotient Lemma) Let $f\colon X \to Y$ be a continuous map between two topological spaces and let $\sim_X$ and $\sim_Y$ be equivalence relations on X respectively Y.

If $x_1 \sim_X x_2$ implies that $f(x_1) \sim_Y f(x_2)$, then the map

$$\begin{array}{rcl} X/\sim_X & \to & Y/\sim_Y \\ [x] & \mapsto & [f(x)] \end{array}$$

is well-defined and continuous.

Proof. First note that it follows from the Topological-Quotient Proposition 6.1 (1a) that the map $X \xrightarrow{f} Y \xrightarrow{y\mapsto[y]} Y/\sim_Y$ is continuous. Next note that it follows from our hypothesis on f and the Topological-Quotient Proposition 6.1 (1b) that this map descends to a continuous map $X/\sim_X \to Y/\sim_Y$. ■

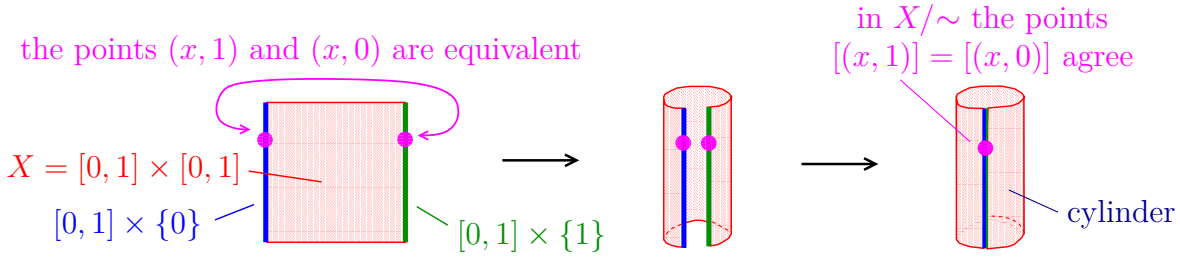
6.2. **Examples: Surfaces I.** In this section, and in its sister Section 6.6, we introduce some of my favorite topological spaces, namely surfaces. In the following we use the word "surface" in a colloquial manner.

(1) We consider $X = [0,1] \times [0,1] \subset \mathbb{R}^2$ and the equivalence relation that is generated by

$$(x,0) \sim (x,1) \quad \text{for all } x \in [0,1].$$

We will give two descriptions of this topological space, there first one is slightly informal, the second one is more rigorous.

(a) The quotient space $X/\sim$ is obtained from the square $X = [0,1]\times[0,1]$ by identifying each point on the upper edge with the corresponding point on the lower edge. In other words, "we glue the upper edge to the lower edge". This interpretation is illustrated in the figure below.



(b) We consider the map

$$\begin{array}{rcl} X/\sim & \to & S^1 \times [0,1] \\ [(s,t)] & \mapsto & (\exp(2\pi\,\mathrm{i}t), s). \end{array}$$

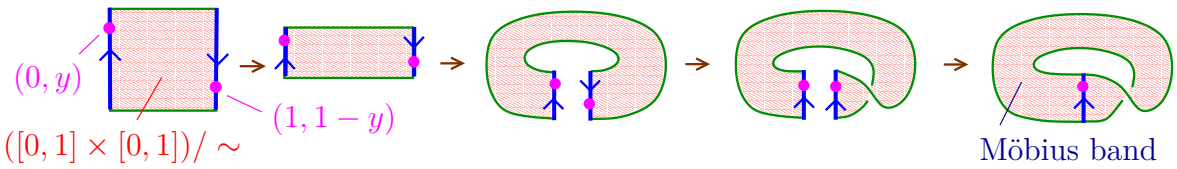
It follows from the Topological-Product Proposition 5.1 (1b) and the Topological-Quotient Proposition 6.1 (1b) that this map is continuous. One can easily verify that the map is a bijection. By the Quotient Topology-Properties Proposition 6.2 (1) we know that $X/\sim$ is compact and by the Product Topology Properties Proposition 5.4 (3) we know that $S^1 \times [0,1]$ is Hausdorff. Thus we obtain from the Compact-Hausdorff Proposition 2.10 (3) that the above map $X/\sim \to S^1 \times [0,1]$ is a homeomorphism. We refer to $X/\sim$ and also to $S^1 \times [0,1]$ as the cylinder or the annulus.

(2) We consider $X = [0,1] \times [0,1] \subset \mathbb{R}^2$, this time with the equivalence relation that is generated by

$$(0,y) \sim (1,1-y) \quad \text{for all } y \in [0,1].$$

We refer to the quotient $X/\sim$ as the Möbius band. Again we give two descriptions:

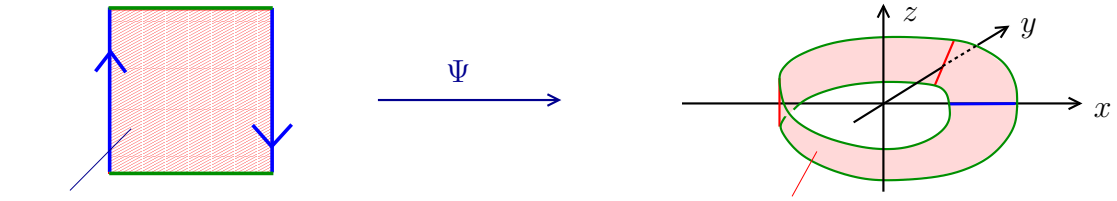
(a) The quotient space $X/\sim$ is obtained from the square X by gluing the edge on the left to the edge on the right, but this time "we glue the edges with a twist".



(b) The name "Möbius band" conjures up a certain picture. We will now see that our name is justified. We consider the map

$$\begin{aligned}\Psi\colon ([0,1]\times[0,1])/\sim \;&\to\; \mathbb{R}^3\\ [(\varphi,r)] \;&\mapsto\; \underbrace{\begin{pmatrix}\cos(2\pi\cdot\varphi) & -\sin(2\pi\cdot\varphi) & 0\\ \sin(2\pi\cdot\varphi) & \cos(2\pi\cdot\varphi) & 0\\ 0 & 0 & 1\end{pmatrix}}_{\text{rotation around } z\text{-axis}} \cdot \underbrace{\begin{pmatrix}2+(r-\frac{1}{2})\cdot\sin(\pi\cdot\varphi)\\ 0\\ (r-\frac{1}{2})\cdot\cos(\pi\cdot\varphi)\end{pmatrix}}_{\substack{\text{describes interval}\\ \text{that makes half a twist}\\ \text{as we go around the } z\text{-axis}}}\end{aligned}$$

Similar to (1b) one can see that Ψ is an embedding. Thus the Möbius band, defined abstractly as above, is indeed homeomorphic to a subspace of $\mathbb{R}^3$ that looks like what we had in mind.



$M = ([0,1]\times[0,1])/(0,y)\sim(1,1-y)$ Möbius band viewed as a subspace of $\mathbb{R}^3$

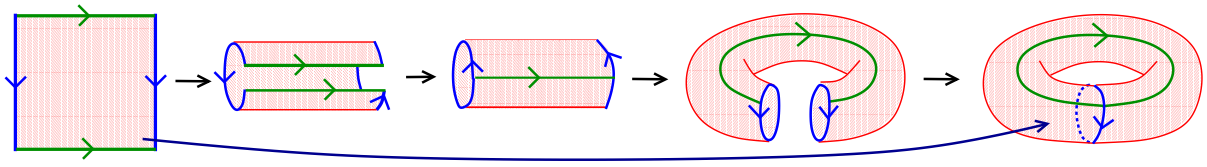
(3) We consider again $X = [0,1]\times[0,1] \subset \mathbb{R}^2$, but this time with the equivalence relation that is generated by

$$(x,0)\sim(x,1) \quad \text{for all } x\in[0,1]$$

and by

$$(0,y)\sim(1,y) \quad \text{for all } y\in[0,1].$$

(a) We obtain the quotient space $X/\sim$ by gluing the top edge to the bottom edge and the left-hand edge to the right-hand edge, each time without adding a twist. The figure below illustrates that the quotient space $X/\sim$ is indeed a torus.



by gluing of the opposite edges of a square we obtain a torus

(b) As in (1b) one can write down an obvious homeomorphism $X/\sim \to S^1\times S^1$. Thus we can make the identification $([0,1]\times[0,1])/\sim =_{\mathrm{i}} S^1\times S^1$.

(4) We consider $X = [0,1]\times[0,1] \subset \mathbb{R}^2$, this time with the equivalence relation that is generated by

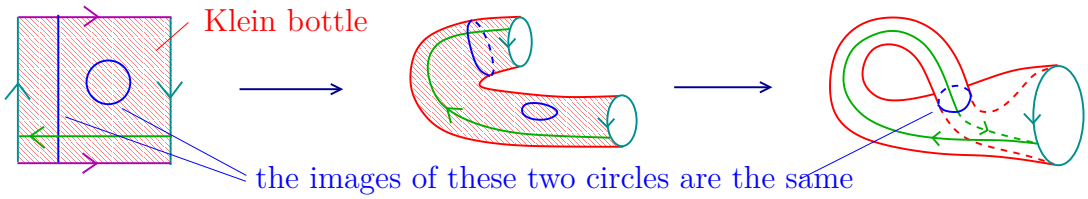
$$(x,0) \sim (x,1) \qquad \text{for all } x \in [0,1]$$

and by

$$(0,y) \sim (1,1-y) \qquad \text{for all } y \in [0,1].$$

The resulting topological space is called the Klein bottle.

(a) We obtain the quotient space $X/\sim$ by gluing the upper edge to the lower edge and by gluing the edge on the left to the edge on the right with a twist. In the figure below we show the Klein bottle to the left. Furthermore we indicate a map from the Klein bottle to $\mathbb{R}^3$. This particular map is not injective.



(b) Let $0 < b < a$ be real numbers. It is elementary, but painful, to show that the map

$$\begin{array}{rcl} ([0,1]\times[0,1])/\sim & \to & \mathbb{R}^4 \\ [(x,y)] & \mapsto & \begin{pmatrix} (a+b\cdot\cos(y\cdot 2\pi))\cdot\cos(x\cdot 2\pi) \\ (a+b\cdot\cos(y\cdot 2\pi))\cdot\sin(x\cdot 2\pi) \\ b\cdot\sin(y\cdot 2\pi)\cdot\cos(x\cdot\pi) \\ b\cdot\sin(y\cdot 2\pi)\cdot\sin(x\cdot\pi) \end{pmatrix} \end{array}$$

is an embedding. Thus we can view the Klein bottle as a subspace of $\mathbb{R}^4$.

We will introduce more surfaces a little later in Section 6.6.

6.3. **Quotients by subspaces.** In this section we will study another important source of quotient spaces.

Definition. Let X be a topological space and let $A \subset X$ be a subset.

(1) First we suppose that A is a non-empty subset. For $P, Q \in X$ we define

$$P \sim_A Q \quad :\Longleftrightarrow \quad P = Q \text{ or } P, Q \text{ both lie in } A.$$

This is easily seen to be an equivalence relation on X. We write $X/A := X/\sim_A$ and we always equip X/A with the quotient topology. We refer to X/A as the quotient of X by A. Note that $A \subset X$ is an equivalence class, thus it defines a point of X/A. In an attempt to avoid confusion we write $[A]$ for the point in X/A that is given by the equivalence class A.

(2) If $A = \varnothing$ is the empty set, then we define[68] $X/\varnothing := X \sqcup \{X\}$, i.e. $X/\varnothing$ equals the disjoint union of the topological space X with the topological space consisting of a single point $\{X\}$. For consistency we denote this point by $[\varnothing]$ as well.[69][70]

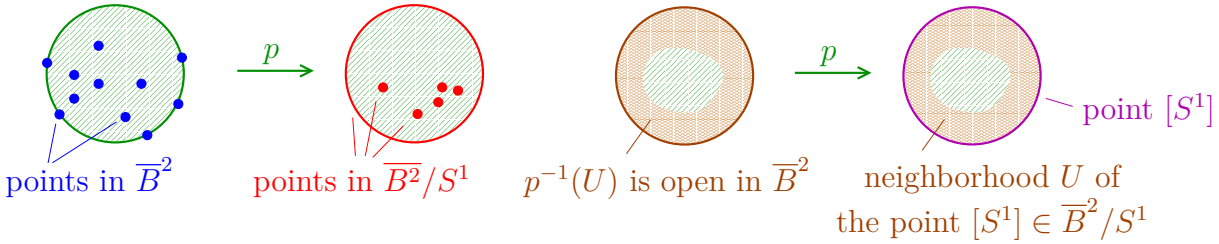
[68] Note that according to [**Cie97**, Therem 4.1.1] $X \notin X$.

[69] In particular $\varnothing/\varnothing$ is the topological space consisting of the single point $\{\varnothing\}$.

[70] The reader will have spotted that the case $A = \varnothing$ is rather ad hoc. On page 143 we will give a less ad hoc interpretation. In general the case $A = \varnothing$ is a major nuisance. It is conceivable that later on in proofs we will subconsciously suppress this special case. Whenever this occurs we leave it to the reader to deal with the case $A = \varnothing$ separately.

Remark. Let us attempt to digest the definition of X/A. As a set X/A corresponds precisely to the points in $X \setminus A$ together with the point $[A]$. Furthermore, by definition of the quotient topology a set $U \subset X/A$ is open if and only if $p^{-1}(U)$ is an open subset of X. Note that given any subset $Z \subset X/A = (X \setminus A) \cup \{[A]\}$ we have[71]

$$p^{-1}(Z) = \begin{cases} Z, & \text{if } [A] \notin Z, \\ Z \cup A, & \text{if } [A] \in Z. \end{cases}$$

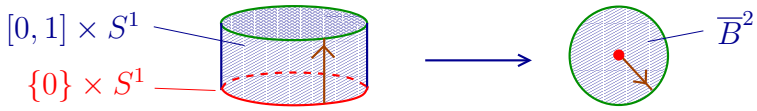


This shows that the open subsets in X/A correspond precisely to the open subsets of X that lie in $X \setminus A$ and the open subsets of X that contain A. □

Lemma 6.4. (Cylinder Quotient-Ball Lemma) The map

$$\begin{array}{rcl} ([0,1] \times S^n)/(\{0\} \times S^n) & \to & \overline{B}^{n+1} \\ [(r,v)] & \mapsto & r \cdot v \end{array}$$

is a homeomorphism.



Proof. As in the proof of the Torus Embedding Lemma 5.5 it follows from the Topological-Quotient Proposition 6.1 (1b) that the given map is continuous. It is basically clear that the map is a bijection. The left-hand side is compact by the Topological-Quotient Proposition 6.1 (1b). It follows from the Compact-Hausdorff Proposition 2.10 (3) that the map is a homeomorphism. ■

Lemma 6.5. (Ball Quotient-Sphere Lemma) Let $n \in \mathbb{N}$. The map[72]

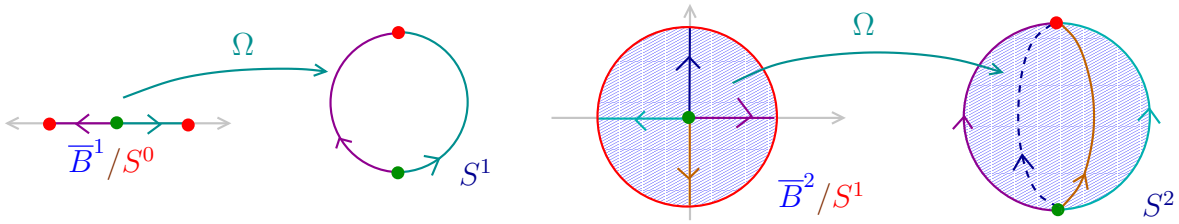
$$\begin{array}{rcl} \Omega\colon \overline{B}^n/S^{n-1} & \to & S^n \\ [r \cdot v] & \mapsto & \underbrace{\begin{pmatrix} 0 & (-1)^{n-1} & 0 \\ 0 & 0 & \mathrm{id}_{n-1} \\ -1 & \dots & 0 \end{pmatrix}}_{\in \mathrm{SO}(n+1)} \cdot \underbrace{\begin{pmatrix} \cos(\pi \cdot r) \\ \sin(\pi \cdot r) \cdot v \end{pmatrix}}_{\in \mathbb{R} \times \mathbb{R}^n = \mathbb{R}^{n+1}} \end{array} \quad \text{with } r \in [0,1] \text{ and } v \in S^{n-1}$$

[71] In an attempt to lighten the notation, given $x \in X \setminus A$ we do not distinguish between the point $x \in X \setminus A$ and the equivalence class $\{x\} \subset X$.

is a homeomorphism. Furthermore, for $n = 0$ the map

$$\begin{array}{rcl} \Omega\colon \overline{B}^0/S^{-1} & \to & S^0 \\ [x] & \mapsto & \begin{cases} -1, & \text{if } x = 0 \in \overline{B}^0, \\ 1, & \text{if } [x] = [S^{-1}] \end{cases} \end{array}$$

is a homeomorphism.



Identification. For each $n \in \mathbb{N}_0$ we use the homeomorphism from the Ball Quotient-Sphere Lemma 6.5 to make the identification $S^n =_{\mathrm{i}} \overline{B}^n/S^{n-1}$.

Proof of Lemma 6.5. Evidently we first deal with the case $n \neq 0$. It follows quite easily from the Cylinder Quotient-Ball Lemma 6.4 (2), the Topological-Quotient Proposition 6.1 (1b), the Quotient Topology-Properties Proposition 6.2 (1) and the Compact-Hausdorff Proposition 2.10 (3) that the map $\Omega\colon \overline{B}^n/S^{n-1} \to S^n$ is a homeomorphism.

Finally note that the case $n = 0$ follows easily from the observation that in the definition on page 131 we defined $X/\varnothing \coloneqq X \sqcup \{\varnothing\}$. ■

6.4. **Group actions on topological spaces.** We recall the following definitions from pages 48 and 49.

Definition. Let X be a set and let G be a group with trivial element e.

(1) An action of G on X is a map

$$\begin{array}{rcl} G \times X & \to & X \\ (g, x) & \mapsto & g \cdot x \end{array}$$

with the following properties:

$$\begin{array}{rcll} e \cdot x & = & x, & \text{for all } x \in X, \\ g \cdot (h \cdot x) & = & (g \cdot_G h) \cdot x, & \text{for all } x \in X \text{ and } g, h \in G. \end{array}$$

group action; multiplication in the group G

(2) The action is called free, if $g \cdot x = x$ for some $x \in X$ implies that $g = e$.

(3) If X is a topological space, then we say that the action is continuous, if for every $g \in G$ the map

$$\begin{array}{rcl} X & \to & X \\ x & \mapsto & g \cdot x \end{array}$$

is continuous.

(4) Let X be a topological space and let G be a group that acts on X. By the Group Action-Equivalence Relation Lemma 0.13 we know that

$$x \sim_G y \quad :\Longleftrightarrow \quad \text{there exists a } g \in G \text{ such that } g \cdot x = y$$

[72] The weird signs come from the fact that we want the map $B^n \to S^n$ to be orientation-preserving, whatever that means.

is an equivalence relation on X. We set $X/G := X/\sim_G$ and we view X/G as a topological space equipped with the quotient topology.

We formulate our first example as a lemma:

Lemma 6.6. (Torus-as-Quotient Lemma)

(1) The map
$$\begin{array}{rcl} \mathbb{Z}^n \times \mathbb{R}^n & \to & \mathbb{R}^n \\ (z, v) & \mapsto & z + v \end{array}$$
defines a free and continuous action of the group $(\mathbb{Z}^n, +)$ on the topological space $X = \mathbb{R}^n$.

(2) The map
$$\begin{array}{rcl} \mathbb{R}^n/\mathbb{Z}^n & \to & (S^1)^n \\ [(t_1, \ldots, t_n)] & \mapsto & \big(\exp(2\pi \mathrm{i} t_1), \ldots, \exp(2\pi \mathrm{i} t_n)\big) \end{array}$$
is a homeomorphism.

Proof of Lemma 6.6. We start out with the following observations:

(1) This statement follows easily from the definitions.

(2) (a) The map is continuous by the Topological-Product Proposition 5.1 (1b) and the Topological-Quotient Proposition 6.1 (1b). Furthermore it is straightforward to see that it is a bijection.

(b) The obvious map $[0,1]^n \to \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$ is continuous and it is easily seen to be surjective. By the Heine-Borel Theorem 1.12 we know that $[0,1]^n$ is compact. It follows from the Compact Image Lemma 2.8 that $\mathbb{R}^n/\mathbb{Z}^n$ is compact.

(c) From the Product Topology-Properties Proposition 5.4 (3) we obtain that $(S^1)^n$ is Hausdorff.

It follows from the above together with the Compact-Hausdorff Proposition 2.10 (3) that the given map $\mathbb{R}^n/\mathbb{Z}^n \to (S^1)^n$ is indeed a homeomorphism. ■

Example. If X is a Hausdorff space and if G acts continuously on X, then it is unfortunately not necessarily true that the quotient space X/G itself is Hausdorff. For example, consider the obvious action of $G = (\mathbb{Q}, +)$ on $X = \mathbb{R}$. It is a moderately insightful exercise to verify that the quotient space $\mathbb{R}/\mathbb{Q}$ is not Hausdorff. □

This example leads us to the following definition:

Definition. Let X be a topological space and let G be a group that acts on X. We say that G acts properly if for every two points[73] x and y in X there exist open neighborhoods U of x and V of y such that the set $\{g \in G \,|\, gU \cap V \neq \varnothing\}$ is finite.[74]

Examples.

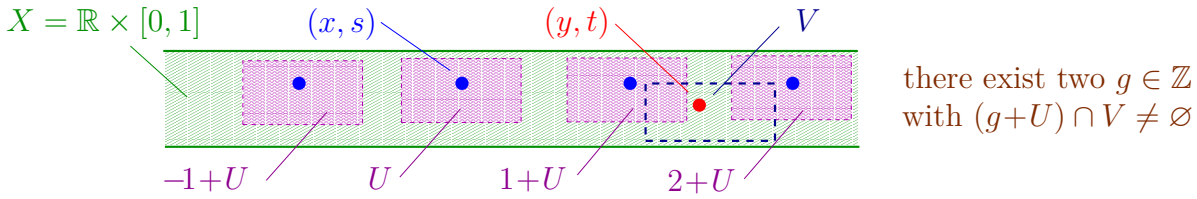
(1) The action of any finite group is proper by definition.

(2) We consider the action of the group $G = \mathbb{Z}$ on $X = \mathbb{R} \times [0,1]$ that is given by
$$\begin{array}{rcl} \mathbb{Z} \times (\mathbb{R} \times [0,1]) & \to & \mathbb{R} \times [0,1] \\ (n, (x, y)) & \mapsto & (x + n, y). \end{array}$$
This action is proper. For example if we are given by $(x, s), (y, t) \in \mathbb{R} \times [0,1]$, then $U := \{(a, b) \in \mathbb{R} \times [0,1] \,|\, |a - x| < 1\}$ and $V := \{(a, b) \in \mathbb{R} \times [0,1] \,|\, |a - y| < 1\}$ have

[73] Note that we allow $x = y$.

[74] The notion of a "proper action" is defined in different ways by different authors.

the desired properties. With some artistic freedom this situation is shown in the figure below.



(3) Similar to (2) one can easily verify that the action of $G = (\mathbb{Z}^n, +)$ on $X = \mathbb{R}^n$ is proper.
(4) One can easily show that the obvious action of $G = (\mathbb{Q}, +)$ on $X = \mathbb{R}$ is not proper. □

Proposition 6.7. (Group Action-Hausdorff Quotient Proposition) Let G be a group that acts on a topological space X. If X is Hausdorff and if the action is proper and continuous, then the quotient space X/G is also Hausdorff.

Proof. We outsource the uninspiring proof to [**Fri24**]. ■

Instead of dealing with the proof of the above proposition, let us look at an example:

Lemma 6.8. (Möbius Band-as-Quotient Lemma)

(1) The map

$$\begin{array}{ccc} \mathbb{Z} \times (\mathbb{R} \times [-1,1]) & \to & \mathbb{R} \times [-1,1] \\ (n,(x,y)) & \mapsto & (x+n, (-1)^n \cdot y) \end{array}$$

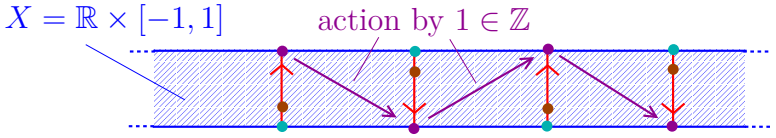
defines an action of the group $(\mathbb{Z}, +)$ on the topological space $X = \mathbb{R} \times [-1,1]$ that is free, continuous and proper.

(2) As on page 129 let $([0,1] \times [0,1])/\sim$ be the Möbius band, where $(0,y) \sim (1,1-y)$. The map

$$\begin{array}{ccc} ([0,1] \times [0,1])/\sim & \to & (\mathbb{R} \times [-1,1])/\sim \\ [(x,y)] & \mapsto & [(x, 2y-1)] \end{array}$$

is a homeomorphism.

(3) The Möbius band $M = ([0,1] \times [0,1])/\sim$ is compact and Hausdorff.



Identification. We will use the homeomorphism from the Möbius Band-as-Quotient Lemma 6.8 to identify the Möbius band $([0,1] \times [0,1])/\sim$ with $(\mathbb{R} \times [-1,1])/\sim$.

Proof of Lemma 6.8.

(1) It is clear that the action is free and continuous. The fact that the action is proper is proved as in the above example.

(2) It follows from the Twice Quotient Lemma 6.3 that the map is well-defined and continuous. One can easily verify that the map is a bijection. By the Quotient Topology-Properties Proposition 6.2 (1) we know that $([0,1] \times [0,1])/\sim$ is compact. Furthermore it follows from (1) and the Group Action-Hausdorff Quotient Proposition 6.7 that $(\mathbb{R} \times [0,1])/\sim$ is Hausdorff. Thus it follows from the Compact-Hausdorff Proposition 2.10 (3) that the map $([0,1] \times [0,1])/\sim \to (\mathbb{R} \times [0,1])/\sim$ is a homeomorphism.

(3) We showed in the proof of (2) that the Möbius band is compact and Hausdorff. ■

6.5. **Projective spaces.** In this section we introduce two related families of topological spaces, namely the real and complex projective spaces, which initially might look artificial, but which will play an essential role throughout these notes.

Definition.

(1) The map
$$\begin{array}{rcl}(\mathbb{R}\setminus\{0\})\times(\mathbb{R}^{n+1}\setminus\{0\}) & \to & \mathbb{R}^{n+1}\setminus\{0\}\\ (z,P) & \mapsto & z\cdot P\end{array}$$
defines a continuous action of the multiplicative group $(\mathbb{R}\setminus\{0\},\cdot)$ on the topological space $\mathbb{R}^{n+1}\setminus\{0\}$. The quotient space $(\mathbb{R}^{n+1}\setminus\{0\})/(\mathbb{R}\setminus\{0\})$ is called the n-dimensional real projective space $\mathbb{R}\mathrm{P}^n$. For $n=2$ we sometimes refer to $\mathbb{R}\mathrm{P}^2$ as the real projective plane.

(2) In (1) we can replace "$\mathbb{R}$" by "$\mathbb{C}$" and we obtain the n-dimensional complex projective space $\mathbb{C}\mathrm{P}^n := (\mathbb{C}^{n+1}\setminus\{0\})/(\mathbb{C}\setminus\{0\})$.

Remark. Informally speaking we can view $\mathbb{R}\mathrm{P}^n$ as the set of lines in $\mathbb{R}^{n+1}$ through the origin, or equivalently as the set of 1-dimensional vector subspaces of $\mathbb{R}^{n+1}$.

In our treatments of real and complex projective spaces we will often use the following notation:

Notation. For $(x_0,\dots,x_n)\in\mathbb{R}^{n+1}\setminus\{0\}$ we denote by $[x_0:\dots:x_n]$ the corresponding point in $\mathbb{R}\mathrm{P}^n$. We also use the same notation for points in complex projective spaces.

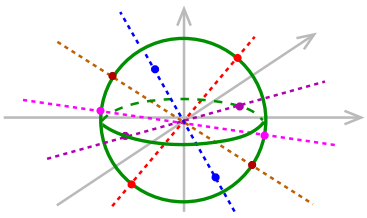
The following lemma gives us a convenient alternative description of projective spaces.

Lemma 6.9. (Projective Space-Sphere Quotient Lemma) Let $n\in\mathbb{N}_0$.

(1) The map
$$\begin{array}{rcl}\{\pm1\}\times S^n & \to & S^n\\ (w,(x_1,\dots,x_{n+1})) & \mapsto & (w\cdot x_1,\dots,w\cdot x_{n+1})\end{array}$$
defines a continuous action of the group $\{\pm1\}$ on S^n. The maps
$$\begin{array}{rcl}S^n/\{\pm1\} & \to & \mathbb{R}\mathrm{P}^n\\ [x] & \mapsto & [x]\end{array}\quad\text{and}\quad\begin{array}{rcl}\mathbb{R}\mathrm{P}^n & \to & S^n/\{\pm1\}\\ [x] & \mapsto & \left[\frac{x}{\|x\|}\right]\end{array}$$
are inverses of one another and they are homeomorphisms.

(2) The map
$$\begin{array}{rcl}S^1\times\overbrace{S^{2n+1}}^{\subset\mathbb{R}^{2n+2}=\mathbb{C}^{n+1}} & \to & S^{2n+1}\\ (w,(z_1,\dots,z_{n+1})) & \mapsto & (w\cdot z_1,\dots,w\cdot z_{n+1})\end{array}$$
defines a continuous action of the group $(S^1,\cdot)$ on S^{2n+1}. The maps
$$\begin{array}{rcl}S^{2n+1}/S^1 & \to & \mathbb{C}\mathrm{P}^n\\ [z] & \mapsto & [z]\end{array}\quad\text{and}\quad\begin{array}{rcl}\mathbb{C}\mathrm{P}^n & \to & S^{2n+1}/S^1\\ [z_1:\dots:z_{n+1}] & \mapsto & \left[\frac{(z_1,\dots,z_{n+1})}{\sqrt{|z_1|^2+\dots+|z_{n+1}|^2}}\right]\end{array}$$
are inverses of one another and they are homeomorphisms.

$S^2/\{\pm1\}$ is the "set of pairs of antipodal points on S^2"



$\mathbb{R}\mathrm{P}^2=(\mathbb{R}^3\setminus\{0\}/(\mathbb{R}\setminus\{0\})$ is the "set of lines through 0"

Identification. We use the homeomorphisms from the Projective Space-Sphere Quotient Lemma 6.9 to make the identifications $\mathbb{R}\mathrm{P}^n =_{\mathrm{i}} S^n/\{\pm 1\}$ and $\mathbb{C}\mathrm{P}^n =_{\mathrm{i}} S^{2n+1}/S^1$.

Proof of Lemma 6.9. It follows easily from the Twice Quotient Lemma 6.3 that all four maps are continuous. It is easy to verify that in (1) and (2) the maps are inverses of one another. ■

The following lemma shows that real and complex projective spaces have some of our favorite properties.

Proposition 6.10. (Projective Spaces-Compact+Hausdorff Proposition) The real projective and the complex projective spaces $\mathbb{R}\mathrm{P}^n$ and $\mathbb{C}\mathrm{P}^n$ are compact, path-connected and Hausdorff.

Example. Let $n \in \mathbb{N}_0$ and let $m \leq n$. We consider the maps

$$\begin{array}{rcl} i\colon \mathbb{R}\mathrm{P}^m & \to & \mathbb{R}\mathrm{P}^n \\ [x_0\colon\ldots\colon x_m] & \mapsto & [x_0\colon\ldots\colon x_m\colon 0\colon\ldots\colon 0] \end{array} \quad \text{and} \quad \begin{array}{rcl} j\colon \mathbb{C}\mathrm{P}^m & \to & \mathbb{C}\mathrm{P}^n \\ [z_0\colon\ldots\colon z_m] & \mapsto & [z_0\colon\ldots\colon z_m\colon 0\colon\ldots\colon 0]. \end{array}$$

It follows from the Twice Quotient Lemma 6.3, the Projective Spaces-Compact+Hausdorff Proposition 6.10 and the Compact-Hausdorff Proposition 2.10 (2) that both maps are embeddings. □

Example. For $n = 1$ we obtain homeomorphisms

$$\begin{array}{ccccc} & \text{homeomorphism from the Quotient Lemma 6.9} & & \text{homeomorphism by the Compact-Hausdorff Proposition 2.10 (3)} & \\ & \downarrow\cong & & \downarrow\cong & \\ \mathbb{R}\mathrm{P}^1 & \longrightarrow & S^1/\{\pm 1\} & \longrightarrow & S^1 \\ [x] & \mapsto & \left[\frac{x}{\|x\|}\right] & \mapsto & \frac{x}{\|x\|} \end{array} \xrightarrow{\cong} S^2.$$

Proof. Let $n \in \mathbb{N}_0$. By the Projective Space-Sphere Quotient Lemma 6.9 it suffices to show that $S^n/\{\pm 1\}$ and S^{2n+1}/S^1 are compact, path-connected and Hausdorff. In fact it follows immediately from the fact that S^n and S^{2n+1} are compact and path-connected together with the Quotient Topology-Properties Proposition 6.2 (1) and (2) that both quotients are compact and path-connected. Furthermore note that it follows from the Group Action-Hausdorff Quotient Proposition 6.7 that $S^n/\{\pm 1\}$ is Hausdorff. In [**Fri24**] it is shown, using the Quotient Topology-Properties Proposition 6.2 (3), that S^{2n+1}/S^1 is also Hausdorff. ■

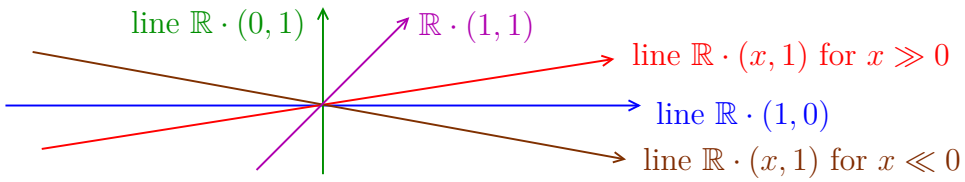
In the following lemma we give alternative descriptions of $\mathbb{R}\mathrm{P}^n$ and $\mathbb{C}\mathrm{P}^1$ that will be used on many occasions. For technical reasons the lemma is split into three parts.

Lemma 6.11. (Projective Space-Avatar Lemma)

(1) The maps

$$\begin{array}{rcl} \overbrace{\mathbb{C}\cup\{\infty\}}^{=_{\mathrm{i}}S^2} & \to & \mathbb{C}\mathrm{P}^1 \\ z & \mapsto & \begin{cases} [z:1], & \text{if } z \in \mathbb{C}, \\ [1:0], & \text{if } z = \infty \end{cases} \end{array} \quad \text{and} \quad \begin{array}{rcl} \mathbb{C}\mathrm{P}^1 & \to & \overbrace{\mathbb{C}\cup\{\infty\}}^{=_{\mathrm{i}}S^2} \\ [z_0:z_1] & \mapsto & \begin{cases} z_0\cdot z_1^{-1}, & \text{if } z_1 \neq 0, \\ \infty, & \text{if } [z_0:z_1]=[1:0] \end{cases} \end{array}$$

are inverses of one another and both are homeomorphisms. Similarly we see that $\mathbb{R}\mathrm{P}^1$ is homeomorphic to $\mathbb{R}\cup\{\infty\}$, which in turn is homeomorphic to S^1.



Lemma 6.11. (Projective Space-Avatar Lemma)

(2) Let $n \in \mathbb{N}_0$. If we denote by $\sim$ the equivalence relation on $\overline{B}^n$ that is generated by $P \sim -P$ for $P \in S^{n-1} = \partial\overline{B}^n$, then the maps[75]

$$\begin{array}{rcl} \overline{B}^n/\sim & \to & \mathbb{R}\mathrm{P}^n \\ [x] & \mapsto & [(x, \sqrt{1-\|x\|^2})] \end{array} \quad \text{and} \quad \begin{array}{rcl} \mathbb{R}\mathrm{P}^n =_{\mathrm{i}} S^n/\{\pm 1\} & \xrightarrow{\cong} & \overline{B}^n/\sim \\ [\underbrace{(v, x_{n+1})}_{v\in\mathbb{R}^n,\, x_{n+1}\in\mathbb{R}}] & \mapsto & \begin{cases} [v], & \text{if } x_{n+1} \geq 0, \\ [-v], & \text{if } x_{n+1} \leq 0 \end{cases} \end{array}$$

are homeomorphisms that are inverses of one another.

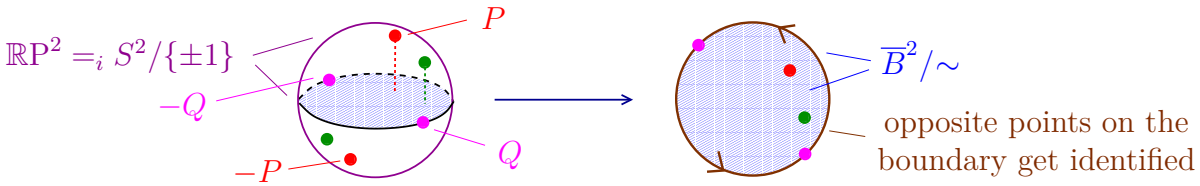


Illustration of the homeomorphism $\mathbb{R}\mathrm{P}^2 \to \overline{B}^2/\sim$

Lemma 6.11. (Projective Space-Avatar Lemma)

(3) As in (2) let $\sim$ be the equivalence relation on $\overline{B}^3$ that is generated by $x \sim -x$ for $x \in S^2$. The map

$$\begin{array}{rcl} F\colon \overline{B}^3/\sim & \to & \mathrm{SO}(3) \\ [\underbrace{r\cdot v}_{r\in[0,1] \text{ and } v\in S^2}] & \mapsto & \text{matrix in SO(3) which represents rotation by the angle } \pi\cdot r \\ & & \text{around the axis } \mathbb{R}_{\geq 0}\cdot v \text{ where the rotation} \\ & & \text{direction is given by the right-hand rule} \end{array}$$

is a homeomorphism. In particular SO(3) is homeomorphic to $\mathbb{R}\mathrm{P}^3$.

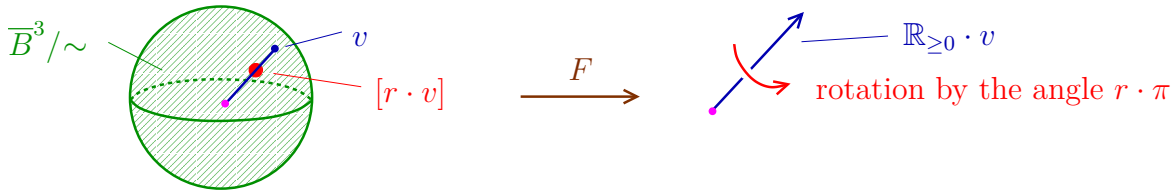


Illustration of the homeomorphism $\overline{B}^3/\sim \to \mathrm{SO}(3)$

Identification.

(1) We will use the homeomorphism from the Projective Space-Avatar Lemma 6.11 (1) to make the identification $S^1 =_{\mathrm{i}} \mathbb{R}\mathrm{P}^1$.

[75] Note that the map to the left is in fact well-defined even if $x_{n+1} = 0$.

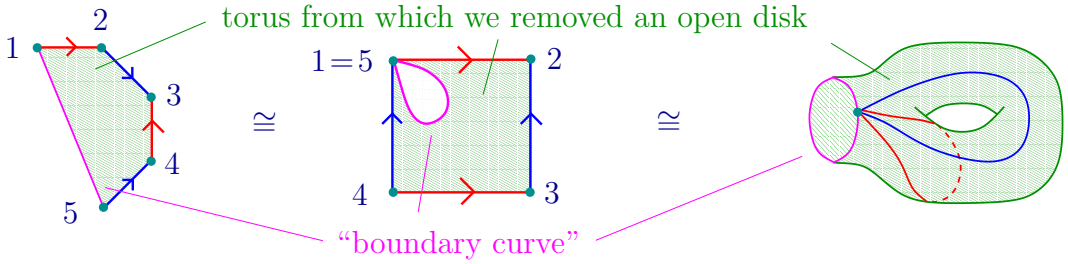
(2) Recall that in the Stereographic Projection Lemma 2.11 (1) we gave an explicit homeomorphism $S^2 \to \mathbb{R}^2 \cup \{\infty\}$. Combining this homeomorphism with the homeomorphism from the Projective Space-Avatar Lemma 6.11 (1) we obtain a homeomorphism $S^2 \to \mathbb{C}\mathrm{P}^1$ which we use to make the identification $S^2 =_{\mathrm{i}} \mathbb{C}\mathrm{P}^1$. Note that under this identification the North Pole $(0,0,1) \in S^2$ corresponds to the point $[1:0] \in \mathbb{C}\mathrm{P}^1$.
(3) We will use the homeomorphism from the Projective Space-Avatar Lemma 6.11 (2) to make the identification $\mathbb{R}\mathrm{P}^n =_{\mathrm{i}} \overline{B}^n/\!\sim$.

Proof of Lemma 6.11.

(1) One can easily verify that the given maps are inverses of one another. Next we show that the given map $\mathbb{C}\mathrm{P}^1 \to \mathbb{C} \cup \{\infty\}$ is continuous. First note that it follows from the Topological-Quotient Proposition 6.1 (1b) that it suffices to show that the corresponding map $\Theta\colon \mathbb{C}^2 \setminus \{(0,0\} \to \mathbb{C} \cup \{\infty\}$ is continuous. Next note that for any $C \in \mathbb{R}$ we have
$$\Theta^{-1}(\text{complement of } \overline{B}^n_C) \;=\; \{(z_0,z_1) \in \mathbb{C}^2 \setminus \{(0,0)\} \,|\, |z_0| > |z_1|\}$$
which is an open subset of $\mathbb{C}^2 \setminus \{(0,0)\}$. It follows quite easily from this discussion together with the Subbasis-Continuity Proposition 2.2 and together with the discussion of the basis of $\mathbb{R}^2 \cup \{\infty\}$ on page 65 that the map Θ is continuous. By the Projective Spaces-Compact+Hausdorff Proposition 6.10 we know that each $\mathbb{C}\mathrm{P}^1$ is compact. We leave it to the reader to verify that $\mathbb{R}^2 \cup \{\infty\}$ is Hausdorff. Thus it follows from the Compact-Hausdorff Proposition 2.10 (3) that the map $\mathbb{C}\mathrm{P}^1 \to \mathbb{C} \cup \{\infty\}$ is a homeomorphism.
(2) It is elementary to verify that the given continuous maps are inverses of one another, in particular they are bijections. We still need to show that the maps are continuous.
  (a) We consider the map $S^n/\{\pm 1\} \to \overline{B}^n/\!\sim$. By the Topological-Quotient Proposition 6.1 (1b) it suffices to show that the corresponding map $S^n \to \overline{B}^n/\!\sim$ is continuous. One can easily verify that the restrictions of the map $S^n \to \overline{B}^n/\!\sim$ to the closed subsets $S^n_{\geq 0}$ and $S^n_{\leq 0}$ are continuous. Thus it follows from the Pasting Proposition 2.4 (2) that the map $S^n \to \overline{B}^n/\!\sim$ is continuous.
  (b) It follows almost immediately from the Twice Quotient Lemma 6.3 that the given map $\overline{B}^n/\!\sim \to \mathbb{R}\mathrm{P}^n$ is continuous.
(3) The proof that $F\colon \overline{B}^3/\!\sim \to \mathrm{SO}(3)$ is a homeomorphism is given in [**Fri24**]. ■

6.6. **Examples: Surfaces II.** In Section 6.2 we already introduced some surfaces. It is time to introduce a few more.

Example. We start out with a slightly informal discussion of an example. In the figure below we see to the left a pentagon, where we identify two pairs of sides.



In the figure we illustrate that this topological space is homeomorphic to the torus minus an open disk.

□

The above example motivates our next example:

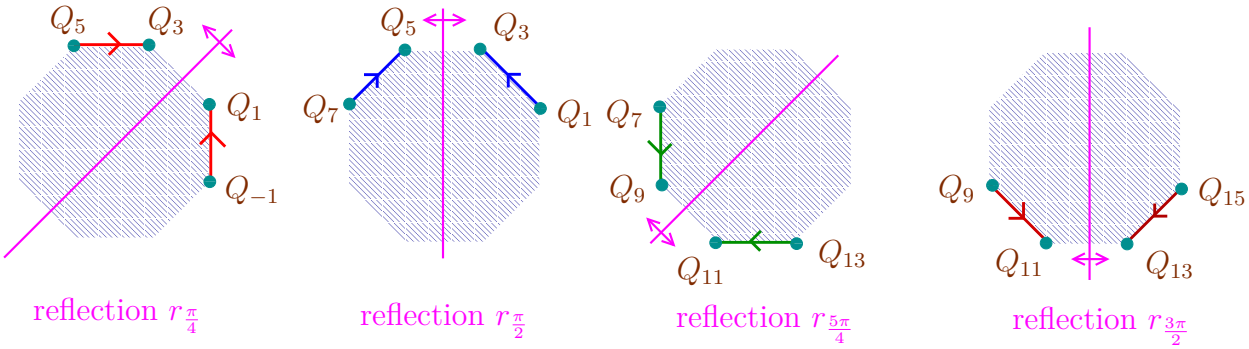
Example. We introduce the following notation:

(1) Let E_8 be the octagon in $\mathbb{C}$ with the vertices $Q_k = \exp(2\pi \mathrm{i} k/16)$, $k = 1, 3, 5, 7, \ldots, 15$, i.e. E_8 is the convex hull of these eight points in $\mathbb{R}^2$.
(2) Given two points $A, B \in \mathbb{R}^2 =_{\mathrm{i}} \mathbb{C}$ let $\overline{AB}$ be the Euclidean segment from A to B.
(3) Given $\varphi \in \mathbb{R}$ let $r_\varphi \colon \mathbb{C} \to \mathbb{C}$ be the reflection in the Euclidean line $\{t \cdot \exp(\mathrm{i}\varphi) \,|\, t \in \mathbb{R}\}$.

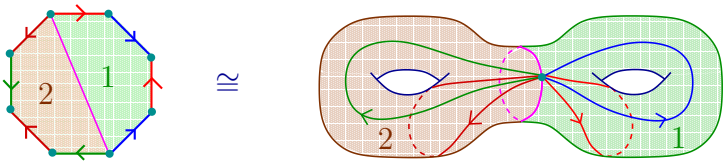
We denote by $\sim$ the equivalence relation on E_8 that is generated by

$$P \in \overline{Q_{2k-1}Q_{2k+1}} \;\sim\; r_{2\pi(2k+2)/16}(P) \;\in\; \overline{Q_{2k+3}Q_{2k+5}}$$

for $k = 0, 1, 4, 5$. We consider the quotient space $E_8/\!\sim$.



Note that $E_8/\!\sim$ is given by two copies of the previous example glued together along the "boundary curves".



two copies of torus minus an open disk glued along the boundary

In other words, $E_8/\!\sim$ is homeomorphic to the "physical" surface of genus 2 in $\mathbb{R}^3$. □

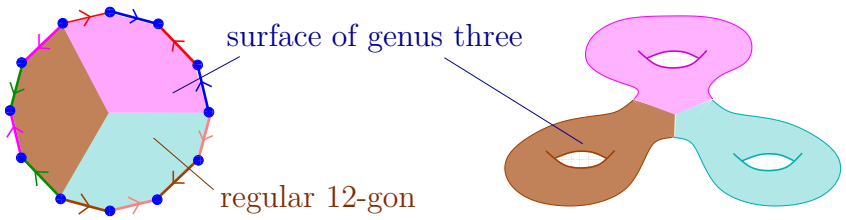
The above example leads us to the following definition:

Definition.

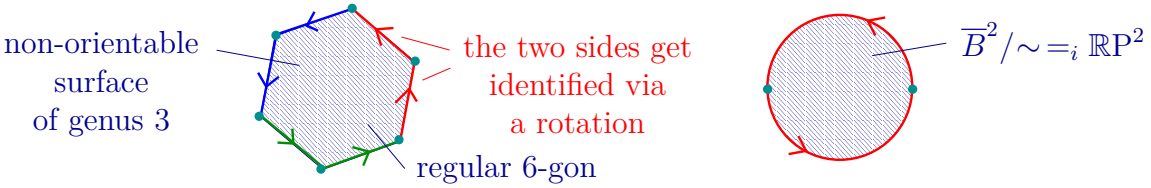
(1) Let $g \in \mathbb{N}$. We take a regular $4g$-gon E_{4g} in $\mathbb{C}$ and for $j = 0, \ldots, g-1$ we use reflections to identify the $(4j+1)$-st edge with the $(4j+3)$-rd edge and the $(4j+2)$-nd edge with the $(4j+4)$-th edge. We refer to the quotient space $E_{4g}/\!\sim$ as the surface of genus g.
(2) We refer to the sphere S^2 as the surface of genus 0.

Example. On page 130 we saw that the surface of genus 1 is homeomorphic to the torus $S^1 \times S^1$. □

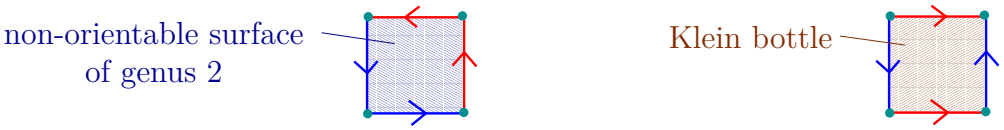
We move on to a closely related definition.



Definition.
(1) Given $g \in \mathbb{N}_{\geq 2}$ we consider the regular $2g$-gon E_{2g} with the identification of the boundary segments shown in the figure below. We refer to this topological space as the non-orientable surface N_g of genus g.
(2) We refer to the real projective space $\mathbb{R}\mathrm{P}^2$ as the non-orientable surface of genus 1.



Example. We leave it to the reader to show that the non-orientable surface of genus 2 is homeomorphic to the Klein bottle. □



6.7. **The pushout in the topological category.** We conclude our long list of various types of quotients of topological spaces with a discussion of pushouts in the category of topological spaces. The following proposition is formulated in the tradition of the Topological-Product Proposition 5.1,f the Topological-Coproduct Proposition 5.12 and the Topological-Quotient Proposition 6.1.

Proposition 6.12. (Topological-Pushout Proposition) Let $f\colon A \to Y$ and $g\colon A \to Z$ be two continuous maps between topological spaces. We consider the topological space

$$Y \cup_A Z \;:=\; (Y \sqcup Z)/\sim \qquad \text{where } f(a) \sim g(a) \text{ for all } a \in A.$$

and the maps

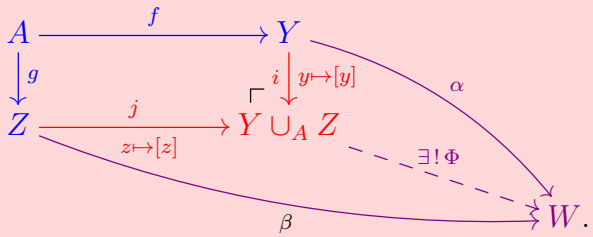
$$\begin{array}{rcl} i\colon Y & \to & Y \cup_A Z \\ y & \mapsto & [y] \end{array} \qquad \text{and} \qquad \begin{array}{rcl} j\colon Z & \to & Y \cup_A Z \\ z & \mapsto & [z] \end{array}$$

The following statements hold:
(0) A subset $U \subset Y \cup_A Z$ is open if and only if $i^{-1}(U) \subset Y$ and $j^{-1}(U) \subset Z$ are open. The same statement holds for "closed" instead of "open".
(1) (a) The maps $i\colon Y \to Y \cup_A Z$ and $j\colon Z \to Y \cup_A Z$ are continuous.

(b) Suppose we are given two continuous maps $\alpha\colon Y \to W$ and $\beta\colon Z \to W$ to some topological space W such that $\alpha \circ f = \beta \circ g\colon A \to W$. The map

$$\begin{array}{rcl} \Phi\colon Y \cup_A Z & \to & W \\ [P] & \mapsto & \begin{cases} \alpha(y) & \text{if } P = [y] \text{ for some } y \in Y, \\ \beta(z) & \text{if } P = [z] \text{ for some } z \in Z \end{cases} \end{array}$$

is well-defined and it is the unique map $\Phi\colon Y \cup_A Z \to W$ with $\Phi \circ i = \alpha$ and $\Phi \circ j = \beta$. Furthermore, if α and β are continuous, then Φ is also continuous. In particular we obtain the following commutative diagram:



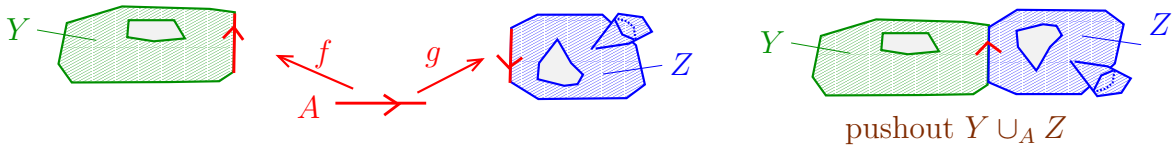
(∗) A map $h\colon Y \cup_A Z \to W$ to some topological space W is continuous if and only if the maps $h \circ i\colon Y \to W$ and $h \circ j\colon Z \to W$ are continuous.

(2) The topological space $Y \cup_A Z$ with the natural maps $Y \to Y \cup_A Z$ and $Z \to Y \cup_A Z$ is the pushout, in the sense of the definition on page 111, in the category $\mathcal{Top}$ of topological spaces.

Definition. Let $f\colon A \to Y$ and $g\colon A \to Z$ be two continuous maps between topological spaces. We refer to

$$Y \cup_A Z \;\coloneqq\; (Y \sqcup Z)/\!\sim \qquad \text{where } f(a) \sim g(a) \text{ for all } a \in A$$

as the pushout of f and g and we refer to the maps $i\colon Y \to Y \cup_A Z$ and $j\colon Z \to Y \cup_A Z$ as the two natural maps.



Proof.

(0) This statement follows from the Topological-Coproduct Proposition 5.12 (0) and the Topological-Quotient Proposition 6.1 (0).

(1) (a) This statement follows immediately from (0). Alternatively this statement follows from the Topological-Coproduct Proposition 5.12 (1a) and the Topological-Quotient Proposition 6.1 (1a).

(b) This existence of the continuous map follows immediately from the Topological-Coproduct Proposition 5.12 (1b) and the Topological-Quotient Proposition 6.1 (1b). We leave the proof of the uniqueness to the reader.

(∗) The "only if"-statement follows from (1a) and the fact that the composition of two continuous maps is continuous. The "if"-statement follows easily from (1b).

(2) This statement follows easily from the definitions and Statement (1). We leave it to the reader to fill in the details. ∎

Example. Let X be a topological space, let $A \subset X$ be a subspace, let $f\colon A \to \{*\}$ be the constant map to the topological space consisting of a single point and let $g\colon A \to X$ be the inclusion map. Using the uniqueness statement of the Quotient-Factorization Lemma 0.5 and using the Topological-Quotient Proposition 6.1 one can show that the quotient X/A, as defined on page 131, is a pushout of the pushout diagram. In summary we have the following pushout diagram:

$$\begin{array}{ccc} A & \xrightarrow{f} & \{*\} \\ {\scriptstyle g}\downarrow & & \downarrow \\ X & \xrightarrow{x\mapsto[x]} & X/A. \end{array}$$

From the uniqueness statements for pushouts, see the Colimit Uniqueness Lemma 4.7, it follows that X/A is homeomorphic to the pushout $X \cup_A \{*\}$.[76] □

The following proposition is the analogue of the Disjoint Union Topology-Properties Lemma 5.13 and the Quotient Topology-Properties Proposition 6.2.

Proposition 6.13. (Pushout-Properties Proposition) Let $f\colon A \to Y$, $g\colon A \to Z$ be two continuous maps between topological spaces. We consider the corresponding pushout $Y \cup_A Z := (Y \sqcup Z)/\sim$.

(1) If Y and Z are compact, then so is the pushout $Y \cup_A Z$.
(2) If Y and Z are (path-) connected and if A is non-empty, then the pushout $Y \cup_A Z$ is also (path-) connected.
(3) If A, Y and Z are compact, if Y and Z are Hausdorff and if $f\colon A \to Y$ and $g\colon A \to Z$ are injective, then $Y \cup_A Z$ is also Hausdorff.

Proof.

(1) If Y and Z are compact, then it follows from the Disjoint Union Topology-Properties Lemma 5.13 (1) and the Quotient Topology-Properties Proposition 6.2 (1) that $Y \cup_A Z$ is compact.
(2) If Y and Z are (path-) connected and if A is non-empty, then it follows easily from the Topological Pushout Proposition 6.12 Image-Connected Lemma 2.16 and the Union Connected Lemma 2.19 that $Y \cup_A Z$ is also (path-) connected.
(3) Since f and g are injective we see that the relation on $Y \sqcup Z$ is given by
$$\{(f(a), g(a)) \,|\, a \in A\} \cup \{(w, w) \,|\, w \in Y \sqcup Z\} \subset (Y \sqcup Z) \times (Y \sqcup Z).$$
It follows from the Compact Image Lemma 2.8 and the Compact-Closed Lemma 1.9 (2), together with the hypotheses that A, Y, Z are compact and Y, Z are Hausdorff, that this is a closed subset. It now follows from the Quotient Topology-Properties Proposition 6.2 (3) that $Y \cup_A Z$ is Hausdorff. ■

Example. Let $n \in \mathbb{N}_0$. Let $\overline{B}^n_\pm$ be two copies of $\overline{B}^n$. We consider the two obvious inclusion maps $S^{n-1} \to \overline{B}^n_\pm$ and the corresponding pushout $\overline{B}^n_+ \cup_{S^{n-1}} \overline{B}^n_-$. As in the Stereographic

[76] Note that this even works if $A = \varnothing$.

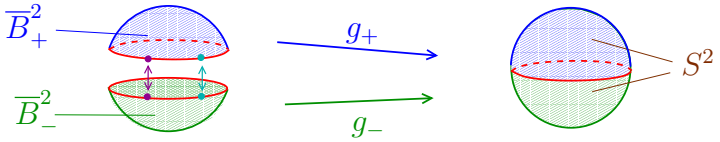
Projection Lemma 2.11 we consider for $\epsilon = -1, 1$ the homeomorphisms[77]

$$\begin{array}{rcl} g_\epsilon \colon \overline{B}^n_\epsilon & \to & S^n_{\geq 0} \\ x = (x_1, \ldots, x_n) & \mapsto & \Big(((-1)^{n+1} \cdot x_1, x_2, \ldots, x_n) \cdot \frac{2}{\|x\|^2+1}, \epsilon \cdot \frac{\|x\|^2-1}{\|x\|^2+1} \Big). \end{array}$$

Note that $g_+ = g_-$ on S^{n-1}. Thus it follows from the Topological Pushout Proposition 6.12 (1b) that the map

$$\begin{array}{rcl} \overline{B}^n_+ \cup_{S^{n-1}} \overline{B}^n_- & \to & S^n \\ [P] & \mapsto & \begin{cases} g_+(P), & \text{if } P \in \overline{B}^n_+, \\ g_-(P), & \text{if } P \in \overline{B}^n_- \end{cases} \end{array}$$

is continuous. One can easily verify that it is a bijection.



It follows from the Pushout-Properties Proposition 6.13 (1) combined with the Compact-Hausdorff Proposition 2.10 (3) that the map is a homeomorphism.[78][79] □

Identification. We use the homeomorphism from the above example to make the identification $\overline{B}^n_+ \cup_{S^{n-1}} \overline{B}^n_- =_{\mathrm{i}} S^n$.

Lemma 6.14. (Topological Pushout-Maps Lemma) Let $f \colon A \to Y$ and $g \colon A \to Z$ be two continuous maps between topological spaces. We obtain the following commutative diagram:

$$\begin{array}{ccc} A & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle i} \; {\scriptstyle y \mapsto [y]} \\ Z & \xrightarrow[z \mapsto [z]]{j} & Y \cup_A Z. \end{array}$$

The following statements hold:

(1) If f is a homeomorphism, then the "opposite map" j is a homeomorphism.
(2) If f is injective, then the "opposite map" j is injective. The same statement holds for "surjective" and "bijective".[80]

[77] We put in the superfluous looking sign $(-1)^{n+1}$ for the following reason: once we have introduced orientations of smooth manifolds and once we have introduced the standard orientations of $\overline{B}^n$ and S^n this sign assures us that the map $g_+ \colon \overline{B}^n \to S^n$ is orientation-preserving and it means that $g_- \colon \overline{B}^n \to S^n$ is orientation-reversing.

[78] In [**Lip14**, Chapter 2] and [**Pet79**] the argument is made that the universe in Dante's Divina Commedia consists of two 3-balls (corresponding to heaven and hell) that are glued along the boundary. In other words, Dante describes the 3-sphere.

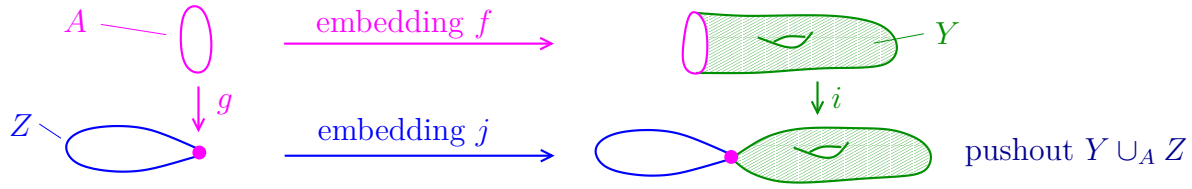
[79] We could also have worked with the maps

$$\begin{array}{rcl} h_\pm \colon \overline{B}^n = \overline{B}^n_\pm & \to & S^n \\ (x_1, \ldots, x_n) & \mapsto & \big((-1)^n \cdot x_1, x_2, \ldots, x_n, \pm\sqrt{1 - x_1^2 - \cdots - x_n^2} \big). \end{array}$$

The advantage of our maps $g_\pm$ is that they are not only homeomorphisms but they are actually "diffeomorphisms".

(3) If f is an embedding, then the "opposite map" j is an embedding. In fact if f is an open (closed) embedding, then the "opposite map" j is also an open (closed) embedding.

By symmetry the statements in (1), (2) and (3) hold also for g and its "opposite map" i.



Proof. Throughout the proof we denote by $p\colon Y \sqcup Z \to Y \cup_A Z$ the natural projection.

(1) This statement is a special case of the Pushout Lemma 4.8 (2).

(2) First we assume that $f\colon A \to Y$ is injective. We need to show that $j\colon Z \to Y \cup_A Z$ is injective. Let $z \in Z$. Note that

$$\begin{aligned} j^{-1}(j(z)) \;=\; \{z' \in Z \,|\, j(z') = j(z)\} \;&=\; \text{set of elements in } Z \text{ equivalent to } z \\ &\qquad \text{with respect to } \sim \text{ on } Y \sqcup Z \\ &= \bigcup_{k\in\mathbb{N}_0} \underbrace{g(f^{-1}(f(g^{-1}(\dots g(f^{-1}(f(g^{-1}}_{\text{applied } k \text{ times}}(\{z\}))))\dots)))) \\ &= \bigcup_{k\in\mathbb{N}_0} g(g^{-1}(\dots g(g^{-1}(\{z\}))\dots)) \;=\; \{z\}. \end{aligned}$$

by the Image-Preimage Lemma 0.3 (1b), since f is injective — Image-Preimage Lemma 0.3 (2a)

This shows that the map j is injective.

Now we assume that $f\colon A \to Y$ is surjective. We need to show that $j\colon Z \to Y \cup_A Z$ is surjective. Recall that by definition $Y \cup_A Z := (Y \sqcup Z)/\!\sim$ where $f(a) \sim g(a)$ for all $a \in A$. Let $[w] \in Y \cup_A Z$. If $w \in Z$, then evidently $j(w) = [w]$. Now consider the case that $w \in Y$. Since $f\colon A \to Y$ is surjective there exists an $a \in A$ with $f(a) = w$. But now we see that $j(g(a)) = i(f(a)) = i(w) = [w]$.

(3) We suppose that $f\colon A \to Y$ is an embedding. We need to show that $j\colon Z \to Y \cup_A Z$ is an embedding. By the Topological Pushout Proposition 6.12 (1a) we know that j is continuous and by (2) we know that $j\colon Z \to Y \cup_A Z$ is injective. By the Open-Injective Lemma 2.9 (2) and by definition of the subspace topology on $j(Z)$ it remains to prove the following claim:

Claim. If U is an open subset of Z, then there exists an open subset W of $Y \cup_A Z$ such that $j(U) = W \cap j(Z)$.

Proof. Let U be an open subset of Z. Since g is continuous we know that $g^{-1}(U)$ is an open subset of A. Since $f\colon X \to Y$ is an embedding we know that there exists an open subset $V \subset Y$ with $f(g^{-1}(U)) = f(A) \cap V$. We set $W := p(U \sqcup V)$. An elementary argument as in (2), that uses that f is injective, shows that $p^{-1}(W) = U \sqcup V$. Thus it follows from the definition of the quotient topology and the disjoint union topology that W is indeed an open subset of $Y \cup_A Z$. Basically by construction we have $j(U) = W \cap j(Z)$. ∎

[80] This statement has nothing to do with topological spaces, the statement also holds for sets.

CHAPTER 7

Graphs

In this chapter we will first introduce the purely combinatorial concept of an abstract graph. Afterwards we introduce the corresponding topological realizations. These notions are of interest for various reasons. Abstract graphs are ubiquitous in mathematics and at times they will also play a useful role in our discussions. The corresponding topological realizations form fairly simple, but nonetheless interesting examples of topological spaces.

7.1. Abstract graphs.

We start out with the following rather abstract definition.

Definition. An abstract graph G is a quadruple (V, E, i, t) where V is a set, E is a set and $i, t\colon E \to V$ are maps from E to V.

(1) The elements of V are called the vertices of G and the elements of E are called the edges of G.
(2) Given $e \in E$ the point $i(e)$ is called the initial point of e and $t(e)$ is called the terminal point of e. We refer to $i(e)$ and $t(e)$ as the vertices of e.
(3) We say that an edge $e \in E$ is a loop if $i(e) = t(e)$.
(4) We say that G is finite (countable) if both V and E are finite (countable).

Example. We take $V = \{A, B, C\}$ and $E = \{e, f, g, h\}$. Furthermore let $i\colon E \to V$ and $t\colon E \to V$ be the maps that are given by the following table:

	e	f	g	h
i	A	B	B	A
t	B	C	C	A

The quadruple (V, E, i, t) is an abstract graph. We can try to visualize such an abstract graph by drawing a point for each vertex and by drawing an arrow for each edge e that points from the initial point $i(e)$ to the endpoint $t(e)$.



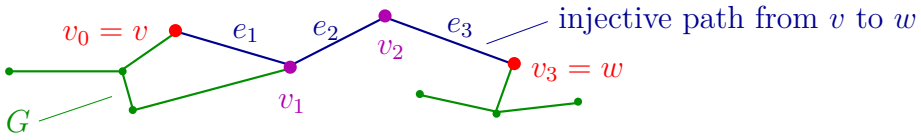
The above figure already indicates that abstract graphs give rise to topological spaces. The following definition is already somewhat topology-like.

Definition. Let $G = (V, E, i, t)$ be an abstract graph.

(1) Let $v, w \in V$. A path of length $k \in \mathbb{N}$ from v to w is a sequence of edges $e_1, \ldots, e_k$ such that there exist vertices $v = v_0, v_1, \ldots, v_k = w$ such that for each $i \in \{1, \ldots, k\}$

the vertices of e_i are v_{i-1}, v_i. We say that the path $e_1, \dots, e_k$ is injective if the vertices $v_0, \dots, v_k$ are distinct. If $v = w$, then we refer to the empty path as an injective path of length zero from v to v.

(2) We say that G is connected if for every pair of vertices $v, w \in V$ there exists a path from v to w. One can easily show that if G is connected, then any two vertices can also be connected via an injective path.



Example. The above abstract graph is connected, for example if we take $v = A$ and $w = C$, then $v_0 = A$, $v_1 = B$ and $v_2 = C$ and $e_0 = e$ and $e_1 = f$ have the desired property. □

Definition.

(1) Let
$$\begin{aligned} G &= (V, E, i\colon E \to V, t\colon E \to V) \\ \text{and} \quad \widetilde{G} &= (\widetilde{V}, \widetilde{E}, \widetilde{i}\colon \widetilde{E} \to \widetilde{V}, \widetilde{t}\colon \widetilde{E} \to \widetilde{V}) \end{aligned}$$
be two abstract graphs in the sense of the definition on page 146. We define a morphism $G \to \widetilde{G}$ of abstract graphs to be a pair $(\alpha\colon V \to \widetilde{V}, \beta\colon E \to \widetilde{E})$ of maps such that for each $e \in E$ we have $\widetilde{i}(\beta(e)) = \alpha(i(e))$ and $\widetilde{t}(\beta(e)) = \alpha(t(e))$.

(2) We refer to the category $\mathcal{AbsGraph}$ with
$$\begin{aligned} \mathrm{Ob}(\mathcal{AbsGraph}) &:= \text{all abstract graphs,} \\ \mathrm{Mor}(G, \widetilde{G}) &:= \text{all maps from } G \text{ to } \widetilde{G} \text{ in the above sense} \end{aligned}$$
with the usual composition of maps as the category of abstract graphs.

Example. The maps
$$\begin{aligned} (V, E, i\colon E \to V, t\colon E \to V) &\mapsto (V, E, t\colon E \to V, i\colon E \to V) \\ (f\colon G \to \widetilde{G}) &\mapsto (f\colon G \to \widetilde{G}) \end{aligned}$$
define a covariant functor from the category $\mathcal{AbsGraph}$ to itself. The effect on each abstract graph is that it swaps the roles of the initial points and the endpoints. Pictorially speaking the functor "inverts the orientations of the arrows". □

The following definition is almost self explanatory.

Definition. Let $G = (V, E, i, t)$ be an abstract graph. A subgraph of $G = (V, E, i, t)$ is an abstract graph $G' = (V', E', i', t')$ such that $V' \subset V$ and $E' \subset E$ are subsets such that $i' = i|_{E'}$ and $t' = t|_{E'}$. Since i' and t' are determined by V' and E' we often suppress both of them from the notation.

Example.

(1) Every $v \in V$ gives rise to the subgraph with a single vertex v and no edge.

(2) Let $\widetilde{G} = (\widetilde{V}, \widetilde{E}, \widetilde{i}, \widetilde{t})$ be an abstract graph and let $G = (V, E, i, t)$ be a subgraph. It follows immediately from the definitions that the inclusions $V \to \widetilde{V}$ and $E \to \widetilde{E}$ define a map $\iota\colon G \to \widetilde{G}$ of abstract graphs in the above sense. □

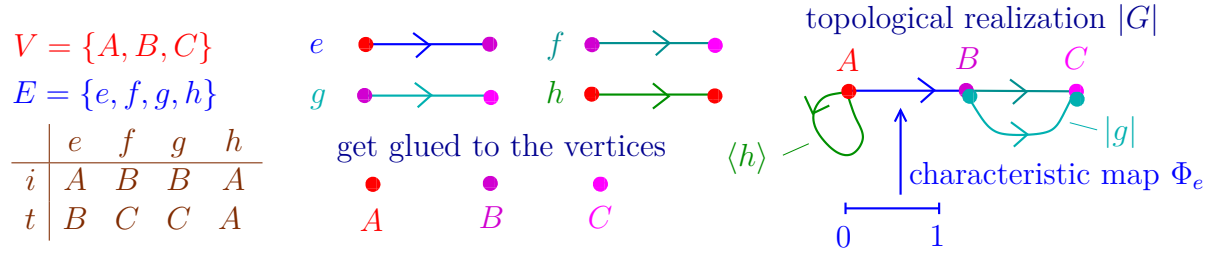
7.2. **The topological realization of an abstract graph.** Surely the reader is bothered by the absence of topology. We will change this sorry state of affairs by introducing the topological realization of a graph. The idea is, basically, that the topological realization of an abstract graph $G = (V, E, i, t)$ is given by one point for each vertex $v \in V$ and one interval for each edge $e \in E$ with endpoints given by $i(e)$ and $t(e)$. The definition below is the formalization of that idea.

Definition. Let $G = (V, E, i, t)$ be an abstract graph.

(1) We define the topological realization of G as the topological space

$$|G| \;\coloneqq\; \Big(\underbrace{V}_{\substack{\uparrow\\ \text{here } V \text{ is equipped}\\ \text{with the } \textit{discrete} \text{ topology}}} \sqcup \bigsqcup_{e\in E} [0,1]\Big)\Big/\sim \qquad \begin{array}{l}\text{where for any } e \in E \text{ we have}\\ (e,0) \sim i(e) \text{ and } (e,1) \sim t(e).\end{array}$$

(2) Let $e \in E$.
   (a) We refer to the map $\Phi_e \colon [0,1] \xrightarrow{x\mapsto(e,x)} \{e\} \times [0,1] \to |G|$ as the characteristic map of the edge e.
   (b) We write $|e| = \Phi_e([0,1])$ and we write $\langle e\rangle \coloneqq \Phi_e((0,1))$.

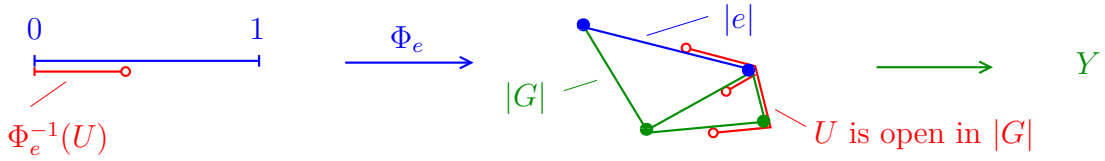


The following lemma gives us a practical way to understand the topology on the topological realization of an abstract graph.

Lemma 7.1. (Abstract Graph-Topology Lemma) Let $G = (V, E, i, t)$ be an abstract graph.

(1) For each $e \in E$ the characteristic map $\Phi_e \colon [0,1] \to |G|$ is continuous.
(2) Let $U \subset |G|$ be a subset.
   (a) $U \subset |G|$ is open $\iff$ for each $e \in E$ the preimage $\Phi_e^{-1}(U)$ is open in $[0,1]$.
   (b) $U \subset |G|$ is closed $\iff$ for each $e \in E$ the preimage $\Phi_e^{-1}(U)$ is closed in $[0,1]$.
(3) Let $f \colon |G| \to X$ be a map to some topological space X.

$$f \text{ is continuous} \iff \text{for each } e \in E \text{ the map } f \circ \Phi_e \colon [0,1] \to X \text{ is continuous.}$$



Example. Let $G = (V, E, i, t)$ be an abstract graph. The following statements follow easily from the Abstract Graph-Topology Lemma 7.1:

(1) The obvious map $V \to |G|$ is injective and the image is a discrete subset of $|G|$.[81]
(2) Let $e \in E$. Let $\Phi_e \colon [0,1] \to |G|$ be the characteristic map.
  (a) $|e|$ is a closed subset of $|G|$ and $\langle e \rangle$ is an open subset of $|G|$.
  (b) If $i(e) \neq t(e)$, then Φ_e restricts to a homeomorphism $[0,1] \to |e|$.
  (c) If $i(e) = t(e)$, then Φ_e induces a homeomorphism $[0,1]/0 \sim 1 \to |e|$. □

Proof of Lemma 7.1.

(1) This statement follows immediately from the Topological-Coproduct Proposition 5.12 (1a) and the Topological-Quotient Proposition 6.1 (1a).
(2) (a) This statement follows easily from the Topological-Coproduct Proposition 5.12 (0a) and the Topological-Quotient Proposition 6.1 (0a) and the observation that any subspace of a topological space consisting of a single point is open.
  (b) This statement follows easily from the Topological-Coproduct Proposition 5.12 and the Topological-Quotient Proposition 6.1 and the observation that any subspace of a topological space consisting of a single point is closed.
(3) This statement follows immediately from (1) and (2a). ■

The following lemma makes the unsurprising assertion that the above assignment of a topological space to an abstract graph complex is functorial.

Lemma 7.2. (Topological Realization–Functor Lemma) The following two statements hold:

(1) Let $G = (V, E, i, t)$ and $\widetilde{G} = (\widetilde{V}, \widetilde{E}, \widetilde{i}, \widetilde{t})$ be two abstract graphs. Furthermore let $f := (\alpha \colon V \to \widetilde{V}, \beta \colon E \to \widetilde{E})$ be a map between the two abstract graphs. We consider the following map between the topological realizations of G and $\widetilde{G}$:

$$\begin{array}{ccc} & \text{where } (e,0) \sim i(e) \text{ and } (e,1) \sim t(e) & \text{similarly } (e,0) \sim \widetilde{i}(e) \text{ and } (e,1) \sim \widetilde{t}(e) \\ |f| \colon |G| = \Big(V \sqcup \bigsqcup_{e \in E} [0,1]\Big) \Big/ \overset{\downarrow}{\sim} & \to & |\widetilde{G}| = \Big(\widetilde{V} \sqcup \bigsqcup_{e \in \widetilde{E}} [0,1]\Big) \Big/ \overset{\downarrow}{\sim} \\ [x] & \mapsto & \begin{cases} \alpha(x), & \text{if } x \in V, \\ (\beta(e), t), & \text{if } x = (e,t) \in E \times [0,1]. \end{cases} \end{array}$$

This map $|f| \colon |G| \to |\widetilde{G}|$ is well-defined and continuous.

(2) The map

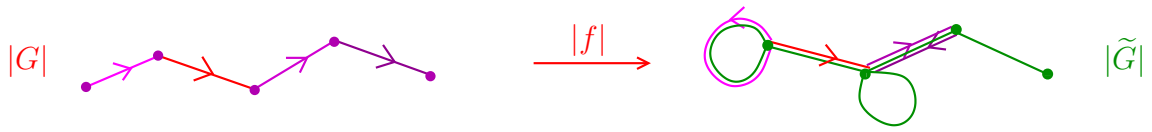
$$\begin{array}{rcl} F \colon \mathrm{Ob}(\mathcal{AbsGraph}) & \to & \mathrm{Ob}(\mathcal{Top}) \\ G & \mapsto & |G|, \end{array}$$

and the maps

$$\begin{array}{rcl} \mathrm{Mor}_{\mathcal{AbsGraph}}(G, \widetilde{G}) & \to & \mathrm{Mor}_{\mathcal{Top}}(|G|, |\widetilde{G}|) \\ f & \mapsto & |f| \end{array}$$

form a covariant functor from the category $\mathcal{AbsGraph}$ of abstract graphs to the category $\mathcal{Top}$ of topological spaces.

[81] Here are the details why $V \subset |G|$ is discrete. Let $v \in V$. We consider $U := |G| \setminus (V \setminus \{v\})$. Using the Abstract Graph-Topology Lemma 7.1 (2a) one can easily verify that U is an open neighborhood of $v \in |G|$. By construction U does not contain any other point of $V \subset |G|$. This shows that $V \subset |G|$ is discrete.



Example. Let $\widetilde{G} = (\widetilde{V}, \widetilde{E}, \widetilde{i}, \widetilde{t})$ be an abstract graph and let $G = (V, E, i, t)$ be a subgraph. The inclusions $V \to \widetilde{V}$ and $E \to \widetilde{E}$ define a map $\iota\colon G \to \widetilde{G}$ of abstract graphs in the sense of the definition on page 147. Using the Topological Realization–Functor Lemma 7.2 (1), the Abstract Graph-Topology Lemma 7.1 (2a) and the Open-Injective Lemma 2.9 one can easily show that the induced map $|\iota|\colon |G| \to |\widetilde{G}|$ between the topological realizations is a closed embedding. In the following we will sometimes use this observation as an excuse to view $|G|$ as a closed subset of $|\widetilde{G}|$. □

Proof of Lemma 7.2.

(1) It follows almost immediately from the definitions that the map is well-defined. Furthermore it follows easily from the Abstract Graph-Topology Lemma 7.1 (1) and (3) that the map is continuous.

(2) This statement follows easily from the definitions. ■

The following lemma can be viewed as the analogue of the Product Topology-Properties Proposition 5.4 and the Disjoint Union Topology-Properties Lemma 5.13.

Lemma 7.3. (Topological Realization–Properties Lemma) Let $G = (V, E, i, t)$ be an abstract graph.

(1) (a) Every compact subspace of $|G|$ is contained in the topological realization of a finite subgraph.
   (b) $|G|$ is compact if and only if G is finite.

(2) The following statements are equivalent:
   (a) G is connected.
   (b) Given any two distinct vertices v, w there exists an embedding $\gamma\colon [0,1] \to |G|$ with $\gamma(0) = v$, $\gamma(1) = w$ and such that $\gamma([0,1]) \subset |G|$ is the union of edges.
   (c) The topological realization $|G|$ is path-connected.
   (d) The topological realization $|G|$ is connected.

(3) $|G|$ is Hausdorff.

Proof.

(1) (a) Let $K \subset |G|$ be a compact subspace. We set

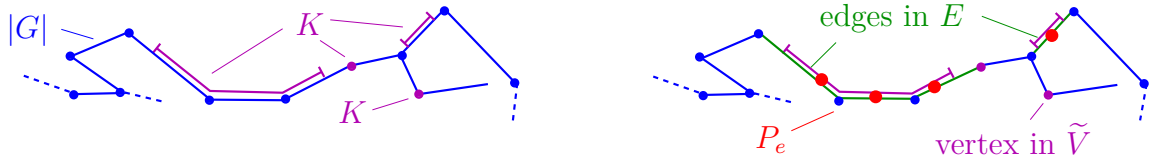
$$\begin{aligned} \widetilde{E} &:= \{e \in E \,|\, K \cap \langle e\rangle \neq \varnothing\} \\ \widetilde{V} &:= \{v \in V \,|\, v \in K \text{ and there is no } e \in \widetilde{E} \text{ with } v \in |e|\}. \end{aligned}$$

We do the following:

- For each $e \in \widetilde{E}$ we pick a $P_e \in K \cap \langle e\rangle$ and we set $A_e := \langle e\rangle$.
- For each $v \in \widetilde{V}$ we set $B_v := \{v\} \cup \bigcup_e \langle e\rangle$, where we run over all $e \in E$ such that v is a vertex of e.
- We set $C := (|G| \setminus \bigcup_{e\in\widetilde{E}} \{P_e\}) \setminus \widetilde{V}$.

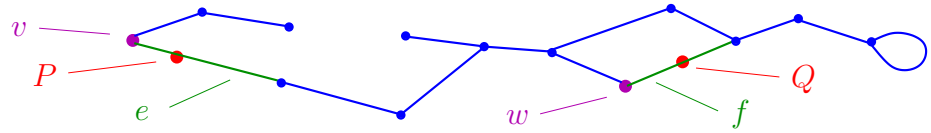
It follows easily from the Topological Realization–Properties Lemma 7.3 that each A_e, B_v and C is an open subset of $|G|$. Note that $\{A_e\}_{e\in\widetilde{E}}$, $\{B_v\}_{v\in\widetilde{V}}$ and C form

an open cover of K. Since K is compact we see that we can cover K by finitely many of these open subsets. But each A_e contains a point of K, namely P_e, that does not lie in any of the other sets in the open cover. Similarly we see that each B_v contains a point in K, namely v, that does not lie in any of the other sets in the open cover. This shows that $\widetilde{E}$ and $\widetilde{V}$ are finite. Now we see that K is contained in the topological realization of the finite abstract graph that is formed by $\widetilde{E}$, the vertices of the $e \in \widetilde{E}$ and by $\widetilde{V}$.



(b) If $|G|$ is compact, then it follows from (a) that G is finite. Next we assume that $G = (V, E, i, t)$ is finite. It follows from the Compact Image Lemma 2.8 that each $|e|$ is compact. Since there are only finitely edges we see that $|G|$ is compact.

(2) (a)⇒(b). Let $v \neq w \in V$. Since we assume that G is connected there exists an injective path $e_1, \ldots, e_k$ from v to w. Recall that this means that there exists distinct vertices $v = v_0, \ldots, v_k = w$ such that the vertices of each e_i are v_{i-1}, v_i. We obtain a map $\gamma\colon [0,1] \to |G|$ with $\gamma(0) = v$ such that on each subinterval $[\frac{i-1}{n}, \frac{i}{n}]$ the map γ is just a reparametrization of the characteristic map Φ_{e_i}, which now goes from v_{i-1} to v_i. It follows from the Abstract Graph-Topology Lemma 7.1 (1) and the Pasting Proposition 2.4 (2) that γ is continuous. Using the Abstract Graph-Topology Lemma 7.1 (2b) one can easily show that γ is a closed embedding.
(b)⇒(c). Let $P, Q \in |G|$. We pick edges e, f with $P \in |e|$ and $Q \in |f|$ and we pick vertices v of e and w of f. Since the characteristic maps of e and f are continuous one sees that there exists a path from P to v and a path from Q to w. By (b) there exists also a path from v to w. It follows from the Path-Equivalence Lemma 2.20 that there exists a path from P to Q.



(c)⇒(d). This follows from the Path-Connected-Implies-Connected Corollary 2.15.
(d)⇒(a). It follows easily from the definitions that G can be written as the disjoint union of subgraphs $\{G_i\}_{i\in I}$ such that each G_i is connected. It follows that $|G| = \bigsqcup_{i\in I} |G_i|$ and it follows easily from the Abstract Graph-Topology Lemma 7.1 (2a) that each $|G_i| \subset |G|$ is open. Since $|G|$ is connected we see that $\#I = 1$, i.e. G is connected.

(3) We leave it to the reader to prove this statement. ∎

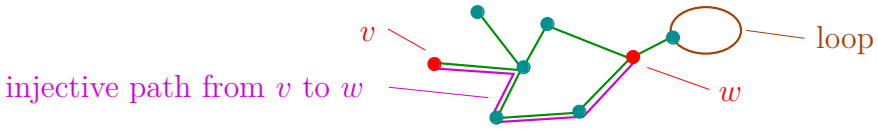
We conclude this section with the following definition:

Definition. We say that a topological space X is a topological graph if there exists an abstract graph G such that X is homeomorphic to $|G|$.

7.3. **Trees.** In this and the following section we again leave the realm of topology and we stay strictly in the world of abstract graphs. For the reader's convenience we recall the following definition from page 147:

Definition. Let G be an abstract graph.

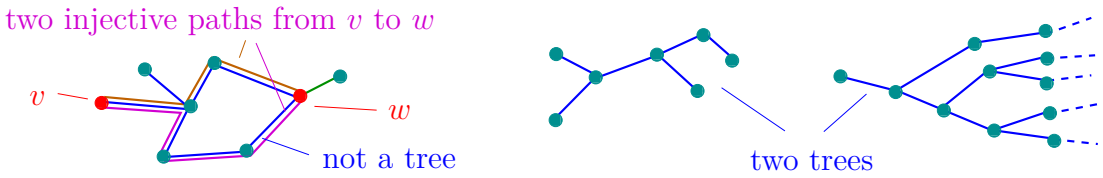
(1) An edge is called a loop if it has precisely one vertex.
(2) Let $v, w \in V$. A path of length k from v to w is a sequence of edges $e_1, \dots, e_k$ such that there exist vertices $v = v_0, v_1, \dots, v_k = w$ such that for each $i \in \{1, \dots, k\}$ the vertices of e_i are v_{i-1}, v_i. We say that the path $e_1, \dots, e_k$ is injective if the vertices $v_0, \dots, v_k$ are distinct.



Now we can introduce the following basic but surprisingly useful class of abstract graphs.

Definition. Let G be an abstract graph.

(1) We say that G is a tree if G is non-empty, if G does not contain a loop, and if given any $v, w \in V$ there exists a unique injective path from v to w.
(2) A subgraph that is a tree is called a subtree.



Lemma 7.4. (Tree–Basics Lemma)

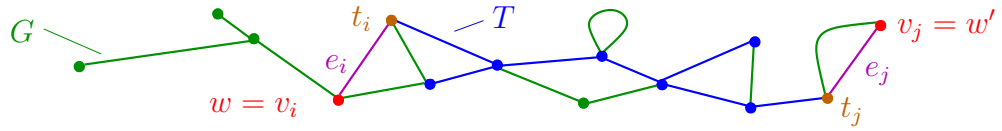
(1) Any connected non-empty subgraph of a tree is again a tree.
(2) Let G be a graph. Let $T = (V, E)$ be a subtree of G and let $\{e_i\}_{i \in I}$ be a family of edges of G with the following properties:
   (a) The edge e_i has two vertices, one vertex t_i of e_i lies in T and the other vertex v_i does not lie in T.
   (b) The vertices v_i are distinct.
   Then $(V \cup \bigcup_{i \in I} \{v_i\}, E \cup \bigcup_{i \in I} \{e_i\})$ is also a subtree of G.
(3) Let G be a graph. If $T_1 \subset T_2 \subset \dots$ is a sequence of subtrees of G, then $\bigcup_{i \in \mathbb{N}} T_i$ is also a subtree of G.

Proof.

(1) This statement follows easily from the definitions.
(2) We set $W := V \cup \bigcup_{i \in I} \{v_i\}$ and $F := E \cup \bigcup_{i \in I} \{e_i\}$. We claim that (W, F) is a subtree. Since none of the e_i's is a loop we see that (W, F) has no loops. Next let $w, w' \in W$ be two vertices. We need to show that there exists a unique injective path from w to w'. There are several cases to distinguish.

   First assume that there exist $i, j \in I$ with $w = v_i$ and $w' = v_j$. Since (V, E) is a subtree we know that there exists an injective path in (V, E) from t_i to t_j. Together with e_i and e_j these form an injective path in (W, F) from $w = v_i$ to $w' = v_j$. We

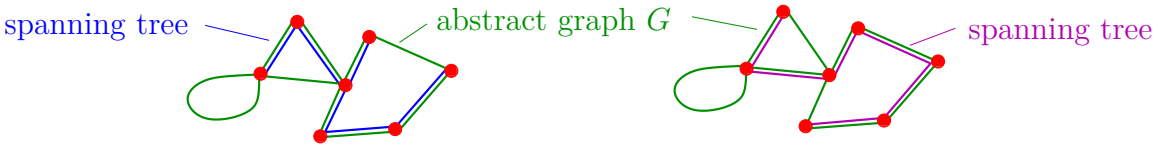
need to show that this is the unique injective path in (W, F) from $w = v_i$ to $w' = v_j$. Thus let $f_1, \dots, f_k$ be an injective path in (W, F) from w to w'. It follows from our hypothesis (a) and (b) that $f_1 = e_i$ and $f_k = e_j$ and that $f_2, \dots, f_{k-1}$ is an injective path which cannot involve any of the other e_l. Thus it is an injective path in (V, E), which is unique since T is a subtree. All other cases are treated in a similar fashion.



(3) We leave the proof of this statement as an instructive exercise to the reader. ■

Now we provide a few more definitions on abstract graphs.

Definition. Let G be an abstract graph. A spanning tree for G is a subtree of G that contains all vertices of G.

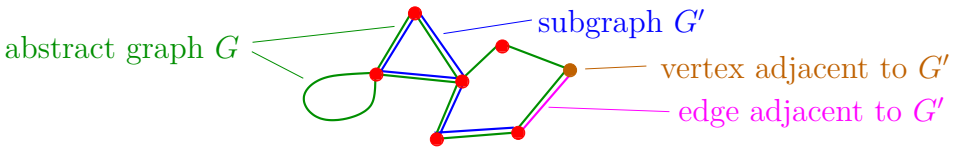


Remark. The example of the figure above shows that an abstract graph usually has many different spanning trees. □

Definition. Let G be an abstract graph and let G' be a subgraph.

(1) We say that an edge e of G is adjacent to G' if e does not lie in G' and if at least one vertex of e lies in G'.

(2) We say that a vertex v of G is adjacent to G' if v does not lie in G' and if v is a vertex of an edge that is adjacent to G'.



Proposition 7.5. (Spanning Tree–Characterization Proposition) Let G be a connected abstract graph and let $T \subset G$ be a tree. The following statements are equivalent:

(1) T is a spanning tree.

(2) T is maximal in the sense that if T is contained in a subtree T' of G, then $T = T'$.[82]

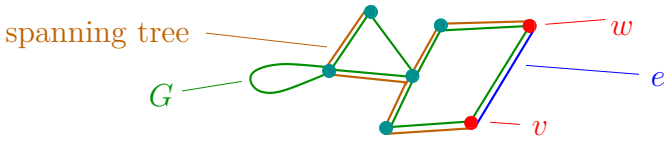
(3) T does not admit an adjacent vertex.

Proof.

(1)⇒(2). Let T be a spanning tree. We claim that T is maximal. Let H be a subgraph of G with $T \subsetneq H$. We need to show that H cannot be a tree. Since T contains all vertices and since $T \subsetneq H$ we see that there exists an edge e in H that does not lie in T. If e is a loop, then we know already that H is not a tree. Thus it remains to consider the case that e is not a loop. This means that e has two vertices v and w. Since T is connected there

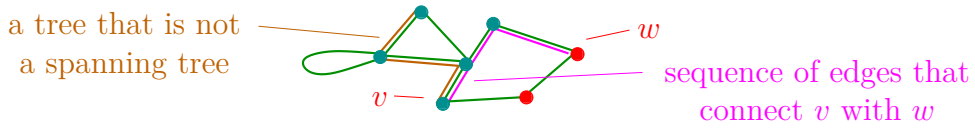
[82]The equivalence of (1) and (2) explains why a spanning tree is often also called a *maximal tree*.

exists an injective path from v to w in T. On the other hand e is an injective path from v to w in H. Thus we see that there exist two injective paths in H from v to w. Thus H is not a tree.



(2)⇒(3). Let $T = (W, F)$ be a subtree of G that admits an adjacent vertex x. We need to show that T is not a maximal subtree. Let f be an edge such that one vertex is x and such that the other vertex lies in W. It follows from the Tree–Basics Lemma 7.4 (2) that $(W \cup \{x\}, F \cup \{f\})$ is a subtree of G that contains T.

(3)⇒(1). Let T be a subtree of G. We will show that if T is not a spanning tree, then T admits an adjacent vertex. Thus suppose that there exists a vertex w that does not lie in T. Let v be a vertex in T. Since G is connected there exists a path $e_1, \ldots, e_n$ from v to w.

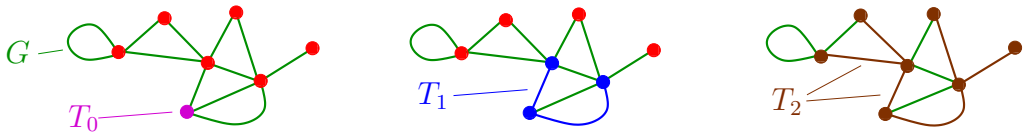


By definition this means that there exist vertices $v = v_0, \ldots, v_n = w$ such that the vertices of each e_i are v_{i-1}, v_i. The first vertex that does not lie in T is adjacent to T. ■

We conclude this section with the following proposition:

Proposition 7.6. (Spanning Tree–Existence Proposition) Every connected non-empty abstract graph admits a spanning tree.

Proof. Let $G = (V, E)$ be a connected non-empty abstract graph. Since G is non-empty we can pick a vertex $v \in V$. We set $T_0 := \{v\}$ and view T_0 as a subtree of G. Now we construct iteratively a sequence of subtrees $T_0 \subset T_1 \subset \ldots$ as follows: Suppose we have a subtree $T_i = (V_i, E_i)$. We denote by W_i the set of all vertices of G that are adjacent to T_i. For each $w \in W_i$ we pick an edge e_w that is adjacent to T_i and such that w is a vertex of e_w.[83] We set $V_{i+1} := V_i \cup W_i$ and $E_{i+1} := E_i \cup \bigcup_{w \in W_i} \{e_w\}$. It follows from the Tree–Basics Lemma 7.4 (2) that T_{i+1} is also a tree.



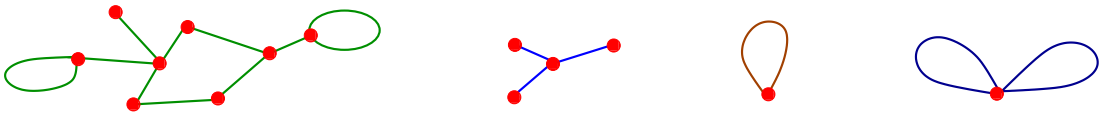
We set $T := \bigcup_{i \in \mathbb{N}_0} T_i$. It follows from the Tree–Basics Lemma 7.4 (3) that T is a tree. We claim that T is a spanning tree. By the (3)⇒(1)-Implication of the Spanning Tree–Characterization Proposition 7.5 it suffices to show that there is no vertex of G that is adjacent to T. In other words, we have to show that if there is an edge e of G with one vertex v in T, then the other vertex w also lies in T. If we are given such an edge e with one vertex v in T, then there exists an $i \in \mathbb{N}_0$ with $v \in T_i$, but then it follows from the definition of T_{i+1} that $w \in T_{i+1} \subset T$. ■

[83] Note that here we secretly use the Axiom of Choice 0.1.

7.4. **The Euler characteristic.** The following essential definition introduces a concept that will stay with us, in many different guises, throughout these lecture notes.

Definition. Let $G = (V, E)$ be a finite abstract graph. The Euler characteristic of G is defined as

$$\chi(G) \;:=\; \#V \,-\, \#E \;=\; \text{number of vertices} \,-\, \text{number of edges}.$$



Euler characteristic of G equals $\chi(G) = 8 - 10 = -2$

$\chi = 1$

$\chi = 0$

$\chi = -1$

The following proposition gives a characterization of finite trees in terms of the Euler characteristic.

Proposition 7.7. (Tree Characterization Proposition) Let $G = (V, E)$ be a finite abstract graph. The following statements are equivalent:

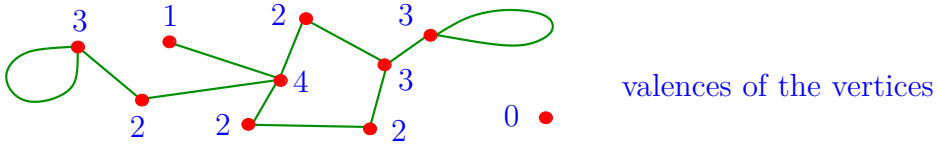
(1) G is a tree.
(2) G is connected and $\chi(G) = 1$.

The proof of the Tree Characterization Proposition 7.7 rests on a lemma which is of independent interest. To formulate the lemma we need to introduce the following natural definition.

Definition. If $G = (V, E, i, t)$ is an abstract graph, then the valence of a vertex v is defined as

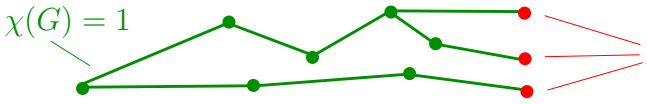
$$\text{valence}(v) \;:=\; \#\{e \in E \,|\, i(e) = v\} + \#\{e \in E \,|\, t(e) = v\}.$$

In a slightly less formal way, the valence of a vertex v is the number of edges at v, but hereby we have to count any loop at v twice, since it "goes twice into v".



valences of the vertices

Lemma 7.8. (Tree–Valence One Edge Lemma)[84] Let $G = (V, E)$ be a finite abstract graph. If $\chi(G) = 1$ and if G has at least one edge, then G admits at least two vertices of valence one.



vertices of valence one

Proof of Lemma 7.8. Let $G = (V, E, i, t)$ be a finite abstract graph with $\chi(G) = 1$ that has at least one edge.

(a) Since G has at least one edge and since it is by definition connected, we see that the abstract graph G does not have a vertex of valence 0.

(b) We calculate that

$$\begin{aligned}\sum_{\text{vertex } v} \underbrace{\text{valence}(v)}_{\geq 1 \text{ by (a)}} &= \sum_{\text{vertex } v} (\#\{e \in E \,|\, i(e) = v\} + \#\{e \in E \,|\, t(e) = v\}) \\ &\underset{\text{each edge contributes 2 to the sum}}{=} 2 \cdot \#\text{ edges} \underset{\text{by hypothesis we have } \chi(G)=1}{=} 2 \cdot \#\text{vertices} - 2.\end{aligned}$$

By (a) the equality can only hold if there are at least two vertices with valence one. ■

Proof of the Tree Characterization Proposition 7.7. Let $G = (V, E)$ be a finite abstract graph.

(2)⇒(1). We now assume that G is connected and that $\chi(G) = 1$. We need to show that G is a tree. We will do so by inducting on the number of edges of G. If G has no edge, then it follows from $\chi(G) = 1$ that G has one vertex. Clearly this implies that G is a tree.

Now suppose that the statement holds for all graphs with n edges. Let $G = (V, E)$ be a connected graph with $n + 1$ edges and with $\chi(G) = 1$. It follows from the Tree–Valence One Edge Lemma 7.8 that G admits a vertex v of valence one. Let e be the corresponding edge. Note that $G' = (V \setminus \{v\}, E \setminus \{e\})$ is connected and that

$$\chi(G') \;=\; (\#V - 1) - (\#E - 1) \;=\; \#V - \#E \;=\; \chi(G) \;=\; 1.$$

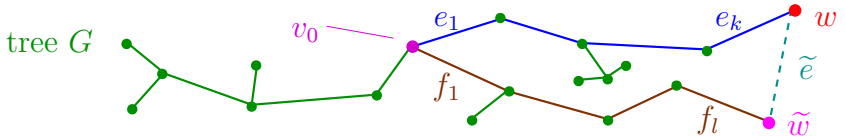
By induction G' is a tree. But then it follows from the Tree–Basics Lemma 7.4 (2) that G itself is also a tree.

(1)⇒(2). We suppose that G is a tree. If follows immediately from the definition of a tree that G is connected. We still need to show that $\chi(G) = 1$. We prove this statement by induction on the number of edges. If there is no edge then it follows from the fact that trees are non-empty and connected that there is precisely one vertex, thus $\chi(G) = 1$.

Next suppose that the statement holds if we have n edges. Let G be a tree with $n + 1$ edges. We pick a vertex $v_0 \in V$. Let $w \in V$ be a vertex of maximal distance to v_0.

Claim. There exists precisely one edge e that has w as a vertex.

Proof. Let $e_1, \ldots, e_k$ be a path of minimal length from v_0 to w. Since the path has minimal length we see that it is injective. Evidently w is a vertex of e_k. Suppose that there was a second edge $\widetilde{e}$ for which w is a vertex. As we mentioned before, a tree does not contain a loop. This implies that the edge $\widetilde{e}$ has a second vertex besides w. We denote this second vertex by $\widetilde{w}$. Since G is connected and since w has maximal distance from v_0 there exists an injective path $f_0, \ldots, f_l$ from v_0 to $\widetilde{w}$ of length $\leq k$. Thus this path cannot be the path $e_1, \ldots, e_k, \widetilde{e}$, which has length $k + 1$.



We have thus found two different injective paths from v_0 to $\widetilde{w}$, namely $e_1, \ldots, e_k, \widetilde{w}$ and $f_1, \ldots, f_l$. But this contradicts the fact that we started out with a tree. ⊞

We can now consider the subgraph G' that is given by $V \setminus \{w\}$ and $E \setminus \{e\}$. One can easily show that this graph is again connected, thus it follows from the Tree–Basics

Lemma 7.4 (1) that G' is a tree. We now see that

$$\chi(G) \;=\; \#V - \#E \;=\; (\#V-1)-(\#E-1) \;=\; \chi(G') \;\underset{\text{by induction hypothesis}}{=}\; 1.$$

■

7.5. **Quotients of abstract graphs.** In this section we will define and discuss quotients of abstract graphs and of the corresponding topological realizations.

Definition. Let $G=(V,E,i,t)$ be an abstract graph and let $H=(W,F)$ be a non-empty subgraph.

(1) We define $\sim$ to be the equivalence relation generated by $v\sim v'$ if v and v' are both vertices in H. We denote by $*$ the equivalence class defined by the vertices in H.
(2) We denote by $\pi\colon V\to V/\!\sim$ the map that is given by $v\mapsto[v]$.
(3) Finally we define the abstract graph

$$G/H \;:=\; (V/\!\sim, E\setminus F, \pi\circ i\colon E\setminus F\to V/\!\sim, \pi\circ t\colon E\setminus F\to V/\!\sim).$$

We refer to G/H as the quotient of the abstract graph G by the abstract graph H.

The following lemma summarizes a few properties of the abstract graph G/H.

Lemma 7.9. (Abstract Graph–Quotient Lemma) Let $G=(V,E,i,t)$ be an abstract graph and let $H=(W,F)$ be a non-empty subgraph. We continue with the above notation.

(1) If G is finite, then we have $\chi(G/H)=\chi(G)-\chi(H)+1$.
(2) We consider the map

$$\begin{array}{ccc} p\colon |G| = (V\sqcup(E\times[0,1]))/\!\sim & \to & |G/H| = (V/\!\sim \sqcup((E\setminus F)\times[0,1]))/\!\sim \\ \text{where } (e,0)\sim i(e) \text{ and } (e,1)\sim t(e) & & \text{similarly } (e,0)\sim\pi(i(e)) \text{ and } (e,1)\sim\pi(t(e)) \end{array}$$

$$[x]\;\mapsto\; \begin{cases} [*], & \text{if } x\in|H|,\\ [\pi(x)], & \text{if } x\in V,\\ [(e,t)], & \text{if } x=(e,t) \text{ with } e\in E\setminus F \text{ and } t\in[0,1]. \end{cases}$$

The induced map $\overline{p}\colon |G|/|H|\to|G/H|$ is well-defined and it is a homeomorphism.



Proof.

(1) We calculate that

$$\chi(G/H)=\#(V/\!\sim)-\#(E\setminus F)=(\#V-\#W+1)-(\#E-\#F)=\chi(G)-\chi(H)+1.$$

(2) First note that it follows easily from the Abstract Graph-Topology Lemma 7.1 (3) and the Topological-Quotient Proposition 6.1 (1b) that the map $\overline{p}\colon|G|/|H|\to|G/H|$ is continuous. One can easily verify that the map is a bijection. We refer to [**Fri24**] for the proof that the map is actually a homeomorphism. ■

[84] It follows from the discussion on page 150 that it is legitimate to view $|H|$ as a subspace of $|G|$.

CHAPTER 8

Homotopies

Loosely speaking two continuous maps $f, g\colon X \to Y$ between topological spaces are called homotopic if one can be deformed "continuously" into the other. In the subsequent chapter we will use to this notion of homotopic maps to explain what it means for two topological spaces to be "homotopy equivalent".

8.1. Homotopic maps.

In its most basic form a homotopy is a continuous deformation of continuous maps between two topological spaces. To make this notion precise it is convenient to introduce the following notation:

Notation. Let X and Y be two topological spaces and let $I \subset \mathbb{R}$ be a subset. Given a continuous map $F\colon X \times I \to Y$ and given $t \in I$ we denote by F_t the continuous map[85]

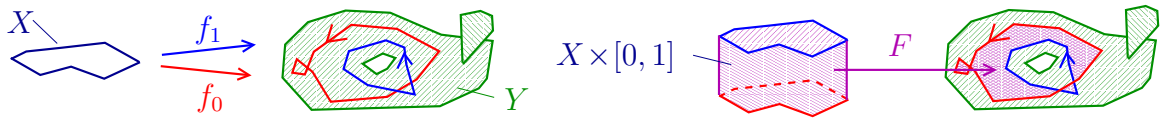
$$\begin{array}{rccc} F_t\colon & X & \to & Y \\ & x & \mapsto & F(x,t). \end{array}$$

Now we move on to the actual definition we are interested in.

Definition. Let X and Y be two topological spaces.

(1) Let $f_0, f_1\colon X \to Y$ be two continuous maps. A homotopy between the maps f_0 and f_1 is a continuous map $F\colon X \times [0,1] \to Y$ with $F_0 = f_0$ and $F_1 = f_1$. If there exists a homotopy between the maps f_0 and f_1, then we say that f_0 and f_1 are homotopic and sometimes we write $f_0 \simeq f_1$. If every F_t is an embedding, then we call F an isotopy.

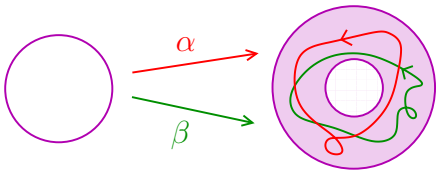
(2) A continuous map $f\colon X \to Y$ is called null-homotopic if it is homotopic to a constant map.



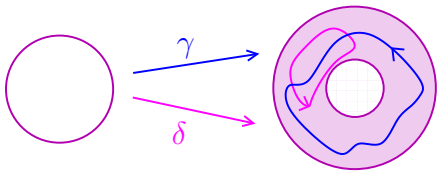
Examples.

(1) In the figure below we show four maps α, β, γ and δ from the circle S^1 to the annulus $Y = B_2(0) \setminus B_1(0)$. The maps α and β are homotopic whereas the maps γ and δ look as if they are not homotopic.

[85] Note that it follows from the Product Topology-Basics Lemma 5.3 (1∗) that the map F_t is indeed continuous.



the maps α and β are homotopic

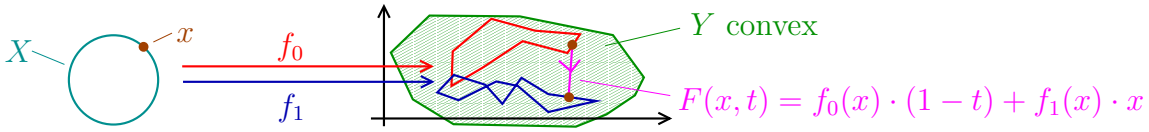


the maps γ and δ appear not to be homotopic

(2) Given an orthogonal matrix $A \in \mathrm{O}(n)$ we denote by $\rho(A)$ the map $S^{n-1} \to S^{n-1}$ that is given $x \mapsto A \cdot x$.
   (a) Let $\gamma\colon [0,1] \to \mathrm{SO}(n)$ be a path from some matrix A to some matrix B. The map[86]
$$\begin{array}{rcl} S^{n-1} \times [0,1] & \to & S^{n-1} \\ (x,t) & \mapsto & \gamma(t) \cdot x \end{array}$$
   is a homotopy from $\rho(A)$ to $\rho(B)$.
   (b) Let $A \in \mathrm{O}(n)$. If $\det(A) = 1$, i.e. if $A \in \mathrm{SO}(n)$, then the Matrix Group-π_0-Proposition 2.23 (3) provides us with a path $\gamma\colon [0,1] \to \mathrm{SO}(n)$ from $\gamma(0) = A$ to $\gamma(1) = \mathrm{id}$. It follows from (a) that $\rho(A)\colon S^{n-1} \to S^{n-1}$ is homotopic to the identity.
(3) It follows immediately from the definitions that a topological space X is path-connected if and only if any two maps $\{*\} \to X$ are homotopic. □

The next example of a homotopy is written down as a lemma.

Lemma 8.1. (Convex Homotopy Lemma) Let X be a topological space. Given any two continuous maps $f_0, f_1\colon X \to Y$ to a *convex* subset $Y \subset \mathbb{R}^n$ the map
$$\begin{array}{rcl} F\colon X \times [0,1] & \to & Y \\ (x,t) & \mapsto & \underbrace{f_0(x) \cdot (1-t) + f_1(x) \cdot t}_{\in Y \text{ since } Y \text{ is convex}} \end{array}$$
is a homotopy between f_0 and f_1.



Proof. Basically by definition we have $F_0 = f_0$ and $F_1 = f_1$. But we are not done yet, the conscientious reader will not have failed to notice that we need to show that the map F is actually continuous. The map F can be written as the composition of the following two maps:
$$X \times [0,1] \xrightarrow{(x,t)\mapsto(f_0(x),f_1(x),t)} \mathbb{R}^n \times \mathbb{R}^n \times [0,1] \xrightarrow{(x,y,t)\mapsto(x\cdot(1-t)+y\cdot t)} \mathbb{R}^n.$$
It follows from the Topological-Product Proposition 5.1 (1) that the map to the left is continuous. As in the proof of the Torus Embedding Lemma 5.5 we see that the map to the right is continuous. ■

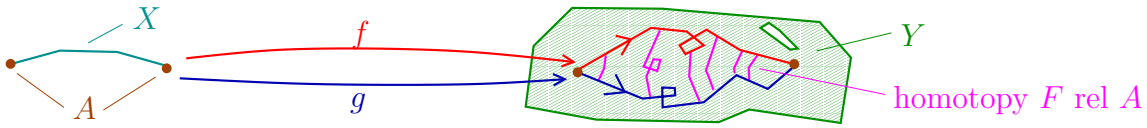
Often it is more interesting to consider homotopies that fix a given subset. This leads us to the following definition:

[86] The argument of the proof of the Torus Embedding Lemma 5.5 shows that the map $S^{n-1} \times [0,1] \to S^{n-1}$ is continuous.

Definition. Let X and Y be topological spaces and let $A \subset X$ be a subset.

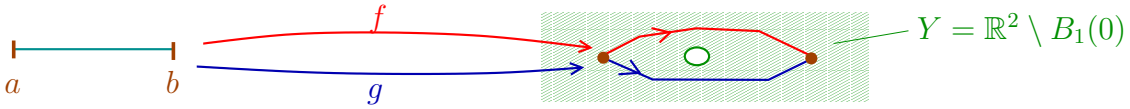
(1) We say that a continuous map $F\colon X \times [0,1] \to Y$ is a homotopy rel A if for all $a \in A$ and $s, t \in [0,1]$ we have $F(a,s) = F(a,t)$.

(2) We say that two continuous maps $f, g\colon X \to Y$ with $f|_A = g|_A$ are homotopic rel A if there exists a homotopy F rel A with $F_0 = f$ and $F_1 = g$.

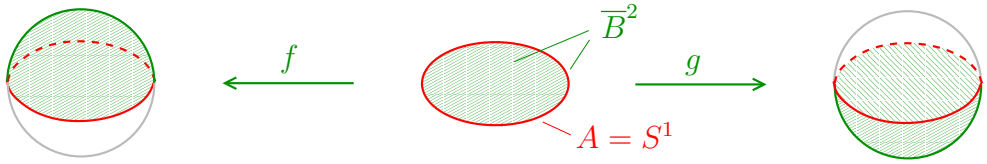


Examples.

(1) Let $f, g\colon [a,b] \to Y = \mathbb{R}^2 \setminus B_1(0)$ be the two continuous maps from an interval to the topological space Y that are shown in the figure below. Note that $f(a) = g(a)$ and $f(b) = g(b)$. One can easily show that any two continuous maps from an interval to a path-connected space are homotopic. On other hand it looks rather unlikely that the two maps are homotopic rel $\{a, b\}$, i.e. it does not look like we can deform f into g while keeping the endpoints of the interval fixed.



(2) We consider the two maps $f, g\colon \overline{B}^2 \to S^2$ that are shown in the figure below:



These maps agree on S^1. It is fairly easy to see that the two maps are homotopic but it seems rather unlikely that they are homotopic rel S^1. Unfortunately, as in (1), right now we lack the tools for showing the latter statement.

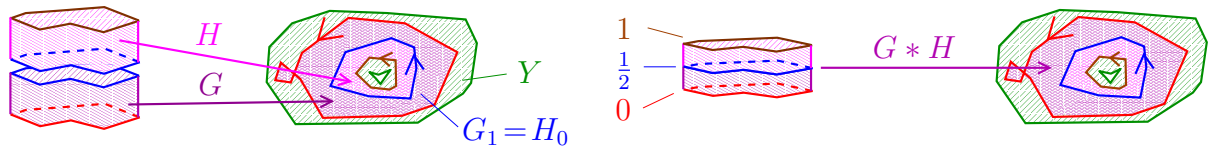
After these examples we return to developing the general theory of homotopies. On many occasions it will be useful to combine two homotopies. This leads us to the following lemma.

Lemma 8.2. (Homotopy Combination Lemma) Let X and Z be topological spaces. Let $A \subset X$ be a (possibly empty) subset of X and let $G, H\colon X \times [0,1] \to Z$ be two homotopies rel A. If $G_1 = H_0$, then the map

$$\begin{aligned} G * H\colon X \times [0,1] &\to Z \\ (x,t) &\mapsto \begin{cases} G(x,2t), & \text{if } t \in [0, \tfrac{1}{2}], \\ H(x,2t-1), & \text{if } t \in [\tfrac{1}{2}, 1]. \end{cases} \end{aligned}$$

is a homotopy rel A from G_0 to H_1.

Proof. It follows from the Pasting Proposition 2.4 (2) together with the Product Topology-Basics Lemma 5.3 (1a) that the map is continuous. ∎



Lemma 8.3. (Homotopy-Equivalence Relation Lemma) Let X and Y be topological spaces. Then "homotopy of maps" forms an equivalence relation on the set of continuous maps from X to Y.

Proof. We have to show that "homotopy" satisfies the three properties of an equivalence relation. In the following let $f, g, h\colon X \to Y$ be three continuous maps.

(1) The "constant homotopy" given by $F(x,t) \coloneqq f(x)$ is a homotopy between f and f.
(2) Let F be a homotopy from f to g. Then it is clear that the map F' that is defined by $F'(x,t) \coloneqq F(x,1-t)$ is a homotopy from g to f.
(3) Let F be a homotopy from f to g and let G be a homotopy from g to h. We consider the map

$$\begin{array}{rcl} F * G\colon X \times [0,1] & \to & Y \\ (x,t) & \mapsto & \begin{cases} F(x,2t), & \text{if } t \in \left[0,\frac{1}{2}\right], \\ G(x,2t-1), & \text{if } t \in \left[\frac{1}{2},1\right]. \end{cases} \end{array}$$

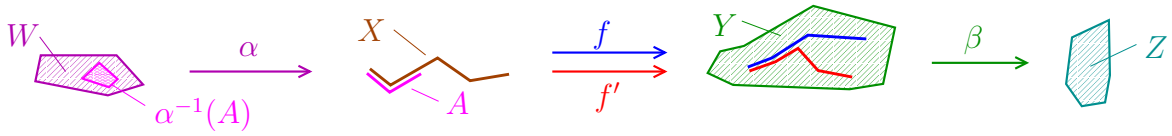
By the Homotopy Combination Lemma 8.2 (2) this is a homotopy from f to h. ■

Lemma 8.4. (Homotopy Composition Lemma)

(1) Let $f, f'\colon X \to Y$, let $\alpha\colon W \to X$ and $\beta\colon Y \to Z$ be continuous maps between topological spaces. Furthermore let $A \subset X$ be a (possibly empty) subset. The following conclusion holds:

$$\begin{array}{rcl} f \text{ is homotopic to } f' \text{ rel } A & \implies & f \circ \alpha \text{ is homotopic to } f' \circ \alpha \text{ rel } \alpha^{-1}(A) \\ f \text{ is homotopic to } f' \text{ rel } A & \implies & \beta \circ f \text{ is homotopic to } \beta \circ f' \text{ rel } A \end{array}$$

(2) Let X, Y and Z be topological spaces and let $A \subset X$ and $B \subset Y$ be subsets. We suppose that $f, f'\colon X \to Y$ and $g, g'\colon Y \to Z$ are continuous maps such that f and f' are homotopic rel A and such that g and g' are homotopic rel B. If $f(A) = f'(A) \subset B$, then $g \circ f$ is homotopic to $g' \circ f'$ rel A.



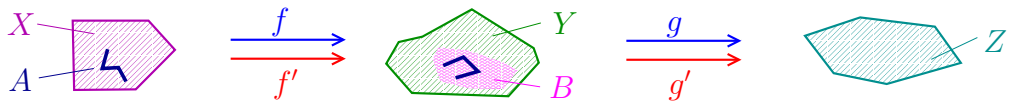
Proof.

(1) This statement is basically obvious, we just need to precompose and postcompose a homotopy $H\colon X \times [0,1] \to Y$ between f and g by $\alpha \times \mathrm{id}_{[0,1]}\colon W \times [0,1] \to X \times [0,1]$ respectively by $\beta\colon Y \to Z$.
(2) Note that:

$$g \circ f \underset{\uparrow}{\simeq} g' \circ f \underset{\uparrow}{\simeq} g' \circ f'.$$

both follow from (1)

Furthermore one can easily verify that the "rel" business also works out. ■



8.2. Homotopies on quotients.

Theorem 8.5. (Homotopy Quotient Theorem) Let X and Y be topological spaces. Furthermore let $\sim_X$ be an equivalence relation on X and let $\sim_Y$ be an equivalence relation on Y. If $F\colon X\times[0,1]\to Y$ is a homotopy such that $F(x,t)\sim_Y F(x',t)$ whenever $x\sim_X x'$, then the map[87]

$$\begin{array}{rcl}\varphi\colon (X/\sim_X)\times[0,1] & \to & Y/\sim_Y\\ ([x],t) & \mapsto & [F(x,t)]\end{array}$$

is continuous.

Example. Let $n\in\mathbb{N}$ and let $\varphi\colon[0,1]\to \mathrm{GL}(n,\mathbb{R})$ be a continuous map. It follows from the argument of the proof of the Torus Embedding Lemma 5.5 that the map

$$\begin{array}{rcl}F\colon (\mathbb{R}^n\setminus\{0\})\times[0,1] & \to & \mathbb{R}^n\setminus\{0\}\\ (P,t) & \mapsto & \varphi(t)\cdot P\end{array}$$

is continuous. Next recall that $\mathbb{R}\mathrm{P}^{n-1}=(\mathbb{R}^n\setminus\{0\})/(\mathbb{R}\setminus\{0\})$. It follows from the Homotopy Quotient Theorem 8.5 that the map

$$\begin{array}{rcl}\mathbb{R}\mathrm{P}^{n-1}\times[0,1] & \to & \mathbb{R}\mathrm{P}^{n-1}\\ ([P],t) & \mapsto & [\varphi(t)\cdot P]\end{array}$$

is continuous. The same conclusion also holds if we replace "$\mathbb{R}$" by "$\mathbb{C}$". □

Proof of Theorem 8.5. The result is called a theorem, since the proof is indeed rather tricky. We refer to [**Fri24**] for details. Below we will deal with a special case. ■

Proof of Theorem 8.5 if X is compact and if $X/\sim_X$ is Hausdorff. We now assume that X is compact and that the quotient space $X/\sim_X$ is Hausdorff. On $X\times[0,1]$ we consider the equivalence relation $\sim$ given by

$$(x,t)\sim(x',t')\iff x\sim_X x' \text{ and } t=t'.$$

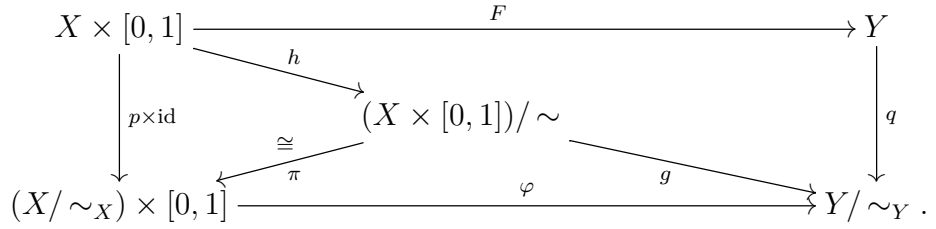
We denote by $h\colon X\times[0,1]\to(X\times[0,1])/\sim$ the natural projection. We consider the maps

$$\begin{array}{rcl}p\times\mathrm{id}\colon X\times[0,1] & \to & X/\sim_X\times[0,1]\\ (x,t) & \mapsto & ([x],t)\end{array}\qquad\text{and}\qquad\begin{array}{rcl}\pi\colon (X\times[0,1])/\sim & \to & X/\sim_X\times[0,1]\\ [(x,t)] & \mapsto & ([x],t).\end{array}$$

By the Topological-Quotient Proposition 6.1 (1a) the projection $p\colon X\to X/\sim_X$ is continuous. This shows that the map to the left is continuous. By the Topological-Quotient Proposition 6.1 (1b) this implies that the induced map π to the right is also continuous. One can easily verify that π is a bijection. By hypothesis X is compact. By the Product Topology-Properties Proposition 5.4 (1) this implies that $X\times[0,1]$ is compact. It follows from the Topological-Quotient Proposition 6.1 (1a) and the Compact Image Lemma 2.8 that $(X\times[0,1])/\sim$ is compact. Next recall that by hypothesis $X/\sim_X$ is Hausdorff. By the Product Topology-Properties Proposition 5.4 (3) this implies that $X/\sim_X\times[0,1]$ is Hausdorff. In summary it follows from the Compact-Hausdorff Proposition 2.10 (3) that π is a homeomorphism.

[87] It follows from our hypothesis on F that the map φ is well-defined.

Now we consider the following diagram:



We make the following clarifications and observations:

(1) We denote by $q\colon Y \to Y/\sim_Y$ the natural projection.

(2) First note that the map $q \circ F\colon X \times [0,1] \to Y/\sim_Y$ factors by our hypothesis through the map $h\colon X \times [0,1] \to (X \times [0,1])/\sim$. It follows again from the Topological-Quotient Proposition 6.1 (1b) that the induced map $g\colon (X \times [0,1])/\sim \to Y/\sim_Y$ is continuous.

(3) The map $\varphi\colon (X/\sim_X) \times [0,1] \to Y/\sim_Y$ equals the map $g \circ \pi^{-1}$. We just showed that the two maps π and g are continuous. Thus φ is continuous. ■

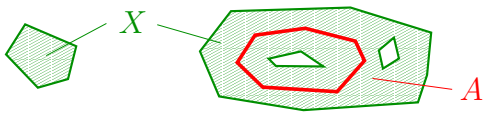
CHAPTER 9

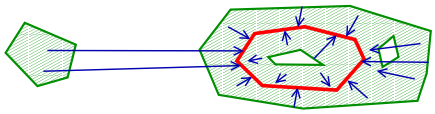
Homotopy equivalences and homotopy equivalent topological spaces

This chapter is a direct continuation of the previous chapter. In the last chapter we defined what it means for *maps* between topological spaces to be *homotopic*. Now in this chapter we will use this notion to define what it means for *topological spaces* to be *homotopy equivalent*. This notion is rather subtle and - at least for this author - not entirely intuitive.

9.1. **Deformation retractions.** Before we introduce the notion of a homotopy equivalence we want to introduce the easier notions of a (deformation) retraction.

Definition. Let X be a topological space. We say that a subset $A \subset X$ is a retract of X if there exists a retraction $r\colon X \to A$, i.e. a continuous map with $r(a) = a$ for all $a \in A$.





retraction r from X to A

Let us recall the following definition from page 158:

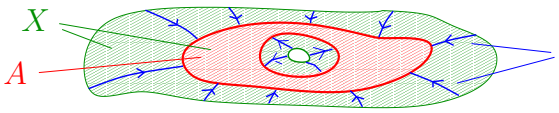
Definition. Let X and Y be two topological spaces.

(1) Given a continuous map $F\colon X \times [0,1] \to Y$ and given $t \in [0,1]$ we denote by F_t the continuous map
$$\begin{array}{rccc} F_t\colon & X & \to & Y \\ & x & \mapsto & F(x,t). \end{array}$$
(2) A continuous map $F\colon X \times [0,1] \to Y$ map is called a homotopy from F_0 to F_1.

Next we introduce an interesting special case of a homotopy.

Definition. Let X be a topological space and let A be a subset.

(1) A deformation retraction from X to A is a homotopy $F\colon X \times [0,1] \to X$ which has the following three properties:
   (a) $F_0 = \mathrm{id}_X$,
   (b) F is a homotopy rel A, i.e. for all $a \in A$ and all $t \in [0,1]$ we have $F(a,t) = a$,
   (c) $F_1(X) = A$.
(2) We say that A is a deformation retract of X if there exists a deformation retraction from X to A. We say a point $a \in X$ is a deformation retract of X if the subset $\{a\}$ is a deformation retract of X.



deformation retraction from X to A

Examples. Let $n \in \mathbb{N}$.

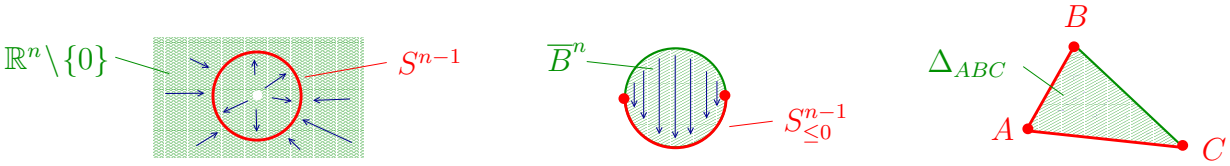
(1) (a) The map[88]

$$\begin{array}{rcl}(\mathbb{R}^n \setminus \{0\}) \times [0,1] & \to & \mathbb{R}^n \setminus \{0\} \\ (v,t) & \mapsto & v \cdot (1 - t + t \cdot \frac{1}{\|v\|})\end{array}$$

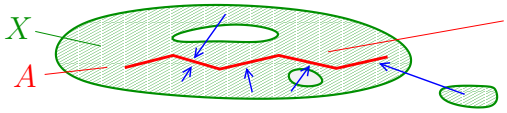
is a deformation retraction from $\mathbb{R}^n \setminus \{0\}$ to the sphere S^{n-1}.

(b) The hemisphere $S^{n-1}_{\leq 0}$ is a deformation retract of $\overline{B}^n$.

(c) Given any three non-collinear points $A, B, C \in \mathbb{R}^n$ the triangle Δ_{ABC} admits a deformation retraction to the union of any two sides of the triangle.

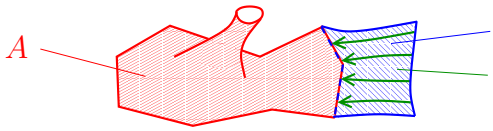


(2) If $F\colon X \times [0,1] \to X$ is a deformation retraction from X to A, then $F_1\colon X \to A$ is by definition a retraction from X to A. But, as we see in the figure below, not every retract is a deformation retract.



A is a retract of X
if it was a deformation retract, any point $x \in X$
would be connected to A via the path $R(x,t)$

(3) Let X be a topological space. Suppose that we are given a decomposition $X = A \cup B$ into two *closed* subsets. If $A \cap B$ is a deformation retract of B, then A is a deformation retract of X.



Indeed, if $F\colon B \times [0,1] \to B$ is a deformation retraction from B to $A \cap B$, then it follows from the Homotopy Combination Lemma 8.2 and our hypothesis that A and B are closed subsets of X, that the map

$$\begin{array}{rcl} G\colon X \times [0,1] & \to & X \\ (x,t) & \mapsto & \begin{cases} x, & \text{if } x \in A, \\ F(x,t), & \text{if } x \in B \end{cases}\end{array}$$

is continuous. It is now clear that G is the desired deformation retraction from X to A. We will use this remark on several occasions. □

We continue with an example of a deformation retraction which is so interesting, that it gets formulated as a proposition.

Proposition 9.1. (Matrix Groups Deformation Retract Proposition) Let $n \in \mathbb{N}_0$. There exists a deformation retraction

$$R\colon \mathrm{GL}(n,\mathbb{R}) \times [0,1] \to \mathrm{GL}(n,\mathbb{R})$$

[88] The argument of the proof of the Torus Embedding Lemma 5.5 shows that this map is continuous.

from $\mathrm{GL}(n,\mathbb{R})$ to $\mathrm{O}(n)$ such that $R_1\colon \mathrm{GL}(n,\mathbb{R}) \to \mathrm{O}(n)$ is the map that is given by the Gram-Schmidt Orthonormalization Process 0.12. The analogous statement holds if we replace the pair $\mathrm{GL}(n,\mathbb{R})$ and $\mathrm{O}(n)$ by one of the following four pairs:

(a)	$\mathrm{GL}_+(n,\mathbb{R})$	and	$\mathrm{SO}(n)$
(b)	$\mathrm{SL}(n,\mathbb{R})$	and	$\mathrm{SO}(n)$
(c)	$\mathrm{GL}(n,\mathbb{C})$	and	$\mathrm{U}(n)$
(d)	$\mathrm{SL}(n,\mathbb{C})$	and	$\mathrm{SU}(n)$.

Proof of Proposition 9.1. For $i = 1,\dots,n$ we consider the following subspaces of $\mathrm{GL}(n,\mathbb{R})$:

$$\begin{aligned} X(i) &:= \{(v_1 \dots v_n) \in \mathrm{GL}(n,\mathbb{R}) \,|\, v_1,\dots,v_{i-1} \text{ are orthonormal and } v_i \perp v_1,\dots,v_{i-1}\} \\ Y(i) &:= \{(v_1 \dots v_n) \in \mathrm{GL}(n,\mathbb{R}) \,|\, v_1,\dots,v_i \text{ are orthonormal}\}. \end{aligned}$$

In the proof of the Matrix Group Path-Component Proposition 2.23, we implicitly already showed that the $2k$ steps of the Gram-Schmidt Orthogonalization Process define deformation retractions from $X(i)$ to $Y(i)$ and $Y(i)$ to $X(i+1)$. Since $Y(1) = \mathrm{GL}(n,\mathbb{R})$ and since $Y(n) = \mathrm{O}(n)$ we can combine these $2k-1$ deformation retractions to obtain a deformation retraction

$$R\colon\ \mathrm{GL}(n,\mathbb{R}) \times [0,1] \ \to\ \mathrm{GL}(n,\mathbb{R})$$

from $\mathrm{GL}(n,\mathbb{R})$ to $\mathrm{O}(n)$. Furthermore note that it follows from the Gram-Schmidt Orthonormalization Process 0.12 (2) and (3) that this deformation retraction R restricts to a deformation retraction from $\mathrm{GL}_+(n,\mathbb{R})$ to $\mathrm{SO}(n)$ and from $\mathrm{SL}(n,\mathbb{R})$ to $\mathrm{SO}(n)$.

Finally note that the complex cases are treated in a very similar fashion. ■

In the following we will see that deformation retractions descend to quotients under suitable hypotheses.

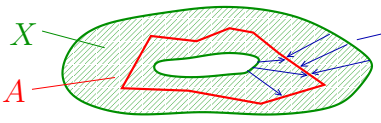
Theorem 9.2. (Deformation Retraction-Quotient Theorem) Let X be a topological space and let $A \subset X$ be a subset.

(1) Let "$\sim$" be an equivalence relation on X and let $R\colon X \times [0,1] \to X$ be a deformation retraction from X to A. If "$\sim$" is compatible with the deformation retraction R in the sense that $R(x,t) \sim R(x',t)$ whenever $x \sim x'$, then the map

$$\begin{aligned} \overline{R}\colon (X/\sim) \times [0,1] &\to X/\sim \\ ([x],t) &\mapsto [R(x,t)] \end{aligned}$$

is a deformation retraction from $X/\sim$ to $A/\sim$.

(2) Let X be a topological space and let A be a subset of X. If A is a deformation retract of X, then the point A/A is a deformation retract of X/A.



deformation retraction $R\colon X \times [0,1] \to X$ from X to A
X/A admits a deformation retract to the point A/A
what does X/A "look like"?

Proof.

(1) It follows from the Homotopy Quotient Theorem 8.5 that $\overline{R}$ is continuous. It is now clear that $\overline{R}$ is a deformation retraction from $X/\sim$ to $A/\sim$.

(2) Let $r\colon X \times [0,1] \to X$ be a deformation retraction from X to A. We consider the equivalence relation "$\sim_A$" on X that is generated by the condition that all points in A

are equivalent. Note that by definition of a deformation retraction the homotopy r is compatible with the equivalence relation "$\sim_A$". Therefore it follows from (1) that the map

$$\begin{array}{rcl}(X/\sim_A)\times[0,1] & \to & X/\sim_A \\ ([x],t) & \mapsto & [r(x,t)].\end{array}$$

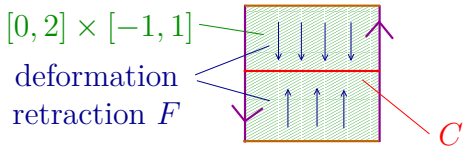
is a deformation retraction from $X/\sim_A = X/A$ to $A/\sim_A = A/A$. ■

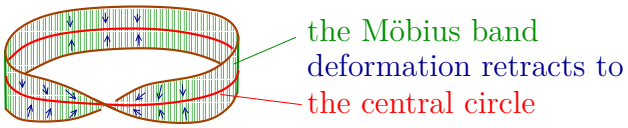
Example. We consider $X = [0,2]\times[-1,1]$ together with the equivalence relation "$\sim$" that is generated by

$$(0,y)\sim(2,-y) \quad \text{for all } y\in[-1,1].$$

It is straightforward to see that $X/\sim$ is homeomorphic to the Möbius band as defined on page 129. We consider the maps

$$\begin{array}{rcl}F\colon X\times[0,1] & \to & X \\ ((x,y),t) & \mapsto & (x, y\cdot(1-t))\end{array} \quad \text{and} \quad \begin{array}{rcl}\overline{F}\colon (X/\sim)\times[0,1] & \to & X/\sim \\ ([(x,y)],t) & \mapsto & [(x, y\cdot(1-t))].\end{array}$$





Note that F is a deformation retraction from X to $C := [0,2]\times\{0\}$. One can easily verify that F is compatible with the equivalence relation "$\sim$". Thus we obtain from the Deformation Retraction-Quotient Theorem 9.2 (1) that $\overline{F}$ is a deformation retraction from the Möbius band to $C/\sim$. Note that $C/\sim$ is homeomorphic to S^1. □

9.2. **Homotopy equivalent topological spaces.** For the reader's convenience we recall the following definition from page 158:

Definition. Let X and Y be two topological spaces. We say that two maps $f_0, f_1\colon X\to Y$ are homotopic, written $f_0\simeq f_1$, if there exists a continuous map $F\colon X\times[0,1]\to Y$ with $F_0 = f_0$ and with $F_1 = f_1$.

The following definition is one of the key definitions in topology. Unfortunately it usually takes some time to fully digest it.

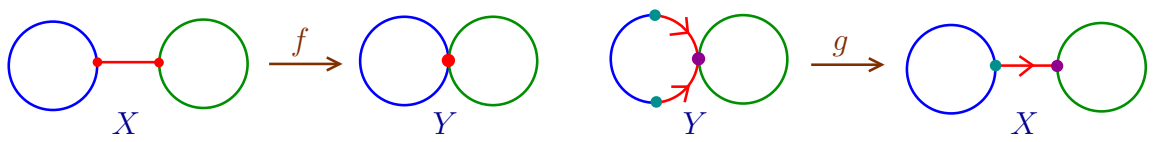
Definition. Let X be and Y be topological spaces.

(1) A continuous $f\colon X\to Y$ is a homotopy equivalence between X and Y, if f admits a homotopy inverse, i.e. a continuous map $g\colon Y\to X$ with $g\circ f\simeq \mathrm{id}_X$ and $f\circ g\simeq \mathrm{id}_Y$.
(2) If there exists a homotopy equivalence between X and Y, then we say that X and Y are homotopy equivalent.

Example.

(1) Basically by definition a homeomorphism is a homotopy equivalence.
(2) In the figure we see two subspaces X and Y of $\mathbb{R}^2$ and we indicate continuous maps $f\colon X\to Y$ and $g\colon Y\to X$.
With some effort one can show that $g\circ f\simeq \mathrm{id}_X$ and $f\circ g\simeq \mathrm{id}_Y$. Thus f and g are homotopy equivalences and X and Y are homotopy equivalent. □

The following lemma shows that in the previous section we already acquired a long list of homotopy equivalences.



Lemma 9.3. (Deformation Retract–Homotopy Equivalence Lemma) Let X be a topological space and let A be a subset. If $F\colon X \times [0,1] \to A$ is a deformation retraction from X to A, then the inclusion map

$$\begin{array}{rcl} i\colon A & \to & X \\ a & \mapsto & a \end{array} \qquad \text{and the map} \qquad \begin{array}{rcl} r\colon X & \to & A \\ x & \mapsto & F(x,1) \end{array}$$

are homotopy inverses of one another. In particular both maps are homotopy equivalences and in particular the deformation retract A is homotopy equivalent to the total space X.

Example. It follows from the Deformation Retract–Homotopy Equivalence Lemma 9.3 and the discussions on page 165 and 167 that the sphere S^{n-1} is homotopy equivalent to $\mathbb{R}^n \setminus \{0\}$ and that the Möbius band is homotopy equivalent to S^1. □

Proof. By definition of a deformation retraction the map $r \circ i = F_1 \circ i\colon A \to A$ is the identity id_A. Thus it remains to show that $i \circ r\colon X \to X$ is homotopic to the identity id_X. A homotopy between id_X and $i \circ r$ is given by $F\colon X \times [0,1] \to X$. ■

The name "homotopy equivalence" already suggests that this is an equivalence relation on the class of topological spaces. It is elementary, but slightly messy, to prove this by hand. Instead of proving this by hand we will first reformulate the notion of being homotopy equivalent in a more categorical language, and then the statement will become trivial.

9.3. **The homotopy category.** In this section we introduce the homotopy category. Initially this might look like a detour, but in fact it simplifies and clarifies the discussion of homotopy equivalences.

Definition. Given two topological spaces X and Y we set

$$[X,Y] \coloneqq \{\text{continuous maps } X \to Y\}/\sim$$

where $f \sim g$ if there exists a homotopy between f and g. Sometimes, given a continuous map $f\colon X \to Y$ we denote by $[f]$ the equivalence class it represents in $[X,Y]$.

Example. It follows easily from the definitions that given a topological space the set $[\{*\}, X]$ is in bijection to $\pi_0(X)$, i.e. in bijection to the set of path-components of X. □

Definition. We define the homotopy category $\mathcal{HomTop}$ of topological spaces by[89]

$$\begin{array}{rcl} \mathrm{Ob}(\mathcal{HomTop}) & \coloneqq & \text{all topological spaces,} \\ \mathrm{Mor}(X,Y) & \coloneqq & [X,Y] \end{array}$$

with the composition[90]

$$\begin{array}{rcl} \mathrm{Mor}(X,Y) \times \mathrm{Mor}(Y,Z) & \to & \mathrm{Mor}(X,Z) \\ ([f],[g]) & \mapsto & [g \circ f]. \end{array}$$

The following lemma changes subtly our point of view on homotopy equivalences.

Lemma 9.4. (Homotopy Equivalence-Reformulation Lemma) Let $f\colon X \to Y$ be a continuous map between topological spaces. The following statements are equivalent:

(1) The element $[f] \in \mathrm{Mor}_{\mathcal{HomTop}}(X, Y)$ is an isomorphism in the homotopy category $\mathcal{HomTop}$.
(2) f is a homotopy equivalence.

In particular the following statements are equivalent:

(1′) X and Y are isomorphic in the homotopy category $\mathcal{HomTop}$.
(2′) X and Y are homotopy equivalent.

Proof. First we show the "(1)⇒(2)"-direction. Let $f\colon X \to Y$ be a continuous map. If $[f]$ is an isomorphism in the category $\mathcal{HomTop}$, then there exists a morphism $\varphi \in \mathrm{Mor}_{\mathcal{HomTop}}(Y, X)$ with $[f] \circ \varphi = \mathrm{id}_Y$ and $\varphi \circ [f] = \mathrm{id}_X$. We pick a continuous map $g\colon Y \to X$ with $\varphi = [g]$. Then we see that $[f \circ g] = [f] \circ [g] = [f] \circ \varphi = \mathrm{id}_Y = [\mathrm{id}_Y]$. But this means that $f \circ g \simeq \mathrm{id}_Y$. The same argument shows that $g \circ f \simeq \mathrm{id}_X$. Thus we have shown that f is a homotopy equivalence.

The argument for "(2)⇒(1)" follows almost immediately from the definitions. Finally note that the equivalence of (1′) and (2′) is an immediate consequence of the equivalence of (1) and (2). ■

Lemma 9.5. (Homotopy-Equivalence Basics Lemma)

(1) The relation "homotopy equivalence" is an equivalence relation on the class of all topological spaces.
(2) Let $f, g\colon X \to Y$ be two continuous maps between topological spaces. If f is a homotopy equivalence and if g is homotopic to f, then g is also a homotopy equivalence.

Proof. All the statements follow easily from the Homotopy Equivalence-Reformulation Lemma 9.4 and the Isomorphism Lemma 3.3. We leave it to the reader to fill in the details. ■

Example. We now have three methods for showing that topological spaces are homotopy equivalent:

(1) Homeomorphisms are almost by definition homotopy equivalences.
(2) The Homotopy-Equivalence Basics Lemma 9.5 (1) says that being homotopy equivalent is an equivalence relation.
(3) The Deformation Retract–Homotopy Equivalence Lemma 9.3 says that if A is a deformation retract of some topological space X, then the inclusion $i\colon A \to X$ is a homotopy equivalence.

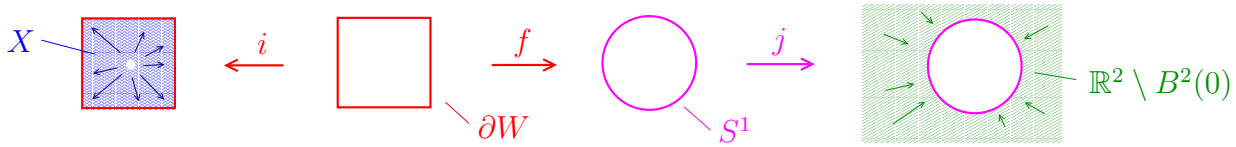
The combination of these three methods can be very powerful: We set $W = [-1, 1] \times [-1, 1]$. We want to show that the topological spaces $X := W \backslash \{0\}$ and $Y := \mathbb{R}^2 \backslash B^2(0)$ are homotopy

[89] Note that in this category the morphisms are not maps but they are equivalence classes of continuous maps.
[90] It follows from the Homotopy Composition Lemma 8.4 (2) that this composition map is well-defined. It follows immediately from the definitions that this composition satisfies the two conditions on a category, as defined on page 88, namely associativity and existence of an identity morphism.

equivalent. To do so we consider the maps

$$X = W \setminus \{0\} \xhookleftarrow{\ i\ } \partial W \xrightarrow[f]{x \mapsto \frac{x}{\|x\|}} S^1 \xhookrightarrow{\ j\ } \mathbb{R}^2 \setminus B^2(0).$$

As on page 165 one can easily show that ∂W is a deformation retract of $X = W \setminus \{0\}$ and that S^1 is a deformation retract of $\mathbb{R}^2 \setminus B^2(0)$. Furthermore f is a homeomorphism.



It follows from the combination of (1), (2) and (3) that $X = W \setminus \{0\}$ is homotopy equivalent to $Y = \mathbb{R}^2 \setminus B^2(0)$. □

We recall the following definitions from pages 82 and 96.

Definition. Let X be a topological space.
(1) We denote by $\pi_0(X)$ the set of path-components of X.
(2) Given a continuous map $f\colon X \to Y$ we write

$$\begin{array}{rcl} f_*\colon \pi_0(X) & \to & \pi_0(Y) \\ C & \mapsto & \text{unique path-component of } Y \text{ that contains } f(C). \end{array}$$

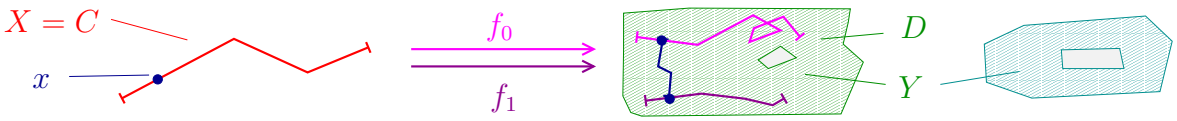
Lemma 9.6. (Homotopy-π_0-Lemma)
(1) The maps

$$\begin{array}{rcl} X & \mapsto & \pi_0(X) \\ [f\colon X \to Y] & \mapsto & [f_*\colon \pi_0(X) \to \pi_0(Y)] \end{array}$$

form a covariant functor from the homotopy category *HomTop* to the category *Set* of sets.
(2) (a) Let $f\colon X \to Y$ be a continuous map between topological spaces. If f is a homotopy equivalence, then f induces a bijection $f_*\colon \pi_0(X) \to \pi_0(Y)$.
(b) If X and Y are homotopy equivalent topological spaces, then X is path-connected if and only if Y is path-connected.

Proof.

(1) We only really need to show that the functor is well-defined. More precisely, we need to show that if $f_0, f_1\colon X \to Y$ are homotopic, then $f_{0*} = f_{1*}\colon \pi_0(X) \to \pi_0(Y)$. Thus let C be a path-component of X and let D be the unique path-component of Y that contains $f_0(C)$. Since f_0 and f_1 are homotopic we see that for every $x \in X$ the points $f_0(x)$ and $f_1(x)$ are path-equivalent in Y. But that means, by definition of a path-component as a path-equivalence class, that $f_1(C) \subset D$. This implies that $f_{1*}(C) = D = f_{0*}(C)$.



(2) (a) This statement follows from (1), the Homotopy Equivalence-Reformulation Lemma 9.4 and the fact that functors send isomorphisms to isomorphisms.
(b) This statement is a special case of (a). ■

9.4. **Contractible topological spaces.** We turn to an important special case of homotopy equivalences:

Definition. Let X be a topological space. We say X is contractible, if X is homotopy equivalent to a topological space which consists of precisely one point.

Remark. Note that the empty topological space is *not* contractible. □

The following straightforward lemma gives an easy criterion for showing that a topological space is contractible.

Lemma 9.7. (Point Deformation Retract Lemma) If a topological space X admits a deformation retraction to a point $x_0 \in X$, then X is homotopy equivalent to $\{x_0\}$, in particular X is contractible.

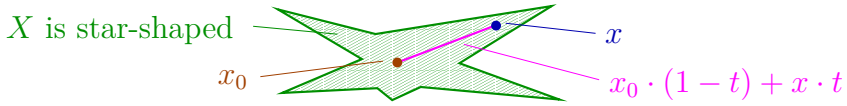
Proof. This lemma is just a special case of the Deformation Retract–Homotopy Equivalence Lemma 9.3. ■

Example.

(1) We say that a subset X of $\mathbb{R}^n$ is star-shaped if there exists a point $x_0 \in X$ such that for every $x \in X$ the segment from x_0 to x lies in X. Basically by the same argument as in the Convex Homotopy Lemma 8.1 we see that the map

$$\begin{array}{rcl} f\colon X \times [0,1] & \to & X \\ (x,t) & \mapsto & \underbrace{x \cdot (1-t) + x_0 \cdot t}_{\in X, \text{ since } X \text{ is star-shaped}} \end{array}$$

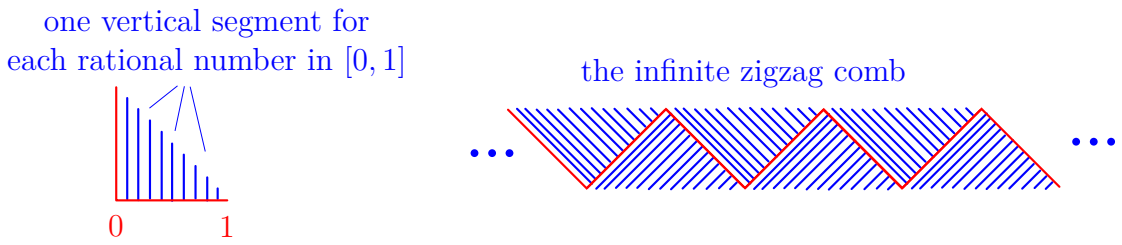
is deformation retraction from X to the point x_0. It follows from the Point Deformation Retract Lemma 9.7 that X is contractible.



(2) Let $n \in \mathbb{N}_0$ and $k \in \mathbb{N}_0$. We just learned that the topological spaces $\mathbb{R}^n$ and $\mathbb{R}^k$ are contractible. It follows from the Homotopy-Equivalence Basics Lemma 9.5 (2) that $\mathbb{R}^n$ and $\mathbb{R}^k$ are homotopy equivalent. □

In the Point Deformation Retract Lemma 9.7 we saw that if a topological space X admits a deformation retraction to a point $x_0 \in X$, then X is contractible. The following lemma shows that in general the converse of this statement does *not* hold.

Lemma 9.8. (Zigzag Comb-Lemma) The "infinite zigzag comb" shown in the figure below is contractible, but it does not admit a deformation retraction to a single point.



Proof. We leave it to the reader to provide the proof. ■

Exercise 9.1. Let $n \in \mathbb{N}$. We consider the topological space $\mathbb{R}^n \cup \{\infty\}$. Let $I \subset \mathbb{R}$ be an interval. We consider $B_I \coloneqq \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n \in I\} \cup \{\infty\}$. We equip B_I with the subspace topology coming from $\mathbb{R}^n \cup \{\infty\}$.

(a) For which intervals I is the map

$$\begin{array}{rcl} F\colon B_I \times [0,1] & \to & \mathbb{R}^n \cup \{\infty\} \\ (P,t) & \mapsto & \begin{cases} \infty, & \text{if } P = \infty \text{ or } t = 1, \\ P + (\frac{1}{1-t} - 1) \cdot e_n, & \text{otherwise} \end{cases} \end{array}$$

continuous?

(b) For which intervals I does F define a deformation retraction from B_I to $\{\infty\}$?

Part II

Fundamental groups and higher homotopy groups

This part of the lecture notes is organized as follows:

- In Chapter 10 we will give a short motivation and introduction to the techniques that we will develop in these lecture notes.
- In Chapter 11 we will introduce the fundamental group $\pi_1(X, x_0)$ of a topological space together with a base point x_0. We will calculate the fundamental group of the circle S^1 and of the higher-dimensional spheres S^n, $n \geq 2$.
- In Chapter 12 we study first basic applications of fundamental groups. For example we will give a proof of the Fundamental Theorem of Algebra using fundamental groups.
- In Chapter 13 we will see that fundamental groups form a covariant functor from the category of pointed topological spaces to the category of groups. The functoriality is the key for developing the full potential of fundamental groups.
- In Chapter 14 we introduce the notion of a covering of a topological space and we will study the relationship between coverings and fundamental groups.
- In Chapter 15 we will use the results from Chapter 14 to prove the Action-π_1-Theorem 15.1 which explains the relationship between group actions and fundamental groups. This theorem allows us to calculate the fundamental groups of tori, Klein bottles, real projective spaces and lens spcaes.
- In Chapter 16 we will study semi-serious applications of fundamental groups to "real life problems".
- In Chapter 17 we will introduce wedges of pointed topological spaces and we will introduce topological graphs.
- In Chapter 18 we will recall some basic concepts and facts about abelian groups.
- In Chapter 19 we turn to the purely algebraic study of (possibly non-abelian) groups. In particular we will introduce the notion of free and amalgamated products of groups.
- In Chapter 20 we will state the Seifert–van Kampen Theorem 20.1 which says that, under suitable hypothesis, the fundamental group of a union of two topological spaces can be written as an amalgamated product of the fundamental groups of the two given topological spaces.
- In Chapter21 we will learn about presentations of groups. These allow us to express many of the groups we are interested in, in finite data.
- In Chapter 22 we will calculate the fundamental groups of surfaces and we will state the classification of compact 2-dimensional manifolds.
- In Chapter 23 we will prove a variation on the Seifert–van Kampen Theorem 20.1 which is especially suitable for manifolds. We will use it to show that given any finitely presented group π and given any $n \geq 4$ there exists a closed orientable connected non-empty n-dimensional smooth manifold M with $\pi_1(M) \cong \pi$.
- In Chapter 24 we will discuss decision problems. For example we will discuss the question whether it is possible to show that two finitely presented groups are isomorphic.
- In Chapter 25 we will discuss the categorical notion of direct colimits.
- In Chapter 26 we will use direct colimits to determine fundamental groups of non-compact topological spaces that can be written as the union of compact topological spaces.

CHAPTER 10

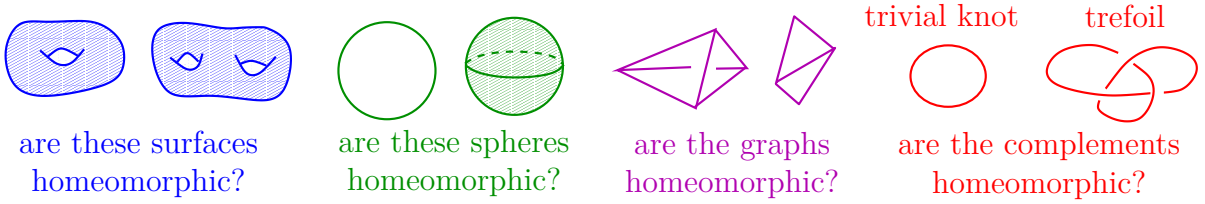
How to show that topological spaces are (not) homeomorphic?

In the previous chapters we collected many examples of topological spaces. In many cases it was relatively straightforward to see that two topological spaces are homeomorphic. For example we wrote down an explicit homeomorphism from $\mathbb{R}^n/\mathbb{Z}^n$ to $(S^1)^n$. Sometimes it is rather painful to explicitly write down a homeomorphism, for example it was not particularly easy to find a homeomorphism from the open cube $(0,1)^n$ to the open ball $B^n = \{x \in \mathbb{R}^n \,|\, \|x\| < 1\}$. Nonetheless, if two topological spaces are homeomorphic, then usually one can write down an explicit homeomorphism.

On the other hand, how can we tell that two topological spaces are not homeomorphic? There are a couple of trivial criteria: being compact, connected, Hausdorff are properties of topological spaces that are shared by homeomorphic topological spaces. This implies for example that the closed cube $[0,1]^n$ is not homeomorphic to the open cube $(0,1)^n$, since the former topological space is compact and the latter is not.

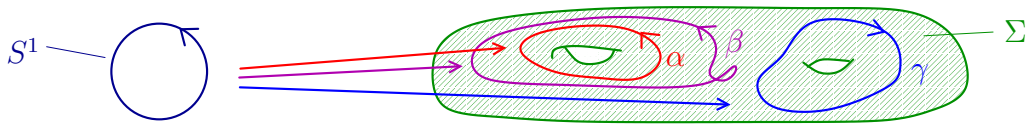
But these criteria are of no use when it comes to studying more subtle examples and questions. For example they do not suffice to address the following questions:

Question 10.1.
(1) Can surfaces with different genera be homeomorphic?
(2) Given $k \neq l \in \mathbb{N}$, is it possible that the spheres S^k and S^l are homeomorphic?
(3) Are the two graphs shown below homeomorphic?
(4) Is the complement of the trefoil homeomorphic to the complement of the trivial knot?
In each case we suspect that the answer is no. Certainly we do not succeed in finding a homeomorphism, but how can we show that this is not due to lack of imagination, but that there is indeed no homeomorphism?



Here is another type of question that we cannot really address right now.

Question 10.2. We consider the maps $S^1 \to \Sigma$ from S^1 to the surface Σ that are shown in the figure below. The curves α and β can be deformed into one another. But it does not look like one can deform α into γ. But how can one prove the non-existence of a deformation?



The reader might have encountered some ideas for tackling these questions in one or both of the following topics:

(a) Differential forms on smooth manifolds, in particular de Rham cohomology, as developed say in [**BT82, Lee02**].
(b) Complex analysis, see e.g. [**Lan99, Jän11**].

First let's talk about approach (a). For the reader who has seen differential forms and de Rham cohomology we recall that to each smooth manifold M and each $k \in \mathbb{Z}_{\geq 0}$ we can assign the de Rham cohomology group $\mathrm{H}^k_{\mathrm{dR}}(M)$. It follows more or less from the definition that diffeomorphic smooth manifolds have isomorphic de Rham cohomology groups. This approach has several disadvantages:

(1) The de Rham cohomology groups are only defined for smooth manifolds.
(2) The de Rham cohomology groups can only be used to show that two smooth manifolds are not *diffeomorphic*, a priori they cannot be used to show that two smooth manifolds are not *homeomorphic*.
(3) At least just starting from the definition it is not clear how one can calculate the de Rham cohomology groups of a given smooth manifold.

Later on we will introduce "singular cohomology" which will take care of all of these three issues.

We turn to the second approach from complex analysis. We recall some definitions that are familiar to the reader who has taken a course on complex analysis.

Definition.

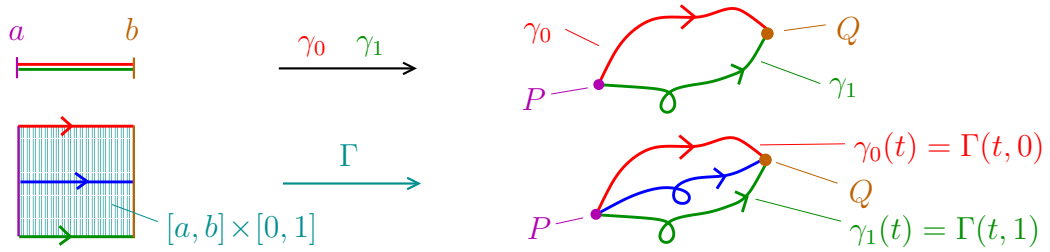
(1) Given a subset U of $\mathbb{C}$, we define a path in U to be a continuous map $\gamma\colon [a,b] \to U$. If $\gamma(a) = \gamma(b)$, then we say that γ is a loop.
(2) Two paths $\gamma_0, \gamma_1\colon [a,b] \to U$ with the same starting point $P := \gamma_0(a) = \gamma_1(a)$ and the same endpoint $Q := \gamma_0(b) = \gamma_1(b)$ are called path-homotopic in U, if there exists a homotopy rel $\{a,b\}$ from γ_0 to γ_1. In other words, if there exists a continuous map

$$\begin{array}{rcl} \Gamma\colon [a,b]\times[0,1] & \to & U \\ (t,s) & \mapsto & \Gamma(t,s), \end{array}$$

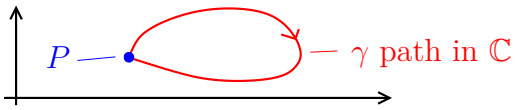
with the following properties:

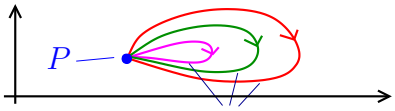
(a) for every $t \in [a,b]$ we have $\Gamma(t,0) = \gamma_0(t)$ and $\Gamma(t,1) = \gamma_1(t)$,
(b) for every $s \in [0,1]$ we have $\Gamma(a,s) = P$ and $\Gamma(b,s) = Q$.

In the figure below we illustrate a path-homotopy between two paths.
(3) A loop $\gamma\colon [a,b] \to U$ is called null-homotopic if it is path-homotopic to the constant path given by $\delta(t) := \gamma(a)$ for all $t \in [a,b]$.
(4) We say U is simply connected if U is non-empty, if U is path-connected and if each loop is null-homotopic.



Example. Every loop in $\mathbb{C}$ is null-homotopic. Indeed, we let $\gamma\colon [0,1]\to\mathbb{C}$ be a loop with starting and endpoint P. Then
$$\begin{array}{rccl}\Gamma\colon & [a,b]\times[0,1] & \to & \mathbb{C}\\ & (t,s) & \mapsto & \gamma(t)\cdot(1-s)+P\cdot s\end{array}$$
is a path-homotopy between the loop γ and the constant path at P. In particular $\mathbb{C}$ is simply connected.





path-homotopy from γ to a constant path

Definition. Let $U\subset\mathbb{C}$ be an open subset and let $f\colon U\to\mathbb{C}$ be a continuous map. Furthermore let $\gamma\colon[a,b]\to U$ a smooth path. We define the path integral of f along γ as follows:
$$\int_\gamma f(z)\,dz \;:=\; \int_{t=a}^{t=b} \underbrace{f(\gamma(t))}_{\in\mathbb{C}} \cdot \underbrace{\gamma'(t)}_{\in\mathbb{C}}\,dt \;\in\mathbb{C}.$$
product of complex numbers

Examples.

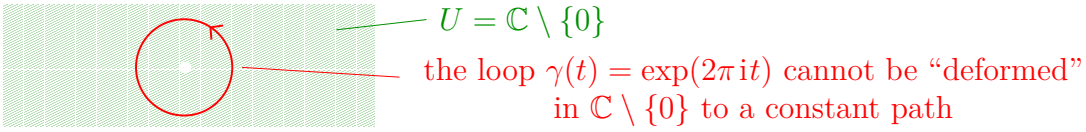
(1) Let $U=\mathbb{C}\setminus\{0\}$, let $f(z)=\frac{1}{z}$ and let $\gamma\colon[0,1]\to\mathbb{C}\setminus\{0\}$ be the loop that is given by $\gamma(t)=\exp(2\pi\,\mathrm{i}t)$, i.e. γ is a loop that "goes once around the origin". Then
$$\int_\gamma f(z)\,dz \;=\; \int_{t=0}^{t=1} f(\gamma(t))\cdot\gamma'(t)\,dt \;=\; \int_{t=0}^{t=1} \frac{1}{\exp(2\pi\,\mathrm{i}t)}\cdot\exp(2\pi\,\mathrm{i}t)\cdot 2\pi\,\mathrm{i}\,dt \;=\; 2\pi\,\mathrm{i}.$$

(2) If γ is a constant path, then $\gamma'(t)\equiv 0$. This shows that the path-integral along a constant path is zero. □

Theorem 10.3. (Cauchy Theorem)[91] Let $U\subset\mathbb{C}$ be an open subset, let $f\colon U\to\mathbb{C}$ be a holomorphic function and let $\gamma,\delta\colon[a,b]\to U$ be two smooth paths. If γ and δ are path-homotopic, then
$$\int_\gamma f(z)\,dz \;=\; \int_\delta f(z)\,dz.$$

Proof. This theorem is proved in complex analysis. ■

Corollary 10.4. The loop $\gamma\colon [0,1] \to \mathbb{C} \setminus \{0\}$ given by $\gamma(t) = \exp(2\pi\,\mathrm{i}t)$ is *not* path-homotopic to a constant path, i.e. γ is not null-homotopic.



Proof. In the above example we saw that $\int_\gamma \frac{1}{z}\,dz \neq 0$ and we saw that for a constant loop δ in $\mathbb{C} \setminus \{0\}$ we have $\int_\delta \frac{1}{z}\,dz = 0$. Thus it follows from the Cauchy Theorem 10.3 that the loop γ is not path-homotopic to a constant loop. ■

Example. If two subsets of $\mathbb{C}$ are homeomorphic, then evidently either both are simply connected or none is. It follows from the discussion on page 177 and Corollary 10.4 that $\mathbb{R}^2 =_{\mathrm{i}} \mathbb{C}$ and $\mathbb{R}^2 \setminus \{(0,0)\} = \mathbb{C} \setminus \{0\}$ are not homeomorphic. □

This second approach to studying topological spaces, namely through path-homotopies of loops, also has several disadvantages:

(1) A priori these notions are defined only for subsets of $\mathbb{C}$.
(2) This approach does not allow us to show that the two subsets $U = \mathbb{C} \setminus \{0\}$ and $V = \mathbb{C} \setminus \{0,1\}$ are not homeomorphic, since neither is simply connected.

The first objection is of course easy to rectify, since the definition of being simply connected makes sense for any topological space. But if we work with more general situations a new problem arises: how can we show, without using complex analysis, that a topological space is not simply connected? In the following chapters we will address all these issues.

Regarding the second objection, here is one idea for distinguishing $U = \mathbb{C} \setminus \{0\}$ and $V = \mathbb{C} \setminus \{0,1\}$: naively one could say that V has "more" loops that are not null-homotopic than U. In the Chapter 11 we will in particular make precise what "more" means.

[91] We will not really make use of the Cauchy Theorem 177. But it motivates us to study path-homotopies in general topological spaces.

CHAPTER 11

The fundamental group

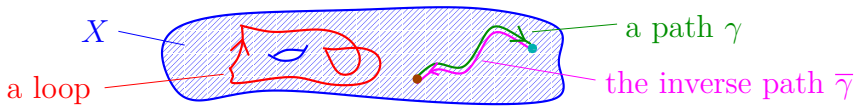
11.1. **Path-homotopy.** In this chapter we extend some of the ideas and notions about paths that we introduced in the previous section to the more general setting of paths in topological spaces. The following definition is a slight generalization of the concept of a path that we initially introduced on page 78.

Definition. Let X be a topological space.

(1) We define a path in X as a continuous map $\gamma\colon [a,b] \to X$. We call $\gamma(a)$ the starting point of γ and we call $\gamma(b)$ the endpoint of γ. Often we say that γ is a path from $\gamma(a)$ to $\gamma(b)$.

(2) A loop is a path for which the starting point and the endpoint coincide.

(3) Given a path $\gamma\colon [a,b] \to X$ we refer to the path $\overline{\gamma}\colon [a,b] \to X$ that is defined by $\overline{\gamma}(t) = \gamma(b+a-t)$ as the inverse path of γ.



Definition. Let X be a topological space.

(1) Let $\gamma_0, \gamma_1\colon [a,b] \to X$ be two paths with the same starting point P and the same endpoint Q. A path-homotopy between the paths γ_0 and γ_1 is a homotopy rel $\{a,b\}$ between γ_0 and γ_1. In other words, spelling out the definition on page 160, a path-homotopy is a continuous map

$$\begin{array}{rcl} \Gamma\colon [a,b]\times[0,1] & \to & X \\ (t,s) & \mapsto & \Gamma(t,s), \end{array}$$

with the following properties:

(a) for every $t \in [a,b]$ we have $\Gamma(t,0) = \gamma_0(t)$ and $\Gamma(t,1) = \gamma_1(t)$,

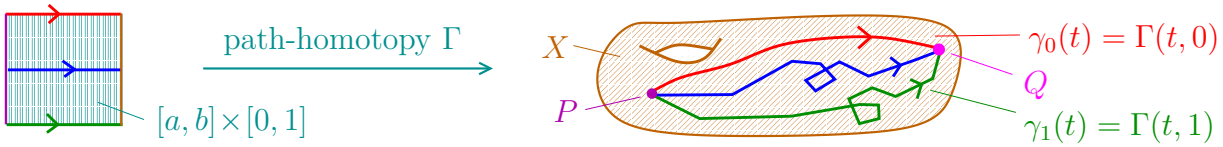
(b) for every $s \in [0,1]$ we have $\Gamma(a,s) = P$ and $\Gamma(b,s) = Q$.

(2) If there exists a path-homotopy between two paths γ_0 and γ_1, then we say that γ_0 and γ_1 are path-homotopic, and we write $\gamma_0 \simeq_{\mathrm{p}} \gamma_1$.

(3) We say that a loop $\gamma\colon [a,b] \to X$ is null-homotopic, if it is path-homotopic to the constant path δ given by $\delta(t) := \gamma(a)$ for all $t \in [a,b]$.

Example.

(A) Let C be a convex subset of $\mathbb{R}^n$ and let $\gamma_0, \gamma_1\colon [a,b] \to C$ be two paths in C with same starting point and the same endpoint. As in the Convex Homotopy Lemma 8.1 we see that



$$\begin{aligned}\Gamma\colon [a,b]\times[0,1] &\to C\\ (t,s) &\mapsto \underbrace{\gamma_0(t)\cdot(1-s)+\gamma_1(t)\cdot s}_{\in C,\text{ since } C \text{ is convex}}\end{aligned}$$

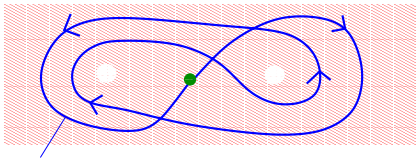
is a path-homotopy between γ_0 and γ_1.[92]

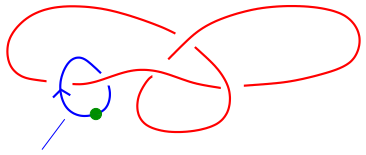
(B) Let $X=\mathbb{C}\setminus\{0\}=\mathbb{R}^2\setminus\{(0,0)\}$. We consider the loop

$$\begin{aligned}\gamma\colon [0,1] &\to X\\ t &\mapsto \exp(2\pi\,\mathrm{i}t)=(\cos(2\pi t),\sin(2\pi t)).\end{aligned}$$

In Corollary 10.4 we showed that the path is not null-homotopic in X. We will soon give a different proof of this statement.

(C) We consider the two loops in the figure below. In both cases it is hard to see how these loops could possibly be path-homotopic to a constant path, but it is also difficult to find an argument why that should not be possible.[93] We will return to this example on page 295.

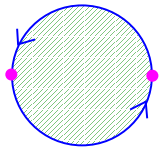

loop in $\mathbb{C}\setminus\{0,1\}$

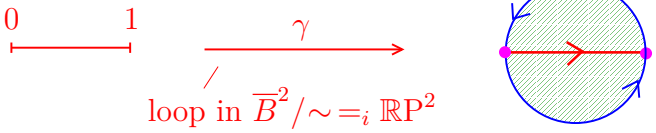

loop in the complement of a knot

(D) We consider the real projective space $\mathbb{R}\mathrm{P}^2 =_{\mathrm{i}} \overline{B}^2/\!\sim$ where $P\sim -P$ for $P\in S^1=\partial\overline{B}^2$. Note that

$$\begin{aligned}\gamma\colon [0,1] &\to \overline{B}^2/\!\sim\\ t &\mapsto [(2t-1,0)]\end{aligned}$$

is a loop in $\mathbb{R}\mathrm{P}^2 =_{\mathrm{i}} \overline{B}^2/\!\sim$.[94] But is it null-homotopic? □


$\overline{B}^2/\!\sim =_i \mathbb{R}\mathrm{P}^2$



Our first lemma of this chapter will provide us with a rich source of path-homotopies. To formulate the lemma we need the following definition:

[92] We had already illustrated this fact in the figure on page 177.

[93] For example the given loop in $\mathbb{C}\setminus\{0,1\}$ is actually null-homotopic in $\mathbb{C}\setminus\{0\}$ and it is null-homotopic in $\mathbb{C}\setminus\{1\}$. In particular our trick of using Corollary 10.4 does not work in this case.

[94] Why is it a loop?

Definition. Let $\gamma\colon [a,b] \to X$ be a path in a topological space X and let $\varphi\colon [a,b] \to [a,b]$ be a continuous map with $\varphi(a) = a$ and $\varphi(b) = b$. We refer to $\gamma \circ \varphi\colon [a,b] \to X$ as a reparametrization of γ.

The following lemma says that reparametrizing does not change the path-homotopy class of a path.

Lemma 11.1. (Reparametrization-Homotopy Lemma) Let $\gamma\colon [a,b] \to X$ be a path in a topological space X. If a path $\delta\colon [a,b] \to X$ is obtained from γ by a reparametrization, then γ and δ are path-homotopic.

Proof. Let $\gamma\colon [a,b] \to X$ be a path in a topological space and let $\varphi\colon [a,b] \to [a,b]$ be a continuous map with $\varphi(a) = a$, $\varphi(b) = b$. A path-homotopy between γ and $\gamma \circ \varphi$ is given by[95]

$$\begin{array}{rccl} P\colon [a,b] \times [0,1] & \to & X \\ (t,s) & \mapsto & \gamma(t \cdot s + \varphi(t) \cdot (1-s)). \end{array}$$

■

The following lemma is a special case of the Homotopy-Equivalence Relation Lemma 8.3.

Lemma 11.2. (Path-Homotopy–Equivalence Relation Lemma) Let X be a topological space and let $x, y \in X$. Then "path-homotopy" is an equivalence relation on the set of paths $\gamma\colon [a,b] \to X$ from x to y.

Proof. This lemma is just the special case of the Homotopy-Equivalence Relation Lemma 8.3 with $X = [a,b]$ and $A = \{a,b\}$. ■

The Path-Homotopy–Equivalence Relation Lemma 11.2 allows us to make the following definition:

Definition. Given a path $f\colon [a,b] \to X$ in a topological space X we denote by $[f]$ the equivalence class of f with respect to the equivalence relation which is given by path-homotopies. We call $[f]$ the path-homotopy class of f.

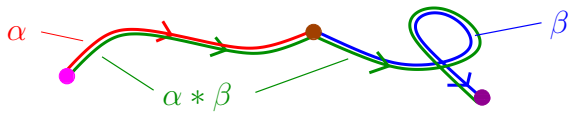
We now recall the following definition from page 81:

Definition. Let X be a topological space and let $\alpha\colon [a,b] \to X$ and $\beta\colon [c,d] \to X$ be two paths with $\alpha(b) = \beta(c)$. We define the concatenation of α and β as the path $\alpha * \beta$ which is given by

$$\begin{array}{rccl} \alpha * \beta\colon [0,1] & \to & X \\ t & \mapsto & \begin{cases} \alpha(a + 2t(b-a)), & \text{if } t \in \left[0, \frac{1}{2}\right] \\ \beta(c + (2t-1)(d-c)), & \text{if } t \in \left[\frac{1}{2}, 1\right]. \end{cases} \end{array}$$

Similarly we also define the concatenation of a finite number of paths.

The concatenation of two paths is thus given by "first running along α" and then "running along β". Here we reparametrized the path so that the domain of $\alpha * \beta$ is $[0,1]$. This convention might look slightly odd, but later on we will mostly work with paths that are defined on the interval $[0,1]$.



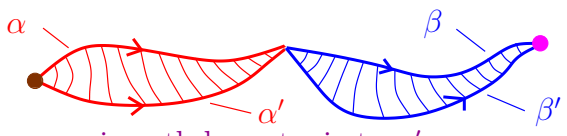
[95] Where do we actually use the hypothesis that $\varphi(a) = a$ and $\varphi(b) = b$?

Lemma 11.3. (Path-Homotopy–Concatenation Lemma) Let X be a topological space. Let $\alpha, \alpha' \colon [a,b] \to X$ and $\beta, \beta' \colon [c,d] \to X$ be paths with $\alpha(b) = \alpha'(b) = \beta(c) = \beta'(c)$. Then

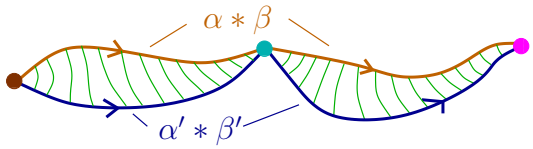
$$\alpha \simeq_{\mathrm{p}} \alpha' \text{ and } \beta \simeq_{\mathrm{p}} \beta' \implies \alpha * \beta \simeq_{\mathrm{p}} \alpha' * \beta'.$$

Put differently, if $[\alpha] = [\alpha']$ and $[\beta] = [\beta']$, then $[\alpha * \beta] = [\alpha' * \beta']$.

Proof. We leave it to the reader to provide the fairly elementary proof. ∎



α is path-homotopic to α'
and β is path-homotopic to β'



then $\alpha * \beta$ is path-homotopic to $\alpha' * \beta'$

Definition. Let $[\alpha]$ and $[\beta]$ be two equivalence classes of paths in a topological space X with $\alpha(1) = \beta(0)$. We define the[96] product of the equivalence classes $[\alpha]$ and $[\beta]$ as

$$[\alpha] \cdot [\beta] \;:=\; [\alpha * \beta].$$

The following proposition says that the product of equivalence classes of paths shares many properties of the product structure of a group.

Proposition 11.4. (Path-Homotopy Product Proposition) Let X be a topological space. Given any three paths $\alpha, \beta, \gamma \colon [0,1] \to X$ the following hold:

(1) If $\alpha(1) = \beta(0)$ and $\beta(1) = \gamma(0)$, then[97]

$$[\alpha] \cdot ([\beta] \cdot [\gamma]) \;=\; ([\alpha] \cdot [\beta]) \cdot [\gamma].$$

(2) For $x \in X$ let e_x be the constant path that is given by $e_x(t) := x$. The following equality holds:

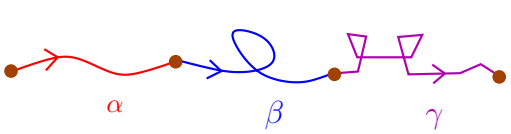
$$[e_{\alpha(0)}] \cdot [\alpha] \;=\; [\alpha] \;=\; [\alpha] \cdot [e_{\alpha(1)}].$$

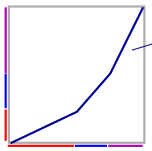
(3) We denote by $\overline{\alpha}$ the inverse path. Then the following equalities hold:

$$[\alpha] \cdot [\overline{\alpha}] \;=\; [e_{\alpha(0)}] \quad \text{and} \quad [\overline{\alpha}] \cdot [\alpha] \;=\; [e_{\alpha(1)}].$$

Proof. Let $\alpha, \beta, \gamma \colon [0,1] \to X$ be three paths in a topological space X.

(1) Suppose that $\alpha(1) = \beta(0)$ and $\beta(1) = \gamma(0)$. We consider the map $\varphi \colon [0,1] \to [0,1]$ that is shown in the figure below. It follows immediately from the definitions that $(\alpha * (\beta * \gamma)) \circ \varphi = (\alpha * \beta) * \gamma$. Thus it follows from the Reparametrization-Homotopy Lemma 11.1 that $\alpha * (\beta * \gamma) \simeq_{\mathrm{p}} (\alpha * \beta) * \gamma$, which is what we need to show.





graph of $\varphi \colon [0,1] \to [0,1]$
which is used to reparametrize
the path $\alpha * (\beta * \gamma)$ to
obtain the path $(\alpha * \beta) * \gamma$

[96] According to the Path-Homotopy–Concatenation Lemma 11.3 this definition does not depend on the choice of the representatives α and β of the equivalence classes.

[97] Put differently, the paths $\alpha * (\beta * \gamma)$ and $(\alpha * \beta) * \gamma$ are path-homotopic. Note that these paths are similar in the sense that both paths "run through" the points on α, β and γ, but at different speeds. In general the paths are not equal, for example we have $(\alpha * (\beta * \gamma))(\frac{1}{2}) = \alpha(1)$ and $((\alpha * \beta) * \gamma)(\frac{1}{2}) = \beta(1)$.

(2) We consider the map

$$\begin{array}{rcl} \varphi\colon [0,1] & \to & [0,1] \\ t & \mapsto & \begin{cases} 0, & \text{if } t \in [0,\frac{1}{2}], \\ 2t-1, & \text{if } t \in [\frac{1}{2},1]. \end{cases} \end{array}$$

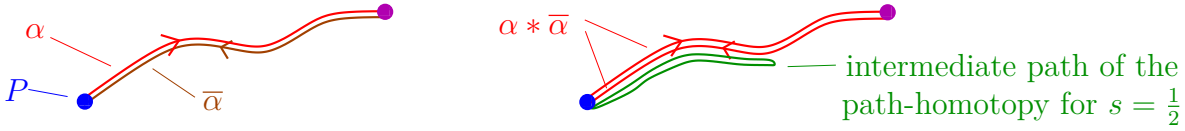
Then $e_{\alpha(0)} * \alpha \underset{\text{by definition}}{=} \alpha \circ \varphi \underset{\text{Reparametrization-Homotopy Lemma 11.1}}{\simeq_{\mathrm{p}}} \alpha.$

This shows that $[e_{\alpha(0)}] \cdot [\alpha] = [e_{\alpha(0)} * \alpha] = [\alpha]$. Almost the same argument also implies that $[\alpha] = [\alpha] \cdot [e_{\alpha(1)}]$.

(3) We write $P = \alpha(0)$. Note that $\alpha * \overline{\alpha}$ is a loop in P. We consider the map

$$\begin{array}{rcl} F\colon [0,1] \times [0,1] & \to & X \\ (t,s) & \mapsto & \begin{cases} \alpha(2t \cdot s), & \text{if } t \in [0,\frac{1}{2}], \\ \alpha(2s - 2t \cdot s), & \text{if } t \in [\frac{1}{2},1]. \end{cases} \end{array}$$

It follows from the Homotopy Combination Lemma 8.2 (1) that this map is indeed continuous.



For $s = 0$ we obtain the constant path at P and for $s = 1$ we obtain the path $\alpha * \overline{\alpha}$. For any $s \in [0,1]$ we furthermore have $F(0,s) = F(1,s) = P$. Thus F is a path-homotopy from the constant path to $\alpha * \overline{\alpha}$. This implies that $[e_{\alpha(0)}] = [\alpha * \overline{\alpha}] = [\alpha] \cdot [\overline{\alpha}]$. Almost the same argument shows that $[\overline{\alpha} * \alpha] = [e_{\alpha(1)}]$. ■

11.2. The fundamental group of a pointed topological space.

In the Path-Homotopy Product Proposition 11.4 we saw that the concatenation of path-homotopy classes of paths shares many properties of the multiplication in a group. Nonetheless the path-homotopy classes do not form a group, since the concatenation of two paths is not defined, unless the endpoint of the first path agrees with the starting point of the second path.

We will resolve this problem by restricting our attention to loops that start and end at a fixed base point.

Definition. Let X be a topological space and let $x_0 \in X$ be a point. A loop in (X, x_0) is a path $f\colon [0,1] \to X$[98] with $f(0) = f(1) = x_0$, i.e. x_0 is the starting as well as the endpoint of f. If X is understood, then often we just say "loop in x_0".

Proposition 11.5. (π_1–Group Proposition) Let X be a topological space and $x_0 \in X$. The set

$$\pi_1(X, x_0) \;:=\; \{\text{path-homotopy classes of loops in } (X, x_0)\}$$

together with the product map $[\alpha] \cdot [\beta] := [\alpha * \beta]$ forms a group where the neutral element is represented by the constant path at x_0 and where the inverse of $[\gamma]$ is given by $[\overline{\gamma}]$.

Proof. It is clear that the concatenation of any two loops in (X, x_0) is defined. According to the Path-Homotopy–Concatenation Lemma 11.3 the concatenation of paths descends to a well-defined map

[98] Note that part of the definition of "loop in (X, x_0)" is that the domain is the interval $[0,1]$.

$$\begin{array}{rcl} \pi_1(X,x_0)\times\pi_1(X,x_0) & \to & \pi_1(X,x_0) \\ ([\alpha],[\beta]) & \mapsto & [\alpha]\cdot[\beta] \coloneqq [\alpha * \beta]. \end{array}$$

The Path-Homotopy Product Proposition 11.4 (1)–(3) gives us the following:

(1) The above product map satisfies the associativity axiom.
(2) The path-homotopy class of the constant loop $[e_{x_0}]$ is a neutral element.
(3) The inverse of the path-homotopy class of a loop $\gamma\colon [0,1]\to X$ in x_0 is given by the path-homotopy class of the inverse loop defined by $\overline{\gamma}(t) \coloneqq \gamma(1-t)$, $t\in[0,1]$. ■

Definition. Let (X,x_0) be a pointed topological space. We call

$$\pi_1(X,x_0) \;\coloneqq\; \{\text{path-homotopy classes of loops in } (X,x_0)\}$$

the fundamental group of X with respect to the base point x_0. If $\pi_1(X,x_0)$ is the trivial group, then we usually write $\pi_1(X,x_0)=0$.[99]

One of our main goals in the intermediate future will be to determine the fundamental group of the topological spaces that we introduced in the earlier chapters, for example spheres, surfaces, the projective spaces $\mathbb{R}\mathrm{P}^n$ and so on. We start with the simplest example.

Example.

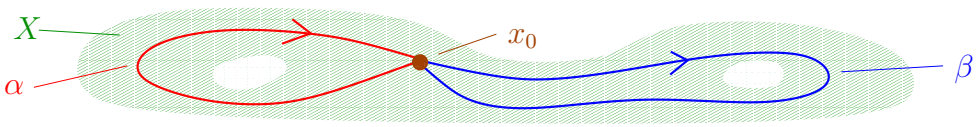
(1) We say that a subset $X\subset\mathbb{R}^n$ is star-shaped with respect to $x_0\in X$ if for all $y\in X$ the segment $\{x_0\cdot(1-t)+y\cdot t\,|\,t\in[0,1]\}$ lies in X. As on page 179 we see that all loops in (X,x_0) are null-homotopic. More precisely, if $\gamma\colon[0,1]\to\mathbb{R}^n$ is a loop in (X,x_0), then

$$\begin{array}{rcl} F\colon [0,1]\times[0,1] & \to & X \\ (t,s) & \mapsto & \gamma(t)\cdot(1-s)+x_0\cdot s \end{array}$$

is a path-homotopy in X between the loop γ and the constant loop e_{x_0}. Thus it follows that $\pi_1(X,x_0)=0$.



(2) Let X be a convex subset of $\mathbb{R}^n$. (For example open and closed balls and open and closed rectangles are convex.) Note that X is star-shaped with respect to any $x_0\in X$, thus we obtain from (1) that $\pi_1(X,x_0)=0$ for all $x_0\in X$.
(3) The π_1–Group Proposition 11.5 says that $\pi_1(X,x_0)$ is a group, but it is not clear whether the group is necessarily abelian. In fact, in the figure below we see two loops α, β in (X,x_0) where it is not clear at all whether $[\alpha]\cdot[\beta]\overset{?}{=}[\beta]\cdot[\alpha]\in\pi_1(X,x_0)$.



□

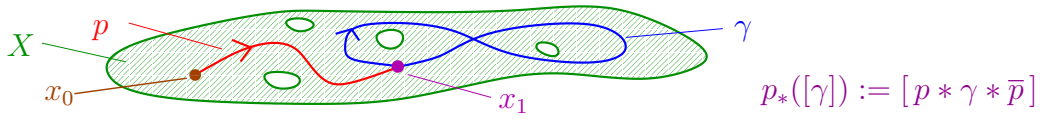
[99] This notation is commonly used, even though it is not entirely logical: we use the multiplicative notation for the product structure on the group $\pi_1(X,x_0)$, and it would therefore make more sense to write $\pi_1(X,x_0)=1$ if the group $\pi_1(X,x_0)$ is trivial.

11.3. **Change of base points.** The definition of the fundamental group of a topological space X relies on the choice of a base point x_0. The following proposition shows that, at least for path-connected topological spaces, the choice of the base point does not affect the isomorphism type of the fundamental group.

Proposition 11.6. (Change-of-Base Point Proposition) Let X be a topological space, let x_0 and x_1 be two points in X and let $p\colon [0,1] \to X$ be a path from x_0 to x_1. The map

$$\begin{array}{rcl} p_*\colon \pi_1(X,x_1) & \to & \pi_1(X,x_0) \\ [\gamma] & \mapsto & [p*\gamma*\overline{p}\,] \end{array}$$

is well-defined and it is a group isomorphism.



Proof. Let X be a topological space and let x_0 and x_1 be two points in X that are connected via a path $p\colon [0,1] \to X$ from x_0 to x_1. Let γ be a loop in (X,x_1). Then $p*\gamma*\overline{p}$ is evidently a loop in (X,x_0). From the Path-Homotopy–Concatenation Lemma 11.3 and the Path-Homotopy Product Proposition 11.4 it follows that the path-homotopy class of $p*\gamma*\overline{p}$ depends only on the choice of the path-homotopy class of γ, i.e. the map

$$\begin{array}{rcl} \Phi\colon \pi_1(X,x_1) & \to & \pi_1(X,x_0) \\ [\gamma] & \mapsto & [p*\gamma*\overline{p}\,] \end{array}$$

is well-defined. Furthermore for two loops γ, δ in (X,x_1) we have

$$\begin{array}{rcl} \Phi([\gamma])\cdot\Phi([\delta]) & = & [p*\gamma*\overline{p}\,]\cdot[p*\delta*\overline{p}\,] \\ & \underset{\uparrow}{=} & [p*\gamma*e_{x_1}*\delta*\overline{p}\,] \end{array} \quad \begin{array}{rcl} & = & [p*\gamma*\overline{p}*p*\delta*\overline{p}] \\ & \underset{\uparrow}{=} & [p*(\gamma*\delta)*\overline{p}\,] \end{array} = \Phi([\gamma*\delta]) = \Phi([\gamma]\cdot[\delta]).$$

the two statements follow from the Path-Homotopy Concatenation Proposition 11.4 (3) and (2) together with the Path-Homotopy–Concatenation Lemma 11.3

Thus we have shown that Φ is a group homomorphism. Furthermore Φ is even a group isomorphism, since an inverse map is given by[100]

$$\begin{array}{rcl} \Psi\colon \pi_1(X,x_0) & \to & \pi_1(X,x_1) \\ [\delta] & \mapsto & [\overline{p}*\delta*p]. \end{array}$$

■

We obtain the following corollary:

Corollary 11.7. (Base Point Independence Corollary) Let X be a topological space and let $x_0, x_1 \in X$. If X is path-connected, then the groups $\pi_1(X,x_0)$ and $\pi_1(X,x_1)$ are isomorphic.

Proof. This statement follows easily from the Change-of-Base Point Proposition 11.6. ■

Example. Let X be a star-shaped subset of $\mathbb{R}^n$. Above we saw that there exists an $x_0 \in X$ such that $\pi_1(X,x_0) = 0$. It now follows from the Base Point Independence Corollary 11.7 that $\pi_1(X,x_1) = 0$ for any point $x_1 \in X$. Put differently, the fundamental group of every star-shaped subset of $\mathbb{R}^n$ with respect to any base point is trivial. □

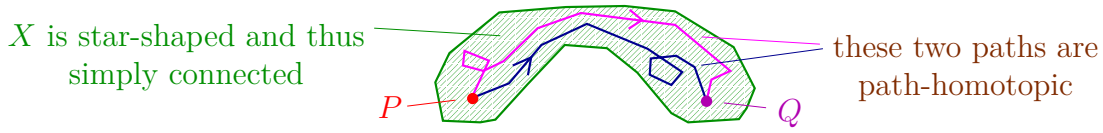
[100] Why is $\Phi\circ\Psi$ the identity on $\pi_1(X,x_0)$ and why is $\Psi\circ\Phi$ the identity on $\pi_1(X,x_1)$?

If X is a path-connected topological space, then the Base Point Independence Corollary 11.7 states that the isomorphism type of the fundamental group does not depend on the choice of the base point. This leads us to the following definition:

Definition. Let X be a path-connected non-empty topological space.
(1) We denote by $\pi_1(X)$ the isomorphism type of the fundamental group.
(2) If $\pi_1(X) = 0$, then we say that X is simply connected.[101]

Example. On page 184 we saw that star-shaped subsets of $\mathbb{R}^n$ are simply connected. □

Lemma 11.8. (Paths-in-Simply Connected Space Lemma) Let X be a topological space and let P, Q be two points in X. If X is simply connected, then any two paths $\alpha, \beta\colon [0,1] \to X$ from P to Q are path-homotopic.



Proof. We leave the proof as an exercise to the reader. ■

In the following two sections we will first calculate the isomorphism type of the fundamental groups of the circle S^1 and then we will calculate the isomorphism type of the fundamental groups of the spheres S^n with $n \geq 2$. Both calculations require some effort. But the effort will pay off in multiple ways:

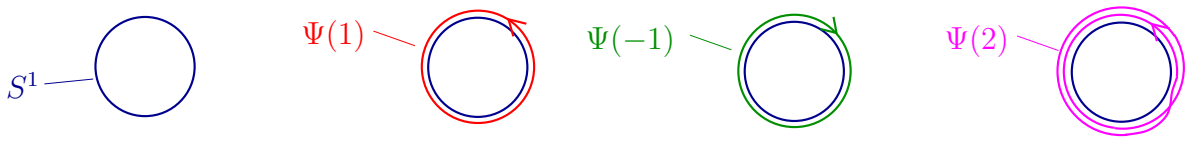
(1) Using these calculations we can already prove several interesting results, like the Fundamental Theorem of Algebra and (with a little bit of extra theory) the Borsuk-Ulam Theorem for S^2.
(2) The techniques we will employ in these special cases will give us the ideas for tackling much more general computational problems later on.

11.4. **The fundamental group of the circle.** The goal of this section is to prove the following important theorem, which gives us our first calculation of a non-trivial fundamental group.

Theorem 11.13. (Circle-π_1-Theorem) The map

$$\begin{array}{rcl} \Psi\colon \mathbb{Z} & \to & \pi_1(S^1, 1) \\ n & \mapsto & \left[\begin{array}{rcl} [0,1] & \to & S^1 \\ t & \mapsto & \exp(2\pi\,\mathrm{i}\cdot n\cdot t)\end{array}\right] \end{array}$$

is an isomorphism of groups.



[101] It is straightforward to see that for subsets of $\mathbb{C}$ we obtain the same notion of "simply connectedness" as on page 176.

The proof of the Circle-π_1-Theorem 11.13 requires several preparations, which are actually interesting in their own right. In particular, as many other calculations of fundamental groups the proof of the Circle-π_1-Theorem 11.13 makes use of the Lebesgue Lemma:

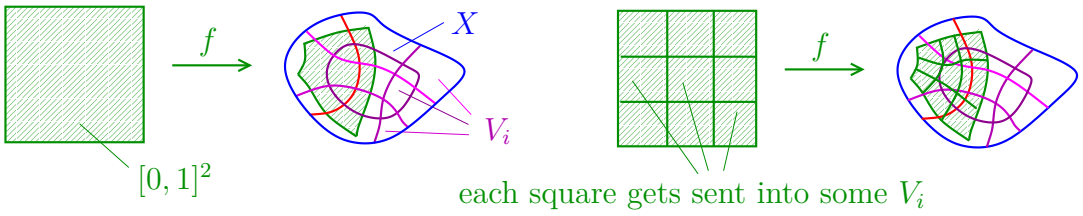
Lemma 11.9. (Lebesgue Lemma)

(1) Let $K \subset \mathbb{R}^n$ be a compact subset and let $\{U_i\}_{i\in I}$ be an open cover of K. There exists a $\delta > 0$ such that for every subset A of K with diameter $\operatorname{diam}(A) < \delta$ there exists an $i \in I$ with $A \subset U_i$.

(2) Let $f\colon [0,1]^m \to X$ be a continuous map from the cube $[0,1]^m$ to a topological space X and let $\{V_i\}_{i\in I}$ be an open cover of X. Then there exists an $N \in \mathbb{N}$ such that for any $a_1, \dots, a_m \in \{0, \dots, N-1\}$ there exists an $i \in I$ such that

$$f\big(\big[\tfrac{a_1}{N}, \tfrac{a_1+1}{N}\big] \times \cdots \times \big[\tfrac{a_m}{N}, \tfrac{a_m+1}{N}\big]\big) \;\subset\; V_i.$$

Pictorially the Lebesgue Lemma 11.9 (2) says that if $f\colon [0,1]^m \to X$ is a continuous map, and if we are given an open cover of X, then we can always find a small grid on the cube, such that each cube of the grid gets mapped into one of the open subsets V_i of the given open cover of X.



Proof. We proved Statement (1) in Analysis III. Statement (2) is a straightforward consequence of Statement (1). Full details can also be found in [**Fri24**]. ■

The next definition is the key tool to studying the fundamental group of S^1:

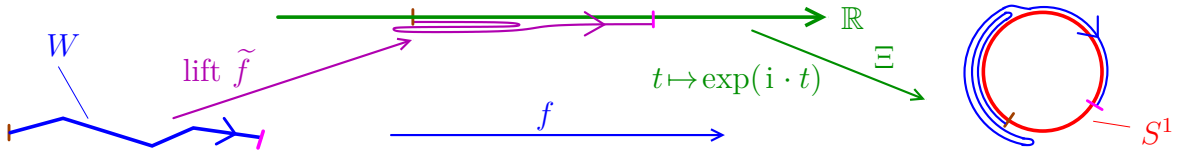
Definition.

(1) We work with the map

$$\begin{array}{rcl} \Xi\colon \mathbb{R} & \to & S^1 \\ t & \mapsto & \exp(\mathrm{i}\cdot t). \end{array}$$

(2) Let W be a topological space and let $f\colon W \to S^1$ be a continuous map. A lift of f is a continuous map $\widetilde{f}\colon W \to \mathbb{R}$ such that $\Xi \circ \widetilde{f} = f$, in other words, such that for all $w \in W$ we have

$$f(w) \;=\; \exp\big(\mathrm{i}\cdot \widetilde{f}(w)\big).$$



Example. For $n \in \mathbb{Z}$ we consider the map $f\colon [0,1] \to S^1$ with $f(t) := \exp(2\pi\,\mathrm{i}\cdot n\cdot t)$. A lift of f is given by the map

$$\begin{array}{rcl} \widetilde{f}\colon [0,1] & \to & \mathbb{R} \\ t & \mapsto & 2\pi\cdot n\cdot t. \end{array}$$

This lift is not unique though since we can "shift" the lift by any $k\cdot 2\pi$ with $k \in \mathbb{Z}$. □

Lemma 11.10. (Lift Uniqueness Lemma) Let W be a connected topological space, let $f\colon W \to S^1$ be a continuous map and let $\widetilde{\varphi}, \widetilde{\psi}\colon W \to \mathbb{R}$ be two lifts of f. If $\widetilde{\varphi}$ and $\widetilde{\psi}$ agree at some point $w_0 \in W$, then they agree everywhere.

Proof. We consider the map

$$\begin{array}{rcl} \theta\colon W & \to & \mathbb{R} \\ w & \mapsto & \theta(w) \coloneqq \widetilde{\varphi}(w) - \widetilde{\psi}(w). \end{array}$$

By hypothesis we have $\theta(w_0) = 0$. Next note that for each $w \in W$ we have

$$\underbrace{\exp\big(\mathrm{i}\cdot(\widetilde{\varphi}(w) - \widetilde{\psi}(w))\big)}_{=\exp(\mathrm{i}\cdot\theta(w))=\Xi(\theta(w))} = \exp\big(\mathrm{i}\cdot\widetilde{\varphi}(w)\big)\cdot\exp\big(\mathrm{i}\cdot\widetilde{\psi}(w)\big)^{-1} \underset{\substack{\uparrow \\ \text{since } \widetilde{\varphi},\widetilde{\psi}\colon W\to\mathbb{R} \text{ are lifts of } f}}{=} f(w)\cdot f(w)^{-1} = 1.$$

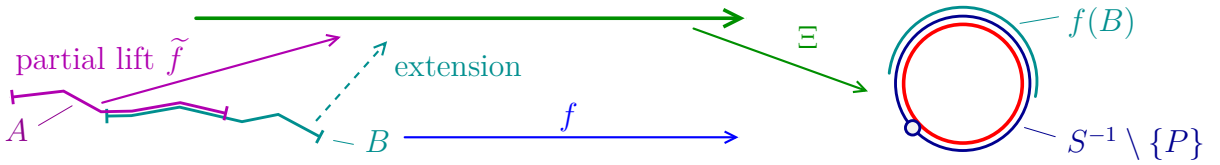
This shows that the continuous map $s \mapsto \theta(w) \coloneqq \widetilde{\varphi}(w) - \widetilde{\psi}(w)$ takes values in the discrete subspace $2\pi\cdot\mathbb{Z} = \Xi^{-1}(1) \subset \mathbb{R}$. Since W is connected it follows from the Connected-to-Discrete Lemma 2.18 that the map θ is constant. Since the map θ vanishes at w_0 we see that it is zero everywhere. This implies that $\widetilde{\varphi} = \widetilde{\psi}$. ∎

Our next goal is to prove the existence of lifts of continuous maps $X \to S^1$ in suitable settings. The key tool for constructing lifts is the following lemma:

Lemma 11.11. (Lift-Extension Lemma) Let X be a topological space. We suppose that we have a decomposition $X = A \cup B$ such that A and B are closed subsets. Let $f\colon X \to S^1$ be a continuous map and let $\widetilde{f}\colon A \to \mathbb{R}$ be a lift of $f|_A$. We suppose that the following statements hold:

(1) There exists a $P \in S^1$ with $f(B) \subset S^1 \setminus \{P\}$.
(2) The intersection $A \cap B$ is connected and non-empty.

Then there exists a lift $\widetilde{f}\colon X \to \mathbb{R}$ of f that agrees with the given lift on A.



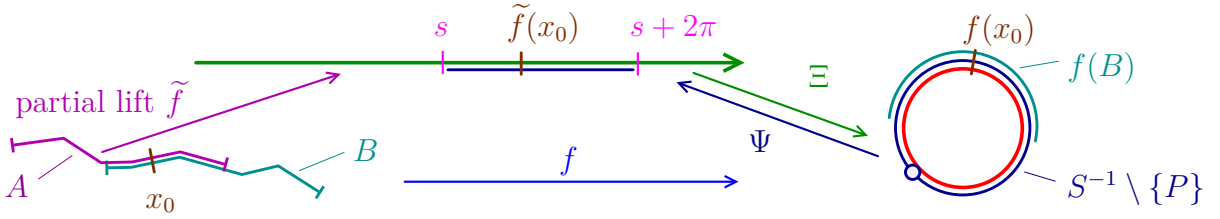
Proof. We start with a few preparations:

(1) We pick $x_0 \in A \cap B$ and we pick $s \in \mathbb{R}$ with $\Xi(s) = P$.
(2) Since $f(x_0) \in f(A\cap B) \subset f(B) \subset S^1 \setminus \{P\}$ and since $\widetilde{f}\colon A \to \mathbb{R}$ is a lift of $f\colon A \to S^1$ we see that $\widetilde{f}(x_0) \notin \Xi^{-1}(P)$.
(3) It follows from (2) that there exists an $m \in \mathbb{Z}$ such that $\widetilde{f}(x_0) \in (s+m\cdot 2\pi, s+m\cdot 2\pi+2\pi)$.
(4) One can easily verify that the restriction of Ξ to $(s+m\cdot 2\pi, s+m\cdot 2\pi+2\pi) \to S^1\setminus\{P\}$ is a homeomorphism. We denote by Ψ the inverse.

We set

$$\begin{array}{rcl} \widetilde{f}\colon X & \to & \mathbb{R} \\ x & \mapsto & \begin{cases} \widetilde{f}(x), & \text{if } x \in A, \\ \Psi(\underbrace{f(x)}_{\in S^1\setminus\{P\}}), & \text{if } x \in B. \end{cases} \end{array}$$

Note that on $A \cap B$ we have a priori two different definitions of $\widetilde{f}$. But both definitions are lifts of $f|_{A\cap B}$ and by design they agree on x_0. By hypothesis the intersection $A \cap B$



is connected. Thus we obtain from the Lift Uniqueness Lemma 11.10 that the two maps actually agree on the overlap $A\cap B$. This shows that $\widetilde{f}\colon X\to\mathbb{R}$ is well-defined.

Next note that since A and B are closed it follows from the Pasting Proposition 2.4 that this map $\widetilde{f}\colon X\to\mathbb{R}$ is continuous. It is now clear that it is a lift of $f\colon X\to\mathbb{R}$. ■

Lemma 11.12. (Lift-from-S^1-to-$\mathbb{R}$ Lemma) Given any continuous map $f\colon [0,1]\to S^1$ and given any $\widetilde{y}_0\in\mathbb{R}$ with $f(0)=\exp(\mathrm{i}\cdot\widetilde{y}_0)=:\Xi(\widetilde{y}_0)$ there exists a *unique* lift $\widetilde{f}\colon[0,1]\to\mathbb{R}$ of f with $\widetilde{f}(0)=\widetilde{y}_0$.

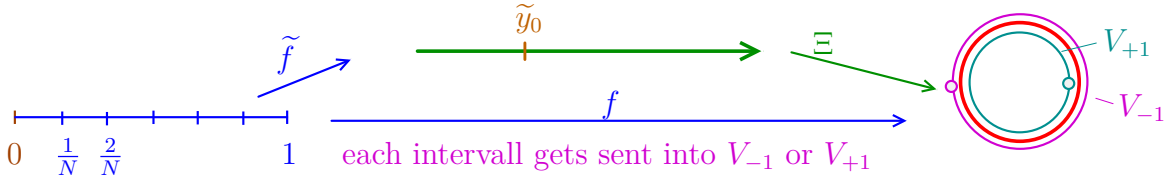
Proof of Lemma 11.12. Let $f\colon[0,1]\to S^1$ be a continuous map and let $\widetilde{y}_0\in\mathbb{R}$ such that $f(0)=\exp(\mathrm{i}\cdot\widetilde{y}_0)$. We need to show that there exists a *unique* lift $\widetilde{f}\colon[0,1]\to\mathbb{R}$ of f with $\widetilde{f}(0)=\widetilde{y}_0$.

The uniqueness statement follows immediately from the Lift Uniqueness Lemma 11.10. Thus it remains to show the existence statement.

The idea is quite simple: We want to apply the Lift-Extension Lemma 11.11. To do so we need to decompose the interval $[0,1]$ into smaller intervals, such that on each subinterval the image of f misses at least one point in S^1.

We set $V_{\pm1}:=S^1\setminus\{\pm1\}$. By the Lebesgue Lemma 11.9 there exists an $N\in\mathbb{N}$ such that for every $i\in\{0,\dots,N-1\}$ the image $f([\frac{i}{N},\frac{i+1}{N}])$ is contained in V_{-1} or V_{+1}. It remains to prove the following claim:

Claim. For each $i\in\{0,\dots,N\}$ there exists a lift $\widetilde{f}\colon[0,\frac{i}{N}]\to\mathbb{R}$ of f with $\widetilde{f}(0)=\widetilde{y}_0$.



Proof. For $i=0$ we just set $\widetilde{f}(0):=\widetilde{y}_0$. Now suppose that we have already defined $\widetilde{f}$ on some interval $[0,\frac{i}{N}]$ with $i<N$. Note that by construction there exists an $\epsilon\in\{\pm1\}$ with $f([\frac{i}{N},\frac{i+1}{N}])\in V_\epsilon=S^1\setminus\{\epsilon\}$. Thus it follows from the Lift-Extension Lemma 11.11[102] that we can extend the lift $\widetilde{f}$ to the interval $[0,\frac{i+1}{N}]$. ■

We can now formulate and prove the full version of the Circle-π_1-Theorem 11.13.

Theorem 11.13. (Circle-π_1-Theorem) Let $z_0=\exp(2\pi\,\mathrm{i}\cdot t_0)\in S^1$. Given any continuous map $\gamma\colon[0,1]\to S^1$ with $\gamma(0)=z_0$ we denote by $\widetilde{\gamma}\colon[0,1]\to\mathbb{R}$ the unique lift[103] with

[102] Here we use that the intersection of two adjacent intervals is connected and non-empty.

$\widetilde{\gamma}(0) = t_0 \in \mathbb{R}$. The maps

$$\Psi\colon \mathbb{Z} \to \pi_1(S^1, z_0),\quad n \mapsto \left[\begin{array}{ccc} [0,1] & \to & S^1 \\ t & \mapsto & z_0 \cdot \exp(2\pi \mathrm{i} \cdot n \cdot t) \end{array}\right] \quad \text{and} \quad \Theta\colon \pi_1(S^1, z_0) \to \mathbb{Z},\ [\gamma\colon [0,1] \to S^1] \mapsto \tfrac{1}{2\pi}\cdot(\widetilde{\gamma}(1) - \widetilde{\gamma}(0))$$

are well-defined, they are isomorphisms and they are inverses of one another.

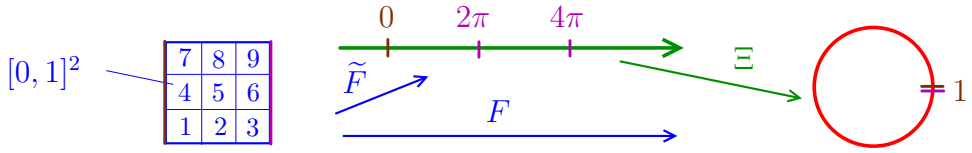
The proof of the Circle-π_1-Theorem 11.13 rests on the following lemma, which is also of independent interest.

Lemma 11.14. (Square Lifting Lemma) Let $F\colon [0,1] \times [0,1] \to S^1$ be a continuous map. Given any $c \in \mathbb{R}$ with $\Xi(c) = F(0,0)$ there exists a unique lift $\widetilde{F}\colon [0,1] \times [0,1] \to \mathbb{R}$ of F with $\widetilde{F}(0,0) = c$.

Proof of the Square Lifting Lemma 11.14. Let $F\colon [0,1]\times[0,1] \to S^1$ be a continuous map and let $c \in \mathbb{R}$ with $\Xi(c) = F(0,0)$. As in the proof of the Lift-from-S^1-to-$\mathbb{R}$ Lemma 11.12 we set $V_{\pm 1} := S^1 \setminus \{\pm 1\}$. By the Lebesgue Lemma 11.9 there exists an $N \in \mathbb{N}$ such that for any $i, j \in \{0, \dots, N-1\}$ there exists an $\epsilon \in \{\pm 1\}$ with

$$f\big(\big[\tfrac{i}{N}, \tfrac{i+1}{N}\big] \times \big[\tfrac{j}{N}, \tfrac{j+1}{N}\big]\big) \subset V_\epsilon.$$

The same way as we lifted the map $[0,1] \to S^1$ to $\mathbb{R}$ by lifting the map "intervall-by-intervall" starting from the left, we now lift the map $F\colon [0,1]^2 \to S^1$ to $\mathbb{R}$ by lifting the map "square-by-square" starting from the "bottom left".[104] We leave it to the reader to fill in the details.



The uniqueness statement of the lemma is just a special case of the Lift Uniqueness Lemma 11.10. ■

Proof of the Circle-π_1-Theorem 11.13. To simplify the notation we only consider the case that $z_0 = 1$ and $t_0 = 0$. The following claim shows that $\Theta\colon \pi_1(S^1, 1) \to \mathbb{Z}$ is well-defined.

Claim 1. Let $\alpha, \beta\colon [0,1] \to S^1$ be two loops in $(S^1, 1)$. If α and β are path-homotopic, then $\widetilde{\alpha}(1) = \widetilde{\beta}(1) \in \mathbb{R}$.

Proof. Let $F\colon [0,1] \times [0,1] \to S^1$ be a path-homotopy between α and β. By the Square Lifting Lemma 11.14 we know that F lifts to a continuous map $\widetilde{F}\colon [0,1] \times [0,1] \to \mathbb{R}$.

Next note that the fact that F is a path-homotopy means that $F(1,t) = 1$ for all $t \in [0,1]$. This in turn implies that $\widetilde{F}(1,t) \in 2\pi \cdot \mathbb{Z}$ for all $t \in [0,1]$. It follows from the Connected-to-Discrete Lemma 2.18 that the map $t \mapsto \widetilde{F}(1,t)$ is constant. Therefore we have shown that $\widetilde{\alpha}(1) = \widetilde{F}(1,0) = \widetilde{F}(1,1) = \widetilde{\beta}(1)$. ⊞

[104] The key observation is that if for any square that we add to the domain the intersection with the previous domain is always connected and non-empty.

Next note that it follows almost immediately from the definitions that $\Psi\colon \mathbb{Z} \to \pi_1(S^1, 1)$ is a homomorphism. Furthermore note that it follows easily from the definitions[105] that $\Theta \circ \Psi\colon \mathbb{Z} \to \mathbb{Z}$ is the identity. Thus it remains to prove the following claim:

Claim 2. The map $\Theta\colon \pi_1(S^1, 1) \to \mathbb{Z}$ is a bijection.

Proof. Since $\Theta \circ \Psi = \mathrm{id}_{\mathbb{Z}}$ we see that Θ is surjective. It remains to show that Θ is injective. So suppose that we are given $[\alpha], [\beta] \in \pi_1(S^1, 1)$ with $\Theta([\alpha]) = \Theta([\beta])$, i.e. with $\widetilde{\alpha}(1) = \widetilde{\beta}(1)$. Note that the maps $\widetilde{\alpha}, \widetilde{\beta}\colon [0,1] \to \mathbb{R}$ have the same starting point and the same endpoint in $\mathbb{R}$. *Since* $\mathbb{R}$ *is convex* we see, as in the Convex Homotopy Lemma 8.1, that

$$\begin{array}{rcl} F\colon [0,1] \times [0,1] & \to & \mathbb{R} \\ (t,s) & \mapsto & \widetilde{\alpha}(t) \cdot (1-s) + \widetilde{\beta}(t) \cdot s \end{array}$$

is a path-homotopy in $\mathbb{R}$ between $\widetilde{\alpha}$ and $\widetilde{\beta}$. But this implies that

$$\begin{array}{rcl} \Xi \circ F\colon [0,1] \times [0,1] & \to & S^1 \\ (t,s) & \mapsto & \Xi(\widetilde{\alpha}(t) \cdot (1-s) + \widetilde{\beta}(t) \cdot s) \end{array}$$

is a path-homotopy in S^1 between $\alpha = \Xi \circ \widetilde{\alpha}$ and $\beta = \Xi \circ \widetilde{\beta}$. Thus $[\alpha] = [\beta] \in \pi_1(S^1, 1)$. ∎

11.5. **The fundamental group of spheres.** In the Circle-π_1-Theorem 11.13 we just calculated the fundamental group of S^1. In this section we will deal with the higher-dimensional spheres.

Proposition 11.15. (Sphere-π_1-Proposition) For any $n \geq 2$ the sphere S^n is simply connected, i.e. we have $\pi_1(S^n) = 0$.

In the proof of the Sphere-π_1-Proposition 11.15 we will need the following lemma:

Lemma 11.16. (Sphere-Minus-Point Lemma) For every $P \in S^n$ the topological space $S^n \setminus \{P\}$ is simply connected.

Proof of Lemma 11.16. Recall that on page 184 we saw that $\mathbb{R}^n$ is simply connected. Thus it suffices to show that $S^n \setminus \{P\}$ is homeomorphic to $\mathbb{R}^n$.

Recall that in the Stereographic Projection Lemma 2.11 we showed that there exists a homeomorphism $f\colon S^n \to \mathbb{R}^n \cup \{\infty\}$ with $f^{-1}(\infty) = N := (0, \ldots, 0, 1)$. Thus f restricts to a homeomorphism $f\colon S^n \setminus \{N\} \to \mathbb{R}^n$. Now let P be any other point on S^n. There exists an orthogonal matrix A with $A \cdot P = N$. This shows that there exists a homeomorphism $g\colon S^n \to S^n$ with $g(P) = N$. Then $f \circ g\colon S^n \to \mathbb{R}^n \cup \{\infty\}$ restricts to a homeomorphism $S^n \setminus \{P\} \to \mathbb{R}^n$. ∎

Now we can turn to the proof of the Sphere-π_1-Proposition 11.15.

> Here is a very short "proof": Let $x_0 \in S^n$ and let $\gamma\colon [0,1] \to S^n$ be a loop in x_0. We have to show that γ is null-homotopic. We pick a point P that does not lie in the image of γ. By the Sphere-Minus-Point Lemma 11.16 the topological space $S^n \setminus \{P\}$ is simply connected, hence γ is null-homotopic in $S^n \setminus \{P\}$, in particular it is null-homotopic in S^n.
>
> This sounds convincing, except for one crucial gap: we did not justify why there

[105] To be slightly more precise, this follows from the fact that the lift of the loop $t \mapsto \exp(2\pi\,\mathrm{i} \cdot n \cdot t)$ is given by $t \mapsto 2\pi \cdot n \cdot t$.

exists a point P that does not lie in the image of γ. In fact such a point does not need to exist, since there are surjective continuous maps from an interval to any S^n.

Proof of the Sphere-π_1-Proposition 11.15. Let $n \geq 2$. We write

$$N \coloneqq (0,\dots,0,1) \quad \text{“North Pole”} \quad \text{and} \quad U \coloneqq S^n \setminus \{N\},$$

and also

$$S \coloneqq (0,\dots,0,-1) \quad \text{“South Pole”} \quad \text{and} \quad V \coloneqq S^n \setminus \{S\}.$$

We fix a base point $x_0 \in U \cap V$. In the following we will only use the following three facts about U and V:

(I) U and V form an open cover of S^n.
(II) By the Sphere-Minus-Point Lemma 11.16 we know that U and V are simply connected.
(III) Our hypothesis $n \geq 2$ implies that the intersection $U \cap V = S^n \setminus \{N, S\}$ is path-connected.[106]

It follows from (II) together with the Path-Homotopy–Equivalence Relation Lemma 11.2 that it suffices to show that any loop in (S^n, x_0) is path-homotopic to a loop in U. Thus let $\gamma\colon [0,1] \to S^n$ be a loop in (S^n, x_0).

Claim 1. There exists a subdivision

$$0 = t_0 < t_1 < \dots < t_{k-1} < t_k = 1,$$

such that the following hold:
(1) for each $i \in \{0,\dots,k-1\}$ we have $\gamma([t_i, t_{i+1}]) \subset U$ or $\gamma([t_i, t_{i+1}]) \subset V$, and
(2) for each $i \in \{1,\dots,k-1\}$ we have $\gamma(t_i) \in U \cap V$.

Proof. By (I) we can apply the Lebesgue Lemma 11.9 to $S^n = U \cup V$ and to $\gamma\colon [0,1] \to S^n$. Thus there exists a $k \in \mathbb{N}$ such that for any $i \in \{0,\dots,k-1\}$ the image $\gamma([\frac{i}{k}, \frac{i+1}{k}])$ lies in U or in V. In particular the subdivision $t_i = \frac{i}{k}$, $i = 0,\dots,k$ satisfies condition (1).

Now suppose that there exists an $i \in \{1,\dots,k-1\}$ with $\gamma(t_i) \notin U$. Then neither $\gamma([t_{i-1}, t_i])$ nor $\gamma([t_i, t_{i+1}])$ are contained in U. So by (1) both have to be contained in V, in particular we have $\gamma([t_{i-1}, t_{i+1}]) \subset V$. But then we can remove t_i from the subdivision. The same way we deal with the case that $\gamma(t_i) \notin V$. Iterating this process we can assume that for each every i we have $\gamma(t_i) \in U \cap V$. ⊞

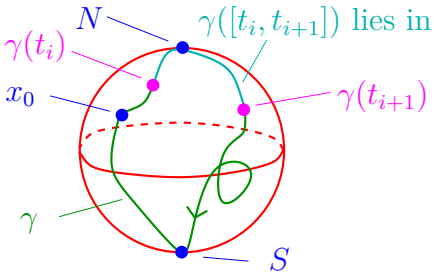


illustration for Claim 1

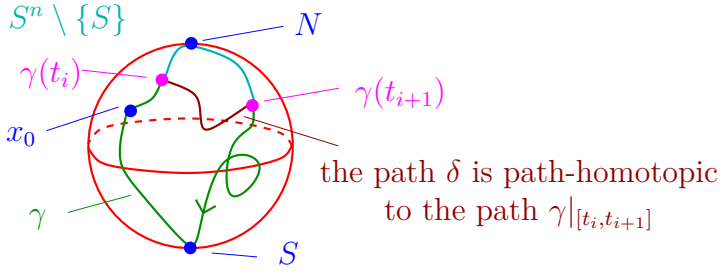


illustration for the proof of Claim 2

As mentioned before, it now, suffices to prove the following claim:

[106] This can be seen as follows: $S^n \setminus \{N, S\}$ is homeomorphic to $S^{n-1} \times (-1, 1)$, which in turn is clearly path-connected.

Claim 2. The loop $\gamma\colon [0,1] \to S^n$ in (S^n, x_0) is path-homotopic to a path in U.

Proof. We pick a subdivision of $[0,1]$ as in Claim 1. Suppose that there exists an $i \in \{0,\dots,k-1\}$ such that $\gamma([t_i, t_{i+1}])$ is not contained in U. By the choice of our subdivision we know that this implies that $\gamma([t_i, t_{i+1}])$ lies entirely in V. Since $U \cap V$ is path-connected and since $\gamma(t_i)$ and $\gamma(t_{i+1})$ both lie in $U \cap V$ we can pick a path $\delta\colon [t_i, t_{i+1}] \to U \cap V$ that connects $\gamma(t_i)$ and $\gamma(t_{i+1})$. Since the paths $\gamma|_{[t_i,t_{i+1}]}$ and δ both lie in the simply connected set V they are path-homotopic by the Paths-in-Simply Connected Space Lemma 11.8. In the loop γ we now replace $\gamma|_{[t_i,t_{i+1}]}$ by δ. It is straightforward to see that this way we obtain a new loop that is path-homotopic to the original loop and that stays within $U \cap V \subset U$ on the interval $[t_i, t_{i+1}]$. Iterating this procedure gives us the desired path-homotopy to a loop that is contained in U. ∎

11.6. **Alternative description of fundamental groups.** In this short section we will present a slightly different way to think about fundamental groups, namely we will see that we can think of fundamental groups also as defined via continuous maps $S^1 \to X$. We start out with the following identification.

Identification. Note that the map

$$\begin{array}{rcl} [0,1]/0\sim 1 & \to & S^1 \\ [t] & \mapsto & \exp(2\pi \mathrm{i}\cdot t) \end{array}$$

is a homeomorphism. We use it to make the identification $S^1 =_{\mathrm{i}} [0,1]/0 \sim 1$.

This identification allows us to introduce the following convention:

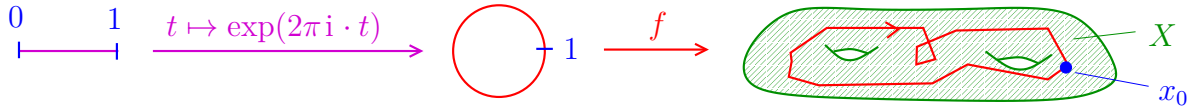
Convention. Let X be a topological space and let $x_0 \in X$.

(1) Using the above identification, using the Topological-Quotient Proposition 6.1 (1b) and using the Homotopy Quotient Theorem 8.5 we see that we can make the natural identification

$$\pi_1(X, x_0) \ =_{\mathrm{i}} \ \{\text{all continuous maps } (S^1, 1) \to (X, x_0)\}/\sim$$

where $f \sim g$ if there exists a homotopy $F\colon S^1 \times [0,1] \to X$ such that for all $t \in [0,1]$ we have $F_t(1) = x_0$.

(2) We refer to continuous maps $(S^1, 1) \to (X, x_0)$ also as loops in (X, x_0).



CHAPTER 12

Applications of fundamental groups

In this short chapter we present a few applications of the results on fundamental groups that we proved in the previous chapter:

(1) We will prove the Fundamental Theorem of Algebra.
(2) We prove the Borsuk-Ulam Theorem for S^2.

We will not use any of the results of this chapter later on. The reader who is eager to move forward with the development of the theory can safely skip this chapter.

12.1. The Fundamental Theorem of Algebra.

As a first application of our methods we give a proof of the Fundamental Theorem of Algebra that does not rely on complex analysis:

Theorem 12.1. (Fundamental Theorem of Algebra) Every nonconstant polynomial with coefficients in $\mathbb{C}$ has a zero in $\mathbb{C}$.

Proof. We start out with some notation: Let $q(z)$ be a polynomial with complex coefficients. If $q(z) \neq 0$ for all $z \in \mathbb{C}$ with $|z| = r$, then we can consider the path

$$\begin{array}{rccl} \gamma_r^{q(z)}\colon [0,1] & \to & S^1 \\ s & \mapsto & \dfrac{q(r \cdot \exp(2\pi \mathrm{i} \cdot s))}{\left|q(r \cdot \exp(2\pi \mathrm{i} \cdot s))\right|} \cdot \dfrac{|q(r)|}{q(r)}. \end{array}$$

Note that $\gamma_r^{q(z)}(0) = \gamma_r^{q(z)}(1) = 1$, i.e. $\gamma_r^{q(z)}$ is a loop in $(S^1, 1)$.

Now we start out with the actual proof of the theorem. Thus let

$$p(z) \;=\; a_n \cdot z^n + a_{n-1} \cdot z^{n-1} + \cdots + a_1 \cdot z + a_0$$

be a polynomial with $a_n \neq 0$. We assume that $p(z)$ has no zeros in $\mathbb{C}$. We need to show that $n = 0$. For $t \in [0,1]$ we also consider the polynomial

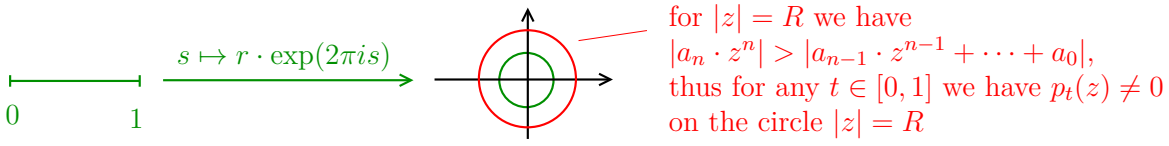
$$p_t(z) \;:=\; a_n \cdot z^n + t \cdot (a_{n-1} \cdot z^{n-1} + \cdots + a_1 \cdot z + a_0).$$

Note that $p_1(z) = p(z)$ and $p_0(z) = a_n \cdot z^n$. Since $a_n \neq 0$ there exists an $R > 0$ such that[107]

$$|a_n \cdot z^n| \;>\; |\underbrace{a_{n-1} \cdot z^{n-1} + \cdots + a_1 \cdot z + a_0}_{=0 \text{ if } n=0}|$$

for any $z \in \mathbb{C}$ with $|z| = R$. This inequality implies that for any $t \in [0,1]$ and any $z \in \mathbb{C}$ with $|z| = R$ we have $p_t(z) \neq 0$, i.e. $p_t(z)$ has no zeros on the circle $|z| = R$.

[107] In fact we can write it down explicitly: it follows easily from the triangle inequality that any $R \in \mathbb{R}$ with $R \cdot |a_n| > |a_{n-1}| + \cdots + |a_0|$ will work.



We obtain the following sequence of path-homotopic loops:

$$\text{constant loop at } 1 \overset{\text{follows from the definition}}{=} \gamma_0^{p(z)} \underset{\uparrow}{\simeq_{\mathrm{p}}} \gamma_R^{p(z)} = \gamma_R^{p_1(z)} \underset{\uparrow}{\simeq_{\mathrm{p}}} \gamma_R^{p_0(z)} \overset{\text{follows from } p_0(z) = a_n z^n}{=} \text{the loop } \begin{array}{l}[0,1] \to S^1 \\ t \mapsto \exp(2\pi i n t).\end{array}$$

maps $\gamma_r^{p(z)}$ with $r \in [0,R]$ are defined, since the polynomials $p(z)$ has no zeros, they form a path-homotopy from $\gamma_0^{p(z)}$ to $\gamma_R^{p(z)}$

maps $\gamma_R^{p_t(z)}$ with $t \in [0,1]$ are defined, since polynomials $p_t(z)$ have no zeros on $|z|=R$, and and they form a path-homotopy from $\gamma_R^{p_1(z)}$ to $\gamma_R^{p_0(z)}$

This shows that the constant loop is path-homotopic to the loop on the right hand side. By the Circle-π_1-Theorem 11.13 this is only possible if $n = 0$. ■

12.2. **The 2-dimensional Borsuk-Ulam Theorem.** The following theorem goes back to the Polish mathematicians Karol Borsuk (1905-1982) and Stanislaw Ulam (1909-1984). It was first proved in 1932.

Theorem 12.2. (2D-Borsuk-Ulam Theorem) For every continuous map $f\colon S^2 \to \mathbb{R}^2$ there exists a pair of antipodal points x and $-x$ in S^2 with $f(x) = f(-x)$.

Example.

(1) It follows from the 2D-Borsuk-Ulam Theorem 12.2 that there exists a pair of antipodal points on earth with the same temperature and the same barometric pressure.
(2) The 2D-Borsuk-Ulam Theorem 12.2 implies in particular that there is no injective continuous map from $S^2 \to \mathbb{R}^2$. □

Our proof of the 2D-Borsuk-Ulam Theorem 12.2 makes use of the concept of the degree of a continuous map $S^1 \to S^1$. In the following we develop this concept, which is of independent interest. At the end of this section we return to the proof of the 2D-Borsuk-Ulam Theorem 12.2.

Definition.

(1) As on page 187 we work with the map

$$\begin{array}{rcl}\Xi\colon \mathbb{R} & \to & S^1 \\ t & \mapsto & \exp(\mathrm{i} \cdot t).\end{array}$$

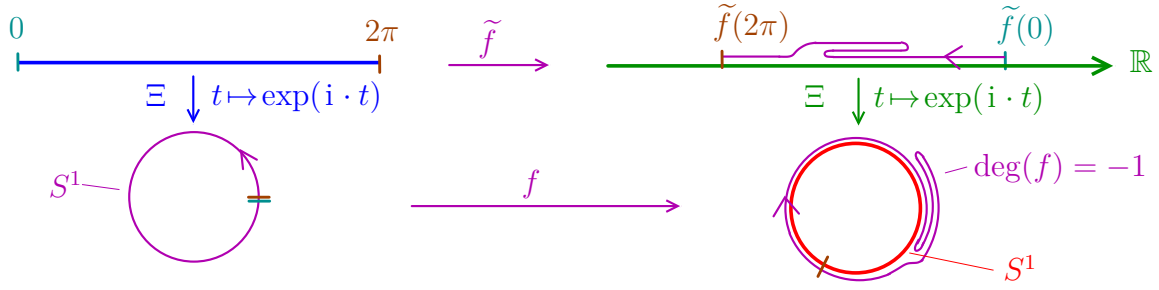
(2) Let $f\colon S^1 \to S^1$ be a continuous map. By the Lift-from-S^1-to-$\mathbb{R}$ Lemma 11.12 we can lift the path[108] $f \circ \Xi\colon [0,2\pi] \to S^1$ to a path $\widetilde{f}\colon [0,2\pi] \to \mathbb{R}$. We refer to

$$\deg(f) \coloneqq \frac{1}{2\pi} \cdot \big(\widetilde{f}(2\pi) - \widetilde{f}(0)\big) \in \mathbb{Z}$$

as the degree of f.

Lemma 12.3. (Degree–Well-Defined Lemma) The above degree of a continuous map $S^1 \to S^1$ is well-defined.

[108] The fact that $1 \neq 2\pi$ is one of the major curses of mathematics.



Proof. Let $f\colon S^1 \to S^1$ be a continuous map. Let $\widetilde{f}_1, \widetilde{f}_2\colon [0, 2\pi] \to \mathbb{R}$ be two lifts of the map $f \circ \Xi\colon [0, 2\pi] \to S^1$. Since $\exp(\mathrm{i}\widetilde{f}_1(0)) = \Xi(\widetilde{f}_1(0)) = f(\Xi(0)) = \Xi(\widetilde{f}_2(0)) = \exp(\mathrm{i}\widetilde{f}_2(0))$ we see that $\widetilde{f}_2(0) - \widetilde{f}_1(0) \in 2\pi \cdot \mathbb{Z}$. It follows that the map $t \mapsto \widetilde{f}_1(t) + \widetilde{f}_2(0) - \widetilde{f}_1(0)$ is also a lift of $f \circ \Xi$. Since this lift has the same starting point as $\widetilde{f}_2$ it follows from the uniqueness statement of the Lift-from-S^1-to-$\mathbb{R}$ Lemma 11.12 that for all $t \in [0, 2\pi]$ we have $\widetilde{f}_2(t) = \widetilde{f}_1(t) + \widetilde{f}_2(0) - \widetilde{f}_1(0)$. Using this observation we now see that

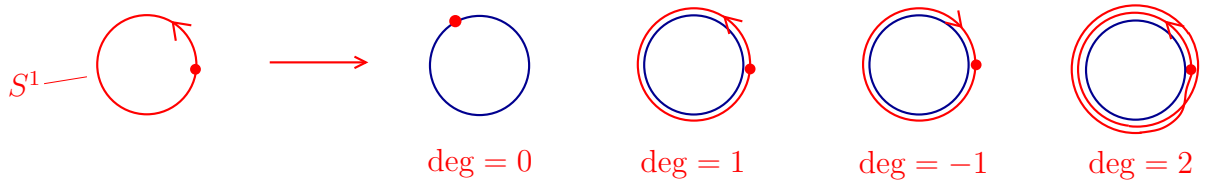
$$\frac{1}{2\pi} \cdot \big(\widetilde{f}_2(2\pi) - \widetilde{f}_2(0)\big) \underset{\text{by the above discussion}}{\overset{\uparrow}{=}} \frac{1}{2\pi} \cdot \big(\underbrace{\widetilde{f}_1(2\pi) + \widetilde{f}_2(0) - \widetilde{f}_1(0)}_{=\widetilde{f}_2(2\pi)} - \widetilde{f}_2(0)\big) \underset{\text{simplifying}}{\overset{\uparrow}{=}} \frac{1}{2\pi} \cdot \big(\widetilde{f}_1(2\pi) - \widetilde{f}_1(0)\big).$$

This shows that the definition of the degree of f is independent of the choice of the lift of $f \circ \Xi$. ∎

Now that we know that the notion of a degree is well-defined it is worth looking at a few examples.

Example.

(1) Let $c\colon S^1 \to S^1$ be a constant map. Since any lift of the constant map $c \circ \Xi\colon [0, 2\pi] \to S^1$ is also constant we deduce that the degree of the constant map $c\colon S^1 \to S^1$ is zero.
(2) Let $k \in \mathbb{Z}$. We consider the self-map of S^1 that is given by $z \mapsto z^k$. A lift of the map $(z \mapsto z^k) \circ \Xi$ is given by the map $t \mapsto 2\pi \cdot t \cdot k$. We deduce that $\deg(z \mapsto z^k) = k$.



The following lemma provides us with one of the key properties of degrees.

Lemma 12.4. (Degree Homotopy Lemma) If $f, g\colon S^1 \to S^1$ are two homotopic maps, then $\deg(f) = \deg(g)$.

Proof. Let $F\colon S^1 \times [0, 1] \to S^1$ be a homotopy between two maps $f, g\colon S^1 \to S^1$. By the Square Lifting Lemma 11.14 we know that the map $F \circ (\Xi \times \mathrm{id}_{[0,1]})\colon [0, 2\pi] \times [0, 1] \to S^1$ lifts to a continuous map $\widetilde{F}\colon [0, 2\pi] \times [0, 1] \to \mathbb{R}$. Next note that for any $t \in [0, 2\pi]$ we have $\Xi(\widetilde{F}_t(0)) = F_t(\Xi(0)) = F_t(1) = F_t(\Xi(2\pi)) = \Xi(\widetilde{F}_t(2\pi))$. This implies that for each $t \in [0, 2\pi]$ we have $\widetilde{F}_t(2\pi) - \widetilde{F}_t(0) \in 2\pi \cdot \mathbb{Z}$. It follows from the Connected-to-Discrete

Lemma 2.18 that the map $t \mapsto \widetilde{F}_t(2\pi) - \widetilde{F}_t(0)$ is constant. We now see that

$$\deg(f) \underset{\text{by definition, since } F_0 = f}{=} \tfrac{1}{2\pi} \cdot \big(\widetilde{F}_0(2\pi) - \widetilde{F}_0(0)\big) \underset{\text{by the above discussion}}{=} \tfrac{1}{2\pi} \cdot \big(\widetilde{F}_1(2\pi) - \widetilde{F}_1(0)\big) \underset{\text{by definition, since } F_1 = g}{=} \deg(g).$$

■

In the proof of the 2D-Borsuk-Ulam Theorem 12.2 we will need the following notion of an odd self-map of S^1:

Definition. A map $f\colon S^1 \to S^1$ is called odd, if $f(-z) = -f(z)$ for all $z \in S^1$.

Example. Let $k \in \mathbb{Z}$. The map $S^1 \to S^1$ that is given by $z \mapsto z^k$ is odd if and only if k is odd. □

The statement of the following proposition is mercifully easy to internalize.

Proposition 12.5. (Odd Degree Proposition) Let $n \in \mathbb{N}$ and let $f\colon S^1 \to S^1$ be a continuous map. If f is odd, then $\deg(f)$ is odd.

Proof. Let $f\colon S^1 \to S^1$ be a continuous odd map. We pick a lift $\widetilde{f}\colon [0, 2\pi] \to \mathbb{R}$ of the map $f \circ \Xi\colon [0, 2\pi] \to S^1$. Recall that this means that $\Xi(\widetilde{f}(t)) = f(\Xi(t))$ for any $t \in [0, 2\pi]$.

Note that for any $t \in [0, \pi]$ we have

$$\begin{array}{ccccccccc} & & \text{by def. of } \Xi & & \text{since } \widetilde{f} \text{ is a lift} & & \text{by def. of } \Xi & & \text{since } f \text{ is odd} \\ \Xi(\widetilde{f}(t+\pi)+\pi) & \overset{\downarrow}{=} & -\Xi(\widetilde{f}(t+\pi)) & \overset{\downarrow}{=} & -f(\Xi(t+\pi)) & \overset{\downarrow}{=} & -f(-\Xi(t)) & \overset{\downarrow}{=} & -(-f(\Xi(t))) \\ & & & & & & & = & f(\Xi(t)). \end{array}$$

We have thus shown that $t \mapsto \widetilde{f}(t+\pi) + \pi$ is also a lift of $f|_{[0,\pi]}$. It follows from the uniqueness statement of the Lift-from-S^1-to-$\mathbb{R}$ Lemma 11.12 that there exists an $n \in \mathbb{Z}$ such that for all $t \in [0, \pi]$ we have $\widetilde{f}(t+\pi) + \pi = \widetilde{f}(t) + n \cdot 2\pi$. We now see that

$$\deg(f) = \tfrac{1}{2\pi} \cdot \big(\widetilde{f}(2\pi) - \widetilde{f}(0)\big) = \tfrac{1}{2\pi} \cdot \Big(\underbrace{\widetilde{f}(2\pi) - \widetilde{f}(\pi)}_{= n\cdot 2\pi - \pi, \text{ by the above applied to } t = \pi} + \underbrace{\widetilde{f}(\pi) - \widetilde{f}(0)}_{= n\cdot 2\pi - \pi, \text{ by the above applied to } t = 0} \Big) = 2n - 1.$$

We have thus shown that $\deg(f)$ is odd. ■

Using the above results on degrees we can now give a concise proof of the 2D-Borsuk-Ulam Theorem 12.2.

Proof of the 2D-Borsuk-Ulam Theorem 12.2. Let $f\colon S^2 \to \mathbb{R}^2$ be a continuous map. Recall that we need to show that there exists an $x \in S^2$ with $f(-x) = f(x)$.

Suppose that there exists no pair of antipodal points x and $-x$ in S^2 with $f(x) = f(-x)$. Put differently, suppose that for every $x \in S^2$ we have $f(x) \neq f(-x)$. We consider the composition of the following two maps

$$\begin{array}{rccccc} \Phi\colon S^1 \times [0,1] & \to & S^2_{\geq 0} & \to & S^1. \\ (y,t) & \mapsto & (y \cdot \sqrt{1-t^2}, t) & & \\ & & x & \mapsto & \underbrace{\frac{f(x)-f(-x)}{\|f(x)-f(-x)\|}}_{\text{defined since } f(x) \neq f(-x)} \end{array}$$

A quick look at the definitions shows that Φ_0 is odd and it shows that Φ_1 is a constant map. Thus we see that

by the Degree Homotopy Lemma 12.4, since Φ_0 and Φ_1 are homotopic

$$\text{odd number} \underset{\uparrow}{=} \deg(\Phi_0) \overset{\downarrow}{=} \deg(\Phi_1) \underset{\uparrow}{=} \deg(\text{constant map}) \underset{\uparrow}{=} 0.$$

since Φ_0 is odd we obtain from the Odd Degree Proposition 12.5 that $\deg(\Phi_0)$ is odd

by definition of Φ_1

on page 196 we saw that the degree of any constant map is zero

We have thus obtained a contradiction. ■

CHAPTER 13

Functoriality of fundamental groups

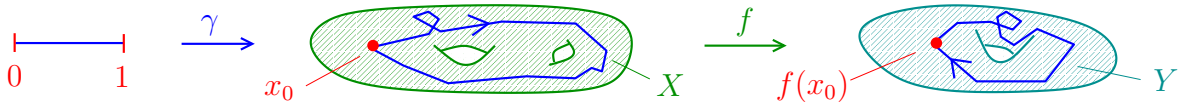
To unlock the full power of fundamental groups we need to view the fundamental group as a covariant functor.

13.1. The fundamental group as functor.

Definition. Let $f\colon X \to Y$ be a continuous map between topological spaces and let $x_0 \in X$ be a base point. If $\gamma\colon [0,1] \to X$ is a loop in the point x_0, then $f \circ \gamma\colon [0,1] \to Y$ is a loop in the point $f(x_0)$. It follows from the Homotopy Composition Lemma 8.4 that

$$\begin{array}{rcl} f_*\colon \pi_1(X,x_0) & \to & \pi_1(Y,f(x_0)) \\ [\gamma] & \mapsto & [f\circ\gamma] \end{array}$$

is a well-defined map, i.e. $f_*([\gamma])$ is independent of the choice of the representative of the path-homotopy class. We call f_* the induced map.



Example.

(1) If $f\colon X \to Y$ is a constant map, then the induced map $f_*\colon \pi_1(X,x_0) \to \pi_1(Y,f(x_0))$ is the trivial group homomorphisms.

(2) Let $n \in \mathbb{Z}$. We consider the map $f\colon S^1 \to S^1$ that is given by $z \mapsto z^n$. We want to understand the map $f_*\colon \pi_1(S^1,1) \to \pi_1(S^1,1)$. Here is one possible answer: It follows immediately from the definitions that the following diagram commutes:

$$\begin{array}{ccc} \pi_1(S^1,1) & \xrightarrow{f_*} & \pi_1(S^1,1) \\ {\scriptstyle k\mapsto[t\mapsto\exp(2\pi\mathrm{i}kt)]}\uparrow\cong & & \cong\uparrow{\scriptstyle k\mapsto[t\mapsto\exp(2\pi\mathrm{i}kt)]} \\ \mathbb{Z} & \xrightarrow{g\mapsto n\cdot g} & \mathbb{Z}. \end{array}$$

(Here the vertical isomorphisms are given by the Circle-π_1-Theorem 11.13.) □

The induced maps look they should define a covariant functor. We just need to figure out what the categories are supposed to be. This leads us to the following definition:

Definition. A pointed topological space is a pair (X,x_0), where X is a topological space and x_0 is a point in X. We refer to the category $\mathcal{PT}op$ with

$$\begin{array}{rcl} \mathrm{Ob}(\mathcal{PT}op) & := & \text{all pointed topological spaces,} \\ \mathrm{Mor}((X,x_0),(Y,y_0)) & := & \text{all continuous maps } f \text{ from } X \text{ to } Y \text{ with } f(x_0)=y_0 \end{array}$$

with the usual composition of maps as the category of pointed topological spaces.

Proposition 13.1. (π_1-Functor Proposition)

(1) Let *PTop* be the category of pointed topological spaces and let *Group* be the category *Group* of groups. The map

$$\begin{array}{rcl} \mathrm{Ob}(\mathcal{PT}op) & \to & \mathrm{Ob}(\mathcal{G}roup) \\ (X, x_0) & \mapsto & \pi_1(X, x_0) \end{array}$$

and the maps

$$\begin{array}{rcl} \mathrm{Mor}((X,x_0),(Y,y_0)) & \to & \mathrm{Mor}(\pi_1(X,x_0),\pi_1(Y,y_0)) \\ (f\colon (X,x_0)\to(Y,y_0)) & \mapsto & \left(\begin{array}{rcl} f_*\colon \pi_1(X,x_0) & \to & \pi_1(Y,y_0) \\ [\gamma] & \mapsto & [f\circ\gamma] \end{array} \right) \end{array}$$

form a covariant functor from the category *PTop* of pointed topological spaces to the category *Group* of groups.

(2) If $f\colon X \to Y$ is a homeomorphism between two topological spaces, then for any $x_0 \in X$ the induced map $f_*\colon \pi_1(X,x_0) \to \pi_1(Y, f(x_0))$ is a group isomorphism.

Example. On page 71 we saw that the upper and lower hemispheres $S^n_{\geq 0}$ and $S^n_{\leq 0}$ are homeomorphic to the closed ball $\overline{B}^n$. It follows from the π_1-Functor Proposition 13.1 and the discussion on page 184 that the hemispheres are simply connected. □

Proof. Statement (2) follows immediately from Statement (1). Thus it remains to prove Statement (1).

Let $f\colon (X,x_0) \to (Y,y_0)$ be a continuous map between pointed topological spaces. It follows easily from the definitions that for any two loops γ and δ in (X,x_0) we have $(f\circ\gamma) * (f\circ\delta) = f\circ(\gamma * \delta)$. But this means that $f_*\colon \pi_1(X,x_0) \to \pi_1(Y,y_0)$ is a group homomorphism. Furthermore one can easily verify that

$$\begin{array}{rcll} (\mathrm{id}_X)_* & = & \mathrm{id}_{\pi_1(X,x_0)}, & \text{for all pointed pairs } (X,x_0), \\ (g\circ f)_* & = & g_*\circ f_*, & \text{for all continuous maps } f\colon (X,x_0)\to(Y,y_0), g\colon (Y,y_0)\to(Z,z_0). \end{array}$$

But this means precisely that the given maps define a covariant functor from the category *PTop* of pointed topological spaces to the category *Group* of groups. ■

In many applications we will want to know whether a given map between topological spaces induces an isomorphism of fundamental groups. The following proposition, which can be viewed as an addendum to the Change-of-Base Point Proposition 11.6, says that in most settings the property of being an isomorphism is independent of the choice of base point.

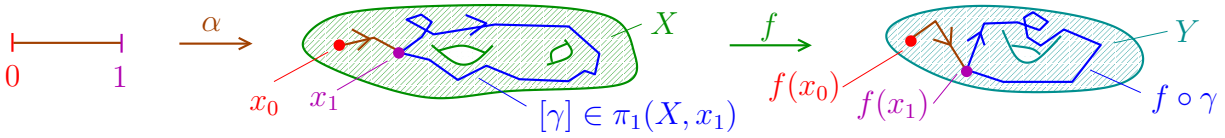
Proposition 13.2. (Change-of-Base Point Proposition) Let $f\colon X \to Y$ be a continuous map between topological spaces and let $x_0 \in X$.

(1) If $\alpha\colon [0,1] \to X$ is a path from x_0 to some point x_1, then the following diagram commutes:

$$\begin{array}{ccc} \pi_1(X,x_1) & \xrightarrow{f_*} & \pi_1(Y,f(x_1)) \\ \cong\downarrow{\scriptstyle \alpha_*} & & \cong\downarrow{\scriptstyle (f\circ\alpha)_*} \\ \pi_1(X,x_0) & \xrightarrow{f_*} & \pi_1(Y,f(x_0)). \end{array}$$

(The vertical maps are the isomorphisms that we introduced in the Change-of-Base Point Proposition 11.6 (1).)

(2) If $f_*\colon \pi_1(X,x_0) \to \pi_1(Y,f(x_0))$ is an isomorphism (epimorphism, monomorphism) for some base point $x_0 \in X$, then for any other base point x_1 in the same path component of X the map $f_*\colon \pi_1(X,x_1) \to \pi_1(Y,f(x_1))$ is also an isomorphism (epimorphism, monomorphism).



Proof.

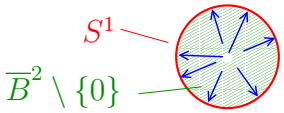
(1) This statement follows immediately from the definitions.
(2) This statement follows from (1) and the fact that by the Change-of-Base Point Proposition 11.6 (1) the vertical maps are isomorphisms. ■

13.2. **Retractions and fundamental groups.** In this section we will see that the functoriality of the fundamental groups can be helpful for showing the non-existence of retractions. Let us recall the following definition from page 164:

Definition. Let X be a topological space and let $A \subset X$ be a subspace. We say that A is a retract of X if there exists a retraction $r\colon X \to A$, i.e. a continuous map with $r(a) = a$ for all $a \in A$.

Examples.

(1) The circle $S^1 \times \{1\}$ is a retract of the torus $S^1 \times S^1$, in fact a retraction is given by $r(z,w) = (z,1)$.
(2) On page 165 we saw that S^{n-1} is a retract of $\overline{B}^n \setminus \{0\}$.



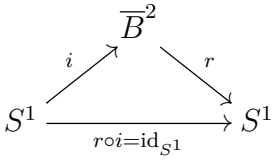
$r(x) := \frac{x}{\|x\|}$ defines a retraction from $\overline{B}^2 \setminus \{0\}$ to S^1

(3) The set $S^0 = \{-1,1\}$ is not a retract of $\overline{B}^1 = [-1,1]$. Indeed, since $[-1,1]$ is connected and since S^0 is discrete it follows from the Connected-to-Discrete Lemma 2.18 that any continuous map from $[-1,1]$ to S^0 has to be constant. Thus there does not exist a continuous map $f\colon [-1,1] \to S^0$ with $f(-1) = -1$ and $f(1) = 1$.

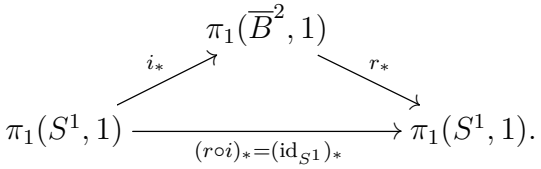
Now we can push last example up by one dimension.

Proposition 13.3. (Circle–Non Retract Proposition) The circle S^1 is not a retract of $\overline{B}^2$.

Proof. Let $i\colon S^1 \to \overline{B}^2$ be the inclusion map. We now suppose there exists a retraction $r\colon \overline{B}^2 \to S^1$. By definition of a retraction this means that $r \circ i = \mathrm{id}_{S^1}$. We obtain the following commutative diagram of continuous maps between topological spaces



which induces



It follows from the functoriality of fundamental groups that $r_* \circ i_* = (r \circ i)_*$, i.e. the diagram on the right-hand side commutes. By the functoriality we also know that $(\mathrm{id}_{S^1})_*$ is the identity of $\pi_1(S^1, 1)$.

We saw on page 184 that $\pi_1(\overline{B}^2, 1)$ is the trivial group whereas we saw in the Circle-π_1-Theorem 11.13 that $\pi_1(S^1, 1)$ is not the trivial group. The lower horizontal map in the second diagram is thus an isomorphism of a non-trivial group, but the upper map factors through the trivial group, i.e. the composition of i_* and r_* cannot be an isomorphism of a non-trivial group. Thus we have obtained a contradiction. ■

13.3. **The fundamental groups of homotopy equivalent topological spaces.** Now we will see that homotopy equivalent topological spaces have isomorphic fundamental groups. On numerous occasions this fact will greatly simplify the calculation of fundamental groups.

Proposition 13.4. (Homotopy-π_1-Proposition)

(1) Let $f, g\colon X \to Y$ be two continuous maps between topological spaces and let $x_0 \in X$. If f and g are homotopic, then there exists a path $\alpha\colon [0,1] \to Y$ from $f(x_0)$ to $g(x_0)$ such that the following diagram commutes:

$$\begin{array}{ccc} & \overset{g_*}{\nearrow} & \pi_1(Y, g(x_0)) \\ \pi_1(X, x_0) & & \Big\downarrow{\scriptstyle \alpha_*\ [\gamma] \mapsto [\alpha * \gamma * \overline{\alpha}]} \\ & \underset{f_*}{\searrow} & \pi_1(Y, f(x_0)). \end{array}$$

(Here the vertical map is the isomorphism given by the Change-of-Base Point Proposition 11.6 (1).) In particular

$$\begin{array}{c} f_*\colon \pi_1(X, x_0) \to \pi_1(Y, f(x_0)) \\ \text{is an isomorphism} \end{array} \iff \begin{array}{c} g_*\colon \pi_1(X, x_0) \to \pi_1(Y, g(x_0)) \\ \text{is an isomorphism.} \end{array}$$

The analogous statement holds if we replace "isomorphism" by "epimorphism" or "monomorphism".

(2) (a) Let $f\colon X \to Y$ be a continuous map between topological spaces and let $x_0 \in X$. If f is a homotopy equivalence, then the induced map

$$f_*\colon \pi_1(X, x_0) \ \to \ \pi_1(Y, f(x_0))$$

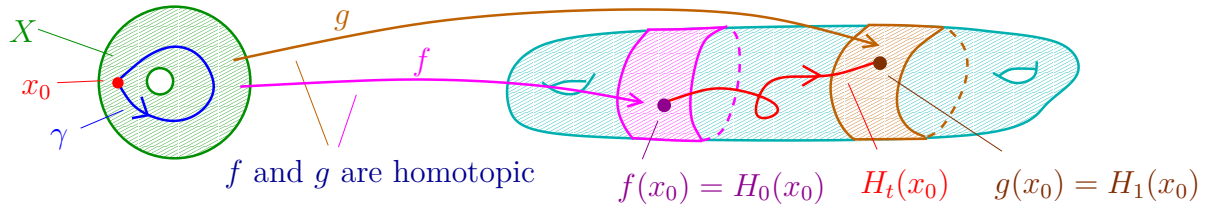
is an isomorphism.

(b) Let X be a topological space and let $A \subset X$ be a subspace. We denote by $i\colon A \to X$ the inclusion. If A is a deformation retract of X, then for any $a_0 \in A$ the induced map

$$i_*\colon \pi_1(A, a_0) \ \to \ \pi_1(X, a_0)$$

is an isomorphism.

(3) If X is a contractible topological space, then for any $x_0 \in X$ we have $\pi_1(X, x_0) = 0$.



Example. For any $n \geq 2$ and any $Q \in \mathbb{R}^n$ we have

on page 165 we saw that S^{n-1} is a deformation retract of $\mathbb{R}^n \setminus \{0\}$, thus we get an isomorphism from the Homotopy-π_1-Proposition 13.4 (2b)

$$\pi_1(\mathbb{R}^n \setminus \{Q\}) \underset{\uparrow}{\cong} \pi_1(\mathbb{R}^n \setminus \{0\}) \underset{\downarrow}{\cong} \pi_1(S^{n-1}) \underset{\uparrow}{\cong} \begin{cases} \mathbb{Z}, & \text{if } n = 2, \\ 0, & \text{if } n \geq 3. \end{cases}$$

follows from the π_1-Functor Proposition 13.1 (2) since $\mathbb{R}^n \setminus \{Q\}$ is homeomorphic to $\mathbb{R}^n \setminus \{0\}$

by the Circle-π_1-Theorem 11.13 and the Sphere-π_1-Proposition 11.15 □

Proof.

(1) Let $f, g\colon X \to Y$ be two continuous maps between topological spaces, let $x_0 \in X$ and let $H\colon X \times [0,1] \to Y$ be a homotopy between f and g. Let

$$\begin{array}{rcl} \alpha\colon [0,1] & \to & Y \\ t & \mapsto & H(x_0, t) \end{array}$$

be the path with which the homotopy H connects the two points $H(x_0, 0) = f(x_0)$ and $H(x_0, 1) = g(x_0)$.

Claim. The maps

$$\begin{array}{rcl} f_*\colon \pi_1(X, x_0) & \to & \pi_1(Y, f(x_0)) \\ [\gamma] & \mapsto & [f \circ \gamma] \end{array} \quad \text{and} \quad \begin{array}{rcl} \pi_1(X, x_0) & \to & \pi_1(Y, f(x_0)) \\ [\gamma] & \mapsto & [\alpha * (g \circ \gamma) * \overline{\alpha}] \end{array}$$

are identical.

Proof. Let $\gamma\colon [0,1] \to X$ be a loop in (X, x_0). We need to show that the loops $f \circ \gamma$ and $\alpha * (g \circ \gamma) * \overline{\alpha}$ are path-homotopic. We consider the map

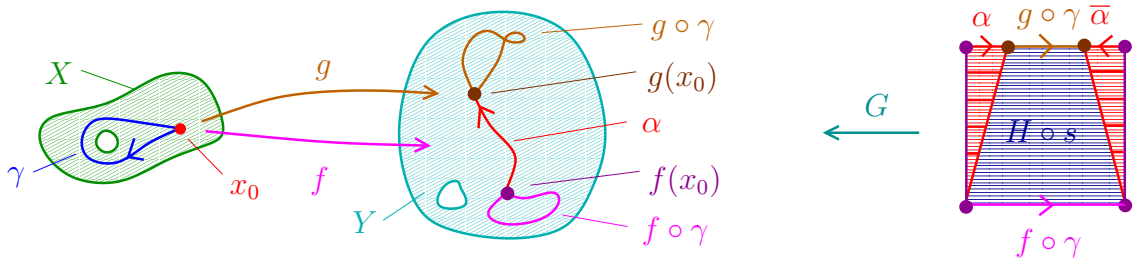
$$\begin{array}{rcl} G\colon [0,1] \times [0,1] & \to & Y \\ (r,t) & \mapsto & \begin{cases} \alpha(4r), & \text{if } r \in \left[0, \frac{1}{4}t\right], \\ H\left(\gamma\left(\frac{r - \frac{1}{4}t}{1 - \frac{1}{2}t}\right), t\right), & \text{if } r \in \left[\frac{1}{4}t, 1 - \frac{1}{4}t\right], \\ \overline{\alpha}(4r - 3), & \text{if } r \in \left[1 - \frac{1}{4}t, 1\right]. \end{cases} \end{array}$$

Using the Pasting Proposition 2.4 (2) it is straightforward to verify that G is indeed a continuous map. Furthermore it follows easily from the definition that the map G satisfies $G_0(t) = G_1(t) = f(x_0)$ for all $t \in [0,1]$ and that it satisfies

$$G(r,0) = (f \circ \gamma)(r) \quad \text{and} \quad G(r,1) = (\alpha * (g \circ \gamma) * \overline{\alpha})(r) \quad \text{for all } r \in [0,1].$$

This observation shows that G is in fact a path-homotopy between the loops $f \circ \gamma$ and $\alpha * (g \circ \gamma) * \overline{\alpha}$[109] in $(Y, f(x_0))$. ⊞

[109] To be precise, G is a homotopy between $f \circ \gamma$ and a suitable parametrization of $\alpha * (g \circ s) * \overline{\alpha}$.



As in the statement of the proposition we consider the following diagram:

$$\begin{array}{ccc} & \overset{g_*}{\nearrow} & \pi_1(Y, g(x_0)) \\ \pi_1(X, x_0) & & \Big\downarrow{\scriptstyle \alpha_*}\ [\gamma] \mapsto [\alpha * \gamma * \overline{\alpha}] \\ & \underset{f_*}{\searrow} & \pi_1(Y, f(x_0)). \end{array}$$

We make the following observations:

(i) It follows from the claim that the diagram commutes.

(ii) By the Change-of-Base Point Proposition 11.6 (1) we know that the vertical map is an isomorphism.

It follows from (i) and (ii) that f_* is an isomorphism if and only if g_* is an isomorphism. Evidently the analogous statement holds if we replace "isomorphism" by "epimorphism" or "monomorphism".

(2) (a) Let $f\colon X \to Y$ be a continuous map between topological spaces which is a homotopy equivalence. Let $x_0 \in X$ be a point. We pick a homotopy inverse $g\colon Y \to X$ of f. By definition of a homotopy inverse this means that $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$.

Next note that it follows from (1) and the functoriality of fundamental groups that the induced maps

$$\begin{array}{rcl} \alpha := g_* \circ f_* = (g \circ f)_*\colon \pi_1(X, x_0) & \to & \pi_1(X, g(f(x_0)) \\ \beta := f_* \circ g_* = (f \circ g)_*\colon \pi_1(Y, f(x_0)) & \to & \pi_1(Y, f(g(f(x_0)))) \end{array}$$

are isomorphisms. This shows that f_* has the left inverse $\alpha^{-1} \circ g_*$ and it has the right inverse $g_* \circ \beta^{-1}$. It follows from the Inverse Uniqueness Lemma 3.2 (1) that f_* is an isomorphism. Similarly we see that g_* is an isomorphism.

(b) This statement follows immediately from (a) and the Deformation Retract–Homotopy Equivalence Lemma 9.3.

(3) This statement is just a frequently used special case of Statement (2). ■

In an exercise sheet we already saw that the topological spaces $\mathbb{R}$ and $\mathbb{R}^2$ are not homeomorphic. Now we can prove a similar statement in one dimension higher.

Lemma 13.5. ($\mathbb{R}^2$-$\mathbb{R}^3$-Non Homeomorphism–Lemma) The topological spaces $\mathbb{R}^2$ and $\mathbb{R}^3$ are not homeomorphic.

Proof. Let us suppose that there exists a homeomorphism $f\colon \mathbb{R}^2 \to \mathbb{R}^3$. Let $P \in \mathbb{R}^2$ be a point. We put $Q := f(P)$. Note that f restricts to a homeomorphism from $\mathbb{R}^2 \setminus \{P\}$ to

$\mathbb{R}^3 \setminus \{Q\}$. But then we obtain that

$$\mathbb{Z} \underset{\text{page 203}}{\cong} \pi_1(\mathbb{R}^2 \setminus \{P\}) \underset{\text{since } f \text{ is a homeomorphism}}{\cong} \pi_1(\mathbb{R}^3 \setminus \{Q\}) \underset{\text{page 203}}{=} 0.$$

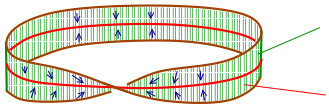
Thus we obtained a contradiction. ■

We conclude this section with one more calculation of a fundamental group.

Corollary 13.6. (Möbius Band-π_1-Corollary) We have

$$\pi_1(\text{Möbius band }) \cong \mathbb{Z}.$$

Proof. On page 167 we saw that the Möbius band admits a deformation retraction onto the "central circle" C.



Thus we see that $\pi_1(\text{Möbius band }) \underset{\text{Homotopy-}\pi_1\text{-Proposition 13.4}}{\cong} \pi_1(C) \cong \pi_1(S^1) \underset{\text{Circle-}\pi_1\text{-Theorem 11.13}}{\cong} \mathbb{Z}.$ ■

13.4. **The fundamental group of the product of topological spaces.** In Chapters 5 and 6 we got to know several ways to construct new (pointed) topological spaces out of given (pointed) topological spaces. It is natural to ask how these constructions affect the fundamental groups. In this section we consider the product of two pointed topological spaces.

Before we state the relevant proposition it is worth recalling the definition of the direct product of two groups.

Definition. Let G and H two groups. We can form the cartesian product

$$G \times H \coloneqq \{(g,h) \,|\, g \in G \text{ and } h \in H\}.$$

This is again a group with the group structure given by

$$(g_1,h_1) \cdot (g_2,h_2) \coloneqq (g_1 \cdot g_2, h_1 \cdot h_2)$$

for $g_1, g_2 \in G$ and $h_1, h_2 \in H$. We call the group $G \times H$ the direct product of the groups G and H.

Now we can formulate the promised proposition.

Proposition 13.7. (Product-π_1-Proposition) Let $n \in \mathbb{N}$.

(1) Let $(X_i, x_i)_{i \in I}$ be a family of pointed topological spaces. For each $j \in I$ we denote by $p_j \colon \prod_{i \in I} X_i \to X_j$ the natural projection. The induced natural homomorphism[110]

$$\begin{aligned} \Phi \colon \pi_1\Big(\prod_{i\in I} X_i, (x_i)_{i\in I}\Big) &\to \prod_{i\in I} \pi_1(X_i, x_i) \\ [f] &\mapsto (p_{i*}([f]))_{i\in I} \end{aligned}$$

is an isomorphism.

(2) If we are given a finite family $(X_1, x_1), \ldots, (X_k, x_k)$ of pointed topological spaces, then the inverse to the map from (1) is given by the natural homomorphism

$$\begin{array}{rcl} \Psi\colon \pi_1(X_1, x_1) \times \cdots \times \pi_1(X_k, x_k) & \to & \pi_1(X_1 \times \cdots \times X_k, (x_1, \ldots, x_k)) \\ (g_1, \ldots, g_k) & \mapsto & \iota_{1*}(g_1) \cdot \ldots \cdot \iota_{k*}(g_k) \end{array}$$

where $\iota_i\colon X_i \to X_1 \times \cdots \times X_k$ is given by $y \mapsto (x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_k)$.

Example.

(1) For $n \in \mathbb{N}$ we obtain from the Product-π_1-Proposition 13.7 together with the Circle-π_1-Theorem 11.13 that $\pi_1((S^1)^n) \cong \mathbb{Z}^n$.

(2) From the Product-π_1-Proposition 13.7 together with the Circle-π_1-Theorem 11.13 we also obtain that $\pi_1((S^1)^{\mathbb{N}}) \cong \mathbb{Z}^{\mathbb{N}}$. This result is interesting, since by Tychonoff's Theorem 5.7 we know that $(S^1)^{\mathbb{N}}$ is compact. Since $\mathbb{Z}^{\mathbb{N}}$ is uncountable we have shown that there exists a compact topological space with uncountable fundamental group. □

Remark.

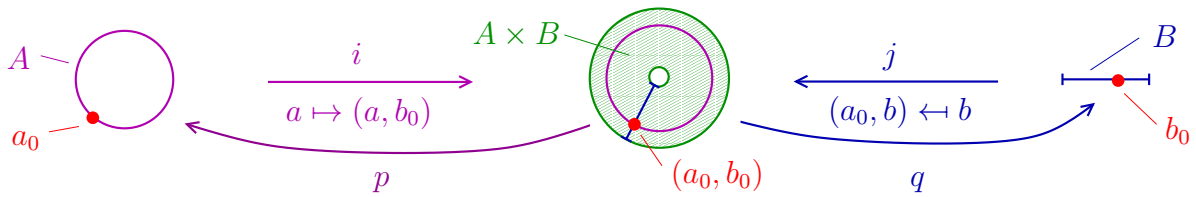
(1) The Product-π_1-Proposition 13.7 can be summarized as saying that the π_1-functor turns a product in the category of pointed topological spaces into a product in the category *Group* of groups. Put more succinctly: the fundamental group preserves products.

(2) In most cases we apply the Product-π_1-Proposition 13.7 to the case of two pointed topological spaces (A, a_0) and (B, b_0). Let us spell out this special case. We consider the projections

$$\begin{array}{rcl} p\colon A \times B & \to & A \\ (a, b) & \mapsto & a \end{array} \quad \text{and} \quad \begin{array}{rcl} q\colon A \times B & \to & B \\ (a, b) & \mapsto & b \end{array}$$

and the inclusions

$$\begin{array}{rcl} i\colon A & \to & A \times B \\ a & \mapsto & (a, b_0) \end{array} \quad \text{and} \quad \begin{array}{rcl} j\colon B & \to & A \times B \\ b & \mapsto & (a_0, b). \end{array}$$



The maps

$$\begin{array}{rcl} \Phi\colon \pi_1(A \times B, (a_0, b_0)) & \to & \pi_1(A, a_0) \times \pi_1(B, b_0) \\ \gamma & \mapsto & (p_*(\gamma), q_*(\gamma)) \\ i_*(\alpha) \cdot j_*(\beta) & \leftarrow\!\shortmid & (\alpha, \beta) \end{array}$$

are both natural[111] isomorphisms. □

[110] The projections p_i induce homomorphisms on fundamental groups which in turn, by the Product-of-Sets-and-Groups Lemma 4.2, induce the given homomorphism.

[111] Here "natural" means the following. If $f\colon (A, a_0) \to (\widetilde{A}, \widetilde{a}_0)$ and $g\colon (B, b_0) \to (\widetilde{B}, \widetilde{b}_0)$ are continuous maps of pointed topological spaces, then it follows easily from the definitions that the following diagrams commute:

$$\begin{array}{ccc} \pi_1(A\times B, (a_0, b_0)) & \xrightarrow{\Phi} & \pi_1(A, a_0)\times\pi_1(B, b_0) \\ \downarrow{\scriptstyle (f\times g)_*} & & \downarrow{\scriptstyle f_*\times g_*} \\ \pi_1(\widetilde{A}\times\widetilde{B}, (\widetilde{a}_0, \widetilde{b}_0) & \xrightarrow{\Phi} & \pi_1(\widetilde{A}, \widetilde{a}_0)\times\pi_1(\widetilde{B}, \widetilde{b}_0)) \end{array} \quad \text{and} \quad \begin{array}{ccc} \pi_1(A, a_0)\times\pi_1(B, b_0) & \xrightarrow{\Psi} & \pi_1(A\times B, (a_0, b_0)) \\ \downarrow{\scriptstyle f_*\times g_*} & & \downarrow{\scriptstyle (f\times g)_*} \\ \pi_1(\widetilde{A}, \widetilde{a}_0)\times\pi_1(\widetilde{B}, \widetilde{b}_0) & \xrightarrow{\Psi} & \pi_1(\widetilde{A}\times\widetilde{B}, (\widetilde{a}_0, \widetilde{b}_0)). \end{array}$$

Proof.

(1) First we introduce some general notation:

- Given two pairs of topological spaces (W, A) and (X, B) we denote by $C((W, A), (X, B))$ the set of all continuous maps $f\colon W \to X$ with $f(A) \subset B$.
- We say that two continuous maps $f, g \in C((W, A), (X, B))$ are *homotopic* if there exists a homotopy $H\colon W \times [0, 1] \to X$ with $H_0 = f$ and $H_1 = g$ such that for all $t \in [0, 1]$ we have $H_t(A) \subset B$.
- We denote by $[(W, A), (X, B)]$ the set of all homotopy classes of continuous maps $(W, A) \to (X, B)$.
- By definition we have $\pi_1(Z, z_0) = [([0, 1], \{0, 1\}), (Z, \{z_0\})]$ for any pointed topological space (Z, z_0).

Let $(X_i, x_i)_{i\in I}$ be a family of pointed topological spaces. We set $x := (x_i)_{i\in I}$. It follows from the Topological-Product Proposition 5.1 (1) that the map

$$\begin{array}{rcl} C\Big(([0, 1], \{0, 1\}), \Big(\prod\limits_{i\in I} X_i, x\Big)\Big) & \to & \prod\limits_{i\in I} C(([0, 1], \{0, 1\}), (X_i, x_i)) \\ f & \mapsto & (p_i \circ f)_{i\in I} \end{array}$$

is a bijection. The analogous statement holds if we replace the pair $([0, 1], \{0, 1\})$ by the pair $([0, 1] \times [0, 1], \{0, 1\} \times [0, 1])$. This shows that the map

$$\begin{array}{rcl} \pi_1\Big(\prod\limits_{i\in I} X_i, x\Big) = \Big[([0, 1], \{0, 1\}), \Big(\prod\limits_{i\in I} X_i, x\Big)\Big] & \to & \prod\limits_{i\in I} [([0, 1], \{0, 1\}), (X_i, x_i)] = \prod\limits_{i\in I} \pi_1(X_i, x_i) \\ [f] & \mapsto & \underbrace{[(p_i \circ f)_{i\in I}]}_{=p_{i*}(f)} \end{array}$$

is a bijection. But this is precisely the homomorphism that we are considering.

(2) To simplify the notation we just work with two pointed topological spaces (X, x_0) and (Y, y_0) and we denote the two inclusions by $i\colon X \to X \times Y$ and $j\colon Y \to X \times Y$. Since by (1) we already know that Φ is an isomorphism it remains to prove the following claim:

Claim. We have $\Phi \circ \Psi = \mathrm{id}$ as self-maps of $\pi_1(X, x_0) \times \pi_1(Y, y_0)$.

Proof. Given $(\alpha, \beta) \in \pi_1(X, x_0) \times \pi_1(Y, y_0)$ we obtain the following equalities in the group $\pi_1(X, x_0) \times \pi_1(Y, y_0)$:

$$\begin{array}{rcccccl} & & \text{definition of } \Psi & & \text{definition of } \Phi & & \text{since } p_* \text{ and } q_* \text{ are homomorphisms} \\ (\Phi \circ \Psi)(\alpha, \beta) & \overset{\downarrow}{=} & \Phi(i_*(\alpha) \cdot j_*(\beta)) & \overset{\downarrow}{=} & (p_*(i_*(\alpha) \cdot j_*(\beta)), q_*(i_*(\alpha) \cdot j_*(\beta))) & \overset{\downarrow}{=} & \\ & = & (\underbrace{p_*(i_*(\alpha))}_{=(p\circ i)_*=\mathrm{id}_*} \cdot \underbrace{p_*(j_*(\beta))}_{=(p\circ j)_*=0}, \underbrace{q_*(i_*(\alpha))}_{=(q\circ i)_*=0} \cdot \underbrace{q_*(j_*(\beta))}_{=(q\circ j)_*=\mathrm{id}_*}) & = & (\alpha, \beta). & & \end{array}$$

■

We record one interesting application of the Product-π_1-Proposition 13.7 as a corollary:

Corollary 13.8. (Torus-π_1-Corollary) Let $n \in \mathbb{N}$.

(1) We consider the n-dimensional torus $(S^1)^n = S^1 \times \cdots \times S^1$. Given $j \in \{1, \dots, n\}$ we consider the loop

$$\begin{array}{rcl} v_j\colon [0, 1] & \to & S^1 \times \cdots \times S^1 \\ t & \mapsto & (1, \dots, 1, \underbrace{\exp(2\pi\,\mathrm{i} \cdot t)}_{\uparrow}, 1, \dots, 1). \end{array}$$

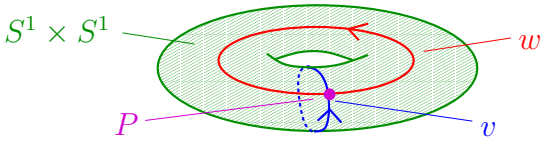
j-th coordinate

The map

$$\begin{array}{rcl} \mathbb{Z}^n & \mapsto & \pi_1(S^1 \times \cdots \times S^1, (1, \ldots, 1)) \\ (r_1, \ldots, r_n) & \mapsto & [v_1]^{r_1} \cdot \ldots \cdot [v_n]^{r_n} \end{array}$$

is an isomorphism.

(2) For $n \geq 2$ the n-dimensional torus is not homeomorphic to the sphere S^n.



the map $\begin{array}{rcl} \mathbb{Z}^2 & \to & \pi_1(S^1 \times S^1, P) \\ (r, s) & \mapsto & [v]^r \cdot [w]^s \end{array}$

is an isomorphism

Proof.

(1) The statement follows immediately from the Product-π_1-Proposition 13.7 and the explicit isomorphism $\Psi\colon \mathbb{Z} \to \pi_1(S^1, 1)$ given in the Circle-π_1-Theorem 11.13.

(2) This statement follows from the π_1-Functor Proposition 13.1 (2), the above calculation of $\pi_1((S^1)^n)$ and the fact, shown in the Sphere-π_1-Proposition 11.15 that for $n \geq 2$ we have $\pi_1(S^n) = 0$. ■

CHAPTER 14

Coverings and fundamental groups

Arguably our greatest success so far in the theory of fundamental groups was the calculation of $\pi_1(S^1)$ in the Circle-π_1-Theorem 11.13. The calculation relied on a detailed study of the map $\Xi\colon \mathbb{R} \to S^1$ that is given by $\Xi(t) = \exp(\mathrm{i}t)$. In particular this setup had two interesting features:

(1) Given any $z \in S^1$ we could find an open neighborhood U (called $V_\pm$) such that the preimage of the open neighborhood consists of disjoint open subsets $\{U_i\}_{i\in\mathbb{Z}}$ of $\mathbb{R}$ such that for each $i \in \mathbb{Z}$ the map Ξ restricts to a homeomorphism $U_i \to U$.
(2) $\mathbb{R}$ is simply connected.

The idea now is to generalize the above approach of calculating $\pi_1(S^1, 1)$ to more general settings. The key to doing so is the concept of a "covering" $p\colon X \to B$ between topological spaces. In this chapter we will introduce coverings, we will discuss many examples and we will relate coverings to fundamental groups.

In the following chapter we will use the results of this chapter to calculate in particular the fundamental groups of the real projective spaces and the Klein bottle.

14.1. **Group actions.** The main topic of this chapter are "coverings". As we will see shortly, a main source of "coverings" is the already familiar concept of group actions. In this section we consider group actions in their own right. In particular we will give a new description of the Klein bottle and we will introduce new, fascinating topological spaces, namely the "lens spaces". In the subsequent sections we turn to "coverings" and their interaction with groups actions and fundamental groups.

We recall two definitions about actions from page 48 and we introduce one new definition.

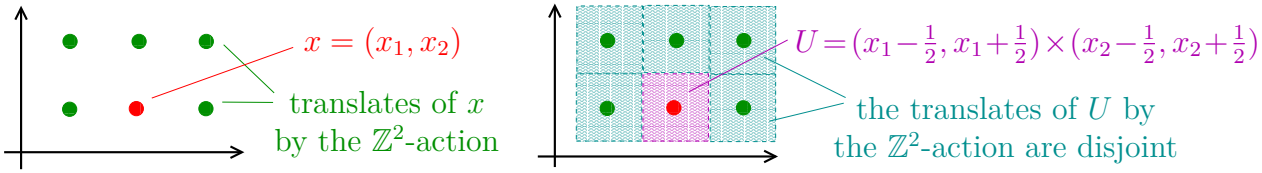
Definition. Let G be a group that acts continuously on a topological space X.

(1) We say that G acts freely if $g \cdot x = x$ for some $g \in G$ and some $x \in X$ implies that $g = e$.
(2) We G acts properly if for every two points[112] x and y in X there exist open neighborhoods U of x and V of y such that the set $\{g \in G \,|\, g \cdot U \cap V \neq \varnothing\}$ is finite.
(3) We say that G acts discretely, if for each $x \in X$ there exists an open neighborhood U of x so that[113 114 115]
$$U \cap g{\cdot}U \;=\; \varnothing \quad \text{for all } g \neq e.$$

Examples.

[114]Note that if G acts discretely, then it also acts freely.
[115]It follows easily from the properties of an action that if $U \cap g \cdot U = \varnothing$ for all $g \neq e$, then $g \cdot U \cap k \cdot U = \varnothing$ for all $k \neq g$. On several occasions we will make use of this observation.

(1) The group $G = \mathbb{Z}^n$ acts on $X = \mathbb{R}^n$ by addition. This action is discrete. Indeed, for any $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ the open subset $U := (x_1 - \frac{1}{2}, x_1 + \frac{1}{2}) \times \cdots \times (x_n - \frac{1}{2}, x_n + \frac{1}{2})$ has the desired property.



(2) The action
$$\begin{array}{rcl} \mathbb{Z} \times (\mathbb{R} \times [-1,1]) & \to & \mathbb{R} \times [-1,1] \\ (n, (x,y)) & \mapsto & (x+n, (-1)^n \cdot y) \end{array}$$
of the group $\mathbb{Z}$ on the topological space $\mathbb{R} \times [-1,1]$ is discrete. Indeed, for any (x,y) in X the open neighborhood $U := (x - \frac{1}{2}, x + \frac{1}{2}) \times [-1,1]$ has the desired property.

(3) The map
$$\begin{array}{rcl} \{\pm 1\} \times S^n & \to & S^n \\ (\epsilon, P) & \mapsto & \epsilon \cdot P \end{array}$$
defines an action of the group $\{\pm 1\}$ on S^n that is discrete. Indeed, for any $P \in S^n$ the open hemisphere $U := \{Q \in S^n \mid \underbrace{P \cdot Q}_{\text{scalar product}} > 0\}$ has the desired property.

(4) Evidently every discrete action is free. But the converse does not hold. For example the action of the group $(\mathbb{R}, +)$ on $\mathbb{R}$ by addition is clearly free but one can easily verify that it is not discrete. □

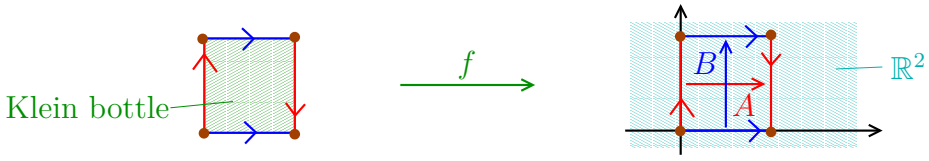
We consider our last two types of examples in greater detail.

Lemma 14.1. (Klein Bottle-as-Quotient Lemma) We consider the following two self-homeomorphisms of $\mathbb{R}^2$:
$$\begin{array}{rcl} A\colon \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (x,y) & \mapsto & (x+1, 1-y) \end{array} \quad \text{and} \quad \begin{array}{rcl} B\colon \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (x,y) & \mapsto & (x, y+1). \end{array}$$
We denote by G the subgroup of all homeomorphisms of $\mathbb{R}^2$ that is generated by A and B. This means that
$$G := \begin{array}{l} \text{all self-homeomorphisms of } \mathbb{R}^2 \text{ that can be written as} \\ \text{a finite composition of the maps } A, B, A^{-1} \text{ and } B^{-1}. \end{array}$$
The group G acts by definition on $\mathbb{R}^2$.

(1) The elements A and B of G do not commute, in particular the group G is non-abelian.

(2) The action of G on $\mathbb{R}^2$ is proper and discrete.

(3) We consider the Klein bottle $K = ([0,1] \times [0,1])/\sim$ as defined on page 131. The map
$$\begin{array}{rcl} f\colon K = ([0,1] \times [0,1])/\sim & \to & \mathbb{R}^2/G \\ [(x,y)] & \mapsto & [(x,y)] \end{array}$$
is a homeomorphism.



Proof. In the group G we use the product symbol "$\cdot$" to indicate the group structure instead of the "$\circ$" symbol.

(1) We calculate that

$$(A \cdot B)(0,0) \;=\; A(0,1) \;=\; (1,0) \;\neq\; (1,2) \;=\; B(1,1) \;=\; (B \cdot A)(0,0).$$

This shows that A and B do not commute in G. This shows that G is not abelian.

(2) We start out with the following claim:

Claim 1. Given $\epsilon, \eta \in \{-1, 1\}$ we have $A^\epsilon \cdot B^\eta = B^{-\eta} \cdot A^\epsilon$.

Proof. First let $\epsilon = 1$. We perform the following calculation:

$$\begin{aligned}(A\cdot B)(x,y) &= A(x,y+1) = (x+1,-y) &&= B^{-1}(x+1,1-y) &&= (B^{-1}\cdot A)(x,y),\\ (A\cdot B^{-1})(x,y) &= A(x,y-1) = (x+1,2-y) &&= B(x+1,1-y) &&= (B\cdot A)(x,y).\end{aligned}$$

We just showed that $A \cdot B^\eta = B^{-\eta} \cdot A$. By taking inverses on both sides we get the promised equalities $A^{-1} \cdot B^\eta = B^{-\eta} \cdot A^{-1}$ for $\epsilon = -1$. ⊞

Claim 2.

(a) Any $g \in G$ is of the form $A^k \cdot B^l$ for some $k, l \in \mathbb{Z}$.

(b) For $k, l \in \mathbb{Z}$ we have

$$(A^k \cdot B^l)(x,y) \;=\; \begin{cases} (x+k, y+l), & \text{if } k \text{ is even,} \\ (x+k, 1-y-l), & \text{if } k \text{ is odd.} \end{cases}$$

Proof.

(a) Let $g \in G$. By definition we can write g as a product of powers of A and B. Using the equality $A^\epsilon \cdot B^\eta = B^{-\eta} \cdot A^\epsilon$ from Claim 1 we can swap the order of powers of A with powers of B, at the price of changing the exponent of B. This means that we can bring all $A^{\pm 1}$ to the left and get the desired result.

(b) This statement follows from an elementary straightforward calculation. ⊞

Claim 3. The group G acts properly on $\mathbb{R}^2$.

Proof. Let $P, Q \in \mathbb{R}^2$. We set $U := B^2_{\frac{1}{2}}(P)$ and $V := B^2_{\frac{1}{2}}(Q)$. It follows easily from Claim 2 (a) and (b) that the set $\{g \in G \mid g \cdot U \cap V \neq \varnothing\}$ is finite. ⊞

Claim 4. The group G acts discretely on $\mathbb{R}^2$.

Proof. Let $(x, y) \in \mathbb{R}^2$. We set $U := (x - \frac{1}{2}, x + \frac{1}{2}) \times (y - \frac{1}{2}, y + \frac{1}{2})$. It follows easily from Claim 2 (a) and (b) that for any non-trivial $g \in G$ we have $U \cap g \cdot U = \varnothing$. ⊞

(3) Using Claim 2 from (2) one can easily show that the map $f\colon ([0,1] \times [0,1])/\sim \to \mathbb{R}^2/G$ is a bijection. It follows from the Twice Quotient Lemma 6.3 that f is continuous. By the Quotient Topology-Properties Proposition 6.2 (1) we know that $([0,1] \times [0,1])/\sim$ is compact. By (2) we know that the action of G is proper. Thus it follows from the Group Action-Hausdorff Quotient Proposition 6.7 that $\mathbb{R}^2/\sim$ is Hausdorff. In summary it follows from the Compact-Hausdorff Proposition 2.10 (3) that f is a homeomorphism. ■

We turn to our final example of a group acting on a topological space.

Lemma 14.2. (Lens Space-Discrete Action Lemma) We consider the 3-dimensional sphere $S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \,|\, |z_1|^2 + |z_2|^2 = 1\}$. Let $p \in \mathbb{Z} \setminus \{0\}$ and let $q \in \mathbb{Z}$ with $\gcd(p, q) = 1$. We set $\xi := \exp(2\pi \mathrm{i}/p)$.

(1) The map
$$\begin{array}{rcl} \mathbb{Z}_p \times S^3 & \to & S^3 \\ (k + p\mathbb{Z}, (z_1, z_2)) & \mapsto & (\xi^k \cdot z_1, \xi^{q \cdot k} \cdot z_2) \end{array}$$
defines a smooth, orientation-preserving and discrete action of $\mathbb{Z}_p$ on S^3.

(2) The quotient space $S^3/\mathbb{Z}_p$ is a closed 3-dimensional topological manifold that has a unique structure of an oriented smooth manifold such that the projection $S^3 \to S^3/\mathbb{Z}_p$ is a local diffeomorphism and orientation-preserving.

Proof.

(1) It is straightforward to verify that the map $\mathbb{Z}_p \times S^3 \to S^3$ does indeed define an action of the group $\mathbb{Z}_p$ on the topological space S^3. Evidently the action is smooth and a direct calculation shows that the action is orientation-preserving.

Claim. The action is discrete.

Proof. Let $(z_1, z_2) \in S^3$. First we assume that $z_1 \neq 0$. We set
$$U := \{(u_1, u_2) \in S^3 \,|\, u_1 = z_1 \cdot r \cdot \exp(2\pi \mathrm{i} t) \text{ with } r \in (0, \infty), t \in (-\tfrac{1}{2p}, \tfrac{1}{2p})\}.$$
One can easily verify that U is an open neighborhood of $(z_1, z_2) \in S^3$.[116] Next note that $\xi^k = 1$ if and only if $k \in p \cdot \mathbb{Z}$ and note that our hypothesis $\gcd(p, q) = 1$ implies that $\xi^{k \cdot q} = 1$ if and only if $k \in p \cdot \mathbb{Z}$. Using these two observations one can easily show that all non-trivial translates of U are disjoint.

If $z_2 \neq 0$, then we just swap the roles of the first and second coordinate in the definition of U. ⊞

(2) We refer to [**Fri24**] for details. ■

Definition. Let $p \in \mathbb{Z} \setminus \{0\}$ and let $q \in \mathbb{Z}$ with $\gcd(p, q) = 1$.[117] We refer to
$$L(p, q) := S^3/\mathbb{Z}_p,$$
with the action from the previous lemma, as a lens space.[118]

Example. We consider two special cases of lens spaces.

(1) For $p = 1$ we only have the action of the trivial group on S^3 and the quotient space $L(1, q)$ is therefore just the original sphere S^3.

(2) For $p = 2$ we have q is odd, so for $k = 0$ we have $\exp(2\pi \mathrm{i} k/2) = \exp(2\pi \mathrm{i} kq/2) = 1$ and for $k = 1$ we have $\exp(2\pi \mathrm{i} k/2) = \exp(2\pi \mathrm{i} kq/2) = -1$. Put differently, the action of $\mathbb{Z}_2$ on S^3 is the same as the action of $\mathbb{Z}_2 = \{\pm 1\}$ that we had used in the definition of the real projective space $S^2/\sim =_{\mathrm{i}} \mathbb{R}\mathrm{P}^3$. This shows that $L(2, q) = S^3/\sim =_{\mathrm{i}} \mathbb{R}\mathrm{P}^3$. □

[116] We set $\widetilde{U} = \{(u_1, u_2) \in \mathbb{C}^2 \,|\, u_1 = z_1 \cdot r \cdot \exp(2\pi \mathrm{i} t) \text{ with } r \in (0, \infty), t \in (-\tfrac{1}{2p}, \tfrac{1}{2p})\}$. Note that $\widetilde{U}$ is an open neighborhood of $(z_1, z_2) \in \mathbb{C}^2$. Thus $U = \widetilde{U} \cap S^3$ is an open neighborhood of $(z_1, z_2) \in S^3$.

[117] By definition we have $L(p, q + k \cdot p) = L(p, q)$. It follows that any lens space is of the form $L(p, q)$ with $q \in \{0, \ldots, p - 1\}$.

[118] The name "lens space" comes from an alternative description of these smooth manifolds.

For $p \in \mathbb{N}$ and $q \in \mathbb{N}$ with $\gcd(p,q) = 1$ we just introduced a new topological space, namely the lens space $L(p,q)$. It is natural to ask: For which parameters $(p,q), (\widetilde{p},\widetilde{q})$ are these topological spaces homeomorphic? We will now provide a partial answer:

Lemma 14.3. (Lens Spaces-Diffeomorphisms Lemma) Let $p \in \mathbb{N}$ and furthermore let $q, r \in \mathbb{Z}$ with $\gcd(p,q) = \gcd(p,r) = 1$. If $q \equiv \pm r^{\pm 1} \bmod p$,[119] then there exists a diffeomorphism $L(p,q) \to L(p,r)$.

Proof. Let $p \in \mathbb{N}$ and let $q, r \in \mathbb{Z}$ with $\gcd(p,q) = \gcd(p,r) = 1$. We pick $s \in \mathbb{Z}$ with $r \cdot s \equiv 1 \bmod p$. We suppose that $q \equiv \pm r \bmod p$ or that $q \equiv \pm s \bmod p$.

Claim. The map

$$\begin{array}{rcl} \Phi\colon S^3 & \to & S^3 \\ (z_1, z_2) & \mapsto & \begin{cases} (z_1, z_2), & \text{if } q \equiv r \bmod p, \\ (z_1, \overline{z_2}), & \text{if } q \equiv -r \bmod p, \\ (z_2, z_1), & \text{if } q \equiv s \bmod p, \\ (z_2, \overline{z_1}), & \text{if } q \equiv -s \bmod p \end{cases} \end{array}$$

descends to a well-defined map $\overline{\Phi}\colon L(p,q) \to L(p,r)$ and this map is a bijection.

Proof. We set $\xi := \exp(2\pi \mathrm{i}/p)$. If $q \equiv \pm r \bmod p$, then one can easily verify that Φ descends to a well-defined map $\overline{\Phi}\colon L(p,q) \to L(p,r)$. Now let us consider the case that $q \equiv s \bmod p$. Given $(z_1, z_2), (\widetilde{z_1}, \widetilde{z_2}) \in S^3$ we see that :

$$\begin{array}{rcl} [(z_1,z_2)] = [(\widetilde{z_1},\widetilde{z_2})] \in L(p,q) & \Longrightarrow & \text{there exists } k \in \mathbb{Z} \text{ with } (\xi^k \cdot z_1, \xi^{q\cdot k} \cdot z_2) = (\widetilde{z_1}, \widetilde{z_2}) \\ & \overset{rs \equiv 1 \bmod p}{\Longrightarrow} & \text{we have } (\xi^{k\cdot r\cdot s} \cdot z_1, \xi^{q\cdot k} \cdot z_2) = (\widetilde{z_1}, \widetilde{z_2}) \\ & \overset{s \equiv q \bmod p}{\Longrightarrow} & \text{we have } (\xi^{k\cdot r\cdot q} \cdot z_1, \xi^{q\cdot k} \cdot z_2) = (\widetilde{z_1}, \widetilde{z_2}) \\ & \Longrightarrow & \text{with } l = k \cdot q \text{ we have } (\xi^{r\cdot l} \cdot z_1, \xi^{l} \cdot z_2) = (\widetilde{z_1}, \widetilde{z_2}) \\ & \Longrightarrow & \text{with } l = k \cdot q \text{ we have } (\xi^{l} \cdot z_2, \xi^{r\cdot l} \cdot z_1) = (\widetilde{z_2}, \widetilde{z_1}) \\ & \Longrightarrow & [(z_2,z_1)] = [(\widetilde{z_2},\widetilde{z_1})] \in L(p,r) \\ & \Longrightarrow & [\Phi(z_1,z_2)] = [\Phi(\widetilde{z_1},\widetilde{z_2})] \in L(p,r). \end{array}$$

This argument shows that we get a well-defined map $\overline{\Phi}\colon L(p,q) \to L(p,r)$. Evidently the map $\overline{\Phi}\colon L(p,q) \to L(p,r)$ is surjective. Furthermore, running the above argument backwards one see that the map $\overline{\Phi}\colon L(p,q) \to L(p,r)$ is injective. In summary we see that the map $\overline{\Phi}\colon L(p,q) \to L(p,r)$ is a bijection. Finally we note that the case $r \equiv -s \bmod p$ is dealt with in a similar fashion. ⊞

One can now easily verify that the induced map $\overline{\Phi}\colon L(p,q) \to L(p,r)$ is a diffeomorphism. We refer to [**Fri24**] for full details. ■

14.2. **Coverings of topological spaces.** After this short excursion into group actions on topological spaces we turn to the main topic of this chapter, namely coverings:

Definition. Let $p\colon X \to B$ be a continuous map between topological spaces.

(1) An open subset $U \subset B$ is called uniformly covered, if $p^{-1}(U) \subset X$ is the union of disjoint open subsets $\{V_i\}_{i\in I}$ of X with the property such that for each $i \in I$ the map $p|_{V_i}\colon V_i \to U$ is a homeomorphism.[120]

[119] By our hypothesis we have $\gcd(p,r) = 1$. This implies that there exists a multiplicative inverse of r modulo p, i.e. there exists an s with $r \cdot s = 1 \bmod p$. We write $r^{-1} := s$.

(2) We say that the map $p\colon X \to B$ is a[121] covering, if p is surjective and if for every $b \in B$ there exists an open neighborhood U of b which is uniformly covered.
(3) Sometimes we call X a covering space of B.

Example.

(1) The map

$$\begin{array}{rccc} p\colon & \mathbb{R} & \to & S^1 \\ & \varphi & \mapsto & \exp(\mathrm{i}\varphi) \end{array}$$

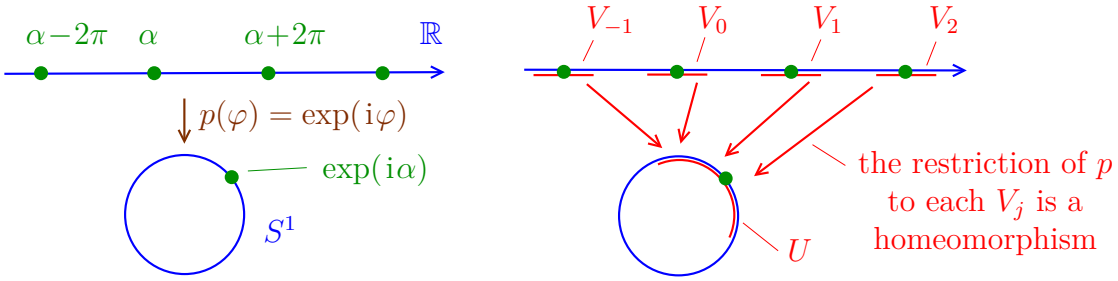
is a covering. Indeed, let $P = \exp(\mathrm{i}\alpha)$ be a point in S^1. We pick the open neighborhood

$$U \;:=\; \left\{ \exp(\mathrm{i}\varphi) \,\middle|\, \varphi \in \left(\alpha - \tfrac{\pi}{4}, \alpha + \tfrac{\pi}{4}\right)\right\}.$$

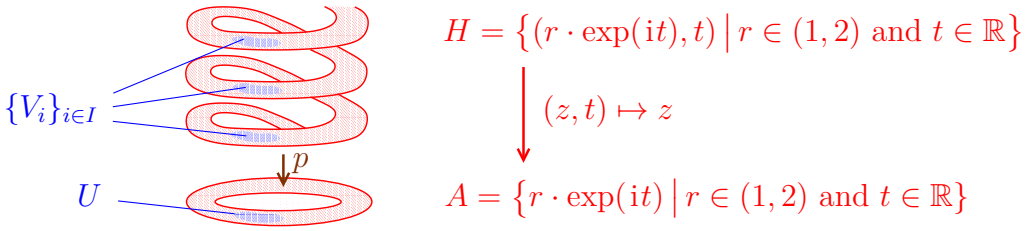
Then

$$p^{-1}(U) \;=\; \bigsqcup_{j\in\mathbb{Z}} \underbrace{\left(\alpha - \tfrac{\pi}{4} + 2\pi j, \alpha + \tfrac{\pi}{4} + 2\pi j\right)}_{=:V_j},$$

and for each $j \in \mathbb{Z}$ the restriction of p to $V_j \to U$ is a homeomorphism.



(2) In the figure below we illustrate another covering, which gives a good pictorial way to think about coverings.



Lemma 14.4. (Covering–Preimage Lemma) Let $p\colon X \to B$ be a covering of topological spaces. If B is a connected topological space, then for any two points P and Q in B we have[122]

$$\#p^{-1}(P) \;=\; \#p^{-1}(Q).$$

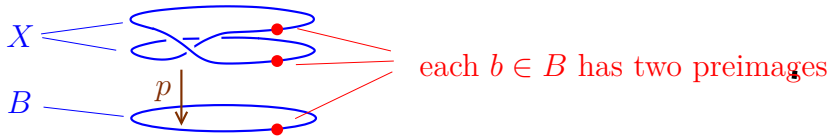
[120] Let $b \in B$. It follows immediately from the definition that each $x \in p^{-1}(b)$ is contained in precisely one V_i. This implies that we can arrange that the index set is $p^{-1}(b)$. We will often make use of this observation.

[121] On page 58 we introduced the notion of a "cover". As we mentioned on page 58, such a "cover" is sometimes also called a "covering". Now we introduce the notion of a "covering". Rather maddeningly, in the literature this notion often also gets called a "cover". Fortunately, from the context, it is usually clear what is meant by "cover" and "covering". These issues do not arise in other languages, e.g. in German and French what we call "cover" is referred to as "Überdeckung" and "recouvremement" and what we call "covering" is referred to as "Überlagerung" and "revêtement".

Proof. The proof of the lemma is left as an exercise to the reader. ■

Example.

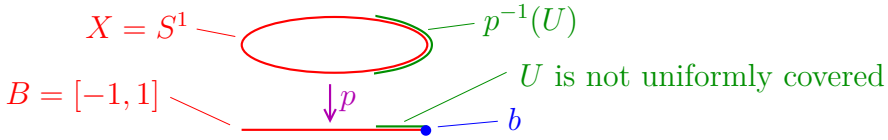
(1) We consider the map $p\colon X \to B$ that is shown in the figure.[123] This map is a covering and every $b \in B$ has precisely two preimages.



(2) The map

$$\begin{array}{rcl} p\colon S^1 & \to & [-1,1] \\ (x,y) & \mapsto & x. \end{array}$$

is *not* a covering since we see that the conclusion of the Covering–Preimage Lemma 14.4 does not hold.



In fact one can show that $b = 1 \in [-1,1]$ does not admit a neighborhood that is uniformly covered. □

Definition. Let $p\colon X \to B$ be a covering of a connected non-empty topological space.

(1) We define the degree $[X : B]$ of p as the cardinality of $\#p^{-1}(b)$ for some $b \in B$.[124]
(2) If the degree is a finite number n, then we say that p is an n-fold covering. Otherwise we say that p is an infinite covering.
(3) If $p\colon X \to B$ is a covering over a possibly disconnected topological space, then we say that p is a finite covering if for each component C of B the covering $p\colon p^{-1}(C) \to C$ is a finite covering in the sense of (2).

Examples.

(1) As usual we consider $S^1 = \{z \in \mathbb{C} \,|\, |z| = 1\}$ as a subset of $\mathbb{C}$. Given any $n \in \mathbb{N}$ the map $p\colon S^1 \to S^1$ given by $p(z) := z^n$ is an n-fold covering.[125]

[122] Recall that given a set S we denote by $\#S$ the cardinality of S, as defined on page 41. In particular $\#p^{-1}(P) = \#p^{-1}(Q)$ means that there exists a bijection between the sets $p^{-1}(P)$ and $p^{-1}(Q)$.

[123] A mathematically precise description would be to consider the Möbius band $([0,1] \times [0,1])/\sim$ and to consider the map from the boundary to the central circle that is given by $[(x,y)] \mapsto [(x, \frac{1}{2})]$.

[124] By the Covering–Preimage Lemma 14.4 this definition does not depend on the choice of b.

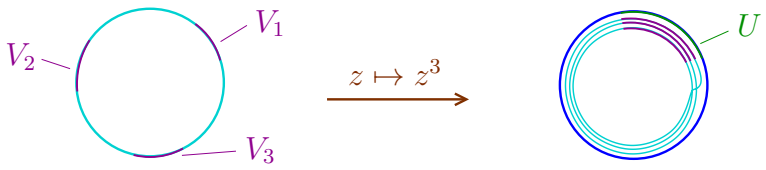
[125] This can be seen as follows: Let $z = \exp(2\pi \mathrm{i}\varphi)$ be a point in S^1. We set

$$U := \big\{ \exp(2\pi \mathrm{i}\psi) \,\big|\, \psi \in \big(\varphi - \tfrac{1}{4}, \varphi + \tfrac{1}{4}\big) \big\}.$$

Note that

$$p^{-1}(U) = \bigsqcup_{j=1}^{n} \underbrace{\big\{ \exp(2\pi \mathrm{i}\psi) \,\big|\, \psi \in \big(\tfrac{1}{n}(\varphi - \tfrac{1}{4}) + \tfrac{j}{n}, \tfrac{1}{n}(\varphi + \tfrac{1}{4}) + \tfrac{j}{n}\big) \big\}}_{=:V_j}.$$

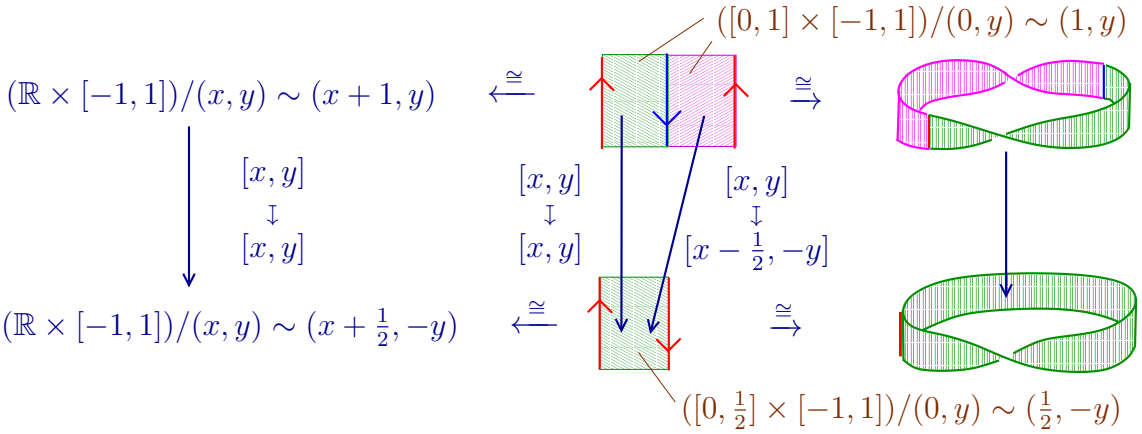
One can easily verify that the restriction of p to each V_j is a homeomorphism.



(2) We consider the map

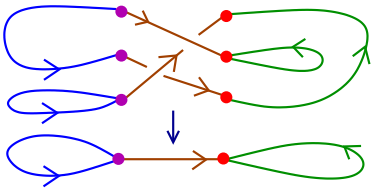
$$\begin{array}{rccc} p\colon & \overbrace{(\mathbb{R}\times[-1,1])/(x,y)\sim(x+1,y)}^{\text{annulus}} & \to & \overbrace{(\mathbb{R}\times[-1,1])/(x,y)\sim(x+\frac{1}{2},-y)}^{\text{Möbius band}} \\ & [(x,y)] & \mapsto & [(x,y)]. \end{array}$$

It is straightforward to show that p is a 2-fold covering. Thus we can view the annulus as a 2-fold covering of the Möbius band.



(3) If in the previous example we replace the interval $[-1,1]$ by the circle $[-1,1]/-1\sim 1$, then on the left-hand side we obtain the torus and on the right-hand side we obtain the Klein bottle. This shows that there exists a 2-fold covering from the torus to the Klein bottle.

(4) In the figure below we sketch three coverings of topological graphs:



The coverings are hereby given by sending edges of a certain color, with the indicated direction, to the edge of the same color with the indicated direction.[126] □

The following proposition connects the notion of discrete actions to the theory of coverings.

Proposition 14.5. (Action–Covering Proposition) Let X be a topological space together with a discrete and continuous action by a group G.

(1) The natural projection $p\colon X\to X/G$ is a covering where the degree is given by $\#G$.

[126] It is a good exercise to convince oneself, that the indicated maps are in fact coverings.

(2) Let H be a subgroup of G. The natural projection

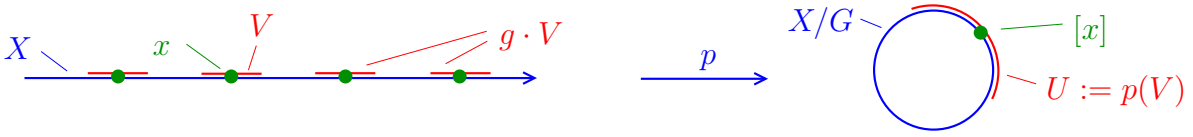
$$\begin{array}{rcl} p\colon X/H & \to & X/G \\ [x] & \mapsto & [x] \end{array}$$

is a covering of degree $[G : H]$[127].

Example. The Action–Covering Proposition 14.5 (1), applied to the examples on page 210, shows that $\begin{array}{rcl} p\colon \mathbb{R}^n & \to & \mathbb{R}^n/\mathbb{Z}^n \\ x & \mapsto & [x] = x + \mathbb{Z}^n \end{array}$ and $\begin{array}{rcl} p\colon S^n & \to & \mathbb{R}\mathrm{P}^n =_{\mathrm{i}} S^n/x \sim -x \\ x & \mapsto & [x] \end{array}$ is an infinite respectively 2-fold covering. □

Proof of Proposition 14.5. In the following we prove Statement (1). We leave it to the reader to generalize the proof to the more general setting of Statement (2).

Let G be a group that acts discretely and continuously on a topological space X. We want to show that the projection $p\colon X \to X/G$ is a covering. Evidently the map is surjective. Next let $[x] \in X/G$. Since the group acts discretely there exists, by definition, an open neighborhood V of $x \in X$ such that $V \cap g{\cdot}V = \varnothing$ for all $g \neq e$. We put $U := p(V)$.



Claim. U is a uniformly covered open neighborhood of $[x]$.

Proof. Using the fact that G acts continuously one can easily show that $p\colon X \to X/G$ is an open map. Thus it follows that the set U is an open neighborhood of $[x] = p(x)$.

Next note that the preimage $p^{-1}(U)$ is the disjoint union of the subsets $g \cdot V, g \in G$. Since G acts continuously we know that each $g \cdot V$ is again open. The restriction of the map p to any $g \cdot V$ is a continuous and bijective map $g \cdot V \to U$. Since $p\colon X \to X/G$ is open and since $g \cdot V \subset X$ is open we see that the restriction of p to $g \cdot V$ is also open. It follows from the Open-Injective Lemma 2.9 that the map $p|_{g\cdot V}\colon g \cdot V \to U$ is in fact a homeomorphism. Thus we have shown that U is uniformly covered. ⊞

Since this holds for any point in X/G we have shown that the projection $p\colon X \to X/G$ is a covering. Finally note that $[X : X/G] = p^{-1}([x]) = \#G$. ■

Example. Let $X = \mathbb{R} \times [-1, 1]$ and $G = \mathbb{Z}$. Once again we consider the discrete action

$$\begin{array}{rcl} \mathbb{Z} \times (\mathbb{R} \times [-1,1]) & \to & \mathbb{R} \times [-1,1] \\ (n, (x, y)) & \mapsto & \big(x + \frac{1}{2}n, (-1)^n \cdot y\big). \end{array}$$

Let $H = 2\mathbb{Z}$. According to the Action–Covering Proposition 14.5 (2) the projection $p\colon X/H \to X/G$ is a 2-fold covering. Put differently, all the horizontal maps in the following

[127]Here $[G : H]$ denotes the index of the subgroup $H \subset G$, i.e. the cardinality of G/H. We define this notion in detail on page 225.

commutative diagram are coverings:

$$\begin{array}{ccc}
(\mathbb{R}\times[-1,1])/2\mathbb{Z} & \longrightarrow & (\mathbb{R}\times[-1,1])/\mathbb{Z} \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} \\
(\mathbb{R}\times[-1,1])/(x,y)\sim(x+\tfrac{1}{2}2k,(-1)^{2k}\cdot y) & \longrightarrow & (\mathbb{R}\times[-1,1])/(x,y)\sim(x+\tfrac{1}{2}k,(-1)^{k}\cdot y) \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} \\
\underbrace{(\mathbb{R}\times[-1,1])/(x,y)\sim(x+k,y)}_{=\text{the annulus}} & \longrightarrow & \underbrace{(\mathbb{R}\times[-1,1])/(x,y)\sim(x+\tfrac{1}{2}k,(-1)^{k}\cdot y).}_{=\text{Möbius band}}
\end{array}$$

Thus we recover the result from page 216 that the annulus is a 2-fold covering of the Möbius band.

14.3. **Basic properties of coverings.** The following lemma summarizes some basic facts about coverings.

Lemma 14.6. (Covering–Basics Lemma) Given any covering $p\colon X\to B$ of topological spaces the following statements hold:

(1) The map $p\colon X\to B$ is open.[128]
(2) For every $b\in B$ the preimage $p^{-1}(b)$ is a discrete subset of X.
(3) If B is Hausdorff, then so is X.
(4) The following two statements are equivalent:
   (a) X is compact.
   (b) p is a finite covering and B is compact.

Proof. Let $p\colon X\to B$ be a covering of topological spaces.

(1) We will not really make use of this statement, thus we refer to [**Fri24**] for the proof.
(2) Let $b\in B$. Since $p\colon X\to B$ is a covering there exists an open neighborhood U of b such that $p^{-1}(U)$ is the union of disjoint open subsets $\{V_i\}_{i\in I}$ with the property that each $p|_{V_i}\colon V_i\to U$ is a homeomorphism. For each $i\in I$ there exists a unique point $x_i\in V_i$ with $p(x_i)=b$ and the preimage $p^{-1}(b)$ consists of precisely the x_i, $i\in I$. Since each V_i is open in X it follows from the definition of the subspace topology that each $x_i\in p^{-1}(b)$ is an open subset of $p^{-1}(b)$. Put differently, the subspace topology on $p^{-1}(b)$ is the discrete topology. But this is equivalent to saying that $p^{-1}(b)$ is a discrete subset of X.
(3) This statement is relatively easy to prove. We leave it to the reader to fill in the details.
(4) We first prove the(a)⇒(b)-direction. Thus we assume that X is compact. By definition the covering $p\colon X\to B$ is surjective and continuous. Thus it follows from the Compact Image Lemma 2.8 that B is also compact.

   It remains to show that the covering has finite degree.[129] By definition we can cover the topological space B by open subsets that are uniformly covered. Since B is compact we can cover B by finitely many open subsets $U_1,\dots,U_k$ that are uniformly covered.

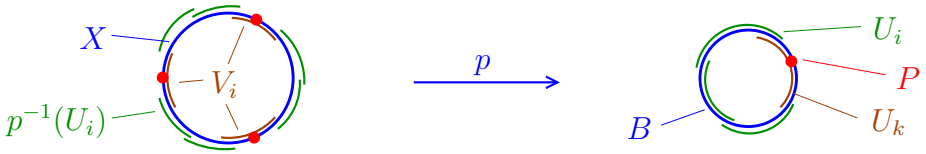
[128]Recall that this means that the image of each open subset in X is open in B.

[129]If B is Hausdorff the argument is quite simple: Let $b\in B$. By the Points-in-Hausdorff Lemma 1.8 we know that $\{b\}\subset B$ is a closed subset. Since p is continuous we see that $p^{-1}(b)$ is a closed subset of X. By (2) we know that $p^{-1}(b)$ is a discrete subset. Since X is compact it now follows from an elementary argument that $[X:B]:=p^{-1}(b)$ is finite.

Without loss of generality we can assume that U_k is not contained in $U_1 \cup \cdots \cup U_{k-1}$.[130] Now let P be a point in $U_k \setminus (U_1 \cup \cdots \cup U_{k-1})$. Furthermore let V_i, $i \in I$ be the disjoint open subsets of X such that $p^{-1}(U_k)$ is the union of the V_i's and such that each $p|_{V_i} \colon V_i \to U_k$ is a homeomorphism. For $i \in I$ we denote by $Q_i \in V_i$ the unique point with $p(Q_i) = P$. Now we consider the open cover

$$X \;=\; p^{-1}(U_1) \cup \ldots \cup p^{-1}(U_{k-1}) \cup \bigcup_{i \in I} V_i.$$

Note that for $i \in I$ the point Q_i is contained only in V_i. This means that we cannot remove any of the open subsets V_i, $i \in I$ from the open cover. Since X is compact, we see that I is finite. But $\#I$ is precisely the degree of the covering. Thus we have shown that the covering has finite degree.



We turn to the proof of the (b)⇒(a)-direction. Let $p \colon X \to B$ be an n-fold covering of a compact space B. We need to show that X is compact. Thus let $\{U_i\}_{i \in I}$ be an open cover of X.

We start out with some preparations: Let $b \in B$. We pick an open neighborhood M of b that is uniformly covered. Recall that by definition this means that $p^{-1}(M)$ is the disjoint union of open subsets $\{N_x\}_{x \in p^{-1}(b)}$ with the following two properties:

- For each $x \in p^{-1}(b)$ we have $x \in N_x$.
- The restriction of p to each N_x is a homeomorphism.

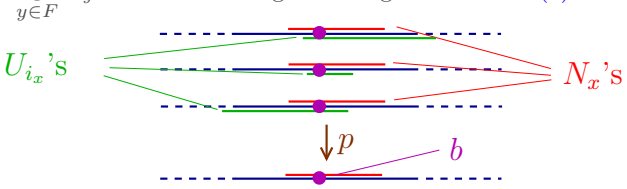
For each $x \in p^{-1}(b)$ we pick an $i_x \in I$ with $x \in U_{i_x}$. We set

$$W_b \;\coloneqq\; \bigcap_{x \in p^{-1}(b)} p(U_{i_x} \cap N_x).$$

Claim. The preimage $p^{-1}(W_b)$ is contained in the union of $n = \#p^{-1}(b)$ subsets of our open cover $\{U_i\}_{i \in I}$.

Proof. Let $x \in p^{-1}(b)$. We denote by $p_x \coloneqq p|_{N_x} \colon N_x \to M$ the homeomorphism that is given by restricting p to N_x. We write $F \coloneqq p^{-1}(b)$. We see that:

$$p^{-1}(W_b) \underset{\substack{\uparrow \\ \text{since } p^{-1}(W_b) \subset \bigcup_{y \in F} N_y}}{=} \bigcup_{y \in F} p_y^{-1}\Big(\bigcap_{x \in F} p_x(U_{i_x} \cap N_x)\Big) \underset{\substack{\uparrow \\ \text{Image-Preimage Lemma 0.3 (7)}}}{=} \bigcup_{y \in F} \overbrace{\bigcap_{x \in F} \underbrace{p_y^{-1}(p_x(U_{i_x}))}_{= U_{iy} \text{ if } y = x} \cap \underbrace{p_y^{-1}(p_x(N_x))}_{= N_y}}^{\subset U_{i_y} \cap N_y} \subset \bigcup_{y \in F} U_{i_y}.$$

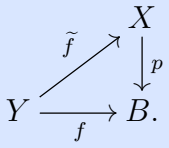


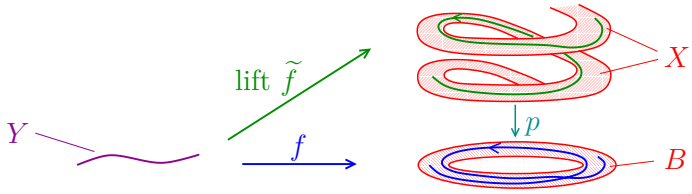
[130] Indeed, if U_k was contained in $U_1 \cup \cdots \cup U_{k-1}$, then we could remove U_k from the collection of open subsets and we continue with the collection $U_1, \ldots, U_{k-1}$.

By (1) each $p(U_{i_x} \cap N_x)$ is an open subset of B and since $p^{-1}(b)$ is a finite set we see that W_b is the intersection of finitely many open subsets of B, thus W_b is an open subset of B. Furthermore b is contained in each $p(U_{i_x} \cap N_x)$, so W_b is in fact an open neighborhood of $b \in B$. Since B is compact we can cover B by finitely many of these open subsets, i.e. there exist $b_1, \dots, b_k \in B$ with $B = W_{b_1} \cup \cdots \cup W_{b_k}$. The preimages $p^{-1}(W_{b_1}), \dots, p^{-1}(W_{b_k})$ cover X. Note that by the above claim each preimage $p^{-1}(W_{b_i})$ is contained in the union of $n = \#p^{-1}(b_i)$ subsets of our open cover $\{U_i\}_{i\in I}$. Summarizing we have shown that we can cover X by $n \cdot k$ sets out of the collection $\{U_i\}_{i\in I}$. ■

14.4. **Lifts of maps.** The following definition generalizes the notion of a lift that we introduced on page 187.

Definition. Let $p\colon X \to B$ be a covering and let $f\colon Y \to B$ be a continuous map between topological spaces. A lift of f to X is a continuous map $\widetilde{f}\colon Y \to X$ such that $p \circ \widetilde{f} = f$, i.e. such that the following diagram of maps commutes:





Similar to the discussion in Chapter 11.4 we want to prove the existence and uniqueness of lifts of paths $p\colon [0,1] \to B$ or more generally, of continuous maps $[0,1]^m \to B$. We start out with the following proposition which is a generalization of the Lift Uniqueness Lemma 11.10.

Proposition 14.7. (Lift Uniqueness Lemma) Let $p\colon X \to B$ be a covering and let Y be a connected topological space. Furthermore let $f\colon Y \to B$ be a continuous map and let $\widetilde{f}, \widehat{f}\colon Y \to X$ be two lifts of f. If $\widetilde{f}$ and $\widehat{f}$ agree at some point $y_0 \in Y$, then they agree everywhere.

Proof. We set

$$\begin{aligned} U &:= \{y \in Y \mid \widetilde{f}(y) = \widehat{f}(y)\} \quad \subset Y \\ V &:= \{y \in Y \mid \widetilde{f}(y) \neq \widehat{f}(y)\} \quad \subset Y. \end{aligned}$$

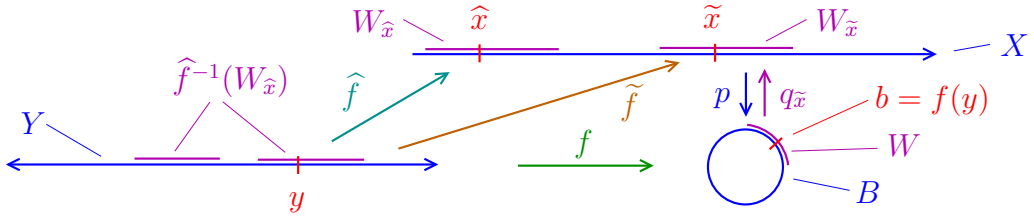
Note that by hypothesis $U \neq \varnothing$. We want to show that $U = Y$. Since Y is connected it suffices to prove that U and V are open subsets of Y. By the JH-Openness Criterion 1.1 it suffices to prove the following claim:

Claim. Every $y \in Y$ admits an open neighborhood N such that $N \subset U$ or $N \subset V$.

Proof. Let $y \in Y$. We set $b := f(y)$ and we set $\widetilde{x} := \widetilde{f}(y)$ and $\widehat{x} := \widehat{f}(y)$. Since $p\colon X \to B$ is a covering we know that b admits an open neighborhood W that is uniformly covered,

i.e. we know that we can write $p^{-1}(W) = \bigsqcup_{x\in p^{-1}(b)} W_x$ such that for each $x \in p^{-1}(b)$ the following statements hold:

(1) W_x is an open neighborhood of $x \in X$.
(2) The map $p|_{W_x}\colon W_x \to W$ is a homeomorphism. We denote by $q_x\colon W \to W_x$ the inverse.



Note that on $\widetilde{f}^{-1}(W_{\widetilde{x}})$ we have

$$\widetilde{f} = \underbrace{q_{\widetilde{x}} \circ p|_{W_{\widetilde{x}}}}_{=\mathrm{id}_{W_{\widetilde{x}}}} \circ \widetilde{f} = q_{\widetilde{x}} \circ \underbrace{p|_{W_{\widetilde{x}}} \circ \widetilde{f}}_{=f} = q_{\widetilde{x}} \circ f\colon \widetilde{f}^{-1}(W_{\widetilde{x}}) \to W_{\widetilde{x}}.$$

Similarly we see that on $\widehat{f}^{-1}(W_{\widehat{x}})$ we have $\widehat{f} = q_{\widehat{x}} \circ f$. The claim follows from the following observations:

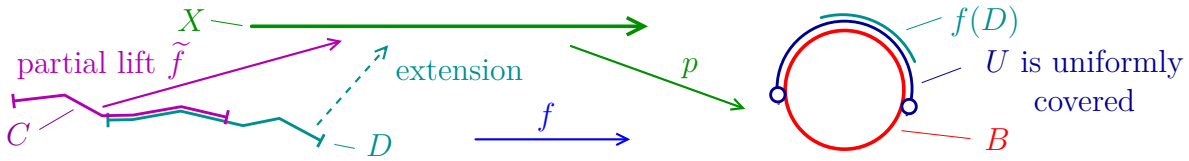
- If $y \in U$, i.e. if $\widetilde{x} = \widehat{x}$, then we see that $\widetilde{f} = \widehat{f}$ on $\widetilde{f}^{-1}(W_{\widetilde{x}}) \cap \widehat{f}^{-1}(W_{\widehat{x}})$, i.e. we see that $\widetilde{f}^{-1}(W_{\widetilde{x}}) \cap \widehat{f}^{-1}(W_{\widehat{x}}) \subset U$.
- If $y \notin U$, i.e. if $\widetilde{x} \neq \widehat{x}$, then we see that the images of $\widetilde{f}$ and $\widehat{f}$ on $\widetilde{f}^{-1}(W_{\widetilde{x}}) \cap \widehat{f}^{-1}(W_{\widehat{x}})$ are disjoint, thus we see that $\widetilde{f}^{-1}(W_{\widetilde{x}}) \cap \widehat{f}^{-1}(W_{\widehat{x}}) \subset V$. ∎

As in Section 11.4 our next goal is to prove the existence of lifts in suitable settings. The key tool for constructing lifts is the following generalization of the Lift-Extension Lemma 11.11.

Lemma 14.8. (Lift-Extension Lemma) Let $p\colon X \to B$ be a covering and let Y be a topological space. We suppose that we have a decomposition $Y = C \cup D$ such that C and D are closed subsets. Let $f\colon Y \to B$ be a continuous map and let $\widetilde{f}\colon C \to X$ be a lift of $f|_C$. We suppose that the following statements hold:

(1) The image $f(D)$ is contained in a uniformly covered open subset of B.
(2) The intersection $C \cap D$ is connected and non-empty.

Then there exists a lift $\widetilde{f}\colon Y \to X$ of f that agrees with the given lift on C.

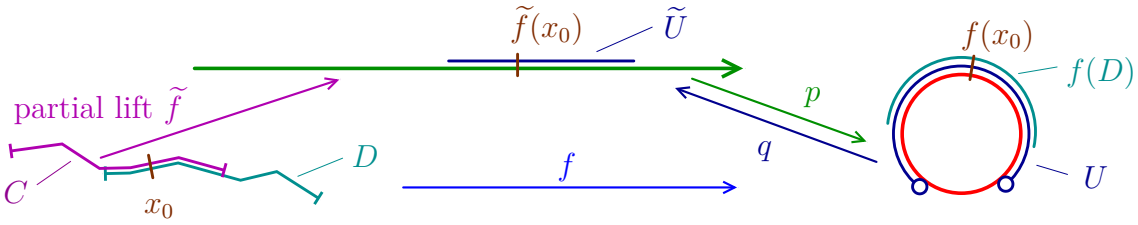


Proof. We start with a few preparations:

(a) Since $C \cap D = \varnothing$ we can pick $x_0 \in C \cap D$.
(b) We pick a uniformly covered open subset U of B that contains $f(D)$.
(c) Since U is uniformly covered open subset it thus follows from $p(\widetilde{f}(x_0)) = f(x_0) \in f(D)$ that there exists an open neighborhood $\widetilde{U}$ of $\widetilde{f}(x_0)$ such that $p\colon \widetilde{U} \to U$ is a homeomorphism.
(d) We denote by $q\colon U \to \widetilde{U}$ the inverse of $p\colon \widetilde{U} \to U$.

We set

$$\begin{aligned} \widetilde{f}\colon Y &\to X \\ x &\mapsto \begin{cases} \widetilde{f}(x), & \text{if } x \in C, \\ q(\underbrace{f(x)}_{\in U}), & \text{if } x \in D. \end{cases} \end{aligned}$$



Note that on $C \cap D$ we have a priori two different definitions of $\widetilde{f}$. But both definitions are lifts of $f|_{C\cap D}$ and by design they agree on x_0. By hypothesis the intersection $C \cap D$ is connected. Thus we obtain from the Lift Uniqueness Lemma 11.10 that the two maps actually agree on the overlap $C \cap D$. This shows that $\widetilde{f}\colon Y \to X$ is well-defined.

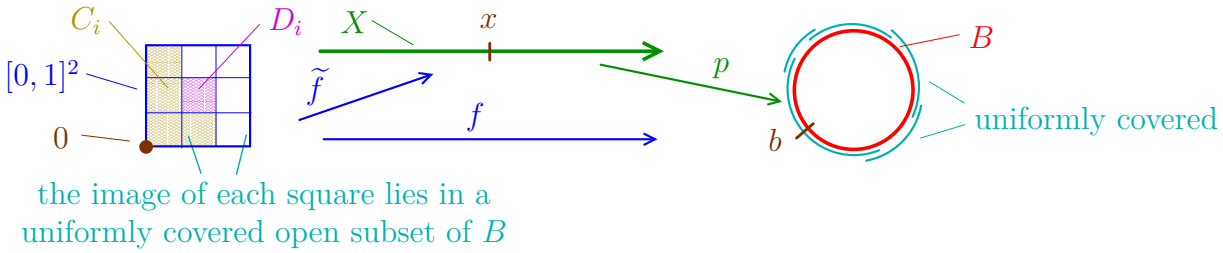
Finally note that since C and D are closed it follows from the Pasting Proposition 2.4 that this map $\widetilde{f}\colon Y \to X$ is continuous. It is now clear that it is a lift of $f\colon Y \to X$. ■

The following lemma is in particular a generalization of the Lift-from-S^1-to-$\mathbb{R}$ Lemma 11.12.

Lemma 14.9. (Lift-Existence Lemma) Let $p\colon X \to B$ be a covering of topological spaces. Given any continuous map $f\colon [0,1]^m \to B$ and given any $x \in X$ with $p(x) = f(0)$ there exists a unique lift $\widetilde{f}\colon [0,1]^m \to X$ with $\widetilde{f}(0) = x$.

Proof. [131] Let $p\colon X \to B$ be a covering of topological spaces, let $f\colon [0,1]^m \to B$ be a continuous map and let $x \in X$ with $p(x) = f(0)$. Since $p\colon X \to B$ is a covering there exists an open cover of B by uniformly covered open subsets of B. It follows from the Lebesgue Lemma 11.9 that there exists an $N \in \mathbb{N}$ such that for any $(i_1, \dots, i_m) \in \{0, \dots, N-1\}^m$ the image $f([\frac{i_1}{N}, \frac{i_1+1}{N}] \times \cdots \times [\frac{i_m}{N}, \frac{i_m+1}{N}])$ is contained in a uniformly covered open subset of B. The idea is now to lift the map $f\colon [0,1]^m \to B$ as follows:

(1) We set $\widetilde{f}(0) \coloneqq x$.
(2) We apply the Lift-Extension Lemma 14.8 iteratively to the N^m cubes of $[0,1]^m$ starting from the "bottom left".



We now carry out the technical details. We introduce the following notation:

(a) We write $I \coloneqq \{0, \dots, N-1\}^m$. We give I the lexicographic order.
(b) Given $i \coloneqq (i_1, \dots, i_m) \in I$ we write $D_i \coloneqq [\frac{i_1}{N}, \frac{i_1+1}{N}] \times \cdots \times [\frac{i_m}{N}, \frac{i_m+1}{N}]$.

[131] The proof of the proposition is a generalization of the proof of Claim 1 in the proof of the Circle-π_1-Theorem 11.13.

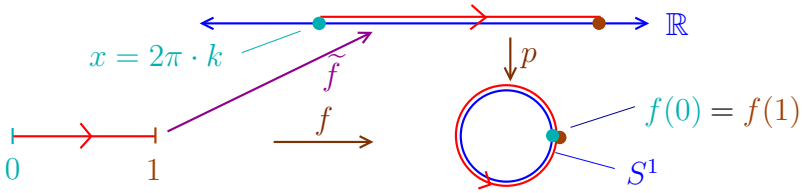
(c) Given $i \in I$ we write $C_i := \bigcup_{h<i} D_h$.

We set $\widetilde{f}(0) := x$. Now we use the Lift-Extension Lemma 14.8 to add the cubes $\{D_i\}_{i\in I}$ in lexicographic order to the domain of $\widetilde{f}$. This is possible since the image of each D_i is contained in a uniformly covered open subset of B and since given any $i \in I$ the intersection $C_i \cap D_i$ is easily seen to be non-empty and connected. ■

14.5. **Lifts of paths.** We now want to apply the results from the previous sections to paths and path-homotopies. It is convenient to introduce the following definition:

Definition. Let $p\colon X \to B$ be a covering and let $f\colon [0,1] \to B$ be a path. Let $x \in X$ be a point with $p(x) = f(0)$. A path $\widetilde{f}\colon [0,1] \to X$ is called a lift of f to the starting point x, if $\widetilde{f}$ is a lift of f with $\widetilde{f}(0) = x$.

Example. Again we consider the covering $p\colon \mathbb{R} \to S^1$ that is given by $t \mapsto \exp(it)$. We consider the path $f\colon [0,1] \to S^1$ that is given by $t \mapsto \exp(2it)$. Given any $k \in \mathbb{Z}$ the map $\widetilde{f}\colon [0,1] \to \mathbb{R}$ given by $t \mapsto 2\pi \cdot k + 2t$ is a lift of f to the starting point $2\pi \cdot k$.

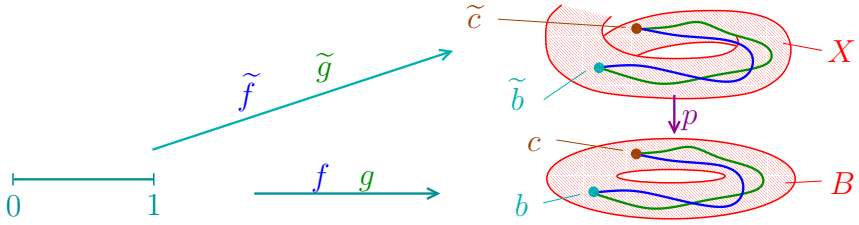


Note that the path $f\colon [0,1] \to S^1$ is a loop but the lift $\widetilde{f}\colon [0,1] \to \mathbb{R}$ is not a loop. □

Proposition 14.10. (Path-Lifting Proposition) Let $p\colon X \to B$ be a covering.

(1) Let $f\colon [0,1] \to B$ be a path. Given any $\widetilde{b} \in X$ with $p(\widetilde{b}) = f(0)$ there exists a unique lift $\widetilde{f}\colon [0,1] \to X$ of f to the starting point $\widetilde{b}$.

(2) Let $f, g\colon [0,1] \to B$ be two paths and let $\widetilde{f}, \widetilde{g}\colon [0,1] \to X$ be two lifts with the same starting point. If f and g are path-homotopic, then the endpoints of $\widetilde{f}$ and $\widetilde{g}$ agree and the paths $\widetilde{f}$ and $\widetilde{g}$ are path-homotopic.



Proof.

(1) This statement follows immediately from the Lift-Existence Lemma 14.9.

(2) We set $b := f(0) = g(0)$ and $c := f(1) = g(1)$. Furthermore let $\widetilde{b} := \widetilde{f}(0) = \widetilde{g}(0)$ be the common starting point of the lifts. Let

$$\begin{aligned} H\colon [0,1] \times [0,1] &\to B \\ (t,s) &\mapsto H(t,s) \end{aligned}$$

be a path-homotopy between the paths f and g. By the Lift-Existence Lemma 14.9 we can lift H to a continuous map $\widetilde{H}\colon [0,1] \times [0,1] \to X$ with $\widetilde{H}(0,0) = \widetilde{f}(0)$.

Claim. The map $\widetilde{H}\colon [0,1]\times[0,1]\to X$ is a path-homotopy from $\widetilde{f}$ to $\widetilde{g}$. In particular the endpoints of $\widetilde{f}$ and $\widetilde{g}$ agree.

Proof.

(a) For each $s\in[0,1]$ we have $p(\widetilde{H}(0,s)) = H(0,s) = b$, thus the map $s\mapsto \widetilde{H}(0,s)$ takes values in $p^{-1}(b)$. By the Covering–Basics Lemma 14.6 (2) we know that $p^{-1}(b)$ is a discrete subspace. It follows from the Connected-to-Discrete Lemma 2.18 that the map $s\mapsto \widetilde{H}(0,s)$ is constant. Similarly we see that the $s\mapsto \widetilde{H}(1,s)$ is constant. Thus we have shown that $\widetilde{H}$ is a path-homotopy.

(b) Since $\widetilde{H}_0$ and $\widetilde{f}$ are both lifts of $H_0 = f$ to the starting point $\widetilde{b}$ we obtain from the Lift Uniqueness Proposition 14.7 that $\widetilde{H}_0 = \widetilde{f}$.

(c) It follows from (a) that $\widetilde{H}_1$ and $\widetilde{g}$ are both lifts of $H_1 = g$ to the starting point $\widetilde{b}$. Thus we obtain again from the Lift Uniqueness Proposition 14.7 that $\widetilde{H}_1 = \widetilde{g}$.

In summary, we have shown that $\widetilde{H}$ is a path-homotopy from $\widetilde{f}$ to $\widetilde{g}$. ■

Proposition 14.11. (Covering-π_1-Monomorphism Proposition) Let $p\colon X\to B$ be a covering. Given any $x_0\in X$ the induced map $p_*\colon \pi_1(X,x_0)\to\pi_1(B,p(x_0))$ is a monomorphism.

Example. In the figure below we see one of the coverings from page 216:



Even though we cannot compute the fundamental groups of X and B it is clear that the fundamental group of X is "bigger" in a suitable sense than the fundamental group of B. But counter intuitively the Covering-π_1-Monomorphism Proposition 14.11 shows that there exists a monomorphism, namely p_*, from $\pi_1(X,x_0)$ into $\pi_1(B,p(x_0))$. □

Proof. We set $b_0 \coloneqq p(x_0)$. We see that

$$[f]\in\ker(p_*\colon \pi_1(X,x_0)\to\pi_1(B,b_0)) \implies [p\circ f] = e\in\pi_1(B,b_0) \implies p\circ f\simeq_{\mathrm{p}} \mathrm{const}_{b_0}$$
$$\implies f \simeq_{\mathrm{p}} \mathrm{const}_{x_0} \implies [f] = e\in\pi_1(X,x_0).$$

this follows from the Path-Lifting Proposition 14.10 (2), since f is the lift of $p\circ f$ to the starting point x_0 and const_{x_0} is the lift of const_{b_0} to the starting point x_0 ■

Example. By the Sphere-π_1-Proposition 11.15 we know that $\pi_1(S^2) = 0$ whereas we saw in the Torus-π_1-Corollary 13.8 that $\pi_1(S^1\times S^1)\cong\mathbb{Z}^2$ is non-trivial. It follows from the Covering-π_1-Monomorphism Proposition 14.11 that there is no covering map from $S^1\times S^1$ to S^2. □

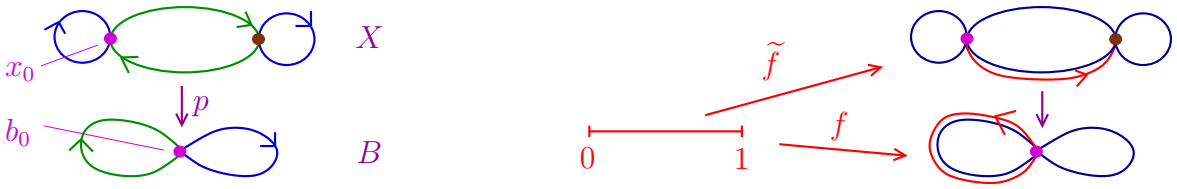
Proposition 14.12. (Loop–Lifting Proposition) Let $p\colon X\to B$ be a covering. Let $b_0\in B$ and let $x_0\in X$ be a point with $p(x_0) = b_0$. Let $f\colon[0,1]\to B$ be a loop in (B,b_0). We denote by $\widetilde{f}\colon[0,1]\to X$ the lift[132] of f to the starting point x_0. The following holds:

$$\widetilde{f} \text{ is a loop in } (X,x_0) \iff [f] \text{ lies in } p_*(\pi_1(X,x_0))\subset\pi_1(B,b_0).$$

In particular, if $\widetilde{f}$ is not a loop in (X,x_0), then $[f]$ is a non-trivial element of $\pi_1(B,b_0)$.

[132] Note that such a lift exists, and is unique, by the Path-Lifting Proposition 14.10 (1).

Example. We consider the pointed topological space (B, b_0) and the loop $f\colon [0,1] \to B$ that are shown in the figure below. We also show a covering $p\colon X \to B$ together with the lift $\widetilde{f}$ of f to the starting point x_0:



Since $\widetilde{f}$ is not a loop it follows from the Loop–Lifting Proposition 14.12 that $[f] \in \pi_1(B, b_0)$ is non-trivial. □

Proof. Let f be a loop in (B, b_0). We denote by $\widetilde{f}$ the lift of f to the starting point x_0.

(⇒) If $\widetilde{f}$ is a loop in (X, x_0), then it defines an element in $\pi_1(X, x_0)$. Therefore we see that $[f] = [p \circ \widetilde{f}] = p_*([\widetilde{f}]) \in p_*(\pi_1(X, x_0))$.

(⇐) We assume that $[f]$ lies in $p_*(\pi_1(X, x_0)) \subset \pi_1(B, b_0)$. We make two observations:

- The hypothesis that $[f] \in p_*(\pi_1(X, x_0))$ implies that there exists a loop g in (X, x_0) with $[f] = p_*([g]) = [p \circ g]$. In particular we see that f and $p \circ g$ are path-homotopic.
- Note that $\widetilde{f}$ is a lift of f and that g is a lift of $p \circ g$ and note that these two lifts have the same starting point x_0.

It follows from these two observations and the Path-Lifting Proposition 14.10 (2) that the endpoint of $\widetilde{f}$ agrees with the endpoint of $\widetilde{g}$, which is x_0. Thus we have shown that $\widetilde{f}$ is a loop in (X, x_0). ■

The following definition should be familiar from a (linear) algebra course.

Definition. Let G be a group and let H be a subgroup of G.

(1) We get an equivalence relation $\sim$ on G by setting $g_1 \sim g_2$ if there exists an $h \in H$ with $g_1 \cdot h = g_2$. We denote by G/H the set of equivalence classes.

(2) If in (1) we use left multiplication by $h \in H$, then also get an equivalence relation and we denote the set of equivalence classes by $H\backslash G$. Note that the map $[g] \mapsto [g^{-1}]$ defines a bijection $H\backslash G \to G/H$.

(3) We write $[G : H] := \#G/H = \#H\backslash G$.

Proposition 14.13. (Covering-Degree Proposition) Let $p\colon X \to B$ be a covering. Let $b_0 \in B$ and let $x_0 \in X$ be a point with $p(x_0) = b_0$. If X is path-connected, then[133]

$$\underbrace{[X : B]}_{\substack{\text{degree as defined} \\ \text{on page 215}}} = [\pi_1(B, b_0) : p_*(\pi_1(X, x_0))].$$

Example. Let $n \geq 2$.

(1) On page 217 we saw that we have a 2-fold covering $p\colon S^n \to S^n/\{\pm 1\} =_{\mathrm{i}} \mathbb{R}\mathrm{P}^n$.

(2) It follows from the Covering-Degree Proposition 14.13 that $p_*(\pi_1(S^n))$ is subgroup of index two of $\pi_1(S^n/\{\pm 1\})$.

(3) Since $n \geq 2$ we know by the Sphere-π_1-Proposition 11.15 that $\pi_1(S^n) = 0$.

[133] Recall that the left-hand side and the right-hand side are defined as cardinalities of sets.

(4) It follows from (1)–(3) that $\pi_1(S^n/\{\pm 1\})$ is a group with two elements.
(5) It is elementary to see that (4) implies that $\pi_1(\mathbb{R}\mathrm{P}^n) = \pi_1(S^n/\{\pm 1\}) \cong \mathbb{Z}_2$.

In the next chapter we will return to this calculation in a more systematic fashion. □

As we will see shortly, the Covering-Degree Proposition 14.13 follows almost immediately from the following lemma:

Lemma 14.14. (Covering–Bijection Lemma) Let $p\colon X \to B$ be a covering. Let $b_0 \in B$ and let $x_0 \in X$ be a point with $p(x_0) = b_0$. If X is path-connected, then the map

$$\begin{array}{rcl} \Theta\colon p^{-1}(b_0) & \to & p_*(\pi_1(X,x_0))\backslash\pi_1(B,b_0) \\ y_0 & \mapsto & [\,p \circ (\underbrace{\text{path } f \text{ in } X \text{ from } x_0 \text{ to } y_0}_{\text{exists since } X \text{ is path-connected}})\,] \end{array}$$

is well-defined and it is a bijection.



Proof of Lemma 14.14. Let $p\colon X \to B$ be a covering. Let $b_0 \in B$ and let $x_0 \in X$ be a point with $p(x_0) = b_0$. We assume that X is path-connected.

(1) First we show that the map Θ is well-defined. Thus let $f, g\colon [0,1] \to X$ be two paths from x_0 to y_0. Note that

$$\underbrace{[p \circ f]}_{\in \pi_1(B,b_0)} = [p \circ f * \underbrace{p \circ \overline{g} * p \circ g}_{\substack{\text{null-homotopic by} \\ \text{the Path-Homotopy} \\ \text{Product Proposition 11.4}}}] = \underbrace{[p \circ (f * \overline{g})]}_{\substack{\text{loop in } (X,x_0) \\ \text{since the endpoint} \\ \text{of } f \text{ equals the} \\ \text{starting point of } \overline{g}}} \cdot [p \circ g] = p_*(\underbrace{[f * \overline{g}]}_{\in \pi_1(X,x_0)}) \cdot \underbrace{[p \circ g]}_{\in \pi_1(B,b_0)}.$$



(2) Next we show that the map Θ is surjective. Thus let $[f] \in \pi_1(B, b_0)$. By the Path-Lifting Proposition 14.10 (1) there exists a lift $\widetilde{f}$ of f to the starting point x_0. Note that $p(\widetilde{f}(1)) = f(1) = b_0$. Thus $\widetilde{f}(1) \in p^{-1}(b_0)$. Evidently $\widetilde{f}$ is a path in X from x_0 to $\widetilde{f}(1)$. Thus we see that $\Theta(\widetilde{f}(1)) = [p \circ \widetilde{f}] = [f]$.

(3) Finally we show that the map Θ is injective. Suppose we have $y_0, z_0 \in p^{-1}(b_0)$ such that $\Theta(y_0) = \Theta(z_0)$.

(a) We pick a path $f_0\colon [0,1] \to X$ from x_0 to y_0 and we pick a path $g_0\colon [0,1] \to X$ from x_0 to z_0.



(b) The fact that $\Theta(y_0) = \Theta(z_0) \in p_*(\pi_1(X, x_0))\backslash\pi_1(B, b_0)$ implies that there exists a loop $h\colon [0,1] \to X$ in (X, x_0) such that $[p \circ h] \cdot [p \circ f_0] = [p \circ g_0]$.

(c) Note that $h * f_0$ is defined since the endpoint of h equals x_0, which is the starting point of f_0.[134] Now we see that

$$[p \circ (h * f_0)] \;=\; [(p \circ h) * (p \circ f_0)] \;=\; [p \circ h] \cdot [p \circ f_0] \;=\; [p \circ g_0].$$

(d) By (c) we know that $p \circ (h * f_0) \simeq_{\mathrm{p}} p \circ g_0$.

(e) Since $h * f_0$ is a lift of $p \circ (h * f_0)$ to the starting point x_0 and since g_0 is a lift of $p \circ g_0$ to the starting point x_0 we obtain from (d) and the Path-Lifting Proposition 14.10 (2) that the endpoints of $h * f_0$ and g_0 agree. But these are precisely y_0 and z_0. ■

Proof of Proposition 14.13. Let $p\colon X \to B$ be a covering. Let $b_0 \in B$ and let $x_0 \in X$ be a point with $p(x_0) = b_0$. We assume that X is path-connected. We see that

$$[X : B] \underset{\text{definition of degree}}{=} \#p^{-1}(b_0) \underset{\text{Covering–Bijection Lemma 14.14}}{=} \#p_*(\pi_1(X, x_0))\backslash\pi_1(B, b_0) \underset{\text{definition of index}}{=} [\pi_1(B, b_0) : p_*(\pi_1(X, x_0))].$$

■

14.6. **Pullbacks of coverings.** We conclude this chapter with a lemma on pullbacks of coverings. This lemma will play a minor role here and there, but it can also be skipped at a first reading.

Lemma 14.15. (Pullback-Covering Lemma) Let $p\colon X \to B$ be a continuous map of topological spaces, let C be a topological space and let $f\colon C \to B$ be a continuous map. As in the Topological-Pullback Proposition 5.10 we consider the pullback

$$f^*X \;:=\; \{(c, x) \in C \times X \mid f(c) = p(x)\}$$

which gives us the following commutative diagram:

$$\begin{array}{ccc} f^*X & \xrightarrow{\;\tilde{f}\;} & X \\ {\scriptstyle q}\downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ C & \xrightarrow[\;f\;]{} & B. \end{array}$$

If $p\colon X \to B$ is in fact a covering, then the following statements hold:

(1) The map $q\colon f^*X \to C$ is also a covering.

(2) If B and C are connected, then

$$\deg(q\colon f^*X \to C) \;=\; \deg(p\colon X \to B).$$

(3) Let $c \in C$. We write $b = f(c)$ and we pick[135] $x \in X$ such that $p(x) = b$. The following equality holds:

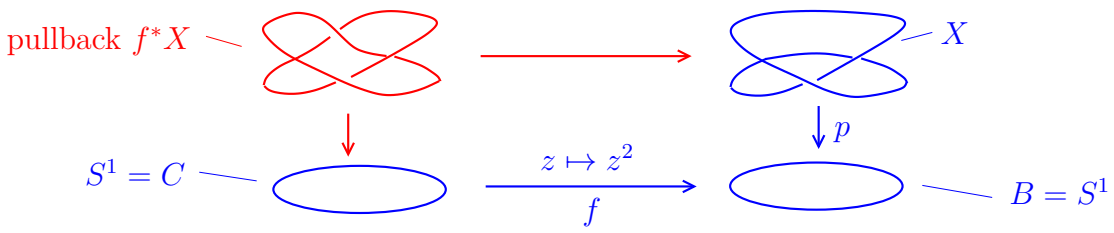
$$q_*(\pi_1(f^*X, (c, x))) \;=\; f_*^{-1}(p_*(\pi_1(X, x))) \;\subset\; \pi_1(C, c).$$

Example. Similar to the discussion on page 122 we consider the covering $p\colon X \to B = S^1$ that is shown in the figure below and we consider the map $f\colon C = S^1 \to S^1 = B$ that is given by $f(z) = z^2$.

The pullback covering $f^*X \to C = S^1$ is a product, i.e. there exists a homeomorphism $f^*X \cong S^1 \times \{-1, 1\}$ that commutes with the projections to S^1. □

[134] This is the reason why we work with right cosets.

[135] Recall that covering maps are in particular surjective, in particular such x exists.



Proof. Let $p\colon X \to B$ be a covering. We write $Z := f^*X$ and $z := (c, x)$. Finally we denote by $q\colon f^*X \to C$ the map that is given by $(c, x) \mapsto c$.

(1) First we show that $q\colon f^*X \to C$ is surjective. Thus let $c \in C$. Since $p\colon X \to B$ is in particular surjective we see that there exists an $x \in X$ with $p(x) = f(c)$. Note that $(c, x) \in f^*X$. Since $q(c, x) = c$ we have shown that q is surjective.

Next let $c \in C$. We need to show that c admits a uniformly covered neighborhood. Since p is a covering the point $f(c) \in B$ admits a uniformly covered neighborhood U. This means that $p^{-1}(U) = \bigsqcup_{i \in I} A_i$ such that each A_i is open in X and such that each $p|_{A_i}\colon A_i \to U$ is a homeomorphism.

We set $V := f^{-1}(U)$. We claim that V is uniformly covered. Given $i \in I$ we consider $B_i := \{(c, x) \in f^*X \mid c \in V, x \in U_i\}$. Since $B_i = i^*X \cap (V \times X_i)$ we see that B_i is an open subset of i^*X. Finally one can easily verify that each restriction $q|_{B_i}\colon B_i \to V$ is a homeomorphism.

(2) Since B and C are connected it makes sense to speak of the degrees of p and q. For any $c \in C$ we have a bijection

$$q^{-1}(c) \;=\; \{(c, x) \in C \times X \mid f(c) = p(x)\} \xrightarrow{\text{bijection}} p^{-1}(f(c)) \;=\; \deg(p).$$

This shows that the degree of q equals the degree of p.

(3) We need to show that $q_*(\pi_1(Z, z)) = f_*^{-1}(\pi_1(X, x))$. We show the two inclusions "$\subset$" and "$\supset$" separately.

"$\subset$" Let $\widetilde{f}\colon Z \to X$ be the map that is given by $(c, x) \to x$. By construction we have $f \circ q = \widetilde{f} \circ p$. But this implies immediately that $f_*(q_*(\pi_1(Z, z)))$ is contained in $p_*(\pi_1(X, x))$.

"$\supset$" Let $\gamma\colon [0, 1] \to C$ be a loop in (C, c) with $[f \circ \gamma] = f_*([\gamma]) \in p_*(\pi_1(X, x))$. We write $g := f \circ \gamma$. Let $\widetilde{g}\colon [0, 1] \to X$ be the lift with $\widetilde{g}(0) = x$. Since $[g] \in p_*(\pi_1(X, x))$ we obtain from the Loop–Lifting Proposition 14.12 that $\widetilde{g}$ is actually a loop. Thus the map $\alpha\colon [0, 1] \to C \times X$ that is given by $t \mapsto (\gamma(t), \widetilde{g}(t))$ is a loop. By construction the loop takes values in Z and evidently we have $q_*([\alpha]) = [\gamma]$. This shows that $[\gamma] \in q_*(\pi_1(Z, z))$. ∎

CHAPTER 15

Group actions and fundamental groups

In this chapter we will build on the techniques and results of the last chapter to calculate the fundamental groups of many topological spaces, in particular of the real projective spaces, lens spaces and the Klein bottle. We will also consider mapping tori and their fundamental groups.

15.1. **The main theorem on group actions and fundamental groups.** The following theorem will allow us to determine many fundamental groups. This proposition is the reward for the many rather theoretical results of the previous chapter.

Theorem 15.1. (Action–π_1-Theorem) Let X be a topological space and let G be a group that acts continuously and discretely[136] on X. We choose an $x_0 \in X$. We denote by $p\colon X \to X/G$ the natural projection.

(1) If X is simply connected, then the map

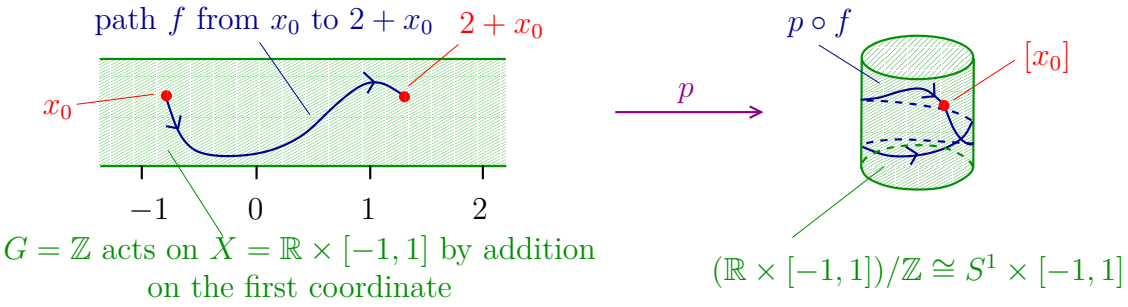
$$\begin{array}{rcl} G & \to & \pi_1(X/G, [x_0]) \\ g & \mapsto & [p \circ (\text{path in } X \text{ from } x_0 \text{ to } g \cdot x_0)] \end{array}$$

is a well-defined isomorphism of groups. In particular we have $\pi_1(X/G) \cong G$.

(2) [137] If X is path-connected (but not necessarily simply connected), then $p_*(\pi_1(X, x_0))$ is a normal subgroup of $\pi_1(X/G, [x_0])$ and the map

$$\begin{array}{rcl} G & \to & \pi_1(X/G, [x_0])/p_*(\pi_1(X, x_0)) \\ g & \mapsto & [p \circ (\text{path in } X \text{ from } x_0 \text{ to } g \cdot x_0)] \end{array}$$

is a well-defined isomorphism of groups.



[136] Recall that we say that G acts discretely, if for each $x \in X$ there exists an open neighborhood U such that $U \cap g \cdot U = \varnothing$ for all $g \neq e$.

[137] Statement (2) is a generalization of Statement (1). For many, but not all, applications Statement (1) is good enough.

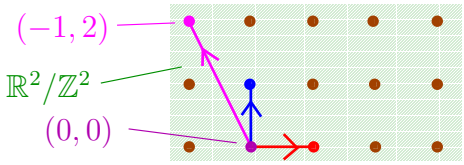
Before we provide the proof of the Action–π_1-Theorem 15.1 it is perhaps motivating to consider a few basic examples. In the following examples we will see that the Action–π_1-Theorem 15.1 can be used to recover some of our calculations, namely we can use it to calculate the fundamental groups of tori and the Möbius band. After the proof we will give more interesting applications.

Example.

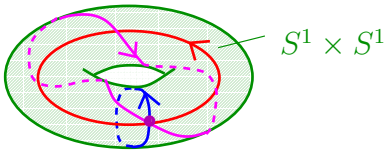
(1) Let $n \in \mathbb{N}$. It follows from the discussion on page 184 that $\mathbb{R}^n$ is simply connected. We saw on page 210 that the usual continuous action of $(\mathbb{Z}^n, +)$ on $\mathbb{R}^n$ is discrete. It follows from the Action–π_1-Theorem 15.1 (1) that the map

$$\begin{array}{rcl} \mathbb{Z}^n & \to & \pi_1(\mathbb{R}^n/\mathbb{Z}^n, [(0)]) \\ v & \mapsto & \left[\begin{array}{rcl} [0,1] & \to & \mathbb{R}^n/\mathbb{Z}^n \\ t & \mapsto & [t \cdot v] \end{array}\right] \end{array}$$

is an isomorphism. Using the homeomorphism $(S^1)^n \to \mathbb{R}^n/\mathbb{Z}^n$ from the Torus-as-Quotient Lemma 6.6 this also gives a new calculation of the fundamental group of $(S^1)^n$. In particular we obtain a new proof of the Circle-π_1-Theorem 11.13 and the Torus-π_1-Corollary 13.8.



which corresponds to



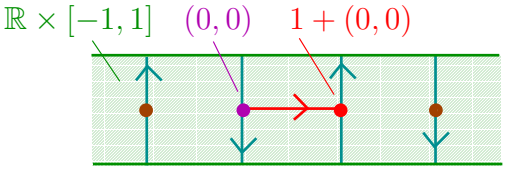
(2) We consider $X = \mathbb{R} \times [-1, 1]$ with the continuous $\mathbb{Z}$-action

$$\begin{array}{rcl} \mathbb{Z} \times (\mathbb{R} \times [-1,1]) & \to & \mathbb{R} \times [-1,1] \\ (n, (x, y)) & \mapsto & (x + n, (-1)^n \cdot y). \end{array}$$

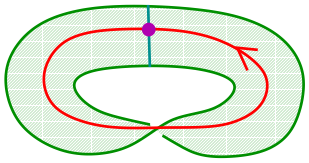
On page 210 we saw that this action is discrete. Note that on page 184 we saw that $\mathbb{R} \times [-1, 1]$ is simply connected. It follows from the Action–π_1-Theorem 15.1 (1) that the map

$$\begin{array}{rcl} \mathbb{Z} & \to & \pi_1((\mathbb{R} \times [-1,1])/\mathbb{Z}, ([(0,0)]) \\ n & \mapsto & \left[\begin{array}{rcl} [0,1] & \to & (\mathbb{R} \times [-1,1])/\mathbb{Z} \\ t & \mapsto & [(n \cdot t, 0)] \end{array}\right] \end{array}$$

is an isomorphism. Recall that in the Möbius Band-as-Quotient Lemma 6.8 we saw that $(\mathbb{R} \times [-1, 1])/\mathbb{Z}$ is homeomorphic to the Möbius band.



which corresponds to



Thus we obtain a new proof of the Möbius Band-π_1-Corollary 13.6. □

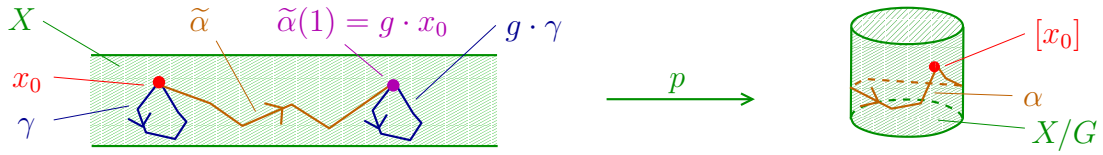
Now we finally provide the proof of the Action–π_1-Theorem 15.1

Proof of the Action–π_1-Theorem 15.1. Note that Statement (1) is just a user-friendly special case of Statement (2). Thus it suffices to prove Statement (2). Let X be a path-connected topological space and let G be a group that acts continuously and discretely

on X. Note that it follows from the Action–Covering Proposition 14.5 that the natural projection $p\colon X \to X/G$ is a covering. Finally let $x_0 \in X$. We start out with the following claim:

Claim 1. The image $p_*(\pi_1(X, x_0)) \subset \pi_1(X/G, [x_0])$ is a normal subgroup.

Proof. Let $[\gamma] \in \pi_1(X, x_0)$ and let $[\alpha] \in \pi_1(X/G, [x_0])$. We denote by $\widetilde{\alpha}\colon [0,1] \to X$ the lift of α with the starting point x_0. Note that $p(\widetilde{\alpha}(1)) = \alpha(1) = [x_0]$. This shows that there exists a $g \in G$ with $\widetilde{\alpha}(1) = g \cdot x_0$. We denote by $g \cdot \gamma$ the loop in X that is given by $t \mapsto g \cdot \gamma(t)$. Note that we can form the loop $\widetilde{\alpha} * g \cdot \gamma * \overline{\widetilde{\alpha}}$ since the endpoints always match the starting point.



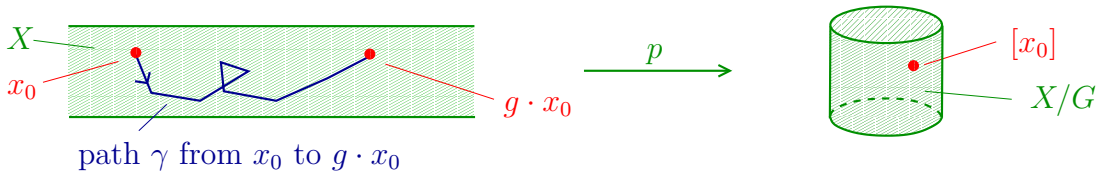
Now we see that

$$\begin{aligned}
[\alpha] \cdot p_*([\gamma]) \cdot [\alpha]^{-1} &= [p \circ \widetilde{\alpha}] \cdot [p \circ \gamma] \cdot [p \circ \overline{\widetilde{\alpha}}] \quad \overset{\text{since } p(g\cdot x) = p(x) \in X/G \text{ for all } x \in X}{=} \quad [p \circ \widetilde{\alpha}] \cdot [p \circ (g \cdot \gamma)] \cdot [p \circ \overline{\widetilde{\alpha}}] \\
&= [p \circ (\underbrace{\widetilde{\alpha} * g \cdot \gamma * \overline{\widetilde{\alpha}}}_{\substack{\text{defined since the endpoints} \\ \text{always match the starting points}}})] \qquad = p_*([\underbrace{\widetilde{\alpha} * g \cdot \gamma * \overline{\widetilde{\alpha}}}_{\text{loop in } (X, x_0)}]) \in p_*(\pi_1(X, x_0)).
\end{aligned}$$

Thus we have shown that $p_*(\pi_1(X, x_0))$ is a normal subgroup of $\pi_1(X/G, [x_0])$. ⊞

Similar to the Covering–Bijection Lemma 14.14 we consider the map

$$\begin{aligned}
\Phi\colon G &\to \pi_1(X/G, [x_0])/p_*(\pi_1(X, x_0)) \\
g &\mapsto [p \circ (\text{path in } X \text{ from } x_0 \text{ to } g \cdot x_0)].
\end{aligned}$$



Claim 2. The map Φ is well-defined and it is a bijection.

Proof. We consider the maps

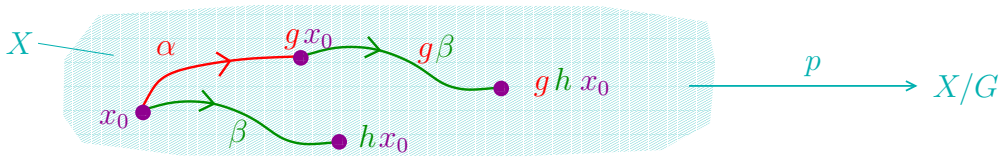
$$\begin{array}{ccccc}
G & \to & p^{-1}([x_0]) & \to & \pi_1(X/G, [x_0])/p_*(\pi_1(X, x_0)) \\
g & \mapsto & g \cdot x_0 & & \\
 & & y_0 & \mapsto & [p \circ (\text{path in } X \text{ from } x_0 \text{ to } y_0)].
\end{array}$$

The map to the left is clearly a bijection. The claim now follows from the observation that we showed in the Covering–Bijection Lemma 14.14 that the map to the right is well-defined and that it is a bijection. ⊞

It remains to prove the following claim:

Claim 3. The map $\Phi\colon G \to \pi_1(X/G, [x_0])/p_*(\pi_1(X, x_0))$ is a group homomorphism.

Proof. Let $g, h \in G$. We pick a path α from x_0 to $g \cdot x_0$ and we pick a path β from x_0 to $h \cdot x_0$. Then $g \cdot \beta$[138] is a path from $g \cdot x_0$ to $(g \cdot (h \cdot x_0)) = gh \cdot x_0$ and $\alpha * g \cdot \beta$ is a path from x_0 to $gh \cdot x_0$.



It follows that

$$\Phi(g \cdot h) = [p \circ (\alpha * g \cdot \beta)] \overset{\text{since } p \circ (\gamma * \delta) = p \circ \gamma * p \circ \delta}{=} [(p \circ \alpha) * (p \circ g \cdot \beta)] \overset{\text{since } p(g \cdot x) = p(x) \in X/G \text{ for all } x \in X}{=} [(p \circ \alpha) * (p \circ \beta)]$$
$$= [p \circ \alpha] \cdot [p \circ \beta] = \Phi(g) \cdot \Phi(h).$$

(since $\alpha * g \cdot \beta$ is a path from x_0 to $gh \cdot x_0$; definition of $\cdot$ in $\pi_1(X/G, [x_0])$) ■

15.2. **Applications of the Action–π_1-Theorem 15.1.** Using the Action–π_1-Theorem 15.1 we can now determine the fundamental groups of many of the topological spaces that we have encountered so far.

Proposition 15.2. (Lens Space-π_1-Proposition)

(1) Let $p \in \mathbb{Z} \setminus \{0\}$ and let $q \in \mathbb{Z}$ with $\gcd(p, q) = 1$. If $L(p, q) = S^3/\mathbb{Z}_p$ is a lens space as defined on page 212, then
$$\pi_1(L(p, q)) \cong \mathbb{Z}_p.$$

(2) Let $p, \widetilde{p} \in \mathbb{Z} \setminus \{0\}$ and let $q, \widetilde{q} \in \mathbb{Z}$ with $\gcd(p, q) = 1$ and $\gcd(\widetilde{p}, \widetilde{q}) = 1$. If $L(p, q)$ is homotopy equivalent (e.g. homeomorphic) to $L(\widetilde{p}, \widetilde{q})$, then $p = \widetilde{p}$.

Proof.

(1) First we note that by the Sphere-π_1-Proposition 11.15 we know that S^3 is simply connected. Next note that in the Lens Space-Discrete Action Lemma 14.2 we showed that the action of $\mathbb{Z}_p$ on S^3 is discrete. Thus the statement follows from the Action–π_1-Theorem 15.1 (1).

(2) This statement follows from (1) and the Homotopy-π_1-Proposition 13.4 (2a). ■

We move on to the only slightly less bewildering real projective spaces. The following proposition can be viewed as a refinement of the argument on page 226.

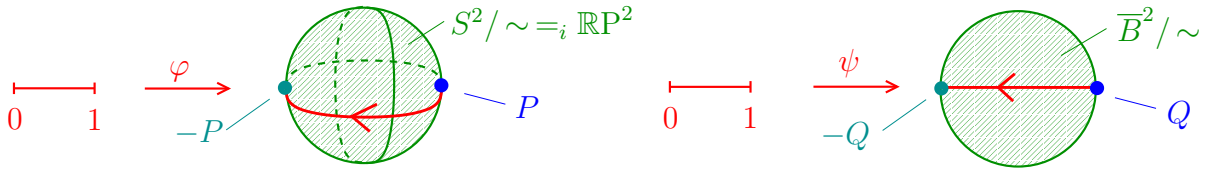
Proposition 15.3. (Real Projective Space–π_1-Proposition) Let $n \in \mathbb{N}_{\geq 2}$.[139]

(1) We consider the real projective space $\mathbb{R}\mathrm{P}^n =_{\mathrm{i}} S^n/\{\pm 1\}$.
   (a) $\pi_1(\text{real projective space } \mathbb{R}\mathrm{P}^n) =_{\mathrm{i}} \pi_1(S^n/\{\pm 1\}) \cong \mathbb{Z}_2$.
   (b) We set $P := (1, 0, \ldots, 0) \in S^n$. The unique non-trivial element of the fundamental group $\pi_1(\mathbb{R}\mathrm{P}^n, [P]) =_{\mathrm{i}} \pi_1(S^n/\{\pm 1\}, [P])$ is represented by the loop
$$\begin{aligned} \varphi\colon [0, 1] &\to \mathbb{R}\mathrm{P}^n =_{\mathrm{i}} S^n/\{\pm 1\} \\ t &\mapsto [(\cos(\pi t), \sin(\pi t), 0, \ldots, 0)]. \end{aligned}$$

[138] Here $g \cdot \beta$ denotes the path $[0, 1] \to X$ that is given by $t \mapsto g \cdot \beta(t)$.

(2) Let "$\sim$" be the equivalence relation on $\overline{B}^n$ that is generated by $x \sim -x$ for $x \in S^{n-1}$.
   (a) $\pi_1(\overline{B}^n/\sim) \cong \mathbb{Z}_2$.
   (b) With $Q = (1, 0, \dots, 0) \in \overline{B}^n$ the unique non-trivial element of $\pi_1(\overline{B}^n/\sim, [Q])$ is represented by the loop

$$\begin{array}{rcl} \psi\colon [0,1] & \to & \overline{B}^n/\sim \\ t & \mapsto & [2t-1, 0, \dots, 0)]. \end{array}$$



Proof.

(1) These two statements follows almost immediately from the following observations:
   (i) We consider the usual continuous action of $\{\pm 1\}$ on S^n. On page 210 we saw that the action of $\{\pm 1\}$ on S^n is discrete. We denote by $p\colon S^n \to S^n/\{\pm 1\}$ the projection.
   (ii) Since $n \geq 2$ we obtain from the Sphere-π_1-Proposition 11.15 that the sphere S^n is simply connected.
   (iii) Note that it follows from (i) and (ii) together with the Action–π_1-Theorem 15.1 (1) that the map

$$\begin{array}{rcl} \{\pm 1\} & \to & \pi_1(S^n/\sim) \\ \epsilon & \mapsto & [p \circ (\text{path in } S^n \text{ from } P \text{ to } \epsilon \cdot P)] \end{array}$$

   is an isomorphism. Since $t \mapsto (\cos(\pi t), \sin(\pi t), 0, \dots, 0)$ is precisely a path from P to $-P$ we see that φ represents the non-trivial element of $\pi_1(S^n/\{\pm 1\}, [P])$.
   (iv) Recall that in the Projective Space-Sphere Quotient Lemma 6.9 we saw that the inclusion $S^n \to \mathbb{R}^{n+1} \setminus \{0\}$ induces a homeomorphism $S^n/\{\pm 1\} \to \mathbb{RP}^n$.

(2) In the Projective Space-Avatar Lemma 6.11 (2) we showed that the map

$$\begin{array}{rcl} f\colon \mathbb{RP}^n =_{\mathrm{i}} S^n/\{\pm 1\} & \xrightarrow{\cong} & \overline{B}^n/\sim \\ [\underbrace{(v, x_{n+1})}_{v \in \mathbb{R}^n,\, x_{n+1} \in \mathbb{R}}] & \mapsto & \begin{cases} [v], & \text{if } x_{n+1} \geq 0, \\ [-v], & \text{if } x_{n+1} \leq 0 \end{cases} \end{array}$$

is a homeomorphism with $f([P]) = [Q]$. It follows from this observation together with the results in (1) that $\pi_1(\overline{B}^n/\sim) \cong \mathbb{Z}_2$ and that the unique non-trivial of $\pi_1(\overline{B}^n/\sim, [Q])$ is presented by the loop $f \circ \varphi$, i.e. by the loop $t \mapsto [(\cos(\pi t), \sin(\pi t), 0, \dots, 0)]$. It is straightforward to see that this loop is path-homotopic to the loop ψ. ■

Proposition 15.4. (SO(3)–π_1**-Proposition**)
(1) $\pi_1(\mathrm{SO}(3), \mathrm{id}) \cong \mathbb{Z}_2$.

[139] Recall that in the Projective Space-Avatar Lemma 6.11 (or alternatively on page 137) we showed that $\mathbb{RP}^1$ is homeomorphic to S^1. Thus it follows from the Circle-π_1-Theorem 11.13 that $\pi_1(\mathbb{RP}^1) \cong \mathbb{Z}$.

(2) The unique non-trivial element is represented by the loop

$$\begin{array}{rcl} \gamma\colon [0,1] & \to & \mathrm{SO}(3) \\ t & \mapsto & \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(2\pi t) & \sin(2\pi t) \\ 0 & -\sin(2\pi t) & \cos(2\pi t) \end{pmatrix}}_{\text{rotation around the } x\text{-axis}} \end{array}$$

at the base point $\mathrm{id} \in \mathrm{SO}(3)$.

Proof.

(1) (a) Let $\sim$ be the equivalence relation on $\overline{B}^3$ that is generated by $x \sim -x$ for $x \in S^2$.

(b) We set $Q := (1,0,0) \in \overline{B}^3$.

(c) In the Projective Space-Avatar Lemma 6.11 (3) we saw that the map

$$\begin{array}{rcl} f\colon \overline{B}^3/\!\sim & \to & \mathrm{SO}(3) \\ [\underbrace{r \cdot v}_{r \in [0,1] \text{ and } v \in S^2}] & \mapsto & \text{matrix in SO(3) which represents rotation by the angle } \pi \cdot r \\ & & \text{around the axis } \mathbb{R}_{\geq 0} \cdot v \text{ where the rotation} \\ & & \text{direction is given by the right-hand rule} \end{array}$$

is a homeomorphism. Note that $f(0) = \mathrm{id}$.

(d) In the Real Projective Space–π_1-Proposition 15.3 (2) we showed that $\pi_1(\overline{B}^3/\sim, [Q]) \cong \mathbb{Z}_2$.

(e) It follows from (c) and (d) that $\pi_1(\mathrm{SO}(3), f([Q])) \cong \mathbb{Z}_2$.

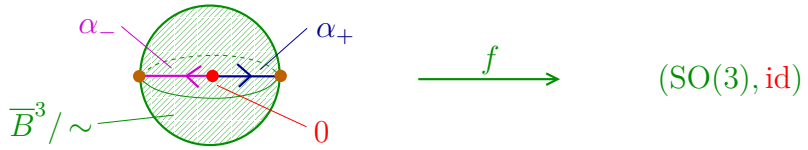
(2) We still need to show that γ represents the unique non-trivial element of $\pi_1(\mathrm{SO}(3), \mathrm{id})$.

(f) Given $\epsilon \in \{-1, 1\}$ let $\alpha_\epsilon\colon [0,1] \to \overline{B}^3$ be the path that is given by $t \mapsto [t \cdot Q \cdot \epsilon]$.

(g) In the Real Projective Space–π_1-Proposition 15.3 (2) we showed that the unique non-trivial element of $\pi_1(\overline{B}^3/\sim, [Q])$ is represented by the loop $\overline{\alpha}_+ * \alpha_-$.

(h) Note that α_+ is a path from 0 to Q. It follows from (d) and the Change-of-Base Point Proposition 11.6 that the unique non-trivial element of $\pi_1(\overline{B}^3/\sim, 0)$ is represented by $\alpha_+ * (\overline{\alpha}_+ * \alpha_-) * \overline{\alpha}_+ \simeq_{\mathrm{p}} \alpha_- * \overline{\alpha}_+$.

(i) It follows from (c) and (h) and the fact that $f(0) = \mathrm{id}$ that the unique non-trivial element of $\pi_1(\mathrm{SO}(3), \mathrm{id})$ is represented by $\varphi := f \circ (\overline{\alpha}_+ * \alpha_-)$.



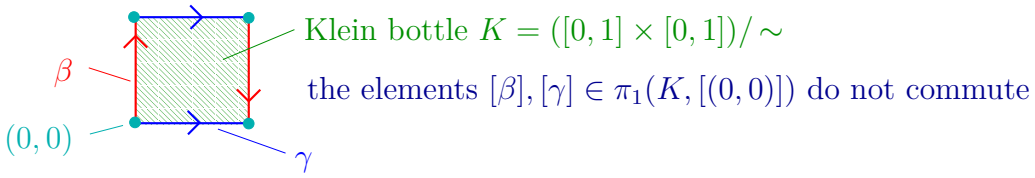
(j) The loop that we gave in the statement of the proposition is precisely $\overline{\varphi}$. But if φ represents a non-trivial element of $\pi_1(\mathrm{SO}(3), \mathrm{id})$, then so does $\overline{\varphi}$. ■

Proposition 15.5. (Klein Bottle–π_1-Proposition) As in the Klein Bottle-as-Quotient Lemma 14.1 we consider the group G that is given by all self-homeomorphisms of $\mathbb{R}^2$ that can be written as concatenations of the two self-homeomorphisms

$$\begin{array}{rcl} A\colon \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (x,y) & \mapsto & (x+1, 1-y) \end{array} \quad \text{and} \quad \begin{array}{rcl} B\colon \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (x,y) & \mapsto & (x, y+1) \end{array}$$

and their inverses.

(1) The fundamental group of the Klein bottle $K = ([0,1] \times [0,1])/\sim$ is isomorphic to G.
(2) The fundamental group of the Klein bottle K is non-abelian, in particular the two elements $[\beta]$ and $[\gamma]$ shown in the figure below do not commute.



Proof.

(1) In the Klein Bottle-as-Quotient Lemma 14.1 we showed the following two facts:
  (a) The group G acts discretely on $\mathbb{R}^2$.
  (b) The inclusion-induced map $f\colon ([0,1] \times [0,1])/\sim \to \mathbb{R}^2/G$ is a homeomorphism.
  It follows from (a) together with the Action–π_1-Theorem 15.1 (1) that the map

$$\begin{array}{rcl} \Theta\colon G & \to & \pi_1(\mathbb{R}^2/G, [(0,0)]) \\ g & \mapsto & \left[\begin{array}{rcl} [0,1] & \to & \mathbb{R}^2/G \\ t & \mapsto & [t \cdot \underbrace{g(0,0)}_{\in \mathbb{R}^2}] \end{array}\right] \end{array}$$

  is an isomorphism. If we combine this calculation with (b) we see that the fundamental group of the Klein bottle is isomorphic to G.
(2) (a) We set $C := B^{-1} \cdot A$. An easy calculation shows that the two homeomorphisms $B(x,y) = (x, y+1)$ and $C(x,y) = (x+1, -y)$ do not commute.
  (b) It follows from (a) that $(f_*)^{-1}(\Theta(B)), (f_*)^{-1}(\Theta(C)) \in \pi_1(K, [(0,0)])$ do not commute.
  (c) We consider the two loops β, γ in $(K, [(0,0)])$ that are shown in the above figure. Note that it follows easily from the definitions that $(f_*)^{-1}(\Theta(B)) = [\beta]$ and that $(f_*)^{-1}(\Theta(C)) = [\gamma]$.
  (d) It follows from (b) and (c) that $[\beta]$ and $[\gamma]$ do not commute. ∎

15.3. **The fundamental group of mapping tori.** We start out with the following definition which we will use on several occasions:

Definition. Let X be a topological space and let $f\colon X \to X$ be a homeomorphism. We refer to

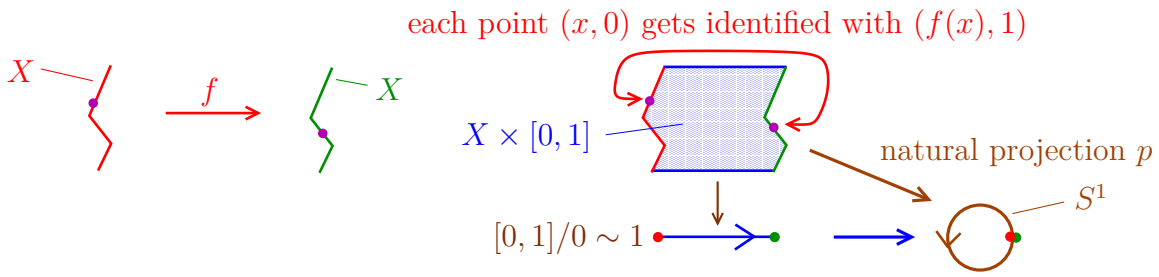
$$\mathrm{Tor}(X,f) := (X \times [0,1])/\sim \qquad \text{where } (x,0) \sim (f(x),1) \text{ for all } x \in X,$$

as the mapping torus of (X,f). We introduce two maps:

$$\underbrace{\begin{array}{rcl} X & \to & \mathrm{Tor}(X,f) \\ x & \mapsto & [(x,0)] \end{array}}_{\text{the natural injection}} \quad \text{and} \quad \underbrace{\begin{array}{rcl} \mathrm{Tor}(X,f) & \to & S^1 \\ [(x,t)] & \mapsto & \exp(2\pi i t). \end{array}}_{\text{the natural projection}}$$

Examples.

[139] It follows from the Topological-Quotient Proposition 6.1 (1b) that the natural projection is indeed continuous.

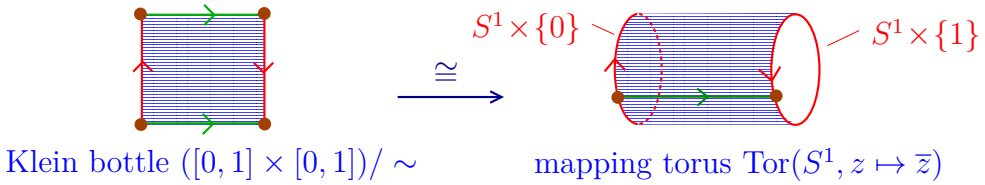


(1) Let X be a topological space. In [**Fri24**] it is shown that the map

$$\begin{array}{rccl} \Phi\colon & \mathrm{Tor}(X,\mathrm{id}) & \to & X \times S^1 \\ & [(x,t)] & \mapsto & (x, \exp(2\pi\,\mathrm{i}t)) \end{array}$$

is a homeomorphism.

(2) We consider the homeomorphism $f\colon S^1 \to S^1$ that is given by $z \mapsto \overline{z}$. We also consider the Klein bottle $([0,1]\times[0,1])/\sim$, as defined on page 131.



It is not difficult to show that the map

$$\begin{array}{rccl} & ([0,1]\times[0,1])/\sim & \to & \mathrm{Tor}(S^1, z \mapsto \overline{z}) \\ & [(x,y)] & \mapsto & [(\exp(2\pi\,\mathrm{i}x), y)] \end{array}$$

is a homeomorphism. □

Our goal in this section is to determine the fundamental groups of mapping tori. Before we can do so we need to introduce a new concept from group theory.

Definition. Let N be a group, let $k \in \mathbb{N}_0$ and let $\varphi\colon N \to N$ be an isomorphism such that $\varphi^k = \mathrm{id}$.[140] We define the semidirect product of N and $\mathbb{Z}_k$ with respect to φ as the group[141] $N \rtimes_\varphi \mathbb{Z}_k$ where the underlying set is given by the direct product $N \times \mathbb{Z}_k$,[142] but where the group multiplication[143] is given by

$$(h,m)\cdot(h',m') := (h\cdot\varphi^m(h'), m+m').$$

If φ is understood from the context, then we drop it from the notation, i.e. we write $N \rtimes \mathbb{Z}_k$ instead of $N \rtimes_\varphi \mathbb{Z}_k$.

Examples.

(1) If N is a group and if $\varphi = \mathrm{id}_N$, then $N \rtimes_{\mathrm{id}} \mathbb{Z} = N \times \mathbb{Z}$ is the direct product of N and $\mathbb{Z}$.

[140] Note that this condition is automatically satisfied if $k = 0$, i.e. for any isomorphism φ of any group we always have $\varphi^0 = \mathrm{id}$.

[141] One can easily show that $N = \{(h,0) \mid h \in N\} \subset N \rtimes \mathbb{Z}_k$ is a normal subgroup of $N \rtimes \mathbb{Z}_k$. Put differently, we have $N \lhd N \rtimes \mathbb{Z}_k$. This is the reason for the symbol "$\rtimes$" in the notation of the semidirect product.

[142] In particular, if N is finite and $k \in \mathbb{N}$, then the cardinality of $N \rtimes \mathbb{Z}_k$ is given by $|N| \cdot k$.

[143] It is an elementary argument to show that $N \rtimes_\varphi \mathbb{Z}_k$ with the given multiplication is indeed a group.

(2) Let $N = \mathbb{Z}_n$ for some $n \in \mathbb{N}_0$. We denote by $\varphi\colon \mathbb{Z}_n \to \mathbb{Z}_n$ the isomorphism that is given by multiplication by -1. Clearly this isomorphism satisfies $\varphi^2 = \mathrm{id}$, thus we can form the semidirect product $\mathbb{Z}_n \rtimes_\varphi \mathbb{Z}_2$, which is called the n-th dihedral group D_n. It has $2n$ elements. As an example, the third dihedral group D_3 is isomorphic to the permutation group S_3 via the isomorphism [144]

$$\begin{array}{rccl} \Psi\colon \mathbb{Z}_3 \rtimes \mathbb{Z}_2 & \to & S_3 \\ (h, m) & \mapsto & (1\,2\,3)^h \cdot (1\,2)^m. \end{array}$$

□

The following lemma shows that many groups can be written as a semidirect product $N \rtimes \mathbb{Z}$.

Lemma 15.6. (Semidirect Product Lemma) Let G be a group, let $\xi\colon G \to \mathbb{Z}$ be an epimorphism and let $t \in G$ with $\xi(t) = 1$. We write $N \coloneqq \ker(\xi)$ and we denote by φ the isomorphism

$$\begin{array}{rccl} \varphi\colon N & \to & N \\ h & \mapsto & t \cdot h \cdot t^{-1}. \end{array}$$

Then the map[145]

$$\begin{array}{rccl} \Theta\colon G & \to & N \rtimes_\varphi \mathbb{Z} \\ g & \mapsto & \big(g \cdot t^{-\xi(g)}, \xi(g)\big) \end{array}$$

is an isomorphism of groups.

Proof. It is straightforward to verify that Θ is injective and surjective. It remains to show that Θ is a homomorphism: Given $g, h \in G$ we have indeed

$$\begin{aligned} \Theta(g\cdot h) &= (g\cdot h\cdot t^{-\xi(g\cdot h)}, \xi(g\cdot h)) = (g\cdot t^{-\xi(g)}\cdot \underbrace{t^{\xi(g)}\cdot h\cdot t^{-\xi(h)}}_{=\varphi^{\xi(g)}(h)\,t^{-\xi(h)}}\cdot t^{-\xi(g)}, \xi(g)+\xi(h)) \\ &= (g\cdot t^{-\xi(g)}, \xi(g))\cdot(h\cdot t^{-\xi(h)}, \xi(h)) = \Theta(g)\cdot\Theta(h). \end{aligned}$$

■

The following proposition shows that the fundamental group of a mapping torus $\mathrm{Tor}(X, f)$ is isomorphic to some semidirect product $\pi_1(X) \rtimes \mathbb{Z}$.

Proposition 15.7. (Mapping Torus-π_1-Proposition) Let (X, x_0) be a pointed path-connected topological space, let $f\colon X \to X$ be a homeomorphism and let γ be a path from x_0 to $f(x_0)$. We denote by $\gamma_*\colon \pi_1(X, f(x_0)) \to \pi_1(X, x_0)$ the isomorphism given by the Change-of-Base Point Proposition 11.6 and we define

$$\varphi \coloneqq \gamma_* \circ f_*\colon \pi_1(X, x_0) \xrightarrow[\cong]{f_*} \pi_1(X, f(x_0)) \xrightarrow[\cong]{\gamma_*} \pi_1(X, x_0).$$

Let $i\colon X \to \mathrm{Tor}(X, f)$ be the natural inclusion and let $p\colon \mathrm{Tor}(X, f) \to S^1$ be the natural projection. There exists a natural isomorphism

$$\Theta\colon \pi_1(\mathrm{Tor}(X, f), [(x_0, 0)]) \xrightarrow{\cong} \pi_1(X, x_0) \rtimes_\varphi \mathbb{Z}$$

---

[144] First note that a straightforward calculation shows that $(1\,2)^{-1}\cdot(1\,2\,3)\cdot(1\,2) = (1\,3\,2) = (1\,2\,3)^{-1}$. This implies that also for any $m \in \mathbb{Z}_2$ and $s \in \mathbb{Z}$ we have $(1\,2)^{-m}\cdot(1\,2\,3)^s\cdot(1\,2)^m = (1\,2\,3)^{s(-1)^m}$. Now for any $(h, m), (h', m') \in \mathbb{Z}_n \rtimes_\varphi \mathbb{Z}_2$ we have

$$\begin{aligned} \Psi(h,m)\Psi(h',m') &= (1\,2\,3)^h\cdot(1\,2)^m\cdot(1\,2\,3)^{h'}\cdot(1\,2)^{m'} = (1\,2\,3)^h\cdot(1\,2)^m\cdot(1\,2)^{-m}\cdot(1\,2\,3)^{(-1)^m h'}\cdot(1\,2)^m\cdot(1\,2)^{m'} \\ &= (1\,2\,3)^{h+(-1)^m h'}\cdot(1\,2)^{m+m'} = \Psi(h+(-1)^m h', m+m') = \Psi((h,m)\cdot(h',m')). \end{aligned}$$

We have thus shown that Ψ is a homomorphism. It is now straightforward to verify that Ψ is in fact an isomorphism.

[145] Note that $\xi(g\cdot t^{-\xi(g)}) = \xi(g) + \xi(t)(-\xi(g)) = 0$, i.e. $g\cdot t^{-\xi(g)}$ does indeed lie in $N = \ker(\xi)$.

such that the following diagram commutes:[146]

$$\begin{array}{ccccc} \pi_1(X,x_0) & \xrightarrow{i_*} & \pi_1(\mathrm{Tor}(X,f),[(x_0,0)]) & \xrightarrow{p_*} & \pi_1(S^1,1) \\ \downarrow{\scriptstyle \mathrm{id}} & & \cong\downarrow\Theta & & \downarrow\cong \\ \pi_1(X,x_0) & \xrightarrow{g\mapsto(g,0)} & \pi_1(X,x_0)\rtimes_\varphi\mathbb{Z} & \xrightarrow{(g,n)\mapsto n} & \mathbb{Z}. \end{array}$$

(Here the vertical isomorphism to the right is the standard isomorphism from the Circle-π_1-Theorem 11.13.)

Proof. Let X be a topological space and let $f\colon X\to X$ be a homeomorphism. One can easily show, that the following two statements hold:

(1) The map
$$\begin{array}{rcl} \mathbb{Z}\times(X\times\mathbb{R}) & \to & X\times\mathbb{R} \\ (n,(x,t)) & \mapsto & (f^n(x),t+n) \end{array}$$
defines a continuous, proper and discrete action of the group $\mathbb{Z}$ on $X\times\mathbb{R}$.

(2) The map
$$\begin{array}{rccl} \Psi\colon & \mathrm{Tor}(X,f) & \to & (X\times\mathbb{R})/\mathbb{Z} \\ & [(x,t)] & \mapsto & [(x,t)] \end{array}$$
is a homeomorphism.

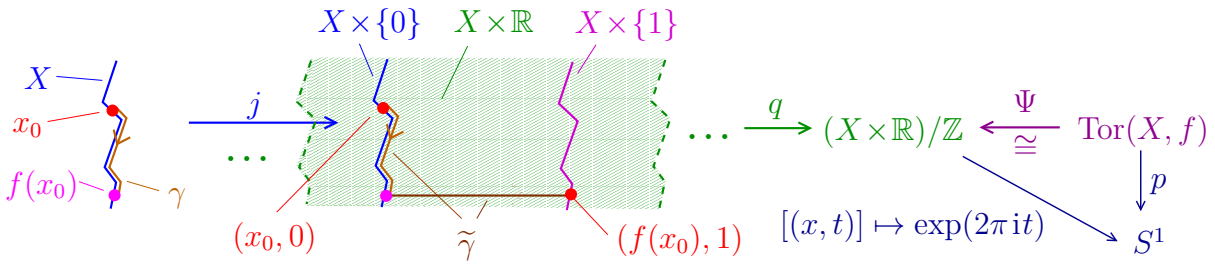
We introduce the following notation:

(a) We denote by $q\colon X\times\mathbb{R}\to(X\times\mathbb{R})/\mathbb{Z}$ the natural projection.

(b) We denote by $j\colon X\to X\times\mathbb{R}$ the inclusion that is given by $x\mapsto(x,0)$. Using the fact that $\{0\}$ is a deformation retract of $\mathbb{R}$ and using the Deformation Retract–Homotopy Equivalence Lemma 9.3 one can easily show that $j\colon X\to X\times\mathbb{R}$ is a homotopy equivalence.

(c) We denote by $\widetilde{\gamma}\colon[0,1]\to X\times\mathbb{R}$ the path from $(x_0,0)$ to $(f(x_0),1))$ that is given by the concatenation of the path $i\circ\gamma$ and the "horizontal" path $t\mapsto(x_0,t)$.

It follows from (1) and the Action–π_1-Theorem 15.1 (2) that the map
$$\begin{array}{rcl} \mathbb{Z} & \to & \pi_1((X\times\mathbb{R})/\mathbb{Z},[(x_0,0)])/q_*(\pi_1(X\times\mathbb{R},(x_0,0))) \\ n & \mapsto & [(q\circ\widetilde{\gamma})^n] \end{array}$$
is an isomorphism. This map gives rise to an epimorphism $\xi\colon\pi_1((X\times\mathbb{R})/\mathbb{Z},[(x_0,0)])\to\mathbb{Z}$ with kernel $q_*(\pi_1(X\times\mathbb{R},(x_0,0)))$. Note that $\xi([q\circ\widetilde{\gamma}])=1$. From the Semidirect Product Lemma 15.6 we obtain an explicit isomorphism
$$\Theta\colon\pi_1((X\times\mathbb{R})/\mathbb{Z},[(x_0,0)]) \to q_*(\pi_1(X\times\mathbb{R},(x_0,0)))\rtimes\mathbb{Z}$$



[146]Note that both maps on the bottom are group homomorphisms.

In summary we get the following diagram:

$$\begin{array}{ccccc}
\pi_1(X,x_0) & \xrightarrow{i_*} & \pi_1(\mathrm{Tor}(X),f),[(x_0,0)]) & \xrightarrow{q_*} & \pi_1(S^1,1) \\
\cong\downarrow j_* & & \cong\downarrow \Psi_* & & \downarrow \mathrm{id} \\
\pi_1(X\times\mathbb{R},(x_0,0)) & \longrightarrow & \pi_1((X\times\mathbb{R})/\mathbb{Z},[(x_0,0)]) & \xrightarrow[[(x,t)]\mapsto \exp(2\pi\,\mathrm{i}t)]{\text{induced by}} & \pi_1(S^1,1) \\
\downarrow \mathrm{id} & & \cong\downarrow \Theta & & \uparrow n\mapsto[t\mapsto\exp(2\pi\,\mathrm{i}n)] \\
\pi_1(X\times\mathbb{R},(x_0,0)) & \xrightarrow{g\mapsto(g,0)} & q_*(\pi_1(X\times\mathbb{R},(x_0,0)))\rtimes\mathbb{Z} & \xrightarrow{(g,n)\mapsto n} & \mathbb{Z} \\
\cong\uparrow j_* & & \cong\uparrow \text{induced by } \psi_*{}^{147} & & \cong\downarrow \mathrm{id} \\
\pi_1(X,x_0) & \xrightarrow{g\mapsto(g,0)} & \pi_1(X,x_0)\rtimes_\varphi\mathbb{Z} & \xrightarrow{(g,n)\mapsto n} & \mathbb{Z}.
\end{array}$$

It follows fairly easily from the definitions that the diagram commutes.[148] Therefore the composition of the three middle vertical isomorphisms gives us the promised isomorphism $\pi_1(\mathrm{Tor}(X,f),[(x_0,0)]) \to \pi_1(X,x_0)\rtimes_\varphi\mathbb{Z}$. ■

We conclude this section with a discussion of several examples.

Example.

(1) Let $K = \mathrm{Tor}(S^1, f(z) \coloneqq \overline{z})$ be the Klein bottle. It follows from the discussion on page 199 that the automorphism $f_* =: \varphi$ of $\pi_1(S^1) = \mathbb{Z}$ is multiplication by -1. It follows from the Mapping Torus-π_1-Proposition 15.7 that

$$\pi_1(\text{Klein bottle}) \;\cong\; \mathbb{Z}\rtimes_\varphi\mathbb{Z}.$$

(2) Let X be a path-connected topological space. We calculate that

$$\pi_1(X\times S^1) \;\cong\; \pi_1(\mathrm{Tor}(X,\mathrm{id}_X)) \underset{\uparrow}{\cong} \pi_1(X)\rtimes_{\mathrm{id}}\mathbb{Z} = \pi_1(X)\times\mathbb{Z} = \pi_1(X)\times\pi_1(S^1).$$

by the Mapping Torus-π_1-Proposition 15.7

So in this special case we obtain the same result as in the Product-π_1-Proposition 13.7.

(3) Let $A \in \mathrm{GL}(n,\mathbb{Z})$. We leave it as an exercise to the reader to prove the following two statements:
- The map $f(A)\colon \mathbb{R}^n/\mathbb{Z}^n \to \mathbb{R}^n/\mathbb{Z}^n$ given by $[v]\mapsto[Av]$ is a homeomorphism.
- We use the Action–π_1-Theorem 15.1 to make the identification $\pi_1(\mathbb{R}^n/\mathbb{Z}^n,0) = \mathbb{Z}^n$. We will see that the induced map $f(A)_*$ on $\pi_1(\mathbb{R}^n/\mathbb{Z}^n,0) = \mathbb{Z}^n$ is given by multiplication by A.

It follows from these two facts together with the Mapping Torus-π_1-Proposition 15.7 that $\pi_1(\mathrm{Tor}(\mathbb{R}^n/\mathbb{Z}^n, f(A))) \cong \mathbb{Z}^n\rtimes_A\mathbb{Z}$. □

[147] Note that it follows from the Covering-π_1-Monomorphism Proposition 14.11 that the homomorphism $\psi_*\colon \pi_1(X,x_0)\to q_*(\pi_1(X\times\mathbb{R},(x_0,0)))$ is in fact an isomorphism.

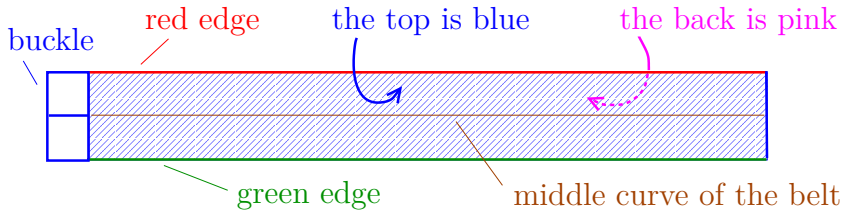
[148] Perhaps the most opaque statement is that the two semidirect products match up. But this also follows once one writes down the definitions in detail.

CHAPTER 16

Applied topology

16.1. **The belt trick.** In this section we will see that one can visualize the non-trivial element in $\pi_1(\mathrm{SO}(3)) \cong \mathbb{Z}_2$ using a belt. We proceed as follows.

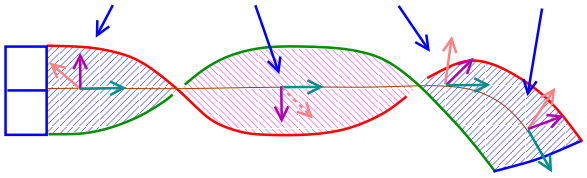
(1) First of all note that a matrix in SO(3) is precisely given by an ordered orthonormal basis of $\mathbb{R}^3$ that obeys the "right hand rule", i.e. by a choice of an ordered set of three orthogonal vectors v_1, v_2, v_3 of length 1 in $\mathbb{R}^3$ with $\det(v_1\ v_2\ v_3) = 1$.
(2) We equip SO(3) with the base point given by the identity matrix, i.e. by the standard basis $\{e_1, e_2, e_3\}$.
(3) Now suppose that we are given a belt with a buckle, a red edge and a green edge, with a blue top side and a pink lower side.



(4) Any point on a twisted belt defines a triple of orthogonal vectors:
  (a) The first vector lies in the plane defined by the belt and points away from the buckle.
  (b) The second vector also lies in the plane defined by the belt and points towards the red edge.
  (c) The third is orthogonal to the belt and it is attached to the top side of the belt.
(5) In the remainder of the discussion we fix the buckle in the xy-plane such that it points in the x-direction, so that to a point at the buckle we assign the identity matrix.
(6) We say that the end of the belt is in *standard position* if it lies in a plane parallel to the plane of the buckle and if it points into the x-direction. Put differently, the end of the belt is in standard position precisely if the matrix assigned to the endpoint is the identity matrix.
(7) When we walk on the middle curve of the belt from the buckle towards the other end of the belt, then we obtain a path $[0, 1] \to \mathrm{SO}(3)$. In particular, if the end of the belt is in standard position, then the path defined by the belt is in fact a loop in SO(3) at the base point id.

(8) When we hold the belt tightly without twisting it, then we assign the identity matrix to every point on the belt, i.e. the belt defines the constant loop $t \mapsto \mathrm{id} \in \mathrm{SO}(3)$.
(9) Now we give the belt, that is still fixed at the buckle, a full twist. As in (7) this defines a loop $\gamma\colon [0, 1] \to \mathrm{SO}(3)$, in fact it is straightforward to see that this loop is precisely

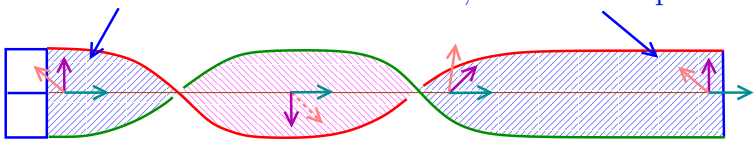
every point on a twisted belt defines a matrix in SO(3)



moving along the middle curve of the belt we get a path $[0,1] \to \mathrm{SO}(3)$

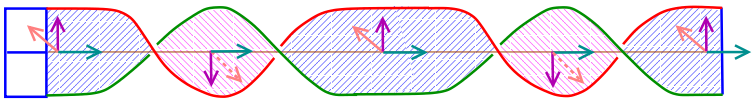
the loop that we considered in the SO(3)–π_1-Proposition 15.4 (2). In particular we have visualized the non-trivial element $[\gamma] \in \pi_1(\mathrm{SO}(3), \mathrm{id}) \in \mathbb{Z}_2$.

both points define the standard basis of $\mathbb{R}^3$, i.e. the base point id $\in \mathrm{SO}(3)$



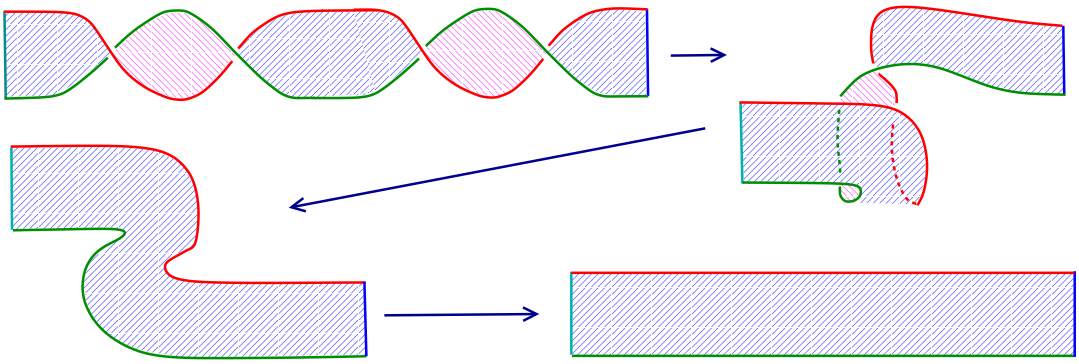
the path $\gamma\colon [0,1] \to \mathrm{SO}(3)$ defined by the matrices along the middle curve is a loop and thus defines an element $[\gamma] \in \pi_1(\mathrm{SO}(3), \mathrm{id})$

(10) The loop $\gamma * \gamma\colon [0,1] \to \mathrm{SO}(3)$ is given by performing another full twist along the belt. It follows from the SO(3)–π_1-Proposition 15.4 that $[\gamma * \gamma] = [\gamma]^2$ is trivial in $\pi_1(\mathrm{SO}(3), \mathrm{id}) \cong \mathbb{Z}_2$, i.e. $\gamma * \gamma$ is null-homotopic.



the double full twist defines the loop $\gamma * \gamma$ which represents $[\gamma]^2 \in \pi_1(\mathrm{SO}(3), \mathrm{id})$

(10) One can see that the loop $\gamma * \gamma$ is null-homotopic as follows: one can untwist the belt while keeping the end in standard position, put differently, we can find a path-homotopy from the loop $\gamma * \gamma$ to the trivial loop. This homotopy is illustrated in the figure below.



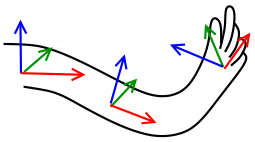
16.2. **The plate trick.** The plate trick is a variation on the belt trick. It describes the same phenomenon as the belt trick. The plate trick has the great advantage that the

only tool it requires is the body of a volunteer. The disadvantage is that it is perhaps mathematically slightly less compelling.

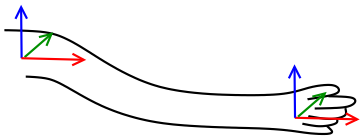
The basic idea is as in the belt trick. Namely we want to describe paths and loops in SO(3). This time we think of a path from the shoulder to ones left hand as a path in SO(3):

(a) the first vector lies on the arm and it points from the shoulder to the hand,
(b) the second vector also lies on the arm and it points from one side to the other,
(c) the third vector points away from the body.

If the fingers of the left hands are parallel to the shoulder, then we see a loop in SO(3).



the path from the shoulder
to the hand defines a path in SO(3)

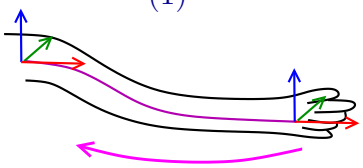


hand parallel to the shoulder
defines a loop in SO(3)

Now you, or your volunteer, does the following:

(1) You stretch your left arm and hands outward such that the fingers are parallel to the shoulder.
(2) You rotate the left arm counterclockwise. The physiology of most arms will force the forearm to go under the arm.
(3) One continues till one does a 2π turn and the left arm is outstretched again. At this time we are describing a loop β in SO(3) which is non-constant since the arm is twisted. (If it doesn't hurt slightly, then the experiment is done incorrectly).
(4) Now one rotates the left arm still counterclockwise. The usual physiology will force the movement of the forearm to go over the arm.
(5) Again one continues till one has done another 2π turn. Up to a homotopy, keeping the end points fixed, the arm now describes $\beta * \beta$. But the arm turns out to be untwisted (which feels much better). This illustrates that β represents an element of order 2 in $\pi_1(\mathrm{SO}(3))$.

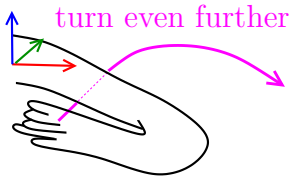
(1)



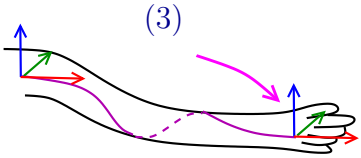
turn arm this direction

(2)

turn even further



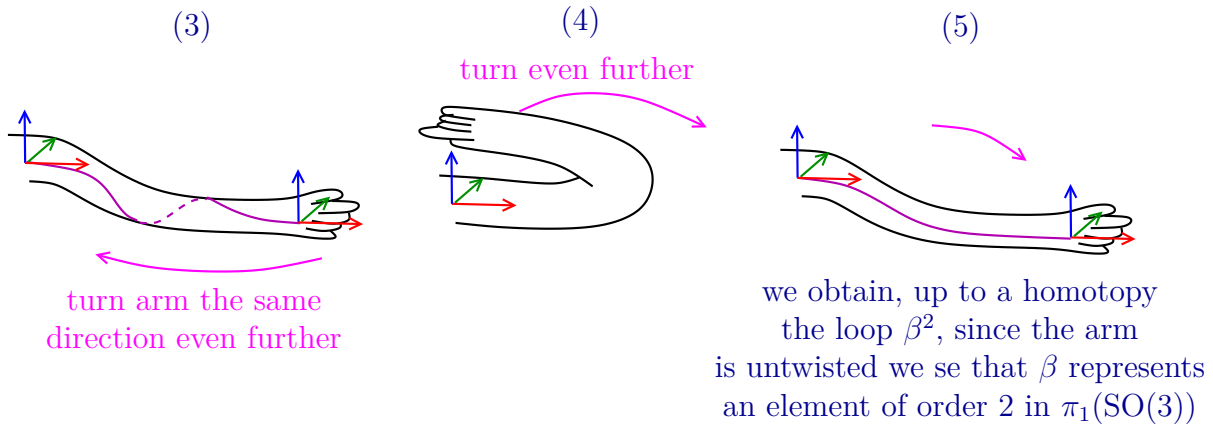
(3)



we obtain a loop β in SO(3),
it is not the standard loop
since the arm is twisted (it hurts!)

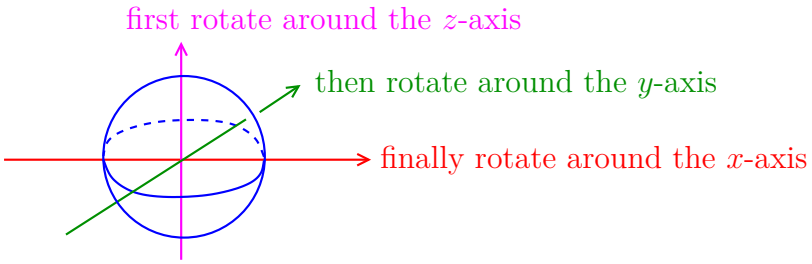
There are many more or less convincing illustrations and explanations of the plate trick. The most charming way of doing might be this one:

`http://ariwatch.com/VS/Algorithms/PlateTrick.htm`



16.3. **Topological robotics.** In this section we want to discuss a "real-life" application of the SO(3)–π_1-Proposition 15.4. Let us assume that we are given a solid ball and we are allowed to perform the following operations:

(1) first we can rotate the ball around the z-axis by an angle $\gamma \in S^1$,
(2) then we can rotate the ball around the y-axis by an angle $\beta \in S^1$, finally
(3) we can rotate the ball around the x-axis by an angle $\alpha \in S^1$.



Using these three operations, in that order, we can rotate the ball into any position. More precisely, we have the following lemma:

Lemma 16.1. (Torus-to-SO(3)-Lemma) The map

$$\begin{array}{rcl} \Phi\colon S^1\times S^1\times S^1 & \to & \mathrm{SO}(3) \\ (\alpha,\beta,\gamma) & \mapsto & \begin{array}{c}\text{rotation by } \alpha\\ \text{around the } x\text{-axis}\end{array} \circ \begin{array}{c}\text{rotation by } \beta\\ \text{around the } y\text{-axis}\end{array} \circ \begin{array}{c}\text{rotation by } \gamma\\ \text{around the } z\text{-axis}\end{array} \end{array}$$

is a surjection.[149]

Sketch of proof. Let $A \in \mathrm{SO}(3)$ be matrix with columns v_1, v_2, v_3. After a rotation around the z-axis we can assume that v_1 lies in the xz-plane. After a rotation around the y-axis we can then arrange that v_1 lies in fact on the x-axis, i.e. we can arrange that $v_1 = e_1$. Since $v_1 = e_1, v_2, v_3$ is an orthonormal basis we see that v_2, v_3 lie in the yz-plane. After a rotation around the x-axis we can finally arrange that $v_2 = e_2$ and $v_3 = e_3$. ■

This lemma gives rise to the following practical problem: Given a matrix $P \in \mathrm{SO}(3)$, can we continuously find parameters $\alpha(P), \beta(P), \gamma(P) \in S^1$ such that these parameters give back the matrix P?

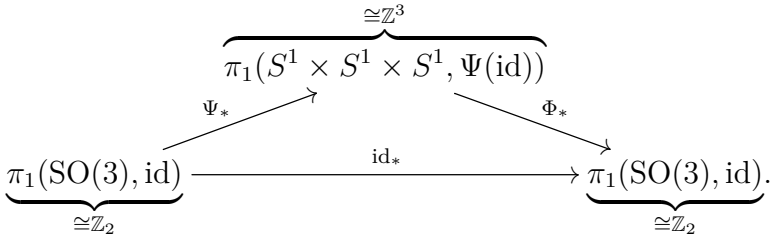
[149]The map is also continuous. Why is that?

It turns out that the answer is negative. More precisely, we have the following proposition:

Proposition 16.2. (No Splitting-SO(3)-Torus Proposition) There is no continuous map $\Psi\colon \mathrm{SO}(3) \to S^1 \times S^1 \times S^1$ such that $\Phi \circ \Psi = \mathrm{id}_{\mathrm{SO}(3)}$.

Remark. This proposition falls into the area of "topological robotics", more information on this example and on topological robotics can be found in [**Ghr14**, p. 76]. Other aspects of topological robotics are discussed in the monograph [**Far08**]. □

Proof. Let us suppose that there exists a continuous map $\Psi\colon \mathrm{SO}(3) \to S^1 \times S^1 \times S^1$ such that $\Phi \circ \Psi = \mathrm{id}_{\mathrm{SO}(3)}$. Then we get the following commutative diagram:



By the SO(3)–π_1-Proposition 15.4 we have that $\pi_1(\mathrm{SO}(3)) \cong \mathbb{Z}_2$ and by the Torus-π_1-Corollary 13.8 we know that $\pi_1(S^1 \times S^1 \times S^1) \cong \mathbb{Z}^3$. Since $\mathbb{Z}_2$ is a finite group and since $\mathbb{Z}^3$ is torsion-free we see that the homomorphism $\Psi_*\colon \pi_1(\mathrm{SO}(3)) \to \pi_1(S^1 \times S^1 \times S^1)$ is the trivial map. But that is not possible since the horizontal map is an isomorphism of a non-trivial group. ■

CHAPTER 17

Wedges and topological graphs

In the previous chapters we used coverings and group actions to calculate fundamental groups of many topological spaces, e.g. of the n-dimensional torus $\mathbb{R}^n/\mathbb{Z}^n \cong (S^1)^n$, of the real projective spaces $\mathbb{R}\mathrm{P}^n$, of the Klein bottle and the matrix group $\mathrm{SO}(3)$. There are still many topological spaces that elude us, for example so far we cannot calculate the fundamental groups of the surfaces of genus $g \geq 2$ or of topological graphs.

The next big goal will be to formulate and prove the Seifert–van Kampen Theorem which will allow us to do many more calculations. In this intermediate chapter we will introduce the wedge of pointed topological spaces, which is the same as the coproduct of pointed topological spaces. Furthermore we will discuss topological graphs and we will relate topological graphs to wedges of circles. Later in the Seifert–van Kampen Theorem we will then learn how to calculate fundamental groups of wedges and of topological graphs.

17.1. **The wedge of pointed topological spaces.** In Section 5.6 we introduced the disjoint union of topological spaces and we saw that the disjoint union is the coproduct in the category of topological spaces. In this section we now consider analogous constructions and results in the context of pointed topological spaces.

Definition. Let $\{(X_i, x_i)\}_{i\in I}$ be a family of pointed topological spaces.

(1) We refer to[150]
$$\bigvee_{i\in I} (X_i, x_i) \coloneqq \Big(\bigsqcup_{i\in I} X_i\Big)\Big/\bigsqcup_{i\in I} \{x_i\}$$
as the wedge of the pointed topological spaces (X_i, x_i). We refer to the point given by the identifications as the wedge point. If we do not say anything else it will be denoted by $*$. We equip the wedge with this base point to view the wedge as a pointed topological space.

(2) Given $j \in I$ we refer to the map
$$\begin{aligned} \iota_j \colon X_j &\to \bigvee_{i\in I} (X_i, x_i) \\ x &\mapsto [x] \end{aligned}$$
as the natural injection.

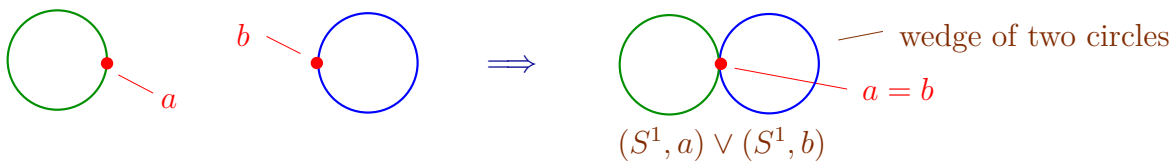
Example.

(1) It follows from the definition of the quotient topological space on page 131 that the wedge of the empty family of pointed topological spaces is $\varnothing/\varnothing$, which is defined as the topological space consisting of a single point $\{\varnothing\}$.

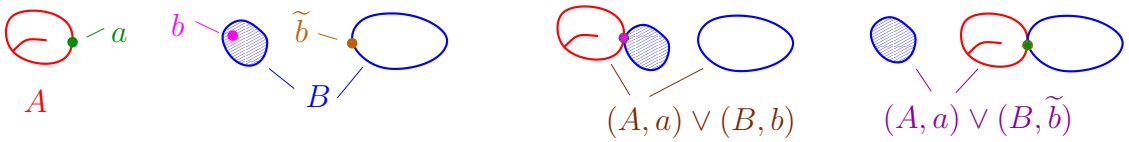
(2) The wedge $(A, a) \vee (B, b)$ of two pointed topological spaces (A, a) and (B, b) should be viewed as the result of gluing A and B along the points a and b.

(3) In general the homeomorphism type of the wedge depends on the choice of base points. This becomes particularly clear if we consider disconnected spaces. For example it is

[150] We use obvious adaptions of the notation for finite families of pointed topological spaces.



pretty clear that the wedges $(A, a) \vee (B, b)$ and $(A, a) \vee (B, \widetilde{b})$ considered in the figure below are not homeomorphic. □



The following proposition can be viewed as an analogue of the Topological-Coproduct Proposition 5.12.

Proposition 17.1. (Wedge Proposition) Let $\{(X_i, x_i)\}_{i\in I}$ be a family of pointed topological spaces. We consider the wedge

$$\bigvee_{i\in I} (X_i, x_i) \;:=\; \Big(\bigsqcup_{i\in I} X_i\Big)/\sim \qquad \text{where } x_i \sim x_j \text{ for all } i, j \in I.$$

(0) A subset $U \subset \bigvee_{i\in I} (X_i, x_i)$ is open (closed) if and only if for each $i \in I$ the preimage $\iota_i^{-1}(U)$ is an open (closed) subset of X_i.

(1) (a) Let $j \in J$. The natural injection $\iota_j \colon X_j \to \bigvee_{i\in I} (X_i, x_i)$ is continuous and it is an embedding.[151]

(b) If (Z, z_0) is a pointed topological space and if $\{f_i \colon (X_i, x_i) \to (Z, z_0)\}_{i\in I}$ is a family of continuous maps, then the map

$$\begin{array}{rcl} \bigvee_{i\in I} f_i \colon \bigvee_{i\in I} (X_i, x_i) & \to & Z \\ [(x, i)] & \mapsto & f_i(x) \end{array}$$

is continuous and it sends the wedge point to z_0.

(∗) A map $f \colon \bigvee_{i\in I} (X_i, x_i) \to Z$ is continuous if and only if each $f \circ \iota_i \colon X_i \to Z$ is continuous.

(2) The pointed topological space $\Big(\bigvee_{i\in I} (X_i, x_i), *\Big)$ together with the natural injections ι_i is a coproduct of $\{(X_i, x_i)\}_{i\in I}$ in the category $\mathcal{PTop}$ of pointed topological spaces, that we defined on page 199.



Proof. Let $\{(X_i, x_i)\}_{i\in I}$ be a family of pointed topological spaces.

(0) Both statements follow easily from the Topological-Coproduct Proposition 5.12 (0) together with the Topological-Quotient Proposition 6.1 (0).

[151] In most cases this statement allows us to safely view each X_j as a subspace of the wedge $\bigvee_{i\in I} (X_i, x_i)$.

(1) (a) Let $j \in I$. It follows immediately from the Topological-Quotient Proposition 6.1 (1a) and the Topological-Coproduct Proposition 5.12 (1a) that the natural injection $\iota_j \colon X_j \to X := \bigvee_{i\in I} (X_i, x_i)$ is continuous. It also pretty clear that this map is injective. It remains to prove the following claim:

Claim. The map $\iota_j \colon X_j \to \iota_j(X_j)$ is open.

Proof. Let $U \subset X_j$ be an open subset. First suppose that $x_j \notin U$. Using (0) one can now easily verify that $\iota_j(U)$ is in fact open in X, in particular it is open in $\iota_j(X_j)$. Finally we suppose that $x_j \in U$. We set $V := \iota_j(U) \cup \bigcup_{i\neq j} \iota_i(X_i)$. Using (0) one can again easily verify that V is an open subset of X. Furthermore it follows easily from the definitions that $\iota_j(U) = \iota_j(X_j) \cap V$. Thus we have shown that $\iota_j(U)$ is indeed open in $\iota_j(X_j)$.

(b) This statement follows immediately from the Topological-Quotient Proposition 6.1 (1b) and the Topological-Coproduct Proposition 5.12 (1b).

(∗) The "only if"-statement follows from (1a) and the fact that the composition of two continuous maps is continuous. The "if"-statement follows easily from (1b).

(2) This statement follows easily from (1a) and (1b). We leave it to the reader to fill in the few remaining details. ■

Example.

(1) Let $(X_i, x_i)_{i\in I}$ be a family of pointed topological spaces. Let $J \subset I$. Using the Wedge Proposition 17.1 one can easily show that the natural map $\bigvee_{j\in J} (X_j, x_j) \to \bigvee_{i\in I} (X_i, x_i)$ is an embedding. Furthermore we see that a retraction from $\bigvee_{i\in I} (X_i, x_i)$ to the subspace $\bigvee_{i\in J} (X_j, x_j)$ is given by sending all points in X_i with $i \neq j$ to the wedge point.

(2) Let $G = (V = \{v\}, E, i, t)$ be an abstract graph with a *single* vertex. We consider the bijection

$$f \colon |G| = \Big(V \sqcup \bigsqcup_{e\in E} [0,1]\Big)/\sim \;\to\; \bigvee_{e\in E} (S^1_e, 1)$$
$$[x] \;\mapsto\; \begin{cases} *, & \text{if } x = v, \\ \exp(2\pi \mathrm{i} t) \in S^1_e, & \text{if } x = (e,t) \text{ with } e \in E,\ t \in [0,1]. \end{cases}$$

It follows from the Abstract Graph-Topology Lemma 7.1 (3) that this map is continuous. Using the Wedge Proposition 17.1, and using the fact that the map $[0,1]/0 \sim 1 \to S^1$ given by $[t] \mapsto \exp(2\pi \mathrm{i} t)$ is a homeomorphism, one can show that the inverse f^{-1} is also continuous. In summary, the above map f is a homeomorphism. □

The following lemma plays the role of the Disjoint Union Topology-Properties Lemma 5.13. It is entirely possible that later on we use the lemma below without giving it credit.

Lemma 17.2. (Wedge–Properties Lemma) Let $\{(X_i, x_i)\}_{i\in I}$ be a family of pointed topological spaces.

(1) If I is finite and if each X_i is compact, then $\bigvee_{i\in I} (X_i, x_i)$ is compact.

(2) If each X_i is (path)-connected, then the wedge $\bigvee_{i\in I} (X_i, x_i)$ is (path)-connected.

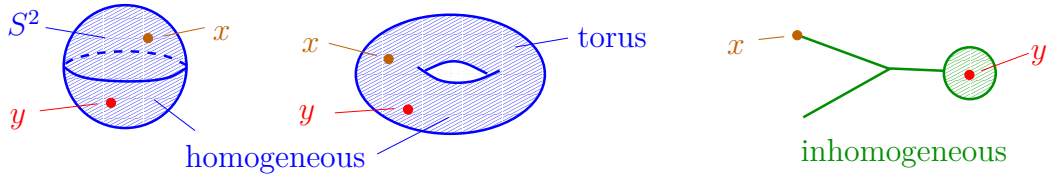
(3) The wedge $\bigvee_{i\in I} (X_i, x_i)$ is Hausdorff if and only if each X_i is Hausdorff.

Proof.

(1) This statement follows immediately from the Disjoint Union Topology-Properties Lemma 5.13 (1) together with the Quotient Topology-Properties Proposition 6.2 (1).
(2) This statement follows immediately from the Union Connected Lemma 2.19 together with the Wedge Proposition 17.1 (1a).
(3) This statement can be proved easily using the Wedge Proposition 17.1 (0a). ■

On page 246 we saw that in general the homeomorphism type of the wedge of two topological spaces depends on the choice of the base points. Fortunately we will now see that in many settings the choice of base point is largely irrelevant.

Definition. We say that a topological space X is homogeneous if given any two points $x, y \in X$ there exists a homeomorphism $f\colon X \to X$ with $f(x) = y$.



Example.

(1) It is clear that the topological space $\mathbb{R}^n$ is homogeneous. One can also easily show "by hand" that the sphere S^n is homogeneous.
(2) In [**Fri24**] it is shown that every connected topological manifold with empty boundary is homogeneous. □

Lemma 17.3. (Wedging Point–Irrelevance Lemma) Let $\{X_i\}_{i\in I}$ be a family of topological spaces. If each X_i is homogeneous, then different choices of base points still lead to homeomorphic wedges.

Proof. Suppose that for each $i \in I$ we have base points $x_i, y_i \in X_i$. Since each X_i is homogeneous we know that for each $i \in I$ there exists a homeomorphism $f_i\colon X_i \to X_i$ with $f_i(x_i) = y_i$. It follows from the Wedge Proposition 17.1 (1) that the maps

$$\bigvee_{i\in I}(X_i, x_i) \to \bigvee_{i\in I}(X_i, y_i) \qquad \text{and} \qquad \bigvee_{i\in I}(X_i, y_i) \to \bigvee_{i\in I}(X_i, x_i)$$

induced by the maps f_i and by the maps f_i^{-1} are continuous. Clearly they are inverses of one another, so they are homeomorphisms. ■

This lemma leads us to the following slightly sloppy definition.

Definition. Let $\{X_i\}_{i\in I}$ be a family of non-empty homogeneous topological spaces (e.g. a family of spheres). For each $i \in I$ we pick a point $x_i \in X_i$ and we write

$$\bigvee_{i\in I} X_i \;:=\; \bigvee_{i\in I}(X_i, x_i).$$

By the Wedging Point–Irrelevance Lemma 17.3 the homeomorphism type of $\bigvee_{i\in I} X_i$ is well-defined.

17.2. **Abstract graphs and wedges of circles.** In this section we have another look at abstract graphs and their topological incarnations. First we recall the following definitions from pages 152 and 155.

Definition. Let $G = (V, E)$ be an abstract graph.

(1) If G is finite, i.e. if V and E are finite, then we refer to

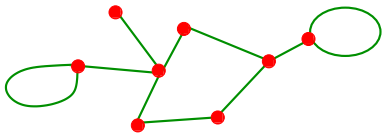
$$\chi(G) \;:=\; \#V \;-\; \#E \;=\; \text{number of vertices} \;-\; \text{number of edges}$$

as the Euler characteristic of G.

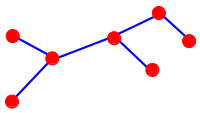
(2) We say that G is a tree if G is non-empty, if it does not contain a loop and if given any $v, w \in V$ there exists a unique injective path from v to w. If G is finite, then it follows from the Tree Characterization Proposition 7.7 that G is a tree if and only if G is connected and if $\chi(G) = 1$.

(3) The valence of a vertex v is defined as

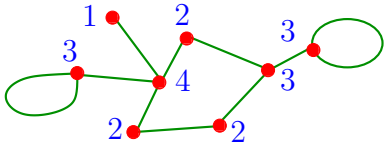
$$\text{valence}(v) \;:=\; \#\{e \in E \,|\, i(e) = v\} + \#\{e \in E \,|\, t(e) = v\}.$$



Euler characteristic of G equals $\chi(G) = 8 - 10 = -2$



example of a tree



valences of the vertices

Proposition 17.4. (Finite Tree–Contractible Proposition) Let T be an abstract graph. If T is a finite tree, then the following hold:

(1) The topological realization $|T|$ admits a deformation retraction to a vertex, in particular it is contractible.

(2) The fundamental group $\pi_1(|T|)$ is trivial.

Proof. Note that Statement (2) is an immediate consequence of Statement (1) together with the Homotopy-π_1-Proposition 13.4 (3). Thus it suffices to prove Statement (1).

We will prove that any tree admits a deformation retraction to a vertex. We will prove this statement by induction on the number of edges. If a tree has zero edges, then it follows from the connectedness of T that T consists of precisely one vertex. So we are done with this special case.

Now suppose that we know that all trees with n edges admit a deformation retraction to a vertex. Let $T = (V, E)$ be a tree with $n + 1$ edges. By the Tree–Valence One Edge Lemma 7.8 we know that T admits a vertex v of valence one. We denote by e the unique edge with v as an endpoint. We set $T' = (V \setminus \{v\}, E \setminus \{e\})$. Note that T' is a non-empty connected abstract subgraph of T. It follows from the Tree–Basics Lemma 7.4 (1) that T' is again a tree.

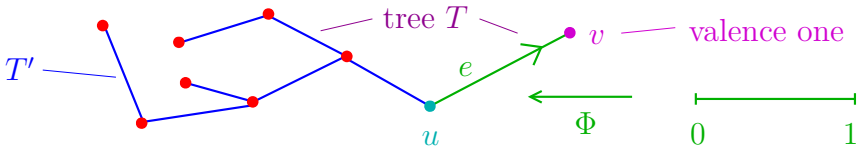
Claim. $|T'|$ is a deformation retract of $|T|$.

Proof. We denote by u the other vertex of e. Since $u \neq v$ we obtain from the Abstract Graph-Topology Lemma 7.1, see also the discussion on page 149, that there exists a closed

embedding $\Phi\colon [0,1] \to |T|$ with $\Phi(0) = u$ and $\Phi(1) = v$. We consider the map

$$\begin{array}{rcl} H\colon |T| \times [0,1] & \to & |T| \\ (x,t) & \mapsto & \begin{cases} x, & \text{if } x \in |T'|, \\ \Phi(t \cdot y), & \text{if } x = \Phi(y) \text{ for some } y \in [0,1]. \end{cases} \end{array}$$

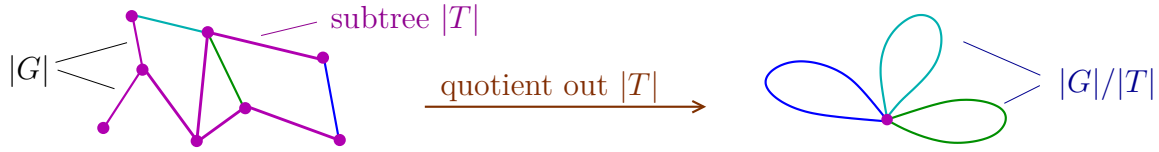
It follows from the Abstract Graph-Topology Lemma 7.1 that $|T'|$ and $|e| = \Phi([0,1])$ are closed subsets of $|T|$. It thus follows from the Homotopy Combination Lemma 8.2 that the map H is continuous. It is now clear that H is a deformation retraction from $|T|$ to $|T'|$. ⊞



We have thus shown that $|T'|$ is a deformation retract of $|T'|$. By our induction hypothesis $|T'|$ admits a deformation retraction to a vertex $v \in T'$. It follows that $|T|$ also admits a deformation retraction to the vertex v. Finally it follows from this fact together with the Point Deformation Retract Lemma 9.7 that T is contractible. ■

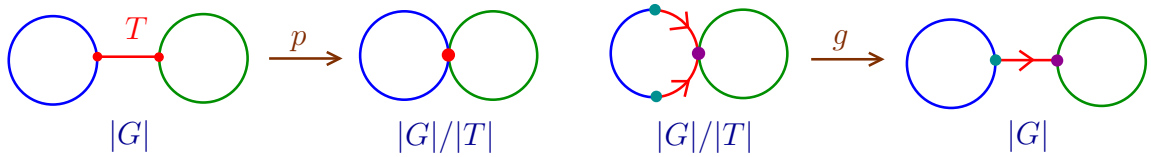
We conclude this discussion of abstract and topological graphs with the following proposition and a corresponding corollary.

Proposition 17.5. (Graph-mod-Tree Proposition) Let G be an abstract graph. For every finite subtree T of G the projection $|G| \to |G|/|T|$ is a homotopy equivalence.



Proof. We outsource the proof to the next section. ■

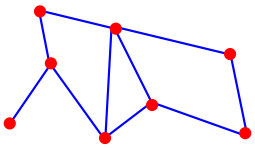
Example. We consider the abstract graph G and the subtree T that are shown in the figure below.



It follows from the Graph-mod-Tree Proposition 17.5 that the projection $p\colon |G| \to |G|/|T|$ is a homotopy equivalence. On page 167 we already saw that the map $g\colon |G|/|T| \to |G|$ that is also sketched in the figure is a homotopy inverse to p. The Graph-mod-Tree Proposition 17.5 can be viewed as a notationally slightly heroic generalization of this example. □

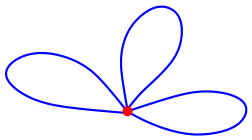
Corollary 17.6. (Graph-Homotopy Type Corollary)

(1) Let $G = (V, E)$ be a finite abstract graph. If G is connected and non-empty, then the topological realization $|G|$ is homotopy equivalent to the wedge of $1 - \chi(G)$ circles.
(2) Let G, G' be two finite abstract connected non-empty graphs. If $\chi(G) = \chi(G')$, then the topological realizations $|G|$ and $|G'|$ are homotopy equivalent.



topological graph of Euler characteristic $8 - 10 = -2$

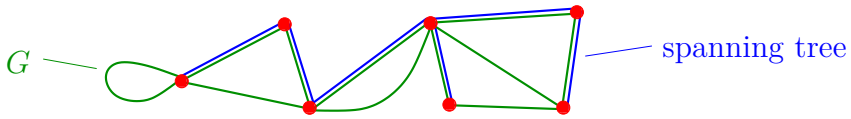
is homotopy equivalent to



wedge of three circles

Proof. Let G be a finite abstract graph.

(1) By the Spanning Tree–Existence Proposition 7.6 (1) we know that G admits a spanning tree T. Next note that by the Graph-mod-Tree Proposition 17.5 we know that the map $p\colon |G| \to |G|/|T|$ is a homotopy equivalence. By the Abstract Graph–Quotient Lemma 7.9 (2) we know that $|G|/|T|$ is homeomorphic to $|G/T|$. Thus it remains to show that $|G/T|$ is homeomorphic to the wedge of $1 - \chi(G)$ circles. We make the following observation:
   (a) Note that by definition the spanning tree T contains all vertices of G. In other words, the abstract graph G/T contains only a single vertex.
   (b) Note that
$$\chi(G/T) \underset{\uparrow}{=} \chi(G) - \chi(T) + 1 \underset{\uparrow}{=} \chi(G) - 1 + 1 = \chi(G).$$
   Abstract Graph–Quotient Lemma 7.9 (1) — Tree Characterization Proposition 7.7
   (c) By (a) and (b) we know that G/T has $1 - \chi(G)$ edges.
   (d) It follows from (a) and (c), together with the discussion on page 247 that $|G/T|$ is indeed homeomorphic to the wedge of $1 - \chi(G)$ circles.



(2) The last statement is an immediate consequence of (1) and the Homotopy-Equivalence Basics Lemma 9.5. ∎

17.3. **Proof of the Graph-mod-Tree Proposition 17.5.** The sole purpose of this section is to provide the following proof:

Proof of the Graph-mod-Tree Proposition 17.5. Let $G = (V, E, i, t)$ be an abstract graph and let $T = (W, F)$ be a finite subtree of G. We consider the quotient abstract graph G/T given by identifying all vertices in T and by removing all edges in T. We consider the map

where $(e,0) \sim i(e)$ and $(e,1) \sim t(e)$ — similarly $(e,0) \sim \rho(i(e))$ and $(e,1) \sim \rho(t(e))$

$$p\colon |G| = (V \sqcup (E \times [0,1]))/\!\underset{\downarrow}{\sim} \to |G/T| = (V/\!\sim \sqcup ((E \setminus F) \times [0,1]))/\!\underset{\downarrow}{\sim}$$

$$[x] \mapsto \begin{cases} [*], & \text{if } x \in |T|, \\ [x], & \text{if } x \in V, \\ [(e,t)], & \text{if } x = (e,t) \text{ with } e \in E \setminus F \text{ and } t \in [0,1]. \end{cases}$$

One can easily show that the induced map $p\colon |G|/|T| \to |G/T|$ is a homeomorphism. Note that p fits into the following commutative diagram:

$$\begin{array}{ccc} |G| & \xrightarrow{\quad p \quad} & |G/T| \\ & {\scriptstyle x\mapsto[x]}\searrow \quad \nearrow & \\ & |G|/|T|. & \end{array}$$

We need to show that the projection $|G| \to |G|/|T|$ to the left is a homotopy equivalence. As we just mention, the diagonal map to the right is a homeomorphism. Thus it suffices to show that the horizontal map $p\colon |G| \to |G/T|$ is a homotopy equivalence.

Note that by the Finite Tree–Contractible Proposition 17.4 we know that $|T|$ admits a deformation retraction $R\colon |T| \times [0,1] \to |T|$ to some vertex $t_0 \in T$. Put differently, there exists a homotopy $R\colon |T| \times [0,1] \to |T|$ such that $R_1 = \mathrm{id}_{|T|}$ and such that $R_0 = \mathrm{const}_{t_0}$ is the constant map. We extend R somewhat by setting $R(v,t) \coloneqq v$ for all vertices v that do not lie in $|T|$.

Let f be an edge of G that is not contained in T.

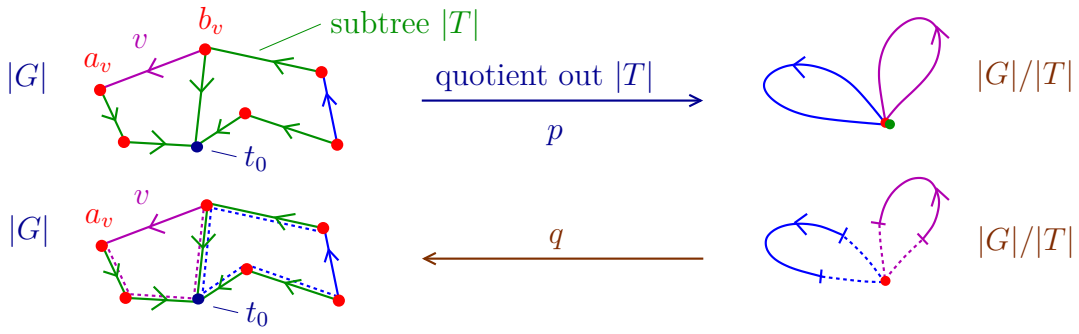
- If $i(f) = t(f)$, then it follows from the discussion on page 149 that the characteristic map of f gives rise to a continuous map $\Phi_f\colon [-2,2] \to |G|$ that induces a homeomorphism $[-2,2]/-2 \sim 2 \to |f|$.
- If $i(f) \neq t(f)$, then the same discussion on page 149 shows that there exists a continuous map $\Phi_f\colon [-2,2] \to |G|$ which restricts to a homeomorphism $[-2,2] \to |f|$.[152]

We write $a_f = \Phi_f(-2)$ and $b_f = \Phi_f(2)$.

Next we consider the quotient graph G/T. Let $p\colon |G| \to |G/T|$ be the projection. For every edge f of G/T, i.e. for every $f \in E \setminus F$, we denote by $\Psi_f\colon [-2,2] \to |G/T|$ the map $p \circ \Phi_f$. We consider the map

$$q\colon |G/T| \to |G|$$
$$x \mapsto \begin{cases} R(a_f, -s-1), & \text{if } x = \Psi_f(s) \text{ for some } s \in [-2,-1], \\ \Psi_f(2 \cdot s), & \text{if } x = \Psi_f(s) \text{ for some } s \in [-1,1], \\ R(b_f, s-1), & \text{if } x = \Psi_f(s) \text{ for some } s \in [1,2]. \end{cases}$$

Using the Abstract Graph-Topology Lemma 7.1 one can show that q is continuous.



It remains to prove the following claim:

Claim. The maps p and q are homotopy inverses of one another.

[152] In other words, the map Φ_f is just a rescaling of the characteristic maps of f.

Proof. The proof of the claim is fairly elementary, just notationally a little messy. First we consider the following map

$$\begin{aligned} H\colon |G/T| \times [0,1] &\to |G/T| \\ (x,t) &\mapsto \begin{cases} *, & \text{if } x = \Psi_f(s) \text{ for some } s \in [-2,-1-t], \\ \Psi_f(\frac{2s}{1+t}), & \text{if } x = \Psi_f(s) \text{ for some } s \in [-1-t,1+t], \\ *, & \text{if } x = \Psi_f(s) \text{ for some } s \in [1+t,2]. \end{cases} \end{aligned}$$

Using the Abstract Graph-Topology Lemma 7.1 and the Homotopy Combination Lemma 8.2 one can easily show that H is continuous. It follows immediately from the definitions that $H_0 = p \circ q$ and $H_1 = \mathrm{id}_{|G/T|}$. Next we consider the following map

$$\begin{aligned} \widetilde{H}\colon |G| \times [0,1] &\to |G| \\ (x,t) &\mapsto \begin{cases} R(x,t), & \text{if } x \in T, \\ R(a_f, s+2+t), & \text{if } x = \Phi_f(s) \text{ for some } s \in [-2,-1-t], \\ \Phi_f(\frac{2s}{1+t}), & \text{if } x = \Phi_f(s) \text{ for some } s \in [-1-t,1+t], \\ R(b_f, -s+2+t), & \text{if } x = \Phi_f(s) \text{ for some } s \in [1+t,2]. \end{cases} \end{aligned}$$

Again, using the Abstract Graph-Topology Lemma 7.1 and the Homotopy Combination Lemma 8.2 one can show that $\widetilde{H}$ is continuous. Furthermore it follows again immediately from the definitions that $\widetilde{H}_0 = q \circ p$ and $\widetilde{H}_1 = \mathrm{id}_{|G|}$. ∎

17.4. **Outlook.** Summarizing we have now finally managed to calculate the fundamental groups of the following topological spaces:

(1) the circle S^1 and all spheres S^n with $n \geq 2$,
(2) the n-dimensional torus $(S^1)^n \cong \mathbb{R}^n/\mathbb{Z}^n$,
(3) the Klein bottle,
(4) the real projective spaces $\mathbb{R}\mathrm{P}^n$,
(5) the lens spaces $L(p,q)$,
(6) trees,
(7) products of the above.

Nonetheless, the list of topological spaces for which we cannot yet calculate the fundamental group is still pretty long, it includes

(1) topological graphs,
(2) the complement of more than one point in $\mathbb{C}$, e.g. $\mathbb{C} \setminus \{0,1\}$,
(3) surfaces of genus ≥ 2,
(4) knot complements,
(5) wedges of topological spaces.

In the following two chapters we will first build up our knowledge of group theory. Then we will state and prove the Seifert-van Kampen Theorem in various flavors which will help us with calculating the open fundamental groups.

CHAPTER 18

Basics of group theory I: Abelian groups

In the Product-π_1-Proposition 13.7 we showed that for any two pointed topological spaces (X, x_0) and (Y, y_0) we have an isomorphism $\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$. In more formal terms, we showed the following:

(∗) The fundamental group of the product of two pointed topological spaces is isomorphic to the product of the fundamental groups of the two pointed topological spaces.

In the Wedge Proposition 17.1 we just determined the coproduct of two pointed topological spaces. It is natural to ask, whether in (∗) we can replace product by coproduct.

To answer this question we first need to understand the coproduct in the category *Group* of groups. Also, more generally it is worthwhile to build up our understanding of groups.

Since general groups are quite difficult we will first build up our understanding of *abelian* groups. More precisely, in this chapter we study the coproduct in the category *AbGroup* of abelian groups, we introduce free abelian groups, we study the rank of (free) abelian groups and we will state and prove the classification of finitely generated abelian groups.

In the next chapter we will then see to what degrees these concepts and results on abelian groups generalize to the context of (potentially non-abelian) groups.

18.1. The direct sum of abelian groups.

We start out by recalling the following definitions from the Product-of-Sets-and-Groups Lemma 4.2 and from page 105.

Definition. Let $\{X_s\}_{s\in S}$ be a family of abelian groups.

(1) If S is non-empty, then we define the product of the X_s to be the group

$$\prod_{s\in S} X_s \coloneqq \left\{ f\colon S \to \bigcup_{s\in S} X_s \;\middle|\; \text{for every } s \in S \text{ we have } f(s) \in X_s \right\},$$

where the group structure is given by the obvious "component wise multiplication". If $S = \varnothing$, then we define the product to be the trivial group.

(2) We define the direct sum of the X_s to be the group

$$\bigoplus_{s\in S} X_s \coloneqq \left\{ (g_s)_{s\in S} \in \prod_{s\in S} X_s \;\middle|\; \text{all but finitely many } g_s \text{ are trivial} \right\}.$$

(3) Given $t \in S$ we refer to

$$\begin{array}{rccl} \iota_t\colon X_t & \to & \displaystyle\bigoplus_{s\in S} X_s & \\ x & \mapsto & \left(\begin{array}{ccl} S & \to & \displaystyle\bigcup_{s\in S} X_s \\ s & \mapsto & \begin{cases} x, & \text{if } s = t, \\ 0_{X_t}, & \text{if } s \neq t. \end{cases} \end{array} \right) \end{array}$$

as the natural homomorphism.

By the Product-of-Sets-and-Groups Lemma 4.2 we know that $\prod_{s\in S} X_s$ is the product of the X_s in the category *AbGroup* of abelian groups and by the Coproduct-of-Sets-and-Abelian Groups Lemma 4.4 we know that $\bigoplus_{s\in S} X_s$ is the coproduct of the X_s in the category *AbGroup* of abelian groups. For the reader's convenience we recall this second statement in greater detail:

Lemma 18.1. (Coproduct-of-Abelian Groups Lemma) Let $\{X_s\}_{s\in S}$ be a family of abelian groups.

(1) Let Z be an abelian group and let $\{f_s\colon X_s \to Z\}_{s\in S}$ be a family of homomorphisms. The map

$$\begin{array}{rcl} \bigoplus_{s\in S} f_s\colon \bigoplus_{s\in S} X_s & \to & Z \\ (g_s)_{s\in S} & \mapsto & \sum\limits_{\substack{s\in S \text{ with} \\ g_s \neq e}} f_s(g_s) \end{array}$$

is the unique homomorphism[153] such that for each $s \in S$ we have $f_s = \Phi \circ \iota_s$.

(2) The coproduct in the category *AbGroup* of *abelian* groups of $\{X_s\}_{s\in S}$ is given by $\bigoplus_{s\in S} X_s$ together with the above natural homomorphisms $\iota_t\colon X_t \to \bigoplus_{s\in S} X_s$.

Proof. This statement is basically the content of the Coproduct-of-Sets-and-Abelian Groups Lemma 4.4 (2). But of course one can also prove it easily by hand. ■

Convention.

(1) Let $\{X_s\}_{s\in S}$ be a family of abelian groups. For $t \in S$ we use the natural injection $\iota_s\colon X_t \to \bigoplus_{s\in S} X_s$ from the Coproduct-of-Abelian Groups Lemma 18.1 to view X_t as a subgroup of the direct sum $\bigoplus_{s\in S} X_s$.

(2) Let G be an abelian group and let $\{X_s\}_{s\in S}$ be a family of subgroups. The inclusions $X_s \to G$ define a natural homomorphism

$$\bigoplus_{s\in S} X_s \to G.$$

If this natural homomorphism is a monomorphism, then we refer to the image also as the direct sum of the X_s.[154]

Example. Let G be an abelian group and let A and B be two subgroups. If $A \cap B = \{0\}$ then the natural homomorphism $A \oplus B \to G$ is easily seen to be a monomorphism. Thus we can view the direct sum $A \oplus B$ as a subgroup of G. □

Theorem 18.2. (Chinese Remainder Theorem) If $r, s \in \mathbb{N}$ are coprime, then the map

$$\begin{array}{rcl} \Phi\colon \mathbb{Z}_{r\cdot s} & \to & \mathbb{Z}_r \times \mathbb{Z}_s = \mathbb{Z}_r \oplus \mathbb{Z}_s \\ [k] & \mapsto & ([k],[k]) \end{array}$$

[153] Note that in the verification of that assertion that Φ is a homomorphism it is essential that Z is abelian.

[154] Sometimes in the literature this image is called the *internal direct sum*. In practice it becomes a mess to distinguish in the notation between the "direct sum" and the "internal direct sum", thus we will not try to distinguish the two concepts.

is an isomorphism.

Proof. Since $r, s \in \mathbb{N}$ are coprime there exist $a, b \in \mathbb{Z}$ with $a \cdot r + b \cdot s = 1$. We consider the map

$$\begin{array}{rcl} \Psi\colon \mathbb{Z}_r \times \mathbb{Z}_s & \to & \mathbb{Z}_{r\cdot s} \\ ([e]_{\mathbb{Z}_r}, [f]_{\mathbb{Z}_s}) & \mapsto & [e \cdot b \cdot s + f \cdot a \cdot r]_{\mathbb{Z}_{rs}}. \end{array}$$

One can easily verify that this map is well-defined. We claim that it is an inverse to Ψ. Indeed, we have

$$\Psi(\Phi([k]_{\mathbb{Z}_{rs}})) = \Psi([k]_{\mathbb{Z}_r}, [k]_{\mathbb{Z}_s}) = [k \cdot b \cdot s + k \cdot a \cdot r]_{\mathbb{Z}_{rs}} = [k \cdot \underbrace{(b \cdot s + a \cdot r)}_{=1}]_{\mathbb{Z}_{rs}} = [k]_{\mathbb{Z}_{rs}}$$

and we have

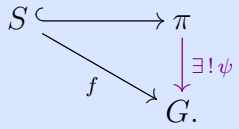
$$\begin{aligned} \Phi(\Psi([e]_{\mathbb{Z}_r}, [f]_{\mathbb{Z}_r})) &= \Phi([e \cdot b \cdot s + f \cdot a \cdot r]_{\mathbb{Z}_{rs}}) \\ &= ([e \cdot b \cdot s + f \cdot a \cdot r]_{\mathbb{Z}_r}, [e \cdot b \cdot s + f \cdot a \cdot r]_{\mathbb{Z}_s}) \\ &= ([e \cdot \underbrace{b \cdot s}_{=1-a\cdot r}]_{\mathbb{Z}_r}, [f \cdot \underbrace{a \cdot r}_{=1-b\cdot s}]_{\mathbb{Z}_s}) = ([e]_{\mathbb{Z}_r}, [f]_{\mathbb{Z}_s}). \end{aligned}$$

■

18.2. **Free abelian groups.** We start out with the following definition, which in principle can be modified to many different settings.

Definition.

(1) We say that a group π is a free abelian group on a subset $S \subset \pi$, if the following universal property holds: For any *map* $f\colon S \to G$ of the set S to an abelian group G there exists a unique *homomorphism* $\psi\colon \pi \to G$ that makes the following diagram commute:



We call S a basis of the free abelian group π.

(2) We say that a group π is free abelian if it is a free abelian group on some subset S.

Example. One can easily show that the group

$$\mathbb{Z}^\infty := \{(x_1, x_2, x_3, \dots) \mid x_1, x_2, \dots \in \mathbb{Z} \text{ and all but finitely many } x_i \text{ are zero}\}$$

is a free abelian group on the subset $\{e_1, e_2, \dots\}$ where $e_i := (0, \dots, 0, 1, 0, \dots)$ has a unique 1 in the i-th entry. □

As always, it is easy to prove a suitable uniqueness of an object that is described by a universal property:

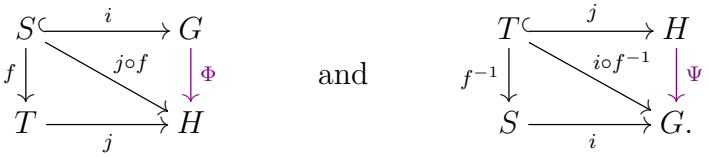
Lemma 18.3. (Free Abelian Group–Uniqueness Lemma) Let G be a free abelian group on a subset $S \subset G$ and let H be a free abelian group on a subset $T \subset H$. Given any bijection $f\colon S \to T$ there exists a unique isomorphism $\psi\colon G \to H$ such that $\psi(s) = f(s)$ for all $s \in S$.

Example.

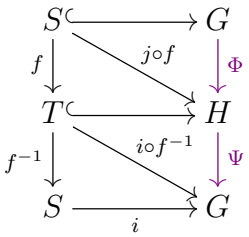
(1) Let $m \in \mathbb{N}_0$. One can easily verify that $\mathbb{Z}^m$ is a free abelian group on the subset $e_1, \dots, e_m$ given by the m standard basis vectors of $\mathbb{Z}^m$. It follows from the Free Abelian Group–Uniqueness Lemma 18.3 that if G is a free abelian group on a finite set S, then G is isomorphic to $\mathbb{Z}^{\#S}$.

(2) Let π be a free abelian group with basis S. If $\Phi\colon \pi \to \pi$ is a homomorphism such that $\Phi(s) = s$ for all $s \in S$, then it follows immediately from the uniqueness statement of the above definition, that $\Phi = \mathrm{id}$. □

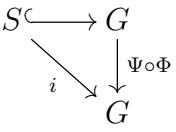
Proof of Lemma 18.3. We denote by $i\colon S \to G$ and $j\colon T \to H$ the inclusion maps. Furthermore we consider the maps $f\colon S \to T$ and $f^{-1}\colon T \to S$. Since G is a free abelian group on the subset $S \subset G$ and since H is a free abelian group on the subset $T \subset H$ we see that there exist homomorphisms $\Phi\colon G \to H$ and $\Psi\colon H \to G$ that make the following diagrams commute:



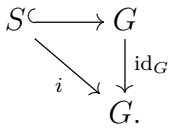
In particular we get the commutative diagram



which simplifies to



and of course we also have the commutative diagram



It follows from the uniqueness part of the universal property applied to the two diagrams on the right that $\Psi \circ \Phi = \mathrm{id}_G$. The same way one shows that $\Phi \circ \Psi = \mathrm{id}_H$. In particular Φ and Ψ are isomorphisms. ■

To prove existence of free abelian groups it is useful to specialize the definition of direct sums from the previous section to the case that $X_s = \mathbb{Z}$ for all $s \in S$. This leads us to the following definition, which will stay with us throughout these notes.

Definition. Let S be a set.

(1) We define

$$\mathbb{Z}^{(S)} := \bigoplus_{s\in S} \mathbb{Z} = \{\text{all maps } S \to \mathbb{Z} \text{ that are zero for all but finitely many } s \in S\}.$$

(2) Let $s \in S$. The map

$$\begin{array}{rcl} S & \to & \mathbb{Z} \\ t & \mapsto & \begin{cases} 1, & \text{if } t = s, \\ 0, & \text{otherwise.} \end{cases} \end{array}$$

lies in $\mathbb{Z}^{(S)}$ and it is also called s. Thus we can view S as a subset of $\mathbb{Z}^{(S)}$.

(3) Given elements $s_1, \dots, s_k \in S$ and $n_1, \dots, n_k \in \mathbb{Z}$ we obtain the corresponding element $n_1 \cdot s_1 + \cdots + n_k \cdot s_k$ of $\mathbb{Z}^{(S)}$. Every element of $\mathbb{Z}^{(S)}$ can be written in a unique way as such a linear combination of finitely many elements of S. Indeed, for $f \in \mathbb{Z}^{(S)}$ we can uniquely write[155]

$$f = \sum_{s \in S \text{ with } f(s) \neq 0} f(s) \cdot s.$$

The following lemma will not really be a surprise to the alert reader.

[155]It is easy to see that both sides define the same map $S \to \mathbb{Z}$, so they are the same element of $\mathbb{Z}^{(S)}$.

Lemma 18.4. (Free Abelian Group–Existence Lemma) Let S be a set. The group $\mathbb{Z}^{(S)}$ is a free abelian group on the subset $S \subset \mathbb{Z}^{(S)}$.

The Free Abelian Group–Existence Lemma 18.4 suggests the following definition:

Definition. Let S be a set. We refer to $\mathbb{Z}^{(S)}$ as *the* free abelian group generated by S.

Proof of Lemma 18.4. Let $f\colon S \to G$ be a map to an abelian group. We consider the map[156]

$$\begin{array}{rcl} \psi\colon \mathbb{Z}^{(S)} & \to & G \\ \underbrace{\sum_{i=1}^{k} n_i \cdot s_i}_{\substack{\text{we saw above that} \\ \text{every element of } \mathbb{Z}^{(S)} \text{ is} \\ \text{uniquely of that form}}} & \mapsto & \sum_{i=1}^{k} n_i \cdot f(s_i). \end{array}$$

It is straightforward to verify that ψ is in fact a homomorphism, that ψ has the desired property and that ψ is the unique such homomorphism. ■

The following lemma gives a typical application of free abelian groups.

Lemma 18.5. (Free Abelian Group–Splitting Lemma) Let $\varphi\colon G \to A$ be an epimorphism between abelian groups.

(1) If A is a free abelian group, then there exists a right inverse $\sigma\colon A \to G$ of φ, this means that there exists a homomorphism $\sigma\colon A \to G$ with $\varphi \circ \sigma = \mathrm{id}_A$.[157]
(2) If $\sigma\colon A \to G$ is a right inverse of φ, then $G = \ker(\varphi) \oplus \sigma(A)$.[158]

Proof. Let $\varphi\colon G \to A$ be an epimorphism between abelian groups.

(1) We assume that A is a free abelian group. We pick a basis B for the free abelian group A. Since $\varphi\colon G \to A$ is an epimorphism we can choose for each $b \in B$ an element $\widetilde{b} \in G$ with $\varphi(\widetilde{b}) = b$.

Since A is a free abelian group on B there exists a unique homomorphism $\sigma\colon A \to G$ with $\sigma(b) = \widetilde{b}$ for all $b \in B$. For each $b \in \widetilde{B}$ we have $(\varphi \circ \sigma)(b) = \varphi(\widetilde{b}) = b$. It follows from the discussion on page 257 that $\varphi \circ \sigma = \mathrm{id}_A$.

(2) Let $\sigma\colon A \to G$ be a right inverse of φ. We start out with the following claim:

Claim. We have $\ker(\varphi) \cap \sigma(A) = \{0\}$.

Proof. If $g = \sigma(a) \in \ker(\varphi) \cap \sigma(A)$, then

$$g = \sigma(a) = (\sigma \circ \mathrm{id}_A)(a) \underset{\substack{\uparrow \\ \text{since } \varphi \circ \sigma = \mathrm{id}_A}}{=} \sigma(\varphi(\sigma(a))) \underset{\substack{\uparrow \\ \text{since } g = \sigma(a)}}{=} \sigma(\varphi(g)) \underset{\substack{\uparrow \\ \text{since } g \in \ker(\varphi)}}{=} \sigma(0) = 0.$$ ⊞

As we discussed on page 255, it follows from the above claim that the natural homomorphism

$$\begin{array}{rcl} \Theta\colon \ker(\varphi) \oplus \sigma(A) & \to & G \\ g \oplus h & \mapsto & g + h \end{array}$$

[156]Here we assume that G is written as an additive group. If G is a multiplicative group we would need to replace $n_i \cdot \phi(s_i)$ by $\phi(s_i)^{n_i}$ and we would need to replace the sum by a product.
[157]Note that any such right inverse $\sigma\colon A \to G$ is in particular a monomorphism.
[158]Here we use the convention from page 255 to view the direct sum $\ker(\varphi) \oplus \sigma(A)$ as a subgroup of G.

is a monomorphism. It remains to show that Θ is an epimorphism. Let $g \in G$. Note that

$$\varphi(g - \sigma(\varphi(g))) = \varphi(g) - \varphi(\sigma(\varphi(g))) \underset{\substack{\uparrow \\ \text{since } \varphi\sigma = \mathrm{id}_A}}{=} \varphi(g) - \varphi(g) = 0.$$

This shows that $g - \sigma(\varphi(g)) \in \ker(\varphi)$. It follows that $g \in \mathrm{im}(\sigma) \oplus \ker(\varphi)$. ■

Theorem 18.6. (Basis Cardinality Theorem) Let F be an abelian group. If F is free on two subsets S and T, i.e. if S and T are bases for F, then $\#S = \#T$.

Proof. The proof of this statement is basically the same as the proof that in a vector space any two bases have the same cardinality. This proof is given in detail in [**Loe14**, Chapter 16.7]. ■

The Basis Cardinality Theorem 18.6 allows us to make the following definition:

Definition. If F is a free abelian group on a subset S, then we define[159] [160]

$$\mathrm{rank}(F) := \#S \quad \in \mathbb{N}_0 \cup \{\infty\}.$$

Example.

(1) The rank of the free abelian group $\mathbb{Z}^m$ is m.
(2) Let G be a free abelian group on a subset S and let H be a free abelian group on a subset T. It follows easily from the definitions and the Coproduct-of-Abelian Groups Lemma 18.1, that $G \oplus H$ is a free abelian group of rank $\mathrm{rank}(G) + \mathrm{rank}(H) \in \mathbb{N}_0 \cup \{\infty\}$.

□

We conclude this section with the following non-trivial theorem.

Theorem 18.7. (Free Abelian Subgroup Theorem) Let F be a free abelian group. Every subgroup H of F is a free abelian subgroup with $\mathrm{rank}(H) \leq \mathrm{rank}(F)$.

Proof. We only provide the proof in the special case that the rank of F is finite. The general case is proved in [**Fri24**]. Thus let F be an abelian group of rank n and let H be a subgroup of F. We need to show that H is free of rank $\leq n$.

We prove the statement by induction on n. If $n = 1$, then $F \cong \mathbb{Z}$. Since $\mathbb{Z}$ is a PID (principal ideal domain) we know that every subgroup of $\mathbb{Z}$ is of the form $k \cdot \mathbb{Z}$ for some $k \in \mathbb{N}_0$. Thus we see that every subgroup is a free abelian group of rank ≤ 1.

Now we suppose that the statement holds for all free abelian groups of rank $< n$ with $n \in \mathbb{N}_{\geq 2}$. Let F be a free abelian group of rank n and let $H \subset F$ be a subgroup. Let $t_1, \ldots, t_n$ be a basis for F. Let $F_n \subset F$ be the free abelian group on $t_1, \ldots, t_{n-1}$. We set $H_n := H \cap F_n$. By induction H_n is a free abelian group of rank $\leq n - 1$. It remains to prove the following claim:

Claim. Either $H = H_n$ or there exists an $x \in H \setminus H_n$ with[161] $H = H_n \oplus \mathbb{Z} \cdot x$.

---

[159] Here we denote by $\#S$ the cardinality of the set S, as defined on page 41.
[160] For simplicity we only view the cardinalities as elements of $\mathbb{N}_0 \cup \{\infty\}$. In other words, for the rank of an abelian group we do not distinguish between the different infinite cardinalities. This simplification is good enough for our purpose and it prevents us from having to deal with thorny set theoretic issues.
[161] It follows from the above example, that $\mathrm{rank}(H_n \oplus \mathbb{Z} \cdot x) = \mathrm{rank}(H_n) + \mathrm{rank}(\mathbb{Z} \cdot x) = n - 1 + 1 = n$.

Proof. We consider the following sequence of homomorphisms:

$$\begin{array}{ccccccc} H/H_n & = & H/(H\cap F_n) & \overset{\cong}{\to} & (H+F_n)/F_n & \subset & F/F_n & \overset{\cong}{\leftarrow} & \mathbb{Z}. \\ & & [h] & \mapsto & [h] & & [m\cdot t_n] & \mapsto & m. \end{array}$$

If the resulting homomorphism $H/H_n \to \mathbb{Z}$ is trivial, then $H = H_n$, and we are done. Now suppose that the homomorphism $\varphi\colon H/H_n \to \mathbb{Z}$ is non-trivial. Then there exists an $r \in \mathbb{N}$ such that the map $\varphi\colon H/H_n \to \mathbb{Z}\cdot r$ is an epimorphism. It follows from the Free Abelian Group–Splitting Lemma 18.5, applied to the free abelian group $\mathbb{Z}\cdot r$, that $H = H_n \oplus \mathbb{Z}\cdot x$ for some $x \in H \setminus H_n$. ■

18.3. **Finitely generated abelian groups.** In this section we will introduce the notion of a "finitely generated abelian group" and we will see that these are among the rare classes of mathematical objects which are completely classified.

Definition.

(1) Given an abelian group G we say that $S \subset G$ is a generating set if the natural homomorphism $\Xi_S\colon \mathbb{Z}^{(S)} \to G$ is an epimorphism.
(2) We call an abelian group finitely generated if it admits a finite generating set.

Lemma 18.8. (Finite Generation Lemma) Let $\varphi\colon A \to B$ be a homomorphism of abelian groups.

(1) If φ is an epimorphism and if A is finitely generated, then so is B.
(2) If φ is a monomorphism and if B is finitely generated, then so is A.

Proof.

(1) Let $\varphi\colon A \to B$ be an epimorphism of abelian groups. If S is a finite generating set for A, then clearly the finite set $\varphi(S)$ is a generating set for B.
(2) We only need to show that if B is a finitely generated group, then so is every subgroup A of B. Let T be a finite generating set for B. We denote by $\Xi_T\colon \mathbb{Z}^{(T)} \to B$ the natural epimorphism. Note that $\Xi_T^{-1}(A)$ is a subgroup of $\mathbb{Z}^{(T)}$. We can summarize the maps in the following commutative diagram:

$$\begin{array}{ccc} \Xi_T^{-1}(A) & \hookrightarrow & \mathbb{Z}^{(T)} \\ {\scriptstyle \Xi_T}\downarrow & & \downarrow{\scriptstyle \Xi_T} \\ A & \hookrightarrow & B. \end{array}$$

By the Free Abelian Subgroup Theorem 18.7 we know that $\Xi_T^{-1}(A)$ is a free abelian group of rank $\leq \#T$. Let S be a basis for $\Xi_T^{-1}(A)$. It follows immediately from the definitions that $\Xi_T(S)$ is a generating set for A. Since $\#\Xi_T(S) \leq \#S \leq \#T$ we see that A is finitely generated. ■

Lemma 18.9. (Matrix Quotient Lemma)

(1) Let H be an abelian group with finite generating set $S = \{s_1, \dots, s_m\}$. There exists an $n \in \mathbb{N}_0$ and an $m \times n$–matrix A over $\mathbb{Z}$ such that the map

$$\begin{array}{rcl} \mathbb{Z}^m/A \cdot \mathbb{Z}^n & \to & H \\ \left[\begin{pmatrix} v_1 \\ \vdots \\ v_m \end{pmatrix}\right] & \mapsto & \sum_{i=1}^{m} v_i \cdot s_i \end{array}$$

is a well-defined isomorphism.

(2) If $A \in \mathrm{M}(n \times n, \mathbb{Z})$ is a diagonal matrix with entries $d_1, \dots, d_n \in \mathbb{Z}$, then

$$\mathbb{Z}^n/A \cdot \mathbb{Z}^n \cong \bigoplus_{i=1}^{n} \mathbb{Z}_{d_i}.$$

(3) Let A be an $m \times n$–matrix over $\mathbb{Z}$. For matrices $P \in \mathrm{GL}(m, \mathbb{Z})$ and $Q \in \mathrm{GL}(n, \mathbb{Z})$ the map

$$\begin{array}{rcl} \mathbb{Z}^m/A \cdot \mathbb{Z}^n & \cong & \mathbb{Z}^m/PAQ \cdot \mathbb{Z}^n \\ [v] & \mapsto & [P \cdot v] \end{array}$$

is well-defined and it is an isomorphism.

Proof.

(1) Let H be an abelian group with finite generating set $S = \{s_1, \dots, s_m\}$. By definition the natural homomorphism $\Xi_S \colon \mathbb{Z}^{(S)} \to H$ is an epimorphism. Let $\Theta \colon \mathbb{Z}^m \to \mathbb{Z}^{(S)}$ be the natural isomorphism given by $\Theta(e_i) = s_i$. Note that $\Xi_S \circ \Theta \colon \mathbb{Z}^m \to H$ is also an epimorphism. By the Free Abelian Subgroup Theorem 18.7 we know that $\ker(\Xi_s) \subset \mathbb{Z}^m$ is a finitely generated free abelian subgroup. Let $a_1, \dots, a_n \in \mathbb{Z}^m$ be a basis for this free abelian group. We set $A \coloneqq (a_1 \ \dots \ a_n) \in \mathrm{M}(m \times n)$. One can easily verify that this matrix has the desired properties.

(2) This statement follows easily from the definitions.

(3) First note that for $Q \in \mathrm{GL}(n, \mathbb{Z})$ we have $Q \cdot \mathbb{Z}^n = \mathbb{Z}^n$. It follows that the given map is well-defined. An inverse is given by $[w] \mapsto [P^{-1} \cdot w]$. This shows that the given map is an isomorphism. ∎

Theorem 18.10. (Smith Normal Form Theorem) Let A be an $m \times n$–matrix over $\mathbb{Z}$. There exist two matrices $P \in \mathrm{GL}(m, \mathbb{Z})$ and $Q \in \mathrm{GL}(n, \mathbb{Z})$ and $d_1, \dots, d_t \in \mathbb{N}$ such that[162]

$$P \cdot A \cdot Q = \begin{pmatrix} d_1 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & 0 & d_t & 0 \\ 0 & 0 & 0 & 0_{(m-t)\times(n-t)} \end{pmatrix} \in \mathrm{M}(m \times n, \mathbb{Z}).$$

The matrix on the right is sometimes called a Smith normal form of A.

Proof. We postpone the proof to the next section. Alternatively we refer to [**New72**, Chapter II.15] and [**Nor12**, Chapter 1] for a proof. ∎

Now we can state and prove one of the best-loved theorems in mathematics.

[162] Here the last row and column of zeros can be arbitrarily large.

Theorem 18.11. (Abelian Group-Classification Theorem) Let H be a finitely generated abelian group. There exist prime powers[163] $q_1, \dots, q_k$ such that

$$H \;\cong\; \mathbb{Z}^r \oplus \bigoplus_{i=1}^{k} \mathbb{Z}_{q_i}.$$

Furthermore r is uniquely determined and the prime powers $q_1, \dots, q_k$ are uniquely determined up to permutation.

Sketch of proof. [164] Let H be a finitely generated abelian group. We first prove the existence of the promised isomorphism. Note that it follows from the Matrix Quotient Lemma 18.9 (1), the Smith Normal Form Theorem 18.10 and the Matrix Quotient Lemma 18.9 (2) and (3) that there exist $r \in \mathbb{N}$ and $d_1, \dots, d_t \in \mathbb{N}$ such that

$$H \;\cong\; \mathbb{Z}^r \oplus \bigoplus_{i=1}^{t} \mathbb{Z}_{d_i}.$$

The existence of the promised decomposition follows from the Chinese Remainder Theorem 18.2 which tells us, that each summand $\mathbb{Z}_{d_i}$ is isomorphic to the direct sum of quotients of $\mathbb{Z}$ by prime powers.

It remains to prove the uniqueness of the prime power q_i up to permutation. One can do so by introducing the following invariant of abelian groups: Given a prime p, given $k \in \mathbb{N}$ and given an abelian group V we consider

$$T_{p,k}(V) \;:=\; \#\{v \in V \,|\, p^k \cdot v = 0\}.$$

These numbers are clearly isomorphism invariants of abelian groups. By studying these invariants one can easily prove that the q_i are unique up to permutation. We leave it to the reader to fill in the details. ∎

18.4. Appendix: Proof of the Smith Normal Form Theorem 18.10.

In this appendix we will provide a sketch of the proof of the Smith Normal Form Theorem 18.10.

Proof of the Smith Normal Form Theorem 18.10. First note that if an $m \times n$–matrix P is obtained from an $m \times n$–matrix Q by swapping two rows, then P is obtained by left-multiplying Q by an $m \times m$–matrix of the form[165]

$$\begin{pmatrix} \mathrm{id} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \mathrm{id} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathrm{id} \end{pmatrix}.$$

[163] By a prime power we mean any natural number of the form p^s where p is a prime and $s \in \mathbb{N}$.
[164] We provide a sketch of a proof, full details are for example given in [**Hun80**, Chapter II.2].
[165] Here is the reality check whether that is really correct: we have

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;=\; \begin{pmatrix} c & d \\ a & b \end{pmatrix}.$$

Similarly if P is obtained from an $m \times n$–matrix Q by adding the k-th row to the l-th row, then P is obtained by left-multiplying Q by an $m \times m$–matrix of the form

$$\begin{pmatrix} \mathrm{id} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & \mathrm{id} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \mathrm{id} \end{pmatrix}$$

where the extra 1 sits in the $(1,k)$-entry. Both these $m \times m$-matrices lie in $\mathrm{GL}(m,\mathbb{Z})$. If we do the same operations for columns, then this corresponds to right multiplication by the same type of matrices in $\mathrm{GL}(n,\mathbb{Z})$ as above.

Now let A be an $m \times n$–matrix over $\mathbb{Z}$. It follows from the above discussion that it suffices to show that we can turn A into the desired form using row and column operations.

If A is the zero matrix, then there is nothing to show. So suppose that A is not the zero matrix. We perform the following steps:

(a) After swapping rows and columns we can arrange that the absolute value of the $(1,1)$-entry is less or equal than the absolute value of all other entries. After possibly multiplying the first column by -1 we can assume that the $(1,1)$-entry v_{11} is positive.
(b) By adding and subtracting the first column from the other columns we can arrange that all entries in the first row, except for the $(1,1)$-entry, lie between 0 and $v_{11} - 1$.
(c) The same way as in (2) we can arrange that all entries in the first column, except for the $(1,1)$-entry, lie between 0 and $v_{11} - 1$.

Now we have to distinguish two cases:

(1) If there exists outside of the $(1,1)$-entry another entry in the first row or column that is non-zero, then we start again with (1). Note that the new entry $(1,1)$-entry will be less than before. So this process will come to a halt after finitely many steps.
(2) If all entries in the first row and column, except for the $(1,1)$-entry are zero, then

$$A = \begin{pmatrix} v_{11} & 0 \\ 0 & A' \end{pmatrix}$$

for some $(m-1)\times(n-1)$–matrix A' and we perform the same operations on the smaller matrix A'. ∎

18.5. **Appendix: Group rings.** In this chapter we learned about the free abelian group generated by a set. If we look at the free abelian group generated by group, we actually get a new and interesting mathematical object, namely a ring:

Definition. Let G be a group and let R be a commutative ring (in most cases $R = \mathbb{Z}$). We denote the free R-module generated[166] by the set G by $R[G]$. Put differently we have

$$R[G] \coloneqq \left\{ \sum_{i=1}^{m} r_i \cdot g_i \,\middle|\, r_1, \dots, r_m \in R \text{ and } g_1, \dots, g_m \in G \right\}.$$

We equip $R[G]$ with the multiplication that is given by[167]

$$\left(\sum_{i=1}^{m} r_i \cdot g_i \right) \cdot \left(\sum_{j=1}^{n} s_j \cdot h_j \right) \coloneqq \sum_{i=1}^{m} \sum_{j=1}^{n} \underbrace{r_i \cdot_R s_j}_{\in R} \cdot \underbrace{(g_i \cdot_G h_j)}_{\in G}.$$

We refer to $R[G]$ as the group ring of G with R-coefficients. (For $R = \mathbb{Z}$ we refer to $\mathbb{Z}[G]$ as the group ring of G.)

The following lemma summarizes a few basic facts about group rings.

Lemma 18.12. (Group Ring Lemma) Let G be a group and let R be a commutative ring.

(1) The group ring $R[G]$ is an associative ring with a multiplicatively neutral element, namely $1_R \cdot e$ where e denotes the trivial element in G.
(2) The group ring $R[G]$ is commutative if and only if G is an abelian group.
(3) If R is a field, then $R[G]$ is an algebra over the field R.
(4) Given a unit $r \in R$ and an element $g \in G$ the product $r \cdot g$ is a unit in $R[G]$.
(5) If G has a torsion element, i.e. if there exists a non-trivial $g \in G$ such that $g^k = e$ for some $k \in \mathbb{N}$, then $R[G]$ has zero divisors.

Proof. All the statements are verified easily. Perhaps the most interesting statement is (5). We will not rob the reader of the pleasure of proving this statement on their own. ■

Example. Let R be any commutative ring.

(1) We consider the infinite cyclic group $\langle t \rangle := \{\dots t^{-2}, t^{-1}, 1, t, t^2, \dots\}$. In this case the group ring $R[\langle t \rangle]$ is isomorphic to the ring $R[t^{\pm 1}]$ of Laurent polynomials, more precisely the "obvious" map
$$\begin{array}{rcl} R[t^{\pm 1}] & \to & R[\langle t \rangle] \\ \sum\limits_{i=r}^{i=s} a_i \cdot t^i & \mapsto & \sum\limits_{i=r}^{i=s} a_i \cdot t^i \end{array}$$
is clearly a ring isomorphism.
(2) If H is a free abelian group of rank m, then any choice of a basis $t_1, \dots, t_m$ for H gives rise to a natural isomorphism from the multivariable Laurent polynomial ring $R[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$ to the group ring $R[H]$. □

Lemma 18.13. (Group Ring Functor Lemma) Let R be a commutative ring. The map
$$\begin{array}{rcl} \mathrm{Ob}(\mathcal{Group}) & \to & \mathrm{Ob}(\mathcal{Ring}) \\ G & \mapsto & R[G] \end{array}$$
and the maps
$$\begin{array}{rcl} \mathrm{Mor}_{\mathcal{Group}}(G, H) & \to & \mathrm{Mor}_{\mathcal{Ring}}(R[G], R[H]) \\ (\varphi\colon G \to H) & \mapsto & \left(\begin{array}{rcl} \varphi_*\colon R[G] & \to & R[H] \\ \sum\limits_{i=1}^{m} r_i \cdot g_i & \mapsto & \sum\limits_{i=1}^{m} r_i \cdot \varphi(g_i) \end{array} \right) \end{array}$$
define a covariant functor from the category $\mathcal{Group}$ of groups to the category $\mathcal{Ring}$ of rings.

Proof. One can easily verify that the induced map $\varphi_*\colon R[G] \to R[H]$ is indeed a ring homomorphism. All other statements are basically trivial. ■

We continue with the following elementary lemma:

Lemma 18.14. (Group Ring–Module Lemma) Let G be a group and let A be an abelian group (with additive notation for the group structure). Suppose we are given a

[166]The free R-module generated by a set is defined in the same way as we defined the free abelian group generated by a set.
[167]The multiplication is basically given by naive "multiplying out".

G-action on the group A. Then the map[168]

$$\begin{array}{rcl} \mathbb{Z}[G] \times A & \to & A \\ \left(\sum_{i=1}^{m} r_i \cdot g_i, a \right) & \mapsto & \sum_{i=1}^{m} r_i \cdot (\underbrace{g_i \cdot a}_{\in A}) \end{array}$$

defines a natural left $\mathbb{Z}[G]$-module[169] structure on A.

Proof. The lemma is indeed elementary and trivial. We leave it to the reader to ponder what "natural" might mean in this context. ■

As a side remark we mention some (partially obsolete) conjectures. These say (or said) in particular that in some sense the statements of the Group Ring Lemma 18.12 (4) and (5) are optimal.

Conjecture 18.15. (Kaplansky Conjecture) Let G be a torsion-free group and let $\mathbb{F}$ be a field.

(1) The elements 0 and 1 are the only idempotents of $\mathbb{F}[G]$, i.e. they are the only elements x in $\mathbb{F}[G]$ with $x^2 = x$.
(2) The group ring $\mathbb{F}[G]$ is a domain, i.e. it has no non-trivial zero divisors.

Historic Conjecture 18.15. (Kaplansky Conjecture) Let G be a torsion-free group and let $\mathbb{F}$ be a field.

(3) The only units in $\mathbb{F}[G]$ are the trivial units, i.e. the elements of the form $f \cdot g$ with $f \in \mathbb{F} \setminus \{0\}$ and $g \in G$.

Conjecture 18.16. (Skew Field Conjectures) Given any torsion-free group G the group ring $\mathbb{C}[G]$ can be embedded in a skew field.

Remark.

(1) The three conjecture in the Kaplansky Conjecture 18.15 are named after the Canadian mathematician Irving Kaplansky [**Kap, Kap70**]. The Kaplansky Conjecture 18.15 (1) is often also referred to as the *zero divisor conjecture*. These conjectures are formulated in [**Lüc02**, Conjecture 10.14], [**Lam91**, p. 95] and [**FG05**, p. 729]. It is elementary to see that the Skew Field Conjectures 18.16 implies the Kaplansky Conjecture 18.15 (2) and that the Kaplansky Conjecture 18.15 (2) implies the Kaplansky Conjecture 18.15 (1).

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[168]Here we use a somewhat perilous notation, namely we use the symbol "$\cdot$" with two different meanings. First of all, for $g \in G$ and $a \in A$ we denote by $g \cdot a \in A$ the element defined by the G-action on A. Secondly, for $r \in \mathbb{Z}$ and $b \in A$ we write, as usual, $r \cdot b = \underbrace{b + \cdots + b}_{r\text{-times}}$ for $r \geq 0$ and $r \cdot b = \underbrace{-b - \cdots - b}_{(-r)\text{-times}}$ for $r < 0$.

[169]If R is a non-commutative ring, then one needs to be a little careful when one talks about "R-modules". More precisely, there are now two different concepts one can consider: Given an abelian group M one can demand that for any $r, s \in R$ and $m \in M$ one has $(rs) \cdot m = r \cdot (s \cdot m)$ in which case one says that M is an "R-left module". Alternatively one could demand that for any $r, s \in R$ and $m \in M$ one has $(rs) \cdot m = s \cdot (r \cdot m)$ in which case one says that M is an "R-right module". This discussion is the analogue of the discussion on page 48.

(2) The Kaplansky Conjecture 18.15 (3) is by now obsolete since Giles Gardam [**Gar21**, Theorem A][**?**, Theorem A] and Alan Murray [**Mur21**] showed that for any characteristic there exists a field $\mathbb{F}$ and a torsion-free group G such that $\mathbb{F}[G]$ has non-trivial units.

(3) The Kaplansky Conjectures are very attractive since they are easy to state. But it is usually very very hard to prove non-trivial group-theoretic statements that are valid for all (torsion-free) groups. In particular the Kaplansky Conjectures 18.15 (1) and (2) are still wide open. We refer to [**Lüc09**, Chapter 4.5] for a summary of what is known regarding these two conjectures. □

CHAPTER 19

Basics of group theory II: The free & amalgamated product of groups

19.1. **The free product of groups.** In the Coproduct-of-Abelian Groups Lemma 18.1 we saw that the coproduct of a family $\{X_s\}_{s\in S}$ of *abelian* groups in the category $\mathcal{AbGroup}$ of *abelian* groups is given by the direct sum $\bigoplus_{s\in S} X_s$. In this section we want to determine the coproduct of any family of groups in the category $\mathcal{Group}$ of all groups. The following definition is the key to doing so.

Definition. Let $\{X_s\}_{s\in S}$ be a family of groups. (After possibly replacing the family $\{X_s\}_{s\in S}$ by the family $\{X_s \times \{s\}\}_{s\in S}$ we can and will assume that the groups X_s are pairwise disjoint.)

(1) A reduced word in $\{X_s\}_{s\in S}$ is a finite sequence $(x_1, \dots, x_m)$ such that the following conditions are satisfied:
   (a) each x_i lies in one of the groups X_s,[170]
   (b) no x_i is the neutral element of any X_s,
   (c) any two consecutive x_j's lie in two different groups.
   Note that we also allow the "empty sequence" ().

(2) We denote by $\ast_{s\in S} X_s$ the set of all reduced words.

(3) Let $(x_1, \dots, x_m)$ and $(y_1, \dots, y_n)$ be two elements in $\ast_{s\in S} X_s$. We define the product of these two elements as follows:[171]
   (a) First suppose that x_m and y_1 lie in different groups. Then we define
   $$(x_1, \dots, x_m) \cdot (y_1, \dots, y_n) \;\coloneqq\; (x_1, \dots, x_m, y_1, \dots, y_n).$$
   (b) Now suppose that x_m and y_1 lie in the same group. Let $j \in \{1, \dots, \min\{m, n\}\}$ be the maximal element such that x_{m+1-i} and y_i lie in the same group and such that $x_{m+1-i} = y_i^{-1}$ for $i = 1, \dots, j$. If $j < \min\{m, n\}$, then we set[172]
   $$(x_1, \dots, x_m) \cdot (y_1, \dots, y_n) \;\coloneqq\; (x_1, \dots, \underbrace{x_{m-j} \cdot y_{j+1}}_{\neq e}, \dots, y_n).$$
   If $j = m$, we set $\quad (x_1, \dots, x_m) \cdot (y_1, \dots, y_n) \;\coloneqq\; (y_{m+1}, \dots, y_n).$
   We proceed similarly if $j = n$.

(4) We refer to $\ast_{s\in S} X_s$ together with this product structure as the free product of the X_s. For a finite family $X_1, \dots, X_k$ of groups we denote the free product by $X_1 * \cdots * X_k$.

To consider our first example of a free product we introduce one new definition.

Definition. Let t be a symbol. We define

$$\langle t\rangle \;:=\; \{\dots, t^{-2}, t^{-1}, e, t, t^2, \dots\}.$$

The set $\langle t\rangle$ admits an obvious group structure that is given by $t^i \cdot t^j := t^{i+j}$. This group $\langle t\rangle$ is called the infinite cyclic group generated by t. The map $\mathbb{Z} \to \langle t\rangle$, $n \mapsto t^n$ is evidently an isomorphism.

Example. We consider the free product $\langle s\rangle * \langle t\rangle$. Examples of elements in this free product are (s^3, t^{-5}, s^2) and $(s^{-2}, t^4, s^2, t^{-1})$. The product of these two elements is given by

$$(s^3, t^{-5}, s^2) \cdot (s^{-2}, t^4, s^2, t^{-1}) \;=\; (s^3, \underbrace{t^{-5} \cdot t^4}_{=t^{-1}}, s^2, t^{-1}).$$

Note that $(s) \cdot (t) = (s,t) \neq (t,s) = (t) \cdot (s)$. Thus we see that the free product $\langle s\rangle * \langle t\rangle$ is very much non-abelian. □

We have the following important lemma, which allows us to view each free product of groups as a group in its own right.

Lemma 19.1. (Free Product Lemma) Let $\{X_s\}_{s\in S}$ be a family of groups.

(1) The free product $\ast_{s\in S} X_s$ with the above product is again a group.

(2) The neutral element of $\ast_{s\in S} X_s$ is given by the empty sequence $()$.

(3) The inverse of an element $(x_1, \dots, x_m) \in \ast_{s\in S} X_s$ is given by $(x_m^{-1}, \dots, x_1^{-1}) \in \ast_{s\in S} X_s$.

Proof. It is clear that the empty sequence is a neutral element. It follows immediately from the definition of the product that the inverse of an element $(x_1, \dots, x_m) \in \ast_{s\in S} X_s$ is given by $(x_m^{-1}, \dots, x_1^{-1})$. It remains to show that the multiplication satisfies associativity.

In principle one can show associativity naively "by hand", but there are surprisingly many cases one needs to distinguish, so that the proof becomes rather painful.

We will therefore follow a different approach. We write $W := \ast_{s\in S} X_s$ and we denote by $(\mathrm{Bij}(W), \circ)$ the group of bijections $W \to W$, where the group structure is given by

[170] Since the groups X_s are disjoint we know that for each x_i there exists a unique $s \in S$ with $x_i \in X_s$.

[171] A more informal way of stating the product formula is as follows: Given two reduced words $(x_1, \dots, x_m)$ and $(y_1, \dots, y_n)$ we put them next to another to obtain $(x_1, \dots, x_m, y_1, \dots, y_n)$ and we do the following:

(a) We delete any occurrence of a subsequence of the form a, a^{-1} for $a \in X_s$.
(b) If there exists a subsequence of the form a, b with $a, b \in X_s$ and $a \neq b^{-1}$, then we replace it by $a \cdot b \in X_s$.

This way we end up with a reduced word. The problem with this informal definition is that it is not entirely clear that mixing (a) and (b) we have a well-defined procedure.

[172] Since x_m and y_1 lie in the same group, the elements x_{m-j} and y_{j+1} also lie in the same group, so it makes sense to consider $x_{m-j} \cdot y_{j+1}$.

composition of maps. For $g \in X_s$ we consider the map[173]

$$\begin{array}{rcl} \lambda_g \colon W & \to & W \\ (x_1, \ldots, x_m) & \mapsto & \begin{cases} (g, x_1, x_2, \ldots, x_m), & \text{if } x_1 \notin X_s, \\ (g \cdot x_1, x_2, \ldots, x_m), & \text{if } x_1 \in X_s \text{ and } g \neq x_1^{-1}, \\ (x_2, \ldots, x_m), & \text{if } g = x_1^{-1}. \end{cases} \end{array}$$

We make the following two observations:

(a) If $() \in W$ is the empty word, then $\lambda_{()} = \mathrm{id}$.

(b) It is straightforward to see that for any $g, g' \in X_s$ we have $\lambda_g \circ \lambda_{g'} = \lambda_{g \cdot g'} \colon W \to W$.

It follows from (a) and (b) that for any $g \in X_s$ we have $\lambda_{g^{-1}} \circ \lambda_g = \mathrm{id}_W$. This shows in particular that each λ_g is a bijection. Next we consider the map

$$\begin{array}{rcl} \lambda \colon W & \to & \mathrm{Bij}(W) \\ (x_1, \ldots, x_n) & \mapsto & \lambda_{x_1} \circ \cdots \circ \lambda_{x_n}. \end{array}$$

This map is injective since $\lambda(x_1, \ldots, x_n)$ applied to the empty word $()$ gives the word $(x_1, \ldots, x_n)$. Thus different elements of W define different bijections of W.

Using the above two observations (a) and (b) it is straightforward to verify that the map $\lambda \colon W \to \mathrm{Bij}(W)$ satisfies $\lambda(x \cdot y) = \lambda_x \circ \lambda_y$. Put differently, λ defines an injective map $\lambda \colon (W, \cdot\,) \to (\mathrm{Bij}(W), \circ)$ that preserves the product structure. But since $(\mathrm{Bij}(W), \circ)$ satisfies the associativity law we now see that $(W, \cdot\,)$ also satisfies the associativity law. ■

Example. Let G and H be two groups. We consider the corresponding free product $G * H$. Let $g \in G$ and $h \in H$ be two non-trivial elements. Note that in $G * H$ we have $(g) \cdot (h) = (g, h)$ and $(h) \cdot (g) = (h, g)$. These are different elements of the free product $G * H$. This shows, similar to the discussion on page 268, that the free product of at least two non-trivial groups is non-abelian. □

The following lemma, which can be viewed as an analogue of the Coproduct-of-Abelian Groups Lemma 18.1, summarizes the main properties of the free product of groups.

Lemma 19.2. (Coproduct-of-Groups Lemma) Let $\{X_s\}_{s \in S}$ be a family of groups.

(1) (a) Let $t \in S$. The map

$$\begin{array}{rcl} \iota_t \colon X_t & \to & \ast_{s \in S} X_s \\ x & \mapsto & \begin{cases} (x), & \text{if } x \neq e, \\ (), & \text{if } x = e \end{cases} \end{array}$$

is a monomorphism.

(b) Let Z be a group and let $\{f_s \colon X_s \to Z\}_{s \in S}$ be a family of homomorphisms. The map

$$\begin{array}{rcl} \Phi \colon \ast_{s \in S} X_s & \to & Z \\ (x_1, \ldots, x_m) & \mapsto & \underbrace{f_{s_1}(x_1) \cdot \ldots \cdot f_{s_m}(x_m)}_{\text{where } x_i \in X_{s_i}} \end{array}$$

is a homomorphism and it is the unique homomorphism such that for each $s \in S$ we have $f_s = \Phi \circ \iota_s$.

[173] It is straightforward to see that the sequences on the right do indeed lie in $\ast_{s \in S} X_s$.

(2) The coproduct in the category *Group* of groups of $\{X_s\}_{s\in S}$ is given by $\ast_{s\in S} X_s$ together with the homomorphisms $\iota_t\colon X_t \to \ast_{s\in S} X_s$ from (1a).

Proof.

(1) (a) Let $t \in T$. It is clear that the map $\iota_t\colon X_t \to \ast_{s\in S} X_s$ is injective and it follows immediately from the definition of the group structure on $\ast_{s\in S} X_s$ that the map is a group homomorphism.

(b) Let Z be a group and let $\{f_s\colon X_s \to Z\}_{s\in S}$ be a family of homomorphisms. One can easily verify that the given map $\Phi\colon \ast_{s\in S} X_s \to Z$ is a homomorphism and that it satisfies $f_s = \Phi \circ \iota_s$ for each $s \in S$. It remains to show the uniqueness of Φ. Any element in $\ast_{s\in S} X_s$ is a product of elements of the form $\iota_t(x)$ with $x \in X_t$. It follows that any homomorphism $\delta\colon \ast_{s\in S} X_s \to Z$ is already uniquely determined by the values of δ on all elements of the form $\iota_t(x)$ with $x \in X_t$. Put differently, there is at most one homomorphism $\delta\colon \ast_{s\in S} X_s \to Z$ with $\delta(\iota_t(x)) = f_t(\iota_t(x)) = (f_t \circ \iota_t)(x)$ for all $t \in T$ and all $x \in X_t$. This implies the uniqueness of Φ.

(2) This statement follows immediately from (1a) and (1b) and the definition of the coproduct on page 104. ∎

Convention. Let $\{X_s\}_{s\in S}$ be a family of groups.

(1) Let $t \in T$. We refer to

$$\begin{aligned} \iota_t\colon X_t &\to \ast_{s\in S} X_s \\ x &\mapsto \begin{cases} (x), & \text{if } x \neq e, \\ (), & \text{if } x = e \end{cases} \end{aligned}$$

as the natural monomorphism $\iota_t\colon X_t \to \ast_{s\in S} X_s$.

(2) For $t \in S$ we use the natural monomorphism $\iota_s\colon X_t \to \ast_{s\in S} X_s$ to view X_t as a subgroup of the free product $\ast_{s\in S} X_s$.

(3) Let $(x_1, \dots, x_m) \in \ast_{s\in S} X_s$. Since $(x_1, \dots, x_m) = (x_1)\cdot\ldots\cdot(x_m)$ we can use (2) to denote this element by $x_1 \cdot \ldots \cdot x_m$.

Example. We consider again the free product $\langle s\rangle * \langle t\rangle$. Using the above convention we now write $s^3 \cdot t^{-5} \cdot s^2 = (s^3, t^{-5}, s^2)$ and $s^{-2} \cdot t^4 \cdot s^2 \cdot t^{-1} = (s^{-2}, t^4, s^2, t^{-1})$. The product of these two elements is given by very reasonable looking

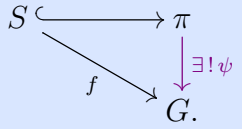
$$(s^3 \cdot t^{-5} \cdot s^2) \cdot (s^{-2} \cdot t^4 \cdot s^2 \cdot t^{-1}) \;=\; s^3 \cdot \underbrace{t^{-5} \cdot t^4}_{=t^{-1}} \cdot s^2 \cdot t^{-1}.$$

Also note that the discussion on page 268 shows that $s \cdot t \neq t \cdot s$, in particular we see that the free product $\langle s\rangle * \langle t\rangle$ is non-abelian. □

19.2. **Free groups.** In Section 18.2 we introduced the notion of a free *abelian* group. In this section we will modify the definition to introduce the notion of a free group. We start out with the following definition which is the analogue of the definition on page 256.

Definition.

(1) We say that a group π is a free group on a subset $S \subset \pi$, if the following universal property holds: For any *map* $f\colon S \to G$ of the set S to a group G there exists a unique *homomorphism* $\psi\colon \pi \to G$ that makes the following diagram commute:



We call S a basis of the free group π.

(2) We say that a group π is free if it is a free group on some subset S.

Lemma 19.3. (Free Group–Uniqueness Lemma) Let G be a free group on a subset $S \subset G$ and let H be a free group on a subset $T \subset H$. Given any bijection $f\colon S \to T$ there exists a unique isomorphism $\psi\colon G \to H$ such that $\psi(s) = f(s)$ for all $s \in S$.

Proof. The proof of this lemma is basically identical to the proof of the Free Abelian Group–Uniqueness Lemma 18.3. ∎

Definition. For a set S we write $\langle S\rangle \coloneqq \mathop{\ast}_{s\in S} \langle s\rangle$.

The lemma below allows us to call $\langle S\rangle$ the free group on the set S. The map $S \to \langle S\rangle$ that is given by $s \mapsto (s)$ is an injection and we use it to view S as a subset of $\langle S\rangle$.

The following lemma is now the analogue of the Free Abelian Group–Existence Lemma 18.4.

Lemma 19.4. (Free Group–Existence Lemma) Let S be a set. The group $\langle S\rangle$ is a free group on the subset $S \subset \langle S\rangle$.

Proof. Let G be a group and let $f\colon S \to G$ be a map. For $s \in S$ the map

$$\begin{array}{rcl} \psi_s\colon \langle s\rangle & \to & G \\ s^n & \mapsto & f(s)^n \end{array}$$

is evidently the unique homomorphism $\psi_s\colon \langle s\rangle \to G$ with $\psi_s(s) = f(s)$. It now follows from the Coproduct-of-Groups Lemma 19.2 (2) that the map

$$\begin{array}{rcl} \psi\colon \langle S\rangle = \mathop{\ast}_{s\in S} \langle s\rangle & \to & G \\ s_{i_1}^{n_1} \cdot \ldots \cdot s_{i_k}^{n_k} & \mapsto & f(s_{i_1})^{n_1} \cdot \ldots \cdot f(s_{i_k})^{n_k} \end{array}$$

is the unique homomorphism $\langle S\rangle \to G$ with $\psi(s) = \psi_s(s) = f(s)$ for $s \in S$. ∎

The Free Group–Existence Lemma 19.4 motivates the following definition:

Definition.

(1) For a set S we refer to $\langle S\rangle = \mathop{\ast}_{s\in S} \langle s\rangle$ as *the* free group on the set S.

(2) Let $s_1, \ldots, s_k$ be distinct elements of a set. We refer to

$$\langle s_1, \ldots, s_k\rangle \coloneqq \langle s_1\rangle * \cdots * \langle s_k\rangle$$

as the free group on the generators $s_1, \dots, s_k$. Note that each element of the free group $\langle s_1, \dots, s_k \rangle$ is uniquely of the form

$$s_{r_1}^{n_1} \cdot \ldots \cdot s_{r_m}^{n_m}$$

with $r_i \neq r_{i+1}$ for $i = 1, \dots, m-1$ and $s_i \neq 0$ for $i = 1, \dots, m$.

19.3. **The abelianization of a group.** It is pretty clear that abelian groups are much easier to handle than non-abelian groups. In this section we will get to know the abelianization of a group, which allows us in many settings to turn questions about arbitrary groups into questions about abelian groups.

Definition. Let G be a group.

(1) Given $x, y \in G$ we refer to $[x, y] := x \cdot y \cdot x^{-1} \cdot y^{-1}$ as the commutator of x and y.

(2) We refer to

$$[G, G] := \left\{ \prod_{i=1}^{n} [x_i, y_i] \,\middle|\, x_1, y_1, \dots, x_n, y_n \in G \right\}$$

as the commutator subgroup of G.

Remark. Note that in general not every element of the commutator subgroup is a commutator. For example, if $G = \langle a, b, c, d \rangle$ is the free group on four generators a, b, c, d, then we have $[a, b] \cdot [c, d] \in [G, G]$. But one can easily show that $[a, b] \cdot [c, d]$ is not a commutator in G. □

Lemma 19.5. (Commutator Subgroup Lemma) Let G be a group.

(1) The commutator subgroup $[G, G]$ is a normal subgroup of G.

(2) The group $G_{\mathrm{ab}} := G/[G, G]$ is abelian.

(3) Let $\alpha \colon G \to H$ be a homomorphism to an abelian group. There exists a unique homomorphism $\beta \colon G_{\mathrm{ab}} = G/[G, G] \to H$ such that the following diagram commutes:[174]

$$\begin{array}{ccc} G & & \\ {\scriptstyle g \mapsto [g]}\downarrow & \searrow^{\alpha} & \\ G_{\mathrm{ab}} = G/[G,G] & \xrightarrow[\exists!\,\beta]{} & H \text{ abelian}. \end{array}$$

This homomorphism β is given by $\beta([g]) = \alpha(g)$.

Proof. Let G be a group.

(1) First we show that $[G, G]$ is a subgroup of G:

- Evidently the subset $[G, G]$ is closed under products.
- With $n = 0$ we see that $e \in [G, G]$.
- The inverse of a commutator is again a commutator; indeed given $x, y \in G$ we have

$$[x, y]^{-1} = (xyx^{-1}y^{-1})^{-1} = yxy^{-1}x^{-1} = [y, x].$$

It follows easily that $[G, G]$ is closed under taking inverses.

[174] Another way of formulating this statement is to say that for any group G and any abelian group H the map

$$\begin{array}{ccc} \mathrm{Hom}(G_{\mathrm{ab}}, H) & \to & \mathrm{Hom}(G, H) \\ (\alpha \colon G_{\mathrm{ab}} \to H) & \mapsto & \left(G \xrightarrow{g \mapsto [g]} G_{\mathrm{ab}} \xrightarrow{\alpha} H\right) \end{array}$$

is a bijection.

We have thus shown that $[G,G]$ is a subgroup of G. It remains to show that $[G,G]$ is a normal subgroup of G. The key to doing so is the following claim:

Claim. A conjugate of a commutator is again a commutator.

Proof. Given $x, y, g \in G$ we have

$$\begin{aligned} g[x,y]g^{-1} = gxyx^{-1}y^{-1}g^{-1} &= gxg^{-1}gyg^{-1}gx^{-1}g^{-1}gy^{-1}g^{-1} \\ &= (gxg^{-1})(gyg^{-1})(gxg^{-1})^{-1}(gyg^{-1})^{-1} = [gxg^{-1}, gyg^{-1}]. \end{aligned}$$ ⊞

Given $x_1, \dots, x_n, y_1, \dots, y_n \in G$ and given $g \in G$ we now see that

$$g \cdot \Big(\prod_{i=1}^{n} [x_i, y_i] \Big) g^{-1} \underset{\substack{\uparrow \\ \text{since } g(ab)g^{-1} = gag^{-1} \cdot gbg^{-1}}}{=} \prod_{i=1}^{n} g \cdot [x_i, y_i] \cdot g^{-1} \underset{\substack{\uparrow \\ \text{by the claim}}}{=} \prod_{i=1}^{n} [gx_ig^{-1}, gy_ig^{-1}] \in [G,G].$$

We have thus shown that $[G,G]$ is a normal subgroup.

(2) For any two elements $x, y \in G$ we have

$$xy\,[G,G] = yx \cdot x^{-1}y^{-1}xy\,[G,G] = yx \cdot \underbrace{[x^{-1}, y^{-1}]}_{\in [G,G]}\,[G,G] = yx\,[G,G].$$

This shows that $G_{\mathrm{ab}} = G/[G,G]$ is indeed abelian.

(3) Let G be a group and let $\alpha\colon G \to H$ be a homomorphism to an abelian group. Using the fact that H is abelian one can easily verify that $[G,G]$ lies in the kernel of α. The desired statement now follows immediately from the universal property of quotient groups. ■

Definition. Let G be a group.

(1) We call $G_{\mathrm{ab}} := G/[G,G]$ the abelianization of G.

(2) We denote by
$$\begin{aligned} \Phi_G\colon G &\to G_{\mathrm{ab}} = G/[G,G] \\ g &\mapsto [g] \end{aligned}$$
the natural projection.

(3) Let $\alpha\colon G \to H$ be a homomorphism. By the Commutator Subgroup Lemma 19.5 (3), applied to the abelian group H_{ab} and the homomorphism $\Phi_H \circ \alpha\colon G \to H_{\mathrm{ab}}$, there exists a unique homomorphism $\alpha_*\colon G_{\mathrm{ab}} \to H_{\mathrm{ab}}$ such that the following diagram commutes:

$$\begin{array}{ccc} G & \xrightarrow{\alpha} & H \\ {\scriptstyle \Phi_G}\downarrow & & \downarrow{\scriptstyle \Phi_H} \\ G_{\mathrm{ab}} & \xrightarrow{\alpha_*} & H_{\mathrm{ab}}. \end{array}$$

We refer to α_* as the induced homomorphism. Note that α_* is given by $\alpha_*([g]) = [\alpha(g)]$.

Example. Given $k \in \mathbb{N}$ we denote by S_k the permutation group on k elements and we denote by $A_k \subset S_k$ the alternating group, i.e. the group of all permutations with positive sign. In basically every standard algebra book (see [**Rob95**, p. 69] or [**Bog08**, Theorem 11.3]) it is shown that for $k \geq 5$ the group A_k is simple, i.e. the only normal subgroups of A_k are the trivial group and the group A_k itself. We just saw in the Commutator Subgroup Lemma 19.5 (1) that the commutator subgroup $[A_k, A_k]$ is normal. Since A_k is not abelian we cannot have $[A_k, A_k] = \{e\}$. Thus we see that $[A_k, A_k] = A_k$, which implies that the abelianization of A_k is the trivial group. □

Proposition 19.6. (Abelianization Functor Proposition)

(1) The maps
$$\begin{array}{rcl} G & \mapsto & G_{\mathrm{ab}} = G/[G,G] \\ (\alpha\colon G \to H) & \mapsto & (\alpha_*\colon G_{\mathrm{ab}} \to H_{\mathrm{ab}}) \end{array}$$
define a covariant functor from the category *Group* of groups to the category *AbGroup* of abelian groups.

(2) If $\varphi\colon G \to H$ is an isomorphism of groups, then the induced map $\varphi_*\colon G_{\mathrm{ab}} \to H_{\mathrm{ab}}$ is also an isomorphism. In particular, isomorphic groups have isomorphic abelianizations.

Proof.

(1) This statement follows easily from the definitions.

(2) This statement follows from (1) and the fact that functors turn isomorphisms into isomorphisms. ■

The following lemma will be used on many occasions.

Lemma 19.7. (Abelianization-Epimorphism Lemma) If $\alpha\colon G \to H$ is an epimorphism, then the induced map $\alpha_*\colon G_{\mathrm{ab}} \to H_{\mathrm{ab}}$ is also an epimorphism.

Proof. By definition of α_* we have the following commutative diagram:
$$\begin{array}{ccc} G & \xrightarrow{\alpha} & H \\ {\scriptstyle \Phi_G}\downarrow & & \downarrow{\scriptstyle \Phi_H} \\ G_{\mathrm{ab}} & \xrightarrow{\alpha_*} & H_{\mathrm{ab}}. \end{array}$$
By hypothesis $\alpha\colon G \to H$ is an epimorphism. It follows that $\Phi_H \circ \alpha\colon G \to H_{\mathrm{ab}}$ is an epimorphism, which implies that α_* is an epimorphism. ■

The following proposition allows us to calculate the abelianization in several basic situations.

Proposition 19.8. (Abelianization Calculation Proposition)

(1) If π is abelian, then the natural homomorphism $\Phi_\pi\colon \pi \to \pi_{\mathrm{ab}} = \pi/[\pi,\pi]$ is an isomorphism.

(2) Let $\{X_s\}_{s\in S}$ be a family of groups. We have natural homomorphisms
$$\begin{array}{rcl} \left(\underset{s\in S}{*} X_s\right)_{\mathrm{ab}} & \to & \bigoplus\limits_{s\in S} (X_s)_{\mathrm{ab}} \\ [(x_1,\dots,x_k)] & \mapsto & \sum\limits_{i=1}^{k} [x_i] \end{array} \qquad \text{and} \qquad \begin{array}{rcl} \bigoplus\limits_{s\in S} (X_s)_{\mathrm{ab}} & \to & \left(\underset{s\in S}{*} X_s\right)_{\mathrm{ab}} \\ \sum\limits_{i=1}^{k} [x_i] & \mapsto & [x_1 \cdot \ldots \cdot x_k]. \end{array}$$
These are inverses of one another, in particular they are isomorphisms.

(3) Let S be a set. The map
$$\begin{array}{rcl} \langle S\rangle_{\mathrm{ab}} & \to & \mathbb{Z}^{(S)} \\ [s_{i_1}^{n_1} \cdot \ldots \cdot s_{i_k}^{n_k}] & \mapsto & \sum\limits_{i=1}^{k} n_i \cdot s_i \end{array}$$
is a natural isomorphism.

Proof of the Abelianization Calculation Proposition 19.8.

(1) If π is abelian, then every commutator is trivial, thus $[\pi,\pi] = \{e\}$. This implies that the natural homomorphism $\Phi_\pi\colon \pi \to \pi/[\pi,\pi]$ is an isomorphism.

(2) It is worth explaining where the two maps in the proposition come from.
   (a) By the Coproduct-of-Abelian Groups Lemma 18.1 (1a) we have for each $s \in S$ a natural homomorphism $X_s \to (X_s)_{\mathrm{ab}} \to \bigoplus_{s\in S}(X_s)_{\mathrm{ab}}$. By the Coproduct-of-Groups Lemma 19.2 these give rise to a homomorphism $\ast_{s\in S} X_s \to \bigoplus_{s\in S}(X_s)_{\mathrm{ab}}$. Since the target is abelian we obtain from the Commutator Subgroup Lemma 19.5 (3) an induced homomorphism $\left(\ast_{s\in S} X_s\right)_{\mathrm{ab}} \to \bigoplus_{s\in S}(X_s)_{\mathrm{ab}}$.
   (b) By the Coproduct-of-Groups Lemma 19.2 (1a) we have for each $s \in S$ a natural homomorphism $X_s \to \ast_{s\in S} X_s$. This homomorphism induces a homomorphism $(X_s)_{\mathrm{ab}} \to \left(\ast_{s\in S} X_s\right)_{\mathrm{ab}}$. By the Coproduct-of-Abelian Groups Lemma 18.1 (1b) these homomorphisms gives rise to a homomorphism $\bigoplus_{s\in S}(X_s)_{\mathrm{ab}} \to \left(\ast_{s\in S} X_s\right)_{\mathrm{ab}}$.

   If one goes through the explicit descriptions of the homomorphisms one sees that the above two homomorphisms are precisely the homomorphisms we wrote down in the proposition. This shows that the maps of the proposition are well-defined and that they are homomorphisms. Now that we know that the given maps are well-defined it is pretty clear that they are inverses of one another.
(3) This statement follows immediately from the statements (1) and (2) together with the observation that by definition we have $\langle S\rangle = \ast_{s\in S}\langle s\rangle$ and $\mathbb{Z}^{(S)} = \bigoplus_{s\in\mathbb{Z}}\mathbb{Z}$. ■

We conclude this section with a proposition which be viewed as an analogue of the Basis Cardinality Theorem 18.6 for free abelian groups.

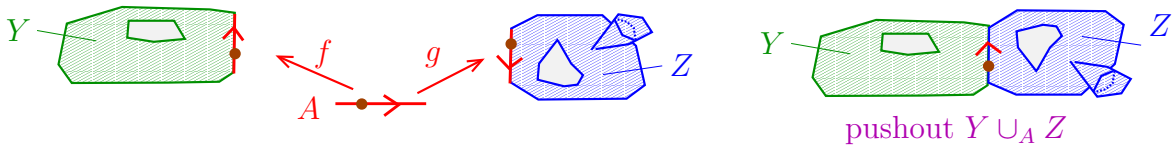
Proposition 19.9. (Basis Cardinality Proposition) Let G be a group that is free on a subset S and let H be a group that is free on a subset $T \subset H$. If G and H are isomorphic, then $\#S = \#T$.

Proof. It follows immediately from the Free Group–Uniqueness Lemma 19.3 that we can assume that $G = \langle S\rangle$ and that $H = \langle T\rangle$.
(a) Let $\alpha\colon \langle S\rangle \to \langle T\rangle$ be an isomorphism.
(b) It follows from the Abelianization Functor Proposition 19.6 (2) that $\alpha_*\colon \langle S\rangle_{\mathrm{ab}} \to \langle T\rangle_{\mathrm{ab}}$ is also an isomorphism.
(c) It follows from (b) and the Abelianization Calculation Proposition 19.8 (3) that $\mathbb{Z}^{(S)}$ and $\mathbb{Z}^{(T)}$ are isomorphic.
(d) It follows from (c) and the Basis Cardinality Theorem 18.6 that $\#S = \#T$. ■

19.4. **The amalgamated product of groups.** Suppose we are given two continuous maps $f\colon (A, a_0) \to (Y, y_0)$ and $g\colon (A, a_0) \to (Z, z_0)$ between pointed topological spaces. Note that by the same argument as in the Topological Pushout Proposition 6.12 we see that the topological space

$$Y \cup_A Z \;:=\; (Y \sqcup Z)/\!\sim \qquad \text{where } f(a) \sim g(a) \text{ for all } a \in A$$

together with the base points $[y_0] = [z_0]$ is the pushout of the two maps in the category of pointed topological spaces.
In Section 6.7 we saw that the pushout is a useful tool to construct and describe (pointed) topological spaces. Two questions now arise:
(1) Does the fundamental group preserve pushouts?
(2) What on earth is the pushout in the category *Group* of groups?



In the remainder of this section we will answer Question (2). We will return to Question (1) in the next chapter, when we will formulate and prove the Seifert–van Kampen Theorem.

The construction of the pushout in the category *Group* of groups will be formally quite similar to the above explicit description of the pushout in the category of pointed topological spaces. We first take a coproduct, afterwards we take a quotient. One problem arises: Given a group G and a subgroup A the quotient G/A is in general not a group. Only if A is a normal subgroup, then G/A is in fact a group. This leads us to the following definition:

Definition. Let G be a group and let $A \subset G$ be a subset. We define

$$\text{subgroup of } G \text{ normally generated by } A \;:=\; \langle\langle A \rangle\rangle \;:=\; \bigcap_{\substack{A \subset H \lhd G \\ H \text{ normal subgroup}}} H,$$

i.e. $\langle\langle A \rangle\rangle$ is the intersection of all normal subgroups H of G that contain A. This is again a normal[175] subgroup of G.

Lemma 19.10. (Normal Closure–Factorization Lemma) Let G be a group and let $A \subset G$ be a subset. Let $\alpha\colon G \to \Gamma$ be a group homomorphism such that $\alpha(a) = e \in \Gamma$ for all $a \in A$. Then there exists a unique group homomorphism $\beta\colon G/\langle\langle A \rangle\rangle \to \Gamma$ such that the following diagram commutes:

$$\begin{array}{ccc} G & & \\ \downarrow & \searrow^{\alpha} & \\ G/\langle\langle A \rangle\rangle & \xrightarrow[\exists!\,\beta]{} & \Gamma. \end{array}$$

Proof. Let G be a group, let $A \subset G$ be a subset and let $\alpha\colon G \to \Gamma$ be a group homomorphism such that $\alpha(a) = e \in \Gamma$ for all $a \in A$. This means that $A \subset \ker(\alpha)$. Since $\ker(\alpha)$ is a normal subgroup of G it follows immediately from the definition of $\langle\langle A \rangle\rangle$ that $\langle\langle A \rangle\rangle \subset \ker(\alpha)$. The lemma follows from the universal property of the quotient group. ∎

The following proposition can be viewed as an analogue of the Topological Pushout Proposition 6.12 and the above discussion of the pushout in the category of pointed topological spaces.

Proposition 19.11. (Group Pushout Proposition) Let $\alpha\colon G \to A$ and $\beta\colon G \to B$ be two group homomorphisms. We consider the group

$$A *_G B \;:=\; (A * B)/\langle\langle \{\alpha(g) \cdot \beta(g)^{-1}\}_{g \in G} \rangle\rangle$$

and the maps

$$\begin{array}{rcl} \mu\colon A & \to & A *_G B \\ a & \mapsto & [(a)] \end{array} \qquad \text{and} \qquad \begin{array}{rcl} \nu\colon B & \to & A *_G B \\ b & \mapsto & [(b)] \end{array}$$

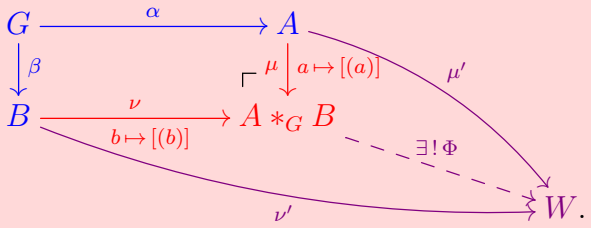
The following statements hold:

(1) (a) The maps $\mu\colon A \to A *_G B$ and $\nu\colon B \to A *_G B$ are homomorphisms.

(b) Suppose we are given two homomorphisms $\mu'\colon A \to W$ and $\nu'\colon B \to W$ to some group W such that $\mu' \circ \alpha = \nu' \circ \beta\colon G \to W$. By the Coproduct-of-Groups Lemma 19.2 the homomorphisms μ' and ν' induce a homomorphism $A * B \to W$. This homomorphism descends to a homomorphism

$$\Phi\colon A *_G B \;\to\; W.$$

This homomorphism is the unique homomorphism $\Phi\colon A *_G B \to W$ with $\Phi \circ \mu = \mu'$ and $\Phi \circ \nu = \nu'$. In particular we obtain the following commutative diagram:



(2) The group $A *_G B$ with the natural homomorphisms $\alpha\colon A \to A *_G B$ and $\beta\colon B \to A *_G B$ is the pushout, in the sense of the definition on page 111, in the category *Group* of groups.

Definition. Let $\alpha\colon G \to A$ and $\beta\colon G \to B$ be two group homomorphisms. We write

$$A *_G B \;:=\; (A * B)/\langle\langle \{\alpha(g) \cdot \beta(g)^{-1}\}_{g \in G} \rangle\rangle$$

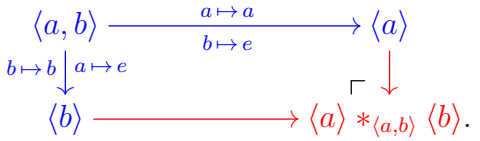
and we refer to $A *_G B$ as the amalgamated product of A and B with amalgam G.[176] We refer to the obvious homomorphisms $A \to A *_G B$ and $B \to A *_G B$ as the natural homomorphisms.

Proof.

(1) (a) The statement follows from the Coproduct-of-Groups Lemma 19.2 (1a).
(b) This statement follows easily from the Coproduct-of-Groups Lemma 19.2 (1b). together with the Normal Closure–Factorization Lemma 19.10.
(2) This statement follows easily from the definitions and Statement (1). We leave it to the reader to fill in the details. ∎

Example.

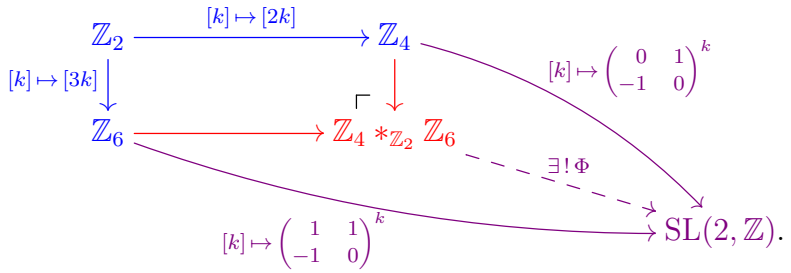
(1) We consider the following pushout diagram



It follows easily from the universal property of a pushout or from the explicit description of the amalgamated product that $\langle a \rangle *_{\langle a,b \rangle} \langle b \rangle = \{e\}$.

[176] The notation is a little misleading since the homomorphisms α and β do not get mentioned in the notation, even though they play of course an important role. In particular different choices of α and β can lead to non-isomorphic amalgamated products.

(2) We consider the following diagram:



Since the two maps from $\mathbb{Z}_2$ to $\mathrm{SL}(2, \mathbb{Z})$ agree we obtain from the Group Pushout Proposition 6.12 (1b) a unique homomorphism $\Phi \colon \mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6 \to \mathrm{SL}(2, \mathbb{Z})$ that makes the diagram commute. More surprisingly, this homomorphism is in fact an isomorphism. We refer to [**Ros94**, p. 218] and [**Ser80**, p. 35] for a proof. □

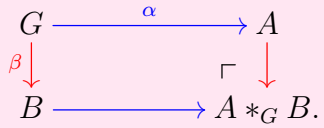
The following lemma summarizes some formal properties of the amalgamated product.

Lemma 19.12. (Amalgamated Product Lemma)

(1) For two groups A and B and the trivial group $G = \{e\}$, the corresponding amalgamated product is the usual free product, i.e.

$$A *_{\{e\}} B \ = A * B.$$

(2) Let $\alpha \colon G \to A$ and $\beta \colon G \to B$ be group homomorphisms. We consider the diagram



The following two statements hold:

(a) If the left vertical map β is an isomorphism, then the right vertical map is also an isomorphism and the inverse $A *_G B \to A$ is given by the identity on A and $\beta^{-1} \colon B \to G$.

(b) If the left vertical map β is an epimorphism, then the right vertical map is also an epimorphism.

The analogous statements of course hold with the roles of "A" and "B" swapped.

Proof. As we will now see, all statements follow from formal properties of pushouts:

(1) This statement follows from the fact that $\{e\}$ is the initial object of the category *Group* of groups and together with the Pushout Lemma 4.8.

(2) These two statements follows from the Pushout Lemma 4.8 (2a) and (2b). ■

We conclude this discussion of amalgamated products with the following which is specific to the group theoretic setting.[177]

Lemma 19.13. (Amalgamated Product–Quotient Lemma) If $\alpha \colon G \to A$ is a group homomorphism and if $\beta \colon G \to \{e\}$ is the trivial homomorphism, then the projection

[177] Note thought that the lemma is analogous to the discussion on page 143 where we discussed the pushout in the category of topological spaces where one of the two maps is given by the unique map $A \to \{*\}$ to the terminal object in the category *Top* of topological spaces.

$A \to A/\langle\langle \alpha(G) \rangle\rangle$ induces a natural isomorphism

$$A *_G \{e\} \xrightarrow{\cong} A/\langle\langle \alpha(G) \rangle\rangle.$$

Proof. The statement follows immediately from the definition of the amalgamated product and the natural isomorphism $A * \{e\} \xrightarrow{\cong} A$. ■

CHAPTER 20

The Seifert–van Kampen Theorem

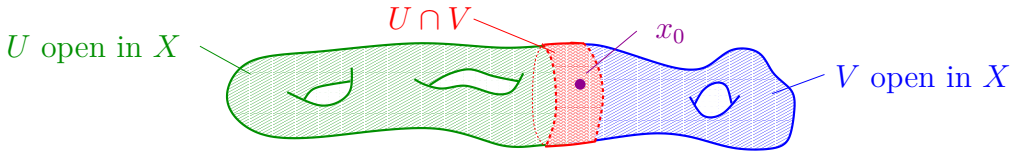
We are still lacking the computation of fundamental groups of even fairly simple topological spaces, for example we do not know the fundamental groups of wedges of circles, of topological graphs and of surfaces. In this chapter we will take a new approach to calculating fundamental groups, namely we will formulate and prove the Seifert–van Kampen Theorem which allow us to calculate the fundamental group of a given topological space in terms of the fundamental groups of suitable subspaces and in terms of inclusion-induced maps. Using the Seifert–van Kampen Theorem we will be able to calculate many of the missing fundamental groups.

20.1. **The formulation of the general Seifert–van Kampen Theorem.** The idea of the Seifert–van Kampen theorem is to reduce the calculation of the fundamental group of a topological space X to the determination of fundamental groups of "simpler subsets" of X.

More precisely: Let X be a topological space and let $U, V \subset X$ be two subsets such that $X = U \cup V$ and such that $U \cap V \neq \varnothing$. We choose a base point $x_0 \in U \cap V$ and we consider the following inclusion maps and the corresponding induced homomorphisms of the fundamental groups:

$$\begin{array}{ccc} U \cap V & \xrightarrow{\ i\ } & U \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle k} \\ V & \xrightarrow[\ l\ ]{} & X = U \cup V \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \pi_1(U \cap V, x_0) & \xrightarrow{\ i_*\ } & \pi_1(U, x_0) \\ {\scriptstyle j_*}\downarrow & & \downarrow{\scriptstyle k_*} \\ \pi_1(V, x_0) & \xrightarrow[\ l_*\ ]{} & \pi_1(X, x_0) = \pi_1(U \cup V, x_0). \end{array}$$

It is not difficult to show that if U and V are *open* subsets of X, then the diagram to the left is a pushout. The following theorem now says that if U and V are open and if $U \cap V$ is path-connected, then the diagram to the right is also a pushout.

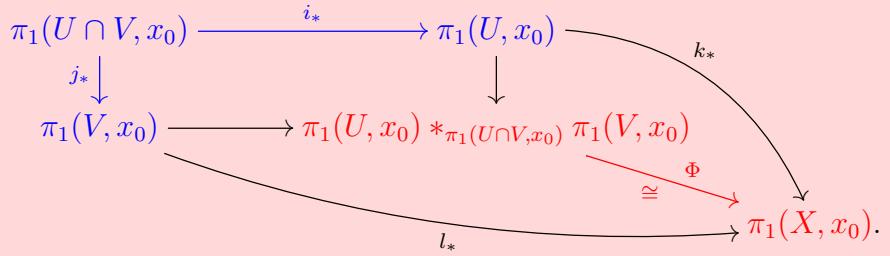


Theorem 20.1. (Seifert–van Kampen Theorem)[178] Let X be a topological space and let $X = U \cup V$ be a decomposition of X in two *open* subsets U and V such that $U \cap V$ is non-empty and *path-connected*. Let $x_0 \in U \cap V$. We denote by $i\colon U \cap V \to U$ and $j\colon U \cap V \to V$ the inclusions.

(1) (a) The inclusions $k\colon U \to X$ and $l\colon V \to X$ induce an isomorphism

$$\Phi\colon \pi_1(U, x_0) *_{\pi_1(U \cap V, x_0)} \pi_1(V, x_0) \xrightarrow{\ \cong\ } \pi_1(X, x_0)$$

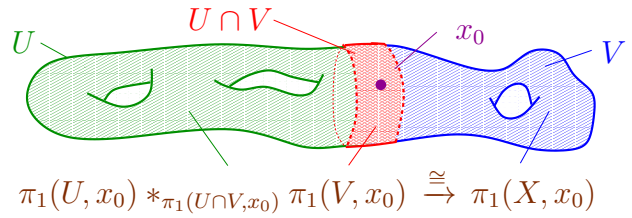
which has the property that the following diagram commutes:

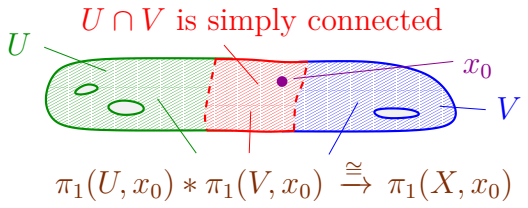


Here the "undecorated" maps are natural homomorphism to the amalgamated product.

(b) The group $\pi_1(X, x_0)$ is a pushout.

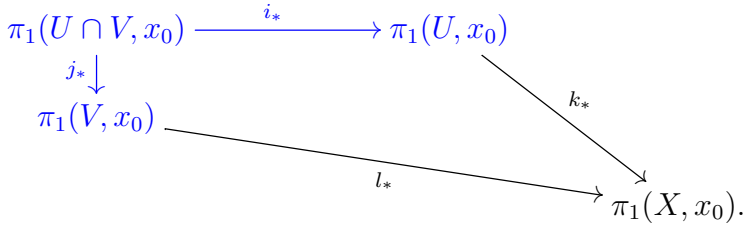
(2) If $U \cap V$ is simply connected, then the inclusions $U \to X$ and $V \to X$ induce an isomorphism $\Phi\colon \pi_1(U, x_0) * \pi_1(V, x_0) \xrightarrow{\cong} \pi_1(X, x_0)$





Summary of proof. First note that Statement (2) is a special case of Statement (1a). Furthermore note that Statement (1b) follows from Statement (1a) and the Group Pushout Proposition 6.12 (2),

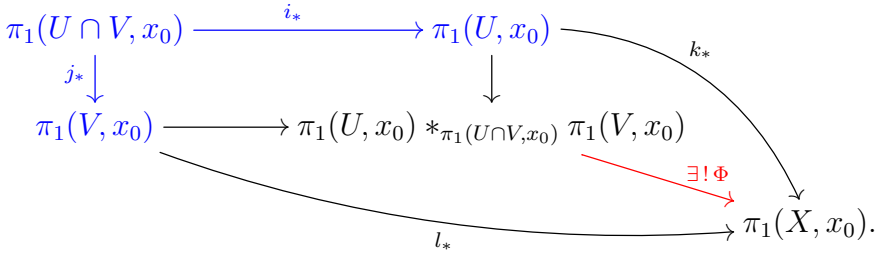
Thus it remains to prove Statement (1a). Note that the various inclusions of topological spaces give rise to the following commutative diagram:



By the Group Pushout Proposition 6.12 (1b) there exists a unique homomorphism

$$\Phi\colon \pi_1(U, x_0) *_{\pi_1(U\cap V, x_0)} \pi_1(V, x_0) \to \pi_1(X, x_0)$$

such that the following diagram commutes:



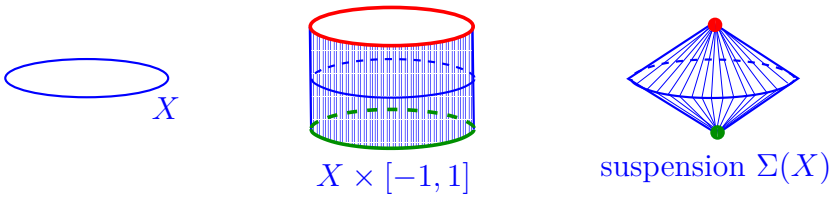
[178]The theorem was proved independently by Herbert Seifert (1907-1996) and Egbert van Kampen (1908-1942).

It remains to show that Φ is surjective and that Φ is injective. The plan is as follows:

(a) In Section 20.2 we will see that Φ is surjective.
(b) In Section 20.4 we will show that Φ is injective by showing that there exists a map $\Psi\colon \pi_1(X,x_0) \to \pi_1(U,x_0) *_{\pi_1(U\cap V,x_0)} \pi_1(V,x_0)$ with $\Psi \circ \Phi = \mathrm{id}$. To construct the map Ψ it is convenient to work with an alternative description of free and amalgamated products. We will explain this alternative point of view in Section 20.3. ■

As we outlined above, we will provide the longish proof of the Seifert–van Kampen Theorem 20.1 in the next three sections. But let us first consider an example to get a feel for the Seifert–van Kampen Theorem 20.1.

Definition. Let X be a non-empty topological space. We define the suspension of X to be the topological space
$$\Sigma(X) \;:=\; (X \times [-1,1])/\sim$$
where we identify all points in $X \times \{-1\}$ to a single point and we identify all points in $X \times \{1\}$ to a single point.



As a first application of the Seifert–van Kampen Theorem 20.1 we can now prove the following lemma:

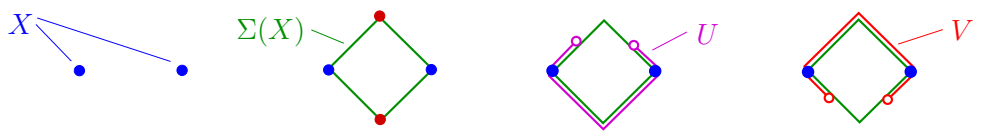
Lemma 20.2. (Suspension-π_1-Lemma) If X is a path-connected non-empty topological space, then its suspension $\Sigma(X) = (X \times [-1,1])/\sim$ is simply connected.

Example. One can easily show that the suspension $\Sigma(S^n)$ of the n-dimensional sphere is homeomorphic to the $(n+1)$-dimensional sphere S^{n+1}.

(1) By the Ball-Sphere-Connected Lemma 2.17 we know that for $n \geq 1$ the sphere S^n is path-connected. Thus it follows from the Suspension-π_1-Lemma 20.2 that for $n \geq 1$ we have $\pi_1(S^{n+1}) = 0$. In other words, we have obtained a new proof of the Sphere-π_1-Proposition 11.15.
(2) The example $n = 0$ together with the Circle-π_1-Theorem 11.13 shows that in the Suspension-π_1-Lemma 20.2 we cannot drop the hypothesis that the topological space X is path-connected.

□

Proof. Let X be a path-connected non-empty topological space. In the following we denote by $p\colon X \times [-1,1] \to \Sigma(X)$ the natural projection. We consider the two subsets $U = p(X \times [-1, \frac{1}{2}))$ and $V = p(X \times (-\frac{1}{2}, 1])$. It follows from the definition of the quotient topology on $\Sigma(X)$ that U and V are open subsets of $\Sigma(X)$. We pick a base point x on $X = X \times \{0\} \subset \Sigma(X)$.



We calculate that

$$\pi_1(\Sigma(X), x) \underset{\text{by the Seifert–van Kampen -Theorem 20.1 which we can apply since } U \cap V \cong X \times (-\frac{1}{2}, \frac{1}{2}) \text{ is path-connected}}{\cong} \pi_1(U, x) *_{\pi_1(U \cap V, x)} \pi_1(V, x) \underset{\text{one can easily show that } U \text{ and } V \text{ are contractible, hence their fundamental groups are trivial by the Homotopy-}\pi_1\text{-Proposition 13.4 (3)}}{=} \underset{\text{of trivial groups}}{\text{amalgamated product}} = \text{trivial group.}$$

We have thus shown that $\Sigma(X)$ is indeed simply connected. ■

In the following three sections we provide the proof of the Seifert–van Kampen-Theorem 20.1. Afterwards we will consider many more applications of the Seifert–van Kampen-Theorem 20.1.

20.2. Proof of the Seifert–van Kampen-Theorem 20.1: Surjectivity.

Proof of the Seifert–van Kampen-Theorem 20.1: Surjectivity. Let X be a topological space and let $U, V \subset X$ be two open subsets with $X = U \cup V$ and such that $U \cap V$ is path-connected and non-empty. We choose a base point $x_0 \in U \cap V$. We need to show that the map

$$\Phi \colon \pi_1(U, x_0) *_{\pi_1(U \cap V, x_0)} \pi_1(V, x_0) \to \pi_1(X, x_0)$$

from page 281 is surjective. We consider the following commutative diagram of homomorphisms:

$$\begin{array}{ccc} \pi_1(U, x_0) * \pi_1(V, x_0) & & \\ \downarrow & \overset{\Xi}{\searrow} & \\ \pi_1(U, x_0) *_{\pi_1(U \cap V, x_0)} \pi_1(V, x_0) & \xrightarrow[\Phi]{} & \pi_1(X, x_0) \end{array}$$

where the horizontal homomorphism Ξ and the left diagonal homomorphism are given by the Coproduct-of-Groups Lemma 19.2 (1b). It remains to prove that the horizontal homomorphism $\Xi \colon \pi_1(U, x_0) * \pi_1(V, x_0) \to \pi_1(X, x_0)$ is surjective. Clearly it suffices to prove the following claim:

Claim. Every loop $s \colon [0, 1] \to X = U \cup V$ in x_0 is path-homotopic to a loop which is the concatenation of finitely many loops, each of which lies in (U, x_0) or in (V, x_0).

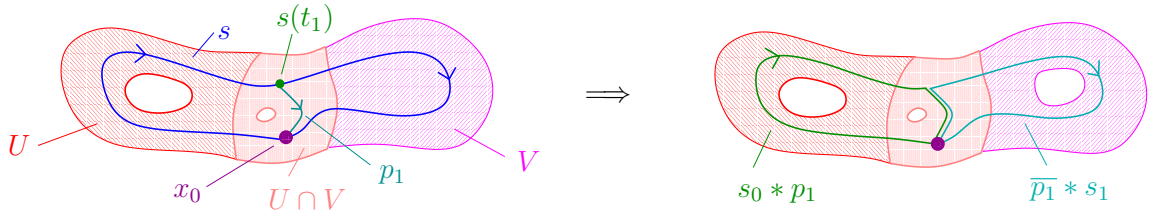
Proof. Let $s \colon [0, 1] \to X = U \cup V$ be a loop in x_0. Since U and V are *open* we can use the Lebesgue Lemma 11.9 to find a subdivision

$$0 = t_0 < t_1 < t_2 < \ldots < t_k < t_{k+1} = 1$$

of the interval $[0, 1]$ such that the image of each $s_i := s|_{[t_i, t_{i+1}]}$ lies in U or in V. The same argument as in the proof of the Sphere-π_1-Proposition 11.15 shows that we can assume that each $s(t_i)$ lies in $U \cap V$.[179] The problem is that even though the paths s_i lie in U respectively in V, they are not necessarily loops in x_0. But since $U \cap V$ is path-connected

[179] Indeed, if $s(t_i)$ lies in $U \setminus V$, then $s([t_{i-1}, t_i])$ and $s([t_i, t_{i+1}])$ cannot lie in V, hence they both lie in U. This implies that $s_{([t_{i-1}, t_{i+1}])}$ lies in U, and therefore we can remove t_i from the subdivision. We proceed similarly if $s(t_i)$ lie in $V \setminus U$.

we can find for each $i \in \{1, \dots, k\}$ a path p_i in $U \cap V$ that connects $s(t_i) \in U \cap V$ to the base point $x_0 \in U \cap V$.



Now we have the following path-homotopies:

$$s \underset{\text{p}}{\simeq} s_0 * \cdots * s_k \underset{\text{p}}{\simeq} s_0 * p_1 * \overline{p_1} * s_1 * p_2 * \overline{p_2} * s_2 * p_3 * \overline{p_3} * \ldots * \overline{p_k} * s_k$$

(by the Reparametrization–Homotopy Lemma 11.1, since s is a reparametrization of $s_0 * \cdots * s_k$; since by the Path-Homotopy Product Proposition 11.4 (3) we know that each $p_i * \overline{p_i}$ is null-homotopic)

$$\simeq_{\text{p}} (\underbrace{s_0 * p_1}_{\substack{\text{loop in } (U,x_0)\\ \text{or in } (V,x_0)}}) * (\underbrace{\overline{p_1} * s_1 * p_2}_{\substack{\text{loop in } (U,x_0)\\ \text{or in } (V,x_0)}}) * (\underbrace{\overline{p_2} * s_2 * p_3}_{\substack{\text{loop in } (U,x_0)\\ \text{or in } (V,x_0)}}) * (\overline{p_3} * \ldots * (\overline{p_k} * s_k).$$

Thus we have shown that s is path-homotopic to a concatenation of loops, each of which lies in (U, x_0) or in (V, x_0). ■

20.3. **Interlude: An alternative definition of the free product of groups.** In our proof of the injectivity of the map in the Seifert–van Kampen-Theorem 20.1 it is convenient to work with a different model for the free and the amalgamated product of two groups. We will present this alternative approach in this purely algebraic section. We return to the proof of the Seifert–van Kampen-Theorem 20.1 in the next section.

Definition. Let A and B be two groups.

(1) We set[180]
$$S(A,B) \;:=\; \{\text{all finite sequences } (g_1, \dots, g_k) \text{ with each } g_i \text{ in } A \text{ or } B\}.$$
We equip $S(A,B)$ with the product structure given by
$$(x_1, \dots, x_k) \cdot (y_1, \dots, y_l) \;:=\; (x_1, \dots, x_k, y_1, \dots, y_l).$$
This turns $S(A,B)$ into a monoid.[181]

(2) Let $\alpha\colon G \to A$ and $\beta\colon G \to B$ be two group homomorphisms. We denote by $\sim_G$ the equivalence relation on $S(A,B)$ that is generated by the relations

(a) $$(x_1, \dots, x_l, e, x_{l+2}, \dots, x_k) \;\sim_G\; (x_1, \dots, x_l, x_{l+2}, \dots, x_k)$$

(b) $$(x_1, \dots, x_l, x_{l+1}, \dots, x_k) \;\sim_G\; (x_1, \dots, x_l \cdot x_{l+1}, \dots, x_k).$$
(if both x_l and x_{l+1} lie in A or both lie in B)

(c) $$(x_1, \dots, x_l, \alpha(g), x_{l+2}, \dots, x_k) \;\sim_G\; (x_1, \dots, x_l, \beta(g), x_{l+2}, \dots, x_k).$$
(for any $g \in G$)

If $G = \{e\}$, then we just write $\sim$ instead of $\sim_G$.

(3) The monoid structure on $S(A,B)$ descends to a well-defined map

$$\begin{array}{rcl} S(A,B)/\sim_G \times S(A,B)/\sim_G & \to & S(A,B)/\sim_G \\ ([(g_1,\dots,g_k)],[h_1,\dots,h_l]) & \mapsto & [(g_1,\dots,g_k,h_1,\dots,h_l)]. \end{array}$$

One can easily verify that this defines a group structure on $S(A,B)/\sim_G$.[182]

Proposition 20.3. (Alternative–Amalgamated Product Proposition) Let A and B be two groups and let $\alpha\colon G \to A$ and $\beta\colon G \to B$ be two group homomorphisms. The map[183]

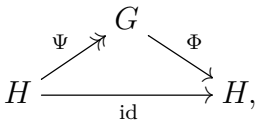
$$\begin{array}{rcl} \Phi\colon S(A,B)/\sim_G & \to & A *_G B = (A*B)/\langle\langle\{\alpha(g)\cdot\beta(g)^{-1}\}_{g\in G}\rangle\rangle \\ [\underbrace{(x_1,\dots,x_k)}_{\in S(A,B)}] & \mapsto & [\underbrace{x_1\cdot\ldots\cdot x_k}_{\in A*B}] \end{array}$$

is well-defined and it is a group isomorphism.

In the proof of the Alternative–Amalgamated Product Proposition 20.3 we will need the following little lemma.

Auxiliary Lemma 20.4. Let $\Phi\colon G \to H$ and $\Psi\colon H \to G$ be two group homomorphisms such that $\Phi\circ\Psi = \mathrm{id}_H$ and such that Ψ is an epimorphism. Then Φ is an isomorphism.

The lemma can be summarized as follows: If we are given a commutative diagram of the form



then Φ is an isomorphism.

Proof of Lemma 20.4. It follows from $\Phi\circ\Psi = \mathrm{id}_H$ that Φ is an epimorphism. It remains to show that Φ is a monomorphism. So let $g \in \ker(\Phi)$. Since Ψ is an epimorphism there exists an $h \in H$ with $\Psi(h) = g$. Since $\Phi\circ\Psi = \mathrm{id}_H$ we see that $h = \Phi(\Psi(h)) = \Phi(g) = e$. Thus $h = e$ and thus $g = \Psi(h) = \Psi(e) = e$. ∎

Proof of Proposition 20.3. We start out with the following claim:

Claim 1. The map

$$\begin{array}{rcl} \Phi\colon S(A,B)/\sim_G & \to & A *_G B = (A*B)/\langle\langle\{\alpha(g)\cdot\beta(g)^{-1}\}_{g\in G}\rangle\rangle \\ [(x_1,\dots,x_k)] & \mapsto & [x_1\cdot\ldots\cdot x_k] \end{array}$$

is well-defined, it is a homomorphism and it is surjective.

Proof. We consider the map

$$\begin{array}{rcl} \widetilde{\Phi}\colon S(A,B) & \to & A *_G B \\ (x_1,\dots,x_k) & \mapsto & [x_1\cdot\ldots\cdot x_k]. \end{array}$$

[180]The definition of $S(A,B)$ is similar to the definition of $A*B$, but now we allow that elements of the sequence are trivial and we allow that consecutive elements lie in the same group.

[181]Loosely speaking a monoid is like a group, except that we do not demand the existence of inverses. In our case the neutral element is again represented by the empty sequence ( ).

[182]Since we have a monoid structure on $S(A,B)$ we see that this multiplication defines a monoid structure on $S(A,B)/\sim_G$. Furthermore it follows from (2a) and (2b) that the inverse of $[(g_1,\dots,g_k)]$ is given by $[(g_k^{-1},\dots,g_1^{-1})]$. Thus we have in fact a group structure.

[183]Recall that we use the natural monomorphisms from the Coproduct-of-Groups Lemma 19.2 (1a) to view A and B as subgroups of $A*B$. If we want to be more precise, then we could say that Φ is given by $[(x_1,\dots,x_k)] \mapsto [(x_1)\cdot\ldots\cdot(x_k)]$.

As above we view $S(A,B)$ as a monoid. With this structure it is clear that $\widetilde{\Phi}$ is a monoid homomorphism. Since $A*B$ is a subset of $S(A,B)$ we see that $\widetilde{\Phi}$ is surjective.

It remains to show that $\widetilde{\Phi}$ descends to a well-defined map on $S(A,B)/\sim_G$. The first two relations in the definition of $\sim_G$ do not pose a problem. But let us consider relation (c). Let $w=(x_1,\dots,x_l)$, $\widetilde{w}=(x_{l+2},\dots,x_k)$ in $S(A,B)$ and let $g\in G$. Note that we have the following equality in $A*_G B$:

$$\widetilde{\Phi}(\underbrace{(w,\alpha(g),\widetilde{w})}_{\in S(A,B)}) \overset{\text{since } \widetilde{\Phi} \text{ is a monoid homomorphism}}{=} \widetilde{\Phi}(w)\cdot\alpha(g)\cdot\widetilde{\Phi}(\widetilde{w}) = \widetilde{\Phi}(w)\cdot\overbrace{\alpha(g)\cdot\beta(g)^{-1}}^{=e\in A*_G B}\cdot\beta(g)\cdot\widetilde{\Phi}(\widetilde{w}) = \widetilde{\Phi}(w)\cdot\beta(g)\cdot\widetilde{\Phi}(\widetilde{w}) = \widetilde{\Phi}(\underbrace{(w,\beta(g),\widetilde{w})}_{\in S(A,B)}).$$

We have thus verified that $\widetilde{\Phi}$ factors through the relation given by (d). ⊞

Claim 2. The map
$$\begin{array}{rcl}\Psi\colon A*_G B & \to & S(A,B)/\sim_G\\ \underbrace{[(x_1,\dots,x_k)]}_{\in A*B} & \mapsto & \underbrace{[(x_1,\dots,x_k)]}_{\in S(A,b)}\end{array}$$
is well-defined and it is a homomorphism.

Proof. First we consider the map
$$\begin{array}{rcl}\widetilde{\Psi}\colon A*B & \to & S(A,B)/\sim_G\\ (x_1,\dots,x_k) & \mapsto & [(x_1,\dots,x_k)].\end{array}$$
One can easily verify that this is in fact a group homomorphism. It follows immediately from the definitions that for any $g\in G$ we have

$$\widetilde{\Psi}(\alpha(g)\cdot\beta(g)^{-1}) = [(\alpha(g),\beta(g)^{-1})] \underset{\uparrow}{=} [(\beta(g),\beta(g)^{-1})] \underset{\uparrow}{=} [(\beta(g)\cdot\beta(g)^{-1})] \underset{\uparrow}{=} [()] = e.$$
follows from part (c), (b) and (a) of the definition of $\sim_G$

Thus it follows from the Normal Closure–Factorization Lemma 19.10 that $\widetilde{\Psi}$ descends to a well-defined homomorphism $\Psi\colon A*_G B\to S(A,B)/\sim_G$. ⊞

It is clear that $\Phi\circ\Psi=\mathrm{id}_{A*_G B}$. It is not so obvious that $\Psi\circ\Phi=\mathrm{id}_{S(A,B)/\sim_G}$. In fact we will proceed in a slightly different fashion: to show that Φ is an isomorphism it suffices, according to Lemma 20.4, to show that $\Psi\colon A*_G B\to S(A,B)/\sim_G$ is surjective. This in turn follows from the following claim:

Claim 3. Any element in $S(A,B)$ is equivalent, via $\sim_G$, to an element in $A*B$.

Proof. We prove the claim by induction on the length of a sequence in $S(A,B)$. Clearly the claim holds for the unique sequence in $S(A,B)$ of length 0. Suppose we already know the claim for all sequences of length $\le k-1$. Let $(x_1,\dots,x_k)\in S(A,B)$ be a sequence of length k. If $(x_1,\dots,x_k)$ is a reduced word then by definition it lies in $A*B$ and we are done. Now we suppose that $(x_1,\dots,x_k)$ is not reduced. Then, by the definition on page 267, one of the following occurs:

(1) there exists an $i\in\{1,\dots,k\}$ with $x_i=e$, or
(2) there exists an $i\in\{1,\dots,k-1\}$ with $x_i,x_{i+1}\in A$, or
(3) there exists an $i\in\{1,\dots,k-1\}$ with $x_i,x_{i+1}\in B$.

In the first case the sequence is equivalent to $(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_k)$. In the second and third case the sequence is equivalent to $(x_1, \dots, x_{i-1} \cdot x_{i+1}, \dots, x_k)$. In all three cases we have thus shown that our original sequence is equivalent to a sequence of length $k-1$. So we are done by our induction hypothesis. ■

Remark. Let A and B be two groups. We have now seen that there are two ways to think about the free product $A * B$. We can either take the original approach by viewing elements of $A * B$ as reduced words. Or we can use the Alternative–Amalgamated Product Proposition 20.3 to identify $A * B$ with the group $S(A, B)/\sim$. Both approaches to defining the free product of A and B have their advantages:

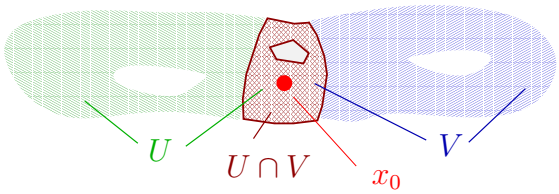
(1) In the group $S(A, B)/\sim$ it is easier to write down the product of two elements, we just need to juxtapose sequences of elements in A and B.
(2) On the other hand in the group $A * B$ it is trivial to check whether two elements are the same. For example for two non-trivial elements $a \in A$ and $b \in B$ the elements (a, b) and (b, a) of $A * B$ are by definition different. But from the definition of $S(A, B)/\sim$ it is not immediately clear that $[(a, b)]$ and $[(b, a)]$ are indeed different. □

20.4. **Proof of the Seifert–van Kampen-Theorem 20.1: Injectivity.** We return to the proof of the Seifert–van Kampen-Theorem 20.1. We recall the setup. Let X be a topological space and let $U, V \subset X$ be two open subsets with $X = U \cup V$ and with $U \cap V \neq \varnothing$ such that $U \cap V$ is non-empty. We choose a base point $x_0 \in U \cap V$. We write $A := \pi_1(U, x_0)$, $B := \pi_1(V, x_0)$ and $G := \pi_1(U \cap V, x_0)$. On page 281 we defined a homomorphism

$$\Phi\colon A *_G B = \pi_1(U, x_0) *_{\pi_1(U\cap V, x_0)} \pi_1(V, x_0) \to \pi_1(X, x_0).$$

In Section 20.2 we showed that this homomorphism is surjective. It remains to show that this homomorphism is injective. This is what we will do now:

Proof of the Seifert–van Kampen-Theorem 20.1: Injectivity. We want to show that the map $\Phi\colon A *_G B = \pi_1(U, x_0) *_{\pi_1(U\cap V, x_0)} \pi_1(V, x_0) \to \pi_1(X, x_0)$ which is induced by the inclusions $i\colon U \to X$ and $j\colon V \to X$ is injective.



To prove injectivity of Φ it is enough to show that there exists a map $\Psi\colon \pi_1(X, x_0) \to A *_G B$ such that $\Psi \circ \Phi = \mathrm{id}_{A*_G B}$. We will prove the existence of such a map Ψ in three steps:

(1) First we construct a map $\overline{\Psi}\colon \{\text{loops in } (X, x_0)\} \to A *_G B$.
(2) Then we show that $\overline{\Psi}$ descends to a well-defined map

$$\begin{aligned} \Psi\colon \pi_1(X, x_0) &\to A *_G B \\ [g] &\mapsto \overline{\Psi}(g). \end{aligned}$$

(3) Finally we show that $\Psi \circ \Phi = \mathrm{id}_{A*_G B}$.

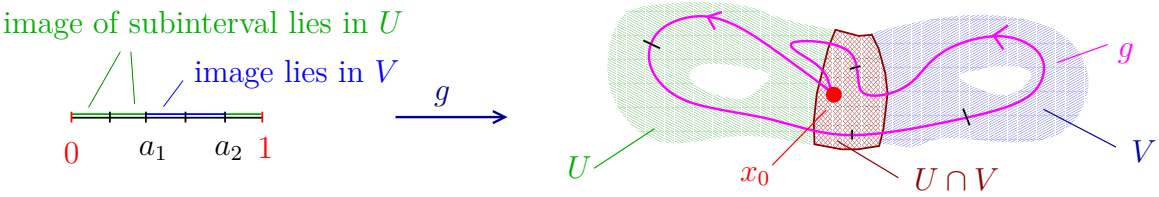
For the remainder of the proof we will identify $A *_G B$ with $S(A, B)/\sim_G$ using the isomorphism given by the Alternative–Amalgamated Product Proposition 20.3.

Step (1). Let $g\colon [0,1] \to X$ be a loop in (X, x_0). We apply the Lebesgue Lemma 11.9 to the map $g\colon [0,1] \to X$ and the open cover $X = U \cup V$. We obtain an N such that for every $i \in \{0, \dots, N-1\}$ the image of the interval $[\frac{i}{N}, \frac{i+1}{N}]$ under the map g lies in U or in V. We introduce a few definitions.

(a) We say that a subinterval of $[0,1]$ can be *colored green* if the image of the subinterval under the map g lies in U and we say that an interval can be *colored blue* if the image of the subinterval lies in V. Note that the image of a subinterval lies in $U \cap V$ if and only if the subinterval can be colored green as well as blue.

(b) A *coloring* is a map $\alpha\colon \{0, \frac{1}{N}, \dots, 1 - \frac{1}{N}\} \to \{\text{blue}, \text{green}\}$ such that each subinterval $[i, i + \frac{1}{N}]$ can be colored by $\alpha(i)$.

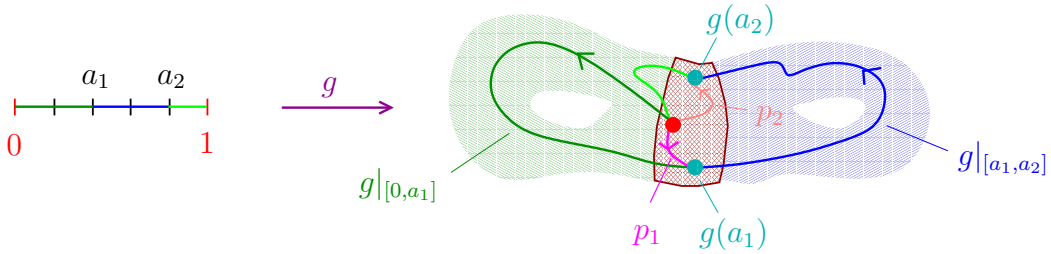
(c) Given a coloring α we denote by $0 < a_1 < \dots < a_{k-1} < 1$ the numbers where the coloring of the subintervals changes from one color to another. Note that for each a_i the intervals $[a_i - \frac{1}{N}, a_i]$ and $[a_i, a_i + \frac{1}{N}]$ can be colored in two different colors, which implies that $g(a_i) \in U \cap V$.



(d) A *basing* for a coloring α is a collection of paths $p_1, \dots, p_{k-1}$ in $U \cap V$ such that each path p_i goes from x_0 to $g(a_i)$.[184]

Given a coloring α and a basing $p_1, \dots, p_{k-1}$ we set[185] [186]

$$\overline{\Psi}(g) \;:=\; \big[\big[\underbrace{g|_{[0,a_1]} * \overline{p}_1}_{\substack{\text{loop in } (U,x_0) \text{ or} \\ \text{loop in } (V,x_0)}}\big], \big[\underbrace{p_1 * g|_{[a_1,a_2]} * \overline{p}_2}_{\substack{\text{loop in } (U,x_0) \text{ or} \\ \text{loop in } (V,x_0)}}\big], \dots, \big[\underbrace{p_{k-1} * g|_{[a_{k-1},1]}}_{\substack{\text{loop in } (U,x_0) \text{ or} \\ \text{loop in } (V,x_0)}}\big]\big] \;\in\; S(A,B)/\sim_G \;=\; A *_G B.$$



Our next goal is to prove the following claim:

Claim 1. The definition of $\overline{\Psi}(g) \in S(A,B)/\sim_G$ does not depend on any of the choices made.

[184] Such basing exists since each $g(a_i)$ lies in $U \cap V$ and since $U \cap V$ is path-connected.

[185] Each path $p_m * g|_{[a_m,a_{m+1}]} * \overline{p}_{m+1}$ is a loop in (U,x_0) or (V,x_0), so each path represents an element in $\pi_1(U,x_0)$ or $\pi_1(V,x_0)$. Thus we obtain a sequence of elements in $\pi_1(U,x_0)$ and $\pi_1(V,x_0)$. The equivalence class of this sequence defines an element in $S(A,B)/\sim_G = A *_G B$.

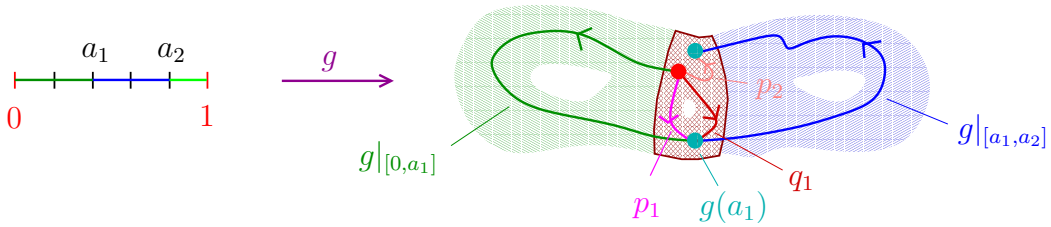
[186] This definition is almost the same as in the proof on page 283.

Proof. We need to prove the following three statements:

(a) Given N and a coloring the definition does not depend on the choice of the basings.
(b) Given N the definition does not depend on the choice of the coloring.
(c) The definition does not depend on the choice of N.

We now prove these three statements:

(a) We first fix a coloring α and we show that the definition of $\overline{\Psi}(g)$ does not depend on the choice of the basing. Thus let us pick a different path q_1 from x_0 to $g(a_1)$.



In this case we have the following equality in $S(A,B)/\sim_G$:

$$\begin{aligned}
&\text{definition of } \overline{\Psi}(g) \text{ using the basing } p_1, p_2, \ldots, p_{k-1} \;=\\
&= \big[[g|_{[0,a_1]} * \overline{p}_1],\ [p_1 * g|_{[a_1,a_2]} * \overline{p}_2], \ldots,\ [p_{k-1} * g|_{[a_{k-1},1]}]\big]\\
&= \big[[g|_{[0,a_1]} * \overline{p}_1 * \overbrace{(p_1 * \overline{q}_1) * (\underbrace{q_1 * \overline{p}_1}_{\text{loop in } U\cap V})}^{\text{null-homotopic loop}}],\ [p_1 * g|_{[a_1,a_2]} * \overline{p}_2], \ldots,\ [p_{k-1} * g|_{[a_{k-1},1]}]\big]\\
&= \big[[g|_{[0,a_1]} * \overline{p}_1 * (p_1 * \overline{q}_1)],\ [(q_1 * \overline{p}_1) * p_1 * g|_{[a_1,a_2]} * \overline{p}_2], \ldots,\ [p_{k-1} * g|_{[a_{k-1},1]}]\big]
\end{aligned}$$

note that $q_1 * \overline{p}_1$ is a loop in $U \cap V$, thus it defines elements in $G = \pi_1(U \cap V, x_0)$, thus we get this equality in $S(A,B)/\sim_G$

$$\begin{aligned}
&= \big[[g|_{[0,a_1]} * (\underbrace{\overline{p}_1 * p_1}_{\text{null-homotopic}}) * \overline{q}_1)],\ [q_1 * (\underbrace{\overline{p}_1 * p_1}_{\text{null-homotopic}}) * g|_{[a_1,a_2]} * \overline{p}_2], \ldots,\ [p_{k-1} * g|_{[a_{k-1},1]}]\big]\\
&= \big[[g|_{[0,a_1]} * \overline{q}_1],\ [q_1 * g|_{[a_1,a_2]} * \overline{p}_2], \ldots,\ [p_{k-1} * g|_{[a_{k-1},1]}]\big]\\
&= \text{definition of } \overline{\Psi}(g) \text{ using the basing } q_1, p_2, \ldots, p_{k-1}.
\end{aligned}$$

Basically the same argument (with slightly more messy notation) shows that the choices of the other paths p_i are also irrelevant.

(b) Now we show that the definition does not depend on the choice of the coloring. So suppose we have two colorings α and β. We can turn one coloring into the other by a finite sequence of color swaps of a single subinterval. Therefore it suffices to consider the case that there exists a single $i \in \{0, \frac{1}{N}, \ldots, 1 - \frac{1}{N}\}$ where the values for $\alpha(i)$ and $\beta(i)$ differ.

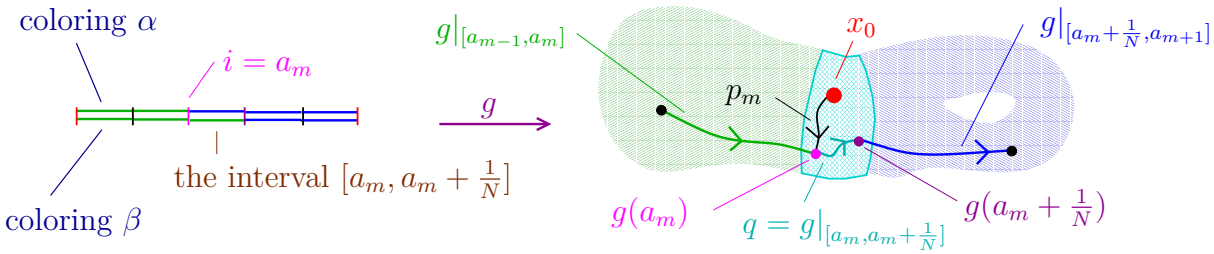
We consider the case, sketched in the figure below, that $\alpha(i) =$ blue, $\beta(i) =$ green and such that the common coloring to the left is green and the common coloring to the right is blue. All other cases are dealt with in a similar way.[187]

Let $0 = a_0 < a_1 < \cdots < a_k = 1$ be the decomposition corresponding to the coloring α. We pick a basing $p_1, \ldots, p_{k-1}$ for α. There exists a unique m with $a_m = i$. The

[187] In fact the only case which requires a slightly modified argument is that the coloring to the left and right are the same, i.e. either both are blue or both are green. We leave it to the reader to perform that argument.

corresponding decomposition for the coloring β is given by $0 = a_0 < a_1 < \cdots < a_m + \frac{1}{N} < a_{m+1} < a_k = 1$. We write $q = g_{[a_m, a_m+\frac{1}{N}]}$. Since $[a_m, a_m + \frac{1}{N}]$ can be colored green and blue we see that q lies actually in $U \cap V$. We have

$$\begin{aligned}
&\text{definition of } \overline{\Psi}(g) \text{ using the coloring } \alpha \text{ and the basing } p_1, \dots, p_{k-1} \;=\\
&\quad= \big[[g|_{[0,a_1]} * \overline{p}_1], \dots, [p_{m-1} * g|_{[a_{m-1},a_m]} * \overline{p}_m], [p_m * g|_{[a_m,a_{m+1}]} * \overline{p}_{m+1}], \dots\big]\\
&\quad= \big[[g|_{[0,a_1]} * \overline{p}_1], \dots, [p_{m-1} * g|_{[a_{m-1},a_m+\frac{1}{N}]} * \overline{q} * \overline{p}_m], [p_m * q * g|_{[a_m+\frac{1}{N},a_{m+1}]} * \overline{p}_{m+1}], \dots\big]\\
&\qquad\uparrow\\
&\text{since } g_{[a_{m-1},a_m]} \simeq_{\mathrm{p}} g_{[a_{m-1},a_m+\frac{1}{N}]} * \overline{q} \text{ and } g_{[a_m,a_m+\frac{1}{N}]} \simeq_{\mathrm{p}} q * g_{[a_m+\frac{1}{N},a_{m+1}]}\\
&\quad= \text{definition of } \overline{\Psi}(g) \text{ using the coloring } \beta \text{ and basing } p_1, \dots, p_{m-1}, p_m * q, p_{m+1}, \dots, p_{k-1}.
\end{aligned}$$



(c) First suppose that instead of dividing $[0,1]$ into N intervals we divide it into $m \cdot N$ intervals for some $m \in \mathbb{N}$. A coloring for the subdivision into N subintervals then also gives rise to a coloring into $m \cdot N$ subintervals, and it follows immediately from the definitions that we end up with the same element of $A *_G B$. Now if we had chosen a different N' instead, then we can replace N and N' by $N \cdot N'$ and by the argument we just made we obtain the same element of $A *_G B$. This concludes the proof of (c). ⊞

Step (2). We need to prove the following claim:

Claim 2. If $g, g' \colon [0,1] \to X$ are two path-homotopic loops in (X, x_0), then $\overline{\Psi}(g) = \overline{\Psi}(g')$.

Proof. Let $g, g' \colon [0,1] \to X$ be two path-homotopic loops in (X, x_0). We pick a path-homotopy

$$\begin{array}{rccc} H \colon & [0,1] \times [0,1] & \to & X \\ & (t,s) & \mapsto & H(t,s) \end{array}$$

from the loop g to the loop g'. This time we apply the Lebesgue Lemma 11.9 to the map $H \colon [0,1] \times [0,1] \to X$ and the open cover $X = U \cup V$. We obtain an $N \in \mathbb{N}$ such that for every choice of $i, j \in \{0, \frac{1}{N}, \dots, 1 - \frac{1}{N}\}$ the image of the square $[i, i + \frac{1}{N}] \times [j, j + \frac{1}{N}]$ under the map H lies in U or in V. Given $a \in [0,1]$ we introduce the two paths

$$\underbrace{\begin{array}{rccc} H_a \colon & [0,1] & \to & X \\ & t & \mapsto & (t,a) \end{array}}_{\text{“horizontal path”}} \quad \text{and} \quad \underbrace{\begin{array}{rccc} V_a \colon & [0,1] & \to & X \\ & t & \mapsto & (a,t). \end{array}}_{\text{“vertical path”}}$$

Note that each H_a is a loop in x_0 and that $H_0 = g$ and $H_1 = g'$. Therefore it suffices to show that for every $j \in \{0, \frac{1}{N}, \dots, 1 - \frac{1}{N}\}$ we have $\overline{\Psi}(H_j) = \overline{\Psi}(H_{j+\frac{1}{N}})$. So let $j \in \{0, \frac{1}{N}, \dots, 1 - \frac{1}{N}\}$.

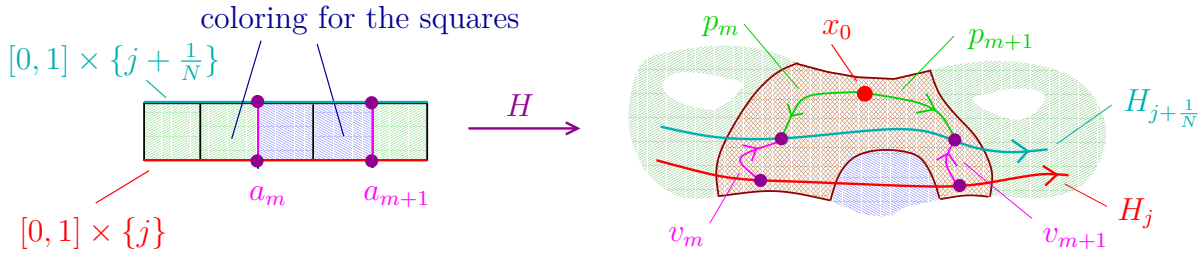
As above we say that a subset of $[0,1] \times [0,1]$ can be *colored green* if the image of the subset under the map H lies in U and we say that the subset can be *colored blue* if the image lies in V. We pick a map $\alpha \colon \{0, \frac{1}{N}, \dots, 1 - \frac{1}{N}\} \to \{\text{blue}, \text{green}\}$ such that for each i

the value $\alpha(i)$ is a color with which the square $[i, i+\frac{1}{N}] \times [\frac{j}{N}, j+\frac{1}{N}]$ can be colored. Note that α also defines a coloring for the loops H_j and $H_{j+\frac{1}{N}}$.

We define $0 = a_0 < a_1 < \cdots < \ldots a_k = 1$ as above for the coloring α. We pick a basing p_i, $i = 1, \ldots, k-1$ for the loop H_j. For $i = 0, \ldots, k$ we write $v_i = V_{a_i}|_{[j, j+\frac{1}{N}]}$. Put differently, v_j connects the point $H_j(a_i)$ to the corresponding point $H_{j+\frac{1}{N}}(a_i)$. Now we have

$$\begin{aligned}
\overline{\Psi}(H_j) &= \overline{\Psi}(H_j) \text{ defined using the coloring } \alpha \text{ and the basing } p_1, \ldots, p_{k-1} \\
&= \big[\ldots\big] * \big[p_m * H_j|_{[a_m, a_{m+1}]} * \overline{p}_{m+1}\big] * \big[\ldots\big] \\
&= \big[\ldots\big] * \big[p_m * v_m * \overline{v}_m * H_j|_{[a_m, a_{m+1}]} * v_{m+1} * \overline{v}_{m+1} * \overline{p}_{m+1}\big] * \big[\ldots\big] \\
&\quad \uparrow \\
&\quad \text{since each } v_m * \overline{v}_m \text{ is null-homotopic} \\
&= \big[\ldots\big] * \big[p_m * v_m * H_{j+\frac{1}{N}}|_{[a_m, a_{m+1}]} * \overline{v}_{m+1} * \overline{p}_{m+1}\big] * \big[\ldots\big] \\
&\quad \uparrow \\
&\quad \text{since } H \text{ gives rise to path-homotopies from } \overline{v}_m * H_j|_{[a_m, a_{m+1}]} * v_{m+1} \text{ to } H_{j+\frac{1}{N}}|_{[a_m, a_{m+1}]} \text{ for } m = 0, \ldots, k-1 \\
&= \overline{\Psi}(H_{j+\frac{1}{N}}) \text{ defined using } \alpha \text{ and the basing } p_1 * v_1, \ldots, p_{k-1} * v_{k-1}.
\end{aligned}$$

This concludes the proof of the claim and thus of part (2) of our three step program. ⊞



Summarizing we now showed that the map $\overline{\Psi}$ descends to a well-defined map

$$\begin{aligned}
\Psi\colon \pi_1(X, x_0) &\to A *_G B \\
[g] &\mapsto \big[\overline{\Psi}(g)\big].
\end{aligned}$$

Step (3). It remains to prove the following claim:

Claim 3. We have $\Psi \circ \Phi = \mathrm{id}_{A *_G B}$.

Proof. So let $[y_1] \cdot \ldots \cdot [y_k] \in A *_G B$ where $y_1, \ldots, y_k$ are loops in (U, x_0) and (V, x_0). Then

$$\begin{aligned}
(\Psi \circ \Phi)([y_1] \cdot \ldots \cdot [y_k]) &\overset{\text{definition of } \Phi\colon A *_G B = \pi_1(U, x_0) * \pi_1(V, x_0) \to \pi_1(X, x_0)}{=} \Psi([y_1 * \cdots * y_k]) = \big[\overline{\Psi}(y_1 * \cdots * y_k)\big] \\
&= [y_1] \cdot \ldots \cdot [y_k]. \qquad = [y_1 * \cdots * y_k] \\
&\qquad\qquad \uparrow \\
&\qquad\qquad \text{by definition of } \overline{\Psi}, \text{ where we use } N = k \text{ and where} \\
&\qquad\qquad \text{we can use the constant paths as basings since} \\
&\qquad\qquad y_1, \ldots, y_k \text{ are already loops in } (U, x_0) \text{ and } (V, x_0)
\end{aligned}$$

This concludes the proof of the claim and thus also the proof of the Seifert–van Kampen-Theorem 20.1 ■

20.5. **Fundamental groups of wedges.** We recall the following definition from page 245:

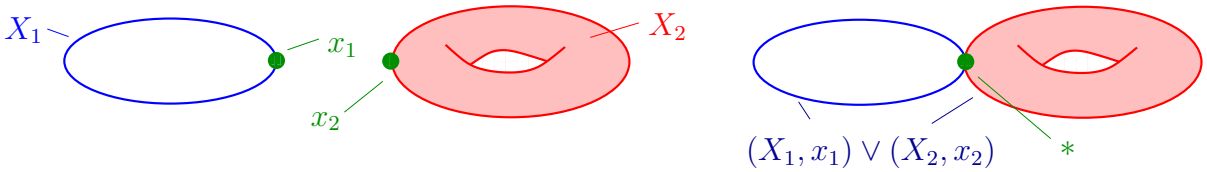
Definition. Let $\{(X_i, x_i)\}_{i \in I}$ be a y family of pointed topological spaces.
(1) We refer to $\bigvee_{i \in I} (X_i, x_i) := \Big(\bigsqcup_{i \in I} X_i\Big) \Big/ \bigsqcup_{i \in I} \{x_i\}$

as the wedge of the pointed topological spaces (X_i, x_i). We refer to the point given by the identifications as the wedge point and we denote it by $*$.

(2) Let $j \in I$. We refer to

$$\begin{aligned} \iota_j \colon X_j &\to \bigvee_{i\in I} (X_i, x_i) \\ x &\mapsto [x] \end{aligned}$$

as the natural inclusion. Note that in the Wedge Proposition 17.1 we showed that each natural inclusion is an embedding, thus we can view each X_j as a subspace of the wedge.



Let $(X_i, x_i)_{i\in I}$ be a family of pointed topological spaces. The natural inclusions ι_{i*} induce a homomorphism

$$\underbrace{\ast_{i\in I}\, \pi_1(X_i, x_i)}_{\substack{\text{coproduct of the} \\ \text{fundamental groups}}} \to \pi_1\Big(\underbrace{\bigvee_{i\in I} (X_i, x_i)}_{\substack{\text{coproduct of} \\ \text{the } (X_i, x_i)}} , * \Big).$$

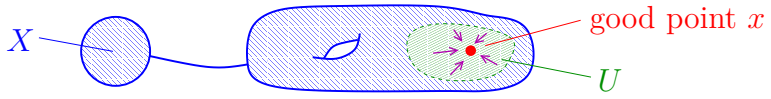
It seems reasonable to expect that this homomorphism is an isomorphism. For simplicity let us restrict ourself for a second to the case that we are dealing with just two topological spaces X_1 and X_2. It is now very tempting to apply the Seifert–van Kampen-Theorem 20.1 to the decomposition $X_1 \vee X_2 = X_1 \cup X_2$. Unfortunately this approach hits a snag: in the Seifert–van Kampen Theorem we have to deal with decompositions of a topological space X into *open* subsets. But in most cases X_1 and X_2 will certainly not be open subspaces of $X_1 \vee X_2$.

The idea now is to replace each X_i by an *open* subset U_i which is "close" to X_i. This idea leads us to the following somewhat technical definition.

Definition. We say that a point x in a topological space X is good, if the following two conditions are satisfied:

(1) $\{x\}$ is a closed subset of X[188].

(2) There exists an open neighborhood U of x such that x is a deformation retract of U.



Remark. What we call a "good point" is sometimes called a "non-degenerate point", see e.g. [**Lee00**, p. 212]. On the other hand, the term "non-degenerate point" is defined in a subtly different way, see e.g. [**Spa95**, p. 380] or [**Bre93**, p. 435]. Also note that a point $x \in X$ is good if the pair $(X, \{x\})$ is good in the sense of [**Hat02**, p. 114]. □

The following lemma gives us two important types of examples of good points, which will take care of most examples that we will have to deal with.

[188]Recall that if X is Hausdorff, then we know by the Points-in-Hausdorff Lemma 1.8 (1) that any subset consisting of a single point is closed. Thus the first condition in the definition of a good point is often satisfied "for free".

Lemma 20.5. (Good Point-Lemma)

(1) Every point in a topological manifold (e.g. on a sphere) is good.

(2) Let (X_i, x_i) be a family of pointed topological spaces. If each $x_i \in X_i$ is good, then the wedge point $*$ is a good point in $\bigvee_{i\in I}(X_i, x_i)$.

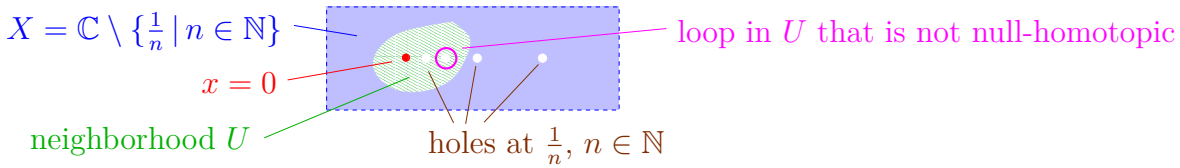


Proof.

(1) This follows easily from the fact that topological manifolds are by definition Hausdorff and that they are locally homeomorphic to open (half-) balls.

(2) Let (X_i, x_i) be a family of pointed topological spaces such that each $x_i \in X_i$ is good. It follows easily from the Wedge Proposition 17.1 (0) that the wedge point $*$ defines a closed subset of the wedge. For each $i \in I$ let U_i be an open neighborhood of $x_i \in X_i$ such that x_i is a deformation retract of U_i. We can think of $U := \bigvee_{i\in I} U_i$ as an open neighborhood of the wedge point $*$. One can easily show that U admits a deformation retraction to $*$. ■

Non-Example. We consider $X = \mathbb{C} \setminus \{\frac{1}{n} \mid n \in \mathbb{N}\}$. Morally speaking the point $x = 0$ is not good, "since any open neighborhood of 0 has non-trivial fundamental group".



We leave leave it to the reader to turn this slogan into a rigorous proof.

We can now state the main result of this section:

Proposition 20.6. (Wedge-π_1-Proposition) Let $\{(X_i, x_i)\}_{i\in I}$ be a family of pointed topological spaces. We suppose that each x_i is a good point in X_i. Given $j \in I$ we denote by $\iota_j \colon X_j \to \bigvee_{i\in I} X_i$ the natural inclusion. We denote by $*$ the wedge point of $\bigvee_{i\in I}(X_i, x_i)$. The map

$$\ast_{i\in I} \pi_1(X_i, x_i) \;\to\; \pi_1\Big(\bigvee_{i\in I}(X_i, x_i), *\Big),$$

which is induced by the inclusions ι_i together with the Coproduct-of-Groups Lemma 19.2 (1b), is an isomorphism.

Proof. We postpone the proof of the Wedge-π_1-Proposition 20.6 to the next section. ■

Using the Wedge-π_1-Proposition 20.6 we can now provide the long overdue calculation of the fundamental groups of the wedge of circles:

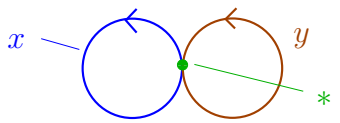
Lemma 20.7. (Wedge-of-Circles–π_1-Lemma) Let S be a set. The map[189]

$$\langle S\rangle \;\to\; \pi_1\Big(\bigvee_{s\in S}(S^1,1),*\Big)$$

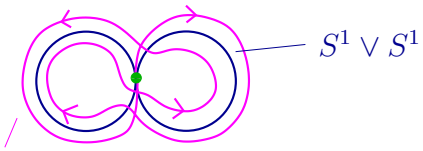
that is determined by

$$s \;\mapsto\; \left[\begin{array}{ccc}[0,1] & \to & \bigvee\limits_{s\in S}(S^1,1)\\ t & \mapsto & [\exp(2\pi\,\mathrm{i}t),s]\end{array}\right]$$

is an isomorphism.



$\langle x,y\rangle \xrightarrow{\cong} \pi_1(S^1\vee S^1,*)$



$y^{-1}x^{-1}yx$ is a non-trivial element in $\pi_1(S^1\vee S^1,*)=\langle x,y\rangle$

Proof. We consider the following maps:

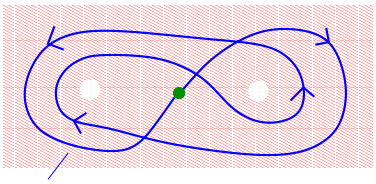
$$\langle S\rangle \underset{\text{(a)}}{=} \ast_{s\in S}\langle s\rangle \xrightarrow[\text{(b)}]{\cong} \ast_{s\in S}\pi_1(S^1,1) \xrightarrow[\text{(c)}]{\cong} \pi_1\Big(\bigvee_{s\in S}(S^1,1),*\Big)$$

We now clarify what we mean by the above maps:

(a) By the definition on page 271 we have $\langle S\rangle = \ast_{s\in S}\langle s\rangle$ where $\langle s\rangle = \{\dots,s^{-1},e,s,s^2,\dots\}$.
(b) It follows from the Circle-π_1-Theorem 11.13 that for each $s\in S$ we have an isomorphism from the infinite cyclic group $\langle s\rangle$ to $\pi_1(S^1,1)$ by sending the generator s of $\langle s\rangle$ to the path-homotopy class of the loop given by $t\mapsto\exp(2\pi\,\mathrm{i}t)$.
(c) By the Good Point-Lemma 20.5 (or alternatively by a completely elementary argument) we know that $1\in S^1$ is a good point. The isomorphism is thus given by the Wedge-π_1-Proposition 20.6.

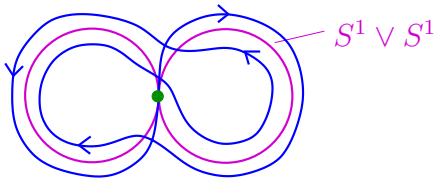
We have now seen that all of the above maps are indeed isomorphisms. The composition is precisely the map that we claimed was an isomorphism. ■

Example. We consider the loop γ from page 180 in $\mathbb{C}\setminus\{\pm1\}$ that is also illustrated in the figure below.



loop γ in $\mathbb{C}\setminus\{\pm1\}$

retraction f from $\mathbb{C}\setminus\{\pm1\}$ to $S^1\vee S^1$



$[f\circ\gamma]=y^{-1}x^{-1}yx\in\pi_1(S^1\vee S^1,*)$

One can easily show that there exists a deformation retraction

$$f\colon \mathbb{C}\setminus\{\pm1\} \;\to\; S^1\vee S^1 \;\overset{\downarrow}{=}\; \underbrace{\{z\in\mathbb{C}\mid |z+1|=1\}\cup\{z\in\mathbb{C}\mid |z-1|=1\}}_{\text{two circles that meet at the origin}}.$$

(we use the homeomorphism from page 180 to make this identification)

Note that

[189] Here $\langle S\rangle$ denotes the free group on the set S as defined on page 271.

$$f_*([\gamma]) \;=\; [f\circ\gamma] \;=\; y^{-1}x^{-1}yx \;\in \langle x,y\rangle \;=\; \pi_1(S^1\vee S^1,*),$$

we use the Wedge-of-Circles–π_1-Lemma 20.7 to make this identification

which is non-trivial. Thus we see that $f\circ\gamma$ is not null-homotopic in $S^1\vee S^1$. This implies that γ is not null-homotopic in $\mathbb{C}\setminus\{\pm 1\}$. □ □

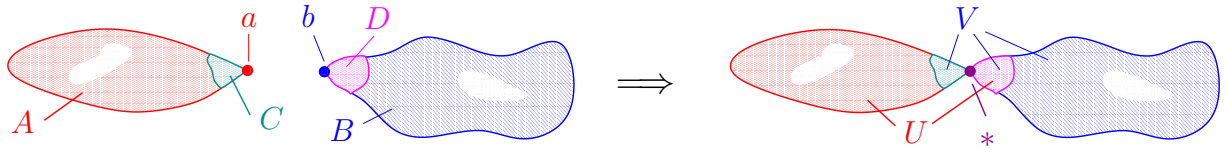
20.6. **Proof of the Wedge-π_1-Proposition 20.6.** In this section we provide the proof of the Wedge-π_1-Proposition 20.6. We proceed as follows:

(1) In the Finite Wedge-π_1-Lemma 20.8 we first deal with the case of the wedge of two pointed topological spaces. By applying this lemma iteratively we get the desired statement also for the wedge of finitely many pointed topological spaces.
(2) We reduce the general case of the Wedge-π_1-Proposition 20.6 via a compactness argument to the case of the wedge of finitely many pointed topological spaces.

Lemma 20.8. (Finite Wedge-π_1-Lemma) Let A and B be two path-connected topological spaces and let $a\in A$ and $b\in B$ be good points. The inclusion maps induce an isomorphism[190]
$$\pi_1(A,a)*\pi_1(B,b) \;\xrightarrow{\cong}\; \pi_1((A,a)\vee(B,b),*).$$

Proof. Let A and B be two path-connected topological spaces and let $a\in A$ and $b\in B$ be good points. All wedges below will be with respect to these two base points, thus, to simplify the notation we will drop them from the notation. In this spirit we write $X=A\vee B$. As usual we denote by $*$ the wedge point in $A\vee B$. By the Wedge Proposition 17.1 (1b) we can view A and B as subspaces of $X=A\vee B$ in the obvious way.

Since a is a good point we can pick an open neighborhood C in A of a that deformation retracts to a. We can also pick an open neighborhood D of b in B that deformation retracts to b. We consider $U:=A\vee D$ and $V:=B\vee C$.



Next we consider the following diagram:

$$\begin{array}{ccc}\pi_1(A,a)*\pi_1(B,b) & \longrightarrow & \pi_1(X,*)\\ \downarrow & & \downarrow{\scriptstyle \mathrm{id}}\\ \pi_1(U,*)*\pi_1(V,*) & \longrightarrow & \pi_1(X,*).\end{array}$$

We make the following observations:

(1) The vertical and horizontal maps are all induced by the various inclusions of topological spaces. In particular the diagram commutes.
(2) We can view $U=A\vee D$ and $V=B\vee C$ as open subsets of $X=A\vee B$.
(3) One can easily show that A is a deformation retract of $U=A\vee D$.
(4) As in (3) we see that B is a deformation retract of $V=B\vee C$.

[190] In the definition of a "good point" we demanded that the one-point subset given by the point is closed. Strictly speaking we do not need this fact in the proof of the Finite Wedge-π_1-Lemma 20.8.

(5) As in (3), we see that $A \cap B = \{*\}$ is a deformation retract of $U \cap V = C \vee D$. It follows from the Homotopy-π_1-Proposition 13.4 (2b) that $U \cap V$ is simply connected.
(6) By (2) and (5) we can apply the Seifert–van Kampen-Theorem 20.1 to the decomposition $A \vee B = U \cup V$ and we obtain that the bottom horizontal map of the above diagram is an isomorphism.
(7) It follows from (3), (4) and the Deformation Retract–Homotopy Equivalence Lemma 9.3 that the inclusions $A \to U = A \cup D$ and $B \to V = B \cup C$ are homotopy equivalences.
(8) It follows from (7) together with the Homotopy-π_1-Proposition 13.4 (2) that the two inclusion-induced maps $\pi_1(A, a) \to \pi_1(U, *)$ and $\pi_1(B, b) \to \pi_1(V, *)$ are isomorphisms. This implies that the left vertical map in the above commutative diagram is an isomorphism.
(9) We obtain from (1), (6) and (8) that the top horizontal map is, as desired, also an isomorphism. ■

Proof of the Wedge-π_1-Proposition 20.6. Let $\{(X_i, x_i)\}_{i \in I}$ be a family of pointed topological spaces such that each x_i is a good point in X_i. We use these points to form the wedge $\bigvee_{i \in I} X_i$. Recall that the fact that x_i is good means that

(1) $\{x_i\}$ is a closed subset of X_i and
(2) that there exists an open neighborhood U_i of x_i in X_i such that x_i is a deformation retract of U_i.

We need to show that the homomorphism

$$\Phi \colon \ast_{i \in I} \pi_1(X_i, x_i) \ \to \ \pi_1\Big(\bigvee_{i \in I} X_i, * \Big)$$

that is induced by the natural inclusions ι_i is an isomorphism. We will show that the homomorphism Φ is surjective and injective. In both cases the trick is to reduce the statement to the case of the Finite Wedge-π_1-Lemma 20.8.

Claim 1. The map

$$\Phi \colon \ast_{i \in I} \pi_1(X_i, x_i) \ \to \ \pi_1\Big(\bigvee_{i \in I} X_i \Big)$$

is injective.

Proof. Given a finite subset $J \subset I$ we consider the following diagram:

$$\begin{array}{ccc} \ast_{i \in J} \pi_1(X_i, x_i) & \xrightarrow{\ \cong\ } & \pi_1\Big(\bigvee_{i \in J} X_i, * \Big) \\ \downarrow & & \downarrow{\scriptstyle \iota_*} \\ \ast_{i \in I} \pi_1(X_i, x_i) & \longrightarrow & \pi_1\Big(\bigvee_{i \in I} X_i, * \Big). \end{array}$$

We make the following clarifications and observations:

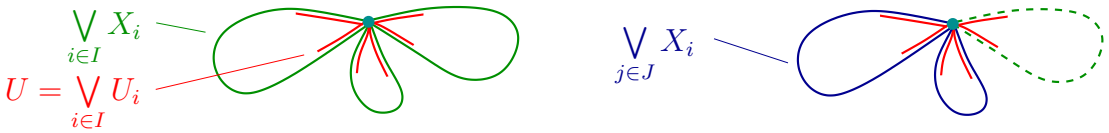
(1) The maps are induced by the various obvious inclusion. It follows easily from the definitions that the diagram commutes.
(2) Since J is finite we know by the Finite Wedge-π_1-Lemma 20.8, applied finitely many times, that the top horizontal homomorphism is an isomorphism.
(3) Note that as on page 247 we see that $\bigvee_{i \in J} X_i \subset \bigvee_{i \in I} X_i$ is a retract. It follows from the functoriality of fundamental groups that the right vertical map is a monomorphism.

(4) Note that it follows easily from the explicit description of a free product, that given any $g = (g_1, \dots, g_m) \in \ast_{i\in I} \pi_1(X_i, x_i)$ there exists a finite subset $J \subset I$ such that g lies in the image of the left vertical map.
(5) The claim follows from the above observations. ⊞

The proof that Φ is also surjective requires more of an effort.

Claim 2. We set $U := \bigcup_{i\in I} U_i$. Given any compact subset $K \subset \bigvee_{i\in I} X_i$ there exists a finite subset $J \subset I$ with $K \subset U \cup \bigvee_{i\in J} X_i$.

Proof. Given $i \in I$ we set $V_i := X_i \setminus \{x_i\}$. Since x_i is a good point we see that V_i is an open subset of X_i. Using the Wedge Proposition 17.1 (0) one can now easily see that V_i is in fact an open subset of $\bigvee_{i\in I} X_i$. Evidently the V_i together with U form an open cover of $\bigvee_{i\in I} X_i$. Since K is compact we see that there exists a finite subset $J \subset I$ with $K \subset U \cup \bigcup_{j\in J} V_j$. But this set is precisely $U \cup \bigvee_{i\in J} X_i$. ⊞



Claim 3. The map

$$\Phi\colon \ast_{i\in I} \pi_1(X_i, x_i) \;\to\; \pi_1\Big(\bigvee_{i\in I} X_i, *\Big)$$

is surjective.

Proof. Given a finite subset $J \subset I$ we consider the following commutative diagram:

$$\begin{array}{ccccc}
\ast_{j\in J} \pi_1(X_j, *) & \xrightarrow[\Phi_J]{\cong} & \pi_1\Big(\bigvee_{i\in J} X_i, *\Big) & \xrightarrow{\cong} & \\
\downarrow & & \downarrow & & \pi_1\Big(U \cup \bigvee_{i\in J} X_i, *\Big) \\
\ast_{i\in I} \pi_1(X_i, x_i) & \xrightarrow{\Phi} & \pi_1\Big(\bigvee_{i\in I} X_i, *\Big). & \leftarrow &
\end{array}$$

We make the following observations and comments:

(1) The maps are either the obvious algebraic inclusion maps or they are the obvious maps induced by inclusions of topological spaces. It follows almost immediately from the definitions that the diagram commutes.
(2) Since J is finite we obtain, from applying the Finite Wedge-π_1-Lemma 20.8 finitely many times, that the top horizontal map is an isomorphism.
(3) Note that $\bigvee_{i\in J} X_i$ is a deformation retract of $U \cup \bigvee_{i\in J} X_i$. Thus it follows from the Homotopy-π_1-Proposition 13.4 (2b) that the top diagonal map is an isomorphism.

Now we turn to the actual proof that the above map Φ is surjective. So suppose that we are given some $[\gamma] \in \pi_1\Big(\bigvee_{i\in I} X_i, *\Big)$. It follows from the Compact Image Lemma 2.8 together with Claim 2 that there exists a finite subset $J \subset I$ such that the image of γ is contained in $U \cup \bigvee_{i\in J} X_i$. But considering the above diagram, and using the fact that the top horizontal map is an isomorphism we see that $[\gamma]$ actually lies in the image of the bottom horizontal map Φ. ■

20.7. **Fundamental groups of topological graphs.** Now we have all the tools to calculate the fundamental groups of the topological realizations of finite abstract graphs.

Proposition 20.9. (Graph-π_1-Proposition)

(1) Let $G = (V, E, i, t)$ be a finite connected non-empty abstract graph and let T be a spanning tree[191] of G. Let $e_1, \dots, e_n$ be the edges not contained in T. We pick some $v \in V$. For $j = 1, \dots, n$ we pick a path[192] $\alpha_j \colon [0,1] \to |T|$ from v to $i(e_j)$, we denote by $\beta_j \colon [0,1] \to |e_j| \subset |G|$ the obvious path from $i(e_j)$ to $t(e_j)$, and we pick a path $\gamma_j \colon [0,1] \to |T|$ from $t(e_j)$ to v. The map

$$\Theta_G \colon \langle x_1, \dots, x_n \rangle \;\to\; \pi_1(|G|, v)$$

that is given by

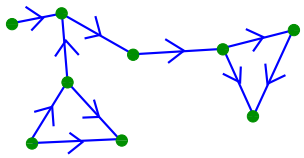
$$x_j \;\mapsto\; [\alpha_j * \beta_j * \gamma_j]$$

is an isomorphism.

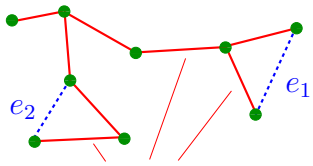
(2) Let X be a finite connected non-empty topological graph[193] with v vertices and e edges. Then

$$\pi_1(X) \;\cong\; \text{free group on } 1 - \chi(X) = 1 + e - v \text{ generators.}$$

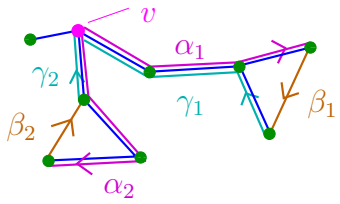
(3) Two finite connected topological graphs are homotopy equivalent if and only if they have the same Euler characteristic.



topological graph $|G|$



spanning tree $|T|$



Proof.

(1) We consider the maps

isomorphism induced by the homeomorphism on page 247

$$\langle x_1, \dots, x_n \rangle \xrightarrow{\Theta_G} \pi_1(|G|, [v]) \;\underset{\uparrow}{\to}\; \pi_1(|G|/|T|, v) \;\overset{\downarrow}{\underset{\cong}{\leftarrow}}\; \pi_1\Big(\bigvee_{j=1}^{n} S^1_j, *\Big) \;\underset{\uparrow}{\overset{\cong}{\leftarrow}}\; \langle y_1, \dots, y_n \rangle.$$

isomorphism by the Graph-Homotopy Type Corollary 17.6 (1) and the Homotopy-π_1-Proposition 13.4 (2)

isomorphism given by the Wedge-of-Circles–π_1-Lemma 20.7

It follows fairly easily from the explicit descriptions of the various maps that for each $j \in \{1, \dots, n\}$ the images of x_j and y_j in $\pi_1(|G|/|T|, v)$ are represented by path-homotopic loops, in particular the images of x_j and y_j in $\pi_1(|G|/|T|, [v])$ are the same. It follows from the above and the fact that all other maps are isomorphisms that Θ_G itself is indeed an isomorphism.

(2) Basically by definition we only need to prove the statement if $X = |G|$ is the topological realization of a finite connected non-empty abstract graph $G = (V, E, i, t)$. By the

[191] Recall that according to the definition on page 153 a spanning tree for G is a subtree of G that contains all vertices of G.

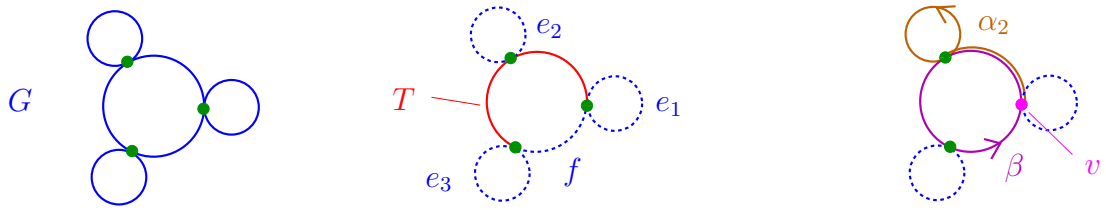
[191] A topological graph is a topological space that is homeomorphic to the topological realization of an abstract graph.

[192] Note that it follows from the Abstract Graph-Topology Lemma 7.1 that $|T|$ is path-connected.

Spanning Tree–Existence Proposition 7.6 we know that G admits a spanning tree $T = (W, F)$. By definition T contains all vertices of V. Furthermore, since T is a tree we know by the Tree Characterization Proposition 7.7 that $\#W - \#F = 1$. Thus it follows that there are precisely $\#E - \#F = e - (\#W - 1) = e - v + 1$ edges of G that are not contained in T. The desired statement now follows from (1).

(3) Let X and X' be two finite connected topological graphs. If $\chi(X) = \chi(X')$, then we already saw in the Graph-Homotopy Type Corollary 17.6 (2) that X and X' are homotopy equivalent. On the other hand, if $\chi(X) \neq \chi(X')$, then it follows from (1) together with the Basis Cardinality Proposition 19.9 that X and X' are not homotopy equivalent. ■

Example. We look at the graph G shown in the figure below. We also show a spanning tree T and a "base vertex v". We make the following observations:

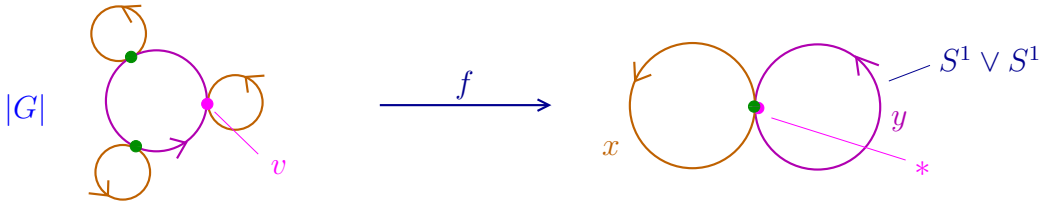


(1) The three small loops e_1, e_2, e_3 and the edge f are not contained in T.
(2) Each loop e_i gives rise to a loop α_i with base point v, by traveling along T, going once around e_i and going back to v along T.
(3) The edge f gives rise to a loop β by going once around the central circle.

It follows from the Graph-π_1-Proposition 20.9 that the map

$$\begin{array}{rcl} \langle a_1, a_2, a_3, b\rangle & \to & \pi_1(|G|, v) \\ a_i & \mapsto & [\alpha_i] \\ b & \mapsto & [\beta] \end{array}$$

is an isomorphism. Next we do something much more interesting: We consider the map $f\colon |G| \to S^1 \vee S^1$ shown in the figure below.



One can easily verify that f is a covering of degree 3. We use the Wedge-of-Circles–π_1-Lemma 20.7 to make the usual identification $\pi_1(S^1 \vee S^1, *) = \langle x, y\rangle$. By working directly with the definition of f_* one can see that f_* is given by

$$\begin{array}{rcl} f_*\colon \langle a_1, a_2, a_3, b\rangle = \pi_1(|G|, v) & \to & \pi_1(S^1 \vee S^1, *) = \langle x, y\rangle \\ a_i & \mapsto & y^{i-1} \cdot x \cdot y^{-(i-1)} \\ b & \mapsto & y^3. \end{array}$$

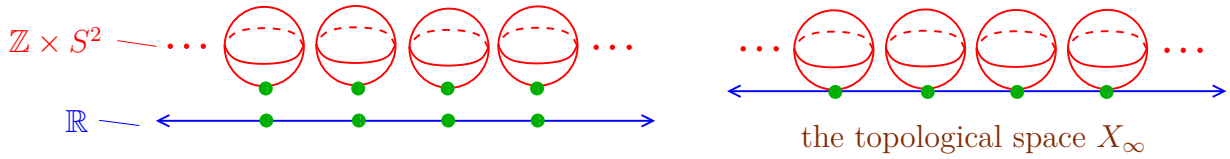
Since $f\colon |G| \to S^1 \vee S^1$ is a covering of degree 3 we obtain from the Covering-π_1-Monomorphism Proposition 14.11 together with the Covering-Degree Proposition 14.13 that the

induced map f_* is a monomorphism and that the image of f_* is a subgroup of index three. In summary, we have now shown that the free group $\langle x, y\rangle$ on two generators contains the subgroup $f_*(\langle a_1, a_2, a_2, b\rangle) = \langle x, yxy^{-1}, y^2xy^{-2}, y\rangle$ which is a free group on four generators. Thus we see that the free group of rank two can contain "larger" free groups. This is in complete contrast to the situation in free abelian groups which we studied in the Free Abelian Subgroup Theorem 18.7. □

20.8. **Exhaustions of topological spaces.** In many situations one is dealing with non-compact topological spaces where one would be tempted to apply the Seifert–van Kampen-Theorem 20.1 or the Wedge-π_1-Proposition 20.6 "infinitely many times", which does not really make sense. For example let us consider the topological space

$$X_\infty \;=\; \big(\mathbb{R} \sqcup (\mathbb{Z} \times S^2)\big) \,/\, n \sim (n, (0, 0, -1)).$$

The topological space X consists of an infinite line where to each $n \in \mathbb{Z}$ we attach a 2-dimensional sphere. We would like to show that X_∞ is simply connected. The following lemma lets us get around the issue of "infinite operations".



Proposition 20.10. (Exhaustion–Simply Connected Proposition) Let X be a topological space and let

$$X_0 \;\subset\; X_1 \;\subset\; X_2 \;\subset\; X_3 \;\subset\; \dots$$

be a sequence of subspaces of X such that the following three conditions are satisfied:

(1) We have $\bigcup\limits_{i \in \mathbb{N}_0} X_i = X$.

(2) For each $i \in \mathbb{N}$ we have $X_i \subset \mathring{X}_{i+1}$.

(3) Each X_i is simply connected.

Then for each $x_0 \in X$ we have $\pi_1(X, x_0) = 0$.

The proof of the Exhaustion–Simply Connected Proposition 20.10 rests on the following lemma which will come in handy on several occasions.

Lemma 20.11. (Compact Subspace–Exhaustion Lemma) Let X be a topological space and let

$$X_0 \;\subset\; X_1 \;\subset\; X_2 \;\subset\; X_3 \;\subset\; \dots$$

be a sequence of subspaces of X such that the following two conditions are satisfied:

(1) We have $\bigcup\limits_{i \in \mathbb{N}_0} X_i = X$.

(2) For each $i \in \mathbb{N}_0$ we have $X_i \subset \mathring{X}_{i+1}$.

Given any compact subset K of X there exists an $i \in \mathbb{N}_0$ with $K \subset X_i$.

Proof of the Compact Subspace–Exhaustion Lemma 20.11. We start out with the following little claim.

Claim. We have $\bigcup\limits_{i \in \mathbb{N}} \mathring{X}_i \;=\; X.$

Proof. We have the following equalities and inclusions

$$X \underset{\text{by hypothesis (1)}}{=} \bigcup_{i\in\mathbb{N}} X_i \underset{\text{by hypothesis (2)}}{=} \bigcup_{i\in\mathbb{N}} \mathring{X}_i \subset X.$$

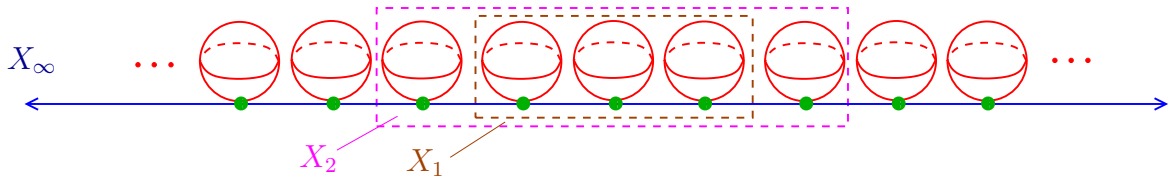
We see that the inclusion to the right is in fact an equality. ⊞

Now let $K \subset X$ be a compact subspace. Recall that by the Interior-Closure Lemma 1.6 (1) we know that interiors are always open subsets. It follows from the compactness of K and the claim that K is contained in the union of finitely many of the $\mathring{X}_i$. It follows from our hypothesis on the X_i that K is in fact contained in a single X_i. ■

Proof of the Exhaustion–Simply Connected Proposition 20.10. Let $x_0 \in X$ and let $f\colon [0,1] \to X$ be a loop in x_0. By the Compact Image Lemma 2.8 we know that $f([0,1])$ is a compact subset of X. By the Compact Subspace–Exhaustion Lemma 20.11 we know that $f([0,1])$ is contained in a single X_i. It follows from Hypothesis (3) that f is null-homotopic in X_i. It is clear that this implies that f is also null-homotopic in X. ■

Example. We return to the topological space X_∞ from the beginning of the section. Given $k \in \mathbb{N}$ we set

$$X_k \;=\; \big((-k-1, k+1) \sqcup (\{-k, \dots, k\} \times S^2)\big) \,/\, n \sim (n, (0,0,-1)).$$



Using the Wedge-π_1-Proposition 20.6 one can easily show that each X_k is simply connected. It follows from the Exhaustion–Simply Connected Proposition 20.10 that X_∞ is simply connected. □

Using the Exhaustion–Simply Connected Proposition 20.10 we can now give a characterisation of arbitrary trees in terms of their fundamental groups.

Proposition 20.12. (Tree–π_1-Proposition) Let $T = (V, E, \varphi)$ be a non-empty connected undirected abstract graph. The following statements are equivalent:[193]

(1) T is a tree.
(2) $\pi_1(|T|)$ is the trivial group.

Proof. Let $T = (V, E, \varphi)$ be a non-empty connected undirected abstract graph.

We start out with a proof of the (1)⇒(2)-direction. Since T is non-empty we can pick a vertex $v \in V$. We consider the subgraph given by $T_0 \coloneqq \{v\}$. Iteratively we build a sequence of subgraphs $T_0, T_1, T_2, \dots$ of T as follows: given T_i we add all edges (and the corresponding vertices) that have at least one vertex in common with T_i. We make the following observations:

(a) One can easily verify that each T_i is connected.
(b) It follows from (a) and the Tree–Basics Lemma 7.4 (1) that each T_i is a tree.
(c) It follows from (b) and Finite Tree–Contractible Proposition 17.4 that each T_i is simply connected.
(d) One can easily verify that for each $i \in \mathbb{N}_0$ we have $T_i \subset \mathring{T}_{i+1}$.

(e) We have $T = \bigcup_{i\in\mathbb{N}_0} T_i$.

It follows from the above together with the Exhaustion–Simply Connected Proposition 20.10 that T is simply connected.

We turn to the proof of the (2)⇒(1)-direction. By the Spanning Tree–Existence Proposition 7.6 we know that T admits a spanning tree S. It remains to show that if $S \neq T$, then $\pi_1(|T|)$ is non-trivial. Thus suppose that there exists an edge $e \in T \setminus S$. We denote by $|e| \subset |T|$ the corresponding subspace. By definition there exist $v, w \in V$ with $\varphi(e) = \{v, w\}$.[194] By definition of the topological realization there exists a continuous map $\alpha\colon [0,1] \to |e|$ with $\alpha(0) = v$ and $\alpha(1) = w$ such that α restricts to a homeomorphism $(0,1) \to |e| \setminus \{v, w\}$. Since S is a spanning tree we know that $v, w \in S$. Also note that trees are connected. Thus it follows from the Abstract Graph-Topology Lemma 7.1 that $|S|$ is connected. Therefore there exists a continuous map $\beta\colon [0,1] \to |S|$ with $\beta(0) = v$ and $\beta(1) = w$. Next we consider the map

$$\begin{array}{rcl} f\colon |T| & \to & S^1 \\ x & \mapsto & \begin{cases} 1, & \text{if } x \notin \alpha(0), \\ \exp(2\pi\,\mathrm{i}t), & \text{if } x = \alpha(t). \end{cases} \end{array}$$

It follows from the Topological-Pushout Proposition 6.12 that f is continuous.



Note that $\alpha * \beta$ is a loop in v and note that $f \circ (\alpha * \beta)\colon [0,1] \to S^1$ is path-homotopic to the loop $t \mapsto \exp(2\pi\,\mathrm{i}t)$. It follows from these observations together with the Circle-π_1-Theorem 11.13 that $f_*([\beta * \alpha]) \in \pi_1(S^1, 1)$ is non-trivial. But this shows that $\pi_1(|T|, v)$ is non-trivial. ∎

[194] Note that it is possible that $v = w$.

CHAPTER 21

Presentations of groups

In the Seifert–van Kampen Theorem 20.1 we just learned how to express fundamental groups of suitable topological spaces as amalgamated products. But it is not always clear in what sense we calculated a fundamental group and what such a description as an amalgamated product is trying to tell us.

In this chapter we introduce the notion of a (finite) presentation of a group. We will see in this and the following chapters that we can describe (fundamental) groups as finitely presented groups, i.e. we can give a finite description of these groups.

21.1. **Generating sets of groups.** On page 267 we introduced the notion of a "reduced word", which begs the question, what is a "word". Here is the answer.

Definition. Let G be a group and let $S \subset G$ be a subset.
(1) A word in S is a finite sequence $(s_1^{\epsilon_1}, \dots, s_k^{\epsilon_k})$ with $s_1, \dots, s_k \in S$ and $\epsilon_1, \dots, \epsilon_k \in \{-1, 1\}$. We refer to k as the length of the word.
(2) A word $(s_1^{\epsilon_1}, \dots, s_k^{\epsilon_k})$ defines an element $s_1^{\epsilon_1} \cdot \ldots \cdot s_k^{\epsilon_k}$ in G. In casual language we also refer to an expression of the form $s_1^{\epsilon_1} \cdot \ldots \cdot s_k^{\epsilon_k}$ as a word.

Definition.
(1) Let G be a group and let $S \subset G$ be a subset. We denote by $\Xi_S \colon \langle S \rangle \to G$ the natural homomorphism (given by the Free Group–Existence Lemma 19.4) from the free group $\langle S \rangle$ to G that is given by $s \mapsto s$.
(2) Given a group G we say that $S \subset G$ is a generating set if the natural homomorphism $\Xi_S \colon \langle S \rangle \to G$ is an epimorphism.
(3) We say that a group G is finitely generated if it admits a finite generating set.

Lemma 21.1. (Generating Set–Characterization Lemma) Let G be a group and let $S \subset G$ be a subset. The following statements are equivalent:
(1) S is a generating set for G.
(2) Every element of G can be represented by a word in S, i.e. we have

$$G \;=\; \big\{ s_1^{\epsilon_1} \cdot \ldots \cdot s_k^{\epsilon_k} \mid s_1, \dots, s_k \in S \text{ and } \epsilon_1, \dots, \epsilon_k \in \{-1, 1\} \big\}.$$

(3) If H is a subgroup of G that contains S, then $H = G$.

Proof.
(1)⇒(2): This statement follows immediately from the definitions and the trivial observation that for $s \in S$ and $n = \epsilon \cdot m$ with $\epsilon \in \{-1, 1\}$ and $m \in \mathbb{N}$ we have $s^{\epsilon \cdot m} = (s^{\epsilon})^m$.
(2)⇒(3): This statement follows from the observation that if $H \subset G$ is subgroup that contains S, then H also contains all words in S.

(3)⇒(1): This statement follows from the observation that the image of the natural homomorphism $\Xi_S\colon \langle S\rangle \to G$ is a subgroup of G that contains S. ■

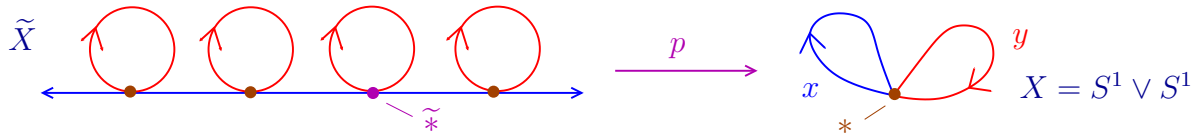
Example. It follows from the Abelianization-Epimorphism Lemma 19.7 together with the Abelianization Calculation Proposition 19.8 (2) that any finitely generated abelian group in the sense of page 260 is also a finitely generated group in the above sense. □

Lemma 21.2. (Generating Set–Lemma)
(1) If G is a free group on a subset S, then S is a generating set for G.
(2) If $\varphi\colon G \to H$ is an epimorphism between two groups and if S is a generating set for G, then $\varphi(S)$ is a generating set of H. In particular, if G is finitely generated, then so is H.

Proof. Both statements follow easily from the definitions. ■

Example. First recall that it follows from the Wedge-of-Circles–π_1-Lemma 20.7 that we can make the identification $\pi_1(S^1 \vee S^1, *) = \langle x, y\rangle$. In the figure below we show a covering $p\colon \widetilde{X} \to S^1 \vee S^1$. It follows from the Covering-π_1-Monomorphism Proposition 14.11 that $\pi_1(\widetilde{X}, \widetilde{*})$ is isomorphic to a subgroup of $\pi_1(S^1 \vee S^1, *) = \langle x, y\rangle$.

Using the Graph-π_1-Proposition 20.9 one can show that $\pi_1(\widetilde{X}, \widetilde{*})$ admits epimorphisms onto free groups of any rank $n \in \mathbb{N}$. Thus it follows from the Generating Set–Lemma 21.2 that $\pi_1(\widetilde{X}, \widetilde{*})$ is not finitely generated.



We have thus shown that there exists a finitely generated group that admits a generated subgroup that is not finitely generated. This is in contrast to the situation in *abelian* groups, where in the Finite Generation Lemma 18.8 (2) we showed that every subgroup of a finitely generated *abelian* subgroup is again finitely generated. □

Lemma 21.3. (Generating Set Properties–Lemma) Let G be a group and let S be a generating set for G.
(1) If $\varphi\colon G \to G$ is a homomorphism such that $\varphi(s) = s$ for all $s \in S$, then $\varphi = \mathrm{id}$.
(2) If for all $s, t \in S$ we have $s \cdot t = t \cdot s$, then G is abelian.
(3) If $\varphi\colon H \to G$ is a group homomorphism with $S \subset \varphi(H)$, then φ is an epimorphism.

Proof.

(1) Let $\varphi\colon G \to G$ be a homomorphism such that $\varphi(s) = s$ for every $s \in S$. We consider $H := \{g \in G \,|\, \varphi(g) = g\}$. It is straightforward to verify that H is a subgroup of G. Since H contains S it follows from the Generating Set–Characterization Lemma 21.1 (1)⇒(3) that $H = G$, i.e. $\varphi = \mathrm{id}$.
(2) We assume that for all $s, t \in S$ we have $s \cdot t = t \cdot s$. We want to show that G is abelian. We start out with the following claim:

Claim. For all $s, t \in S$ and all $\mu, \nu \in \{-1, 1\}$ we have $s^\mu \cdot t^\nu = t^\nu \cdot s^\mu$.

Proof. First we consider the case $\mu = 1$ and $\nu = -1$:

$$s \cdot t^{-1} \;=\; \underbrace{t^{-1} \cdot t}_{=e} \cdot s \cdot t^{-1} \;\underset{\substack{\uparrow \\ \text{since } s \cdot t = t \cdot s}}{=}\; t^{-1} \cdot s \cdot t \cdot t^{-1} \;=\; t^{-1} \cdot s.$$

By swapping the roles of s and t we see that the above also takes care of the case $\mu = -1$ and $\nu = 1$. Thus it remains to deal with the case $\mu = -1$ and $\nu = -1$. By hypothesis we have $t \cdot s = s \cdot t$. Taking inverses we see that $(t \cdot s)^{-1} = (s \cdot t)^{-1}$. But this means precisely that $s^{-1} \cdot t^{-1} = t^{-1} \cdot s^{-1}$. ⊞

Now let $g, h \in G$. By the Generating Set–Characterization Lemma 21.1 (1)⇒(2) we can write $g = s_1^{\epsilon_1} \cdot \ldots \cdot s_k^{\epsilon_k}$ and $h = t_1^{\tau_1} \cdot \ldots \cdot t_l^{\tau_l}$ for some $s_1, \ldots, s_k, t_1, \ldots, t_l \in S$ and $\epsilon_1, \ldots, \epsilon_k, \tau_1, \ldots, \tau_l \in \{-1, 1\}$. By applying the above claim altogether $k \cdot l$ times we see that $g \cdot h = h \cdot g$.

(3) Let $\varphi\colon H \to G$ be a group homomorphism with $S \subset \varphi(G)$. Note that $\varphi(H)$ is a subgroup of G. Since $S \subset \varphi(G)$ it thus follows from the Generating Set–Characterization Lemma 21.1 (1)⇒(3) that $\varphi(H) = G$. ■

21.2. **Presentations of groups.** In this section we introduce presentations of groups. Before we do so it is convenient to recall the following definition from page 276:

Definition. Let G be a group and let $S \subset G$ be a subset. We define

$$\text{subgroup of } G \text{ normally generated by } S \;:=\; \langle\langle S \rangle\rangle \;:=\; \bigcap_{S \subset H \lhd G} H.$$

If $\langle\langle S \rangle\rangle = G$, then we say that S normally generates G

Now we can provide the main definition of this section.

Definition. Let X be a set and let R be a subset of the free group $\langle X \rangle$ generated by the set X. We write

$$\langle X \mid R \rangle \;:=\; \langle X \rangle \,/\, \langle\langle R \rangle\rangle.$$

If $X = \{x_1, \ldots, x_k\}$ and $R = \{r_1, \ldots, r_l\}$, then we use the convenient shorthand

$$\langle x_1, \ldots, x_k \,|\, r_1, \ldots, r_l \rangle \;:=\; \langle \{x_1, \ldots, x_k\} \,|\, \{r_1, \ldots, r_l\} \rangle.$$

On many occasions we will use that the Generating Set–Lemma 21.2 implies that X is a generating set of $\langle X \mid R \rangle$.

Examples.

(1) For $k \in \mathbb{N}_0$ we have

$$\langle g \mid g^k \rangle \;=\; \langle g \rangle / \langle\langle g^k \rangle\rangle \;\underset{\substack{\uparrow \\ \text{this follows easily from the observation that } \langle g \rangle \text{ is abelian}}}{=}\; \langle g \rangle / \{g^{k \cdot n} \mid n \in \mathbb{Z}\} \;\cong\; \mathbb{Z}_k.$$

(2) The group $\langle x_1, \ldots, x_k \mid \rangle$ with $R = \varnothing$ is just the free group on $x_1, \ldots, x_k$. □

Convention. Given a group $\langle X \mid R \rangle$ and given $g \in \langle X \rangle$ we denote the images of the g in the quotient $\langle x_1, \ldots, x_k \,|\, r_1, \ldots, r_l \rangle$ again by g.

As we will now see, finite presentations of groups are useful for finding group homomorphisms. To formulate the corresponding lemma we need the following definition:

Definition. Let $r \in \langle x_1, \ldots, x_k \rangle$. Furthermore let G be a group and let $g_1, \ldots, g_k$ be elements of G. We say that the elements $g_1, \ldots, g_k$ satisfy the relation r if the obvious

homomorphism[195] $\langle x_1, \dots, x_k \rangle \to G$ that is given by $x_i \mapsto g_i$, $i = 1, \dots, k$ sends r to the trivial element.[196]

Example.

(1) We consider the word $r = [x_1, x_2] = x_1 x_2 x_1^{-1} x_2^{-1} \in \langle x_1, x_2 \rangle$. If a group G is abelian, then any two elements $g_1, g_2 \in G$ satisfy the relation $x_1 x_2 x_1^{-1} x_2^{-1}$.

(2) Let G be a finite group with d elements. It follows from Lagrange's Theorem, see e.g. [**KS04**, p. 8], that every element in G satisfies the relation $x^d \in \langle x \rangle$. □

Lemma 21.4. (Free Abelian Group-Presentation Lemma) Let $m \in \mathbb{N}$. There exists a unique isomorphism[197]

$$\alpha \colon \langle x_1, \dots, x_m \mid [x_i, x_j] = e \text{ for all } 1 \leq i < j \leq m \rangle \xrightarrow{\cong} \mathbb{Z}^m$$

which sends each x_i to the i-th standard basis vector e_i of $\mathbb{Z}^m$.

Proof. We set $X = \{x_1, \dots, x_m\}$ and $R = \{[x_i, x_j] \mid 1 \leq i < j \leq m\}$. Since $\mathbb{Z}^m$ is abelian we obtain from the Free Group–Existence Lemma 19.4 together with the Normal Closure–Factorization Lemma 19.10 that there exists a unique homomorphism $\alpha \colon \langle X \mid R \rangle \to \mathbb{Z}^m$ with $\alpha(x_i) = e_i$. Since the image of α contains the generating set $\{e_1, \dots, e_m\}$ of $\mathbb{Z}^m$ we obtain from the Generating Set Properties–Lemma 21.3 (3) that α is an epimorphism. Thus it remains to show that α is a monomorphism.

Claim. The group $\langle X \mid R \rangle$ is abelian.

Proof. Recall that it follows from the Generating Set–Lemma 21.2 that $X = \{x_1, \dots, x_m\}$ is a generating set for $\langle X \mid R \rangle$. Let $i < j$. We have the following equality in $\langle X \mid R \rangle$:

$$x_j \cdot x_i \;=\; \underbrace{x_i \cdot x_j \cdot x_i^{-1} \cdot x_j^{-1}}_{=[x_i, x_j] = e \in \langle X \mid R \rangle} \cdot x_j \cdot x_i \;=\; x_i \cdot x_j.$$

This shows that the elements of the generating set commute. Thus it follows from the Generating Set Properties–Lemma 21.3 (2) that the group $\langle X \mid R \rangle$ is abelian. ⊞

As we mentioned on page 256 (2), the group $\mathbb{Z}^m$ is free abelian on the set $e_1, \dots, e_m$. We just learned in the above claim that $\langle X \mid R \rangle$ is abelian. This implies in particular that there exists a unique homomorphism $\beta \colon \mathbb{Z}^m \to \langle X \mid R \rangle$ with $\beta(e_i) = x_i$. Note that for each $i \in \{1, \dots, m\}$ we have $(\beta \circ \alpha)(x_i) = \beta(\alpha(x_i)) = \beta(e_i) = x_i$. Thus it follows from the Generating Set Properties–Lemma 21.3 (1) that $\beta \circ \alpha = \mathrm{id}$. We have thus shown that $\alpha \colon \langle X \mid R \rangle \to \mathbb{Z}^m$ is a monomorphism. ■

We continue with the following definitions.

Definition.

(1) A presentation is a pair (X, R) consisting of a set X together with a subset $R \subset \langle X \rangle$. We call the elements of X the generators and we call the elements of R the relators[198] of the presentation.

---

[195] The fact that there exists a unique such homomorphism is a consequence of the Free Group–Existence Lemma 19.4.

[196] More colloquially speaking: $g_1, \dots, g_k$ satisfy a relation r if replacing the x_i by g_i in the word r_i returns the trivial element in G.

[197] On page 272 we introduced for two elements g, h of a group G the commutator $[x, y] \coloneqq xyx^{-1}y^{-1}$.

(2) Let π be a group. A presentation for π is a presentation (X, R) together with an isomorphism
$$\langle X \mid R\rangle \xrightarrow{\cong} \pi.$$
Very often one just says $\langle X \mid R\rangle$ is a presentation for π and the explicit isomorphism is suppressed from the notation.

(3) We say that a group π is finitely presented, if π admits a presentation with finitely many generators and finitely many relators.

Example.

(1) Every group admits a presentation. Indeed, let π be a group. Then we set $X := \pi$, i.e. X is the set of elements of π. By the Free Group–Existence Lemma 19.4 there exists a unique homomorphism $\alpha\colon \langle X\rangle \to \pi$ with $\alpha(g) = g$ for all $g \in X = \pi$. We set $R := \ker(\alpha)$. Then
$$\langle X \mid R\rangle \;=\; \langle X\rangle/\langle\langle R\rangle\rangle \;=\; \langle X\rangle/\langle\langle \ker(\alpha)\rangle\rangle \underset{\text{since } R=\ker(\alpha) \text{ is a normal subgroup}}{=} \langle X\rangle/\ker(\alpha) \xrightarrow[\alpha]{\cong} \pi.$$

(2) Using a modification of the argument in (1) one sees that a group π is finitely generated, in the sense of the definition on page 303, if and only if π admits a presentation $\langle X \mid R\rangle \to \pi$ such that X is finite.

(3) We had just seen in the Free Abelian Group-Presentation Lemma 21.4 that
$$\mathbb{Z}^n \;\cong\; \langle x_1, \dots, x_n \mid [x_i, x_j] = e \text{ with } 1 \le i < j \le n\rangle.$$
This shows that $\mathbb{Z}^n$ is a finitely presented group.

(4) There exist groups that are finitely generated but not finitely presented. In fact it follows easily from the Countability Lemma 0.6 that there are only countably many isomorphism types of finitely presented groups. On the other hand it is shown in [**Rob95**, Chapter 14] that there are uncountably many isomorphism types of finitely generated groups.[199][200] □

Let us recall the following definition from page 273:

Definition. Let π be a group. The commutator subgroup of π is defined as
$$[\pi, \pi] \;:=\; \left\{ \prod_{i=1}^{n} [x_i, y_i] \;\middle|\; x_1, y_1, \dots, x_n, y_n \in \pi \right\}$$
We call $\pi_{\mathrm{ab}} := \pi/[\pi, \pi]$ the abelianization of π.

Proposition 21.5. (Abelianization-of-Presentation Proposition) Suppose we are given a finitely presented group $\pi = \langle x_1, \dots, x_m \mid r_1, \dots, r_n\rangle$. We denote by
$$\Phi\colon \langle x_1, \dots, x_m\rangle \;\to\; \mathbb{Z}^m$$

[198] The relators are often also called relations.

[199] The idea is that in a free group $\langle x_1, \dots, x_k\rangle$ there are uncountably many infinite sequences of elements $\{r_1, r_2, \dots\}$. The difficulty is to show that there exist uncountably many sequences that give raise to non-isomorphic groups.

[200] Note that this also shows that any free group on ≥ 2 generators contains uncountably many different subgroups.

the homomorphism that is given by $\Phi(x_i) = e_i$. There exists a unique isomorphism[201]

$$\Psi\colon \underbrace{\pi/[\pi,\pi]}_{=\pi_{\mathrm{ab}}} \xrightarrow{\cong} \mathbb{Z}^m/\operatorname{span}(\Phi(r_1),\dots,\Phi(r_n))$$

such that the following diagram commutes:

$$\begin{array}{ccc}
\langle x_1,\dots,x_m\rangle & \xrightarrow[\Phi]{x_i\mapsto e_i} & \twoheadrightarrow \mathbb{Z}^m \\
\big\downarrow & & \\
\pi = \langle x_1,\dots,x_m\rangle/\langle\!\langle r_1,\dots,r_n\rangle\!\rangle & & \big\downarrow \\
\big\downarrow & & \\
\pi_{\mathrm{ab}} = \pi/[\pi,\pi] & \xrightarrow[\cong]{\Psi} & \mathbb{Z}^m/\operatorname{span}(\Phi(r_1),\dots,\Phi(r_n)).
\end{array}$$

Example. We consider the group $\pi = \langle x, y \mid x^2y^{-2}x, y^{-1}x^3yx\rangle$.

Let $\Phi\colon \langle x,y\rangle \to \mathbb{Z}^2$ be the epimorphism given by $\Phi(x) = (1,0)$ and $\Phi(y) = (0,1)$. It follows from the Abelianization-of-Presentation Proposition 21.5 that the abelianization of π is given by

$$\pi_{\mathrm{ab}} = \mathbb{Z}^2/\operatorname{span}(\underbrace{\Phi(x^2y^{-2}x)}_{\binom{3}{-2}}, \underbrace{\Phi(y^{-1}x^3yx)}_{=\binom{4}{0}}) = \mathbb{Z}^2/\begin{pmatrix}3 & 4\\ -2 & 0\end{pmatrix}\mathbb{Z}^2 \cong \mathbb{Z}_8.$$

(the isomorphism can be calculated using the Smith normal form of the matrix) □

Proof. It is straightforward to see that there exists a unique homomorphism

$$\Psi\colon \pi/[\pi,\pi] \to \mathbb{Z}^m/\operatorname{span}(\Phi(r_1),\dots,\Phi(r_n))$$

with $\Psi([x_i]) = [e_i]$, i.e. that makes the diagram in the proposition commute. It remains to show that Ψ is an isomorphism. We write $F = \langle x_1,\dots,x_m\rangle$ and we write $G = \langle y_1,\dots,y_n\rangle$. We consider the following diagram:

$$\begin{array}{ccc}
G_{\mathrm{ab}} = G/[G,G] & \xrightarrow[\cong]{[y_i]\mapsto e_i} & \mathbb{Z}^n \\
{\scriptstyle [y_i]\mapsto [r_i]}\big\downarrow & & \big\downarrow{\scriptstyle e_i\mapsto \Phi(r_i)} \\
F_{\mathrm{ab}} = F/[F,F] & \xrightarrow[\cong]{[x_i]\mapsto e_i} & \mathbb{Z}^m \\
\big\downarrow & & \big\downarrow \\
\pi_{\mathrm{ab}} = \pi/[\pi,\pi] & \xrightarrow{\Psi} & \mathbb{Z}^m/\operatorname{span}(\Phi(r_1),\dots,\Phi(r_n)).
\end{array}$$

Note that the diagram commutes. It follows from the Abelianization Calculation Proposition 19.8 (4) that the top two horizontal maps are isomorphisms. The bottom vertical maps are clearly epimorphisms and it is clear that the images of the top vertical maps lie in the kernel of the bottom vertical maps. It now follows from an elementary "diagram

[201] By $\operatorname{span}(v_1,\dots,v_n)$ we mean the set of all linear combinations $\sum_{i=1}^{n} \lambda_i \cdot v_i$ with $\lambda_1,\dots,\lambda_n \in \mathbb{Z}$.

chase" that the bottom horizontal map is also an isomorphism.[202] We leave the details to the reader. ■

21.3. **Presentations of groups and constructions of groups.** In this section we will see how various familiar constructions of groups interact with presentations.

Lemma 21.6. (Quotient Presentation–Lemma) Let $G = \langle X \,|\, R\rangle$ be a group and let $\varphi\colon G \to H$ be an epimorphism. If $S \subset \langle X\rangle$ is a subset, such that $\ker(\varphi) = \langle\langle\{[s] \,|\, s \in S\}\rangle\rangle$, then

$$\begin{array}{rcl} \langle X \,|\, R \cup S\rangle & \to & H \\ x & \mapsto & \varphi([x]) \end{array}$$

is a presentation for H.

Proof. We consider the following diagram:

$$\begin{array}{ccccc} \langle X\rangle & \xrightarrow{\quad\mathrm{id}\quad} & \langle X\rangle & & \\ \downarrow & & \downarrow & & \\ & & \langle X \,|\, R\rangle & & \\ & & \downarrow & & \\ \langle X \,|\, R \cup S\rangle & \longrightarrow & \langle X \,|\, R\rangle / \ker(\varphi) & \xrightarrow{[g]\mapsto\varphi(g)} & H. \end{array}$$

Note that the identity homomorphism on the top (in both directions) descends to homomorphisms (in both directions) between the groups on the bottom left. It follows from the Generating Set Properties–Lemma 21.3 (1) that the two homomorphisms are inverses of one another, in particular the homomorphism from the left to the center is an isomorphism. The lemma now follows from the observation that the homomorphism to the bottom right is an isomorphism, since φ is an epimorphism. ■

Before we give our next example, and before we formulate further results on propositions let us introduce the following intuitive notation.

Notation. Given a set X we sometimes write a relator in the form $a = b$ for $a, b \in \langle X\rangle$, this just means that the relator is given by $a \cdot b^{-1} \in \langle X\rangle$.

Example. Let A be a finitely generated abelian group. By the Abelian Group-Classification Theorem 18.11 there exist $a_1, \ldots, a_n \in \mathbb{N}_0$ such that $A \cong \mathbb{Z}_{a_1} \oplus \cdots \oplus \mathbb{Z}_{a_n}$. In other words, we can write A as a quotient of $\mathbb{Z}^n$. Using the Abelianization-of-Presentation Proposition 21.5 and the Quotient Presentation–Lemma 21.6 one can now easily verify that

$$A \cong \bigoplus_{i=1}^{n} \mathbb{Z}_{a_i} \cong \langle x_1, \ldots, x_n \,|\, [x_i, x_j] = e \text{ for } 1 \le i < j \le n \text{ and } x_i^{a_i} = e \text{ for } i = 1, \ldots, n\rangle.$$

[202] More precisely, we use the following fact: suppose that we are given a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\cong} & A' \\ \downarrow & & \downarrow \\ B & \xrightarrow{\cong} & B' \\ \twoheadrightarrow & & \twoheadrightarrow \\ C & \longrightarrow & C' \end{array}$$

of group homomorphisms where the bottom vertical maps are epimorphisms, where the top two horizontal maps are isomorphisms and where the kernels of the bottom vertical maps are the images of the top vertical maps. Then the bottom horizontal map is also an isomorphism. This general statement can be proved by a "diagram chase".

This shows that every finitely generated abelian group is also finitely presented. □

The following lemma gives us a convenient way to find a presentation of the amalgamated product of two groups.

Lemma 21.7. (Amalgamated Product–Presentation Lemma) Let $A = \langle X \mid R\rangle$ and $B = \langle Y \mid S\rangle$ be two groups with $X \cap Y = \varnothing$. Furthermore let G be a group with generating set Z and let $\alpha\colon G \to A$ and $\beta\colon G \to B$ be two group homomorphisms. There is a natural isomorphism

$$\langle X \sqcup Y \mid R \sqcup S \sqcup \{\alpha(z) = \beta(z) \mid z \in Z\}\rangle \xrightarrow{\cong} A *_G B.$$

In particular A and B are finitely presented and if G is finitely generated, then $A *_G B$ is finitely presented.

Example.

(1) The case $G = \{e\}$ in the Amalgamated Product–Presentation Lemma 21.7, together with the Amalgamated Product Lemma 19.12 (1), gives us presentations of free products of two groups.

(2) Let $A = \langle a\rangle$ and let $B = \langle b\rangle$ be two infinite cyclic groups and let $G = \langle x, y \mid [x, y]\rangle$ be a group that is isomorphic to $\mathbb{Z}^2$. We consider the homomorphism $\alpha\colon G \to A$ that is given by $\alpha(x) = a$ and $\alpha(y) = e$ and we consider the homomorphism $\beta\colon G \to B$ that is given by $\beta(x) = e$ and $\beta(y) = b$. Then, similar to the discussion on page 277, we see that

$$A *_G B \underset{\substack{\uparrow\\ \text{the Amalgamated Product–Presentation Lemma 21.7}}}{=} \langle a, b \mid \alpha(x){=}\beta(x), \alpha(y){=}\beta(y)\rangle = \langle a, b \mid a{=}e, e{=}b\rangle = \langle a, b \mid a, b\rangle = \{e\}.$$

This shows in particular that the amalgamated product of two non-trivial groups can in fact be trivial. □

Proof of the Amalgamated Product–Presentation Lemma 21.7. We first consider the setting of the free product $\langle X \mid R\rangle * \langle Y \mid S\rangle$. We consider the following diagram:

$$\begin{array}{ccccc} \langle X \sqcup Y\rangle & \longrightarrow & \langle X\rangle * \langle Y\rangle & \longrightarrow & \langle X \sqcup Y\rangle \\ \downarrow & & \downarrow & & \downarrow \\ \langle X \sqcup Y \mid R \cup S\rangle & \xrightarrow{\alpha} & \langle X \mid R\rangle * \langle Y \mid S\rangle & \xrightarrow{\beta} & \langle X \sqcup Y \mid R \cup S\rangle. \end{array}$$

We make the following observations and clarifications:

(1) The top horizontal maps are given by the universal properties of free groups and free products.

(2) The vertical maps are (induced by) the natural projections.

(3) The bottom left horizontal map is induced by the Normal Closure–Factorization Lemma 19.10.

(4) It follows from the Normal Closure–Factorization Lemma 19.10 that the homomorphism $\langle X\rangle \to \langle X \sqcup Y \mid R \cup S\rangle$ descends to a homomorphism starting at $\langle X \mid R\rangle$. The same way we obtain a homomorphism starting at $\langle Y \mid S\rangle$. These two homomorphisms give us the bottom right horizontal homomorphism.

(5) Using the Generating Set–Lemma 21.2 and the Generating Set Properties–Lemma 21.3 (1) one can easily verify that $\beta \circ \alpha = \mathrm{id}$ and $\alpha \circ \beta = \mathrm{id}$.

We have thus proved the desired statement for the case of a free product. The case of a non-trivial amalgamated product follows easily from the above discussion together with the Quotient Presentation–Lemma 21.6. ■

We conclude this section with an interesting little group-theoretic lemma. It will play an important role later in the proof of the Topological Manifolds-Invariants Proposition 60.10 (4b). But it is also of independent interest. For the formulation of the lemma we need the following harmless definition.

Definition. Let G and H be groups. We say that H is a retract of G if there exist a group homomorphism $\varphi\colon H \to G$ and $\rho\colon G \to H$ such that $\rho \circ \varphi = \mathrm{id}_H$.

Lemma 21.8. (Finite Presentation-Retract Lemma) Every retract of a finitely presented group is also finitely presented.

Proof. Let H be a group, let G be a finitely presented group and let $\varphi\colon H \to G$ and $\rho\colon G \to H$ be group homomorphisms such that $\rho \circ \varphi = \mathrm{id}_H$. We need to show that H is also finitely presented.

To do so we pick a finite presentation $G = \langle g_1, \dots, g_k \,|\, r_1, \dots, r_l \rangle$ for the group G. We write $F = \langle g_1, \dots, g_k \rangle$. In the following we denote by $p\colon F \to G$ the obvious epimorphism. For each $i \in \{1, \dots, k\}$ we pick $w_i \in F$ with $p(w_i) = (\varphi \circ \rho \circ p)(g_i) \in G$. It suffices to prove the following claim:

Claim. A finite presentation for H is given by $\langle g_1, \dots, g_k \,|\, r_1, \dots, r_l, g_1^{-1}w_1, \dots, g_k^{-1}w_k \rangle$.

Proof. Let L be the group that is defined by the finite presentation of the claim. We denote by $u\colon G \to L$ and $q\colon F \to L$ the natural projections. Note that for each $j \in \{1, \dots, l\}$ we have $\rho(p(r_i)) = \rho(e) = e$ and for each $i \in \{1, \dots, k\}$ we have

$$\rho(p(g_i^{-1}w_i)) \;=\; \rho(p(g_i^{-1})) \cdot \rho(p(w_i)) \;\underset{\substack{\uparrow \\ \text{since } p(w_i) = (\varphi \circ \rho \circ p)(g_i) \text{ and since } \rho \circ \varphi = \mathrm{id}_H}}{=}\; \rho(p(g_i^{-1})) \cdot \rho(p(g_i)) \;=\; e.$$

It follows from the Normal Closure–Factorization Lemma 19.10 that the homomorphism $p \circ \varphi\colon F \to H$ factors through a homomorphism $v\colon L \to H$.

We want to show that $u \circ \varphi$ is an isomorphism. First note that

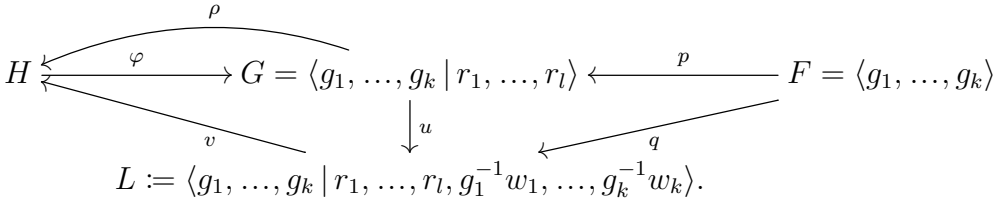
$$v \circ (u \circ \varphi) \;=\; (v \circ u) \circ \varphi \;\underset{\substack{\uparrow \\ \text{since } \rho = v \circ u}}{=}\; \rho \circ \varphi \;=\; \mathrm{id}_H\,.$$

By Lemma 20.4 it remains to show that $u \circ \varphi$ is an epimorphism. Note that for $i = 1, \dots, k$ we have

$$(u \circ \varphi)(\underbrace{(\rho \circ p)(w_i)}_{\in H}) \;\underset{\substack{\uparrow \\ \text{since } \varphi \circ \rho = \mathrm{id}_H}}{=}\; u(p(w_i)) \;=\; q(w_i) \;\underset{\substack{\uparrow \\ \text{since } q(w_i) = g_i \in L}}{=}\; g_i \;\in\; L.$$

Since $g_1, \dots, g_k$ is a generating set for L we see that $u \circ \varphi$ is indeed an epimorphism.

For convenience we summarize all objects in the following diagram:



■

21.4. **Tietze transformations.** We have now seen that many groups can be described by finite presentations. The question arises, to what degree are finite presentations unique? To answer this question we introduce the following definition:

Definition. Let
$$\phi\colon \langle x_1,\dots,x_k \,|\, r_1,\dots,r_l\rangle \xrightarrow{\cong} \pi$$
be a presentation of a group π. We obtain new presentations of π as follows:

(T1) For any element s of $\langle\langle r_1,\dots,r_l\rangle\rangle \subset \langle X\rangle$ we obtain the presentation
$$\langle x_1,\dots,x_k \mid r_1,\dots,r_l, s\rangle \underset{\uparrow}{=} \langle x_1,\dots,x_k \mid r_1,\dots,r_l\rangle \xrightarrow{\phi} \pi.$$
since $s \in \langle\langle r_1,\dots,r_l\rangle\rangle$ the groups are identical

(T2) For any $x \notin \{x_1,\dots,x_k\}$ and any two words $s,t \in \langle x_1,\dots,x_k\rangle$ we obtain the presentation[203]
$$\begin{aligned}\langle x_1,\dots,x_k, x \mid r_1,\dots,r_l, s\cdot x\cdot t\rangle &\to \langle x_1,\dots,x_k \mid r_1,\dots,r_l\rangle \xrightarrow{\phi} \pi.\\ x_i &\mapsto x_i\\ x &\mapsto s^{-1}\cdot t^{-1}\end{aligned}$$

The above two methods, and their inverses, of obtaining new presentations out of a given one are called Tietze transformations.[204]

Example. We want to show that the group $\langle a,b,c \,|\, (ab)^2ab^2\rangle$ is a free group on two generators. Borrowing an argument from [**MKS04**, p. 50] we see this as follows:
$$\begin{aligned}\langle a,b,c \,|\, (ab)^2ab^2\rangle &\overset{(T2),(T2)}{\cong} \langle a,b,c,x,y \,|\, (ab)^2ab^2, x=ab, y=ab^2\rangle\\ &\overset{(T1)}{\cong} \langle a,b,c,x,y \,|\, (ab)^2ab^2, x=ab, y=ab^2, x^2y\rangle\\ &\overset{(T1)^{-1}}{\cong} \langle a,b,c,x,y \,|\, x=ab, y=ab^2, x^2y\rangle\\ &\overset{(T1),(T1)}{\cong} \langle a,b,c,x,y \,|\, x=ab, y=ab^2, x^2y, b=x^{-1}y, a=xy^{-1}x\rangle\\ &\overset{(T2)^{-1},(T2)^{-1}}{\cong} \langle c,x,y \,|\, x=xy^{-1}x\cdot x^{-1}y, y=xy^{-1}x(x^{-1}y)(x^{-1}y), x^2y\rangle\\ &\overset{(T2)^{-1}}{\cong} \langle c,x,y \,|\, x^2y\rangle \overset{(T1),(T1)^{-1}}{\cong} \langle c,x,y \,|\, y=x^{-2}\rangle \overset{(T2)^{-1}}{\cong} \langle c,x \,|\,\rangle.\end{aligned}$$
As we can see, working with Tietze transformations leads too long and unreadable arguments. □

The above Tietze transformations show that from a given finite presentation we can obtain infinitely many new finite presentations. This fact is not that surprising. What is much more interesting is the following theorem which was first proved by Heinrich Tietze [**Tie08**] in 1908.

Theorem 21.9. (Tietze Theorem) Any two finite presentations for a given group are related by a finite sequence of Tietze transformations.

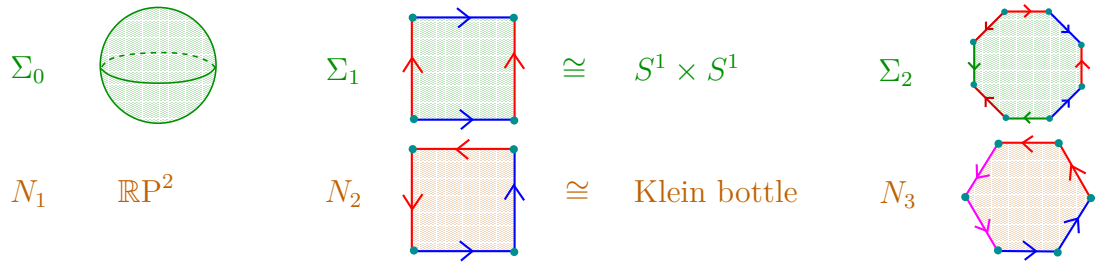
[204] First note that the map to the left is well-defined. This can be seen using the observation that under the assignment $x_i \mapsto x_i$ and $x \mapsto s^{-1}\cdot t^{-1}$ the relation $s\cdot x\cdot t$ gets sent to $s\cdot s^{-1}\cdot t^{-1}\cdot t = e$. The map to the left is also an isomorphism. In fact, as one can verify using the Generating Set Properties–Lemma 21.3 (1), an inverse is given by $x_i \mapsto x_i$.

Proof. We outsource the proof to [**Fri24**]. ■

CHAPTER 22

The classification of compact 2-dimensional manifolds

Given $g \in \mathbb{N}_0$ we introduced on page 140 the surface Σ_g of genus g and given $g \in \mathbb{N}$ we introduced on page 141 the non-orientable surface N_g of genus g.[205]

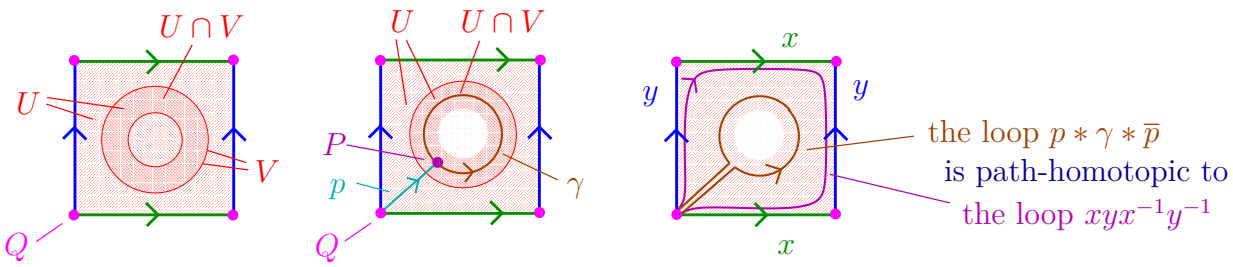


In this chapter we will first use the Seifert–van Kampen Theorem 20.1 to provide the long overdue calculation of the fundamental groups of the (non-orientable) surfaces Σ_g and N_g. Afterwards, in Section 22.5 we will use these calculations to give the classification of compact 2-dimensional smooth and topological manifolds.

22.1. **The fundamental groups of the surfaces Σ_g.** We want to use the Seifert–van Kampen-Theorem 20.1 to determine the fundamental groups of the surfaces Σ_g of genus $g \geq 2$. Before we do so we consider the torus, the Klein bottle and the real projective plane to gain some confidence in our newly acquired methods.

(1) We first consider the torus $T = ([0,1] \times [0,1])/\sim$. We will use the following notation:
   (a) We denote by Q the point on T corresponding to one of the four vertices of the square.
   (b) We set
   $$U = T \setminus \overline{B_{\frac{1}{8}}(\tfrac{1}{2},\tfrac{1}{2})} \subset T \quad \text{and } V = B_{\frac{1}{4}}(\tfrac{1}{2},\tfrac{1}{2}) \subset T,$$
   i.e. U is the complement of a closed disk and V is an open disk. Note that $T = U \cup V$ and note that $U \cap V$ is an open annulus, i.e. homeomorphic to $S^1 \times (0,1)$.
   (c) We denote by $i\colon U \cap V \to U$ the inclusion.
   (d) We let $P = (\frac{1}{3},\frac{1}{3}) \in U \cap V$ and we denote by p the direct path from Q to P.
   (e) We pick a loop γ in $(U \cap V, P)$ that "goes counterclockwise around the annulus". Since $U \cap V$ has an obvious deformation retraction to a circle we obtain from the Circle-π_1-Theorem 11.13 together with the Homotopy-π_1-Proposition 13.4 (2b) that γ represents a generator of the infinite cyclic group $\pi_1(U \cap V, P) \cong \mathbb{Z}$.
   (f) We denote by x and y the two loops in (T, Q) corresponding to the horizontal and the vertical edge of the square. By a slight abuse of notation we denote the corresponding elements in $\pi_1(T, Q)$ by x and y as well.

[205] In these notes, the notion of "(non-orientable) surface of genus g" is properly defined. At times we will also use the word "surface" in a colloquial sense, even though as a standalone term we never introduced it.



Now we see that

$$\pi_1(T,P) \underset{\uparrow}{\cong} \pi_1(U,P) *_{\pi_1(U\cap V,P)} \pi_1(V,P) \underset{\uparrow}{=} \pi_1(U,P) *_{\pi_1(U\cap V,P)} \{e\}$$

Seifert–van Kampen-Theorem 20.1 — V is a disk, hence $\pi_1(V,P) = \{e\}$

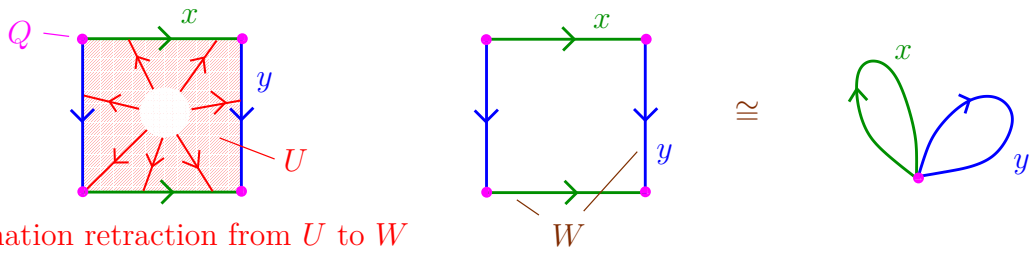
$$\underset{\uparrow}{=} \pi_1(U,P)/\langle\langle i_*(\pi_1(U\cap V,P))\rangle\rangle \underset{\uparrow}{=} \pi_1(U,P)/\langle\langle [\gamma]\rangle\rangle$$

by the Amalgamated Product–Quotient Lemma 19.13 we have $A *_G \{e\} = A/\langle\langle \alpha(G)\rangle\rangle$ — since $\pi_1(U\cap V,P) = \langle[\gamma]\rangle$

$$\underset{\uparrow}{\overset{\cong}{\to}} \pi_1(U,Q)/\langle\langle [p*\gamma*\overline{p}]\rangle\rangle \underset{\uparrow}{=} \pi_1(U,Q)/\langle\langle xyx^{-1}y^{-1}\rangle\rangle = (*).$$

by the Change-of-Base Point Proposition 11.6 (1) the path p gives rise to an isomorphism p_* — since $p*\gamma*\overline{p}$ is path-homotopic to $xyx^{-1}y^{-1}$

One can easily show that $W := \partial([0,1]\times[0,1])/\sim$ is a deformation retract of U[206] and that W is homeomorphic to the wedge of two circles[207]. By the Wedge-of-Circles–π_1-Lemma 20.7 we can therefore make the identification $\pi_1(W,P) = \langle x,y\rangle$.



Now we continue with the above calculation of $\pi_1(T,P)$:

$$(*) = \pi_1(U,Q)/\langle\langle xyx^{-1}y^{-1}\rangle\rangle \underset{\uparrow}{\overset{\cong}{\to}} \pi_1(W,Q)/\langle\langle xyx^{-1}y^{-1}\rangle\rangle$$

by the Homotopy-π_1-Proposition 13.4 (2b), since W is a deformation retract of U

$$\underset{\uparrow}{=} \langle x,y\rangle/\langle\langle xyx^{-1}y^{-1}\rangle\rangle = \langle x,y \mid [x,y]\rangle \underset{\uparrow}{\cong} \mathbb{Z}^2.$$

by the Wedge-of-Circles–π_1-Lemma 20.7 since W is the wedge of the circles x and y — by the Free Abelian Group-Presentation Lemma 21.4

[206] We consider the map

$$\begin{aligned} r\colon U &\to \partial([0,1]\times[0,1]) \\ x &\mapsto \text{unique point in } \mathbb{R}_{>0}\cdot x \cap \partial([0,1]\times[0,1]). \end{aligned}$$

A deformation retraction is given by

$$\begin{aligned} U\times[0,1] &\to U \\ ([P],t) &\mapsto [P\cdot(1-t) + r(P)\cdot t]. \end{aligned}$$

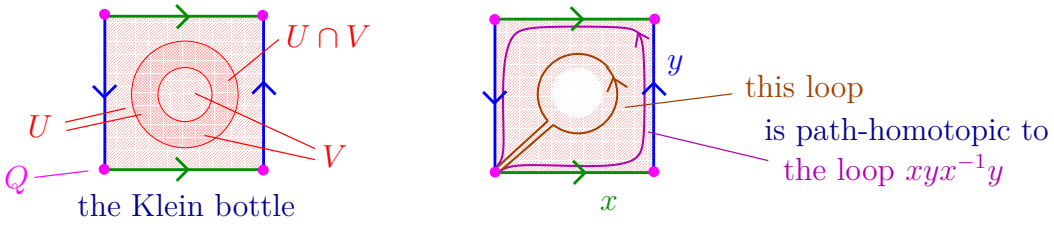
(This map is continuous by the popular Homotopy Quotient Theorem 8.5.)

[207] A homeomorphism is given by sending the point on W that corresponds to the vertices to the wedge point of the two circles, and by sending the edges that get identified to the same circle. This idea can be made precise using the Twice Quotient Lemma 6.3 and the Compact-Hausdorff Proposition 2.10 (3).

Thus we have reproved the fact, first shown in the Torus-π_1–Corollary 13.8, that the fundamental group of the torus is isomorphic to the free abelian group $\mathbb{Z}^2$.

(2) We turn to the Klein bottle $K = ([0,1] \times [0,1])/\sim$ with base point $Q = [(0,0)]$. The approach to calculating the fundamental group of the Klein bottle is almost identical to the above approach to calculating the fundamental group of the torus. We just replace the situation illustrated in the figure above by the situation illustrated in the figure below. The same argument as above shows that [208]

$$\pi_1(K,Q) \;\cong\; \langle x, y \mid xyx^{-1}y\rangle.$$



(3) Next we consider again the real projective space $\mathbb{R}\mathrm{P}^2$. Here we use the description of $\mathbb{R}\mathrm{P}^2$ from the Projective Space-Avatar Lemma 6.11 (2) as

$$\mathbb{R}\mathrm{P}^2 \;=_{\mathrm{i}}\; \overline{B}^2/\!\sim \qquad \text{where for } z \in S^1 \text{ we have } z \sim -z.$$

As in the previous two examples we decompose $\mathbb{R}\mathrm{P}^2$ into an open disk V and the complement of a closed disk U such that $U \cap V$ is an annulus. We consider the base point $Q = [1] = [-1] \in \mathbb{R}\mathrm{P}^2$ and we denote by α the loop

$$\begin{array}{rccc} \alpha\colon [0,1] & \to & \mathbb{R}\mathrm{P}^2 \\ t & \mapsto & [\exp(\mathrm{i}\pi t)]. \end{array}$$

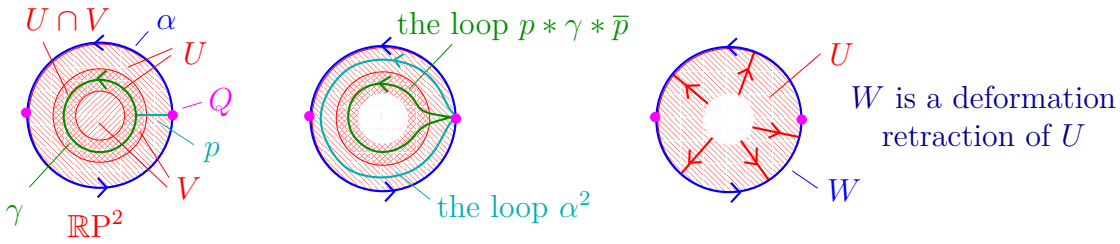
By a rather slight abuse of notation we denote by α also the corresponding element in $\pi_1(\mathbb{R}\mathrm{P}^2, Q)$. Furthermore we denote by W the image of α in $\mathbb{R}\mathrm{P}^2$. One can easily verify that W is homeomorphic to S^1. It follows from the Circle-π_1-Theorem 11.13 that α is a generator of $\pi_1(W,Q)$. With similar arguments as in the torus case, we see that

$$\pi_1(\mathbb{R}\mathrm{P}^2,Q) \;\underset{\text{by the Seifert–van Kampen-Theorem 20.1 and the Amalgamated Product–Quotient Lemma 19.13}}{\cong}\; \pi_1(U,Q)/\langle\langle p*\gamma*\overline{p}\rangle\rangle \;\underset{\text{by the Homotopy-}\pi_1\text{-Proposition 13.4 (2b), since } W \text{ is a deformation retract of } U}{\cong}\; \pi_1(W,Q)/\langle\langle p*\gamma*\overline{p}\rangle\rangle \;\underset{\text{since } \pi_1(W,Q)=\langle\alpha\rangle \text{ and } [p*\gamma*\overline{p}]=\alpha^2}{\cong}\; \langle\alpha\rangle/\langle\langle\alpha^2\rangle\rangle \;\cong\; \mathbb{Z}_2.$$

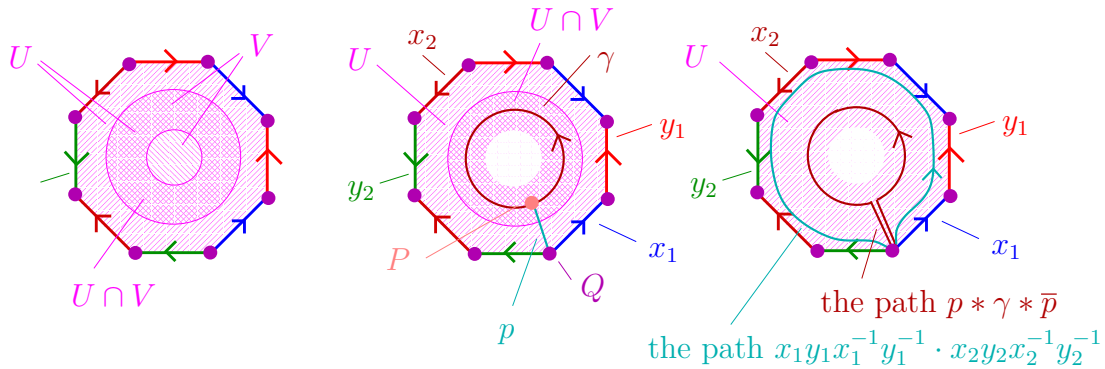
In particular we obtain the same result, as we should, as in the Real Projective Space–π_1-Proposition 15.3.

(4) Now that we have gained some confidence that the approach of determining fundamental groups using the Seifert–van Kampen-Theorem 20.1 does indeed give the right fundamental groups we turn to the surfaces of higher genus that we could not handle with our previous methods.

[208] Using this presentation and using the Abelianization-of-Presentation Proposition 21.5 we see that the abelianization of $\pi_1(K,Q)$ is given by $\pi_1(K,Q)_{\mathrm{ab}} \;\cong\; \langle x,y \mid xyx^{-1}y\rangle_{\mathrm{ab}} \;\cong\; \mathbb{Z}^2/\begin{pmatrix}0\\2\end{pmatrix}\cdot\mathbb{Z} \;\cong\; \mathbb{Z}\oplus\mathbb{Z}_2$.



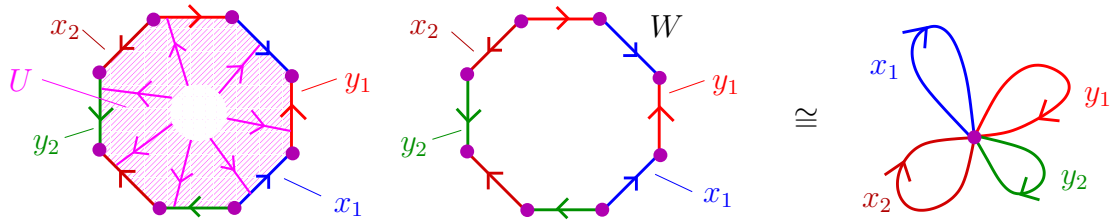
So let us consider the surface $F = E_8/\sim$ of genus 2. As in the torus case we write $F = U \cup V$ where V is an open disk and where U is the complement of a closed disk such that $U \cap V$ is an annulus. Similar to the torus case we write $W = \partial E_8/\sim$. As in the torus case we see that W is a deformation retraction of U and that W is homeomorphic to the wedge of four circles corresponding to x_1, y_1, x_2, y_2. Finally almost



the same argument as in the torus case shows that

$$\begin{aligned}
\pi_1(E_8/\sim, P) &\cong \pi_1(U, Q)/\langle\langle \overbrace{\underbrace{x_1 y_1 x_1^{-1} y_1^{-1} \cdot x_2 y_2 x_2^{-1} y_2^{-1}}_{=[x_1,y_1]\cdot[x_2,y_2]}}^{\text{path-homotopic to } p*\gamma*\overline{p}} \rangle\rangle \\
&\cong \pi_1(W, Q)/\langle\langle [x_1, y_1] \cdot [x_2, y_2] \rangle\rangle \\
&\cong \langle x_1, y_1, x_2, y_2 \rangle / \langle\langle [x_1, y_1] \cdot [x_2, y_2] \rangle\rangle \;=\; \langle x_1, y_1, x_2, y_2 \mid [x_1, y_1] \cdot [x_2, y_2] \rangle.
\end{aligned}$$

To summarize: we have finally succeeded in calculating the fundamental group of the surface of genus 2.



Proposition 22.1. (Surface-π_1-Proposition)

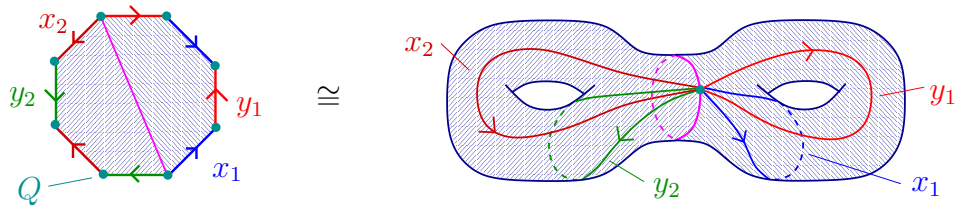
(1) Let $g \in \mathbb{N}$. Let Q be the point in $\Sigma_g = E_{4g}/\sim$ corresponding to the vertex of E_{4g}. We have an isomorphism[209]

$$\pi_1(\text{surface } \Sigma_g \text{ of genus } g, Q) \;\cong\; \langle x_1, y_1, \ldots, x_g, y_g \mid [x_1, y_1] \cdot \ldots \cdot [x_g, y_g] \rangle$$

where the $x_1, y_1, \ldots, x_g, y_g$ correspond to edges of E_{4g}. Furthermore we have

$$\pi_1(\text{surface } \Sigma_g \text{ of genus } g, Q)_{\text{ab}} \;\cong\; \mathbb{Z}^{2g}.$$

(2) For $g \neq h$ the surfaces of genus g and h are not homotopy equivalent, in particular they are not homeomorphic.



Proof.

(1) Let $g \in \mathbb{N}_0$. For $g = 0$ we obtain the fundamental group from the Sphere-π_1-Proposition 11.15 and for $g = 1$ we obtain the presentation from the discussion on page 315. For $g \geq 2$ the presentation follows from a straightforward generalization of the calculation preceding the proposition. Finally the stated abelianization of the fundamental group is an immediate consequence of the Abelianization-of-Presentation Proposition 21.5.

(2) For $g \neq h$ we obtain from the Abelian Group-Classification Theorem 18.11 that the abelianizations of the fundamental groups of the surfaces of genus $g \neq h$ are not isomorphic. It follows from the Homotopy-π_1-Proposition 13.4 that the surfaces are not homotopy equivalent. ■

22.2. **The fundamental groups of the non-orientable surfaces N_g.** In this section we will discuss the fundamental groups of the non-orientable surfaces N_g of some genus $g \in \mathbb{N}$. The following proposition is the analogue of the Surface-π_1-Proposition 22.1.

Proposition 22.2. (Surface-π_1-Proposition II)

(1) For any $g \in \mathbb{N}$ we have

$$\pi_1(\text{non-orientable surface } N_g \text{ of genus } g) \;\cong\; \langle x_1, \ldots, x_g \mid x_1^2 \cdot \ldots \cdot x_g^2 \rangle$$

and

$$\pi_1(\text{non-orientable surface } N_g \text{ of genus } g)_{\text{ab}} \;\cong\; \mathbb{Z}^{g-1} \oplus \mathbb{Z}_2.$$

(2) For $g \neq h$ the non-orientable surfaces of genus g and h are not homeomorphic.

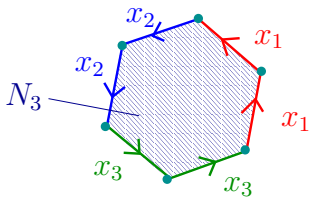
Proof.

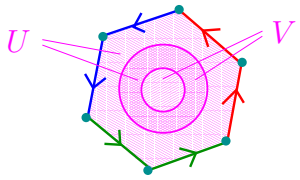
(1) Let $g \in \mathbb{N}$. Recall that by the definition on page 141 the non-orientable surface of genus one is the real projective plane. In the Real Projective Space–π_1-Proposition 15.3 (or alternatively above on page 316) we saw that $\pi_1(\mathbb{R}\text{P}^2) \cong \mathbb{Z}_2 \cong \langle x \mid x^2 \rangle$. The case $g \geq 2$ follows from a straightforward modification of the calculation of the fundamental group of the surface of genus 2 that we gave on page 317. Using the figure below the reader

[209] This statement also makes sense for $g = 0$.

should have no troubles with assembling a full argument. Finally note that

$$\pi_1(N_g)_{\text{ab}} \underset{\text{by (1)}}{\overset{\uparrow}{\cong}} \langle x_1,\dots,x_g \mid x_1^2\cdot\ldots\cdot x_g^2\rangle_{\text{ab}} \underset{\text{by the Abelianization-of-Presentation Proposition 21.5}}{\overset{\uparrow}{\cong}} (\mathbb{Z}^g)/\mathbb{Z}\cdot\begin{pmatrix}2\\ \vdots\\ 2\end{pmatrix} \cong \mathbb{Z}^{g-1}\oplus\mathbb{Z}_2.$$





(2) For $g \neq h$ we obtain from the above calculation and from the Abelian Group-Classification Theorem 18.11 that the abelianizations of the fundamental groups of the non-orientable surfaces of genus g and h are not isomorphic. Hence the non-orientable surfaces of genus $g \neq h$ are not homeomorphic. ■

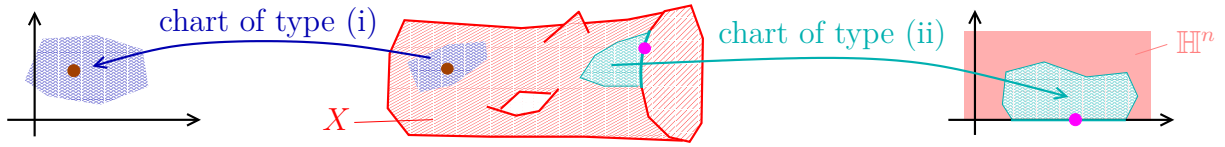
22.3. **Interlude: The boundary of 2-dimensional topological manifolds.** In the next section we will study the fundamental groups of 2-dimensional topological manifolds with boundary. Before we get there we need to prove the subtle statement that the boundary of a 2-dimensional topological manifold is what you think it should be.

Definition. Let X be a topological space.

(a) An n-dimensional chart for X at a point $x \in X$ is a homeomorphism $\Phi\colon U \to V$ where U is an open neighborhood of x and V is one of the following:
  (i) V is an open subset of $\mathbb{R}^n$ or
  (ii) V is an open subset of the upper half-space $\mathbb{H}^n := \{(x_1,\dots,x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$ and $\Phi(x)$ lies on $\partial\mathbb{H}^n = \{(x_1,\dots,x_n) \in \mathbb{R}^n \mid x_n = 0\}$.
  In the former case we say that Φ is a chart of type (i), in the latter case we say that Φ is a chart of type (ii).

(b) We say that X is an n-dimensional topological manifold if every point admits a chart and if X is Hausdorff and second countable.

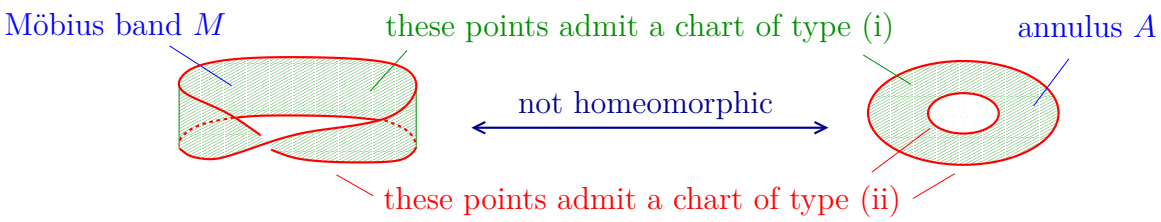
(c) We say that a point x on a topological manifold X is a boundary point if x does not admit a chart of type (i). We denote by ∂X the set of all boundary points of X.



Proposition 22.3. (Topological Surface–Boundary Proposition) Let F be a 2-dimensional topological manifold. Every point on F admits either a chart of type (i) or a chart of type (ii).

Remark. We will prove the high-dimensional analogue of the above proposition in algebraic topology II, once we have introduced and studied homology groups. □

Example. In the figure below we show the Möbius band M and the annulus A.

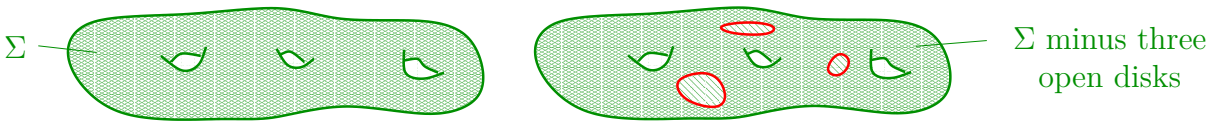


The Topological Surface–Boundary Proposition 22.3 says that the boundary of the Möbius band M and the annulus A, *as a topological manifold*, is exactly what we think it should be, namely precisely the points that "obviously" admit a chart of type (ii). In particular we see that ∂M consists of one component whereas ∂A consists of two components. Since homeomorphisms of topological manifolds induce homeomorphisms of their boundaries and since ∂M is not homeomorphic to ∂A we see that the Möbius band M is not homeomorphic to the annulus A. □

Proof. We outsource the proof of the Topological Surface–Boundary Proposition 22.3 (which uses the fundamental group) to [**Fri24**]. ■

22.4. **The fundamental groups of surfaces with boundary.** As mentioned above, one goal of any classification theorem is to provide a "complete" list of objects. We already have a good list of closed surfaces. Now we need to add surfaces with non-empty boundary to our arsenal.

Definition. Let Σ be a closed connected non-empty 2-dimensional smooth manifold and let $n \in \mathbb{N}_0$. We pick smooth embeddings $\varphi_1, \dots, \varphi_n \colon \overline{B}^2 \to \Sigma$ with disjoint images. We refer to $\Sigma \setminus (\varphi_1(B^2) \cup \dots \cup \varphi_n(B^2))$ as Σ minus n open disks.



The following lemma captures some of the main features of the above definition.

Lemma 22.4. (Surface-Minus-Open Disks Lemma) Let Σ be a closed connected non-empty 2-dimensional smooth manifold and let $n \in \mathbb{N}_0$.

(1) The topological space Σ minus n open disks is a compact connected 2-dimensional topological manifold with n boundary components, which are given by the boundaries of the closed disks whose interiors were removed.
(2) Σ minus n open disks is a submanifold of the smooth manifold Σ.
(3) The diffeomorphism type of the smooth manifold Σ minus n open disks does not depend on the choice of the disks.

Proof. The lemma follows from basic facts about smooth manifolds. For example (3) follows from the fact, proved in the Smooth Ball Embedding Theorem of [**Fri24**], that any two choices of smooth embeddings $\overline{B}^2 \to \Sigma$ are smoothly isotopic (up to a possible reflection of $\overline{B}^2$.) We refer to [**Fri24**] for details. ■

The Surface-Minus-Open Disks Lemma 22.4 allows us to introduce the following notation:

Notation. Given $g, n \in \mathbb{N}_0$ we write

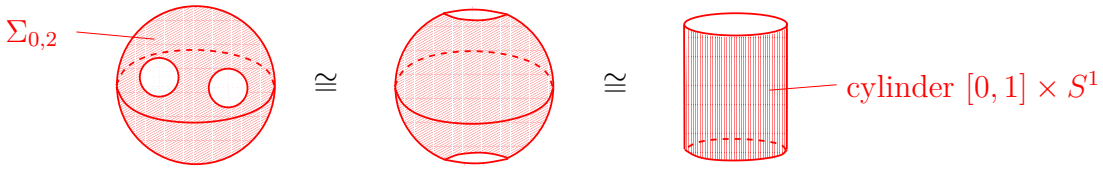
$$\Sigma_{g,n} \coloneqq \text{the surface of genus } g \text{ minus } n \text{ open disks.}$$

Furthermore for $g \in \mathbb{N}$ and $n \in \mathbb{N}_0$ we write

$$N_{g,n} \coloneqq \text{the non-orientable surface of genus } g \text{ minus } n \text{ open disks.}$$

Example.

(1) The surface $\Sigma_{0,2}$ is the result of removing the interiors of two disjoint closed disks from the 2-sphere. It is pretty elementary to show that the resulting smooth manifold is diffeomorphic to the "cylinder" $[0,1] \times S^1$.



(2) We saw in some exercise that $\mathbb{R}\mathrm{P}^2$ minus an open disk is diffeomorphic to the Möbius band, i.e. $N_{1,1}$ is diffeomorphic to the Möbius band. □

The following lemma shows that the fundamental groups of our new friends are rather dull and relatively uninformative.

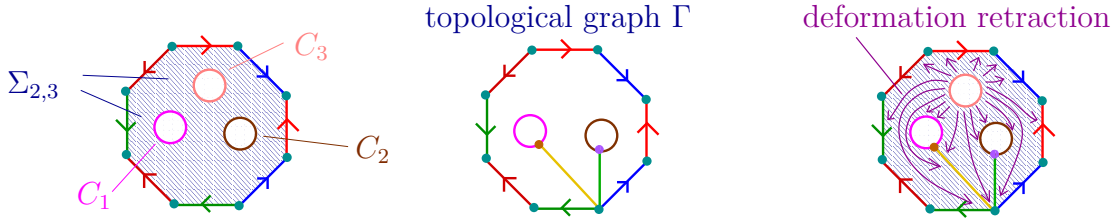
Lemma 22.5. (Surface-with-Boundary–π_1-Lemma)

(1) Given $g \in \mathbb{N}_0$ and $n \in \mathbb{N}$ the surface $\Sigma_{g,n}$ admits a deformation retract to a topological graph of Euler characteristic $2 - 2g - n$. In particular $\pi_1(\Sigma_{g,n})$ is a free group on $2g + n - 1$ generators.

(2) Given $k \in \mathbb{N}$ and $n \in \mathbb{N}$ the surface $N_{k,n}$ admits a deformation retract to a topological graph of Euler characteristic $2-k-n$. In particular $\pi_1(N_{k,n})$ is a free group on $k+n-1$ generators.

Sketch of proof.

(1) We denote by $C_1, \ldots, C_n$ the boundary components of $\Sigma_{g,n}$. Let Γ be the topological space that is given by the wedge of $2g$-circles (which is obtained from identifying as usual the edges of the $4g$-gon pairwise), together with the boundary components $C_1, \ldots, C_{n-1}$ and together with edges $e_1, \ldots, e_{n-1}$ that connect the wedge point to $C_1, \ldots, C_{n-1}$. The deformation retraction from $\Sigma_{g,n}$ to Γ is given by pushing the remaining boundary component C_n "inward". We illustrate this in the figure below. (This is one of the rare occasions where we provide a "proof by picture". It is of course possible to give a completely rigorous proof, but one does not gain anything from doing so.)



We equip Γ with the obvious structure of a topological graph which has $n = 1+(n-1)$ vertices and $2g + 2(n-1)$ edges. Note that $\chi(\Gamma) = 2 - 2g - n$. Finally we see that

$$\pi_1(\Sigma_{g,n}) \underset{\uparrow}{\cong} \pi_1(\Gamma) \cong \pi_1\Big(\begin{array}{c}\text{connected topological graph } \Gamma \\ \text{with } \chi(\Gamma) = 2-2g-n\end{array}\Big) \underset{\uparrow}{\cong} \begin{array}{c}\text{free group on} \\ 2g+n-1 \text{ generators}\end{array}$$

by the above together with the Homotopy-π_1-Proposition 13.4 (2b) — by the Graph-π_1-Proposition 20.9

(2) The proof of this statement is almost identical to the proof of Statement (1). ■

In the following we will see that we can "recover" the closed surfaces Σ_g and N_g from any $\Sigma_{g,n}$ respectively any $N_{g,n}$. The key to doing so is the following definition:

Definition. Let M be a topological manifold. Let "$\sim$" be the equivalence relation on M that is generated by $x \sim y$ whenever x and y lie on the same component of ∂M. We write $\widehat{M} := M/\sim$.

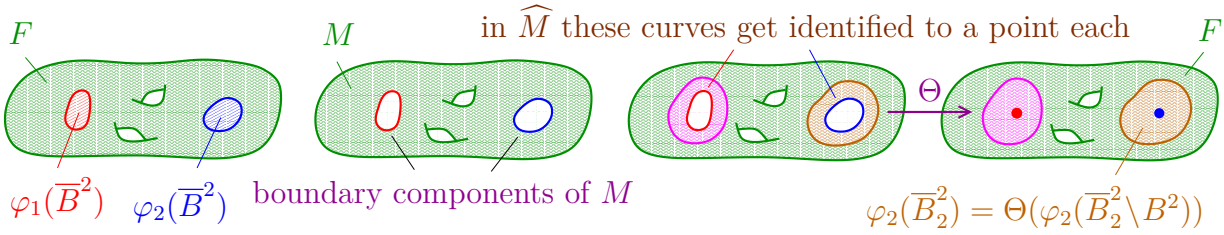


The following lemma shows in particular that $\widehat{\Sigma_{g,n}}$ is homeomorphic to Σ_g and that $\widehat{N_{g,n}}$ is homeomorphic to N_g.

Lemma 22.6. (Surface Recovery Lemma) Let F be a closed connected non-empty 2-dimensional smooth manifold and let $n \in \mathbb{N}_0$. If we set M to be F minus n open disks, then $\widehat{M}$ is homeomorphic to F.

Proof. By the Surface-Minus-Open Disks Lemma 22.4 we can work with any smooth embeddings. Thus let $\varphi_1, \ldots, \varphi_n \colon \overline{B}^2_2 \to F$ be smooth embeddings of the closed disks *of radius* 2 with disjoint images. We need to show that for $M := F \setminus (\varphi_1(B^2) \cup \cdots \cup \varphi_n(B^2))$ there exists a homeomorphism $\widehat{M} \to F$. We consider the following map:[210]

$$\begin{array}{rcl}\Theta \colon \widehat{M} & \to & F \\ [P] & \mapsto & \begin{cases} \varphi_i(Q \cdot 2 \cdot (\|Q\| - 1)), & \text{if } P = \varphi_i(Q) \text{ for some } i \in \{1, \ldots, n\} \text{ and } Q \in \overline{B}^2_2 \setminus B^2, \\ [P], & \text{otherwise.} \end{cases}\end{array}$$



It follows from the Pasting Proposition 2.4 (2) and the Topological-Quotient Proposition 6.1 (1b) that Θ is continuous. Note that by the Compact Image Lemma 2.8 we know that $\widehat{M}$ is compact. Now it follows almost immediately from the Compact-Hausdorff Proposition 2.10 (3) that the given map $\Theta \colon \widehat{M} \to F$ is indeed a homeomorphism. ■

[210] Note that discretely we are using the Surface-Minus-Open Disks Lemma 22.4 (1) which says that the boundary of M, viewed as a topological manifold, is indeed what we think it should be, namely it is given by $\varphi_1(S^1) \sqcup \cdots \sqcup \varphi_n(S^1)$.

22.5. **The Surface Classification Theorem.** One of the goals of mathematicians is to classify objects of one's favorite category $\mathcal{C}$. We could for example consider the category of (finitely presented) groups, the category of (finite-dimensional) $\mathbb{K}$-vector spaces or the category of (compact) n-dimensional topological (or smooth) manifolds. But what does "classify" mean? One interpretation would be the following:

(I) Provide a collection $\{X_i\}_{i\in I}$ of objects with the following two properties:
  (a) Any other object of $\mathcal{C}$ is isomorphic to some X_i.
  (b) If $i \neq j$, then X_i and X_j are not isomorphic.
(II) Provide an algorithm which determines, given an object in $\mathcal{C}$, to which X_i it is isomorphic to.

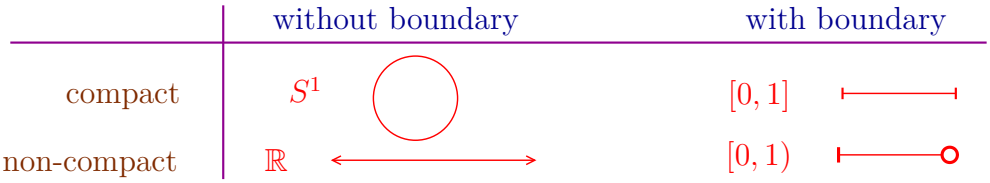
For example in the category of finite-dimensional $\mathbb{K}$-vector spaces a complete list is given by $\{\mathbb{K}^i\}_{i\in\mathbb{N}_0}$ and the calculation of the dimension gives us (II). Furthermore the Abelian Group-Classification Theorem 18.11 gives a classification of the objects in the category of finitely generated abelian groups.

In these notes we are mostly interested in the classification of connected n-dimensional smooth and topological manifolds. The following theorem gives the classification of 1-dimensional topological manifolds.

Theorem 22.7. (Topological 1-Dimensional Manifold Classification Theorem) Let M be a connected non-empty 1-dimensional topological manifold. The homeomorphism type of M is given by the following table:

	with empty boundary	with non-empty boundary
compact	the circle S^1	the closed interval $[0,1]$
non-compact	the real line $\mathbb{R}$	the half-open interval $[0,1)$.

An analogous statement holds for the diffeomorphism type of smooth manifolds.



Proof. The proof is relatively elementary, but it is nonetheless not totally obvious. For example it is not entirely clear where in the proof one uses that topological manifolds are Hausdorff and second countable. We provide the proof in [**Fri24**]. ■

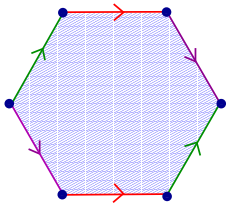
We turn our gaze to the 2-dimensional setting. We will state the classification theorem for compact connected non-empty 2-dimensional topological and smooth manifolds and we will prove (I b) and (II) in the above setting. For the proof of (I a) we will need some patience though, we will only provide it in a later topology course.

Theorem 22.8. (Surface Classification Theorem) Every compact connected non-empty 2-dimensional topological manifold is homeomorphic to either

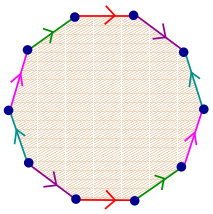
(1) the surface $\Sigma_{g,n}$ for *unique* $g, n \in \mathbb{N}_0$ or
(2) to the surface $N_{k,n}$ for *unique* $k \in \mathbb{N}$ and $n \in \mathbb{N}_0$.

Furthermore, if we are given a smooth manifold, then we can upgrade "homeomorphic" to "diffeomorphic".

Example. At least for closed orientable connected 2-dimensional smooth manifolds it is perhaps tempting to dismiss the Surface Classification Theorem 22.8 (1) as "obvious". After all, what other closed orientable connected 2-dimensional smooth manifold could there possibly be? It is thus instructive to consider the figure below. It shows a hexagon and a decagon where *opposite sides* are identified. The quotient spaces are closed orientable 2-dimensional smooth manifolds. To which of the standard surfaces Σ_g are they diffeomorphic to?



hexagon with opposite sides identified



dodecagon with opposite sides identified

The Surface Classification Theorem 22.8 is an immediate consequence of the following three theorems.

Theorem 22.9. (Diffeomorphism-to-Standard Surfaces Theorem) Let Σ be a compact connected non-empty 2-dimensional smooth manifold.

(1) If Σ is orientable, then Σ is diffeomorphic to the surface $\Sigma_{g,n}$ for some $g, n \in \mathbb{N}_0$.
(2) If Σ is non-orientable, then Σ is diffeomorphic to the surface $N_{k,n}$ for some $k \in \mathbb{N}$ and $n \in \mathbb{N}_0$.

Theorem 22.10. (Surface-Smooth Structure Existence Theorem) Every compact 2-dimensional topological manifold admits a smooth structure.

Theorem 22.11. (Surfaces-Non Homeomorphic Theorem) The topological spaces $\Sigma_{g,m}$ and $N_{h,n}$ are pairwise non-homeomorphic.

We now turn to the discussion of the proofs of the three theorems.

Proof of the Diffeomorphism-to-Standard Surfaces Theorem 22.9. We will prove this theorem later in a few semesters time using "Morse theory". The impatient student can also look up the proof in [**Fri24**]. ■

Proof of the Surface-Smooth Structure Existence Theorem 22.10. This theorem is technically quite tricky. A proof is for example given in [**Hat13**, Theorem A]. ■

Since we did not really prove the first two theorems we should at least provide full details for the third theorem. The idea is to find a complete set of invariants to distinguish the $\Sigma_{g,m}$ and $N_{h,n}$ pairwise. The first invariant we want to use is of course the (abelianization) of the fundamental group. But since this invariant is not enough we have to look for further invariants. One idea would be to use the number of boundary components. But even that is not good enough: In the following figure we see two compact connected 2-dimensional topological manifolds such that the fundamental groups are both isomorphic to the free group on two generators and such that both have one boundary component.



It is rather unlikely though that they are homeomorphic.[211] This example strongly suggests that the fundamental group and the number of boundary components is not enough to distinguish all examples.

Proof of the Surfaces-Non Homeomorphic Theorem 22.11. Let M be a topological manifold. As before let "$\sim$" be the equivalence relation on M that is generated by $x \sim y$ whenever x and y lie on the same component of ∂M. As before we write $\widehat{M} \coloneqq M/\sim$.

Claim. For $g \in \mathbb{N}_0$ and $k \in \mathbb{N}$ we have the following table of invariants:

X	Σ_g	$\Sigma_{g,m}$ with $m \geq 1$	N_k	$N_{k,n}$ with $n \geq 1$
isomorphism type of $\pi_1(X)_{\mathrm{ab}}$	$\mathbb{Z}^{2g}$	$\mathbb{Z}^{2g+m-1}$	$\mathbb{Z}^{k-1} \oplus \mathbb{Z}_2$	$\mathbb{Z}^{k+n-1}$
number of boundary components	0	m	0	n
isomorphism type of $\pi_1(\widehat{X})_{\mathrm{ab}}$	$\mathbb{Z}^{2g}$	$\mathbb{Z}^{2g}$	$\mathbb{Z}^{k-1} \oplus \mathbb{Z}_2$	$\mathbb{Z}^{k-1} \oplus \mathbb{Z}_2$.

Proof. The number of boundary components can be determined using the Topological Surface–Boundary Proposition 22.3. The abelianizations of the fundamental groups of X are given by the Surface-π_1-Propositions 22.1 and 22.2 and the Surface-with-Boundary–π_1-Lemma 22.5. Finally the abelianizations of the fundamental of $\widehat{X}$ are given by the above calculation and the fact, shown in the Surface Recovery Lemma 22.6, that $\widehat{\Sigma_{g,m}} \cong \Sigma_g$ and $\widehat{N_{k,n}} \cong N_k$. ⊞

It follows from the second and third row that the $\Sigma_{g,m}$, $g \in \mathbb{N}_0, m \in \mathbb{N}_0$ and $N_{k,n}$, $k \in \mathbb{N}$ and $n \in \mathbb{N}_0$ are pairwise non-homeomorphic. ■

[211] As a smooth manifold X is orientable and Y is non-orientable. But as of right now we do not know whether orientability is preserved under homeomorphisms.

CHAPTER 23

The Seifert–van Kampen Theorem applied to Manifolds

In this chapter we will first prove a variation on the Seifert–van Kampen-Theorem 20.1 which is more suitable to smooth manifolds. We will use this variation to prove that every finitely presented group occurs as the fundamental group of a closed orientable smooth manifold of dimension ≥ 4.

23.1. **The Seifert-van Kampen Theorem for manifolds.** For many applications the following theorem, which is a variation on the Seifert–van Kampen-Theorem 20.1, is very useful.

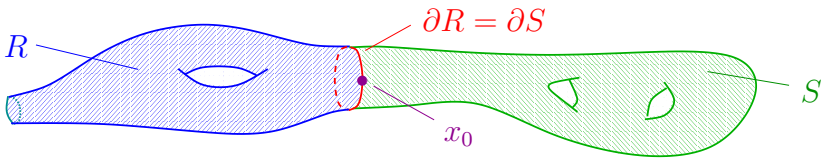
Theorem 23.1. (Manifold–Seifert-van Kampen Theorem) Let M be an m-dimensional smooth or topological manifold and let $R, S \subset M$ be two codimension zero submanifolds such that the following hold:

(1) $M = R \cup S$.
(2) R and S are closed subsets of M. (Note that by the Compact-Closed Lemma 1.9 (2) this condition is satisfied if R and S are compact.)
(3) $R \cap S$ is a single component of ∂R and it is a single component of ∂S.

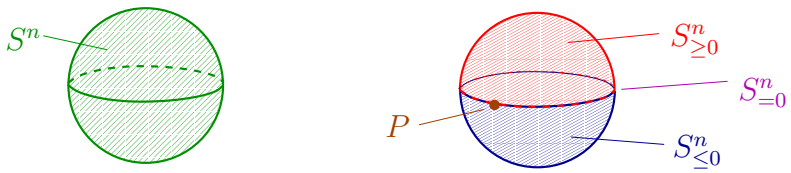
For every base point $x_0 \in R \cap S$ the inclusion-induced map

$$\pi_1(R, x_0) *_{\pi_1(R\cap S, x_0)} \pi_1(S, x_0) \ \to \ \pi_1(M, x_0)$$

is an isomorphism.



Example. Given $n \in \mathbb{N}_0$ we denote as usual by $S^n_{\geq 0}$ the upper hemisphere of S^n and we denote by $S^n_{\leq 0}$ the lower hemisphere of S^n. □



Clearly the two hemispheres are closed subsets and one can show easily that both are codimension-zero submanifolds of S^n. We denote by $S^n_{=0} = S^n_{\geq 0} \cap S^n_{\leq 0}$ the "equator" of S^n, which is of course homeomorphic to S^{n-1}. Given any $n \in \mathbb{N}_{\geq 2}$ and any $P \in S^n_{=0}$ we can

perform the following calculation:

$$\pi_1(S^n,P) \underset{\uparrow}{\cong} \pi_1(S^n_{\leq 0},P) *_{\pi_1(S^n_{=0},P)} \pi_1(S^n_{\geq 0},P) \underset{\downarrow}{\cong} \{e\} *_{\pi_1(S^{n-1},P)} \{e\} \underset{\uparrow}{\cong} \{e\}.$$

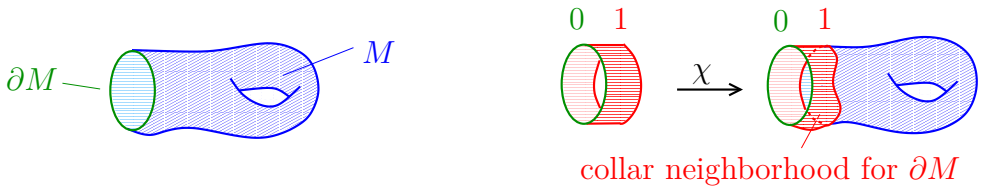
by page 71 the hemispheres are homeomorphic to $\overline{B}^n$, hence the fundamental groups are trivial

by the Manifold–Seifert-van Kampen Theorem 23.1, note that we use that our hypothesis $n \in \mathbb{N}_{\geq 2}$ implies that S^{n-1} is path-connected

by definition or by the Amalgamated Product–Quotient Lemma 19.13

This calculation gives another proof of the Sphere-π_1-Proposition 11.15.[212]
Of course the idea is to prove the Manifold–Seifert-van Kampen-Theorem 23.1 by reducing it to the Seifert-van Kampen-Theorem 20.1. But note that we cannot apply the Seifert-van Kampen-Theorem 20.1 directly, since R and S are not open. We will apply the same trick as in the proof of the Wedge-π_1-Proposition 20.6, namely we will expand R and S to open subsets U and V to which we can apply the Seifert-van Kampen-Theorem 20.1. The key to carrying out this idea is the following theorem.

Theorem 23.2. (Collar Neighborhood Theorem) Given any smooth or topological manifold M and given any union R of components of ∂M there exists an embedding $\chi\colon [0,1] \times R \to M$ with the following two properties:
(1) For every $P \in R$ we have $\chi(0,P) = P$.
(2) The image of $[0,1) \times R$ is an open subset of M.
The image of χ is sometimes called a collar neighborhood of R.



Proof. The proof is provided in [**Fri24**]. ■

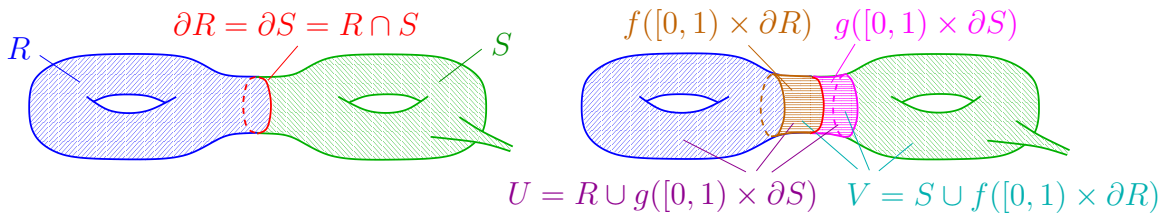
Proof of Theorem 23.1. Let M be an m-dimensional smooth or topological manifold. Furthermore let $R, S \subset M$ be two codimension-zero submanifolds such that the following hold:

(1) $M = R \cup S$.
(2) R and S are closed subsets of M.
(3) $R \cap S$ is a component of ∂R and it is a component of ∂S.

In fact, to simplify the notation a little bit we assume that $R \cap S = \partial R = \partial S$. Let $x_0 \in R \cap S$. We pick embeddings $f\colon [0,1] \times \partial R \to R$ and $g\colon [0,1] \times \partial S \to S$ that are provided by the Collar Neighborhood Theorem 23.2. We set

$$U \coloneqq R \cup \underbrace{g([0,1) \times \partial S)}_{\subset S} \quad \text{and} \quad V \coloneqq S \cup \underbrace{f([0,1) \times \partial R)}_{\subset R}.$$

[212] This argument is at the end quite similar to the proof of the Suspension-π_1-Lemma 20.2.



We consider the following diagram:

$$\begin{array}{ccc} \pi_1(R,x_0) *_{\pi_1(R\cap S,x_0)} \pi_1(S,x_0) & \longrightarrow & \pi_1(M,x_0) \\ \downarrow & & \downarrow \mathrm{id} \\ \pi_1(U,x_0) *_{\pi_1(U\cap V,x_0)} \pi_1(V,x_0) & \longrightarrow & \pi_1(M,x_0). \end{array}$$

The following steps are similar to the ones we performed in the proof of the Wedge-π_1-Proposition 20.6.

(a) Since the vertical maps are induced by inclusions we see that the diagram commutes.

(b) We claim that U is an open subset of $M = R \cup S$. Since R and S are by our hypothesis (2) closed subsets of M we obtain from the Cover-Open Subset Lemma 1.5 (2a) that it suffices to show that $U \cap R$ is open in R and that $U \cap S$ is open in S. Evidently $U \cap R = R$ is open in R and $U \cap S = g([0,1) \times \partial S)$ is open in S by the second property of g, as stated in the Smooth Collar Neighborhood Theorem 23.2.

(c) We want to show that R is a deformation retract of U. To do we consider the following map:

$$\begin{array}{rcl} U \times [0,1] & \to & U \\ (u,t) & \mapsto & \begin{cases} u, & \text{if } u \in R, \\ g(s \cdot t, y), & \text{if } u = g(s,y) \in g([0,1) \times \partial S). \end{cases} \end{array}$$

It remains to show that this map is continuous. By the Homotopy Combination Lemma 8.2 (1) it suffices to show that R and $g([0,1) \times \partial S)$ are closed subsets of U. We see this as follows: By our hypothesis R and S are closed subsets of M. It follows that $R = R \cap U$ and $g([0,1) \times \partial S) = S \cap U$ are closed subsets of U.

(d) The obvious analogues of (b) and (c) hold for V and S.

(e) Similar to (c) one can show that $R \cap S$ is a deformation retract of $U \cap V$.

(f) We consider the following diagram:

$$\begin{array}{ccccc} \pi_1(R,x_0) & \longleftarrow & \pi_1(R\cap S,x_0) & \longrightarrow & \pi_1(S,x_0) \\ \downarrow & & \downarrow & & \downarrow \\ \pi_1(U,x_0) & \longleftarrow & \pi_1(U\cap V,x_0) & \longrightarrow & \pi_1(V,x_0). \end{array}$$

Here all maps are induced by inclusions. It follows that the diagram commutes. Furthermore we obtain from (c), (d), (e) and the Homotopy-π_1-Proposition 13.4 (2) that the vertical maps are all isomorphisms. It follows that the induced map between amalgamated products, shown in the initial diagram to the left, is an isomorphism.

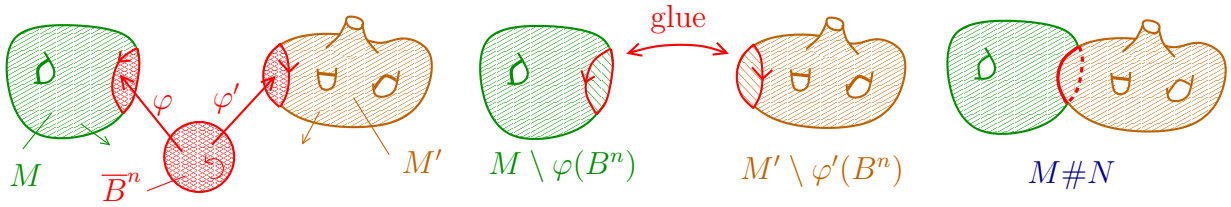
(g) By (b) and (e) we can apply the Seifert–van Kampen-Theorem 20.1 to the decomposition $M = U \cup V$ and we know that the bottom horizontal map of the initial commutative diagram is an isomorphism.

(h) Since the vertical maps are isomorphisms we see that the top horizontal map of the initial diagram is also an isomorphism. ■

23.2. **Connected sum of smooth manifolds.** The following construction gives us a very general approach to constructing new examples of smooth manifolds.

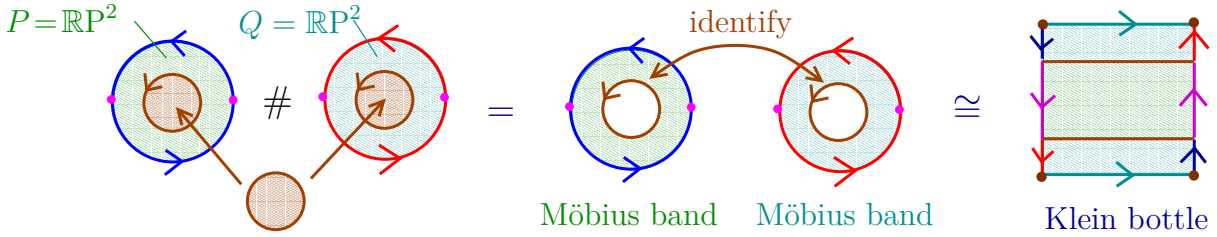
Definition. Let $n \in \mathbb{N}$. Let M and M' be two connected non-empty n-dimensional smooth manifolds. Let $\varphi\colon \overline{B}^n \to M \setminus \partial M$ and $\varphi'\colon \overline{B}^n \to M' \setminus \partial M'$ be smooth embeddings. We define the connected sum of the smooth manifolds M and M' as

$$M\#M' := \big(M \setminus \varphi(B^n)\big) \sqcup \big(M' \setminus \varphi'(B^n)\big)/\sim \quad \text{where } \varphi(P) \sim \varphi'(P) \text{ for all } P \in S^{n-1}.$$



Example. Note that the connected sum of g tori is diffeomorphic to the surface Σ_g of genus g. Furthermore note that the connected sum of g copies of $\mathbb{R}\mathrm{P}^2$ is diffeomorphic to the non-orientable surface N_g of genus g. □

Example. Note that $\mathbb{R}\mathrm{P}^2\#\mathbb{R}\mathrm{P}^2$ is homeomorphic to the Klein bottle K. Just for fun and for practice we provide the direct calculation $\pi_1(K) \cong \pi_1(\mathbb{R}\mathrm{P}^2\#\mathbb{R}\mathrm{P}^2)$ using the Manifold–Seifert-van Kampen Theorem 23.1. In order to distinguish the two copies of $\mathbb{R}\mathrm{P}^2 =_{\mathrm{i}} \overline{B}^2/\sim$ we write $P = \mathbb{R}\mathrm{P}^2$ and $Q = \mathbb{R}\mathrm{P}^2$.



We view $\frac{1}{2} \in \overline{B}^2 \subset \mathbb{C}$ as a common base point. We have

$$\pi_1(\mathbb{R}\mathrm{P}^2\#\mathbb{R}\mathrm{P}^2, \tfrac{1}{2}) = \pi_1(P\#Q, \tfrac{1}{2}) \underset{\uparrow}{\cong} \pi_1\big(P \setminus \tfrac{1}{2}B^2, \tfrac{1}{2}\big) *_{\pi_1(\frac{1}{2}S^1, \frac{1}{2})} \pi_1\big(Q \setminus \tfrac{1}{2}B^2, \tfrac{1}{2}\big).$$

by the definition of $P\#Q$ and the Manifold–Seifert-van Kampen Theorem 23.1

We pick a generator g for $\pi_1(\frac{1}{2}S^1, \frac{1}{2})$. In the calculation of $\pi_1(\overline{B}^2/\sim)$ on page 316 we saw that $\pi_1(P \setminus \frac{1}{2}B^2, \frac{1}{2}) \cong \langle x \rangle$ and that the inclusion-induced map $\pi_1(\frac{1}{2}S^1) \to \pi_1(P \setminus \frac{1}{2}B^2, \frac{1}{2})$ is given by $g \mapsto x^2$. Evidently the same holds for Q with $\pi_1(Q \setminus \frac{1}{2}B^2) \cong \langle y \rangle$. Summarizing we see that

$$\begin{aligned}
\pi_1(K) \cong \pi_1(\mathbb{R}\mathrm{P}^2\#\mathbb{R}\mathrm{P}^2, \tfrac{1}{2}) &\cong \pi_1\big(P \setminus \tfrac{1}{2}B^2\big) *_{\pi_1(\frac{1}{2}S^1, \frac{1}{2})} \pi_1\big(Q \setminus \tfrac{1}{2}B^2, \tfrac{1}{2}\big) \\
&\cong \text{amalgamated product of } \begin{pmatrix} \langle g \rangle \to \langle x \rangle \\ g \mapsto x^2 \end{pmatrix} \text{ and } \begin{pmatrix} \langle g \rangle \to \langle y \rangle \\ g \mapsto y^2 \end{pmatrix} \\
&\underset{\uparrow}{\cong} \langle x, y \mid x^2 = y^2 \rangle = \langle x, y \mid x^2 \cdot y^{-2} \rangle.
\end{aligned}$$

the Amalgamated Product–Presentation Lemma 21.7

□

We now turn to the study of the fundamental groups of the connected sum of two smooth manifolds of dimension ≥ 3.

Proposition 23.3. (Connected Sum-π_1-Proposition) Let $n \geq 3$ and let M and M' be two connected non-empty n-dimensional smooth manifolds. Then

$$\pi_1(M \# M') \;\cong\; \pi_1(M) * \pi_1(M').$$

Example. Let $m \geq 2$. We consider the connected sum of k copies of $S^1 \times S^m$. We see that:

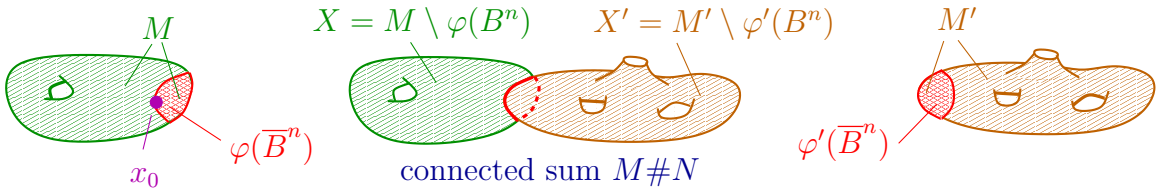
$$\pi_1((S^1 \times S^m) \# \ldots \# (S^1 \times S^m)) \underset{\text{by the Connected Sum-}\pi_1\text{-Proposition 23.3, since } m \geq 2}{\cong} \pi_1(S^1 \times S^m) * \ldots * \pi_1(S^1 \times S^m) \underset{\text{by the Product-}\pi_1\text{-Proposition 13.7 we know that } \pi_1(S^1 \times S^m) \cong \langle g_i \rangle}{\cong} \overbrace{\langle g_1 \rangle * \ldots * \langle g_k \rangle}^{= \langle g_1, \ldots, g_k \rangle}.$$

□

Proof of the Connected Sum-π_1-Proposition 23.3. Let M and M' be two connected non-empty n-dimensional smooth manifolds, and let $\varphi\colon \overline{B}^n \to M$ and $\varphi'\colon \overline{B}^n \to M'$ be two smooth embeddings. Recall that the connected sum of M and M' is defined as

$$M \# M' \;:=\; \big(\underbrace{(M \setminus \varphi(B^n))}_{=:X} \sqcup \underbrace{(M' \setminus \varphi'(B^n))}_{=:X'} \big) / \sim \qquad \text{where } \varphi(P) \sim \varphi'(P) \text{ for } P \in S^{n-1}.$$

We pick a base point x_0 that lies in the intersection of the images of X and X' in $M \# M'$.



Claim. The inclusion-induced maps $\pi_1(X, x_0) \to \pi_1(M, x_0)$ and $\pi_1(X', x_0) \to \pi_1(M', x_0)$ are isomorphisms.

Proof. Note that by symmetry it suffices to show that the inclusion-induced map $\pi_1(X, x_0) \to \pi_1(M, x_0)$ is an isomorphism. Note that X and $\varphi(\overline{B}^n)$ are n-dimensional submanifolds of M and that they are closed subsets of M. Note that $X \cap \varphi(\overline{B}^n) = \varphi(S^{n-1})$. From the Manifold–Seifert-van Kampen Theorem 23.1 we obtain the following pushout diagram:

$$\begin{array}{ccc} \pi_1(\varphi(S^{n-1}), x_0) & \longrightarrow & \pi_1(X, x_0) \\ \downarrow & & \ulcorner \downarrow \\ \pi_1(\varphi(\overline{B}^n), x_0) & \longrightarrow & \pi_1(M, x_0). \end{array}$$

Since $n \geq 3$ we obtain from the Sphere-π_1-Proposition 11.15 that $\pi_1(\varphi(S^{n-1}), x_0) = \{e\}$. Thus the vertical map to the left is an isomorphism. It follows from the Amalgamated Product Lemma 19.12 (2a) that the right vertical map is also an isomorphism. ⊞

Now we see that

$$\pi_1(M \# M', x_0) \underset{\text{by the Manifold–Seifert-van Kampen Theorem 23.1, since } X \cap X' \cong S^{n-1} \text{ is simply connected}}{\overset{\cong}{\leftarrow}} \pi_1(X, x_0) * \pi_1(X', x_0) \underset{\text{by the above claim}}{\overset{\cong}{\rightarrow}} \pi_1(M, x_0) * \pi_1(M', x_0).$$

■

23.3. **Fundamental groups of high-dimensional smooth manifolds.** Later we will see that the fundamental group of a compact connected smooth manifold is necessarily finitely presented. In this section we want to study the converse: which finitely presented groups can occur as fundamental groups of compact smooth manifolds?

First note that it follows from the Topological 1-Dimensional Manifold Classification Theorem 22.7 that only the trivial group and $\mathbb{Z}$ occur as the fundamental group of a compact connected 1-dimensional topological manifold.

Similarly it follows from the Surface Classification Theorem 22.8, the Surface-π_1-Propositions 22.1 and 22.2 and the Surface-with-Boundary–π_1-Lemma 22.5 that only "few" finitely presented groups occur as the fundamental group of compact connected 2-dimensional topological manifolds. In contrast we will now see that once we go to dimension ≥ 4 any finitely presented group occurs as the fundamental group of a closed orientable smooth manifold.

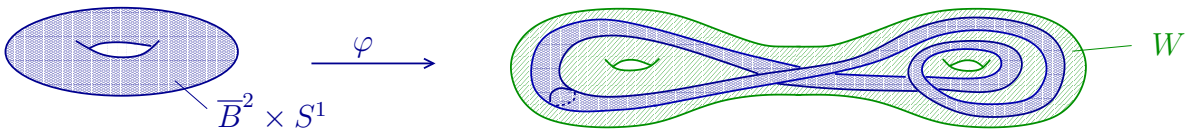
Proposition 23.4. (Manifold-π_1-Realization Proposition) Let π be a finitely presented group. Given any $n \geq 4$ there exists a closed orientable connected non-empty n-dimensional smooth manifold M with $\pi_1(M) \cong \pi$.

The idea of the proof is as follows. Given a finite presentation $\langle g_1, \dots, g_k \,|\, r_1, \dots, r_l \rangle$ we start out with the connected sum $X = (S^1 \times S^{n-1}) \# \dots \# (S^1 \times S^{n-1})$ of k copies of $S^1 \times S^{n-1}$. Above we showed that the fundamental group of X is isomorphic to the free group $\langle g_1, \dots, g_k \rangle$. We will modify X by removing "solid tori $\overline{B}^{n-1} \times S^1$" and gluing in copies of $S^{n-2} \times \overline{B}^2$ to modify the fundamental group.

To carry out this idea we need to formulate and prove two lemmas. The first lemma is actually quite interesting in its own right.

Lemma 23.5. (Drilling Out–π_1–Lemma) Let W be an n-dimensional smooth manifold. Let $k \in \mathbb{N}$ and let $\varphi\colon \overline{B}^{n-k} \times S^k \to W \setminus \partial W$ be a smooth embedding.

(1) If $n \geq k+3$, then for every base point $w_0 \in W \setminus \varphi(B^{n-k} \times S^k)$ the inclusion-induced map $\pi_1(W \setminus \varphi(B^{n-k} \times S^k), w_0) \to \pi_1(W, w_0)$ is an isomorphism.
(2) If $n = k+2$, then for every base point $w_0 \in W \setminus \varphi(B^{n-k} \times S^k)$ the inclusion-induced map $\pi_1(W \setminus \varphi(B^{n-k} \times S^k), w_0) \to \pi_1(W, w_0)$ is an epimorphism.



Proof of the Drilling Out–π_1–Lemma 23.5. First note that it follows from the Change-of-Base Point Proposition 13.2 that it suffices to prove the statement for some base point $w_0 \in \varphi(S^{n-k-1} \times S^k)$. We consider the following pushout diagram:

$$\begin{array}{ccc} \pi_1(\varphi(S^{n-k-1} \times S^k), w_0) & \longrightarrow & \pi_1(W \setminus \varphi(B^{n-k} \times S^k), w_0) \\ \downarrow & & \downarrow \\ \pi_1(\varphi(B^{n-k} \times S^k), w_0) & \longrightarrow & \pi_1(W, w_0) \end{array}$$

It follows from the Manifold–Seifert-van Kampen Theorem 23.1 that this is a pushout diagram.

If $n \geq k+3$, then $n-k-1 \geq 2$. Thus we obtain from the Sphere-π_1-Proposition 11.15 that $\pi_1(S^{n-k-1})$ is trivial. It follows from the Product-π_1-Proposition 13.7 that the left vertical map is an isomorphism[213], thus it follows from the Amalgamated Product Lemma 19.12 (2a) that the right vertical map is – as promised – an isomorphism.

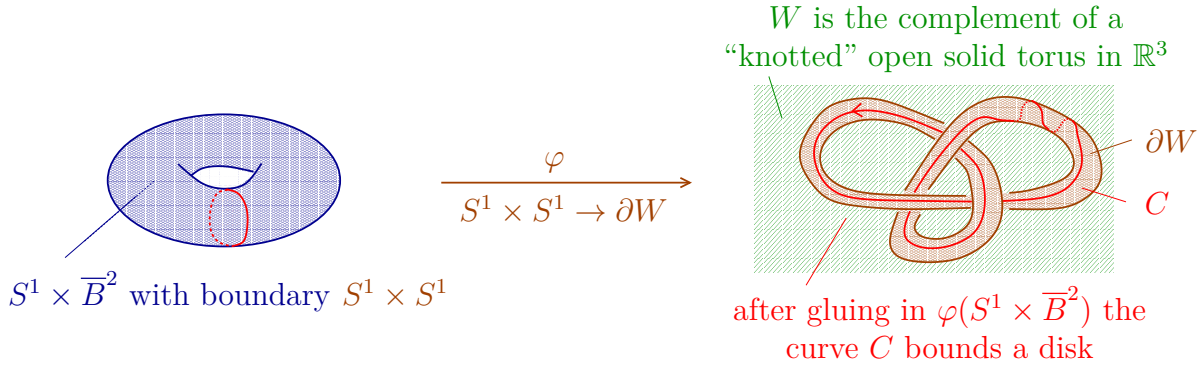
If $n = k+2$, then the left vertical map is an epimorphism. It follows from the Amalgamated Product Lemma 19.12 (2b) that the right vertical map is also an epimorphism. ■

Lemma 23.6. (Fill-in-Solid Torus-π_1-Lemma) Let W be an n-dimensional smooth manifold. Let $\varphi\colon S^{n-2}\times S^1 \to \partial W$ be a diffeomorphism onto a component of ∂W. We set

$$X \;:=\; (W \sqcup (S^{n-2}\times \overline{B}^2))/\sim \qquad \text{where } \varphi(x,y)\sim(x,y) \text{ for } (x,y)\in S^{n-2}\times S^1.$$

We pick $x \in S^{n-2}$ and we set $C := \varphi(\{x\}\times S^1) \subset \partial W$. If $n \geq 3$, then for any $w_0 \in W$ the inclusion-induced map $\pi_1(W,w_0) \to \pi_1(X,w_0)$ induces an isomorphism

$$\pi_1(W,w_0)/\langle\langle C\rangle\rangle \;\xrightarrow{\cong}\; \pi_1(X,w_0).$$



Proof of the Fill-in-Solid Torus-π_1-Lemma 23.6. We will only need the case $n \geq 4$. Thus in the following we will provide the (slightly easier) proof of the special case $n \geq 4$. We leave it to the reader to modify the argument below to deal with the case $n = 3$. Note that it follows from the Change-of-Base Point Proposition 13.2 that it suffices to prove the statement for some base point $w_0 \in \varphi(S^{n-2}\times S^1)$. Next note that we have the following two isomorphisms:

$$\pi_1(W,w_0)/\langle\langle C\rangle\rangle \quad \underset{\uparrow}{\xrightarrow{\cong}} \quad \pi_1(W,w_0) *_{\pi_1(S^{n-2}\times S^1,w_0)} \pi_1\big(S^{n-2}\times \overline{B}^2, w_0\big) \quad \underset{\uparrow}{\xrightarrow{\cong}} \quad \pi_1(X,w_0).$$

an isomorphism by the Amalgamated Product–Quotient Lemma 19.13, here we use that $\pi_1(S^{n-2}\times\overline{B}^2,w_0)=0$ and that $\pi_1(S^{n-2}\times S^1,w_0)=\langle C\rangle$, note that for the latter statement we used that $n\geq 4$

an isomorphism by the Manifold–Seifert-van Kampen Theorem 23.1

Since the horizontal maps are induced by the inclusions we see that the inclusion-induced map $\pi_1(W,w_0)/\langle\langle C\rangle\rangle \to \pi_1(X,w_0)$ is indeed an isomorphism. ■

Now we are fully equipped for proving the Manifold-π_1-Realization Proposition 23.4.

Proof of the Manifold-π_1-Realization Proposition 23.4. Throughout the proof we will mostly ignore base points. But at the end of the proof we will argue, why being a little sloppy with base points in between does not affect the outcome.

Let $\pi = \langle g_1,\dots,g_k \,|\, r_1,\dots,r_l\rangle$ be a finitely presented group. Furthermore let $n \geq 4$. We consider

[213] Note that this argument also applies for $k=1$

$$X \coloneqq \underbrace{(S^1 \times S^{n-1}) \# \ldots \# (S^1 \times S^{n-1})}_{k \text{ copies of } S^1 \times S^{n-1}}.$$

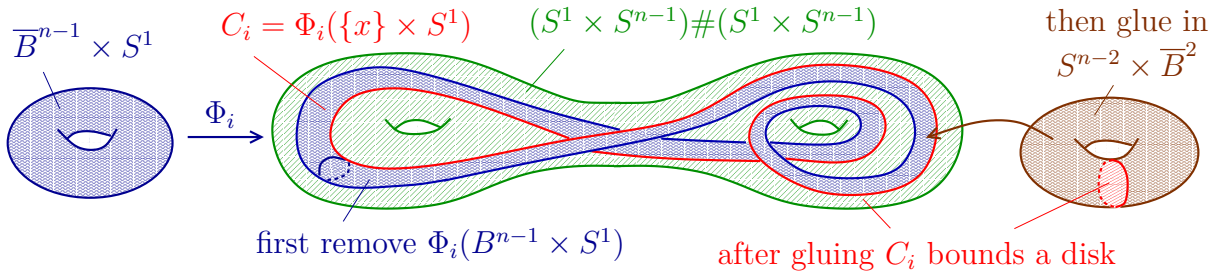
Note that X is a closed orientable connected n-dimensional smooth manifold. As on page 330 we see that

$$\pi_1(\overbrace{(S^1 \times S^{n-1}) \# \ldots \# (S^1 \times S^{n-1})}^{=X}) \underset{\text{by the Connected Sum-}\pi_1\text{-Proposition 23.3, since } n \geq 3}{\cong} \pi_1(S^1 \times S^{n-1}) * \ldots * \pi_1(S^1 \times S^{n-1}) \underset{\text{by the Product-}\pi_1\text{-Proposition 13.7 and since } n \geq 3 \text{ we know that } \pi_1(S^1 \times S^{n-1}) \cong \mathbb{Z}}{\cong} \overbrace{\langle g_1 \rangle * \ldots * \langle g_k \rangle}^{=\langle g_1, \ldots, g_k \rangle}.$$

We use this isomorphism to make the identification $\pi_1(X, x_0) = \langle g_1, \ldots, g_k \rangle$. Next we pick loops $b_1, \ldots, b_l \colon (S^1, *) \to (X, x_0)$ that represent $r_1, \ldots, r_l \in \pi_1(X, x_0) = \langle g_1, \ldots, g_k \rangle$.

Since $\dim(X) = n \geq 3$ we can apply the Circle–Smooth Embedding Theorem from [**Fri24**] and we obtain that the map $b_1 \sqcup \cdots \sqcup b_l \colon S^1 \sqcup \cdots \sqcup S^1 \to X$ is in fact homotopic to a smooth embedding $c_1 \sqcup \cdots \sqcup c_l \colon S^1 \sqcup \cdots \sqcup S^1 \to X$. It follows from the Smooth Embedding Theorem and the Tubular Neighborhood Theorem from [**Fri24**] that there exist smooth embeddings $\Phi_i \colon \overline{B}^{n-1} \times S^1 \to X$, $i = 1, \ldots, l$ with disjoint images such that for each $i \in \{1, \ldots, l\}$ and each $y \in S^1$ we have $\Phi_i(0, y) = c_i(y)$. Next we consider

$$Y \coloneqq X \setminus \bigcup_{i=1}^{l} \Phi_i(B^{n-1} \times S^1) \qquad \text{and} \qquad Z \coloneqq \Big(Y \sqcup \bigsqcup_{i=1}^{l} (S^{n-2} \times \overline{B}^2)\Big) \Big/ \underset{\text{where } \Phi_i(x,y) \sim (x,y) \text{ for } x \in S^{n-2} \text{ and } y \in S^1}{\sim}$$



Finally we see that

$$\pi_1(Z) \underset{\text{the Fill-in-Solid Torus-}\pi_1\text{-Lemma 23.6 applied } l \text{ times, hereby we set } C_i \coloneqq \Phi_i(\{x\} \times S^1) \text{ for some } x \in S^{n-2}}{\overset{\downarrow \cong}{\leftarrow}} \pi_1(Y) / \langle\langle C_1, \ldots, C_l \rangle\rangle \underset{\text{by the above, the } x_i \text{ are necessary since we lost track of the base points}}{\cong} \langle g_1, \ldots, g_k \rangle / \langle\langle x_1 r_1 x_1^{-1}, \ldots, x_l r_l x_l^{-1} \rangle\rangle \underset{\text{the Drilling Out–}\pi_1\text{–Lemma 23.5 applied } k \text{ times}}{\overset{\downarrow \cong}{\leftarrow}} \pi_1(X) / \langle\langle C_1, \ldots, C_l \rangle\rangle = \underbrace{\langle g_1, \ldots, g_k \rangle / \langle\langle r_1, \ldots, r_l \rangle\rangle}_{= \langle g_1, \ldots, g_k \,|\, r_1, \ldots, r_l \rangle}.$$

We have thus found the promised smooth manifold. ∎

CHAPTER 24

Decision problems

In the last chapters we developed several techniques for determining the fundamental groups of topological spaces. In particular we managed to find finite presentations for the fundamental groups of large classes of topological spaces, e.g. topological graphs, surfaces and mapping tori. In fact we now have enough tools to determine, with enough effort and perseverance, a presentation for the fundamental groups of most "nice" spaces.

The question that now arises is, what information can we extract from a finite presentation of a group? Here are a couple of natural questions.

Question 24.1.

(1) Triviality Problem: Does there exist an algorithm that can decide whether or not a given finite presentation $\pi = \langle g_1, \dots, g_k \mid r_1, \dots, r_l \rangle$ represents the trivial group?
(2) Isomorphism Problem: Does there exist an algorithm that can decide whether or not two given finite presentations $\pi = \langle g_1, \dots, g_k \mid r_1, \dots, r_l \rangle$ and $\Gamma = \langle h_1, \dots, h_m \mid s_1, \dots, s_n \rangle$ represent isomorphic groups?
(3) Word Problem: Does there exist an algorithm that can determine, given
   (a) a finite presentation $\pi = \langle g_1, \dots, g_k \mid r_1, \dots, r_l \rangle$ and
   (b) a word w in $g_1, \dots, g_k$, i.e. an element $w \in \langle g_1, \dots, g_k \rangle$,
   whether or not w represents the trivial element in the group π?
(4) Conjugacy Problem: Does there exist an algorithm that can determine, given
   (a) a finite presentation $\pi = \langle g_1, \dots, g_k \mid r_1, \dots, r_l \rangle$ and
   (b) and two words v and w in $g_1, \dots, g_k$,
   whether or not v and w represent conjugate elements in the group π?

Remark.

(1) These above questions were first posed by Max Dehn [**Deh11**] in 1911.
(2) The above questions are formulated in a slightly informal way. After all, what is an "algorithm" supposed to be? The questions are made more precise in [**CZ93**, Chapter 7], [**Sti82**] and [**III89**]. □

On page 312 we introduced the Tietze transformations on presentations. In the Tietze Theorem 21.9 we showed that any two finite presentations for a given group are in fact related by a finite sequence of Tietze transformations. In particular, if $\pi = \langle g_1, \dots, g_k \mid r_1, \dots, r_l \rangle$ is a presentation for the trivial group, then we can turn the presentation after finitely many Tietze transformations into the trivial presentation $\langle | \rangle$. The problem is that at times it can be very hard to find the correct Tietze transformations.

Example. We consider the presentation

$$\pi \;=\; \langle a, b, c \mid a^3, b^3, c^4, ac = ca^{-1}, aba^{-1} = bcb^{-1} \rangle.$$

This is a presentation of the trivial group as can be seen as follows:[214] [215]

(1) from $aba^{-1} \sim bcb^{-1}$ we get $(aba^{-1})^3 \sim (bcb^{-1})^3$,
(2) since $a^{-1}a \sim 1$ and $b^{-1}b \sim 1$ it follows that $(aba^{-1})^3 \sim ab^3a^{-1}$ and $(bcb^{-1})^3 \sim bc^3b^{-1}$,
(3) we have $b^3 \sim 1$, hence $ab^3a^{-1} \sim 1$, therefore we get from (1) and (2) that $bc^3b^{-1} \sim 1$ hence $c^3 \sim 1$,
(4) we have $c^3 \sim 1$ but we also have the relation $c^4 \sim 1$, so $c \sim 1$,
(5) since $aba^{-1} \sim bcb^{-1}$ it follows that $aba^{-1} \sim 1$ and so $b \sim 1$,
(6) from $ac \sim ca^{-1}$ and $c \sim 1$ it follows that $a \sim a^{-1}$, hence $a^2 \sim 1$,
(7) together with $a^3 \sim 1$ we obtain $a \sim 1$.
(8) we now showed that a, b, c are all equivalent to 1, thus we see that the group defined by the presentation is the trivial group.

This example shows that if a presentation corresponds to the trivial group, then it can be surprisingly complicated to verify this statement. □

Now suppose that we are given a finite presentation and we have to decide whether or not it is a presentation of the trivial group. If we are given a finite presentation of the trivial group and if we systematically run through all finite sequences of Tietze transformations, then sooner or later we end up with the empty presentation.

But what happens if after many attempts at modifying the presentation using Tietze transformations we still do not have the trivial presentation? Does that mean that the presentation corresponds to a non-trivial group? Or does it just mean that we did not try long enough?

So the real question is, does there exist an algorithm, that given a presentation which defines a *non-trivial* group, actually verifies that the group is *non-trivial*?

Unfortunately the answer to this question and in fact all of the above questions is quite sobering.

Theorem 24.2. (Groups–Non-Solvability Theorem) The answer to all four questions raised in Question 24.1 is no.

Remark. Here "no" means that one can show that no such algorithm exists. This is a much stronger statement than saying that we do not know of an example of such an algorithm. □

Proof.

(1) Sergei Adyan [**Ady55**] and Michael Rabin [**Rab58**] showed in 1955 that the answer to the first question is no. Alternatively see [**III89**, Theorem 3.3] for a proof.
(2) An algorithm that performs (2) would also give an algorithm for determining whether or not a given presentation is isomorphic to the trivial group with the empty presentation $\langle | \rangle$. But since the answer to (1) is no, it follows that the answer to (2) is also no.
(3) If there did exist an algorithm that dealt with the third problem, then we could apply it in particular to the generators. Since a presentation presents the trivial group if and only if all generators are trivial, it follows that an algorithm for (3) would also give an algorithm for (1). Since the latter cannot exist, the former cannot exist either.

[214] We leave it to the reader to show that these steps (1)–(7) can be realized by Tietze transformations.
[215] For elements g, h of $\langle a, b, c \rangle$ we write $g \sim h$ if they represent the same element in π.

Historically, it was first proved by Sergey Novikov [**Nov55**] and William Boone [**Boo58**] that (3) has a negative answer before a negative answer to (1) was given.

(4) An element in a group is trivial if and only if it is conjugate to the trivial element. Hence the fact that (3) has a negative answer leads to a negative answer to (4). ■

Remark.

(1) There are many other non-existence results in the vein of the Groups–Non-Solvability Theorem 24.2. For example, [**III89**, Corollary 3.4] says the following:
   (a) it is undecidable whether a finitely presented group is free,
   (b) it is undecidable whether a finitely presented group is abelian,
   (c) it is undecidable whether a finitely presented group is nilpotent,
   (d) it is undecidable whether a finitely presented group is solvable,
   (e) it is undecidable whether a finitely presented group is simple,
   (f) it is undecidable whether a finitely presented group is torsion-free.

(2) The statement of the Groups–Non-Solvability Theorem 24.2 that the answer to all four questions of Question 24.1 is no, is of course somewhat disappointing. But this raises the question whether there are interesting classes of groups for which the answer is actually yes. For example Mikhail Gromov [**Gro87**][216] introduced the notion of a "word hyperbolic group". Examples of word hyperbolic groups are given by
   (a) finite groups,
   (b) free groups, fundamental groups of surfaces of genus ≥ 2,
   (c) fundamental groups of hyperbolic smooth manifolds of any dimension,
   (d) fundamental groups of "Riemannian manifolds with negative sectional curvature".
   In fact in a precise sense a "random" finitely presented group is word hyperbolic, see e.g. [**Oll05**] for more details. If we restrict ourselves to the class of all word hyperbolic groups, then it is known that the answer to all four questions posed in Question 24.1 turns out to be yes. We refer to [**BH99, Löh17, DG11, Sel95**] for an introduction to word hyperbolic groups and for details on the various algorithms that deal with the problems in the setting of word hyperbolic groups. □

Of course our main goal is not to classify groups, but our goal is to understand "nice" topological spaces. For example ideally we would like to classify compact topological and smooth manifolds.

Given $n \in \mathbb{N}$ we ideally would like to classify closed orientable connected n-dimensional smooth manifolds up to diffeomorphism. For $n = 1$ we did so in the Smooth 1-Dimensional Manifold Classification Theorem 22.7 and for $n = 2$ we did so in the Surface Classification Theorem 22.8. In particular we saw that in these dimensions only "few groups" appear as fundamental groups.

We also proved the following result which shows that in dimension ≥ 4 the situation is dramatically different.

[216] Mikhail Gromov (1943-) is a French-Russian mathematician famous for his many contributions to geometry and group theory. He was the first person to view groups as "geometric objects", giving rise to the subject of "geometric group theory".

Proposition 23.4. (Manifold-π_1-Realization Proposition) Given any finitely presented group π and any $n \geq 4$ there exists a closed orientable connected n-dimensional smooth manifold M with $\pi_1(M) \cong \pi$.

The Non-SolvabilityTheorem 24.2 and the Manifold-π_1-Realization Proposition 23.4 are at the heart of the proof of the following theorem:

Theorem 24.3. (Manifold–Non-Classification Theorem) Let $n \geq 4$.

(1) There is no algorithm that can decide whether or not two closed orientable n-dimensional smooth manifolds are homeomorphic.
(2) There is no algorithm that can decide whether or not two closed orientable n-dimensional smooth manifolds are diffeomorphic.

Proof. Statement (1) was first proved by Aleksandr Markov [**Mar58**]. We refer to [**BHP68**, Theorem 1] and [**Hak73**] for a detailed discussion of the proofs of both statements. In particular these references also explain in more detail how one can describe a closed smooth manifold with finite data. We also refer to [**Sti93**, Chapter 9.4.2], [**CZ93**, Chapter 7.2] and [**MG21**] for more information and variations on the above results. ■

This leaves us with the case of 3-dimensional smooth manifolds. It turns out that in this dimension there does in fact exist such an algorithm.

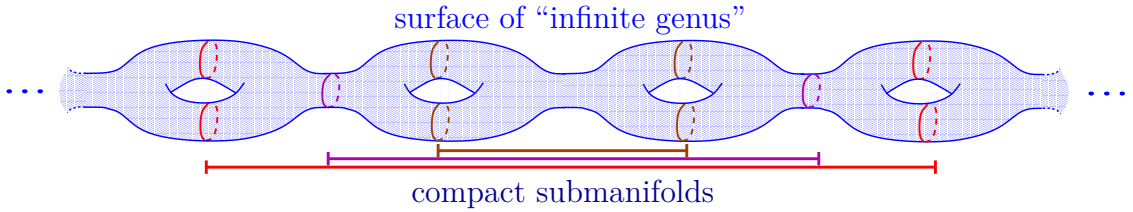
Theorem 24.4. (Smooth 3-Manifold Recognition Theorem) There exists an algorithm that can decide whether or not two closed orientable connected 3-dimensional smooth manifolds are diffeomorphic.

Proof. The proof of this theorem is considerably more complicated than in the 1-dimensional and the 2-dimensional case. The proof that such an algorithm exists is due to the work of many mathematicians, first and foremost William Thurston [**Thu82**] and Grigori Perelman. The proof of the theorem was completed about 2003. It is impossible to mention here all the work that goes into proving Theorem 24.4. We refer to [**AFW**, Theorem 4.27] for details and precise references. ■

CHAPTER 25

Direct colimits

The goal of this and the next chapter is to determine the fundamental group of "unbounded spaces", e.g. we would like to determine the fundamental group of the surface of "infinite genus" illustrated in the figure below.[217] One idea might be to view the surface of "infinite genus" as a "limit" of a direct sequence of compact surfaces. But then we need to make precise what we mean by a "limit of a direct sequence of topological spaces" and "limit of a direct sequence of (fundamental) groups". We will introduce these notions in this chapter. In the next chapter we will use this language of limits to calculate, among many other things, the fundamental group of the surface of infinite genus.



25.1. **Directed sets.** The following definition partly recalls and extends some notions that we introduced on page 37.

Definition.

(1) Let X be a set and let $\leq$ be a relation on X.
   (a) The relation is called reflexive if for every $x \in X$ we have $x \leq x$.
   (b) The relation is called transitive if for every $x, y, z \in X$ we have
   $$x \leq y \text{ and } y \leq z \implies x \leq z.$$
   (c) The relation is called antisymmetric if $x \leq y$ and $y \leq x$ implies $x = y$.
   (d) A relation that is reflexive, transitive and antisymmetric is called a partial order.

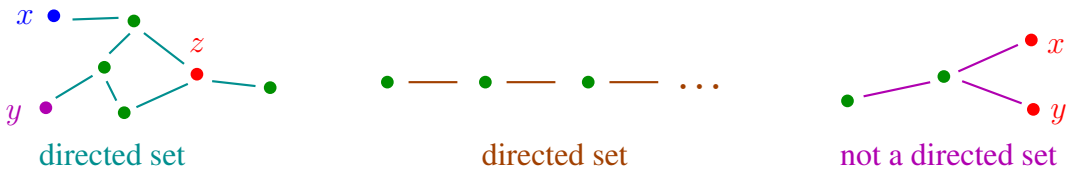
(2) A directed set is a set X together with a partial order $\leq$ that has the extra property, that for any $x, y \in X$ there exists a $z \in X$ with $x \leq z$ and $y \leq z$.

Examples.

(1) We consider the three partial orders that are illustrated in the figure below. Here it is understood that $P \leq Q$ if P is connected to Q via a path that goes throughout from left to right. We see that the two partial orders to the left form directed sets whereas the one on the right does not form a directed set.

(2) Let X be a subset of $\mathbb{R}$. If we denote by $\leq$ the usual "less or equal" relation on the real numbers, then $(X, \leq)$ is a directed set. In particular $(\mathbb{N}, \leq)$ is a directed set. □

[217] Actually we also need to make precise what we mean by the surface of "infinite genus".



25.2. **Direct systems and direct colimits.** Now that we know what a directed set is, we can define the notion of a direct system in a category:

Definition. Let $(I, \leq)$ be a directed set and let $\mathcal{C}$ be a category. A direct system in the category $\mathcal{C}$ over $(I \leq)$ is a diagram (in the sense of the definition on page 106) in $\mathcal{C}$ over the directed set $(I, \leq)$. In other words, a direct system in $\mathcal{C}$ over $(I, \leq)$ is a family of objects $\{X_i\}_{i\in I}$ in $\mathcal{C}$, together with a family of morphisms $\{f_{ji}\colon X_i \to X_j\}_{i\leq j}$ such that the following two conditions are satisfied:

(1) $f_{ii} = \mathrm{id}_{X_i}$ for all $i \in I$,
(2) $f_{ki} = f_{kj} \circ f_{ji}$ for all $i, j, k \in I$ with $i \leq j \leq k$.

For the most part we are interested in the following special case of a direct system:

Definition. Let $\mathcal{C}$ be a category and let $w \in \mathbb{Z}$. A direct sequence[224] in the category $\mathcal{C}$ consists of a family $\{X_i\}_{i\in\mathbb{Z}_{\geq w}}$ of objects and a family of morphisms $\{\varphi_i\colon X_i \to X_{i+1}\}_{i\in\mathbb{Z}_{\geq w}}$. Evidently we usually write a direct sequence as follows:

$$X_w \xrightarrow{\varphi_w} X_{w+1} \xrightarrow{\varphi_{w+1}} X_{w+2} \xrightarrow{\varphi_{w+2}} \dots$$

A direct sequence gives rise to a direct system over the directed set $(\mathbb{Z}_{\geq w}, \leq)$ by considering for $i \leq j$ the morphism $f_{ji} \coloneqq \varphi_{j-1} \circ \dots \circ \varphi_{i+1} \circ \varphi_i\colon X_i \to X_j$.

Examples.

(A) In the category $\mathcal{AbGroup}$ of abelian groups we consider the direct sequence

$$\mathbb{Z} \xrightarrow{v\mapsto(v,0)} \mathbb{Z}^2 \xrightarrow{v\mapsto(v,0)} \mathbb{Z}^3 \xrightarrow{v\mapsto(v,0)} \dots.$$

(B) In the category $\mathcal{Group}$ of groups we consider the direct sequence

$$\langle x_1\rangle \hookrightarrow \langle x_1, x_2\rangle \hookrightarrow \langle x_1, x_2, x_3\rangle \hookrightarrow \dots$$

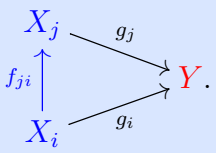
of free groups where the morphisms are the obvious inclusion homomorphisms. □

The experienced reader will by now have anticipated the following definition:

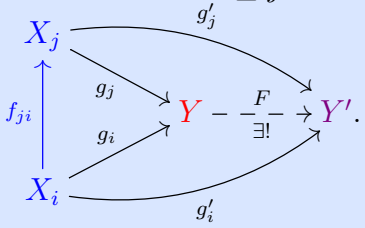
Definition. Let $(I, \leq)$ be a directed set and let $\mathcal{C}$ be a category. Suppose we are given a direct system $(\{X_i\}_{i\in I}, \{f_{ji}\colon X_i \to X_j\}_{i\leq j})$ in the category $\mathcal{C}$. A direct colimit[225] of the direct system is a colimit[226], in the sense of the definition on page 110, of the direct system viewed as a diagram. In other words, a direct colimit is an object Y in $\mathcal{C}$ together with a family of morphisms $\{g_i\colon X_i \to Y\}_{i\in I}$ in the category $\mathcal{C}$ such that the following two conditions are satisfied:

(1) For all $i \leq j$ we have $g_j \circ f_{ji} = g_i\colon X_i \to Y$, i.e. for all $i \leq j$ the following diagram commutes:

[224] Later, on page 1224 we will introduce the notion of an "inverse sequence".

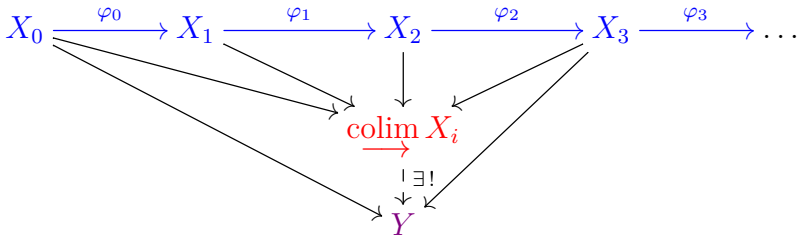


(2) If we are given another object Y' and a family of morphisms $\{g'_i \colon X_i \to Y'\}_{i\in I}$ that satisfy (1), then there exists a unique morphism $F \colon Y \to Y'$ such that for all $i \in I$ we have $g'_i = F \circ g_i \colon X_i \to Y'$. In particular for all $i \le j$ the following diagram commutes:



By the Colimit Uniqueness Lemma 4.7 we know that if a direct colimit exists, then it is unique in the usual sense. We denote the direct colimit, if it exists, by $\varinjlim X_i$ or sometimes by $\varinjlim_I X_i$ or $\varinjlim_i X_i$. [227]

Remark. In the case of a direct sequence $X_0 \xrightarrow{\varphi_0} X_1 \xrightarrow{\varphi_1} X_2 \xrightarrow{\varphi_2} \dots$ of morphisms the notion of a direct colimit can be summarized in the following diagram:



More precisely, a direct colimit $\varinjlim X_i$, if it exists, comes with morphisms $X_i \to \varinjlim X_i$ which form commutative diagrams, furthermore given morphisms $X_i \to Y$ which form commutative diagrams, we get a unique morphism $\varinjlim X_i \to Y$ which makes everything commute.

Examples. We discuss the direct colimits of the direct systems we had discussed above.

(A) As on page 339 we consider in the category *AbGroup* of abelian groups the direct sequence of monomorphisms $\mathbb{Z} \to \mathbb{Z}^2 \to \mathbb{Z}^3 \to \dots$ that is given by the inclusion maps $\varphi_i(v) := (v, 0)$. We claim that

$$\varinjlim \mathbb{Z}^i \;=\; \mathbb{Z}^\infty \;=\; \{(x_1, x_2, \dots) \mid x_i \in \mathbb{Z} \text{ but only finitely many } x_i\text{'s are non-zero}\},$$

where the maps $g_j \colon \mathbb{Z}^j \to \varinjlim \mathbb{Z}^i = \mathbb{Z}^\infty$, $j \in \mathbb{N}$ are the obvious monomorphisms. Indeed, if we are given an abelian group Y' and homomorphisms $g'_j \colon \mathbb{Z}^j \to Y'$ that

[225] Somewhat confusingly a "direct colimit" is often also called an "inductive limit".

[226] It is somewhat unfortunate that what historically is called a direct colimit is actually a colimit. It is important to remember this, since as a general principle colimits tend to commute with colimits, but colimits do not commute with limits. For example in [**Fri24**] it is shown that in general colimits do not commute with products.

[227] Here we ignore the maps f_{ji} and g_i in the notation.

satisfy $g'_{j+1} \circ \varphi_j = g'_j$ for all $j \in \mathbb{N}$, then we set

$$\begin{array}{rcccl} F\colon \varinjlim \mathbb{Z}^i & = & \mathbb{Z}^\infty & \to & Y' \\ (x_1, \dots, x_j, 0, \dots) & & & \mapsto & g'_j(x_1, \dots, x_j). \end{array}$$

It is straightforward to see that this definition is well-defined, i.e. independent of the choice of the x_j.[228] It is clear that for all $i \in \mathbb{N}$ we have $g'_i = F \circ g_i$. Furthermore, since any $x \in \mathbb{Z}^\infty$ is contained in the image of some g_j, we see that F is the unique homomorphism that makes the diagram (2) commute.

Thus we have shown that the direct colimit is the free abelian group on an infinite countable generating set.

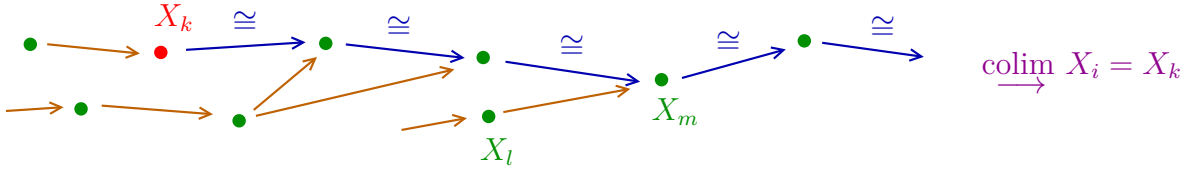
(B) As on page 339 we consider in the category *Group* of groups the direct sequence of monomorphisms $\langle x_1 \rangle \to \langle x_1, x_2 \rangle \to \dots$. Similar to the argument in (A) one can show that

$$\varinjlim \langle x_1, x_2, \dots, x_i \rangle \;=\; \langle x_1, x_2, \dots \rangle \;=\; \text{free group on the generators } x_1, x_2, x_3, \dots$$

where the maps $F_j \to \varinjlim F_i = \langle x_1, x_2, \dots \rangle$, $j \in \mathbb{N}$ are the obvious homomorphisms. Put differently, we have shown that the direct colimit is the free group on an infinite countable generating set. □

The following lemma gives a simple but frequently applicable approach to calculating a direct colimit.

Lemma 25.1. (Direct Colimit–Stabilization Lemma) Let $(I, \leq)$ be a directed set, let $\mathcal{C}$ be a category and let $(\{X_i\}_{i \in I}, \{f_{ji}\}_{i \leq j})$ be a direct system in the category $\mathcal{C}$. Suppose that there exists a $k \in I$ such that for all $l \geq k$ the morphism $f_{lk}\colon X_k \to X_l$ is in fact an isomorphism. Then $\varinjlim X_i$ exists and it is naturally isomorphic to X_k. More precisely, a direct colimit is given by the object $\varinjlim X_i := X_k$ and where for each $l \in I$ the morphism $X_l \to X_k$ is given by $f_{mk}^{-1} \circ f_{ml}\colon X_l \to X_k$ where $m \in I$ is chosen such that $m \geq k$ and $m \geq l$.



Proof. We leave the proof as an exercise to the reader. ■

Example.

(1) If $(I, \leq)$ is a finite directed set, then there exists a maximal element $i_{\max}$. It follows from the Direct Colimit–Stabilization Lemma 25.1 that the direct colimit is given by $X_{i_{\max}}$.

(2) We have $\varinjlim \left(\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cdot(-1)} \mathbb{Z} \xrightarrow{\mathrm{id}} \mathbb{Z} \xrightarrow{\cdot(-1)} \mathbb{Z} \xrightarrow{\mathrm{id}} \dots \right) \underset{\text{by the Direct Colimit–Stabilization Lemma 25.1}}{\cong} \mathbb{Z}.$ □

[228] Note that we did not demand that $x_j \neq 0$.

25.3. **Direct colimits in the category of sets and (abelian) groups.** In the following we prove the existence of direct colimits in some of our favorite categories. We start out with the category of sets, which in its own might not be so interesting, but which will serve as a guide for other categories.

Proposition 25.2. (Sets–Direct Colimit Existence Proposition) Let $(I, \leq)$ be a directed set and let $(\{X_i\}_{i\in I}, \{f_{ji}\colon X_i \to X_j\}_{i\leq j})$ be a direct system in the category $\mathcal{Set}$ of sets.

(1) The direct colimit is given by the set

$$\varinjlim X_i \;:=\; \Big(\bigsqcup_{i\in I} X_i\Big)/\sim \quad \text{where}^{229}\ x \sim f_{ji}(x) \text{ for all } i \leq j,\ x \in X_i$$

together with the natural maps from each X_i into $\varinjlim X_i$.

(2) If all of the maps $f_{ji}\colon X_i \to X_j$ are inclusion maps, i.e. if for each $i \leq j$ we have $X_i \subset X_j$ and $f_{ji}\colon X_i \to X_j$ is the inclusion, then we can also take

$$\varinjlim X_i \;:=\; \bigcup_{i\in I} X_i$$

together with the natural maps from each X_i into $\bigcup_{i\in I} X_i$.

Proof. Both statements follow easily from the definitions and elementary arguments. For peace of mind we carry out the argument in detail for (2).

(2) We now assume that all of the maps $f_{ji}\colon X_i \to X_j$ are inclusion maps. For $i \in I$ we denote by $g_i\colon X_i \to \bigcup_{i\in I} X_i$ the natural inclusion. Since we only deal with inclusion maps it is clear that for any $i \leq j$ we have $g_j \circ f_{ji} = g_i$. Now suppose that we are given a set Y' and maps $g'_i\colon X_i \to Y'$ such that for all $i \leq j$ we have $g'_j \circ f_{ji} = g'_i\colon X_i \to Y$.

Claim. The map

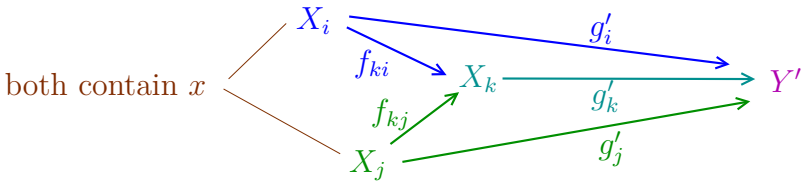
$$\begin{aligned} F\colon \bigcup_{i\in I} X_i &\to Y' \\ x &\mapsto g'_i(x) \qquad \text{if } x \in X_i \end{aligned}$$

is well-defined.

Proof. Let $x \in \bigcup_{i\in I} X_i$. Suppose that $x \in X_i$ and $x \in X_j$. Since we are dealing with a directed set there exists a $k \in I$ with $k \geq i$ and $k \geq j$. Now we see that

$$g'_i(x) = (g'_k \circ f_{ki})(x) = g'_k(f_{ki}(x)) \underset{\uparrow}{=} g'_k(x) \underset{\uparrow}{=} g'_k(f_{kj}(x)) = (g'_k \circ f_{kj})(x) = g'_j(x).$$

since f_{ki} and f_{kj} are inclusion maps



It follows immediately from the claim that the map $F\colon \bigcup_{i\in I} X_i \to Y'$ has the property that for all $i \in I$ we have $g'_i = F \circ g_i\colon X_i \to Y'$.

[229] Of course, as always, we mean by "$\sim$" the equivalence relation that is generated, in the sense of the definition on page 40, by the relation $x \sim f_{ji}(x)$.

It remains to prove the uniqueness of F. Let $x \in \bigcup_{i\in I} X_i$. We pick $i \in I$ with $x \in X_i$. We necessarily have $F(x) = (F \circ g_i)(x) = g'_i(x)$.

(1) This statement follows easily from a modification of the argument in (2), using basic observations about disjoint unions and using the Quotient-Factorization Lemma 0.5. We leave the details to the reader. ■

Next we turn to our favorite algebraic categories. The Sets–Direct Colimit Existence Proposition 25.2 suggests that one can construct direct colimits as quotients of coproducts. This is indeed the case.

Proposition 25.3. (Algebraic–Direct Colimit Existence Proposition)

(1) Let $(I, \leq)$ be a directed set and let $\mathcal{C}$ be one of the following two categories:
   (a) the category *Group* of groups,
   (b) the category *AbGroup* of abelian groups.
   Let $(\{X_i\}_{i\in I}, \{f_{ji}\colon X_i \to X_j\}_{i\leq j})$ be a direct system in $\mathcal{C}$. A direct colimit is given by[230]

$$\text{(a)} \quad \varinjlim X_i = \Big(\ast_{i\in I} X_i\Big) / \langle\langle f_{ji}(x) \cdot x^{-1} \mid i \leq j \text{ and } x \in X_i \rangle\rangle,$$

$$\text{(b)} \quad \varinjlim X_i = \Big(\bigoplus_{i\in I} X_i\Big) / \langle f_{ji}(x) - x \mid i \leq j \text{ and } x \in X_i \rangle,$$

   with the obvious algebraic structure. Alternatively in each of the two categories a direct colimit can be chosen whose underlying set is given by

$$\varinjlim X_i = \Big(\bigsqcup_{i\in I} X_i\Big) / \sim \quad \text{where } x \sim f_{ji}(x) \text{ for all } i \leq j,\ x \in X_i.$$

   In particular in either of the two categories any direct system admits a direct colimit.

(2) Suppose we are in the setting of (1). If all of the homomorphisms $f_{ji}\colon X_i \to X_j$ are inclusion maps, then as underlying set we can always take

$$\varinjlim X_i = \bigcup_{i\in I} X_i.$$

Proof. Let $(I, \leq)$ be a directed set and let $\mathcal{C}$ be one of the two given categories. Furthermore let $(\{X_i\}_{i\in I}, \{f_{ji}\colon X_i \to X_j\}_{i\leq j})$ be a direct system in $\mathcal{C}$.

(1) Statement (1) for the category *Group* of groups follows easily from the Coproduct-of-Groups Lemma 19.2. Similarly we see that Statement (1) for the category *AbGroup* of abelian groups follows from the Coproduct-of-Abelian Groups Lemma 18.1.

Now we turn to Statement (2). We set

$$\varinjlim X_i := \Big(\bigsqcup_{i\in I} X_i\Big) / \sim \quad \text{where } x \sim f_{ji}(x) \text{ for all } i \leq j,\ x \in X_i$$

and we denote by $g_j\colon X_j \to \varinjlim X_i$, $j \in I$, the obvious maps. If we work in the category of (abelian) groups, then $\varinjlim X_i = \big(\bigsqcup_{i\in I} X_i\big)/\sim$ is naturally an (abelian) group as follows: Let $a, b \in \varinjlim X_i = \big(\bigsqcup_{i\in I} X_i\big)/\sim$. This means that there exist

[230] Here given a group G and a subset A of G we denote by $\langle\langle A \rangle\rangle \subset G$ the subgroup normally generated by A, see page 276 and $\langle A \rangle$ denotes the subgroup generated by A. If G is abelian, then basically by definition we have $\langle\langle A \rangle\rangle = \langle A \rangle$.

$i, j \in I$ and $\widetilde{a} \in X_i$ and $\widetilde{b} \in X_j$ with $a = [\widetilde{a}]$ and $b = [\widetilde{b}]$. By definition of a directed set there exists a $k \in I$ with $i \leq k$ and $j \leq k$. Using multiplicative notation we set

$$a \cdot b \;:=\; \big[\underbrace{f_{ki}(\widetilde{a})}_{\in X_k} \cdot \underbrace{f_{kj}(\widetilde{b})}_{\in X_k}\big] \;\in\; \varinjlim X_i.$$

It follows easily from the definition of a direct system that this definition of the product is independent of the choice of k and that this defines indeed an (abelian) group structure on $\varinjlim X_i$.

(2) This statement follows easily from the definitions. ■

The following lemma allows us often to get a handle on elements in direct colimits:

Lemma 25.4. (Elements-in-Direct Colimit Lemma) Let $(I, \leq)$ be a directed set and let $(\{X_i\}_{i\in I}, \{f_{ji}\colon X_i \to X_j\}_{i\leq j})$ be a direct system over I in the category *AbGroup* of abelian groups or the category *Group* of groups. Then the following three statements hold:

(1) Every element in $\varinjlim X_i$ lies in the image of one of the maps $X_j \to \varinjlim X_i$.
(2) Let $a_j \in X_j$ and let $a_k \in X_k$. The images of a_j and a_k in $\varinjlim X_i$ agree if and only if there exists an $l \in I$ with $l \geq j$ and $l \geq k$ such that $f_{lj}(a_j) = f_{lk}(a_k) \in X_l$.
(3) Let $a_j \in X_j$. The image of a_j in $\varinjlim X_i$ is zero if and only if there exists a $k \geq j$ such that $f_{kj}(a_j) = 0 \in X_k$.

Proof.

(1) Statement (1) follows immediately from the explicit description of the direct colimit given in the Algebraic–Direct Colimit Existence Proposition 25.3.
(2) The proof of Statement (2) is left as an exercise. For the exercise it is useful to note that the symbol "$\sim$" in the definition of $\varinjlim X_i$ in the Algebraic–Direct Colimit Existence Proposition 25.3 is only a relation and we really have to work with the equivalence relation generated by this relation, see page 40 for more information.
(3) Statement (3) is a special case of Statement (2). ■

We conclude this section with a general definition.

Definition. Let $(I, \leq)$ be a directed set and let $(\{K_i\}_{i\in I}, \{\chi_{ji}\}_{i\leq j})$ and $(\{L_i\}_{i\in I}, \{\lambda_{ji}\}_{i\leq j})$ be direct systems in a category $\mathcal{C}$.

(1) We define a morphism between the two direct systems to be a family $\{f_i\colon K_i \to L_i\}_{i\in I}$ of morphisms that has the property that for any $i \leq j$ the following diagram commutes:

$$\begin{array}{ccc} K_i & \xrightarrow{f_i} & L_i \\ {\scriptstyle \chi_{ji}}\downarrow & & \downarrow{\scriptstyle \lambda_{ji}} \\ K_j & \xrightarrow{f_j} & L_j. \end{array}$$

We say that the morphism is an isomorphism if all f_i are isomorphisms.
(2) By the universal property of limits the morphisms $K_i \to L_i \to \varinjlim L_i$ give rise to a uniquely defined morphism

$$\varinjlim f_i\colon \varinjlim K_i \;\to\; \varinjlim L_i.$$

Example. We consider the two direct systems that are given by $\{\mathbb{Z}^i\}_{i\in\mathbb{N}}$ and $\{\mathbb{Z}^i \oplus 0\}_{i\in\mathbb{N}}$ where $\mathbb{Z}^i \oplus 0 \subset \mathbb{Z}^\infty$. We have the following isomorphisms:

$$\begin{array}{ccccc} \varinjlim \mathbb{Z}^i & \cong & \varinjlim (\mathbb{Z}^i \oplus 0) & \cong & \mathbb{Z}^\infty. \\ & \uparrow & & \uparrow & \\ & \text{since the two direct systems} & & \text{Algebraic–Direct Colimit Existence} & \\ & \text{are isomorphic} & & \text{Proposition 25.3} & \end{array}$$

We have thus obtained the result from page 341 in a more systematic fashion. □

25.4. **The direct colimit in the topological category.** We now turn to the direct colimit in the topological category. In this setting we will only be interested in the case of a direct sequence of embeddings.

Proposition 25.5. (Topological-Direct Colimit Proposition) Let

$$X_0 \xrightarrow{\varphi_0} X_1 \xrightarrow{\varphi_1} X_2 \xrightarrow{\varphi_2} \dots$$

be a direct sequence of topological spaces such that each $\varphi_i \colon X_i \to X_{i+1}$ is an embedding.

(0) We consider the set

$$\varinjlim X_i \coloneqq \Big(\bigsqcup_{i\in\mathbb{N}_0} X_i\Big)/\sim \quad \text{where } x \sim \varphi_i(x) \text{ for all } i \in \mathbb{N}_0 \text{ and } x \in X_i.$$

Given $i \in \mathbb{N}_0$ we denote by $g_i \colon X_i \to \varinjlim X_i$ the natural map.

(a) We say that $U \subset \varinjlim X_i$ is open if and only if for each $i \in \mathbb{N}_0$ the preimage $g_i^{-1}(U)$ is an open subset of X_i. This defines a topology on $\varinjlim X_i$. In the remainder of this proposition we equip $\varinjlim X_i$ with this topology.

(b) A subset $U \subset \varinjlim X_i$ is closed with respect to the above topology if and only if for each $i \in \mathbb{N}_0$ the preimage $g_i^{-1}(U)$ is a closed subset of X_i.

(1) (a) For each $i \in \mathbb{N}_0$ the natural map $g_i \colon X_i \to \varinjlim X_i$ is an embedding.

(b) Let Y' be a topological space. Given a family of continuous maps $g_i' \colon X_i \to Y'$ with the property that for any $i \in \mathbb{N}_0$ we have $g_{i+1}' \circ \varphi_i = g_i'$, the map

$$\begin{array}{rccl} G' \colon \varinjlim X_i & \to & Y' & \\ & x & \mapsto & g_i'(x_i) \qquad \text{if } x = g_i(x_i) \text{ for some } x_i \in X_i \end{array}$$

is well-defined and it is continuous.

(∗) A map $h \colon \varinjlim X_i \to Y'$ to some topological space is continuous if and only if each map $h \circ g_i \colon X_i \to Y'$ is continuous.

(2) The topological space $\varinjlim X_i$, together with the natural inclusions $g_i \colon X_i \to \varinjlim X_i$, is the direct colimit of the above direct sequence in the category of topological spaces.

If all the embeddings $\varphi_i \colon X_i \to X_{i+1}$ are in fact inclusion maps, i.e. if for each $i \in \mathbb{N}_0$ we have $X_i \subset X_{i+1}$ and $\varphi_i \colon X_i \to X_{i+1}$ is the inclusion, then we can replace (0) by

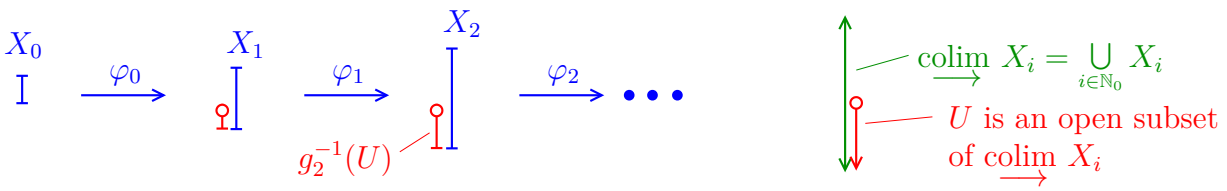
(0′) We consider the set

$$\varinjlim X_i \coloneqq \bigcup_{i\in\mathbb{N}_0} X_i.$$

We say that $U \subset \varinjlim X_i$ is open if and only if for each $i \in \mathbb{N}_0$ the intersection $U \cap X_i$ is an open subset of X_i. This defines a topology on X which again satisfies (1a), (1b) and (2).

Proof. Let

$$X_0 \xrightarrow{\varphi_0} X_1 \xrightarrow{\varphi_1} X_2 \xrightarrow{\varphi_2} \dots$$



be a direct sequence of topological spaces such that each $\varphi_i\colon X_i \to X_{i+1}$ is an embedding. We set

$$\operatorname{colim} X_i \coloneqq \Big(\bigsqcup_{i\in\mathbb{N}_0} X_i\Big)/\sim \quad \text{where } x \sim \varphi_i(x) \text{ for all } i \in \mathbb{N}_0 \text{ and } x \in X_i.$$

For each $i \in \mathbb{N}_0$ we consider the natural inclusion $g_i\colon X_i \to \operatorname{colim} X_i$.

(0) (a) We set

$$\mathcal{T} \coloneqq \big\{U \subset \operatorname{colim} X_i \,\big|\, \text{for each } i \in \mathbb{N} \text{ the preimage } g_i^{-1}(U) \text{ is open in } X_i\big\}.$$

One can easily verify, say using the Image-Preimage Lemma 0.3, that $\mathcal{T}$ defines a topology on $\operatorname{colim} X_i$.

(b) This statement follows immediately from the definition of being closed and the observation that by the Image-Preimage Lemma 0.3 (8) we always have the equality $g_i^{-1}(\operatorname{colim} X_i \setminus U) = g_i^{-1}(\operatorname{colim} X_i) \setminus g_i^{-1}(U) = X_i \setminus g_i^{-1}(U)$.

(1) (a) Let $j \in \mathbb{N}_0$. We need to show that the natural inclusion $g_j\colon X_j \to \operatorname{colim} X_i$ is an embedding. The map is evidently injective. It follows immediately from the definition of $\mathcal{T}$ that $g_j\colon X_j \to \operatorname{colim} X_i$ is continuous. To show that $g_j\colon X_j \to X$ is actually an embedding it remains to prove the following claim:

Claim. Given any open subset $U \subset X_j$ there exists an open subset $V \subset \operatorname{colim} X_i$ with $V \cap g_j(X_j) = g_j(U)$.

Proof. Let $U_j \subset X_j$ be an open subset. Since $\varphi_j\colon X_j \to X_{j+1}$ is an embedding there exists an open subset $U_{j+1} \subset X_{j+1}$ such that $\varphi_j(X_j) \cap U_{j+1} = \varphi_j(U_j)$. We iterate this procedure to obtain subsets $U_j, U_{j+1}, U_{j+2}, \ldots$. We set $V \coloneqq \bigcup_{k\geq j} g_k(U_k)$. Evidently we have $V \cap g_j(X_j) = g_j(U_j)$. It remains to show that V is open in $\operatorname{colim} X_i$. This means that we need to show that for every $i \in \mathbb{N}_0$ the preimage $g_i^{-1}(V)$ is an open subset of X_i. For $i \geq j$ this follows immediately from the construction of V. It remains to consider the case $i < j$. First note that we have $g_{j-1}^{-1}(V) = (g_j \circ \varphi_{j-1})^{-1}(V) = \varphi_{j-1}^{-1}(g_j^{-1}(V)) = \varphi_{j-1}^{-1}(U_j)$. Since $U_j \subset X_j$ is open and since $\varphi_{j-1}\colon X_{j-1} \to X_j$ is continuous we see that $g_{j-1}^{-1}(U_j)$ is indeed an open subset of X_{j-1}. Iterating this argument we see that for all $i \in \{0, \ldots, j-1\}$ the preimage $g_i^{-1}(V)$ is an open subset of X_i. ⊞

(b) As in the proof of the Sets–Direct Colimit Existence Proposition 25.2 we see that the map $G'\colon \operatorname{colim} X_i \to Y'$ is well-defined. By construction we have $G' \circ g_i = g_i'$ for each $i \in \mathbb{N}_0$. It remains to show that the map G' is continuous. Thus let V be an open subset of Y'. We need to show that $(G')^{-1}(V)$ is an open subset of $\operatorname{colim} X_i$. By definition of the topology on $\operatorname{colim} X_i$ this means that we have to show that for each $i \in \mathbb{N}_0$ the preimage $g_i^{-1}((G')^{-1}(V))$ is an open subset of X_i. But by construction of G' we have $G' \circ g_i = g_i'$. Thus we see that $g_i^{-1}((G')^{-1}(V)) = (g_i')^{-1}(V)$, which is of course an open subset of X_i since g_i' is continuous.

(∗) The "only if"-statement follows from (1a) and the fact that the composition of two continuous maps is continuous. The "if"-statement follows easily from (1b).

(2) This statement follows from (1a) and (1b) together with the observation, which we proved in the Sets–Direct Colimit Existence Proposition 25.2, that the map G in (1b) is the unique map which satisfies $G \circ g_i = g'_i$ for all $i \in \mathbb{N}_0$.

Finally note that if all the maps $f_{ji} \colon X_i \to X_j$ are in fact inclusion maps, then it follows by the same argument as in the Sets–Direct Colimit Existence Proposition 25.2 (2) that we can work with $\varinjlim X_i := \bigcup_{i \in \mathbb{N}_0} X_i$. We leave it to the reader to make the few adjustments to finish off this proof. ∎

In many settings the following lemma gives us a convenient way to think about the direct colimit of topological spaces:

Lemma 25.6. (Direct Colimit-Characterization Lemma) Let X be a topological space and let $X_0 \subset X_1 \subset X_2 \subset \dots$ be a sequence of subspaces such that $X = \bigcup_{i \in \mathbb{N}_0} X_i$.

Then the following three statements are equivalent:

(a) We have $X = \varinjlim X_i$.

(b) For every subset $U \subset X$ we have

$$A \subset X \text{ is open} \iff \text{for all } i \in \mathbb{N} \text{ the subset } A \cap X_i \text{ is open in } X_i$$

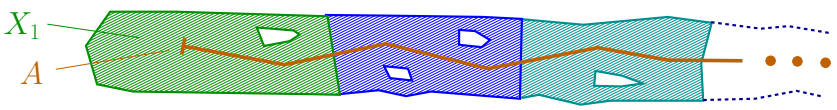
(c) For every subset $A \subset X$ we have

$$A \subset X \text{ is closed} \iff \text{for all } i \in \mathbb{N} \text{ the subset } A \cap X_i \text{ is closed in } X_i$$

Proof of Lemma 25.6. First of all note that it follows from the Iterated Subspace Lemma **??** that each inclusion $X_i \to X_{i+1}$ is an embedding. We now see that the equivalence of (a) and (b) is just a reformulation of the Topological-Direct Colimit Proposition 25.5 (0′).

Next we prove the (b)⇒(c)-direction. For each subset $A \subset X$ we now see that

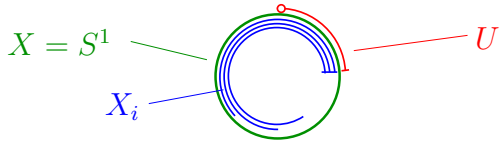
$$\begin{aligned} A \subset X \text{ is closed in } X &\iff X \setminus A \text{ is open in } X \\ &\iff \text{for all } i \in \mathbb{N} \text{ the subset } (X \setminus A) \cap X_i \text{ is open in } X_i \\ &\quad \uparrow \\ &\quad \text{since we assume that (b) holds} \\ &\iff \text{for all } i \in \mathbb{N} \text{ the subset } A \cap X_i \text{ is closed in } X_i \\ &\quad \uparrow \\ &\quad \text{follows from } (X \setminus A) \cap X_i = X_i \setminus (A \cap X_i) \text{ and the definition of closed subsets} \end{aligned}$$



The proof of the (c)⇒(b)-direction is basically the same as the proof of the (b)⇒(c)-direction, since a subset is open if and only if its complement is closed. ∎

Example. The Topological-Direct Colimit Proposition 25.5 raises the following question: If we are given a topological space X and subspaces $X_1 \subset X_2 \subset \dots$ with $\bigcup_{i \in \mathbb{N}} X_i = X$, does the direct colimit topology $\varinjlim X_i$ necessarily agree with the original topology on X? It turns out that in general this is *not* the case. For example we consider the circle $X = S^1$ and the subspaces $X_i := \{\exp(2\pi\,\mathrm{i}t) \mid t \in [0, 1 - \frac{1}{n})\}$.

For $U = \{\exp(2\pi\,\mathrm{i}t) \,|\, t \in [0, \frac{1}{4})\}$ we see that each $U \cap X_i$ is an open subset of X_i, but $U \subset S^1$ is of course not open.



Thus it follows from the Direct Colimit-Characterization Lemma 25.6 (a)$\Longleftrightarrow$(b) that X is not the direct limit of the X_i. □

In contrast to the above example, the following lemma gives a useful criterion for when we have $X = \operatorname{colim}_{\longrightarrow} X_i$.

Criterion 25.7. (Union-equals-Direct Colimit Criterion) Let X be a topological space and let $X_0 \subset X_1 \subset X_2 \subset \ldots$ be subspaces of X. We assume that the following two conditions are satisfied:

(1) We have $\bigcup\limits_{i\in\mathbb{N}_0} X_i = X$.

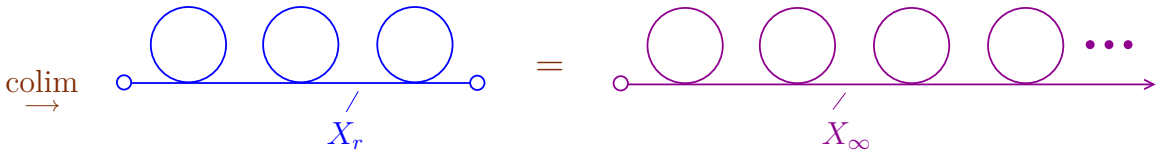
(2) For each $i \in \mathbb{N}_0$ we have $X_i \subset \mathring{X}_{i+1}$.

Then $X = \operatorname{colim}_{\longrightarrow} X_i$ as topological spaces.

Example. Given $r \in \mathbb{N}_0$ we set

$$X_r \;:=\; \big(0, r + \tfrac{1}{2}\big) \;\cup\; \bigcup_{j=1}^{r} \underbrace{\big\{z \in \mathbb{C} \,\big|\, \big|z - (j + \tfrac{1}{4}\,\mathrm{i})\big| = \tfrac{1}{4}\big\}}_{\text{circle of radius } \frac{1}{4} \text{ around } j + \frac{1}{4}\,\mathrm{i} \,\in\, \mathbb{C}} \;\subset\; \mathbb{C}.$$

In other words, for $r \in \mathbb{N}_0$ the set X_r is the subspace of $\mathbb{C} =_{\mathrm{i}} \mathbb{R}^2$ that is given by the interval $(0, r + \frac{1}{2})$ with r circles attached. We obtain a direct sequence of continuous maps by considering the inclusions $X_r \to X_{r+1}$.



It follows fairly easily from the Union-equals-Direct Colimit Criterion 25.7 that the direct colimit is the interval $(0, \infty)$ with infinitely many circles attached, i.e.

$$\operatorname{colim}_{\longrightarrow} X_r \;=\; X_\infty \;:=\; \big(0, \infty\big) \;\cup\; \bigcup_{j=1}^{\infty} \big\{z \in \mathbb{C} \,\big|\, \big|z - (j + \tfrac{1}{4}\,\mathrm{i})\big| = \tfrac{1}{4}\big\} \;\subset\; \mathbb{C},$$

where the maps $X_s \to \operatorname{colim}_{\longrightarrow} X_r = X_\infty$, $s \in \mathbb{N}_0$ are the obvious inclusion maps. □

Proof. By the Direct Colimit-Characterization Lemma 25.6 (a)$\Leftrightarrow$(b) it suffices to prove the following claim:

Claim. For each subset $U \subset X$ we have

$$U \subset X \text{ is open} \quad\Longleftrightarrow\quad \text{for each } i \in \mathbb{N}_0 \text{ the subset } U \cap X_i \subset X_i \text{ is open.}$$

Proof. First we prove the "$\Rightarrow$"-direction. Thus let $U \subset X$ be open. But then then, by definition of the subspace topology, each $U \cap X_i$ is open in X_i.

We turn to the proof of the "$\Leftarrow$"-direction. We now assume that each $U \cap X_i$ is open in X_i. By definition of the subspace topology this means that for each $i \in \mathbb{N}_0$ there exists an open subset $V_i \subset X$ such that $V_i \cap X_i = U \cap X_i$. Now we see that

$$\begin{array}{ccccccc} \bigcup\limits_{i\in\mathbb{N}_0} (V_i \cap \mathring{X}_i) & \subset & \bigcup\limits_{i\in\mathbb{N}_0} (V_i \cap X_i) & \overset{\text{since } V_i\cap X_i = U\cap X_i}{=} & \bigcup\limits_{i\in\mathbb{N}_0} (U \cap X_i) & = & U \cap \bigcup\limits_{i\in\mathbb{N}_0} X_i \overset{\text{follows from (1)}}{=} U \\ & = & U \cap \bigcup\limits_{i\in\mathbb{N}_0} \mathring{X}_i & = & \bigcup\limits_{i\in\mathbb{N}_0} (U \cap \mathring{X}_i) & = & \bigcup\limits_{i\in\mathbb{N}_0} (V_i \cap \mathring{X}_i). \\ & \uparrow & & & & \uparrow & \\ & \text{since it follows from (1) and (2) that } X = \bigcup\limits_{i\in\mathbb{N}_0} \mathring{X}_i & & & \text{since } V_i\cap X_i = U\cap X_i \text{ and since } \mathring{X}_i \subset X_i & & \end{array}$$

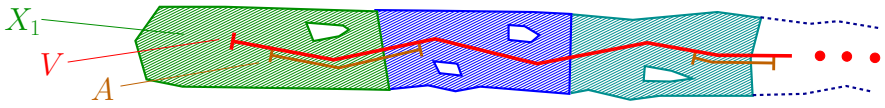
We see that all inclusions are equalities. This implies that $U = \bigcup_{i\in I} (V_i \cap \mathring{X}_i)$. Thus we have shown that U is the union of open subsets, which implies that U itself is open. ∎

Lemma 25.8. (Direct Colimit–Subspace Lemma) Let X be a topological space and let $X_0 \subset X_1 \subset X_2 \subset \dots$ be closed subsets of X such that $X = \operatorname{colim} X_i$. Then for each closed subset $V \subset X$ we have $V = \operatorname{colim} (V \cap X_i)$ as topological spaces.

Proof. Let X be a topological space, let $X_0 \subset X_1 \subset X_2 \subset \dots$ be closed subsets of X such that $X = \operatorname{colim} X_i$ and let $V \subset X$ be a closed subset. For $i \in \mathbb{N}_0$ we set $V_i := V \cap X_i$. We want to show that $V = \operatorname{colim} V_i$. By the Direct Colimit-Characterization Lemma 25.6 (a)$\Longleftrightarrow$(c) it suffices to prove the following claim:

Claim. For every subset $A \subset V$ the following statement holds:

$$A \subset V \text{ is closed} \iff \text{for all } i \in \mathbb{N} \text{ the subset } A \cap V_i \text{ is closed in } V_i$$



Proof. The "$\Rightarrow$"-direction is a consequence of the Iterated Subspace Lemma **??**. Now we turn to the "$\Leftarrow$"-direction. We see that:

$$\begin{array}{l} \text{for all } i \in \mathbb{N} \text{ the subset } A \cap V_i \text{ is closed in } V_i \\ \quad \Longrightarrow \text{ for all } i \in \mathbb{N} \text{ the subset } A \cap (V \cap X_i) \text{ is closed in } V \cap X_i \\ \quad \underset{\uparrow}{\Longrightarrow} \text{ for all } i \in \mathbb{N} \text{ the subset } A \cap (V \cap X_i) = (A \cap V) \cap X_i \text{ is closed in } X \\ \qquad \text{follows from the Open-Closed Inclusion Lemma 2.7, here we use that it follows from the hypothesis on } V \text{ and } X_i \text{ that } V \cap X_i \text{ is a closed subset of } X \\ \quad \underset{\uparrow}{\Longrightarrow} \text{ the subset } A \cap V \text{ is closed in } X \\ \qquad \text{by the Direct Colimit-Characterization Lemma 25.6 (a)}\Longleftrightarrow\text{(c) applied to } X = \operatorname{colim} X_i \\ \quad \underset{\uparrow}{\Longrightarrow} \text{ the subset } V \text{ is closed in } A. \\ \qquad \text{follows from the Subspace-Characterization Lemma ?? (2)} \end{array}$$

∎

We conclude this section with the following lemma, which we will use later on to build open subsets in direct limits:

Lemma 25.9. (Direct Colimit–Open Subset Lemma) Let X be a topological space and let $X_0 \subset X_1 \subset X_2 \subset \dots$ be subspaces of X with $\varinjlim X_i = X$. Suppose that we are given $w \in \mathbb{N}_0$ and suppose that for each $i \geq w$ we are given an open subset $U_i \subset X_i$. If for each $i \geq w$ we have $U_i \subset U_{i+1}$, then $\bigcup_{i \geq w} U_i$ is an open subset of X.

Proof. We set $U := \bigcup_{i \geq w} U_i$. Since $X = \varinjlim X_i$ It follows from the Direct Colimit-Characterization Lemma 25.6 (a)$\Longleftrightarrow$(b) that it suffices to prove the following claim:

Claim. For each $j \in \mathbb{N}_0$ the intersection $X_j \cap U$ is open in X_j.

Proof. Thus let $j \in \mathbb{N}_0$. We set $m := \max\{j, w\}$. Note that

$$X_j \cap U \;=\; X_j \cap \bigcup_{i \geq w} U_i \;\underset{\substack{\uparrow \\ \text{since for each } i \geq w \text{ we have } U_i \subset U_{i+1}}}{=}\; X_j \cap \bigcup_{i \geq m} U_i \;=\; \bigcup_{i \geq m} (X_j \cap U_i).$$

Next note that for $i \geq m$ it follows from the fact that U_i is open in X_i that $X_j \cap U$ is open in X_j. In summary we have shown that $X_j \cap U_i$ is a union of open subsets of X_j, thus $X_j \cap U_i$ is an open subset of X_j. ■

25.5. **Examples and properties of the direct colimit of topological spaces.** In this section we study several direct colimits of natural direct sequences of embeddings of topological spaces which lead to new and interesting topological spaces. Along the way we also develop the theory of direct colimits of direct sequences of topological spaces a little further.

Definition.

(1) We consider the *set*

$$\mathbb{R}^\infty \;:=\; \mathbb{R}^{(\mathbb{N})} \;=\; \{(x_1, x_2, \dots) \mid x_i \in \mathbb{R} \text{ but only finitely many } x_i\text{'s are non-zero}\}$$

(2) Given a subset $X \subset \mathbb{R}^k$ we write

$$X \times \{0\} \;:=\; \{(x, 0, 0, \dots) \mid x \in X\} \;\subset\; \mathbb{R}^\infty.$$

We have an obvious bijection of $X \times \{0\}$ to X and we use it to equip $X \times \{0\}$ with a topology.

(3) Unless we explicitly say something else we equip the set $\mathbb{R}^\infty$ with the topology given by $\mathbb{R}^\infty = \varinjlim (\mathbb{R}^i \times \{0\})$. In other words, $U \subset \mathbb{R}^\infty$ is open if and only if for each $i \in \mathbb{N}_0$ the intersection $U \cap (\mathbb{R}^i \times \{0\})$ is open in $\mathbb{R}^i \times \{0\}$.

(4) Similarly we define $\mathbb{C}^\infty := \varinjlim (\mathbb{C}^i \times \{0\})$ as a topological space.

Example.

(1) To practice the above definition we will show that any linear map $\mathbb{R}^\infty \to \mathbb{R}^\infty$ is continuous with respect to the above topology. The proof proceeds in two steps:

(a) By the Topological-Direct Colimit Proposition 25.5 (1b) it suffices to that show that for each $n \in \mathbb{N}_0$ the map

$$\begin{array}{rccc} f_n \colon & \mathbb{R}^n \times \{0\} & \to & \mathbb{R}^\infty \\ & x & \mapsto & f(x, 0) \end{array}$$

is continuous.

(b) Let $n \in \mathbb{N}_0$. Note that there exists a $k \in \mathbb{N}$ such that $f(e_1), \dots, f(e_n) \in \mathbb{R}^k \times \{0\}$. Thus we obtain the following commutative diagram:

$$\begin{array}{ccc} \mathbb{R}^n \times \{0\} & \xrightarrow{x \mapsto f(x,0)} & \mathbb{R}^\infty \\ \downarrow{\scriptstyle (x,0)\mapsto x} & & \uparrow{\scriptstyle y\mapsto(y,0)} \\ \mathbb{R}^n & \xrightarrow{\widetilde{f}} & \mathbb{R}^k \end{array}$$

for a suitable linear map $\widetilde{f}\colon \mathbb{R}^n \to \mathbb{R}^k$. We make three observations:

- The left vertical map is a homeomorphism.
- Each linear map $\mathbb{R}^n \to \mathbb{R}^k$ is continuous.
- By the Topological-Direct Colimit Proposition 25.5 (1a) we know that the right vertical map is continuous.

It follows that the top horizontal map is continuous.

(2) We consider the map

$$\begin{array}{rcl} \mathbb{C}^\infty & \to & \mathbb{R}^\infty \\ (x_1 + \mathrm{i}y_1, x_2 + \mathrm{i}y_2, \dots) & \mapsto & (x_1, y_1, x_2, y_2, \dots). \end{array}$$

This map has an obvious inverse. Using the approach from (1) one can easily show that both maps are continuous, thus they are both homeomorphisms. At times we will use these homeomorphisms to make the identification $\mathbb{C}^\infty =_{\mathrm{i}} \mathbb{R}^\infty$ □

The above definition motivates the following related definition.

Definition. We consider the topological space

$$S^\infty := \left\{(x_1, x_2, \dots) \in \mathbb{R}^\infty \,\middle|\, \sum_{j=1}^{\infty} x_j^2 = 1\right\} = \varinjlim (S^{i-1} \times \{0\}).$$

Thus $U \subset S^\infty$ is open if and only if for each $i \in \mathbb{N}_0$ the intersection $U \cap (S^{i-1} \times \{0\})$ is an open subset of $S^{i-1} \times \{0\}$.

Before we can formulate our next lemma it is helpful to introduce the following definition:

Definition. Let G be a group that acts on two sets X and Y. A map $f\colon X \to Y$ is called G-equivariant if given any $x \in X$ and $g \in G$ we have $f(g \cdot x) = g \cdot f(x)$.

Lemma 25.10. (Direct Colimit–Action Lemma) Let

$$X_0 \xrightarrow{\varphi_0} X_1 \xrightarrow{\varphi_1} X_2 \xrightarrow{\varphi_2} \dots$$

be a direct sequence of topological spaces such that each $\varphi_i\colon X_i \to X_{i+1}$ is an embedding. Let G be a group. Suppose that for each $i \in \mathbb{N}_0$ we are given a continuous action of G on X_i such that each φ_i is G-equivariant. It follows almost immediately from this data that G also acts continuously on $\varinjlim X_i$.

(1) The natural maps $g_i\colon X_i \to \varinjlim X_i$ are G-equivariant.

(2) The resulting natural map $\varinjlim (X_i/G) \to \left(\varinjlim X_i\right)/G$ is a homeomorphism.

Sketch of proof. For $g \in G$ let $\theta_i^g\colon X_i \to X_i$ be the continuous map that is given by multiplication by g. We consider the continuous maps $X_i \xrightarrow{\theta_i^g} X_i \to \varinjlim X_i$. By our hypothesis on the actions and the universal property of direct colimits we get a unique induced continuous map

$$\theta^g\colon \varinjlim X_i \to \varinjlim X_i.$$

Using the uniqueness of the universal property one can easily verify that these maps define a continuous G-action on $\varinjlim X_i$ and that the natural inclusions $X_i \to \varinjlim X_i$ are G-equivariant. Thus we get induced maps

$$X_i/G \;\to\; (\varinjlim X_i)/G.$$

It follows from the Twice Quotient Lemma 6.3 that these maps are continuous. By the universal property of direct colimits we obtain a continuous map

$$\Xi\colon \varinjlim (X_i/G) \;\to\; \big(\varinjlim X_i\big)/G.$$

It follows from the explicit description of the direct colimit in the category of topological spaces that we gave in the Topological-Direct Colimit Proposition 25.5 that this map Ξ is a bijection. We want to show that Ξ is a homeomorphism. It remains to prove the following claim:

Claim. The map $\Xi\colon \varinjlim (X_i/G) \to \big(\varinjlim X_i\big)/G$ is an open map.

Proof. For $i \in \mathbb{N}_0$ we denote by $p_i\colon X_i \to X_i/G$ the natural projection. Furthermore we denote by $q\colon \varinjlim X_i \to \big(\varinjlim X_i\big)/G$ the natural projection. Let $U \subset \varinjlim (X_i/G)$ be an open subset. By definition of the direct colimit topology this means that for each $i \in \mathbb{N}_0$ the intersection $U_i := (X_i/G) \cap U$ is an open subset of X_i/G. Note that by definition of the quotient topology this means that $\widetilde{U}_i := p_i^{-1}(U_i)$ is an open subset of X_i. We set $\widetilde{U} := \bigcup_{i\in\mathbb{N}_0} \widetilde{U}_i \subset \varinjlim X_i$. It follows easily from the definition of the direct colimit topology that $\widetilde{U}$ is an open subset of $\varinjlim X_i$. Next note that it follows from an elementary argument that $q(\widetilde{U})$ is an open subset of $\big(\varinjlim X_i\big)/G$. Finally note that it follows easily from the various definitions that $\Xi(U) = q(\widetilde{U})$. We have thus shown that Ξ is indeed an open map. ∎

We consider one last example of an interesting direct sequence of topological spaces.

Definition.

(1) We consider the direct sequence

$$\mathbb{R}\mathrm{P}^0 \xrightarrow{[x]\mapsto[x:0]} \mathbb{R}\mathrm{P}^1 \xrightarrow{[x]\mapsto[x:0]} \mathbb{R}\mathrm{P}^2 \xrightarrow{[x]\mapsto[x:0]} \dots$$

of embeddings. We refer to

$$\mathbb{R}\mathrm{P}^\infty := \varinjlim \mathbb{R}\mathrm{P}^i$$

as the infinite-dimensional real projective space.

(2) The same way we also define the infinite-dimensional complex projective space $\mathbb{C}\mathrm{P}^\infty$.

At this point there is not much to say about the infinite-dimensional projective spaces, except that we can show that there exist several different ways to describe these topological spaces:

Lemma 25.11. (Infinite Projective Space-Sphere Quotient Lemma)

(1) The obvious action of the group $\{\pm 1\}$ on S^∞ is continuous and discrete.

(2) There exists a homeomorphism $S^\infty/\{\pm 1\} \to \mathbb{R}\mathrm{P}^\infty$.

Proof.

(1) We leave the verification of this statement to the reader.

(2) We consider the following commutative diagram:

$$\begin{array}{ccc} \varinjlim (S^n/\{\pm 1\}) & \longrightarrow & \varinjlim \mathbb{R}\mathrm{P}^n \\ \downarrow & & \downarrow \\ S^\infty/\{\pm 1\} & \longrightarrow & \mathbb{R}\mathrm{P}^\infty. \end{array}$$

It follows from the Projective Space-Sphere Quotient Lemma 6.9 that the top horizontal map is a homeomorphism. The right vertical map is a homeomorphism by definition. Finally the left vertical map is a homeomorphism by the Direct Colimit–Action Lemma 25.10. It follows that the bottom map is a homeomorphism. ■

CHAPTER 26

Fundamental groups and direct colimits

We already showed that fundamental groups behave well with respect to several operations:[231]

(1) In the Product-π_1-Proposition 13.7 we showed that fundamental groups preserve products.
(2) In the Wedge-π_1-Proposition 20.6 we showed that, under slight technical hypotheses, fundamental groups preserve coproducts.
(3) In the Seifert–van Kampen Theorem 20.1 we showed that, again under fairly mild technical hypotheses, fundamental groups preserve pushouts.

This discussion, and the definitions from the last chapter, raise the question, whether fundamental groups preserve direct colimits. In this chapter we will see that, under suitable hypotheses, this is indeed the case.

In the second half of this chapter we will use this result to calculate the fundamental group of the surface of infinite genus and to prove several results on fundamental groups of finite genus.

The following proposition gives the promised statement about the relationship between fundamental groups and direct colimits.

Proposition 26.1. (Direct Colimit-π_1-Proposition) Let

$$(X_0, w_0) \xrightarrow{\varphi_0} (X_1, w_1) \xrightarrow{\varphi_1} (X_2, w_2) \xrightarrow{\varphi_2} \dots$$

be a sequence of pointed topological spaces such that each $\varphi_i \colon X_i \to X_{i+1}$ is an embedding with $\varphi_i(w_i) = w_{i+1}$. For each $j \in \mathbb{N}_0$ we denote by $g_j \colon X_j \to \operatorname{colim} X_i$ the natural map. If for each $i \in \mathbb{N}_0$ we have $g_i(X_i) \subset \operatorname{int}(g_{i+1}(X_{i+1}))$, then we have a natural isomorphism

$$\operatorname{colim} \pi_1(X_i, w_i) \xrightarrow{\cong} \pi_1\big(\operatorname{colim} X_i, g_0(w_0)\big).$$

Before we turn to the proof of the Direct Colimit-π_1-Proposition 26.1 we state a corollary, which is the result which in practice will be more relevant.

Proposition 26.2. (Exhaustion-π_1-Proposition) Let X be a topological space and let

$$X_0 \subset X_1 \subset X_2 \subset X_3 \subset \dots$$

be a sequence of subspaces of X. We assume that the following two conditions are satisfied:

(1) We have $\bigcup_{i \in I} X_i = X$.
(2) For each $i \in \mathbb{N}_0$ we have $X_i \subset \mathring{X}_{i+1}$.

[231] To smoothen the exposition we are a little ambiguous about whether we are dealing with the category of topological spaces or the category of pointed topological spaces.

Then for any $w \in X_0$ we have a natural isomorphism

$$\operatorname*{colim}_{\longrightarrow} \pi_1(X_i, w) \xrightarrow{\cong} \pi_1(X, w).$$

Example. We assume that we are in the setting of the Exhaustion-π_1-Proposition 26.2. If all the X_i are simply connected, then we see that

$$\begin{array}{ccccccc} \pi_1(X,w) & \xleftarrow{\cong} & \operatorname*{colim}_{\longrightarrow} \pi_1(X_i,w) & \cong & \operatorname*{colim}_{\longrightarrow} \{e\} & = & \{e\}. \\ \uparrow & & \uparrow & & \uparrow & & \\ \text{Exhaustion-}\pi_1\text{-Proposition 26.2} & & \text{since we assume that the } X_i \text{ are simply connected} & & \text{by the Direct Colimit Stabilization Lemma 25.1} & & \end{array}$$

Thus we reproved the statement of the Exhaustion–Simply Connected Proposition 20.10. □

Proof of Proposition 26.2 assuming Proposition 26.1. We consider the following natural diagram:

$$\begin{array}{ccc} \operatorname*{colim}_{\longrightarrow} \pi_1(X_i, w) & \longrightarrow & \pi_1\big(\operatorname*{colim}_{\longrightarrow} X_i, w\big) \\ & \searrow \quad \swarrow & \\ & \pi_1(X, w). & \end{array}$$

First note that it follows from the universal property of direct colimits that the diagram commutes. Next note that it follows from our hypothesis that we can apply the Direct Colimit-π_1-Proposition 26.1, which then tells us, that the horizontal map is an isomorphism. Since we know that for each $i \in \mathbb{N}_0$ we have $X_i \subset \overset{\circ}{X}_{i+1}$ we obtain from the Union-equals-Direct Colimit Criterion 25.7 that we have a natural homeomorphism $\operatorname*{colim}_{\longrightarrow} X_i \to X$. Thus the right diagonal map is an isomorphism. It follows from this discussion that the left diagonal map is an isomorphism. ■

The proof of the Direct Colimit-π_1-Proposition 26.1 rests on the following elementary lemma from earlier on.

Lemma 20.11. (Compact Subspace–Exhaustion Lemma) Let X be a topological space and let

$$X_0 \subset X_1 \subset X_2 \subset X_3 \subset \dots$$

be a sequence of subspaces of X such that the following two conditions are satisfied:

(1) We have $\bigcup_{i \in \mathbb{N}_0} X_i = X$.

(2) For each $i \in \mathbb{N}_0$ we have $X_i \subset \overset{\circ}{X}_{i+1}$.

Given any compact subset K of X there exists an $i \in \mathbb{N}_0$ with $K \subset X_i$.

Now we can provide the proof of the Direct Colimit-π_1-Proposition 26.1.

Proof of the Direct Colimit-π_1-Proposition 26.1. Let

$$(X_0, w_0) \xrightarrow{\varphi_0} (X_1, w_1) \xrightarrow{\varphi_1} (X_2, w_2) \xrightarrow{\varphi_2} \dots$$

be a sequence of topological spaces such that each $\varphi_i \colon X_i \to X_{i+1}$ is an embedding. For each $i \in \mathbb{N}_0$ we denote by $g_i \colon X_i \to \operatorname*{colim}_{\longrightarrow} X_i$ the natural map. We assume that for each $i \in \mathbb{N}_0$ we have $g_i(X_i) \subset \operatorname{int}(g_{i+1}(X_{i+1}))$. We set $X \coloneqq \operatorname*{colim}_{\longrightarrow} X_i$. It follows from the Topological-Direct Colimit Proposition 25.5 that the natural maps $g_i \colon X_i \to X$ are embeddings and that the union of the images of the X_i is all of X. Thus we can and will view each X_i as a

subspace of X. In summary we can now assume that we have a sequence

$$X_0 \subset X_1 \subset X_2 \subset X_3 \subset \dots$$

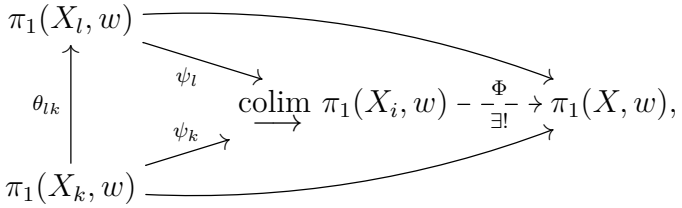
of subspaces of X such that the following two conditions are satisfied:

(a) We have $\bigcup_{i\in\mathbb{N}_0} X_i = X$.

(b) For each $i \in \mathbb{N}_0$ we have $X_i \subset \mathring{X}_{i+1}$.

Let $w \in X_0$. Given $k < l$ we denote by $\theta_{lk}\colon \pi_1(X_k, w) \to \pi_1(X_l, w)$ the inclusion-induced homomorphism. Given $k \in \mathbb{N}_0$ we denote by $\psi_k\colon \pi_1(X_k, w) \to \operatorname{colim} \pi_1(X_i, w)$ the natural homomorphism.

Note that by the universal property of a direct colimit there exists a unique homomorphism $\Phi\colon \operatorname{colim} \pi_1(X_i, w) \to \pi_1(X, w)$ that makes the following diagram commute: For any $k < l$:



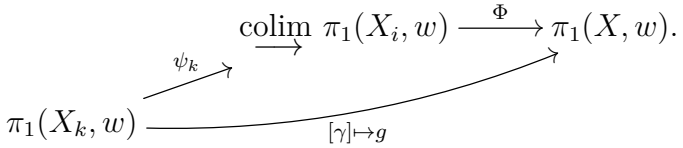
where the curved maps are induced by the inclusions $X_k \to X$ and $X_l \to X$. We need to show that Φ is an isomorphism.

(1) First we show that the map Φ is surjective. Let $g \in \pi_1(X, w)$.

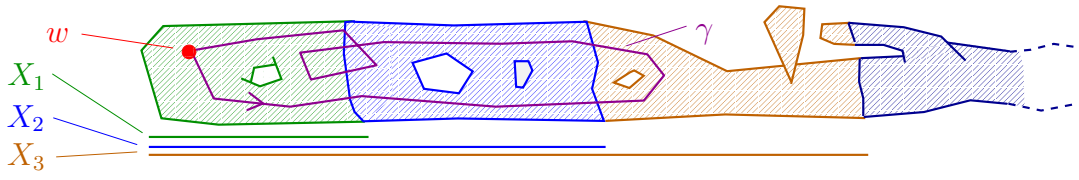
Claim. There exists a compact subset K of X with $w \in K$ such that g lies in the image of the inclusion-induced map $\pi_1(K, w) \to \pi_1(X, w)$.

Proof. We pick a loop $\gamma\colon [0, 1] \to X$ that represents $g \in \pi_1(X, w)$. By the Compact Image Lemma 2.8 the image $\gamma([0, 1])$ is compact. Thus $K \coloneqq \gamma([0, 1])$ has the desired property. ⊞

Note that it follows from the claim together with the Compact Subspace–Exhaustion Lemma 20.11 that there exists a $k \in \mathbb{N}_0$ such that g lies in the image of the inclusion-induced map $\pi_1(X_k, w) \to \pi_1(X, w)$. We consider the following commutative diagram:



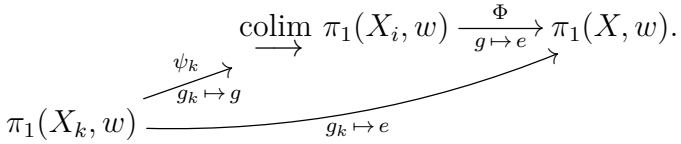
By our choice of k we see that g lies in the image of the curved map, hence it also lies in the image of Φ. This concludes the proof that Φ is surjective.



(2) Now we show that Φ is injective. So let $g \in \ker\big(\Phi\colon \operatorname{colim} \pi_1(X_i, w) \to \pi_1(X, w)\big)$. By the Algebraic–Direct Colimit Existence Proposition 25.3 we can write

$$\operatorname*{colim} \pi_1(X_i, w) \;=\; \Big(\bigsqcup_{i\in\mathbb{N}_0} \pi_1(X_i, w)\Big)/\sim \quad \text{where } h \sim \theta_{lk}(h) \text{ for all } k \le l,\ h \in \pi_1(X_k, w).$$

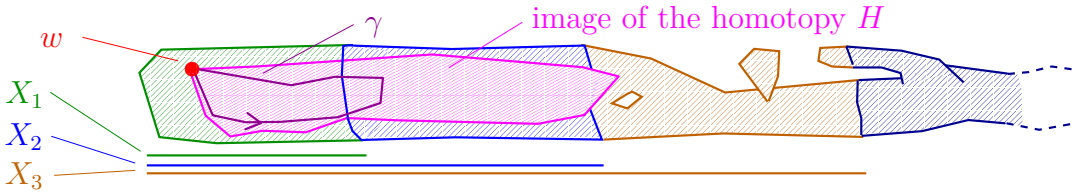
It follows in particular from this explicit description that there exists a $k \in \mathbb{N}_0$ and a $g_k \in \pi_1(X_k, w)$ with $g = \psi_k(g_k)$. In summary, we have the following situation:



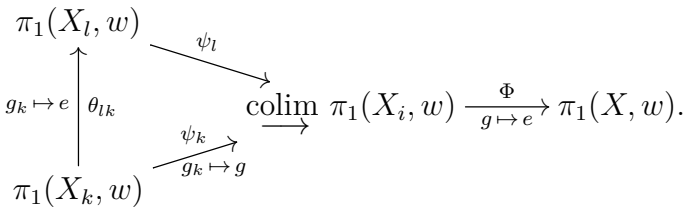
Claim. There exists a compact subset K of X such that g_k lies in the kernel of the inclusion-induced map $\pi_1(X_k, w) \to \pi_1(X_k \cup K, w)$.

Proof. We pick a loop $\gamma\colon [0,1] \to X_k$ that represents g_k. By the above we know that $[\gamma] \in \ker(\pi_1(X_k, w) \to \pi_1(X, w))$. By definition this means that there exists a path-homotopy $H\colon [0,1]\times[0,1] \to X$ from the path γ to the constant path. We appeal again to the Compact Image Lemma 2.8 to deduce that $K := H([0,1] \times [0,1])$ is a compact subset of X. Evidently K has the desired property. ⊞

It follows from the claim, the Compact Subspace–Exhaustion Lemma 20.11 and the fact that we always have $X_i \subset X_{i+1}$ that there exists an $l \ge k$ such that $\theta_{lk}([\gamma])$ is trivial in $\pi_1(X_l, w)$.



We obtain the following diagram:



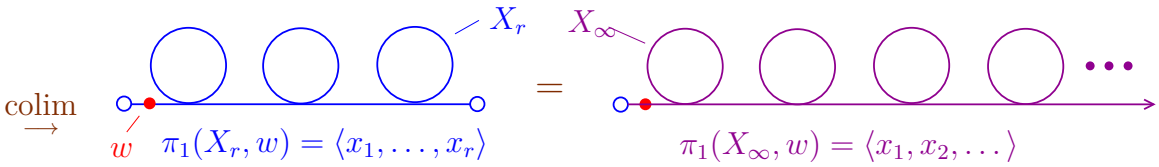
It follows immediately that g itself is trivial. This concludes the proof that Φ is injective. ■

Example.

(1) As on page 348 we denote by X_∞ the interval $(0,\infty)$ with infinitely many circles attached. We pick the base point $w = \frac{1}{4}$. We obtain that [232]

$$\pi_1\big(X_\infty, w\big) \underset{\text{see page 341}}{=} \pi_1\big(\operatorname*{colim}_{\longrightarrow} X_r, w\big) \underset{\text{Direct Colimit-}\pi_1\text{-Proposition 26.1}}{\overset{\cong}{\longleftarrow}} \operatorname*{colim}_{\longrightarrow} \pi_1\big(X_r, w\big) \overset{\text{Graph-}\pi_1\text{-Proposition 20.9}}{\overset{\cong}{\longrightarrow}} \operatorname*{colim}_{\longrightarrow} \langle x_1, \dots, x_r\rangle \underset{\text{see page 341}}{=} \langle x_1, x_2, \dots\rangle.$$

Furthermore the proof of the Graph-π_1-Proposition 20.9 and the above argument show that the generators $x_1, x_2, \dots$ of $\pi_1(X_\infty)$ can be chosen such that x_n, $n \in \mathbb{N}$, corresponds to a loop which goes "once around the n-th circle".



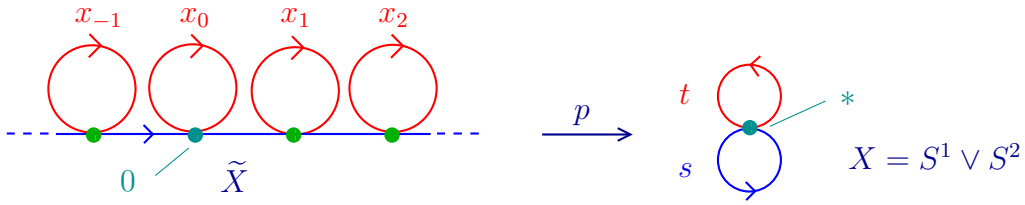
(2) We consider

$$\widetilde{X} \coloneqq \mathbb{R} \cup \bigcup_{j\in\mathbb{Z}} \underbrace{\big\{z \in \mathbb{C} \,\big|\, \big|z - (j + \tfrac{1}{4}\mathrm{i})\big| = \tfrac{1}{4}\big\}}_{\text{circle of radius } \frac{1}{4} \text{ around } j+\frac{1}{4}\mathrm{i} \in \mathbb{C}} \subset \mathbb{C}.$$

Put differently, $\widetilde{X}$ is the real line with infinitely many circles attached. The same argument as in (1) shows that $\pi_1(\widetilde{X}, 0)$ is the free group on infinitely many generators $\{x_n\}_{n\in\mathbb{Z}}$. In the figure below we illustrate a covering map $p\colon \widetilde{X} \to X = S^1 \vee S^1$. By the Covering-$\pi_1$-Monomorphism Proposition 14.11 the induced homomorphism $p_*\colon \pi_1(\widetilde{X}, 0) \to \pi_1(X, *)$ is a monomorphism. We had just seen that $\pi_1(\widetilde{X})$ is a free group on infinitely many generators and by the Wedge-of-Circles-π_1-Lemma 20.7 we know that $\pi_1(X)$ is a free group on two generators. In fact, with appropriate orientations and conventions this shows that the homomorphism

$$\begin{array}{ccccccc} \pi_1(\widetilde{X}, 0) & \overset{\cong}{\longleftarrow} & \langle \dots, x_{-2}, x_{-1}, x_0, x_1, x_2, \dots\rangle & \to & \langle s, t\rangle & \overset{\cong}{\longrightarrow} & \pi_1(X, *) \\ & & x_n & \mapsto & s^n \cdot t \cdot s^{-n} & & \end{array}$$

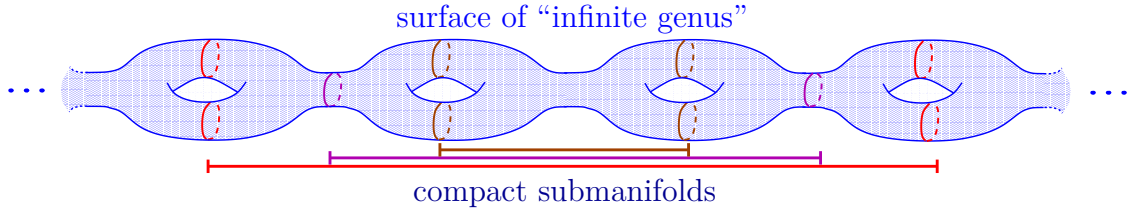
is a monomorphism. In particular we see that the free group $\langle s, t\rangle$ on two generators contains a free group on infinitely many generators. □



[232] The third equality follows from the Graph-π_1-Proposition 20.9, to be more precise, it follows from the fact that for any $r \le s$ the inclusion map $\iota_{sr}\colon X_r \to X_s$ and the isomorphisms of the Graph-π_1-Proposition 20.9 give rise to a commutative diagram

$$\begin{array}{ccc} \pi_1(X_r, w) & \xrightarrow{\cong} & \langle x_1, \dots, x_r\rangle \\ {\scriptstyle (\iota_{sr})_*}\downarrow & & \downarrow{\scriptstyle x_i \mapsto x_i} \\ \pi_1(X_s, w) & \xrightarrow{\cong} & \langle x_1, \dots, x_s\rangle. \end{array}$$

Example. In [**Fri24**] we will use the Direct Colimit-π_1-Proposition 26.1 to show that the fundamental group of the surface of infinite genus is the free group on infinitely many generators. □



CHAPTER 27

Universal coverings

After spending a lot of time on determining the fundamental groups of topological spaces we now want to return to the study of coverings. For the reader's convenience we recall the definition from page 213.

Definition. Let $p\colon X \to B$ be a continuous map between topological spaces.

(1) An open subset $U \subset B$ is called uniformly covered, if $p^{-1}(U) \subset X$ is the union of disjoint open subsets $\{V_i\}_{i\in I}$ of X with the property that for each $i \in I$ the map $p|_{V_i}\colon V_i \to U$ is a homeomorphism.

(2) We say that the map $p\colon X \to B$ is a covering, if p is surjective and if for every $b \in B$ there exists an open neighborhood U of b that is uniformly covered.

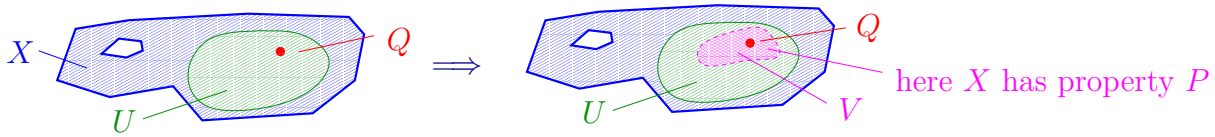
(3) A covering $p\colon (X, x_0) \to (B, b_0)$ of pointed topological spaces is a covering $p\colon X \to B$ with $p(x_0) = b_0$.

(4) We say a covering $p\colon X \to B$ is path-connected, if X is path-connected. We adopt obvious variations on this language.

We would like answer the following question: Does every path-connected non-empty topological space B admit a path-connected covering $p\colon X \to B$ such that X is simply connected? We know of course that the answer is yes for the torus, the Klein bottle and the Möbius band. But does such a covering exist for the surface of genus 2? How about the wedge $S^1 \vee S^1$ of two circles?

27.1. **Local properties of topological spaces.** Before we can address the above question we need to recall the definition from Section 2.9 of a local property of topological spaces.

Definition. Let P be a property of topological spaces. We say that a topological space X is locally P if given any $Q \in X$ and any neighborhood U of Q there exists an open neighborhood V of Q that is contained in U and that has the property P.



Examples.

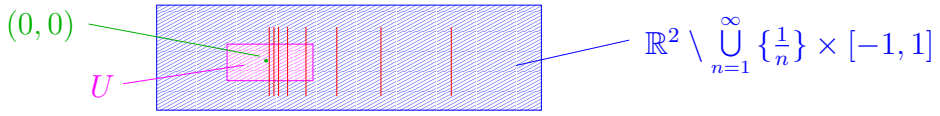
(1) Note that every topological manifold (in particular every open subset of $\mathbb{R}^n$) is locally contractible.

(2) One can easily show that every topological graph is locally contractible.

(3) One can easily show that any wedge of topological manifolds (e.g. any wedge of spheres) is locally contractible.

(4) By the Homotopy-π_0-Lemma 9.6 and the Homotopy-π_1-Proposition 13.4 we know that every contractible topological space is path-connected and simply connected. Thus every locally contractible topological space is also locally path-connected and locally simply connected.

(5) We consider
$$X := \mathbb{R}^2 \setminus \bigcup_{n=1}^{\infty} \{\tfrac{1}{n}\} \times [-1,1],$$
with the usual subspace topology coming from $\mathbb{R}^2$. Then X is path-connected but not locally path-connected. Indeed, we consider the point $(0,0) \in X$ and the neighborhood $U = (-\frac{1}{2},\frac{1}{2}) \times (-\frac{1}{2},\frac{1}{2}) \cap X$. It is straightforward to show that there does not exist a path-connected neighborhood V of $(0,0)$ which is contained in U.



(6) On numerous occasions we will use the following elementary fact: If $p\colon X \to B$ is a covering and if B is locally P, then so is X. □

27.2. **Lifting maps to coverings.** The following definition is a mild variation on the definition on page 220.

Definition. Let $p\colon (X,x_0) \to (B,b_0)$ be a covering of pointed topological spaces. Furthermore let $f\colon (Z,z_0) \to (B,b_0)$ be a continuous map. A lift of f is a continuous map $\widetilde{f}\colon (Z,z_0) \to (X,x_0)$ that makes the following diagram commute:

$$\begin{array}{ccc} & & (X,x_0) \\ & \overset{\widetilde{f}}{\nearrow} & \downarrow p \\ (Z,z_0) & \xrightarrow[f]{} & (B,b_0). \end{array}$$

It is worth recalling some earlier results on lifts of maps. First of all we have the following proposition that deals with any type of covering.

Proposition 14.7. (Lift Uniqueness Proposition) Let $p\colon X \to B$ be a covering and let Z be a connected topological space. Furthermore let $f\colon Z \to B$ be a continuous map and let $\widetilde{f},\widehat{f}\colon Z \to X$ be two lifts of f. If $\widetilde{f}$ and $\widehat{f}$ agree at some point $z_0 \in Z$, then they agree everywhere.

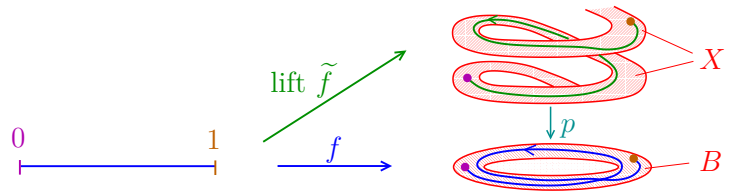
We are already experts on lifts of paths, i.e. of continuous maps $[0,1] \to B$. We now recall two of the main results.

Proposition 14.10. (Path-Lifting Proposition) Let $p\colon X \to B$ be a covering.

(1) Let $f\colon [0,1] \to B$ be a path. Given any $\widetilde{b} \in X$ with $p(\widetilde{b}) = f(0)$ there exists a unique lift $\widetilde{f}\colon [0,1] \to X$ of f to the starting point $\widetilde{b}$, i.e. with $\widetilde{f}(0) = \widetilde{b}$.

(2) Let $f,g\colon [0,1] \to B$ be two paths and let $\widetilde{f},\widetilde{g}\colon [0,1] \to X$ be two lifts with the same starting point. If f and g are path-homotopic, then the endpoints of $\widetilde{f}$ and $\widetilde{g}$ agree and the paths $\widetilde{f}$ and $\widetilde{g}$ are path-homotopic.

Proposition 14.12 (Loop–Lifting Proposition) Let $p\colon (X,x_0) \to (B,b_0)$ be a covering of pointed topological spaces. Let $f\colon [0,1] \to B$ be a loop in (B,b_0). We denote by



$\widetilde{f}\colon [0,1] \to \widetilde{B}$ the lift of f to the starting point x_0. The following holds:

$$\widetilde{f} \text{ is a loop in } (X, x_0) \iff [f] \text{ lies in } p_*(\pi_1(X, x_0)) \subset \pi_1(B, b_0).$$

Given a covering $p\colon (X, x_0) \to (B, b_0)$ we now want to study the existence of lifts of arbitrary continuous maps $(Z, z_0) \to (B, b_0)$. We start out with a very instructive example.

Example.

(1) We consider the maps:

$$\begin{array}{ccc} & & (S^1,1) \\ & \overset{\text{¿¿¿ lift } \widetilde{f} \text{ ???}}{\nearrow} & \big\downarrow{\scriptstyle p\ z\mapsto z^3} \\ (S^1,1) & \xrightarrow[f]{z\mapsto z^2} & (S^1,1). \end{array}$$

Here the map p is a 3-fold covering. Does the map f lift to a continuous map $\widetilde{f}$?

(2) To answer the question we just posed in (1) it is helpful to consider a more general setup. Thus let $p\colon (X, x_0) \to (B, b_0)$ be a covering of pointed topological spaces and let $f\colon (Z, z_0) \to (B, b_0)$ be a continuous map. Let us suppose that there exists a lift $\widetilde{f}\colon (Z, z_0) \to (X, x_0)$. We obtain the commutative diagram

$$\begin{array}{ccc} & & (X,x_0) \\ & \overset{\widetilde{f}}{\nearrow} & \big\downarrow{\scriptstyle p} \\ (Z,z_0) & \xrightarrow[f]{} & (B,b_0) \end{array}$$

from the functoriality of the fundamental group it follows that $f_* = \big(p \circ \widetilde{f}\big)_* = p_* \circ \widetilde{f}_*$, i.e. we get the commutative diagram

$$\begin{array}{ccc} & & \pi_1(X,x_0) \\ & \overset{\widetilde{f}_*}{\nearrow} & \big\downarrow{\scriptstyle p_*} \\ \pi_1(Z,z_0) & \xrightarrow[f_*]{} & \pi_1(B,b_0). \end{array}$$

Thus it follows that if f lifts, then

$$\operatorname{im}(f_*) \;=\; \operatorname{im}((p \circ \widetilde{f})_*) \;=\; \operatorname{im}(p_* \circ \widetilde{f}_*) \;\subset\; \operatorname{im}(p_*).$$

(3) In the above example (1) we have $\pi_1(S^1, 1) =_{\mathrm{i}} \mathbb{Z}$. By the discussion on page 199 we have $\operatorname{im}(f_*) = 2\mathbb{Z}$ and we have $\operatorname{im}(p_*) = 3\mathbb{Z}$. Since $2\mathbb{Z} \not\subset 3\mathbb{Z}$ we see that f does *not* lift. □

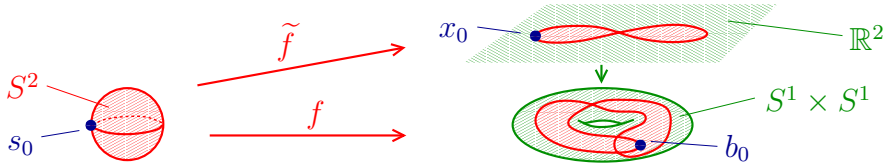
The following proposition says, that the above conclusion has a converse for "reasonably" topological spaces.

Proposition 27.1. (Map Lifting Criterion) Let $p\colon (X, x_0) \to (B, b_0)$ be a covering of pointed topological spaces. Furthermore let Z be a path-connected topological space and let $f\colon (Z, z_0) \to (B, b_0)$ be a continuous map. If Z is locally path-connected, then we have the following equivalence of statements:[233]

$$\begin{array}{ccc} & & (X,x_0) \\ & \overset{\text{there exists a lift } \widetilde{f}}{\nearrow} & \big\downarrow{\scriptstyle p} \\ (Z,z_0) & \xrightarrow[f]{} & (B,b_0) \end{array} \qquad \Longleftrightarrow \qquad \operatorname{im}(f_*) \subset \operatorname{im}(p_*).$$

Example. Let $p\colon (X, x_0) \to (B, b_0)$ be a covering of pointed topological spaces.

(1) The condition on the fundamental groups in the Map Lifting Criterion 27.1 is automatically satisfied if Z is simply connected.
(2) Let $f\colon (S^n, s_0) \to (B, b_0)$ be a continuous map. The topological manifold S^n is path-connected and locally path-connected. By the Sphere-π_1-Proposition 11.15 the sphere S^n is simply connected if $n \geq 2$.



Therefore it follows from the observation in (1) that there exists a uniquely determined lift $\widetilde{f}\colon (S^n, s_0) \to (X, x_0)$ of the map $f\colon (S^n, s_0) \to (B, b_0)$.

□

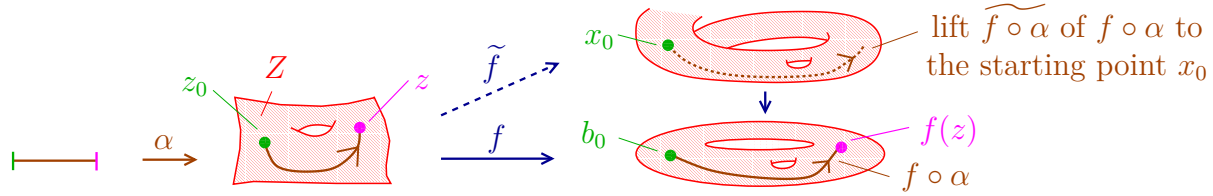
Proof of the Map Lifting Criterion 27.1. Let $p\colon (X, x_0) \to (B, b_0)$ be a covering of pointed topological spaces. Furthermore let Z be a path-connected and locally path-connected topological space and let $f\colon (Z, z_0) \to (B, b_0)$ be a continuous map.

By the discussion on page 362 we only have to prove the "$\Longleftarrow$"-direction of the proposition. So suppose that we have

$$\operatorname{im}(f_*\colon \pi_1(Z, z_0) \to \pi_1(B, b_0)) \subset \operatorname{im}(p_*\colon \pi_1(X, x_0) \to \pi_1(B, b_0)).$$

We have to construct a lift $\widetilde{f}\colon (Z, z_0) \to (X, x_0)$. We will do so as follows. Let $z \in Z$ be a point. Since Z is path-connected we can find a path $\alpha\colon [0, 1] \to Z$ with $\alpha(0) = z_0$ and $\alpha(1) = z$. According to the Path-Lifting Proposition 14.10 (1) we can lift the path $f \circ \alpha\colon [0, 1] \to B$ to a path $\widetilde{(f \circ \alpha)}\colon [0, 1] \to X$ with starting point x_0. We set

$$\widetilde{f}(z) := \widetilde{(f \circ \alpha)}(1).$$



It remains to prove the following claim:

Claim.
(1) The map $\widetilde{f}\colon Z \to X$ is well-defined.
(2) The map $\widetilde{f}$ is continuous.
(3) The map $\widetilde{f}$ is a lift.

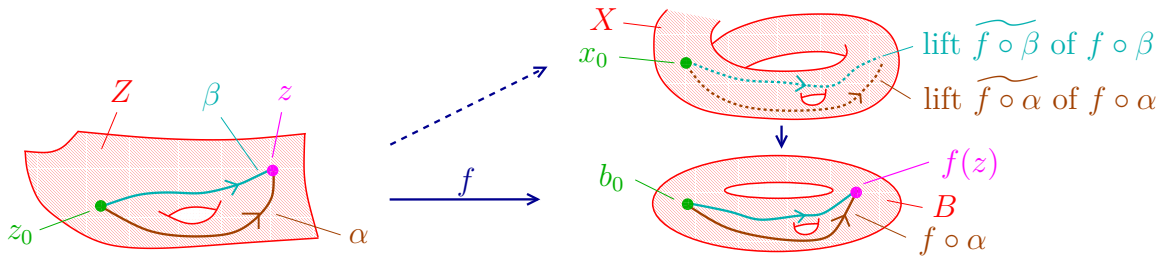
Proof.

[233]It is worth recalling that since Z is in particular connected, it follows from the Lift Uniqueness Lemma 11.10 that a lift $(Z, z_0) \to (X, x_0)$, if it exists, is unique.

(1) We have to show that for $z \in Z$ the point $\widetilde{f}(z) \in X$ is well-defined, i.e. independent of the choice of the path α. Thus let $\beta\colon [0,1] \to Z$ be another path with $\beta(0) = z_0$ and $\beta(1) = z$. It follows that

$$\begin{aligned}
\widetilde{(f\circ\alpha)}(t=1) &= (\text{lift of } f\circ\alpha \text{ to the starting point } x_0)(t=1)\\
&= (\text{lift of } (f\circ\alpha) * \overline{(f\circ\beta)} * (f\circ\beta) \text{ to the starting point } x_0)(t=1)\\
&\quad \uparrow \\
&\quad \text{by the Path-Lifting Proposition 14.10 (2) since } f\circ\alpha \text{ and } (f\circ\alpha) * \overline{(f\circ\beta)} * (f\circ\beta) \text{ are path-homotopic}\\
&= (\text{lift of } (f\circ(\underbrace{\alpha * \overline{\beta}}_{=:\gamma,\ \text{a loop in } (Z,z_0)})) * (f\circ\beta) \text{ to the starting point } x_0)(t=1)\\
&= (\text{lift of } f\circ\beta \text{ to the starting point } \widetilde{(f\circ\gamma)}(1))(t=1)\\
&= (\text{lift of } f\circ\beta \text{ to starting point } x_0)(t=1) \;=\; \widetilde{(f\circ\beta)}(t=1).\\
&\quad \uparrow \\
&\quad \text{by our hypothesis we have } [f\circ\gamma] = f_*([\gamma]) \in \operatorname{im}(f_*) \subset p_*(\pi_1(X,x_0)),\\
&\quad \text{hence } \widetilde{f\circ\gamma} \text{ is a loop in } x_0 \text{ by the Loop–Lifting Proposition 14.12}
\end{aligned}$$

Thus we have shown that the map $\widetilde{f}$ is well-defined.



(2) We need to show that $\widetilde{f}$ is continuous. By the Pasting Proposition 2.4 it suffices to show that $\widetilde{f}$ is locally continuous. So let $z \in Z$. We pick a path α from z_0 to z. Since p is a covering there exists an open neighborhood V of $\widetilde{f}(z)$ such that the restriction of p to V is an embedding. We denote by $q\colon p(V) \to V$ the inverse map. Since Z is locally path-connected there exists an open neighborhood U of z that is contained in $f^{-1}(p(V))$ and that is path-connected. Since f and q are continuous it suffices to prove the following claim:

Claim. On the open neighborhood U of z the map $\widetilde{f}$ agrees with $q \circ f$.

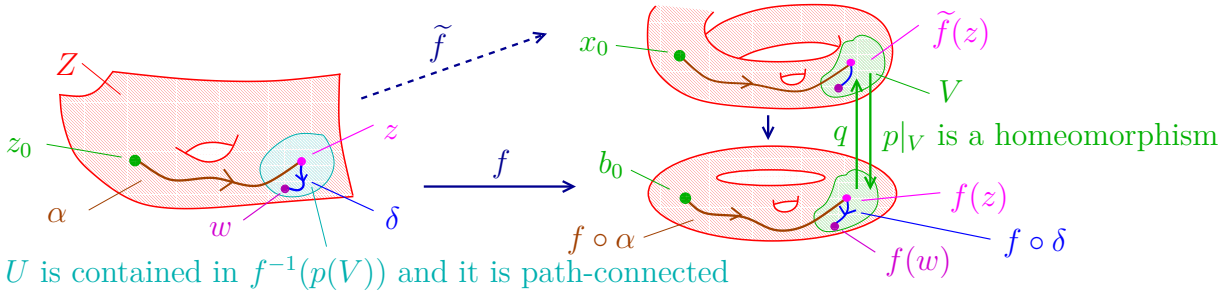
Proof. So let $w \in U$. Since U is path-connected there exists a path δ from z to w that lies in U. Then

$$\begin{aligned}
\widetilde{f}(w) &= \text{endpoint of the lift of } f\circ(\alpha*\delta) = (f\circ\alpha)*(f\circ\delta) \text{ to the starting point } x_0\\
&= \text{endpoint of the lift of } f\circ\delta \text{ to the starting point } \widetilde{f}(z)\\
&= \text{endpoint of } q\circ(f\circ\delta) \;=\; (q\circ f)(\delta(1)) \;=\; (q\circ f)(w).\\
&\quad \uparrow \\
&\quad \text{since } q = p^{-1} \text{ we know that } q\circ(f\circ\delta) \text{ is the lift of } f\circ\delta \text{ to the starting point } f(z)
\end{aligned}$$

(3) We need to show that the map $\widetilde{f}\colon Z \to X$ is a lift of $f\colon Z \to B$, i.e. we need to show that $p \circ \widetilde{f} = f$. Given any $z \in Z$ we have indeed the following equalities:

$$\begin{aligned}
p\big(\widetilde{f}(z)\big) &= p\big(\widetilde{(f\circ\alpha)}(1)\big) \;=\; (f\circ\alpha)(1) \;=\; f(\alpha(1)) \;=\; f(z).\\
&\quad \uparrow \\
&\quad \text{we pick a path } \alpha \text{ from } z_0 \text{ to } z \text{ and we denote by } \widetilde{f\circ\alpha} \text{ the lift of } f\circ\alpha
\end{aligned}$$

■



27.3. **Existence of covering spaces.** Our next goal will be to show that every "reasonable" path-connected non-empty topological spaces admits a covering that is simply connected.

Let $p\colon (X, x_0) \to (B, b_0)$ be a covering of pointed topological spaces. In the Covering-π_1-Monomorphism Proposition 14.11 we saw that $p_*\colon \pi_1(X, x_0) \to \pi_1(B, b_0)$ is an injective homomorphism. Thus we can view $\pi_1(X, x_0)$ as a subgroup of $\pi_1(B, b_0)$. One can now ask, whether to each subgroup Γ of $\pi_1(B, b_0)$ there exists a corresponding covering with $p_*(\pi_1(X, x_0)) = \Gamma$. We will see that this is indeed the case, at least under the rather technical assumption that the topological space B is locally simply connected.

Theorem 27.2. (Covering Existence Theorem) Let Y be a topological space that is path-connected and locally simply connected, let $y_0 \in Y$ and let $\Gamma \subset \pi = \pi_1(Y, y_0)$ be a subgroup.

(1) There exists a path-connected covering $p\colon (X, x_0) \to (Y, y_0)$ of pointed topological spaces such that $p_*(\pi_1(X, x_0)) = \Gamma$.[234]

(2) For the covering p from (1) we have

$$\underbrace{[X : Y]}_{\substack{\text{degree as defined} \\ \text{on page 215}}} = [\pi_1(Y, y_0) : \Gamma].$$

We will provide the proof of the Covering Existence Theorem 27.2 in the next two sections. In the following we discuss a few examples.

Example. We consider $(Y, y_0) = (S^1, 1)$. We pick an identification[235] $\pi_1(S^1, 1) = \mathbb{Z}$. Given $n \in \mathbb{N}$ we consider $\Gamma = n\mathbb{Z} \subset \mathbb{Z} =_{\mathrm{i}} \pi_1(S^1, 1)$. It follows from the discussion on page 199 that the covering

$$\begin{array}{rcl} p\colon S^1 & \to & S^1 \\ z & \mapsto & z^n \end{array}$$

has the property that $p_*(\pi_1(S^1, 1)) = n \cdot \mathbb{Z}$. Another example is given by the covering

$$\begin{array}{rcl} q\colon \mathbb{R}/n\mathbb{Z} & \to & S^1 \\ t + n\mathbb{Z} & \mapsto & \exp(2\pi\,\mathrm{i}t). \end{array}$$

Here we also have $q_*(\pi_1(\mathbb{R}/n\mathbb{Z}, [0])) = n\mathbb{Z}$. □

27.4. **Proof of the Covering Existence Theorem 27.2 I.** We turn to the proof of the Covering Existence Theorem 27.2. Note that we only need to show the existence of such

[234]Note that it follows from the Covering-π_1-Monomorphism Proposition 14.11 that $\pi_1(X, x_0) \cong \Gamma$.

[235]This means that we pick an isomorphism $\pi_1(S^1, 1) \xrightarrow{\cong} \mathbb{Z}$ and then ignore it in the notation.

a covering, the statement about the degree of the covering is an immediate consequence of the Covering-Degree Proposition 14.13.

We start out with a general discussion of the construction of the covering. Only afterwards we will turn to the nitty-gritty of the proof. In particular for the time being let Y be any path-connected topological space, without any other adjectives. Furthermore let $y_0 \in Y$ and let $\Gamma \subset \pi_1(Y, y_0)$ be a subgroup.

To get an idea for the construction, let us first consider the case that Γ is the trivial group. In this case we want to construct a covering $p\colon X \to Y$ such that X is simply connected. Let us assume for a second that we have such an X. We pick $x_0 \in X$ with $p(x_0) = y_0$. Then, somewhat similar to the statement of the Action–π_1-Theorem 15.1, we have a map

$$\begin{array}{rcl} \Phi\colon X & \to & \{\text{path-homotopy classes of paths } [0,1] \to Y \text{ starting in } y_0\} \\ x & \mapsto & p \circ (\text{any path from } x_0 \text{ to } x). \end{array}$$

In fact the proof of the Action–π_1-Theorem 15.1 comes close to showing that the map Φ is a bijection. Hence Φ is in fact a homeomorphism, provided we equip the right-hand side with an appropriate topology.

Now that we are not given X, the idea is, that given (Y, y_0) we can define the right-hand side of the above bijection, and with a suitable topology we will have found our X.

This discussion leads us to the following definition:

Definition. Let Y be a topological space, let $y_0 \in Y$ and let $\Gamma \subset \pi_1(Y, y_0)$ be a subgroup.

(1) We denote by $Y^{[0,1]}$ the set of all continuous maps $[0,1] \to Y$.

(2) We consider the path space

$$P_{y_0} \coloneqq \{\gamma \in Y^{[0,1]} \mid \gamma(0) = y_0\} = \{\text{all paths in } Y \text{ with starting point } y_0\}.$$

(3) For $u, v \in P_{y_0}$ we write

$$u \sim v \quad :\Longleftrightarrow \quad u \text{ and } v \text{ have the same endpoint and } [\ \underbrace{u * \overline{v}}_{\text{loop in } (Y, y_0)}\ ] \in \Gamma.$$

Using the fact that $\Gamma \subset \pi_1(Y, y_0)$ is a subgroup one can show easily that $\sim$ is indeed an equivalence relation on P_{y_0}.

(4) We write $X_{y_0,\Gamma} \coloneqq P_{y_0}/\sim$ and $x_0 \coloneqq [e_{y_0}]$, i.e. x_0 is the equivalence class of the constant path.

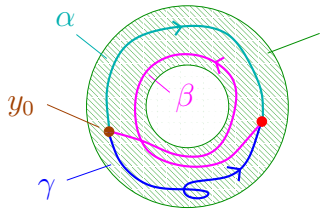
(5) We consider the evaluation map

$$\begin{array}{rcl} p\colon X_{y_0,\Gamma} = P_{y_0}/\sim & \to & Y \\ [v] & \mapsto & v(1). \end{array}$$

Note that this map is well-defined, i.e. it is independent of the choice of the representative v.

It remains to equip $X_{y_0,\Gamma} = P_{y_0}/\sim$ with a sensible topology and to show that the map $p\colon X_{y_0,\Gamma} = P_{y_0}/\sim \to Y$ has all the desired properties.

The following lemma defines such a topology and thus completes the proof of the Covering Existence Theorem 27.2.



Y is an annulus
we consider $\Gamma = 2\mathbb{Z} \subset \pi_1(Y, y_0) = \mathbb{Z}$,
then α and β are equivalent,
but α and γ are inequivalent

Lemma 27.3. (Whisker Topology Lemma) Let Y be a path-connected topological space, let $y_0 \in Y$ and let $\Gamma \subset \pi_1(Y, y_0)$ be a subgroup. We write $P := P_{y_0}$ and we write $X := X_{y_0,\Gamma}$. Given $x = [f] \in X = P/\sim$ and given an open neighborhood V of $p(x) = f(1) \in Y$ we consider

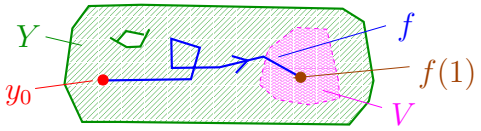
$$U(x, V) := \{[f * u] \mid u \text{ is a path in } V \text{ with starting point } f(1)\} \subset X = P/\sim .$$

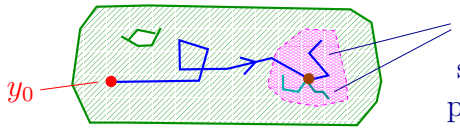
With this notation the following statement holds:

(0) The family of all such subsets $U(x, V)$ of X satisfy the "basis property" that we introduced on page 65.

The topology that is generated on $X = X_{y_0,\Gamma}$ by the sets $U(x, V)$ is called the whisker topology. If Y is locally simply connected, and if we equip $X = X_{y_0,\Gamma}$ with the whisker topology, then the following statements hold:

(1) The above evaluation map $p\colon X \to Y$ is continuous and open.
(2) The map $p\colon X \to Y$ is a covering.
(3) The topological space X is path-connected.
(4) We have $p_*(\pi_1(X, x_0)) = \Gamma$.





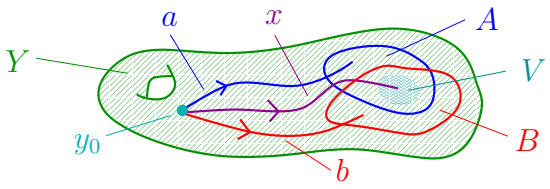
all these paths
starting at y_0 are
points in $U([f], V)$

Proof. Let Y be a path-connected topological space, let $y_0 \in Y$ and let $\Gamma \subset \pi_1(Y, y_0)$ be a subgroup. We write $P := P_{y_0}$ and we write $X := X_{y_0,\Gamma}$. Given $x = [f] \in X = P/\sim$ and given an open neighborhood V of $p(x) = f(1) \in Y$ we consider

$$U(x, V) := \{[f * u] \mid u \text{ is a path in } V \text{ with starting point } f(1)\} \subset X = P/\sim .$$

We denote by $\mathcal{B}$ the family of all such subsets of X.

(0) We need to show that $\mathcal{B}$ has the basis property.
  (B1) For each $x \in X$ we have $x \in U(x, X)$. This shows that $\mathcal{B}$ satisfies (B1).
  (B2) Now let $x = [f]$ be a point in the intersection of $U(a, A)$ and $U(b, B)$. Let V be any open neighborhood V of $f(1)$ with $V \subset A \cap B$, e.g. we could take $V = A \cap B$. It is straightforward to see that $U(x, V) \subset U(a, A) \cap U(b, B)$.



In the following we assume that Y is locally path-connected and that Y is locally simply connected. We equip $X = X_{y_0,\Gamma}$ with the *whisker topology*, i.e. we equip it with the topology generated by $\mathcal{B}$. Since Y is locally path-connected it follows easily that the whisker topology is also generated by

$$\mathcal{C} \coloneqq \{U(x,V) \in \mathcal{B} \,|\, x \in X \text{ and } V \text{ is a } \textit{path-connected} \text{ open neighborhood of } p(x)\}.$$

(1) We need to show that the map $p\colon X = P/\sim \to Y$ is continuous and open. We start with the following observation: For every $x \in X$ and every open path-connected neighborhood V of $p(x)$ we have $p(U(x,V)) = V$.[236]

Now we turn to the actual proof of continuity. So let $A \subset Y$ be an open subset and let $x = [f] \in p^{-1}(A)$. It suffices to show that there exists an open subset B of Y such that $U(x,B) \subset p^{-1}(A)$. Since Y is locally path-connected, there exists an open neighborhood B of x that is contained in A and that is path-connected. By the above observation we have $p(U(x,B)) = B \subset A$, i.e. $U(x,B) \subset p^{-1}(A)$.

Finally we show that p is also open. It suffices to verify that images of the basis $\mathcal{C}$ of the topology of X are open. But this statement we had just verified in the above observation.

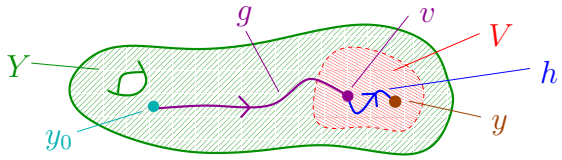
(2) Now we want to prove that $p\colon X = P/\sim \to Y$ is a covering map. First note that it follows from the hypothesis that Y is path-connected that p is surjective. Next let $y \in Y$ be a point. We need to show that there exists a connected open neighborhood V of y such that the restriction of p to each component of $p^{-1}(V)$ is a homeomorphism.

Since Y is locally simply connected, there exists an open neighborhood V of y such that the inclusion-induced map $\pi_1(V,x) \to \pi_1(Y,y)$ is the trivial map. We claim that V already has the desired property.

Claim.
$$p^{-1}(V) = \bigsqcup_{\widetilde{y} \in p^{-1}(y)} U(\widetilde{y}, V).$$

Proof. It is clear that the right-hand side is contained in the left-hand side. Next we want to show that the left-hand side is contained in the right-hand side. So let $[g] \in p^{-1}(V)$. Here g is a path in Y from y_0 to a point $v \in V$. Since V is in particular path-connected there exists a path h in V from v to y. We then have

$$[g] = [g * h * \overline{h}] \in U(\underbrace{[g*h]}_{\in p^{-1}(y)}, V).$$



It remains to show that the sets on the right-hand side of the claim are disjoint. So let $[v'], [v''] \in p^{-1}(y)$ and let $[g] \in U([v'],V) \cap U([v''],V)$. Then there exist paths u' and u''

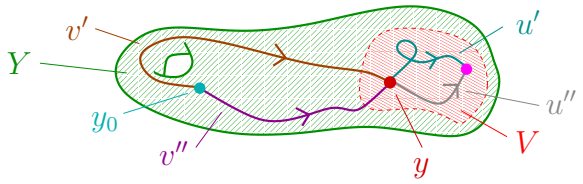
[236] By definition we have $p(U(x,V)) \subset V$. But since V is path-connected we also have $V \subset p(U(x,V))$.

in V with $[v' * u'] = [v'' * u'']$. It follows that

$$[v'] \;=\; [v' * u' * \overline{u}'] \;=\; [v'' * u'' * \overline{u}'] \;\underset{\uparrow}{=}\; [v'']$$

$u'' * \overline{u}'$ is null-homotopic since the map $\pi_1(V, y) \to \pi_1(Y, y)$ is the trivial map

Thus $U([v'], V) \cap U([v''], V)$ intersect only if $[v'] = [v'']$. ⊞



The sets $U(\widetilde{y}, V)$ are by definition open. By the above claim the sets are also disjoint. Thus it suffices to prove the following claim:

Claim. For each $\widetilde{y} \in p^{-1}(y)$ the restriction of the projection $p\colon X = P/\!\sim \to Y$ to the map $p\colon U(\widetilde{y}, V) \to V$ is a homeomorphism.

Proof. So let $\widetilde{y} \in p^{-1}(y)$. As we had just pointed out in (1), the map $p\colon U(\widetilde{y}, V) \to V$ is surjective. From the fact that the inclusion-induced map $\pi_1(V, y) \to \pi_1(Y, y)$ is trivial it also follows immediately that p is injective. Furthermore we had just verified in (1) that p is continuous and open. Therefore $p\colon U(\widetilde{y}, V) \to V$ is a homeomorphism. ⊞

(3) Now we want to show that $X = P/\!\sim$ is path-connected. We start out with the following claim:

Claim. For any path $\alpha\colon [0,1] \to Y$ the following map is continuous:

$$\begin{array}{rcl} \widetilde{\alpha}\colon [0,1] & \to & X = P/\!\sim \\ t & \mapsto & \left[\begin{array}{rcl} [0,1] & \to & Y \\ s & \mapsto & \alpha(t \cdot s) \end{array}\right]. \end{array}$$

Proof. By the Subbasis-Continuity Proposition 2.2 it suffices to show that for each $U(x, V)$ in the basis $\mathcal{C}$ of the topology on X the preimage $\widetilde{\alpha}^{-1}(U(x, V))$ is open in $[0, 1]$. So suppose that we are given $U(x, V)$. We denote by $\{I_j\}_{j \in J}$ the path-components of $\alpha^{-1}(V)$. Since α is continuous each I_j is an open subset of $[0, 1]$. Note that for each $j \in J$ we either have[237] $\widetilde{\alpha}(I_j) \subset U(x, V)$ or $\widetilde{\alpha}(I_j) \cap U(x, V) = \varnothing$. It follows that

$$\widetilde{\alpha}^{-1}(U(x, V)) \;=\; \text{union of all the } I_j \text{ with the property that } \widetilde{\alpha}(I_j) \subset U(x, V).$$

Thus we see that $\widetilde{\alpha}^{-1}(U(x, V))$ is a union of open subsets of $[0, 1]$, hence the preimage itself is open. ⊞

To show that X is path-connected it suffices to show that for every point $x \in X$ there exists a path in X from $x_0 = [e_{y_0}]$ to x. So let $x = [\alpha] \in X$. Then such a path is given by $\widetilde{\alpha}\colon [0, 1] \to X$, where $\widetilde{\alpha}$ is defined as in the claim. By the claim $\widetilde{\alpha}$ is indeed continuous.

[237] This might require a few minutes of thought.

(4) It remains to show that $p_*(\pi_1(X,x_0)) = \Gamma$. Let $g \in \pi_1(Y,y_0)$. We choose a representatives $\alpha\colon [0,1] \to Y$ of g. We consider the following path[238] in $X = P/\sim$

$$\begin{array}{rcl} \widetilde{\alpha}\colon [0,1] & \to & X = P/\sim \\ t & \mapsto & \left[\begin{array}{ccc} [0,1] & \to & Y \\ s & \mapsto & \alpha(t\cdot s) \end{array}\right]. \end{array}$$

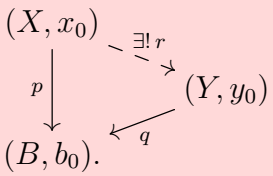
Note that $\widetilde{\alpha}$ is evidently the lift of α to the starting point x_0. Then we have

$$g \in p_*(\pi_1(X,x_0)) \underset{\uparrow}{\Leftrightarrow} \widetilde{\alpha} \text{ is a loop} \Leftrightarrow \text{in } X = P/\sim \text{ we have } \widetilde{\alpha}(1) = x_0 = [e_{y_0}] \underset{\uparrow}{\Leftrightarrow} \alpha \in \Gamma.$$

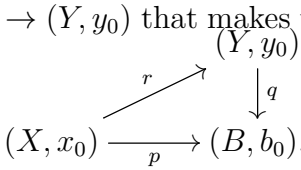
the Loop–Lifting Proposition 14.12 — definition of $\sim$ on P ■

27.5. **Uniqueness of coverings.** Let (B,b_0) be a path connected topological space. We just saw, that under a few mild technical hypothesis, any subgroup $\Gamma \subset \pi_1(B,b_0)$ gives rise to a covering. The following question arises: to what degree is the covering corresponding to the subgroup Γ unique? To answer this question it is helpful to first prove the following universal property of coverings corresponding to a subgroup.

Theorem 27.4. (Covering–Covering Theorem) Let B be a locally path-connected topological space, let $b_0 \in B$ and let $p\colon (X,x_0) \to (B,b_0)$ be a path-connected covering. For every other path-connected covering $q\colon (Y,y_0) \to (B,b_0)$ with $p_*(\pi_1(X,x_0)) \subset q_*(\pi_1(Y,y_0))$ there exists a *unique* covering $r\colon (X,x_0) \to (Y,y_0)$ such that the following diagram commutes:



Proof. Let B be a locally path-connected topological space and let $b_0 \in B$ and let $p\colon (X,x_0) \to (B,b_0)$ and $q\colon (Y,y_0) \to (B,b_0)$ be two path-connected coverings such that $p_*(\pi_1(X,x_0)) \subset q_*(\pi_1(Y,y_0))$. Note that X is also locally path-connected. Thus it follows from the Map Lifting Criterion 27.1 together with the Lift Uniqueness Lemma 11.10 that there exists a unique map $r\colon (X,x_0) \to (Y,y_0)$ that makes the following diagram commute:



It remains to prove that $r\colon X \to Y$ is a covering. Recall that this means that we need to prove the following two statements:

(a) Each $y_1 \in Y$ admits a uniformly covered open neighborhood.
(b) The map $r\colon X \to Y$ is surjective.

We turn to the proofs of these two statements:

(a) Let $y_1 \in Y$. We set $b_1 := q(y_1)$. Since B is locally path-connected and since $p\colon X \to B$ and $q\colon Y \to B$ are coverings there exists a path-connected open neighborhood W of b that is uniformly covered. Since $p\colon X \to B$ and $r\colon Y \to B$ are coverings we can

[238] Note that this map is continuous by the claim in (3).

write $p^{-1}(W) = \bigsqcup_{x\in p^{-1}(b_1)} U_x$ where each $U_x \subset X$ is open and each $p\colon U_x \to W$ is a homeomorphism and we can write $q^{-1}(W) = \bigsqcup_{y\in q^{-1}(b_1)} V_y$ where each $V_y \subset Y$ is open and each $q\colon V_y \to W$ is a homeomorphism.

Claim. Let $x \in p^{-1}(b_1)$ and let $y \in q^{-1}(b_1)$. If $r(U_x) \cap V_y \neq \varnothing$, then $r(x) = y$ and $r(U_x) = V_y$.

Proof. We make the following observations:

(i) Since W is path-connected it is in particular connected. Since U_x is homeomorphic to W we see that U_x is connected.

(ii) Since $p = q \circ r$ we see that $r(U_x) \subset q^{-1}(q(r(U_x)) = q^{-1}(p(U_x)) = q^{-1}(W)$.

(iii) It follows from (i), (ii), the fact $q^{-1}(W) = \bigsqcup_{y\in q^{-1}(b_1)} V_y$ and the fact that the V_y are open that there exists a $y \in q^{-1}(b_1)$ with $r(U_x) \subset V_y$.

(iv) It follows from (iii) together with $p(x) = b_1 = q(y)$, $p^{-1}(b_1) \cap U_x = \{x\}$, $q^{-1}(b_1) \cap V_y = \{y\}$ and $p = q \circ r$ that $r(x) = y$.

(v) Since $p\colon U_x \to W$ and $q\colon V_y \to W$ are homeomorphisms and since $p = q \circ r$ we obtain from (iii) that $r\colon U_x \to V_y$ is a homeomorphism. ⊞

Next note that

$$\bigsqcup_{x\in r^{-1}(y)} \bigsqcup_{y\in q^{-1}(b)} U_x = \bigsqcup_{x\in (q\circ r)^{-1}(b)} U_x = \bigsqcup_{x\in p^{-1}(b)} U_x = p^{-1}(W) = (q\circ r)^{-1}(W)$$
$$= r^{-1}(q^{-1}(W)) = r^{-1}\Big(\bigsqcup_{y\in q^{-1}(b)} V_y\Big) \underset{\uparrow}{=} \bigsqcup_{y\in q^{-1}(b)} r^{-1}(V_y).$$

Image-Preimage Lemma 0.3 (6)

It follows from this equality and the above claim that

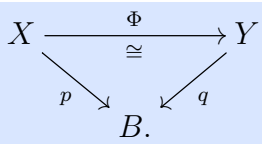
$$\bigsqcup_{x\in r^{-1}(y_1)} U_x = r^{-1}(V_{y_1}).$$

Next let $x \in r^{-1}(y_1)$. We already know that U_x is an open subset of X. Note that the restriction of r to U_x is given by $(q|_{V_y\to W})^{-1} \circ (p\colon U_x \to W)$. This shows that $r\colon U_x \to V_{y_1}$ is a homeomorphism. In summary, we have shown that V_{y_1} is uniformly covered.

(b) We want to show that $r\colon X \to Y$ is surjective. Let $y_1 \in Y$. We just showed in (a) that there exists an open neighborhood V_{y_1} that is uniformly covered with respect to the map $r\colon X \to Y$. But this means that either $V_{y_1} \subset r(X)$ or $V_{y_1} \cap r(X) = \varnothing$. It follows from the JH-Openness Criterion 1.1 that $r(X)$ and $Y \setminus r(X)$ are both open. Since Y is by hypothesis path-connected (and thus connected) and since $X \neq \varnothing$ we see that $r(X) = Y$. ■

Now we want to formulate the promised uniqueness statement for coverings. This statement builds on the following natural definition.

Definition. Let $p\colon X \to B$ and $q\colon Y \to B$ be two coverings of a topological space B. We say that p and q are equivalent, if there exists a homeomorphism $\Phi\colon X \to Y$ such that the following diagram commutes:



Similarly we define the equivalence of coverings of pointed topological spaces.

Example. Given $n \in \mathbb{N}$ the two coverings

$$\begin{array}{rcl} q\colon \mathbb{R}/n\mathbb{Z} & \to & S^1 \\ t + n\mathbb{Z} & \mapsto & \exp(2\pi \mathrm{i} t) \end{array} \qquad \text{and} \qquad \begin{array}{rcl} p\colon S^1 & \to & S^1 \\ z & \mapsto & z^n \end{array}$$

are equivalent. In fact the desired map $\Phi\colon \mathbb{R}/n\mathbb{Z} \to S^1$ is given by $[t] \mapsto \exp(2\pi \mathrm{i} \frac{t}{n})$. □

Now we can formulate the promised uniqueness theorem:

Theorem 27.5. (Covering Unique Existence Theorem) Let B be a path-connected topological space that is locally simply connected. Furthermore let $b_0 \in B$ and let $\Gamma \subset \pi_1(B, b_0)$ be a subgroup. There exists, up to equivalence, a *unique* path-connected covering $p\colon (X, x_0) \to (B, b_0)$ of pointed topological spaces such that $p_*(\pi_1(X, x_0)) = \Gamma$.

Proof. Let $b_0 \in B$ and let $\Gamma \subset \pi_1(B, b_0)$ be a subgroup. The existence of such a covering is just the statement of the Covering Existence Theorem 27.2.

Now suppose that we are given two path-connected coverings $p\colon (X, x_0) \to (B, b_0)$ and $q\colon (Y, Y_0) \to (B, b_0)$ with $p_*(\pi_1(X, x_0)) = q_*(\pi_1(Y, y_0)) = \Gamma$. There are two approaches to showing that p and q are equivalent:

(1) The uniqueness follows immediately from the Covering–Covering Theorem 27.4 and the usual proof of uniqueness for an object satisfying a universal property. We leave it to the reader to fill in the details.

(2) By the Covering–Covering Theorem 27.4 there exists a covering $f\colon (X, x_0) \to (Y, y_0)$ with $p = q \circ f$. Next note that it follows from our hypotheses and the Covering-π_1-Monomorphism Proposition 14.11 that $p_*\colon \pi_1(X, x_0) \to \Gamma$ and $q_*\colon \pi_1(X, x_0) \to \Gamma$ are isomorphisms. Since $p_* = q_* \circ f_*$ we see that f_* is an isomorphism. It follows from the Covering-Degree Proposition 14.13 that $f\colon X \to Y$ is a covering of degree 1. This implies that f is a homeomorphism. ■

27.6. **The universal covering.** In the Covering Unique Existence Theorem 27.5 we saw that subgroups of fundamental groups give rise to coverings. An obvious choice of a subgroup is the trivial group and we immediately obtain the following corollary to the Covering Unique Existence Theorem 27.5.

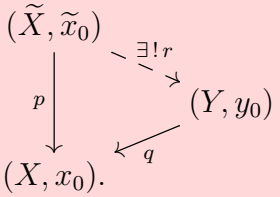
Proposition 27.6. (Universal Covering Proposition) Let X be a topological space that is path-connected and locally simply connected. Furthermore let $x_0 \in X$. The following statements hold:

(1) There exists, up to equivalence, a unique path-connected covering

$$p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$$

such that $\widetilde{X}$ is simply connected.

(2) Let $q\colon (Y, y_0) \to (X, x_0)$ be another path-connected covering. There exists a unique covering $r\colon (\widetilde{X}, \widetilde{x}_0) \to (Y, y_0)$ such that the following diagram commutes:



Proof. Let X be a topological space that is path-connected and locally simply connected. Furthermore let $x_0 \in X$.

(1) Note that if we apply the Covering Unique Existence Theorem 27.5 to the trivial subgroup of $\pi_1(X, x_0)$ and we obtain, up to equivalence, a unique path-connected covering $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ such that $p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))$ is the trivial group. It follows from the Covering-π_1-Monomorphism Proposition 14.11 that $\widetilde{X}$ is simply connected.
(2) Statement (2) is an immediate consequence of the Covering–Covering Theorem 27.4. ■

Definition. Let X be a topological space that is path-connected, locally simply connected[239] and let $x_0 \in X$. We refer to any covering $p\colon (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ such that $\widetilde{X}$ is simply connected as a universal covering[240] of (X, x_0).

Example.

(1) For any $n \in \mathbb{N}$ the map
$$\begin{aligned} p\colon \mathbb{R}^n &\to (S^1)^n \\ (t_1, \dots, t_n) &\mapsto \big(\exp(2\pi \mathrm{i} t_1), \dots, \exp(2\pi \mathrm{i} t_n)\big) \end{aligned}$$
is a universal covering of the n-dimensional torus $(S^1)^n$.
(2) Let $p, q \in \mathbb{N}$ be coprime. The natural projection $S^3 \to S^3/\mathbb{Z}_p = L(p, q)$ is a universal covering of the lens space $L(p, q)$. □

Convention. In practice one often suppresses the actual map p from the language. For example, often one says that "$\mathbb{R}^n$ is a universal covering of $(S^1)^n$".

Example.

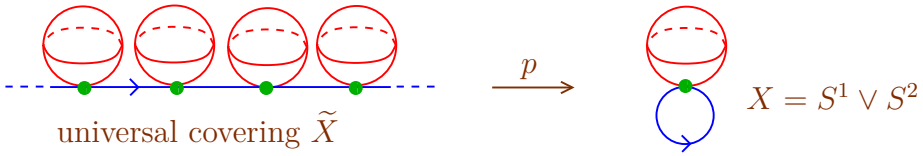
(1) Let $n \geq 2$. It follows from the Projective Space-Sphere Quotient Lemma 6.9 together with the Sphere-π_1-Proposition 11.15 that the universal covering of $\mathbb{R}\mathrm{P}^n$ is given by S^n.
(2) It follows from the Projective Space-Avatar Lemma 6.11 that a universal covering of $\mathrm{SO}(3)$ is given by $\mathrm{SU}(2)$.
(3) By the Klein Bottle-as-Quotient Lemma 14.1 we know that a universal covering of the Klein bottle is given by $\mathbb{R}^2$. □

[239] Recall that most "reasonable" topological spaces (e.g. topological manifolds) are locally simply connected.
[240] The name "universal covering" stems from the fact, proved in the Universal Covering Proposition 27.6 (2), that it has a universal property, namely the universal covering of a topological space X "covers" every other path-connected covering of X.

27.7. **Explicit descriptions of the universal covering.** We have seen in the Universal Covering Proposition 27.6 that every "reasonable" topological space X admits a universal covering $p\colon \widetilde{X} \to X$. In the proof we gave abstract construction. In the previous section we also gave an explicit description of the universal covering of some popular topological spaces, namely the n-dimensional torus $(S^1)^n$, the lens spaces, the projective spaces $\mathbb{R}\mathrm{P}^n$ with $n \geq 2$, $\mathrm{SO}(3)$ and the Klein bottle. In this section we will give explicit descriptions of the universal coverings of more complicated examples, namely of $X = S^1 \vee S^2$ and of $X = S^1 \vee S^1$.

Example. We consider $X = S^1 \vee S^2$ where we identify $\mathrm{i} \in S^1$ with $(0,0,-1) \in S^2$. We also consider the topological space

$$\widetilde{X} \;=\; \big(\mathbb{R} \cup (\mathbb{Z} \times S^2)\big) \,/\, n \sim (n,(0,0,-1)), \quad n \in \mathbb{Z}.$$



It is pretty straightforward to see that the map

$$\begin{array}{rcl} p\colon \big(\mathbb{R} \cup (\mathbb{Z} \times S^2)\big) \,/\, n \sim (n,(0,0,-1)) & \to & S^1 \vee S^2 \\ [P] & \mapsto & \begin{cases} \exp(\mathrm{i}(2\pi P + \frac{\pi}{2})), & \text{if } P \in \mathbb{R} \\ Q, & \text{if } P = (n,Q) \in \mathbb{Z} \times S^2 \end{cases} \end{array}$$

is well-defined and in fact a covering. On page 301 we showed that $\widetilde{X}$ is simply connected. Therefore $p\colon \widetilde{X} \to X$ is a universal covering of X. □

Now we turn to the universal covering of the wedge $X = S^1 \vee S^1$. By the discussion on page 247 we know that X is homeomorphic to the topological realization of the unique abstract graph G that is given by the single vertex $*$ and the edge set $\{x,y\}$.[241] It is convenient to work in the following in the setup of abstract graphs.

Convention. Let V be a set and let $E \subset V \times V$ be a subset. The pair (V,E) defines an abstract graph (V,E,i,t) by setting $i(g,h) = g$ and $t(g,h) = h$. We use this observation to view such (V,E) as an abstract graph.

Proposition 27.7. (Wedge-of-Circles–Universal Covering Proposition) We consider the abstract graph $G = (\{*\},\{x,y\})$. We also consider the abstract graph

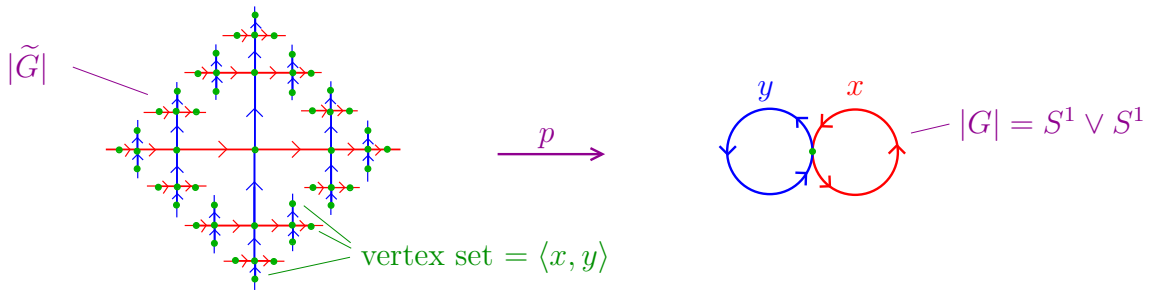
$$\widetilde{G} \;=\; (\underbrace{\widetilde{V} = \text{ free group } \langle x,y\rangle}_{\text{vertex set}}, \underbrace{\widetilde{E} = \{(g,h) \in \widetilde{V} \times \widetilde{V} \,|\, h = g \cdot x \text{ or } h = g \cdot y\}}_{\text{edge set}}).$$

The maps

$$\begin{array}{rcl} \text{vertex set of } \widetilde{G} & \to & \text{vertex set of } G \\ g & \mapsto & * \end{array} \quad \text{and} \quad \begin{array}{rcl} \text{edge set of } \widetilde{G} & \to & \text{edge set of } G \\ (g,h) & \mapsto & g^{-1} \cdot h \end{array}$$

define a morphism of graphs in the sense of the definition on page 147. The corresponding map $p\colon |\widetilde{G}| \to |G|$ is a universal covering of $|G| \cong S^1 \vee S^1$.

[241] Since we have a unique vertex we do not need to specify the maps from the edge set to the vertex set.



Proof. We leave it to the reader to verify that the map $p\colon |\widetilde{G}| \to |G|$ is a covering. It remains to show that $|\widetilde{G}|$ is simply connected. This requires some preparations.

Given $g \in \widetilde{V} = \langle x, y\rangle$ we define its length to be

$$l(g) \;:=\; \min\{k \in \mathbb{N}_0 \,|\, g = a_1 \cdot \ldots \cdot a_k \text{ for some } a_1, \ldots, a_k \in \{x^{\pm 1}, y^{\pm 1}\}\}.$$

Given $k \in \mathbb{N}_0$ we consider the subgraph $\widetilde{G}_k = (\widetilde{V}_k, \widetilde{E}_k)$ of $\widetilde{G} = (\widetilde{V}, \widetilde{E})$ that is given by $\widetilde{V}_k := \{g \in \widetilde{V} = \langle x, y\rangle \,|\, l(g) \leq k\}$ and $\widetilde{E}_k := \widetilde{E} \cap \mathcal{P}(\widetilde{V}_k)$.

Claim. For each $k \in \mathbb{N}_0$ we have $\chi(\widetilde{G}_k) = 1$.[242]

Proof. We prove the claim by induction on $k \in \mathbb{N}_0$. First note that $\widetilde{G}_0$ has a single vertex and no edge. Thus we see that $\chi(\widetilde{G}_0) = 1$. Now suppose that we know that $\chi(\widetilde{G}_k) = 1$. We make the following two observations:

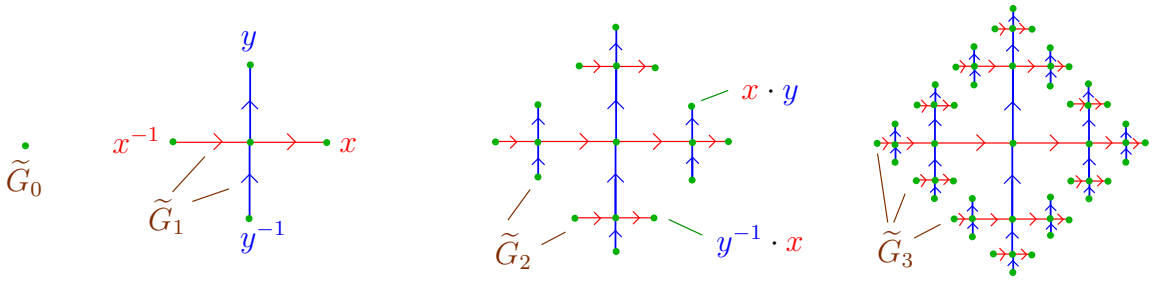
(1) It follows easily from the definition of the free product $\langle x, y\rangle = \langle x\rangle * \langle y\rangle$ as a set of reduced words, see page 267, that each $g \in \widetilde{V}_{k+1} \setminus \widetilde{V}_k$ is uniquely of the form $a_1 \cdot \ldots \cdot a_{k+1}$ with $a_1, \ldots, a_{k+1} \in \{x^{\pm 1}, y^{\pm 1}\}$ and with $g := a_1 \cdot \ldots \cdot a_k \in \widetilde{V}_k$. Note that *precisely one* of $(g, g \cdot a_{k+1})$ or $(g \cdot a_{k+1}, g)$ lies in $\widetilde{E}_{k+1}$.

(2) Let $(g, h) \in \widetilde{E}_{k+1}$. It follows again easily from the definition of $\langle x, y\rangle = \langle x\rangle * \langle y\rangle$ as a set of reduced words that precisely one of g and h lies in $\widetilde{V}_k$ and $\widetilde{V}_{k+1}$.

Thus we see that

$$\chi(\widetilde{G}_{k+1}) \;=\; \chi(\widetilde{V}_k) + \underbrace{\#(\widetilde{V}_{k+1} \setminus \widetilde{V}_k) - \#(\widetilde{E}_{k+1} \setminus \widetilde{E}_k)}_{\substack{= 0, \text{ since by the above two observations} \\ \text{we have a bijection between the} \\ \text{sets } \widetilde{V}_{k+1} \setminus \widetilde{V}_k \text{ and } \widetilde{E}_{k+1} \setminus \widetilde{E}_k}} \;=\; \chi(\widetilde{G}_k) \;=\; 1.$$

⊞



It follows from the claim together with the Tree Characterization Proposition 7.7 that each $\widetilde{G}_k$ is a tree. The Finite Tree–Contractible Proposition 17.4 together with the Homotopy-π_1-Proposition 13.4 now implies that each $|\widetilde{G}_k|$ is simply connected. It follows easily from

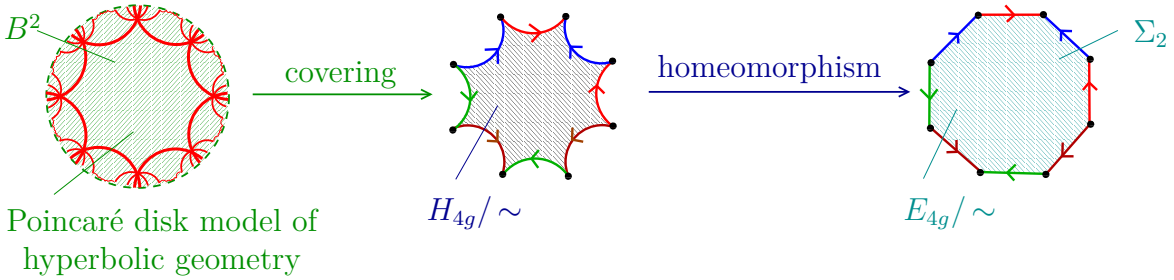
[242] Here χ denotes the Euler characteristic, i.e. the number of vertices minus number of edges.

the Exhaustion–Simply Connected Proposition 20.10 (or alternatively the Exhaustion-π_1-Proposition 26.2) that $|\widetilde{G}| = \bigcup_{k\in\mathbb{N}} |\widetilde{G}_k|$ itself is simply connected. ■

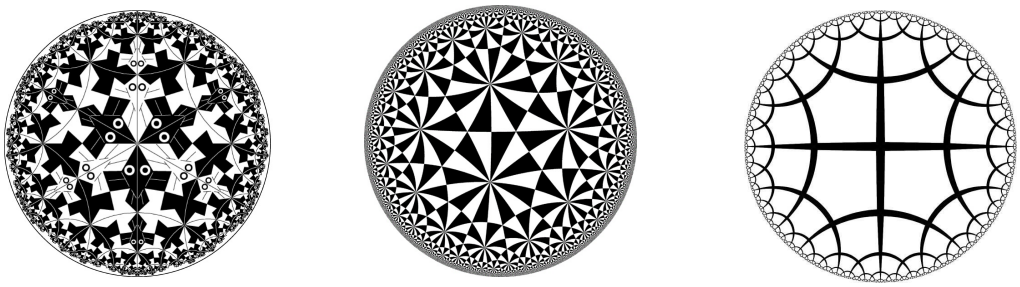
The surfaces of genus $g \geq 2$ are arguably one of the most interesting types of topological manifolds. Evidently we also would like to understand their universal coverings.

Theorem 27.8. (Hadamard's Theorem) Given any $g \geq 2$ there exists a covering map $B^2 \to \Sigma_g = E_{4g}/\sim$.

Proof. The key idea of the proof is to replace the "Euclidean" octagon E_{4g} by a "hyperbolic" octagon H_{4g} and to equip the open disk B^2 with a "hyperbolic structure" via the Poincaré disk model of hyperbolic geometry. The long and difficult details are worked out in [**Fri24**]. ■



Example. We do not have the time and space to explain the Poincaré disk model of hyperbolic geometry. In the figure below we see three different decompositions of $\mathbb{D}$ into subsets (the pictures are of course due to M.C. Escher). In each of the three cases these subsets have the same hyperbolic size, even though they differ in the usual Euclidean sense. □



CHAPTER 28

Higher homotopy groups: Definition and basics

28.1. **Definition of the higher homotopy groups.** We start out with the following variation on earlier definitions.

Definition. Let W and X be topological spaces and let $A \subset W$ and $B \subset X$ be subsets.

(1) A continuous map $(W, A) \to (X, B)$ of pairs of topological spaces is a continuous map $f\colon W \to X$ with $f(A) \subset B$.

(2) Let $f, g\colon (W, A) \to (X, B)$ be two continuous maps of pairs of topological spaces.

(a) A homotopy between f and g is a continuous map

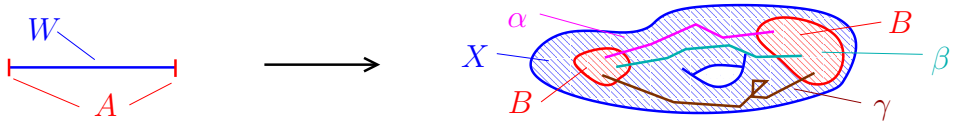
$$H\colon W \times [0,1] \to X,$$

such that $H_0 = f$ and $H_1 = g$ and such that each H_t is a map $(W, A) \to (X, B)$.

(b) If there exists a homotopy, then we say that f and g are homotopic and we write $f \simeq g$.

(3) We set

$$[(W, A), (X, B)] \coloneqq \{\text{continuous maps } (W, A) \to (X, B)\}/\simeq .$$



$\alpha, \beta\colon (W, A) \to (X, B)$ are homotopic but $\alpha, \gamma\colon (W, A) \to (X, B)$ are not homotopic

Notation. Throughout this chapter we write $I = [0, 1]$ for the unit interval. Given $n \in \mathbb{N}$ we denote by $I^n \subset \mathbb{R}^n$ the cube in $\mathbb{R}^n$ and we denote by ∂I^n its boundary as a subspace of $\mathbb{R}^n$.

Let (X, x_0) be a pointed topological space. With the above definitions we can write[243]

$$\pi_1(X, x_0) \;=\; [(I, \partial I), (X, x_0)].$$

This definition can be easily generalized to higher dimensions:

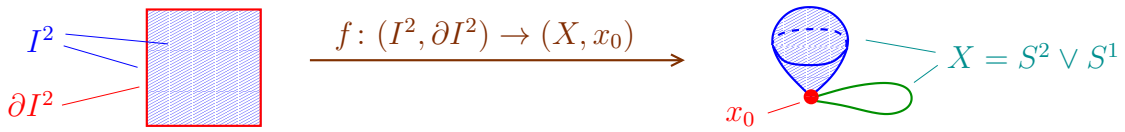
Definition. Given a pointed topological space (X, x_0) and given $n \in \mathbb{N}_0$ we define

$$\begin{aligned} \pi_n(X, x_0) &\coloneqq [(I^n, \partial I^n), (X, x_0)] \\ &= \text{homotopy classes of continuous maps } (I^n, \partial I^n) \to (X, x_0). \end{aligned}$$

For $n \geq 1$ we refer to $\pi_n(X, x_0)$ as the n-th homotopy group. We refer to any $\pi_n(X, x_0)$ with $n \geq 2$ as a higher homotopy group.

Note that $\pi_n(X, x_0)$ is a priori only a set. So to justify the name "homotopy group" we need to define a group structure on $\pi_n(X, x_0)$ which for $n = 1$ agrees with our previous definition of the group structure. Here is what we do:

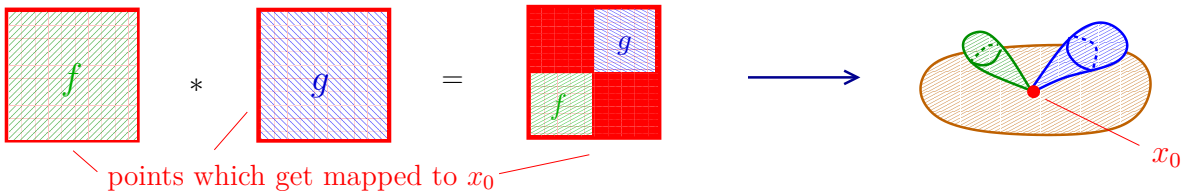
[243] For a point $x_0 \in X$ we write (X, x_0) instead of the more correct $(X, \{x_0\})$.



Definition. Let (X, x_0) be a pointed topological space and let $n \geq 1$. Furthermore let $f, g\colon (I^n, \partial I^n) \to (X, x_0)$ be two continuous maps. We define

$$\begin{array}{rcl} f * g\colon (I^n, \partial I^n) & \to & (X, x_0) \\ (x_1, \ldots, x_n) & \mapsto & \begin{cases} f(2x_1, \ldots, 2x_n), & \text{if } (x_1, \ldots, x_n) \in [0, \frac{1}{2}]^n, \\ g(2x_1 - 1, \ldots, 2x_n - 1), & \text{if } (x_1, \ldots, x_n) \in [\frac{1}{2}, 1]^n, \\ x_0, & \text{otherwise.} \end{cases} \end{array}$$

Using the Pasting Proposition 2.4 (2) one can easily show that this map is indeed continuous.



It is straightforward to see that the homotopy class of $f * g$ only depends on the homotopy classes of f and g. This leads us to the following definition:

Definition. Let (X, x_0) be a pointed topological space and let $n \geq 1$. We define

$$\begin{array}{rcl} \pi_n(X, x_0) \times \pi_n(X, x_0) & \to & \pi_n(X, x_0) \\ ([f], [g]) & \mapsto & [f] \cdot [g] \coloneqq [f * g]. \end{array}$$

Note that for $n = 1$ this is precisely the same definition of the product structure that we already introduced in Chapter 11.2.

Now we can formulate the following proposition:

Proposition 28.1. (π_n-Group Proposition) Let (X, x_0) be a pointed topological space and let $n \geq 1$. The following two statements hold:

(1) The map

$$\begin{array}{rcl} \pi_n(X, x_0) \times \pi_n(X, x_0) & \to & \pi_n(X, x_0) \\ ([f], [g]) & \mapsto & [f] \cdot [g] \coloneqq [f * g] \end{array}$$

defines a group structure on $\pi_n(X, x_0)$.

(2) Given a continuous map $f\colon (I^n, \partial I^n) \to (X, x_0)$ the inverse of $[f] \in \pi_n(X, x_0)$ is represented by the map

$$\begin{array}{rcl} \overline{f}\colon I^n & \to & X \\ (x_1, x_2, \ldots, x_n) & \mapsto & f(1 - x_1, x_2, \ldots, x_n). \end{array}$$

Example. There is one non-empty topological space for which we can immediately compute the higher homotopy groups, namely the topological space consisting of a single point P. In this case the only map $I^n \to \{P\}$ is the constant map, hence we see that $\pi_n(\{P\}, P) = 0$ for any $n \geq 1$. □

The proof of the π_n-Group Proposition 28.1 requires the remainder of this section. As we will see, the proof of the π_n-Group Proposition 28.1 is quite similar to the proof of the Path-Homotopy Product Proposition 11.4 and the proof of the π_1–Group Proposition 11.5.

The technicalities are evidently more difficult than before. We will provide (most) of the gory details. But most readers will be best served by just glimpsing at the pictures in the proof and by ignoring the technical details.

After this preamble let us turn to the actual proof of the π_n-Group Proposition 28.1. To deal with the technicalities we need a few more definitions and one lemma.

Definition.

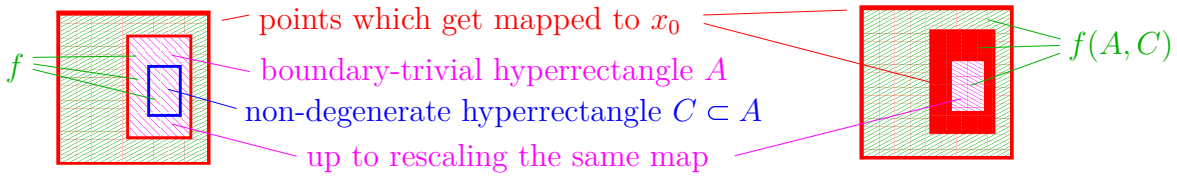
(1) A hyperrectangle is any subset of $\mathbb{R}^n$ of the form $[c_1, d_1] \times \cdots \times [c_n, d_n]$.
(2) We say that a hyperrectangle $[c_1, d_1] \times \cdots \times [c_n, d_n]$ is non-degenerate if for all $i = 1, \ldots, n$ we have $d_i > c_i$.

Let X be a topological space, let $x_0 \in X$ and let $f\colon I^n \to X$ be a continuous map.

(3) We say that a hyperrectangle $A \subset I^n$ is boundary-trivial if f sends all points on the boundary ∂A of A to x_0.
(4) Let $A = [a_1, b_1] \times \cdots \times [a_n, b_n] \subset I^n$ be a boundary-trivial hyperrectangle. Furthermore let $C = [c_1, d_1] \times \cdots \times [c_n, d_n]$ be a non-degenerate hyperrectangle that is contained in A. We define

$$\begin{array}{rcl} f(A,C)\colon I^n & \to & X \\ x = (x_1, \ldots, x_n) & \mapsto & \begin{cases} f(x), & \text{if } x \notin A, \\ x_0, & \text{if } x \in A \setminus C, \\ f\left(a_1 + (x_1 - c_1) \cdot \frac{b_1 - a_1}{d_1 - c_1}, \cdots, a_n + (x_n - c_n) \cdot \frac{b_n - a_n}{d_n - c_n}\right) & \text{if } x \in C. \end{cases} \end{array}$$

Using the Pasting Proposition 2.4 (2) one can verify that this map is continuous.



Lemma 28.2. (Shrinking Lemma) Let (X, x_0) be a pointed topological space. Furthermore let $f\colon (I^n, \partial I^n) \to (X, x_0)$ be a continuous map. Furthermore let $A \subset I^n$ be a boundary-trivial hyperrectangle and let C be a non-degenerate hyperrectangle that is contained in A. Then the maps f and $f(A, C)$ from $(I^n, \partial I^n)$ to (X, x_0) are homotopic.

Proof. Let $f\colon (I^n, \partial I^n) \to (X, x_0)$ be a continuous map, let $A = [a_1, b_1] \times \cdots \times [a_n, b_n] \subset I^n$ be a boundary-trivial hyperrectangle and let $C = [c_1, d_1] \times \cdots \times [c_n, d_n]$ be a non-degenerate hyperrectangle that is contained in A. For $t \in [0, 1]$ we write

$$C_t = [a_1 \cdot (1-t) + c_1 \cdot t, b_1(1-t) + d_1 \cdot t] \times \cdots \times [a_n \cdot (1-t) + c_n \cdot t, b_n \cdot (1-t) + d_n \cdot t].$$

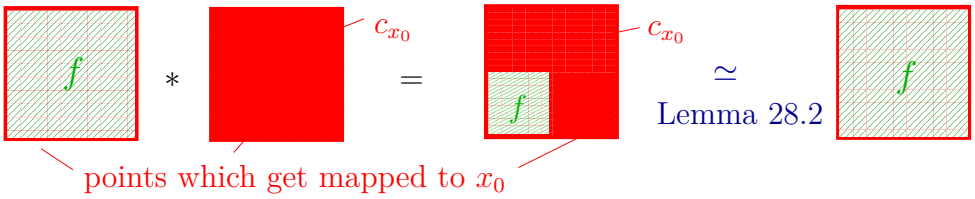
Note that $C_0 = A$ and $C_1 = C$. Using the Pasting Proposition 2.4 (2) one can show fairly easily that the map

$$\begin{array}{rcl} F\colon I^n \times [0,1] & \to & X \\ (x, t) & \mapsto & f(A, C_t)(x) \end{array}$$

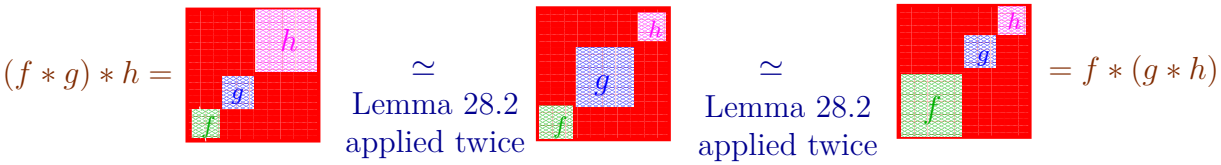
is continuous. By construction we have $F_0 = F$ and $F_1 = f(A, C)$, thus F is a homotopy between f and $f(A, C)$. ∎

Proof of the π_n-Group Proposition 28.1.

- The neutral element of the product structure is given by the equivalence class of the constant map
$$\begin{array}{rcl} c_{x_0}\colon I^n & \to & X \\ x & \mapsto & x_0. \end{array}$$
Indeed, it follows immediately from the Shrinking Lemma 28.2, applied to $A = [0,1]^n$ and $C = [0, \frac{1}{2}]^n$, that given any continuous map $f\colon (I^n, \partial I^n) \to (X, x_0)$ we have $f * c_{x_0} \simeq f$. Basically the same argument shows also that $c_{x_0} * f \simeq f$.



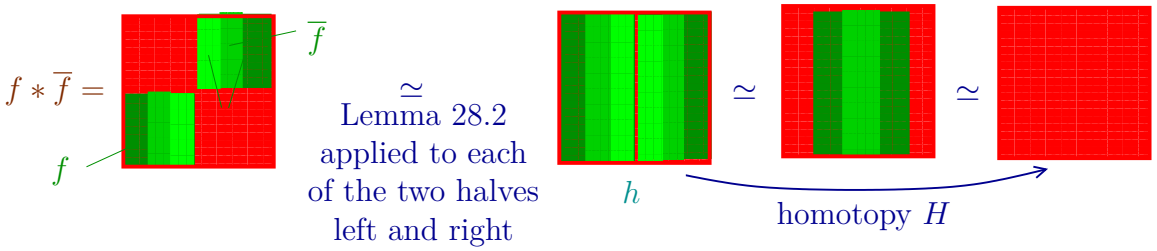
- We consider three maps $f, g, h\colon (I^n, \partial I^n) \to (X, x_0)$. By applying the Shrinking Lemma 28.2 several times one can easily show that $(f * g) * h \simeq f * (g * h)$, see e.g. the figure below. This proves that the multiplication on $\pi_n(X, x_0)$ is associative.



- Now we turn to the most interesting question, namely finding the inverse of a given element in $\pi_n(X, x_0)$. Thus suppose that we are given a continuous map $f\colon (I^n, \partial I^n) \to (X, x_0)$. We consider the map
$$\begin{array}{rcl} \overline{f}\colon I^n & \to & X \\ (x_1, \dots, x_n) & \mapsto & f(1 - x_1, x_2, \dots, x_n). \end{array}$$
We claim that $[f] \cdot [\overline{f}] = [f * \overline{f}] = [c_{x_0}]$. Thus we need to show that $f * \overline{f}$ is homotopic to c_{x_0}. It follows from the Shrinking Lemma 28.2 that $f * \overline{f}$ is homotopic to the map
$$\begin{array}{rcl} h\colon (I^n, \partial I^n) & \to & (X, x_0) \\ (x_1, \dots, x_n) & \mapsto & \begin{cases} f(2x_1, x_2, \dots, x_n), & \text{if } x_1 \in [0, \frac{1}{2}], \\ \underbrace{\overline{f}(2x_1 - 1, x_2, \dots, x_n)}_{= f(2 - 2x_1, x_2, \dots, x_n)}, & \text{if } x_1 \in [\frac{1}{2}, 1]. \end{cases} \end{array}$$
Thus it suffices to show that h is homotopic to c_{x_0}. It is in fact straightforward to see that the map[244] [245]
$$\begin{array}{rcl} H\colon I^n \times [0,1] & \to & X \\ (x_1, \dots, x_n, t) & \mapsto & \begin{cases} f(2(x_1 - \frac{t}{2}), x_2, \dots, x_n), & \text{if } x_1 \in [\frac{t}{2}, \frac{1}{2}], \\ f(2 - 2(x_1 + \frac{t}{2}), x_2, \dots, x_n), & \text{if } x_1 \in [\frac{1}{2}, 1 - \frac{t}{2}], \\ x_0, & \text{otherwise} \end{cases} \end{array}$$

[244] It follows from the Pasting Proposition 2.4 (2) that this map is continuous.
[245] Even for $n = 1$ this homotopy is different from the homotopy used in the Path-Homotopy Product Proposition 11.4 (3). Why is that?

is a homotopy from h to c_{x_0}. We refer to the figure below for an illustration of the argument.



Basically the same argument also shows that $[\overline{f}] \cdot [f] = [\overline{f} * f] = [c_{x_0}]$. ■

28.2. **Alternative descriptions of homotopy groups.** In Section 11.6 we showed that there are two different but equivalent ways to define fundamental groups, namely we can either consider the set $[(I, \partial I), (X, x_0)]$ or we can consider the set $[(S^1, 1), (X, x_0)]$. Similarly, in this section we will see that there are several alternative points of view regarding elements in homotopy groups.

Identification. Let $n \in \mathbb{N}$.

(1) We consider the homeomorphism

$$\begin{array}{rcl} \Phi\colon (I^n, \partial I^n) & \to & (\overline{B}^n, S^{n-1}) \\ x & \mapsto & \begin{cases} \frac{x}{\|x\|} \cdot \max\{|x_1|, \ldots, |x_n|\}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases} \end{array}$$

(2) We equip S^n throughout with the base point $* = (0, \ldots, 0, 1) \in S^n$.[246]

(3) We denote by $\Psi\colon (\overline{B}^n / S^{n-1}, [S^{n-1}]) \to (S^n, *)$ the explicit homeomorphism from the Ball Quotient-Sphere Lemma 6.5.

(4) Let X be a topological space and let $x_0 \in X$. It follows easily from the Topological-Quotient Proposition 6.1 (1) together with the Homotopy Quotient Theorem 8.5 that the above homeomorphisms Φ and Ψ define bijections

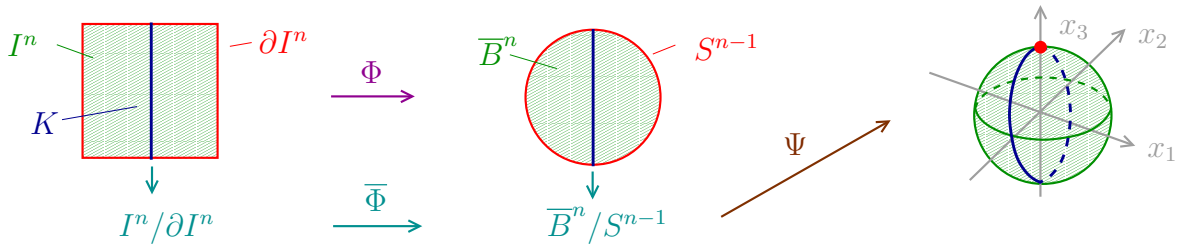
$$\begin{array}{rl} & \text{(homotopy classes) of continuous maps } (I^n, \partial I^n) \to (X, x_0) \\ \iff & \text{(homotopy classes) of continuous maps } (\overline{B}^n, S^{n-1}) \to (X, x_0) \\ \iff & \text{(homotopy classes) of continuous maps } (\overline{B}^n / S^{n-1}, [S^{n-1}]) \to (X, x_0) \\ \iff & \text{(homotopy classes) of continuous maps } (S^n, *) \to (X, x_0). \end{array}$$

The four equivalences combined say that we can view elements in $\pi_n(X, x_0)$ as homotopy classes of continuous maps $(S^n, *) \to (X, x_0)$. This point of view is perhaps closest to our visual approach to thinking about topological spaces. But as we have seen above, for technical purposes it is often much easier to work with the maps $(I^n, \partial I^n) \to (X, x_0)$.

[246] Here is a slight nuisance: For S^1, viewed as a subspace of $\mathbb{C}$, it seems most reasonable to pick the base point $1 = (1, 0) \in \mathbb{C} =_{\text{i}} \mathbb{R}^2$. On the other hand for $S^n \subset \mathbb{R}^{n+1}$ it seems most reasonable to pick the North Pole $(0, \ldots, 0, 1)$ as a base point. Of course mathematically there is no difference and we are fully confident that the reader can handle the fact that for S^1 we use at times two different base points.

Remark. It is visually most natural to work with the last point of view, namely to think of elements in $\pi_n(X, x_0)$ as homotopy classes of continuous maps $(S^n, *) \to (X, x_0)$. Unfortunately with this definition it is a trifle less clear how the group structure on $\pi_n(X, x_0)$ is defined. To visualize the group structure we need to work a little:

(1) Note that the homeomorphism $\Phi\colon (I^n, \partial I^n) \to (\overline{B}^n, S^{n-1})$ descends to a homeomorphism $\overline{\Phi}\colon I/\partial I^n \to \overline{B}^n/S^{n-1}$.
(2) We consider the homeomorphism $\Theta := \overline{\Phi}^{-1} \circ \Psi^{-1}\colon S^n \to I^n/\partial I^n$. Note that it identifies the North Pole $(0, \dots, 0, 1)$ with the point $[\partial I^n]$.
(3) We write $K := \{(x_1, \dots, x_n) \in I^n \mid x_1 = \frac{1}{2}\}$.



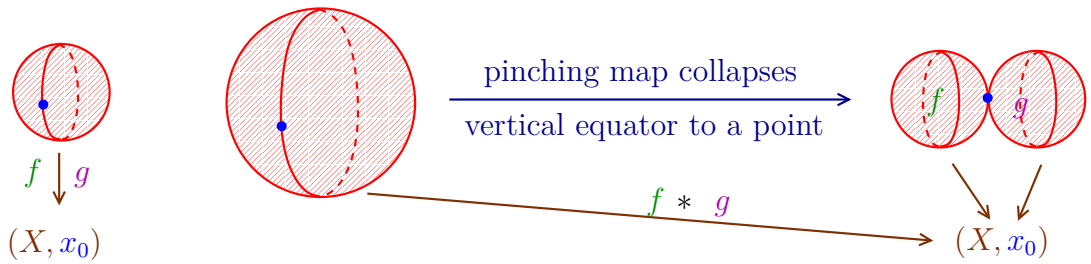
(4) We denote by $S^n_1 \vee S^n_2$ the wedge of two copies of S^n along the point $* \in S^n$.
(5) We refer to the map[247]

$$\begin{aligned} S^n \xrightarrow{\Theta} I^n/\partial I^n \;&\to\; I^n/(\partial I^n \cup K) \;\to\; S^n_1 \vee S^n_2 \\ [x_1, \dots, x_n] \;&\mapsto\; \begin{cases} \Theta^{-1}([2x_1, x_2, \dots, x_n]) \in S^n_1, & \text{if } x_1 \in [0, \frac{1}{2}], \\ \Theta^{-1}([2x_1 - 1, x_2, \dots, x_n]) \in S^n_2, & \text{if } x_1 \in [\frac{1}{2}, 1] \end{cases} \end{aligned}$$

as the *pinching map*.
(6) It follows basically immediately from the definitions that the group structure on $\pi_n(X, x_0)$ is given by

$$\begin{aligned} \pi_n(X, x_0) \times \pi_n(X, x_0) \;&\to\; \pi_n(X, x_0) \\ ([f\colon S^n \to X], [g\colon S^n \to X]) \;&\mapsto\; \big[S^n \to S^n_1 \vee S^n_2 \xrightarrow{f \vee g} X\big]. \end{aligned}$$

In the figure below we sketch the pinching map and we sketch the corresponding definition of the group structure on $\pi_n(X, x_0)$.[248]

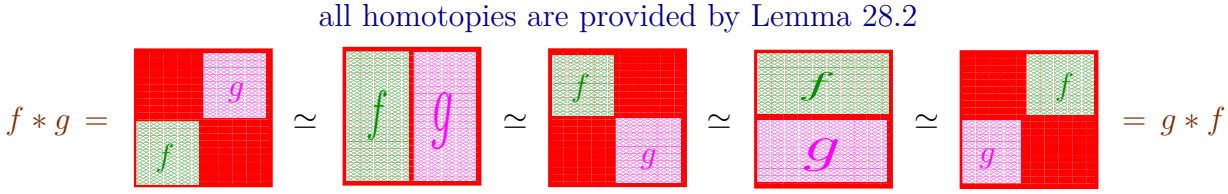


[247] It follows easily from the Pasting Proposition 2.4 (2′), the Topological-Quotient Proposition 6.1 (1b) and the Wedge Proposition 17.1 (1a) that this map is continuous.
[248] The meticulous reader will notice that it is perhaps not completely clear how the maps f and g are defined on the two smaller spheres.

28.3. **Properties and basic calculations of higher homotopy groups.** Now we want to understand some of the basic properties of higher homotopy groups. The higher homotopy groups $\pi_n(X, x_0)$ seem even more complicated than the fundamental group. Surprisingly at least their algebraic structure is much simpler namely, with a great sight of relief, we will now see that higher homotopy groups are abelian.

Proposition 28.3. (π_n-Abelian Proposition) For any pointed topological space (X, x_0) and any $n \geq 2$ the homotopy group $\pi_n(X, x_0)$ is abelian.

Proof. Suppose we are given two continuous maps $f, g\colon (I^n, \partial I^n) \to (X, x_0)$. We have to show that $f * g$ is homotopic to $g * f$. The homotopy is described in the figure below for $n = 2$.



It is straightforward to write down an explicit homotopy between $f * g$ and $g * f$ by composing four instances of the homotopy provided by the Shrinking Lemma 28.2. We leave the details to the reader. ■

The following proposition can be viewed as a generalization of the key statements of the Change-of-Base Point Proposition 11.6.

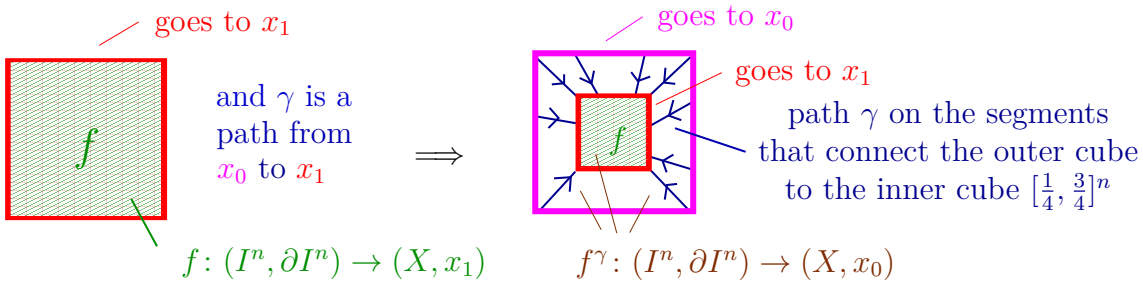
Proposition 28.4. (Change-of-Base Point Proposition) Let X be a topological space, let x_0 and x_1 be two points in X and let $\gamma\colon [0,1] \to X$ be a path from x_0 to x_1. Given a continuous map $f\colon (I^n, \partial I^n) \to (X, x_1)$ we consider the map[249]

$$\begin{array}{rcl} f^\gamma\colon (I^n, \partial I^n) & \to & (X, x_0) \\ (t_1, \dots, t_n) & \mapsto & \begin{cases} f\big(2t_1 - \frac{1}{2}, \dots, 2t_n - \frac{1}{2}\big), & \text{if } (t_1, \dots, t_n) \in \big[\frac{1}{4}, \frac{3}{4}\big]^n, \\ \gamma(2-4t), & \text{if } t := \max\{|t_1 - \frac{1}{2}|, \dots, |t_n - \frac{1}{2}|\} \in [\frac{1}{4}, \frac{1}{2}]. \end{cases} \end{array}$$

The map

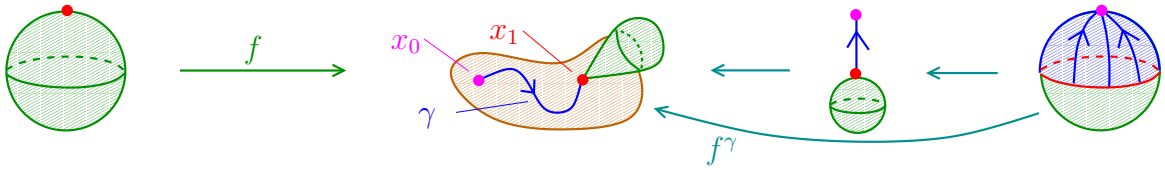
$$\begin{array}{rcl} \gamma_*\colon \pi_n(X, x_1) & \to & \pi_n(X, x_0) \\ [f] & \mapsto & [f^\gamma] \end{array}$$

is well-defined and it is a group isomorphism.



[249] Using the Pasting Proposition 2.4 (2) one can easily verify that the map is continuous.

Remark. In the figure below we show the induced map $\gamma_*\colon \pi_n(X,x_1)\to\pi_n(X,x_0)$ that is given by $f\mapsto f^\gamma$, if in the definition of homotopy groups we work with $(S^n,*)$ instead of $(I^n,\partial I^n)$.



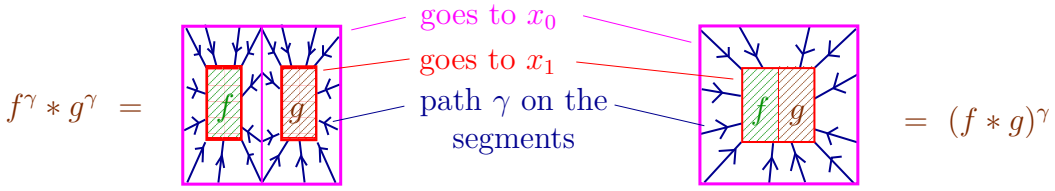
Proof of the Change-of-Base Point Proposition 28.4. Let X be a topological space, let x_0 and x_1 be two points in X and let $\gamma\colon[0,1]\to X$ be a path from x_0 to x_1. We need to prove the following claim:

Claim.

(1) The map $\Phi\colon\pi_n(X,x_1)\to\pi_n(X,x_0)$ given by $[f]\mapsto[f^\gamma]$ is well-defined.
(2) The map Φ is a bijection.
(3) The map Φ is a group homomorphism.

Proof.

(1) This statement can be verified easily by hand.
(2) As usual we denote by $\overline{\gamma}$ the path given by $\overline{\gamma}(t)=\gamma(1-t)$. It is straightforward to verify that the map $\Psi\colon\pi_n(X,x_0)\to\pi_n(X,x_1)$ given by $[g]\mapsto[f^{\overline{\gamma}}]$ is an inverse to Φ.
(3) Let $f,g\colon(I^n,\partial I^n)\to(X,x_0)$ be continuous maps. We need to show that $f^\gamma * g^\gamma$ is homotopic to $(f*g)^\gamma$.



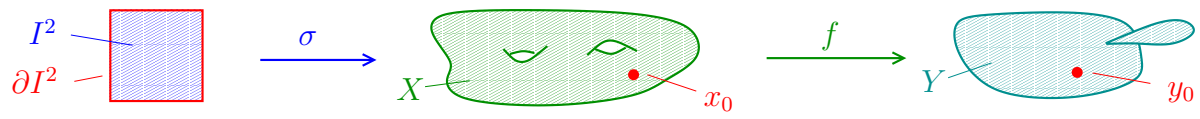
It should be clear that verifying this statement is elementary, in the sense that one can write down an explicit homotopy. On the other hand it should also be clear that writing down an explicit homotopy is supremely painful and highly non-instructive. Thus we will skip this step and leave it to the eager reader to fill in the details. ■

28.4. **Functoriality of homotopy groups.** Next we will see that higher homotopy groups are covariantly functorial.

Definition. Let $f\colon X\to Y$ be a continuous map between two topological spaces, let $n\in\mathbb{N}$ and let $x_0\in X$. We write $y_0=f(x_0)$. One can easily verify that the map

$$\begin{array}{rcl} f_*\colon \pi_n(X,x_0) & \to & \pi_n(Y,y_0) \\ [\sigma\colon(I^n,\partial I^n)\to(X,x_0)] & \mapsto & [f\circ\sigma\colon(I^n,\partial I^n)\xrightarrow{\sigma}(X,x_0)\xrightarrow{f}(Y,y_0)] \end{array}$$

is well-defined. As usual we say that f_* is the map induced by f.



The next proposition is a straightforward generalization of the π_1-Functor Proposition 13.1.

Proposition 28.5. (π_n-Functor Proposition) Let $n \geq 1$. As usual we denote by $\mathcal{PT}op$ the category of pointed topological spaces and that we denote by $\mathcal{G}roup$ the category $\mathcal{G}roup$ of groups. The map[250]

$$\begin{array}{rcl} \mathrm{Ob}(\mathcal{PT}op) & \to & \mathrm{Ob}(\mathcal{G}roup) \\ (X, x_0) & \mapsto & \pi_n(X, x_0) \end{array}$$

and the maps

$$\begin{array}{rcl} \mathrm{Mor}(\mathcal{PT}op) & \to & \mathrm{Mor}(\mathcal{G}roup) \\ (f\colon (X, x_0) \to (Y, y_0)) & \mapsto & (f_*\colon \pi_n(X, x_0) \to \pi_n(Y, y_0)) \end{array}$$

form a covariant functor.

Proof. First note that it follows immediately from the definitions that given a continuous map $f\colon (X, x_0) \to (Y, y_0)$ the induced map $f_*\colon \pi_n(X, x_0) \to \pi_n(Y, y_0)$ is actually a group homomorphism. Furthermore note that it follows immediately from the definitions that

$$\begin{array}{rcll} \mathrm{id}_* & = & \mathrm{id} & \text{for all pointed pairs } (X, x_0) \\ (g \circ f)_* & = & g_* \circ f_* & \text{for all continuous maps } f\colon (X, x_0) \to (Y, y_0) \text{ and } g\colon (Y, y_0) \to (Z, z_0). \end{array}$$

But this is exactly what we needed to show. ■

We continue with the following proposition, which generalizes the essentials of the Homotopy-π_1-Proposition 13.4.

Proposition 28.6. (Homotopy-π_n-Proposition) Let $n \geq 1$.

(1) Let $f, g\colon X \to Y$ be two continuous maps between topological spaces and let $x_0 \in X$. If f and g are homotopic, then

$$\begin{array}{c} f_*\colon \pi_n(X, x_0) \to \pi_n(Y, f(x_0)) \\ \text{is an isomorphism} \end{array} \iff \begin{array}{c} g_*\colon \pi_n(X, x_0) \to \pi_n(Y, g(x_0)) \\ \text{is an isomorphism.} \end{array}$$

The analogous statement holds if we replace "isomorphism" by "epimorphism" or "monomorphism".

(2) (a) Let $f\colon X \to Y$ be a continuous map between topological spaces and let $x_0 \in X$ be a point. If f is a homotopy equivalence, then

$$f_*\colon \pi_n(X, x_0) \to \pi_n(Y, f(x_0))$$

is an isomorphism.

(b) Let X be a topological space and let $A \subset X$ be a subspace. We denote by $i\colon A \to X$ the inclusion. If A is a deformation retract of X, then for any $a_0 \in A$ the induced map

$$i_*\colon \pi_n(A, a_0) \to \pi_n(X, a_0)$$

is an isomorphism.

(3) If X is a contractible topological space, then for any $x_0 \in X$ we have $\pi_n(X, x_0) = 0$.

Proof. The proof of Statement (1) is quite similar to the proof of Statement (1) of the Homotopy-π_1-Proposition 13.4. The only problem is that the notation gets a little more messy. As in the Homotopy-π_1-Proposition 13.4 one can easily deduce Statements (2) and (3) from Statement (1). ■

[250]It follows from the π_n-Abelian Proposition 28.3 that for $n \geq 2$ this is also a covariant functor to the category $\mathcal{A}b\mathcal{G}roup$ of abelian groups.

Example. The topological space $\mathbb{R}^k$ is homotopy equivalent to a point $\{P\}$. Thus it follows from the Homotopy-π_n-Proposition 28.6 (2) and the discussion on page 378 that $\pi_n(\mathbb{R}^k) \cong \pi_n(\{P\}) \cong 0$. Exactly the same way, using the discussion on page 171, one can show that all the higher homotopy groups of convex subsets of $\mathbb{R}^k$ vanish, e.g. we have $\pi_n\big(\overline{B}^k\big) = \pi_n(B^k) = 0$. □

The next proposition is the high-dimensional analogue of the Product-π_1-Proposition 13.7.

Proposition 28.7. (Product-π_n-Proposition) Let $n \in \mathbb{N}$. Let $(X_i, x_i)_{i\in I}$ be a family of pointed topological spaces. For each $j \in I$ we denote by $p_j \colon \prod_{i\in I} X_i \to X_j$ the natural projection. The induced homomorphism[251]

$$\begin{array}{rcl} \Phi \colon \pi_n\Big(\prod_{i\in I} X_i, (x_i)_{i\in I}\Big) & \to & \prod_{i\in I} \pi_n(X_i, x_i) \\ [f] & \mapsto & (p_{i*}([f]))_{i\in I} \end{array}$$

is an isomorphism.

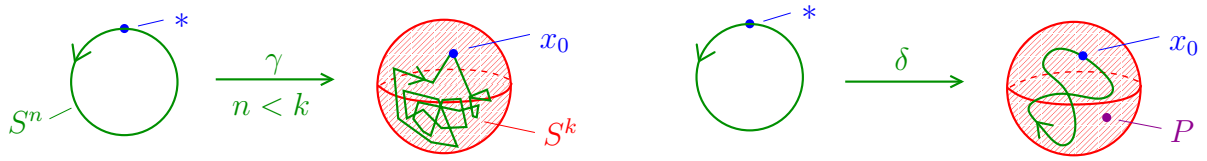
Proof. The proof is basically identical to the proof of the Product-π_1-Proposition 13.7. ■

In the Sphere-π_1-Proposition 11.15 we saw that $\pi_1(S^n) = 0$ for $n \geq 2$. The following proposition generalizes this results to suitable higher dimensions.

Proposition 28.8. (Sphere–Low-π_n-Proposition) Let $k \in \mathbb{N}$. For any $n < k$ we have

$$\pi_n(S^k) \;=\; 0.$$

Proof. Let $k \geq 2$, let $x_0 \in S^k$ and let $\gamma \colon (S^n, *) \to (S^k, x_0)$ be a continuous map. It follows immediately from the Whitney Approximation Theorem [**Fri24**], applied to the closed subset $\{*\} \subset S^n$, that there exists a homotopy rel $\{*\}$ from $\gamma \colon S^n \to S^k$ to a smooth map $\delta \colon S^n \to S^k$ with $\delta(*) = x_0$. Since δ is smooth and since $k > n$ we see that the image of δ has measure zero, which implies that the smooth map $\delta \colon S^n \to S^k$ is not surjective. We pick $P \in S^n \setminus \delta(S^n)$.



Note that $S^k \setminus \{P\}$ is homeomorphic to $\mathbb{R}^k$, in particular contractible, which implies that $[\delta] = 0 \in \pi_n(S^k \setminus \{P\}, *)$, which implies that $[\delta] = 0 \in \pi_n(S^k, *)$. ■

The following lemma is the obvious extension of the Exhaustion–Simply Connected Proposition 20.10.

Proposition 28.9. (Exhaustion-π_n-Zero-Proposition) Let $n \in \mathbb{N}$, let X be a path-connected topological space and let

$$X_0 \;\subset\; X_1 \;\subset\; X_2 \;\subset\; X_3 \;\subset\; \ldots$$

[251] The projections p_i induce homomorphisms on fundamental groups which in turn, by the Product-of-Sets-and-Groups Lemma 4.2, induce the given homomorphism.

be a sequence of subspaces of X such that the following two conditions are satisfied:
(1) We have $\bigcup_{i\in\mathbb{N}_0} X_i = X$.
(2) For each $i \in \mathbb{N}_0$ we have $X_i \subset \mathring{X}_{i+1}$.
(3) For each $i \in \mathbb{N}_0$ the following holds: X_i is path-connected and we have $\pi_n(X_i) = 0$.
Then for each $x_0 \in X$ we have $\pi_n(X, x_0) = 0$.

Proof. Let $x_0 \in X$ and let $f\colon (I^n, \partial I^n) \to (X, x_0)$ be a continuous map. By the Compact Image Lemma 2.8 we know that $f([0,1]^n)$ is a compact subset of X. By the Compact Subspace–Exhaustion Lemma 20.11 we know that $f([0,1]^n)$ is contained in a single X_i. But then it follows from Hypothesis (1) that $[f] = 0 \in \pi_n(X_i, x_0)$, which implies that $[f] = 0 \in \pi_n(X, x_0)$. ∎

28.5. **Covering spaces and higher homotopy groups.** Let $p\colon (X, x_0) \to (B, b_0)$ be a covering of pointed topological spaces. In the Covering-π_1-Monomorphism Proposition 14.11 we saw that the induced map $p_*\colon \pi_1(X, x_0) \to \pi_1(B, b_0)$ is a monomorphism. We saw in many examples that the map is in general not an isomorphism. Perhaps somewhat surprisingly, for the homotopy groups π_n with $n \geq 2$ the following proposition, which was first proved by Witold Hurewicz [**Hur35a**, Satz II], shows that the induced map is in fact always an isomorphism:

Proposition 28.10. (Covering-π_n-Isomorphism Proposition) Let $p\colon (X, x_0) \to (B, b_0)$ be a covering of pointed topological spaces. Given any $n \geq 2$ the following two statements hold:
(1) The map $p_*\colon \pi_n(X, x_0) \to \pi_n(B, b_0)$ is an isomorphism.
(2) The map[252] [253]
$$\begin{array}{rcl} \pi_n(B, b_0) & \to & \pi_n(X, x_0) \\ [f\colon (S^n, *) \to (B, b_0)] & \mapsto & [\text{unique lift of } f \text{ to a map } (S^n, *) \to (X, x_0)] \end{array}$$
is well-defined and it is the inverse to p_*.

Before we provide the proof of the Covering-π_n-Isomorphism Proposition 28.10 let us first give a few examples. We will consider a few more examples after the proof of the Covering-π_n-Isomorphism Proposition 28.10.

Examples.
(1) For $n \geq 2$ we have
$$\pi_n((S^1)^k) \quad \cong \quad \pi_n(\mathbb{R}^k/\mathbb{Z}^k) \quad \xleftarrow[\uparrow]{\cong} \quad \pi_n(\mathbb{R}^k) \quad \underset{\uparrow}{=} \quad 0.$$
Covering-π_n-Isomorphism Proposition 28.10 page 386

In particular we see that all of the higher homotopy groups of S^1 and the torus $S^1 \times S^1$ vanish.
(2) Let Σ be a surface of genus ≥ 2. For any $n \geq 2$ we see that
$$\pi_n(\Sigma) \quad \underset{\uparrow}{\cong} \quad \pi_n(\text{universal covering of } \Sigma) \quad \underset{\uparrow}{\cong} \quad \pi_n(B^2) \quad \underset{\uparrow}{=} \quad 0.$$
Covering-π_n-Isomorphism Proposition 28.10 Hadamard's Theorem 27.8 see page 386

[252] Here we take the point of view from page 381 where we saw that we view the elements of $\pi_n(B, b_0)$ as homotopy classes of continuous maps $(S^n, *) \to (B, b_0)$.
[253] We will discuss this map $\pi_n(B, b_0) \to \pi_n(X, x_0)$ in greater detail in the proof.

Together with the example on page 387 we have now shown that the higher homotopy groups of every surface of genus ≥ 1 vanish. □

Proof of the Covering-π_n-Isomorphism Proposition 28.10. Let $p\colon (X, x_0) \to (B, b_0)$ be a covering of pointed topological spaces and let $n \geq 2$. Our goal is to construct an inverse to the map $p_*\colon \pi_n(X, x_0) \to \pi_n(B, b_0)$.

We saw on page 381 that we can think of elements in $\pi_n(B, b_0)$ as homotopy classes of continuous maps $(S^n, *) \to (B, b_0)$. Thus let $f\colon (S^n, *) \to (B, b_0)$ be a continuous map. We are in the following setting[254] [255]

$$\begin{array}{ccc} & & (X, x_0) \\ & & \downarrow p \\ (S^n, *) & \xrightarrow[f]{} & (B, b_0). \end{array} \qquad \begin{array}{c} \text{by the Map Lifting Criterion 27.1} \\ \text{since by } n \geq 2 \text{ we have } \pi_1(S^n) = 0 \end{array} \qquad \begin{array}{ccc} & \exists!\, \widetilde{f} \nearrow & (X, x_0) \\ & \nearrow & \downarrow p \\ (S^n, *) & \xrightarrow[f]{} & (B, b_0). \end{array}$$

Furthermore it follows again from the Map Lifting Criterion 27.1, applied to the pointed topological space $(Z, z_0) = (S^n \times [0,1], (*, 0))$, that the lifts of two homotopic maps $(S^n, *) \to (B, b_0)$ are two homotopic maps $(S^n, *) \to (X, x_0)$.[256] Summarizing we just constructed a map

$$\begin{array}{rccc} \lambda\colon & \pi_n(B, b_0) & \to & \pi_n(X, x_0) \\ & [f] & \mapsto & [\widetilde{f}]. \end{array}$$

We have to show that λ is an inverse to p_*. First of all, given any $[f] \in \pi_n(B, b_0)$ we have

$$(p_* \circ \lambda)([f]) \;=\; p_*\big([\widetilde{f}]\big) \;=\; [p \circ \widetilde{f}] \;=\; [f].$$

Now let $[g] \in \pi_n(X, x_0)$. We then have

$$(\lambda \circ p_*)([g]) \;=\; \lambda([p \circ g]) \underset{\uparrow}{=} [g].$$

since g is the unique lift of $p \circ g$ to x_0.

■

Now we give a few more applications of the Covering-π_n-Isomorphism Proposition 28.10.

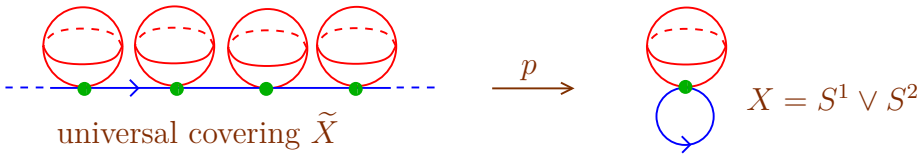
Example.

(1) We consider $X = S^1 \vee S^2$. On page 374 we determined its universal covering $\widetilde{X}$, which is given by the wedge of $\mathbb{R}$ with infinitely many spheres. It follows from the Covering-π_n-Isomorphism Proposition 28.10 that $\pi_2(S^1 \vee S^2)$ is isomorphic to $\pi_2(\widetilde{X})$. At the moment we cannot determine $\pi_2(\widetilde{X})$, but since $\widetilde{X}$ has "infinitely many spheres" one might expect $\pi_2(\widetilde{X})$ to be pretty big.

[254] In the Map Lifting Criterion 27.1 we need that the sphere S^n is path-connected and locally path-connected, but there are no hypotheses on B or X.

[255] We also use throughout the proof that the Lift Uniqueness Lemma 11.10 guarantees that the lifts, if they exist, are unique.

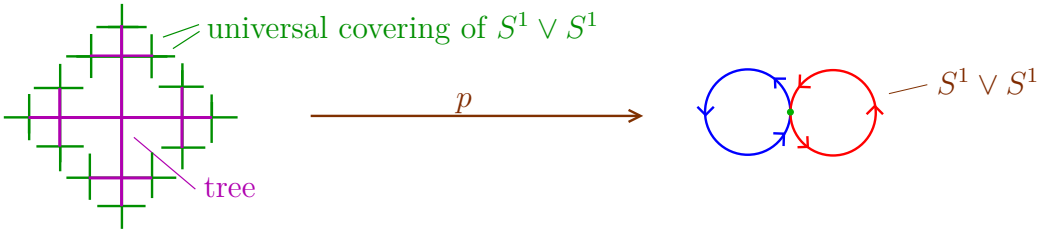
[256] Note that there is a little hidden subtlety: if $F\colon S^n \times [0,1] \to B$ is a homotopy between two maps $f, g\colon (S^n, *) \to (B, b_0)$, then we can lift f, g to maps $\widetilde{f}, \widetilde{g}\colon (S^n, *) \to (X, x_0)$. Furthermore we can apply the Map Lifting Criterion 27.1 to lift F to a continuous map $\widetilde{F}\colon S^n \times [0,1] \to X$ such that $\widetilde{F}(*, 0) = x_0$. We still need to show that $\widetilde{F}(*, t) = x_0$ for all $t \in [0,1]$. We know that for every $t \in [0,1]$ we have $\widetilde{F}(*, t) \in p^{-1}(b_0)$. By the Covering–Basics Lemma 14.6 (2) the preimage $p^{-1}(b_0)$ is discrete. Since $[0,1]$ is connected it follows that the map $p\colon [0,1] \to p^{-1}(b_0)$ is constant by the Connected-to-Discrete Lemma 2.18. But since it takes the value x_0 for $t = 0$ it takes the value x_0 for all $t \in [0,1]$.



(2) We have

$$\pi_n(S^1 \vee S^1) \underset{\text{Covering-}\pi_n\text{-Isomorphism Proposition 28.10}}{\cong} \pi_n(\text{universal covering of } S^1 \vee S^1) \underset{\text{Wedge-of-Circles–Universal Covering Proposition 27.7}}{\cong} \pi_n\Big(\bigcup_{k\in\mathbb{N}} \text{trees}\Big) \underset{\text{by the Exhaustion-}\pi_n\text{-Zero-Proposition 28.9, since trees are contractible}}{=} 0.$$

We will generalize this argument in the 1-Complex-π_n-Lemma 32.2.



□

28.6. **Are there any higher homotopy groups that are non-trivial?** We have now computed lots of higher homotopy groups, but so far the results have been rather dull, all the groups we have computed so far are trivial.

So are there are any higher homotopy groups that are non-trivial? A natural candidate for a non-trivial group is surely $\pi_n(S^n)$. The following question naturally arises:

Question 28.11. Let $n \geq 2$.
(1) Does the identity map $\mathrm{id}\colon S^n \to S^n$ define a non-trivial element in the homotopy group $\pi_n(S^n, *)$?
(2) Do we have an isomorphism $\pi_n(S^n, *) \cong \mathbb{Z}$?

We can also ask, what are the higher homotopy groups of spheres, i.e. what can we say about $\pi_k(S^n)$ for $k > n$? On page 387 we saw that all these groups are trivial for $n = 1$. But what happens for $n > 1$? Can the higher homotopy groups of spheres be non-trivial? We will answer this question in Algebraic Topology II using singular homology.

To construct our first example of a potentially interesting map from a higher-dimensional to a lower-dimensional sphere we recall that on page 139 we gave an identification $\mathbb{C}\mathrm{P}^1 =_{\mathrm{i}} S^2$. which identifies $[0:1]$ with the North Pole $N = (0,0,1)$.

Definition. We refer to

$$\begin{array}{rcl} H\colon S^3 = \{(z_1,z_2) \in \mathbb{C}^2 \,|\, |z_1|^2 + |z_2|^2 = 1\} & \to & S^2 =_{\mathrm{i}} \mathbb{C}\mathrm{P}^1 \\ (z_1,z_2) & \mapsto & [z_1 : z_2] \end{array}$$

as the Hopf map[257]. The homotopy class of H defines an element in $\pi_3(S^2, N)$.

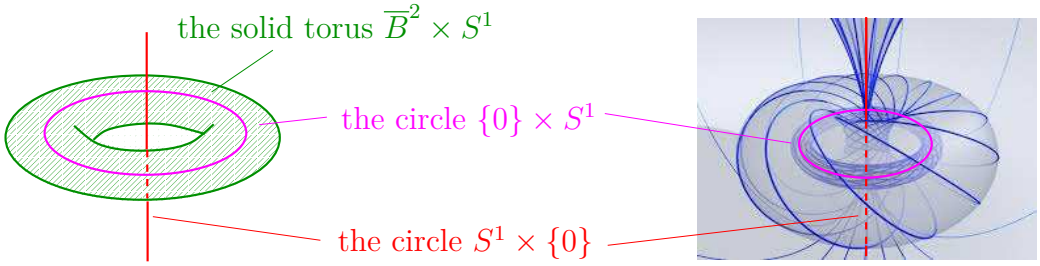
[257] Heinz Hopf (1894-1971) was a German mathematician.

Question 28.12. Does the Hopf map $H\colon S^3 \to S^2$ represent a non-trivial element in the homotopy group $\pi_3(S^2, N)$?

We will answer this question in Algebraic topology III

Remark. The definition of the Hopf map is very elegant, but unfortunately we pay for the elegance by loosing any intuition for what the map looks like. So how are supposed to think about the Hopf map? A straightforward calculation shows that for each $P \in S^2$ the preimage $H^{-1}(P)$ is a circle. Furthermore, if we use the explicit homeomorphism from the Stereographic Projection Lemma 2.11 to make the identification of S^3 with $\mathbb{R}^3 \cup \{\infty\}$ then we can make the following observations:

(1) The circle $S^1 \times \{0\}$ corresponds to the points $(z, 0) \in S^3$, in particular the circle gets sent to the point $[1:0] \in \mathbb{C}\mathrm{P}^1$.
(2) The circle that is given by the z-axis together with ∞ corresponds to the points $(0, z) \in S^3$, in particular the circle gets sent to the point $[0:1] \in \mathbb{C}\mathrm{P}^1$.



One can show that there exists a homeomorphism

$$(S^1 \times \overline{B}^2) \cup_{S^1 \times S^1} (\overline{B}^2 \times S^1) \xrightarrow{\cong} S^3.$$

It follows from the definition of this identification that on the torus $S^1 \times S^1$ the "diagonal circle" $\{(z, z) \,|\, z \in S^1\}$ is the preimage of the point $[1:1] \in \mathbb{C}\mathrm{P}^1$. In the figure above we show the preimages of H for some points in $\mathbb{C}\mathrm{P}^2 = S^2$. □

We introduce one more construction of continuous maps that represent potentially non-trivial elements in homotopy groups of spheres.

Definition.

(1) Let X be a non-empty topological space. We define the suspension of X to be the topological space

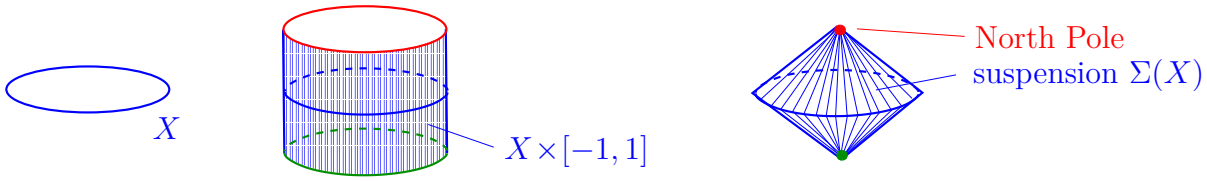
$$\Sigma(X) \;:=\; (X \times [-1,1])/(X \times \{-1\})/\sim$$

where we identify all points in $X \times \{-1\}$ to a single point and we identify all points in $X \times \{1\}$ to a single point. We refer to the point $[X \times \{1\}]$ as the North Pole of $\Sigma(X)$.

(2) Given a continuous map $f\colon X \to Y$ between topological spaces we denote by

$$\begin{array}{rccc} \Sigma(f)\colon & \Sigma(X) & \to & \Sigma(Y) \\ & [(x,t)] & \mapsto & [(f(x),t)] \end{array}$$

the suspension of f.



Remark. Let $n \in \mathbb{N}_0$. One can easily write down an explicit homeomorphism $\Sigma(S^n) \to S^{n+1}$ that sends the North Pole of $\Sigma(S^n)$ to the North Pole $(0, \dots, 0, 1)$ of S^{n+1}. We will use this homeomorphism to identify $\Sigma(S^n)$ with S^{n+1}. □

Now we have the following elementary lemma:

Lemma 28.13. (Suspension Homomorphism Lemma) Let $k \in \mathbb{N}_0$.

(1) Let (X, x_0) be a pointed topological space. We denote by $*$ the North Pole of $\Sigma(X)$. The map
$$\begin{array}{rcl} \pi_k(X, x_0) & \to & \pi_{k+1}(\Sigma(X), *) \\ [f\colon S^k \to X] & \mapsto & \big[\Sigma(f)\colon \Sigma(S^k) = S^{k+1} \to \Sigma(X)\big] \end{array}$$
is a well-defined group homomorphism. We refer to it as the suspension homomorphism.

(2) Let $n \in \mathbb{N}_0$. We denote by $*$ the North Pole of S^n and S^{n+1}. The map
$$\begin{array}{rcl} \pi_k(S^n, *) & \to & \pi_{k+1}(S^{n+1}, *) \\ [f\colon S^k \to S^n] & \mapsto & \big[\Sigma(f)\colon \Sigma(S^k) =_{\mathrm{i}} S^{k+1} \to \Sigma(S^n) =_{\mathrm{i}} S^{n+1}\big] \end{array}$$
is a well-defined group homomorphism. We refer to it also as the suspension homomorphism.

Proof.

(1) We leave the task of verifying that the map is in fact a group homomorphism as a somewhat challenging exercise to the reader. Alternatively we refer to [**Hil53**, Theorem 2.1] for a proof.

(2) This statement follows immediately from (1). ■

We conclude this chapter with the following question:

Question 28.14. Let $H\colon S^3 \to S^2$ be the Hopf map. Is the image of $[H] \in \pi_3(S^2)$ under the iterated suspension homomorphisms $\pi_3(S^2, x_0) \to \pi_{3+k}(S^{2+k}, x_1)$ non-trivial?

Unfortunately at the moment we do not have many tools to address the above questions. In general it is very hard to compute higher homotopy groups as there are few computational techniques that can be used to show that higher homotopy groups are non-zero. One of the key problems is that there is no analogue to the Seifert–van Kampen-Theorem 20.1. We will get to know a few techniques to compute higher homotopy groups of spheres, but we will cover these only in much later chapters, since the calculations will require the use of singular homology groups. But even then we will only be able to determine some homotopy groups of spheres. But note that the full computation of homotopy groups of spheres is still an open problem.

This sorry state of affairs means that as of right now we cannot use higher homotopy groups to distinguish any of the topological spaces that we could not distinguish already

using the fundamental group. In particular we still cannot show that for $k \neq l$ the topological spaces $\mathbb{R}^k$ and $\mathbb{R}^l$ are not homeomorphic. We will only be able to do so after developing a new tool, namely the singular homology groups, in the next semester.

Part III

CW-complexes

CHAPTER 29

Cell attachments

In this chapter we introduce the concept of attaching a cell to a topological space. We will see that many topological spaces arise that way and we will see that attaching cells changes the fundamental group in a controlled way. This allows us to give the overdue calculation of the fundamental groups of the complex projective spaces $\mathbb{C}\mathrm{P}^n$. More importantly though the study of cell attachments will be essential in the follow-up chapter where will graduate to the more general concept of a CW-complex.

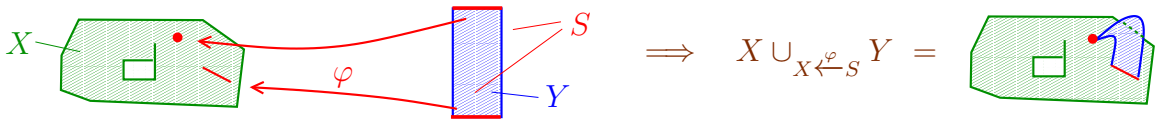
29.1. **Attaching a cell to a topological space.** We start out with a very general definition.

Definition. Let X and Y be topological spaces, let $S \subset Y$ be a subspace and let $\varphi\colon S \to X$ be a continuous map. (Note that φ does not need to be injective.) We write

$$X \cup_\varphi Y \;:=\; X \cup_{X \xleftarrow{\varphi} S} Y \;:=\; (X \sqcup Y)/\sim$$

where $\sim$ is the equivalence relation generated by $\varphi(s) \sim s$ for $s \in S$. We call $X \cup_\varphi Y$ the topological space obtained by gluing Y to X via the attaching map φ. Note that according to the Topological Pushout Proposition 6.12 we obtain a pushout diagram

$$\begin{array}{ccc} S & \xrightarrow{\varphi} & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \cup_\varphi Y. \end{array}$$



In this chapter we will mostly be interested in the following special case.

Definition. Let X be a topological space and let $\varphi\colon S^{n-1} \to X$ be a continuous map. We consider the topological space

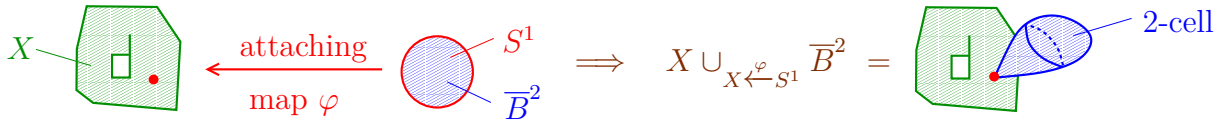
$$X \cup_{X \xleftarrow{\varphi} S^{n-1}} \overline{B}^n.$$

We introduce the following language:

(1) We refer to the image of B^n as an open n-cell.
(2) We refer to the image of $\overline{B}^n$ as an n-cell.
(3) We refer to φ as the attaching map of the n-cell.

We say that $X \cup_\varphi \overline{B}^n$ is obtained from X by attaching an n-cell along the attaching map φ. It follows from the fact that the inclusion $S^{n-1} \to \overline{B}^n$ is a closed embedding together with the Topological Pushout-Maps Lemma 6.14 (4) that the natural map $X \to X \cup_\varphi \overline{B}^n$

is a closed embedding. Often we will use this fact as an excuse to view X as a closed subset of $X \cup_\varphi \overline{B}^n$.



In the following we provide a very long list of examples of cell attachments. These examples train our geometric intuition and they will come in super handy in the next chapter, when we study CW-complexes.

Examples.

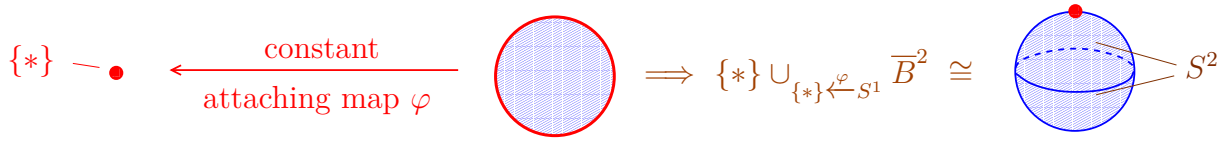
(1) We consider the effect of adding a 0-cell to a topological space X. Since $S^{-1} = \varnothing$ and since $\overline{B}^0 = \{0\}$ we see that attaching a 0-cell is the same as taking the disjoint union of X with a point.

(2) Let $X = \{*\}$ and let $\varphi\colon S^{n-1} \to \{*\}$ be the constant map. We have homeomorphisms

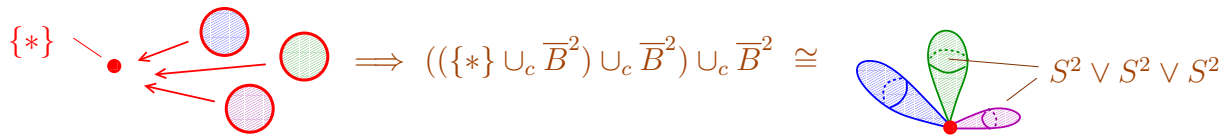
$$\{*\} \cup_{\{*\} \xleftarrow{\varphi} S^{n-1}} \overline{B}^n \xrightarrow{\cong} \overline{B}^n / S^{n-1} \xrightarrow{\cong} S^n.$$

see the homeomorphism from page 143 — identification from the Ball Quotient-Sphere Lemma 6.5

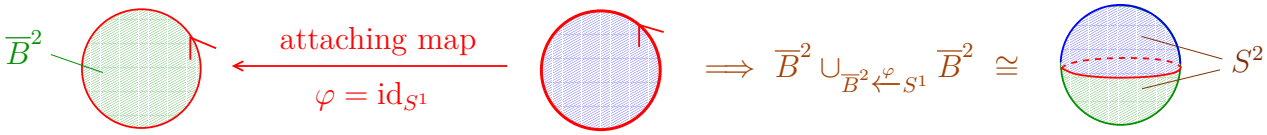
In fact, looking at these explicit homeomorphisms we see that these homeomorphisms send $\{*\}$ to the North Pole $(0, \dots, 0, 1)$.



(3) Again let $X = \{*\}$. Iteratively we attach k n-cells according to the constant attaching map $c\colon S^{n-1} \to \{*\}$. It follows easily from the Topological Pushout Proposition 6.12 (1$*$), the Compact-Hausdorff Proposition 2.10 (3), and the Wedge-Properties Lemma 17.2 (3) that the resulting topological space is homeomorphic to the wedge of k copies of S^n.



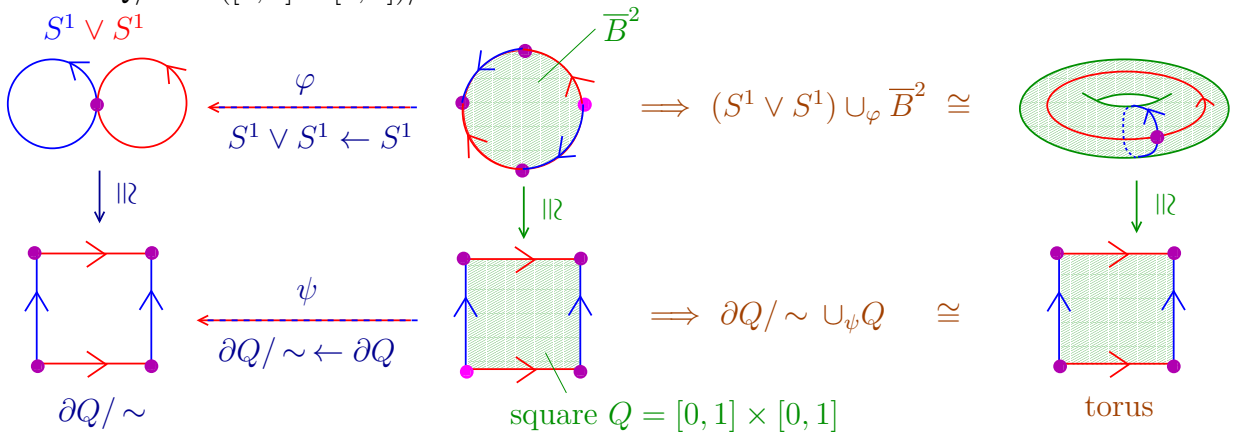
(4) Let $n \in \mathbb{N}_0$. We consider the n-ball $\overline{B}^n$ and we attach an n-cell to it along the attaching map $\varphi\colon S^{n-1} \to \overline{B}^n$ that is given by the inclusion. On page 144 we already saw that $\overline{B}^n \cup_{\overline{B}^n \xleftarrow{\varphi} S^{n-1}} \overline{B}^n$ is homeomorphic to the n-sphere S^n.



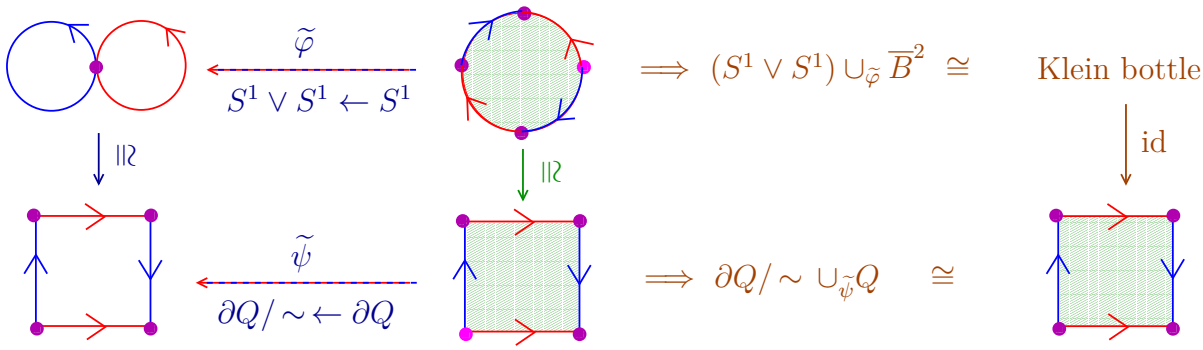
(5) We consider $X = S^1 \vee S^1$.

(a) We attach a 2-cell along the attaching map $\varphi\colon S^1 \to S^1 \vee S^1$ that is illustrated below. The Convex-to-Ball Proposition 2.12 gives us an explicit homeomorphism

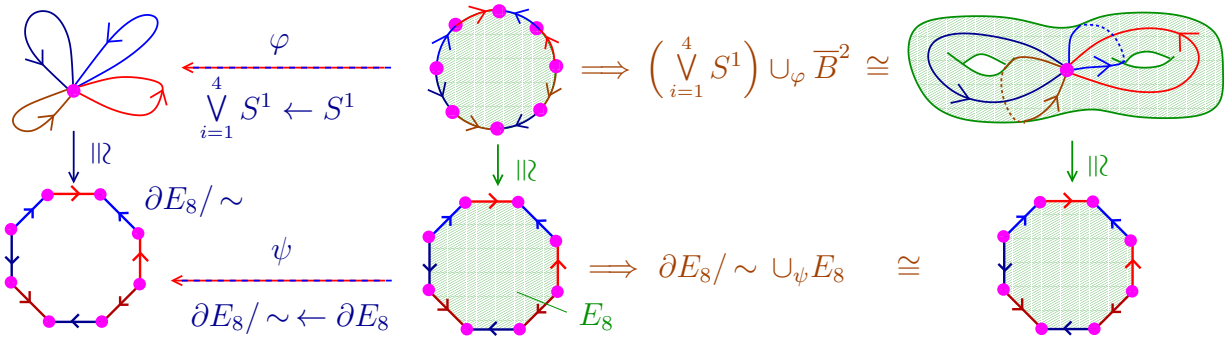
$\overline{B}^2 \to Q := [0,1] \times [0,1]$. This homeomorphism, together with the Topological Pushout Proposition 6.12 (1∗) and the Compact-Hausdorff Proposition 2.10 (3) can be used to show that $(S^1 \vee S^1) \cup_{S^1 \vee S^1 \xleftarrow{\varphi} S^1} \overline{B}^2$ is homeomorphic to the 2-torus $Q/\sim \cong ([0,1] \times [0,1])/\sim$.



(b) If instead we consider the attaching map $\widetilde{\varphi}\colon S^1 \to S^1 \vee S^1$ that is shown in the figure below, then one easily sees that $(S^1 \vee S^1) \cup_{S^1 \vee S^1 \xleftarrow{\widetilde{\varphi}} S^1} \overline{B}^2$ is homeomorphic to the Klein bottle.



(c) In the figure below we show that if we attach a 2-cell in a suitable way to the wedge of four circles, then we obtain a topological space that is homeomorphic to the surface of genus 2.



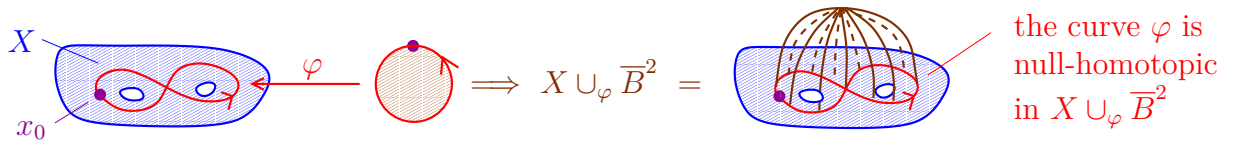
29.2. **Attaching cells and fundamental groups.** The following proposition tells us how attaching a cell affects the fundamental group of a given topological space.

Proposition 29.1. (Cell Attachment-π_1-Proposition) Let X be a topological space, let $n \geq 1$ and let $\varphi\colon S^{n-1} \to X$ be a continuous map. We set $x_0 \coloneqq \varphi(\underbrace{(0,\ldots,0,1)}_{\in S^{n-1}})$. The following statements hold:

($n > 2$) The inclusion-induced map $\pi_1(X, x_0) \to \pi_1(X \cup_\varphi \overline{B}^n, x_0)$ is an isomorphism.

($n = 2$) The map $[0,1] \to X$ given by $t \mapsto \varphi(\exp(2\pi i t))$ defines an element in $\pi_1(X, x_0)$ that we denote by $[\varphi]$. The inclusion-induced map $\pi_1(X, x_0) \to \pi_1(X \cup_\varphi \overline{B}^n, x_0)$ descends to an isomorphism

$$\pi_1(X, x_0)/\langle\langle [\varphi] \rangle\rangle \xrightarrow{\cong} \pi_1\big(X \cup_\varphi \overline{B}^2, x_0\big).$$



Proposition 29.1. (Cell Attachment-π_1-Proposition) We continue with the above notation.

($n = 1$) We set $x_0 \coloneqq \varphi(1)$ and $y_0 \coloneqq \varphi(-1)$.

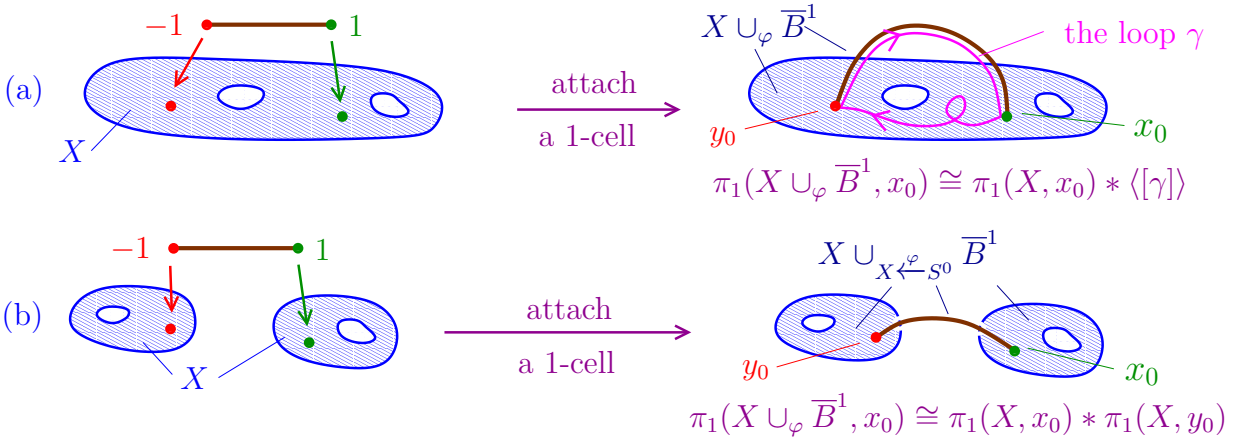
(a) Suppose that x_0 and y_0 lie in the same path-component of X. Let γ be a loop in $X \cup_\varphi \overline{B}^1$ that is given by the concatenation of the obvious path in the 1-cell from $x_0 = \varphi(1)$ to $y_0 = \varphi(-1)$ and some path in X from $y_0 = \varphi(-1)$ to $x_0 = \varphi(1)$. With this notation the map

$$\pi_1(X, x_0) * \langle t \rangle \;\to\; \pi_1(X \cup_\varphi \overline{B}^1, x_0)$$

that is given by the inclusion-induced map $\pi_1(X, x_0) \to \pi_1(X \cup_\varphi \overline{B}^1, x_0)$ and given by $t \mapsto [\gamma]$ is an isomorphism.

(b) Suppose that x_0 and y_0 lie in two different path components of X. Then we get an isomorphism

$$\pi_1(X, x_0) * \pi_1(X, y_0) \;\to\; \pi_1(X \cup_\varphi \overline{B}^1, x_0).$$



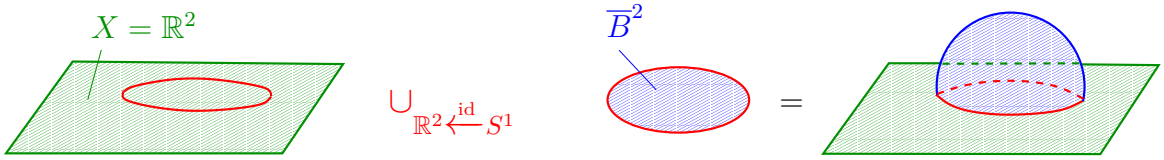
Examples.

(1) Let $X = S^1$ and let $n \in \mathbb{N}$. We attach to S^1 a 2-cell via the attaching map $\varphi\colon S^1 \to S^1$ that is given by $\varphi(z) = z^n$. Then

$$\pi_1\big(S^1 \cup_\varphi \overline{B}^2, 1\big) \underset{\uparrow}{=} \pi_1(S^1, 1)/\langle\langle[\varphi]\rangle\rangle \underset{\uparrow}{=} \langle x\rangle/\langle\langle x^n\rangle\rangle \cong \mathbb{Z}_n.$$

Cell Attachment-π_1-Proposition 29.1 — see the Circle-π_1-Theorem 11.13

(2) We attach a 2-cell to $X = \mathbb{R}^2$ via the attaching map $\varphi := \mathrm{id}_{S^1}\colon S^1 \to S^1$ that is given by the identity. By the Cell Attachment-π_1-Proposition 29.1 the new topological space $\mathbb{R}^2 \cup_\varphi \overline{B}^2$ is again simply connected

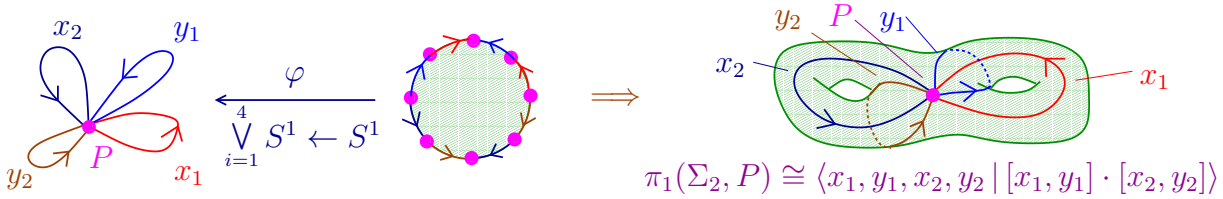


(3) In the previous section we already saw that many interesting topological spaces can be built out of easier topological spaces by attaching cells. In many cases we can use the Cell Attachment-π_1-Proposition 29.1 to calculate the fundamental group. For example, on page 396 we showed that we can obtain the surface of genus 2 from the wedge of four circles by attaching a 2-cell. With the notation from the figure below we now see that

$$\begin{aligned}\pi_1(\Sigma_2, P) \cong \pi_1\Big(\Big(\bigvee_{i=1}^4 S^1\Big) \cup_\varphi \overline{B}^2, P\Big) &\underset{\downarrow}{\cong} \pi_1\Big(\bigvee_{i=1}^4 S^1, P\Big)/\langle\langle[\varphi]\rangle\rangle \\ &\underset{\uparrow}{\cong} \langle x_1, y_1, x_2, y_2\rangle/\langle\langle[x_1, y_1]\cdot[x_2, y_2]\rangle\rangle.\end{aligned}$$

by the Cell Attachment-π_1-Proposition 29.1

by the Wedge-π_1-Proposition 20.6 and since $[\varphi] = [x_1, y_1]\cdot[x_2, y_2]$



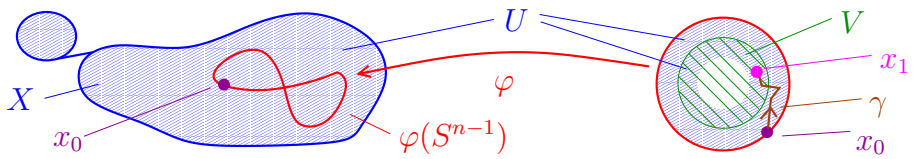
Of course we obtained the same result much earlier. The advantage of the current approach is that the Cell Attachment-π_1-Proposition 29.1 absorbs many of the technical arguments we had to deal with in the proof of the Surface-π_1-Proposition 22.1, i.e. in our initial calculation of $\pi_1(\Sigma_2, P)$. □

Proof of the Cell Attachment-π_1-Proposition 29.1 for $n \geq 2$. Let X be a topological space, let $n \geq 2$ and let $\varphi\colon S^{n-1} \to X$ be a continuous map. We set $x_0 := \varphi((0, \dots, 0, 1))$. The key idea of the proof is to apply the Seifert–van Kampen-Theorem 20.1 to the following two subsets of $X \cup_{X \xleftarrow{\varphi} S^{n-1}} \overline{B}^n$

$$U = X \cup_{X \xleftarrow{\varphi} S^{n-1}} \big\{z \in \overline{B}^n \,\big|\, \|z\| > \tfrac{1}{4}\big\} \quad \text{and} \quad V = \big\{z \in \overline{B}^n \,\big|\, \|z\| < \tfrac{3}{4}\big\}.$$

The remainder of the proof consists in showing that the resulting isomorphism implies the desired, a priori slightly different, isomorphism.

We pick a point $x_1 \in U \cap V$ and we pick a path γ in U from x_0 to x_1.



Claim. The following diagram of maps

$$\begin{array}{ccc}
\pi_1(X,x_0) & \longrightarrow & \pi_1(X\cup_\varphi \overline{B}^n,x_0) \\
\cong\downarrow & & \downarrow\mathrm{id} \\
\pi_1(U,x_0) & \longrightarrow & \pi_1(X\cup_\varphi \overline{B}^n,x_0) \\
\cong\uparrow\gamma_* & & \cong\uparrow\gamma_* \\
\pi_1(U,x_1) & \longrightarrow & \pi_1(X\cup_\varphi \overline{B}^n,x_1) \\
\downarrow & & \downarrow\mathrm{id} \\
\pi_1(U,x_1)/\langle\langle\pi_1(U\cap V,x_1)\rangle\rangle & \longrightarrow & \pi_1(X\cup_\varphi \overline{B}^n,x_1) \\
\cong\uparrow & \text{Amalgamated Product–Quotient Lemma 19.13} & \downarrow\mathrm{id} \\
\pi_1(U,x_1)*_{\pi_1(U\cap V,x_1)} \underbrace{\pi_1(V,x_1)}_{=\{e\}} & \xrightarrow[\text{Seifert–van Kampen Theorem 20.1}]{\cong} & \pi_1(X\cup_\varphi \overline{B}^n,x_1)
\end{array}$$

has the following three properties:

(i) The diagram commutes.
(ii) All vertical maps except possibly the third vertical map on the left are isomorphisms.
(iii) The bottom horizontal map is an isomorphism.

Proof.

(i) The two vertical maps from the third to the second level are the homomorphisms induced by the choice of base point coming from the path γ from x_0 to x_1. It follows from the Change-of-Base Point Proposition 13.2 that the second square commutes. All the other vertical and horizontal maps are induced by inclusions or they are the obvious algebraic map. It follows easily from functoriality that the diagram commutes.

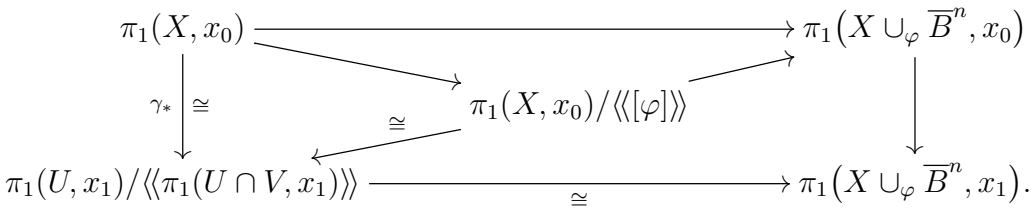
(ii) First note that it follows from the Change-of-Base Point Proposition 11.6 (1) that the vertical maps from the third to the second level are isomorphisms. Using the Homotopy Pushout Lemma from [**Fri24**] one can easily show that X is a deformation retract of U. Thus it follows from the Homotopy-π_1-Proposition 13.4 (2b) that the top left vertical map is an isomorphism.

The natural map $\{z\in B^n \mid \|z\| < \frac{3}{4}\} \to X\cup_{X\xleftarrow{\varphi} S^{n-1}} \overline{B}^n$ is easily seen to be an embedding. It follows that V is homeomorphic to an open ball. Hence $\pi_1(V,x_1)$ is the trivial group. It follows from this observation and the Amalgamated Product–Quotient Lemma 19.13 that the bottom vertical map on the left is an isomorphism.

(iii) It follows easily from the Topological-Pushout Proposition 6.12 (0) and the Topological-Coproduct Proposition 5.12 (0) that U and V are indeed open subsets of $X\cup_{X\xleftarrow{\varphi} S^{n-1}}\overline{B}^n$. We can therefore apply the Seifert–van Kampen-Theorem 20.1 and we obtain that the bottom horizontal map is an isomorphism. ⊞

Now we consider the case $n \geq 3$. In this case $\pi_1(U \cap V, x_1)$ is trivial since $U \cap V$ is homotopy equivalent to S^{n-1}. It follows from the Homotopy-π_1-Proposition 13.4 and the Sphere-π_1-Proposition 11.15 that $\pi_1(U \cap V, x_1) \cong \pi_1(S^{n-1}) = 0$. It is now clear that the one missing vertical map on the left is also an isomorphism. Since all vertical maps are isomorphisms, since the diagram commutes and since the bottom horizontal map is an isomorphism it follows that the top horizontal map is an isomorphism.

Finally we consider the case $n = 2$. In this case $U \cap V$ is homotopy equivalent to S^1 which implies, by the explicit isomorphism from the Circle-π_1-Theorem 11.13 and the Homotopy-π_1-Proposition 13.4, that $\pi_1(U \cap V, x_1) \cong \mathbb{Z}$. Note that under the explicit isomorphism $\Phi\colon \pi_1(X, x_0) \to \pi_1(U, x_1)$ this infinite cyclic subgroup corresponds to $\langle [\varphi] \rangle$. It is straightforward to see that we get a commutative diagram



This concludes the proof of the proposition. ■

Proof of the Cell Attachment-π_1-Proposition 29.1 for $n = 1$.

(a) We consider the case $\varphi(-1)$ and $\varphi(1)$ lie in the same path-component of X. This case is treated in [**Fri24**]. We leave it to the reader to fill in the details.
(b) We consider the case $\varphi(-1)$ and $\varphi(1)$ lie in different path-components of X. This case follows once again fairly easily from the Seifert–van Kampen-Theorem 20.1. Again we leave it to the reader to fill in the details. ■

29.3. **Projective spaces.** We just saw that the Cell Attachment-π_1-Proposition 29.1 can be very useful for calculating fundamental groups. Before we had already been quite successful with calculating fundamental groups, but it turns out that there is one interesting example of topological spaces that we could not handle yet, namely the complex projective spaces $\mathbb{C}\mathrm{P}^n$.

For the reader's convenience we recall the definition of real and complex projective spaces which we defined on page 136.

Definition. Let $\mathbb{A} = \mathbb{R}$ or $\mathbb{A} = \mathbb{C}$.
(1) For $n \in \mathbb{N}_0$ we set
$$\mathbb{A}\mathrm{P}^n := (\mathbb{A}^{n+1} \setminus \{0\})/\sim \quad \text{where } v \sim \lambda \cdot w \text{ for } \lambda \in \mathbb{A} \setminus \{0\}.$$
Given a point $(v_0, \ldots, v_n) \in \mathbb{A}^{n+1} \setminus \{0\}$ we denote by $[v_0 : \ldots : v_n]$ its equivalence class.
(2) For $k \leq n$ we refer to the map
$$\begin{array}{rcl} \iota\colon \mathbb{A}\mathrm{P}^k & \to & \mathbb{A}\mathrm{P}^n \\ [v_0 : \ldots : v_k] & \mapsto & [v_0 : \ldots : v_k : 0 : \ldots : 0] \end{array}$$
as the standard embedding. We saw on page 137 that this map is indeed an embedding.

The following lemma shows that we can obtain $\mathbb{R}\mathrm{P}^n$ from $\mathbb{R}\mathrm{P}^{n-1}$ by attaching a single n-cell and that we can obtain $\mathbb{C}\mathrm{P}^n$ from $\mathbb{C}\mathrm{P}^{n-1}$ by attaching a single $2n$-cell.

Lemma 29.2. (Projective Space–Cell Attachment Lemma) Let $n \in \mathbb{N}_0$.

(1) We consider the continuous map

$$\begin{array}{rcl} \varphi\colon S^{n-1} & \to & \mathbb{R}\mathrm{P}^{n-1} \\ (x_1, \ldots, x_n) & \mapsto & [x_1 : \ldots : x_n]. \end{array}$$

There exists a homeomorphism $\mathbb{R}\mathrm{P}^{n-1} \cup_{\mathbb{R}\mathrm{P}^{n-1} \xleftarrow{\varphi} S^{n-1}} \overline{B}^n \to \mathbb{R}\mathrm{P}^n$ such that the restriction to $\mathbb{R}\mathrm{P}^{n-1}$ is given by the standard embedding $\mathbb{R}\mathrm{P}^{n-1} \to \mathbb{R}\mathrm{P}^n$.

(2) We think of $\overline{B}^{2n}$ and S^{2n-1} as subspaces of $\mathbb{C}^n =_{\mathrm{i}} \mathbb{R}^{2n}$. We consider the attaching map

$$\begin{array}{rcl} \varphi\colon S^{2n-1} & \to & \mathbb{C}\mathrm{P}^{n-1} \\ (z_1, \ldots, z_n) & \mapsto & [z_1 : \ldots : z_n]. \end{array}$$

There exists a homeomorphism $\mathbb{C}\mathrm{P}^{n-1} \cup_{\mathbb{C}\mathrm{P}^{n-1} \xleftarrow{\varphi} S^{2n-1}} \overline{B}^{2n} \to \mathbb{C}\mathrm{P}^n$ such that the restriction to $\mathbb{C}\mathrm{P}^{n-1}$ is given by the standard embedding $\mathbb{C}\mathrm{P}^{n-1} \to \mathbb{C}\mathrm{P}^n$.

Proof.

(1) We consider the map

$$\begin{array}{rcl} \Theta\colon \mathbb{R}\mathrm{P}^{n-1} \cup_\varphi \overline{B}^n & \to & \mathbb{R}\mathrm{P}^n \\ [P] & \mapsto & \begin{cases} [x_1 : \cdots : x_n : 0], & \text{if } P = [x_1 : \cdots : x_n] \in \mathbb{R}\mathrm{P}^{n-1}, \\ [P : \sqrt{1 - \|P\|^2}], & \text{if } P \in \overline{B}^n. \end{cases} \end{array}$$



It follows immediately from the definition of φ that the map Θ is well-defined. Furthermore it follows easily from the Topological Pushout Proposition 6.12 (1∗) that the map Θ is continuous. It is basically clear that the map Θ is a surjection and it is straightforward to verify that the map Θ is an injection. Using the Topological-Quotient Proposition 6.1 (1b), the Projective Spaces-Compact+Hausdorff Proposition 6.10 and the Compact-Hausdorff Proposition 2.10 (3) one can easily verify that the map is a homeomorphism. Finally it is clear that the restriction of the map Θ to $\mathbb{R}\mathrm{P}^{n-1}$ is the standard embedding.

(2) Similar to (1) we see that the following map is a homeomorphism:

$$\begin{array}{rcl} \Theta\colon \mathbb{C}\mathrm{P}^{n-1} \cup_\varphi \overline{B}^{2n} & \to & \mathbb{C}\mathrm{P}^n \\ [P] & \mapsto & \begin{cases} [z_1 : \cdots : z_n : 0], & \text{if } P = [z_1 : \cdots : z_n] \in \mathbb{C}\mathrm{P}^{n-1}, \\ [P : \underbrace{\sqrt{1 - \|P\|^2}}_{\in \mathbb{R} \subset \mathbb{C}}], & \text{if } P \in \overline{B}^{2n}. \end{cases} \end{array}$$

■

The combination of the Cell Attachment-π_1-Proposition 29.1 together with the Projective Space–Cell Attachment Lemma 29.2 allows us to (re-) calculate the fundamental groups of projective spaces.

Proposition 29.3. (Projective Space–π_1-Proposition)

(1) We have $\pi_1(\mathbb{R}\mathrm{P}^1) \cong \mathbb{Z}$ and for $n \geq 2$ we have $\pi_1(\mathbb{R}\mathrm{P}^n) \cong \mathbb{Z}_2$.

(2) For any $n \in \mathbb{N}_0$ we have $\pi_1(\mathbb{C}\mathrm{P}^n) = 0$.

Proof.

(1) Since we already calculated the fundamental groups of $\mathbb{R}\mathrm{P}^n$ in the Real Projective Space–π_1-Proposition 15.3 we outsource this part as an exercise to the reader.

(2) Note that $\mathbb{C}\mathrm{P}^0$ consists of a single point. It follows from the Projective Space–Cell Attachment Lemma 29.2 that we obtain $\mathbb{C}\mathrm{P}^n$ from $\mathbb{C}\mathrm{P}^0$ by iteratively attaching cells of dimension $2, 4, \ldots, 2n$. It follows from the Cell Attachment-π_1-Proposition 29.1 that $\pi_1(\mathbb{C}\mathrm{P}^n) \cong \pi_1(\mathbb{C}\mathrm{P}^0) = 0$. ∎

29.4. **Attaching arbitrarily many cells.** In the following we slightly generalize the concepts from page 394 to the following notationally less pleasant definition.

Definition. Let X be a topological space and let $\{\varphi_i \colon S^{n_i-1} \to X\}_{i\in I}$ be a family of continuous maps.

(1) We consider the topological space

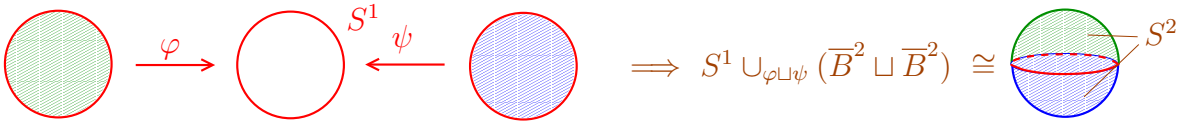
$$X \cup_{X \leftarrow \coprod_{i\in I} S^{n_i-1} \,:\, \coprod_{i\in I} \varphi_i} \coprod_{i\in I} \overline{B}^{n_i}.$$

As before we introduce the notion of an (open) cell and of an attaching map of a cell. We say that the above topological space is obtained from X by attaching cells along the attaching maps φ_i.

(2) It follows from the fact that the inclusions $S^{n-1} \to \overline{B}^n$ are closed embeddings together with the Topological Pushout-Maps Lemma 6.14 (4) that the natural map of X into the above topological space is a closed embedding. Often we will use this fact as an excuse to view X as a closed subset of the above topological space.

Example.

(1) Let $n \in \mathbb{N}_0$. We attach two n-cells to S^{n-1} where in both cases the attaching map is given by the identity map $\varphi := \psi := \mathrm{id} \colon S^{n-1} \to S^{n-1}$. Using the Compact-Hausdorff Proposition 2.10 (3) one can easily show that there exists a homeomorphism $S^{n-1} \cup_{\varphi\sqcup\psi} (\overline{B}^n \sqcup \overline{B}^n) \to S^n$ which on S^{n-1} is given by $x \mapsto (x, 0)$.

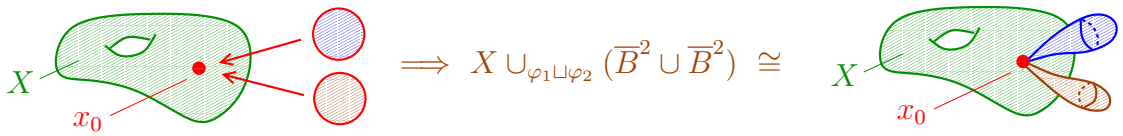


(2) Let (X, x_0) be a pointed topological space. Furthermore let I be a set and for each $i \in I$ let $n_i \in \mathbb{N}_0$. Given $i \in I$ we denote by $\varphi_i \colon S^{n_i-1} \to X$ the constant map that sends all points to x_0. Using the Ball Quotient-Sphere Lemma 6.5 one can show that there exists a homeomorphism

$$X \cup_{X \leftarrow \coprod_{i\in I} S^{n_i-1} \,:\, \coprod_{i\in I} \varphi_i} \coprod_{i\in I} \overline{B}^{n_i} \xrightarrow{\cong} \underbrace{(X, x_0) \vee \bigvee_{i\in I} (S^{n_i}, N)}_{\substack{\text{wedge of pointed topological} \\ \text{spaces where } N \in S^{n_i} \text{ is the} \\ \text{North Pole}}}$$

that is the identity on X. [258]

[258] For fans of checking continuity let us elaborate a little bit. Let $p_n \colon \overline{B}^n \to \overline{B}^n/S^{n-1}$ be the projection and let $\Theta_n \colon \overline{B}^n/S^{n-1} \to S^n$ be the homeomorphism from the Ball Quotient-Sphere Lemma 6.5. The maps



The following proposition shows that the Hausdorff property is preserved under cell attachments.

Proposition 29.4. (Cell Attachment–Hausdorff Proposition) Let X be a topological space and let $\{\varphi_i \colon S^{n_i-1} \to X\}_{i\in I}$ be a family of continuous maps. We consider the topological space

$$\widehat{X} \;:=\; X \cup_{X \leftarrow \bigsqcup_{i\in I} S^{n_i-1} \,:\, \bigsqcup_{i\in I} \varphi_i} \bigsqcup_{i\in I} \overline{B}^{n_i}$$

that we obtain from X by attaching cells along the attaching maps φ_i.

(1) If U, V are disjoint open subset of X, then there exist disjoint open subsets $\widehat{U}$ and $\widehat{V}$ of $\widehat{X}$ with $X \cap \widehat{U} = U$ and $X \cap \widehat{V} = V$.

(2) If X is Hausdorff, then $\widehat{X}$ is also Hausdorff.

Proof. For each $i \in I$ we consider the natural map $\Phi_i \colon \overline{B}^{n_i} \to \widehat{X}$. It follows immediately from the Topological-Pushout Proposition 6.12 (0) and the Topological-Coproduct Proposition 5.12 (0) that given any subset $U \subset \widehat{X}$ the following statement holds:

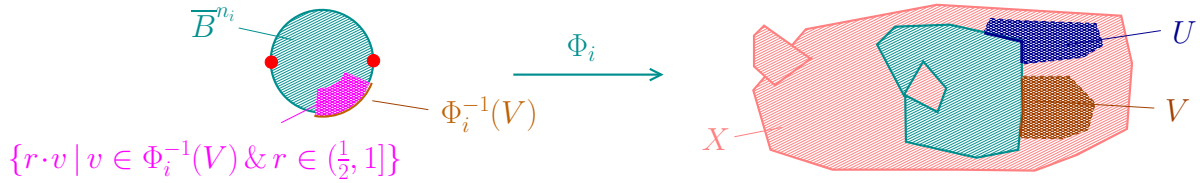
$$U \text{ is open in } \widehat{X} \iff \begin{array}{l} \text{(a) the intersection } U \cap X \text{ is open in } X \text{ and} \\ \text{(b) for all } i \in I \text{ the preimage } \Phi_i^{-1}(U) \text{ is open in } \overline{B}^{n_i}. \end{array}$$

After this preparation we turn to the actual proof of the two statements:

(1) Let U and V be two disjoint open subsets of X. We set

$$\begin{aligned} \widehat{U} &:= U \cup \bigcup_{i\in I} \Phi_i(\{r \cdot u \,|\, u \in \Phi_i^{-1}(U) \text{ and } r \in (\tfrac{1}{2}, 1]\}). \\ \widehat{V} &:= V \cup \bigcup_{i\in I} \Phi_i(\{r \cdot v \,|\, v \in \Phi_i^{-1}(V) \text{ and } r \in (\tfrac{1}{2}, 1]\}). \end{aligned}$$

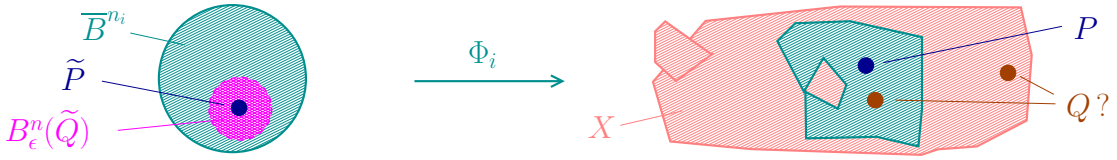
It follows immediately from the above discussion that $\widehat{U}$ and $\widehat{V}$ are open subsets of $\widehat{X}$. Furthermore it is clear that they are disjoint.



$\Theta_{n_i} \circ p_{n_i} \colon \overline{B}^{n_i} \to S^{n_i}$ together with the identity on X define, by the Topological-Coproduct Proposition 5.12 (1b), Topological Pushout Proposition 6.12 (1b) and the Wedge Proposition 17.1 (1a), a continuous map $\alpha \colon X \cup \bigcup_{i\in I} \overline{B}^{n_i} \to X \vee \bigvee_{i\in I} S^{n_i}$. Conversely note that the natural map $\overline{B}^{n_i} \to X \cup \bigsqcup_{i\in I} \overline{B}^{n_i}$ descends to a continuous map $f_i \colon \overline{B}^{n_i}/S^{n_i-1} \to X \cup \bigsqcup_{i\in I} \overline{B}^{n_i}$. The maps $f_i \circ \Theta_{n_i}^{-1} \colon S^n \to X \cup \bigsqcup_{i\in I} \overline{B}^{n_i}$ and the identity on X define, by the Wedge Proposition 17.1 (1a), a continuous map $\beta \colon X \vee \bigvee_{i\in I} S^{n_i} \to X \cup \bigsqcup_{i\in I} \overline{B}^{n_i}$. One can easily verify that these two maps α and β are inverses of one another.

(2) Let $P, Q \in \widehat{X}$ be two distinct points. If P and Q lie in X, then it follows from the hypothesis that X is Hausdorff and (1) that we can find disjoint open neighborhoods $\widehat{U}$ of P and $\widehat{V}$ of Q in $\widehat{X}$.

It remains to consider the case that at least one of P and Q lies in $\widehat{X} \setminus X$. Let us assume that $P \in \widehat{X} \setminus X$. This means that there exists an $i \in I$ and a $\widetilde{P} \in B^{n_i}$ such that $P = \Phi_i(\widetilde{P})$. Since $Q \neq P$ there exists an $\epsilon > 0$ such that $B^{n_i}_\epsilon(\widetilde{P}) \subset B^{n_i}$ and such that $Q \notin \Phi_i(B^{n_i}_\epsilon(\widetilde{P}))$.



We set $U := \Phi_i(B^{n_i}_{\frac{\epsilon}{2}}(\widetilde{Q}))$ and $V := \widehat{X} \setminus \Phi_i(\overline{B}^{n_i}_{\frac{\epsilon}{2}}(\widetilde{Q}))$. It is clear that U and V are disjoint. It follows from the discussion at the beginning of the proof that U and V are open subsets of $\widehat{X}$. ■

CHAPTER 30

CW-complexes: Definitions and basic properties

In this chapter we introduce CW-complexes, we will see that many popular topological spaces have the structure of a CW-complex and we discuss basic properties of CW-complexes. The notion of a CW-complex will play an essential role throughout these lecture notes.

30.1. **Definition of CW-complexes and examples.** After the long discussion of cell attachments in the last chapter we are now mentally ready for the all-important notion of a CW-complex:

Definition. A CW-complex X[259] [260] is a topological space X together with a filtration

$$\varnothing = X^{-1} \subset X^0 \subset X^1 \subset \cdots \subset X$$

by subspaces which has the following properties:

(CW1) We have $X = \operatorname{colim} X^n$, as described in the Topological-Direct Colimit Proposition 25.5. In other words, as a set we have $X = \bigcup_{n\in\mathbb{N}_0} X^n$ and $U \subset X$ is open if and only if for each $n \in \mathbb{N}_0$ the intersection $U \cap X^n$ is open in X^n.

(CW2) For each $n \in \mathbb{N}_0$ there exists a set I, a family $\{\varphi_i \colon S^{n-1} \to X^{n-1}\}_{i\in I}$ of continuous maps and a homeomorphism[261] [262]

$$\Xi_n \colon X^{n-1} \cup_{X^{n-1} \leftarrow \bigsqcup_{i\in I} S^{n-1} \,:\, \bigsqcup_{i\in I} \varphi_i} \bigsqcup_{i\in I} \overline{B}^n \;\to\; X^n$$

which is the inclusion on X^{n-1}.

A CW-structure on a topological space X is a filtration as above.

[259] CW-complexes were introduced by John H. C. Whitehead[260]. The name comes from "C" for "closure-finite", and the "W" for "weak topology":

(1) "Closure finite" means that the closure of each open cell is contained in the union of finitely many open cells. It follows from the Contained-in-Finite Subcomplex Lemma 30.12 (3) and the Cell-Compact Proposition 30.4 (2) that this is in fact the case.

(2) "Weak topology" means that a subset is open if and only if each intersection $U \cap X^n$ is open.

Some people also say that the name CW goes back to the initials C. W. of Whitehead.

[260] At first glance the definitions of CW-complexes that appear in the literature look widely different. Some differences just stem from different notation. Some authors, see e.g. [**Lee00**, p. 132] and [**Mun84**, p. 214] require "closure-finiteness" as part of the definition, but as we will see in the Contained-in-Finite Subcomplex Lemma 30.12 (3) and the Cell-Compact Proposition 30.4 (2), this property follows from the definition as given. One could also make the case that the definitions of CW-complexes in some references are a little vague.

[261] In other words, X^n is obtained from attaching n-cells to X^{n-1} along the maps φ_i.

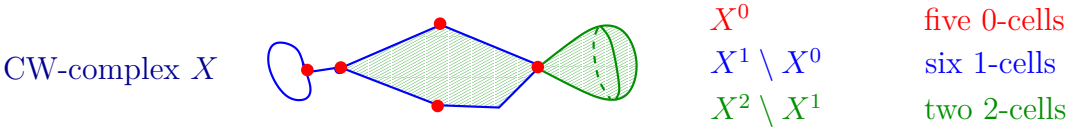
[262] Note that the φ_c and the homeomorphism are *not* part of the data of a CW-complex.

Before we discuss examples of CW-complexes it is useful to already introduce some extra language.

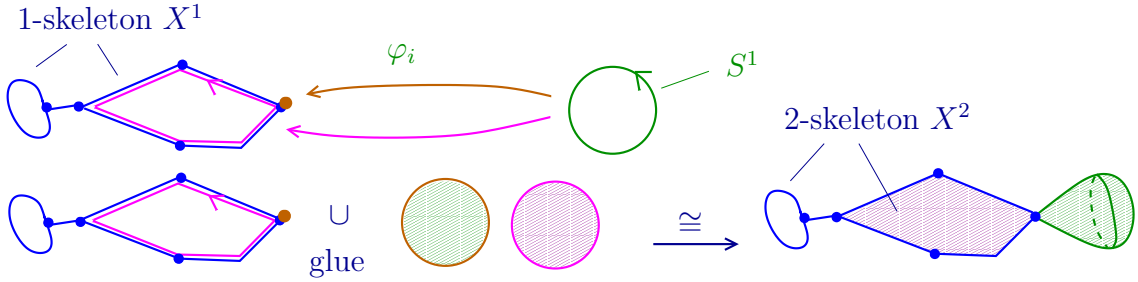
Definition. Using the notation from above we introduce the following language:

(1) Given $n \in \mathbb{N}_{\geq 0}$ we refer to X^n as the n-skeleton of X.
(2) The components of $X^n \setminus X^{n-1}$ are called the open n-cells of X.[263] [264]
(3) For each open n-cell σ of X we refer to its closure $\overline{\sigma}$ in X as an n-cell of X.
(4) We set
$$\dim(X) \;:=\; \sup\{n \in \mathbb{Z}_{\geq -1} \,|\, X^{n-1} \neq X^n\} \;\in\; \mathbb{Z}_{\geq -1} \cup \{\infty\}.$$
(5) We say that the CW-complex is finite if it has only finitely many cells and we say that it is countable if it has only countably many cells.

Illustration. In the figure below we show a 2-dimensional CW-complex X with its filtration $X^0 \subset X^1 \subset X^2$.



In the next figure we indicate how we obtain the 2-skeleton from the 1-skeleton by gluing on 2-disks.



Our long list of examples of cell attachments from page 395 provides us, almost for free, with a long list of CW-complexes.

Examples.

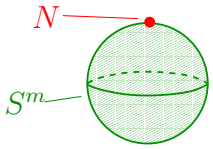
(1) The one type of CW-complexes that is easy to understand are 0-dimensional CW-complexes. Similar to the discussion on page 395 it follows from the observation that $\overline{B}^0$ consists of a point and that $S^{-1} = \varnothing$ that a CW-complex of dimension ≤ 0 is essentially the same as a set of points with the discrete topology.

[263] It follows from the Disjoint Union Topology-Properties Lemma 5.13 (2b) and the Topological Pushout Proposition 6.12 (0) that, in the notation of (CW2), the open cells are precisely the images of the B^n_c and that each open n-cell is an open subset of X^n.
[264] It follows almost immediately from the definitions that, *as a set*, any CW-complex is the disjoint union of all the open cells.

(2) Let $m \in \mathbb{N}_0$. As usual we denote by $N := (0, \dots, 0, 1)$ the North Pole of S^m. We consider the filtration $\varnothing \subset \{N\} \subset S^m$. That this is a CW-structure can be seen as follows:

(CW1) This condition is vacuous since the m-skeleton equals S^m.

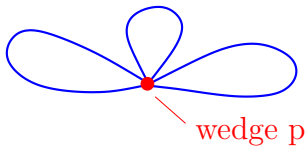
(CW2) The case $n = 0$ is clear since $\{N\}$ has the discrete topology. For $n = 1, \dots, m-1$ there is nothing to show. Finally note that on page 395 we showed that we obtain S^m from $\{N\}$ by attaching an m-cell.



the filtration $\varnothing \subset \{N\} \subset S^m$ is a CW-structure with
one 0-cell
one m-cell

(3) Let $n \in \mathbb{N}_0$ and let $X = \bigvee_{i=1}^{n} S^1$ be the wedge of n circles with wedge point $*$. The discussion on page 395 implies that the filtration $\varnothing \subset \{*\} \subset \bigvee_{i=1}^{n} S^1$ is a CW-complex with one 0-cell and n 1-cells.

$X = S^1 \vee S^1 \vee S^1$

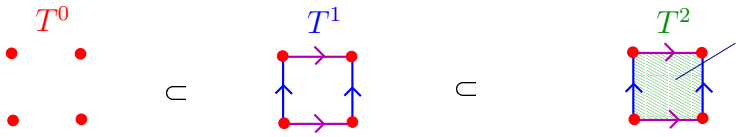


the filtration $\varnothing \subset \{*\} \subset S^1 \vee S^1 \vee S^1$
is a CW-complex with
one 0-cell
three 1-cells

(4) The figure below shows a filtration of the torus $T = ([0,1] \times [0,1])/\sim$ that is a CW-structure with one 0-cell, two 1-cells and one 2-cell. That this is a CW-structure can be seen as follows:

(CW1) This condition is vacuous since $T^2 = T$.

(CW2) The case $n = 0$ is clear since T^0 has the discrete topology. For $n = 1$ note that T^1 is clearly homeomorphic to the wedge of two circles along the unique point of T^0. Thus we obtain from the previous example that (CW2) is satisfied for $n = 1$. Finally note that on page 396 we showed that we obtain T from attaching a 2-cell to $T^1 \cong S^1 \vee S^1$.



torus $T = ([0,1] \times [0,1])/\sim$ with
one 0-cell
two 1-cells
one 2-cell

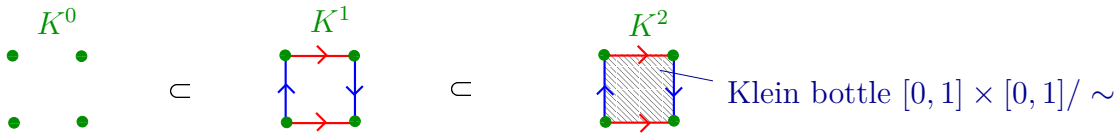
For fun we also show the filtration if we view the torus as a subspace of $\mathbb{R}^3$:

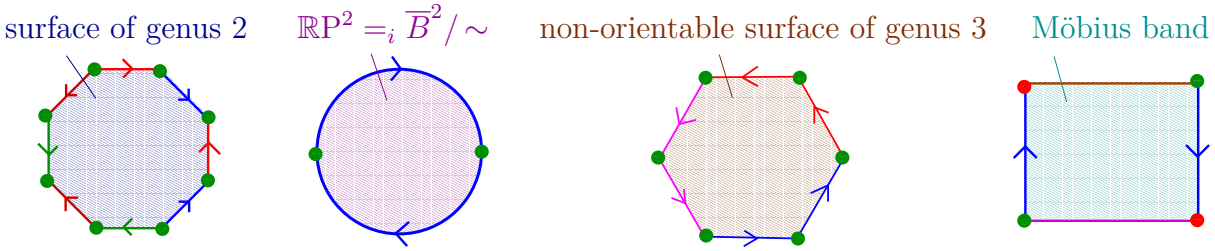


torus as subspace of $\mathbb{R}^3$

(5) The figure below shows a filtration of the Klein bottle that is a CW-structure with one 0-cell, two 1-cells and one 2-cell. We see that the two CW-structures of the torus and the Klein bottle have the same 1-skeleton, but the 2-cells have different attaching maps.

(6) In the figure below we sketch CW-structures for the surface of genus 2, the projective plane $\mathbb{R}\mathrm{P}^2 =_{\mathrm{i}} \overline{B}^2/\sim$, the non-orientable surface of genus 3 and the Möbius band as CW-complexes.





Generalizing these examples one can easily show, using the Surface Classification Theorem 22.8, that every compact 2-dimensional topological manifold admits a CW-structure. □

So far we only considered CW-complexes with finitely many cells and for the most part we just recycled material from the previous chapter. We now move on to CW-complexes with infinitely many cells:

Examples.

(1) (a) Let us consider the topological space $\mathbb{R}$. We equip it with the filtration $\varnothing \subset \mathbb{Z} \subset \mathbb{R}$. We consider the family $\{\varphi_i \colon S^0 \to \mathbb{Z}\}_{i\in\mathbb{Z}}$ of maps given by $\varphi_i(\epsilon) = i + \frac{1+\epsilon}{2}$. Using the Cover-Open Subset Lemma 1.5 (2) one can verify that the map

$$\begin{aligned} \mathbb{Z} \cup_{\mathbb{Z} \leftarrow \bigsqcup_{i\in\mathbb{Z}} S_i^0 \,:\, \bigsqcup_{i\in\mathbb{Z}} \varphi_i} \bigsqcup_{i\in\mathbb{Z}} \overline{B}_i^1 \;&\to\; \mathbb{R} \\ P \;&\mapsto\; \begin{cases} n, & \text{if } P = [n] \text{ for some } n \in \mathbb{Z} \\ i + \frac{1+t}{2}, & \text{if } P = [(t,i)] \text{ for some } t \in \overline{B}^1 = [-1,1] \text{ and } i \in \mathbb{Z} \end{cases} \end{aligned}$$

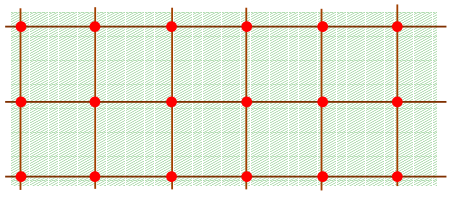
is a homeomorphism. Thus the filtration $\varnothing \subset \mathbb{Z} \subset \mathbb{R}$ is a CW-structure on $\mathbb{R}$.

(b) Using the Convex-to-Ball Proposition 2.12 one can easily generalize the above example and one can show that the subspaces

$$X^i \;:=\; \{(x_1,\ldots,x_n) \in \mathbb{R}^n \,|\, \text{at least } n-i \text{ coordinates lie in } \mathbb{Z}\} \text{ with } i = 0,\ldots,n$$

form a CW-structure on $\mathbb{R}^n$.

CW-structure on $\mathbb{R}$

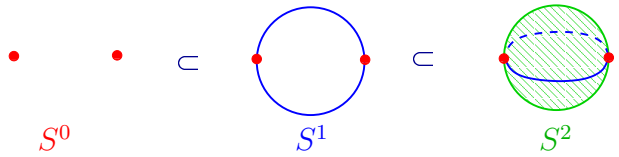


CW-structure on $\mathbb{R}^2$

(2) Let I be a set and for each $i \in I$ let $n_i \in \mathbb{N}$. We consider the wedge $\bigvee_{i\in I} S_i^{n_i}$. Given $n \in \mathbb{N}_0$ we denote by X_n the wedge of all spheres of dimension $\leq n$. It follows from the discussion on page 402 that the filtration $\varnothing \subset X_0 \subset X_1 \subset \dots$ is a CW-structure on $\bigvee_{i\in I} S_i^{n_i}$. □

Next we want to consider the infinite families S^n, $\mathbb{R}\mathrm{P}^n := (\mathbb{R}^{n+1} \setminus \{0\})/(\mathbb{R} \setminus \{0\})$ and $\mathbb{C}\mathrm{P}^n := (\mathbb{C}^{n+1} \setminus \{0\})/(\mathbb{C} \setminus \{0\})$. To study these family we introduce, with great reluctance, the following convention.

Convention. For $k < n$ we use in the following discussion the embedding $S^k \to S^n$ given by $x \mapsto (x, 0)$ to think of S^k as a subspace of S^n. Similarly we view $\mathbb{R}\mathrm{P}^k$ as a subspace of $\mathbb{R}\mathrm{P}^n$ and we view $\mathbb{C}\mathrm{P}^k$ as a subspace of $\mathbb{C}\mathrm{P}^n$.



It is natural to try to find CW-structures which are "compatible" with these inclusions. For example we could try to find CW-complexes such that the "smaller family members" are "subcomplexes". This leads us to the following definition:

Definition. We say that a CW-complex A is a subcomplex of a CW-complex X if $A \subset X$ and if the following two conditions are satisfied:

(1) For each $n \in \mathbb{N}_0$ we have $A^n = A \cap X^n$.

(2) Each open n-cell of A is also an open n-cell of X.

(For example, it follows immediately from the definitions that any skeleton is a subcomplex.)

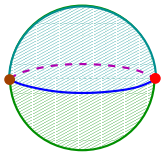
Examples.

(1) We consider the spheres S^n.

(a) On page 407 we found a "tiny" CW-structure for S^n with just one 0-cell and one n-cell. But this CW-structure does not have any interesting subcomplexes. To fix this, recall that on page 402 we saw that we obtain S^{k+1} from S^k by attaching two $(k+1)$-cells. It follows that a CW-structure for S^n is given by

$$\varnothing \subset S^0 \subset S^1 \subset S^2 \subset \dots \subset S^n.$$

With this CW-structure we can view each S^k with $k < n$ as a subcomplex of S^n.



the filtration $\varnothing \subset S^0 \subset S^1 \subset S^2$ is a CW-structure with
two 0-cells
two 1-cells
two 2-cells

(b) The new CW-structure on spheres gives us an immediate bonus: As on page 351 we consider the topological space

$$S^\infty := \left\{(x_1, x_2, \dots) \in \mathbb{R}^\infty \,\middle|\, \sum_{i=1}^{\infty} x_i^2 = 1\right\} = \operatorname*{colim}_{\longrightarrow} S^{i-1}.$$

It follows immediately from (a) and the definitions that a CW-structure for S^∞ is given by

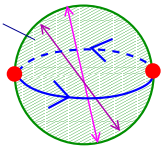
$$\varnothing \subset S^0 \subset S^1 \subset S^2 \subset \ldots \subset S^\infty.$$

(2) We consider the real projective spaces $\mathbb{R}\mathrm{P}^n$.

(a) In the Projective Space–Cell Attachment Lemma 29.2 (1) we showed that we can obtain $\mathbb{R}\mathrm{P}^{k+1}$ from $\mathbb{R}\mathrm{P}^k$ by attaching a single $(k+1)$-cell. It follows that a CW-structure for $\mathbb{R}\mathrm{P}^n$ is given by

$$\varnothing \subset \mathbb{R}\mathrm{P}^0 \subset \mathbb{R}\mathrm{P}^1 \subset \mathbb{R}\mathrm{P}^2 \subset \ldots \subset \mathbb{R}\mathrm{P}^n$$

and that this CW-structure has precisely one cell in dimensions $0, 1, 2, \ldots, n$.

$\mathbb{R}\mathrm{P}^2 =_i S^2/\sim$ 

the filtration $\varnothing \subset \mathbb{R}\mathrm{P}^0 \subset \mathbb{R}\mathrm{P}^1 \subset \mathbb{R}\mathrm{P}^2$ is a CW-structure with
one 0-cell
one 1-cell
one 2-cell

(b) As on page 352 we consider $\mathbb{R}\mathrm{P}^\infty = \operatorname{colim} \mathbb{R}\mathrm{P}^i$. As in (1b) we see that

$$\varnothing \subset \mathbb{R}\mathrm{P}^0 \subset \mathbb{R}\mathrm{P}^1 \subset \mathbb{R}\mathrm{P}^2 \subset \ldots \subset \mathbb{R}\mathrm{P}^\infty$$

is a CW-structure which has precisely one cell in each dimension.

(3) Finally we consider the complex projective spaces $\mathbb{C}\mathrm{P}^n$.

(a) As in (2a) we see, this time using the Projective Space–Cell Attachment Lemma 29.2 (2), that a CW-structure for $\mathbb{C}\mathrm{P}^n$ is given by[265]

$$\varnothing \subset \underset{\substack{\uparrow\\ \text{0-skeleton}}}{\mathbb{C}\mathrm{P}^0} \subset \underset{\substack{\uparrow\\ \text{2-skeleton}}}{\mathbb{C}\mathrm{P}^1} \subset \underset{\substack{\uparrow\\ \text{4-skeleton}}}{\mathbb{C}\mathrm{P}^2} \subset \ldots \subset \underset{\substack{\uparrow\\ 2n\text{-skeleton}}}{\mathbb{C}\mathrm{P}^n}$$

and that this CW-structure has precisely one cell in dimensions $0, 2, 4, \ldots, 2n$.

(b) As on page 352 we consider $\mathbb{C}\mathrm{P}^\infty = \operatorname{colim} \mathbb{C}\mathrm{P}^i$ and we see that

$$\varnothing \subset \mathbb{C}\mathrm{P}^0 \subset \mathbb{C}\mathrm{P}^1 \subset \mathbb{C}\mathrm{P}^2 \subset \ldots \subset \mathbb{C}\mathrm{P}^\infty$$

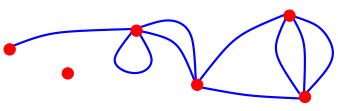
is a CW-structure which has precisely one cell in each even dimension. □

The following lemma shows that topological graphs and CW-complexes of dimension ≤ 1 are essentially the same.

Lemma 30.1. (Graph–CW-Complex Lemma)

(1) If $(X, G = (V, E, \theta), \Theta\colon |G| \to X)$ is an undirected topological graph then $\varnothing \subset V \subset X$ is a CW-structure.

(2) Let $\varnothing \subset X^0 \subset X^1 = X$ be a CW-complex of dimension ≤ 1. There exists an undirected abstract graph (V, E, θ) and a homeomorphism $f\colon |G| \to X$ such that $f(V) = X^0$.



vertices form the 0-skeleton
edges correspond to 1-cells

Sketch of proof.

(1) This statement follows basically immediately from the definitions.

[265] For better readability we suppress the odd skeleta from the notation.

(2) By definition of a CW-complex there exists a set C, a family $\{\varphi_c\colon S^0 \to X^0\}_{c\in C}$ of continuous maps and a homeomorphism

$$\Xi\colon X^0 \cup_{X^0 \leftarrow \bigsqcup_{c\in C} S^0_c \,:\, \bigsqcup_{c\in C}\varphi_c} \bigsqcup_{c\in C} \overline{B}^1_c \;\to\; X^1 = X$$

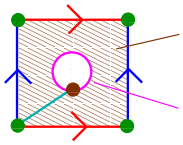
which is the inclusion on X^0. Given $c \in C$ we set $\theta(c) \coloneqq \{\varphi_c(-1), \varphi_c(1)\} \subset \mathcal{P}(X^0)$. It is clear that $G \coloneqq (X^0, C, \theta)$ is an undirected abstract graph. The map Ξ gives rise to a homeomorphism $f\colon |G| \to X$ with $f(V) = X^0$. ■

We conclude our long list of examples of CW-complexes with a sneak preview of a later result.

Theorem 30.2. (Smooth Manifold-CW-Theorem) Every compact connected n-dimensional smooth manifold M admits a finite CW-structure with the following properties:
(1) The maximal dimension of a cell is n.
(2) The boundary ∂M is a subcomplex.

Proof. This theorem is proved in [**Fri24**]. ■

Example. In the figure below we consider the smooth manifold M that is given by removing an open disk from a torus.



We see that M has a CW-structure such that the boundary ∂M is a subcomplex. □

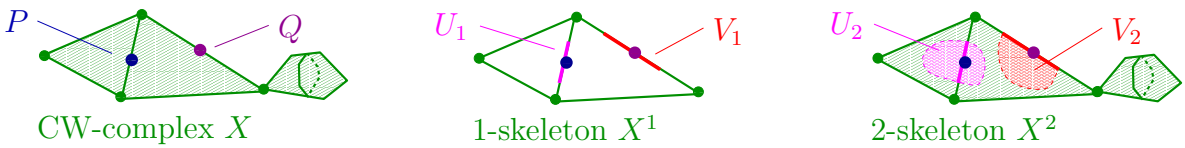
30.2. **Attaching maps and characteristic maps.** Before we continue in developing our language for CW-complexes it is essential to get the following proposition out of the way.

Proposition 30.3. (CW-Complex–Hausdorff Proposition) Every CW-complex is Hausdorff.

Proof. Let X be a CW-complex and let P, Q be two distinct points in X. Recall that $X = \bigcup_{n\in\mathbb{N}_0} X^n$. Thus there exists an $n \in \mathbb{N}_0$ with $P, Q \in X^n$.

Since the empty topological space is Hausdorff we obtain from iteratively applying the Cell Attachment–Hausdorff Proposition 29.4 (2) that $X^0, \dots, X^n$ are Hausdorff. In particular we see that there exist disjoint open neighborhoods U_n of $P \in X^n$ and V_n of $Q \in X^n$.

It follows from iteratively applying the Attachment–Hausdorff Proposition 29.4 (1) that for each $d \geq n$ we can find disjoint open neighborhoods U_d of $P \in X^d$ and V_d of $Q \in X^d$ such that for $d \geq n$ we have $U_d \subset U_{d+1}$ and $V_d \subset V_{d+1}$.



It remains to prove the following claim:

Claim. The sets $U := \bigcup_{d\geq n} U_d$ and $V := \bigcup_{d\geq n} V_d$ are disjoint open neighborhoods of P and Q in X.

Proof. Note that for each $d \geq n$ we have $(U \cap X^d) \cap (V \cap X^d) = U^d \cap V^d = \varnothing$. This shows that U and V are disjoint. It follows immediately from the Direct Colimit–Open Subset Lemma 25.9 that U and V are open subsets of X. ∎

Before we can continue with the theory of CW-complexes we need to introduce the following definition:

Definition. Let X be a CW-complex

(1) Given $n \in \mathbb{N}_0$ we denote by $\mathrm{Cell}_n(X)$ the set of n-cells of X. Furthermore we denote by $\mathrm{Cell}(X) := \bigcup_{n\in\mathbb{N}_0} \mathrm{Cell}_n(X)$ the set of all cells of X.

(2) Let $c \in \mathrm{Cell}_n(X)$ be an n-cell of X.

(a) An attaching map for c is a continuous map $\varphi\colon S^{n-1} \to X^{n-1}$ such that there exists a homeomorphism

$$X^{n-1} \cup_{X^{n-1}\leftarrow S^{n-1}\,:\varphi} \overline{B}^n \;\to\; \underbrace{X^{n-1} \cup c}_{\text{subspace of } X^n}$$

which is the identity on X^{n-1}.

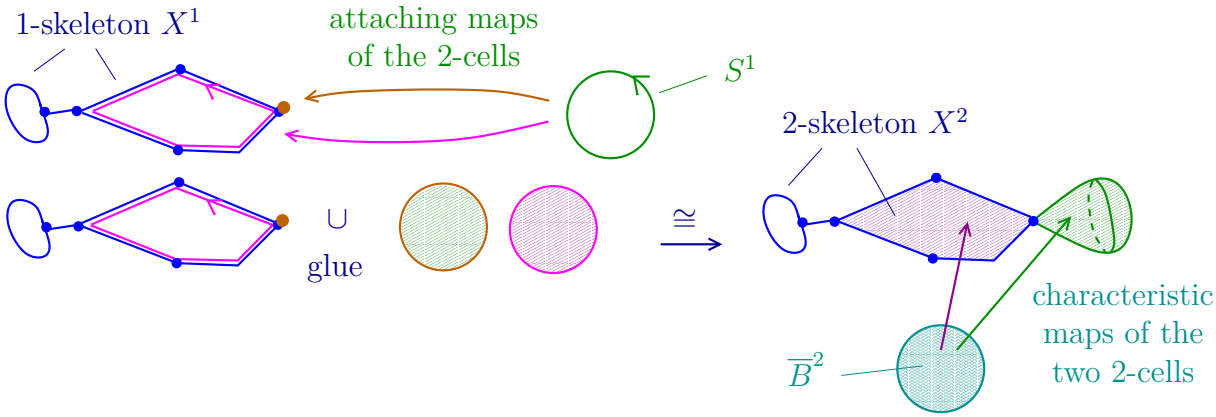
(b) A characteristic map for c is a continuous map $\Phi\colon \overline{B}^n \to X$ such that the map

$$X^{n-1} \cup_{X^{n-1}\leftarrow S^{n-1}\,:\Phi|_{S^{n-1}}} \overline{B}^n \;\to\; X^{n-1} \cup c$$

induced by the identity on X^{n-1} and by the map Φ on $\overline{B}^n$ is well-defined and it is a homeomorphism.

It follows fairly easily from the definitions that a characteristic map, and thus also an attaching map, always exists.

(3) A full set of attaching maps for the cells of the CW-complex X is defined as a family of continuous maps $\{\varphi_c\colon S^{\dim(c)-1} \to X^{\dim(c)-1}\}_{c\in\mathrm{Cell}(X)}$ such that each φ_c is an attaching map for the cell c. Similarly we define a full set of characteristic maps for the n-cells of X.



Using the CW-Complex–Hausdorff Proposition 30.3 we can prove the following essential proposition.

Proposition 30.4. (Cell-Compact Proposition) Let X be a CW-complex.
(1) Let c be an n-cell of X. If $\Phi\colon \overline{B}^n \to X$ is a characteristic map for c, then $\Phi(\overline{B}^n) = c$.
(2) Every cell is compact.

Proof. Let c be an n-cell of X.

(1) Let $\Phi\colon \overline{B}^n \to X$ be a characteristic map for c. We start out with the following claim:

Claim. The image $\Phi(\overline{B}^n)$ is a closed subset of X.

Proof. Since characteristic maps are continuous and since $\overline{B}^n$ is compact we obtain from the Compact Image Lemma 2.8 that $\Phi(\overline{B}^n)$ is compact. By the CW-Complex–Hausdorff Proposition 30.3 we know that X is Hausdorff. Thus it follows from the Compact-Closed Lemma 1.9 (2) that $\Phi(\overline{B}^n)$ is a closed subset of X. ⊞

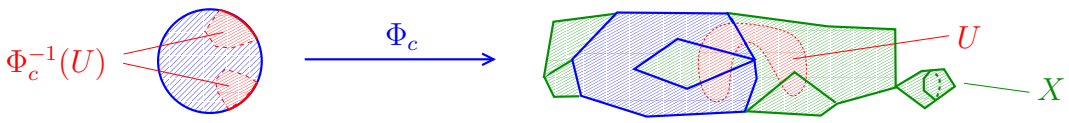
Note that we have the following inclusions

$$\Phi(\overline{B}^n) \quad \underset{\text{by an elementary argument}}{\subset} \quad \underbrace{\overline{\Phi(B^n)}}_{=c=\overline{f}} \quad \underset{\text{definition of the closure}}{=} \quad \bigcap \text{closed subsets containing } \Phi(B^n) \quad \underset{\text{by the claim}}{\subset} \quad \Phi(\overline{B}^n).$$

It follows that $\overline{\Phi(B^n)} = \Phi(\overline{B}^n)$. Next recall that by definition of an n-cell there exists an open n-cell $f \subset X$ with $\overline{f} = c$. It follows easily from the definition of a characteristic map that the restriction of Φ to B^n defines a homeomorphism $\Phi\colon B^n \to f$. It follows from the above that we have $c = \overline{f} = \overline{\Phi(B^n)} = \Phi(\overline{B}^n)$.

(2) This statement follows from (1) and the Compact Image Lemma 2.8. ■

Proposition 30.5. (Characteristic Maps Proposition) Let X be a CW-complex. Furthermore let $\{\Phi_c\colon \overline{B}^{\dim(c)} \to X^{\dim(c)}\}_{c\in \mathrm{Cell}(X)}$ be a full set of characteristic maps for the cells of X.
(1) Given any subset $U \subset X$ the following statements are equivalent:
  (a) U is open in X.
  (b) For every $c \in \mathrm{Cell}(X)$ the intersection $c \cap U$ is open in c.
  (c) For every $c \in \mathrm{Cell}(X)$ the preimage $\Phi_c^{-1}(U)$ is open in $\overline{B}^{\dim(c)}$.
  The same statements also hold if we replace "open" by "closed".
(2) Let $f\colon X \to Y$ be a map to a topological space Y. The following statements are equivalent:
  (a) f is continuous.
  (b) For every $c \in \mathrm{Cell}(X)$ the restriction of f to c is continuous.
  (c) For every $c \in \mathrm{Cell}(X)$ the map $f \circ \Phi_c\colon \overline{B}^{\dim(c)} \to Y$ is continuous.



Proof. Let X be a CW-complex. We write $C := \mathrm{Cell}(X)$ and given $n \in \mathbb{N}_0$ we write $C_n := \mathrm{Cell}_n(X)$. By definition of a CW-complex we know that for each $n \in \mathbb{N}_0$ there exists

a family $\{\varphi_c\colon S^{n-1} \to X^{n-1}\}_{c\in C_n}$ of continuous maps and a homeomorphism

$$\Xi_n\colon X^{n-1} \cup_{X \leftarrow \bigsqcup_c S_c^{n-1} \,:\bigsqcup_c \varphi_c} \bigsqcup_{c\in C_n} \overline{B}_c^n \;\to\; X^n$$

which is the inclusion on X^{n-1}. We start out with the following technical claim:

Claim. For each $n \in \mathbb{N}_0$ the map

$$\Omega_n\colon X^{n-1} \cup_{X \leftarrow \bigsqcup_{c\in C} S_c^{n-1} \,:\bigsqcup_{c\in C} \Phi_c|_{S^{n-1}}} \bigsqcup_{c\in C} \overline{B}_c^n \;\to\; X^n$$

induced by the identity on X^{n-1} and the maps $\Phi_c\colon \overline{B}_c^n \to X^n$ is well-defined and it is a homeomorphism.

Proof. We consider the map

$$\underbrace{X^{n-1} \cup_{X \leftarrow \bigsqcup_{c\in C_n} S_c^{n-1} \,:\bigsqcup_{c\in C_n} \Phi_c|_{S^{n-1}}} \bigsqcup_{c\in C_n} \overline{B}_c^n}_{=:\widetilde{S}} \xrightarrow{\Omega_n} X^n \xrightarrow[\cong]{\Xi_n^{-1}} \underbrace{X^{n-1} \cup_{X \leftarrow \bigsqcup_{c\in C_n} S_c^{n-1} \,:\bigsqcup_{c\in C_n} \varphi_c} \bigsqcup_{c\in C_n} \overline{B}_c^n}_{=:S}.$$

Since Ξ_n is a homeomorphism it suffices to show that $\Xi_n^{-1}\circ\Omega_n\colon \widetilde{S} \to S$ is a homeomorphism.

First note that one can easily verify that $\Xi_n^{-1} \circ \Omega_n\colon \widetilde{S} \to S$ is a bijection. Since each Φ_c is continuous we obtain from the Topological-Pushout Proposition 6.12 (1b) and the Topological-Coproduct Proposition 5.12 (1b) that $\Xi_n^{-1} \circ \Omega_n\colon \widetilde{S} \to S$ is continuous.

Given $c \in C_n$ we denote by $\widetilde{c} \subset \widetilde{S}$ the image of $\overline{B}_c^n$ on the left hand side. Note that $\Xi_n^{-1} \circ \Omega_n\colon \widetilde{c} \to c$ is a continuous bijection. By the Compact Image Lemma 2.8 we know that $\widetilde{c}$ is compact and by the CW-Complex–Hausdorff Proposition 30.3 we know that c is Hausdorff. Therefore it follows from the Compact-Hausdorff Proposition 2.10 (3) that $\Xi_n^{-1} \circ \Omega_n\colon \widetilde{c} \to c$ is a homeomorphism. We denote by $g_c\colon c \to \widetilde{c}$ its inverse. Furthermore we denote by $p_c\colon \overline{B}_c^n \to S$ the natural map. Since each $g_c \circ p_c$ is continuous we obtain from the Topological-Pushout Proposition 6.12 (1b) and the Topological-Coproduct Proposition 5.12 (1b) that $\Omega_n^{-1} \circ \Xi_n\colon S \to \widetilde{S}$ is continuous. ⊞

Now we turn to the actual proof of the statements of the proposition.

(1) Let $U \subset X$ be a subset. The (a)⇒(b)-direction follows from the definition of the subspace topology and the (b)⇒(c)-direction follows from the continuity of the attaching maps. It remains to deal with the (c)⇒(a)-direction. Thus we assume that U satisfies (c). It follows from the claim and the Topological-Pushout Proposition 6.12 (0) and the Topological-Coproduct Proposition 5.12 (0), applied iteratively for $n = 0, 1, 2, \ldots$, that each $X^n \cap U$ is open. But by definition of a CW-complex this means that U is open in X.

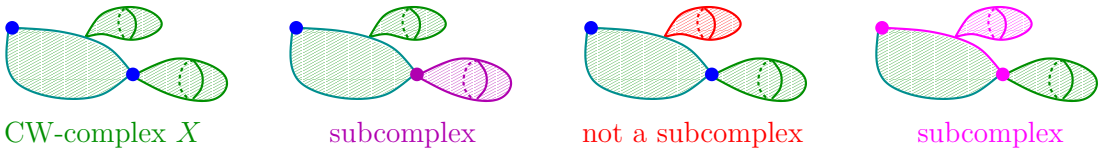
(2) This statement follows immediately from (1). ■

30.3. **Subcomplexes.** We recall the following definition from page 409:

Definition. We say that a CW-complex A is a subcomplex of a CW-complex X if $A \subset X$ and if the following two conditions are satisfied:

(1) For each $n \in \mathbb{N}_0$ we have $A^n = A \cap X^n$.
(2) Each open n-cell of A is also an open n-cell of X.

Example. If X is a CW-complex, then any skeleton X^n, with the obvious filtration, is basically by definition a subcomplex of X. □



The above definition of a subcomplex is short and succinct. But in practice it is not always that easy to work with it. In the following lemma we first show that subcomplexes are always closed subsets. Afterwards we will give a convenient characterization of subcomplexes, which in practice is quite helpful.

Lemma 30.6. (Subcomplex–Closed Lemma) Each subcomplex of a CW-complex X is a closed subset of X.

Proof. We provide the somewhat technical proof in [**Fri24**]. ■

The following proposition gives a very convenient characterisation of subcomplexes.

Proposition 30.7. (Subcomplex–Characterization Proposition) Let X be a CW-complex.

(1) Every subcomplex of X is the union of open cells of X.
(2) Let $\{\sigma_i\}_{i\in I}$ be a family of open cells of X. We set $A := \bigcup_{i\in I} \sigma_i$. The following statements are equivalent:
   (a) A is a subcomplex.
   (b) A is a closed subset of X.
   (c) For every $i \in I$ the closure $\overline{\sigma}_i$ of $\sigma_i \subset X$ is contained in A.

Proof. Also in this case we outsource the rather technical proof to [**Fri24**]. ■

The Subcomplex–Characterization Proposition 30.7 allows us to prove the following frequently used lemma.

Lemma 30.8. (Subcomplex–Intersection-Union Lemma) Let X be a CW-complex. The union and also the intersection of arbitrarily many subcomplexes of X is again a subcomplex of X.

Proof. Let X be a CW-complex and let $\{A_k\}_{k\in K}$ be a family of subcomplexes of X. For each $k \in K$ let $\{\sigma_i\}_{i\in I_K}$ be the corresponding family of open cells of X.

(1) It follows easily from the Subcomplex–Characterization Proposition 30.7 (1), together with the(a)⇔(c)-direction of the Subcomplex–Characterization Proposition 30.7 (2) that $\bigcup_{k\in K} A_k = \bigcup_{k\in K} \bigcup_{i\in I_K} \sigma_i$ is a subcomplex of X.
(2) It follows from the Subcomplex–Characterization Proposition 30.7 (1), together with the(a)⇔(b)-direction of the Subcomplex–Characterization Proposition 30.7 (2) and the fact that the intersection of closed sets is always closed that $\bigcap_{k\in K} A_k = \bigcup_{i\in \bigcap_{k\in K} I_K} \sigma_i$ is a subcomplex of X. ■

Lemma 30.9. (CW-Complex Pasting Lemma) Let X be a CW-complex and let $\{X_i\}_{i\in I}$ be a family of subcomplexes such that $\bigcup_{i\in I} X_i = X$. The following three statements hold:

(1) (a) A subset $U \subset X$ is open if and only if each $X_i \cap U$ is an open subset of X_i.
(b) A subset $A \subset X$ is closed if and only if each $X_i \cap A$ is a closed subset of X_i.
(2) A map $f\colon X \to Y$ to a topological space Y is continuous if and only if each map $f|_{X_i}\colon X_i \to Y$ is continuous.

Proof.

(1) (a) Given any subset $U \subset X$ we have

$$U \subset X \text{ is open} \overset{\text{by the Characteristic Maps Proposition 30.5 (1)}}{\Leftrightarrow} \forall_{c\in \mathrm{Cell}(X)}\, c\cap U \text{ is open in } c \overset{\text{follows from } \cup X_i = X \text{ and the observation that it is a consequence of the Subcomplex–Characterization Proposition 30.7 (a)}\Rightarrow\text{(c) that every cell of some } X_i \text{ is a cell of } X}{\Leftrightarrow} \forall_{i\in I}\forall_{c\in\mathrm{Cell}(X_i)}\, c\cap U \text{ is open in } c$$
$$\overset{\text{follows from the Characteristic Maps Proposition 30.5 (1) applied to the CW-complex } X_i}{\Leftrightarrow} \forall_{i\in I}\, X_i\cap U \text{ is open in } X_i.$$

(b) The same logic as in (a) also applies for "closed" instead of "open".
This statement is an easy consequence of the Subcomplex–Characterization Proposition 30.7 (a)⇒(c) and the Characteristic Maps Proposition 30.5 (1).

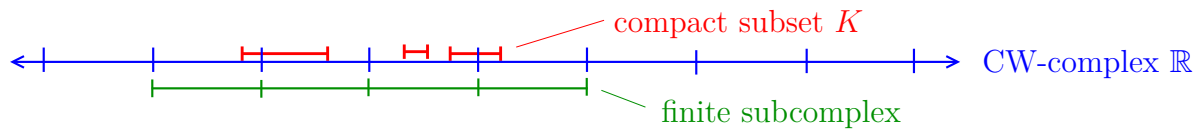
(2) The "only if"-direction is an immediate consequence of the Basics-of-Continuity Lemma 2.1 (3). The "if"-direction follows easily from (1a). ■

30.4. **The Finiteness Theorem for CW-complexes.** The following proposition is one of the key technical results about infinite CW-complexes.

Theorem 30.10. (CW-Complex-Finiteness Theorem) Every compact (connected) subset of a CW-complex X is contained in a finite (connected) subcomplex.

Example.

(1) We consider the wedge $X := \bigvee_{i\in\mathbb{N}_0} S^1_i$. As discussed on page 409, we can view X as a CW-complex with one 0-cell, given by the wedge point, and 1-cells indexed by $c \in \mathbb{N}_0$. It follows from the CW-Complex-Finiteness Theorem 30.10 that each compact subset of X is contained in the union of finitely many S^1_i.
(2) We consider the real line $\mathbb{R}$ with the CW-structure $\varnothing \subset \mathbb{Z} \subset \mathbb{R}$.



In this context the CW-Complex-Finiteness Theorem 30.10 gives us the familiar statement that any compact subset of $\mathbb{R}$ is bounded. □

Before we start out with the proof of the CW-Complex-Finiteness Theorem 30.10 we formulate the following pretty corollary.

Corollary 30.11. (CW-Complex Compactness Corollary) A CW-complex is compact if and only if it is finite.

Proof. A finite CW-complex is by definition the union of finitely many cells. Thus it follows from the Cell-Compact Proposition 30.4 (2) that a finite CW-complex is compact.

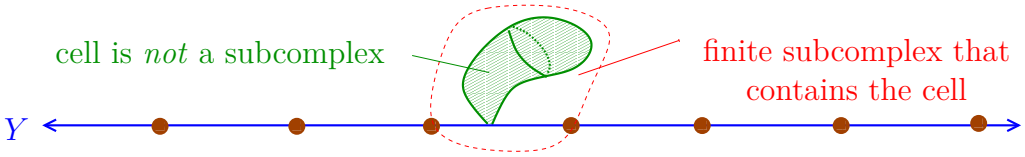
Conversely let X be a compact CW-complex. It follows from the CW-Complex-Finiteness Theorem 30.10 that X is contained in a finite subcomplex Y of X. But from $X \subset Y \subset X$ it follows that $X = Y$, i.e. X is a finite CW-complex. ■

The proof of the CW-Complex-Finiteness Theorem 30.10 will be a fairly straightforward consequence of the following rather technical lemma.

Lemma 30.12. (Contained-in-Finite Subcomplex Lemma) Let X be a CW-complex.

(1) Let $A \subset X$ be a subset. If every open cell of X contains at most one point of A, then A is closed and discrete.
(2) If $C \subset X$ is a compact subset, then C is contained in a finite union of open cells.
(3) Every cell of X is contained in a finite subcomplex of X.

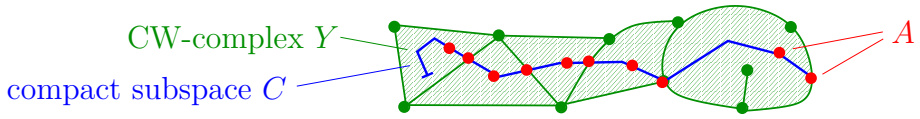
Example. Let X be a CW-complex. The example below shows that a cell itself is not necessarily a subcomplex, but, as predicted by the Contained-in-Finite Subcomplex Lemma 30.12, we see in our example that every cell is contained in a finite subcomplex.



Proof of Lemma 30.12. We start out with the following claim:

Claim. For any CW-complex Y the statements (1), (2) and (3) of the lemma are equivalent.

Proof. Let Y be a CW-complex.
(1)⇒(2). Let $C \subset Y$ be a compact subset. For each open cell σ with $C \cap \sigma \neq \varnothing$ we pick a single point in $C \cap \sigma$. We denote the resulting set of points by A. By (1) the set A is closed and discrete. Since A is a closed subset of the compact set C we obtain from the Compact-Closed Lemma 1.9 (1) that A is compact. Therefore A is in fact finite. But that means that C intersects only finitely many cells, i.e. C is contained in a finite union of open cells. This is precisely what we had to show.

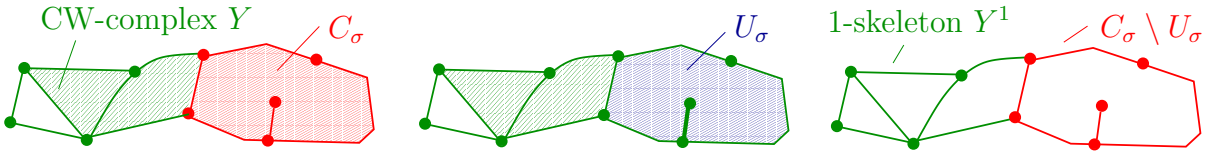


(2)⇒(3). This direction follows from the following stronger subclaim.

Subclaim. For every $n \in \mathbb{N}_0$ the following holds: If Y^{n-1} satisfies (2), then Y^n satisfies (3).

Proof of subclaim. We prove the claim by induction on $n \in \mathbb{N}_0$. For $n = 0$ there is nothing really to show. Now suppose that the statement holds for some $n \in \mathbb{N}_0$. Let $C_\sigma \subset Y$ be an $(n+1)$-cell. We denote by U_σ the corresponding open cell. Note that by the Cell-Compact Proposition 30.4 (2) we know that C_σ is compact. Furthermore note that it follows from the discussion on page 406 that U_σ is an open subset of Y^{n+1}. In particular we see that U_σ is an open subset of C_σ. Finally note that it follows from these observations and the Compact-Closed Lemma 1.9 that $C_\sigma \setminus U_\sigma$ is a compact subset of the n-skeleton Y^n.

Since we assume that (2) holds for n it follows that the subset $C_\sigma \setminus U_\sigma$ is contained in a finite union of open cells of dimension $\leq n$. By induction each of these open cells is contained in a finite subcomplex of Y. By the Subcomplex–Intersection-Union Lemma 30.8 the union of these finitely many finite subcomplexes forms a subcomplex of Y. This subcomplex, together with U_σ, forms a finite subcomplex which contains C_σ. □



(3)⇒(1). We start out with the following subclaim.

Subclaim. If $B \subset Y$ is a subset such that no two points lie in the same open cell, then B is a closed subset.

Proof of subclaim. Let $B \subset Y$ be a subset such that no two points lie in the same open cell. We need to show that B is a closed subset of X. By the Characteristic Maps Proposition 30.5 it suffices to show that the intersection of B with each cell c is closed. Thus let c be a cell of X. By (3) we know that c is contained in a finite subcomplex Y'. Since B has at most one point in common with any open cell we see that the intersection $B \cap Y'$ must be finite. Thus the intersection $B \cap c$ is finite. It follows from the CW-Complex–Hausdorff Proposition 30.3 that $B \cap c$ is a closed subset of c. □

Now let $A \subset Y$ be a subset such that no two points lie in the same open cell. By the subclaim we know that A is a closed subset of Y. We still need to show that A is discrete. Note that for any point $x \in A$ the subset $A \setminus \{x\}$ also satisfies the hypothesis of the subclaim. Thus $A \setminus \{x\}$ is a closed subset of X. Hence $\{x\}$ is open in A, so A is discrete. ⊞

Finally we turn to the actual proof of the lemma. Let X be a CW-complex. We need to show that for X all three statements hold. First note that Statement (1) holds for X^0 by definition of the topology on X^0. Now suppose that for some $n \in \mathbb{N}_0$ we have (1) for the n-skeleton X^n By the claim we know that (2) and (3) also hold for X^n. By the subclaim in the proof of (2)⇒(3) we know that (3) actually holds for X^{n+1}. Thus we also have (1) for X^{n+1}. Summarizing we have all three statements for X^n for all $n \in \mathbb{N}_0$. But any cell lies in some X^n, and so we know (3) for X itself, and by the claim this implies that (1) and (2) also hold for X. ■

Proof of the CW-Complex-Finiteness Theorem 30.10. Let X be a CW-complex and let $C \subset X$ be a compact subset. By the Contained-in-Finite Subcomplex Lemma 30.12 (2) the subset C is contained in a union of finitely many open cells. By Lemma 30.12 (3) each of these open cells is contained in a finite subcomplex. The union of these finitely many finite subcomplexes is a finite subcomplex Z which contains C.

If C is connected, then we take the component of Z that contains C. ■

30.5. **Fundamental groups of CW-complexes.** Before we head to the fundamental groups of CW-complexes let us quickly study the path-connectedness of CW-complexes. The following lemma says that the path-connectedness depends only on the 1-skeleton.

Lemma 30.13. (CW-Skeleton-π_0-Lemma) Let X be a CW-complex.
(1) The inclusion-induced map $\pi_0(X^0) \to \pi_0(X)$ is a surjection.
(2) For every $k \geq 1$ the inclusion-induced map $\pi_0(X^k) \to \pi_0(X)$ is a bijection.

Proof. We leave the proof as an exercise to the reader. ■

Example. We equip S^∞, $\mathbb{R}\mathrm{P}^\infty$ and $\mathbb{C}\mathrm{P}^\infty$ with the usual CW-structures. Their 1-skeleta are $S^1, \mathbb{R}\mathrm{P}^1$ and $\mathbb{C}\mathrm{P}^0$. Since these are path-connected we obtain from the CW-Skeleton-π_0-Lemma 30.13 that S^∞, $\mathbb{R}\mathrm{P}^\infty$ and $\mathbb{C}\mathrm{P}^\infty$ are path-connected. □

In a similar fashion the following proposition says that for CW-complexes the fundamental group is determined by the fundamental group of its 2-skeleton.

Proposition 30.14. (CW-Skeleton-π_1-Proposition) Let X be a connected CW-complex and let x_0 be a point in the 0-skeleton of X.
(1) The inclusion-induced map $\pi_1(X^1, x_0) \to \pi_1(X, x_0)$ is an epimorphism.
(2) For any $k \geq 2$ the inclusion-induced map $\pi_1(X^k, x_0) \to \pi_1(X, x_0)$ is an isomorphism.

Example. It follows from the CW-Skeleton-π_1-Proposition 30.14 together with some of our earlier calculations that

$$\begin{array}{lcllcll} \pi_1(S^\infty) & \cong & \pi_1(\text{2-skeleton of } S^\infty) & \cong & \pi_1(S^2) & = & 0, \\ \pi_1(\mathbb{R}\mathrm{P}^\infty) & \cong & \pi_1(\text{2-skeleton of } \mathbb{R}\mathrm{P}^\infty) & \cong & \pi_1(\mathbb{R}\mathrm{P}^2) & = & \mathbb{Z}_2, \\ \pi_1(\mathbb{C}\mathrm{P}^\infty) & \cong & \pi_1(\text{2-skeleton of } \mathbb{C}\mathrm{P}^\infty) & \cong & \pi_1(\mathbb{C}\mathrm{P}^1) & = & 0. \end{array}$$

□

Proof.
(1) Let $\gamma\colon [0,1] \to X$ be a loop in (X, x_0). It follows from the Compact Image Lemma 2.8 and the CW-Complex-Finiteness Theorem 30.10 that there exists a finite connected subcomplex A of X with $\gamma([0,1]) \subset A$. We consider the following commutative diagram:

$$\begin{array}{ccc} \pi_1(A^1, x_0) & \longrightarrow & \pi_1(X^1, x_0) \\ \downarrow\!\!\!\downarrow & & \downarrow \\ \pi_1(A, x_0) & \longrightarrow & \pi_1(X, x_0). \end{array}$$

Since A is a finite CW-complex we see that A is obtained from its 1-skeleton A^1 by attaching finitely many cells of dimension ≥ 2. We obtain from iteratively applying the Cell Attachment-π_1-Proposition 29.1 that the left vertical map is an epimorphism. By choice of A we see that $[\gamma]$ lies in the image of the bottom horizontal map. Since the diagram commutes we see that $[\gamma]$ lies in the image of the right vertical map. We have thus shown that the right vertical map is indeed an epimorphism.
(2) The proof of this statement is a variation on the argument of (1) and it follows from the same logic as in the proof of the Direct Colimit-π_1-Proposition 26.1. We leave it to the reader to fill in the details. ■

Proposition 30.15. (CW-π_1-Finite Presentation–Proposition) Let X be a connected CW-complex and let $x_0 \in X$.
(1) If there are only finitely many cells in dimensions 0 and 1, then $\pi_1(X, x_0)$ is finitely generated.

(2) If there are only finitely many cells in dimensions 0, 1 and 2, then $\pi_1(X, x_0)$ is finitely presented.

Proof.

(1) By the CW-Skeleton-π_1-Proposition 30.14 it suffices to show that the fundamental group of the 1-skeleton X^1 is finitely generated. Note that by hypothesis and by the CW-Skeleton-π_0-Lemma 30.13 we know that the 1-skeleton is a finite connected 1-dimensional CW-complex. It follows now easily from the Cell Attachment-π_1-Proposition 29.1, or alternatively from the Graph–CW-Complex Lemma 30.1 together with the Graph-π_1-Proposition 20.9, that $\pi_1(X^1)$ is a finitely generated free group.

(2) As in (1) we see that by the CW-Skeleton-π_1-Proposition 30.14 it suffices to show that a finite connected 2-dimensional CW-complex X has a finitely presented fundamental group. We pick $x_0 \in X$. Let $\varphi_1, \dots, \varphi_k \colon S^1 \to X$ be attaching maps of all of the 2-cells. For $i = 1, \dots, k$ we consider the corresponding path $\widetilde{\varphi}_i \colon [0,1] \to X$ that is given by $\widetilde{\varphi}_i(t) \coloneqq \varphi(\exp(2\pi \mathrm{i} \cdot t))$. Furthermore, for $i = 1, \dots, k$ we pick a path p_i from x_0 to the point $\widetilde{\varphi}_i(0) = \widetilde{\varphi}_i(1)$ and we write $\psi_i = p_i * \widetilde{\varphi}_i * \overline{p}_i$. Now we see that

$$\pi_1(X) \underset{\uparrow}{\cong} \pi_1(X^1)/\langle\langle [\psi_1] \rangle\rangle \dots /\langle\langle [\psi_k] \rangle\rangle = \pi_1(X^1)/\langle\langle [\psi_1], \dots, [\psi_k] \rangle\rangle$$

by the Change-of-Base Point Proposition 11.6 and by the Cell Attachment-π_1-Proposition 29.1 — follows from the Quotient Presentation–Lemma 21.6

$$= \underset{\uparrow}{} \{\text{finitely generated free group}\}/\langle\langle [\psi_1], \dots, [\psi_k] \rangle\rangle.$$

see the argument in (1)

We have thus shown that $\pi_1(X)$ is a finitely presented group. ∎

30.6. **CW-complexes and direct colimits.** The following propositions give a convenient variation on the Exhaustion-π_1-Proposition 26.2.

Proposition 30.16. (CW-Direct Colimit-π_n-Proposition) Let X be a CW-complex and let $X_1 \subset X_2 \subset X_3 \subset \dots$ be a sequence of subcomplexes such that $X = \bigcup_{i \in \mathbb{N}} X_i$.

(1) We have $X = \operatorname{colim} X_i$.

(2) For any $n \in \mathbb{N}_0$ and any $w \in X_1$ we have a natural isomorphism[266]

$$\operatorname{colim} \pi_n(X_i, w) \xrightarrow{\cong} \pi_n\big(\underbrace{\operatorname{colim} X_i}_{=X}, w\big).$$

Sketch of proof.

(1) This statement follows immediately from the characterization of the direct colimit $\operatorname{colim} X_i$ in the Topological-Direct Colimit Proposition 25.5 together with the CW-Complex Pasting Lemma 30.9.

(2) The proof of this proposition is basically identical to the proof of the Exhaustion-π_1-Proposition 26.2. We just need to replace the Compact Subspace–Exhaustion Lemma 20.11 by the CW-Complex-Finiteness Theorem 30.10. ∎

Examples.

(1) For every $n \in \mathbb{N}_0$ we have

[266] For $n = 0$ this is an isomorphism of sets, i.e. a bijection. For $n \geq 1$ this is an isomorphism of groups.

$$\begin{array}{ccccccc} \pi_n(S^\infty) & \cong & \varinjlim \pi_n(S^i) & = & \varinjlim (\,? \to \ldots \to ? \to \overset{\pi_n(S^{n+1})}{\overset{\downarrow}{0}} \to 0 \to \ldots) & \cong & 0. \\ \uparrow & & & \uparrow & & \uparrow & \end{array}$$

by the Direct Colimit-π_n Proposition 30.16

since by the Sphere-Low-π_n Proposition 28.8 we have $\pi_n(S^i) = 0$ for $i > n$

by the CW-Direct Colimit Stabilization Lemma 25.1

(2) Recall that by the Infinite Projective Space-Sphere Quotient Lemma 25.11 we know that the group $\{\pm 1\}$ acts continuously and discretely on S^∞ and that $S^\infty/\{\pm 1\}$ is homeomorphic to $\mathbb{R}\mathrm{P}^\infty$. Therefore it follows from (1) and the Action-π_1–Theorem 15.1 that $\pi_1(\mathbb{R}\mathrm{P}^\infty) \cong \pi_1(S^\infty/\{\pm 1\}) \cong \mathbb{Z}_2$. Furthermore it follows from (1), the Action–Covering Proposition 14.5 and the Covering-π_n-Isomorphism Proposition 28.10 that for every $n \in \mathbb{N}_{\geq 2}$ we have $\pi_n(\mathbb{R}\mathrm{P}^\infty) \cong \pi_n(S^\infty) = 0$.

CHAPTER 31

Homotopies on CW-complexes

In this chapter we study homotopies on CW-complexes. The technical tool for doing so is the all important CW×Interval Proposition 31.1. We will use this result to show that the infinite dimensional sphere S^∞ is contractible and we will use this result to show that CW-complexes are locally contractible.

31.1. **The topology of the product of a CW-complex with an interval.** We want to study homotopies of continuous maps $X \to Y$ between topological spaces. In other words, we want to study *continuous* maps $X \times [0,1] \to Y$. In general it is quite difficult to show that a given map out of a product is continuous. Fortunately the following important proposition simplifies life quite a bit if X is a CW-complex.

Proposition 31.1. (CW×Interval Proposition) Let X be a CW-complex.

(1) The following two topologies on $X \times [0,1]$ agree:
   (a) The product topology on the product $X \times [0,1]$.
   (b) The topology on $X \times [0,1]$ viewed as the direct colimit $\operatorname{colim}_{\longrightarrow} \underbrace{(X^k \times [0,1])}_{\text{product topology}}$.

(2) Let $f\colon X \times [0,1] \to Y$ be a map to a topological space Y. The following statements are equivalent:
   (a) The map f is continuous where $X \times [0,1]$ is equipped with the product topology.
   (b) For every $k \in \mathbb{N}_0$ the restriction of f to $X^k \times [0,1]$ is continuous.

Proof. First we show that any subset of $X \times [0,1]$ that is open with respect to (a) is also open with respect to (b). Thus let $U \subset X \times [0,1]$ be a subset that is open with respect to the product topology on $X \times [0,1]$. We need to show that U is open in $\operatorname{colim}_{\longrightarrow} (X^k \times [0,1])$. By definition of the direct colimit topology this means that we need to show that for each $k \in \mathbb{N}_0$ the intersection $U \cap (X^k \times [0,1])$ is open in $X^k \times [0,1]$ where we equip $X^k \times [0,1]$ with the product topology. By the Product Topology-Basics Lemma 5.3 (1a) we know that the subspace topology is compatible with taking products. In our context this implies that the product topology on $X^k \times [0,1]$ conveniently enough agrees with the subspace topology coming from $X \times [0,1]$. It follows from this discussion that $U \cap (X^k \times [0,1])$ is indeed open in $X^k \times [0,1]$.

The converse, that every subset of $X \times [0,1]$ that is open with respect to (a) is also open with respect to (b) is much more difficult and technical. It is not a particularly interesting task to read the proof, but it is good to know that this direction is non-trivial and that it requires some thought. In this spirit we outsource the proof to [**Fri24**]. ■

We now consider the infinite-dimensional sphere S^∞ which is defined as[267]

$$S^\infty \;:=\; \left\{(x_1, x_2, \dots) \in \mathbb{R}^\infty \;\middle|\; \sum_{j=1}^{\infty} x_j^2 = 1\right\} \;=\; \operatorname*{colim}_{\longrightarrow} S^i.$$

On page 420 we showed that $\pi_n(S^\infty) = 0$ for every $n \in \mathbb{N}$. Thus none of the homotopy groups prevent S^∞ from being contractible. Indeed the next mind-bending lemma shows that S^∞ is, in contrast to the finite dimensional spheres, contractible.

Lemma 31.2. (Infinite Sphere–Contractibility Lemma) The infinite-dimensional sphere S^∞ is contractible.

Proof. We want to show that S^∞ is contractible. It suffices to show that there exists a homotopy H from id_{S^∞} to a constant map. Our goal is to find such H. We set

$$X \;:=\; \{(0, x_2, x_3, \dots) \in \mathbb{R}^\infty \,|\, x_2, x_3, \dots \in \mathbb{R}\}.$$

Next we consider the maps

$$\begin{aligned} F\colon \mathbb{R}^\infty \times [0,1] &\to \mathbb{R}^\infty \\ (P,t) &\mapsto \underbrace{P\cdot(1-t) + (0,P)\cdot t}_{\neq 0 \text{ if } P \neq 0} \end{aligned} \quad \text{and} \quad \begin{aligned} G\colon X \times [0,1] &\to \mathbb{R}^\infty \\ (Q,t) &\mapsto \underbrace{Q\cdot(1-t) + (1,0,\dots)\cdot t}_{\neq 0 \text{ if } Q \neq 0}. \end{aligned}$$

Note that F_1 takes values in X and note that G_1 is a constant map. For non-zero $P \in \mathbb{R}^\infty$ and $P \in X$ we define $\widetilde{F}(P,t) \in S^\infty$ and $\widetilde{G}(P,t) \in S^\infty$ to be the normalizations of $F(P,t)$ and $G(P,t)$ to length one. Next we consider the map[268]

$$\begin{aligned} H\colon S^\infty \times [0,1] &\to S^\infty \\ (P,t) &\mapsto \begin{cases} \widetilde{F}(P, 2t), & \text{if } t \in [0, \frac{1}{2}], \\ \widetilde{G}(\underbrace{\widetilde{F}(P,1)}_{\in X}, 2t-1), & \text{if } t \in [\frac{1}{2}, 1]. \end{cases} \end{aligned}$$

Clearly $H_0 = \mathrm{id}$ and it is straightforward to see that H_1 is a constant map. It is tempting to conclude that we are done. But in fact we still need to show the following claim:

Claim. The map H is continuous.

Proof. The proof proceeds in several steps:

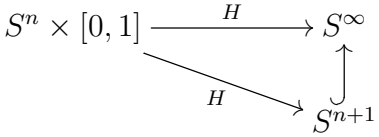
(1) On page 409 we saw that we can equip S^∞ with a CW-structure where each i-skeleton is given by $S^i \subset S^\infty$.

(2) By (1) and by the CW×Interval Proposition 31.1 (2) it suffices to show that for each $n \in \mathbb{N}$ the map

$$\begin{aligned} H\colon S^n \times [0,1] &\to S^\infty \\ (x,t) &\mapsto H((x,0),t) \end{aligned}$$

is continuous.

(3) Let $n \in \mathbb{N}_0$. Note that $H(S^n \times [0,1]) \subset S^{n+1}$. Thus we obtain the following commutative diagram:



[267] To simplify the notation we think of S^i as a subset of S^∞ via the injective map $x \mapsto (x, 0)$.

[268] One can easily verify that the map is well-defined for $t = \frac{1}{2}$.

By the Topological-Direct Colimit Proposition 25.5 (1a) we know that the vertical map is a continuous. It follows from (2) and this discussion that it suffices to show that for each $n \in \mathbb{N}_0$ the diagonal map is a continuous.

(4) By (3) and by the Homotopy Combination Lemma 8.2 (1) it suffices to show that for each $n \in \mathbb{N}_0$ the restriction of $H\colon S^n \times [0,1] \to S^{n+1}$ to the subspaces $S^n \times [0, \frac{1}{2}]$ and $S^n \times [\frac{1}{2}, 1]$ are continuous. But now we are in familiar territory and the continuity is a consequence of elementary arguments. ■

31.2. **Trees.** In this section we want to return to the study of "trees", which we first encountered on page 152. Our main result on trees builds on the following proposition, which is of interest in its own right, and which is an application of the CW×Interval Proposition 31.1 (2).

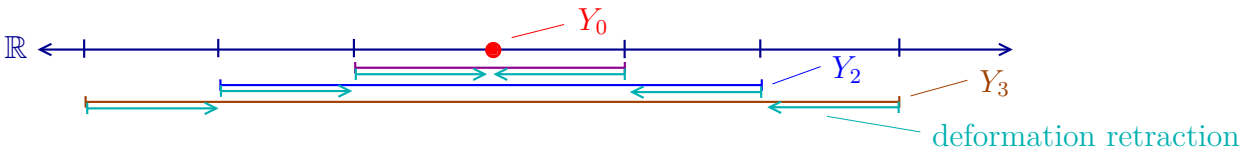
Proposition 31.3. (Iterated Deformation Retract Proposition) Let X be a CW-complex and let $Y_0 \subset Y_1 \subset Y_2 \subset \dots$ be a sequence of subcomplexes of X such that the following two conditions are satisfied:

(1) we have $X = \bigcup_{k\in\mathbb{N}_0} Y_k$,

(2) each Y_k is a deformation retract of Y_{k+1}.

Then Y_0 is a deformation retract of X.

Example. We consider the CW-complex $\mathbb{R}$ with the usual CW-structure that is given by $\varnothing \subset \mathbb{Z} \subset \mathbb{R}$. For $k \in \mathbb{N}_0$ we consider the subcomplex $Y_k := [-k, k]$. Evidently each Y_k is a deformation retract of Y_{k+1}. Thus we obtain from the Iterated Deformation Retract Proposition 31.3 that $\{0\}$ is a deformation retraction of $\mathbb{R}$.



This statement is not very surprising. But it is perhaps somewhat puzzling how one can get out of this infinite family of deformation retractions from Y_{k+1} to Y_k a deformation retraction from $\mathbb{R}$ to $\{0\}$. □

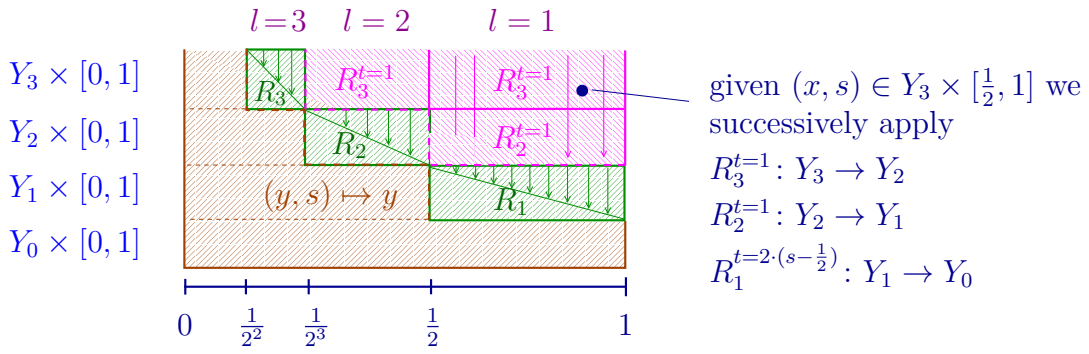
Proof. By hypothesis there exists for each $k \in \mathbb{N}_0$ a deformation retraction

$$R_k\colon Y_k \times [0,1] \;\to\; Y_k$$

from Y_k to Y_{k-1}. We consider the map

$$F\colon X\times[0,1] \to X$$
$$(x,s) \mapsto \begin{cases} x, & \text{if } x\in Y_k \text{ and } s\in[0,\frac{1}{2^k}], \\ \underbrace{(R_l^{t=2^l\cdot(s-\frac{1}{2^l})} \circ R_{l+1}^{t=1} \circ \dots \circ R_k^{t=1})(x)}_{\in Y_{l-1}}, & \text{if } x\in Y_k \text{ and } s\in[\tfrac{1}{2^l},\tfrac{1}{2^{l-1}}] \text{ with } 1\le l\le k. \end{cases}$$

One can easily verify that this map is well-defined. It follows from the Homotopy Combination Lemma 8.2 (1) that the restriction of F to each $Y_k \times [0,1]$ is continuous. Thus it follows from the CW×Interval Proposition 31.1 (2) that $F\colon X \times [0,1] \to X$ is continuous. This map is the desired deformation retraction from X to Y_0. ■



Next we turn to the study of trees. On page 152 we defined what it means for an undirected abstract graph to be a tree. We now define what it means for a CW-complex to be a tree:

Definition. A simply connected CW-complex of dimension ≤ 1 is called a tree.

Remark. As we just mentioned, on page 152 we defined what it means for an undirected abstract graph to be a tree. In the Graph–CW-Complex Lemma 30.1 we saw that undirected abstract graphs and CW-complexes of dimension ≤ 1 are essentially the same. It follows from the Tree–π_1-Proposition 20.12 that the two definitions of a tree are compatible. □

In the Finite Tree–Contractible Proposition 17.4 we showed that finite trees are contractible. Now we can generalize this statement to arbitrary trees:

Proposition 31.4. (Tree–Contractible Proposition) Let T be a non-empty connected CW-complex of dimension ≤ 1. The following statements are equivalent:

(1) T is a tree.
(2) Each 0-cell is a deformation retract of T.
(3) T is contractible.

Proof. Let T be a non-empty connected CW-complex of dimension ≤ 1. The (2)⇒(3)-direction follows from the Deformation Retract–Homotopy Equivalence Lemma 9.3 and the (3)⇒(1)-direction follows from the Homotopy-π_1-Proposition 13.4.

Thus at this stage it remains to prove the (1)⇒(2)-direction. Let $v_0 \in T$ be a 0-cell. By the Iterated Deformation Retract Proposition 31.3 it suffices to prove the following claim:

Claim. There exists a sequence $Y_0 \subset Y_1 \subset Y_2 \subset \ldots$ of subcomplexes of T with the following properties:

(1) We have $\bigcup_{i \in \mathbb{N}_0} Y_i = T$.
(2) Each Y_i is a deformation retract of Y_{i+1}.

Proof. We construct a sequence $\{v_0\} = Y_0 \subset Y_1 \subset Y_2 \subset \ldots$ of subcomplexes of T as follows:[269]

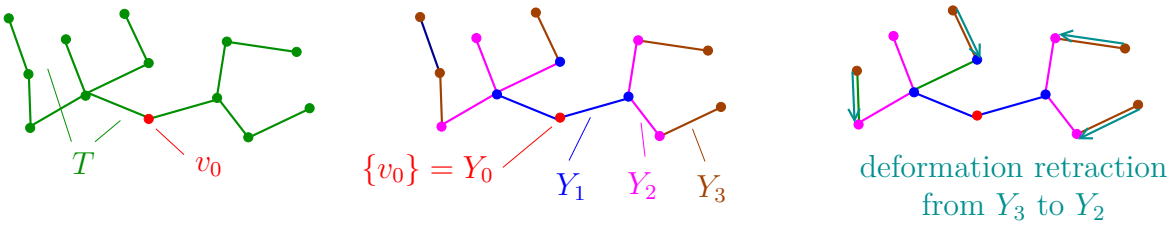
(a) We set $Y_0 := \{v_0\}$.
(b) Given Y_i we define Y_{i+1} to be the union of Y_i with all 1-cells which contain at least one 0-cell of Y_i.

[269]The construction is very similar to a construction that we gave in the proof of the Spanning Tree–Existence Proposition 7.6.

We make the following observations:

(i) Note that every 1-cell of any CW-complex defines a subcomplex of T.[270] Thus it follows from the Subcomplex–Intersection-Union Lemma 30.8 that Y_{i+1} is a subcomplex of T.

(ii) We set $Y \coloneqq \bigcup_{i\in\mathbb{N}} Y_i$. We claim that $T = Y$. This can be seen as follows. First note that, basically by construction of Y, we have for each characteristic map[271] $\Phi\colon \overline{B}^n \to T$, $n = 0, 1$ that $\Phi^{-1}(Y) = \varnothing$ or $\Phi^{-1}(Y) = \overline{B}^n$. Since T is 1-dimensional it follows from the Characteristic Maps Proposition 30.5 (1) that Y is a subset of T that is closed and that is open. Since T is connected and since Y is non-empty we see that $Y = T$.



It remains to show that each Y_i is a deformation retract of Y_{i+1}. As we will see, this is almost obvious, except that we need to worry a little bit about continuity issues. Let V be the set of 0-cells in Y_{i+1} that are not contained in Y_i. Note that for each $v \in V$ the corresponding 1-cell contains two 0-cells, namely v and one 0-cell w in Y_i. Finally we consider the map

$$F\colon \Big(\overbrace{\Big(Y_i \sqcup \bigsqcup_{v\in V} [0,1]_v\Big)/v \sim (0,v)}^{=Y_{i+1}}\Big) \times [0,1] \to Y_{i+1} = \Big(Y_i \sqcup \bigsqcup_{v\in V} [0,1]_v\Big)/v \sim (0,v)$$

$$([P],t) \mapsto \begin{cases} P, & \text{if } P \in Y_i, \\ [(x\cdot(1-t),v)], & \text{if } P = (x,v) \in [0,1]_v. \end{cases}$$

It follows almost immediately from the Homotopy Pushout Lemma from [**Fri24**] that the map F is continuous. It is now basically clear that F defines a deformation retraction from Y_{i+1} to Y_i. ■

31.3. **The fundamental group of 1-dimensional CW-complexes.** Our next goal is to use the results from the previous section to calculate the fundamental groups of 1-dimensional CW-complexes. To formulate our theorem we need to introduce the following definition:

Definition. Let X be a 1-dimensional CW-complex. A spanning tree is a subcomplex T of X that is a tree and that contains all 0-cells of X.[272]

The following theorem can be viewed as a generalization of the Graph-π_1-Proposition 20.9 and the Graph-Homotopy Type Corollary 17.6.

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[270] This follows from the observation that any non-empty subset of a 0-cell is all of the 0-cell.

[271] For $n = 0$ the statement is a tautology. For $n = 1$ note that if $\Phi^{-1}(Y) \neq \varnothing$, then there exists an $i \in \mathbb{N}_0$ with $\Phi^{-1}(Y_i) \neq \varnothing$. Since Y_i is a subcomplex we see that $\Phi^{-1}(Y_i)$ contains at least one 0-cell. But by construction this means that $\Phi^{-1}(Y_{i+1}) = \overline{B}^1$.

[272] On page 153 we introduced the notion of a spanning tree for an undirected abstract graph. It follows immediately from the definitions together with the Tree–π_1-Proposition 20.12 that the two notions of spanning tree match under the correspondence given by the Graph–CW-Complex Lemma 30.1.



Theorem 31.5. (1-Complex-π_1-Theorem) Let X be a connected non-empty 1-dimensional CW-complex X.

(1) X admits a spanning tree.

Let T be a spanning tree for X and let I be the set of 1-cells of X that are not contained in T.

(2) (a) X is homotopy equivalent to $\bigvee_{i\in I} S^1$.
(b) Given any $x_0 \in X$ the group $\pi_1(X, x_0)$ is isomorphic to the free group on the set I.

(3) If the CW-complex X is finite, then we have $\#I = 1 - \chi(X)$, where the Euler characteristic $\chi(X)$ of X is defined as $\chi(X) := \#\{\text{0-cells}\} - \#\{\text{1-cells}\}$.

The proof of the 1-Complex-π_1-Theorem 31.5 rests on the following proposition, which is perhaps of independent interest.

Proposition 31.6. (1-Complex-mod-Tree Proposition) Let X be a 1-dimensional CW-complex. For every subtree T of X the projection $X \to X/T$ is a homotopy equivalence.

Proof of the 1-Complex-mod-Tree Proposition 31.6. The proof of the proposition is basically identical to the proof of the Graph-mod-Tree Proposition 17.5. We just need to replace the Finite Tree–Contractible Proposition 17.4 by the Tree–Contractible Proposition 31.4. ■

Proof of the 1-Complex-π_1-Theorem 31.5. Let X be a connected non-empty 1-dimensional CW-complex X.

(1) It follows almost immediately from the Spanning Tree–Existence Proposition 7.6 together with the correspondence given by the Graph–CW-Complex Lemma 30.1 that X admits a spanning tree.

Let T be a spanning tree for X and let I be the set of 1-cells of X that are not contained in T.

(2) (a) By the 1-Complex-mod-Tree Proposition 31.6 we know that the projection $X \to X/T$ is a homotopy equivalence. It follows from the Abstract Graph–Quotient Lemma 7.9 that X/T admits the structure of a 1-dimensional CW-complex with a single 0-cell and one 1-cell for each $i \in I$. It follows from the discussion on page 247 that X/T is homeomorphic to the wedge $\bigvee_{i\in I} S^1$.
(b) This statement follows from (a) together with the Homotopy-π_1-Proposition 13.4 and the Wedge-of-Circles–π_1-Lemma 20.7.

(3) This statement follows from the Graph-Homotopy Type Corollary 17.6 together with the Graph–CW-Complex Lemma 30.1. ■

[272] The two paths exist since a tree is in particular connected.

31.4. **Local properties of CW-complexes.** The following, rather technical proposition, says that CW-complexes are locally in many ways very well-behaved.

Proposition 31.7. (CW-Complex–Local Properties Proposition)

(1) Let X be a CW-complex, let $x \in X$ and let U be an open neighborhood of x. There exists an open neighborhood V of x that is contained in U and that admits a deformation retraction to $\{x\}$.

(2) CW-complexes are locally contractible, locally path-connected and locally simply-connected.



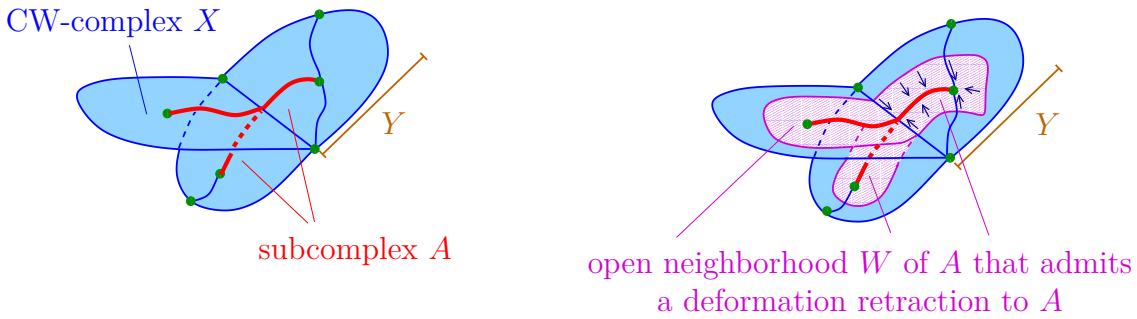
Proof. The proof of this proposition is technically fairly demanding and it also rests on the CW×Interval Proposition 31.1 (2). Since the proof, despite or because of the difficulty, is not very interesting to read we hide it in [**Fri24**]. ■

On several occasions we will need the following variation on the CW-Complex–Local Properties Proposition 31.7.

Proposition 31.8. (Subcomplex–Neighborhood Proposition) Let X be a CW-complex.

(1) For each subcomplex A of X there exists an open neighborhood W of A and a deformation retraction $F\colon W \times [0,1] \to W$ from W to A.

(2) One can choose the deformation retraction such that for each subcomplex Y of X the map F restricts to a deformation retraction $F\colon (Y \cap W) \times [0,1] \to Y \cap W$ from $Y \cap W$ to $Y \cap A$.



Proof. The proof of this proposition is similar to the proof of the CW-Complex–Local Properties Proposition 31.7. Again we refer to [**Fri24**] for details. ■

CHAPTER 32

Coverings of CW-complexes

Let X be a connected CW-complex, let $x_0 \in X$ and let $\Gamma \subset \pi_1(X, x_0)$ be a subgroup. By the CW-Complex–Local Properties Proposition 31.7 we know that X is locally simply connected. It follows from the Covering Existence Theorem 27.2 that there exists an (essentially unique) covering $p\colon \widetilde{X} \to X$ with $p_*(\pi_1(\widetilde{X}, \widetilde{x})) = \Gamma$. This discussion shows that CW-complexes tend to have many coverings. In this short chapter we will see that any covering of a CW-complex inherits a natural CW-structure. We will use this topological result to prove several "topology free" statements in group theory.

32.1. **Coverings of CW-complexes.** In this section we will see that, as promised above, a covering of a CW-complex inherits a natural CW-structure.

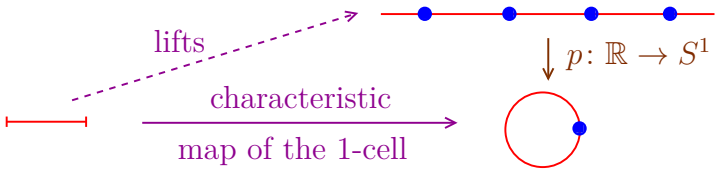
Proposition 32.1. (CW-Complex Covering Proposition) Let $p\colon \widetilde{X} \to X$ be a connected covering of a non-empty CW-complex X.

(1) The filtration
$$\varnothing = p^{-1}(X^{-1}) \subset p^{-1}(X^0) \subset p^{-1}(X^1) \subset \ldots \subset \widetilde{X}$$
defines a CW-structure for $\widetilde{X}$ which has the property that for any $k \in \mathbb{N}_0$ we have[273][274]
$$\#k\text{-cells of } \widetilde{X} \;=\; \big[\widetilde{X} : X\big] \cdot \#k\text{-cells of } X.$$

(2) (a) Each characteristic map $\Phi\colon \overline{B}^n \to X^n$ admits precisely $[\widetilde{X} : X]$ lifts.
(b) Let $\{\Phi_c\colon \overline{B}^{\dim(c)} \to X\}_{c \in \mathrm{Cell}(X)}$ be a full set of characteristic maps for X. The set of all lifts of all these characteristic maps is a full set of characteristic maps for $\widetilde{X}$.

Examples.

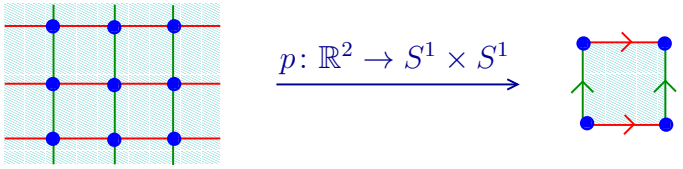
(1) We equip S^1 with the CW-structure $\varnothing \subset \{1\} \subset S^1$ consisting of one 0-cell and one 1-cell. We denote by $p\colon \mathbb{R} \to S^1$, $t \mapsto \exp(2\pi i t)$ the universal covering of S^1. It follows from the CW-Complex Covering Proposition 32.1, that a CW-structure on $\mathbb{R}$ is given by $\varnothing \subset p^{-1}(\{1\}) \subset S^1$, i.e. by $\varnothing \subset \mathbb{Z} \subset \mathbb{R}$.



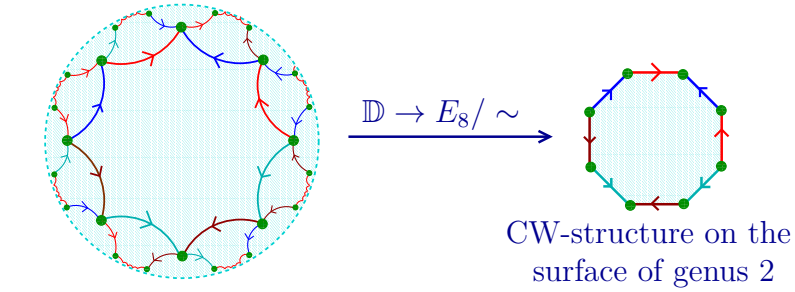
[273] Recall that on page 215 we defined $[\widetilde{X} : X] := \#p^{-1}(x)$ where x is any point in X. Since X is non-empty and connected this definition makes sense.

[274] Note that this is an equality of cardinalities. For example, if $[\widetilde{X} : X]$ is infinite countable and if X has finitely many k-cells, then $\widetilde{X}$ has infinitely many, but countably many, k-cells.

(2) We equip $S^1 \times S^1 =_{\mathrm{i}} ([0,1]\times[0,1])/\sim$ with the usual CW-structure with one 0-cell, two 1-cells and one 2-cell. We consider the universal covering $\mathbb{R}^2 \to S^1 \times S^1$. The resulting CW-structure on $\mathbb{R}^2$ is illustrated in the figure below.



(3) We consider $\mathbb{D} = \{z \in \mathbb{C} \,|\, |z| < 1\}$. Recall that Hadamard's Theorem 27.8 says that there exists a covering map $\mathbb{D} \to H_8/\sim \cong E_8/\sim$. If we equip $E_8/\sim$ with the "usual" CW-structure then we obtain from the CW-Complex Covering Proposition 32.1 the CW-structure on $\mathbb{D}$ that is shown in the figure below.



Proof. We start out with an observation. Let $\Phi\colon \overline{B}^n \to X$ be any continuous map. Since each $\overline{B}^n$ is simply connected and locally path-connected we obtain from the Map Lifting Criterion 27.1 and the Lift Uniqueness Proposition 14.7 that for each $y \in p^{-1}(\Phi(0))$ there exists a unique lift $\widetilde{\Phi}_y\colon \overline{B}^n \to \widetilde{X}$ of $\Phi\colon \overline{B}^n \to X$ with $\widetilde{\Phi}_y(0) = y$. Note that this observation already implies (2a).

We turn to the proof of (1) and (2b). We introduce the following notation:

- For $n \in \mathbb{N}_0$ we set $\widetilde{X}^n := p^{-1}(X^n)$.
- Let $\{\Phi_c\colon \overline{B}^{\dim(c)} \to X\}_{c\in \mathrm{Cell}(X)}$ be a full set of characteristic maps for the CW-complex X. By the above observation we obtain for each $c \in \mathrm{Cell}(X)$ the family of continuous maps $\{\widetilde{\Phi}_{c,y}\colon \overline{B}^{\dim(c)} \to \widetilde{X}\}_{y\in p^{-1}(\Phi_c(0))}$.

Statements (1) and (2b) follow from the following two claims.

Claim 1. A set $U \subset \widetilde{X}$ is open if and only if for each $n \in \mathbb{N}_0$ the intersection $\widetilde{X}^n \cap U$ is open in $\widetilde{X}^n$.

Claim 2. Let $n \in \mathbb{N}_0$. We consider the map

$$\Theta\colon \widehat{\widetilde{X}^n} := \widetilde{X}^{n-1} \cup \bigcup_{c\in \mathrm{Cell}_n(X)} \bigcup_{y\in p^{-1}(\Phi_c(0))} \overline{B}^n_{c,y} \;\to\; \widetilde{X}^n$$

that is induced by the identity on $\widetilde{X}^{n-1}$ and by the maps $\widetilde{\Phi}_{c,y}\colon \overline{B}^n \to \widetilde{X}^n$. This map is a homeomorphism.

Proof of Claim 1. Since $p\colon \widetilde{X} \to X$ is a covering there exists an open cover $\{\widetilde{W}_i\}_{i\in I}$ of $\widetilde{X}$ such that for each $i \in I$ the image $p(\widetilde{W}_i) =: W_i$ is an open subset of X and that the map

$p_i := p|_{\widetilde{W}_i} : \widetilde{W}_i \to W_i$ is a homeomorphism. Now we see that

$$
\begin{aligned}
& \bigwedge_{n\in\mathbb{N}_0} \widetilde{X}^n \cap U \text{ is open in } \widetilde{X}^n \\
\iff & \bigwedge_{n\in\mathbb{N}_0} \bigwedge_{i\in I} \widetilde{W}_i \cap (\widetilde{X}^n \cap U) \text{ is open in } \widetilde{W}_i \cap \widetilde{X}^n && \text{(Cover-Open Subset Lemma 1.5)} \\
\iff & \bigwedge_{n\in\mathbb{N}_0} \bigwedge_{i\in I} p_i(\widetilde{W}_i \cap (\widetilde{X}^n \cap U)) \text{ is open in } p_i(\widetilde{W}_i \cap \widetilde{X}^n) && \text{(since each } p_i \text{ is a homeomorphism)} \\
\iff & \bigwedge_{i\in I} \bigwedge_{n\in\mathbb{N}_0} (W_i \cap p_i(U)) \cap X^n \text{ is open in } W_i \cap X^n && \text{(since } \widetilde{X}^n = p^{-1}(X^n) \text{ and } p(\widetilde{W}_i) = W_i) \\
\iff & \bigwedge_{i\in I} \bigwedge_{n\in\mathbb{N}_0} (W_i \cap p_i(U)) \cap X^n \text{ is open in } X^n && \text{(since } W_i \text{ is open in } X) \\
\iff & \bigwedge_{i\in I} W_i \cap p_i(U) \text{ is open in } W_i && \text{(since } X \text{ is a CW-complex)} \\
\iff & \bigwedge_{i\in I} \widetilde{W}_i \cap U \text{ is open in } \widetilde{W}_i && \text{(since } p_i \text{ is a homeomorphism)} \\
\iff & U \text{ is open in } \widetilde{X} && \text{(Cover-Open Subset Lemma 1.5).}
\end{aligned}
$$

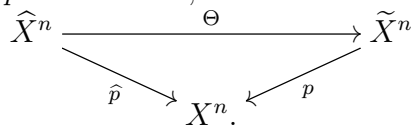
⊞

Proof of Claim 2. First note that one can easily verify that Θ is a bijection. Also note that it follows from the Topological-Pushout Proposition 6.12 (0) that the map Θ is continuous. To show that it is actually a homeomorphism we have to work a little harder.

To do so we consider the map

$$\widehat{p}\colon \widehat{X}^n = \widetilde{X}^{n-1} \cup \bigcup_{c\in \mathrm{Cell}_n(X)} \bigcup_{y\in p^{-1}(\Phi_c(0))} \overline{B}^n_{c,y} \;\to\; X^n$$

that is induced by the projection $p\colon \widetilde{X}^{n-1} \to X^{n-1}$ and the maps $\overline{B}^n_{c,y} \to \overline{B}^n_c \xrightarrow{\Phi_c} X^n$. It follows again from the Topological-Pushout Proposition 6.12 (0) that $\widehat{p}$ is continuous. Also note that p satisfies $p \circ \Theta = \widehat{p}\colon \widehat{X}^n \to X^n$, i.e. we have the following little commutative diagram:

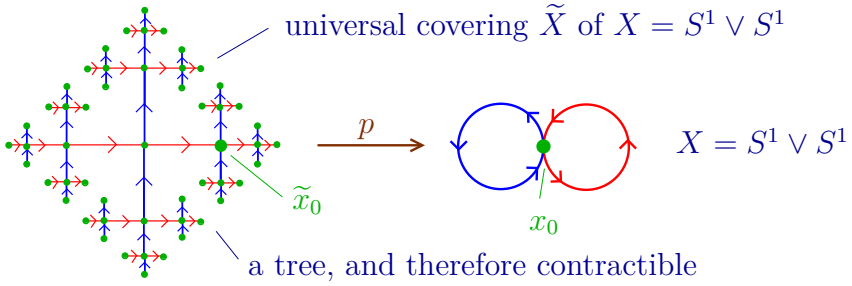


It is not hard to show that $\widehat{p}\colon \widehat{X}^n \to X^n$ is a covering. (We leave it to the reader to write down the details.) We now see that p and $\widehat{p}$ are both local homeomorphisms. It follows from $p \circ \Theta = \widehat{p}$ that Θ is a local homeomorphism. Since Θ is a bijection it follows that Θ is a homeomorphism. ■

The CW-Complex Covering Proposition 32.1, together with the Tree–Contractible Proposition 31.4, allows us to calculate the higher homotopy groups of 1-dimensional CW-complexes.

Lemma 32.2. (1-Complex-π_n-Lemma) Let X be a connected 1-dimensional CW-complex and let $x_0 \in X$ be a base point. Then $\pi_n(X, x_0) = 0$ for every $n \geq 2$.

Proof. Let X be a connected 1-dimensional CW-complex, let $x_0 \in X$ and let $n \geq 2$. By the CW-Complex–Local Properties Proposition 31.7 we know that X is locally simply connected. It follows from the Covering Existence Theorem 27.2 that there exists a universal covering $p\colon \widetilde{X} \to X$, i.e. a covering such that $\widetilde{X}$ is simply connected. We pick $\widetilde{x}_0 \in \widetilde{X}$ with $p(\widetilde{x}_0) = x_0$.



We perform the following calculation:

$$\begin{array}{ccccc} & & \text{by the CW-Complex Covering} & & \\ \text{by the Covering-}\pi_n\text{-Isomorphism} & & \text{Proposition 32.1} & & \\ \text{Proposition 28.10, since } n \geq 2 & & \downarrow & & \\ \pi_n(X, x_0) & \xleftarrow{\cong} & \pi_n(\widetilde{X}, \widetilde{x}_0) & = & \pi_n(\text{simply connected 1-dimensional CW-complex}) \\ & & & = & \pi_n(\text{contractible topological space}) \quad = \quad 0. \\ & & & & \uparrow \\ & & \text{by the Tree–Contractible Proposition 31.4} & & \text{by the Homotopy-}\pi_n\text{-Proposition 28.6} \end{array}$$

∎

32.2. **Subgroups of finitely presented groups.** We conclude this chapter with two purely group theoretic results, which we will prove using topological methods. In particular the CW-Complex Covering Proposition 32.1 will play a prominent role in both proofs. We start out with the following result.

Proposition 32.3. (Finite-Index Subgroup–Presentation Proposition) Every finite-index subgroup of a finitely presented group is finitely presented.

We will prove the Finite-Index Subgroup–Presentation Proposition 32.3 by switching from the group-theoretic setting to the topological setting. The key to doing so is the following construction, which associates to any finite presentation a 2-dimensional CW-complex.

Construction. Let $\langle g_1, \ldots, g_k \,|\, r_1, \ldots, r_m \rangle$ be a finite presentation. For $i = 1, \ldots, k$ let S^1_i be a copy of S^1. For $j = 1, \ldots, m$ we do the following:

(1) We write the relation r_j as a reduced word $g_{s_1}^{\epsilon_1} \cdot \ldots \cdot g_{s_n}^{\epsilon_n}$ with $\epsilon_1, \ldots, \epsilon_n \in \{-1, 1\}$.

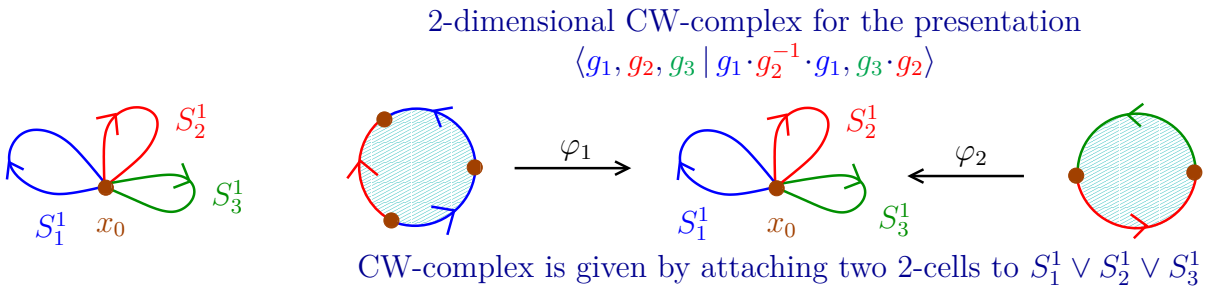
(2) We denote by $\varphi_j \colon S^1 =_{\mathrm{i}} [0,1]/0 \sim 1 \to \bigvee_{i=1}^{k} S^1_i$ the (hopefully obvious) path which on the interval $[\frac{i-1}{n}, \frac{i}{n}]$ travels through $S^1_{s_i}$ at constant speed in the direction determined by ϵ_i.

We denote by Y the CW-complex that is given by attaching m 2-cells to $\bigvee_{i=1}^{k} S^1_i$ along the attaching maps $\varphi_1, \ldots, \varphi_m$. We refer to Y as the CW-complex associated to the presentation.

Now we can state the following unsurprising lemma.

Lemma 32.4. (Presentation–CW Lemma) Let $\langle g_1, \ldots, g_k \,|\, r_1, \ldots, r_m \rangle$ be a finite presentation for a group π. If Y is the associated CW-complex, then $\pi_1(Y) \cong \pi$.

Proof of the Presentation–CW Lemma 32.4. Let $\pi = \langle g_1, \ldots, g_k \,|\, r_1, \ldots, r_m \rangle$ be a finitely presented group. We use the notation introduced in the above construction of the CW-complex associated to π. We denote by X the wedge of the k circles and we denote by x_0 the wedge point. By the Wedge-of-Circles–π_1-Lemma 20.7 we can make the identification



$\pi_1(X, x_0) = \langle g_1, \ldots, g_k \rangle$. This identification has the property that for $j = 1, \ldots, m$ the corresponding loop $\varphi_j \colon S^1 \to X$ represents $r_j \in \pi_1(X, x_0) = \langle g_1, \ldots, g_k \rangle$. We obtain that

$$\begin{array}{lcll} & \text{Cell Attachment-}\pi_1\text{-Proposition 29.1, applied } m \text{ times} & & \text{since } [\varphi_j] = r_j \\ & \downarrow & & \downarrow \\ \pi_1(Y, x_0) & \cong & \langle g_1, \ldots, g_k \rangle / \langle\langle [\varphi_1] \rangle\rangle \ldots / \langle\langle [\varphi_m] \rangle\rangle & = & \langle g_1, \ldots, g_k \rangle / \langle\langle r_1 \rangle\rangle \ldots / \langle\langle r_m \rangle\rangle \\ & = & \langle g_1, \ldots, g_k \rangle / \langle\langle r_1, \ldots, r_m \rangle\rangle & = & \langle g_1, \ldots, g_k \,|\, r_1, \ldots, r_m \rangle. \\ & \text{follows from the Quotient Presentation–Lemma 21.6} & & \text{by definition} & \end{array}$$

■

Proof of the Finite-Index Subgroup–Presentation Proposition 32.3. Suppose we are given a finitely presented group $\pi = \langle g_1, \ldots, g_k \,|\, r_1, \ldots, r_m \rangle$. By the Presentation–CW Lemma 32.4 there exists a finite connected 2-dimensional CW-complex Y with $\pi_1(Y) \cong \pi$. Now let Γ be a finite-index subgroup of π. By the CW-Complex–Local Properties Proposition 31.7 (2) together with the Covering Existence Theorem 27.2 there exists a finite covering $p \colon \widetilde{Y} \to Y$ such that $\pi_1(\widetilde{Y}) \cong \Gamma$. By the CW-Complex Covering Proposition 32.1 (1) the 2-complex $\widetilde{Y}$ has finitely many 1-cells and 2-cells. Thus it follows from the CW-π_1-Finite Presentation–Proposition 30.15 that $\Gamma \cong \pi_1(\widetilde{Y})$ is a finitely presented group. ■

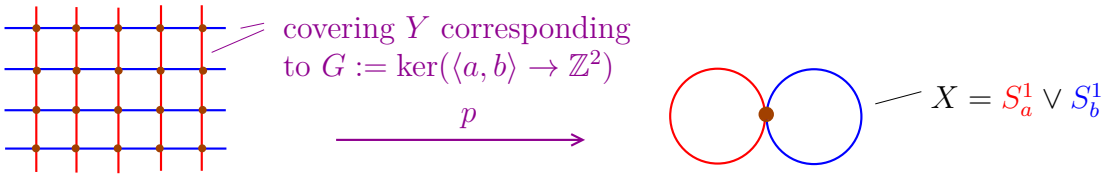
32.3. **Subgroups of free groups.** We conclude this chapter with our second application of our topological methods and results to group theory:

Proposition 32.5. (Free Subgroup Proposition)
(1) Every subgroup of a free group (of any rank) is again a free group.
(2) If F is a finitely generated free group and if G is a finite-index subgroup of F, then

$$\operatorname{rank}(G) - 1 \;=\; [F : G] \cdot (\operatorname{rank}(F) - 1).$$

Proof. Let $F = \langle S \rangle$ be a free group on a set S and let G be a subgroup of F.

(1) We consider the 1-dimensional CW-complex $X = \bigvee_{s \in S} S_s^1$. By the 1-Complex-$\pi_1$-Theorem 31.5 we have $\pi_1(X, P) \cong F$. By the CW-Complex–Local Properties Proposition 31.7 (2) and the Covering Existence Theorem 27.2 there exists a covering $p \colon (Y, Q) \to (X, P)$ of degree $[F : G]$ such that $p_*(\pi_1(Y, Q)) = G$.



Recall that by the Covering-π_1-Monomorphism Proposition 14.11 the induced map $p_* \colon \pi_1(Y, Q) \to \pi_1(X, P)$ is injective. Thus we see that $\pi_1(Y, Q)$ itself is isomorphic to

G. By the CW-Complex Covering Proposition 32.1 we can equip Y with the structure of a 1-dimensional CW-complex. We deduce from the 1-Complex-π_1-Theorem 31.5 that $G \cong \pi_1(Y, Q)$ is a free group.

(2) Now we suppose that F is a free group of finite rank and we suppose that G is a finite-index subgroup of F. Then

$$\begin{array}{rl} & \text{by the 1-Complex-}\pi_1\text{-Theorem 31.5} \\ \operatorname{rank}(G) & \overset{\downarrow}{=} \#\{\text{1-cells of } Y\} - \#\{\text{0-cells of } Y\} + 1 \\ & = [F:G]\cdot(\underbrace{\#\{\text{1-cells of } X\}}_{=\operatorname{rank}(F)} - \underbrace{\#\{\text{0-cells of } X\}}_{=1}) + 1 \;=\; [F:G]\cdot(\operatorname{rank}(F)-1)+1. \\ & \text{by the CW-Complex Covering Proposition 32.1} \end{array}$$

■

CHAPTER 33

Constructions of CW-complexes

In this chapter we will introduce and discuss several ways to construct CW-complexes out of given CW-complexes. In particular we will consider the disjoint union, the pushout, the quotient and the wedge of CW-complexes. We will also consider the somewhat tricky topic of products of CW-complexes.

33.1. **The two categories of CW-complexes.** As fans of category theory it is natural to view CW-complexes as objects of a category. We start out with one obvious way of doing so.

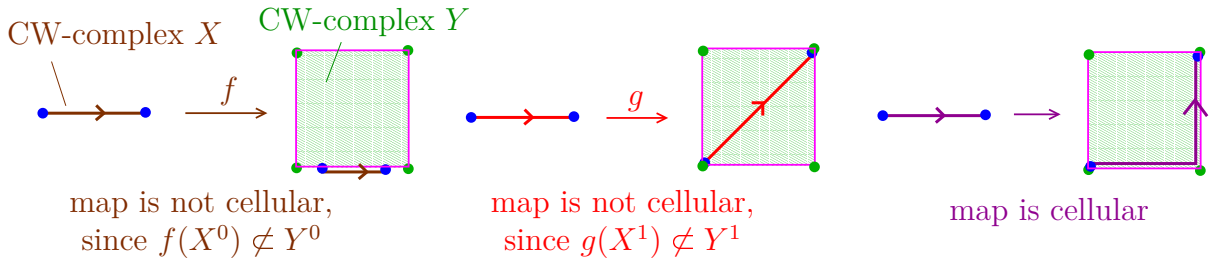
Definition. The full subcategory $\mathcal{CW}$ of CW-complexes is the full subcategory[275] $\mathcal{FullCW}$ of the category $\mathcal{Top}$ of topological spaces that is given by

$$\begin{aligned} \mathrm{Ob}(\mathcal{FullCW}) &:= \text{all CW-complexes,} \\ \mathrm{Mor}(X,Y) &:= \text{all } \textit{continuous} \text{ maps from } X \text{ to } Y. \end{aligned}$$

As we will see, in many settings it is reasonable to consider maps between CW-complexes that respect the CW-structure to a certain degree. This leads us to the following definition:

Definition. A map $f\colon X \to Y$ between two CW-complexes is called cellular if f is continuous and if for each $n \in \mathbb{N}_0$ we have $f(X^n) \subset Y^n$.

Example. In the figure below we consider maps from the CW-complex $X = [0,1]$ to the CW-complex $Y = [0,1] \times [0,1]$, where each is equipped with the "obvious" CW-structure. The first two maps are not cellular, the one to the right is cellular. □



It is clear that the composition of two cellular maps is again cellular. This leads us to the following definition:

Definition. The category $\mathcal{CW}$ of CW-complexes is the category $\mathcal{CW}$ that is given by

$$\begin{aligned} \mathrm{Ob}(\mathcal{CW}) &:= \text{all CW-complexes,} \\ \mathrm{Mor}(X,Y) &:= \text{all } \textit{cellular} \text{ maps from } X \text{ to } Y. \end{aligned}$$

[275] Recall that on page **??** we introduced the notion of a full subcategory of a category.

33.2. **Constructions of CW-structures: Pushouts.** In this section we finally turn to constructions of CW-complexes. We start out with the following elementary lemma.

Lemma 33.1. (CW-Disjoint Union Proposition) Let $\{X_i\}_{i\in I}$ be a family of CW-complexes. The disjoint union $\bigsqcup_{i\in I} X_i$ admits a natural CW-structure where each n-skeleton is given by $\bigsqcup_{i\in I} X_i^n$.

(a) The natural maps $X_j \to \bigsqcup_{i\in I} X_i$ are cellular.

(b) If Z is a CW-complex and if $f_i\colon X_i \to Z$, $i \in I$ are continuous (cellular) maps, then the corresponding map $\bigsqcup_{i\in I} f_i\colon \bigsqcup_{i\in I} X_i \to Z$ is also continuous (cellular).

Proof. The proof is elementary, and the more one says, the more confusing it gets. ■

We continue with the following much more interesting proposition.

Proposition 33.2. (CW-Gluing Proposition) Let X and Y be CW-complexes, let A be a subcomplex of X and let $f\colon A \to Y$ be a cellular map. As on page 142 we consider the following topological space:

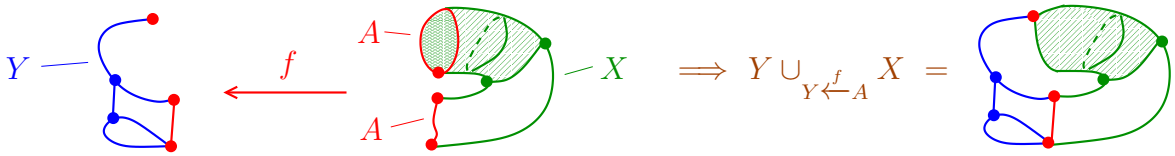
$$Y \cup_{Y\xleftarrow{f} A} X \;:=\; Y \cup_f X \;:=\; (Y \sqcup X)/\!\sim \qquad \text{where } f(a) \sim a \text{ for } a \in A$$

By the Topological-Pushout Proposition 6.12 the following diagram is a pushout diagram in the category *Top* of topological spaces:

$$\begin{array}{ccc} A & \hookrightarrow & X \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle p}\ {\scriptstyle x\mapsto[x]} \\ Y & \xrightarrow[q]{y\mapsto[y]} & Y \cup_f X. \end{array}$$

(1) The subspaces $(Y \cup_f X)^n := q(Y^n) \cup p(X^n)$ define a CW-structure on $Y \cup_f X$.

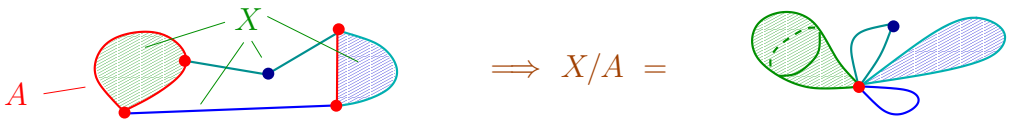
(2) If $\{\Phi_c\colon \overline{B}^{\dim(c)} \to X\}_{c\in \mathrm{Cell}(X)}$ and $\{\Phi_d\colon \overline{B}^{\dim(d)} \to Y\}_{d\in \mathrm{Cell}(Y)}$ are full sets of characteristic maps of X and Y, then $\{p\circ\Phi_c\colon \overline{B}^{\dim(c)} \to Y \cup_f X\}_{c\in \mathrm{Cell}(X)\setminus \mathrm{Cell}(A)}$ together with $\{q\circ\Phi_d\colon \overline{B}^{\dim(d)} \to Y \cup_f X\}_{d\in \mathrm{Cell}(Y)}$ is a full set of characteristic maps of $Y \cup_f X$.



Proof. The proof is somewhat lengthy, but at the end of the day there are no real technical difficulties. Thus we outsource the proof to [**Fri24**]. ■

In the following lemmas we will spell out some important special cases of the CW-Gluing Proposition 33.2.

Lemma 33.3. (CW-Quotient Lemma) Let X be a CW-complex and let A be a non-empty subcomplex. We denote by $p\colon X \to X/A$ the natural projection. The subsets $p(X^n)$, $n \in \mathbb{N}_0$, define a CW-structure on X/A.



Proof. On page 143 we pointed out that we have the following pushout diagram:

$$\begin{array}{ccc} A & \hookrightarrow & X \\ \downarrow & & \downarrow \\ \{*\} & \longrightarrow & X/A. \end{array} \qquad x \mapsto [x]$$

The lemma is thus an almost immediate consequence of the CW-Gluing Proposition 33.2. ■

Lemma 33.4. (CW-Wedge Lemma) Let $\{X_i\}_{i\in I}$ be a family of CW-complexes and suppose that for each $i \in I$ we are given a point $x_i \in X_i^0$. The corresponding wedge $\bigvee_{i\in I}(X_i, x_i)$, admits a natural CW-complex structure.

Proof. It follows from the CW-Disjoint Union Lemma 33.1 that $X := \bigsqcup_{i\in I} X_i$ has a natural CW-structure such that $A := \bigsqcup_{i\in I}\{x_i\}$ is a subcomplex of X. It follows from the CW-Quotient Lemma 33.3 that $X/A = \bigvee_{i\in I}(X_i, x_i)$ inherits a natural CW-structure. ■

Later on we will make good use of the following lemma:

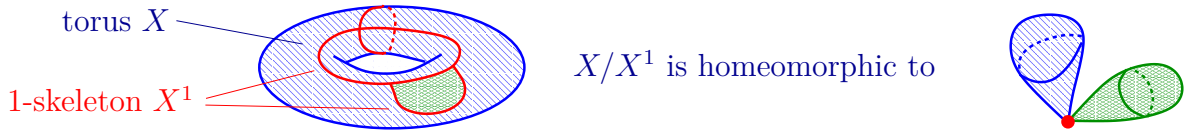
Lemma 33.5. (Skeleton-Quotient Lemma) Let X be a CW-complex and let $n \in \mathbb{N}_0$. Let $\{\Phi_c \colon \overline{B}^{n+1} \to X^n\}_{c\in \mathrm{Cell}(X)}$ be a full set of characteristic maps of the $(n+1)$-cells of X. The map

$$\begin{array}{rccl} \displaystyle\bigvee_{c\in\mathrm{Cell}(X)} & \overbrace{(\overline{B}_c^{n+1}/S_c^n, [S_c^n])} & \to & X^{n+1}/X^n \\ & [x] & \mapsto & [\Phi_c(x)] \quad \text{if } x \in \overline{B}_c^{n+1} \end{array}$$

(by the Ball Quotient-Sphere Lemma 6.5 the pointed topological space $(\overline{B}^{n+1}/S^n, [S^n])$ is homeomorphic to (S^{n+1}, N))

is a homeomorphism.

Example. In the figure below we see the torus together with a CW-structure X that has two 2-cells.



It follows from the Skeleton-Quotient Lemma 33.5 that $X/X^1 = X^2/X^1$ is homeomorphic to the wedge of two spheres. □

Proof. It is basically clear that the given map $f \colon \bigvee_{c\in\mathrm{Cell}(X)} (\overline{B}_c^{n+1}/S_c^n, [S_c^n]) \to X^{n+1}/X^n$ is a bijection. Next note that it follows immediately from the Twice Quotient Lemma 6.3, the Wedge Proposition 17.1 (1b) and the Topological-Quotient Proposition 6.1 (1) that f is continuous. To upgrade this bijective continuous map to a homeomorphism we have to work a little harder.

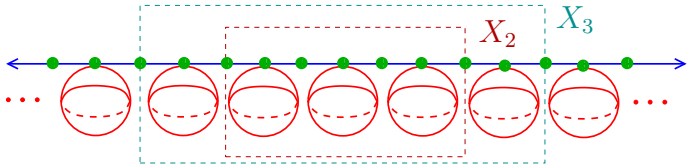
To do so note that by the Ball Quotient-Sphere Lemma 6.5 the pointed topological space $(\overline{B}^{n+1}/S^n, [S^n])$ is homeomorphic to (S^{n+1}, N). It follows from the discussion on page 407 that we can view $\overline{B}^{n+1}/S^n$ as a CW-complex where the point $[S^n]$ is the unique 0-cell.

Therefore we can use the CW-Wedge Lemma 33.4 and the CW-Quotient Lemma 33.3 to view both sides of the Skeleton-Quotient Lemma 33.5 as CW-complexes. The map f gives a bijection between the cells of the two sides. Using the Compact-Hausdorff Proposition 2.10 (3) one can show that the restriction of f^{-1} to each cell of X^{n+1}/X^n is continuous. It follows from the Characteristic Maps Proposition 30.5 that f^{-1} is indeed continuous. ■

We conclude this section with the following lemma:

Lemma 33.6. (Direct Colimit–CW Lemma) Let $\{f_i \colon X_i \to X_{i+1}\}_{i\in\mathbb{N}_0}$ be a sequence of cellular embeddings between CW-complexes. The direct colimits $\operatorname{colim}\limits_{\longrightarrow} X_i^n$ of the skeleta of the X_i define a CW-structure on $\operatorname{colim}\limits_{\longrightarrow} X_i$.

Example. We consider the line with infinitely many spheres attached.



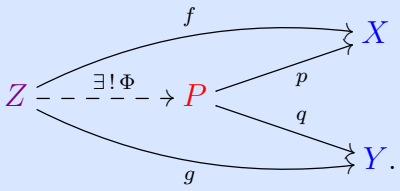
the topological space
$X = \bigcup\limits_{i\in\mathbb{N}_0} X_i = \operatorname{colim}\limits_{\longrightarrow} X_i$

The Direct Colimit–CW Lemma 33.6 allows us to view the line with infinitely many spheres attached as a CW-complex. □

Proof. As so often, it is hard to tell whether a proof is elementary or not. This proof turns out is not that elementary, but it is certainly not very enlightening. Thus we hide the proof in [**Fri24**]. ■

33.3. **Products of CW-complexes.** The reader will have noticed that a discussion of products is long overdue. As we will see, there was a reason why we were procrastinating. In fact it is perhaps helpful to reflect about what "product" is supposed to mean in the first place. Thus let us recall the following from page 101.

Definition. Let $\mathcal{C}$ be a category and let $X, Y \in \operatorname{Ob}(\mathcal{C})$ be two objects. A product of X and Y in the category $\mathcal{C}$ is some object $P \in \operatorname{Ob}(\mathcal{C})$ together with two morphisms $p \colon P \to X$ and $q \colon P \to Y$ such that the following property is satisfied: Whenever we are given two morphisms $f \colon Z \to X$ and $g \colon Z \to Y$ there exists a unique morphism $\Phi \colon Z \to P$ such that $f = p \circ \Phi$ and $g = q \circ \Phi$. The situation is summarized in the following diagram:



By the Product Uniqueness Lemma 4.1 we know that if the product exists, then it is unique in a suitable sense.

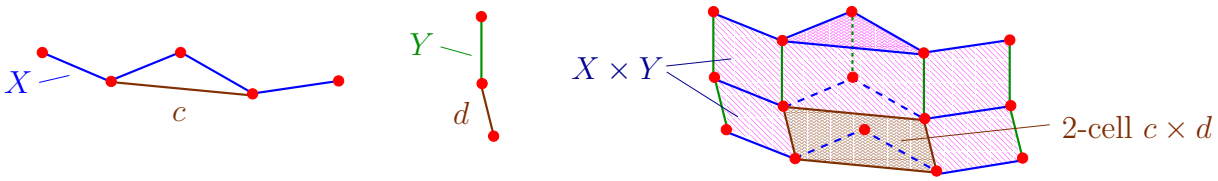
We remind the reader of the following simplified version of the Topological-Product Proposition 5.1.

Proposition 5.1. (Topological-Product Proposition) Let X and Y be two topological spaces. We consider the topology given by

$$\mathcal{TP} := \left\{ U \subset X \times Y \;\middle|\; \begin{array}{l} \text{for all } (x,y) \in U \text{ there exist open neighborhoods} \\ A \subset X \text{ of } x \text{ and } B \subset Y \text{ of } y \text{ with } A \times B \subset U \end{array} \right\}.$$

(1) (a) The natural projections $X \times Y \to X$ and $X \times Y \to Y$ are continuous.
(b) Given a topological space Z and given continuous maps $f\colon Z \to X$ and $g\colon Z \to Y$ the map $(f,g)\colon Z \to (X \times Y, \mathcal{TP})$ that is given by $z \mapsto (f(z), g(z))$ is continuous.
(2) The topological space $X \times Y$ with the natural projections is a product of X and Y in the category $\mathcal{Top}$ of topological spaces.

Now let X and Y be CW-complexes. We want to equip $X \times Y$ with a CW-structure. Perhaps initially it is easier to think in terms of cells instead of filtrations. If c is a p-cell of X and if d is a q-cell of Y, then it seems reasonable that $c \times d$ should be a $(p+q)$-cell of $X \times Y$.



If that is the idea, then it follows from the Characteristic Maps Proposition 30.5 that $X \times Y$ should have the topology that is described in the following definition:

Definition. Let X and Y be two CW-complexes.
(1) Given a p-cell c for X and given a q-cell d for Y we equip c and d with the subspace topology and we equip $c \times d$ with the product topology.
(2) We denote by $\mathcal{CWP}$ the topology on $X \times Y$ that is defined by the condition that $U \subset X \times Y$ is open if and only if each intersection $(c \times d) \cap U \subset c \times d$ is open.

Proposition 33.7. (CW-Category–Product Proposition) Let X and Y be two CW-complexes.
(0) (a) A CW-structure for the topological space $(X \times Y, \mathcal{CWP})$ is given by the filtration

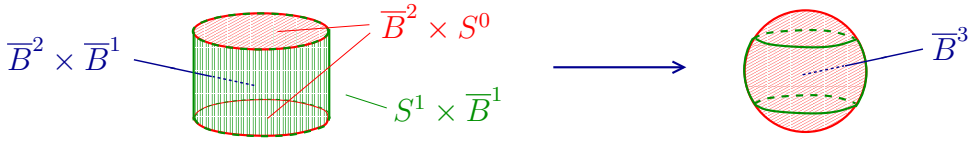
$$(X \times Y)^n := \bigcup_{\substack{p,q \in \mathbb{N}_0 \text{ with} \\ p+q=n}} (X^p \times Y^q) \qquad \text{with } n \in \mathbb{N}_0.$$

(b) The n-cells of $X \times Y$ are precisely the sets of the form $c \times d$ where c is a p-cell of X and d is an $(n-p)$-cell of Y.
(1) (a) The natural projections $X \times Y \to X$ and $X \times Y \to Y$ are cellular, in particular they are continuous with respect to $\mathcal{CWP}$.
(b) If Z is a CW-complex and if $f\colon Z \to X$ and $g\colon Z \to Y$ are continuous maps, then the map $Z \to (X \times Y, \mathcal{CWP})$ that is given by $z \mapsto (f(z), g(z))$ is also continuous.[276]
(2) The CW-complex $(X \times Y, \mathcal{CWP})$ with the natural projections is a product of the CW-complexes X and Y in the full category $\mathcal{FullCW}$ of CW-complexes.

[275] It follows easily from the Image-Preimage Lemma 0.3 that this defines a topology on $X \times Y$.
[276] Note that if f and g are cellular, this does *not* mean that the map $(f,g)\colon Z \to X \times Y$ is cellular.

In the proof of the CW-Category–Product Proposition 33.7 we will need the following elementary lemma:

Lemma 33.8. (Ball-times-Ball Lemma) Let $p, q \in \mathbb{N}_0$. There exists a homeomorphism $\overline{B}^p \times \overline{B}^q \to \overline{B}^{p+q}$ that restricts to a homeomorphism $(\overline{B}^p \times S^{q-1}) \cup (S^{p-1} \times \overline{B}^q) \to S^{p+q-1}$.



Proof of the Ball-times-Ball Lemma 33.8. This lemma is proved in [**Fri24**]. ■

Proof of the CW–Product Proposition 33.7. Let X and Y be two CW-complexes.

(0) We start out with the following claim:

Claim 0. Let $p, q \in \mathbb{N}_0$. We view the skeleta X^p and Y^q as CW-complexes in their own right. The $\mathcal{CWP}$-topology on $X^p \times Y^q$ agrees with the subspace topology coming from $(X \times Y, \mathcal{CWP})$

Proof. We consider the inclusion map $f\colon (X^p \times Y^q, \mathcal{CWP}) \to (X \times Y, \mathcal{CWP})$. Note that it follows from the Subcomplex–Characterization Proposition 30.7 that every cell of X^p is a cell of X and that every cell of Y^q is a cell of Y. It follows basically immediately from this observation and the definition of the CW-product topology that the map $f\colon (X^p \times Y^q, \mathcal{CWP}) \to (X \times Y, \mathcal{CWP})$ is continuous. By the Open-Injective Lemma 2.9 it remains to show that f is a closed map. Let A be a subset that is closed in $(X^p \times Y^q, \mathcal{CWP})$. We need to show that A is closed in $(X \times Y, \mathcal{CWP})$. By definition of $(X \times Y, \mathcal{CWP})$ and by an elementary argument it suffices to show that for each cell c of X and each cell d of Y the intersection $(c \times d) \cap A$ is closed in $c \times d$.

Note that it follows from the Cell-Compact Proposition 30.4, the Subcomplex–Closed Lemma 30.6 and the CW-Complex-Finiteness Theorem 30.10 that there exist finitely many cells $x_1, \dots, x_r$ of X^p such that $c \cap X^p \subset x_1 \cup \dots \cup x_r$ and that there exist finitely many cells $y_1, \dots, y_s$ of X^p such that $d \cap Y^q \subset y_1 \cup \dots \cup y_s$. We now see that

$$\begin{aligned}
(c \times d) \cap A &\overset{\text{since } A \subset X^p \times Y^q}{=} (X^p \times Y^q) \cap ((c \times d) \cap A) = ((c \cap X^p) \times (d \cap Y^q)) \cap A \\
&= ((c \cap X^p) \times (d \cap Y^q)) \cap ((x_1 \cup \dots \cup x_r) \times (y_1 \cup \dots \cup y_s)) \cap A \\
&= \underbrace{((c \cap X^p) \times (d \cap Y^q))}_{\substack{\text{closed in } c \times d \text{ since it follows from} \\ \text{the Subcomplex–Closed Lemma 30.6} \\ \text{that } X^p \subset X \text{ and } Y^q \subset Y \text{ are closed}}} \cap \Big(\bigcup_{i=1}^{r} \bigcup_{j=1}^{s} \underbrace{((x_i \times y_j) \cap A)}_{\substack{\text{by hypothesis closed} \\ \text{in } x_i \times y_j, \text{ thus compact,} \\ \text{thus closed in } c \times d}} \Big).
\end{aligned}$$

We have thus shown that $(c \times d) \cap A$ is indeed a closed subset of $c \times d$. ⊞

Now we turn to the actual proof that the filtration $(X \times Y)^n := \bigcup_{p+q=n} (X^p \times Y^q)$ defines a CW-structure on $(X \times Y, \mathcal{CWP})$. The following two claims show that this filtration satisfies the two conditions (CW1) and (CW2) from page 405.

Claim 1. A subset $U \subset (X \times Y, \mathcal{CWP})$ is open if and only if for each $n \in \mathbb{N}_0$ the intersection $(X \times Y)^n \cap U$ is open in $(X \times Y)^n$.

Proof. Let $U \subset X \times Y$. Then:

$$\begin{aligned} U \text{ is open in } \mathcal{CWP} &\iff \bigwedge_{p\in\mathbb{N}_0} \bigwedge_{q\in\mathbb{N}_0} \bigwedge_{c\in \mathrm{Cell}_p(X)} \bigwedge_{d\in\mathrm{Cell}_q(Y)} (c\times d)\cap U \text{ is open in } c\times d \\ &\iff \bigwedge_{p\in\mathbb{N}_0} \bigwedge_{q\in\mathbb{N}_0} (X^p \times Y^q)\cap U \text{ is open in } X^p\times Y^q \\ &\uparrow \text{ follows from Claim 0} \\ &\iff \bigwedge_{n\in\mathbb{N}_0} (X\times Y)^n \cap U \text{ is open in } (X\times Y)^n \\ &\uparrow \text{ follows from the Cover-Open Subset Lemma 1.5 (2), since } (X\times Y)^n \text{ is the union} \\ &\quad \text{of the finitely many closed subsets } X^p\times Y^q, \text{ where } p+q=n \end{aligned}$$

⊞

Claim 2. For each $n \in \mathbb{N}_0$ there exists a set I, a family $\{\varphi_i\colon S^{n-1} \to (X\times Y)^{n-1}\}_{i\in I}$ of continuous maps and a homeomorphism

$$(X\times Y)^{n-1} \cup_{X\times Y \leftarrow \bigsqcup_{i\in I} S_i^{n-1} \,:\, \bigsqcup_{i\in I}\varphi_i} \bigsqcup_{i\in I} \overline{B}_i^n \;\to\; (X\times Y)^n$$

which is the inclusion on $(X\times Y)^{n-1}$.

Proof. Let $\{\Phi_c\colon \overline{B}^{\dim(c)} \to X\}_{c\in\mathrm{Cell}(X)}$ and $\{\Psi_d\colon \overline{B}^{\dim(d)} \to Y\}_{d\in\mathrm{Cell}(Y)}$ be full sets of characteristic maps of the CW-complexes X and Y. Note that for each $p,q\in\mathbb{N}_0$ the restriction of the map $\Phi_c\times\Psi_d\colon \overline{B}^p\times\overline{B}^q \to X\times Y$ to $(\overline{B}^p\times S^{q-1})\cup(S^{p-1}\times\overline{B}^q)$ takes values in $(X^p\times Y^{q-1})\cup(X^{p-1}\cup Y^q)\subset (X\times Y)^{n-1}$. Thus we can consider the topological space

$$(X\times Y)^{n-1} \;\cup_\sim \bigsqcup_{\substack{p,q\in\mathbb{N}_0 \text{ with} \\ p+q=n}} \bigsqcup_{c\in\mathrm{Cell}_p(X)} \bigsqcup_{d\in\mathrm{Cell}_q(Y)} (\overline{B}^p\times\overline{B}^q)$$

where given c,d we use the restrictions of $\Phi_p\times\Psi_q$ to $(\overline{B}^p\times S^{q-1})\cup(S^{p-1}\times\overline{B}^q)$ to glue $\overline{B}^p\times\overline{B}^q$ to $(X\times Y)^{n-1}$. Next we consider the map

$$(X\times Y)^{n-1} \;\cup_\sim \bigsqcup_{\substack{p,q\in\mathbb{N}_0 \text{ with} \\ p+q=n}} \bigsqcup_{c\in\mathrm{Cell}_p(X)} \bigsqcup_{d\in\mathrm{Cell}_q(Y)} (\overline{B}^p\times\overline{B}^q) \;\to\; (X\times Y)^n$$

that is given by the inclusion on $(X\times Y)^{n-1}$ and that is given by all the product maps $\Phi_c\times\Psi_d\colon \overline{B}^p\times\overline{B}^q\to X\times Y$. It follows from the Topological Pushout Proposition 6.12 (0∗) that this map is continuous. It is clearly a bijection. Using the $\mathcal{CWP}$ topology on $(X\times Y)^n$ it is also not difficult to show that the map is closed. It follows from the Open-Injective Lemma 2.9 that the above map is a homeomorphism.

The claim now follows from the above discussion and the observation that the Ball-times-Ball Lemma 33.8 allows us to replace $\overline{B}^p\times\overline{B}^q$ by $\overline{B}^{p+q}=\overline{B}^n$ and it allows us to replace $(\overline{B}^p\times S^{q-1})\cup(S^{p-1}\times\overline{B}^q)$ by $S^{p+q-1}=S^{n-1}$. ⊞

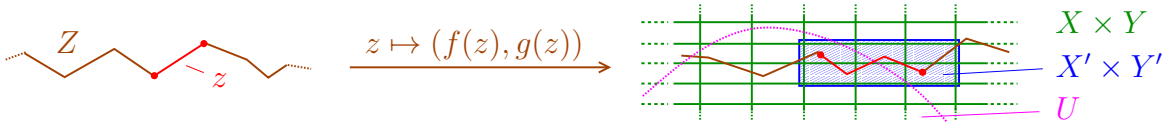
It remains to show that the n-cells of $X\times Y$ are precisely the sets of the form $c\times d$ where c is a p-cell of X and d is an $(n-p)$-cell of Y. But this follows easily from Claim 2.

(1) (a) We consider the natural projection $p\colon (X\times Y,\mathcal{CWP})\to X$. We need to show that this map is continuous. Note that it suffices to show that for each cell $c\subset X$ and each cell $d\subset Y$ the map $c\times d\to X\times Y\to X$ is continuous. But by the Product Topology-Basics Lemma 5.3 (1a) and the Topological-Product Proposition 5.1 (1a) we know that the maps $c\times d\to(X\times Y,\mathcal{TP})$ and $(X\times Y,\mathcal{TP})\to X$ are continuous.

Thus the composition is also continuous. The same argument of course also shows that the natural projection $p\colon X \times Y \to Y$ is continuous.

(b) Let Z be a CW-complex and let $f\colon Z \to X$ and $g\colon Z \to Y$ be two continuous maps. We need to show that the map $Z \to (X \times Y, \mathcal{CWP})$ that is given by $z \mapsto (f(z), g(z))$ is also continuous.[277] Let $U \subset (X \times Y, \mathcal{CWP})$ be an open subset. We need to show that $(f,g)^{-1}(U)$ is an open subset of the CW-complex Z. By the Characteristic Maps Proposition 30.5 it remains to show that for each cell z of Z the intersection $z \cap (f,g)^{-1}(U)$ is open in z.

It follows from the Cell-Compact Proposition 30.4, the Compact Image Lemma 2.8 and the CW-Complex-Finiteness Theorem 30.10 that there exist finite subcomplexes X' of X and Y' of Y with $f(z) \subset X'$ and $g(z) \subset Y'$.



Claim. The intersection $(X' \times Y') \cap U$ is an open subset of $(X' \times Y', \mathcal{TP})$.

Proof. First note that we have the equality

$$(X' \times Y') \cap U \;=\; \bigcup_{c \in \mathrm{Cell}(X')} \bigcup_{d \in \mathrm{Cell}(Y')} (c \times d) \cap U.$$

Since U is open in $\mathcal{CWP}$ we see that each $(c \times d) \cap U$ is open in $c \times d$. Note that each $c \times d$ is a closed subset of $X' \times Y'$. It follows from these observations, and the fact that the finite CW-complexes X' and Y' have only finitely many cells, together with the Cover-Open Subset Lemma 1.5 (2) that $(X' \times Y') \cap U$ is indeed an open subset of $X' \times Y'$. ⊞

Next note that

$$z \cap (f,g)^{-1}(U) \;=\; z \cap (f,g)^{-1}((X' \times Y') \cap U).$$

By the elementary Basics-of-Continuity Lemma 2.1 the map $(f,g)\colon Z \to X' \times Y'$ is also continuous. Thus it follows from the above equality and the claim, that $z \cap (f,g)^{-1}(U)$ is indeed an open subset of z.

(2) As in the case of the Topological-Product Proposition 5.1 this statement follows almost immediately from (1). ■

Let X and Y be two CW-complexes. The obvious question that now arises is, under what circumstances does the CW-product topology $\mathcal{CWP}$ that we used in the CW-Category–Product Proposition 33.7 agree with the usual product topology $\mathcal{TP}$. The following proposition shows that in two settings, which are of relevance to us, this is indeed the case.

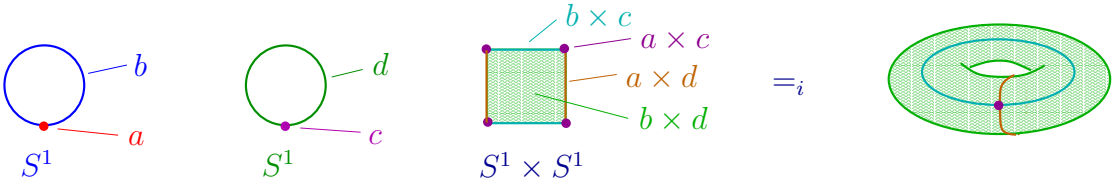
[277] In the CW-Complex Product Proposition 33.9 we will see that $\mathcal{TP} \subset \mathcal{CWP}$. This implies that it is easier to show that maps *out* of $(X \times Y, \mathcal{CWP})$ are continuous than to show that maps *out* of $(X \times Y, \mathcal{TP})$ are continuous. But this fact also implies that it harder to show that maps *to* $(X \times Y, \mathcal{CWP})$ are continuous than to show that maps *to* $(X \times Y, \mathcal{TP})$ are continuous. This observation explains why the proof of (1b) is trickier than the proof of (1a).

Proposition 33.9. (CW-Complex Product Proposition) Let X and Y be two CW-complexes.

(1) The identity map $(X \times Y, \mathcal{CWP}) \to (X \times Y, \mathcal{TP})$ is continuous, in other words, we have the inclusion $\mathcal{TP} \subset \mathcal{CWP}$.
(2) Suppose we are in one of the following two settings:
   (a) X *and* Y are both finite CW-complexes, or
   (b) $Y = [0,1]$ where Y is equipped with the usual CW-structure with two 0-cells and one 1-cell.

   then the identity map $(X \times Y, \mathcal{CWP}) \to (X \times Y, \mathcal{TP})$ is a homeomorphism, in other words, we have $\mathcal{CWP} = \mathcal{TP}$. Thus the CW-product structure from the CW-Category–Product Proposition 33.7 is a CW-structure on $(X \times Y, \mathcal{TP})$.

Example. If we equip $S^1 =_{\mathrm{i}} [0,1]/0 \sim 1$ with the usual CW-structure with one 0-cell and one 1-cell, then the product CW-structure on the torus $S^1 \times S^1 =_{\mathrm{i}} ([0,1] \times [0,1])/\sim$ is precisely the CW-structure that we had already encountered on page 407. We refer to the figure below for an illustration. □



Proof. Let X and Y be two CW-complexes.

(1) In the Product Topology-Basics Lemma 5.3 (1a) we showed that the product topology is compatible with taking the subspace topology. In our context this means that for cells $c \subset X$ and $d \subset Y$ the product topology on $c \times d$ agrees with the subspace topology on $c \times d \subset (X \times Y, \mathcal{TP})$. It follows immediately from this discussion that the identity map $(X \times Y, \mathcal{CWP}) \to (X \times Y, \mathcal{TP})$ is continuous.
(2) (a) Suppose that X and Y are both finite CW-complexes. It follows from the CW-Category–Product Proposition 33.7 that the CW-complex $X \times Y$ has only finitely many cells. It thus follows from the CW-Complex Compactness Corollary 30.11 that $(X \times Y, \mathcal{CWP})$ is compact. Furthermore note that it follows from the CW-Complex–Hausdorff Proposition 30.3 and the Product Topology-Properties Proposition 5.4 that $(X \times Y, \mathcal{TP})$ is Hausdorff. It follows from these facts, the Compact-Hausdorff Proposition 2.10 (3) and Statement (1) that $\mathrm{id}\colon (X \times Y, \mathcal{CWP}) \to (X \times Y, \mathcal{TP})$ is indeed a homeomorphism.
   (b) Suppose we have $Y = [0,1]$ with the obvious CW-structure. By (1) we know that $\mathcal{TP} \subset \mathcal{CWP}$. It remains to show that $\mathcal{CWP} \subset \mathcal{TP}$. Thus let $U \in \mathcal{CWP}$. Let $\{\Phi_c\colon \overline{B}^{\dim(c)} \to X\}_{c \in \mathrm{Cell}(X)}$ be a full set of characteristic maps for the CW-complex X. Since $U \in \mathcal{CWP}$ we know that for each $c \in \mathrm{Cell}(X)$ the intersection $(c \times [0,1]) \cap U$ is open in $c \times [0,1]$. Next note that it follows from the Topological-Product Proposition 5.1 (1) that the map $\Phi_c \times \mathrm{id}_{[0,1]}\colon \overline{B}^{\dim(c)} \times [0,1] \to X \times [0,1]$ is continuous. Thus we see that for each $c \in \mathrm{Cell}(X)$ the preimage $(\Phi_c \times \mathrm{id}_{[0,1]})^{-1}(U)$ is open in $\overline{B}^{\dim(c)} \times [0,1]$. It follows from this discussion together with the (c)⇒(b)-direction of the CW×Interval Proposition 31.1 (1) that $U \in \mathcal{TP}$. ■

Example. The formulation of the CW-Complex Product Proposition 33.9 already suggests that the conclusion does not hold without some assumptions on X and Y. In [**Fri24**] we show that if X is the wedge of uncountably many intervals and if Y is the wedge of countably many intervals, then the topologies $\mathcal{CWP}$ and $\mathcal{TP}$ disagree on $X \times Y$. □

Part IV

Singular Homology

This part of the lecture notes is organized as follows:

- In Chapter 34 we will explain the basic geometric idea behind singular homology.
- In Chapter 35 we will introduce the singular homology groups $\mathrm{H}_k(X)$ of a topological space X, we will show that homology defines a covariant functor from the category *Top* of topological spaces to the category *AbGroup* of abelian groups. We will also compute homology groups in a few basic settings.
- In Chapter 36 we will show in the Homotopy Equivalence-H_*-Corollary 36.7 that the homology groups of homotopy equivalent topological spaces are isomorphic.
- In the purely algebraic Chapter 37 we will introduce the notion of an exact sequence and we will show in the LES-Proposition 37.4 that a short exact sequence of chain complexes gives rise to a long exact sequence in homology.
- In Chapter 38 we introduce, given a pair (X, A) of topological spaces, the relative homology $\mathrm{H}_k(X, A)$. These will play an essential role in the development of the theory, even if one only cares about absolute homology groups.
- In the short Chapter 39 we will introduce the reduced homology groups $\widetilde{\mathrm{H}}_k(X)$ of a topological space. These are a minute, but surprisingly convenient, variation on the ordinary homology groups.
- In Chapter 40 we will prove the Excision Theorem 40.1 which is the key to all non-trivial calculations of homology groups.
- In Chapter 41 we will study applications of the Excision Theorem 40.1. In particular in the Sphere-H_*-Proposition 41.2 we will use it to calculate the homology groups of spheres. This calculation allows us to give a new proof of the Brouwer Fixed Point Theorem 41.6.
- In Chapter 42 we will study the local homology groups of a topological space X at a point $x \in X$ and we will use them to provide explicit generators of homology groups.
- In Chapter 43 we will introduce the degree of a continuous self-map of a sphere and we will calculate it in several instances.
- In Chapter 44 we will introduce local degrees of self-maps of spheres and we will show how to use them to calculate even more degrees of self-maps of spheres.
- In Chapter 45 we will prove the Mayer–Vietoris Theorem 45.1 which can be viewed as the homological analogue of the Seifert-van Kampen Theorem 20.1.
- In Chapter 46 we will discuss the homology groups of a direct colimit of topological spaces.
- In Chapter 47 we will use the results from Chapter 46 to introduce the cellular homology of a CW-complex. We will see that the calculation of cellular homology boils down to calculating degrees of self-maps of spheres. Since we already developed many tools for calculating these degrees we see that cellular homology is imminently computable. Furthermore in the Singular-Cellular-H_*-Proposition 47.3 we will that cellular homology is naturally isomorphic to ordinary homology. This fact opens the door to many calculations of homology groups. For example we will be able to compute homology groups of the real and complex projective spaces and of compact surfaces.
- In Chapter 48 we will introduce the Euler characteristic $\chi(X)$ of a CW-complex X. In the Homology-χ-Proposition 48.1 we will show, using cellular homology, that the Euler characteristic only depends on the homotopy type of X.

- In Chapter 49 we will discuss several fun applications of the Euler characteristic. In particular we will study regular convex polyhedra in $\mathbb{R}^3$ and we will see that not every finite topological graph can be embedded into $\mathbb{R}^2$.

CHAPTER 34

The idea behind singular homology

The definition and the theory of singular homology was developed over several decades in the first half of the 20th century. In this chapter we will try to outline how one might have come up with the idea behind singular homology. The stress in this chapter is on explaining ideas and examples, without getting bogged down with technical discussions. We will not make use of any of the results and definitions of this chapter. We will use this feature as an excuse for taking a few liberties here and there.

So let's start with our story. Historically mathematicians found, and still find it, convenient to work with simplicial complexes. Loosely speaking a simplicial complex is a subset of $\mathbb{R}^n$ that is built out of simplices. Thus we first should tell the reader what we mean by a simplex:

Definition.

(1) We say that $k+1$ points in $\mathbb{R}^n$ are in general position if they are not contained in a $(k-1)$-dimensional affine hyperplane of $\mathbb{R}^n$.[278]

(2) Given any finite subset $S \subset \mathbb{R}^n$ we denote by[279]

$$\text{Hull}(S) \;:=\; \left\{ \sum_{x\in S} \lambda_x \cdot x \;\middle|\; \text{each } \lambda_x \in \mathbb{R}_{\geq 0} \text{ and } \sum_{x\in S} \lambda_x = 1 \right\}$$

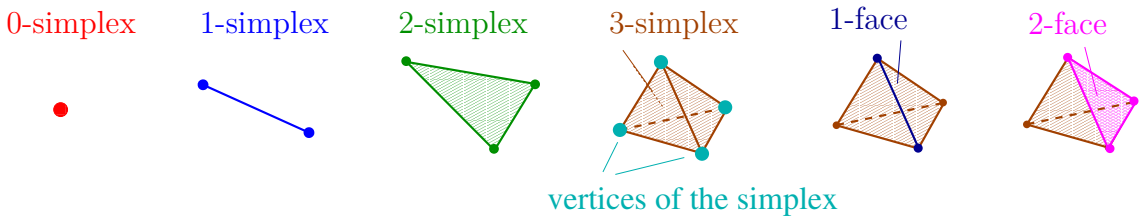
the convex hull of S.

(3) A k-simplex in $\mathbb{R}^n$ is the convex hull of $k+1$ points of $\mathbb{R}^n$ that are in general position.

(4) Let $\sigma \subset \mathbb{R}^n$ be a k-simplex.

(a) There exists a unique[280] subset $V \subset \mathbb{R}^n$ with $k+1$ elements and with $\sigma = \text{Hull}(V)$. We refer to V as the vertex set of σ.

(b) Let V be the vertex set of σ. For any subset $T \subset V$ of with $(l+1)$ elements we refer to $\text{Hull}(T)$ as an l-face of $\sigma = \text{Hull}(V)$.



Simplices can be viewed as convenient building blocks to describe topological spaces. This observation leads us straight to the following definition:

[279] In other words, $P_0, \ldots, P_k \in \mathbb{R}^n$ are in general position if $P_1 - P_0, \ldots, P_k - P_0$ are linearly independent.

[280] It is a nice little exercise to figure out what S is unique.

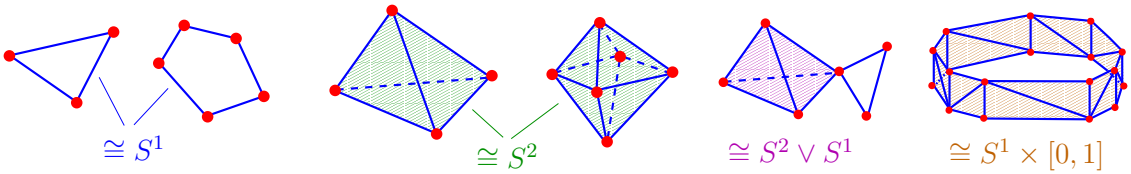
Definition. A simplicial complex in $\mathbb{R}^n$ is a finite family $X = \{\sigma_i\}_{i=1,\dots,k}$ of simplices in $\mathbb{R}^n$ with the following properties:

(1) Any face of a simplex is also contained in the family.
(2) The intersection of any two simplices is either empty or it is a single simplex, which is a face of both simplices.

Given a simplicial complex $X = \{\sigma_i\}_{i=1,\dots,k}$ as above we denote by X also the subspace $\sigma_1 \cup \cdots \cup \sigma_k \subset \mathbb{R}^n$.

Example.

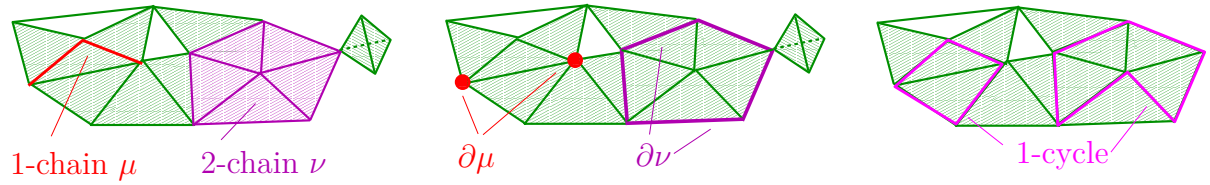
(1) In the figure below we see many examples of simplicial complexes. In particular we see that many topological spaces that we like, are actually homeomorphic to a simplicial complex.



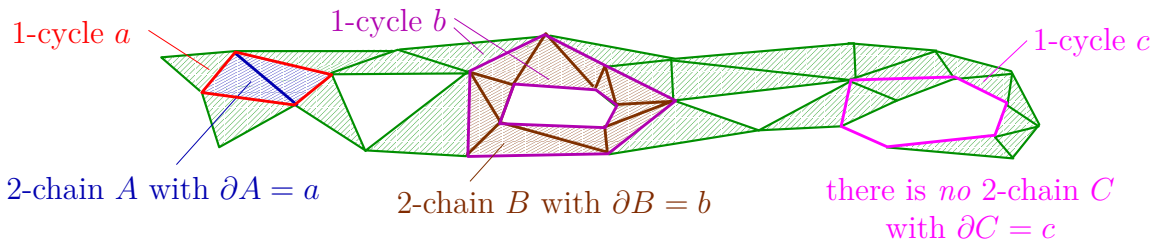
(2) In [**Fri24**] it is shown that every compact smooth manifold is homeomorphic to a simplicial complex and in [**Fri24**] it is also shown that every finite CW-complex is homotopy equivalent to a simplicial complex. □

Definition. Let X be a simplicial complex.

(1) An n-chain in X is a collection of n-simplices of X, in other words, it is a subset of the set of n-simplices of X.
(2) The boundary $\partial\sigma$ of an n-chain σ is the union of all $(n-1)$-simplices that are the faces of an *odd* number of n-simplices of the n-chain.
(3) We say that an n-chain is an n-cycle if its boundary is the empty set of $(n-1)$-simplices.



In the next figure we see that cycles come in two different flavors. Some n-cycles are the boundary of an $(n+1)$-chain and are thus not very interesting. On the other hand there are n-cycles which are not the boundary of an $(n+1)$-chain, and these non-bounding cycles seem to detect something interesting, like the presence of a "hole".



The idea now is to "measure" how many n-cycles are not the boundary of an $(n+1)$-cycle by introducing some algebra. This leads us to the following definition:

Definition. Let X be a simplicial complex.

(1) We define the n-th simplicial chain group of X as

$$\mathrm{C}_n^{\mathrm{simp}}(X) \;:=\; \text{set of all } n\text{-chains} \;=\; \mathbb{Z}_2^{\text{set of } n\text{-simplices of } X}.$$

for any set W we have $\mathcal{P}(W) = \mathbb{Z}_2^W$, we apply this logic to the set of all n-simplices of X

In other words the n-th chain group $\mathrm{C}_n^{\mathrm{simp}}(X)$ of X is defined as the free $\mathbb{Z}_2$-vector space on the set of all n-simplices of X. More informally speaking, $\mathrm{C}_n^{\mathrm{simp}}(X)$ is the set of all finite $\mathbb{Z}_2$-linear combinations of n-simplices of X.

(2) We define the boundary map

$$\begin{aligned} \partial_n \colon \mathrm{C}_n^{\mathrm{simp}}(X) &\to \mathrm{C}_{n-1}^{\mathrm{simp}}(X) \\ \sum_{i=1}^{r} a_i \cdot \sigma_i &\mapsto \sum_{i=1}^{r} a_i \cdot \underbrace{\partial \sigma_i}_{\uparrow} \end{aligned}$$

we view $\partial\sigma_i$ as a linear combination of $(n-1)$-simplices

(3) We refer to the sequence

$$\ldots \xrightarrow{\partial_3} \mathrm{C}_2^{\mathrm{simp}}(X) \xrightarrow{\partial_2} \mathrm{C}_1^{\mathrm{simp}}(X) \xrightarrow{\partial_1} \mathrm{C}_0^{\mathrm{simp}}(X) \to 0$$

as the simplicial chain complex of X.

Remark. Note that it follows easily from the definitions that an n-chain μ is an n-cycle if and only if $\partial_n(\mu) = 0$.[281] □

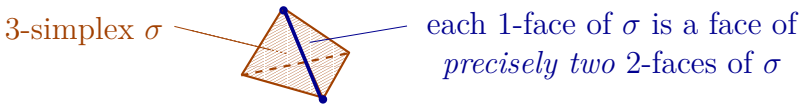
The following lemma is the crucial observation, which makes the whole theory come alive.

Lemma 34.1. (Simplicial–Double Boundary Lemma) Let X be a simplicial complex. For every $n \in \mathbb{N}_0$ the composition $\partial_n \circ \partial_{n+1} \colon \mathrm{C}_{n+1}^{\mathrm{simp}}(X) \to \mathrm{C}_{n-1}^{\mathrm{simp}}(X)$ of two consecutive boundary maps is the zero map.

Sketch of proof. Since $\mathrm{C}_{n+1}^{\mathrm{simp}}(X)$ is a vector space with basis given by the $(n+1)$-simplices of X it suffices to prove that for each $(n+1)$-simplex of X we have $\partial_n(\partial_{n+1}(\sigma)) = 0$. Thus let σ be an $(n+1)$-simplex with vertex set $V = \{x_0, \ldots, x_{n+1}\}$. We compute[282]

$$\begin{aligned} \partial_n(\partial_{n+1}(\sigma)) &= \partial_n\Big(\sum_{i=0}^{n+1} \mathrm{Hull}(\{x_0, \ldots, \widehat{x_i}, \ldots, x_{n+1}\}\Big) \\ &= \sum_{i=0}^{n+1} \sum_{\substack{j=0 \\ i \neq j}}^{n+1} \mathrm{Hull}(\{x_0, \ldots, \widehat{x_i}, \ldots, \widehat{x_j}, x_{n+1}\}) \;=\; 0. \end{aligned}$$

since each $(n-1)$-face appears twice and since we work with $\mathbb{Z}_2$-coefficients



[281] Here the odd "odd" condition in the definition of the boundary of an n-chain on page 449 is absolutely crucial.

[282] Here $\widehat{x_i}$ and $\widehat{x_j}$ means that we leave out this point.

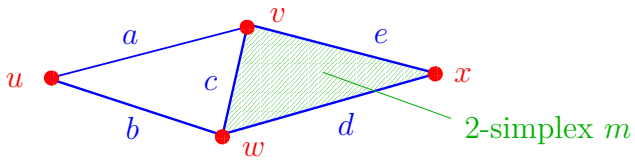
Another way of stating this argument is to say that each $(n-1)$-face of σ is the face of precisely two n-faces of σ. ■

The Simplicial–Double Boundary Lemma 34.1 allows us to introduce the following definition:

Definition. Let X be a simplicial complex. Furthermore let $n \in \mathbb{N}_0$. It follows immediately from the above Simplicial–Double Boundary Lemma 34.1 that we have the inclusion $\operatorname{im}(\partial_{n+1}\colon \mathrm{C}_{n+1}^{\mathrm{simp}}(X) \to \mathrm{C}_n^{\mathrm{simp}}(X)) \subset \ker(\partial_n\colon \mathrm{C}_n^{\mathrm{simp}}(X) \to \mathrm{C}_{n-1}^{\mathrm{simp}}(X))$. Thus it makes sense to define the n-th simplicial homology group as follows:

$$\mathrm{H}_n^{\mathrm{simp}}(X) \coloneqq \frac{\ker(\partial_n\colon \mathrm{C}_n^{\mathrm{simp}}(X) \to \mathrm{C}_{n-1}^{\mathrm{simp}}(X))}{\operatorname{im}(\partial_{n+1}\colon \mathrm{C}_{n+1}^{\mathrm{simp}}(X) \to \mathrm{C}_n^{\mathrm{simp}}(X))}.$$

Example. We consider the simplicial complex X that is shown below. It has four 0-simplices u, v, w, x, five 1-simplices a, b, c, d, e and one 2-simplex m.



One can easily verify that the corresponding simplicial chain complex is given as follows:

$$0 \to \underbrace{\mathbb{Z}_2 \cdot m}_{=\mathrm{C}_2^{\mathrm{simp}}(X)} \xrightarrow{\begin{pmatrix}0\\0\\1\\1\\1\end{pmatrix}} \underbrace{\begin{matrix}\mathbb{Z}_2 \cdot a\\ \mathbb{Z}_2 \cdot b\\ \mathbb{Z}_2 \cdot c\\ \mathbb{Z}_2 \cdot d\\ \mathbb{Z}_2 \cdot e\end{matrix}}_{=\mathrm{C}_1^{\mathrm{simp}}(X)} \xrightarrow{\begin{pmatrix}1&1&0&0&0\\1&0&1&0&1\\0&1&1&1&0\\0&0&0&1&1\end{pmatrix}} \underbrace{\begin{matrix}\mathbb{Z}_2 \cdot u\\ \mathbb{Z}_2 \cdot v\\ \mathbb{Z}_2 \cdot w\\ \mathbb{Z}_2 \cdot x\end{matrix}}_{=\mathrm{C}_0^{\mathrm{simp}}(X)} \to 0.$$

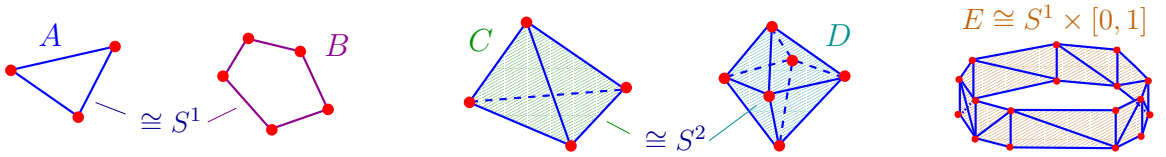
An elementary, albeit rather painful calculation shows that

$$\mathrm{H}_k^{\mathrm{simp}}(X) \cong \begin{cases} \mathbb{Z}_2, & \text{if } k = 0,\\ \mathbb{Z}_2, & \text{if } k = 1,\\ 0, & \text{if } k \geq 2. \end{cases}$$

□

The previous example should convince the reader that simplicial homology groups can be computed, with enough patience, for any given simplicial complex. This is a good moment to return to some our previous examples. With sufficient energy one obtains the following simplicial homology groups:
Using these examples we can already make a couple of educated guesses:

- The simplicial complexes A and B are homeomorphic and the corresponding simplicial homology groups $\mathrm{H}_k^{\mathrm{simp}}(A)$ and $\mathrm{H}_k^{\mathrm{simp}}(B)$ are isomorphic. We see the same behavior for the simplicial complexes C and D which are also homeomorphic. This suggests that the isomorphism type of simplicial homology might only depend on the homeomorphism type of the simplicial complex.
- If we look at the simplicial homology groups of the simplicial complexes that are homeomorphic to S^1 and that are homeomorphic to S^2 we might spot a pattern, namely one



$\mathrm{H}_0^{\mathrm{simp}}(A) \cong \mathbb{Z}_2$
$\mathrm{H}_1^{\mathrm{simp}}(A) \cong \mathbb{Z}_2$
$\mathrm{H}_2^{\mathrm{simp}}(A) = 0$

$\mathrm{H}_0^{\mathrm{simp}}(B) \cong \mathbb{Z}_2$
$\mathrm{H}_1^{\mathrm{simp}}(B) \cong \mathbb{Z}_2$
$\mathrm{H}_2^{\mathrm{simp}}(B) = 0$

$\mathrm{H}_0^{\mathrm{simp}}(C) \cong \mathbb{Z}_2$
$\mathrm{H}_1^{\mathrm{simp}}(C) = 0$
$\mathrm{H}_2^{\mathrm{simp}}(C) \cong \mathbb{Z}_2$
$\mathrm{H}_3^{\mathrm{simp}}(C) = 0$

$\mathrm{H}_0^{\mathrm{simp}}(D) \cong \mathbb{Z}_2$
$\mathrm{H}_1^{\mathrm{simp}}(D) = 0$
$\mathrm{H}_2^{\mathrm{simp}}(D) \cong \mathbb{Z}_2$
$\mathrm{H}_3^{\mathrm{simp}}(D) = 0$

$\mathrm{H}_0^{\mathrm{simp}}(E) \cong \mathbb{Z}_2$
$\mathrm{H}_1^{\mathrm{simp}}(E) \cong \mathbb{Z}_2$
$\mathrm{H}_2^{\mathrm{simp}}(E) = 0$

could make the guess that for any[283] $n \geq 1$ and any $k \in \mathbb{N}_0$ we have

$$\mathrm{H}_k^{\mathrm{simp}}(\text{simplicial complex homeomorphic to } S^n) \cong \begin{cases} \mathbb{Z}_2, & \text{if } k = 0, n, \\ 0, & \text{otherwise.} \end{cases}$$

- The simplicial complexes $A \cong S^1$ and $E \cong S^1 \times [0,1]$ are homotopy equivalent. We notice that the simplicial homology groups are isomorphic. This suggests, on admittedly rather tenuous evidence, that the isomorphism type of simplicial homology might actually only depend on the homotopy type of the simplicial complex.

All this sounds very promising. But there are serious issues with the concept of simplicial homology:

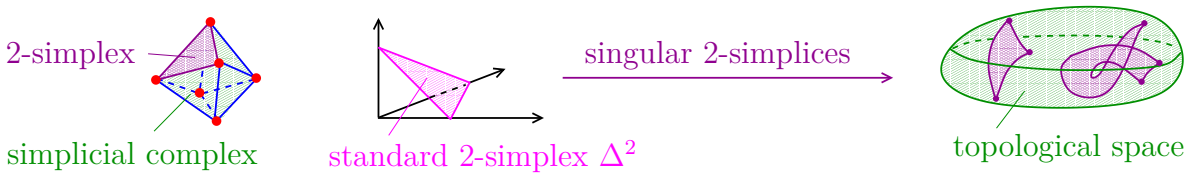
- First of all it seems rather restrictive to work just with simplicial complexes. Even though many topological spaces are homeomorphic to simplicial complexes, already for simple topological spaces, like the torus $S^1 \times S^1$, it is quite painful and unnatural to represent them as a simplicial complex.
- For simplicial homology to be really useful as a topological tool one would need to show that simplicial homology groups depend only on the homeomorphism type (or better homotopy type) of the simplicial complex. The longer one thinks about it, the harder it seems to prove such a statement.
- When we studied fundamental groups we saw that in applications it was essential that fundamental groups are covariantly functorial, i.e. that continuous maps induce maps on fundamental groups. Given a continuous map between two simplicial complexes there is no reasonable way how such a map will induce a map on the corresponding simplicial homology groups.

Because of all these issues we have to go back to the drawing board and come up with a better concept. The key idea is as follows:

- In *simplicial* homology $\mathrm{H}_k^{\mathrm{simp}}(X)$ we consider only simplicial complexes X and we study the simplices contained in them.
- In *singular* homology $\mathrm{H}_k(X)$ we allow any topological space X and this time we take the drastic step of considering "singular simplices", namely we study all continuous maps of the standard simplices Δ^n to X, regardless of whether or not the maps are injective.

This new concept will eventually solve all of the technical problems. For example we will see that, almost by definition, singular homology groups are covariantly functorial under continuous maps and thus homeomorphism invariants. Furthermore it will also not be

[283] What about $n = 0$?



particularly difficult to show that singular homology groups depend only on the homotopy type of the topological space. The problem with singular homology is that in our attempt to solve the technical issues of simplicial homology we threw computability out of the window. We will have to work quite hard to calculate singular homology groups. But eventually we will see that singular homology encompasses simplicial homology and that it is the best of two worlds: it has very nice theoretical properties and highly computable.

CHAPTER 35

The singular homology groups of a topological space

In the previous chatty chapter we gave a very short introduction to simplicial homology, we calculated a few examples and we discovered serious issues with the concept of simplicial homology. As indicated towards the end of the previous chapter the solution to these problems is to introduce a "homology theory" where we study "singular simplices". In this chapter we will follow up on this idea, we will introduce singular homology, we will see it is covariantly functorial and we will calculate singular homology in a few very basic examples. In the coming chapters we will then develop the theory of singular homology groups in its full glory.

35.1. **Algebraic chain complexes.** In the previous chapter, in the discussion of simplicial homology, we implicitly briefly encountered the purely algebraic notion of a chain complex. This notion will play a prominent role in the discussion of homology and it thus useful to introduce this algebraic notion before we return to topology and before we get to the actual definition of singular homology.

Definition.

(1) An chain complex (C_*, ∂_*) is a sequence

$$\ldots \to C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \ldots C_1 \xrightarrow{\partial_1} C_0 \to 0$$

of homomorphisms between abelian groups such that the composition of any two consecutive boundary maps is the zero map, i.e. for every $i \in \mathbb{N}$ we have $\partial_{i-1} \circ \partial_i = 0$. For $n \in \mathbb{Z}_{<0}$ we always set $C_n := 0$.

(2) Let (C_*, ∂_*) be a chain complex and let $n \in \mathbb{N}_0$. By definition of a chain complex we have $\operatorname{im}(\partial_{n+1} \colon C_{n+1} \to C_n) \subset \ker(\partial_n \colon C_n \to C_{n-1})$. Thus it makes sense to define the n-th homology group of the chain complex as follows:

$$\mathrm{H}_n(C_*) \ := \ \frac{\ker(\partial_n \colon C_n \to C_{n-1})}{\operatorname{im}(\partial_{n+1} \colon C_{n+1} \to C_n)}.$$

(3) (a) We refer to elements in the kernel of $\partial_n \colon C_n \to C_{n-1}$ as cycles.
(b) We say that two elements $c, d \in C_n$ are homologous if there exists an $e \in C_{n+1}$ with $c - d = \partial_{n+1} e$.
(c) We say that a cycle $z \in C_n$ is null-homologous if z is homologous to the zero element of C_n, i.e. if $[z] = 0 \in \mathrm{H}_n(C_*)$.

As always we not only care about objects, but we also want to allow reasonable morphisms between the objects. This quest leads us to the following definition:

Definition. A chain map $f \colon C_* \to D_*$ between two chain complexes[284] consists of a family $\{f_n \colon C_n \to D_n\}_{n \in \mathbb{N}_0}$ of homomorphisms such that for each $n \in \mathbb{N}_0$ the following equality

holds:[285]

$$f_{n-1} \circ \partial_n \;=\; \partial_n \circ f_n.$$

Put differently the homomorphisms f_n have the property that the following diagram commutes:

$$\begin{array}{ccccccccc} \ldots & \xrightarrow{\partial_{n+2}} & C_{n+1} & \xrightarrow{\partial_{n+1}} & C_n & \xrightarrow{\partial_n} & C_{n-1} & \xrightarrow{\partial_{n-1}} & \ldots \\ & & \downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & \\ \ldots & \xrightarrow{\partial_{n+2}} & D_{n+1} & \xrightarrow{\partial_{n+1}} & D_n & \xrightarrow{\partial_n} & D_{n-1} & \xrightarrow{\partial_{n-1}} & \ldots \end{array}$$

The above definitions naturally lead to the following definition:

Definition. We refer to category $\mathcal{ChCplx}$ that is given by

$$\begin{aligned} \mathrm{Ob}(\mathcal{ChCplx}) &:= \text{ all chain complexes,} \\ \mathrm{Mor}(C_*, D_*) &:= \text{ all chain maps from } C_* \text{ to } D_*. \end{aligned}$$

and with morphisms with the obvious composition of morphisms[286] as the category of chain complexes.

Lemma 35.1. (Chain Complex-H_*-Functor Lemma)

(1) If $f\colon C_* \to D_*$ is a chain map between chain complexes C_* and D_*, then the map

$$\begin{array}{rcl} f_*\colon \mathrm{H}_n(C_*) & \to & \mathrm{H}_n(D_*) \\ [c] & \mapsto & [f_n(c)] \end{array}$$

is a well-defined homomorphism.

(2) Given $n \in \mathbb{N}_0$ the maps

$$\begin{array}{rcl} C_* & \mapsto & \mathrm{H}_n(C_*) \\ (f\colon C_* \to D_*) & \mapsto & (f_*\colon \mathrm{H}_n(C_*) \to \mathrm{H}_n(D_*)) \end{array}$$

define a covariant functor from the category $\mathcal{ChCplx}$ of chain complexes to the category $\mathcal{AbGroup}$ of abelian groups.

Definition. In the setting of the Chain Complex-H_*-Functor Lemma 35.1 (1) we refer to the corresponding map $f_*\colon \mathrm{H}_n(C_*) \to \mathrm{H}_n(D_*)$ as the induced map on homology.

Proof.

(1) Let $c \in C_n$ be a cycle. First we have to show that $f_n(c)$ is again a cycle. Indeed we have

$$\partial_n(f_n(c)) \underset{\text{definition of a chain map}}{=} f_{n-1}(\partial_n c) \underset{\text{since } c \text{ is a cycle}}{=} f_n(0) \underset{\substack{\text{since } f_n \text{ is a} \\ \text{homomorphism}}}{=} 0.$$

[284] As usual we suppress bits from the notation which are understood from the context. In this instance we suppress the notation for the boundary maps in the chain complexes. Put differently, instead of saying "let (C_*, ∂_*) be a chain complex" we say simply "let C_* be a chain complex".

[285] More precisely, for any $c \in C_n$ we have $f_{n-1}(\partial_n(c)) = \partial_n(f_n(c))$ where on the left-hand side we consider the boundary map $C_n \to C_{n-1}$, and on the right-hand side we consider the boundary map $D_n \to D_{n-1}$ which we also denote by ∂_n. Usually it is obvious from the context which boundary map we consider.

[286] Here we use the rather elementary observation that the composition of two chain maps $f\colon C_* \to D_*$ and $g\colon D_* \to E_*$ defines a chain map $g \circ f\colon C_* \to E_*$.

Now let $c, d \in C_n$ be two cycles which represent the same element in $\mathrm{H}_n(C_*)$. We have to show that $[f_n(c)] = [f_n(d)]$. Indeed we have

$$[f_n(c)] - [f_n(d)] \;=\; [f_n(c-d)] \;\underset{\uparrow}{=}\; [f_n(\partial_{n+1}(e))] \;\underset{\uparrow}{=}\; [\partial_{n+1}(f_{n+1}(e))] \;\underset{\uparrow}{=}\; 0 \;\in \mathrm{H}_n(D).$$

since c, d are homologous there exists an $e \in C_{n+1}$ with $\partial_{n+1}e = c - d$ — by the definition of a chain map — by the definition of homology

This shows that the map $f_*\colon \mathrm{H}_n(C_*) \to \mathrm{H}_n(D_*)$ is well-defined. It follows easily from the definitions that this map is actually a homomorphism.

(2) The statement follows easily from (1) and the definitions. ∎

At this stage there is only one little lemma on chain complexes that we want to state. Before we can do so, we recall the following definition from page 254:

Definition. Let $\{X_s\}_{s\in S}$ be a family of abelian groups.

(1) We define the product of the X_s to be the abelian group

$$\prod_{s\in S} X_s \;:=\; \Big\{ f\colon S \to \bigcup_{s\in S} X_s \;\Big|\; \text{for every } s \in S \text{ we have } f(s) \in X_s \Big\},$$

where the group structure is given by the obvious "component wise addition".

(2) We define the direct sum of the X_s to be the group

$$\bigoplus_{s\in S} X_s \;:=\; \Big\{ (g_s)_{s\in S} \in \prod_{s\in S} X_s \;\Big|\; \text{all but finitely many } g_s \text{ are trivial} \Big\}.$$

In the Coproduct-of-Abelian Groups Lemma 18.1 we showed that the direct sum with the above natural homomorphisms is the coproduct of $\{X_s\}_{s\in S}$ in the category $\mathcal{AbGroup}$ of abelian groups.

Basically the same way as we defined the direct sum of a family of abelian groups we can also define the direct sum of a family of chain complexes. Now we can formulate the following lemma:

Lemma 35.2. (Direct Sum-H$_*$-Lemma) Let $\{C_a\}_{a\in A}$ be a family of chain complexes. For every $n \in \mathbb{N}_0$ the natural maps $C_a \to \bigoplus_{a\in A} C_a$ induce an isomorphism

$$\begin{array}{ccc} \bigoplus_{a\in A} \mathrm{H}_n(C_a) & \xrightarrow{\cong} & \mathrm{H}_n\Big(\bigoplus_{a\in A} C_a\Big) \\ ([c_a])_{a\in A} & \mapsto & [(c_a)_{a\in A}]. \end{array}$$

Proof. One can easily verify that the map

$$\begin{array}{ccc} \mathrm{H}_n\Big(\bigoplus_{a\in A} C_a\Big) & \xrightarrow{\cong} & \bigoplus_{a\in A} \mathrm{H}_n(C_a) \\ [(c_a)_{a\in A}] & \mapsto & ([c_a])_{a\in A} \end{array}$$

is an inverse to the given map. ∎

We could easily fill the remainder of these notes with the general theory of chain complexes. We will not do so, but instead we will develop the theory of chain complexes along the way, as we develop the theory of the singular chain complexes and singular homology.

35.2. **The singular chain complex.** With this section we return to topology. As a first step towards the definition of singular homology we need to introduce the notion of a (standard) n-simplex:

Definition. Let $n \in \mathbb{N}_0$.

(1) Given $i \in \{0, \dots, n\}$ we write

$$\mathfrak{v}_i \;:=\; (0, \dots, 0, \underset{\substack{\uparrow \\ i\text{-th coordinate}}}{1}, 0, \dots, 0).$$

(2) Given $n \in \mathbb{N}_0$ we refer to the convex hull of $\mathfrak{v}_1, \dots, \mathfrak{v}_{n+1} \in \mathbb{R}^{n+1}$, i.e. to

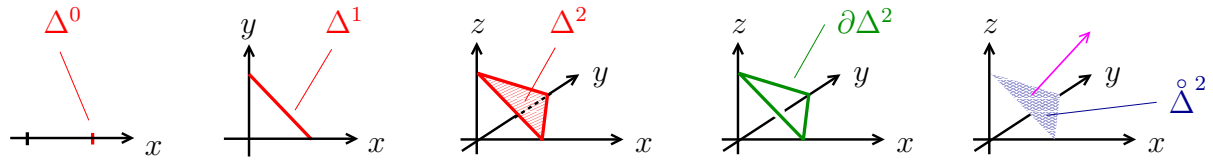
$$\Delta^n \;:=\; \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \,|\, x_0 + \cdots + x_n = 1 \text{ and } x_i \geq 0 \text{ for all } i = 0, \dots, n\}$$

as the standard n-simplex.

(3) Given the standard n-simplex Δ^n we define its boundary as

$$\partial\Delta^n \;:=\; \bigcup_{i=0}^{n} \{(t_0, \dots, t_n) \in \Delta^n \,|\, t_i = 0\}.$$

Sometimes it will be helpful to view $\mathring{\Delta}^n$ as a smooth manifold where a smooth atlas for $\mathring{\Delta}^n$ is given by projection to the first n coordinates. We equip the smooth manifold $\mathring{\Delta}^n$ with the orientation where for each $P \in \mathring{\Delta}^n$ a basis $v_1, \dots, v_n$ of $\widetilde{\mathrm{T}}_P\mathring{\Delta}^n$ is positive if the vectors $\{(1, \dots, 1), v_1, \dots, v_n\}$ are a positive basis for $\mathbb{R}^{n+1}$.



For the record we state the following elementary lemma:

Lemma 35.3. (Δ^n-Homeomorphism Lemma) Let $n \in \mathbb{N}_0$.

(1) We set

$$\triangle_n \;:=\; \left\{(x_1, \dots, x_n) \in [0,1]^n \;\middle|\; \sum_{i=1}^{n} x_i \in [0,1]\right\}$$

and refer to it as the planar n-simplex. The maps

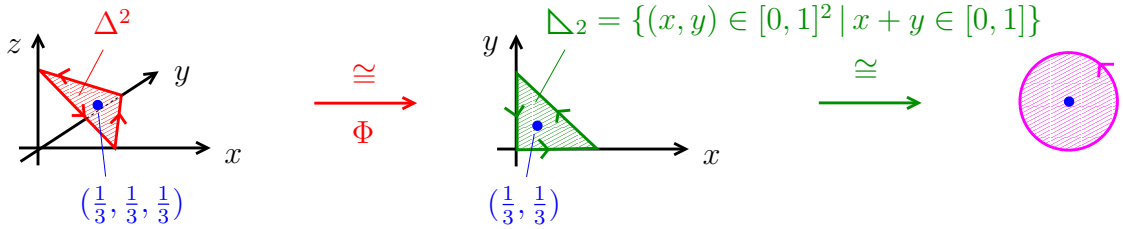
$$\begin{array}{rcl} \Phi\colon \Delta^n & \to & \triangle_n \\ (t_0, \dots, t_n) & \mapsto & (t_0, \dots, t_{n-1}) \end{array} \quad \text{and} \quad \begin{array}{rcl} \Psi\colon \triangle_n & \to & \Delta^n \\ (x_1, \dots, x_n) & \mapsto & \left(x_1, \dots, x_n, 1 - \sum_{i=1}^{n} x_i\right) \end{array}$$

are homeomorphisms that are inverses of one another.

(2) The map

$$\Delta^n \xrightarrow{\Phi} \triangle_n \xrightarrow[\text{applied to } \triangle_n \text{ and } Q = (\frac{1}{n+1}, \dots, \frac{1}{n+1})]{\text{the homeomorphism } \Xi \text{ obtained from the Convex-to-Ball Proposition 2.12 (2)}} \overline{B}^n$$

is a homeomorphism that restricts to a homeomorphism from $\partial\Delta^n$ to the sphere S^{n-1}.



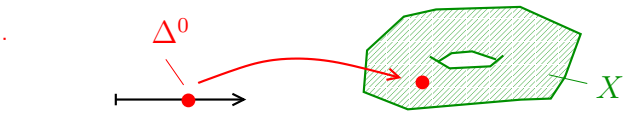
Proof.

(1) Note that Φ extends "by the same formula" to a map $\mathbb{R}^{n+1} \to \mathbb{R}^n$ which is evidently continuous. Hence the restriction to Δ^n is also continuous. Now that this is settled we see that the remaining statements are basically trivial.
(2) The statement follows easily from (1), the observation that $\triangle_n$ is compact (which can be shown easily using the Heine-Borel Theorem 1.12) and the Convex-to-Ball Proposition 2.12 (2). ■

Definition. Let X be a topological space and let $n \in \mathbb{N}_0$. A singular n-simplex is a continuous map $\sigma\colon \Delta^n \to X$.

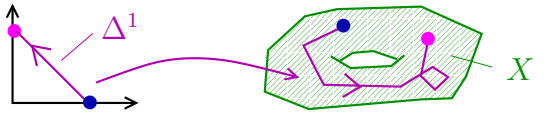
Example. Let X be a topological space.

(1) A singular 0-simplex is a continuous map from Δ^0 to X. Note that $\Delta^0 = \{1\} \subset \mathbb{R}$. Thus a map $\Delta^0 = \{1\} \to X$ is uniquely determined by a point in X. Given a point $P \in X$ we frequently denote by P also the singular 0-simplex $\Delta^0 \to X$ that is given by sending the single point in Δ^0 to P.



a singular 0-simplex in X is essentially the same as a point in X

(2) A singular 1-simplex is a continuous map $\Delta^1 \to X$.



a singular 1-simplex in X is essentially the same as a path in X

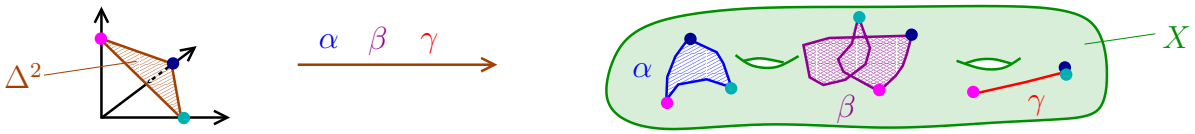
Since the map $[0,1] \to \Delta^1$ given by $t \mapsto (1-t, t)$ is a homeomorphism we see that a singular 1-simplex is essentially the same as a path in X. □

Definition. Let X be a topological space and let $n \in \mathbb{N}_0$.

(1) We define the n-th singular chain group as follows:

$$\mathrm{C}_n(X) \;:=\; \text{free abelian group generated by the set of singular } n\text{-simplices in } X.$$

For $n \in \mathbb{Z}_{<0}$ we set $\mathrm{C}_n(X) := 0$.
(2) We refer to an element in $\mathrm{C}_n(X)$ as a singular n-chain in X.

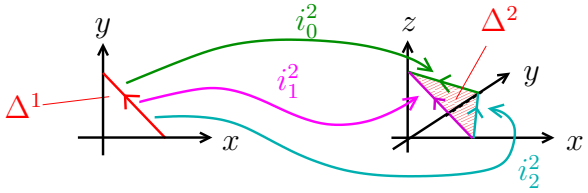


$-2\alpha + \beta + 3\gamma$ is a singular 2-chain

Motivated by the discussion of the last chapter we now want to find suitable boundary maps between the singular chain groups $\mathrm{C}_n(X)$. In the simplicial chain complex our generators were simplices and on page 450 we defined the boundary of a simplex to be a linear combination of codimension-one faces of the simplex. In our current setting generators are singular simplices, i.e. continuous maps $\Delta^n \to X$. To such a singular simplex we now have to associate a singular $(n-1)$-simplex, i.e. a linear combination of continuous maps $\Delta^{n-1} \to X$. To do so it is convenient to introduce the following definition:

Definition. Let $n \in \mathbb{N}_0$. For $j \in \{0, \dots, n\}$ we define the j-th face map as follows:[287][288]

$$\begin{array}{rcl} i_j^n \colon \Delta^{n-1} & \to & \Delta^n \\ (t_0, \dots, t_{n-1}) & \mapsto & (t_0, \dots, t_{j-1}, 0, t_j, \dots, t_{n-1}). \end{array}$$



Now we can define the promised boundary maps $\mathrm{C}_n(X) \to \mathrm{C}_{n-1}(X)$.

Definition. Let X be a topological space and let $\sigma \colon \Delta^n \to X$ be a singular n-simplex. The boundary $\partial_n \sigma \in \mathrm{C}_{n-1}(X)$ of σ is defined as[289]

$$\partial_n \sigma \;:=\; \sum_{j=0}^{n} (-1)^j \cdot \underbrace{\sigma \circ i_j^n}_{\text{map } \Delta^{n-1} \to X}.$$

By the Free Abelian Group–Existence Lemma 18.4 the map $\sigma \mapsto \partial_n \sigma$ extends uniquely to a linear map

$$\begin{array}{rcl} \partial_n \colon \mathrm{C}_n(X) & \to & \mathrm{C}_{n-1}(X) \\ \sum_{r=1}^{k} a_r \cdot \sigma_r & \mapsto & \sum_{r=1}^{k} a_r \cdot \partial \sigma_r, \end{array}$$

which we refer to as the n-th boundary map. If it is clear with what dimension we are working with, then we just write ∂ instead of ∂_n.

Examples.

(1) For any singular 1-simplex $\sigma \colon \Delta^1 \to X$ we have[290]

$$\partial_1 \sigma \;=\; (-1)^0 \cdot \sigma \circ i_0^1 + (-1)^1 \cdot \sigma \circ i_1^1 \;=\; \sigma(0,1) - \sigma(1,0) \;\in\; \mathrm{C}_0(X).$$

Recall that a singular 1-simplex can be viewed as a map σ from the "interval" Δ^1 to X. The boundary of σ is thus the "endpoint" $\sigma(0,1)$ "minus" the "starting point" $\sigma(1,0)$.

(2) In the figure below we sketch the boundary of a singular 2-simplex.

The following proposition, which is an analogue of the Simplicial–Double Boundary Lemma 34.1, shows that these boundary maps have the fundamental property that the composition of two consecutive boundary maps is the zero map.

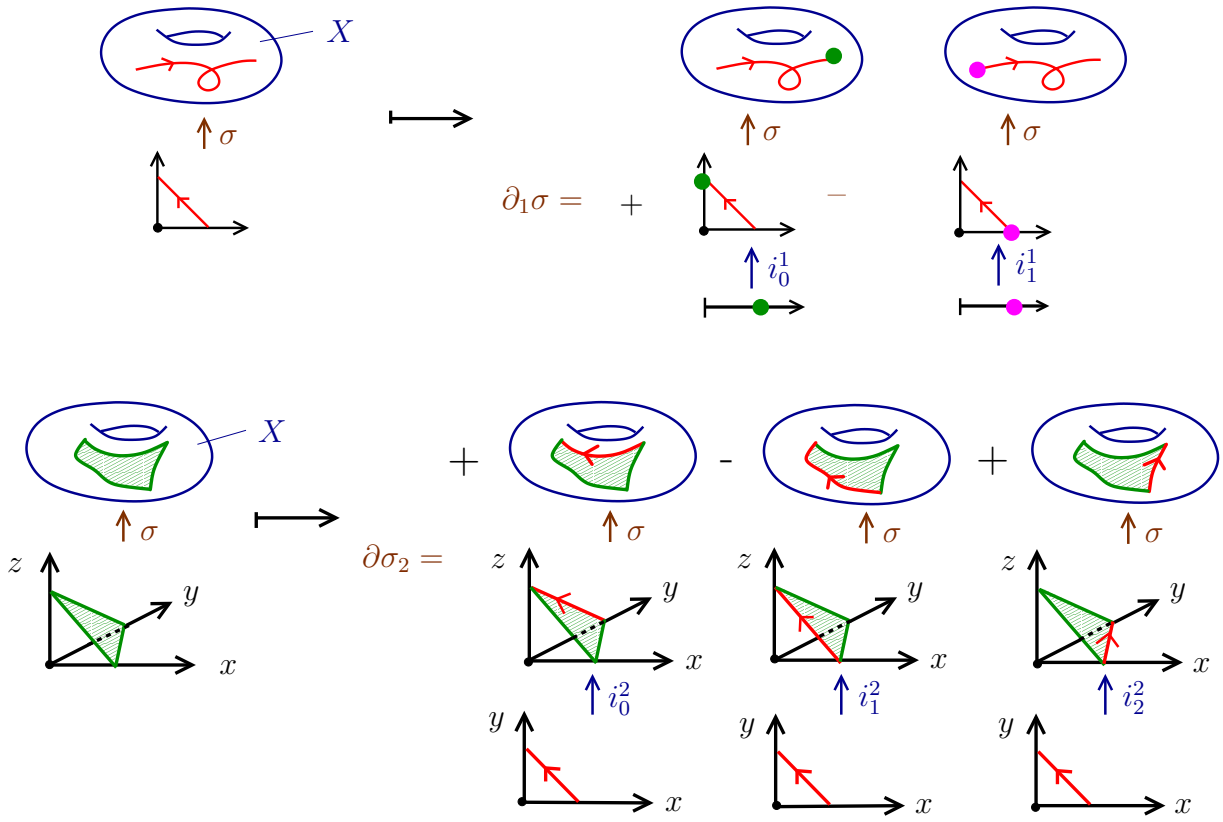
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[288] One can of course easily verify that this map is continuous.

[288] The image of a face map is often called an $(n-1)$-*dimensional face* of the standard simplex. The boundary $\partial\Delta^n$ of the standard n-simplex is the union of all $(n-1)$-dimensional faces of Δ^n.

[289] The sign $(-1)^j$ is surely somewhat unexpected. As we will see shortly, it is necessary to ensure that the composition of two consecutive boundary maps is indeed zero.

[290] Here we again use the convention that given a point $x \in X$ we denote by x also the corresponding singular 0-simplex.



Proposition 35.4. (Double Boundary Proposition) Let X be a topological space. For each $n \in \mathbb{N}_0$ the composition

$$\partial_{n-1} \circ \partial_n \colon \mathrm{C}_n(X) \;\to\; \mathrm{C}_{n-2}(X)$$

is the zero map.

Proof. Let $n \in \mathbb{N}_0$. It suffices to show that for every singular n-simplex $\sigma \colon \Delta^n \to X$ we have $\partial_{n-1}(\partial_n(\sigma)) = 0$. Thus let $\sigma \colon \Delta^n \to X$ be a singular n-simplex. We calculate that

$$\begin{aligned}\partial_{n-1}(\partial_n(\sigma)) &\overset{\text{definition of } \partial_n}{=} \partial_{n-1}\Big(\sum_{j=0}^{n}(-1)^j \cdot \sigma \circ i_j^n\Big) \overset{\text{definition of } \partial_{n-1}}{=} \sum_{k=0}^{n-1}(-1)^k \cdot \sum_{j=0}^{n}(-1)^j \cdot (\sigma \circ i_j^n) \circ i_k^{n-1} \\ &= \sum_{0=k<j\le n}(-1)^{k+j} \cdot \sigma \circ (i_j^n \circ i_k^{n-1}) + \sum_{0=j\le k\le n-1}(-1)^{k+j} \cdot \sigma \circ (i_j^n \circ i_k^{n-1}) =: (*).\end{aligned}$$

Now we want to show that each expression $\sigma \circ (i_j^n \circ i_k^{n-1})$ appears precisely twice, but with opposite signs. From the definitions it follows easily that for $j \le k$ we have[291]

$$i_j^n \circ i_k^{n-1} \;=\; i_{k+1}^n \circ i_j^{n-1}.$$

[291] Indeed, if $j \le k$, then the zero-entries of $(i_j^n \circ i_k^{n-1})(t_0, \ldots, t_{n-1})$ are the entries j and $k+1$.

Now we continue with the above calculation:

$$(*) \underset{\uparrow}{=} \sum_{0=k<j\leq n} (-1)^{k+j} \cdot \sigma \circ (i^n_j \circ i^{n-1}_k) + \sum_{0=j\leq k\leq n-1} (-1)^{k+j} \cdot \sigma \circ (i^n_{k+1} \circ i^{n-1}_j)$$

above calculation of $i^n_j \circ i^{n-1}_k$

$$\underset{\uparrow}{=} \sum_{0=k<j\leq n} (-1)^{k+j} \cdot \sigma \circ (i^n_j \circ i^{n-1}_k) + \sum_{0=j<l\leq n} (-1)^{l+j-1} \cdot \sigma \circ (i^n_l \circ i^{n-1}_j) \underset{\uparrow}{=} 0.$$

substitution $l = k+1$ all summands left and right cancel ■

The Double Boundary Proposition 35.4 says that the map $X \mapsto (\mathrm{C}_*(X), \partial_*)$ assigns to each topological space a chain complex. Of course, as experienced algebraic topologists, we want this assignment to be functorial. This goal leads us to the following definition:

Definition. Let $f\colon X \to Y$ be a continuous map between topological spaces. For each $n \in \mathbb{N}_0$ we refer to

$$\begin{array}{rcl} f_*\colon \mathrm{C}_n(X) & \to & \mathrm{C}_n(Y) \\ \sum_{i=1}^k a_i \cdot \underbrace{\sigma_i}_{\uparrow} & \mapsto & \sum_{i=1}^k a_i \cdot (\underbrace{f \circ \sigma_i}_{\uparrow}). \end{array}$$

map $\Delta^n \to X$ map $\Delta^n \to Y$

as the induced chain map.



The above definition leads to the following lemma:

Lemma 35.5. (C_*–Functor Lemma)

(1) Let $f\colon X \to Y$ be a continuous map between two topological spaces. The induced maps $f_*\colon \mathrm{C}_*(X) \to \mathrm{C}_*(Y)$ form a chain map, i.e. given any $c \in \mathrm{C}_n(X)$ we have

$$f_*(\partial_n c) = \partial_n(f_*(c)) \in \mathrm{C}_n(Y).$$

(2) The maps

$$\text{topological space } X \mapsto (\mathrm{C}_*(X), \partial_*)$$

and

$$(f\colon X \to Y) \mapsto (f_*\colon \mathrm{C}_*(X) \to \mathrm{C}_*(Y))$$

define a covariant functor from the category $\mathcal{T}op$ of topological spaces to the category $\mathcal{ChCplx}$ of chain complexes.

Proof.

(1) Let $f\colon X \to Y$ be a continuous map between topological spaces. Given any singular n-simplex $\sigma\colon \Delta^n \to X$ we have

$$f_*(\partial\sigma) \underset{\uparrow}{=} f_*\Big(\sum_{j=0}^n (-1)^j \cdot \sigma \circ i^n_j\Big) \underset{\uparrow}{=} \sum_{j=0}^n (-1)^j \cdot f \circ (\sigma \circ i^n_j) = \sum_{j=0}^n (-1)^j \cdot \overbrace{(f \circ \sigma)}^{=f_*\sigma} \circ i^n_j \underset{\uparrow}{=} \partial(f_*\sigma).$$

definition of ∂ definition of f_* definition of ∂

Since the singular n-simplices generate $\mathrm{C}_n(X)$ we obtain the desired equality for all singular n-chains. By definition this means that $f_*\colon \mathrm{C}_*(X) \to \mathrm{C}_*(Y)$ is a chain map.

(2) It follows immediately from the definitions that for two continuous maps $f\colon X \to Y$ and $g\colon Y \to Z$ we have $(g \circ f)_* = g_* \circ f_*\colon \mathrm{C}_*(X) \to \mathrm{C}_*(Z)$. With this observation Statement (2) is an almost immediate consequence of Statement (1). ■

For any functor it is interesting to know whether or not products, coproducts and more generally limits and colimits, etc. are preserved. It follows easily from the following lemma that the functor $X \mapsto (\mathrm{C}_*(X), \partial_*)$ preserves colimits.

Lemma 35.6. (Path-Component–C_*–Lemma) Let X be a topological space with path-components $\{X_a\}_{a\in A}$. Given $a \in A$ we denote by $i_a\colon X_a \to X$ the inclusion map. The map

$$\bigoplus_{a\in A} i_a\colon \bigoplus_{a\in A} \mathrm{C}_*(X_a) \;\to\; \mathrm{C}_*(X)$$

is an isomorphism of chain complexes.

Proof. Evidently the standard simplex Δ^n is path-connected. Thus the image of a continuous map $\sigma\colon \Delta^n \to X$ is contained in a path-component X_a. This shows that

$$\bigsqcup_{a\in A} \text{set of singular } n\text{-simplices in } X_a \;=\; \text{set of singular } n\text{-simplices in } X.$$

It follows that the map

$$\bigoplus_{a\in A} i_a\colon \bigoplus_{a\in A} \mathrm{C}_n(X_a) \;\to\; \mathrm{C}_n(X)$$

is an isomorphism of free abelian groups. It follows immediately from the definitions that this a chain map. ■

35.3. **Definition of the singular homology groups.** It follows from the Double Boundary Proposition 35.4 that given a topological space X the sequence $(\mathrm{C}_*(X), \partial_*)$ is a chain complex. This allows us to make the following definition, which is arguably one of the most influential and important definitions of algebraic topology.

Definition. Let X be a topological space and let $n \in \mathbb{N}_0$. We define the n-th singular homology group of X as the n-th homology group of the singular chain complex. In other words, we set

$$\mathrm{H}_n(X) \;:=\; \mathrm{H}_n(\mathrm{C}_*(X)) \;=\; \frac{\ker(\partial_n\colon \mathrm{C}_n(X) \to \mathrm{C}_{n-1}(X))}{\mathrm{im}(\partial_{n+1}\colon \mathrm{C}_{n+1}(X) \to \mathrm{C}_n(X))}.$$

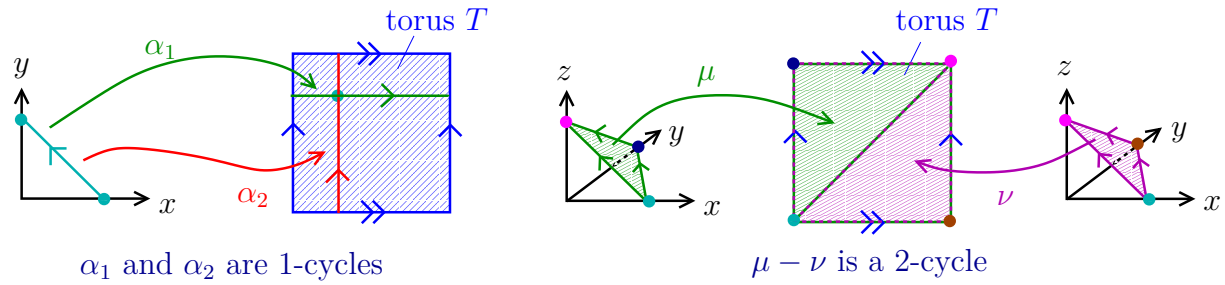
Throughout the lecture notes we will mostly shorten the name "singular homology" to "homology".

Example. There is only one example that we can do immediately: For the empty topological space $\varnothing$ we have for all $n \in \mathbb{N}_0$ that $\mathrm{C}_n(\varnothing) = 0$. It follows that for all $n \in \mathbb{N}_0$ we have $\mathrm{H}_n(\varnothing) = 0$. □

The following definitions are just adaptations of the definitions on page 454.

Definition. Let X be a topological space.

(1) We say that $c \in \mathrm{C}_n(X)$ is a singular n-cycle, or short n-cycle, if $\partial c = 0$.
(2) We say that two singular n-chains c and d are homologous if there exists a singular $(n+1)$-chain $e \in \mathrm{C}_{n+1}(X)$ with $\partial e = c - d$.
(3) An n-cycle c is called null-homologous, if $[c] = 0 \in \mathrm{H}_n(X)$.



Example. In the figure below we show the torus $T = ([0,1] \times [0,1])/\sim$ together with two singular 1-simplices α_1 and α_2 and together with two singular 2-simplices μ and ν.
It follows easily from the definitions that α_1 and α_2 are 1-cycles. It is a good exercise to verify that $\mu - \nu$ is a 2-cycle. With a little bit of practice it is fairly easy to write down cycles, but usually it is very hard to show that a cycle represents a non-trivial element in homology. For example, with our present knowledge it is impossible though to determine whether the cycles $\alpha_1, \alpha_2, \mu - \nu$ represent non-trivial homology classes. □

The following lemma is so fundamental that we will use it constantly without referring to it.

Lemma 35.7. (H_n–Functor Lemma)
(1) Let $f\colon X \to Y$ be a continuous map between topological spaces. The map

$$\begin{array}{rcl} f_*\colon \mathrm{H}_n(X) & \to & \mathrm{H}_n(Y) \\ \left[\sum_{i=1}^{k} a_i \cdot \sigma_i\right] & \mapsto & \left[\sum_{i=1}^{k} a_i \cdot (f \circ \sigma_i)\right] \end{array}$$

is a well-defined homomorphism.
(2) For each $n \in \mathbb{N}_0$ the map

$$X \mapsto \mathrm{H}_n(X)$$

and the map

$$(f\colon X \to Y) \mapsto \left(\begin{array}{rcl} f_*\colon \mathrm{H}_n(X) & \to & \mathrm{H}_n(Y) \\ \left[\sum_{i=1}^{k} a_i \cdot \sigma_i\right] & \mapsto & \left[\sum_{i=1}^{k} a_i \cdot (f \circ \sigma_i)\right] \end{array} \right)$$

define a covariant functor from the category of topological spaces to the category *AbGroup* of abelian groups.

Example. It follows from the H_n–Functor Lemma 35.7 that every homeomorphism induces an isomorphism of homology groups. □

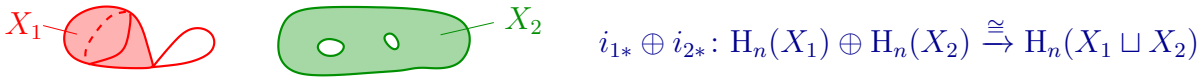
Proof.
(1) This statement follows from the C_*–Functor Lemma 35.5 (1) together with the Chain Complex-H_*-Functor Lemma 35.1 (1).
(2) This statement follows from the combination of (1), the C_*–Functor Lemma 35.5 (2) and the Chain Complex-H_*-Functor Lemma 35.1 (2). ■

Again the question is how singular homology groups behave under coproducts, products, etc. At this stage we have the following very long hanging fruit, from which it follows easily that homology groups preserve coproducts.

Lemma 35.8. (Path-Component–H_*–Lemma) Let X be a topological space with path-components $\{X_a\}_{a\in A}$. Given $a \in A$ we denote by $i_a\colon X_a \to X$ the inclusion map. For each $n \in \mathbb{N}_0$ the map

$$\bigoplus_{a\in A} i_{a*}\colon \bigoplus_{a\in A} \mathrm{H}_n(X_a) \;\to\; \mathrm{H}_n(X)$$

is an isomorphism.



Proof. This lemma is an immediate consequence of the Path-Component–C_*–Lemma 35.6 and the Direct Sum-H_*-Lemma 35.2. ∎

35.4. **First calculations of homology groups.** Given a topological space the corresponding homology groups are defined as the quotient of two abelian groups which in general will not be finitely generated. Somewhat surprisingly we will see in the coming chapters that the homology groups are nonetheless very frequently finitely generated abelian groups.

We start out our computations of homology groups with the most basic calculation, which often gets used implicitly without referring to it:

Lemma 35.9. (Empty Space-H_*-Lemma) For the empty topological space $\varnothing$ we have $\mathrm{H}_k(\varnothing) = 0$ for all $k \in \mathbb{N}_0$.

Proof. This lemma follows immediately from the observation that in this case we have $\mathrm{C}_k(\varnothing) = 0$ for all $k \in \mathbb{N}_0$. ∎

We continue with the somewhat more interesting calculation of the 0-th homology of a path-connected topological space. The key to the calculation is the following definition:

Definition. Given a topological space X we refer to the homomorphism

$$\begin{aligned} \epsilon_X\colon \mathrm{C}_0(X) &\to \mathbb{Z} \\ \sum_{i=1}^{k} a_i \cdot \sigma_i &\mapsto \sum_{i=1}^{k} a_i \end{aligned}$$

as the augmentation homomorphism.

The following lemma shows in particular that the augmentation map is a neat addition to the singular chain complex of a topological space.

Lemma 35.10. (Augmentation Lemma) Let X be a topological space.

(1) The map $\epsilon_X \circ \partial_1\colon \mathrm{C}_1(X) \to \mathbb{Z}$ is the zero map.

(2) The map

$$\begin{aligned} \epsilon_X\colon \mathrm{H}_0(X) &\to \mathbb{Z} \\ [\sigma] &\mapsto \epsilon_X(\sigma) \end{aligned}$$

is well-defined. In the following we refer to this map also as the augmentation map.

(3) The augmentation map is natural in the sense that for any continuous map $f\colon X \to Y$ between topological spaces the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}_0(X) & \xrightarrow{f_*} & \mathrm{H}_0(Y). \\ {\scriptstyle \epsilon_X}\downarrow & & \downarrow{\scriptstyle \epsilon_Y} \\ \mathbb{Z} & \xrightarrow{\mathrm{id}} & \mathbb{Z}. \end{array}$$

Proof. Let X be a topological space.

(1) Since the abelian group $\mathrm{C}_1(X)$ is generated by singular 1-simplices it suffices to prove that for each singular 1-simplex $\sigma\colon \Delta^1 \to X$ we have $(\epsilon_X \circ \partial_1)(\sigma) = 0$. But this is the case since

$$\epsilon_X(\partial_1(\sigma)) \underset{\substack{\uparrow \\ \text{see the discussion on page 459}}}{=} \epsilon_X(\sigma(0,1) - \sigma(1,0)) \underset{\substack{\uparrow \\ \text{by definition of } \epsilon_X}}{=} 1 - 1 = 0.$$

(2) If we have $\sigma, \tau \in \mathrm{C}_0(X)$ with $[\sigma] = [\tau] \in \mathrm{H}_0(X)$, then by definition of $\mathrm{H}_0(X)$ there exists a $\mu \in \mathrm{C}_1(X)$ with $\partial\mu = \sigma - \tau$. We now see that

$$\epsilon_X(\sigma) = \epsilon_X(\tau + \partial\mu) = \epsilon_X(\tau) + \underbrace{\epsilon_X(\partial\mu)}_{=0 \text{ by (1)}} = \epsilon_X(\tau).$$

(3) This statement follows easily from the definitions. ■

Using the augmentation map we can formulate the following proposition:

Proposition 35.11. (H_0-Proposition) For any path-connected non-empty topological space X the augmentation map

$$\epsilon_X\colon \mathrm{H}_0(X) \to \mathbb{Z}$$

is a natural isomorphism.[292] In fact given any $P \in X$ the inverse is given by the map

$$\begin{array}{rcl} \iota\colon \mathbb{Z} & \to & \mathrm{H}_0(X) \\ n & \mapsto & n \cdot [P]. \end{array}$$

Example. Let X be a topological space. As on page 82 we denote by $\pi_0(X)$ the set of path-components of X. It follows from the H_0-Proposition 35.11 together with the Direct Sum-H_*-Lemma 35.2 that we have a natural isomorphism

$$\mathrm{H}_0(X) \xrightarrow{\cong} \mathbb{Z}^{(\pi_0(X))}.$$

In particular, if X consists of n path-components with $n \in \mathbb{N}_0$, then we have an isomorphism

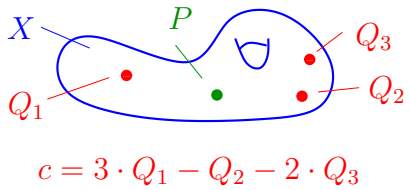
$$\mathrm{H}_0(X) \xrightarrow{\cong} \mathbb{Z}^n.$$ □

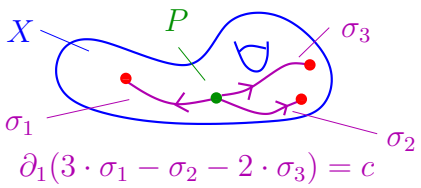
Proof. Let X be a path-connected non-empty topological space. The naturality statement is taken care of by the Augmentation Lemma 35.10 (3). Now let P be a point in X (which exists, since X is non-empty). It is clear that $\epsilon_X \circ \iota = \mathrm{id}_{\mathbb{Z}}$, in particular the homomorphism ϵ_X is surjective. It remains to show that ϵ_X is injective.

[292] Here "natural means that given any continuous map $f\colon X \to Y$ the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}_0(X) & \xrightarrow{f_*} & \mathrm{H}_0(Y). \\ {\scriptstyle \epsilon_X}\downarrow & & \downarrow{\scriptstyle \epsilon_Y} \\ \mathbb{Z} & \xrightarrow{\mathrm{id}} & \mathbb{Z}. \end{array}$$

Thus let $c \in \mathrm{C}_0(X)$ with $\epsilon_X(c) = 0$. By definition of $\mathrm{C}_0(X)$ we can find points $Q_1, \dots, Q_k$ in X and $a_1, \dots, a_k \in \mathbb{Z}$ such that $c = \sum_{i=1}^{k} a_i \cdot Q_i$. Since X is path-connected we can find, for $i = 1, \dots, k$, a singular 1-simplex σ_i with $\sigma_i((0,1)) = Q_i$ and $\sigma_i((1,0)) = P$.





Then we have

$$\partial_1\Big(\sum_{i=1}^{k} a_i \cdot \sigma_i\Big) = \sum_{i=1}^{k} a_i \cdot \partial\sigma_i \underset{\uparrow}{=} \sum_{i=1}^{k} a_i \cdot (Q_i - P) = \sum_{i=1}^{k} a_i \cdot Q_i - \underbrace{\Big(\sum_{i=1}^{k} a_i\Big)}_{=\epsilon_X(c)=0} \cdot P = \underbrace{\sum_{i=1}^{k} a_i \cdot Q_i}_{=c}.$$

see the discussion on page 459

This implies that $[c] = 0 \in \mathrm{H}_0(X)$. Thus we have shown that ϵ_X is injective. ■

We continue with another elementary but essential calculation.

Lemma 35.12. (Singleton-H_*-Lemma) If X is a singleton, i.e. if X is a topological space which consists of a single point, then

$$\mathrm{H}_n(X) \cong \begin{cases} \mathbb{Z}, & \text{if } n = 0, \\ 0, & \text{if } n \neq 0. \end{cases}$$

Proof. Let $X = \{\star\}$ be a topological space consisting of a single point. Given $n \in \mathbb{N}_0$ let $\sigma_n \colon \Delta^n \to \{\star\}$ be the unique map. We calculate that

$$\partial_n(\sigma_n) \underset{\uparrow}{=} \sum_{j=0}^{n} (-1)^j \cdot \underbrace{\sigma_n \circ i_j^n}_{=\sigma_{n-1}} = \sigma_{n-1} \cdot \sum_{j=0}^{n} (-1)^j = \begin{cases} \sigma_{n-1}, & \text{if } n \text{ even}, \\ 0, & \text{if } n \text{ odd}. \end{cases}$$

see the definition of ∂_n on page 459

We see that the singular chain complex is of the form

$$\dots \mathbb{Z} \cdot \sigma_4 \xrightarrow{\cong} \mathbb{Z} \cdot \sigma_3 \xrightarrow{=0} \mathbb{Z} \cdot \sigma_2 \xrightarrow{\cong} \mathbb{Z} \cdot \sigma_1 \xrightarrow{=0} \mathbb{Z} \cdot \sigma_0 \to 0.$$

The lemma follows almost immediately from this observation. ■

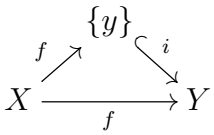
Example.

(1) One of our goals is to calculate the homology groups of the spheres. Now we can at least calculate the homology groups of the 0-dimensional sphere $S^0 = \{\pm 1\}$. More precisely, it follows from the Singleton-H_*-Lemma 35.12 and the Path-Component–H_*–Lemma 35.8 that

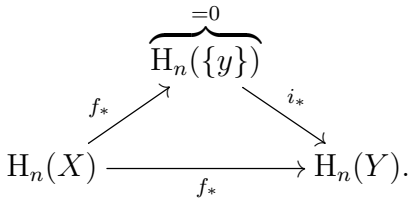
$$\mathrm{H}_n(S^0) \cong \begin{cases} \mathbb{Z}^2, & \text{if } n = 0, \\ 0, & \text{if } n \neq 0. \end{cases}$$

Later we will cleverly reduce the computation of the homology groups of all other spheres to this calculation.

(2) Let X and Y be non-empty topological spaces. Let $f \colon X \to Y$ be a continuous map. We suppose that f is constant, i.e. we suppose that there exists a $y \in Y$ such that $f(X) = \{y\}$. We denote by $i \colon \{y\} \to Y$ the inclusion. Let $n \in \mathbb{N}$. We consider the commutative diagram



which, by the functoriality of homology, gives rise to the commutative diagram



By the Singleton-H_*-Lemma 35.12 we know that $H_n(\{y\}) = 0$. Thus we see that $f_*\colon H_n(X) \to H_n(Y)$ factors through the trivial group, i.e. the map is the trivial map. On many occasions we will make use of this observation without referring to this discussion. □

The calculation of $H_n(\varnothing)$ on page 462, of $H_0(X)$ in the H_0-Proposition 35.11 and the calculation of the homology of a point in the Singleton-H_*-Lemma 35.12 are the only calculations of homology groups of path-connected topological spaces that one can do "by hand". For all other topological spaces we will need to work significantly harder, and this task will keep us busy for many chapters.

35.5. **Singular 1-chains.** It follows from the H_0-Proposition 35.11 together with the Direct Sum-H_*-Lemma 35.2 that we have no troubles calculating 0-th homology groups. Unfortunately, going up by one dimension, things become already much harder. At the moment we do not have the tools to calculate the homology groups in dimension 1. But in this last section we will prove one result on singular 1-chains which will come in handy on numerous occasions. To formulate our lemma we need the following notation:

Notation. Let X be a topological space. Given a singular 1-simplex $\sigma\colon \Delta^1 \to X$ we denote by $\overline{\sigma}$ the singular 1-simplex that is given by $\overline{\sigma}(t, 1-t) = \sigma(1-t, t)$. Informally speaking $\overline{\sigma}$ is given by reversing the orientation of σ.

Lemma 35.13. (Singular 1-Simplex Lemma) Let X be a topological space and let $\sigma\colon \Delta^1 \to X$ be a singular 1-simplex.

(1) σ is a cycle if and only if $\sigma((0,1)) = \sigma((1,0))$.
(2) The singular 1-chains $-\sigma$ and $\overline{\sigma}$ are homologous.
(3) Let $s \in (0,1)$. We consider the singular 1-simplices[293]

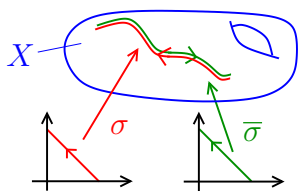
$$\begin{array}{rcl} \alpha\colon \Delta^1 & \to & X \\ (t, 1-t) & \mapsto & \sigma(st, 1-st) \end{array} \quad \text{and} \quad \begin{array}{rcl} \beta\colon \Delta^1 & \to & X \\ (t, 1-t) & \mapsto & \sigma(s+t(1-s), 1-(s+t(1-s))). \end{array}$$

The singular 1-chains σ and $\alpha + \beta$ are homologous.

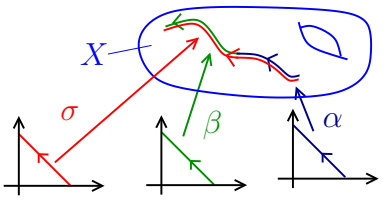
The second part of the lemma says that for a singular 1-simplex the geometric operation of reversing the orientation is (up to the boundary of a simplicial 2-chain) the same as algebraically inverting the sign. Furthermore, the third part of the lemma says that for a singular 1-simplex (again up to the boundary of a simplicial 2-chain) it does not make a difference whether or not we split the singular 1-simplex into two singular 1-simplices.

Proof. Statement (1) follows from the discussion of the boundary of a singular 1-simplex on page 459. We leave the proof of the remaining two statements as an exercise to the reader. ■

[293] Loosely speaking we break the singular 1-simplex σ into two singular 1-simplices α and β.

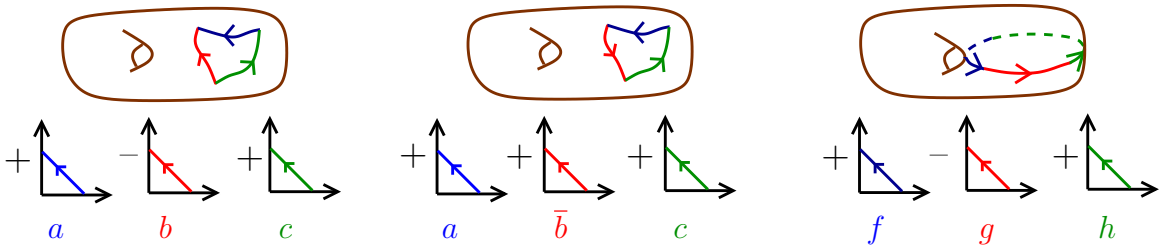


σ and $-\overline{\sigma}$ are homologous

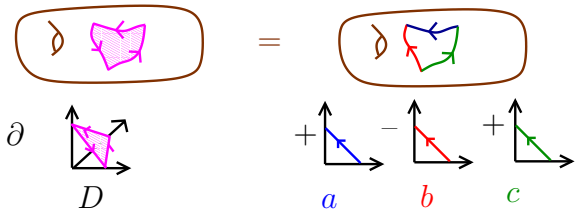


σ and $\alpha+\beta$ are homologous

Example. In the figure below we see three singular chains on the torus. It is straightforward to show that all three singular chains are cycles.



Now the question arises, which of these cycles are in fact null-homologous and which represent non-zero classes in homology? In the figure below we see that the first cycle $a-b+c$ is the boundary of a singular 2-simplex D.



One can easily see that the second cycle, $a+\overline{b}+c$ in the above figure is *not* the boundary of a singular 2-simplex. On the other hand by the Singular 1-Simplex Lemma 35.13 we know that $b+\overline{b}$ is a null-homologous cycle, i.e. there exists an $E \in \mathrm{C}_2(X)$ such that $b+\overline{b} = \partial E \in \mathrm{C}_1(X)$. Thus it follows that

$$a+\overline{b}+c \;=\; a+\partial E-b+c \;=\; a-b+c+\partial E = \partial D+\partial E \;=\; \partial(D+E),$$

i.e. $a+\overline{b}+c$ is null-homologous.

It is not clear whether the third cycle in the figure below is the boundary of a 2-dimensional singular chain. We will see later that this not the case. But it will take us quite some time to develop the techniques which will allow us to prove this statement. □

CHAPTER 36

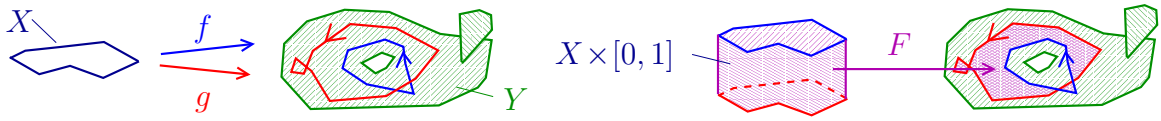
Homology and homotopies

We start out with recalling the following definition from page 158:

Definition. Let $f, g\colon X \to Y$ be two continuous maps between topological spaces. A homotopy between the maps f and g is a continuous map

$$F\colon X \times [0,1] \to Y,$$

such that for all $x \in X$ we have $F(x,0) = f(x)$ and $F(x,1) = g(x)$. If there exists a homotopy between f and g, then we say that f and g are homotopic and we write $f \simeq g$.



We also recall the following related two definitions.

Definition. Let X be and Y be topological spaces.

(1) A continuous map $f\colon X \to Y$ is a homotopy equivalence between X and Y, if f admits a homotopy inverse, i.e. a continuous map $g\colon Y \to X$ with $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$.

(2) If there exists a homotopy equivalence between X and Y, then we say that X and Y are homotopy equivalent and we write $X \simeq Y$.

Example.

(1) In the Deformation Retract–Homotopy Equivalence Lemma 9.3 we showed that if A is a deformation retract of a topological space X, then the inclusion $A \to X$ is a homotopy equivalence.

(2) It follows easily from (1) that each non-empty convex subset of $\mathbb{R}^n$ is homotopy equivalent to a singleton, i.e. to a topological space consisting of a single point. □

Next we recall the following definition from page 168:

Definition. We define the homotopy category $\mathcal{HomTop}$ to be the category given by

$$\begin{aligned} \mathrm{Ob}(\mathcal{HomTop}) &:= \text{all topological spaces,} \\ \mathrm{Mor}(X,Y) &:= \text{set of homotopy equivalence classes of continuous maps } X \to Y \end{aligned}$$

with the composition[294]

$$\begin{array}{rcl} \mathrm{Mor}(X,Y) \times \mathrm{Mor}(Y,Z) & \to & \mathrm{Mor}(X,Z) \\ ([f],[g]) & \mapsto & [g \circ f]. \end{array}$$

It is worth recalling that in the Homotopy Equivalence-Reformulation Lemma 9.4 we proved the elementary fact that a continuous map $f\colon X \to Y$ is a homotopy equivalence if and only if f represents an isomorphism in the homotopy category.

[294] It follows from the Homotopy Composition Lemma 8.4 (2) that this composition map is well-defined.

In the Homotopy-π_1–Proposition 13.4 and in the Homotopy-π_n-Proposition 28.6 we showed that homotopic maps induce essentially the same map on homotopy groups. We used this fact to show that homotopy equivalent path-connected topological spaces have isomorphic homotopy groups.

In this chapter we will see that homotopic maps induce the same map on homology. We will use this result to show that homotopy equivalent topological spaces have isomorphic homology groups. This result, together with the results from the previous chapter, allows us for example to compute the homology groups of closed and open balls.

In the next section we first make some algebraic preparations. In particular we will introduce algebraic analogues of homotopies, of being "homotopy equivalent" and of the homotopy category.

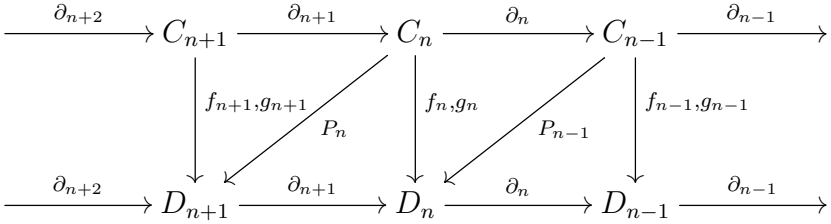
36.1. **Chain homotopies.** In this purely algebraic section we introduce a criterion that ensures that two chain maps $f, g\colon C_* \to D_*$ induce the same maps on the corresponding homology groups. In the next section we will use this to study the effect of homotopic maps on homology groups.

Definition. [295] Let $f, g\colon C_* \to D_*$ be chain maps between two chain complexes (C_*, ∂_*) and (D_*, ∂_*). A chain homotopy $P = \{P_n\}_{n\in\mathbb{N}_0}$ between f and g is defined as a family $\{P_n\colon C_n \to D_{n+1}\}_{n\in\mathbb{N}_0}$ of homomorphisms such that for each $n \in \mathbb{N}_0$ we have

$$\partial_{n+1} \circ P_n + P_{n-1} \circ \partial_n \;=\; g_n - f_n\colon C_n \;\to\; D_n.$$

If there exists a chain homotopy between chain maps f and g, then we say that f and g are chain homotopic and we write $f \simeq g$.

For the definition of a chain homotopy and the subsequent discussion of chain homotopies it is sometimes helpful to keep in mind the following diagram that combines all objects involved in the definition:



The following lemma explains perhaps why we are interested in chain homotopies.

Lemma 36.1. (Chain Homotopy–$\mathbf{H}_*$-Lemma) Let $f, g\colon C_* \to D_*$ be two chain maps between chain complexes. If f and g are chain homotopic, then $f_* = g_*\colon \mathrm{H}_n(C_*) \to \mathrm{H}_n(D_*)$ for all $n \in \mathbb{N}_0$.

Proof. Let $P = \{P_n\}_{n\in\mathbb{N}_0}$ be a chain homotopy between f and g. Then for any cycle $z \in C_n$ we get the equalities

$$\begin{aligned} g(z) - f(z) \;=\; (g-f)(z) \;&=\; (\partial_{n+1} \circ P_n + P_{n-1} \circ \partial_n)(z) \\ &=\; (\partial_{n+1} \circ P_n)(z) + P_{n-1}(\underbrace{\partial_n(z)}_{=0}) \;=\; \partial(P_n(z)), \end{aligned}$$

[295] We apologize for the fact that we present this definition without any motivation. One could spend some time trying to motivate this definition, but we think that this would not really improve matters.

i.e. $f(z)$ and $g(z)$ are homologous. ■

Now that we know that chain homotopies are useful it seems reasonable to develop the theory a bit further. The next lemma can be viewed as an algebraic analogue of the Homotopy-Equivalence Relation Lemma 8.3 and the Homotopy Composition Lemma 8.4.

Lemma 36.2. (Chain Homotopy-Lemma) Let C_* and D_* be two chain complexes.

(1) Chain homotopy defines an equivalence relation on the set of all chain maps from C_* to D_*. In particular, if $f, g, h\colon C_* \to D_*$ are chain maps with $f \simeq g$ and $g \simeq h$, then $f \simeq h$.

(2) Let $f, f'\colon C_* \to D_*$ and $g, g'\colon D_* \to E_*$ be chain maps. Then

$$f \simeq f' \text{ and } g \simeq g' \implies g \circ f \simeq g' \circ f'.$$

Proof. We leave the elementary proof of the lemma to the reader. ■

The following definition is modelled on the definition of the homotopy category $\mathcal{HomTop}$ of topological spaces which we recalled in the beginning of the chapter.

Definition. We define the chain homotopy category $\mathcal{HomCplx}$

$$\begin{aligned} \mathrm{Ob}(\mathcal{HomCplx}) &:= \text{all chain complexes,} \\ \mathrm{Mor}(C_*, D_*) &:= \text{set of chain homotopy equivalence classes of chain maps } C_* \to D_* \end{aligned}$$

with the composition[296]

$$\begin{array}{ccc} \mathrm{Mor}(C_*, D_*) \times \mathrm{Mor}(D_*, E_*) & \to & \mathrm{Mor}(C_*, ED_*) \\ ([f], [g]) & \mapsto & [g \circ f]. \end{array}$$

The following definition is also modelled on a definition which we recalled in the beginning of the chapter.

Definition. Let C_* and D_* be two chain complexes.

(1) A chain map $f\colon C_* \to D_*$ is called a chain homotopy equivalence between C_* and D_*, if it admits a chain homotopy inverse, i.e. a chain map $g\colon D_* \to C_*$ with $g \circ f \simeq \mathrm{id}_{C_*}$ and $f \circ g \simeq \mathrm{id}_{D_*}$.

(2) If there exists a chain homotopy equivalence between C_* and D_*, then we say that the chain complexes C_* and D_* are chain homotopy equivalent and we write $C_* \simeq D_*$.

The following lemma is the algebraic analogue of the Homotopy Equivalence-Reformulation Lemma 9.4.

Lemma 36.3. (Chain Homotopy Equivalence-Reformulation Lemma) Let C_* and D_* be two chain complexes and let $f\colon C_* \to D_*$ be a chain map. The following statements are equivalent:

(1) The element $[f] \in \mathrm{Mor}_{\mathcal{HomCplx}}(C_*, D_*)$ is an isomorphism in the chain homotopy category $\mathcal{HomCplx}$.

(2) f is a chain homotopy equivalence.

In particular the following statements are equivalent:

(1′) C_* and D_* are isomorphic in the chain homotopy category $\mathcal{HomCplx}$.

(2′) C_* and D_* are chain homotopy equivalent.

[296] It follows from the Chain Homotopy-Lemma 36.2 (2) that this composition map is well-defined.

Example. It follows from the Chain Homotopy Equivalence-Reformulation Lemma 36.3 and the Isomorphism Lemma 3.3 that the relation "chain homotopy equivalence" of chain complexes is indeed an equivalence relation on the class of all chain complexes. □

Proof. As in the case of the proof of the Homotopy Equivalence-Reformulation Lemma 9.4 the statements follow almost immediately from the definitions. ■

For later we record the following straightforward corollary to the Chain Homotopy–H_*-Lemma 36.1.

Corollary 36.4. (Chain Homotopy Equivalence-H_*-Corollary) If $f\colon C_* \to D_*$ is a chain homotopy equivalence between two chain complexes, then for each $n \in \mathbb{N}_0$ the induced map $f_*\colon \mathrm{H}_n(C_*) \to \mathrm{H}_n(D_*)$ is an isomorphism.

Proof. In the following we give the same proof twice, but with different language. First the language is a little more high-brow, the second time it is more down-to-earth:

(a) First note that it follows from the Chain Homotopy–H_*-Lemma 36.1 that the homology functor $\mathrm{H}_n\colon \mathcal{ChCplx} \to \mathcal{AbGroup}$ from the category of chain complexes to the category $\mathcal{AbGroup}$ of abelian groups factors through a functor $\mathcal{HomCplx} \to \mathcal{AbGroup}$ from the homotopy category of chain complexes to the category $\mathcal{AbGroup}$ of abelian groups. It follows from this observation, the Chain Homotopy Equivalence-Reformulation Lemma 36.3 and the fact that functors send isomorphisms to isomorphisms that chain homotopies give rise to isomorphisms of abelian groups.

(b) Let $g\colon D_* \to C_*$ be a chain homotopy inverse to f. Then we see that

$$g_* \circ f_* \quad = \quad (g \circ f)_* \quad \underset{\uparrow}{=} \quad (\mathrm{id}_{C_*})_* \quad = \quad \mathrm{id}_{\mathrm{H}_n(C_*)}\,.$$

by the Chain Homotopy–H_*-Lemma 36.1 since $g \circ f$ is chain homotopic to the identity

The same way we see that $f_* \circ g_* = \mathrm{id}_{\mathrm{H}_n(D)}$. ■

36.2. **Homology and homotopic maps.** The name *chain homotopy* introduced in the previous section suggests that this concept is related to the notion of a homotopy between two topological spaces. The following proposition shows that this is indeed the case.

Proposition 36.5. (Homotopy-H_*-Proposition) Let X and Y be two topological spaces and let $f, g\colon X \to Y$ be two continuous maps. If f and g are homotopic, then the following hold:

(1) The induced maps f_* and g_* from $\mathrm{C}_*(X)$ to $\mathrm{C}_*(Y)$ are chain homotopic.
(2) For each $n \in \mathbb{N}_0$ we have $f_* = g_*\colon \mathrm{H}_n(X) \to \mathrm{H}_n(Y)$.

Remark. The proposition can be stated more succinctly as follows:

$$f \simeq g\colon X \to Y \quad \Longrightarrow \quad f_* \simeq g_*\colon \mathrm{C}_*(X) \to \mathrm{C}_*(Y) \quad \Longrightarrow \quad f_* = g_*\colon \mathrm{H}_n(X) \to \mathrm{H}_n(Y). \quad \square$$

We postpone the lengthy proof of the Homotopy-H_*-Proposition 36.5 to Section 36.3. Let us first turn to the fun part, namely let us jot down a few consequences. We start out with the following straightforward consequence.

Corollary 36.6. (Homotopy Category-H_*-Functor Corollary) Each of our homology functors $\mathrm{H}_n\colon \mathcal{Top} \to \mathcal{AbGroup}$ defines a functor $\mathrm{H}_n\colon \mathcal{HomTop} \to \mathcal{AbGroup}$ from the homotopy category of topological spaces to the category $\mathcal{AbGroup}$ of abelian groups.

Proof. This statement follows immediately from the Homotopy-H_*-Proposition 36.5 and the definitions. ■

The following corollary mirrors the fact, proved in the Homotopy-π_1-Proposition 13.4 (2) and the Homotopy-π_n-Proposition 28.6 (2a), that homotopy equivalences induce isomorphisms of fundamental groups and higher homotopy groups.

Corollary 36.7. (Homotopy Equivalence-H_*-Corollary) Let X and Y be topological spaces and let $n \in \mathbb{N}_0$.

(1) (a) If $f\colon X \to Y$ is a homotopy equivalence, then the induced map $f_*\colon \mathrm{H}_n(X) \to \mathrm{H}_n(Y)$ is an isomorphism.
(b) If X is a deformation retract of Y, then the inclusion-induced map $\mathrm{H}_n(X) \to \mathrm{H}_n(Y)$ is an isomorphism.

(2) If X and Y are homotopy equivalent, then $\mathrm{H}_n(X) \cong \mathrm{H}_n(Y)$.

Proof.

(1) (a) This statement follows immediately from the Homotopy Category-H_*-Functor Corollary 36.6 and the fact that functors send isomorphisms to isomorphisms.
(b) This statement follows immediately from (1a) and the Deformation Retract–Homotopy Equivalence Lemma 9.3.

(2) This statement follows immediately from (1a). ■

We can now easily prove the following analogue of the Contractible-$\mathrm{H}^*_{\mathrm{dR}}$-Corollary **??**.

Corollary 36.8. (Contractible-H_*-Corollary) Let X be a topological space. If X is contractible (i.e. if X is homotopy equivalent to a singleton), then we have

$$\mathrm{H}_n(X) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } n = 0, \\ 0, & \text{if } n \neq 0. \end{cases}$$

Proof. This statement follows from the Homotopy Equivalence-H_*-Corollary 36.7 (2) and the Singleton-H_*-Lemma 35.12. ■

Example. On page 171 we saw that every convex subset of $\mathbb{R}^n$, e.g. the open disk B^n, the closed disk $\overline{B}^n$, any simplex or $\mathbb{R}^n$, is contractible. Thus it follows from the Contractible-H_*-Corollary 36.8, that for any $i \in \mathbb{N}_0$ we have

$$\mathrm{H}_i(\underbrace{\text{convex subset of } \mathbb{R}^n}_{\text{e.g. } \mathbb{R}^n,\ \overline{B}^n,\ B^n}) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } i = 0, \\ 0, & \text{if } i > 0. \end{cases}$$

□

36.3. **Proof of the Homotopy-H_*-Proposition 36.5.** For the reader's convenience we recall the proposition that we need to prove.

Proposition 36.5. (Homotopy-H_*-Proposition) Let X and Y be two topological spaces and let $f, g\colon X \to Y$ be two continuous maps. If f and g are homotopic, then the following hold:

(1) The induced maps f_* and g_* from $\mathrm{C}_*(X)$ to $\mathrm{C}_*(Y)$ are chain homotopic, i.e. there exists a family $\{P_n\colon \mathrm{C}_n(X) \to \mathrm{C}_{n+1}(Y)\}_{n\in\mathbb{N}_0}$ of homomorphisms such that for each

$n \in \mathbb{N}_0$ we have

$$\partial_{n+1} \circ P_n + P_{n-1} \circ \partial_n \;=\; g_n - f_n \colon \mathrm{C}_n(X) \;\to\; \mathrm{C}_{n+1}(Y).$$

(2) For each $n \in \mathbb{N}_0$ we have $f_* = g_* \colon \mathrm{H}_n(X) \to \mathrm{H}_n(Y)$.

We start out with a discussion of the key idea behind the proof of the Homotopy-H_*-Proposition 36.5. Afterwards we will fill in the details. So let $f, g \colon X \to Y$ be two continuous maps between topological spaces and let $F \colon X \times [0,1] \to Y$ be a homotopy between the maps f and g. Our goal is to find a chain homotopy between f_* and g_*. Thus let $\sigma \colon \Delta^n \to X$ be a singular n-simplex. We need to find $P(\sigma) \in \mathrm{C}_{n+1}(Y)$ and $P(\partial\sigma) \in \mathrm{C}_n(Y)$ such that

$$\partial P(\sigma) \;=\; -P(\partial\sigma) \,+\, f_*(\sigma) \,-\, g_*(\sigma).$$

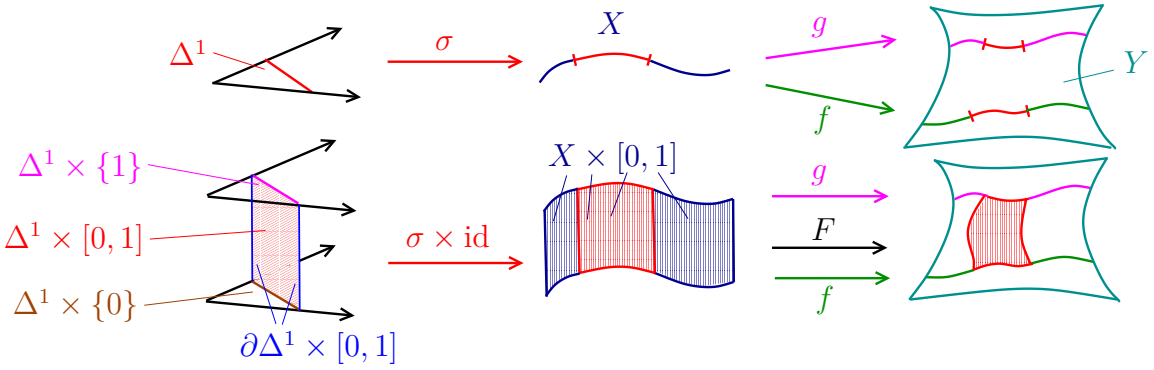
Note that $F \circ (\sigma \times \mathrm{id}_{[0,1]})$ defines a map $\Delta^n \times [0,1] \to Y$. Now we can rewrite the above desired equality as follows:

$$\partial P(\sigma) \;=\; -P(\partial\sigma) + \begin{array}{c}\text{map } F \circ (\sigma \times \mathrm{id}_{[0,1]}) \\ \text{restricted to } \underbrace{\Delta^n \times \{0\}}_{=_{\mathrm{i}}\Delta^n}\end{array} - \begin{array}{c}\text{map } F \circ (\sigma \times \mathrm{id}_{[0,1]}) \\ \text{restricted to } \underbrace{\Delta^n \times \{1\}}_{=_{\mathrm{i}}\Delta^n}.\end{array}$$

To get an idea we consider the figure below. Looking at the figure we note that

$$\partial(\Delta^n \times [0,1]) \;=\; (\partial\Delta^n \times [0,1]) \;\cup\; (\Delta^n \times \{0\}) \;\cup\; (\Delta^n \times \{1\}).$$

Comparing the desired equality of maps and the equality of sets suggests that each map $P_k \colon \mathrm{C}_k(X) \to \mathrm{C}_{k+1}(Y)$ should take the role of replacing a singular k-simplex $\mu \colon \Delta^k \to X$ by the map $F \circ (\mu \times \mathrm{id}_{[0,1]}) \colon \Delta^k \times [0,1] \to Y$. This is a good idea, but the problem is that $\Delta^k \times [0,1]$ is not a $(k+1)$-simplex. The idea now is to write $\Delta^k \times [0,1]$ as the union of $(k+1)$-simplices. In the figure below we see for example that we can view $\Delta^1 \times [0,1]$ as the union of the triangles with the vertices v_0, w_0, w_1 respectively v_0, v_1, w_1.



To generalize this observation we have to introduce some more notation.

Notation. Let $n \in \mathbb{N}_0$.

(1) For $i = 0, \ldots, n$ we write

$$v_i \;:=\; ((0,\ldots,1,0,\ldots,0),0) \;\in\; \mathbb{R}^{n+1} \times \{0\},$$

and

$$w_i \;:=\; ((0,\ldots,1,0,\ldots,0),1) \;\in\; \mathbb{R}^{n+1} \times \{1\}.$$

The points $v_0, \dots, v_n$ thus span the simplex $\Delta^n \times 0$ and the points $w_0, \dots, w_n$ span the simplex $\Delta^n \times 1$.

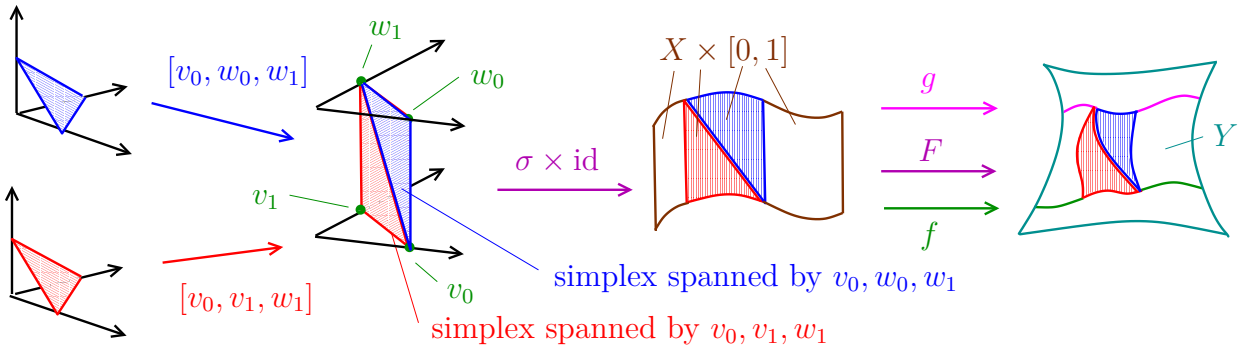
(2) For $a_0, \dots, a_k \in \Delta^n \times [0,1]$ we consider the map

$$\begin{aligned} [a_0, \dots, a_k] \colon \Delta^k &\to \Delta^n \times [0,1] \\ (\lambda_0, \dots, \lambda_k) &\mapsto \sum_{j=0}^{k} \lambda_j \cdot a_j. \end{aligned}$$

The map $[a_0, \dots, a_k]$ thus sends the standard k-simplex Δ^k to the simplex spanned by $a_0, \dots, a_k$ in $\Delta^n \times [0,1]$.

(3) We set

$$\Omega_n \coloneqq \sum_{j=0}^{n} (-1)^j \cdot \underbrace{[v_0, \dots, v_j, w_j, \dots, w_n]}_{\text{map } \Delta^{n+1} \to \Delta^n \times [0,1]} \in \mathrm{C}_{n+1}(\Delta^n \times [0,1]).$$



In the following lemma we compute the boundary of $\Omega_n \in \mathrm{C}_{n+1}(\Delta^n \times [0,1])$. The statement might not be particularly visually appealing, but it will turn out to be very useful.

Lemma 36.9. (Prism Boundary Lemma) Let $n \in \mathbb{N}_0$. Given $t \in \{0,1\}$ we write

$$\begin{aligned} \eta_t \colon \Delta^n &\to \Delta^n \times [0,1] \\ x &\mapsto (x,t). \end{aligned}$$

With this notation we have[297]

$$\partial_{n+1}(\Omega_n) + \sum_{j=0}^{n} (-1)^j \cdot (i_j^n \times \mathrm{id}_{[0,1]})_*\Big(\underbrace{\Omega_{n-1}}_{\in \mathrm{C}_n(\Delta^{n-1} \times [0,1])} \Big) = \eta_1 - \eta_0 \in \mathrm{C}_n(\Delta^n \times [0,1]).$$

Proof. The proof of the lemma is basically an elementary, albeit slightly confusing calculation. First note that

$$\begin{aligned} \partial\Omega_n &= \partial\Big(\sum_{j=0}^{n} (-1)^j \cdot [v_0, \dots, v_j, w_j, \dots, w_n] \Big) \\ &= \sum_{k=0}^{n+1} (-1)^k \cdot \sum_{j=0}^{n} (-1)^j \cdot [v_0, \dots, v_j, w_j, \dots, w_n] \circ i_k^{n+1}. \end{aligned}$$

[297] Here $i_j^n \colon \Delta^{n-1} \to \Delta^n$ denotes the j-th face map as introduced on page 459.

It is straightforward to see that for any $k \in \{0, \dots, n+1\}$ we have[298][299]

$$[v_0, \dots, v_j, w_j, \dots, w_n] \circ i_k^{n+1} = \begin{cases} [v_0, \dots, \widehat{v_k}, \dots, v_j, w_j, \dots, w_n], & \text{if } k \leq j, \\ [v_0, \dots, v_j, w_j, \dots, \widehat{w_{k-1}}, \dots, w_n], & \text{if } k > j. \end{cases}$$

Thus it follows that

$$\begin{aligned} \partial\Omega_n = & \sum_{0 \leq k \leq j \leq n} (-1)^k \cdot (-1)^j \cdot [v_0, \dots, \widehat{v_k}, \dots, v_j, w_j, \dots, w_n] \\ & + \sum_{n+1 \geq k > j \geq 0} (-1)^k \cdot (-1)^j \cdot [v_0, \dots, v_j, w_j, \dots, \widehat{w_{k-1}}, \dots, w_n]. \end{aligned}$$

Now we split the first sum into the cases $k = j$ and $k < j$ and we split the second sum into the cases $k = j+1$ and $k > j+1$. This leads us to the following equality:

$$\begin{aligned} \partial\Omega_n = & \sum_{k=0}^{n} (-1)^{2k} \cdot [v_0, \dots, v_{k-1}, w_k, \dots, w_n] \\ & + \sum_{k=1}^{n+1} (-1)^{2k+1} \cdot [v_0, \dots, v_{k-1}, w_k, \dots, w_n] \\ & + \sum_{0 \leq k < j \leq n} (-1)^k \cdot (-1)^j \cdot [v_0, \dots, \widehat{v_k}, \dots, v_j, w_j, \dots, w_n] \\ & + \sum_{n+1 \geq k > j+1 \geq 1} (-1)^k \cdot (-1)^j \cdot [v_0, \dots, v_j, w_j, \dots, \widehat{w_{k-1}}, \dots, w_n]. \end{aligned}$$

Next we consider the first two sums. The summands cancel each other, except for the summands for $k = 0$ and $k = n+1$. Thus we obtain that

$$\begin{aligned} \partial\Omega_n = & [w_0, \dots, w_n] - [v_0, \dots, v_n] \\ & + \sum_{0 \leq k < j \leq n} (-1)^k \cdot (-1)^j \cdot [v_0, \dots, \widehat{v_k}, \dots, v_j, w_j, \dots, w_n] \\ & + \sum_{n+1 \geq k > j+1 \geq 1} (-1)^k \cdot (-1)^j \cdot [v_0, \dots, v_j, w_j, \dots, \widehat{w_k}, \dots, w_n]. \end{aligned} \tag{$*$}$$

The first summand is just η_1 and the second is $-\eta_0$. It remains to show that the remaining summand corresponds to the remaining term in the formula. Thus we continue our proof with the discussion of the one term in the statement of the lemma that we have not studied yet. We have[300]

$$\sum_{j=0}^{n} (-1)^j \cdot (i_j^n \times \mathrm{id}_{[0,1]})_*(\Omega_{n-1}) \underset{\uparrow}{=} \sum_{r=0}^{n} \sum_{s=0}^{n-1} (-1)^r \cdot (-1)^s \cdot (i_r^n \times \mathrm{id}_{[0,1]}) \circ [v_0, \dots, v_s, w_s, \dots, w_{n-1}].$$

definition of Ω_{n-1} and substitution $r = j$

[298] The first case, $k \leq j$ is slightly easier: one has to figure out, where the h-th vertex e_h of the standard n-simplex gets sent to. If $h < k$, then i_k^{n+1} sends the h-th vertex again to the h-th vertex, and $[v_0, \dots, v_j, w_j, \dots, w_n]$ sends this vertex to v_h. If $h \geq k$, then i_k^{n+1} sends the h-th vertex again to the $(h+1)$-th vertex, and $[v_0, \dots, v_j, w_j, \dots, w_n]$ sends this vertex to v_{h+1}. If $k \leq j$ this corresponds just to the map $[v_0, \dots, \widehat{v_k}, \dots, v_j, w_j, \dots, w_n]$.
The second case, $k > j$ is proved precisely the same way.

[299] Here we use the usual notation that $\widehat{v_k}$ means that we do not plug in $\widehat{v_k}$, the same applies to $\widehat{w_{k-1}}$.

[300] Here $v_0, \dots, v_{n-1}, w_0, \dots, w_{n-1}$ are now, unfortunately somewhat confusingly, the corresponding vertices of $\Delta^{n-1} \times [0,1]$ instead of $\Delta^n \times [0,1]$.

It follows from an elementary (but arguably confusing) argument, using only the definitions, that we have the following equalities of maps $\Delta^n \to \Delta^n \times [0,1]$:

$$(i_r^n \times \mathrm{id}_{[0,1]}) \circ [v_0, \dots, v_s, w_s, \dots, w_{n-1}] \;=\; \begin{cases} [v_0, \dots, \widehat{v_r}, \dots, v_{s+1}, w_{s+1}, \dots, w_n], & \text{if } r \leq s, \\ [v_0, \dots, v_s, w_s, \dots, \widehat{w_{r+1}}, \dots, w_n], & \text{if } r > s. \end{cases}$$

Thus we see that

$$\begin{aligned} \sum_{j=0}^{n} (-1)^j \cdot (i_j^n \times \mathrm{id}_{[0,1]})_*(\Omega_{n-1}) \;=\; & \sum_{0 \leq r \leq s \leq n-1} (-1)^s \cdot (-1)^r \cdot [v_0, \dots, \widehat{v_r}, \dots, v_{s+1}, w_{s+1}, \dots, w_n] \\ & + \sum_{n \geq r > s \geq 0} (-1)^s \cdot (-1)^r \cdot [v_0, \dots, v_s, w_s, \dots, \widehat{w_{r+1}}, \dots, w_n]. \end{aligned}$$

In the first term we perform the substitutions $r = k$ and $s = j - 1$ and in the second term we perform the substitutions $r = k - 1$, $s = j$. We obtain the following equality:

$$\begin{aligned} \sum_{j=0}^{n} (-1)^j \cdot (i_j^n \times \mathrm{id}_{[0,1]})_*(\Omega_{n-1}) \;=\; & \sum_{0 \leq k \leq j-1 \leq n-1} (-1)^{j-1} \cdot (-1)^k \cdot [v_0, \dots, \widehat{v_k}, \dots, v_j, w_j, \dots, w_n] \\ & + \sum_{n \geq k-1 > j \geq 0} (-1)^j \cdot (-1)^{k-1} \cdot [v_0, \dots, v_j, w_j, \dots, \widehat{w_k}, \dots, w_n]. \end{aligned}$$

We see immediately that these terms are just the negatives of the corresponding terms in the above equation (∗). ∎

Now we can finally provide the proof of the Homotopy-H$_*$-Proposition 36.5.

Proof of the Homotopy-H$_*$-Proposition 36.5. Let X be a topological space and let $F\colon X \times [0,1] \to Y$ be a homotopy from a map f to a map g.

(1) This statement follows immediately from the following claim:

Claim. Given any $n \in \mathbb{N}_0$ we define

$$P_n\colon \mathrm{C}_n(X) \;\to\; \mathrm{C}_{n+1}(Y)$$

via

$$(\sigma\colon \Delta^n \to X) \;\mapsto\; (F \circ (\sigma \times \mathrm{id}_{[0,1]}))_*(\Omega_n).$$

we defined $\Omega_n \in \mathrm{C}_{n+1}(\Delta^n \times [0,1])$ on page 475

For any $n \in \mathbb{N}_0$ we have

$$\partial P_n + P_{n-1}\partial \;=\; g_* - f_*$$

as maps from $\mathrm{C}_n(X)$ to $\mathrm{C}_n(Y)$.

Proof. As always it suffices to verify the equality for each singular n-simplex. Thus let $\sigma\colon \Delta^n \to X$ be a singular n-simplex. We calculate that

$$\begin{aligned}
(\partial P_n + P_{n-1}\partial)(\sigma) &= \partial(P_n(\sigma)) + P_{n-1}\Big(\sum_{j=0}^{n}(-1)^j\cdot\sigma\circ i_j^n\Big)\\
&= \partial(P_n(\sigma)) + \sum_{j=0}^{n}(-1)^j\cdot P_{n-1}(\sigma\circ i_j^n)\\
&= \partial((F\circ(\sigma\times \mathrm{id}_{[0,1]}))_*(\Omega_n)) + \sum_{j=0}^{n}(-1)^j\cdot(F\circ(\sigma\circ i_j^n\times \mathrm{id}_{[0,1]}))_*(\Omega_{n-1})\\
&= (F\circ(\sigma\times \mathrm{id}_{[0,1]}))_*\Big(\partial(\Omega_n) + \sum_{j=0}^{n}(-1)^j\cdot(i_j^n\times \mathrm{id}_{[0,1]})_*(\Omega_{n-1})\Big)\\
&\quad\uparrow\\
&\quad\text{by the C}_*\text{–Functor Lemma 35.5 (1) and since we}\\
&\quad\text{have } (\sigma\times\mathrm{id})\circ(i_j^n\times\mathrm{id}) = (\sigma\circ i_j^n)\times\mathrm{id} \text{ as maps } \Delta^n\times[0,1]\times Y\times[0,1]\\
&= (F\circ(\sigma\times \mathrm{id}_{[0,1]}))_*(\eta_1-\eta_0) \;=\; f\circ\sigma - g\circ\sigma \;=\; g_*(\sigma) - f_*(\sigma).\\
&\quad\uparrow \qquad\qquad\uparrow\\
&\quad\text{by the Prism Boundary Lemma 36.9} \qquad \text{since } F \text{ is a homotopy from } f \text{ to } g
\end{aligned}$$

⊞

(2) This statement follows immediately from Statement (1) together with the Chain Homotopy–H$_*$-Lemma 36.1. ■

CHAPTER 37

Exact sequences and long exact sequences

In the next chapter, given a topological space X and a subspace $A \subset X$, we will introduce the relative chain complex given by the groups $\mathrm{C}_n(X, A) := \mathrm{C}_n(X)/\mathrm{C}_n(A)$. To understand the relationship between the homology groups of the chain complexes $\mathrm{C}_*(A)$, $\mathrm{C}_*(X)$ and $\mathrm{C}_*(X, A)$ we will need to develop our algebraic skills.

In this purely algebraic chapter we will therefore introduce the concept of a "(short) exact sequence" and will see that a "short exact sequence" of chain complexes gives rise to a "long exact sequence" of homology groups. These concepts will play an essential role in the theory of homology.

37.1. **Exact sequences.** We start out with the following definition which appears in many parts of mathematics.

Definition. A sequence of homomorphisms of abelian groups

$$\ldots \xrightarrow{f_{n+2}} A_{n+1} \xrightarrow{f_{n+1}} A_n \xrightarrow{f_n} A_{n-1} \xrightarrow{f_{n-1}} A_{n-2} \to \ldots$$

is called exact, if for each n we have

$$\operatorname{im}(f_{n+1}\colon A_{n+1} \to A_n) \;=\; \ker(f_n\colon A_n \to A_{n-1}).$$

In the following lemma we collect a few basic statements about exact sequences which we will use frequently without referring to the lemma.

Lemma 37.1. (Exactness Lemma)

(1) In any exact sequence the composition of two consecutive maps is the zero map.

(2) We have the following equivalences of statements:

$$\begin{aligned}
\text{the sequence } 0 \to A \xrightarrow{f} B \quad &\text{is exact} \iff f \text{ is injective} \\
\text{the sequence } A \xrightarrow{f} B \to 0 \quad &\text{is exact} \iff f \text{ is surjective} \\
\text{the sequence } 0 \to A \xrightarrow{f} B \to 0 \quad &\text{is exact} \iff f \text{ is an isomorphism.}
\end{aligned}$$

(3) Given a monomorphism $i\colon A \to B$ and an epimorphisms $p\colon B \to C$ of groups we have

$$\begin{aligned}
0 \to A \xrightarrow{i} B \xrightarrow{f} C \text{ is exact} &\iff i \text{ induces an isomorphism } A \xrightarrow{\cong} \ker(f) \\
A \xrightarrow{f} B \xrightarrow{p} C \to 0 \text{ is exact} &\iff p \text{ descends to an isomorphism } \operatorname{coker}(f) \xrightarrow{\cong} C.
\end{aligned}$$

Proof. The lemma follows easily from the definitions. We leave it to the reader to provide the details. ∎

[300]Note that we do not assume that the groups are abelian. In most cases the groups will be abelian though, that is why in this section we denote the trivial group by "0".

Example. Let $f\colon A \to B$ be a homomorphism of groups. It follows from the Exactness Lemma 37.1 that the following sequence is exact

$$0 \to \ker(f) \to A \xrightarrow{f} B \to \operatorname{coker}(f) \to 0. \qquad \square$$

37.2. **Split exact sequences.** In many future settings we will be given an exact sequence where we know some of the groups and we will wonder, whether we can determine the missing groups. In this section we will study the simplest instant of such a problem, namely the case of a short exact sequence:

Definition. A short exact sequence is an exact sequence of the form

$$0 \to A \to B \to C \to 0,$$

which consists of at most three non-trivial abelian groups.

Let

$$0 \to A \xrightarrow{\varphi} B \xrightarrow{\psi} C \to 0$$

be a short exact sequence. If we know $\varphi\colon A \to B$ or if we know $\psi\colon B \to C$ then using the Exactness Lemma 37.1 we can read off the third group. Now suppose we know the two "outer" groups A and C. Can we read "read off" B from this short exact sequence? In general the answer is no. For example the short exact sequence

$$0 \to \mathbb{Z}_2 \longrightarrow B \longrightarrow \mathbb{Z}_2 \to 0$$

could be of the form

$$0 \to \mathbb{Z}_2 \xrightarrow{x\mapsto(x,0)} \mathbb{Z}_2 \oplus \mathbb{Z}_2 \xrightarrow{(x,y)\mapsto y} \mathbb{Z}_2 \to 0$$

but it could also be of the form[301]

$$0 \to \mathbb{Z}_2 \xrightarrow{[x]\mapsto[2x]} \mathbb{Z}_4 \xrightarrow{[x]\mapsto[x]} \mathbb{Z}_2 \to 0.$$

In the remainder of this section we will give a criterion for when the naive guess $B \cong A \oplus C$ is actually the correct answer.

Definition. We say that a short exact sequence

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$$

of groups splits, if there exists a splitting $s\colon C \to B$, i.e. a homomorphism $s\colon C \to B$ such that $p \circ s = \mathrm{id}_C$.

Example.

(1) The short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{k\mapsto(k,0)} \mathbb{Z} \oplus \mathbb{Z}_n \xrightarrow{(k,l)\mapsto l} \mathbb{Z}_n \to 0$$

splits. In fact a splitting is given by the homomorphism $b \mapsto (0, b)$. On the other hand the short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{k\mapsto n\cdot k} \mathbb{Z} \to \mathbb{Z}_n \to 0.$$

does not split.

[301] Are there any other possibilities for the isomorphism type of B?

(2) Let $0 \to A \to B \to C \to 0$ be a short exact sequence of abelian groups. It follows from the Free Abelian Group–Splitting Lemma 18.5, that if the last group C is a free abelian group, then the short exact sequence splits. □

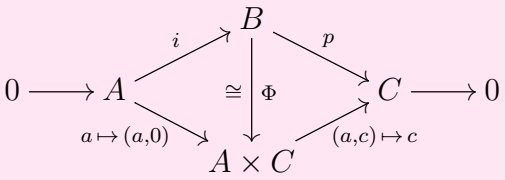
The following lemma will be useful on numerous occasions.

Lemma 37.2. (Splitting Lemma) Let

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$$

be a short exact sequence of abelian groups. The following three statements are equivalent:

(1) The short exact sequence splits, i.e. there exists a homomorphism $s\colon C \to B$ such that $p \circ s = \mathrm{id}_C$.
(2) There exists a homomorphism $t\colon B \to A$ such that $t \circ i = \mathrm{id}_A$.
(3) There exists an isomorphism $\Phi\colon B \to A \times C$ such that the following diagram commutes:



where the lower left diagonal map is the obvious inclusion and the lower right diagonal map is the natural projection.

Proof. Let $0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$ be a short exact sequence of abelian groups.

(1)⇒(3). Suppose that $s\colon C \to B$ is a splitting, i.e. suppose that s is a homomorphism such that $p \circ s = \mathrm{id}_C$. We consider the map

$$\begin{array}{rccl} \Psi\colon & A \times C & \to & B \\ & (a, c) & \mapsto & i(a) + s(c). \end{array}$$

Using the fact that B is abelian one can easily verify that the map is a homomorphism.

Claim. The above homomorphism $\Psi\colon A \times C \to B$ is an isomorphism and the inverse $\Phi := \Psi^{-1}\colon B \to A \times C$ has all desired properties.

Proof. First we show that Ψ is an epimorphism. Let $b \in B$. Note that

$$p(b - s(p(b))) \underset{\substack{\uparrow \\ \text{since } p \text{ is a homomorphism}}}{=} p(b) - \underbrace{(p \circ s)}_{=\mathrm{id}}(p(b)) = p(b) - p(b) = 0.$$

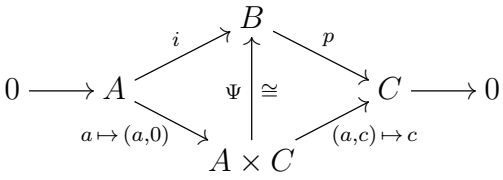
It follows from this calculation and the exactness of the short exact sequence that there exists a unique $a \in A$ with $i(a) = b - s(p(b))$. Then $\Psi(a, p(b)) = i(a) + s(p(b)) = b$.

Next we show that Ψ is a monomorphism. Let $(a, c) \in A \times B$ with $\Psi(a, c) = 0$. Then

$$0 = p(0) = p(\Psi(a, c)) = p(i(a) + s(c)) = \underbrace{(p \circ i)}_{=0}(a) + \underbrace{(p \circ s)}_{=\mathrm{id}_C}(c) = c.$$

Thus we have shown that $c = 0$. It follows that $0 = \Psi(a, c) = i(a) + s(c) = i(a)$. Since i is a monomorphism we see that $a = 0$. In summary we have shown that $(a, c) = 0$.

It remains to show that the following diagram commutes:



It is clear that the left hand triangle commutes. Furthermore it follows from $p \circ i = 0$ and $p \circ s = \mathrm{id}_C$ that the triangle to the right commutes.

(2)⇒(3). Let $t\colon B \to A$ be a homomorphism such that $t \circ i = \mathrm{id}_A$. We define

$$\begin{array}{rccc} \Phi\colon & B & \to & A \times C \\ & b & \mapsto & (t(b), p(b)). \end{array}$$

As in the proof of (1)⇒(3) it follows easily from the exactness of the initial sequence that this map is in fact an isomorphism. It is straightforward to verify that Φ has all the other desired properties.

(3)⇒(1), (2). Now suppose that we are given an isomorphism $\Phi\colon B \to A \times C$ such that the given diagram commutes. It follows immediately from the definitions and given properties that the maps

$$B \xrightarrow{\Phi} A \times C \xrightarrow{(a,c)\mapsto a} A \quad \text{and} \quad C \xrightarrow{c\mapsto(0,c)} A \times C \xrightarrow{\Phi^{-1}} B$$

have the required properties. ■

37.3. **Long exact sequence of homology groups.** In the previous two sections we studied (short) exact sequences of abelian groups. In this section we will study short exact sequences of chain complexes and we will see that they give rise to an exact sequence of homology groups.

Definition. A short exact sequence of chain complexes

$$0 \to A_* \xrightarrow{i} B_* \xrightarrow{p} C_* \to 0$$

consists of chain complexes A_*, B_*, C_* and chain maps $i\colon A_* \to B_*$ and $p\colon B_* \to C_*$ such that for each n the maps

$$0 \to A_n \xrightarrow{i} B_n \xrightarrow{p} C_n \to 0$$

form a short exact sequence.

Definition. Let $0 \to A_* \xrightarrow{i} B_* \xrightarrow{p} C_* \to 0$ be a short exact sequence of chain complexes. Let $n \in \mathbb{N}_0$. We define a homomorphism

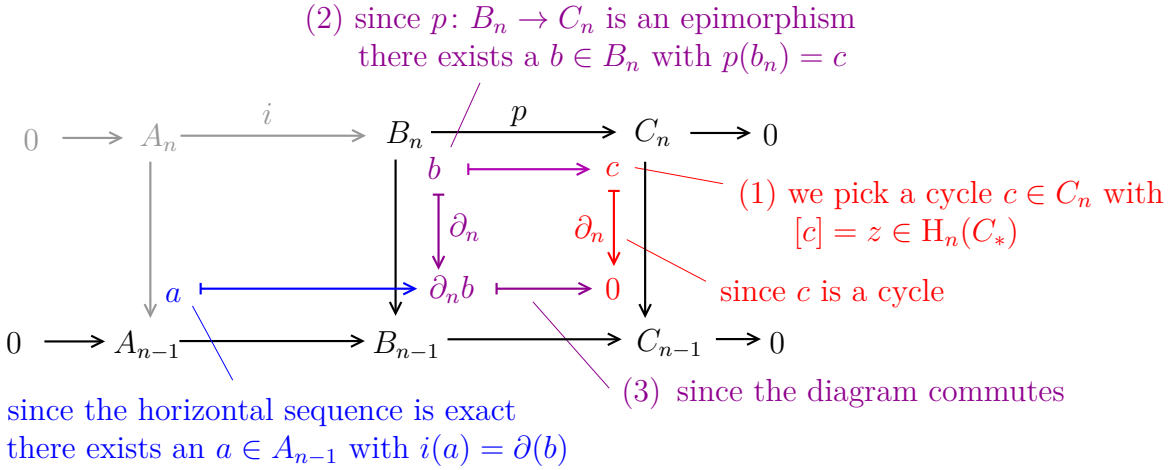
$$\partial_n\colon \mathrm{H}_n(C_*) \to \mathrm{H}_{n-1}(A_*)$$

as follows:

(0) Let $z \in \mathrm{H}_n(C_*)$.
(1) We pick an n-cycle $c \in C_n$ with $[c] = z$.
(2) Since $0 \to A_n \xrightarrow{i} B_n \xrightarrow{p} C_n \to 0$ is exact we know by the Exactness Lemma 37.1 that $p\colon B_n \to C_n$ is an epimorphism. Thus there exists a $b \in B_n$ with $p(b) = c$.
(3) We have

$$p(\partial b) \underset{\text{since } p \text{ is a chain map}}{=} \partial p(b) = \partial c \underset{\text{since } c \text{ is a cycle}}{=} 0.$$

(4) Since $0 \to A_{n-1} \xrightarrow{i} B_{n-1} \xrightarrow{p} C_{n-1} \to 0$ is exact it follows from (3) that there exists a unique $a \in A_{n-1}$ with $i(a) = \partial b$.
(5) We set $\partial_n(z) := [a] \in \mathrm{H}_{n-1}(A_*)$.
We refer to this map $\partial_n \colon \mathrm{H}_n(C_*) \to \mathrm{H}_{n-1}(A_*)$ as the connecting homomorphism. [302]



The following lemma summarizes two of the key properties of the connecting homomorphisms.

Lemma 37.3. (Connecting Homomorphism Lemma)
(1) Let $0 \to A_* \to B_* \to C_* \to 0$ be a short exact sequence of chain complexes. For each $n \in \mathbb{N}_0$ the connecting homomorphism $\partial_n \colon \mathrm{H}_n(C_*) \to \mathrm{H}_{n-1}(A_*)$ is well-defined and it is indeed a homomorphism.
(2) The connecting homomorphism is natural, i.e. if

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_* & \xrightarrow{i} & B_* & \xrightarrow{p} & C_* & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \\ 0 & \longrightarrow & A'_* & \xrightarrow{i'} & B'_* & \xrightarrow{p'} & C'_* & \longrightarrow & 0 \end{array}$$

is a commutative diagram of chain maps where the horizontal sequences are exact, then the following diagram commutes: For every $n \in \mathbb{N}$:

$$\begin{array}{ccc} \mathrm{H}_n(C_*) & \xrightarrow{\partial_n} & \mathrm{H}_{n-1}(A_*) \\ {\scriptstyle c_*}\downarrow & & \downarrow{\scriptstyle a_*} \\ \mathrm{H}_n(C'_*) & \xrightarrow{\partial_n} & \mathrm{H}_{n-1}(A'_*) \end{array}$$

[303] The fact that here the symbol "∂_n" denotes the boundary map of the various chain complexes and also the connecting homomorphism $\mathrm{H}_n(C_*) \to \mathrm{H}_{n-1}(A_*)$ is somewhat unfortunate.

Proof. In the proof of the lemma it is helpful to keep in mind the following commutative diagram of short exact sequences:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A_{n+1} & \xrightarrow{i} & B_{n+1} & \xrightarrow{p} & C_{n+1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \partial_{n+1}} & & \downarrow{\scriptstyle \partial_{n+1}} & & \downarrow{\scriptstyle \partial_{n+1}} & & \\
0 & \longrightarrow & A_n & \xrightarrow{i} & B_n & \xrightarrow{p} & C_n & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \partial_n} & & \downarrow{\scriptstyle \partial_n} & & \downarrow{\scriptstyle \partial_n} & & \\
0 & \longrightarrow & A_{n-1} & \xrightarrow{i} & B_{n-1} & \xrightarrow{p} & C_{n-1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \partial_{n-1}} & & \downarrow{\scriptstyle \partial_{n-1}} & & \downarrow{\scriptstyle \partial_{n-1}} & & \\
0 & \longrightarrow & A_{n-2} & \xrightarrow{i} & B_{n-2} & \xrightarrow{p} & C_{n-2} & \longrightarrow & 0.
\end{array}$$

The proof of this lemma and the following proposition are done by a "diagram chase". This means that one needs to keep the above commutative diagram in mind and on several occasions one has to use that the vertical maps are boundary maps and that the horizontal sequences are exact. In both cases the proofs are quite elementary and utterly unreadable. In both cases it is thus highly recommended to prove the statements on your own and to ignore the proofs provided below.

For the undeterred reader we now give the fully gory details.

(1) We start out with the following claim:

Claim 1. The map $\partial_n \colon \mathrm{H}_n(C_*) \to \mathrm{H}_{n-1}(A_*)$ is well-defined.

Proof. Let $z \in \mathrm{H}_n(C_*)$. We denote by $c \in C_n$, $b \in B_n$ and $a \in A_{n-1}$ the elements that arise in the definition of $\partial_n(z)$. We need to show that the element a is indeed a cycle and that the definition of $\partial_n(z)$ does not depend on the choices of c and b.

The proofs of these two statements are not particularly difficult and they are done via "diagram chasing". First we note that

$$i(\partial_{n-1}(a)) \underset{\substack{\uparrow \\ \text{since } i \text{ is a chain map}}}{=} \partial_{n-1}(i(a)) = \partial_{n-1}(\partial_n(b)) = 0.$$

Since i is injective we obtain that $\partial_{n-1}(a) = 0$, i.e. a is indeed a cycle. Now we show that the homology class $[a] \in \mathrm{H}_{n-1}(A)$ depends only on the homology class $z \in \mathrm{H}_n(C_*)$.

(a) If we chose a different element $\widetilde{b}$ in B_n with $p(\widetilde{b}) = p(b) = c$, then it follows from the fact that the horizontal sequences are exact that this element would be of the form $\widetilde{b} = b + i(a')$ for some $a' \in A_n$. But then we obtain the following equality:

$$\partial_n(\widetilde{b}) = \partial_n(b + i(a')) = \partial_n(b) + \partial_n(i(a')) \overset{\substack{\text{since } i \text{ is a chain map} \\ \downarrow}}{=} \underbrace{\partial_n(b)}_{=i(a)} + i(\partial_n(a')) = i(a + \partial_n(a')) \in B_{n-1}.$$

Since $[a] = [a + \partial_n(a')] \in \mathrm{H}_{n-1}(A)$ this does not change the homology class.

(b) If we picked a different representative $\widetilde{c}$ for $z \in \mathrm{H}_n(C_*)$, then this representative would be of the form $\widetilde{c} = c + \partial_{n+1}(c')$ for some $c' \in C_{n+1}$. Since the map $p \colon B_n \to C_n$

is an epimorphism there exists a $b' \in B_{n+1}$ with $p(b') = c'$. It follows that

$$p(b + \partial_n(b')) \;=\; p(b) + p(\partial_{n+1}(b')) \underset{\text{since } p \text{ is a chain map}}{=} c + \partial_{n+1}(p(b')) \;=\; c + \partial_{n+1}(c') \;=\; \widetilde{c} \in C_n.$$

Since $\partial_n(b+\partial b') = \partial_n b$ we see that we have ended up with the same element in B_{n-1} as before. ⊞

Claim 2. The map

$$\partial_n \colon \mathrm{H}_n(C_*) \;\to\; \mathrm{H}_{n-1}(A_*)$$

is a homomorphism.

Proof. Let $[c_1], [c_2] \in \mathrm{H}_n(C_*)$. For $j = 1, 2$ we choose $b_j \in B_n$ as in the definition of the connecting homomorphism $\partial_n \colon \mathrm{H}_n(C_*) \to \mathrm{H}_{n-1}(A_*)$ and we denote by $a_j \in A_{n-1}$ the uniquely determined element with $i(a_j) = \partial_n(b_j)$. Since i and p are homomorphisms we also have $p(b_1+b_2) = c_1+c_2$ and $i(a_1+a_2) = i(a_1)+i(a_2) = \partial_n(b_1)+\partial_n(b_2) = \partial_n(b_1+b_2)$. It follows that

$$\partial_n([c_1 + c_2]) \;=\; [a_1 + a_2] \;=\; [a_1] + [a_2] \;=\; \partial_n([c_1]) + \partial_n([c_2]).$$ ⊞

(2) It follows easily from the definitions that the connecting homomorphisms are natural. We leave the verification of the details to the reader. ■

Now we can formulate the following proposition which explains the name "connecting homomorphism".

Proposition 37.4. (LES-Proposition)

(1) Let

$$0 \;\to\; A_* \;\xrightarrow{i}\; B_* \;\xrightarrow{p}\; C_* \;\to\; 0$$

be a short exact sequence of chain complexes. The sequence

$$\ldots \to \mathrm{H}_n(A) \xrightarrow{i_*} \mathrm{H}_n(B_*) \xrightarrow{p_*} \mathrm{H}_n(C_*) \xrightarrow{\partial_n} \mathrm{H}_{n-1}(A_*) \xrightarrow{i_*} \mathrm{H}_{n-1}(B_*) \xrightarrow{p_*} \ldots$$

is exact. This sequence is called the long exact sequence in homology corresponding to the short exact sequence of chain complexes.

(2) If

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A_* & \xrightarrow{i} & B_* & \xrightarrow{p} & C_* & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \\
0 & \longrightarrow & A'_* & \xrightarrow{i'} & B'_* & \xrightarrow{p'} & C'_* & \longrightarrow & 0
\end{array}$$

is a commutative diagram of chain maps where the horizontal sequences are exact, then the following diagram commutes:

$$\begin{array}{ccccccccccc}
\ldots & \longrightarrow & \mathrm{H}_n(A_*) & \xrightarrow{i_*} & \mathrm{H}_n(B_*) & \xrightarrow{p_*} & \mathrm{H}_n(C_*) & \xrightarrow{\partial_n} & \mathrm{H}_{n-1}(A_*) & \longrightarrow & \ldots \\
 & & \downarrow{\scriptstyle a_*} & & \downarrow{\scriptstyle b_*} & & \downarrow{\scriptstyle c_*} & & \downarrow{\scriptstyle a_*} & & \\
\ldots & \longrightarrow & \mathrm{H}_n(A'_*) & \xrightarrow{i'_*} & \mathrm{H}_n(B'_*) & \xrightarrow{p'_*} & \mathrm{H}_n(C'_*) & \xrightarrow{\partial_n} & \mathrm{H}_{n-1}(A'_*) & \longrightarrow & \ldots
\end{array}$$

Proof of the LES-Proposition 37.4. First note that Statement (2) is an immediate consequence of the Chain Complex-H_*-Functor Lemma 35.1 (2) together with the Connecting Homomorphism Lemma 37.3. Thus let us turn to the actual meat, namely the proof of Statement (1).

Let

$$0 \;\to\; A_* \;\xrightarrow{i}\; B_* \;\xrightarrow{p}\; C_* \;\to\; 0$$

be a short exact sequence of chain complexes. The sequence

$$\ldots \to \mathrm{H}_{n+1}(C_*) \xrightarrow{\partial_{n+1}} \mathrm{H}_n(A_*) \xrightarrow{i_*} \mathrm{H}_n(B_*) \xrightarrow{p_*} \mathrm{H}_n(C_*) \xrightarrow{\partial_n} \mathrm{H}_{n-1}(A_*) \xrightarrow{i_*} \mathrm{H}_{n-1}(B_*) \xrightarrow{p_*} \ldots$$

is exact. We examine the exactness at the three groups $\mathrm{H}_n(A_*), \mathrm{H}_n(B_*)$ and $\mathrm{H}_n(C_*)$. Put differently, at each of the three groups we want to show the following two statements:

(i) The image of the map to the left is contained in the kernel of the map to the right, or equivalently, the composition of the map to the left and the map to the right is zero,

(ii) The kernel of the map to the right is contained in the image of the map to the left.

Now we perform these steps:

(A) (i) Let $[c] \in \mathrm{H}_{n+1}(C_*)$. We want to show that $i_*(\partial_{n+1}([c])) = 0$. We choose $b \in B_{n+1}$ with $p(b) = c$ and we denote by $a \in A_n$ the element with $i(a) = \partial_{n+1}(b)$. Then we have

$$i_*(\partial_{n+1}([c])) \underset{\substack{\uparrow \\ \text{definition of } \partial_{n+1}\colon \mathrm{H}_{n+1}(C_*) \to \mathrm{H}_n(A_*)}}{=} i_*([a]) = [\partial_{n+1}(b)] = 0 \in \mathrm{H}_n(B_*).$$

(ii) Let $[a] \in \ker(i_*)$. Then $i(a) = \partial_{n+1}(b)$ for some $b \in B_{n+1}$. It follows immediately from the definitions that $\partial_{n+1}([p(b)]) = [a]$.

(B) (i) From $p \circ i = 0$ it follows that $p_* \circ i_* = 0$, i.e. $\mathrm{im}(i_*) \subset \ker(p_*)$.

(ii) Let $b \in \ker(p_*)$. Then $p(b) = \partial_{n+1}(c')$ for some $c' \in C_{n+1}$. Since p is an epimorphism, there exists a $b' \in B_{n+1}$ with $p(b') = c'$. Since p is a chain map we see that $p(\partial_{n+1}(b')) = \partial_{n+1}(p(b')) = \partial_{n+1}(c') = p(b)$. Thus it follows that $p(b - \partial b') = 0$, i.e. $b - \partial b' = i(a)$ for some $a \in A_n$. But then it follows that $[b] = [b - \partial_{n+1}(b')] = [i(a)] \in \mathrm{im}(i_*)$.

(C) (i) Let $[b] \in \mathrm{H}_n(B)$. We have to show that $\partial_n(p_*([b])) = 0$. In the definition of $\partial_n(p_*([b]))$ we need to choose a $\widetilde{b} \in B_n$ with $[p(\widetilde{b})] = p_*([b])$. Of course we take $\widetilde{b} = b$. Since $\partial_n(b) = 0$ it follows immediately that $\partial_n(p_*([b])) = 0$.

(ii) Let $[c] \in \ker(\partial_n\colon \mathrm{H}_n(C_*) \to \mathrm{H}_{n-1}(A_*))$. We choose $b \in B_n$ with $p(b) = c$. Recall that there exists a uniquely determined $a \in A_{n-1}$ with $i(a) = \partial_n(b)$. Since we have $[a] = \partial_n([c]) = 0 \in \mathrm{H}_{n-1}(A_*)$ there exists an $a' \in A_n$ with $\partial_n(a') = a$. It follows that

$$\partial_n(b - i(a')) = \partial_n(b) - \partial_n(i(a')) \underset{\substack{\uparrow \\ \text{since } i \text{ is a chain map}}}{=} \partial_n(b) - i(\partial_n(a')) = \partial_n(b - i(a)) = 0,$$

i.e. $b - i(a') \in B_n$ is a cycle. On the other hand note that we have the equalities $p(b - i(a')) = p(b) - (p \circ i)(a') = p(b)$. Thus we see that $p_*([b - i(a')]) = [c]$. We have thus shown that $\ker(\partial_n) \subset \mathrm{im}(p_*)$.

Thus we have proved the claim and therefore we have also completed the proof of the proposition. ∎

We conclude this chapter with the following lemma, which will often be useful when we compare two long exact sequences of homology groups.

Lemma 37.5. (Five Lemma) Let

$$\begin{array}{ccccccccc}
A & \xrightarrow{i} & B & \xrightarrow{j} & C & \xrightarrow{k} & D & \xrightarrow{l} & E \\
\downarrow{\scriptstyle a} & & \downarrow{\scriptstyle \cong\, b} & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle \cong\, d} & & \downarrow{\scriptstyle e} \\
A' & \xrightarrow{i'} & B' & \xrightarrow{j'} & C' & \xrightarrow{k'} & D' & \xrightarrow{l'} & E'
\end{array}$$

be a diagram of groups (here for once we do not suppose that the groups are abelian) such that the horizontal sequences are exact and such that every square commutes (if all the groups are abelian, then it suffices that the squares commute up to a sign). If a is surjective, if b and d are isomorphisms, and if e is injective, then c is an isomorphism.[304]

Proof. One can prove the lemma via a "diagram chase". We leave the pleasure of the chase as an exercise to the reader. ■

[304]In many applications it suffices to remember the following slightly weaker statement: If the two vertical maps on the left and right are isomorphisms, then the middle map is also an isomorphism.

CHAPTER 38

Relative homology

Let X be a topological space and let $A \subset X$ be a subspace. We denote by $i\colon A \to X$ the inclusion. Does there exist a relationship between the homology groups of X and A? The first, very optimistic, guess might be that perhaps the projection $X \to X/A$ induces an isomorphism $\mathrm{H}_k(X)/i_*(\mathrm{H}_k(A)) \to \mathrm{H}_k(X/A)$. But this idea breaks down almost immediately, for example if we have $X = A$, then for any $k \in \mathbb{N}_0$ we have $\mathrm{H}_k(X)/i_*(\mathrm{H}_k(A)) = 0$ but by the H_0-Proposition 35.11 we have $\mathrm{H}_0(X/A) \cong \mathrm{H}_0(\{*\}) \cong \mathbb{Z}$.

This simple example shows that the covariant functor $X \to \mathrm{H}_k(X)$ does not preserve quotients. But recall that this functor is really the composition of the two covariant functors $X \mapsto (\mathrm{C}_*(X), \partial_*)$ and $(C_*, \partial_*) \mapsto \mathrm{H}_k((C_*, \partial_*))$. In this and the following chapters we will study the question to which degree the covariant functor $X \mapsto (\mathrm{C}_*(X), \partial_*)$ preserves quotients (at least say up to chain homotopy).

38.1. **Definition of the relative homology groups.** As we just discussed it is worth exploring whether the chain complex functor $X \mapsto (\mathrm{C}_*(X), \partial_*)$ preserves taking quotients. This question motivates the following definition, which will play a major role for the remainder of these notes.

Definition. Let (X, A) be a pair of topological spaces.[305] We view each chain group $\mathrm{C}_n(A)$ as a subgroup of $\mathrm{C}_n(X)$. Thus we can define

$$\mathrm{C}_n(X, A) \coloneqq \mathrm{C}_n(X)/\mathrm{C}_n(A)$$

and we consider the boundary map[306]

$$\begin{aligned} \partial_n\colon \mathrm{C}_n(X, A) &\to \mathrm{C}_{n-1}(X, A) \\ [c] &\mapsto [\partial_n c]. \end{aligned}$$

We refer to

$$\ldots \xrightarrow{\partial_{n+2}} \mathrm{C}_{n+1}(X, A) \xrightarrow{\partial_{n+1}} \mathrm{C}_n(X, A) \xrightarrow{\partial_n} \mathrm{C}_{n-1}(X, A) \xrightarrow{\partial_{n-1}} \ldots$$

as the chain complex of the pair (X, A).[307] Given any $n \in \mathbb{N}_0$ we define the n-th relative homology group of (X, A) as follows:

$$\mathrm{H}_n(X, A) \coloneqq \mathrm{H}_n(\mathrm{C}_*(X, A)).$$

Remark. Let (X, A) be a pair of topological spaces and let $n \in \mathbb{N}_0$. We write

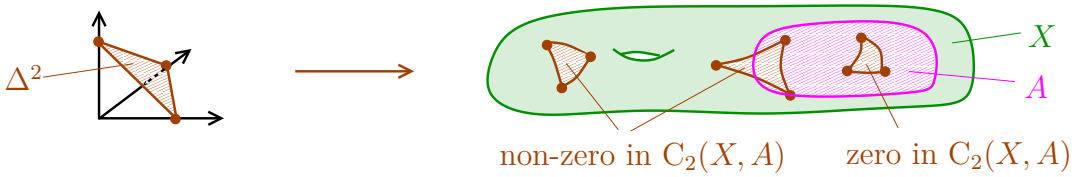
$$S \coloneqq \{\sigma\colon \Delta^n \to X \mid \sigma \text{ is continuous and } \operatorname{im}(\sigma) \cap (X \setminus A) \neq \varnothing\},$$

[305] This means that X is a topological space and $A \subset X$ is a subspace.

[306] The map ∂_n is well-defined, since for any $c \in \mathrm{C}_n(A) \subset \mathrm{C}_n(X)$ the boundary ∂c obviously lies in $\mathrm{C}_{n-1}(A) \subset \mathrm{C}_{n-1}(X)$.

[307] Since $\partial_{n-1} \circ \partial_n\colon \mathrm{C}_n(X) \to \mathrm{C}_{n-2}(X)$ is the zero map it follows immediately that the composition $\partial_{n-1} \circ \partial_n\colon \mathrm{C}_n(X, A) \to \mathrm{C}_{n-2}(X, A)$ is also the zero map.

i.e. S is the set of all singular n-simplices of X whose image is not completely contained in A.



Then it is straightforward to show that the map

$$\begin{array}{rcl} \mathbb{Z}^{(S)} & \to & \mathrm{C}_n(X,A) \\ \sum\limits_{i=1}^{k} r_i \cdot \sigma_i & \mapsto & \left[\sum\limits_{i=1}^{k} r_i \cdot \sigma_i\right] \end{array}$$

is an isomorphism. In particular we see that $\mathrm{C}_n(X,A)$ is a free abelian group. □

Examples.

(1) If $A = \varnothing$, then for all $n \in \mathbb{N}_0$ we have $\mathrm{C}_n(\varnothing) = 0$ and thus we have a natural isomorphism $\mathrm{C}_n(X,\varnothing) = \mathrm{C}_n(X)/\{0\} \xrightarrow{\cong} \mathrm{C}_n(X)$, i.e. we have a natural isomorphism $\mathrm{H}_n(X,\varnothing) \xrightarrow{\cong} \mathrm{H}_n(X)$. In the following we will use these natural isomorphisms to freely go back and forth between the notations $\mathrm{H}_n(X,\varnothing)$ and $\mathrm{H}_n(X)$.

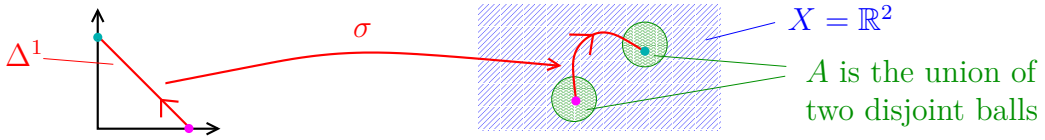
(2) Whereas it is often difficult to write down singular cycles in absolute chain complexes $\mathrm{C}_*(X)$, it can be surprisingly easy to write down cycles in a relative chain complex $\mathrm{C}_*(X,A)$: Let $\sigma\colon \Delta^n \to X$ be a continuous map with $\sigma(\partial\Delta^n) \subset A$. Note that

$$\partial_n(\sigma) \;=\; \sum_{j=0}^{n} (-1)^j \cdot \sigma \circ i_j^n \;\underset{\uparrow}{=}\; 0 \;\in\; \mathrm{C}_{n-1}(X,A) = \mathrm{C}_{n-1}(X)/\mathrm{C}_{n-1}(A).$$

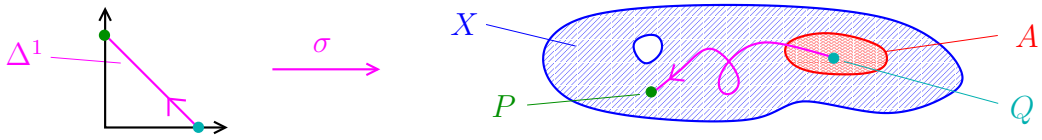
since $\sigma \circ i_j^n\colon \Delta^{n-1}$ takes values in A we see that $\sigma \circ i_j^n \in \mathrm{C}_{n-1}(A)$

Thus we have shown that *any continuous map* $\sigma\colon \Delta^n \to X$ *with* $\sigma(\partial\Delta^n) \subset A$ *defines a cycle in the chain complex* $\mathrm{C}_n(X,A)$.

As an example we consider $X = \mathbb{R}^2$ with the subspace $A = B_1 \sqcup B_2$ that is given by the union of two disjoint closed disks in $\mathbb{R}^2$. Let $\sigma\colon \Delta^1 \to X$ be a singular 1-simplex with $\sigma(0,1) \in B_1$ and $\sigma(1,0) \in B_2$. It follows from the above discussion that σ is a cycle in $\mathrm{C}_*(X,A)$. Later, on page 494, we will see whether or not $[\sigma] \in \mathrm{H}_1(X,A)$ is non-trivial.



(3) If X is a path-connected topological space and if A is non-empty, then $\mathrm{H}_0(X,A) = 0$. This can be seen as follows: First note that $\mathrm{H}_0(X,A)$ is a quotient of $\mathrm{C}_0(X)$, which is generated by singular 0-simplices, i.e. by points. Thus it suffices to show that for each $P \in X$ we have $[P] = 0 \in \mathrm{H}_0(X,A)$. Thus let $P \in X$ be a point. Since A is non-empty we can pick a point $Q \in A$. Since X is path-connected we can pick a continuous map $\sigma\colon \Delta^1 \to X$ such that $\sigma(0,1) = P$ and $\sigma(1,0) = Q$.



We now see that

$$\partial_1([\sigma]) \;=\; [\partial_1(\sigma)] \underset{\text{see page 459}}{\overset{\uparrow}{=}} [P-Q] \;=\; [P]-[Q] \underset{\text{since } Q\in A \text{ we have } [Q]=0\in \mathrm{C}_0(X,A)}{\overset{\uparrow}{=}} [P].$$

We have thus shown that $\partial_1([\sigma]) = [P]$, which implies that $[P] = 0 \in \mathrm{H}_0(X, A)$.

As always we expect that relative homology groups are functorial in a suitable way. This thought leads us to the following definition:

Definition.

(1) We refer to the category that is given by

$$\begin{aligned} \mathrm{Ob}(\mathcal{PairTop}) &:= \text{all pairs of topological spaces,} \\ \mathrm{Mor}((X,A),(Y,B)) &:= \text{all continuous maps } f\colon X\to A \text{ with } f(A)\subset B \end{aligned}$$

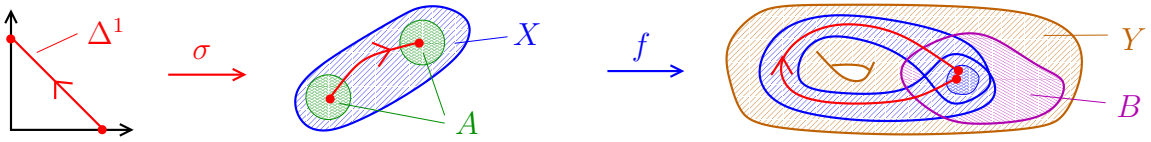
with the usual composition of maps as the category of pairs of topological spaces.

(2) Let $f\colon (X,A)\to(Y,B)$ be a continuous map of pairs of topological spaces, i.e. a continuous map $f\colon X\to Y$ with $f(A)\subset B$. If follows from $f(A)\subset B$ that the induced chain maps $f_*\colon \mathrm{C}_n(X)\to \mathrm{C}_n(Y)$ from the H_n–Functor Lemma 35.7 descends to a chain map

$$f_*\colon \mathrm{C}_n(X,A) \;\to\; \mathrm{C}_n(Y,B).$$

In particular we also get induced homomorphisms

$$f_*\colon \mathrm{H}_n(X,A) \;\to\; \mathrm{H}_n(Y,B).$$



The following lemma is a variation on the C_*–Functor Lemma 35.5 and the H_n–Functor Lemma 35.7. Often we will use the lemma without actually referring to it.

Lemma 38.1. (Relative-H_*–Functor Lemma) The maps

$$\begin{aligned} (X,A) &\mapsto (\mathrm{C}_*(X,A),\partial_*) \\ (f\colon (X,A)\to(Y,B)) &\mapsto (f_*\colon \mathrm{C}_*(X,A)\to \mathrm{C}_*(Y,B)) \end{aligned}$$

define a covariant functor from the category $\mathcal{PairTop}$ of pairs of topological spaces to the category $\mathcal{ChCplx}$ of chain complexes and for each $n\in\mathbb{N}_0$ the maps

$$\begin{aligned} (X,A) &\to \mathrm{H}_n(X,A) \\ (f\colon (X,A)\to(Y,B)) &\mapsto (f_*\colon \mathrm{H}_n(X,A)\to \mathrm{H}_n(Y,B)) \end{aligned}$$

define a covariant functor from the category $\mathcal{PairTop}$ of pairs of topological spaces to the category $\mathcal{AbGroup}$ of abelian groups.

Proof. The lemma follows almost immediately from the definitions. ∎

38.2. **The long exact sequence of a triple.** Let (X, A) be a pair of topological spaces. In the following we want to consider the relationship between the homology groups of A, X and (X, A). Cleverly we prepared the ground in Chapter 37. Using the tools from the previous chapter we can easily show the existence of interesting long exact sequences of homology groups.

In fact it is convenient to study a slightly more general setup, which leads us to the harmless notion of a triple of topological spaces:

Definition.

(1) A triple of topological spaces is a triple (X, B, A) where X is a topological space and $A \subset B \subset X$ are subspaces.
(2) A map $f\colon (X, B, A) \to (X', B', A')$ between triples of topological spaces is a continuous map $f\colon X \to X'$ such that $f(B) \subset B'$ and $f(A) \subset A'$.

Proposition 38.2. (LES-of-Triple Proposition) Let X be a topological space and let $A \subset B \subset X$ be two subspaces. Let $i\colon (B, A) \to (X, A)$ and by $p\colon (X, A) \to (X, B)$ be the obvious maps.

(1) For each $n \in \mathbb{N}_0$ the following map is a well-defined homomorphism:[308]

$$\begin{array}{rcl} \partial_n\colon \mathrm{H}_n(X, B) & \to & \mathrm{H}_{n-1}(B, A) \\ \Big[\sum_{j=1}^{k} s_j \cdot (\sigma_j\colon \Delta^n \to X)\Big] & \mapsto & \Big[\sum_{j=1}^{k} s_j \cdot \underbrace{\partial\sigma_j}_{\substack{\uparrow \\ \text{singular } (n-1)\text{-chain in } B}}\Big]. \end{array}$$

We refer to ∂_n as a connecting homomorphism of the triple (X, B, A).

(2) The connecting homomorphisms are natural. In particular for any continuous map $f\colon (X, A, B) \to (\widetilde{X}, \widetilde{A}, \widetilde{B})$ of triples of topological spaces the following diagram commutes:[309]

$$\begin{array}{ccccccccccc} \ldots & \longrightarrow & \mathrm{H}_n(B, A) & \xrightarrow{i_*} & \mathrm{H}_n(X, A) & \xrightarrow{p_*} & \mathrm{H}_n(X, B) & \xrightarrow{\partial_n} & \mathrm{H}_{n-1}(B, A) & \longrightarrow & \ldots \\ & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \\ \ldots & \longrightarrow & \mathrm{H}_n(\widetilde{B}, \widetilde{A}) & \xrightarrow{\widetilde{i}_*} & \mathrm{H}_n(\widetilde{X}, \widetilde{A}) & \xrightarrow{\widetilde{p}_*} & \mathrm{H}_n(\widetilde{X}, \widetilde{B}) & \xrightarrow{\partial_n} & \mathrm{H}_{n-1}(\widetilde{B}, \widetilde{A}) & \longrightarrow & \ldots. \end{array}$$

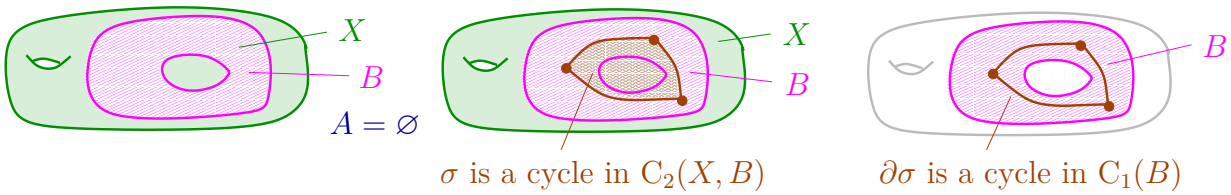
(3) The maps

$$\ldots \to \mathrm{H}_n(B, A) \xrightarrow{i_*} \mathrm{H}_n(X, A) \xrightarrow{p_*} \mathrm{H}_n(X, B) \xrightarrow{\partial_n} \mathrm{H}_{n-1}(B, A) \to \ldots$$

form an exact sequence. It is called the called the long exact sequence (LES) of the triple (X, B, A).

Proof.

[307] Here we use that every homology class in $\mathrm{H}_n(X, B)$ admits by definition a representative in the quotient $\mathrm{C}_n(X, B) = \mathrm{C}_n(X)/\mathrm{C}_n(B)$, so it can be represented by a singular n-chain in X.
[308] We consider the category of triples of topological spaces. The connecting homomorphisms ∂_n define a natural transformation from the functor $(X, B, A) \to \mathrm{H}_n(X, B)$ to the functor $(X, B, A) \to \mathrm{H}_{n-1}(B, A)$.
[309] The fact that ∂_n is a natural transformation means precisely that the third square commutes.



(1) First note that it follows immediately from the definitions and the third isomorphism theorem in group theory[310] that

$$0 \;\to\; \mathrm{C}_*(B,A) \;\xrightarrow{i_*}\; \mathrm{C}_*(X,A) \;\xrightarrow{p_*}\; \mathrm{C}_*(X,B) \;\to\; 0$$

is a short exact sequence of chain complexes.

Claim. The maps ∂_n introduced in Statement (1) of the propositions are precisely the connecting homomorphisms, as defined on page 483, of the above short exact sequence of chain complexes. In particular the maps ∂_n are well-defined homomorphisms.

Proof. The claim follows basically immediately from the definitions. But for completeness' sake we give some details. We consider the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{C}_n(B,A) & \longrightarrow & \mathrm{C}_n(X,A) & \longrightarrow & \mathrm{C}_n(X,B) & \longrightarrow & 0 \\
 & & \downarrow\partial & & \downarrow\partial & & \downarrow\partial & & \\
0 & \longrightarrow & \mathrm{C}_{n-1}(B,A) & \longrightarrow & \mathrm{C}_{n-1}(X,A) & \longrightarrow & \mathrm{C}_{n-1}(X,B) & \longrightarrow & 0.
\end{array}$$

Let $z \in \mathrm{H}_n(X,B)$. Now we will determine the image of z under the connecting homomorphism $\mathrm{H}_n(X,B) \to \mathrm{H}_{n-1}(B,A)$ which during the proof of the claim we denote by ∂^c.

We pick a representative $\sum_{j=1}^{k} s_j \cdot (\sigma_j \colon \Delta^n \to X) \in \mathrm{C}_n(X)$ of $z \in \mathrm{H}_n(X,B)$. Note that $\sum_{j=1}^{k} s_j \cdot (\sigma_j \colon \Delta^n \to X)$ also defines an element in $\mathrm{C}_n(X,A)$. We take the boundary of the above singular n-chain under the boundary map $\partial_n \colon \mathrm{C}_n(X,A) \to \mathrm{C}_{n-1}(X,A)$ and we end up with the singular n-chain $\sum_{j=1}^{k} s_j \cdot \partial\sigma_j \in \mathrm{C}_{n-1}(X,A)$. As we have seen in the proof of the Connecting Homomorphism Lemma 37.3 this singular n-chain lies in fact in $\mathrm{C}_{n-1}(B,A) \subset \mathrm{C}_{n-1}(X,A)$. The homology class thereof is by definition precisely $\partial^c(z) \in \mathrm{H}_{n-1}(B,A)$. Summarizing we have shown that

$$\partial^c(z) \;=\; \Big[\sum_{j=1}^{k} s_j \cdot \partial\sigma_j\Big].$$

This shows that the map defined in (1) agrees with ∂^c. ⊞

(2) It follows from the above claim together with the Connecting Homomorphism Lemma 37.3 that the connecting homomorphisms are natural.

(3) It follows from the above claim together with the LES-Proposition 37.4 that the given sequence is exact. ■

In a second we will state an important special case of the LES-of-Triple Proposition 38.2. This will require the following notation:

[310] The third isomorphism theorem in group theory implies that we have a natural isomorphism $(\mathrm{C}_*(X)/\mathrm{C}_*(A))/(\mathrm{C}_*(B)/\mathrm{C}_*(A)) \cong \mathrm{C}_*(X)/\mathrm{C}_*(B)$.

Notation. Let X be a topological space, let $A \subset X$ be a subspace and let $n \in \mathbb{N}_0$. We introduce the following homomorphisms[311]

$$\mathrm{H}_n(X) \xrightarrow{=} \mathrm{H}_n(X, \varnothing) \longrightarrow \mathrm{H}_n(X, A)$$
$$\underbrace{\qquad\qquad\qquad\qquad}_{=:\text{я}=:\text{Я}_{X,A}}$$

Now we can formulate the promised proposition.

Proposition 38.3. (LES-of-Pair Proposition)

(1) Let X be a topological space and let $B \subset X$ be a subspace. The connecting homomorphism of the LES-of-Triple Proposition 38.2 applied to $A = \varnothing$ gives rise to an exact sequence

$$\begin{array}{ccccccccccc} \ldots & \to & \mathrm{H}_n(B) & \xrightarrow{i_*} & \mathrm{H}_n(X) & \xrightarrow{\text{я}} & \mathrm{H}_n(X,B) & \xrightarrow{\partial_n} & \mathrm{H}_{n-1}(B) & \to & \ldots \\ & & & & & & \left[\sum_{j=1}^{k} s_j \cdot \sigma_j\right] & \mapsto & \left[\sum_{j=1}^{k} s_j \cdot \partial\sigma_j\right] & & \end{array}.$$

This exact sequence is called the long exact sequence of the pair (X, B).

(2) The connecting homomorphisms in (1) are natural. In particular, given a continuous map $f\colon (X, A) \to (Y, B)$ between pairs of topological spaces we obtain a commutative diagram

$$\begin{array}{ccccccccccc} \ldots & \longrightarrow & \mathrm{H}_n(A) & \xrightarrow{i_*} & \mathrm{H}_n(X) & \xrightarrow{\text{я}} & \mathrm{H}_n(X,A) & \xrightarrow{\partial_n} & \mathrm{H}_{n-1}(A) & \longrightarrow & \ldots \\ & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \\ \ldots & \longrightarrow & \mathrm{H}_n(B) & \xrightarrow{i_*} & \mathrm{H}_n(Y) & \xrightarrow{\text{я}} & \mathrm{H}_n(Y,B) & \xrightarrow{\partial_n} & \mathrm{H}_{n-1}(B) & \longrightarrow & \ldots \end{array}$$

Proof. This proposition follows immediately from the LES-of-Triple Proposition 38.2 applied to $A = \varnothing$. ∎

Example. As on page 489 we consider the topological space $X = \mathbb{R}^2$ and the subspace $A = B_1 \sqcup B_2$ that is given by the disjoint union of two closed disks in $\mathbb{R}^2$. We want to determine the relative homology groups of (X, A). We denote by $i\colon A \to X$ the inclusion map. By the LES-of-Pair Proposition 38.3 we have a long exact sequence

$$\mathrm{H}_2(X) \to \mathrm{H}_2(X,A) \to \mathrm{H}_1(A) \xrightarrow{i_*} \mathrm{H}_1(X) \to \mathrm{H}_1(X,A) \to \mathrm{H}_0(A) \to \mathrm{H}_0(X) \to \mathrm{H}_0(X,A) \to 0.$$

By the Contractible-H_*-Corollary 36.8 together with the Path-Component–H_*–Lemma 35.8 we have $\mathrm{H}_i(X) = \mathrm{H}_i(\mathbb{R}^2) = 0$ and also $\mathrm{H}_i(A) = \mathrm{H}_i(B_1) \oplus \mathrm{H}_i(B_2) = 0$ for $i \geq 1$. Thus we see from the above long exact sequence that $\mathrm{H}_i(X, A) = 0$ for $i \geq 2$.

Next let $\epsilon\colon \mathrm{H}_0(\mathbb{R}^2) \to \mathbb{Z}$ be the augmentation map. Since $\mathbb{R}^2$ is path-connected we know by the H_0-Proposition 35.11 that this map is an isomorphism. Similarly we denote by $\mu_1\colon \mathrm{H}_0(B_1) \to \mathbb{Z}$ and $\mu_2\colon \mathrm{H}_0(B_2) \to \mathbb{Z}$ the isomorphisms that are given by the augmentation maps. We obtain the following commutative diagram:

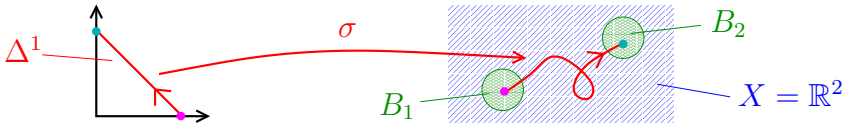
$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathrm{H}_1(X,A) & \xrightarrow{\partial} & \mathrm{H}_0(A) & \xrightarrow{i_*} & \mathrm{H}_0(X) & \longrightarrow & \mathrm{H}_0(X,A) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mu_1\oplus\mu_2}{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong}{\scriptstyle \epsilon} & & \downarrow{\scriptstyle \mathrm{id}} & & \\ 0 & \longrightarrow & \mathrm{H}_1(X,A) & \longrightarrow & \mathbb{Z}^2 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathrm{H}_0(X,A) & \longrightarrow & 0. \end{array}$$

[311] We apologize to whoever wants to teach out of these notes for using Cyrillic symbols.

It follows easily from the Augmentation Lemma 35.10 that the homomorphism $\mathbb{Z}^2 \to \mathbb{Z}$ is given by the matrix $(1\ 1)$. It follows that

$$\mathrm{H}_1(X,A) \underset{\text{by the Exactness Lemma 37.1 (3)}}{\cong} \ker\left(\mathbb{Z}^2 \xrightarrow{(1\ 1)} \mathbb{Z}\right) \cong \mathbb{Z} \quad \text{and} \quad \mathrm{H}_0(X,A) \underset{\text{by the Exactness Lemma 37.1 (3)}}{\cong} \operatorname{coker}\left(\mathbb{Z}^2 \xrightarrow{(1\ 1)} \mathbb{Z}\right) \cong 0.$$

Finally we want to find an explicit generator of the relative homology group $\mathrm{H}_1(X,A) \cong \mathbb{Z}$. As on page 489 let $\sigma\colon \Delta^1 \to X$ be a singular 1-simplex with $\sigma(0,1) \in B_2$ and $\sigma(1,0) \in B_1$.



Note that

$$((\mu_1 \oplus \mu_2) \circ \partial_1)([\sigma]) \underset{\text{see the description of the connecting homomorphism } \partial_1\colon \mathrm{H}_1(X,A) \to \mathrm{H}_0(A) \text{ in the LES-of-Triple Proposition 38.2}}{=} (\mu_1 \oplus \mu_2)([\partial_1 \sigma]) = (\mu_1 \oplus \mu_2)([\underbrace{\sigma(0,1)}_{\in B_2}] - [\underbrace{\sigma(1,0)}_{\in B_1}]) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Since $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is a generator of the kernel of $\mathbb{Z}^2 \xrightarrow{(1\ 1)} \mathbb{Z}$ we see that $[\sigma]$ is in fact a generator of $\mathrm{H}_1(X,A) \cong \mathbb{Z}$. In particular it is non-zero. □

We conclude this section with two more theoretical applications of the LES-of-Pair Proposition 38.3.

Corollary 38.4. (Homotopy Equivalence-Relative-H_*-Corollary) In the following let $f\colon (X,A) \to (Y,B)$ be a continuous map of pairs of topological spaces. If $f\colon X \to Y$ is a homotopy equivalence and if $f\colon A \to B$ is a homotopy equivalence[312], then for every $n \in \mathbb{N}_0$ the induced map $f_*\colon \mathrm{H}_n(X,A) \to \mathrm{H}_n(Y,B)$ is an isomorphism.

Example. It follows from the Homotopy Equivalence-Relative-H_*-Corollary 38.4 that for each $k \in \mathbb{N}_0$ the inclusion-induced homomorphisms $\mathrm{H}_n(\overline{B}^k, S^{k-1}) \to \mathrm{H}_n(\overline{B}^k, \overline{B}^k \setminus \{0\})$ are isomorphisms. □

Proof. We consider the following diagram:

$$\begin{array}{ccccccccccccc}
\ldots & \longrightarrow & \mathrm{H}_n(A) & \xrightarrow{i_*} & \mathrm{H}_n(X) & \xrightarrow{\text{я}} & \mathrm{H}_n(X,A) & \xrightarrow{\partial_n} & \mathrm{H}_{n-1}(A) & \xrightarrow{i_*} & \mathrm{H}_{n-1}(X) & \longrightarrow & \ldots \\
 & & \downarrow f_* & & \downarrow f_* & & \downarrow f_* & & \downarrow f_* & & \downarrow f_* & & \\
\ldots & \longrightarrow & \mathrm{H}_n(B) & \xrightarrow{i_*} & \mathrm{H}_n(Y) & \xrightarrow{\text{я}} & \mathrm{H}_n(Y,B) & \xrightarrow{\partial_n} & \mathrm{H}_{n-1}(B) & \xrightarrow{i_*} & \mathrm{H}_{n-1}(Y) & \longrightarrow & \ldots
\end{array}$$

By the LES-of-Pair Proposition 38.3 (2) we know that the diagram commutes and that the horizontal sequences are exact. Furthermore note that it follows from our hypothesis on f and the Homotopy Equivalence-H_*-Corollary 36.7 that the two vertical maps to the left and the two vertical maps to the right are isomorphisms. It follows from the Five Lemma 37.5 that the middle vertical map is, as promised, an isomorphism. ■

[312] In many applications $f\colon (X,A) \to (Y,B)$ will be an inclusion, where X is a deformation retract of Y and A is a deformation retract of B. It then follows from the Deformation Retract–Homotopy Equivalence Lemma 9.3 that $f\colon X \to Y$ and $f\colon A \to B$ are homotopy equivalences.

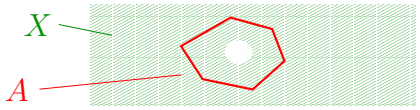
Corollary 38.5. (Deformation Retract-Relative H_*-Corollary) Let X be a topological space. If $A \subset X$ is a deformation retract of X, then $\mathrm{H}_n(X, A) = 0$ for all $n \in \mathbb{N}_0$.

Proof. Let $i\colon A \to X$ be the inclusion map. By the LES-of-Pair Proposition 38.3 we have a long exact sequence

$$\ldots \to \mathrm{H}_n(A) \xrightarrow[\uparrow]{i_*} \mathrm{H}_n(X) \to \mathrm{H}_n(X, A) \xrightarrow{\partial_n} \mathrm{H}_{n-1}(A) \xrightarrow[\uparrow]{i_*} \mathrm{H}_{n-1}(X) \to \ldots$$

it follows from the Deformation Retract–Homotopy Equivalence Lemma 9.3 that $i\colon A \to X$ is a homotopy equivalence, thus it follows from the Homotopy Equivalence-H_*-Corollary 36.7 that i_* is an isomorphism

Since the maps $i_*\colon \mathrm{H}_n(A) \to \mathrm{H}_n(X)$ are isomorphisms and since the sequence is exact we

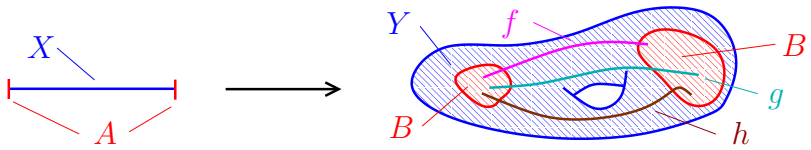


A is a deformation retract of X therefore each $i_*\colon \mathrm{H}_n(A) \to \mathrm{H}_k(X)$ is an isomorphism

see that the relative homology groups $\mathrm{H}_n(X, A)$ are all zero. ■

38.3. Relative homology groups and homotopies.

Definition. We say that two continuous maps $f, g\colon (X, A) \to (Y, B)$ between pairs of topological spaces are homotopic, if there exists a continuous map $F\colon X \times [0, 1] \to Y$ such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$ for all $x \in X$, and such that $F(a, t) \in B$ for all $a \in A$ and all $t \in [0, 1]$.



$f, g\colon (X, A) \to (Y, B)$ are homotopic but $f, h\colon (X, A) \to (Y, B)$ are not homotopic

The following proposition extends the Homotopy-H_*-Proposition 36.5 to the relative case.

Proposition 38.6. (Homotopy-H_*-Proposition) Let $f, g\colon (X, A) \to (Y, B)$ be two continuous maps between pairs of topological spaces. If f and g are homotopic, then the chain maps $f_*, g_*\colon \mathrm{C}_*(X, A) \to \mathrm{C}_*(Y, B)$ are chain homotopic, in particular for every $n \in \mathbb{N}_0$ we have

$$f_* = g_*\colon \mathrm{H}_n(X, A) \to \mathrm{H}_n(Y, B).$$

Proof. Let $f, g\colon (X, A) \to (Y, B)$ be two continuous maps between pairs of topological spaces and let $F\colon X \times [0, 1] \to Y$ be a homotopy between f and g as maps of pairs of topological spaces.[313] Let $\Omega_n \in \mathrm{C}_{n+1}(\Delta^n \times [0, 1])$ be the singular chain that we defined on page 475. On page 477 we showed that the homomorphisms

$$\begin{aligned} P_n\colon \mathrm{C}_n(X) &\to \mathrm{C}_{n+1}(Y) \\ \sum_{i=1}^{k} s_i \cdot (\sigma\colon \Delta^n \to X) &\mapsto \sum_{i=1}^{k} s_i \cdot (F \circ (\sigma \times \mathrm{id}_{[0,1]}))_*(\Omega_n) \end{aligned}$$

[313] In most cases we just want the statement that $f_* = g_*\colon \mathrm{H}_n(X, A) \to \mathrm{H}_n(Y, B)$. This statement follows easily from the LES-of-Pair Proposition 38.3 (2), the Homotopy-H_*-Proposition 36.5 and the Five Lemma 37.5.

define a chain homotopy between the maps $f_*, g_*\colon \mathrm{C}_*(X) \to \mathrm{C}_*(Y)$, i.e. for each $n \in \mathbb{N}_0$ we have

$$\partial_{n+1} \circ P_n + P_{n-1} \circ \partial_n \;=\; f_* - g_*\colon \mathrm{C}_n(X) \;\to\; \mathrm{C}_n(Y).$$

Since F restricts to a map $A \times [0,1] \to B$ we see that these homomorphisms P_n have the property that $\mathrm{C}_n(A)$ gets sent to $\mathrm{C}_{n+1}(B)$. Thus the maps $P_n\colon \mathrm{C}_n(X) \to \mathrm{C}_n(Y)$ descend to homomorphisms $\overline{P}_n\colon \mathrm{C}_n(X,A) \to \mathrm{C}_{n+1}(Y,B)$. These satisfy

$$\partial_{n+1} \circ \overline{P}_n + \overline{P}_{n-1} \circ \partial_n \;=\; f_* - g_*\colon \mathrm{C}_n(X,A) \;\to\; \mathrm{C}_n(Y,B),$$

i.e. the maps $\overline{P}_n$ form a chain homotopy between the maps f_* and g_* from $\mathrm{C}_*(X,A)$ to $\mathrm{C}_*(Y,B)$. It follows from this observation together with the Chain Homotopy–H_*-Lemma 36.1 that $f_* = g_*\colon \mathrm{H}_n(X,A) \to \mathrm{H}_n(Y,B)$. ■

CHAPTER 39

Reduced homology

In this very short chapter we introduce the *reduced* homology groups of topological spaces. These reduced homology groups are almost the same as the by now familiar homology groups. But they will come in handy in the calculations of homology groups in the next chapters, since reduced homology groups are often a little easier to deal with than ordinary homology groups, even though they contain the same information.

39.1. **Definition and basic properties of reduced homology.** The Augmentation Lemma 35.10 (1) motivates the following definition of the reduced homology groups of a topological space.

Definition. Let X be a topological space. By the Augmentation Lemma 35.10 (1) we know that the following is a generalized[314] chain complex

$$\begin{array}{ccccccccccc} & & & & & & \overset{\text{degree } 0}{\downarrow} & & \overset{\text{degree } -1}{\downarrow} & & \\ \ldots & \xrightarrow{\partial_3} & \mathrm{C}_2(X) & \xrightarrow{\partial_2} & \mathrm{C}_1(X) & \xrightarrow{\partial_1} & \mathrm{C}_0(X) & \xrightarrow{\epsilon_X} & \mathbb{Z} & \to & 0. \\ & & & & & & \sum_{i=1}^{k} r_i \cdot \sigma_i & \mapsto & \sum_{i=1}^{k} r_i & & \\ & & & & & & & \underset{\text{augmentation map from page 464}}{\uparrow} & & & \end{array}$$

We introduce the following two definitions.

(1) We refer to this chain complex as the augmented chain complex $\widetilde{\mathrm{C}}_*(X)$ of X.
(2) Given $n \in \mathbb{Z}_{\geq -1}$ we refer to $\widetilde{\mathrm{H}}_n(X) := \mathrm{H}_n(\widetilde{\mathrm{C}}_*(X))$ as the n-th reduced homology group of X.

These reduced homology groups are a minute but at times surprisingly useful variation on the usual homology groups. In the following we will explore the properties of reduced homology groups and we will explore their relationship to the traditional homology groups.

The following lemma is a variation on the by now familiar H_n–Functor Lemma 35.7.

Lemma 39.1. ($\widetilde{\mathrm{H}}_n$–Functor Lemma)

(1) If $f\colon X \to Y$ is a continuous map between two topological spaces, then we denote by $f_*\colon \widetilde{\mathrm{C}}_*(X) \to \widetilde{\mathrm{C}}_*(Y)$ the map that is the usual map on $\widetilde{\mathrm{C}}_n(X) = \mathrm{C}_n(X)$ for $n \geq 0$ and that is the identity on the final $\mathbb{Z}$-term. This map is a chain map and thus induces homomorphisms $f_*\colon \widetilde{\mathrm{H}}_n(X) \to \widetilde{\mathrm{H}}_n(Y)$ on the reduced homology groups.
(2) For any $n \in \mathbb{Z}_{\geq -1}$ the maps

$$\begin{array}{rcl} X & \mapsto & \widetilde{\mathrm{H}}_n(X) \\ (f\colon X \to Y) & \mapsto & (f_*\colon \widetilde{\mathrm{H}}_n(X) \to \widetilde{\mathrm{H}}_n(Y)) \end{array}$$

[314] Recall that a generalized chain complex is like a chain complex, but we also allow non-zero groups in negative degrees.

define a covariant functor from the category *Top* of topological spaces to the category *AbGroup* of abelian groups.

(3) The maps $\mathrm{id}\colon \widetilde{\mathrm{C}}_n(X) \to \mathrm{C}_n(X)$ for $n \geq 0$ and the zero map $\widetilde{\mathrm{C}}_{-1}(X) \to \mathrm{C}_{-1}(X) = 0$ define a natural chain map and for any $n \geq -1$ they induce a natural transformation

$$\text{ж}\colon \widetilde{\mathrm{H}}_n(X) \;\to\; \mathrm{H}_n(X)$$

from the functor $X \mapsto \widetilde{\mathrm{H}}_n(X)$ to the functor $X \mapsto \mathrm{H}_n(X)$.

Proof.

(1) Let $f\colon X \to Y$ be a continuous map between two topological spaces. As in the Augmentation Lemma 35.10 (3) it is a straightforward consequence of the definition of the augmentation maps that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{C}_0(X) & \xrightarrow{f_*} & \mathrm{C}_0(Y) \\ {\scriptstyle \epsilon_X}\downarrow & & \downarrow{\scriptstyle \epsilon_Y} \\ \mathbb{Z} & \xrightarrow{\mathrm{id}} & \mathbb{Z}. \end{array}$$

It follows from this observation together with the C_*–Functor Lemma 35.5 that the induced maps $f_*\colon \widetilde{\mathrm{C}}_*(X) \to \widetilde{\mathrm{C}}_*(Y)$ are chain maps. Furthermore we obtain from the Chain Complex-H_*-Functor Lemma 35.1 (1) that we get induced maps on reduced homology.

(2) This statement is basically obvious.

(3) This statement follows easily from the definitions. ∎

The following elementary lemma explains the relationship between ordinary and reduced homology: Basically they are the same, except that for non-empty topological spaces they "differ by $\mathbb{Z}$" in dimension 0.

Lemma 39.2. (Reduced Homology Lemma) For every topological space X the following statements hold:

(-1) We have[315]

$$\widetilde{\mathrm{H}}_{-1}(X) \;=\; \begin{cases} 0, & \text{if } X \neq \varnothing, \\ \mathbb{Z}, & \text{if } X = \varnothing. \end{cases}$$

(0) (a) If X is non-empty, then the sequence

$$0 \;\to\; \widetilde{\mathrm{H}}_0(X) \xrightarrow{\text{ж}} \mathrm{H}_0(X) \xrightarrow[\text{Augmentation Lemma 35.10}]{\text{augmentation map } \epsilon_X \text{ from the}} \mathbb{Z} \;\to\; 0$$

is exact.

(b) Given any point $P \in X$ the map

$$\begin{array}{rcl} \widetilde{\mathrm{H}}_0(X) \oplus \mathbb{Z} & \to & \mathrm{H}_0(X) \\ [\sigma] \oplus n & \mapsto & [\sigma] + [n \cdot P] \end{array}$$

is an isomorphism.

(c) If X has $n \in \mathbb{N}$ path-components, then $\widetilde{\mathrm{H}}_0(X) \cong \mathbb{Z}^{n-1}$. In particular X is path-connected if and only if $\widetilde{\mathrm{H}}_0(X) = 0$.

(1) For every $n \geq 1$ the natural map $\text{ж}\colon \widetilde{\mathrm{H}}_n(X) \to \mathrm{H}_n(X)$ is the identity.

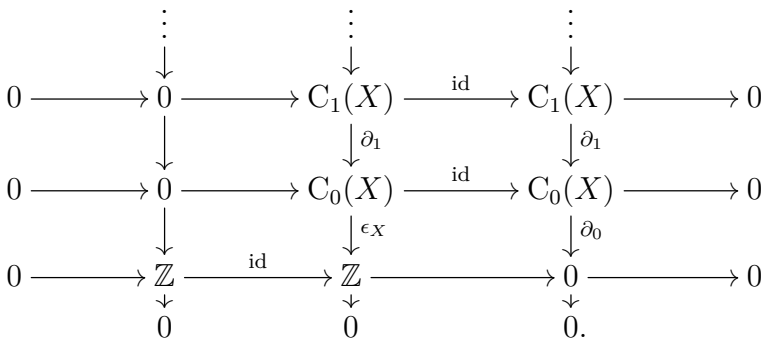
[315] This calculation shows that in almost all cases we can ignore $\widetilde{\mathrm{H}}_{-1}(X)$. Nonetheless, some formulas later on become cleaner if we take into account that $\widetilde{\mathrm{H}}_{-1}(\varnothing) = \mathbb{Z}$.

The lemma shows in particular that reduced homology groups are almost the same as the usual homology groups. So it might sound slightly silly to introduce them. But we will see that on several occasions reduced homology groups make it easier to formulate some statements. Throughout these notes we will often use the Reduced Homology Lemma 39.2 without explicitly mentioning it. For example we will refer to earlier results about homology groups and state the corresponding results for reduced homology groups.

Proof. Let X be a topological space.

(-1) First note that clearly we have $\widetilde{\mathrm{H}}_{-1}(\varnothing) = \mathbb{Z}$. Now we assume that X is non-empty. It follows immediately from the definitions that the augmentation map ϵ_X is an epimorphism which entails that $\widetilde{\mathrm{H}}_{-1}(X) = 0$.

(0) (a) We consider the following short exact sequence of generalized chain complexes:



The desired statement follows easily from the corresponding long exact sequence of homology groups that is given by the LES-Proposition 37.4.

(b) Let $P \in X$ be a point. It is straightforward to show that the maps

$$\begin{array}{rcl} \widetilde{\mathrm{H}}_0(X) \oplus \mathbb{Z} & \to & \mathrm{H}_0(X) \\ [\sigma] \oplus n & \mapsto & [\sigma] + [n \cdot P] \end{array} \quad \text{and} \quad \begin{array}{rcl} \mathrm{H}_0(X) & \to & \widetilde{\mathrm{H}}_0(X) \oplus \mathbb{Z} \\ [\sigma] & \mapsto & [\sigma - \epsilon_X(\sigma) \cdot P] + \epsilon_X(\sigma) \end{array}$$

are inverses of one another. In particular both maps are isomorphisms.

(c) This statement follows almost immediately from (0a), the H_0-Proposition 35.11 together with the Direct Sum-H_*-Lemma 35.2.

(1) This statement is clear. ■

Many statements for ordinary homology groups carry over to the setting of reduced homology groups. For completeness' sake we spell out the analogues of the Homotopy-H_*-Proposition 36.5 and the Homotopy Equivalence-H_*-Corollary 36.7:

Proposition 39.3. (Homotopy-$\widetilde{\mathrm{H}}_*$-Proposition) Let X and Y be two topological spaces and let $f, g\colon X \to Y$ be two continuous maps. If f and g are homotopic, then the following hold:

(1) The induced maps f_* and g_* from $\widetilde{\mathrm{C}}_*(X)$ to $\widetilde{\mathrm{C}}_*(Y)$ are chain homotopic.
(2) For each $n \in \mathbb{N}_0$ we have $f_* = g_*\colon \widetilde{\mathrm{H}}_n(X) \to \widetilde{\mathrm{H}}_n(Y)$.

Sketch of proof.

(1) Recall that by the Homotopy-H_*-Proposition 36.5 (2) we know that there exists a chain homotopy $P_n\colon \mathrm{C}_n(X) \to \mathrm{C}_n(Y)$. We extend this chain homotopy by the zero-map $P_{-1}\colon \widetilde{\mathrm{C}}_{-1}(X) = \mathbb{Z} \to \widetilde{C}_0(Y)$. It is straightforward to verify that this is a chain

homotopy. For convenience and reassurance we show the relevant part in a diagram:

$$\begin{array}{ccccccc} \mathrm{C}_1(X) & \xrightarrow{\partial_1} & \mathrm{C}_0(X) & \xrightarrow{\epsilon_X} & \mathbb{Z} & \longrightarrow & 0 \\ \downarrow{\scriptstyle g_*-f_*} & \swarrow{\scriptstyle P_0} & \downarrow{\scriptstyle g_*-f_*} & \swarrow{\scriptstyle 0} & \downarrow{\scriptstyle \mathrm{id}-\mathrm{id}} & & \\ \mathrm{C}_1(Y) & \xrightarrow[\partial_1]{} & \mathrm{C}_0(Y) & \xrightarrow[\epsilon_Y]{} & \mathbb{Z} & \longrightarrow & 0. \end{array}$$

(2) This statement follows immediately from (1). ■

Corollary 39.4. (Homotopy Equivalence-$\widetilde{\mathrm{H}}_*$-Corollary) Let X and Y be topological spaces and let $n \in \mathbb{N}_0$.

(1) (a) If $f\colon X \to Y$ is a homotopy equivalence, then the induced map $f_*\colon \widetilde{\mathrm{H}}_n(X) \to \widetilde{\mathrm{H}}_n(Y)$ is an isomorphism.
(b) If X is a deformation retract of Y, then the inclusion-induced map $\widetilde{\mathrm{H}}_n(X) \to \widetilde{\mathrm{H}}_n(Y)$ is an isomorphism.

(2) If X and Y are homotopy equivalent, then $\widetilde{\mathrm{H}}_n(X) \cong \widetilde{\mathrm{H}}_n(Y)$.

Sketch of proof. This corollary can be obtained easily from the Homotopy-$\widetilde{\mathrm{H}}_*$-Proposition 39.3. ■

The statement of the following corollary differs subtly from the corresponding "unreduced" version.

Corollary 39.5. (Contractible-$\widetilde{\mathrm{H}}_*$-Corollary) If X is contractible (i.e. if X is homotopy equivalent to a singleton), then we have $\widetilde{\mathrm{H}}_n(X) = 0$ for all $n \in \mathbb{Z}_{\geq -1}$.

Example. On page 171 we saw that every convex subset of $\mathbb{R}^n$ is contractible. Thus it follows from the Contractible-H_*-Corollary 36.8, that for any $i \in \mathbb{Z}_{\geq -1}$ we have

$$\widetilde{\mathrm{H}}_i(\underbrace{\text{convex subset of } \mathbb{R}^n}_{\text{e.g. } \mathbb{R}^n,\ \overline{B}^n,\ B^n \text{ or simplex}}) \ = \ 0.$$

□

Proof. This corollary follows immediately from the Contractible-H_*-Corollary 36.8 together with the Reduced Homology Lemma 39.2. ■

39.2. **The long exact sequence of a pair in reduced homology.** Shortly we will see that a pair of topological spaces gives rise to a long exact sequence that involves reduced homology groups. Before we can formulate this result we need to introduce the following notation:

Notation. Let X be a topological space, let $A \subset X$ and let $n \in \mathbb{N}_0$. We introduce the following map:

$$\widetilde{\mathrm{H}}_n(X) \xrightarrow{\text{ж}} \mathrm{H}_n(X) \xrightarrow{\text{я}} \mathrm{H}_n(X, A), \qquad \text{ж}_{X,A} := \text{я} \circ \text{ж}$$

where $\text{ж}\colon \widetilde{\mathrm{H}}_n(X) \to \mathrm{H}_n(X)$ is the natural homomorphism introduced in the $\widetilde{\mathrm{H}}_n$–Functor Lemma 39.1 and $\text{я}\colon \mathrm{H}_n(X) \to \mathrm{H}_n(X, A)$ is the natural homomorphism that we introduced on page 493.

The following slight variation on the LES-of-Pair Proposition 38.3 will turn out to be surprisingly useful.

Proposition 39.6. (Reduced LES-of-Pair Proposition)

(1) Let X be a topological space and let $A \subset X$ be a subspace. The following sequence is exact[316]

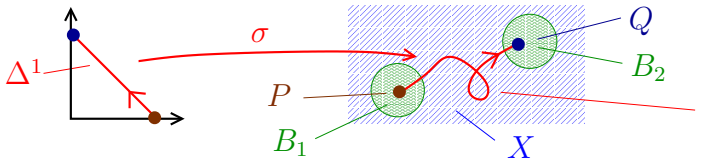
$$\begin{array}{ccccccccccc} \dots & \to & \widetilde{\mathrm{H}}_n(A) & \xrightarrow{i_*} & \widetilde{\mathrm{H}}_n(X) & \xrightarrow{\text{Ж}_{X,A}} & \mathrm{H}_n(X,A) & \xrightarrow{\partial_n} & \widetilde{\mathrm{H}}_{n-1}(A) & \to & \dots \\ & & & & & & \left[\sum_{j=1}^{k} s_j \cdot \sigma_j\right] & \mapsto & \left[\sum_{j=1}^{k} s_j \cdot \partial\sigma_j\right] & & \end{array}$$

This sequence is called the long exact sequence in reduced homology of the pair (X, A).

(2) The connecting homomorphisms in (1) are natural. In particular, given a continuous map $f\colon (X,A) \to (Y,B)$ between pairs of topological spaces we obtain a commutative diagram

$$\begin{array}{ccccccccc} \dots & \longrightarrow & \widetilde{\mathrm{H}}_n(A) & \xrightarrow{i_*} & \widetilde{\mathrm{H}}_n(X) & \xrightarrow{\text{Ж}_{X,A}} & \mathrm{H}_n(X,A) & \xrightarrow{\partial_n} & \widetilde{\mathrm{H}}_{n-1}(A) & \longrightarrow & \dots \\ & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \\ \dots & \longrightarrow & \widetilde{\mathrm{H}}_n(B) & \xrightarrow{i_*} & \widetilde{\mathrm{H}}_n(Y) & \xrightarrow{\text{Ж}_{Y,B}} & \mathrm{H}_n(Y,B) & \xrightarrow{\partial_n} & \widetilde{\mathrm{H}}_{n-1}(B) & \longrightarrow & \dots \end{array}$$

Example. As on pages 489 and 494 we consider the topological space $X = \mathbb{R}^2$ and the subspace $A = B_1 \sqcup B_2$ that is given by the disjoint union of two closed disks in $\mathbb{R}^2$. We want to determine the relative homology groups of (X, A).



We denote by $i\colon A \to X$ the inclusion map. By the Reduced LES-of-Pair Proposition 39.6 we have a long exact sequence

$$\widetilde{\mathrm{H}}_2(X) \to \mathrm{H}_2(X,A) \to \widetilde{\mathrm{H}}_1(A) \xrightarrow{i_*} \widetilde{\mathrm{H}}_1(X) \to \mathrm{H}_1(X,A) \to \widetilde{\mathrm{H}}_0(A) \to \widetilde{\mathrm{H}}_0(X) \to \mathrm{H}_0(X,A) \to 0.$$

By the Contractible-H_*-Corollary 36.8, the Path-Component–H_*–Lemma 35.8 and the Reduced Homology Lemma 39.2 we have $\widetilde{\mathrm{H}}_i(X) = \widetilde{\mathrm{H}}_i(\mathbb{R}^2) = 0$ for all $i \geq 0$, we have $\widetilde{\mathrm{H}}_i(A) = 0$ for $i \geq 1$ and we have $\widetilde{\mathrm{H}}_0(A) \cong \mathbb{Z}$. Thus we see from the above long exact sequence that $\mathrm{H}_i(X,A) = 0$ for $i \neq 1$ and that $\mathrm{H}_1(X,A) \cong \mathbb{Z}$. We get the same result as on page 494, but with much less pain. □

Proof.

[316]Note that if the subspace B is non-empty, then it follows from the Reduced Homology Lemma 39.2 (-1) that $\widetilde{\mathrm{H}}_{-1}(B) = 0$. Thus in this case the last possibly non-zero term is $\mathrm{H}_0(X,B)$. On the other hand, if $B = \varnothing$, then we have $\widetilde{\mathrm{H}}_{-1}(B) = \widetilde{\mathrm{H}}_{-1}(\varnothing) = \mathbb{Z}$, and if $X = \varnothing$, then we have $\widetilde{\mathrm{H}}_{-1}(X) = \widetilde{\mathrm{H}}_{-1}(\varnothing) = \mathbb{Z}$. Therefore, if $B = \varnothing$ the sequence is understood to continue with $\mathrm{H}_0(X,\varnothing) \xrightarrow{\epsilon} \mathrm{H}_{-1}(\varnothing) \to \mathrm{H}_{-1}(X) \to 0$.

(1) We consider the following short exact sequence of chain complexes:

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{C}_1(A) & \longrightarrow & \mathrm{C}_1(X) & \longrightarrow & \mathrm{C}_1(X,A) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \partial_1} & & \downarrow{\scriptstyle \partial_1} & & \downarrow{\scriptstyle \partial_1} & & \\
0 & \longrightarrow & \mathrm{C}_0(A) & \longrightarrow & \mathrm{C}_0(X) & \longrightarrow & \mathrm{C}_0(X,A) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \epsilon_A} & & \downarrow{\scriptstyle \epsilon_X} & & \downarrow{\scriptstyle \partial_0} & & \\
0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\mathrm{id}} & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0. & &
\end{array}$$

From the LES-Proposition 37.4 we obtain a long exact sequence of homology groups, which in this case involve the reduced homology groups of A and X, and the relative homology groups of (X, A). Furthermore, as in the proof of the LES-of-Triple Proposition 38.2 we see that the connecting homomorphism is given by the map

$$\begin{array}{rcl}
\mathrm{H}_n(X,A) & \to & \widetilde{\mathrm{H}}_{n-1}(A) \\
\Big[\sum\limits_{j=1}^{k} s_j \cdot \sigma_j\Big] & \mapsto & \Big[\sum\limits_{j=1}^{k} s_j \cdot \partial\sigma_j\Big]
\end{array}$$

Finally note that one can easily easily verify that the maps $\widetilde{\mathrm{H}}_k(X) \to \mathrm{H}_k(X,A)$ are precisely the above maps $\text{ж}_{X,A}$.

(2) This statement follows easily from the naturality of the connecting homomorphism proved in the LES-Proposition 37.4 ■

We conclude this chapter with the following lemma that elucidates the relationship between reduced and relative homology.

Lemma 39.7. (Reduced-to-Relative-H_*-Lemma) Let X be a topological space and let $n \in \mathbb{N}_0$. For each $x_0 \in X$ the map

$$\text{ж}_{X,\{x_0\}} \colon \widetilde{\mathrm{H}}_n(X) \to \mathrm{H}_n(X, \{x_0\})$$

is a natural isomorphism.

Proof. Let X be a topological space and let $x_0 \in X$. We apply the Reduced LES-of-Pair Proposition 39.6 to the pair $(X, \{x_0\})$ and we obtain a long exact sequence

$$\ldots \to \underbrace{\widetilde{\mathrm{H}}_n(\{x_0\})}_{=0} \xrightarrow{i_*} \widetilde{\mathrm{H}}_n(X) \xrightarrow{\text{ж}_{X,\{x_0\}}} \mathrm{H}_n(X, \{x_0\}) \xrightarrow{\partial_n} \underbrace{\widetilde{\mathrm{H}}_{n-1}(\{x_0\})}_{=0} \to \ldots$$

The lemma follows from the exactness of this sequence and the observation that it follows from the Singleton-H_*-Lemma 35.12 and the Reduced Homology Lemma 39.2 that all reduced homology groups of $\{x_0\}$ vanish. ■

CHAPTER 40

The Excision Theorem

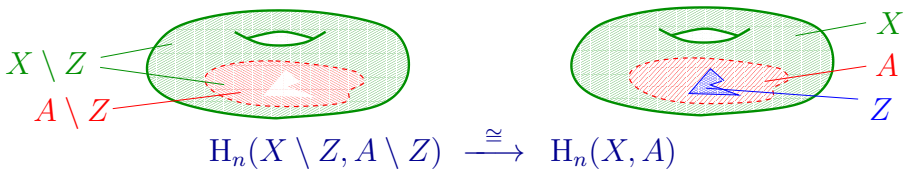
We introduced singular homology on page 462, many chapters ago. Since then we have developed the theory of singular homology groups and related topics. But the reader will have noticed that up to now we have not really calculated any interesting homology groups.

This is about to change. In this chapter we will state and prove the Excision Theorem 40.1, which is easily one of the most essential theorems in algebraic topology. We will use it, and the techniques from the previous chapters, to finally provide the calculation of the homology groups of spheres. In the next chapter we will use the Excision Theorem 40.1 to find, given a suitable pair of topological spaces (X, A), a connection between the groups $\mathrm{H}_n(X, A)$ and $\mathrm{H}_n(X/A)$.

40.1. **The Excision Theorem.** Here is the main theorem of this chapter.

Theorem 40.1. (Excision Theorem) Let X be a topological space and let $Z \subset A \subset X$ be subsets. If the closure of Z is contained in the interior of A, then the inclusion $(X \setminus Z, A \setminus Z) \to (X, A)$ induces for each $n \in \mathbb{N}_0$ an isomorphism

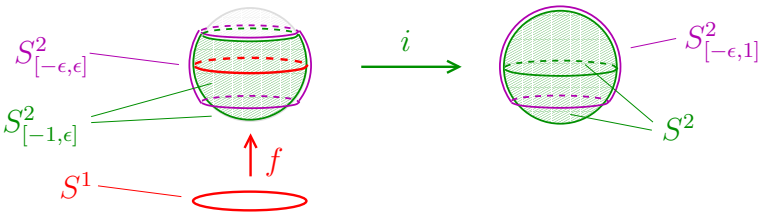
$$\mathrm{H}_n(X \setminus Z, A \setminus Z) \xrightarrow{\cong} \mathrm{H}_n(X, A).$$



The Excision Theorem 40.1 thus says in particular that if we are given a pair of topological spaces (X, A) and if we excise a closed subset Z from the interior of A, then this does not affect the relative homology groups. In the next chapter we will use the Excision Theorem 40.1 to perform many calculations. As an "amuse bouche" we will now compute the homology groups of the spheres S^n:

Example. We want to determine the reduced homology groups of S^n. We pick some $\epsilon \in (0, 1)$ and for each $m \in \mathbb{N}_0$ we consider the following homomorphisms:

$$\begin{array}{ccccccccc}
\longrightarrow \widetilde{\mathrm{H}}_m(S^n_{[-\epsilon,1]}) & \longrightarrow & \widetilde{\mathrm{H}}_m(S^n) & \xrightarrow{\cong} & \mathrm{H}_m(S^n, S^n_{[-\epsilon,1]}) & \xrightarrow{\partial_m} & \widetilde{\mathrm{H}}_{m-1}(S^n_{[-\epsilon,1]}) & \longrightarrow & \dots \\
 & & & & \uparrow{\scriptstyle i_*}\,{\scriptstyle \cong} & & & & \\
\dots \to \widetilde{\mathrm{H}}_{m-1}(S^n_{[-\epsilon,\epsilon]}) \to & \widetilde{\mathrm{H}}_m(S^n_{[-1,\epsilon]}) & \to & \mathrm{H}_m(S^n_{[-1,\epsilon]}, S^n_{[-\epsilon,\epsilon]}) & \xrightarrow[\cong]{\partial_m} & \widetilde{\mathrm{H}}_{m-1}(S^n_{[-\epsilon,\epsilon]}) & \to & \widetilde{\mathrm{H}}_{m-1}(S^n_{[-1,\epsilon]}) & \to \\
 & & & & & \uparrow{\scriptstyle f_*}\,{\scriptstyle \cong} & & & \\
 & & & & & \widetilde{\mathrm{H}}_{m-1}(S^{n-1}) & & &
\end{array}$$



We make the following clarifications and explanations:

(1) For an interval $I \subset [-1,1]$ we write $S^n_I := \{(x_1,\dots,x_{n+1}) \in S^n \,|\, x_{n+1} \in I\}$.
(2) The top horizontal sequence is the long exact sequence in reduced homology of the pair $(S^n, S^n_{[-\epsilon,1]})$ that is given by the Reduced LES-of-Pair Proposition 39.6.
(3) The bottom horizontal sequence is the long exact sequence in reduced homology of the pair $(S^n_{[-1,\epsilon]}, S^n_{[-\epsilon,\epsilon]})$ that is given by the Reduced LES-of-Pair Proposition 39.6.
(4) It follows from the Stereographic Projection Lemma 2.11 that $S^n_{[-\epsilon,1]}$ and $S^n_{[-1,\epsilon]}$ are homeomorphic to the closed n-ball $\overline{B}^n$. It follows from this observation and the Contractible-$\widetilde{\mathrm{H}}_*$-Corollary 39.5 that for all $k \in \mathbb{N}_0$ we have $\widetilde{\mathrm{H}}_k(S^n_{[-\epsilon,1]}) = 0$ and that we have $\widetilde{\mathrm{H}}_k(S^n_{[-1,\epsilon]}) = 0$.
(5) It follows from (4) and the exactness of the horizontal sequences that the horizontal maps shown in red are isomorphisms.
(6) It follows from the Excision Theorem 40.1, applied to $X = S^n$ and the two subspaces $A = S^n_{[-\epsilon,1]}$ and $Z = S^n_{(\epsilon,1]}$, that the vertical map $i_*\colon \mathrm{H}_m(S^n_{[-1,\epsilon]}, S^n_{[-\epsilon,\epsilon]}) \to \mathrm{H}_m(S^n, S^n_{[-\epsilon,1]})$ is an isomorphism.
(7) We denote by $f\colon S^{n-1} \to S^n_{[-\epsilon,\epsilon]}$ the map that is given by $x \mapsto (x,0)$. This map is clearly an embedding. It is not hard to show that $f(S^{n-1})$ is a deformation retract of $S^n_{[-\epsilon,\epsilon]}$. It follows from the Deformation Retract–Homotopy Equivalence Lemma 9.3 that $f\colon S^{n-1} \to S^n_{[-\epsilon,\epsilon]}$ is a homotopy equivalence.
(8) It follows from (7) together with the Homotopy Equivalence-$\widetilde{\mathrm{H}}_*$-Corollary 39.4 for reduced homology that the induced map $f_*\colon \widetilde{\mathrm{H}}_{m-1}(S^{n-1}) \to \widetilde{\mathrm{H}}_{m-1}(S^n_{[-\epsilon,\epsilon]})$ is an isomorphism.

We have now shown that all red homomorphisms are isomorphisms. We now see that for each $m \in \mathbb{Z}_{\geq -1}$ we have[317]

$$\widetilde{\mathrm{H}}_m(S^n) \underset{\uparrow}{\cong} \widetilde{\mathrm{H}}_{m-1}(S^{n-1}) \underset{\uparrow}{\cong} \dots \underset{\uparrow}{\cong} \widetilde{\mathrm{H}}_{m-n}(S^0) \underset{\uparrow}{\cong} \begin{cases} \mathbb{Z}, & \text{if } m = n, \\ 0, & \text{otherwise.} \end{cases}$$

since all red homomorphisms are isomorphisms

iterating the argument

follows from the Singleton-H_*-Lemma 35.12, the Path-Component–H_*–Lemma 35.8 and the Reduced Homology Lemma 39.2

We have thus finally calculated the homology groups of spheres. We will redo the calculation in the Sphere-H_*-Proposition 41.2 using a slightly different approach. Furthermore in Section 41.2 we will discuss several applications of this calculation. □

The plan for the remainder of this chapter is as follows: In the next section we will formulate several variations on the Excision Theorem 40.1. We give a short outline of the

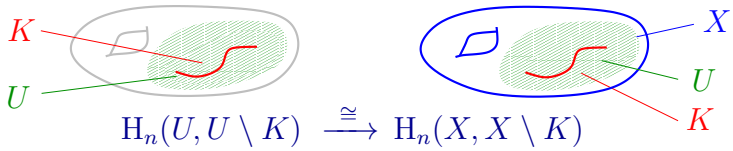
[317] We could even have gone one step further and ended with $S^{-1} = \varnothing$ and the final calculation would have been provided by the Reduced Homology Lemma 39.2 (-1).

proof of the Excision Theorem 40.1 in Section 40.3 and we will provide the full daunting details of the proof of the Excision Theorem 40.1 in Section 40.4. Readers who want to see the Excision Theorem 40.1 in action are advised to move to the next chapter.

40.2. **Variations on the Excision Theorem 40.1.** In this section we will state and prove several variations on the Excision Theorem 40.1. The following theorem is in fact just a convenient reformulation of the Excision Theorem 40.1.

Theorem 40.2. (Excision Theorem) Let X be a topological space, let K be a subset of X and let U be a neighborhood of the closure of K. The inclusion $(U, U \setminus K) \to (X, X \setminus K)$ induces for each $n \in \mathbb{N}_0$ an isomorphism[318]

$$\mathrm{H}_n(U, U \setminus K) \xrightarrow{\cong} \mathrm{H}_n(X, X \setminus K).$$



Proof of Theorem 40.2 assuming Theorem 40.1. We have

$$\mathrm{H}_n(U, U \setminus K) = \mathrm{H}_n(X \setminus Z, A \setminus Z) \xrightarrow{\cong} \mathrm{H}_n(X, A) = \mathrm{H}_n(X, X \setminus K).$$

set $A \coloneqq X \setminus K$ and $Z \coloneqq X \setminus U$

since U is a neighborhood of the closure of K we see that the closure of Z is contained in the interior of A, thus we can apply the Excision Theorem 40.1

■

In many occasions we will deal with manifolds and CW-complexes. In these settings the following two theorems will be useful.

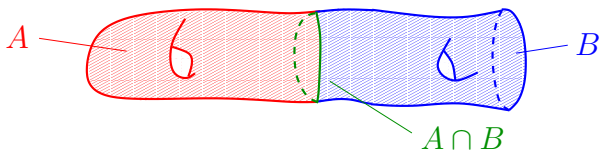
Theorem 40.3. (Manifold–Excision Theorem) Let X be an m-dimensional topological manifold and let $A, B \subset X$ be two codimension-zero submanifolds such that the following statements hold:

(a) $X = A \cup B$.
(b) $A \cap B$ is a union of components of ∂A.
(c) A and B are closed subsets of X.[319]

For every $n \in \mathbb{N}_0$ the inclusion-induced map

$$\mathrm{H}_n(B, A \cap B) \to \mathrm{H}_n(X, A)$$

is an isomorphism.



the inclusion induced map $\mathrm{H}_n(B, A \cap B) \to \mathrm{H}_n(X, A)$ is an isomorphism

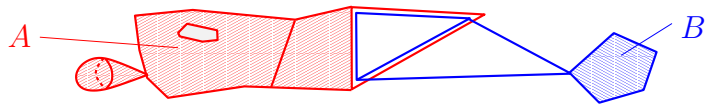
[318] It follows from the Compact-Closed Lemma 1.9 (2) that the hypotheses are for example satisfied if X is Hausdorff (e.g. if X is a topological manifold) and if K is a compact subset that is contained in an open subset U.

[319] Note that by the Compact-Closed Lemma 1.9 (2) this condition is satisfied if A and B are compact.

Theorem 40.4. (CW-Excision Theorem) Let X be a CW-complex and let A and B be two subcomplexes of X with $X = A \cup B$. For every $n \in \mathbb{N}_0$ the inclusion-induced map

$$\mathrm{H}_n(B, A \cap B) \xrightarrow{\cong} \mathrm{H}_n(X, A)$$

is an isomorphism.



the inclusion induced map $\mathrm{H}_n(B, A \cap B) \to \mathrm{H}_n(X, A)$ is an isomorphism

The proofs of the Manifold–Excision Theorem 40.3 and the CW-Excision Theorem 40.4 both rely on the following somewhat technical theorem.

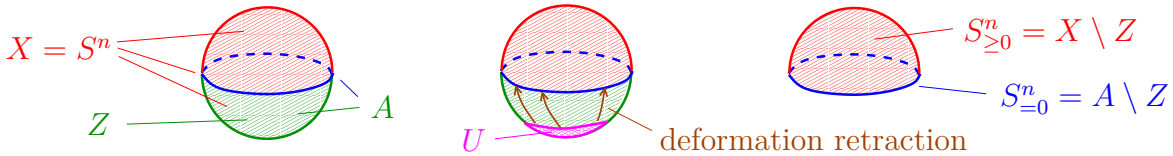
Theorem 40.5. (Excision-via-Deformation Theorem) Let X be a topological space and let $Z \subset A \subset X$ be two subsets. We suppose that there exists a subset $U \subset Z$ such that the closure of U is contained in the interior of A and such that the following statements hold:

(1) $A \setminus Z$ is a deformation retract of $A \setminus U$.
(2) $X \setminus Z$ is a deformation retract of $X \setminus U$

Then the inclusion $(X \setminus Z, A \setminus Z) \to (X, A)$ induces for each $n \in \mathbb{N}_0$ an isomorphism

$$\mathrm{H}_n(X \setminus Z, A \setminus Z) \xrightarrow{\cong} \mathrm{H}_n(X, A).$$

Example. Let $n \in \mathbb{N}$. As usual we denote by $S^n_{\geq 0}$ and $S^n_{\leq 0}$ the upper and lower hemisphere of S^n and we write $S^n_{=0} = S^n_{\geq 0} \cap S^n_{\leq 0}$.



It follows from the Excision-via-Deformation Theorem 40.5 and the choices of X, A, Z and U illustrated in the figure above that for each $k \in \mathbb{N}_0$ the inclusion-induced map

$$\mathrm{H}_k(S^n_{\geq 0}, S^n_{=0}) \to \mathrm{H}_k(S^n, S^n_{\leq 0})$$

is an isomorphism. □

Proof of Theorem 40.5 assuming the Excision Theorem 40.1. We consider the following commutative diagram of inclusion-induced maps:

$$\begin{array}{ccc} \mathrm{H}_n(X \setminus Z, A \setminus Z) & \longrightarrow & \mathrm{H}_n(X, A) \\ \downarrow & \nearrow_{\cong} & \\ \mathrm{H}_n(X \setminus U, A \setminus U). & & \end{array}$$

Since by hypothesis we know that $\overline{U} \subset \mathring{A}$ we obtain from the Excision Theorem 40.1 that the diagonal map is an isomorphism. Furthermore it follows from our hypothesis (1) and (2) together with the Homotopy Equivalence-Relative-H_*-Corollary 38.4 that the vertical map is an isomorphism. It follows that the top horizontal map is an isomorphism. ■

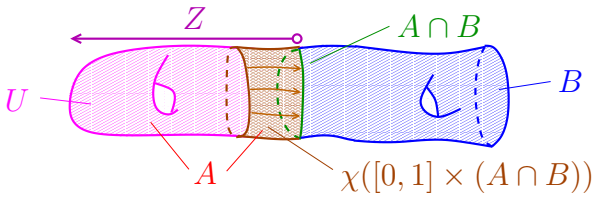
Proof of Theorem 40.3 assuming Theorem 40.5. Let X be an m-dimensional topological manifold and let $A, B \subset X$ be two codimension-zero submanifolds such that the following statements hold:

(a) $X = A \cup B$.
(b) $A \cap B$ is a union of components of ∂A.
(c) A and B are closed subsets of X.

It follows from (b) together with the Collar Neighborhood Theorem 23.2 that there exists a continuous map $\chi\colon [0,1] \times (A \cap B) \to A$ with the following three properties:

(d) The map χ is an embedding.
(e) The image of $[0,1) \times (A \cap B)$ is an open subset of A.
(f) For every $P \in A \cap B$ we have $\chi(0, P) = P$.

We set $Z := X \setminus B$ and we set $U := X \setminus (B \cup \chi([0,1) \times (A \cap B))) = A \setminus \chi([0,1) \times (A \cap B))$.



Claim.

(0) The closure of U is contained in the interior of A.
(1) $A \setminus Z = A \cap B$ is a deformation retract of $A \setminus U = \chi([0,1] \times (A \cap B))$.
(2) $X \setminus Z = B$ is a deformation retract of $X \setminus U = (\chi([0,1] \times (A \cap B)) \cup B$.

Proof.

(0) It follows from (b) that $A \setminus (A \cap B) = X \setminus B$ is an open subset of X. Furthermore it follows from (c), (e), (f) and the Cover-Open Subset Lemma 1.5 (2b) that U is a closed subset of X. It follows from this discussion with the fact that $U \subset X \setminus B$ that the closure of U is contained in the interior of A.
(1) We have an obvious deformation retraction from $[0,1] \times (A \cap B)$ to $\{0\} \times (A \cap B)$. It follows from (d) and (f) that this deformation retraction gives rise to a deformation retraction from $\chi([0,1] \times (A \cap B))$ to $A \cap B$.
(2) It follows from (c) and the Homotopy Combination Lemma 8.2 that we can extend the deformation retraction from (2) in an obvious way to the desired deformation retraction from $X \setminus Z$ to $X \setminus U$. ⊞

It follows from the claim that we can apply the Excision-via-Deformation Theorem 40.5 which tells us that the inclusion-induced map

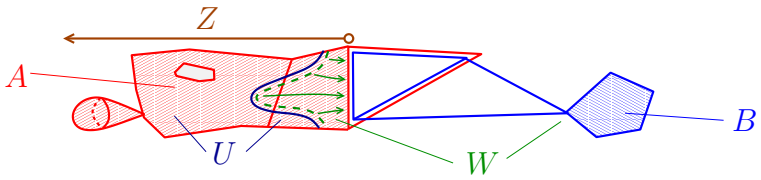
$$\mathrm{H}_n(\underbrace{X \setminus Z}_{=B}, \underbrace{A \setminus Z}_{=A \cap B}) \xrightarrow{\cong} \mathrm{H}_n(X, A)$$

is an isomorphism. But that is exactly what we needed to show. ■

Proof of the CW-Excision Theorem 40.4 assuming Theorem 40.5. Let X be a CW-complex and let A and B be two subcomplexes of X with $X = A \cup B$. Note that by the Subcomplex-Neighborhood Proposition 31.8, applied to the subcomplex B, there exists an open neighborhood W of B and a deformation retraction $F\colon W \times [0,1] \to W$ from

W to B with the extra property that the map F restricts to a deformation retraction $F\colon (A\cap W)\times[0,1]\to A\cap W$ from $A\cap W$ to $A\cap B$.

We set $Z\coloneqq X\setminus B$ and we set $U\coloneqq X\setminus W$. Since W is an open neighborhood of B we see that the closure of U is contained in the interior of A.



Recall that F is a deformation retraction from $W=X\setminus U$ to $B=X\setminus Z$ and that F restricts to a deformation retraction from $A\cap W=A\setminus U$ to $A\cap B=A\setminus Z$. Therefore it follows from the Excision-via-Deformation Theorem 40.5 that the inclusion-induced map

$$\mathrm{H}_n(\underbrace{X\setminus Z}_{=B},\underbrace{A\setminus Z}_{=A\cap B})\xrightarrow{\cong}\mathrm{H}_n(X,A)$$

is an isomorphism. But that is exactly what we needed to show. ∎

40.3. **The proof of the Excision Theorem 40.1: The idea.** In this section we want to come up with the idea for the proof of the Excision Theorem 40.1. The full details of the proof will then be given in the following section.

Thus let X be a topological space and let $Z\subset A\subset X$ be two subsets such that the closure of Z is contained in the interior of A. Our goal is to show that the inclusion $(X\setminus Z,A\setminus Z)\to(X,A)$ induces for every $n\in\mathbb{N}_0$ an isomorphism

$$\mathrm{H}_n(X\setminus Z,A\setminus Z)\ \to\ \mathrm{H}_n(X,A).$$

Let us attempt to prove that the map $\mathrm{H}_n(X\setminus Z,A\setminus Z)\to\mathrm{H}_n(X,A)$ is surjective. We take an element in $\mathrm{H}_n(X,A)=\mathrm{H}_n(\mathrm{C}_*(X,A))=\mathrm{H}_n(\mathrm{C}_*(X)/\mathrm{C}_*(A))$ and we choose a representative $a_1\cdot\sigma_1+\cdots+a_r\cdot\sigma_r\in\mathrm{C}_n(X)$ where $\sigma_i\colon\Delta^n\to X$, $i=1,\dots,r$ are singular n-simplices in X.

We make the following two basic observations:

(1) If the singular n-simplices $\sigma_1,\dots,\sigma_r$ lie in $X\setminus Z$,[320] then $[a_1\cdot\sigma_1+\cdots+a_r\cdot\sigma_r]$ evidently lies in the image of the inclusion-induced map $\mathrm{H}_n(X\setminus Z,A\setminus Z)\to\mathrm{H}_n(X,A)$.
(2) If a singular n-simplex lies entirely in A, then it represents the zero element in the quotient group $\mathrm{C}_n(X,A)=\mathrm{C}_n(X)/\mathrm{C}_n(A)$.

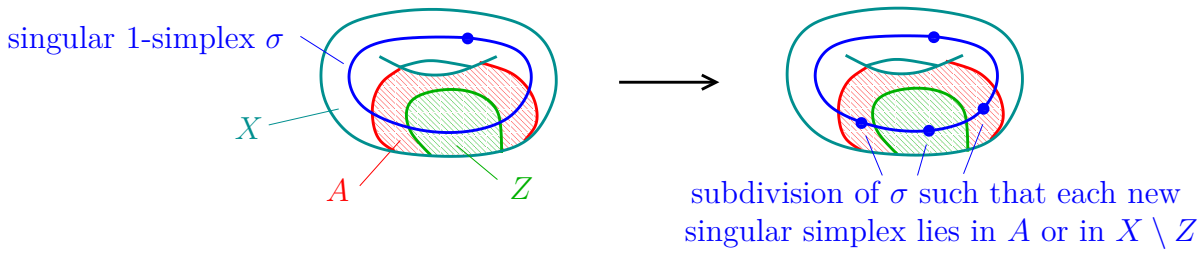
The combination of these two observations proves the following lemma:

Lemma 40.6. Let X be a topological space and let $Z\subset A$ be two subsets of X. Let $c=a_1\cdot\sigma_1+\cdots+a_r\cdot\sigma_r$ be a cycle in X such that each σ_i lies entirely in A or it lies entirely in $X\setminus Z$. Then $[c]$ lies in the image of $\mathrm{H}_n(X\setminus Z,A\setminus Z)\to\mathrm{H}_n(X,A)$.

Unfortunately not every cycle in $\mathrm{C}_n(X)$ satisfies the hypothesis of the lemma. But we will see in the next section that we can subdivide each singular simplex into smaller singular

[320] Given a topological space X and a subset A we say that *a singular simplex* $\sigma\colon\Delta^n\to X$ *lies in* A if the image of σ lies in A.

simplices that lie entirely in A or lie entirely in $X \setminus Z$. This then implies that the inclusion-induced map $\mathrm{H}_n(X \setminus Z, A \setminus Z) \to \mathrm{H}_n(X, A)$ is an epimorphism. A very similar argument then also shows that the map is a monomorphism.



40.4. **The proof of the Excision Theorem 40.1: The full details.** In this chapter we will now turn the vague idea of the last section into a rigorous proof of the Excision Theorem 40.1. This will take a surprisingly long time.

First we introduce a few definitions.

Definition. Let again X be a topological space.

(1) On page 58 we defined a cover of X to be a family $\{U_i\}_{i\in I}$ of subsets of X such that $X = \bigcup_{i\in I} U_i$.

(2) We say that a cover $\{U_i\}_{i\in I}$ is comfortable if $X = \bigcup_{i\in I} \mathring{U}_i$, i.e. if the interiors of the U_i also form a cover of X.

We move straight to the next definition.

Definition. Let again X be a topological space. Given a comfortable cover $\mathcal{U} = \{U_i\}_{i\in I}$ we consider the following subgroup of $\mathrm{C}_n(X)$:

$$\mathrm{C}_n^{\mathcal{U}}(X) := \left\{ \sum_{j=1}^{k} a_j \cdot \sigma_j \;\middle|\; \text{for each } j \text{ there exists a } U_i \in \mathcal{U} \text{ such that the image of } \sigma_j \text{ lies in } U_i \right\}.$$

It follows quite easily from the definitions that the restriction of the usual boundary map $\partial_n \colon \mathrm{C}_n(X) \to \mathrm{C}_{n-1}(X)$ to $\mathrm{C}_n^{\mathcal{U}}(X)$ restricts to a map $\partial_n \colon \mathrm{C}_n^{\mathcal{U}}(X) \xrightarrow{\partial} \mathrm{C}_{n-1}^{\mathcal{U}}(X)$. Since this map is the restriction of the usual boundary map it follows in particular that the composition $\partial_n \circ \partial_{n+1} \colon \mathrm{C}_{n+1}^{\mathcal{U}}(X) \xrightarrow{\partial} \mathrm{C}_{n-1}^{\mathcal{U}}(X)$ is the zero map. Thus we can define

$$\begin{aligned} \mathrm{H}_n^{\mathcal{U}}(X) &:= \mathrm{H}_n\big(\mathrm{C}_*^{\mathcal{U}}(X), \partial_*\big) \\ &= \ker\big(\partial_n \colon \mathrm{C}_n^{\mathcal{U}}(X) \to \mathrm{C}_{n-1}^{\mathcal{U}}(X)\big) \,/\, \mathrm{im}\,\big(\partial_{n+1} \colon \mathrm{C}_{n+1}^{\mathcal{U}}(X) \to \mathrm{C}_n^{\mathcal{U}}(X)\big). \end{aligned}$$

The homology groups $\mathrm{H}_n^{\mathcal{U}}(X)$ are thus defined by singular chains which are "small" in the sense that the image of each singular simplex lies in some U_i.

The following proposition says that we can determine the homology of a topological space through such "small" singular chains. As we will see later on many occasions, this proposition is at the heart of many methods for computing homology groups.

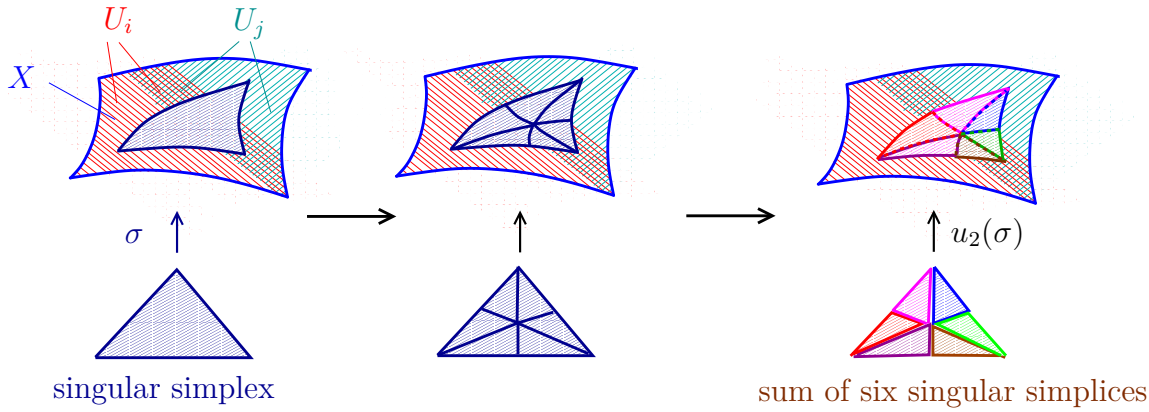
Proposition 40.7. (Comfortable Cover Proposition) Let X be a topological space and let $\mathcal{U} = \{U_i\}_{i\in I}$ be a comfortable cover of X. The inclusion map

$$\mathrm{C}_*^{\mathcal{U}}(X) \;\to\; \mathrm{C}_*(X)$$

induces for every $n \in \mathbb{N}_0$ an isomorphism

$$\mathrm{H}_n^{\mathcal{U}}(X) \;\xrightarrow{\cong}\; \mathrm{H}_n(X).$$

The idea of the proof of the Comfortable Cover Proposition 40.7 is the one we already described in the previous section. We want to replace systematically each singular n-simplex $\sigma\colon \Delta^n \to X$ by a singular n-chain of the form $\sum_{j=1}^{r} a_j \cdot \sigma_j$ such that the image of each singular simplex σ_j lies in some $U_i \in \mathcal{U}$. As we have seen before in the Singular 1-Simplex Lemma 35.13 in the 1-dimensional case, an appropriate subdividing procedure does not change the homology class. The systematic approach we are going to use for the general case is given by "barycentric subdivision". The key idea is sketched in the figure below for the case $n = 2$.



Before we can execute this idea we have to introduce a few more definitions.

(1) Given $n \in \mathbb{N}_0$ we denote by $B_n := (\frac{1}{n+1}, \dots, \frac{1}{n+1}) \in \Delta^n$ the *barycenter*[321] of the standard n-simplex Δ^n.

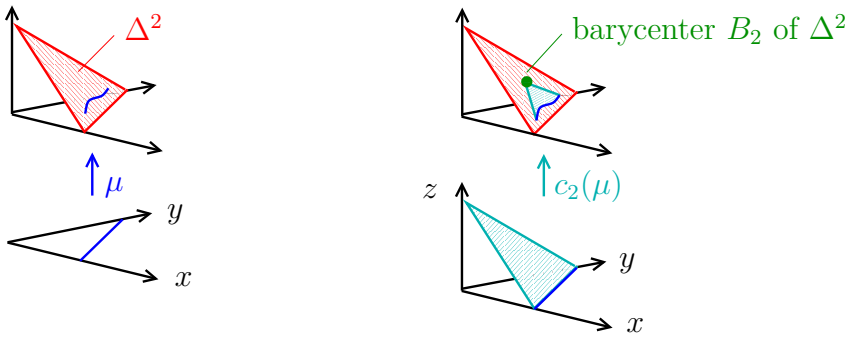
(2) Given a singular l-simplex $\mu\colon \Delta^l \to \Delta^n$ we consider the singular $(l+1)$-simplex $c_{l+1}(\mu)\colon \Delta^{l+1} \to \Delta^n$ which is defined as the composition of the (inverses) of the following two maps:

$$\begin{array}{rcccl} & & \text{homeomorphism by the Compact-Hausdorff Proposition 2.10 (3)} & & \text{continuous by the Topological-Quotient Proposition 6.1 (1b)} \\ & & \downarrow \cong & & \downarrow \\ c_{l+1}(\mu)\colon \Delta^{l+1} & \xleftarrow{} & (\Delta^l \times [0,1])/\Delta^l \times \{1\} & \to & \Delta^n \\ ((t_0,\dots,t_l)\cdot(1-s),s) & \mapsto & [((t_0,\dots,t_l),s)] & & \\ & & [((t_0,\dots,t_l),s)] & \mapsto & \underbrace{\mu(t_0,\dots,t_l)\cdot(1-s)+B\cdot s.}_{\text{contained in } \Delta^n \text{ since } \Delta^n \text{ is convex}} \end{array}$$

Pictorially speaking the singular $(l+1)$-simplex $c_{l+1}(\mu)$ thus corresponds to the "cone" which is given by the singular l-simplex μ and the barycenter B_n. Note that by the Free Abelian Group–Existence Lemma 18.4 this map $\mu \mapsto c_{l+1}(\mu)$ defines a linear map $\mathrm{C}_l(\Delta^n) \to \mathrm{C}_{l+1}(\Delta^n)$ that we also denote by c_{l+1} and which we call the $(l+1)$-st cone map.

(3) For $n = 0, 1, 2, 3, \dots$ we inductively do the following:
   (a) we define an element $\chi_n \in \mathrm{C}_n(\Delta^n)$, and
   (b) for each topological space X we define a homomorphism $u_n\colon \mathrm{C}_n(X) \to \mathrm{C}_n(X)$.

[321] The word "barycenter" comes from the Greek word "barys" which means heavy. In physics the barycenter of an object is the center of mass, in our case the barycenter of the standard n-simplex is the center of mass if we view Δ^k as a physical object of uniform density.



We do this inductive procedure as follows:

(i) We set $\chi_0 = \mathrm{id}\colon \Delta^0 \to \Delta^0$ and for each topological space X we define u_0 to be the identity on $\mathrm{C}_0(X)$.

(ii) Now we suppose that $\chi_{n-1} \in \mathrm{C}_{n-1}(\Delta^{n-1})$ is already defined and we suppose that the homomorphisms $u_{n-1}\colon \mathrm{C}_{n-1}(X) \to \mathrm{C}_{n-1}(X)$ are already defined. Then we set

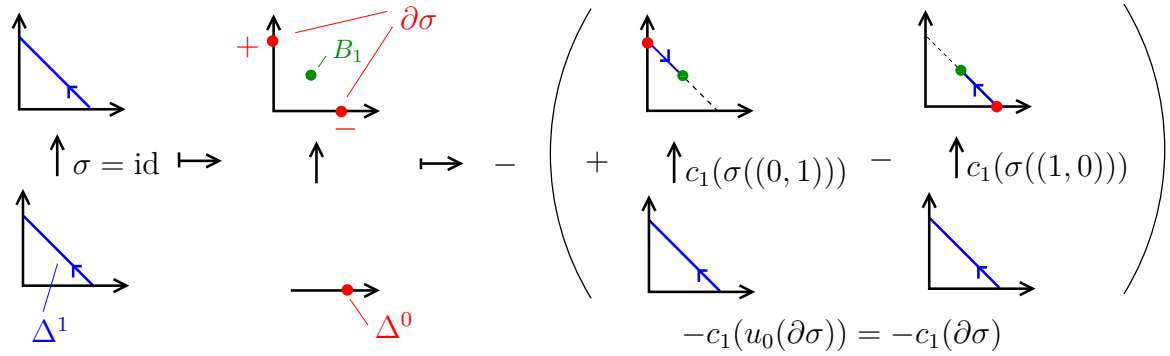
$$\chi_n \;:=\; (-1)^n \cdot c_n(u_{n-1}(\underbrace{\partial\,\mathrm{id}_{\Delta^n}}_{\in \mathrm{C}_{n-1}(\Delta^n)})) \;\in\; \mathrm{C}_n(\Delta^n).$$

Furthermore, for each topological space X we define

$$\begin{array}{rcl} u_n\colon \mathrm{C}_n(X) & \to & \mathrm{C}_n(X) \\ \sum_{l=1}^{r} a_l \cdot (\sigma_l\colon \Delta^n \to X) & \mapsto & \sum_{l=1}^{r} a_l \cdot (\sigma_l)_*(\underbrace{\chi_n}_{\in \mathrm{C}_n(\Delta^n)}). \end{array}$$

map $\sigma_l\colon \Delta^l \to X$ induces a map $(\sigma_l)_*\colon \mathrm{C}_n(\Delta^l) \to \mathrm{C}_n(X)$

Note that with this notation we have $u_n(\mathrm{id}_{\Delta^n}) = \chi_n$. The map u_n is illustrated for $n = 1$ in the figure below. For $n = 2$ we already gave a more schematic sketch in the figure above without considering signs and orientations.



Before we can continue we need to prove several technical lemmas about the subdivision maps. We start out with a straightforward lemma.

Lemma 40.8. (Subdivision–Natural Lemma) The maps $u_n\colon \mathrm{C}_n(X) \to \mathrm{C}_n(X)$ are natural,[322] in particular for any continuous map $f\colon X \to Y$ between topological spaces

and any $n \in \mathbb{N}_0$ we have

$$f_* \circ u_n \;=\; u_n \circ f_* \colon \mathrm{C}_n(X) \to \mathrm{C}_n(Y).$$

Proof. This lemma follows immediately from the definitions. Indeed, let $\sigma \colon \Delta^n \to X$ be a singular n-simplex. Then

$$\begin{array}{ccccc} f_*(u_n(\sigma)) & = & f_*(\sigma_*(\chi_n)) & = & (f \circ \sigma)_*(\chi_n) \quad = \quad u_n(f_*(\sigma)). \\ \uparrow & & \uparrow & & \uparrow \\ \text{by definition of } u_n(\sigma) & & \text{since } (f\circ\sigma)_* = f_* \circ \sigma_* & & \text{by definition of } u_n(f_*(\sigma)) \\ & & \text{as maps } \mathrm{C}_n(\Delta^n) \to \mathrm{C}_n(Y) & & \end{array}$$

Since the desired equality holds for a generating set of $\mathrm{C}_n(X)$ it also holds for all of $\mathrm{C}_n(X)$. ■

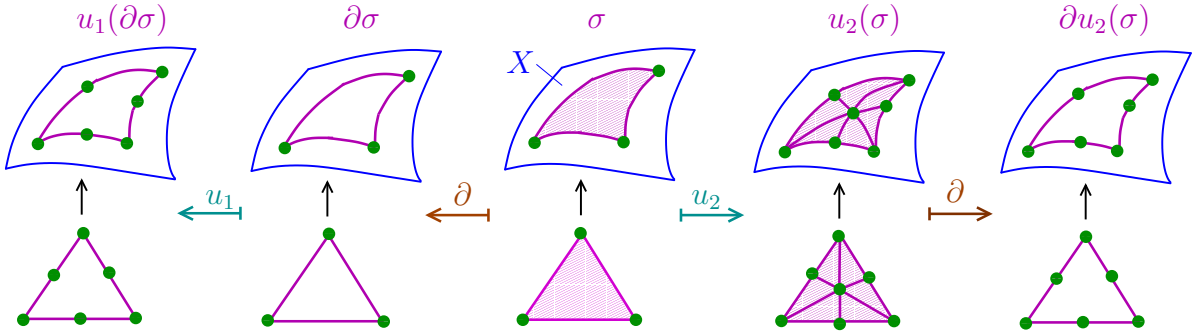
We continue with a lemma which is much harder to prove.

Lemma 40.9. (Chain Subdivision Lemma) Let X be a topological space. The maps $u_n \colon \mathrm{C}_n(X) \to \mathrm{C}_n(X)$, $n \in \mathbb{N}_0$, form a chain map from the chain complex $(\mathrm{C}_*(X), \partial)$ to itself.

The statement of the Chain Subdivision Lemma 40.9 is equivalent to the statement that for every singular n-simplex σ in X we have

$$u_{n-1}(\partial\sigma) \;=\; \partial(u_n(\sigma)).$$

Put differently, the Chain Subdivision Lemma 40.9 says that it does not make a difference whether we first take the boundary and then subdivide, or whether we first subdivide and then take the boundary. This statement is sketched in the figure below.



Proof of the Chain Subdivision Lemma 40.9. [323] We adopt the following notation and we make the following remarks:

(1) We denote the barycenter of Δ^n by b.

[322] Let us explain in what sense these maps deserve to be called "natural". Given $n \in \mathbb{N}_0$ we consider the functor C_* from the category $\mathcal{T}op$ of topological spaces to the category $\mathcal{ChCplx}$ of chain complexes which is given by

$$\begin{array}{rcl} \text{topological space } X & \mapsto & \mathrm{C}_*(X) \\ \text{and} \qquad (f \colon X \to Y) & \mapsto & (f_* \colon \mathrm{C}_*(X) \to \mathrm{C}_*(Y)). \end{array}$$

Given a topological space X we denote by $\Phi_X \colon \mathrm{C}_*(X) \to \mathrm{C}_*(X)$ the map that is given by the n-th subdivision map that we defined on page 511. Lemma 40.9 and the Subdivision–Natural Lemma 40.8 say that these maps define a natural transformation from the functor C_* to itself.

[323] We will not do this proof in the lectures. The proof is somewhat lengthy but straightforward. The cleverness hereby does not lie in the proof, but in the cleverly chosen definitions, which turn the verification into a simple calculation.

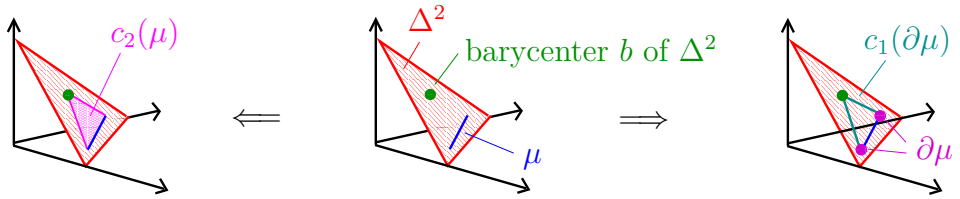
(2) Given points $x_0, \dots, x_k \in \Delta^n$ we denote by $[x_0, \dots, x_k]$ the singular k-chain $\Delta^k \to \Delta^n$ that is given by[324]
$$\begin{array}{ccc} \Delta^k & \to & \Delta^n \\ (t_0, \dots, t_k) & \mapsto & \sum_{i=0}^{k} t_i \cdot x_i, \end{array}$$
we refer to such a singular k-simplex as a *linear* singular simplex.

(3) We denote by $\mathrm{C}_k^{\mathrm{lin}}(\Delta^n)$ the subgroup of $\mathrm{C}_k(\Delta^n)$ generated by all the linear singular k-simplices. Note that the boundary map $\partial\colon \mathrm{C}_k(\Delta^n) \to \mathrm{C}_{k-1}(\Delta^n)$ restricts to a boundary map $\partial\colon \mathrm{C}_k^{\mathrm{lin}}(\Delta^n) \to \mathrm{C}_{k-1}^{\mathrm{lin}}(\Delta^n)$.

(4) For $x_0, \dots, x_k \in \Delta^n$ we write
$$[x_0, \dots, \widehat{x_i}, \dots, x_k] \;:=\; [x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_k].$$

The proof of the lemma will be broken into three claims.

Claim 1. For any $\mu \in \mathrm{C}_k^{\mathrm{lin}}(\Delta^n)$ we have
$$\partial(c_{k+1}(\mu)) \;=\; c_k(\partial\mu) + (-1)^{k+1} \cdot \mu \;\in\; \mathrm{C}_k(\Delta^n).$$



Proof. To prove the claim we first note that for any choice of points $x_0, \dots, x_{l-1} \in \Delta^n$ we have by definition of c_l that
$$c_l([x_0, \dots, x_{l-1}]) \;=\; [x_0, \dots, x_{l-1}, b].$$
It follows that for any $w_0, \dots, w_k \in \Delta^n$ we have the following equality in $\mathrm{C}_k(\Delta^n)$:
$$\begin{aligned} \partial(c_{k+1}([w_0, \dots, w_k])) &\overset{\text{by the above equality}}{\downarrow =} \partial([w_0, \dots, w_k, b]) \\ &= \sum_{j=0}^{k} (-1)^j \cdot [w_0, \dots, \widehat{w_j}, \dots, w_k, b] + (-1)^{k+1} \cdot [w_0, \dots, w_k] \\ &\overset{\text{definition of } \partial \text{ and by the above equality}}{\uparrow =} c_k(\partial([w_0, \dots, w_k])) + (-1)^{k+1} \cdot [w_0, \dots, w_k]. \end{aligned}$$
The claim immediately follows from this equality. ⊞

Claim 2. For any $n \in \mathbb{N}_0$ and any $\nu \in \mathrm{C}_n^{\mathrm{lin}}(\Delta^n)$ we have
$$\partial u_n(\nu) \;=\; u_{n-1}(\partial\nu).$$

Proof. We will prove this claim by induction on n. The case $n = 0$ is trivial. So now suppose we have already shown the case $n - 1$. It suffices to show the claim for all linear

[324] It is straightforward to show that this map does indeed take values in Δ^n.

singular n-simplices. So let $\nu\colon \Delta^n \to \Delta^n$ be a linear singular n-simplex. We then have

$$\begin{aligned}
\partial u_n(\nu) &\overset{\text{definition of } u_n \text{ and } \chi_n, \text{ see page 511}}{=} \partial(\nu_*(\chi_n)) \overset{\downarrow}{=} (-1)^n \cdot \partial(\nu_*(c_n(u_{n-1}(\partial\,\mathrm{id}_{\Delta^n})))) \overset{\text{since } \partial\circ\nu_* = \nu_*\circ\partial}{\overset{\downarrow}{=}} (-1)^n \cdot \nu_*\big(\partial(c_n(\underbrace{u_{n-1}(\partial\,\mathrm{id}_{\Delta^n})}_{=:\mu}))\big)\\
&\overset{\uparrow}{=} (-1)^n \cdot \nu_*\big(c_{n-1}(\partial(u_{n-1}(\partial\,\mathrm{id}_{\Delta^n})) + (-1)^n \cdot u_{n-1}(\partial\,\mathrm{id}_{\Delta^n})\big)\\
&\overset{\text{we can apply the previous claim since } \mu := u_{n-1}(\partial\,\mathrm{id}_{\Delta^n}) \in \mathrm{C}^{\mathrm{lin}}_{n-1}(\Delta^n)}{\overset{\uparrow}{=}} (-1)^n \cdot \nu_*\big(c_{n-1}(u_{n-2}(\underbrace{\partial\partial\,\mathrm{id}_{\Delta^n}}_{=0}))\big) \;+\; \nu_*(u_{n-1}(\partial\,\mathrm{id}_{\Delta^n}))\\
&\overset{\uparrow}{=} u_{n-1}(\nu_*(\partial\,\mathrm{id}_{\Delta^n})) \overset{\uparrow}{=} u_{n-1}(\partial\nu).
\end{aligned}$$

by induction hypothesis we have $\partial u_{n-1}(\rho) = u_{n-2}(\partial\rho)$ for $\rho \in \mathrm{C}^{\mathrm{lin}}_{n-1}(\Delta^n)$, applied here to $\rho = \partial\,\mathrm{id}_{\Delta^n}$

Subdivision–Natural Lemma 40.8 applied to $f = \nu$ — by definition

⊞

The following claim is just a reformulation of the statement of the lemma.

Claim 3. For any singular n-simplex σ in X we have $u_{n-1}(\partial\sigma) = \partial(u_n(\sigma))$.

Proof. So let $\sigma\colon \Delta^n \to X$ be a singular n-simplex in X. Then we have

$$\begin{aligned}
\partial(u_n(\sigma)) &= \partial(u_n(\sigma_*(\mathrm{id}_{\Delta^n}))) \overset{\downarrow}{=} \partial(\sigma_*(u_n(\mathrm{id}_{\Delta^n}))) \overset{\downarrow}{=} \sigma_*(\partial(u_n(\mathrm{id}_{\Delta^n})))\\
&\overset{\uparrow}{=} \sigma_*(u_{n-1}(\partial\,\mathrm{id}_{\Delta^n})) \overset{\uparrow}{=} u_{n-1}(\sigma_*(\partial\,\mathrm{id}_{\Delta^n})) \overset{\uparrow}{=} u_{n-1}(\partial\sigma_*(\mathrm{id}_{\Delta^n})) = u_{n-1}(\partial\sigma).
\end{aligned}$$

by the previous claim — by the Subdivision–Natural Lemma 40.8 applied to $f=\sigma$ — since σ_* is a chain map

■

The proof of the next lemma is again rather involved.

Lemma 40.10. (Subdivision–Chain Homotopy-Lemma) Let X be a topological space. The chain maps

$$u_n\colon \mathrm{C}_n(X) \to \mathrm{C}_n(X)$$

are chain homotopic to the identity.

Proof. Showing that the chain maps $(u_n)_{n\in\mathbb{N}}$ are chain homotopic to the identity requires us to find maps $P_n\colon \mathrm{C}_n(X) \to \mathrm{C}_{n+1}(X), n \in \mathbb{N}_0$ such that for each $n \in \mathbb{N}_0$ the following equality holds:

$$\partial_{n+1}\circ P_n = -P_{n-1}\circ\partial_n + u_n - \mathrm{id}\,.$$

First we consider the case that X is a convex subset of some $\mathbb{R}^m$. Recall that we defined $\mathrm{C}_{-1}(X) = 0$, accordingly we define $P_{-1}\colon \mathrm{C}_{-1}(X) \to \mathrm{C}_0(X)$ to be the zero map. Let us suppose that we have already found maps $P_{-1}, \dots, P_{n-1}$ with the desired properties.

We want to define the next map $P_n\colon \mathrm{C}_n(X) \to \mathrm{C}_{n+1}(X)$. This map has to have the property that for each singular n-simplex σ we have

$$\partial_{n+1}(P_n(\sigma)) = -(P_{n-1}\circ\partial_n)(\sigma) + (u_n - \mathrm{id})(\sigma).$$

Put differently, to define $P_n(\sigma)$ we have to find a singular $(n+1)$-chain $E_\sigma \in \mathrm{C}_{n+1}(X)$ with

$$\partial E_\sigma = -(P_{n-1}\circ\partial)(\sigma) + (u_n - \mathrm{id})(\sigma),$$

so that we can then define $P_n(\sigma) := E_\sigma$. Thus let $\sigma\colon \Delta^n \to X$ be a singular n-simplex.

(1) For $n = 0$ we have

$$\big(\underbrace{P_{-1}}_{=0}\circ\partial_0 + \underbrace{u_0}_{=\mathrm{id}} - \mathrm{id}\big)(\sigma) = 0 \in \mathrm{C}_0(X),$$

and we put $E_\sigma := 0 \in \mathrm{C}_1(X)$.

(2) Now suppose that $n > 0$. Since X is convex we know from the discussion on page 473 that $\mathrm{H}_n(X) = 0$. Therefore to show that an element $c \in \mathrm{C}_n(X)$ is the boundary of some element in $\mathrm{C}_{n+1}(X)$ it suffices to show that $\partial c = 0$. In our specific situation we calculate

$$\begin{aligned} \partial\big(\,(-P_{n-1}\partial_n + u_n - \mathrm{id})\,(\sigma)\big) &\overset{\text{by the Chain Subdivision Lemma 40.9 the subdivision maps form a chain map, i.e. } \partial \circ u_n = u_{n-1} \circ \partial}{=} -\partial P_{n-1}\partial\sigma + u_{n-1}\partial\sigma - \partial\sigma \\ &= \underbrace{(-\partial P_{n-1} + u_{n-1} - \mathrm{id})}_{=P_{n-2}\partial}(\partial\sigma) \;=\; P_{n-2}(\underbrace{\partial\partial\sigma}_{=0}) \;=\; 0. \end{aligned}$$

As we had just remarked, this implies that there exists an $E_\sigma \in \mathrm{C}_{n+1}(X)$ which satisfies $\partial(E_\sigma) = (P_{n-1}\partial_n - u_{n-1} - \mathrm{id})(\sigma)$.

Now we denote by $P_n \colon \mathrm{C}_n(X) \to \mathrm{C}_{n+1}(X)$ a map which assigns to each singular n-simplex σ an element $E_\sigma \in \mathrm{C}_{n+1}(X)$ as above. It follows from the definition of E_σ that

$$\partial_{n+1} \circ P_n + P_{n-1} \circ \partial_n \;=\; u_n - \mathrm{id}$$

as maps $\mathrm{C}_n(X) \to \mathrm{C}_n(X)$. This concludes the proof of the Subdivision–Chain Homotopy-Lemma 40.10 if X is a convex subset of some $\mathbb{R}^m$.

Now we deal with the general case. So let X be a topological space. We consider

$$\begin{aligned} P_n \colon \mathrm{C}_n(X) &\to \mathrm{C}_{n+1}(X) \\ \sum_{l=1}^{r} a_l \cdot (\sigma_l \colon \Delta^n \to X) &\mapsto \sum_{l=1}^{r} a_l \cdot (\sigma_l)_*(\underbrace{P_n(\mathrm{id}_{\Delta^n})}_{\text{defined since } \Delta^n \text{ is convex}}). \end{aligned}$$

It is straightforward to see that this is a chain homotopy from the subdivision map to the identity map. ∎

Definition. Given a topological space X and $m \in \mathbb{N}_0$ we denote by u_n^m the m-fold composition of $u_n \colon \mathrm{C}_n(X) \to \mathrm{C}_n(X)$.

Lemma 40.11. (Iterated Subdivision Lemma) For each topological space X and every $m \in \mathbb{N}$ the maps $\{u_n^m\}_{n\geq 0}$ form a chain map $\mathrm{C}_*(X) \to \mathrm{C}_*(X)$ that is chain homotopic to the identity.

Proof. Recall that in the Chain Subdivision Lemma 40.9 we already saw that the maps $\{u_n\}_{n\geq 0}$ form a chain map $\mathrm{C}_*(X) \to \mathrm{C}_*(X)$. As we remarked on page 455, the composition of chain maps is again a chain map. Thus for any $m \in \mathbb{N}_0$ the maps $\{u_n^m\}_{n\geq 0}$ are also chain maps.

By the Subdivision–Chain Homotopy-Lemma 40.10 we have $u_* \simeq \mathrm{id}$. By the Chain Homotopy-Lemma 36.2 (2) this implies $u_*^2 \simeq u_*$, and by the Chain Homotopy-Lemma 36.2 (1) this implies $u_*^2 \simeq \mathrm{id}$. An induction argument now shows that $u_*^m \simeq \mathrm{id}$ for all m. ∎

Definition. Given a singular chain c we can write it uniquely (up to reordering) as $c = a_1 \cdot \sigma_1 + \cdots + a_s \cdot \sigma_s$ with $a_1, \ldots, a_s \in \mathbb{Z} \setminus \{0\}$ and where the σ_i are pairwise different. We call $\sigma_1, \ldots, \sigma_s$ the pieces of c.

If $\sigma\colon \Delta^n \to X$ is a singular n-simplex, then the pieces of $u_n(\sigma)$ are evidently "smaller" than the pieces of σ, and by iteratively applying u_n one can hope to make the resulting pieces "arbitrary small". In the following two lemmas we will make it precise what we mean by "arbitrary small". Before we formulate the first lemma we need to recall and introduce several definitions.

Definition.

(1) Given a bounded subset A of $\mathbb{R}^{n+1}$ the diameter is defined as

$$\operatorname{diam}(A) \;:=\; \sup\big\{\|a-b\| \,\big|\, a,b\in A\big\} \in \mathbb{R}_{\geq 0}.$$

(2) Given a singular simplex $\sigma\colon \Delta^n \to X$ we denote by $|\sigma|$ the image of σ.

Definition. For points $z_0,\dots,z_k \in \mathbb{R}^n$ we consider the map

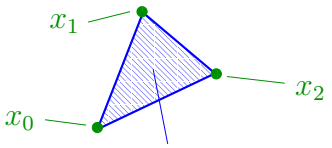
$$\begin{aligned}[z_0,\dots,z_k]\colon \Delta^k &\to \mathbb{R}^n\\ (t_0,\dots,t_k) &\mapsto \sum_{j=0}^{k} t_j\cdot z_j.\end{aligned}$$

The map $[z_0,\dots,z_k]$ thus sends the standard k-simplex Δ^k to the simplex spanned by $z_0,\dots,z_k$ in $\mathbb{R}^n$. Note that $[z_0,\dots,z_k]$ is a singular k-simplex in $\mathbb{R}^n$.

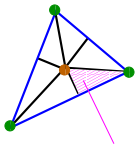
Now we can formulate the following lemma:

Lemma 40.12. (Subdivision–Diameter Lemma) For any $x_0,\dots,x_k \in \mathbb{R}^n$ and any piece τ of $u_k([x_0,\dots,x_k])$ we have

$$\operatorname{diam}(|\tau|) \;\leq\; \tfrac{n}{n+1}\cdot \operatorname{diam}(|[x_0,\dots,x_k]|).$$



the image of $[x_0,x_1,x_2]$



the image of a piece τ of $u_2([x_0,x_1,x_2])$

In the proof of the Subdivision–Diameter Lemma 40.12 we will need the following elementary[325] geometric lemma.

Lemma 40.13. (Simplex–Diameter Lemma) Let $y_0,\dots,y_k \in \mathbb{R}^n$.

(1) We have the equality

$$\operatorname{diam}(|[y_0,\dots,y_k]|) \;=\; \max\big\{\|y_i-y_j\| \,\big|\, i,j=1,\dots,k\big\}.$$

(2) We denote by $b=\frac{1}{k+1}\cdot\sum_{i=0}^{k} y_i$ the barycenter of $y_0,\dots,y_k$. For any $i=0,\dots,k$ we have

$$\|y_i-b\| \;\leq\; \tfrac{k}{k+1}\cdot \operatorname{diam}(|[y_0,\dots,y_k]|).$$

Proof of the Simplex–Diameter Lemma 40.13.

(1) The inequality "$\geq$" follows immediately from the definitions and the observation that each y_i lies in

$$|[y_0,\dots,y_k]| \;=\; \Big\{\sum_{i=0}^{k} t_i\cdot y_i \,\Big|\, t_i\in[0,1] \text{ and } \sum_{i=0}^{k} t_i = 1\Big\}.$$

[325]Here "elementary" means that the statement and the proof are basically high school level mathematics which does not mean that the proofs are entirely obvious.

We turn to the proof of the reverse inequality "$\leq$". So we pick $a, b \in |[y_0, \ldots, y_k]|$. By definition we can write $a = \sum_{i=0}^{k} s_i \cdot y_i$ and $b = \sum_{i=0}^{k} t_i \cdot y_i$ with $s_i, t_i \in [0,1]$ and such that $\sum_{i=0}^{k} s_i = \sum_{i=0}^{k} t_i = 1$. We then see that

$$\begin{array}{rcl} \|a-b\| & = & \left\|a - \sum_{i=0}^{k} t_i \cdot y_i\right\| \\ & \leq & \sum_{i=0}^{k} t_i \cdot \max_{i=0,\ldots,k}\{\|a-y_i\|\} \\ & & \uparrow \text{since } \sum_{i=0}^{k} t_i = 1 \end{array} \quad \begin{array}{rcl} & & \overset{\text{since } \sum_{i=0}^{k} t_i = 1}{\downarrow} \\ & = & \left\|\sum_{i=0}^{k} t_i \cdot (a-y_i)\right\| \leq \sum_{i=0}^{k} \|t_i \cdot (a-y_i)\| \\ & = & \max_{i=0,\ldots,k}\{\|a-y_i\|\} \leq \max_{i,j=0,\ldots,k}\{\|y_j - y_i\|\} \\ & & \text{same argument applied to } a = \sum_{j=0}^{k} s_j \cdot y_j \uparrow \end{array}$$

(2) We denote by b_i the barycenter of $y_0, \ldots, \widehat{y_i}, \ldots, y_k$, i.e.

$$b_i = \tfrac{1}{k} \cdot \sum_{j \neq i} y_j.$$

We then see that

$$b = \tfrac{1}{k+1} \cdot y_i + \tfrac{k}{k+1} \cdot b_i.$$

It follows that

$$\begin{array}{rcl} \|b-y_i\| & = & \left\|\tfrac{1}{k+1} \cdot y_i + \tfrac{k}{k+1} \cdot b_i - y_i\right\| = \left\|\tfrac{k}{k+1} \cdot b_i - \tfrac{k}{k+1} \cdot y_i\right\| \\ & & = \tfrac{k}{k+1} \cdot \|b_i - y_i\| \leq \tfrac{k}{k+1} \cdot \operatorname{diam}(|[y_0, \ldots, y_k]|). \end{array}$$

Thus we have obtained the desired inequality. ■

Now we are ready to provide the proof of the Subdivision–Diameter Lemma 40.12.

Proof of the Subdivision–Diameter Lemma 40.12. Recall that we need to show that for every $x_0, \ldots, x_k \in \mathbb{R}^n$ and every piece τ of $u_k([x_0, \ldots, x_k])$ we have

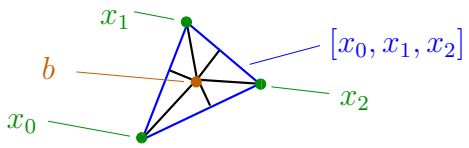
$$\operatorname{diam}(|\tau|) \leq \tfrac{n}{n+1} \cdot \operatorname{diam}(|[x_0, \ldots, x_k]|).$$

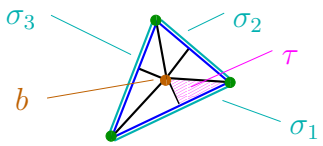
We prove the claim by induction on k. For $k = 0$ all the diameters are zero, so the inequality holds trivially. Now suppose that the claim holds for $k-1$. Let $x_0, \ldots, x_k \in \Delta^n$. We denote by $b = \frac{1}{k} \cdot \sum_{i=0}^{k} x_i$ the barycenter of $x_0, \ldots, x_k$. We denote by $\sigma_1, \ldots, \sigma_r$ the pieces of $\partial[x_0, \ldots, x_k]$.[326]

Now let τ be a piece of $u_k([x_0, \ldots, x_k])$. By the definition of the subdivision map given on page 511 we see that $|\tau| = |[w_0, \ldots, w_{k-1}, b]|$ where $[w_0, \ldots, w_{k-1}]$ is a piece of $u_{k-1}(\sigma_s)$ for some $s \in \{1, \ldots, r\}$. By the Simplex–Diameter Lemma 40.13 (1) it now suffices to prove the following claim:

Claim.
(1) For any $i, j = 0, \ldots, k-1$ we have $\|w_i - w_j\| \leq \frac{k}{k+1} \cdot \operatorname{diam}(|[x_0, \ldots, x_k]|)$.
(2) For any $i = 0, \ldots, k-1$ we have $\|w_i - b\| \leq \frac{k}{k+1} \cdot \operatorname{diam}(|[x_0, \ldots, x_k]|)$.

[326] Why does it is not say that the number of pieces is $k+1$?





Proof. First let $i, j \in \{0, \dots, k-1\}$. Then

$$\|w_i - w_j\| \leq \operatorname{diam}([w_0, \dots, w_{k-1}]) \underset{\uparrow}{\leq} \frac{k-1}{k} \cdot \operatorname{diam}(\sigma_s) \underset{\uparrow}{\leq} \frac{k}{k+1} \cdot \operatorname{diam}([x_0, \dots, x_k]).$$

induction hypothesis applied to σ_s — since $|\sigma_s| \subset |[x_0, \dots, x_k]|$ and $\frac{k-1}{k} \leq \frac{k}{k+1}$

Now let $i \in \{0, \dots, k-1\}$. We then see that

$$\|w_i - b\| \underset{\uparrow}{\leq} \frac{k}{k+1} \cdot \operatorname{diam}(|[w_0, \dots, w_{k-1}]|) \underset{\uparrow}{\leq} \frac{k}{k+1} \cdot \operatorname{diam}(|[x_0, \dots, x_k]|)$$

by the Simplex–Diameter Lemma 40.13 (2) — since $|[w_0, \dots, w_{k-1}]| \subset |[x_0, \dots, x_k]|$ ■

Lemma 40.14. (Simplex–Reduction Lemma) Let X be a topological space and let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of X. For every singular n-simplex $\sigma\colon \Delta^n \to X$ there exists an $m \in \mathbb{N}_0$ such that $u_n^m(\sigma)$ lies in $\mathrm{C}_n^{\mathcal{U}}(X)$.

Proof. Let X be a topological space, let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of X and let $\sigma\colon \Delta^n \to X$ be a singular n-simplex. By the Lebesgue Lemma 11.9 (2) there exists a $\delta > 0$ such that for any subset A of Δ^n with diameter $\operatorname{diam}(A) < \delta$ there exists an $i \in I$ such that $A \subset \sigma^{-1}(U_i)$.

We pick an $m \in \mathbb{N}$ such that $\left(\frac{n}{n+1}\right)^m < \delta$. We calculate that

$$u_n^m(\sigma) = u_n^m(\sigma_*(\mathrm{id}_{\Delta^n})) \underset{\downarrow}{=} \sigma_*(u_n^m(\mathrm{id}_{\Delta^n})) \underset{\downarrow}{=} \sigma_*\begin{pmatrix}\text{simplicial chain such that the image}\\ \text{of each simplicial simplex}\\ \text{has diameter less than } \delta\end{pmatrix}$$

$$\underset{\uparrow}{=} \sigma_*\begin{pmatrix}\text{simplicial chain such that the image}\\ \text{of each piece lies in some } \sigma^{-1}(U_i)\end{pmatrix}.$$

by the Subdivision–Natural Lemma 40.8 — by the Subdivision–Diameter Lemma 40.12 and $\left(\frac{n}{n+1}\right)^m < \delta$ — by definition of δ

But this means precisely that $u_n^m(\sigma) \in \mathrm{C}_n^{\mathcal{U}}(X)$. ■

Now we can prove the Comfortable Cover Proposition 40.7.

Proof of the Comfortable Cover Proposition 40.7. Let X be a topological space and let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of X. We want to show that the inclusion map $\iota\colon \mathrm{C}_*^{\mathcal{U}}(X) \to \mathrm{C}_*(X)$ induces for each $n \in \mathbb{N}_0$ an isomorphism

$$\iota_*\colon \mathrm{H}_n^{\mathcal{U}}(X) \xrightarrow{\cong} \mathrm{H}_n(X).$$

We first show that the map $\iota_*\colon \mathrm{H}_n^{\mathcal{U}}(X) \xrightarrow{\cong} \mathrm{H}_n(X)$ is surjective. Let $z \in \mathrm{H}_n(X)$ be given. We choose a cycle $c = \sum_{j=1}^{r} a_j \cdot \sigma_j \in \mathrm{C}_n(X)$ that represents $z \in \mathrm{H}_n(X)$. It follows from the Simplex–Reduction Lemma 40.14, applied to the singular n-simplices $\sigma_1, \dots, \sigma_r$, that there exists an $m \in \mathbb{N}_0$ such that $u_n^m(c) \in \mathrm{C}_n^{\mathcal{U}}(X)$. By the Iterated Subdivision Lemma 40.11 the maps $\{u_n^m\}_{n \in \mathbb{N}_0}$ form a chain map that is chain homotopic to the identity. By definition this means that there exists a chain homotopy $P_k\colon \mathrm{C}_k(X) \to \mathrm{C}_{k+1}(X)$, $k \in \mathbb{N}_0$. Thus we

obtain that

$$z \;=\; [c] \underset{\substack{\uparrow\\ \text{since } \partial_{n+1}P_n + P_{n-1}\partial_n \;=\; u_n^m - \mathrm{id}}}{=} \big[u_n^m(c) - P_{n-1}\overbrace{\partial_n(c)}^{=0} - \partial_{n+1}P_n(c)\big] \;=\; \big[u_n^m(c) - \partial_{n+1}P_n(c)\big] \underset{\substack{\uparrow\\ \text{equality in } \mathrm{H}_n(X)}}{=} \overbrace{[u_n^m(c)]}^{\in \iota_*(\mathrm{H}_n^{\mathcal{U}}(X))}.$$

Thus we have shown that $\iota_*\colon \mathrm{H}_n^{\mathcal{U}}(X) \xrightarrow{\cong} \mathrm{H}_n(X)$ is surjective.

The argument that the map $\iota_*\colon \mathrm{H}_n^{\mathcal{U}}(X) \xrightarrow{\cong} \mathrm{H}_n(X)$ is injective is very similar. Indeed, let $[c] \in \mathrm{H}_n^{\mathcal{U}}(X)$ be an element such that $[c] = 0 \in \mathrm{H}_n(X)$. This means that there exists an $e \in \mathrm{C}_{n+1}(X)$ with $\partial e = c$. From the Simplex–Reduction Lemma 40.14 it follows, as above, that there exists an $m \in \mathbb{N}_0$ with $u_{n+1}^m(e) \in \mathrm{C}_{n+1}^{\mathcal{U}}(X)$. Again we choose a chain homotopy $\{P_k\}_{k\in\mathbb{N}_0}$ between the chain maps $\{u_k^m\}_{k\in\mathbb{N}_0}$ and id. Then we have

$$c \;=\; \partial e \underset{\substack{\uparrow\\ \text{since } \partial P_{n+1} + P_n\partial \;=\; u_{n+1}^m - \mathrm{id}}}{=} \partial\big(u_{n+1}^m(e) - \partial P_{n+1}(e) - P_n\partial e\big) \;=\; \partial\underbrace{u_{n+1}^m(e)}_{\in \mathrm{C}_{n+1}^{\mathcal{U}}(X)} \;-\; \partial\underbrace{P_n(c)}_{\in \mathrm{C}_{n+1}^{\mathcal{U}}(X)}.$$

Thus have we have shown that c is indeed the boundary of an $(n+1)$-dimensional singular chain in $C_{n+1}^{\mathcal{U}}(X)$ i.e. $[c] = 0 \in \mathrm{H}_n^{\mathcal{U}}(X)$. Put differently, we have now proved that $\iota_*\colon \mathrm{H}_n^{\mathcal{U}}(X) \xrightarrow{\cong} \mathrm{H}_n(X)$ is injective. ∎

Using the Comfortable Cover Proposition 40.7 we can now prove the Excision Theorem 40.1.

Proof of the Excision Theorem 40.1. Let X be a topological space. Furthermore let $Z \subset A \subset X$ be subsets such that the closure of Z is contained in the interior of A. We have to show that the inclusion $(X \setminus Z, A \setminus Z) \to (X, A)$ induces for each $n \in \mathbb{N}_0$ an isomorphism

$$\mathrm{H}_n(X \setminus Z, A \setminus Z) \;\xrightarrow{\cong}\; \mathrm{H}_n(X, A).$$

We put $B \coloneqq X \setminus Z$. Note that $A \cap B = A \setminus Z$. It follows from our hypothesis on Z and A that X is the union of the interiors of A and B. Put differently, the two sets $\{A, B\}$ form a comfortable cover of X.



For each $n \in \mathbb{N}_0$ we consider the following commutative diagram of abelian groups:[327]

$$\begin{array}{ccccccccc}
 & & & & & & \mathrm{C}_n(B, A\cap B) & & \\
 & & & & & & \cong\downarrow{\scriptstyle [\sigma]\mapsto[\sigma]} & & \\
0 & \longrightarrow & \mathrm{C}_n(A) & \longrightarrow & \mathrm{C}_n^{\{A,B\}}(X) & \longrightarrow & \mathrm{C}_n^{\{A,B\}}(X)/\mathrm{C}_n(A) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{C}_n(A) & \longrightarrow & \mathrm{C}_n(X) & \longrightarrow & \mathrm{C}_n(X, A) & \longrightarrow & 0.
\end{array}$$

We make the following two observations:

(1) By definition the horizontal sequences are short exact sequences of chain complexes.

[327] Here we use the definition from page 509 which in this context says that

$$\mathrm{C}_n^{\{A,B\}}(X) \;\coloneqq\; \{\sigma + \tau \,|\, \sigma \in \mathrm{C}_n(A) \text{ and } \tau \in \mathrm{C}_n(B)\}.$$

(2) It follows immediately from the definition of the group $\mathrm{C}_n^{\{A,B\}}(X)$ that the inclusion $\mathrm{C}_n(B) \to \mathrm{C}_n^{\{A,B\}}(X)$ induces a chain map $\mathrm{C}_n(B, A \cap B) \to \mathrm{C}_n^{\{A,B\}}(X)/\mathrm{C}_n(A)$ which is in fact an isomorphism.

Now we consider the long exact sequence of homology groups arising from the above short exact sequences of chain complexes. It follows from the LES-Proposition 37.4, together with the above vertical isomorphism, that we obtain the following commutative diagram where the horizontal sequences are exact:

$$\begin{array}{ccccccccccc}
\ldots & \longrightarrow & \mathrm{H}_n(A) & \longrightarrow & \mathrm{H}_n^{\{A,B\}}(X) & \longrightarrow & \overbrace{\mathrm{H}_n(B, A \cap B)}^{=\mathrm{H}_n(X\setminus Z, A\setminus Z)} & \xrightarrow{\partial_n} & \mathrm{H}_{n-1}(A) & \longrightarrow & \mathrm{H}_{n-1}^{\{A,B\}}(X) \longrightarrow \ldots \\
 & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow \\
\ldots & \longrightarrow & \mathrm{H}_n(A) & \longrightarrow & \mathrm{H}_n(X) & \longrightarrow & \mathrm{H}_n(X, A) & \xrightarrow{\partial_n} & \mathrm{H}_{n-1}(A) & \longrightarrow & \mathrm{H}_{n-1}(X) \longrightarrow \ldots
\end{array}$$

From the Comfortable Cover Proposition 40.7 it follows that the second and fifth vertical map are isomorphisms. Since the first and fourth map are evidently isomorphisms it now follows from the Five Lemma 37.5 that the middle map is also an isomorphism. ■

CHAPTER 41

Applications of the Excision Theorem

In the previous chapter we formulated and proved the Excision Theorem 40.1. The proof turned out to be long and difficult. But in this chapter we will see that this effort really pays off. For example we will use the Excision Theorem 40.1 to compute homology groups of suspensions and wedges of topological spaces.

We will also return to the challenge, posed in Chapter 38, to calculate homology groups of quotient spaces X/A. Using the Excision Theorem 40.1 we will be able, in reasonable settings, to relate the homology groups of A, X and X/A via a long exact sequence.

41.1. **Suspensions and homology groups.** Let us recall the following definitions from pages 282.

Definition. Let X be a non-empty topological space.

(1) We define the suspension of X to be the topological space

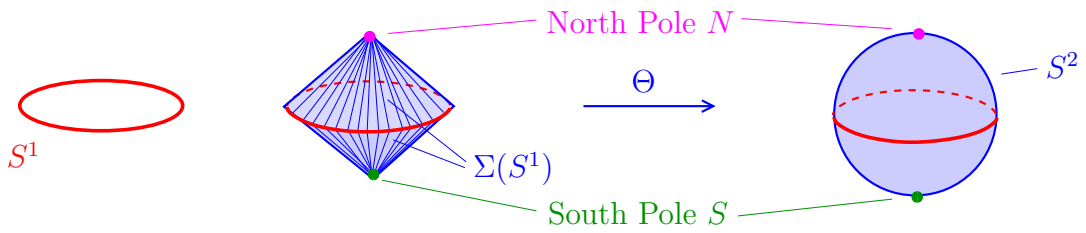
$$\Sigma(X) \;:=\; ((X \times [-1,1])/(X \times \{-1\}))/(X \times \{1\}).$$

(2) We call $S := [X \times \{-1\}]$ the South Pole and we call $N := [X \times \{1\}]$ the North Pole of $\Sigma(X)$.[328]

Example. One can easily verify that the map

$$\begin{array}{rcl} \Theta\colon \Sigma(S^n) & \xrightarrow{\cong} & S^{n+1} \subset \mathbb{R}^{n+1} \times \mathbb{R} \\ [(P,t)] & \mapsto & \big(\underbrace{P \cdot \cos\left(\frac{\pi}{2}t\right)}_{\in \mathbb{R}^{n+1}}, \underbrace{\sin\left(\frac{\pi}{2}t\right)}_{\in \mathbb{R}}\big) \end{array}$$

is a homeomorphism.



In the Suspension-π_1-Lemma 20.2 we calculated the fundamental group of the suspension of a non-empty topological space with finitely many path components. In our next proposition we will determine the homology groups of a suspension. To formulate the proposition we need to introduce the following definition:

[328] If $X = \varnothing$, then $\Sigma(X)$ consists of precisely two points, which we also call North and South Pole.

Definition.

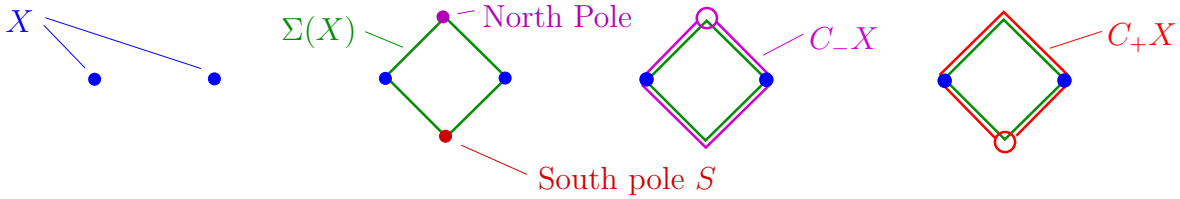
(1) We set

$$C_-X \coloneqq \Sigma(X) \setminus \{N\} \quad \text{and} \quad C_+X \coloneqq \Sigma(X) \setminus \{S\}.$$

(2) Given a continuous map $f\colon X \to Y$ between topological spaces we refer to

$$\begin{array}{rccc} \Sigma(f)\colon & \Sigma(X) & \to & \Sigma(Y) \\ & [(x,t)] & \mapsto & [(f(x),t)] \end{array}$$

as the suspension of f.



Proposition 41.1. (Suspension-H_*-Proposition) Let X be a topological space and let $n \in \mathbb{Z}$.

(1) We write $I \coloneqq (-1,1)$. The following maps are isomorphisms:

$$\widetilde{\mathrm{H}}_{n+1}(\Sigma(X)) \xrightarrow{\text{the natural map defined on page 500}} \mathrm{H}_{n+1}(\Sigma(X), C_-X) \xleftarrow{\text{induced by the inclusion}} \mathrm{H}_{n+1}(C_+X, X\times I) \xrightarrow{\partial_{n+1}} \widetilde{\mathrm{H}}_n(X\times I) \xleftarrow{\text{induced by the embedding } x\mapsto(x,0)} \widetilde{\mathrm{H}}_n(X).$$

connecting homomorphism of the pair $(C_+X, X\times I)$ from the Reduced LES-of-Pair Proposition 39.6

(2) The resulting isomorphism

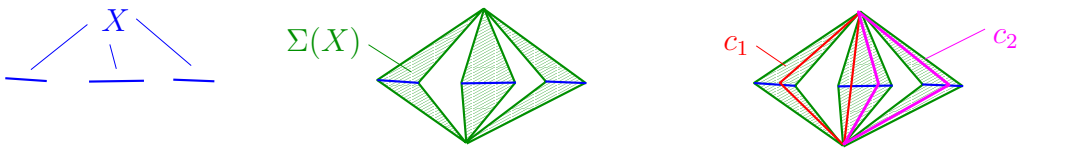
$$\Sigma_X\colon \widetilde{\mathrm{H}}_{n+1}(\Sigma(X)) \xrightarrow{\cong} \widetilde{\mathrm{H}}_n(X)$$

is natural. In other words, given any continuous map $f\colon X \to Y$ between topological spaces the following diagram commutes:

$$\begin{array}{ccc} \widetilde{\mathrm{H}}_{n+1}(\Sigma(X)) & \xrightarrow{\Sigma_X} & \widetilde{\mathrm{H}}_n(X) \\ \downarrow{\scriptstyle \Sigma(f)_*} & & {\scriptstyle f_*}\downarrow \\ \widetilde{\mathrm{H}}_{n+1}(\Sigma(Y)) & \xrightarrow{\Sigma_Y} & \widetilde{\mathrm{H}}_n(Y). \end{array}$$

Example. Let $n \in \mathbb{N}$ and let X be the disjoint union of $n \in \mathbb{N}$ intervals. It follows from the Suspension-H_*-Proposition 41.1 together with the Reduced Homology Lemma 39.2 that we have isomorphisms $\mathrm{H}_1(\Sigma(X)) \cong \widetilde{\mathrm{H}}_0(X) \cong \mathbb{Z}^{n-1}$. In the figure below we show a generating set for $\mathrm{H}_1(\Sigma(X))$ for $n = 3$.
We leave it to the reader to deduce from the proof of the Suspension-H_*-Proposition 41.1 that the given cycles do indeed form a generating set for $\mathrm{H}_1(\Sigma(X))$. □

[328] One can easily show that the maps $X\times(-1,1) \to C_\pm$ and $X\times(-1,1) \to \Sigma(X)$ given by $(x,t) \mapsto [(x,t)]$ are embeddings. We use this embedding to view $X\times(-1,1)$ as a subspace of $C_\pm X$ and of $\Sigma(X)$.



$\{c_1, c_2\}$ is a basis for $\mathrm{H}_1(\Sigma(X))$

Proof. [329]

(1) Let X be a topological space. For each $m \in \mathbb{N}_0$ we consider the following homomorphisms:

$$\begin{array}{c} \longrightarrow \widetilde{\mathrm{H}}_{n+1}(C_-X) \to \widetilde{\mathrm{H}}_{n+1}(\Sigma(X)) \xrightarrow{\cong} \mathrm{H}_{n+1}(\Sigma(X), C_-X) \xrightarrow{\partial_{n+1}} \widetilde{\mathrm{H}}_n(C_-X) \longrightarrow \dots \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \uparrow \cong \\ \dots \to \widetilde{\mathrm{H}}_{n+1}(X \times I) \to \widetilde{\mathrm{H}}_{n+1}(C_+X) \to \mathrm{H}_{n+1}(C_+X, X \times I) \xrightarrow[\cong]{\partial_{n+1}} \widetilde{\mathrm{H}}_n(X \times I) \to \widetilde{\mathrm{H}}_n(C_+X) \to \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\

(2) Let $f\colon X \to Y$ be a continuous map between two topological spaces. We consider the following diagram:

$$\begin{array}{ccccccccc}
 & & & \overset{\Sigma_X}{\frown} & & & & & \\
\widetilde{\mathrm{H}}_{n+1}(\Sigma(X)) & \xrightarrow{\cong} & \mathrm{H}_{n+1}(\Sigma(X), C_-X) & \longleftarrow & \mathrm{H}_{n+1}(C_+X, X\times I) & \xrightarrow{\partial_{n+1}} & \widetilde{\mathrm{H}}_n(X\times I) & \xleftarrow{i_*} & \widetilde{\mathrm{H}}_n(X) \\
\downarrow{\scriptstyle \Sigma(f)_*} & & \downarrow{\scriptstyle \Sigma(f)_*} & & \downarrow{\scriptstyle (f\times \mathrm{id})_*} & & \downarrow{\scriptstyle (f\times \mathrm{id})_*} & & \downarrow{\scriptstyle f_*} \\
\widetilde{\mathrm{H}}_{n+1}(\Sigma(Y)) & \xrightarrow{\cong} & \mathrm{H}_{n+1}(\Sigma(Y), C_-Y) & \longleftarrow & \mathrm{H}_{n+1}(C_+Y, Y\times I) & \xrightarrow{\partial_{n+1}} & \widetilde{\mathrm{H}}_n(Y\times I) & \xleftarrow{i_*} & \widetilde{\mathrm{H}}_n(Y). \\
 & & & \underset{\Sigma_Y}{\smile} & & & & &
\end{array}$$

The first, second and fourth square commutes by functoriality of (reduced) homology. The third square commutes since in the LES-of-Triple Proposition 38.2 (2) we showed that the connecting homomorphism is natural. We have thus shown that the diagram commutes, which implies that Σ_X itself is natural. ■

41.2. **The homology groups of spheres.** Using the Suspension-H_*-Proposition 41.1 we can now redo the calculation of the homology groups of spheres which we first did on page 504.

Proposition 41.2. (Sphere-H_*-Proposition) For any $n \in \mathbb{Z}_{\geq -1}$ we have[331]

$$\widetilde{\mathrm{H}}_k(S^n) \cong \begin{cases} \mathbb{Z}, & \text{if } k = n, \text{ and} \\ 0, & \text{if } k \neq n. \end{cases}$$

Proof. For each $m \in \mathbb{Z}_{\geq -1}$ we have

$$\widetilde{\mathrm{H}}_m(S^n) \underset{\uparrow}{\cong} \widetilde{\mathrm{H}}_m(\Sigma(S^{n-1})) \overset{\downarrow}{\cong} \widetilde{\mathrm{H}}_{m-1}(S^{n-1}) \overset{\downarrow}{\cong} \dots \overset{\downarrow}{\cong} \widetilde{\mathrm{H}}_{m-n}(S^0) \underset{\uparrow}{\cong} \begin{cases} \mathbb{Z}, & \text{if } m-n=0, \\ 0, & \text{otherwise.} \end{cases}$$

by the Suspension-H_*-Proposition 41.1; since S^n is homeomorphic to the suspension $\Sigma(S^{n-1})$; iterating the argument; by the Singleton-H_*-Lemma 35.12, the Path-Component–H_*–Lemma 35.8 and the Reduced Homology Lemma 39.2 ■

In the $\mathbb{R}^2$-$\mathbb{R}^3$-Non Homeomorphism–Lemma 13.5 we used the fundamental group to prove that $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^n$ for $n \geq 3$. Now we can finally show that for any $k \neq l$ the topological spaces $\mathbb{R}^k$ and $\mathbb{R}^l$ are not homeomorphic.

Proposition 41.3. ($\mathbb{R}^k$-$\mathbb{R}^l$-Non Homeomorphism–Proposition) For any $k \neq l$ the topological spaces $\mathbb{R}^k$ and $\mathbb{R}^l$ are not homeomorphic.

Proof. Let $k, l \in \mathbb{N}_0$ and suppose that there exists a homeomorphism $f\colon \mathbb{R}^k \to \mathbb{R}^l$. Then we obtain that

$$\mathbb{Z} \underset{\uparrow}{\cong} \widetilde{\mathrm{H}}_{k-1}(S^{k-1}) \overset{\downarrow}{\cong} \widetilde{\mathrm{H}}_{k-1}(\mathbb{R}^k \setminus \{0\}) \underset{\uparrow}{\xrightarrow{f_*}} \widetilde{\mathrm{H}}_{k-1}(\mathbb{R}^l \setminus \{f(0)\}) \overset{\downarrow}{\cong} \widetilde{\mathrm{H}}_{k-1}(S^{l-1}).$$

by the Sphere-H_*-Proposition 41.2; this follows from the Homotopy Equivalence-$\widetilde{\mathrm{H}}_*$-Corollary 39.4 (1a), since for any $P \in \mathbb{R}^m$ the topological space $\mathbb{R}^m \setminus \{P\}$ is homotopy equivalent to S^{m-1}; isomorphism since f is a homeomorphism

[331] For ordinary homology recall that by the Reduced Homology Lemma 39.2 we know that for $n \geq 0$ we have $\mathrm{H}_0(S^n) \cong \widetilde{\mathrm{H}}_0(S^n) \oplus \mathbb{Z}$ and $\mathrm{H}_k(S^n) \cong \widetilde{\mathrm{H}}_k(S^n)$ for $k > 0$.

So we have $\widetilde{\mathrm{H}}_{k-1}(S^{l-1}) \cong \mathbb{Z}$. But by the Sphere-$\mathrm{H}_*$-Proposition 41.2 that is only possible if $k = l$. ∎

We conclude this section with a calculation of a relative homology group, which we will frequently use.

Lemma 41.4. (Ball-rel-Sphere-H_*-Lemma) Given any $n \in \mathbb{N}_0$ we have

$$\mathrm{H}_k(\overline{B}^n, S^{n-1}) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } k = n, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. We consider the long exact sequence in reduced homology of the pair $(\overline{B}^n, S^{n-1})$ which is provided for us by the Reduced LES-of-Pair Proposition 39.6:

$$\ldots \to \underbrace{\widetilde{\mathrm{H}}_k(\overline{B}^n)}_{=0,\text{ see page 500}} \to \mathrm{H}_k(\overline{B}^n, S^{n-1}) \xrightarrow{\partial} \underbrace{\widetilde{\mathrm{H}}_{k-1}(S^{n-1})}_{\substack{\text{calculated in the} \\ \text{Sphere-H}_*\text{-Proposition 41.2}}} \to \underbrace{\widetilde{\mathrm{H}}_{k-1}(\overline{B}^n)}_{=0,\text{ see page 500}} \to \ldots$$

The statement of the lemma is now an immediate consequence of the Sphere-H_*-Proposition 41.2. ∎

41.3. **The Brouwer Fixed Point Theorem.** In this section we discuss a major application of the calculation of the homology groups of spheres, namely the Brouwer Fixed Point Theorem.

For the reader's convenience we recall the following definition from page 164:

Definition. Let X be a topological space. We say that a subset $A \subset X$ is a retract of X if there exists a retraction $r\colon X \to A$, i.e. a continuous map with $r(a) = a$ for all $a \in A$.

In the Circle–Non Retract Proposition 13.3 we used the fundamental group to show that S^1 is not a retract of $\overline{B}^2$. Precisely the same argument, together with the Sphere-H_*-Proposition 41.2, now gives us the following high-dimensional generalization.

Proposition 41.5. (Sphere-Non Retract Proposition) For every $n \in \mathbb{N}$ the sphere $S^{n-1} = \partial\overline{B}^n$ is not a retract of the closed ball $\overline{B}^n$.

Remark. I found the following amusing description of the Sphere-Non Retract Proposition 41.5 online:
"Suppose a shark jumps into a shoal of fish (a kind of big ball). The small fishes start escaping in all directions towards the border of the shoal, where the fishes stand still. Yet they escape with a certain disposition to follow a continuous flow, as they usually do, since everybody tends to follow its neighborhoods. But since there is no continuous retraction to the boundary, somebody doesn't know where to go, and stays there for a moment, much to the shark's satisfaction. There is also a 2D version, with a wolf entering into a herd of sheep." □

Proof. Suppose that there exists a retraction $r\colon \overline{B}^n \to S^{n-1}$. We denote by $i\colon S^{n-1} \to \overline{B}^n$ the inclusion map. We consider the following diagram:

$$\begin{array}{ccc} & \widetilde{\mathrm{H}}_{n-1}(\overline{B}^n) & \\ {\scriptstyle i_*}\nearrow & & \searrow{\scriptstyle r_*} \\ \widetilde{\mathrm{H}}_{n-1}(S^{n-1}) & \xrightarrow{(r\circ i)_*} & \widetilde{\mathrm{H}}_{n-1}(S^{n-1}). \end{array}$$

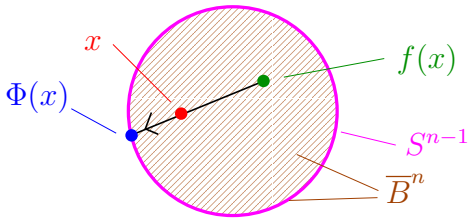
This diagram commutes since it follows from the functoriality of the reduced homology groups that $r_* \circ i_* = (r \circ i)_*$. Since r is a retraction it follows that $r \circ i = \mathrm{id}_{S^{n-1}}$, i.e. $(r \circ i)_* = (\mathrm{id}_{S^{n-1}})_*$ is the identity map on $\widetilde{\mathrm{H}}_{n-1}(S^{n-1})$ which, by the Sphere-H_*-Proposition 41.2, is isomorphic to $\mathbb{Z}$. The lower map is thus an isomorphism. But the upper map factorizes through the group $\widetilde{\mathrm{H}}_{n-1}(\overline{B}^n)$, which is the trivial group by the Contractible-$\widetilde{\mathrm{H}}_*$-Corollary 39.5. This implies that the composition of i_* and r_* cannot be an isomorphism of $\mathbb{Z}$. Thus we have obtained a contradiction. ■

Now we can formulate and prove the famous Brouwer Fixed Point Theorem 41.6.

Theorem 41.6. (Brouwer Fixed Point Theorem) Every continuous map $f\colon \overline{B}^n \to \overline{B}^n$ admits a fixed point, i.e. for every continuous map $f\colon \overline{B}^n \to \overline{B}^n$ there exists a point $x \in \overline{B}^n$ such that $f(x) = x$.

Proof. Suppose that there exists a continuous map $f\colon \overline{B}^n \to \overline{B}^n$ without a fixed point. It follows from the hypothesis that f admits no fixed points that for each $x \in \overline{B}^n$ the open ray $\{f(x) + t \cdot (x - f(x)) \,|\, t > 0\}$ which starts at $f(x)$ and which goes through x contains precisely one point in S^{n-1}. Thus we can consider the following map:

$$\begin{array}{rcl} \Phi\colon \overline{B}^n & \to & \overline{B}^n \\ x & \mapsto & \text{the intersection point of } S^{n-1} \text{ with the} \\ & & \text{uniquely determined open ray from } f(x) \text{ to } x. \end{array}$$



It is elementary to show that Φ is continuous. It is evident that $\Phi(x) = x$ for all $x \in S^{n-1}$ and that $\Phi(x) \in S^{n-1}$ for all $x \in \overline{B}^n$. The map Φ is thus a retraction from $\overline{B}^n$ to S^{n-1}. But according to the Sphere-Non Retract Proposition 41.5 this not possible. Thus we have obtained a contradiction. ■

The Brouwer Fixed Point Theorem pops up many times in mathematics. For example we can use it to prove the following theorem in linear algebra.

Theorem 41.7. (Perron-Frobenius Theorem) Let $A = (a_{ij})$ be an $n \times n$-real matrix. If all entries are positive, then A has a positive eigenvalue λ with an eigenvector v such that all entries of v are non-negative.

Proof. We need to introduce two players:

(1) Given $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$ we write $|v| := \sum_{i=1}^{n} |v_i|$.

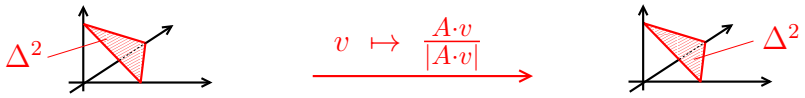
(2) We consider the standard $(n-1)$-simplex

$$\Delta^{n-1} := \{v \in [0,1]^n \,\big|\, |v| = 1\}.$$

We saw in the Δ^n-Homeomorphism Lemma 35.3 (2) that Δ^{n-1} is homeomorphic to $\overline{B}^{n-1}$. It follows immediately from this observation and from the Brouwer Fixed Point Theorem 41.6 that any continuous self-map $f\colon \Delta^{n-1} \to \Delta^{n-1}$ has a fixed point.

Now let $A = (a_{ij})$ be an $n \times n$-real matrix such that all entries are positive. We consider the map

$$\begin{array}{rcl} f\colon \Delta^{n-1} & \to & \Delta^{n-1} \\ v = (v_1, \dots, v_n) & \mapsto & \frac{A \cdot v}{|A \cdot v|}. \end{array}$$



Note that this map is defined since our hypothesis that all a_{ij} are positive, the fact that all entries of v are non-negative and the fact that at least one entry of v is positive imply that $A \cdot v \neq 0$. It is clear that f is continuous. As we had just discussed above, the map $f\colon \Delta^{n-1} \to \Delta^{n-1}$ has a fixed point $v \in \Delta^{n-1}$. But this means that $A \cdot v = |A \cdot v| \cdot v$. Put differently, we have just shown that v is an eigenvector to the positive eigenvalue $|A \cdot v|$. ■

41.4. **Good subsets and relative homology.** A very long time ago, namely in Chapter 38, we started out with our attempt to calculate homology groups of quotients X/A. We developed the concept of relative chain complexes $\mathrm{C}_n(X/A) \coloneqq \mathrm{C}_n(X)/\mathrm{C}_n(A)$ which, as we saw, turned out to be very interesting in its own right.

In this section we return to the original question and we will try to determine homology groups of quotients. There is now an obvious idea: we suspect that in many cases the relative homology groups $\mathrm{H}_n(X, A)$ should be isomorphic to the homology groups of the quotient X/A. In this section we will see that this hunch is basically correct, if A is a "good" subset of X. In the next section we will use this result to determine the homology groups of wedges of topological spaces.

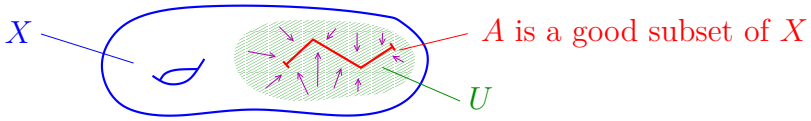
Before we can state the main result of this section we need to introduce the following definition:

Definition. Let X be a topological space and let $A \subset X$ be a[332] subset. We denote by $p\colon (X, A) \to (X/A, A/A)$ the natural projection. For each $n \in \mathbb{N}_0$ we define

$$\mathrm{H}_n(X, A) \xrightarrow{p_*} \mathrm{H}_n(X/A, A/A) \xleftarrow[\cong]{\text{isomorphism } \varkappa \text{ from the Reduced-to-Relative-H}*\text{-Lemma 39.7}} \widetilde{\mathrm{H}}_n(X/A)\,.$$

$$=: \text{ю}$$

Definition. Let X be a topological space and let $A \subset X$ be a subset. We say that A is good if A is closed and if there exists an open neighborhood U of A in X such that A is a deformation retract of U.



Examples.

(1) Let $x \in X$ be a point. Basically by definition the subset $\{x\} \subset X$ is good in the above sense if and only the point is good in the sense of the definition on page 292.

[332]Note that the definition also makes sense if A is the empty set.

(2) The sphere S^{n-1} is a good subset of the closed ball $\overline{B}^n$. Indeed, it is clear that S^{n-1} is a closed subset of $\overline{B}^n$. Furthermore, if we consider the open neighborhood $U = \overline{B}^n \setminus \{0\}$, then a deformation retraction from U to S^{n-1} is given by the map

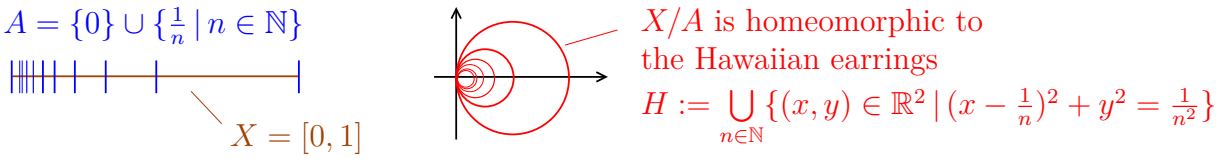
$$\begin{array}{rcl}(\overline{B}^n \setminus \{0\}) \times [0,1] & \to & \overline{B}^n \setminus \{0\} \\ (x,t) & \mapsto & \left(1 - t \cdot \frac{1-\|x\|}{\|x\|}\right) \cdot x.\end{array}$$

(3) It follows from the Subcomplex–Closed Lemma 30.6 and the Subcomplex–Neighborhood Proposition 31.8 (1) that every subcomplex A of a CW-complex X is good. □

Proposition 41.8. (Relative-to-Quotient-H_*-Proposition) Let X be a topological space, let $A \subset X$ be a subset and let $n \in \mathbb{N}_0$. If A is a good subset, then the natural homomorphism $\imath\jmath\colon \mathrm{H}_n(X,A) \to \widetilde{\mathrm{H}}_n(X/A)$ is an isomorphism.

Example. Let X be a topological space, let $A \subset X$ be a subset and let $n \in \mathbb{N}_0$. For better or worse the natural homomorphism $\mathrm{H}_n(X,A) \to \widetilde{\mathrm{H}}_n(X/A)$ is in general not an isomorphism. In fact in general the two groups $\mathrm{H}_n(X,A)$ and $\widetilde{\mathrm{H}}_n(X/A)$ are not even abstractly isomorphic.

For example let us consider the topological space $X = [0,1]$ together with the subspace $A = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\}$.



We leave the proof of the following two statements as an exercise to the reader.

(1) The group $\mathrm{H}_1(X,A)$ is countable.
(2) The group $\mathrm{H}_1(X/A)$ is uncountable. □

The proof of the Relative-to-Quotient-H_*-Proposition 41.8 relies on the following elementary little lemma.

Auxiliary Lemma 41.9. Let X be a topological space and let A be a closed subset of X.
(1) For every open subset $U \subset X \setminus A$ the obvious map $U \to X/A$ is an open embedding.
(2) For every open neighborhood W of A the obvious map $W \setminus A \to (W/A) \setminus (A/A)$ is a homeomorphism.



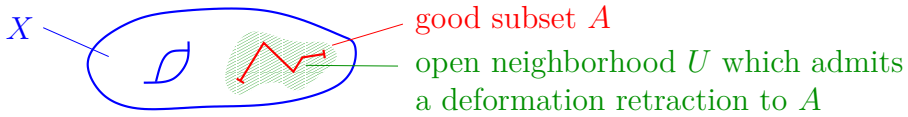
Proof of Lemma 41.9. Let X be a topological space and let A be a closed subset of X. We denote by $p\colon X \to X/A$ the natural projection.

(1) Let $U \subset X \setminus A$ be an open subset. We need to show that the map $q\colon U \to X \to X/A$ is an open embedding. It follows from $U \subset X \setminus A$ that q is an injection. Furthermore note that by the Topological-Quotient Proposition 6.1 (1a) we know that q is continuous. By the Open-Injective Lemma 2.9 (2) it remains to show that $q\colon U \to X/A$ is an open

map. Now let $V \subset U$ be an open subset. We need to show that $q(V) \subset X/A$ is open. We break the argument into three easy steps:
(a) It follows from the Open-Closed Inclusion Lemma 2.7 (1) and the fact that A is by hypothesis closed, that V is also an open subset of X.
(b) Since $A \cap V = \varnothing$ we see that $p^{-1}(p(V)) = V$, which by (a) is in fact an open subset of X.
(c) It follows from (b) and the definition of the quotient topology that $p(V)$ is an open subset of X/A.

(2) Let W be an open neighborhood of A. We set $U \coloneqq W \setminus A$. Since A is closed we see that U is an open subset of $W \setminus A$. Recall that we want to show that the obvious map $q\colon W \setminus A \to (W/A) \setminus (A/A)$ is a homeomorphism. It is clear that q is a bijection and it follows from the Twice Quotient Lemma 6.3 (2) that q is continuous. By the Open-Injective Lemma 2.9 (2) it remains to show that $q\colon W \setminus A \to (W/A) \setminus (A/A)$ is an open map. Thus let V be an open subset of $W \setminus A$. We have $q(V) = p(V) \cap (W/A \setminus A/A)$. It follows from (1c) and the definition of the subspace topology that $q(V)$ is an open subset of $(W/A) \setminus (A/A)$. ■

Proof of the Relative-to-Quotient-H$_*$-Proposition 41.8. Let X be a topological space, let $A \subset X$ be a good subset and let $n \in \mathbb{N}_0$. We make the following simple preparations:
(1) Since A is good we can pick an open neighborhood U of A in X such that A is a deformation retract of U.
(2) We denote by $p\colon X \to X/A$ the projection.
(3) We denote by $q\colon X \setminus A \to (X/A) \setminus (A/A)$ the restriction of p to $X \setminus A$. For completeness' sake we point out that it follows from the Basics-of-Continuity Lemma 2.1 that q is continuous.



Now we consider the following diagram:

$$\begin{array}{ccccc}
\mathrm{H}_n(X, A) & \xrightarrow{(3)} & \mathrm{H}_n(X, U) & \xleftarrow{(1)} & \mathrm{H}_n(X \setminus A, U \setminus A) \\
\downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle q_*} \\
\mathrm{H}_n(X/A, A/A) & \xrightarrow[(4)]{} & \mathrm{H}_n(X/A, U/A) & \xleftarrow[(2)]{} & \mathrm{H}_n((X/A) \setminus (A/A), (U/A) \setminus (A/A)) \\
\cong\uparrow{\scriptstyle \text{ж}} & & & & \\
\widetilde{\mathrm{H}}_n(X/A), & & & &
\end{array}$$

(ю: $\mathrm{H}_n(X, A) \to \widetilde{\mathrm{H}}_n(X/A)$)

where all the horizontal maps are induced by inclusions of pairs of topological spaces. Recall that we need to show that ю$\colon \mathrm{H}_n(X, A) \to \widetilde{\mathrm{H}}_n(X/A)$ is an isomorphism. By the Reduced-to-Relative-H$_*$-Lemma 39.7 we know that ж$\colon \widetilde{\mathrm{H}}_n(X/A) \to \mathrm{H}_n(X/A, A/A)$ is an isomorphism. Thus it suffices to show that the top left vertical map $p_*\colon \mathrm{H}_n(X, A) \to \mathrm{H}_n(X/A, A/A)$ is an isomorphism.

First note that it follows easily from the functoriality of relative homology groups that the two squares of the diagram commute. Next note that it follows from our hypothesis that the subset A is closed, together with the above auxiliary Lemma 41.9 (2), that the

map $q\colon (X \setminus A, U \setminus A) \to ((X/A) \setminus (A/A), (U/A) \setminus (A/A))$ is a homeomorphism of pairs of topological spaces. We obtain from this observation that the vertical map on the right is an isomorphism. At this stage it remains to prove the following claim:

Claim. All horizontal maps are isomorphisms.

Proof.

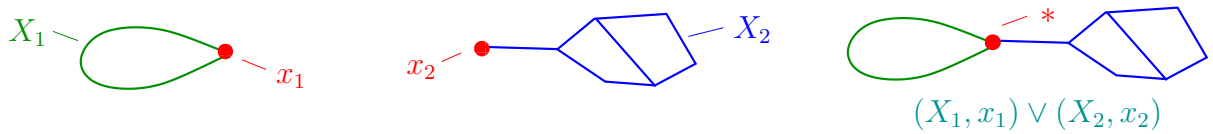
(1) Since $A \subset X$ is closed and since $U \subset X$ is open it follows from the Excision Theorem 40.1 that both the top right horizontal map is an isomorphism.
(2) The hypothesis that $A \subset X$ is closed and that $U \subset X$ is an open neighborhood of $A \subset X$ implies, by definition of the quotient topology, that A/A is a closed subset of X/A and that U/A is an open neighborhood of $A/A \subset X/A$. Thus we can also apply the Excision Theorem 40.1 to the horizontal map on the bottom right, and we see that this map is also an isomorphism.
(3) Since A is deformation retract of U we obtain from the Homotopy Equivalence-Relative-H_*-Corollary 38.4 that the top left horizontal map is an isomorphism.
(4) Recall that A is a deformation retract of U. It follows from the Deformation Retraction-Quotient Theorem 9.2 (2) that A/A is a deformation retract of U/A. As in (3) we now see that the bottom-left horizontal map is an isomorphism. ∎

41.5. **The homology groups of the wedge of topological spaces.** We recall the following definition from page 245:

Definition. Let $\{(X_k, x_k)\}_{k\in K}$ be a family of pointed topological spaces. We refer to

$$\bigvee_{k\in K} (X_k, x_k) \;:=\; \Big(\bigsqcup_{k\in K} X_k\Big)\Big/ \bigsqcup_{k\in K} \{x_k\}$$

as the wedge of the pointed topological spaces (X_k, x_k).



Recall that in the Wedge Proposition 17.1 we showed that the wedge $\bigvee_{k\in K} (X_k, x_k)$ is the coproduct in the category $\mathcal{PTop}$ of *pointed* topological spaces. Also recall that we showed in the Coproduct-of-Sets-and-Abelian Groups Lemma 4.4 that in the category $\mathcal{AbGroup}$ of abelian groups the coproduct is given by the direct sum. It is again natural to ask which functors from the category $\mathcal{PTop}$ of pointed topological groups to the category $\mathcal{AbGroup}$ of abelian groups preserve coproducts. One can see in very easy examples that ordinary homology groups do not quite preserve coproducts. But we have alternative functors, namely for each $n \in \mathbb{N}_0$ we can consider the functor

$$\begin{array}{rcl} \mathcal{PTop} & \to & \mathcal{AbGroup} \\ (X, x_0) & \mapsto & \mathrm{H}_n(X, \{x_0\}) \\ (f\colon (X, x_0) \to (Y, y_0)) & \mapsto & (f_*\colon \mathrm{H}_n(X, \{x_0\}) \to \mathrm{H}_n(Y, \{y_0\})). \end{array}$$

The next shows that these relative homology functors preserve coproducts (at least under rather mild hypotheses on the points $x_k \in X_k$).

Proposition 41.10. (Wedge-H_*-Proposition) Let $\{(X_k, x_k)\}_{k\in K}$ be a family of pointed topological spaces such that each $x_k \in X_k$ is good[333]. We denote the wedge point of $\bigvee_{k\in K}(X_k, x_k)$ by $*$ and given $k \in K$ we denote by $i_k\colon X_k \to \bigvee_{k\in K}(X_k, x_k)$ the natural inclusion that we introduced on page 245. Let $n \in \mathbb{N}_0$.

(1) The natural map

$$\bigoplus_{k\in K} i_{k*}\colon \bigoplus_{k\in K} \mathrm{H}_n(X_k, \{x_k\}) \to \mathrm{H}_n\Big(\bigvee_{k\in K}(X_k, x_k), \{*\}\Big)$$

is an isomorphism.

(2) For $k \in K$ let $p_j\colon \bigvee_{k\in K} x_k \to X_j$ be the natural map that we introduced on page 245. If K is finite, then the map

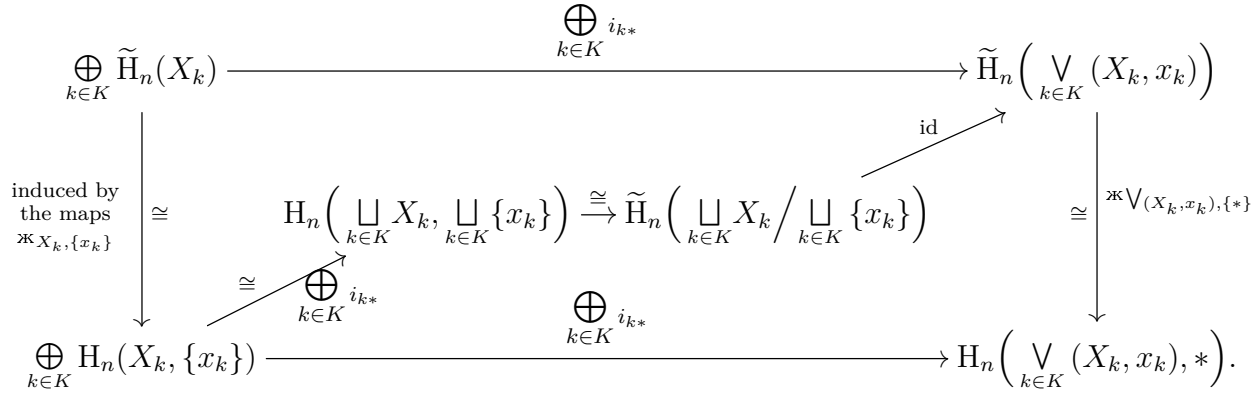
$$\prod_{k\in K} p_k^*\colon \mathrm{H}_n\Big(\bigvee_{k\in K}(X_k, x_k), \{*\}\Big) \to \prod_{k\in K} \mathrm{H}_n(X_k, \{x_k\}) = \bigoplus_{k\in K} \mathrm{H}_n(X_k, \{x_k\})$$

($=$: since K is finite)

is defined and it is the inverse to the isomorphism from (1).

Both statements also hold for reduced homology instead of relative homology.

Proof of the Wedge-H_*-Proposition 41.10 (1). We consider the following diagram:



The proposition follows from the following clarifications and observations:

(a) It follows from the Reduced-to-Relative-H_*-Lemma 39.7 that the vertical maps to the left and right are isomorphisms.

(b) It follows from the obvious generalization of the Path-Component–H_*–Lemma 35.8 that the bottom left diagonal map is an isomorphism.

(c) By hypothesis each $x_k \in X_k$ is good. It follows easily from the definitions that $\bigsqcup_{k\in K}\{x_k\}$ is a good subset of $\bigsqcup_{k\in K} X_k$. Thus we can apply the Relative-to-Quotient-H_*-Proposition 41.8 to see that the middle horizontal map is an isomorphism.

(d) The top right diagonal map is indeed by definition the identity.

(e) One can easily verify that the lower triangle commutes. This implies the desired statement for relative homology groups.

[333]Recall that by the Good Point-Lemma 20.5 (1) we know that any point on a topological manifold is good and that it follows from the CW-Complex-Hausdorff Proposition 30.3 and the CW-Complex–Local Properties Proposition 31.7 (1) that any point on a CW-complex is good.

(f) One can also easily verify that the upper triangle commutes. Thus we also get the desired statement on reduced homology groups. ■

Proof of the Wedge-H$_*$-Proposition 41.10 (2). By (1) it suffices to prove the following claim:

Claim. The composition

$$\bigoplus_{k\in K} \mathrm{H}_n(X_k,\{x_k\}) \xrightarrow{\bigoplus_{k\in K} i_{k*}} \mathrm{H}_n\Big(\bigvee_{k\in K}(X_k,x_k),\{*\}\Big) \xrightarrow{\prod_{k\in K} p_k^*} \bigoplus_{k\in K} \mathrm{H}_n(X_k,\{x_k\})$$

is the identity.

Proof. This claim follows immediately from the observation that $p_k \circ i_k = \mathrm{id}_{X_k}$ and that for $k \neq l$ the map $p_k \circ i_l$ is the constant map $x \mapsto x_k$ which induces the zero map on relative homology. ■

41.6. **The long exact sequence for quotients.** The following theorem gives us, at least in reasonable settings, the promised connection between the homology groups of a topological space X, a subspace $A \subset X$ and the quotient space X/A.

Theorem 41.11. (Quotient–LES Theorem) Let X be a topological space. Furthermore let $A \subset X$ be a good subset. We denote by $i\colon A \to X$ the inclusion map and we denote by $p\colon X \to X/A$ the projection. For each $n \in \mathbb{N}_0$ there exists a natural[334] homomorphism $\partial_n\colon \widetilde{\mathrm{H}}_n(X/A) \to \widetilde{\mathrm{H}}_{n-1}(A)$ such that the sequence

$$\begin{array}{ccccccc} & & & & \dots & \xrightarrow{p_*} & \widetilde{\mathrm{H}}_{n+1}(X/A) \xrightarrow{\partial_{n+1}} \\ \to & \widetilde{\mathrm{H}}_n(A) & \xrightarrow{i_*} & \widetilde{\mathrm{H}}_n(X) & \xrightarrow{p_*} & \widetilde{\mathrm{H}}_n(X/A) & \xrightarrow{\partial_n} \\ \to & \widetilde{\mathrm{H}}_{n-1}(A) & \xrightarrow{i_*} & \dots & & & \end{array}$$

is exact.

Proof. Let X be a topological space and let $A \subset X$ be a good subset. We denote by $i\colon A \to X$ the inclusion and we denote by $p\colon X \to X/A$ the projection. We consider the following diagram:

$$\begin{array}{ccccccccccc} \dots & \longrightarrow & \widetilde{\mathrm{H}}_n(A) & \xrightarrow{i_*} & \widetilde{\mathrm{H}}_n(X) & \xrightarrow{\text{Ж}_{X,A}} & \mathrm{H}_n(X,A) & \xrightarrow{\partial_n} & \widetilde{\mathrm{H}}_{n-1}(A) & \xrightarrow{i_*} & \widetilde{\mathrm{H}}_{n-1}(X) \longrightarrow \dots \\ & & & & & \searrow^{p_*} & \downarrow \text{Ю} & & & & \\ & & & & & & \widetilde{\mathrm{H}}_n(X/A). & & & & \end{array}$$

We make the following comments and observations:

(1) The horizontal sequence is the long exact sequence coming from the Reduced LES-of-Pair Proposition 39.6.

(2) The vertical map is the natural homomorphism that we defined on page 527. It follows easily from the definitions that the triangle commutes.

[334]Here "natural" means the following: If $f\colon (X,A) \to (Y,B)$ is a continuous map between pairs of topological spaces and if A and B are good subsets, then the following diagram commutes:

$$\begin{array}{ccc} \widetilde{\mathrm{H}}_n(X/A) & \xrightarrow{\partial_n} & \widetilde{\mathrm{H}}_{n-1}(A) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_*} \\ \widetilde{\mathrm{H}}_n(Y/B) & \xrightarrow{\partial_n} & \widetilde{\mathrm{H}}_{n-1}(B). \end{array}$$

(3) Since A is good we obtain from the Relative-to-Quotient-H_*-Proposition 41.8 that the vertical map ю is an isomorphism.

Now we consider the homomorphisms $\widetilde{\partial}_n := \partial_n \circ ю^{-1} \colon \widetilde{\mathrm{H}}_n(X/A) \to \widetilde{\mathrm{H}}_{n-1}(A)$. Since ю and ∂ are natural we see that the maps $\widetilde{\partial}_n = \partial_n \circ ю^{-1}$ are also natural. It follows from the above discussion that the sequence

$$\ldots \longrightarrow \widetilde{\mathrm{H}}_n(A) \xrightarrow{i_*} \widetilde{\mathrm{H}}_n(X) \xrightarrow{p_*} \widetilde{\mathrm{H}}_n(X/A) \xrightarrow{\widetilde{\partial}_n} \widetilde{\mathrm{H}}_{n-1}(A) \xrightarrow{i_*} \widetilde{\mathrm{H}}_{n-1}(X) \longrightarrow \ldots$$

is exact. ■

Using the Quotient–LES Theorem 41.11 we can now give a somewhat shorter proof of the Sphere-H_*-Proposition 41.2.

Proposition 41.2. (Sphere-H_*-Proposition) For each $n \in \mathbb{Z}_{\geq -1}$ we have

$$\widetilde{\mathrm{H}}_k(S^n) \cong \begin{cases} \mathbb{Z}, & \text{if } k = n, \text{ and} \\ 0, & \text{if } k \neq n. \end{cases}$$

Proof. We prove the proposition by induction on n. First note that for $n = -1$ we have $S^n = S^{-1} = \varnothing$, thus the statement follows immediately from the Reduced Homology Lemma 39.2 (-1).[335]

Now we suppose that the statement of the proposition holds for $n-1$. We want to determine the reduced homology of S^n. The key idea is to use that by the Ball Quotient-Sphere Lemma 6.5 we know that S^n is homeomorphic to $\overline{B}^n/S^{n-1}$. On page 528 we saw that S^{n-1} is a good subset of $\overline{B}^n$. Therefore we can apply the Quotient–LES Theorem 41.11 to $X = \overline{B}^n$ and the closed subset $A = S^{n-1}$ and we obtain the following exact sequence:

$$\begin{array}{rcccccc} & \ldots & \longrightarrow & \widetilde{\mathrm{H}}_{k+1}(\overline{B}^n) & \xrightarrow{p_*} & \widetilde{\mathrm{H}}_{k+1}(\overline{B}^n/S^{n-1}) & \xrightarrow{\partial_{k+1}} \\ \to & \widetilde{\mathrm{H}}_k(S^{n-1}) & \xrightarrow{i_*} & \widetilde{\mathrm{H}}_k(\overline{B}^n) & \xrightarrow{p_*} & \widetilde{\mathrm{H}}_k(\overline{B}^n/S^{n-1}) & \xrightarrow{\partial_k} \\ \to & \widetilde{\mathrm{H}}_{k-1}(S^{n-1}) & \xrightarrow{i_*} & \ldots & & & \end{array}$$

By the Contractible-$\widetilde{\mathrm{H}}_*$-Corollary 39.5 we know that $\widetilde{\mathrm{H}}_i(\overline{B}^n) = 0$ for all i. Since the sequence is exact we see that all the connecting homomorphisms are isomorphisms. Thus we see that

$$\begin{array}{ccccccc} \widetilde{\mathrm{H}}_k(S^n) & \cong & \widetilde{\mathrm{H}}_k(\overline{B}^n/S^{n-1}) & \cong & \widetilde{\mathrm{H}}_{k-1}(S^{n-1}) & \cong & \begin{cases} \mathbb{Z}, & \text{if } k-1 = n-1 \\ 0, & \text{if } k \neq n. \end{cases} \\ & \uparrow & & \uparrow & & \uparrow & \end{array}$$

in the Ball Quotient-Sphere Lemma 6.5 we showed that $S^n \cong \overline{B}^n/S^{n-1}$

by the above exact sequence

by induction hypothesis

■

[335] To readers who prefer S^0 as a starting point we point out that S^0 consists of precisely two points. Thus the promised isomorphisms follow from the Singleton-H_*-Lemma 35.12, the Path-Component–H_*–Lemma 35.8 and the Reduced Homology Lemma 39.2.

CHAPTER 42

Local homology and explicit generators of homology groups

By the Sphere-H_*-Proposition 41.2 we know that for any $n \in \mathbb{N}_0$ we have

$$\widetilde{\mathrm{H}}_k(S^n) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } k = n, \text{ and} \\ 0, & \text{if } k \neq n. \end{cases}$$

Thus we have determined the isomorphism types of the homology groups of all finite-dimensional spheres. In this chapter we will determine an explicit singular n-cycle in $\mathrm{C}_n(S^n)$ that represents a generator of $\widetilde{\mathrm{H}}_n(S^n)$. Along the way we introduce the concept of the local homology of a topological space X at a point x. This concept will turn out to be quite interesting in its own right.

42.1. **An explicit generator of $\mathbf{H}_n(\Delta^n, \partial\Delta^n)$.** As we mentioned above, in this chapter we eventually want to find explicit generators of $\widetilde{\mathrm{H}}_n(S^n) \cong \mathbb{Z}$ for $n \in \mathbb{N}_0$. We start out with the following lemma, which gives us a closely related statement. In fact the following lemma is the first instance where we can nail down a generator of an interesting homology group.

Lemma 42.1. (Identity-is-Generator Lemma) For every $n \in \mathbb{N}_0$ the following two statements hold:

(1) For $k \in \mathbb{N}_0$ we have

$$\mathrm{H}_n(\Delta^n, \partial\Delta^n) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } k = n, \\ 0, & \text{otherwise.} \end{cases}$$

(2) The identity map $\mathrm{id}\colon \Delta^n \to \Delta^n$ represents a generator of $\mathrm{H}_n(\Delta^n, \partial\Delta^n) \cong \mathbb{Z}$.

Proof. Statement (1) follows immediately from the Ball-rel-Sphere-H_*-Lemma 41.4 and the Δ^n-Homeomorphism Lemma 35.3.

Thus it remains to prove Statement (2). We will prove this statement by induction on $n \in \mathbb{N}_0$. It follows easily from the H_0-Proposition 35.11 that the statement holds for $n = 0$. Now we suppose that the desired statement holds for $n-1$ with $n \in \mathbb{N}$.

Our next goal is to find a connection between the two homology groups $\mathrm{H}_n(\Delta^n, \partial\Delta^n)$ and $\mathrm{H}_{n-1}(\Delta^{n-1}, \partial\Delta^{n-1})$. To do so we introduce the following notation:

(a) We consider

$$\Lambda \;:=\; \bigcup_{i=0}^{n-1} \{(t_0, \ldots, t_n) \in \Delta^n \,|\, t_i = 0\},$$

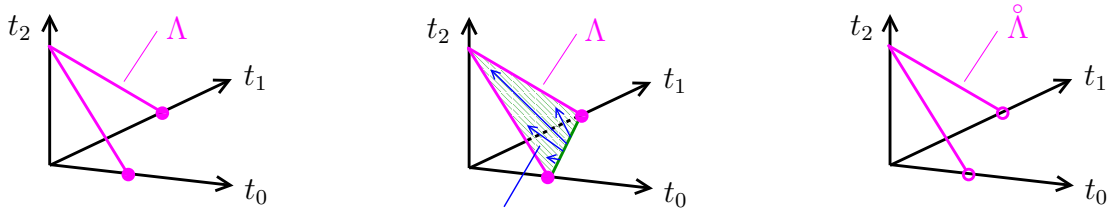
i.e. Λ is the union of all $(n-1)$-dimensional faces of Δ^n except for the face given by $t_n = 0$. Note that Λ is a deformation retract of Δ^n.

(b) We write

$$\mathring{\Lambda} \;:=\; \{(t_0, \ldots, t_n) \in \Lambda \,|\, t_n > 0\}.$$

The set $\mathring{\Lambda}$ can be viewed as the "interior" of Λ.

In the following we view $\mathbb{R}^n$ as the subspace of $\mathbb{R}^{n+1}$ given by setting the last coordinate equal to zero. This point of view allows us to view Δ^{n-1} as a subspace of Δ^n. Note that with this convention we have $\partial\Delta^{n-1} \subset \Lambda$.



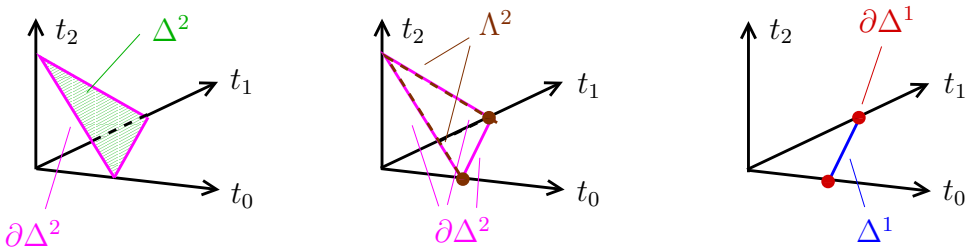
deformation retraction from Δ^2 to Λ

Claim. We consider the maps

$$\mathrm{H}_n(\Delta^n, \partial\Delta^n) \xrightarrow{\partial_n} \mathrm{H}_{n-1}(\partial\Delta^n, \Lambda) \xleftarrow{\iota_*} \mathrm{H}_{n-1}(\Delta^{n-1}, \partial\Delta^{n-1}).$$

Here the map ∂_n on the left is the connecting homomorphism in the long exact sequence of the triple $(\Delta^n, \partial\Delta^n, \Lambda)$ and the right-hand map is induced by the inclusion. Both maps are isomorphisms and the following equality holds:

$$\partial_n([\mathrm{id}_{\Delta^n}]) \;=\; (-1)^n \cdot \iota_*([\mathrm{id}_{\Delta^{n-1}}]).$$



It is clear that a proof of the claim together with the induction hypothesis implies that $\mathrm{H}_n(\Delta^n, \partial\Delta^n) \cong \mathbb{Z}$ and that $[\mathrm{id}_{\Delta^n}]$ is a generator of $\mathrm{H}_n(\Delta^n, \partial\Delta^n)$. In particular a proof of the claim implies the Identity-is-Generator Lemma 42.1.

Proof. We consider the two maps separately.

(a) Of the two maps in the claim we first consider the map ∂_n on the left. It follows from the LES-of-Triple Proposition 38.2 applied to the triple $(\Delta^n, \partial\Delta^n, \Lambda)$ of topological spaces that the following sequence is exact:

$$\mathrm{H}_n(\Delta^n, \Lambda) \xrightarrow{p_*} \mathrm{H}_n(\Delta^n, \partial\Delta^n) \xrightarrow{\partial_n} \mathrm{H}_{n-1}(\partial\Delta^n, \Lambda) \xrightarrow{\iota_*} \mathrm{H}_{n-1}(\Delta^n, \Lambda)$$

where $p\colon (\Delta^n, \Lambda) \to (\Delta^n, \partial\Delta^n)$ is the obvious inclusion. As we remarked above, one can easily show that Λ is a deformation retract of Δ^n. It follows from the Deformation Retract-Relative H_*-Corollary 38.5 that $\mathrm{H}_n(\Delta^n, \Lambda) = 0$, which implies that the connecting homomorphism $\partial_n\colon \mathrm{H}_n(\Delta^n, \partial\Delta^n) \to \mathrm{H}_{n-1}(\partial\Delta^n, \Lambda)$ is an isomorphism. Furthermore

note that we have the following equalities in $\mathrm{H}_n(\partial\Delta^n, \Lambda)$:

$$\partial_n([\mathrm{id}_{\Delta^n}]) \underset{\text{by definition of the connecting homomorphism in the LES-of-Triple Proposition 38.2}}{=} [\partial_n(\mathrm{id}_{\Delta^n})] \overset{\text{see the definition of } \partial_n\colon \mathrm{C}_n(\Delta^n) \to \mathrm{C}_{n-1}(\Delta^n) \text{ on page 459}}{=} \Big[\underbrace{\sum_{j=0}^{n-1} (-1)^j \cdot \mathrm{id} \circ i_j^n}_{\substack{\text{lies in } \mathrm{C}_{n-1}(\Lambda), \\ \text{thus } = 0 \text{ in } \mathrm{C}_{n-1}(\partial\Delta^n, \Lambda)}} + (-1)^n \cdot \underbrace{\mathrm{id} \circ i_n^n}_{=\mathrm{id}_{\Delta^{n-1}}} \Big] \underset{\text{since we work in } \mathrm{H}_{n-1}(\partial\Delta^n, \Lambda)}{=} (-1)^n \cdot [\mathrm{id}_{\Delta^{n-1}}].$$

(b) We turn our attention to the right-hand map $\iota_*\colon \mathrm{H}_{n-1}(\Delta^{n-1}, \partial\Delta^{n-1}) \to \mathrm{H}_{n-1}(\partial\Delta^n, \Lambda)$ in the statement of the claim. It follows from the Excision-via-Deformation Theorem 40.5, applied to $X = \partial\Delta^n$, $A = \Lambda$, $Z = \mathring{\Lambda}$ and $U = (0, \ldots, 0, 1)$[336] that the inclusion-induced map

$$\mathrm{H}_{n-1}(\Delta^{n-1}, \partial\Delta^{n-1}) = \mathrm{H}_{n-1}(\underbrace{\partial\Delta^n \setminus \mathring{\Lambda}}_{=\Delta^{n-1}}, \underbrace{\Lambda \setminus \mathring{\Lambda}}_{=\partial\Delta^{n-1}}) \xrightarrow{\iota_*} \mathrm{H}_{n-1}(\partial\Delta^n, \Lambda)$$

is an isomorphism. Finally note that evidently we have $\iota_*([\mathrm{id}_{\Delta^{n-1}}]) = [\mathrm{id}_{\Delta^{n-1}}]$.

Thus we have proved that both maps in the claim are isomorphisms and that

$$\partial([\mathrm{id}_{\Delta^n}]) = (-1)^n \cdot [\mathrm{id}_{\Delta^{n-1}}] = (-1)^n \cdot \iota_*([\mathrm{id}_{\Delta^{n-1}}]). \qquad \blacksquare$$

42.2. **The local homology groups of a topological space.** We start out this section with the following definition which at first glance might look somewhat artificial.

Definition. Given a topological space X and a point $x \in X$ the relative homology group $\mathrm{H}_n(X, X \setminus \{x\})$ is called the n-th local homology group of X at the point x.

As we will see shortly, the concept of local homology is surprisingly useful. In many settings the following proposition allows us to calculate the local homology of a topological space at a point.

Proposition 42.2. (Local-H_*-Proposition) Let X be a topological space and let $x \in X$.

(1) If $\{x\} \subset X$ is a closed subset[337], then for any neighborhood U of $x \in X$ the inclusion-induced maps $\mathrm{H}_n(U, U \setminus \{x\}) \to \mathrm{H}_n(X, X \setminus \{x\})$ are isomorphisms.[338]

(2) For any homeomorphism $f\colon X \to Y$ the maps $f_*\colon \mathrm{H}_n(X, X \setminus \{x\}) \to \mathrm{H}_n(Y, Y \setminus \{f(x)\})$ are isomorphisms.

(3) (a) If X is an open subset of $\mathbb{R}^n$, then for any $x \in X$ and any $k \in \mathbb{N}_0$ we have

$$\mathrm{H}_k(X, X \setminus \{x\}) \cong \begin{cases} \mathbb{Z}, & \text{if } k = n, \\ 0, & \text{otherwise.} \end{cases}$$

(b) If X is an open subset of $\mathbb{H}^n = \{(t_1, \ldots, t_n) \in \mathbb{R}^n \mid t_n \geq 0\}$ and if $x \in \mathbb{R}^{n-1} \times \{0\}$, then $\mathrm{H}_k(X, X \setminus \{x\}) = 0$ for all $k \in \mathbb{N}_0$.

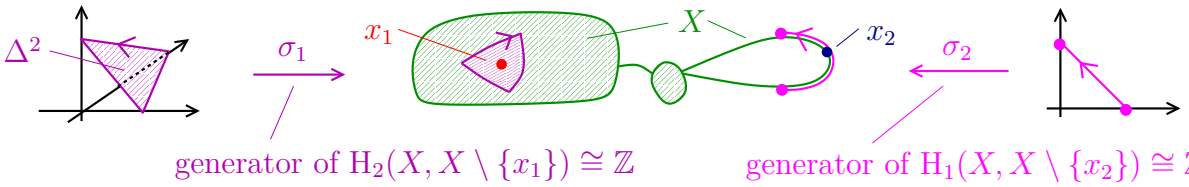
(4) If $\sigma\colon \Delta^n \to X$ is a continuous map such that $\sigma\colon \mathring{\Delta}^n \to X$ is an open embedding, then for any $x \in \sigma(\mathring{\Delta}^n)$ we have $\mathrm{H}_n(X, X \setminus \{x\}) \cong \mathbb{Z}$ and the singular n-simplex $\sigma\colon \Delta^n \to X$ represents a generator of $\mathrm{H}_n(X, X \setminus \{x\}) \cong \mathbb{Z}$.

---

[336] One can easily show that U has the property required in the Excision-via-Deformation Theorem 40.5.

[337] Recall that by the Points-in-Hausdorff Lemma 1.8 we know that this condition is satisfied if X is Hausdorff, e.g. if X is a topological manifold.

[338] This statement hopefully justifies the name "local" homology.



Proof. Let X be a topological space and let $x \in X$.

(1) Let U be a neighborhood of $x \in X$. It follows immediately from the Excision Theorem 40.2 that the inclusion-induced maps $\mathrm{H}_n(U, U \setminus \{x\}) \to \mathrm{H}_n(X, X \setminus \{x\})$ are isomorphisms.

(2) This statement follows from the functoriality of relative homology groups.

(3) (a) It follows easily from (1), applied to $x \in X \subset \mathbb{R}^n$, that it suffices to consider the case that $X = \mathbb{R}^n$. For any $k \in \mathbb{N}_0$ we see that

this is the connecting homomorphism of the long exact sequence of the pair $(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\})$ in reduced homology, it is an isomorphism since on page 500 we showed that $\widetilde{\mathrm{H}}_*(\mathbb{R}^n) = 0$

$$\mathrm{H}_k(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\}) \xrightarrow{\partial_k} \widetilde{\mathrm{H}}_{k-1}(\mathbb{R}^n \setminus \{x\}) \xleftarrow{\cong} \widetilde{\mathrm{H}}_{k-1}(S^{n-1}(x)) \cong \begin{cases} \mathbb{Z}, & \text{if } k = n, \\ 0, & \text{otherwise.} \end{cases}$$

isomorphism since the inclusion $S^{n-1}(x) \to \mathbb{R}^n \setminus \{x\}$ is a homotopy equivalence

Sphere-H_*-Proposition 41.2

(b) It follows easily from (1) that it suffices to consider the case that $X = \mathbb{H}^n$ and that $x \in \mathbb{R}^{n-1} \times \{0\}$. By the Reduced LES-of-Pair Proposition 39.6 we have a long exact sequence of the form

$$\ldots \to \widetilde{\mathrm{H}}_k(\mathbb{H}^n) \to \mathrm{H}_k(\mathbb{H}^n, \mathbb{H}^n \setminus \{x\}) \xrightarrow{\cong} \widetilde{\mathrm{H}}_k(\mathbb{H}^n \setminus \{x\}) \to \ldots$$

It follows easily from the fact that $x \in \mathbb{R}^{n-1} \times \{0\}$ that both $\mathbb{H}^n$ and $\mathbb{H}^n \setminus \{x\}$ are contractible. It follows from the Contractible-H_*-Corollary 39.5 that the reduced homology groups of $\mathbb{H}^n$ and $\mathbb{H}^n \setminus \{x\}$ vanish. But this implies, by exactness, that also the "middle" groups $\mathrm{H}_k(\mathbb{H}^n, \mathbb{H}^n \setminus \{x\})$ have to vanish.

(4) Let X be a topological space and let $\sigma\colon \Delta^n \to X$ be a continuous map such that $\sigma\colon \mathring{\Delta}^n \to X$ is an open embedding. Furthermore let $x \in \sigma(\mathring{\Delta}^n)$. Since $x \in \sigma(\mathring{\Delta}^n)$ and since $\sigma\colon \mathring{\Delta}^n \to X$ is an embedding there exists a unique $a \in \mathring{\Delta}^n$ with $\sigma(a) = x$.

Claim. There exists a continuous map $\tau\colon \Delta^n \to X$ with the following three properties:
(a) $\tau\colon \Delta^n \to X$ is an embedding.
(b) $\tau(\Delta^n)$ is a neighborhood of $x \in X$.
(c) We have $[\sigma] = [\tau] \in \mathrm{H}_n(X, X \setminus \{x\})$.

Proof. We introduce the following notation: We denote by $b := (\frac{1}{n+1}, \ldots, \frac{1}{n+1}) \in \Delta^n$ the barycenter of Δ^n. Furthermore given $\epsilon > 0$ we write

$$\Delta^n_\epsilon = \{(x - b) \cdot (1 - \epsilon) + b \,|\, x \in \Delta^n\}.$$

Note that $\Delta^n_\epsilon \subset \Delta^n$ and note that $\Delta^n_0 = \Delta^n$. It follows from an elementary argument and the hypothesis that $a \in \Delta^n \setminus \partial\Delta^n$ that there exists an $\epsilon > 0$ such that $a \in \Delta^n_\epsilon$.

Next we consider the homotopy

$$\begin{array}{rcl} F\colon \Delta^n \times [0,1] & \to & \Delta^n \\ (x,t) & \mapsto & (x-b)\cdot(1-t+(1-\epsilon)\cdot t)+b. \end{array}$$

Note that $F_0 = \mathrm{id}_{\Delta^n}$ and note that F_1 defines a homeomorphism $\Delta^n \to \Delta^n_\epsilon$. We set $\tau := \sigma \circ F_1 \colon \Delta^n \to X$. We have to verify that τ has the desired properties:

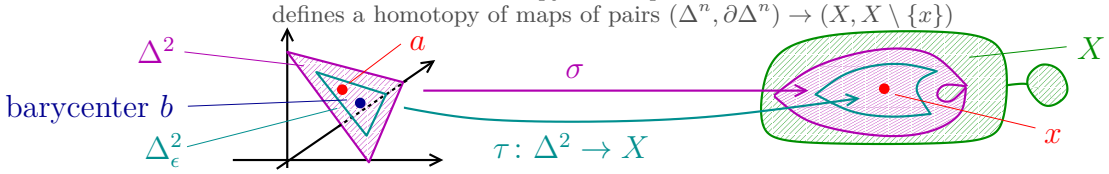
(a) Since $F_1\colon \Delta^n \to \Delta^n$ is an embedding with image Δ^n_ϵ, since $\Delta^n_\epsilon \subset \overset{\circ}{\Delta}{}^n$ and since $\sigma\colon \overset{\circ}{\Delta}{}^n \to X$ is an embedding we see that $\tau = \sigma \circ F_1 \colon \Delta^n \to X$ is an embedding.

(b) It is straightforward to verify that Δ^n_ϵ is a neighborhood of $a \in \Delta^n$. By hypothesis $\sigma(\overset{\circ}{\Delta}{}^n)$ is an open subset of X. It follows from the Open-Closed Inclusion Lemma 2.7 that $\tau(\Delta^n) = \sigma(\Delta^n_\epsilon)$ is a neighborhood of $x = \sigma(a) \in X$.

(c) We have the following equalities in $\mathrm{H}_n(X, X\setminus\{x\})$:

$$[\sigma] \quad = \quad (\sigma\circ F_0)_*([\mathrm{id}_{\Delta^n}]) \quad \underset{\uparrow}{=} \quad (\sigma\circ F_1)_*([\mathrm{id}_{\Delta^n}]) \quad = \quad [\sigma\circ F_1\circ \mathrm{id}_{\Delta^n}] \quad = \quad [\tau].$$

follows from the Homotopy-H_*-Proposition 38.6 since $\sigma\circ F$ defines a homotopy of maps of pairs $(\Delta^n, \partial\Delta^n) \to (X, X\setminus\{x\})$ ⊞



Finally we consider the following sequence of homomorphisms:

since $\tau(\Delta^n)$ is a neighborhood of x it follows from the Excision Theorem 40.2 that this inclusion induced map is an isomorphism

isomorphism since τ is an embedding

$\partial\Delta^n$ is a deformation retract of $\Delta^n \setminus \{a\}$, thus it follows from the Homotopy Equivalence–Relative-H_*-Corollary 38.4 that the map is an isomorphism

$$\begin{array}{ccccccc} \mathrm{H}_n(X, X\setminus\{x\}) & \leftarrow & \mathrm{H}_n(\tau(\Delta^n), \tau(\Delta^n)\setminus\{x\}) & \xleftarrow{\tau_*} & \mathrm{H}_n(\Delta^n, \Delta^n\setminus\{a\}) & \leftarrow & \mathrm{H}_n(\Delta^n, \partial\Delta^n). \\ [\sigma] = [\tau] & \mapsto & [\tau] & \mapsto & [\mathrm{id}_{\Delta^n}] & \mapsto & [\mathrm{id}_{\Delta^n}]. \end{array}$$

By the Identity-is-Generator Lemma 42.1 we know that id_{Δ^n} represents a generator of $\mathrm{H}_n(\Delta^n, \partial\Delta^n)$. It follows from the above explicit isomorphisms that σ represents a generator of $\mathrm{H}_k(X, X\setminus\{x\})$. ■

We will now see that the Local-H_*-Proposition 42.2 can also be used to show that given cycles represent non-trivial homology classes. To state this result it is convenient to introduce the following definition:

Definition. Let $(X, A)_m$ be a pair of topological spaces. In the following we say that a singular n-chain $c = \sum_{i=1}^{m} t_i \cdot \sigma_i \in \mathrm{C}_n(X)$ is non-trivial with respect to $x \in X \setminus A$ if the following statements hold:

(a) We have $\partial_n(c) \in \mathrm{C}_{n-1}(A)$, in other words, c represents an n-cycle in $\mathrm{C}_n(X, A)$.

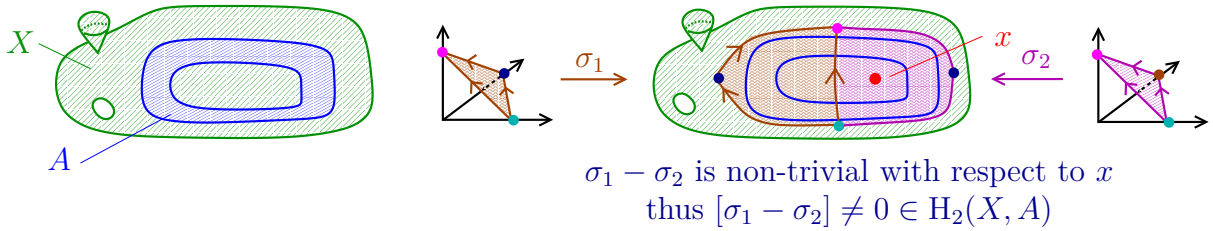
(b) There exists a $j \in \{1, \ldots, m\}$ with $t_j = \{\pm 1\}$ such that the map $\sigma_j\colon \overset{\circ}{\Delta}{}^n \to X$ is an open embedding, such that $x \in \sigma_j(\overset{\circ}{\Delta}{}^n)$ and such that for all $i \neq j$ we have $x \notin \sigma_i(\Delta^n)$.

The following proposition gives a convenient criterion for showing that a given cycle represents a non-trivial element in homology.

Proposition 42.3 (Detect-Cycle Proposition) Let (X, A) be a pair of topological spaces and let $c = \sum_{i=1}^{m} t_i \cdot \sigma_i \in \mathrm{C}_n(X)$ be a singular n-chain that is non-trivial with respect to some $x \in X \setminus A$. The following statements hold:

(1) We have $\mathrm{H}_n(X, X \setminus \{x\}) \cong \mathbb{Z}$ and under the map[339] $\mathrm{H}_n(X, A) \to \mathrm{H}_n(X, X \setminus \{x\})$ the singular n-cycle $c \in \mathrm{C}_n(X, A)$ gets sent to a generator of $\mathrm{H}_n(X, X \setminus \{x\})$.

(2) If $\mathrm{H}_n(X, A) \cong \mathbb{Z}$, then c represents a generator of $\mathrm{H}_n(X, A)$.



Proof. Let (X, A) be a pair of topological spaces and let $c = \sum_{i=1}^{m} t_i \cdot \sigma_i \in \mathrm{C}_n(X)$ be a singular n-chain with the following two properties:

(a) c represents an n-cycle in $\mathrm{C}_n(X, A)$.

(b) There exists an $x \in X \setminus A$ and a $j \in \{1, \dots, m\}$ with $t_j = \{\pm 1\}$ such that the map $\sigma_j \colon \mathring{\Delta}^n \to X$ is an open embedding, such that $x \in \sigma_j(\mathring{\Delta}^n)$ and such that for all $i \neq j$ we have $x \notin \sigma_i(\Delta^n)$.

We turn to the actual proofs of the statements:

(1) First note that it follows from the hypothesis that $x \in \sigma_j(\mathring{\Delta}^n)$ and the hypothesis that $\sigma_j \colon \mathring{\Delta}^n \to X$ is an open embedding, together with the Local-H_*-Proposition 42.2 that $\mathrm{H}_n(X, X \setminus \{x\}) \cong \mathbb{Z}$ and that $[\sigma_j] \in \mathrm{H}_n(X, X \setminus \{x\})$ is a generator. Next we consider the induced homomorphism $\varphi \colon \mathrm{H}_n(X, A) \to \mathrm{H}_n(X, X \setminus \{x\})$. We see that

$$\varphi([c]) = \varphi\Big(\Big[\sum_{i=1}^{m} t_i \cdot \sigma_i\Big]\Big) = \underbrace{t_j}_{=\pm 1} \cdot [\sigma_j] + \varphi\Big(\Big[\sum_{i \neq j} \underbrace{t_j \cdot \sigma_j}_{\substack{= 0 \in \mathrm{C}_n(X, X \setminus \{x\}), \\ \text{since } \sigma_j(\Delta^n) \subset X \setminus \{x\}}}\Big]\Big) = \pm \cdot \underbrace{[\sigma_j]}_{\substack{\uparrow \\ \text{generator of} \\ \mathrm{H}_n(X, X \setminus \{x\})}}.$$

(2) It follows from the hypothesis that $\mathrm{H}_n(X, A) \cong \mathbb{Z}$ and elementary algebra that any epimorphism $\mathrm{H}_n(X, A) \to \mathbb{Z}$ is actually an isomorphism. Statement (2) is an immediate consequence of this observation together with Statement (1). ∎

42.3. **The standard generator of $\mathrm{H}_n(S^n)$.** In this section we will use the Local-H_*-Proposition 42.2 to finally exhibit explicit generators of $\widetilde{\mathrm{H}}_n(S^n) \cong \mathbb{Z}$ for any $n \in \mathbb{N}_0$.

We start out by introducing two singular n-simplices on S^n:

[339] Note that it follows from $x \in X \setminus A$ that $A \subset X \setminus \{x\}$, in particular this map is indeed defined.

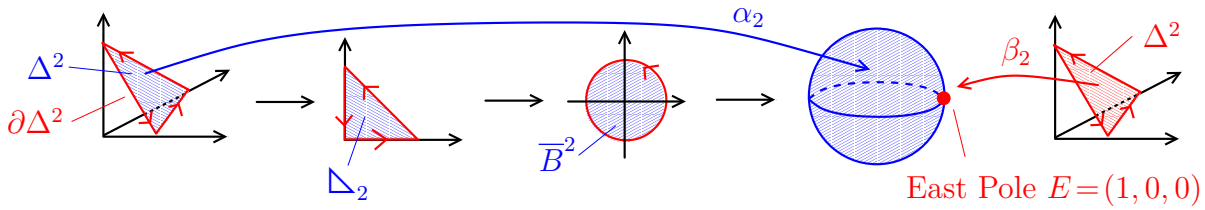
Notation. Let $n \in \mathbb{N}_0$.

(1) We consider the singular n-simplex $\alpha_n \colon \Delta^n \to S^n$ that is given by the concatenation of the following two continuous maps[340]

$$\begin{array}{rcccl} \Delta^n & \xrightarrow[\Delta^n\text{-Homeomorphism Lemma 35.3 (2)}]{\text{homeomorphism given by the}} & \overline{B}^n & \to & S^n \subset \mathbb{R}^{n+1} = \mathbb{R}\times\mathbb{R}^n \\ & & r\cdot v & \mapsto & \underbrace{\begin{pmatrix} -\cos(\pi\cdot r) \\ \sin(\pi\cdot r)\cdot v \end{pmatrix}}_{\in \mathbb{R}\times\mathbb{R}^n = \mathbb{R}^{n+1}}. \end{array}$$

Similar to the Ball Quotient-Sphere Lemma 6.5 we see that this map $\Delta^n \to S^n$ sends $\partial\Delta^n$ to the "East Pole" $E := (1,0,\dots,0)$ and that α_n descends to a homeomorphism $\Delta^n/\partial\Delta^n \to S^n$.

(2) We denote by $\beta_n \colon \Delta^n \to S^n$ the constant map that sends all points to the "East Pole" $(1,0,\dots,0)$.



Proposition 42.4. (Standard Generator-$\mathrm{H}_n(S^n)$-Proposition) Let $n \in \mathbb{N}_0$.

(1) The singular n-chain $\alpha_n - \beta_n \in \widetilde{\mathrm{C}}_n(S^n)$ is a cycle and it represents a generator of $\widetilde{\mathrm{H}}_n(S^n)$. (In particular for $n \geq 1$ it represents a generator of $\mathrm{H}_n(S^n)$.)

(2) If n is odd, then $\alpha_n \in \mathrm{C}_n(S^n)$ is also a cycle with $[\alpha_n] = [\alpha_n - \beta_n] \in \widetilde{\mathrm{H}}_n(S^n)$.

Definition. Let $n \in \mathbb{N}_0$. We refer to[341]

$$[S^n] \;:=\; (-1)^{n-1}\cdot[\alpha_n - \beta_n] \quad \in\; \widetilde{\mathrm{H}}_n(S^n)$$

as the standard generator of $\widetilde{\mathrm{H}}_n(S^n)$.

Example.

(1) For $n=0$ we see that $[S^0] = -[\alpha_0 - \beta_0] = -[(-1)-(1)] = [(1)-(-1)] \in \widetilde{\mathrm{H}}_0(S^0)$.

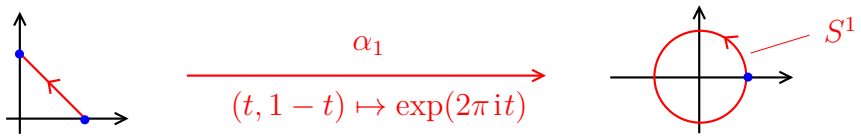
(2) Next we consider the case $n=1$. If one goes carefully through the definition of the map $\alpha_1 \colon \Delta^1 \to S^1$ one sees that

$$\alpha_1(1-t,t) \underset{\substack{\uparrow \\ \text{since the explicit homeomorphism } \Delta^1 \to \overline{B}^1 \text{ is} \\ \text{given by } (t,1-t)\mapsto 1-2t}}{=} (-\cos(\pi(1-2t)),\sin(\pi(1-2t))) \;=\; (\cos(2t\pi),\sin(2t\pi)) \;=\; \exp(2\pi\,\mathrm{i}t).$$

Thus we see that the singular 1-simplex $(1-t,t) \mapsto \exp(2\pi\,\mathrm{i}t)$ represents the standard generator of $\mathrm{H}_1(S^1)$. □

[340] The given map $\overline{B}^n \to S^n$ is a variation on the map given in the Ball Quotient-Sphere Lemma 6.5.

[341] The slightly arbitrary looking factor $(-1)^{n-1}$ will be helpful later to stay consistent with various orientation conventions. In a nutshell it comes from the fact, that if we orient Δ^n with the "outward pointing" normal vector and if we give S^n the standard orientation, then α_n is orientation-preserving if and only if n is odd.



Proof.

(1) We start out with the following claim:

Claim. The singular n-chain $\alpha_n - \beta_n \in \widetilde{\mathrm{C}}_n(S^n)$ is a cycle.

Proof. Let $\gamma\colon \Delta^{n-1} \to S^n$ be the singular $(n-1)$-simplex that sends all points to the "East Pole" $E = (1, 0, \dots, 0)$. For $n \neq 0$ we calculate that

$$\partial_n(\alpha_n - \beta_n) \;=\; \partial_n(\alpha_n) - \partial_n(\beta_n) \;=\; \underset{\uparrow}{\sum_{j=0}^{n} \cdot (-1)^j \cdot \underbrace{\alpha_n \circ i_j^n}_{\substack{=\gamma,\text{ since}\\ \alpha_n(\partial\Delta^n)=\{E\}}}} \;-\; \sum_{j=0}^{n} \cdot (-1)^j \cdot \underbrace{\beta_n \circ i_j^n}_{\substack{=\gamma,\text{ since}\\ \beta_n(\partial\Delta^n)=\{E\}}} \;=\; 0.$$

see the definition of ∂_n on page 459

For $n = 0$ the same argument also shows that $\epsilon_{S^0}(\alpha_0 - \beta_0) = 0$ where $\epsilon_{S^0}\colon \mathrm{C}_0(S^0) \to \mathbb{Z}$ denotes the augmentation map. ⊞

We set $x \coloneqq (-1, 0, \dots, 0)$. We consider the long exact sequence in reduced homology of the pair $(S^n, S^n \setminus \{x\})$:

$$\widetilde{\mathrm{H}}_n(S^n \setminus \{x\}) \;\to\; \widetilde{\mathrm{H}}_n(S^n) \;\xrightarrow{\text{я}}\; \mathrm{H}_n(S^n, S^n \setminus \{x\}) \;\xrightarrow{\partial_n}\; \widetilde{\mathrm{H}}_{n-1}(S^n \setminus \{x\}).$$

It follows from the Stereographic Projection Lemma 2.11 that $S^n \setminus \{x\}$ is homeomorphic to $\mathbb{R}^n$. This implies by the Contractible-$\widetilde{\mathrm{H}}_*$-Corollary 39.5 that $\widetilde{\mathrm{H}}_n(S^n \setminus \{x\}) = \widetilde{\mathrm{H}}_{n-1}(S^n \setminus \{x\}) = 0$. It follows that $\text{я}\colon \widetilde{\mathrm{H}}_n(S^n) \to \mathrm{H}_n(S^n, S^n \setminus \{x\})$ is an isomorphism.

Next note that it follows easily from the definitions that $\alpha_n - \beta_n$ is non-trivial with respect to the given $x \in S^n$. It follows from the Detect-Cycle Proposition 42.3 (1) that $\alpha_n - \beta_n$ represents a generator of $\mathrm{H}_n(S^n, S^n \setminus \{x\})$. Thus by the above it also represents a generator of $\widetilde{\mathrm{H}}_n(S^n)$.[342]

(2) If n is odd, then we know by the simple calculation in the proof of the Singleton-H_*-Lemma 35.12 that β_n is a cycle which is the boundary of the singular $(n+1)$-simplex that is given by the constant map $\gamma\colon \Delta^{n+1} \to (1, 0, \dots, 0)$. Thus we see that $[\alpha_n - \beta_n] = [\alpha_n - \partial\gamma] = [\alpha_n] \in \widetilde{\mathrm{H}}_n(S^n)$. ■

42.4. **Appendix: The topology of topological manifolds.** As the title of this appendix suggests, in this appendix we will consider some applications of homology groups to the study of topological manifolds. In particular we will prove several statements on the topology of topological manifolds which extend the results obtained in Section **??** from the setting of smooth manifolds to the more general context of topological manifolds.

First of all we now have the tools to fix an embarrassing gap in the earlier Topological Manifold–Boundary Proposition **??** and we obtain the following proposition, which establishes an almost impeccable symmetry with the Smooth Manifold–Boundary Proposition **??**:

[342] We could also have argued using the Detect-Cycle Proposition 42.3 (2). But it seems to us that it is more elegant to give a more direct proof.

Proposition 42.5. (Topological Manifold–Boundary Proposition) Let X be an n-dimensional topological manifold. The following statements hold:

(1) (a) Every point on X admits either a chart of type (i) or it admits a chart of type (ii).[343]
   (b) If F is equipped with a smooth atlas, i.e. if we view F as a smooth manifold, then the boundary of F as a topological manifold agrees with the boundary of F viewed as a topological manifold.

(2) The boundary ∂X is an $(n-1)$-dimensional topological manifold with empty boundary, i.e. we have $\partial(\partial X) = \varnothing$.

(3) (a) Any union of components of ∂X is a closed subset of X. In particular ∂X itself is a closed subset of X.
   (b) If X is compact, then ∂X is also compact.

(4) If ∂X is non-empty, then $X \setminus \partial X$ is non-compact.

(5) Let N be a submanifold of X. The following two statements are equivalent:
   (a) N is a proper submanifold.
   (b) The following two conditions are satisfied:
      (i) N is a closed subset of X.
      (ii) We have the equality[344] $\partial N = N \cap \partial X$.

(6) If $N \subset X$ is a proper k-dimensional submanifold, then we have $N \cap \partial X = \partial N$, and $N \cap \partial X = \partial N$ is a proper $(k-1)$-dimensional submanifold of ∂X.

(7) If X is an m-dimensional topological manifold and if Y is an n-dimensional topological manifold, then the product $X \times Y$ is an $(m+n)$-dimensional topological manifold with $\partial(X \times Y) = (\partial X \times Y) \cup (X \times \partial Y)$.

Example. It follows from the Topological Manifold–Boundary Proposition 42.5 and the examples of charts for $\overline{B}^n$ given on page **??** that the boundary of the n-dimensional topological manifold is indeed S^{n-1}. We had suspected this for a long time, but only recently did we acquire the tools for providing a proof. □

Remark.

(1) In the Smooth Manifold Product Propositions **??** and **??** we showed the analogue of the Topological Manifold–Boundary Proposition 42.5 (7) in the setting of smooth manifolds.

(2) Let M and N be two connected n-dimensional topological manifolds with empty boundary. If $M \times [0,1]$ is homeomorphic to $N \times [0,1]$, then evidently the corresponding boundaries are also homeomorphic. It follows from the Topological Manifold–Boundary Proposition 42.5 (7) that $\partial(M \times [0,1]) = M \times \{0,1\}$ and $\partial(N \times [0,1]) = N \times \{0,1\}$. Since M and N are connected we obtain easily from the above that M and N are homeomorphic. Even though this sounds quite reasonable, very similar sounding statements can easily be wrong. For example there exists a 3-dimensional topological manifold W (called the *Whitehead manifold*) which is non-compact and has no boundary, which is

[343] For 2-dimensional topological manifolds we proved this statement in the Topological Surface–Boundary Proposition 22.3.

[344] Here ∂N denotes the boundary of N, viewed as a topological manifold in its own right.

not homeomorphic to $\mathbb{R}^3$ but such that $W \times \mathbb{R}$ is homeomorphic to $\mathbb{R}^3 \times \mathbb{R} = \mathbb{R}^4$. We refer to [**Kir89**, p. 87] and [**AR65**] for details. □

Proof. Let X be an n-dimensional topological manifold.

(1) (a) This statement is precisely the statement of the Manifold-Local-H$_*$-Lemma 61.1 (1). The only reason why we wrote down the statement again is to preserve the analogy with the Smooth Manifold–Boundary Proposition **??**.

(b) Now suppose that X is equipped with a smooth atlas. For the purpose of this proof we write $\partial^{\text{top}}X$ and $\partial^{\text{sm}}X$. By the definitions on pages **??** and **??** we have

$$\begin{aligned} \partial^{\text{top}}X &= \text{all points that do not admit a chart of type (i)} \\ \partial^{\text{sm}}X &= \text{all points that do not admit a chart of type (i) from the smooth atlas.} \end{aligned}$$

By definition we have $\partial^{\text{top}}X \subset \partial^{\text{sm}}X$. Now let $P \in \partial^{\text{sm}}X$. This means that P does not admit a chart of type (i) from the smooth atlas. By definition of a smooth manifold it admits a chart of type (ii). But by (1) this means that it cannot admit any chart of type (i). Thus we see that $P \in \partial^{\text{sm}}X$.

(2) We proved this in the original Topological Manifold–Boundary Proposition **??** (2).

(3) We proved this in the original Topological Manifold–Boundary Proposition **??** (3).

(4) We proved this in the original Topological Manifold–Boundary Proposition **??** (4).

(5) As we already saw in the proof of the Smooth Manifold–Boundary Proposition **??** (5), this statement is straightforward consequence of (1) and (4).

(6) As we already saw in the proof of the Smooth Manifold–Boundary Proposition **??** (6), this statement is straightforward consequence of (1) and (4).

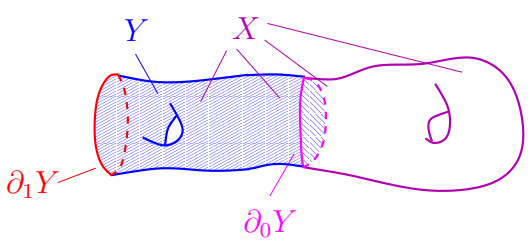
(7) In the Topological Manifold Product Proposition 62.6 we already showed that $X \times Y$ is an $(m+n)$-dimensional topological manifold. Furthermore recall that in the Topological Manifold Product Proposition 62.6 we showed that each point in $(X \times Y) \setminus ((\partial X \times Y) \cup (X \times \partial Y))$ admits a chart of type (i) and we showed that each point in $(\partial X \times Y) \cup (X \times \partial Y)$ admits a chart of type (ii). By (1) and by definition of the boundary of a topological manifold this means that we have indeed $\partial(X \times Y) = (\partial X \times Y) \cup (X \times \partial Y)$. ■

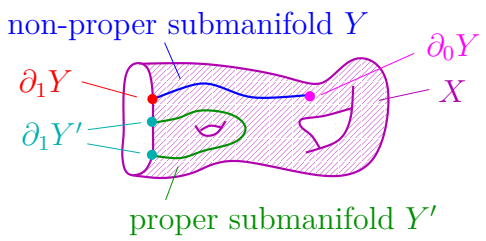
Let X be a topological manifold and let Y be a submanifold. It follows almost immediately from the definitions that Y is a topological manifold in its own right. In particular it makes sense to consider its boundary ∂Y and in particular it makes sense to introduce the following definition which can be viewed as a generalization of the definition on page **??**.

Definition. Let X be a topological manifold and let Y be a submanifold. We write

$$\partial_0 Y \coloneqq \underset{\uparrow}{\partial Y} \setminus \partial X \quad \text{and} \quad \partial_1 Y \coloneqq \underset{\uparrow}{\partial Y} \cap \partial X.$$

here ∂Y denotes the boundary of Y viewed as a topological manifold





The final proposition of this chapter gives us some results on the boundary of submanifolds. It can be viewed as an analogue of the Codimension-Zero Submanifold Proposition **??**, which we had formulated in the smooth context.

Proposition 42.6. (Topological Submanifold Proposition) Let X be an m-dimensional topological manifold and let Y be an n-dimensional submanifold. We assume that Y is a closed subset of X. (In most applications Y will be compact).

(0) (a) $\partial_0 Y$ and $\partial_1 Y$ are disjoint and each of them is a union of components of ∂Y.
(b) $\partial_0 Y$ is a submanifold of X.
(1) If Y is a proper submanifold, then $\partial Y = \partial_1 Y = Y \cap \partial X$.
(2) If Y is of codimension zero, then the interior $\overset{\circ}{Y}$ of Y as a subset of X agrees with $Y \setminus \partial_0 Y$. In particular $Y \setminus \partial_0 Y$ is an open subset of X.
(3) Suppose that Y is of codimension zero and that Y is compact.
(a) The set $Z := X \setminus \overset{\circ}{Y}$ is also a codimension-zero submanifold of X and we have $Y \cap Z = \partial_0 Y = \partial_0 Z$.
(4) If X is compact, then $Z = X \setminus \overset{\circ}{Y}$ is also compact.
(5) If X and ∂Y are connected, then $Z = X \setminus \overset{\circ}{Y}$ is also connected.

Proof. The proposition follows easily from making the obvious modifications to the proof of the Codimension-Zero Submanifold Proposition **??**. The biggest change is that we need to replace the Smooth Manifold–Boundary Proposition **??** by the Topological Manifold–Boundary Proposition 42.5. We leave it to the reader to fill in the details. ■

CHAPTER 43

Homology and self-maps of spheres

To understand the concept of homology we need to be able to compute homology groups of interesting topological spaces *and* we should acquire techniques for calculating induced maps $f_*\colon \mathrm{H}_n(X) \to \mathrm{H}_n(Y)$. In the last chapters we got to know the homology groups of spheres quite well. In this and the following chapter we will get to know several techniques for determining induced maps $f_*\colon \widetilde{\mathrm{H}}_n(S^n) \to \widetilde{\mathrm{H}}_n(S^n)$.

These results will allow us to prove various interesting results, e.g. the mysterious sounding Hairy Ball Theorem.

43.1. **The degree of a self-map of a sphere.** If we want to understand the concept of homology properly, then not only do we need to determine the isomorphism types of homology groups of interesting topological spaces, but we also need to determine induced maps on homology. In this section we will study the maps on homology induced by continuous self-maps of spheres. This topic is interesting in its own right, but the results will also play a major role later on when we will study Mayer-Vietoris sequences and when we try to determine the homology groups of CW-complexes.

Throughout this chapter we will work a lot with the following simple-minded definition:

Definition. Let $(G,+)$ be a group that is isomorphic to $\mathbb{Z}$ and let $\Phi\colon G \to G$ be a homomorphism. Note that Φ is given by multiplication by a unique integer which we refer to as the degree $\deg(\Phi)$.

Next we relate the above definition to topology.

Definition. Let $n \in \mathbb{N}_0$ and let $f\colon S^n \to S^n$ be a continuous map. By the Sphere-H_*-Proposition 41.2 we know that $\widetilde{\mathrm{H}}_n(S^n)$ is isomorphic to $\mathbb{Z}$. Thus we can define the degree of f:

$$\deg(f) \;:=\; \deg\big(f_*\colon \widetilde{\mathrm{H}}_n(S^n) \to \widetilde{\mathrm{H}}_n(S^n)\big).$$

The following lemma summarizes several basic properties of the degree of continuous self-maps of spheres.

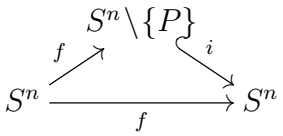
Lemma 43.1. (Sphere–Degree Lemma) Let $n \in \mathbb{N}_0$ and let $f, g\colon S^n \to S^n$ be two continuous maps.

(1) $\deg(\mathrm{id}_{S^n}) = 1$.
(2) If f is not surjective, then $\deg(f) = 0$.
(3) If f is homotopic to g, then $\deg(f) = \deg(g)$.
(4) $\deg(f \circ g) = \deg(f) \cdot \deg(g)$.
(5) If f is a self-homeomorphism of S^n, then $\deg(f^{-1}) = \deg(f)$.

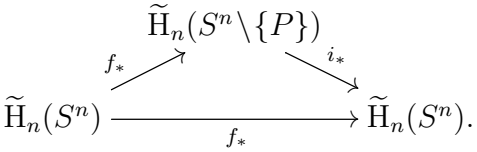
Proof.

(1) Since $(\mathrm{id}_{S^n})_* = \mathrm{id}_{\widetilde{\mathrm{H}}_n(S^n)}$ we know that $\deg(\mathrm{id}_{S^n}) = 1$.

(2) Suppose that the map $f\colon S^n \to S^n$ misses the point $P \in S^n$. We consider the commutative diagram



which gives rise to the commutative diagram



As we already mentioned on page 541, $S^n \setminus \{P\}$ is homeomorphic to $\mathbb{R}^n$, which implies by the Contractible-$\widetilde{\mathrm{H}}_*$-Corollary 39.5 that $\widetilde{\mathrm{H}}_n(S^n \setminus \{P\}) = 0$. But this means that $f_*\colon \widetilde{\mathrm{H}}_n(S^n) \to \widetilde{\mathrm{H}}_n(S^n)$ factors through the zero group, i.e. f_* is the zero map.

(3) If f is homotopic to g, then it follows from the Homotopy-$\widetilde{\mathrm{H}}_*$-Proposition 39.3 that $\deg(f) = \deg(g)$.

(4) We consider the following diagram:

$$\widetilde{\mathrm{H}}_n(S^n) \xrightarrow[\cdot \deg(f)]{f_*} \widetilde{\mathrm{H}}_n(S^n) \xrightarrow[\cdot \deg(g)]{g_*} \widetilde{\mathrm{H}}_n(S^n). \qquad (f\circ g)_*$$

This diagram commutes since $(f \circ g)_* = f_* \circ g_*$. It follows that the curved map is given by multiplication by $\deg(f) \cdot \deg(g)$, i.e. $\deg(f \circ g) = \deg(f) \cdot \deg(g)$.

(5) This statement follows immediately from (1) and (4). ∎

In the following we want to determine the degrees of interesting self-maps of spheres. We start out with self-maps of circles:

Lemma 43.2. (Degree-of-Power Lemma) Given $k \in \mathbb{Z}$ we have[345]

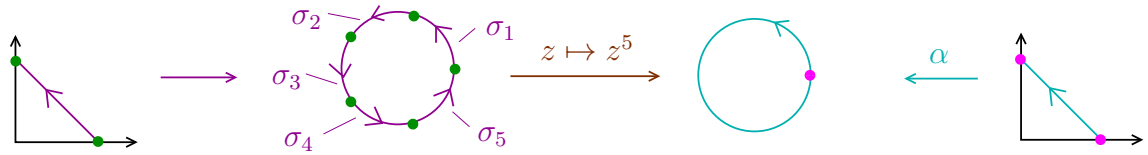
$$\deg \begin{pmatrix} S^1 & \to & S^1 \\ z & \mapsto & z^k \end{pmatrix} = k.$$

Proof. Let $k \in \mathbb{Z}$. We denote by $f\colon S^1 \to S^1$ the map that is given by $z \mapsto z^k$. We need to show that $\deg(f) = k$. Let $\alpha\colon \Delta^1 \to S^1$ be the singular 1-simplex that is given by $\alpha(1-t,t) = \exp(2\pi \mathrm{i} t)$. By the discussion on page 540 we know that $\alpha \in \mathrm{C}_1(S^1)$ is a cycle and $[\alpha] \in \mathrm{H}_1(S^1)$ is a generator.

First we first consider the case $k \geq 1$. For $j = 0, \ldots, k-1$ we denote by $\sigma_j\colon \Delta^1 \to S^1$ the singular 1-simplex that is given by $(1-t,t) \mapsto \exp(2\pi \mathrm{i}(\frac{t+j}{k}))$. Then

$$f_*([\alpha]) \underset{\text{by the Singular 1-Simplex Lemma 35.13 (3)}}{=} f_*([\sigma_0 + \cdots + \sigma_{k-1}]) = [\underbrace{f \circ \sigma_0}_{=\alpha} + \cdots + \underbrace{f \circ \sigma_{k-1}}_{=\alpha}] = k \cdot [\alpha].$$

It follows from this calculation that $\deg(f) = k$.



[345] This result is analogous to the fact, observed on page 199, that for fundamental groups the induced map $f_*\colon \pi_1(S^1,1) \to \pi_1(S^1,1)$ is given by multiplication by k.

The case $k < 0$ is proved is proved in a similar fashion using the Singular 1-Simplex Lemma 35.13 (2). Finally, if $k = 0$, then $f\colon S^1 \to S^1$ is not surjective, thus it follows from the Sphere–Degree Lemma 43.1 (2) that $\deg(f) = 0$. ■

The combination of the Degree-of-Power Lemma 43.2 and the following lemma shows that every $k \in \mathbb{Z}$ is realized as the degree of any sphere S^n of dimension $n \geq 1$.

Lemma 43.3. (Degree-Suspension Lemma) Let $n \in \mathbb{N}_0$.

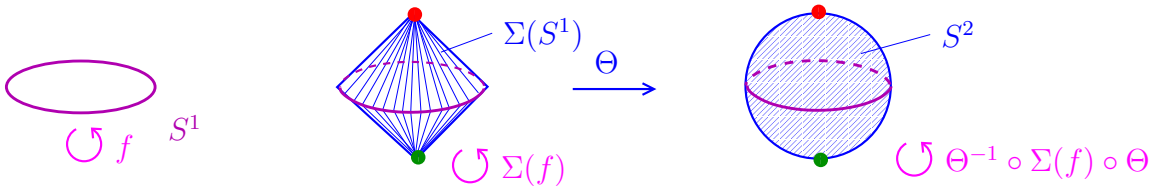
(1) The map

$$\begin{array}{rcl} \Theta\colon \Sigma(S^n) & \to & S^{n+1} \subset \mathbb{R}^{n+1} \times \mathbb{R} \\ [(P,t)] & \mapsto & \big(\underbrace{P \cdot \cos\left(\frac{\pi}{2}t\right)}_{\in \mathbb{R}^{n+1}}, \underbrace{\sin\left(\frac{\pi}{2}t\right)}_{\in \mathbb{R}}\big) \end{array}$$

is a homeomorphism.

(2) Let $f\colon S^n \to S^n$ be a continuous map. Let $\Sigma(f)\colon \Sigma(S^n) \to \Sigma(S^n)$ be the suspension of f that we defined on page 522. Furthermore let $\Theta\colon \Sigma(S^n) \to S^{n+1}$ be any homeomorphism. Then

$$\deg(\Theta \circ \Sigma(f) \circ \Theta^{-1}\colon S^{n+1} \to S^{n+1}) \;=\; \deg(f\colon S^n \to S^n).$$



Proof. Let $n \in \mathbb{N}_0$.

(1) One can easily verify that the given map $\Theta\colon \Sigma(S^n) \to S^{n+1}$ is a homeomorphism.

(2) Let $f\colon S^n \to S^n$ be a continuous map and let $\Theta\colon \Sigma(S^n) \to S^{n+1}$ be a homeomorphism. We consider the following diagram:

$$\begin{array}{ccccc} \widetilde{\mathrm{H}}_n(S^n) & \xleftarrow[\cong]{\Sigma_{S^{n+1}}} & \widetilde{\mathrm{H}}_{n+1}(\Sigma(S^n)) & \xrightarrow[\cong]{\Theta_*} & \widetilde{\mathrm{H}}_{n+1}(S^{n+1}) \\ {\scriptstyle \cdot \deg(f)}\downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle \Sigma(f)_*} & & \downarrow{\scriptstyle (\Theta \circ \Sigma(f) \circ \Theta^{-1})_*} \\ \widetilde{\mathrm{H}}_n(S^n) & \xleftarrow[\cong]{\Sigma_{S^{n+1}}} & \widetilde{\mathrm{H}}_{n+1}(\Sigma(S^n)) & \xrightarrow[\cong]{\Theta_*} & \widetilde{\mathrm{H}}_{n+1}(S^{n+1}). \end{array}$$

First note that it follows from the Suspension-H_*-Proposition 41.1 (2) that the square to the left commutes and that the horizontal maps to the left are isomorphisms. It is clear that the square to the right commutes. Furthermore, since Θ is a homeomorphism we see that the horizontal maps to right are isomorphisms. It follows immediately from the above discussion together with the basic observation that $\deg(\Theta \circ \Sigma(f) \circ \Theta^{-1}) = \deg(f)$. ■

We move on to our final calculation of a degree:

Lemma 43.4. (Sphere–Degree–Matrix Lemma) Given any $A \in \mathrm{O}(n+1)$ we have

$$\deg\begin{pmatrix} S^n & \to & S^n \\ x & \mapsto & A \cdot x \end{pmatrix} \;=\; \underbrace{\det(A)}_{\in\{\pm 1\}}.$$

Example.

(1) If $f\colon S^n \to S^n$ is the reflection in any hyperplane of $\mathbb{R}^{n+1}$, then it follows from elementary linear algebra and the Sphere–Degree–Matrix Lemma 43.4 that $\deg(f) = -1$,
(2) If $f\colon S^n \to S^n$ is the reflection in the origin i.e. if $f(x) = -x$ for all $x \in S^n$, then it follows from the Sphere–Degree–Matrix Lemma 43.4 that $\deg(f) = \det(-\operatorname{id}_{n+1}) = (-1)^{n+1}$. □

Proof. We start out with the following claim:

Claim. Given $n \in \mathbb{N}_0$ we denote by $\rho_n\colon S^n \to S^n$ the reflection in the $(x_1{=}0)$-hyperplane. For any $n \in \mathbb{N}_0$ we have $\deg(\rho_n) = -1$.

Proof. We start out with the case $n = 0$. We denote by $\sigma_+\colon \Delta^0 \to S^0$ and $\sigma_-\colon \Delta^0 \to S^0$ the singular 0-simplices that are given by $\{1\} \mapsto \{1\}$ and $\{1\} \mapsto \{-1\}$. With this notation let us consider the reduced chain complex of S^0:

$$\ldots \to \mathrm{C}_1(S^0) \xrightarrow[\substack{=0,\text{ see}\\ \text{page 459}}]{\partial_1} \underbrace{\mathrm{C}_0(S^0)}_{=\mathbb{Z}\cdot\sigma_+\oplus\mathbb{Z}\cdot\sigma_-} \xrightarrow[=\epsilon_{S^0}]{\substack{\sigma_+ \mapsto 1\\ \sigma_- \mapsto 1}} \mathbb{Z} \to 0.$$

We see that $\widetilde{\mathrm{H}}_0(S^0) = \ker(\epsilon_{S^0}\colon \mathbb{Z}\cdot\sigma_+ \oplus \mathbb{Z}\cdot\sigma_- \to \mathbb{Z}) = \mathbb{Z}\cdot(\sigma_+ - \sigma_-)$. Thus we see that

$$\rho_{0*}(\sigma_+ - \sigma_-) = \underbrace{\rho_0 \circ \sigma_+}_{=\sigma_-} - \underbrace{\rho_0 \circ \sigma_-}_{=\sigma_+} = \sigma_- - \sigma_+ = (-1)\cdot(\sigma_+ - \sigma_-).$$

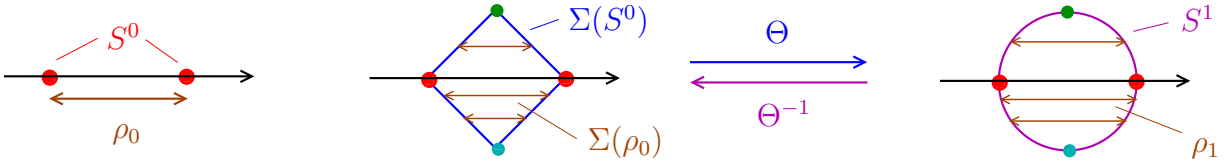
It follows that $\deg(\rho_0) = -1$. We have thus proved the desired statement for $n = 0$.

Now we turn to the proof of the induction step. Thus we assume that we know that the statement holds for some $n \in \mathbb{N}_0$. Let

$$\begin{array}{rcl} \Theta\colon \Sigma(S^n) & \xrightarrow{\cong} & S^{n+1} \subset \mathbb{R}^{n+1} \times \mathbb{R} \\ [(P,t)] & \mapsto & (P\cdot\cos(\frac{\pi}{2}t), \sin(\frac{\pi}{2}t)) \end{array}$$

be the homeomorphism from page 521. It follows immediately from the explicit definitions of the maps that for each $k \in \mathbb{N}_0$ the following diagram commutes:

$$\begin{array}{ccc} \Sigma(S^n) & \xrightarrow{\Theta} & S^{n+1} \\ {\scriptstyle \Sigma(\rho_n)}\downarrow & & \downarrow{\scriptstyle \rho_{n+1}} \\ \Sigma(S^n) & \xrightarrow{\Theta} & S^{n+1}. \end{array}$$



Thus it follows from the Degree-Suspension Lemma 43.3 and the induction hypothesis that $\deg(\rho_n) = \deg(\rho_{n-1}) = -1$. ⊞

Before we get to the denouement of the proof, it is helpful to introduce the following notation: given $B \in \mathrm{O}(n+1)$ we denote by $\mu(B)\colon S^n \to S^n$ the map that is given by $x \mapsto B \cdot x$.

Now let $A \in \mathrm{O}(n+1)$. First we consider the case $\det(A) = 1$, i.e. we assume that $A \in \mathrm{SO}(n+1)$. By the Matrix Group-π_0-Proposition 2.23 we know that $\mathrm{SO}(n+1)$ is

path-connected, so there exists a continuous map $h\colon [0,1] \to \mathrm{SO}(n+1)$ with $h(0) = \mathrm{id}$ and $h(1) = A$. But then $F(x,t) = h(t)\cdot x$ defines a homotopy from $\mathrm{id}_{S^n} = \mu(\mathrm{id})$ to $\mu(A)$. It follows from the Sphere–Degree Lemma 43.1 (1) and (3) that $\deg(\mu(A)) = \deg(\mathrm{id}_{S^n}) = 1$.

Finally we consider the case that $\det(A) \neq 1$. Let $R \in \mathrm{O}(n+1)$ be the matrix such that $\mu(R) = \rho_n$. Now we see that

$$\deg(\mu(A)) \;=\; \deg(\mu(A\cdot R\cdot R^{-1})) \;\underset{\uparrow}{=}\; \deg(\mu(A\cdot R))\cdot\deg(\mu(R)) \;\underset{\uparrow}{=}\; 1\cdot(-1) \;=\; -1.$$

by the Sphere–Degree Lemma 43.1 (4); since $\det(A\cdot R) = 1$ we have $\deg(\mu(A\cdot R)) = 1$ and by the claim we have $\deg(\mu(R)) = -1$ ■

43.2. Application: The Hairy Ball Theorem.

Theorem 43.5. (Sphere-Degree Fixed Point Theorem) Let $f\colon S^n \to S^n$ be a continuous map. If $\deg(f) \neq (-1)^{n+1}$, then f has a fixed point.

Proof. Let $f\colon S^n \to S^n$ be a continuous map that does not admit a fixed point. This means that for all $x \in S^n$ we have $f(x) \neq x$. We need to show that $\deg(f) = (-1)^{n+1}$. We will do this by showing that f is homotopic to $-\mathrm{id}$.

Our hypothesis implies that $(1-t)\cdot f(x) \neq t\cdot x$ for all $t \in [0,1]$ and all $x \in S^n$.[346] Therefore we can consider the map

$$\begin{aligned} F\colon S^n \times [0,1] &\to S^n \\ (x,t) &\mapsto \frac{(1-t)\cdot f(x) - t\cdot x}{\|(1-t)\cdot f(x) - t\cdot x\|}. \end{aligned}$$

This map is a homotopy between the maps $x \mapsto f(x)$ and $x \mapsto -x$. Thus it follows that

$$\deg(f) \;\underset{\uparrow}{=}\; \deg(-\mathrm{id}) \;\underset{\uparrow}{=}\; (-1)^{n+1}.$$

Sphere–Degree Lemma 43.1 (3); Sphere–Degree–Matrix Lemma 43.4 ■

To formulate our next theorem we need to introduce or recall the following definition:

Definition. Let $M \subset \mathbb{R}^n$ be a submanifold. Given $P \in M \setminus \partial M$ we denote by $\mathrm{T}_P M \subset \mathbb{R}^n$ the tangent vector space. A vector field on M is a continuous map $v\colon M \to \mathbb{R}^n$ such that for any $P \in M$ we have $v(P) \in \mathrm{T}_P M$.

Examples.

(1) Let P be a point on S^n. As is explained in [**Fri24**], the tangent space is given by

$$\mathrm{T}_P S^n \;=\; \{v \in \mathbb{R}^{n+1} \mid \langle P, v\rangle = 0\}.$$

(2) For an odd-dimensional sphere S^{2k-1} a vector field is given by[347]

$$\begin{aligned} v\colon S^{2k-1} &\to \mathbb{R}^{2k} \\ (x_1, x_2, \ldots, x_{2k-1}, x_{2k}) &\mapsto (-x_2, x_1, \ldots, -x_{2k}, x_{2k-1}). \end{aligned}$$

This vector field has the property that it is nowhere-vanishing. In the case of the 1-sphere S^1 this is just the vector field of constant length 1 that points in the "counterclockwise direction".

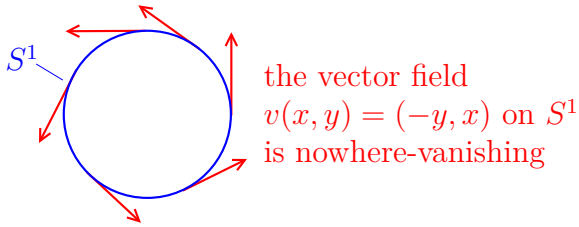
[346] This can be seen as follows: If $t \neq \frac{1}{2}$ we have $\|(1-t)\cdot f(x)\| = 1-t \neq t = \|t\cdot x\|$. On the other hand, for $t = \frac{1}{2}$ we have $(1-t)\cdot f(x) = \frac{1}{2}f(x) \neq \frac{1}{2}x = t\cdot x$.

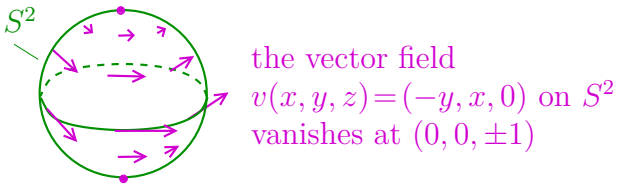
[347] Indeed, for any $P \in S^{2k-1}$ we have $\langle P, v(P)\rangle = 0$, i.e. $v(P)$ does indeed lie in the tangent space $\mathrm{T}_P S^{2k-1}$.

(3) For an even-dimensional sphere S^{2k} a vector field is similarly given by

$$\begin{array}{rcl} v\colon S^{2k} & \to & \mathbb{R}^{2k+1} \\ (x_1, x_2, \ldots, x_{2k-1}, x_{2k}, x_{2k+1}) & \mapsto & (-x_2, x_1, \ldots, -x_{2k}, 0). \end{array}$$

This vector field vanishes at the "North Pole" $(0, \ldots, 0, 1)$ and it also vanishes at the "South Pole" $(0, \ldots, 0, -1)$. □





The above examples raise the question whether it is possible to find a nowhere vanishing vector field on even-dimensional spheres. The following theorem shows that this is *not* possible.

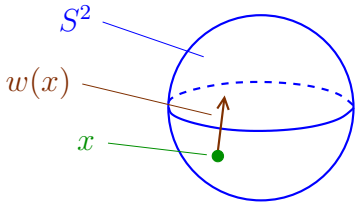
Theorem 43.6. (Hairy Ball Theorem) Every vector field on an even-dimensional sphere S^{2k} vanishes on at least one point.[348]

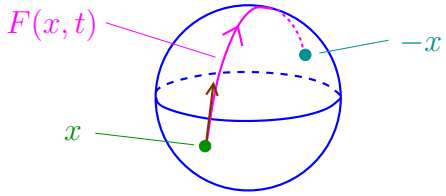
Proof. Let v be a vector field on the sphere S^n which is nowhere-vanishing. We have to show that n is odd. Since v is nowhere vanishing we can consider the vector field w on S^n that is given by rescaling v to length one, i.e. that is given by

$$\begin{array}{rcl} S^n & \to & \mathbb{R}^{n+1} \\ x & \mapsto & w(x) \coloneqq \dfrac{1}{\|v(x)\|} \cdot v(x). \end{array}$$

Furthermore we consider the map[349]

$$\begin{array}{rcl} F\colon S^n \times [0,1] & \to & S^n \\ (x, t) & \mapsto & \cos(\pi t) \cdot x + \sin(\pi t) \cdot w(x). \end{array}$$





The map $F_0 = F(-, 0)$ is the identity and the map $F_1 = F(-, 1)$ equals F is $-\mathrm{id}$. Put differently, the map F is a homotopy between the two maps id and $-\mathrm{id}$. It follows that

$$1 \underset{\text{by the Sphere–Degree Lemma 43.1 (1)}}{=} \deg(\mathrm{id}) \underset{\text{Lemma 43.1 (3)}}{=} \deg(-\mathrm{id}) \underset{\text{by the Sphere–Degree–Matrix Lemma 43.4}}{=} \det(-\mathrm{id}_{n+1}) = (-1)^{n+1}.$$

[348] On S^{2k} we just found a vector field which vanishes at *two* points. Can we find a vector field on S^{2k} which vanishes on only *one* point?

[349] The map does indeed take values in S^n, since

$$\langle \cos(\pi t) \cdot x + \sin(\pi t) \cdot w(x), \cos(\pi t) \cdot x + \sin(\pi t) \cdot w(x) \rangle \underset{\text{since } \langle x, w(x) \rangle = 0}{=} \cos(\pi t)^2 \cdot \|x\|^2 + \sin(\pi t)^2 \cdot \|w(x)\|^2 \underset{\text{since } \|x\| = \|w(x)\| = 1}{=} 1.$$

But this means that n is odd. ■

CHAPTER 44

Local degrees of self-maps of spheres

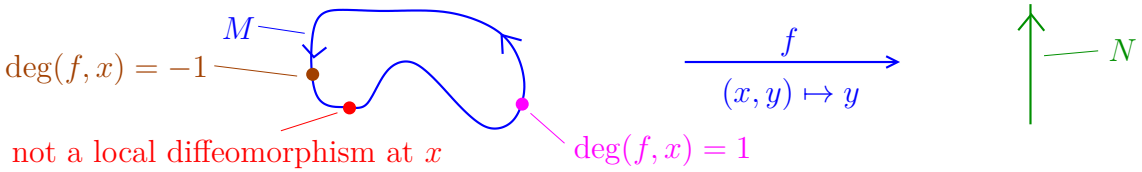
In the previous chapter we introduced the notion of the degree of a continuous self-map of a sphere. We managed to calculate the degree in simple situations, but so far we have no general scheme for calculating degrees. For example in the proof of the Degree-of-Power Lemma 43.2 we had to use a rather ad hoc argument to get the desired result.

In this chapter we will formulate and prove the S^n-Local-Global Degree Proposition 44.2 and the Degree-via-Local Orientations Proposition 44.7. These propositions show how one can determine the degree of a continuous map $f\colon S^n \to S^n$ by considering "local degrees" and "local information" of f.

44.1. **Degrees and local orientations.** We introduce the following harmless definition:

Definition. Let $f\colon M \to N$ be a continuous map between two oriented n-dimensional smooth manifolds and let $x \in M \setminus \partial M$. If f is a local diffeomorphism[350] at x, then we define the local degree of f at the point x as[351]

$$\deg(f,x) \coloneqq \begin{cases} +1, & \text{if } f \text{ is orientation-preserving at } x \\ -1, & \text{if } f \text{ is orientation-reversing at } x. \end{cases}$$



For most computational purposes the following proposition is the only result from this chapter one really needs:

Proposition 44.7. (Degree-via-Local Orientations Proposition) Let $n \in \mathbb{N}$. Let $f\colon S^n \to S^n$ be a continuous map and let $y \in S^n$ such that the following conditions are satisfied:

(1) $f^{-1}(y)$ consists of finitely many points $x_1, \dots, x_m$.
(2) The map f is a local diffeomorphism at each x_i.

Then
$$\deg(f) \;=\; \sum_{i=1}^{m} \deg(f,x_i) \;=\; \sum_{i=1}^{m} \begin{cases} 1, & \text{if } f \text{ is orientation-preserving at } x_i, \\ -1, & \text{if } f \text{ is orientation-reversing at } x_i. \end{cases}$$

As the number of the proposition suggests, it will take a while to get to the proof. Meanwhile let us consider some examples:

[352]By the Inverse Mapping Theorem, see [**Fri24**], the map f is a local diffeomorphism at $x \in M$ if there exists an open neighborhood U around x such that $f\colon U \to N$ is smooth and such that $\mathrm{D}f_x\colon \mathrm{T}_xM \to \mathrm{T}_xN$ is an isomorphism.

Examples.

(1) Let $m \in \mathbb{Z}$. We consider the map

$$\begin{array}{rcl} f\colon S^1 & \to & S^1 \\ z & \mapsto & z^m \end{array}$$

In the Degree-of-Power Lemma 43.2 we already proved that $\deg(f) = m$. Now we will show, just for fun, that the degree of f can also easily be determined using the Degree-via-Local Orientations Proposition 44.7. First let $m \neq 0$. Then we choose $y = 1$. The preimages of y are given by $x_j = \exp(2\pi \mathrm{i} j/m)$ where $j = 1, \ldots, m$. It follows easily from the definitions that for each $j \in \{1, \ldots, m\}$ we have

$$\deg(f, x_j) = \begin{cases} 1, & \text{if } m > 0, \\ -1, & \text{if } m < 0. \end{cases}$$

It follows from this calculation and the Degree-via-Local Orientations Proposition 44.7 that $\deg(f) = |m| \cdot \operatorname{sign}(m) = m$. If $m = 0$, then we take $y = -1$. Note that for $m = 0$ we have $f^{-1}(y) = \varnothing$, thus we see that $\deg(f) = 0$.

(2) If $f\colon S^n \to S^n$ is a diffeomorphism, then we can apply the Degree-via-Local Orientations Proposition 44.7 with any $y \in S^n$ and we obtain that

$$\deg(f) = \begin{cases} 1, & \text{if } f \text{ is orientation-preserving}, \\ -1, & \text{if } f \text{ is orientation-reversing}. \end{cases}$$

□

44.2. **The local degree.** In this section we define, for continuous self-maps $f\colon S^n \to S^n$ and suitable points in S^n a "local degree". We will then prove that the degree of a continuous map $S^n \to S^n$ is determined by the local degrees. In the next section we will relate the new notion of local degree to the notion introduced in the beginning of the chapter. This will then allow us to finally prove the Degree-via-Local Orientations Proposition 44.7 which we formulated in the beginning of the chapter.

Definition. We say that a map $f\colon X \to Y$ between two topological spaces is nice at a point $x \in X$ if there exists a neighborhood U of x such that for all $x' \neq x$ in U we have $f(x') \neq f(x)$. We refer to such a neighborhood as a nice neighborhood of x.

Examples.

(1) Let $f\colon X \to Y$ be a continuous map between two topological spaces and let $x \in X$. If U is a neighborhood such that $f|_U$ is injective, then U is evidently a nice neighborhood of x.

(2) We consider the maps

$$\begin{array}{rcl} f\colon \mathbb{R} & \to & \mathbb{R} \\ t & \mapsto & t^2 \end{array} \quad \text{and} \quad \begin{array}{rcl} g\colon \mathbb{R} & \to & \mathbb{R} \\ t & \mapsto & \begin{cases} 0, & \text{if } t = 0, \\ t \cdot \sin(\frac{1}{t}), & \text{if } t \neq 0. \end{cases} \end{array}$$

For the map f every point on $\mathbb{R}$ is nice, in fact for $x < 0$ we can take $U = (-\infty, 0)$, for $x > 0$ we can take $U = (0, \infty)$ and for $x = 0$ we can take $U = \mathbb{R}$. On the other hand it is straightforward to see that for the map g the point $x = 0$ is not nice. □

The following is the key definition of this section.

Definition. Let $n \geq 1$, let $f\colon S^n \to S^n$ be a continuous map and let $x \in S^n$ be a nice point. Let U be a nice neighborhood of x. The local degree of f at the point x is defined as the degree[353] of the homomorphism

$$\begin{array}{ccccccccc} & & & & \text{isomorphism by the Excision Theorem 40.2} & & & & \\ & & & & \downarrow \cong & & & & \\ \mathrm{H}_n(S^n) & \underset{\cong}{\overset{\text{я}}{\longrightarrow}} & \mathrm{H}_n(S^n, S^n\backslash\{x\}) & \overset{\cong}{\longleftarrow} & \mathrm{H}_n(U, U\backslash\{x\}) & \underset{354}{\overset{f_*}{\longrightarrow}} & \mathrm{H}_n(S^n, S^n\backslash\{f(x)\}) & \underset{\cong}{\overset{\text{я}}{\longleftarrow}} & \mathrm{H}_n(S^n). \\ & \uparrow & & & & & & \uparrow & \\ & \text{the long exact sequence of the pair } (S^n, S^n \setminus \{y\}) \text{ shows that these maps are isomorphisms} & & & & & & & \end{array}$$

We denote this local degree by $\deg(f, x)$.

Lemma 44.1. (Local Degree–Well-Defined Lemma) The local degree $\deg(f, x)$ of a map $f\colon S^n \to S^n$ at a nice point $x \in S^n$ is well-defined.

Proof. Let $n \in \mathbb{N}$, let $f\colon S^n \to S^n$ be a continuous map and let $x \in S^n$ be a nice point. Let U, U' be nice neighborhoods of x. After replacing U' by $U \cap U'$ we can without loss of generality assume that $U' \subset U$. We consider the following diagram:

$$\begin{array}{ccccccc} & & \mathrm{H}_n(U', U'\backslash\{x\}) & \xrightarrow{f_*} & \mathrm{H}_n(S^n, S^n\backslash\{f(x)\}) & & \\ & \overset{\cong}{\swarrow} & \downarrow \cong & & \downarrow \mathrm{id} & \overset{\cong}{\nwarrow} & \\ \mathrm{H}_n(S^n) \longrightarrow \mathrm{H}_n(S^n, S^n\backslash\{x\}) & & & & & & \mathrm{H}_n(S^n). \\ & \underset{\cong}{\nwarrow} & & & & \underset{\cong}{\swarrow} & \\ & & \mathrm{H}_n(U, U\backslash\{x\}) & \xrightarrow{f_*} & \mathrm{H}_n(S^n, S^n\backslash\{f(x)\}). & & \end{array}$$

All the non-decorated maps are induced by inclusions. It follows from the functoriality of relative homology that the diagram commutes. Also note that the left vertical map is an isomorphism by the Excision Theorem 40.2. It follows immediately from these observations that the local degree defined using U' agrees with the local degree defined using U. ■

Using the following proposition we can determine the degree of a continuous map from local degrees.

Proposition 44.2. (S^n-Local-Global Degree Proposition) Let $n \in \mathbb{N}$, let $f\colon S^n \to S^n$ be a continuous map and let $y \in S^n$ such that $f^{-1}(y)$ consists of finitely many points $x_1, \ldots, x_m$. Then all the points $x_1, \ldots, x_m$ are nice and we obtain the equality

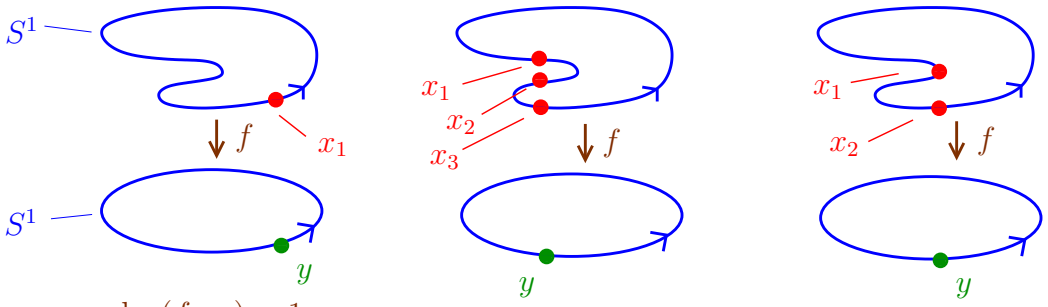
$$\deg(f) \;=\; \sum_{i=1}^{m} \deg(f, x_i).$$

Example. In the figure below we sketch a map $f\colon S^1 \to S^1$ and we choose several points $y \in S^1$.
We see that the number of preimages and the local degrees at the points in the preimages vary, but in all three cases the sum of the local degrees at the points in the preimage is the same. □

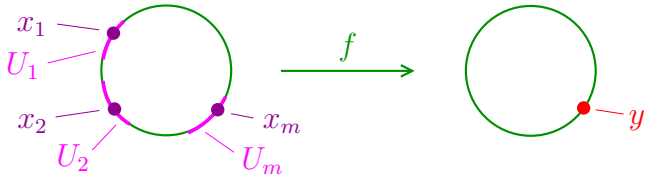
Proof. Let $n \in \mathbb{N}$, let $f\colon S^n \to S^n$ be a continuous map and let $y \in S^n$ such that $f^{-1}(y)$ consists of finitely many points $x_1, \ldots, x_m$. It follows easily from the fact that

[353] By the Sphere-H_*-Proposition 41.2 we know that $\mathrm{H}_n(S^n) \cong \mathbb{Z}$. As on page 545 we note that any homomorphism $\mathrm{H}_n(S^n) \to \mathrm{H}_n(S^n)$ is multiplication by an integer, which we call the degree of this homomorphism.

[354] Note that f induces a map of pairs $(U, U \setminus \{x\}) \to (S^n, S^n \setminus \{f(x)\})$ since U is nice.



S^n is Hausdorff that we can choose pairwise disjoint open neighborhoods $U_1, \dots, U_m$ for $x_1, \dots, x_m$. Note that this observation implies already that all the x_i are nice.



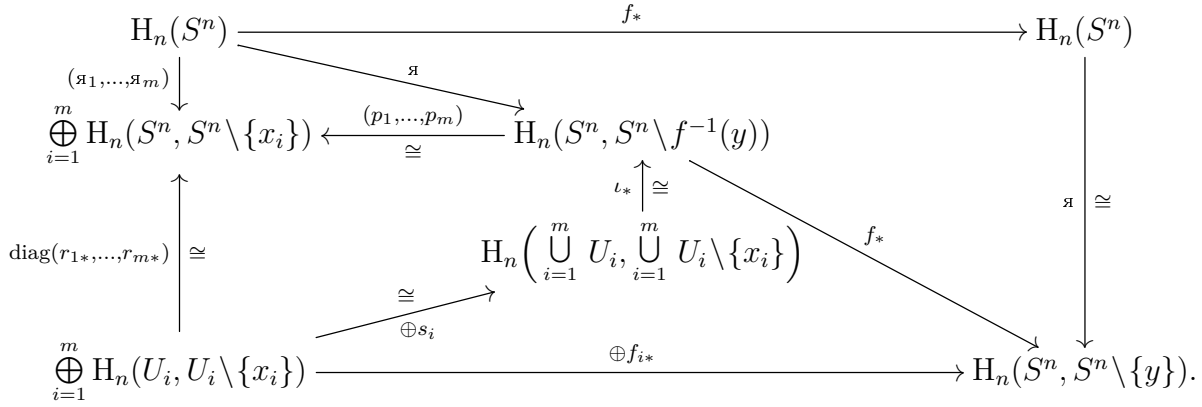
In the following we adopt the following notation:

(1) Given homomorphisms $\varphi_i \colon A_i \to B$ we write $\bigoplus_{i=1}^m \varphi_i \colon \bigoplus_{i=1}^m A_i \to B$.

(2) Given homomorphisms $\varphi_i \colon A \to B_i$ we write $(\varphi_1, \dots, \varphi_m) \colon A \to \bigoplus_{i=1}^m B_i$.

(3) Given homomorphisms $\varphi_i \colon A_i \to B_i$ we write $\operatorname{diag}(\varphi_1, \dots, \varphi_m) \colon \bigoplus_{i=1}^m A_i \to \bigoplus_{i=1}^m B_i$.

We consider the following diagram:



We make the following clarifications and observations:

(1) The maps r_i, s_i, p_i and ι are the maps induced by the obvious inclusions of pairs of topological spaces. Furthermore the maps $f_i \colon (U_i, U_i \setminus \{x_i\}) \to (S^n, S^n \setminus \{y\})$ are given by the restriction of f to U_i.

(2) It follows pretty easily from the functoriality of relative homology groups and elementary considerations that the diagram commutes.

(3) It follows easily from the fact that the U_i are disjoint open subsets of S^n and the Path-Component–H_*–Lemma 35.8 that the diagonal map $\bigoplus_{i=1}^{m} s_i$ to the bottom left is an isomorphism.
(4) It follows from the Excision Theorem 40.2 that the vertical map ι_* and the maps r_{i*} are isomorphisms.
(5) It follows from the fact that the diagram commutes together with (3) and (4) that the horizontal map $(p_1, \dots, p_m)$ is also an isomorphism.
(6) As we pointed out on page 541, it follows from the Contractible-$\widetilde{\mathrm{H}}_*$-Corollary 39.5 that the reduced homology groups of $S^n \setminus \{y\}$ vanish. Since $n \geq 1$ it follows from the LES-of-Pair Proposition 38.3 that $\text{я}\colon \mathrm{H}_n(S^n) \to \mathrm{H}_n(S^n, S^n \setminus \{y\})$ is an isomorphism.

By (6) it suffices to prove the following claim:

Claim. We have $$\text{я}(f_*([S^n])) \;=\; \sum_{i=1}^{m} \deg(f, x_i) \cdot \text{я}([S^n]) \;\in\; \mathrm{H}_n(S^n, S^n \setminus \{y\}).$$

Proof. We verify the claim by performing the following calculation:

$$\begin{aligned}
\text{я}(f_*([S^n])) &\overset{\text{since the diagram commutes and since the map } \mathrm{diag}(r_{1*}, \dots, r_{m*}) \text{ is an isomorphism}}{=} \Big(\bigoplus_{i=1}^{m} f_{i*}\Big) \circ \mathrm{diag}(r_{1*}, \dots, r_{m*})^{-1} \circ (\text{я}_1, \dots, \text{я}_m)([S^n]) \overset{\text{since each } r_{i*} \text{ is an isomorphism}}{=} \\
&= \Big(\bigoplus_{i=1}^{m} f_{i*}\Big) \circ (r_{1*}^{-1} \circ \text{я}_1, \dots, r_{m*}^{-1} \circ \text{я}_m)([S^n]) \\
&\underset{\text{since we always have the equality } \left(\left(\bigoplus_{i=1}^{m} \varphi_i\right) \circ (\psi_1, \dots, \psi_m)\right)(x) = \sum_{i=1}^{m} (\varphi_i \circ \psi_i)(x)}{=} \sum_{i=1}^{m} (f_{i*} \circ r_{i*}^{-1} \circ \text{я}_i)([S^n]) \overset{\text{by the definition of the local degree, see page 554}}{=} \sum_{i=1}^{m} \deg(f, x_i) \cdot \text{я}([S^n]).
\end{aligned}$$

■

44.3. **The local degree and local diffeomorphisms.** In the previous section we introduced the notion of a local degree of a continuous self-map of a sphere. In this section we will see that for local diffeomorphisms the above notion of a local degree matches the definition that we gave in the beginning of this chapter.

Our proof of the equality of the two local degrees relies on the following proposition, which is interesting in its own right.

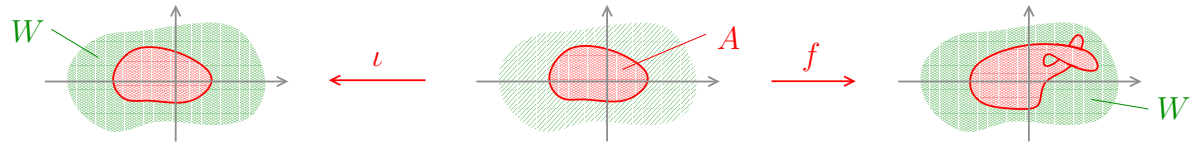
Proposition 44.3. (Local Diffeomorphism-Degree Proposition) Let W be a subset of $\mathbb{R}^n$ and let A be a subset of W such that the origin 0 is contained in the interior of A. We denote by $\iota\colon A \to W$ the inclusion. Let $f\colon A \to W$ be an injective continuous map with $f(0) = 0$. We suppose that f is a local diffeomorphism at 0. Then the degree of the homomorphism[355]

$$\mathrm{H}_n(W, W \setminus \{0\}) \xleftarrow[\cong]{\iota_*} \mathrm{H}_n(A, A \setminus \{0\}) \xrightarrow{f_*} \mathrm{H}_n(W, W \setminus \{0\})$$

equals $+1$ if f is orientation-preserving at 0 and -1 if f is orientation-reversing at 0.

In the proof of the Local Diffeomorphism-Degree Proposition 44.3 we will make use of the following two lemmas which we will also use on several other occasions later on.

[355]By the Local-H_*-Proposition 42.2 we know that $\mathrm{H}_n(W, W \setminus \{0\}) \cong \mathbb{Z}$, thus it makes sense to talk of the degree of an automorphism of $\mathrm{H}_n(W, W \setminus \{0\})$.



Lemma 44.4. (Linearization–Homotopy Lemma) Let $f\colon U \to V$ be a diffeomorphism between two open subsets of $\mathbb{R}^n$. Let $P \in U$. We denote by

$$\begin{array}{rcl} \lambda\colon \mathbb{R}^n & \to & \mathbb{R}^n \\ x & \mapsto & \mathrm{D}f_P \cdot x + f(P) \end{array}$$

the linearization of f at P. There exists an open neighborhood U' of P such that the restrictions of f and λ to maps $(U', U' \setminus \{P\}) \to (\mathbb{R}^n, \mathbb{R}^n \setminus \{f(P)\})$ of pairs of topological spaces are homotopic.

Proof. Let $f\colon U \to V$ be a diffeomorphism between two open subsets of $\mathbb{R}^n$. To simplify the notation we assume that $P = 0$ and that $f(P) = 0$. Furthermore, by restriction to an open ball we might as well assume that U is convex. We write $D \coloneqq \mathrm{D}f_0$. Thus the linearization of f at $P = 0$ is given by the map $\lambda(x) = D \cdot x$. Since f is in particular a local diffeomorphism a $P = 0$ we know by the Chain Rule that D is an invertible matrix.

Since $U \subset \mathbb{R}^n$ is convex we obtain from the Map-as-Matrix Multiplication Proposition [**Fri24**] that there exists a smooth map $A\colon U \to \mathrm{M}(n \times n, \mathbb{R})$ such that $A(0) = \mathrm{D}f_0$ and such that $f(x) = A(x) \cdot x$ for all $x \in U$. Since $\det(A(0)) \neq 0$ we see that there exists an $r > 0$ with $B^n_r(0) \subset U$ such that $\det(A(x)) \neq 0$ for all $x \in B^n_r(0)$. This implies that the map

$$\begin{array}{rcl} B^n_r(0) \times [0,1] & \to & \mathbb{R}^n \\ (x,t) & \mapsto & A(x \cdot (1-t)) \cdot x \end{array}$$

is a homotopy of the maps f and λ as maps $(B^n_r(0), B^n_r(0) \setminus \{0\}) \to (\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$ of pairs of topological spaces. We now see that $U' = B^n_r(0)$ has the desired properties. ∎

Before we can formulate our next lemma we need to introduce the following notation:

Notation.
(1) Given a matrix $P \in \mathrm{GL}(n, \mathbb{R})$ we denote by $\rho(P)\colon \mathbb{R}^n \to \mathbb{R}^n$ the map that is given by $x \mapsto P \cdot x$.
(2) Given a non-zero real number r we denote its sign by $\mathrm{sign}(r) \in \{-1, +1\}$.

Lemma 44.5. (Matrix Multiplication–Degree Lemma) Let $n \in \mathbb{N}$. Given any invertible matrix $A \in \mathrm{GL}(n, \mathbb{R})$ we have[356]

$$\deg\left(\rho(A)_*\colon \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \to \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})\right) = \mathrm{sign}(\det(A)).$$

Proof of the Matrix Multiplication–Degree Lemma 44.5. Let $A \in \mathrm{GL}(n, \mathbb{R})$. We start out by picking an orthogonal matrix $B \in \mathrm{O}(n)$ such that $\mathrm{sign}(\det(B)) = \mathrm{sign}(\det(A))$.

[356]It follows from the Reduced LES-of-Pair Proposition 39.6, the Contractible-$\widetilde{\mathrm{H}}_*$-Corollary 39.5, the Homotopy Equivalence-$\widetilde{\mathrm{H}}_*$-Corollary 39.4 and the Sphere-H_*-Proposition 41.2 that we have isomorphisms $\mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \cong \widetilde{\mathrm{H}}_{n-1}(\mathbb{R}^n \setminus \{0\}) \cong \widetilde{\mathrm{H}}_{n-1}(S^{n-1}) \cong \mathbb{Z}$. Thus the degree of $\rho(A)_*$ is indeed defined.

Next we consider the following diagram:

$$\begin{array}{ccccccc}
\mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) & \xrightarrow{\mathrm{id}} & \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) & \xrightarrow[\cong]{\partial_n} & \widetilde{\mathrm{H}}_{n-1}(\mathbb{R}^n \setminus \{0\}) & \xleftarrow[\cong]{} & \widetilde{\mathrm{H}}_{n-1}(S^{n-1}) \\
\downarrow{\scriptstyle \rho(A)_*} & & \downarrow{\scriptstyle \rho(B)_*} & & \downarrow{\scriptstyle \rho(B)_*} & & \downarrow{\scriptstyle \rho(B)_*} \\
\mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) & \xrightarrow{\mathrm{id}} & \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) & \xrightarrow[\cong]{\partial_n} & \widetilde{\mathrm{H}}_{n-1}(\mathbb{R}^n \setminus \{0\}) & \xleftarrow[\cong]{} & \widetilde{\mathrm{H}}_{n-1}(S^{n-1}).
\end{array}$$

We make the following observations:

(1) Since $\mathrm{sign}(\det(B)) = \mathrm{sign}(\det(A))$ we obtain from the Matrix Group-π_0-Proposition 2.23 (1) that there exists a path $[0,1] \to \mathrm{GL}(n,\mathbb{R})$ from A to B. Similar to the discussion on page 159 this path gives rise to a homotopy between $\rho(A)$ and $\rho(B)$ as maps of pairs of topological spaces $(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \to (\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$. We obtain from the Homotopy-H_*-Proposition 38.6 that $\rho(A)_* = \rho(B)_*$, in other words, the square to the left commutes.
(2) The square in the middle commutes by the naturality of the connecting homomorphism, see the LES-of-Pair Proposition 38.3.
(3) The horizontal maps to the right are induced by the inclusion $S^{n-1} \to \mathbb{R}^n \setminus \{0\}$. Since S^{n-1} is a deformation retract of $\mathbb{R}^n \setminus \{0\}$ we obtain from the Homotopy Equivalence-$\widetilde{\mathrm{H}}_*$-Corollary 39.4 that these horizontal maps are isomorphisms.
(4) Since $B \in \mathrm{O}(n)$ we see that $\rho(B)\colon \mathbb{R}^n \to \mathbb{R}^n$ restricts to a self-map of S^{n-1}. It follows from the functoriality of reduced homology groups that the square to the right commutes.

Now we see that

$$\begin{array}{rcl}
\deg\left(\mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \xrightarrow{\rho(A)_*} \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})\right) & & \\
\quad \overset{\text{by the above commutative diagram}}{=} \deg\left(S^{n-1} \xrightarrow{\rho(B)} S^{n-1}\right) & \overset{\text{by the Sphere–Degree–Matrix Lemma 43.4}}{=} & \det(B) \\
\quad = \mathrm{sign}(\det(B)) & = & \mathrm{sign}(\det(A)). \\
\quad \overset{\text{since } B \in \mathrm{O}(n) \text{ we have } \det(B) \in \{\pm 1\}}{=} & \overset{\text{choice of } B}{=} &
\end{array}$$

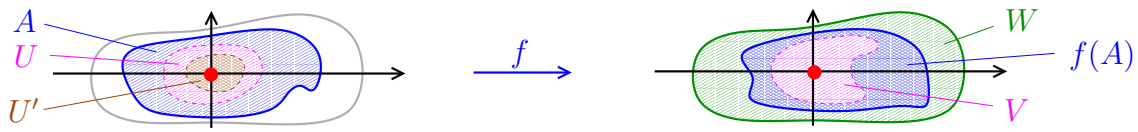
■

Now we are finally in a position to prove the Local Diffeomorphism-Degree Proposition 44.3.

Proof of the Local Diffeomorphism-Degree Proposition 44.3. Let W be a subset of $\mathbb{R}^n$ and let A be a subset of W such that the origin 0 is contained in the interior. Let $f\colon A \to W$ be an injective continuous map with $f(0) = 0$. We suppose that f is a local diffeomorphism at 0. Recall that this means that there exists an open neighborhood U of 0 that is contained in A and an open neighborhood V of 0 such that the map $f\colon U \to V$ is a diffeomorphism.

We start out with the following preparations.

(1) We set
$$\epsilon := \begin{cases} +1, & \text{if } f \text{ is orientation-preserving at } 0, \\ -1, & \text{if } f \text{ is orientation-reversing at } 0. \end{cases}$$
Note that $\epsilon = \mathrm{sign}(\det(\mathrm{D}f_0))$.
(2) Note that by the Linearization–Homotopy Lemma 44.4 there exists an open neighborhood U' of 0 that is contained in U such that the restrictions of f and $\rho(\mathrm{D}f_0)$ to maps $(U', U' \setminus \{0\}) \to (\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$ are homotopic.



Next we consider the following diagram:

$$\begin{array}{ccccc}
\mathrm{H}_n(W, W\setminus\{0\}) & \xleftarrow{\cong} & \mathrm{H}_n(A, A\setminus\{0\}) & \xrightarrow{f_*} & \mathrm{H}_n(W, W\setminus\{0\}) \\
\downarrow\cong & & \uparrow\cong & & \swarrow\cong \\
 & & \mathrm{H}_n(U', U'\setminus\{0\}) & \xrightarrow{f_*} & \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n\setminus\{0\}) \\
\downarrow\cong & & \downarrow\mathrm{id} & & \downarrow\mathrm{id} \qquad \downarrow\cong \\
 & & \mathrm{H}_n(U', U'\setminus\{0\}) & \xrightarrow{\rho(\mathrm{D}f_0)_*} & \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n\setminus\{0\}) \\
 & \swarrow\cong & & & \searrow\cong \\
\mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n\setminus\{0\}) & & \xrightarrow{\rho(\mathrm{D}f_0)_*} & & \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n\setminus\{0\}).
\end{array}$$

We continue with the following clarifications and observations:

(3) The undecorated maps are induced by inclusions. All of them are isomorphisms by the Excision Theorem 40.2.

(4) It follows from (2) and the Homotopy-H_*-Proposition 38.6 that the central square commutes. All other regions of the diagram commute by the functoriality of relative homology groups.

(5) By the Matrix Multiplication–Degree Lemma 44.5 the bottom horizontal map equals multiplication by $\operatorname{sign}(\det(\mathrm{D}f_0))$.

(6) It follows from all of the above that the top map is given by multiplication by ϵ. But that is precisely what we wanted to show. ■

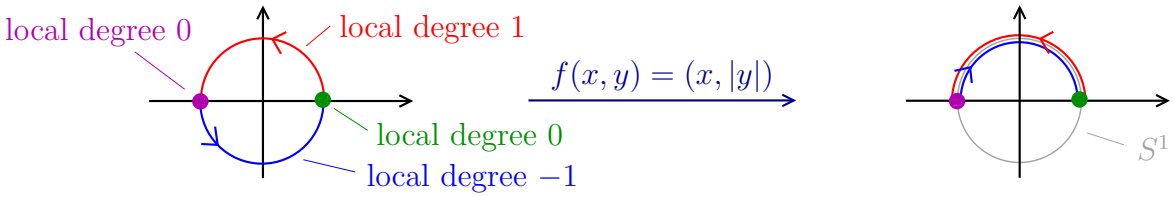
Let $f\colon S^n \to S^n$ be a continuous map with $n \geq 1$ and let $x \in S^n$ be a point such that f is a local diffeomorphism at x. In this setting we now have two different notions of a local degree, namely the notion from page 552 and the notion from page 554. The following proposition shows that the two notions of a local degree agree.

Proposition 44.6. (Local Degree-Calculation Proposition) Let $f\colon S^n \to S^n$ be a continuous map with $n \geq 1$ and let $x \in S^n$. If f is a local diffeomorphism at x, then x is nice and

$$\deg(f, x) \;=\; \begin{cases} 1, & \text{if } f \text{ is orientation-preserving at } x, \\ -1, & \text{if } f \text{ is orientation-reversing at } x. \end{cases}$$

Example. We consider the map $f\colon S^1 \to S^1$ that is given by $(x, y) \mapsto (x, |y|)$. It follows from the Local Degree-Calculation Proposition 44.6 that for every point on the open upper semicircle the local degree is $+1$ and that for every point on the open lower semicircle the local degree is -1.
One can easily show that the local degree at the points ± 1 is in both cases zero. □

Proof. Let $f\colon S^n \to S^n$ be a continuous map with $n \geq 1$ and let $x \in S^n$. We assume that f is a local diffeomorphism at x. By elementary linear algebra we can pick an $A \in \mathrm{SO}(n+1)$ with $A \cdot f(x) = x$. Note that the self-map of S^n given by $z \mapsto A \cdot z$ is orientation-preserving. Furthermore by the Sphere–Degree–Matrix Lemma 43.4 we know that $\deg(z \mapsto A \cdot z) = 1$.

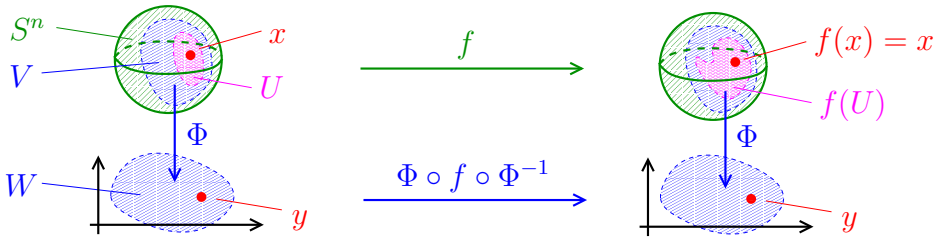


It follows from these observations, together with the Sphere–Degree Lemma 43.1 (4), that after replacing f by the map $z \mapsto A \cdot f(z)$, we might as well assume that $f(x) = x$.

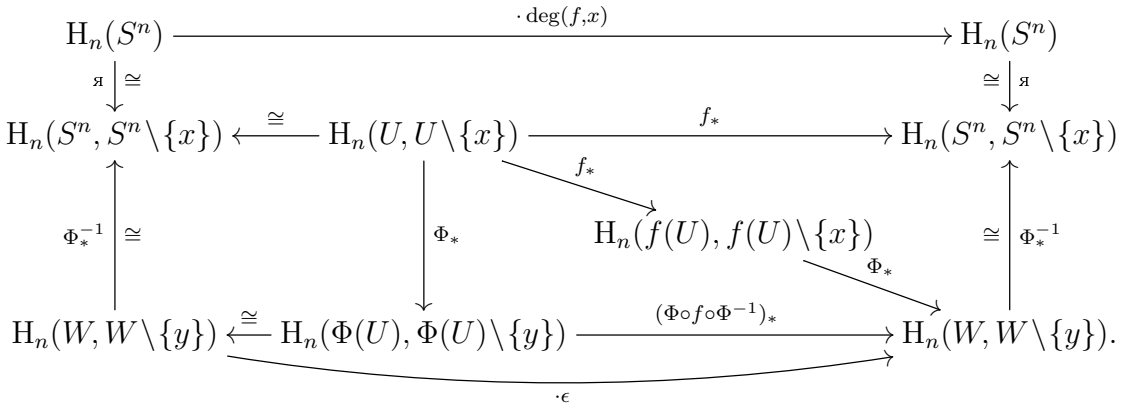
We make a few further preparations.

(1) We pick a chart $\Phi\colon V \to W$, from the smooth atlas, around $x \in S^n$.
(2) Since f is a local diffeomorphism at x there exists an open neighborhood U of x such that $f\colon U \to f(U)$ is a diffeomorphism. Note that after possibly replacing U by the subset $U \cap V \cap f^{-1}(V)$ we can assume that $U \subset V$ and $f(U) \subset V$.
(3) We set $y \coloneqq \Phi(x)$.
(4) We set

$$\epsilon \coloneqq \begin{cases} 1, & \text{if } f \text{ is orientation-preserving at } x, \\ -1, & \text{if } f \text{ is orientation-reversing at } x. \end{cases}$$



Next we consider the following diagram:



We make the following observations and clarifications.

(1) As usual the undecorated maps are induced by inclusions.
(2) The top rectangle commutes by the definition of $\deg(f, x)$. The middle regions of the diagram commute by the functoriality of relative homology. Finally, note that $\Phi \circ f \circ \Phi^{-1}\colon \Phi^{-1}(U) \to W$ is orientation-preserving at y if and only if f is orientation-preserving at x. Thus we see that the bottom part of the diagram commutes by the Local Diffeomorphism-Degree Proposition 44.3.

(3) By the Excision Theorem 40.2 the map $\Phi_*\colon \mathrm{H}_n(W, W \setminus \{y\}) \to \mathrm{H}_n(S^n, S^n \setminus \{x\})$ is an isomorphism.
(4) It follows from the above that $\deg(f, x) = \epsilon$. ■

Now we can finally prove the following proposition, which we already mentioned on page 552, at the very beginning of this chapter.

Proposition 44.7. (Degree-via-Local Orientations Proposition) Let $n \in \mathbb{N}_0$. Let $f\colon S^n \to S^n$ be a continuous map and let $y \in S^n$ such that the following conditions are satisfied:

(1) $f^{-1}(y)$ consists of finitely many points $x_1, \ldots, x_m$.
(2) The map f is a local diffeomorphism at each x_i.

Then
$$\deg(f) \;=\; \sum_{i=1}^{m} \deg(f, x_i) \;=\; \sum_{i=1}^{m} \begin{cases} 1, & \text{if } f \text{ is orientation-preserving at } x_i, \\ -1, & \text{if } f \text{ is orientation-reversing at } x_i. \end{cases}$$

Proof. The proposition is an immediate consequence of the S^n-Local-Global Degree Proposition 44.2 together with the Local Degree-Calculation Proposition 44.6. ■

CHAPTER 45

The Mayer–Vietoris sequence and its applications

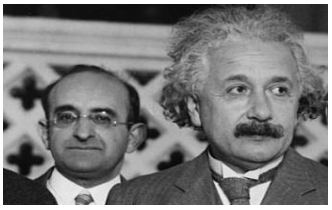
Recall that in the Seifert–van Kampen-Theorem 20.1 we saw how, given a suitable decomposition $X = U \cup V$ of a topological space, one can determine the fundamental group $\pi_1(X, x_0)$ in terms of the two inclusion-induced homomorphisms $\pi_1(U \cap V, x_0) \to \pi_1(U, x_0)$ and $\pi_1(U \cap V, x_0) \to \pi_1(V, x_0)$.

In this chapter we will state and prove the Mayer–Vietoris Theorem 45.1 which gives us a similar result for the homology groups of $X = U \cup V$. As we will see shortly, the Mayer–Vietoris Theorem 45.1 is just a user friendly variation on the Excision Theorem 40.1. But we will also see that it is a very useful tool for computing homology groups. In particular, together with results of the previous chapters, we will be able to determine the homology groups of the torus and the Klein bottle.

45.1. **The Mayer–Vietoris sequence.** In the following theorem and throughout the subsequent discussion we will use the following notation:

Notation. Given a topological space D and a subspace $C \subset D$ we denote the corresponding inclusion map $C \to D$ by i_C^D. To avoid pandemonium with subscripts we often drop the "$*$" when we deal with inclusion-induced maps on homology, e.g. we will write $i_C^D \colon \mathrm{H}_n(C) \to \mathrm{H}_n(D)$ instead of the more accurate $(i_C^D)_* \colon \mathrm{H}_n(C) \to \mathrm{H}_n(D)$. We use similar notation for inclusions of pairs of topological spaces.

The following theorem, which is the main result of this chapter, has its origin in the work of Walther Mayer [**May29**] and Leopold Vietoris [**Vie30**].



Walther Mayer (1887-1948) with some other person



Leopold Vietoris (1891-2002) at 110

Theorem 45.1. (Mayer–Vietoris Theorem) Let X be a topological space and let $A, B \subset X$ be subspaces such that one of the following three conditions holds:

(1) X is the union of the interiors of A and B, i.e. $X = \mathring{A} \cup \mathring{B}$.
(2) X is a CW-complex and A and B are subcomplexes with $X = A \cup B$.
(3) X is a topological manifold and $A, B \subset X$ are two codimension-zero submanifolds such that the following conditions are satisfied:
   - $X = A \cup B$.

- $A \cap B$ is a union of components of ∂A and it is a union of components of ∂B.
- A and B are closed subsets of X.[357]

By (1) the Excision Theorem 40.1, (2) the CW-Excision Theorem 40.4 or (3) the Manifold–Excision Theorem 40.3 each inclusion-induced map $i^{(X,B)}_{(A,A\cap B)}\colon \mathrm{H}_n(A, A\cap B) \to \mathrm{H}_n(X,B)$ is an isomorphism. We refer to the homomorphism

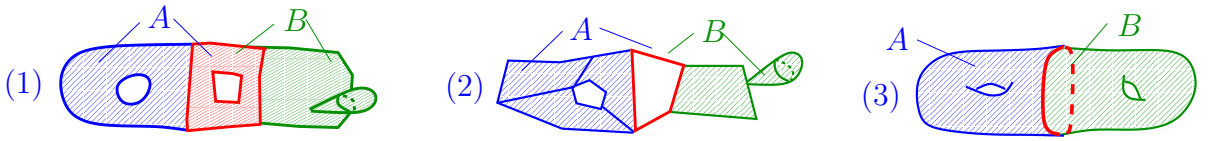
$$\mathrm{H}_n(X) \xrightarrow{\mathfrak{s}} \mathrm{H}_n(X,B) \xrightarrow{\left(i^{(X,B)}_{(A,A\cap B)}\right)^{-1}} \mathrm{H}_n(A, A\cap B) \xrightarrow{\partial_n} \mathrm{H}_{n-1}(A\cap B)$$

connecting homomorphism of the long exact sequence corresponding to the pair $(A, A\cap B)$, provided by the LES-of-Pair Proposition 38.3

as the connecting homomorphism $\partial_n\colon \mathrm{H}_n(X) \to \mathrm{H}_{n-1}(A\cap B)$ corresponding to (X, A, B). The connecting homomorphism is natural in a suitable sense and with this notation the following sequence is exact:

$$\begin{array}{ccccccccc} \ldots \to \mathrm{H}_n(A\cap B) & \xrightarrow{(i^A_{A\cap B}, -i^B_{A\cap B})} & \mathrm{H}_n(A) \oplus \mathrm{H}_n(B) & \xrightarrow{i^X_A \oplus i^X_B} & \mathrm{H}_n(X) & \xrightarrow{\partial_n} & \mathrm{H}_{n-1}(A\cap B) \to \ldots \\ s & \longmapsto & (i^A_{A\cap B}(s), -i^B_{A\cap B}(s)) & & & & \\ & & & & s & \longmapsto & i^X_A(s) + i^X_B(s) \end{array}$$

Totally analogous statements also hold if we replace the homology groups of $A\cap B, A, B$ and X by the corresponding reduced homology groups.[358]



A and B are open — A and B are subcomplexes — A and B are submanifolds

$$\cdots \to \mathrm{H}_n(A\cap B) \xrightarrow{(i^A_{A\cap B}, -i^B_{A\cap B})} \mathrm{H}_n(A) \oplus \mathrm{H}_n(B) \xrightarrow{i^X_A \oplus i^X_B} \mathrm{H}_n(X) \xrightarrow{\partial_n} \mathrm{H}_{n-1}(A\cap B) \to \ldots$$

Definition. The long exact sequence of the Mayer–Vietoris Theorem 45.1 is called the Mayer–Vietoris sequence of the decomposition $X = A \cup B$.

Proof. First note that it follows immediately from the definitions and the LES-of-Pair Proposition 38.3 (2) that the connecting homomorphism $\mathrm{H}_n(X) \to \mathrm{H}_{n-1}(A\cap B)$ is natural in a suitable sense. It remains to show that the given sequence is exact. There are two approaches to proving this statement:

(1) One can deduce the Mayer–Vietoris Theorem 45.1 directly from the Excision Theorem 40.1 using an algebraic manipulation. We will carry out this approach in Section 45.1.1.
(2) One can prove the Mayer–Vietoris Theorem 45.1 using the Comfortable Cover Proposition 40.7. We will carry out this approach in Section 45.1.2.

The first proof is slicker and it has the advantage that it applies to other functors, besides singular homology, that also satisfy a suitable version of the Excision Theorem 40.1. On

[357] Note that by the Compact-Closed Lemma 1.9 (2) this condition is satisfied if A and B are compact.
[358] Note though that for reduced homology groups we might need to use the fact, obtained in the Reduced Homology Lemma 39.2 (-1), that $\widetilde{\mathrm{H}}_{-1}(\varnothing) = \mathbb{Z}$.

the other hand the second proof makes it arguably clearer how the long exact sequence arises. ■

45.1.1. **First proof of the Mayer–Vietoris Theorem 45.1.** Our first proof of the Mayer–Vietoris Theorem 45.1 rests on the shoulders of the following purely algebraic lemma.

Lemma 45.2. (Barratt-Whitehead Lemma) We consider the following commutative diagram of homomorphisms between abelian groups:

$$\begin{array}{ccccccccccc}
\dots & \xrightarrow{\partial_{n+1}} & C'_n & \xrightarrow{i_n} & C_n & \xrightarrow{p_n} & C''_n & \xrightarrow{\partial_n} & C'_{n-1} & \longrightarrow & \dots \\
 & & \downarrow{\scriptstyle f'_n} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f''_n} & & \downarrow{\scriptstyle f'_{n-1}} & & \\
\dots & \xrightarrow{\partial_{n+1}} & D'_n & \xrightarrow{j_n} & D_n & \xrightarrow{q_n} & D''_n & \xrightarrow{\partial_n} & D'_{n-1} & \longrightarrow & \dots
\end{array}$$

We suppose that the horizontal sequences are exact and we suppose that each $f''_n\colon C''_n \to D''_n$ is an isomorphism. For each $n \in \mathbb{N}_0$ we set $\Delta_n := \partial_n \circ (f''_n)^{-1} \circ q_n\colon D_n \to C'_{n-1}$. The following sequence is exact:

$$\dots \longrightarrow C'_n \xrightarrow{(i_n,-f'_n)} C_n \oplus D'_n \xrightarrow{f_n \oplus j_n} D_n \xrightarrow{\Delta_n} C'_{n-1} \longrightarrow \dots$$

Proof of the Barratt-Whitehead Lemma 45.2. Let us prove exactness at D_n. First we show that $\Delta_n \circ (f_n \oplus j_n) = 0$. Thus let $(c,d) \in C_n \oplus D'_n$. Then

$$\begin{array}{lllll}
(\Delta_n \circ (f_n \oplus j_n))((c,d)) & \overset{\text{definition of } \Delta_n}{=} & (\partial_n \circ (f''_n)^{-1} \circ q_n)(f_n(c) + j_n(d)) & \overset{\text{since } q_n \circ j_n = 0}{=} & (\partial_n \circ (f''_n)^{-1} \circ q_n \circ f_n)(c) \\
 & = & (\partial_n \circ (f''_n)^{-1} \circ f''_n \circ p_n)(c) & = & (\partial_n \circ p_n)(c) \quad = \quad 0. \\
 & \uparrow & & & \uparrow \\
 & \text{since the diagram commutes we have } q_n \circ f_n = f''_n \circ p_n & & & \text{since } \partial_n \circ p_n = 0
\end{array}$$

Now suppose we have $d \in D_n$ with $\Delta_n(d) = 0$. Since $0 = \Delta_n(d) = \partial_n((f''_n)^{-1} \circ q_n)(d))$ and since the top horizontal sequence is exact we see that there exists a $c \in C_n$ with $p_n(c) = ((f''_n)^{-1} \circ q_n)(d)$. Next we see that

$$\begin{array}{lllll}
q_n(\underbrace{f_n(c) - d}_{\in D_n}) & = & (q_n \circ f_n)(c) - q_n(d) & \overset{\text{since the diagram commutes}}{=} & (f''_n \circ p_n)(c) - q_n(d) \quad \overset{\text{choice of } c}{=} \quad 0.
\end{array}$$

It follows from this calculation and the hypothesis that the bottom horizontal sequence is exact that there exists a $d' \in D'_n$ with $j_n(d') = f_n(c) - d$. Finally we calculate that

$$(f_n \oplus j_n)((c,-d')) \;=\; f_n(c) - j_n(d') \;=\; f_n(c) - (f_n(c) - d) \;=\; d.$$

We have thus shown that the given sequence is exact at D_n. The proof of the exactness at C'_n and $C_n \oplus D'_n$ is done in a similar fashion. We leave them to the reader. ■

With the Barratt-Whitehead Lemma 45.2 at our disposal we can easily prove the Mayer–Vietoris Theorem 45.1:

Proof of the Mayer–Vietoris Theorem 45.1. Let X be a topological space. Furthermore let $A, B \subset X$ be two subspaces. We assume that one of the following three conditions holds:

(1) $X = \mathring{A} \cup \mathring{B}$.
(2) X is a CW-complex and A and B are subcomplexes with $X = A \cup B$.

(3) X is a topological manifold and $A, B \subset X$ are two codimension-zero submanifolds such that the following conditions are satisfied:
- $X = A \cup B$.
- $A \cap B$ is a union of components of ∂A and it is a union of components of ∂B.
- A and B are closed subsets of X.

We consider the following commutative diagram of continuous maps between pairs of topological spaces:

$$\begin{array}{ccccc} (A\cap B,\varnothing) & \xrightarrow{i^A_{A\cap B}} & (A,\varnothing) & \longrightarrow & (A,A\cap B) \\ \downarrow{\scriptstyle i^B_{A\cap B}} & & \downarrow{\scriptstyle i^X_A} & & \downarrow{\scriptstyle i^{(X,B)}_{(A,A\cap B)}} \\ (B,\varnothing) & \xrightarrow{i^X_B} & (X,\varnothing) & \longrightarrow & (X,B). \end{array}$$

From the LES-of-Pair Proposition 38.3 we get two long exact sequences which, together with the vertical maps, form the following commutative diagram:

$$\begin{array}{ccccccccc} \dots & \longrightarrow & \mathrm{H}_n(A\cap B) & \xrightarrow{i^A_{A\cap B}} & \mathrm{H}_n(A) & \xrightarrow{\text{я}} & \mathrm{H}_n(A,A\cap B) & \xrightarrow{\partial_n} & \mathrm{H}_{n-1}(A\cap B) \longrightarrow \dots \\ & & \downarrow{\scriptstyle i^B_{A\cap B}} & & \downarrow{\scriptstyle i^X_A} & & \downarrow{\scriptstyle i^{(X,B)}_{(A,A\cap B)}} & & \downarrow{\scriptstyle i^B_{A\cap B}} \\ \dots & \longrightarrow & \mathrm{H}_n(B) & \xrightarrow{i^X_B} & \mathrm{H}_n(X) & \xrightarrow{\text{я}} & \mathrm{H}_n(X,B) & \xrightarrow{\partial_n} & \mathrm{H}_{n-1}(B) \longrightarrow \dots. \end{array}$$

Depending on the setting we are in we obtain from
(1) the Excision Theorem 40.1 (applied with $Z = X \setminus A$), or
(2) the Manifold–Excision Theorem 40.3, or
(3) the CW-Excision Theorem 40.4

that the inclusion-induced map $i^{(X,B)}_{(A,A\cap B)} \colon \mathrm{H}_n(A, A\cap B) \to \mathrm{H}_n(X,B)$ is an isomorphism. We apply the Barratt-Whitehead Lemma 45.2 to the above setup and we obtain that the following sequence is exact:

$$\dots \to \mathrm{H}_n(A\cap B) \xrightarrow{(i^A_{A\cap B}, -i^B_{A\cap B})} \mathrm{H}_n(A)\oplus\mathrm{H}_n(B) \xrightarrow{i^X_A \oplus i^X_B} \mathrm{H}_n(X) \xrightarrow{\partial_n \circ (i^{(X,B)}_{(A,A\cap B)})^{-1} \circ \text{я}} \mathrm{H}_{n-1}(A\cap B) \to \dots$$

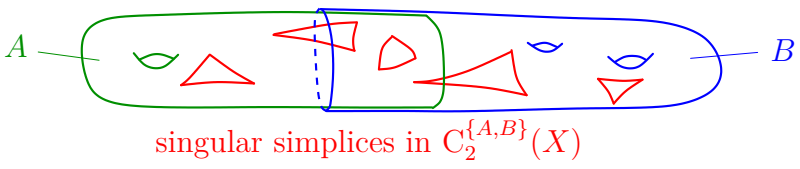
But that was exactly our goal. Finally note that by replacing the LES-of-Pair Proposition 38.3 by the Reduced LES-of-Pair Proposition 39.6 we get the analogous result for reduced homology groups. ∎

45.1.2. **Second proof of the Mayer–Vietoris Theorem 45.1.** In this section we give our second proof of the Mayer–Vietoris Theorem 45.1. Here it is:

Proof of the Mayer–Vietoris Theorem 45.1. Let X be a topological space.

(1) First we suppose that we are given two subsets $A, B \subset X$ such that $X = \mathring{A} \cup \mathring{B}$. We recall the following notation from page 509:

$$\mathrm{C}_n^{\{A,B\}}(X) \;:=\; \{a+b \,|\, a \in \mathrm{C}_n(A) \text{ and } b \in \mathrm{C}_n(B)\}.$$



The inclusion maps induce the following sequence of chain maps:

$$0 \to \mathrm{C}_*(A\cap B) \xrightarrow{(i^A_{A\cap B}, -i^B_{A\cap B})} \mathrm{C}_*(A)\oplus\mathrm{C}_*(B) \xrightarrow{i^X_A \oplus i^X_B} \mathrm{C}^{\{A,B\}}_*(X) \to 0.$$

Claim. The above sequence of chain maps is exact.

Proof. We prove the claim in the following four easy steps:

(a) It is clear that each of the maps $\mathrm{C}_*(A\cap B)\to\mathrm{C}_*(A)$ and $\mathrm{C}_*(A\cap B)\to\mathrm{C}_*(B)$ is already injective, in particular the map $(i^A_{A\cap B}, -i^B_{A\cap B})$ is injective.

(b) It is an immediate consequence of the definition of $\mathrm{C}^{\{A,B\}}_*(X)$ that the map $i^X_A\oplus i^X_B$ is surjective.

(c) It follows immediately from the definitions that the composition of both maps is the zero map,[359] i.e. $\mathrm{im}((i^A_{A\cap B}, -i^B_{A\cap B}))\subset\ker(i^X_A\oplus i^X_B)$.

(d) It remains to show that $\ker(i^X_A\oplus i^X_B)\subset\mathrm{im}((i^A_{A\cap B}, -i^B_{A\cap B}))$. Thus let $x\in\mathrm{C}_n(A)$ and $y\in\mathrm{C}_n(B)$ such that $(i^X_A\oplus i^X_B)((x,y))=0$. By definition this means that x and y, viewed as elements in $\mathrm{C}_n(X)$, satisfy $x=-y$. This means that x lies in $\mathrm{C}_n(A)$ and also in $\mathrm{C}_n(B)$. Put differently, $x\in\mathrm{C}_n(A)\cap\mathrm{C}_n(B)=\mathrm{C}_n(A\cap B)$. But then we have $((i^A_{A\cap B}, -i^B_{A\cap B}))(x)=x+y$. ⊞

Next we consider the following maps:

$$\ldots\longrightarrow \mathrm{H}_n(A\cap B)\xrightarrow{(i^A_{A\cap B},-i^B_{A\cap B})}\mathrm{H}_n(A)\oplus\mathrm{H}_n(B)\xrightarrow{i^X_A\oplus i^X_B}\mathrm{H}^{\{A,B\}}_n(X)\xrightarrow{p_n}\mathrm{H}_{n-1}(A\cap B)\longrightarrow\ldots$$

$$\mathrm{H}_n(A)\oplus\mathrm{H}_n(B)\xrightarrow{i^X_A\oplus i^X_B}\mathrm{H}_n(X),\qquad \mathrm{H}^{\{A,B\}}_n(X)\xrightarrow{\Phi_n}_{\cong}\mathrm{H}_n(X),\qquad \mathrm{H}_n(X)\xrightarrow{:=\partial_n}\mathrm{H}_{n-1}(A\cap B)$$

We make the following clarifications and observations:

(a) Given $n\in\mathbb{N}_0$ we denote by $\mathrm{H}^{\{A,B\}}_n(X)$ the n-th homology group of the chain complex $\mathrm{C}^{\{A,B\}}_*(X)$ and we denote by $p_n\colon \mathrm{H}^{\{A,B\}}_n(X)\to\mathrm{H}_{n-1}(A\cap B)$ the connecting homomorphism of the above short exact sequence of chain complexes, as defined on page 483.

(b) By the LES-Proposition 37.4 the top horizontal sequence is exact.

(c) It follows from the Comfortable Cover Proposition 40.7, together with our hypothesis that $X=\mathring{A}\cup\mathring{B}$, that for every $n\in\mathbb{N}_0$ the inclusion map

$$\mathrm{C}^{\{A,B\}}_*(X) \;\to\; \mathrm{C}_*(X)$$

induces an isomorphism

$$\Phi_n\colon \mathrm{H}^{\{A,B\}}_n(X) \;\xrightarrow{\cong}\; \mathrm{H}_n(X).$$

(d) For each $n\in\mathbb{N}_0$ we write $\partial_n := p_n\circ\Phi_n^{-1}\colon \mathrm{H}_n(X)\to\mathrm{H}_{n-1}(A\cap B)$. Using the explicit description of the connecting homomorphisms on page 483 and in the LES-of-Pair Proposition 38.3 one can show easily that $\partial_n := p_n\circ\Phi_n^{-1}\colon \mathrm{H}_n(X)\to\mathrm{H}_{n-1}(A\cap B)$ equals the connecting homomorphism given in the theorem.

(2) Now we suppose that we are dealing with a CW-complex X and two subcomplexes A and B with $X=A\cup B$. One can use the Subcomplex–Neighborhood Proposition 31.8 together with the Homotopy Equivalence-H_*-Corollary 36.7 to reduce the case of CW-complexes and subcomplexes to the case of an open cover. We will not carry out the

[359] Note that here we use the minus sign that appears in the first map.

technical details for CW-complexes, since we already provided a complete proof of the Mayer–Vietoris Theorem 45.1 (2) in Section 45.1.1.

(3) Similarly to (2), as in the proof of the Manifold–Seifert-van Kampen-Theorem 23.1, we can use the Topological Collar Neighborhood Theorem [**Fri24**] to reduce the manifold case to the case of an open cover. As in (2) we leave the details to the reader.

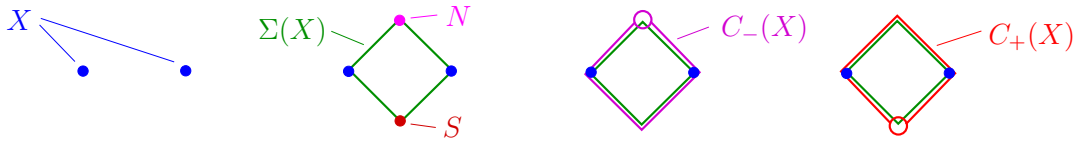
Finally note that the statements for reduced homology can be proved as above, we just need to replace $\mathrm{C}_*^{\{A,B\}}(X)$ by its reduced analogue. Alternatively one can deduce the reduced long exact sequence from the above unreduced long exact sequence and the Reduced Homology Lemma 39.2. Again there is no point in providing the details. ■

45.2. **The Mayer–Vietoris sequence applied to suspensions.** To get used to dealing with the Mayer–Vietoris sequence we will give an alternative proof of a slightly simplified version of the Suspension-H_*-Proposition 41.1:

Proposition 41.1. (Suspension-H_*-Proposition) Given any topological space X and given any $k \in \mathbb{Z}$ we have a natural isomorphism

$$\Sigma_X \colon \widetilde{\mathrm{H}}_{k+1}(\Sigma(X)) \xrightarrow{\cong} \widetilde{\mathrm{H}}_k(X).$$

Proof. Let X be a topological space and let $k \in \mathbb{Z}$. As on page 522 we consider the subspaces $C_+ := C_+(X) := \Sigma(X) \setminus \{S\}$ and $C_- := C_-(X) := \Sigma(X) \setminus \{N\}$, where S and N denote the South and North Pole of the suspension $\Sigma(X)$. Note that $C_+(X)$ and $C_-(X)$ are both open subsets of $\Sigma(X)$ and that $\Sigma(X) = C_+(X) \cup C_-(X)$.



We consider the corresponding long exact sequence in reduced homology from the Mayer–Vietoris Theorem 45.1 (1):

$$\underbrace{\widetilde{\mathrm{H}}_{k+1}(C_+)}_{=0} \oplus \underbrace{\widetilde{\mathrm{H}}_{k+1}(C_-)}_{=0} \longrightarrow \widetilde{\mathrm{H}}_{k+1}(\Sigma(X)) \xrightarrow{\partial_{k+1}} \widetilde{\mathrm{H}}_k(C_+ \cap C_-) \longrightarrow \underbrace{\widetilde{\mathrm{H}}_{k+1}(C_+)}_{=0} \oplus \underbrace{\widetilde{\mathrm{H}}_{k+1}(C_-)}_{=0}$$

$$\widetilde{\mathrm{H}}_k(C_+ \cap C_-) \cong \uparrow i_* \qquad \Sigma_X \swarrow \widetilde{\mathrm{H}}_k(X).$$

We make the following clarifications and observations:

(1) Note that both $C_+ = C_+(X)$ and $C_- = C_-(X)$ are contractible. It follows from the Contractible-$\widetilde{\mathrm{H}}_*$-Corollary 39.5 that the reduced homology groups of $C_+(X)$ and $C_-(X)$ vanish.

(2) Next note that it follows immediately from (1) that the connecting homomorphism $\partial_{k+1} \colon \widetilde{\mathrm{H}}_{k+1}(\Sigma(X)) \to \widetilde{\mathrm{H}}_k(C_+(X) \cap C_-(X))$ is in fact an isomorphism.

(3) We consider the following little diagram:

$$X \xrightarrow{x \mapsto (x,0)} X \times (-1,1) \xrightarrow{(x,t) \mapsto [(x,t)]} C_+(X) \cap C_-(X).$$
$$=: i$$

The map to the left is evidently a homotopy equivalence. Furthermore one can easily show, either by hand, or using Lemma 41.9 (1), that the map to the right is a homeomorphism. We obtain that the map $i\colon X \to C_+(X) \cap C_-(X)$ is a homotopy equivalence. Thus we obtain from the Homotopy Equivalence-$\widetilde{\mathrm{H}}_*$-Corollary 39.4 (2) that the induced map $i_*\colon \widetilde{\mathrm{H}}_k(X) \to \widetilde{\mathrm{H}}_k(C_+(X) \cap C_-(X))$ is an isomorphism.

It follows from the above that we get an isomorphism $\Sigma_X\colon \widetilde{\mathrm{H}}_{k+1}(\Sigma(X)) \xrightarrow{\cong} \widetilde{\mathrm{H}}_k(X)$. It follows easily from the definition of Σ_X and the naturality of the connecting homomorphisms of the Mayer–Vietoris sequence (which is a consequence of the LES-of-Pair Proposition 38.3 (2)), that the isomorphism Σ_X is natural. ■

45.3. **The homology groups of a mapping torus.** For convenience we recall from page 235 the definition of the mapping torus:

Definition. Let X be a topological space and let $f\colon X \to X$ be a continuous map. We refer to

$$\mathrm{Tor}(X, f) \;:=\; (X \times [0,1])/(x,0) \sim (f(x),1)$$

as the mapping torus of (X, f).

Examples.

(1) Let X be a topological space. In [**Fri24**] it is shown that the map

$$\begin{array}{rccc} \Phi\colon & \mathrm{Tor}(X, \mathrm{id}) & \to & X \times S^1 \\ & [(x,t)] & \mapsto & (x, \exp(2\pi\,\mathrm{i}t)) \end{array}$$

is a homeomorphism.

(2) Let $X = [-1,1]$ and let $f\colon [-1,1] \to [-1,1]$ be the map that is given by $x \mapsto -x$.

$\mathrm{Tor}([-1,1], x \mapsto -x)$  $\cong$

The corresponding mapping torus $\mathrm{Tor}([-1,1], x \mapsto -x)$ is clearly homeomorphic to the Möbius band. □

In the Mapping Torus-π_1-Proposition 15.7 we determined the fundamental group of a given mapping torus. In many cases the following proposition allows us to determine the homology groups of a given mapping torus.

Proposition 45.3. (Mapping Torus-H_*-Proposition) Let X be a topological space and let $f\colon X \to X$ be a continuous map. We denote by $i\colon X \to \mathrm{Tor}(X, f)$ the natural map that is given by $x \mapsto [(x,0)]$. For every $n \in \mathbb{N}_0$ there exists a natural homomorphism

$$\partial_n\colon \mathrm{H}_n(\mathrm{Tor}(X, f)) \;\to\; \mathrm{H}_{n-1}(X)$$

(which is described in greater detail in the proof of the proposition) such that the following sequence is exact

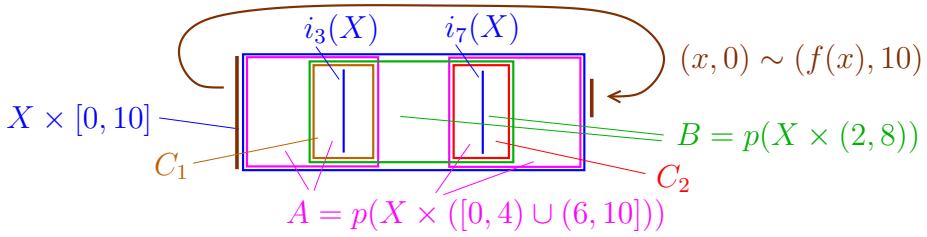
$$\cdots \to \mathrm{H}_n(X) \xrightarrow{f_*-\mathrm{id}} \mathrm{H}_n(X) \xrightarrow{i_*} \mathrm{H}_n(\mathrm{Tor}(X,f)) \xrightarrow{\partial_n} \mathrm{H}_{n-1}(X) \xrightarrow{f_*-\mathrm{id}} \mathrm{H}_{n-1}(X) \to \ldots$$

Proof. Let X be a topological space and let $f\colon X \to X$ be a continuous map. For notational convenience we work in this proof with the following topological space which is clearly naturally homeomorphic to the mapping torus:

$$M \;:=\; (X \times [0,10])/(x,0) \sim (f(x),10).$$

We introduce the following notation:

(1) We denote by $p\colon X \times [0,10] \to M = (X \times [0,1])/\sim$ the natural projection.
(2) We consider the subsets $A \coloneqq p(X \times ([0,4) \cup (6,10])) \subset M$ and $B \coloneqq p(X \times (2,8)) \subset M$.
(3) We consider the two subsets $C_1 = p(X \times (2,4))$ and $C_2 = p(X \times (6,8))$. Note that $A \cap B = C_1 \cup C_2$.
(4) Given $t \in [0,10]$ we consider the map
$$\begin{array}{rcl} i_t\colon X & \to & p(X \times \{t\}) \\ x & \mapsto & p(x,t). \end{array}$$
(5) We write $i_3^A\colon X \to A$ for the composition of $i_3\colon X \to p(X \times \{3\})$ with the inclusion map $p(X \times \{3\}) \to A$. We denote analogous compositions by the obvious variations in the notation.
(6) To simplify the notation we drop the "$*$" when we write down the corresponding induced maps on homology.



We can now consider the following diagram:[360]

$$\begin{array}{ccccccccccc}
\to & \begin{matrix}\mathrm{H}_n(C_1)\\ \mathrm{H}_n(C_2)\end{matrix} & \xrightarrow{\begin{pmatrix} i^A_{C_1} & i^A_{C_2} \\ -i^B_{C_1} & -i^B_{C_2}\end{pmatrix}} & \begin{matrix}\mathrm{H}_n(A)\\ \mathrm{H}_n(B)\end{matrix} & \xrightarrow{(i^M_A \ i^M_B)} & \mathrm{H}_n(M) & \xrightarrow{\partial_n} & \begin{matrix}\mathrm{H}_{n-1}(C_1)\\ \mathrm{H}_{n-1}(C_2)\end{matrix} & \xrightarrow{\begin{pmatrix} i^A_{C_1} & i^A_{C_2} \\ -i^B_{C_1} & -i^B_{C_2}\end{pmatrix}} & \begin{matrix}\mathrm{H}_{n-1}(A)\\ \mathrm{H}_{n-1}(B)\end{matrix} & \to \\
 & \uparrow{\scriptstyle \begin{pmatrix} i_3 & 0 \\ 0 & i_7\end{pmatrix}}_{\cong} & & \uparrow{\scriptstyle \cong \begin{pmatrix} i_7 & 0 \\ 0 & i_7\end{pmatrix}} & & \uparrow{\scriptstyle \mathrm{id}} & & \uparrow{\scriptstyle \begin{pmatrix} i_3 & 0 \\ 0 & i_7\end{pmatrix}}_{\cong} & & \uparrow{\scriptstyle \cong \begin{pmatrix} i_7 & 0 \\ 0 & i_7\end{pmatrix}} & \\
\to & \begin{matrix}\mathrm{H}_n(X)\\ \mathrm{H}_n(X)\end{matrix} & \xrightarrow{\begin{pmatrix} f_* & \mathrm{id} \\ -\mathrm{id} & -\mathrm{id}\end{pmatrix}} & \begin{matrix}\mathrm{H}_n(X)\\ \mathrm{H}_n(X)\end{matrix} & \xrightarrow{(i_7 \ i_7)} & \mathrm{H}_n(M) & \xrightarrow{=:\partial_n} & \begin{matrix}\mathrm{H}_{n-1}(X)\\ \mathrm{H}_{n-1}(X)\end{matrix} & \xrightarrow{\begin{pmatrix} f_* & \mathrm{id} \\ -\mathrm{id} & -\mathrm{id}\end{pmatrix}} & \begin{matrix}\mathrm{H}_{n-1}(X)\\ \mathrm{H}_{n-1}(X)\end{matrix} & \to \\
 & \uparrow{\scriptstyle \begin{pmatrix} -\mathrm{id} \\ \mathrm{id}\end{pmatrix}} & & \uparrow{\scriptstyle \begin{pmatrix} \mathrm{id} \\ 0\end{pmatrix}} & & \uparrow{\scriptstyle \mathrm{id}} & & \uparrow{\scriptstyle \begin{pmatrix} -\mathrm{id} \\ \mathrm{id}\end{pmatrix}} & & \uparrow{\scriptstyle \begin{pmatrix} \mathrm{id} \\ 0\end{pmatrix}} & \\
\to & \mathrm{H}_n(X) & \xrightarrow{\mathrm{id} - f_*} & \mathrm{H}_n(X) & \xrightarrow{i_7} & \mathrm{H}_n(M) & \xrightarrow{=:\partial_n} & \mathrm{H}_{n-1}(X) & \xrightarrow{\mathrm{id} - f_*} & \mathrm{H}_{n-1}(X) & \to
\end{array}$$

[360]Here we write
$$\begin{matrix}\mathrm{H}_i(\mathrm{C}_1)\\ \mathrm{H}_i(\mathrm{C}_2)\end{matrix} \text{ instead of } \mathrm{H}_i(\mathrm{C}_1) \oplus \mathrm{H}_i(\mathrm{C}_2) \quad \text{and} \quad \begin{matrix}\mathrm{H}_i(A)\\ \mathrm{H}_i(B)\end{matrix} \text{ instead of } \mathrm{H}_i(A) \oplus \mathrm{H}_i(B).$$
The reason is that we think of direct sums as columns vectors so that we can use the usual matrix notation for describing the maps between the direct sums.

We make the following clarifications and observations:

(1) Note that $A = p(X \times ([0,4) \cup (6,10])) \subset M$ and $B = p(X \times (2,8)) \subset M$ are open subsets of M that cover all of M. Furthermore note that we have $A \cap B = C_1 \sqcup C_2$.[361] On the top of the diagram we wrote down the corresponding exact Mayer–Vietoris sequence from the Mayer–Vietoris Theorem 45.1 (1).

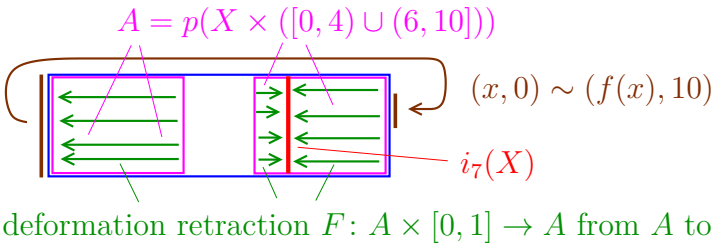
(2) We want to show that the vertical maps from the second to the first row are isomorphisms. In fact this statement follows immediately from the following claim together with the Homotopy Equivalence-H_*-Corollary 36.7 (1a).

Claim. The maps $i_3 \colon X \to C_1$, $i_7 \colon X \to C_2$, $i_7 \colon X \to B$ and $i_7 \colon X \to A$ are homotopy equivalences.

Proof. One can easily verify that the maps $i_t \colon X \to p(X \times \{t\})$, $p \colon X \times (2,4) \to C_1$, $p \colon X \times (6,8) \to C_2$ and $p \colon X \times (2,8) \to B$ are homeomorphisms. Using this observation one can easily verify that the first three maps are homotopy equivalences.

If f is not a homeomorphism, then A is not necessarily homeomorphic to a product. Thus we need to work somewhat harder. Let us consider the following map:

$$\begin{aligned} F \colon A \times [0,1] &\to A \\ ([(x,t)],s) &\mapsto \begin{cases} [(x,t)], & \text{if } s \in [0,\tfrac{1}{2}] \text{ and } t \in (6,10], \\ [(x, t\cdot(1-2s))], & \text{if } s \in [0,\tfrac{1}{2}] \text{ and } t \in [0,4), \\ [(x, t\cdot(2-2s)+7\cdot(2s-1))], & \text{if } s \in [\tfrac{1}{2},1] \text{ and } t \in (6,10], \\ [(f(x), 10\cdot(2-2s)+7\cdot(2s-1))], & \text{if } s \in [\tfrac{1}{2},1] \text{ and } t \in [0,4). \end{cases} \end{aligned}$$



First note that it follows from $(x,0) \sim (f(x),10)$ that this map F is well-defined. One can now easily verify, say using the Pasting Proposition 2.4, that the map F is continuous. It is now clear that the above map F is a deformation retraction from A to $i_7(X)$. This implies, by the Deformation Retract–Homotopy Equivalence Lemma 9.3, that $i_7 \colon X \to A$ is a homotopy equivalence. ⊞

(3) Next we want to show that the top part of the diagram commutes.

(a) We start with the square to the left. We need to show that the following equality holds:

$$\underbrace{\begin{pmatrix} i^A_{C_1} & i^A_{C_2} \\ -i^B_{C_1} & -i^B_{C_2} \end{pmatrix}\begin{pmatrix} i_3^{C_1} & 0 \\ 0 & i_7^{C_2} \end{pmatrix}}_{=\begin{pmatrix} i_3^A & i_7^A \\ -i_3^B & -i_7^B \end{pmatrix}} = \underbrace{\begin{pmatrix} i_7^A & 0 \\ 0 & i_7^B \end{pmatrix}\begin{pmatrix} f_* & \mathrm{id} \\ -\mathrm{id} & -\mathrm{id} \end{pmatrix}}_{=\begin{pmatrix} i_7^A \circ f_* & i_7^A \circ \mathrm{id} \\ i_7^B \circ (-\mathrm{id}) & i_7^B \circ (-\mathrm{id}) \end{pmatrix}}.$$

If we compare the two matrices we see that it remains to prove the following claim:

[361] It follows from the Path-Component-H_*-Lemma 35.8 that we have a natural isomorphism $\mathrm{H}_n(A \cap B) = \mathrm{H}_n(C_1 \sqcup C_2) \xleftarrow{\cong} \mathrm{H}_n(C_1) \oplus \mathrm{H}_n(C_2)$.

Claim. For each $n \in \mathbb{N}_0$ we have $i_3^A = i_7^A \circ f_* \colon \mathrm{H}_n(X) \to \mathrm{H}_n(A)$ and we also have $i_3^B = i_7^B \colon \mathrm{H}_n(X) \to \mathrm{H}_n(B)$.

Proof. We consider the maps[362]

$$\begin{aligned} X \times [0,1] &\to A \\ (x,t) &\mapsto \begin{cases} p(x, 3-6t), & \text{if } t \in [0, \frac{1}{2}], \\ p(f(x), 13-6t) & \text{if } t \in [\frac{1}{2}, 1] \end{cases} \end{aligned} \quad \& \quad \begin{aligned} X \times [0,1] &\to B \\ (x,t) &\mapsto p(x, 3\cdot(1-t) + 7\cdot t). \end{aligned}$$

These two maps are homotopies between the maps $i_3^A, i_7^A \circ f \colon X \to A$ and the maps $i_3^B, i_7^B \colon X \to B$. The claim now follows from the Homotopy-H_*-Proposition 36.5. ⊞

(b) The second square commutes by the functoriality of homology groups.

(c) Recall that in (1) we just showed that the vertical maps from the second to the first row are isomorphisms. Therefore we can define the connecting homomorphism $\partial_n \colon \mathrm{H}_n(M) \to \begin{smallmatrix} \mathrm{H}_{n-1}(X) \\ \mathrm{H}_{n-1}(X) \end{smallmatrix}$ in the second row in such a way that the third square commutes.

(4) It is straightforward to see that the bottom squares in the above diagram commute. We know that the middle sequence is exact. The following claim implies that the bottom sequence is also exact:

Claim. Suppose we are given the following commutative diagram of homomorphisms of abelian groups:

$$\begin{array}{ccccccccccc}
\longrightarrow & \begin{smallmatrix} A_n \\ A_n \end{smallmatrix} & \xrightarrow{\begin{pmatrix} \varphi_n & \mathrm{id} \\ -\mathrm{id} & -\mathrm{id} \end{pmatrix}} & \begin{smallmatrix} A_n \\ A_n \end{smallmatrix} & \xrightarrow{(\alpha_n\ \alpha_n)} & B_n & \xrightarrow{\delta_n} & \begin{smallmatrix} A_{n-1} \\ A_{n-1} \end{smallmatrix} & \xrightarrow{\begin{pmatrix} \varphi_{n-1} & \mathrm{id} \\ -\mathrm{id} & -\mathrm{id} \end{pmatrix}} & \begin{smallmatrix} A_{n-1} \\ A_{n-1} \end{smallmatrix} & \longrightarrow \\
 & \uparrow{\scriptstyle \begin{pmatrix} -\mathrm{id} \\ \mathrm{id} \end{pmatrix}} & & \uparrow{\scriptstyle \begin{pmatrix} \mathrm{id} \\ 0 \end{pmatrix}} & & \uparrow{\scriptstyle \mathrm{id}} & & \uparrow{\scriptstyle \begin{pmatrix} -\mathrm{id} \\ \mathrm{id} \end{pmatrix}} & & \uparrow{\scriptstyle \begin{pmatrix} \mathrm{id} \\ 0 \end{pmatrix}} & \\
\longrightarrow & A_n & \xrightarrow{\mathrm{id} - \varphi_n} & A_n & \xrightarrow{\alpha_n} & B_n & \xrightarrow{\delta_n} & A_{n-1} & \xrightarrow{\mathrm{id} - \varphi_{n-1}} & A_{n-1} & \longrightarrow
\end{array}$$

If the top sequence is exact, then the bottom sequence is also exact.

Proof. We consider the following vertically expanded diagram:

$$\begin{array}{ccccccccccc}
\longrightarrow & A_n & \xrightarrow{-\mathrm{id}} & A_n & \longrightarrow & 0 & \longrightarrow & A_{n-1} & \xrightarrow{-\mathrm{id}} & A_{n-1} & \longrightarrow \\
 & \uparrow{\scriptstyle (\mathrm{id}\ \ \mathrm{id})} & & \uparrow{\scriptstyle (0\ \ \mathrm{id})} & & \uparrow & & \uparrow{\scriptstyle (\mathrm{id}\ \ \mathrm{id})} & & \uparrow{\scriptstyle (0\ \ \mathrm{id})} & \\
\longrightarrow & \begin{smallmatrix} A_n \\ A_n \end{smallmatrix} & \xrightarrow{\begin{pmatrix} \varphi_n & \mathrm{id} \\ -\mathrm{id} & -\mathrm{id} \end{pmatrix}} & \begin{smallmatrix} A_n \\ A_n \end{smallmatrix} & \xrightarrow{(\alpha_n\ \alpha_n)} & B_n & \xrightarrow{\delta_n} & \begin{smallmatrix} A_{n-1} \\ A_{n-1} \end{smallmatrix} & \xrightarrow{\begin{pmatrix} \varphi_{n-1} & \mathrm{id} \\ -\mathrm{id} & -\mathrm{id} \end{pmatrix}} & \begin{smallmatrix} A_{n-1} \\ A_{n-1} \end{smallmatrix} & \longrightarrow \\
 & \uparrow{\scriptstyle \begin{pmatrix} -\mathrm{id} \\ \mathrm{id} \end{pmatrix}} & & \uparrow{\scriptstyle \begin{pmatrix} \mathrm{id} \\ 0 \end{pmatrix}} & & \uparrow{\scriptstyle \mathrm{id}} & & \uparrow{\scriptstyle \begin{pmatrix} -\mathrm{id} \\ \mathrm{id} \end{pmatrix}} & & \uparrow{\scriptstyle \begin{pmatrix} \mathrm{id} \\ 0 \end{pmatrix}} & \\
\longrightarrow & A_n & \xrightarrow{\mathrm{id} - \varphi_n} & A_n & \xrightarrow{\alpha_n} & B_n & \xrightarrow{\delta_n} & A_{n-1} & \xrightarrow{\mathrm{id} - \varphi_{n-1}} & A_{n-1} & \longrightarrow
\end{array}$$

By hypothesis the bottom part of the diagram commutes. It follows easily from the definitions that the top part also commutes. Furthermore note that the vertical sequences, with 0 added on top and bottom, are exact.

[362]Note that it follows from $(x,0) \sim (f(x),10)$ that the map to the left is well-defined. It now follows from the Homotopy Combination Lemma 8.2 that the map to the left is indeed continuous.

Next note that evidently the top sequence is exact. Furthermore by hypothesis the middle sequence is exact. Since the bottom vertical maps are monomorphisms we see that the bottom sequence is a chain complex.

It follows from the above discussion that all these maps form a short exact sequence of chain complexes. Clearly the homology groups of the top chain complex are zero. By hypothesis the homology groups of the middle chain complex are also zero. It follows from the LES-Proposition 37.4 that the homology groups of the bottom chain complex are also zero. But this means that the bottom sequence is exact.

We have thus completed the proof that the bottom horizontal sequence of the diagram on page 570 is exact. If one goes carefully through the definitions one sees that the connecting homomorphism $\partial_n\colon \mathrm{H}_n(\mathrm{Tor}(X,f)) \to \mathrm{H}_{n-1}(X)$ is given by the composition of the homomorphisms in the following unreadable sequence of homomorphisms:[363]

$$\begin{aligned}\mathrm{H}_n(\mathrm{Tor}(X,f)) &\xrightarrow{\text{я}} \mathrm{H}_n(\mathrm{Tor}(X,f),A) \xleftarrow{\cong} \mathrm{H}_n(B, C_1\cup C_2)\\ &\xrightarrow{\partial_n} \mathrm{H}_{n-1}(C_1\cup C_2) \quad \xleftarrow{\cong} \mathrm{H}_{n-1}(C_1)\oplus\mathrm{H}_{n-1}(C_2) \to \mathrm{H}_{n-1}(C_2) \xleftarrow[i_7^{C_2}]{\cong} \mathrm{H}_{n-1}(X). \quad \blacksquare\end{aligned}$$

Now we can easily prove the following proposition.

Proposition 45.4. (Product-with-S^1-H_*-Proposition) Let X be a topological space. We denote by $p\colon X\times S^1 \to X$ the natural projection. For every $n\in\mathbb{N}_0$ there exists a natural[364] homomorphism $\psi_X\colon \mathrm{H}_n(X\times S^1)\to \mathrm{H}_{n-1}(X)$ (which will be described in greater detail in the proof) such that the map

$$\Xi_X \coloneqq p_* \oplus \psi_X\colon \mathrm{H}_n(X\times S^1) \xrightarrow{\cong} \mathrm{H}_n(X)\oplus \mathrm{H}_{n-1}(X)$$

is an isomorphism.[365]

Proof. Let $n\in\mathbb{N}_0$. It is pretty elementary to show that the map

$$\begin{aligned}\Theta\colon \mathrm{Tor}(X,\mathrm{id}) &\to X\times S^1\\ [(x,t)] &\mapsto (x,\exp(2\pi\,\mathrm{i}t))\end{aligned}$$

is a natural homeomorphism. We denote by $i\colon X\to \mathrm{Tor}(X,\mathrm{id})$ the natural map that is given by $x\mapsto[(x,0)]$ and we denote by $j\colon X\to X\times S^1$ the natural embedding that is

[363]The second homomorphism is an isomorphism by the Excision Theorem 40.1, the third homomorphism is the connecting homomorphism of the given pair of topological spaces, the fourth homomorphism is the isomorphism given by the Path-Component-H_*-Lemma 35.8, the fifth homomorphism is the natural projection and the last isomorphism is induced by the map $i_7^{C_2}\colon X\to C_2 = p(X\times(\frac{6}{10},\frac{8}{10}))$, which is easily seen to be a homotopy equivalence.

[364]Here natural means that for any continuous map $f\colon X\to Y$ between topological spaces the following diagram commutes:

$$\begin{array}{ccc}\mathrm{H}_n(X\times S^1) & \xrightarrow[\cong]{\psi_X} & \mathrm{H}_{n-1}(X)\\ \downarrow{\scriptstyle (f\times\mathrm{id})_*} & & {\scriptstyle f_*}\downarrow\\ \mathrm{H}_n(Y\times S^1) & \xrightarrow[\cong]{\psi_Y} & \mathrm{H}_{n-1}(Y).\end{array}$$

[365]The direct sum of natural homomorphisms is of course also natural.

given by $x \mapsto (x, 1)$. Next we consider the following diagram:

$$\begin{array}{ccccccccc}
\mathrm{H}_n(X) & \xrightarrow{\mathrm{id}_* - \mathrm{id}_*} & \mathrm{H}_n(X) & \xrightarrow{i_*} & \mathrm{H}_n(\mathrm{Tor}(X, \mathrm{id})) & \xrightarrow{\partial_n} & \mathrm{H}_{n-1}(X) & \xrightarrow{\mathrm{id}_* - \mathrm{id}_*} & \mathrm{H}_{n-1}(X) \\
& & \downarrow{\scriptstyle \mathrm{id}} & & \cong \downarrow{\scriptstyle \Theta_*} & & \downarrow{\scriptstyle \mathrm{id}} & & \\
0 & \longrightarrow & \mathrm{H}_n(X) & \underset{p_*}{\overset{j_*}{\rightleftarrows}} & \mathrm{H}_n(X \times S^1) & \xrightarrow{\partial_n \circ \Theta_*^{-1}} & \mathrm{H}_{n-1}(X) & \longrightarrow & 0.
\end{array}$$

We make the following clarifications and observations:

(1) The top sequence is the short exact sequence that we obtain from the Mapping Torus-H_*-Proposition 45.3.
(2) Since $\mathrm{id}_* - \mathrm{id}_* = 0$ we see that the homomorphisms to the top left and top right are the zero map.
(3) Since Θ is a homeomorphism and since $\Theta \circ i = j$ we obtain from (1) and (2) that the bottom sequence is also exact.
(4) The map $p\colon X \times S^1 \to X$ is the natural map that is given by projection onto the first factor. Note that $p_* \circ j_* = (p \circ j)_* = \mathrm{id}$.
(5) We set $\psi_X \coloneqq \partial_n \circ \Theta_*^{-1}\colon \mathrm{H}_n(X \times S^1) \to \mathrm{H}_{n-1}(X)$.

Finally note that as in the proof of the Splitting Lemma 37.2 (2)⇒(3) one can show that the map

$$\begin{array}{rcl}
\Xi_X\colon \mathrm{H}_n(X \times S^1) & \to & \mathrm{H}_n(X) \oplus \mathrm{H}_{n-1}(X) \\
\sigma & \mapsto & p_*(\sigma) \oplus \underbrace{(\partial_n \circ \Theta_*^{-1})(\sigma)}_{=\psi_X(\sigma)}
\end{array}$$

is an isomorphism. ∎

Using the Product-with-S^1-H_*-Proposition 45.4 we can now finally compute the homology groups of tori $(S^1)^n$:

Proposition 45.5. (Torus-H_*-Proposition) Let $n \in \mathbb{N}_0$.

(1) For $k = 0, \dots, n$ we have

$$\mathrm{H}_k(n\text{-dimensional torus } (S^1)^n) \;=\; \text{free abelian group of rank } \binom{n}{k}$$

and all other homology groups of $(S^1)^n$ vanish.

(2) For $i = 1, \dots, n$ we consider the singular 1-cycles

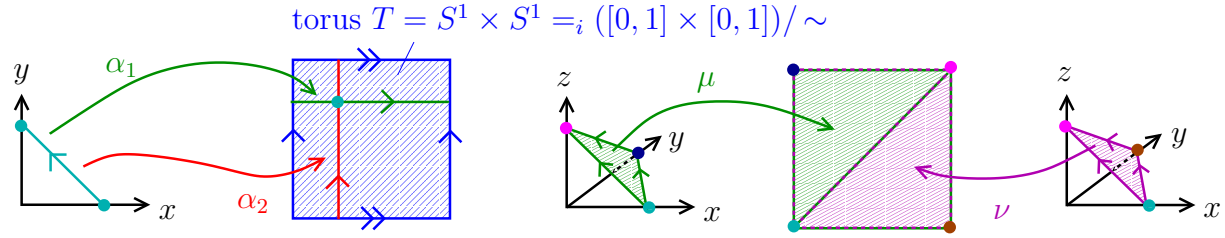
$$\begin{array}{rcl}
\alpha_i\colon \Delta^1 & \to & (S^1)^n \\
(1-t, t) & \mapsto & (1, \dots, 1, \underbrace{\exp(2\pi i t)}_{\substack{\uparrow \\ i\text{-th coordinate}}}, 1, \dots, 1).
\end{array}$$

These singular 1-cycles represent a basis for $\mathrm{H}_1((S^1)^n) \cong \mathbb{Z}^n$.

Example. Let $T = S^1 \times S^1 = ([0,1] \times [0,1])/\sim$ be the 2-dimensional torus. It follows from the Torus-H_*-Proposition 45.5 and the Detect-Cycle Proposition 42.3 that

$$\mathrm{H}_n(T) \;=\; \begin{cases}
\mathbb{Z} \cdot [P], & \text{if } n = 0, \\
\mathbb{Z} \cdot [\alpha_1] \oplus \mathbb{Z} \cdot [\alpha_2], & \text{if } n = 1, \\
\mathbb{Z} \cdot [\mu - \nu], & \text{if } n = 2, \\
0, & \text{otherwise,}
\end{cases}$$

where $P \in T$ is any point and where $\alpha_1, \alpha_2 \colon \Delta^1 \to T$ and $\mu, \nu \colon \Delta^2 \to T$ are indicated in the figure below.[366]



Proof.

(1) We prove this statement by induction on n. For $n = 0$ this statement follows from the Singleton-H_*-Lemma 35.12. The induction step is a straightforward consequence of the Product-with-S^1-H_*-Proposition 45.4. We leave it to the reader to fill in the details.

(2) For $i = 1, \dots, n$ let $p_i \colon (S^1)^n \to S^1$ be the projection onto the i-th factor. We consider the homomorphism

$$\begin{array}{rcl} \Psi \colon \mathrm{H}_1((S^1)^n) & \to & \mathrm{H}_1(S^1) \times \cdots \times \mathrm{H}_1(S^1) \\ \alpha & \mapsto & (p_{1*}(\alpha), \dots, p_{n*}(\alpha)) \end{array}$$

We make the following observations:

(a) By Statement (1) we know that Ψ is a homomorphism between abelian groups that are both isomorphic to $\mathbb{Z}^n$.

(b) On page 540 we showed that the singular 1-cycle $\alpha \colon \Delta^1 \to S^1$ that is given by $(t, 1-t) \mapsto \exp(2\pi\,\mathrm{i}t)$ represents a generator of $\mathrm{H}_1(S^1)$.

(c) It follows easily from (b) and the discussion on page 467 that Ψ is an epimorphism that sends $[\alpha_1], \dots, [\alpha_n]$ to a basis of $\mathrm{H}_1(S^1) \times \cdots \times \mathrm{H}_1(S^1)$.

(d) It follows from (a) and (c)and some elementary algebra that Ψ is an isomorphism.

(e) It follows from (c) and (d) that $[\alpha_1], \dots, [\alpha_n]$ is a basis of $\mathrm{H}_1((S^1)^n)$. ■

The the Mapping Torus-H_*-Proposition 45.3 also provides us with the tools for computing the homology groups of the Klein bottle:

Lemma 45.6. (Klein Bottle-H_*-Lemma) Let

$$K \;=\; ([0,1] \times [0,1])/\sim \qquad \text{where } (x,0) \sim (x,1) \text{ and } (0,y) \sim (1,1-y)$$

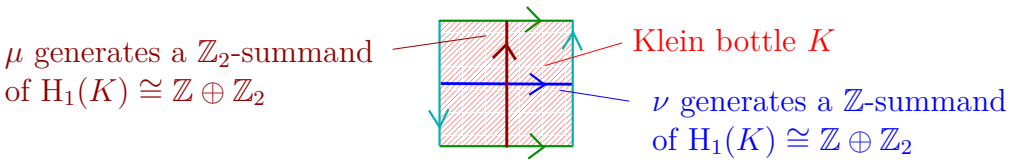
be the Klein bottle as defined on page 131. We consider the two singular 1-cycles

$$\begin{array}{rcl} \mu \colon \Delta^1 & \to & K \\ (t, 1-t) & \mapsto & [(\frac{1}{2}, t)] \end{array} \qquad \text{and} \qquad \begin{array}{rcl} \nu \colon \Delta^1 & \to & K \\ (t, 1-t) & \mapsto & [(t, \frac{1}{2})] \end{array}$$

For any $P \in K$ we have

$$\mathrm{H}_n(K) \;=\; \begin{cases} \mathbb{Z} \cdot [P], & \text{if } n = 0, \\ \mathbb{Z}_2 \cdot [\mu] \oplus \mathbb{Z} \cdot [\nu], & \text{if } n = 1, \\ 0, & \text{otherwise.} \end{cases}$$

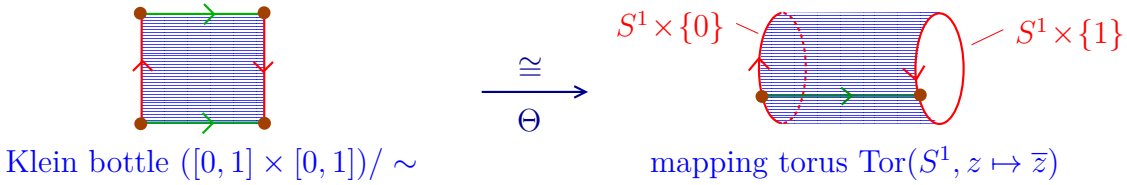
[366] In particular this answers our question from page 463, the cycle $\alpha - \beta$ does indeed represent a non-trivial element in $\mathrm{H}_2(T)$.



Proof. We consider $X = S^1$ and we consider the self-homeomorphism $f\colon S^1 \to S^1$ that is given by $f(z) = \overline{z} = z^{-1}$. It is not difficult to show that the map

$$\begin{array}{rcl} \Theta\colon ([0,1]\times[0,1])/\sim & \to & \mathrm{Tor}(S^1, z \mapsto \overline{z}) \\ [(x,y)] & \mapsto & [(\exp(2\pi\,\mathrm{i}x), y)] \end{array}$$

is a homeomorphism. We will use this homeomorphism as an excuse to study the homology groups of the mapping torus $\mathrm{Tor}(S^1, z \mapsto \overline{z})$.



First note that it follows from the Mapping Torus-H_*-Proposition 45.3 that we get the long exact sequence

$$\begin{array}{ccccccccc} & & & & 0 & \longrightarrow & \mathrm{H}_2(\mathrm{Tor}(S^1,f)) & \xrightarrow{\partial_2} & \\ \to & \mathrm{H}_1(S^1) & \xrightarrow{f_*-\mathrm{id}} & \mathrm{H}_1(S^1) & \xrightarrow{i_*} & \mathrm{H}_1(\mathrm{Tor}(S^1,f)) & \xrightarrow{\partial_1} & & \\ \to & \mathrm{H}_0(S^1) & \xrightarrow{f_*-\mathrm{id}} & \mathrm{H}_0(S^1) & \xrightarrow{i_*} & \mathrm{H}_0(\mathrm{Tor}(S^1,f)) & \longrightarrow & 0. & \end{array}$$

To get our hands on the homology groups of $K \cong \mathrm{Tor}(S^1, f)$ we make the following observations:

(1) By the Sphere-H_*-Proposition 41.2 we know that $\mathrm{H}_0(S^1) \cong \mathbb{Z}$, $\mathrm{H}_1(S^1) \cong \mathbb{Z} \cdot [\alpha]$ where $\alpha(t, 1-t) = \exp(2\pi\,\mathrm{i}t)$, and that $\mathrm{H}_k(S^1) = 0$ for $k \geq 2$.
(2) It follows from the H_0-Proposition 35.11 that the induced map $f_*\colon \mathrm{H}_0(S^1) \to \mathrm{H}_0(S^1)$ is the identity map.
(3) It follows from the Sphere–Degree–Matrix Lemma 43.4 that the induced homomorphism $f_*\colon \mathrm{H}_1(S^1) \to \mathrm{H}_1(S^1)$ is given by multiplication by -1.

It follows easily from these observations that $\mathrm{H}_2(K) \cong \mathrm{H}_2(\mathrm{Tor}(S^1, f)) = 0$ and that $\mathrm{H}_0(K) \cong \mathrm{H}_0(\mathrm{Tor}(S^1, f)) \cong \mathbb{Z}$. Furthermore it follows that we have a short exact sequence

$$0 \to \underbrace{\mathrm{coker}(\mathrm{H}_1(S^1) \xrightarrow{\cdot(-2)} \mathrm{H}_1(S^1))}_{\cong \mathbb{Z}_2} \xrightarrow{i_*} \mathrm{H}_1(\mathrm{Tor}(S^1,f)) \xrightarrow{\partial_1} \underbrace{\mathrm{H}_0(S^1)}_{\cong \mathbb{Z}} \to 0.$$

Note that $i_*([\alpha]) = [\Theta \circ \mu]$. Furthermore, using the explicit description of the connecting homomorphisms ∂_1 on page 572 and using the Singular 1-Simplex Lemma 35.13 one can show that $\partial_1([\Theta \circ \nu])$ is a generator of $\mathrm{H}_0(S^1)$. Using the Free Abelian Group–Splitting Lemma 18.5 together with the Splitting Lemma 37.2 (1)⇒(3) we now deduce that

$$\mathrm{H}_1(\mathrm{Tor}(S^1,f)) \;=\; \mathbb{Z}_2 \cdot \underbrace{[\Theta \circ \mu]}_{=\Theta_*([\mu])} \oplus \mathbb{Z} \cdot \underbrace{[\Theta \circ \nu]}_{=\Theta_*([\nu])}.$$

Finally note that by the isomorphism $\Theta_*\colon \mathrm{H}_1(K) \to \mathrm{H}_1(\mathrm{Tor}(S^1, f))$ this implies that $\mathrm{H}_1(K) = \mathbb{Z}_2 \cdot [\mu] \oplus \mathbb{Z} \cdot [\nu]$. ■

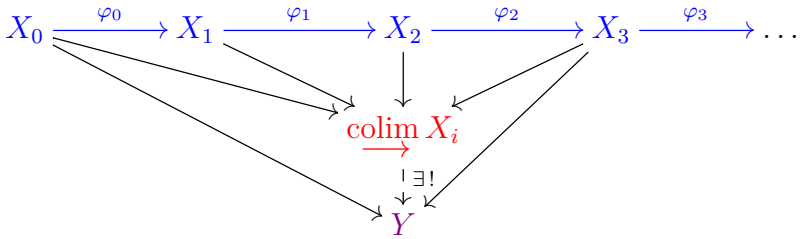
CHAPTER 46

Homology groups and direct colimits

On several occasions we were already wondering whether various functors, e.g. π_1, π_n and H_n, preserve (co-) limits. In particular in the Exhaustion-π_1-Proposition 26.2 and in the CW-Direct Colimit-π_n-Proposition 30.16 we studied the question whether (higher) homotopy groups preserve direct colimits.

In this chapter we will discuss the question to what degree homology groups preserve direct colimits. The results of this chapter will allow us in particular to determine the homology groups of suitable non-compact topological spaces. On a more theoretical level, the results of this chapter will be important in the next chapter when we study cellular homology of CW-complexes.

46.1. **Direct colimits of topological spaces and homology.** First let us remind the reader of the notion of a direct colimit of a sequence of morphisms, as defined on page 339: Given a sequence $X_0 \xrightarrow{\varphi_0} X_1 \xrightarrow{\varphi_1} X_2 \xrightarrow{\varphi_2} \dots$ of morphisms the notion of a direct colimit can be summarized in the following diagram:



More precisely, a direct colimit $\operatorname{colim} X_i$, if it exists, comes with morphisms $X_i \to \operatorname{colim} X_i$ which form a commutative diagram; furthermore given morphisms $X_i \to Y$ which form commutative diagrams, we get a unique morphism $\operatorname{colim} X_i \to Y$ which makes everything commute.

Next let us look at two of our favorite categories and types of sequences:

- First we consider the category $\mathcal{T}op$ of topological spaces. Let X be a topological space and let $X_0 \subset X_1 \subset X_2 \subset \dots$ be a sequence of subspaces of X with $X = \bigcup_{i \in \mathbb{N}_0} X_i$. We suppose that one of the following two conditions is satisfied:
  (1) Each X_i is contained in the interior of X_{i+1}.
  (2) X is a CW-complex and each X_i is a subcomplex of X.
  It follows from (1) the Union-equals-Direct Colimit Criterion 25.7 and (2) the CW-Complex Pasting Lemma 30.9 that in these two settings we have $X = \operatorname{colim} X_i$.
- Next we consider the category $\mathcal{A}b\mathcal{G}roup$ of abelian groups. If we are given a sequence $A_0 \xrightarrow{\varphi_0} A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} \dots$ of homomorphisms between abelian groups, then it follows

easily from the Algebraic–Direct Colimit Existence Proposition 25.3 (1b), that[367] [368]

$$\operatorname*{colim}_{\longrightarrow} A_i \;=\; \Big(\bigoplus_{i\in\mathbb{N}_0} A_i\Big) / \langle \varphi_i(a) - a \mid \; i \in \mathbb{N}_0 \text{ and } a \in A_i\rangle.$$

The following lemma, together with the above discussion, shows that in reasonable circumstances for each $n \in \mathbb{N}_0$ the singular chain group functor $X \mapsto \mathrm{C}_n(X)$ from the category *Top* of topological spaces to the category *AbGroup* of abelian groups preserves direct colimits.

Lemma 46.1. (Exhaustion-C$_*$-Lemma) Let X be a topological space. Furthermore let $X_1 \subset X_2 \subset X_3 \subset \ldots$ be a sequence of subspaces of X with $\bigcup\limits_{i\in\mathbb{N}_0} X_i = X$. Let $n \in \mathbb{N}_0$.

By the universal property of direct colimits the inclusion-induced maps $\mathrm{C}_n(X_i) \to \mathrm{C}_n(X)$ induce a natural homomorphism

$$\operatorname*{colim}_{\longrightarrow i} \mathrm{C}_n(X_i) \;\to\; \mathrm{C}_n(X).$$

If one of the following conditions is satisfied

(1) each X_i is contained in the interior of X_{i+1}, or
(2) X is a CW-complex and each X_i is a subcomplex of X,

then the above map $\operatorname*{colim}\limits_{\longrightarrow i} \mathrm{C}_n(X_i) \to \mathrm{C}_n(X)$ is an isomorphism.

Proof. Let X be a topological space and let $X_1 \subset X_2 \subset X_3 \subset \ldots$ be a sequence of subspaces with $\bigcup\limits_{i\in\mathbb{N}_0} X_i = X$ such that one of the following conditions is satisfied:

(1) each X_i is contained in the interior of X_{i+1}, or
(2) X is a CW-complex and each X_i is a subcomplex of X.

We need to show that for each $n \in \mathbb{N}_0$ the natural map $\operatorname*{colim}\limits_{\longrightarrow} \mathrm{C}_n(X_i) \to \mathrm{C}_n(X)$ is an isomorphism. We start out with the following claim:

Claim. For each compact subset $K \subset X$ there exists an $i \in \mathbb{N}_0$ with $K \subset X_i$.

Proof. Under the hypothesis (1) the claim follows immediately from the elementary Compact Subspace–Exhaustion Lemma 20.11, whereas under the hypothesis (2) the claim follows immediately from the CW-Complex-Finiteness Theorem 30.10. ⊞

In the following, given a topological space Y we denote by $\mathrm{S}_n(Y)$ the set of singular n-simplices in Y. We consider the following diagram:

$$\begin{array}{ccccc}
\operatorname*{colim}\limits_{\longrightarrow i} \mathrm{C}_n(X_i) & & & & \mathrm{C}_n(X) \\
\downarrow{\scriptstyle \mathrm{id}} & & & & \downarrow{\scriptstyle \mathrm{id}} \\
\operatorname*{colim}\limits_{\longrightarrow i} \mathbb{Z}^{(\mathrm{S}_n(X_i))} & \xrightarrow{\;\cong\;} & \mathbb{Z}^{\left(\bigcup\limits_{i\in\mathbb{N}_0} \mathrm{S}_n(X_i)\right)} & \xrightarrow{\;\mathrm{id}\;} & \mathbb{Z}^{(\mathrm{S}_n(X))}.
\end{array}$$

The lemma follows from the following observations:

(a) The top map and the bottom left map are induced by the inclusions $X_i \to X$.

[367] Given a group G and a subset $S \subset G$ we denote by $\langle S\rangle \subset G$ the subgroup generated by S, see page 276 for details.

[368] We use the natural monomorphism $A_j \to \bigoplus\limits_{i\in\mathbb{N}_0} A_i$ to view A_j as a subgroup of $\bigoplus\limits_{i\in\mathbb{N}_0} A_i$.

(b) It follows easily from (a) and the definitions that the diagram commutes.
(c) On page 340 we saw that the bottom left homomorphism is an isomorphism.
(d) It follows immediately from the fact that Δ^n is compact, the Compact Image Lemma 2.8 and the above claim that the image of any continuous map $\Delta^n \to X$ is contained in some X_i. It follows from this observation and the hypothesis that $X = \bigcup_{i\in\mathbb{N}_0} X_i$ that $\mathrm{S}_n(X) = \bigcup_{i\in\mathbb{N}_0} \mathrm{S}_n(X_i)$. In other words, the bottom right map is indeed the identity.
(e) It follows from (b), (c) and (d) that the top homomorphism is also an isomorphism. ■

Next we would like to understand the relationship between direct colimits and homology groups. The key to doing so is the following purely algebraic lemma.

Lemma 46.5. (Direct Colimit-H_*-Lemma) Suppose that for each $i \in \mathbb{N}_0$ we are given a chain complex $(C_{i*}, \partial_*) = (\{C_{in}\}_{n\in\mathbb{N}_0}, \{\partial_n\}_{n\in\mathbb{N}_0})$. Furthermore suppose that we are given a sequence $C_{0*} \to C_{1*} \to C_{2*} \to \dots$ of chain maps.
(1) The groups $\operatorname{colim}_i C_{in}$ with $n \in \mathbb{N}_0$ naturally form a chain complex such that the natural maps $C_{j*} \to \operatorname{colim}_i C_{i*}$ are chain maps.
(2) For each $n \in \mathbb{N}_0$ the homomorphisms $\mathrm{H}_n(C_{j*}) \to \mathrm{H}_n\big(\operatorname{colim}_i C_{i*}\big)$ from (1) induce an isomorphism

$$\operatorname*{colim}_j \mathrm{H}_n(C_{j*}) \xrightarrow{\cong} \mathrm{H}_n\big(\operatorname*{colim}_i C_{i*}\big).$$

Proof. To keep the conversation going we outsource the proof of this algebraic lemma to Section 46.2. In fact in that section we will formulate and prove a slightly more general statement which we will need later on in a different setting. ■

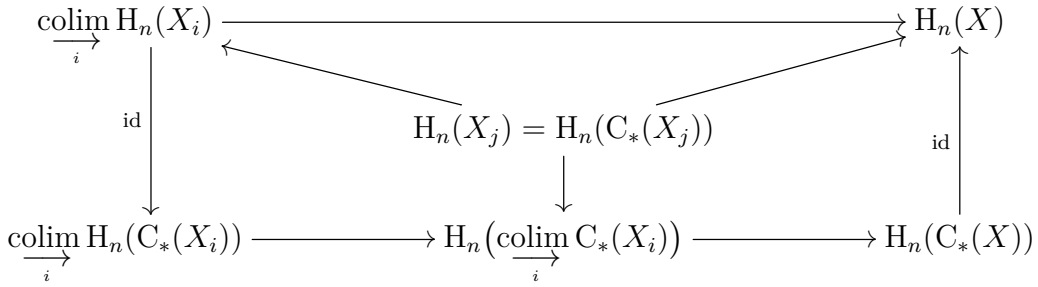
By the discussion on page 577 the following proposition shows in particular that homology groups preserve many direct colimits.

Proposition 46.2. (Exhaustions-H_*-Proposition) Let X be a topological space and let $X_0 \subset X_1 \subset X_2 \subset \dots$ be a sequence of subspaces such that $X = \bigcup_{i\in\mathbb{N}_0} X_i$ and such that one of the following conditions holds:
(1) each X_i is contained in the interior of X_{i+1}, or
(2) X is a CW-complex and each X_i is a subcomplex of X,
then for any $n \in \mathbb{N}_0$ the inclusion-induced maps $\mathrm{H}_n(X_i) \to \mathrm{H}_n(X)$ induce an isomorphism[369]

$$\operatorname*{colim} \mathrm{H}_n(X_i) \xrightarrow{\cong} \mathrm{H}_n(X).$$

[369]This statement is very similar to the statements of the Exhaustion-π_1-Proposition 26.2 and the CW-Direct Colimit-π_n-Proposition 30.16. In fact, as we will see, the proof of our current proposition is at least philosophically also very similar to the earlier proofs.

Proof. For each $j \in \mathbb{N}_0$ we consider the following diagram:



One can easily verify that each segment of the diagram involving $\mathrm{H}_n(X_j)$ commutes. It follows from the universal property of direct colimits that the total diagram commutes. Next note that the bottom left map is an isomorphism by the purely algebraic Direct Colimit-H_*-Lemma 46.5 and that the bottom right map is an isomorphism by the Exhaustion-C_*-Lemma 46.1 (here we use our hypothesis on the sequence of subspaces). It follows that the top map is also an isomorphism. ∎

We conclude this section with an example:

Example. For any $n \in \mathbb{N}_0$ we have

$$\mathrm{H}_n(S^\infty) \;\underset{\uparrow}{=}\; \operatorname*{colim}_k \mathrm{H}_n(S^k) \;\underset{\uparrow}{=}\; \operatorname*{colim}_k \begin{cases} \mathbb{Z}, & \text{if } n = k, 0, \\ 0, & \text{else} \end{cases} \;\underset{\uparrow}{=}\; \begin{cases} \mathbb{Z}, & \text{if } n = 0, \\ 0, & \text{else.} \end{cases}$$

by the Homology-via-Exhaustions Proposition 46.2, where we view S^∞ as a CW-complex as on page 409

by the Sphere-H_*-Proposition 41.2

by the Direct Colimit Stabilization Lemma 25.1

□

46.2. **Direct colimits of chain complexes.** In this section we will provide a proof of the purely algebraic Direct Colimit-H_*-Lemma 46.5. In fact we will consider a slightly more general statement than before, namely we will now consider general direct systems instead of just considering sequences.

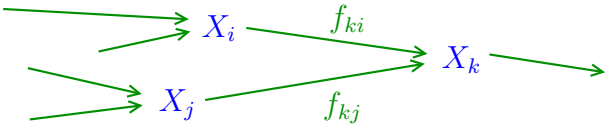
First we recall the following definition from pages 338 and 339:

Definition.

(1) A directed set is a set I together with a partial order $\leq$ that has the extra property that for any $i, j \in I$ there exists a $k \in I$ with $i \leq k$ and $j \leq k$. (In most cases the directed set will just be $(\mathbb{N}_0, \leq)$.)

(2) Let $(I, \leq)$ be a directed set and let $\mathcal{C}$ be a category. A direct system in the category $\mathcal{C}$ over I is a family of objects $\{X_i\}_{i \in I}$ in $\mathcal{C}$, together with a family of morphisms $\{f_{ji} \colon X_i \to X_j\}_{i \leq j}$ such that the following two conditions are satisfied:

(a)	f_{ii}	$= \mathrm{id}_{X_i}$	for all $i \in I$,
(b)	f_{ki}	$= f_{kj} \circ f_{ji}$	for all $i, j, k \in I$ with $i \leq j \leq k$.



Example. Let $\mathcal{C}$ be a category. Any sequence $X_0 \xrightarrow{\varphi_0} X_1 \xrightarrow{\varphi_1} X_2 \xrightarrow{\varphi_2} \dots$ of morphisms in $\mathcal{C}$ gives rise to a direct system over the directed set $(\mathbb{N}_0, \leq)$ by considering for $i \leq j$ the morphism $f_{ji} \coloneqq \varphi_{j-1} \circ \dots \circ \varphi_{i+1} \circ \varphi_i \colon X_i \to X_j$. □

We continue with a new definition:

Definition. Let $(I, \leq)$ be a directed set and let $(\{K_i\}_{i\in I}, \{\kappa_{ji}\}_{i\leq j})$ and $(\{L_i\}_{i\in I}, \{\lambda_{ji}\}_{i\leq j})$ be direct systems in some category $\mathcal{C}$.

(1) A morphism between the above two direct systems is a family $\{f_i \colon K_i \to L_i\}_{i\in I}$ of morphisms such that for any $i \leq j$ the following diagram commutes:

$$\begin{array}{ccc} K_i & \xrightarrow{f_i} & L_i \\ {\scriptstyle \kappa_{ji}}\downarrow & & \downarrow{\scriptstyle \lambda_{ji}} \\ K_j & \xrightarrow{f_j} & L_j. \end{array}$$

(2) By the universal property of direct colimits the morphisms $K_i \to L_i \to \varinjlim L_i$ give rise to a natural morphism

$$\varinjlim f_i \colon \varinjlim K_i \to \varinjlim L_i.$$

Now we can formulate our first result of this section.

Proposition 46.3. (Direct Colimit-Exactness Proposition) Let $(I, \leq)$ be a directed set. Also let $(\{K_i\}_{i\in I}, \{\kappa_{ji}\}_{i\leq j})$, $(\{L_i\}_{i\in I}, \{\lambda_{ji}\}_{i\leq j})$ and $(\{M_i\}_{i\in I}, \{\mu_{ji}\}_{i\leq j})$ be direct systems of abelian groups. Finally suppose that we are given morphisms $f_i \colon K_i \to L_i$ and $g_i \colon L_i \to M_i$ of the direct systems.

(1) If for all $i \in I$ we have $g_i \circ f_i = 0$, then $\varinjlim g_i \circ \varinjlim f_i = 0$.

(2) If for all $i \in I$ the sequence

$$K_i \xrightarrow{f_i} L_i \xrightarrow{g_i} M_i$$

is exact, then the induced sequence

$$\varinjlim K_i \xrightarrow{\varinjlim f_i} \varinjlim L_i \xrightarrow{\varinjlim g_i} \varinjlim M_i$$

is also exact.

Proof. Let $(I, \leq)$ be a directed set. We start out with two general observations regarding direct colimits:

(a) Let $(\{A_i\}_{i\in I}, \{\alpha_{ji}\}_{i\leq j})$ be a direct system of abelian groups. By the Algebraic–Direct Colimit Existence Proposition 25.3 (1b) we know that

$$\varinjlim A_i = \Big(\bigoplus_{i\in I} A_i\Big) / \langle \alpha_{ji}(x) - x \mid \ i \leq j \text{ and } x \in A_i \rangle.$$

We adopt the following notation: Given $a_j \in A_j \subset \bigoplus_{i\in I} A_i$ we denote by $[a_j] \in \varinjlim A_j$ the element represented by a_j.

Claim.

(i) Any element in $\varinjlim A_i$ is of the form $[a_i]$ for some $a_i \in A_i$.

(ii) If we are given $a_j \in A_j$ with $[a_j] = 0 \in \varinjlim A_i$, then there exists a $k \geq j$ such that $\alpha_{kj}(a) = 0 \in A_k$.

Proof.

(i) By the above construction any element in $\varinjlim A_i$ is of the form $[a_{i_1}] + \cdots + [a_{i_m}]$ where $a_{i_j} \in A_{i_j}$. Since $(I, \leq)$ is a directed set we see easily, by considering an $i \in I$ with $i \geq i_1, \ldots, i_m$, that any element of $\varinjlim A_i$ is in fact contained in the image of a *single* A_i, i.e. is of the form $[a_i]$ for some $a_i \in A_i$.

(ii) If we are given $a_j \in A_j$ with $[a_j] = 0 \in \varinjlim A_i$, then it follows from the above description of $\varinjlim A_i$ that in $\bigoplus_{i\in I} A_i$ we can write $a_j = \sum_{r=1}^{m} c_r \cdot (\alpha_{t_r,s_r}(x_r) - x_r) \in \bigoplus_{i\in I} A_i$ where $c_r \in \mathbb{Z}$, $x_r \in A_{s_r}$ and $t_r \geq s_r$. Since we are dealing with a directed set there exists a $k \in I$ with $k \geq j, t_{r_1}, \ldots, t_{r_m}$. Then we obtain the following equality in A_k:

$$\begin{aligned}\alpha_{kj}(a_j) \;=\; \Big(\bigoplus_{i\leq k} \alpha_{ki}\Big)(a_j) \;&=\; \Big(\bigoplus_{i\leq k} \alpha_{ki}\Big)\Big(\sum_{r=1}^{m} c_r \cdot (\alpha_{t_r,s_r}(x_r) - x_r)\Big) \\ &= \sum_{r=1}^{m} c_r \cdot \Big(\underbrace{\Big(\bigoplus_{i\leq k} \alpha_{ki}\Big)(\alpha_{t_r,s_r}(x_r))}_{=\alpha_{k,t_r}(\alpha_{t_r,s_r}(x_r))=\alpha_{k,s_r}(x_r)} - \underbrace{\Big(\bigoplus_{i\leq k} \alpha_{ki}\Big)(x_r)}_{=\alpha_{k,s_r}(x_r)} \Big) \;=\; 0.\end{aligned}$$

⊞

(b) If $(\{A_i\}_{i\in I}, \{\alpha_{ji}\}_{i\leq j})$ and $(\{B_i\}_{i\in I}, \{\beta_{ji}\}_{i\leq j})$ are two direct systems of abelian groups and if $\{f_i \colon A_i \to B_i\}_{i\in I}$ is a morphism between these two direct systems, then it follows easily from the definitions and (a) that the induced map of direct systems is given by

$$\begin{array}{ccc} \varinjlim A_i & \to & \varinjlim B_i \\ \underset{\text{lies in } A_i}{\uparrow}[a_i] & \mapsto & [f_i(a_i)] \end{array}$$

Now we turn to the actual proof of the two statements of the proposition. We write $f \coloneqq \varinjlim f_i$ and $g \coloneqq \varinjlim g_i$.

(1) We assume that for all $i \in I$ we have $g_i \circ f_i = 0$. We need to show that $g \circ f = 0$. Suppose we are given an element of $\varinjlim K_i$. By (a-i) we know that it is of the form $[k_i]$ for some $i \in I$ and some $k_i \in K_i$. We now see that

$$(g \circ f)([k_i]) \;=\; g(f([k_i])) \;\underset{\text{by (b)}}{\uparrow}{=}\; g([f_i(k_i)]) \;\underset{\text{by (b)}}{\uparrow}{=}\; [g_i(f_i(k_i))] \;=\; [(g_i \circ f_i)(k_i)] \;\underset{\text{since } g_i \circ f_i = 0}{\uparrow}{=}\; 0.$$

(2) Now we assume that each $K_j \xrightarrow{f_j} L_j \xrightarrow{g_j} M_j$ is exact. Note that by (1) it remains to prove that $\ker(g) \subset \operatorname{im}(f)$. Thus suppose that we are given an element of $\ker(g) \in \varinjlim L_i$. By (a-i) we know that it is of the form $[l_i]$ for some $i \in I$ and some $l_i \in L_i$. Since $g([l_i]) = 0$ it follows from (a-ii) and (b) that there exists a $j \geq i$ such that $\mu_{ji}(g_i(l_i)) = 0$. Next we consider the following diagram:

$$\begin{array}{ccccc} K_i & \xrightarrow{f_i} & \overset{l_i}{L_i} & \xrightarrow{g_i} & M_i \\ \downarrow{\scriptstyle \kappa_{ji}} & & \downarrow{\scriptstyle \lambda_{ji}} & & \downarrow{\scriptstyle \mu_{ji}} \\ K_j & \xrightarrow{f_j} & L_j & \xrightarrow{g_j} & M_j. \end{array}$$

By hypothesis the diagram commutes, thus we see that $g_j(\lambda_{ji}(l_i)) = \mu_{ji}(g_i(l_i)) = 0$. Since $K_j \xrightarrow{f_j} L_j \xrightarrow{g_j} M_j$ is exact we know that there exists a $k_j \in K_j$ with $f_j(k_j) = \lambda_{ji}(l_i)$.

It follows from (b) that $f([k_j]) = [f_j(k_j)] = [\lambda_{ji}(l_j)] = [l_i] \in \operatorname{colim} L_i$. We have thus shown that $[l_i] \in \operatorname{im}(f)$. ∎

The following corollary says that direct colimits commute with kernels and cokernels.

Corollary 46.4. (Direct Colimit-(Co-)-Kernel Corollary) Let $(I, \leq)$ be a directed set and let $(\{K_i\}_{i\in I}, \{\kappa_{ji}\}_{i\leq j})$ and $(\{L_i\}_{i\in I}, \{\lambda_{ji}\}_{i\leq j})$ be direct systems of abelian groups. Furthermore suppose that we are given a morphism $f_i\colon K_i \to L_i$ between the direct systems. There exist natural isomorphisms

$$\operatorname{colim} \ker(f_i\colon K_i \to L_i) \xrightarrow{\cong} \ker\big(\operatorname{colim} f_i\colon \operatorname{colim} K_i \to \operatorname{colim} L_i\big)$$

and

$$\operatorname{colim} \operatorname{coker}(f_i\colon K_i \to L_i) \xrightarrow{\cong} \operatorname{coker}\big(\operatorname{colim} f_i\colon \operatorname{colim} K_i \to \operatorname{colim} L_i\big).$$

Proof. By the elementary Exactness Lemma 37.1 we know that for each $i \in I$ the sequence

$$0 \to \ker(f_i) \to K_i \xrightarrow{f_i} L_i \to \operatorname{coker}(f_i) \to 0.$$

is exact. One can easily verify that these homomorphisms form morphisms between the various direct systems. It follows from the Direct Colimit-Exactness Proposition 46.3 (2) (applied four times!) that the sequence

$$0 \to \operatorname{colim} \ker(f_i) \to \operatorname{colim} K_i \xrightarrow{\operatorname{colim} f_i} \operatorname{colim} L_i \to \operatorname{colim} \operatorname{coker}(f_i) \to 0$$

is also exact. By the Exactness Lemma 37.1 this observation implies the desired statements. ∎

The following lemma is a slightly more general version of the lemma stated on page 579 in the first section of this chapter.

Lemma 46.5. (Direct Colimit-H_*-Lemma) Let $(I, \leq)$ be a directed set and let $(\{C_i\}_{i\in I}, \{f_{ji}\}_{i\leq j})$ be a direct system of chain complexes.

(1) The groups $\operatorname{colim}_i C_{i*}$ naturally form a chain complex such that the natural maps $C_{j*} \to \operatorname{colim}_i C_{i*}$ are chain maps.

(2) For each $n \in \mathbb{N}_0$ the homomorphisms $\mathrm{H}_n(C_{j*}) \to \mathrm{H}_n\big(\operatorname{colim}_i C_{j*}\big)$ from (1) induce an isomorphism

$$\operatorname*{colim}_j \mathrm{H}_n(C_{j*}) \xrightarrow{\cong} \mathrm{H}_n\big(\operatorname*{colim}_i C_{i*}\big).$$

Proof.

(1) It follows immediately from the Direct Colimit-Exactness Proposition 46.3 (1), applied to the maps $C_{i,n+1} \xrightarrow{\partial_{n+1}} C_{i,n} \xrightarrow{\partial_n} C_{i,n-1}$, that the direct colimits $\operatorname{colim}_i C_{i*}$ form a chain complex. Furthermore note that it follows immediately from the universal properties of direct colimits that the natural maps $C_{j*} \to \operatorname{colim}_i C_{i*}$ are chain maps.

(2) Given any $n \in \mathbb{N}_0$ we see that

$$\begin{aligned}
\operatorname*{colim}_{i} \mathrm{H}_n(C_{i*}) &= \operatorname*{colim}_{i} \overbrace{\operatorname{coker}\left(C_{i,n+1} \xrightarrow{\partial_{n+1}} \ker(C_{i,n} \xrightarrow{\partial_n} C_{i,n-1})\right)}^{=\mathrm{H}_n(C_{i*})} \\
&\overset{\cong}{\uparrow} \operatorname{coker}\left(\operatorname*{colim}_{i} C_{i,n+1} \xrightarrow{\operatorname{colim}\partial_{n+1}} \ker\left(\operatorname*{colim}_{i} C_{i,n} \xrightarrow{\operatorname{colim}\partial_n} \operatorname*{colim}_{i} C_{i,n-1}\right)\right) \\
&\qquad \text{follows from applying the Direct Colimit-(Co-)-Kernel Corollary 46.4 twice} \\
&= \mathrm{H}_n\big(\operatorname*{colim}_{i} C_{i*}\big). \\
&\qquad \text{by definition of the } n\text{-th homology group}
\end{aligned}$$

The fact that this isomorphism is induced by the maps $\mathrm{H}_n(C_{i*}) \to \mathrm{H}_n(\operatorname*{colim}_{\longrightarrow} C_{i*})$ follows easily from the various definitions. We leave the details to the reader. ■

CHAPTER 47

Cellular homology

It follows from the 1-Complex-π_1-Theorem 31.5 and the Cell Attachment-π_1-Proposition 29.1 that it is relatively easy to compute the fundamental group of a finite CW-complex.

In this chapter we will develop an algorithm which reduces the calculation of homology groups of CW-complexes to determining the degrees of suitable self-maps of spheres. We will use this method to finally determine the homology groups of the (non-orientable) surfaces of genus ≥ 2, of all the projective spaces $\mathbb{R}\mathrm{P}^n$ and $\mathbb{C}\mathrm{P}^n$ and even of the infinite-dimensional projective spaces $\mathbb{R}\mathrm{P}^\infty$ and $\mathbb{C}\mathrm{P}^\infty$.

47.1. **Relative homology groups of CW-complexes.** In the following section we will introduce and study the cellular homology groups of a CW-complex X. Throughout this chapter we will use the following familiar notation:

Notation. Let X be a CW-complex
(1) As usual, we denote the filtration of X by $\varnothing = X^{-1} \subset X^0 \subset X^1 \subset X^2 \subset \dots$ and we refer to X^n as the n-skeleton of X.
(2) We denote by $\mathrm{Cell}_n(X)$ the set of n-cells of X.

In preparation for the next section we now prove several basic statements about homology groups of CW-complexes:

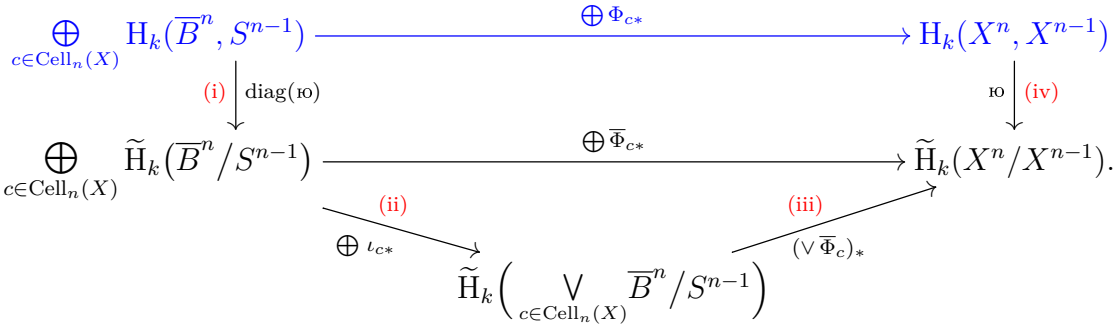
Lemma 47.1. (Skeleton-H_*-Lemma) Let X be a CW-complex and let $n \in \mathbb{N}_0$.
(1) Let $\{\Phi_c \colon \overline{B}^n \to X^n\}_{c \in \mathrm{Cell}_n(X)}$ be a full set of characteristic maps for the n-cells of X.[370] For every $k \in \mathbb{N}_0$ the following map is an isomorphism:
$$\bigoplus_{c \in \mathrm{Cell}_n(X)} \Phi_{c*} \colon \bigoplus_{c \in \mathrm{Cell}_n(X)} \mathrm{H}_k(\overline{B}^n, S^{n-1}) \;\to\; \mathrm{H}_k(X^n, X^{n-1}).$$
In particular the following statements hold:
(a) For $k \neq n$ we have $\mathrm{H}_k(X^n, X^{n-1}) = 0$.
(b) The group $\mathrm{H}_n(X^n, X^{n-1})$ is a free abelian group where the rank equals the cardinality of the set $\mathrm{Cell}_n(X)$ of n-cells in the CW-structure.
(2) For $k > n$ we have $\mathrm{H}_k(X^n) = 0$.
(3) For $k < n$ the inclusion $i \colon X^n \to X$ induces an isomorphism $i_* \colon \mathrm{H}_k(X^n) \xrightarrow{\cong} \mathrm{H}_k(X)$.
(4) The inclusion $i \colon X^n \to X$ induces an epimorphism $i_* \colon \mathrm{H}_n(X^n) \twoheadrightarrow \mathrm{H}_n(X)$.

Remark. The statements (3) and (4) of the Skeleton-H_*-Lemma 47.1 can be viewed as "homological analogues" of the CW-Skeleton-π_1-Proposition 30.14. □

[370] We refer to page 412 for the definition of a full set of characteristic maps and a justification why such a set always exists.

Proof. Let X be a CW-complex and let $\{\Phi_c\colon \overline{B}^n \to X^n\}_{c\in \mathrm{Cell}_n(X)}$ be a full set of characteristic maps for the n-cells of X.

(1) We consider the following diagram:



We make the following clarifications and observations:

- Given a pair (Y, A) of topological space we consider the natural homomorphism $\imath\jmath\colon \mathrm{H}_n(Y, A) \to \widetilde{\mathrm{H}}_n(Y/A)$ from page 527.
- We use (obvious generalizations of) the notation from page 555 regarding maps out of direct sums of abelian groups.
- Given $c \in \mathrm{Cell}_n(X)$ we denote by $\overline{\Phi}_c\colon \overline{B}^n/S^{n-1} \to X^n/X^{n-1}$ the map that is induced by the characteristic map $\Phi_c\colon \overline{B}^n \to X^n$.
- It follows almost immediately from the naturality of the maps $\imath\jmath$ that the rectangle commutes.
- We use the points $[S^{n-1}] \in \overline{B}^n/S^{n-1}$ to form the wedge $\bigvee_{c\in\mathrm{Cell}_n(X)} \overline{B}^n/S^{n-1}$.
- For $j \in \mathrm{Cell}_n(X)$ let $\iota_j\colon \overline{B}^n/S^{n-1} \to \bigvee_{c\in\mathrm{Cell}_n(X)} \overline{B}^n/S^{n-1}$ be the natural inclusion.
- For each $j \in \mathrm{Cell}_n(X)$ we have $\overline{\Phi}_j = \big(\bigvee_{c\in\mathrm{Cell}_n(X)} \overline{\Phi}_c\big) \circ \iota_j$. It follows from this observation and the definition of $\bigoplus_{c\in\mathrm{Cell}_n(X)} \overline{\Phi}_{c*}$ that the triangle commutes.

We want to show that the top horizontal map is an isomorphism. By the above it remains to show that the maps (i)–(iv) are isomorphisms.

(i) This map is an isomorphism by the Relative-to-Quotient-H_*-Proposition 41.8.[371]
(ii) This map is an isomorphism by the Wedge-H_*-Proposition 41.10.
(iii) This map is an isomorphism by the Skeleton-Quotient Lemma 33.5.
(iv) It follows from the Relative-to-Quotient-H_*-Proposition 41.8 together with the Subcomplex–Neighborhood Proposition 31.8 that this map is an isomorphism.

The remaining statements of (1) are now an immediate consequence of the Ball-rel-Sphere-H_*-Lemma 41.4.

(2) We fix $k \in \mathbb{N}$. We need to prove the following claim:

Claim. For $n = 0, \dots, k-1$ we have $\mathrm{H}_k(X^n) = 0$.

Proof. We prove the claim by induction on $n \in \mathbb{N}_0$. First we consider the case $n = 0$. By the discussion on page 406 we know that the 0-skeleton X^0 has the discrete topology.

[371] Here we use that $S^{n-1} \subset \overline{B}^n$ is clearly a good subset.

It follows from the Singleton-H_*-Lemma 35.12, the Path-Component–H_*–Lemma 35.8 and the hypothesis that $k > 0$ that $\mathrm{H}_k(X^0) = 0$.

Now assume that the statement holds for some $n \in \{0, \dots, k-2\}$. We consider the long exact sequence of the pair (X^{n+1}, X^n):

$$\dots \to \mathrm{H}_{k+1}(X^{n+1}, X^n) \to \underbrace{\mathrm{H}_k(X^n)}_{=0 \text{ by induction}} \to \mathrm{H}_k(X^{n+1}) \to \underbrace{\mathrm{H}_k(X^{n+1}, X^n)}_{\substack{=0 \text{ by (1) since } n < k-1 \\ \text{which implies that } k \neq n+1}} \to \dots$$

It follows that we have an epimorphism $\mathrm{H}_k(X^n) \to \mathrm{H}_k(X^{n+1})$. Since $\mathrm{H}_k(X^n) = 0$ we conclude that $\mathrm{H}_k(X^{n+1}) = 0$. ⊞

(3) Let $k \in \mathbb{N}_0$ and $n > k$. Recall that we need to show that the inclusion-induced map $\mathrm{H}_k(X^n) \to \mathrm{H}_k(X)$ is an isomorphism. We consider the following diagram of inclusion-induced homomorphisms:

$$\begin{array}{ccccccccccc} \dots & \longrightarrow & \mathrm{H}_k(X^{n-1}) & \longrightarrow & \mathrm{H}_k(X^n) & \xrightarrow{\cong} & \mathrm{H}_k(X^{n+1}) & \xrightarrow{\cong} & \mathrm{H}_k(X^{n+2}) & \xrightarrow{\cong} & \dots \\ & & & \searrow & \downarrow & \swarrow & & \swarrow & & & \\ & & & & \mathrm{H}_k(X). & & & & & & \end{array}$$

We make the following observations:

(a) The diagram commutes by the functoriality of homology.

(b) We just want to prove the following claim:

Claim. For any $m \geq n$ the inclusion-induced homomorphism $\mathrm{H}_k(X^m) \to \mathrm{H}_k(X^{m+1})$ is an isomorphism.

Proof. This claim follows by considering the following excerpt from the long exact sequence of the pair (X^{m+1}, X^m):

$$\dots \to \underbrace{\mathrm{H}_{k+1}(X^{m+1}, X^m)}_{=0 \text{ by (1) since } k < n \leq m} \to \mathrm{H}_k(X^m) \to \mathrm{H}_k(X^{m+1}) \to \underbrace{\mathrm{H}_k(X^{m+1}, X^m)}_{=0 \text{ by (1) since } k < n \leq m} \to \dots$$

⊞

(c) By the Exhaustions-H_*-Proposition 46.2 (2) we know that $\mathrm{H}_k(X) = \varinjlim \mathrm{H}_k(X^n)$.

(d) It follows from (a)–(c) and the Direct Colimit–Stabilization Lemma 25.1 that the vertical inclusion-induced homomorphism $\mathrm{H}_k(X^n) \to \mathrm{H}_k(X)$ is an isomorphism.

(4) We consider the following excerpt from the long exact sequence of the pair (X^{n+1}, X^n):

$$\begin{array}{ccccccccc} \dots & \longrightarrow & \mathrm{H}_{n+1}(X^{n+1}, X^n) & \longrightarrow & \mathrm{H}_n(X^n) & \longrightarrow & \mathrm{H}_n(X^{n+1}) & \longrightarrow & \underbrace{\mathrm{H}_n(X^{n+1}, X^n)}_{=0 \text{ by (1)}} \longrightarrow \dots \\ & & & & & \searrow & \downarrow \cong \text{ by (3)} & & \\ & & & & & & \mathrm{H}_n(X) & & \end{array}$$

We see that the inclusion-induced map $\mathrm{H}_n(X^n) \to \mathrm{H}_n(X)$ is an epimorphism. ■

47.2. **The definition of cellular homology.** In this section we will introduce the main protagonists of this chapter, namely the cellular homology groups of a CW-complex. We start out with the definition of the cellular boundary map.

Definition. Given a CW-complex X and given $n \in \mathbb{N}_0$ we denote by d_n the homomorphism

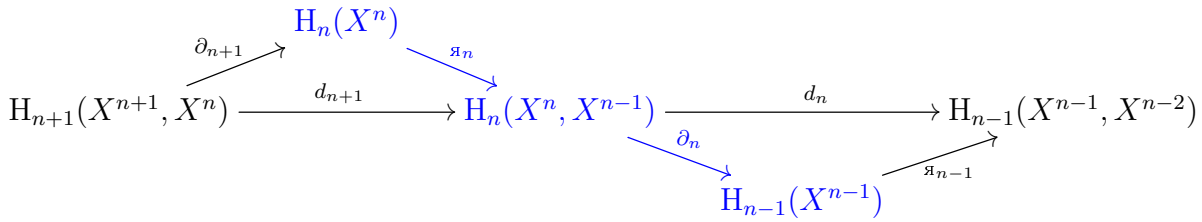
$$\mathrm{H}_n(X^n, X^{n-1}) \xrightarrow{\partial_n} \mathrm{H}_{n-1}(X^{n-1}) \xrightarrow{\text{я}_{n-1}} \mathrm{H}_{n-1}(X^{n-1}, X^{n-2})$$
$$=: d_n$$

where ∂_n is the connecting map in the long exact sequence of the pair (X^n, X^{n-1}) and where $\text{я}_{n-1} \colon \mathrm{H}_{n-1}(X^{n-1}) \to \mathrm{H}_{n-1}(X^{n-1}, X^{n-2})$ is the natural map introduced on page 493. We refer to d_n as the cellular boundary map.

The name "cellular boundary map" already suggests that the maps d_n have the following important property:

Lemma 47.2. (Cellular Boundary Map Lemma) For all $n \in \mathbb{N}_0$ we have $d_n \circ d_{n+1} = 0$.

Proof. We consider the following diagram of homomorphisms:



Note that the diagram commutes by the definition of the cellular boundary maps. The map $\partial_n \circ \text{я}_n \colon \mathrm{H}_n(X^n) \to \mathrm{H}_{n-1}(X^{n-1})$ is the composition of two successive maps in the long exact sequence of the pair (X^n, X^{n-1}). This implies that the map $\partial_n \circ \text{я}_n$ is the zero map. Since the diagram commutes we obtain that $d_n \circ d_{n+1} = 0$. ■

By the Cellular Boundary Map Lemma 47.2 it now makes sense to introduce the following definition:

Definition. Let X be a CW-complex.
(1) Given $n \in \mathbb{N}_0$ we write $\mathrm{C}_n^{\mathrm{CW}}(X) := \mathrm{H}_n(X^n, X^{n-1})$.[372]
(2) We refer to $(\mathrm{C}_*^{\mathrm{CW}}(X), d_*)$ as the cellular chain complex of the CW-complex X.[373]
(3) We denote the homology groups of the cellular chain complex by $\mathrm{H}_n^{\mathrm{CW}}(X)$ and we refer to these groups as the cellular homology groups of the CW-complex X.

Example. For $n \geq 2$ we consider the n-dimensional sphere S^n. On page 407 we showed that we can view S^n as a CW-complex with one 0-cell and one n-cell. It follows from the Skeleton-H_*-Lemma 47.1 (1) that the cellular chain complex $(\mathrm{C}_*^{\mathrm{CW}}(S^n), d_*)$ is of the form

$$0 \to \underbrace{\mathrm{C}_n^{\mathrm{CW}}(S^n)}_{\cong \mathbb{Z}} \to 0 \to \ldots \to 0 \to \underbrace{\mathrm{C}_0^{\mathrm{CW}}(S^n)}_{\cong \mathbb{Z}} \to 0.$$

[372]Note that the Skeleton-H_*-Lemma 47.1 (1) says that the n-th cellular chain group $\mathrm{C}_n^{\mathrm{CW}}(X)$ is a free abelian group where the rank is given by the cardinality of the set of n-cells.
[373]It follows from the Cellular Boundary Map Lemma 47.2 that $(\mathrm{C}_*^{\mathrm{CW}}(X), d_*)$ is indeed a chain complex.

Since $n \geq 2$ the two $\mathbb{Z}$'s are separated by at least one zero group. The cellular boundary maps are therefore necessarily the zero maps. This implies that[374]

$$\mathrm{H}_i^{\mathrm{CW}}(S^n) \cong \begin{cases} \mathbb{Z}, & \text{if } i = 0, n, \\ 0, & \text{otherwise.} \end{cases}$$

These cellular homology groups are of course isomorphic to the "usual" homology groups of S^n. In the next section we will see that this is not a coincidence. □

47.3. **The relationship between cellular and singular homology.** Given a CW-complex X we now have two "competing" homology groups, namely the singular homology groups $\mathrm{H}_n(X)$ and the cellular homology group $\mathrm{H}_n^{\mathrm{CW}}(X)$. The following proposition explains the relationship between these two homology groups.

Proposition 47.3. (Singular-Cellular-H_*-Proposition) Let $n \in \mathbb{N}_0$.

(1) For each CW-complex X there exists a uniquely determined *isomorphism*

$$\Theta_X \colon \mathrm{H}_n(X) \xrightarrow{\cong} \mathrm{H}_n^{\mathrm{CW}}(X)$$

with the property that the following diagram commutes:[375]

$$\begin{array}{ccc} \mathrm{H}_n(X^n) & \twoheadrightarrow & \mathrm{H}_n(X) \\ {\scriptstyle я_n}\downarrow & & \downarrow{\scriptstyle \Theta_X} \\ \ker(d_n) & \xrightarrow{a \mapsto [a]} & \mathrm{H}_n^{\mathrm{CW}}(X). \end{array}$$

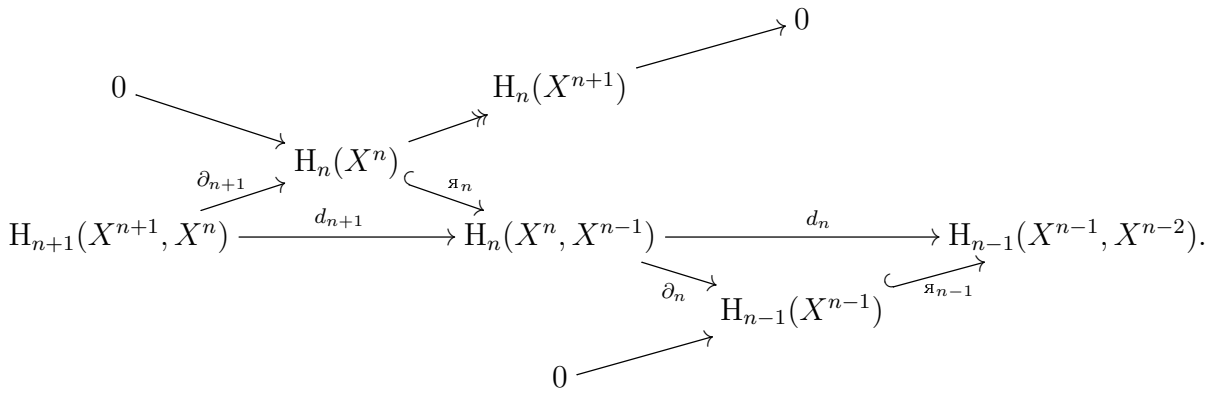
(2) The homomorphisms from (1) define a natural isomorphism between the functor $X \mapsto \mathrm{H}_n(X)$ and the functor $X \mapsto \mathrm{H}_n^{\mathrm{CW}}(X)$. In other words, for every cellular map $f \colon X \to Y$ between CW-complexes the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}_n(X) & \xrightarrow{f_*} & \mathrm{H}_n(Y) \\ {\scriptstyle \Theta_X}\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\downarrow{\scriptstyle \Theta_Y} \\ \mathrm{H}_n^{\mathrm{CW}}(X) & \xrightarrow{f_*} & \mathrm{H}_n^{\mathrm{CW}}(Y). \end{array}$$

[374] It is a fun exercise to calculate the cellular boundary maps of S^1.

[375] Here we again denote by $я_n \colon \mathrm{H}_n(X^n) \to \mathrm{H}_n(X^n, X^{n-1})$ the obvious map. In the long exact sequence of the pair (X^n, X^{n-1}) we see that the image of $\mathrm{H}_n(X^n)$ in $\mathrm{H}_n(X^n, X^{n-1})$ lies in the kernel of the connecting homomorphism ∂_n. Since $d_n = я_n \circ \partial_n$ this implies that the map $я_n \colon \mathrm{H}_n(X^n) \to \mathrm{H}_n(X^n, X^{n-1})$ takes values in $\ker(d_n)$.

Proof of Proposition 47.3 (1). We add a couple of maps to the commutative diagram that we had already encountered in the proof of the Cellular Boundary Map Lemma 47.2:



Claim 1. All the diagonal sequences in the above commutative diagram are exact.

Proof. By the Skeleton-H_*-Lemma 47.1 (1) and (2) we know that

$$\mathrm{H}_n(X^{n+1}, X^n) = 0, \quad \mathrm{H}_n(X^{n-1}) = 0 \quad \text{and} \quad \mathrm{H}_{n-1}(X^{n-2}) = 0.$$

It thus follows that the diagonal sequences in the above commutative diagram are just excerpts from the long exact sequences of the corresponding pairs of topological spaces. In particular all the diagonal sequences are exact. ⊞

Claim 2. The map $\mathrm{я}_n \colon \mathrm{H}_n(X^n) \to \mathrm{H}_n(X^n, X^{n-1})$ induces an isomorphism

$$\mathrm{H}_n(X^n)/\operatorname{im}(\partial_{n+1}) \;\to\; \underbrace{\ker(d_n)/\operatorname{im}(d_{n+1})}_{=\mathrm{H}_n^{\mathrm{CW}}(X)}.$$

Proof.

(1) The given map is well-defined since it follows immediately from $\partial_n \circ \mathrm{я}_n = 0$ that $\mathrm{я}_n(\mathrm{H}_n(X^n)) \subset \ker(d_n)$.

(2) [376] The given map is an epimorphism. This can be seen as follows:
   (a) It follows from Claim 1 that the map $\mathrm{я}_{n-1}$ is injective.
   (b) It follows from (a) and Claim 1 that $\ker(d_n) = \ker(\partial_n) = \operatorname{im}(\mathrm{я}_n)$.
   (c) It follows from (b) that the map $\mathrm{я}_n$ induces an epimorphism $\mathrm{H}_n(X^n) \to \ker(d_n)$.
   (d) Since the left triangle commutes we obtain from (c) that $\mathrm{я}_n$ also induces an epimorphism $\mathrm{H}_n(X^n)/\operatorname{im}(\partial_{n+1}) \to \ker(d_n)/\operatorname{im}(d_{n+1})$.

(3) The given map is a monomorphism: We need to show that if we are given $c \in \mathrm{H}_n(X^n)$ such that $\mathrm{я}_n(c) \in \mathrm{H}_n(X^n, X^{n-1})$ lies in the image of d_{n+1}, then $c \in \operatorname{im}(\partial_{n+1})$. Thus suppose that $\mathrm{я}_n(c) \in \mathrm{H}_n(X^n, X^{n-1})$ lies in the image of d_{n+1}. This means that there exists an $e \in \mathrm{H}_{n+1}(X^{n+1}, X^n)$ with $\mathrm{я}_n(c) = d_{n+1}(e) = \mathrm{я}_n(\partial_{n+1}(e))$. Since by Claim 1 we know that $\mathrm{я}_n$ is injective we obtain that $c = \partial_{n+1}(e)$. ⊞

[376] It remains to show that the given map is an epimorphism and a monomorphism. This is shown by a diagram chase. It is arguably best not to read the remainder of the proof of the claim but to go on the chase on one's own.

Next we consider the following diagram:

$$\begin{array}{ccccc}
 & & \text{isomorphism since in the above diagram the left diagonal sequence is exact} & & \text{isomorphism by the Skeleton-H}_*\text{-Lemma 47.1 (3)} \\
 & & \downarrow & & \downarrow \\
\mathrm{H}_n(X^n)/\operatorname{im}(\partial_{n+1}) & \to & \mathrm{H}_n(X^{n+1}) & \to & \mathrm{H}_n(X) \\
\text{by the claim we know that } \text{я}_n \text{ induces an isomorphism} \Big\downarrow & \cong & & & \Big\downarrow =: \Theta_X \\
\ker(d_n)/\operatorname{im}(d_{n+1}) & & \xrightarrow[\text{id}]{} & & \mathrm{H}_n^{\mathrm{CW}}(X).
\end{array}$$

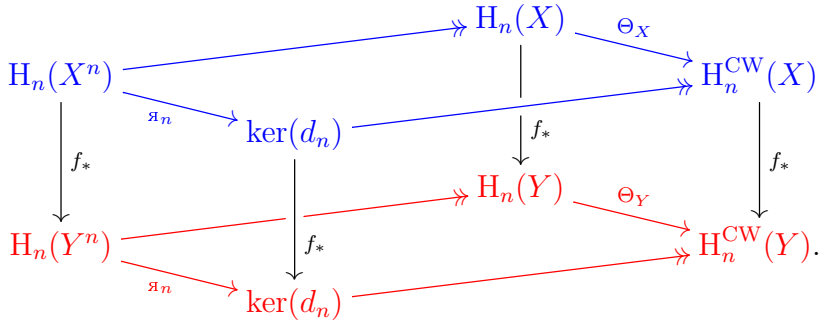
Since all the given maps are isomorphisms we can now define the right vertical map Θ_X as the unique map which makes the diagram commute. Evidently Θ_X is also an isomorphism.

Finally we note that by construction the isomorphism Θ_X has the property that the following diagram commutes:

$$\begin{array}{ccc}
\mathrm{H}_n(X^n) & \twoheadrightarrow & \mathrm{H}_n(X) \\
{\scriptstyle \text{я}_n}\downarrow & & \downarrow{\scriptstyle \Theta_X} \\
\ker(d_n) & \twoheadrightarrow & \mathrm{H}_n^{\mathrm{CW}}(X).
\end{array}$$

Note that by the Skeleton-H$_*$-Lemma 47.1 (4) the top horizontal map is an epimorphism. This implies immediately that Θ_X is the unique homomorphism that makes the diagram commute. ∎

Proof of Proposition 47.3 (2). We need to show that the isomorphisms Θ_X that we just constructed are natural. Thus suppose that we are given a cellular map $f\colon X \to Y$ and let $n \in \mathbb{N}_0$. We consider the following diagram:



We make the following observations:

(a) The sides of the hyperrectangle that do not involve Θ_X and Θ_Y commute basically by definition.
(b) The top and bottom parallelogram commute by definition of Θ_X and Θ_Y.
(c) By the Skeleton-H$_*$-Lemma 47.1 (4) the map $\mathrm{H}_n(X^n) \to \mathrm{H}_n(X)$ is an epimorphism.

It follows from these observation that also the parallelogram involving Θ_X and Θ_Y commutes. But that is exactly, what we had to show. ∎

Even though the cellular boundary maps in the cellular chain complex are still somewhat mysterious, we can already use the above proposition to carry out an interesting calculation, namely we can finally calculate the homology groups of the complex projective spaces $\mathbb{C}\mathrm{P}^n := (\mathbb{C}^{n+1} \setminus \{0\})/(\mathbb{C} \setminus \{0\})$ and of $\mathbb{C}\mathrm{P}^\infty = \operatorname{colim}_{\longrightarrow} \mathbb{C}\mathrm{P}^n$:

Proposition 47.4. ($\mathbb{C}\mathrm{P}^n$-H_*-Proposition)

(1) Given $n \in \mathbb{N}_0$ we have[377]

$$\mathrm{H}_k(\mathbb{C}\mathrm{P}^n) \cong \begin{cases} \mathbb{Z}, & \text{if } k = 0, 2, 4, \dots, 2n \\ 0, & \text{otherwise.} \end{cases}$$

(2) We have

$$\mathrm{H}_k(\mathbb{C}\mathrm{P}^\infty) \cong \begin{cases} \mathbb{Z}, & \text{if } k \text{ is even,} \\ 0, & \text{if } k \text{ is } k \text{ is odd.} \end{cases}$$

Proof.

(1) Let $n \in \mathbb{N}_0$. On page 410 we showed that $\mathbb{C}\mathrm{P}^n$ admits a CW-structure with exactly one cell in dimensions $0, 2, \dots, 2n$ and no other cells. It follows from the Skeleton-H_*-Lemma 47.1 (1) that the corresponding cellular chain complex is of the form

$$0 \to \underbrace{\mathrm{C}^{\mathrm{CW}}_{2n}(\mathbb{C}\mathrm{P}^n)}_{\cong\mathbb{Z}} \to \underbrace{\mathrm{C}^{\mathrm{CW}}_{2n-1}(\mathbb{C}\mathrm{P}^n)}_{=0} \to \dots \to \underbrace{\mathrm{C}^{\mathrm{CW}}_{2}(\mathbb{C}\mathrm{P}^n)}_{\cong\mathbb{Z}} \to \underbrace{\mathrm{C}^{\mathrm{CW}}_{1}(\mathbb{C}\mathrm{P}^n)}_{=0} \to \underbrace{\mathrm{C}^{\mathrm{CW}}_{0}(\mathbb{C}\mathrm{P}^n)}_{\cong\mathbb{Z}} \to 0.$$

Since every other group is trivial it follows that all boundary maps are zero. In particular the cellular homology groups equal the cellular chain groups. It follows that

$$\mathrm{H}_k(\mathbb{C}\mathrm{P}^n) \underset{\uparrow}{\cong} \mathrm{H}^{\mathrm{CW}}_k(\mathbb{C}\mathrm{P}^n) \underset{\uparrow}{\cong} \mathrm{C}^{\mathrm{CW}}_k(\mathbb{C}\mathrm{P}^n) \underset{\uparrow}{\cong} \begin{cases} \mathbb{Z}, & \text{if } k = 0, 2, 4, \dots, 2n, \\ 0, & \text{otherwise.} \end{cases}$$

by the Singular-Cellular Homology Proposition 47.3 — follows from the above discussion

(2) Just for fun we provide two different ways to calculate $\mathrm{H}_k(\mathbb{C}\mathrm{P}^\infty)$:

(a) On page 410 we saw that we can equip the infinite-dimensional complex projective space $\mathbb{C}\mathrm{P}^\infty$ with a CW-structure that has precisely one cell in each even dimension and no cells in odd dimensions. From the same arguments as in (1) it follows that

$$\mathrm{H}_k(\mathbb{C}\mathrm{P}^\infty) \cong \mathrm{H}^{\mathrm{CW}}_k(\mathbb{C}\mathrm{P}^\infty) \cong \mathrm{C}^{\mathrm{CW}}_k(\mathbb{C}\mathrm{P}^\infty) \cong \begin{cases} \mathbb{Z}, & \text{if } k \text{ is even,} \\ 0, & \text{if } k \text{ is odd.} \end{cases}$$

(b) We start out with the following claim:

Claim. Let $j \geq i$. Let $\iota\colon \mathbb{C}\mathrm{P}^i \to \mathbb{C}\mathrm{P}^j$ be the usual embedding that is given by $[x] \mapsto [(x, 0)]$. The induced map $\iota_*\colon \mathrm{H}_k(\mathbb{C}\mathrm{P}^i) \to \mathrm{H}_k(\mathbb{C}\mathrm{P}^j)$ on singular homology is an isomorphism for $k = 0, \dots, 2i$.

Proof. Note that $\iota\colon \mathbb{C}\mathrm{P}^i \to \mathbb{C}\mathrm{P}^j$ is clearly cellular. Thus we obtain from the Singular-Cellular-H_*-Proposition 47.3 (2) the following commutative diagram:

$$\begin{array}{ccc} \mathrm{H}_k(\mathbb{C}\mathrm{P}^i) & \xrightarrow{\iota_*} & \mathrm{H}^{\mathrm{CW}}_k(\mathbb{C}\mathrm{P}^j) \\ {\scriptstyle \Theta_{\mathbb{C}\mathrm{P}^i}}\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\downarrow{\scriptstyle \Theta_{\mathbb{C}\mathrm{P}^j}} \\ \mathrm{H}^{\mathrm{CW}}_k(\mathbb{C}\mathrm{P}^i) & \xrightarrow{\iota_*} & \mathrm{H}^{\mathrm{CW}}_k(\mathbb{C}\mathrm{P}^j) \end{array}.$$

Next note that it follows from inspecting the cellular chain complexes that for any $k \leq 2i$ the bottom horizontal map is an isomorphism. The claim follows immediately from these two observations.[378] ⊞

It follows from the claim, the Exhaustions-H_*-Proposition 46.2 and the Direct Colimit–Stabilization Lemma 25.1 that for any $k \leq 2i$ the inclusion-induced map

[378] Alternatively this claim follows from the Skeleton-H_*-Lemma 47.1 (3).

$\mathrm{H}_k(\mathbb{C}\mathrm{P}^i) \to \mathrm{H}_k(\mathbb{C}\mathrm{P}^\infty)$ is an isomorphism. Together with (1) we thus get the promised result. ∎

47.4. **The cellular boundary maps.** In this section we want to describe an approach to determine the cellular boundary homomorphisms for a given CW-complex. Note that in general an explicit description of a homomorphism $\varphi\colon C \to D$ between two free abelian groups requires the following:

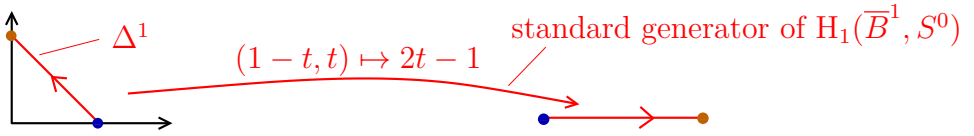
(1) We need bases $\mathcal{C}$ and $\mathcal{D}$ for the free abelian groups.
(2) With respect to these bases we need to express φ by a matrix, i.e. we need to determine the integral matrix $\{a_{dc}\}_{d\in\mathcal{D},c\in\mathcal{C}}$ such that for any $c \in \mathcal{C}$ we have $\varphi(c) = \sum_{d\in\mathcal{D}} a_{dc} \cdot d$.

Now we want to apply this scheme to the cellular boundary maps $d_n\colon \mathrm{C}_n^{\mathrm{CW}}(X) \to \mathrm{C}_{n-1}^{\mathrm{CW}}(X)$. Let us start with Step (1): By the Singular-Cellular-H_*-Proposition 47.3 (1) the key to describing such a basis is to fix a basis for $\mathrm{H}_n(\overline{B}^n, S^{n-1}) \cong \mathbb{Z}$, i.e. we need to fix a preferred generator of $\mathrm{H}_n(\overline{B}^n, S^{n-1})$. We do so in the following definition:

Definition. Let $n \in \mathbb{N}$.
(1) We denote by $[S^{n-1}] \in \widetilde{\mathrm{H}}_{n-1}(S^{n-1})$ the standard generator of $\widetilde{\mathrm{H}}_n(S^{n-1})$, as defined on page 540.
(2) The standard generator $[\overline{B}^n]$ of $\mathrm{H}_n(\overline{B}^n, S^{n-1})$ is defined to be the unique element with $\partial_n([\overline{B}^n]) = [S^{n-1}] \in \widetilde{\mathrm{H}}_{n-1}(S^{n-1})$ where $\partial_n\colon \mathrm{H}_n(\overline{B}^n, S^{n-1}) \to \widetilde{\mathrm{H}}_{n-1}(S^{n-1})$ is the connecting homomorphism of the long exact sequence in reduced homology of the pair $(\overline{B}^n, S^{n-1})$.

Example. Recall that on page 540 we saw that the standard generator of $\widetilde{\mathrm{H}}_0(S^0)$ is given by $[S^0] = [(+1) - (-1)] \in \widetilde{\mathrm{H}}_0(S^0)$. It follows easily from this observation, the explicit description of the connecting homomorphism in the LES-of-Triple Proposition 38.2, and the explicit description of the boundary of a singular 1-simplex on page 459, that the standard generator of $\mathrm{H}_1(\overline{B}^1, S^0)$ is represented by the following cycle in $\mathrm{C}_1(\overline{B}^1, S^0)$:

$$\begin{array}{rcl} \Delta^1 & \to & \overline{B}^1 \\ (1-t,t) & \mapsto & 2t-1. \end{array}$$



Now we can provide a scheme for equipping the cellular chain groups with bases.

Definition.
(1) A charted CW-complex is a pair consisting of a CW-complex X together with a full set $\{\Phi_c\colon \overline{B}^{\dim(c)} \to X^{\dim(c)}\}_{c\in\mathrm{Cell}(X)}$ of characteristic maps of the cells of X.
(2) Let X be a charted CW-complex as in (1). For an n-cell α we write[379] [380]

$$\Xi_\alpha \;:=\; \Phi_{\alpha *}([\overline{B}^n]) \;\in\; \mathrm{H}_n(X^n, X^{n-1}) \;=\; \mathrm{C}_n^{\mathrm{CW}}(X).$$

With the above notation we can now formulate the following essential lemma.

Lemma 47.5. (Cellular Chain Complex–Basis Lemma) Suppose that we are given a charted CW-complex $(X, \{\Phi_c\colon \overline{B}^{\dim(c)} \to X^{\dim(c)}\}_{c\in \mathrm{Cell}(X)})$. For every $n \in \mathbb{N}_0$ the set $\{\Xi_c\}_{c\in \mathrm{Cell}_n(X)}$ is a basis of the free abelian group $\mathrm{C}_n^{\mathrm{CW}}(X) = \mathrm{H}_n(X^n, X^{n-1})$.

Proof. This lemma follows immediately from the Skeleton-H_*-Lemma 47.1 (1). ■

Our goal now is, given a charted CW-complex X, to determine the matrix of the cellular boundary map

$$d_n\colon \underbrace{\mathrm{C}_n^{\mathrm{CW}}(X)}_{\substack{\text{free abelian group with}\\ \text{basis } \{\Xi_c\}_{c\in \mathrm{Cell}_n(X)}}} \to \underbrace{\mathrm{C}_{n-1}^{\mathrm{CW}}(X)}_{\substack{\text{free abelian group with}\\ \text{basis } \{\Xi_c\}_{c\in \mathrm{Cell}_{n-1}(X)}}}$$

As a warm-up we start out with the case $n = 1$. The following lemma shows that it is straightforward to determine the first cellular boundary map:

Lemma 47.6. (Cellular-First Boundary Map Lemma) Let X be a charted CW-complex. For a 1-cell α with characteristic map $\Phi_\alpha\colon \overline{B}^1 \to X^1$ we have

$$\underset{\substack{\uparrow\\ \text{element of } \mathrm{C}_1^{\mathrm{CW}}(X)}}{d_1(\Xi_\alpha)} \;=\; \Xi_{\Phi_\alpha(1)} - \Xi_{\Phi_\alpha(-1)} \;\in\; \mathrm{C}_0^{\mathrm{CW}}(X).$$

In particular, if X has precisely one 0-cell, then the cellular boundary map d_1 is the zero map.

Proof. Recall that the cellular boundary map d_n is defined as follows:

$$\begin{array}{ccc} \mathrm{H}_n(X^n, X^{n-1}) & \xrightarrow{\quad =:d_n \quad} & \mathrm{H}_{n-1}(X^{n-1}, X^{n-2}). \\ & {\scriptstyle\partial_n}\searrow \quad \nearrow{\scriptstyle \text{я}} & \\ & \mathrm{H}_{n-1}(X^{n-1}) & \end{array}$$

In our special case $n = 1$ we see that $X^{n-2} = \varnothing$, thus d_1 equals the connecting homomorphism ∂_1 of the pair (X^1, X^0) as defined in the LES-of-Pair Proposition 38.3. Now we see that

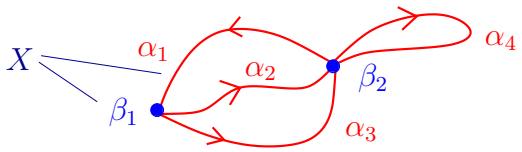
$$\begin{array}{lcll} & \overset{\text{by definition of } \Xi_\alpha \in \mathrm{C}_1^{\mathrm{CW}}(X)}{\downarrow} & \overset{\text{by the above discussion}}{\downarrow} & \overset{\text{see the description of } [\overline{B}^1] \text{ on page 593}}{\downarrow} \\ d_1(\Xi_\alpha) & = & d_1(\Phi_{\alpha*}([\overline{B}^1])) & = & \partial_1(\Phi_{\alpha*}([\overline{B}^1])) & = & \partial_1([(1-t,t)\mapsto \Phi_\alpha(2t-1)]) \\ & & & & & = & \Xi_{\Phi_\alpha(1)} - \Xi_{\Phi_\alpha(-1)}. \\ & & & & & \underset{\text{see the definition of the connecting homomorphism } \partial_1 \text{ in the LES-of-Pair Proposition 38.3 and the discussion of } \partial_1 \text{ on page 459}}{\uparrow} & \end{array}$$

■

Example. We consider the charted 1-dimensional CW-complex X with two 0-cells β_1, β_2 and four 1-cells $\alpha_1, \ldots, \alpha_4$. that is given in the figure below:
By the Cellular-First Boundary Map Lemma 47.6 the cellular chain complex of X is of the form

[379] Note that Φ_α defines a map $(\overline{B}^n, S^{n-1}) \to (X^n, X^{n-1})$ of pairs of topological spaces. Thus we get an induced map $\Phi_{\alpha*}\colon \mathrm{H}_n(\overline{B}^n, S^{n-1}) \to \mathrm{H}_n(X^n, X^{n-1})$, which we can apply to the standard generator $[\overline{B}^n] \in \mathrm{H}_n(\overline{B}^n, S^{n-1})$.

[380] It is quite common to just write α instead of Ξ_α, i.e. it is common to view the cells themselves as elements of the cellular chain complex. With this risque language one can then view $\mathrm{Cell}_n(X)$ as a basis of $\mathrm{C}_n^{\mathrm{CW}}(X)$. For clarity's sake we will not adopt this language.



$$0 \to \underbrace{\begin{matrix}\mathbb{Z}\cdot\Xi_{\alpha_1}\\ \mathbb{Z}\cdot\Xi_{\alpha_2}\\ \mathbb{Z}\cdot\Xi_{\alpha_3}\\ \mathbb{Z}\cdot\Xi_{\alpha_4}\end{matrix}}_{=\mathrm{C}_1^{\mathrm{CW}}(X)} \xrightarrow{\begin{matrix}\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4\end{matrix}\ \begin{matrix}\beta_1\\ \beta_2\end{matrix}\ \begin{pmatrix}1 & -1 & -1 & 0\\ -1 & 1 & 1 & 0\end{pmatrix}} \underbrace{\begin{matrix}\mathbb{Z}\cdot\Xi_{\beta_1}\\ \mathbb{Z}\cdot\Xi_{\beta_2}\end{matrix}}_{=\mathrm{C}_0^{\mathrm{CW}}(X)} \to 0.$$

It follows that

$$\begin{array}{ccccc} \mathrm{H}_n(X) & \cong & \mathrm{H}_n^{\mathrm{CW}}(X) & = & \begin{cases} 0, & \text{if } n \geq 2,\\ \mathbb{Z}^3, & \text{if } n = 1,\\ \mathbb{Z}, & \text{if } n = 0. \end{cases} \\ & \uparrow & & \uparrow & \\ & \text{by the Singular-Cellular-H}_*\text{-} & & \text{by the above calculation} & \\ & \text{Proposition 47.3} & & \text{and elementary linear algebra} & \end{array}$$

□

The next proposition shows that we can calculate the cellular boundary maps d_n in terms of degrees of suitable continuous self-maps of S^{n-1}.

Proposition 47.7. (Cellular Boundary–Degree Proposition) Let X be a charted CW-complex and let $n \in \mathbb{N}$.[381] Given an n-cell α and an $(n-1)$-cell β we consider the following continuous self-map of S^{n-1}:

$$\begin{array}{ccccc} S^{n-1} & \xrightarrow{\Phi_\alpha|_{S^{n-1}}} & X^{n-1} & \xrightarrow{q} & X^{n-1}/X^{n-2} \\ & & & & \cong \uparrow \bigvee_{j\in\mathrm{Cell}_{n-1}(X)} \overline{\Phi}_j \\ & & & & \bigvee_{j\in\mathrm{Cell}_{n-1}(X)} \overline{B}^{n-1}/S^{n-2} \xrightarrow{p_\beta} \overline{B}^{n-1}/S^{n-2} \xrightarrow[\cong]{g} S^{n-1} \end{array}$$

where we adopt the following notation:

- For $c \in \mathrm{Cell}(X)$ we denote by Φ_c the corresponding characteristic map.
- $q\colon X^{n-1} \to X^{n-1}/X^{n-2}$ is the natural projection.
- Let $j \in \mathrm{Cell}_{n-1}(X)$. Since $\Phi_j(S^{n-2}) \subset X^{n-2}$ we see that the characteristic map $\Phi_j\colon \overline{B}^{n-1} \to X^{n-1}$ descends to a continuous map $\overline{\Phi}_j\colon \overline{B}^{n-1}/S^{n-2} \to X^{n-1}/X^{n-2}$.
- $\bigvee_{j\in\mathrm{Cell}_{n-1}(X)} \overline{\Phi}_j\colon \bigvee_{j\in\mathrm{Cell}_{n-1}(X)} \overline{B}^{n-1}/S^{n-2} \to X^{n-1}/X^{n-2}$ is the natural map provided by the Wedge Proposition 17.1. Note that it follows from the Skeleton-Quotient Lemma 33.5 that this map is a homeomorphism.
- $p_\beta\colon \bigvee_{j\in\mathrm{Cell}_n(X)} \overline{B}^{n-1}/S^{n-2} \to \overline{B}^{n-1}/S^{n-2}$ is the natural map that is induced by the identity on the β-copy of $\overline{B}^{n-1}/S^{n-2}$ and the constant map on all other copies of $\overline{B}^{n-1}/S^{n-2}$.
- $g\colon \overline{B}^{n-1}/S^{n-2} \xrightarrow{\cong} S^{n-1}$ denotes the homeomorphism from the Ball Quotient-Sphere Lemma 6.5.

For any n-cell α we have[382]

$$d_n(\Xi_\alpha) \;=\; \sum_{\beta\in \mathrm{Cell}_{n-1}(X)} \deg(\Omega_{\beta,\alpha})\cdot \Xi_\beta \;\in\; \mathrm{C}^{\mathrm{CW}}_{n-1}(X)$$

where[383]

$$\Omega_{\beta,\alpha} \;\coloneqq\; g \circ p_\beta \circ \Big(\bigvee_{j\in \mathrm{Cell}_{n-1}(X)} \overline{\Phi}_j\Big)^{-1} \circ q \circ \Phi_\alpha|_{S^{n-1}} \colon S^{n-1} \;\to\; S^{n-1}.$$

The proof of the Cellular Boundary–Degree Proposition 47.7 requires the following lemma:

Lemma 47.8. (Standard Generator-of-$\overline{B}^n$-Lemma) Let $n \in \mathbb{N}_0$. Furthermore let $ю\colon \mathrm{H}_n(\overline{B}^n, S^{n-1}) \to \widetilde{\mathrm{H}}_n(\overline{B}^n/S^{n-1})$ be the natural map and let $g\colon \overline{B}^n/S^{n-1} \to S^n$ be the homeomorphism from the Ball Quotient-Sphere Lemma 6.5. Then

$$(g_* \circ ю)([\overline{B}^n]) \;=\; [S^n] \quad \in \widetilde{\mathrm{H}}_n(S^n).$$

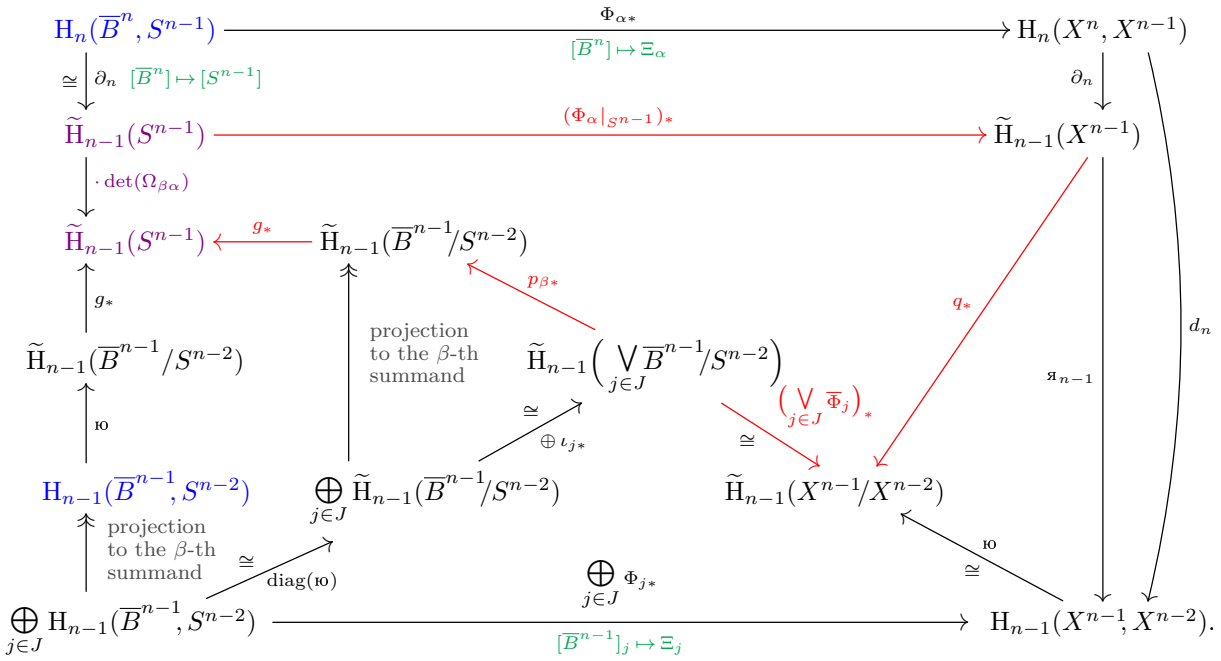
Proof of the Standard Generator-of-$\overline{B}^n$-Lemma 47.8. First note that it follows from the Relative-to-Quotient-H_*-Proposition 41.8 that the map $g_* \circ ю\colon \mathrm{H}_n(\overline{B}^n, S^{n-1}) \to \widetilde{\mathrm{H}}_n(S^n)$ is an isomorphism. Since $[\overline{B}^n] \in \mathrm{H}_n(\overline{B}^n, S^{n-1})$ and $[S^n] \in \widetilde{\mathrm{H}}_n(S^n)$ are generators we thus see that $(g_* \circ ю)([\overline{B}^n]) = \pm[S^n] \in \widetilde{\mathrm{H}}_n(S^n)$. Therefore it remains to determine the precise sign. This is a little tricky and we outsource the proof to [**Fri24**]. For most purposes the precise sign is not really relevant though. For example in the statement of the Cellular Boundary–Degree Proposition 47.7 a different sign in the equality $(g_* \circ ю)([\overline{B}^n]) = \pm[S^n] \in \widetilde{\mathrm{H}}_n(S^n)$ changes at worst all $\deg(\Omega_{\beta,\alpha})$ by the *same* sign, which has no effect later on when we need to determine (co-) kernels. ■

Proof of the Cellular Boundary–Degree Proposition 47.7. We adopt the following notation:

(1) We write $I \coloneqq \mathrm{Cell}_n(X)$ and $J \coloneqq \mathrm{Cell}_{n-1}(X)$.
(2) Given $k \in J$ we denote by $\iota_k\colon \overline{B}^{n-1}/S^{n-2} \to \bigvee_{j\in J} \overline{B}^{n-1}/S^{n-2}$ the natural injection.

[381] It takes a while to see that the statement for $n = 1$ is equivalent to the statement of the Cellular-First Boundary Map Lemma 47.6.
[382] We refer to page 545 for the definition of the degree of a continuous self-map of a sphere.
[383] It follows easily from the CW-Complex-Finiteness Theorem 30.10 that for all but finitely many β the given map $\Omega_{\beta,\alpha}\colon S^{n-1} \to S^{n-1}$ is the constant map, which implies that for all but finitely many β we have $\deg(\alpha_{\beta,\alpha}) = 0$.

For any n-cell α and any $(n-1)$-cell β we consider the following diagram:



We make the following claim:

Claim. The diagram commutes and all the maps decorated with "$\cong$" are indeed isomorphisms.

Proof. We consider the various parts of the diagram.

(1) It follows from the naturality of the connecting homomorphism, see the Reduced LES-of-Pair Proposition 39.6 (2), that the upper rectangle in the diagram commutes.

(2) The red arrows trace out the map $\Omega_{\beta,\alpha}$ of which we take the degree. In other words, the oddly shaped hexagon in the center of the diagram commutes by definition.

(3) We consider the hexagon to the lower left of the diagram. It follows easily from the definition of the diagonal map diag(ю) that the hexagon commutes on each summand, hence it commutes for the direct sum.

(4) (a) Note that the map $p_\beta \circ \iota_j$ equals $\mathrm{id}_{\overline{B}^{n-1}/S^{n-2}}$ if $j = \beta$ and that it is constant otherwise. It follows easily from this observation that the triangle towards the center left, involving the map $p_{\beta *}$ commutes.

(b) It follows from the Wedge-H_*-Proposition 41.10 that the map decorated with "$\cong$" in the triangle is indeed an isomorphism.

(5) In the proof of the Skeleton-H_*-Lemma 47.1 (1) we showed that the large triangle at the bottom commutes and we showed that all the maps involved in the large triangle are indeed isomorphisms.

(6) (a) It follows easily from the definitions that the triangle to the bottom right commutes.

(b) It follows from the Relative-to-Quotient-H_*-Proposition 41.8 that the diagonal map at the very bottom right is an isomorphism. ⊞

Next note that by the definition of the standard generator of $\mathrm{H}_n(\overline{B}^n, S^{n-1})$ on page 593 we have $\partial_n([\overline{B}^n]) = [S^{n-1}] \in \widetilde{\mathrm{H}}_{n-1}(S^{n-1})$ and note that the Standard Generator-of-$\overline{B}^n$-Lemma 47.8 says that $g_*(\mathrm{ю}([\overline{B}^{n-1}])) = [S^{n-1}] \in \widetilde{\mathrm{H}}_{n-1}(S^{n-1})$. The reader who is still awake will notice that the proposition follows immediately from the claim and this observation. ■

47.5. **The homology groups of surfaces.** In this section we will use the Cellular Boundary–Degree Proposition 47.7 to calculate the isomorphism types of the homology groups of compact 2-dimensional topological and smooth manifolds. First let us recall the following standard notation.

Notation. Given $g \in \mathbb{N}_0$ and given $k \in \mathbb{N}$ we write as usual

$$\begin{aligned} \Sigma_g &:= \text{surface of genus } g \\ N_k &:= \text{non-orientable surface of genus } k. \end{aligned}$$

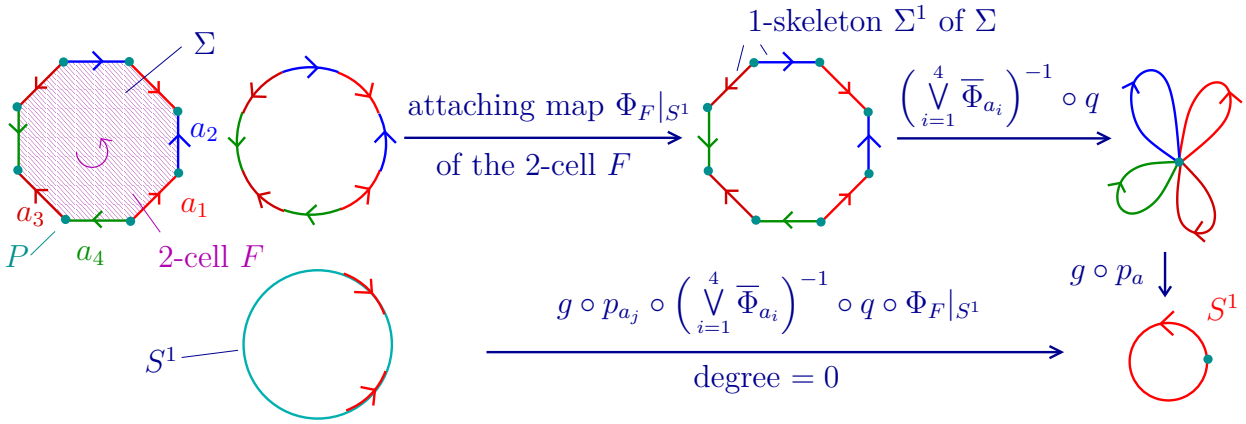
Furthermore given $m \in \mathbb{N}_0$ we write [384] [385]

$$\begin{aligned} \Sigma_{g,m} &:= \text{the surface of genus } g \text{ minus } m \text{ open disks} \\ N_{k,m} &:= \text{the non-orientable surface of genus } k \text{ minus } m \text{ open disks.} \end{aligned}$$

In the following we will first consider some explicit examples before stating a proposition that gives us the calculation of the isomorphism types of the homology groups for all the above surfaces.

Example.

(1) We consider the surface Σ_2 of genus 2. As usual we view Σ_2 as a charted CW-complex with one 0-cell P, four 1-cells a_1, a_2, a_3, a_4 and one 2-cell F.



Note that it follows easily from the definitions, or alternatively from the Degree-via-Local Orientations Proposition 44.7, that for each $j \in \{1, \dots, 4\}$ the degree of the map $\Omega_{F,a_j} = g \circ p_a \circ \left(\bigvee_{i=1}^4 \overline{\Phi}_{a_i}\right)^{-1} \circ q \circ \Phi_F|_{S^1} \colon S^1 \to S^1$ is zero. It follows from the Cellular Boundary–Degree Proposition 47.7 that the cellular boundary map d_2 in the cellular chain complex of Σ_2 is zero. Since Σ_2 has only one 0-cell we know from the Cellular-First Boundary Map Lemma 47.6 that the cellular boundary map d_1 is also zero. Thus

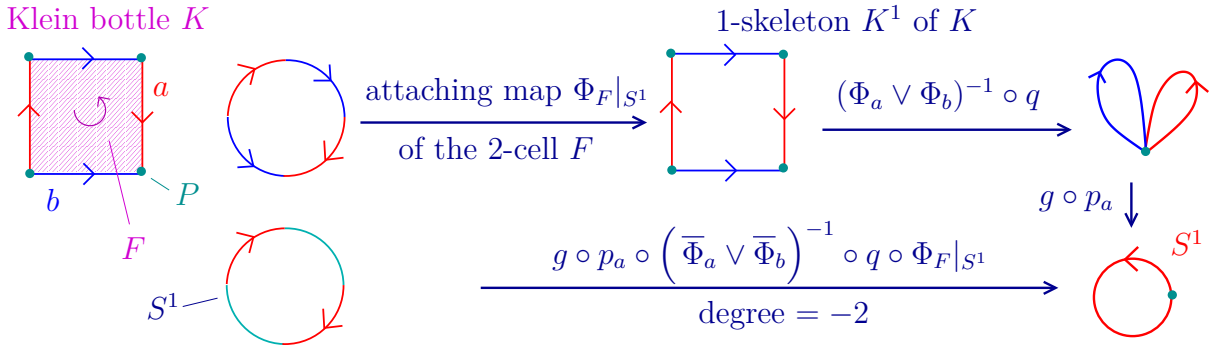
[384] Recall the definition on page 320 of what we mean by a "surface minus m open disks".

[385] Recall that in the Surface Classification Theorem 22.8 we saw that every connected, compact 2-dimensional topological manifold is homeomorphic to either $\Sigma_{g,m}$ for unique $g, m \in \mathbb{N}_0$ or to $N_{k,m}$ for unique $k \in \mathbb{N}$ and $m \in \mathbb{N}_0$.

we see that

$$\mathrm{H}_n(\Sigma_2) \underset{\text{by the Singular-Cellular-H}_*\text{-Proposition 47.3}}{\overset{\cong}{\uparrow}} \mathrm{H}_n^{\mathrm{CW}}(\Sigma_2) \underset{\text{since all the cellular boundary maps are zero}}{\overset{=}{\uparrow}} \mathrm{C}_n^{\mathrm{CW}}(\Sigma_2) \cong \begin{cases} 0, & \text{if } n \geq 3, \\ \mathbb{Z}, & \text{if } n = 2, \\ \mathbb{Z}^4, & \text{if } n = 1, \\ \mathbb{Z}, & \text{if } n = 0. \end{cases}$$

(2) As a reality check we consider once again the Klein bottle $K = N_2$. We view K as a charted CW-complex with one 0-cell P, two 1-cells a, b and one 2-cell F, as shown in the figure below.



It follows from the Cellular Boundary–Degree Proposition 47.7, the Degree-via-Local Orientations Proposition 44.7 and the Cellular-First Boundary Map Lemma 47.6 that the cellular chain complex is given by

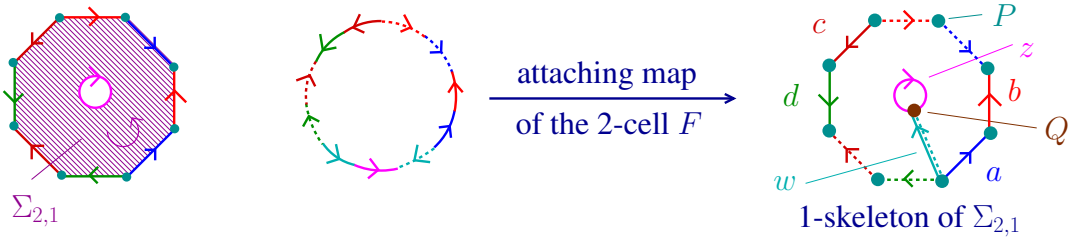
$$0 \to \mathbb{Z} \cdot \Xi_F \xrightarrow{\begin{pmatrix} -2 \\ 0 \end{pmatrix}} \begin{matrix} \mathbb{Z} \cdot \Xi_a \\ \mathbb{Z} \cdot \Xi_b \end{matrix} \xrightarrow{\begin{pmatrix} 0 & 0 \end{pmatrix}} \mathbb{Z} \cdot \Xi_P \to 0.$$

Therefore we see that

$$\mathrm{H}_n(K) \underset{\text{by the Singular-Cellular-H}_*\text{-Proposition 47.3}}{\overset{\cong}{\uparrow}} \mathrm{H}_n^{\mathrm{CW}}(K) \underset{\text{by the above calculation and elementary linear algebra}}{\overset{=}{\uparrow}} \begin{cases} 0, & \text{if } n \geq 2, \\ \mathbb{Z} \oplus \mathbb{Z}_2, & \text{if } n = 1, \\ \mathbb{Z}, & \text{if } n = 0. \end{cases}$$

This is of course the same result that we had already obtained in the Klein Bottle-H$_*$-Lemma 45.6.

(3) We consider $\Sigma_{2,1}$ and we equip it with the charted CW-structure that is illustrated in the figure below.



This CW-structure has two 0-cells P, Q, six 1-cells a, b, c, d, w, z and one 2-cell F. Using the Cellular Boundary–Degree Proposition 47.7 and using the Cellular-First Boundary 47.6 one can now easily show that the cellular chain complex is of the following

form:

$$0 \to \mathbb{Z}\cdot\Xi_F \xrightarrow{\begin{matrix} F \\ a \\ b \\ c \\ d \\ w \\ z \end{matrix}\!\!\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}} \begin{matrix} \mathbb{Z}\cdot\Xi_a \\ \mathbb{Z}\cdot\Xi_b \\ \mathbb{Z}\cdot\Xi_c \\ \mathbb{Z}\cdot\Xi_d \\ \mathbb{Z}\cdot\Xi_w \\ \mathbb{Z}\cdot\Xi_z \end{matrix} \xrightarrow{\begin{matrix} \quad a & b & c & d & w & z \\ P & 0 & 0 & 0 & 0 & -1 & 0 \\ Q & 0 & 0 & 0 & 0 & 1 & 0 \end{matrix}} \begin{matrix} \mathbb{Z}\cdot\Xi_P \\ \mathbb{Z}\cdot\Xi_Q \end{matrix} \to 0.$$

Now we see that

$$\mathrm{H}_n(\Sigma_{2,1}) \cong \mathrm{H}_n^{\mathrm{CW}}(\Sigma_{2,1}) \cong \begin{cases} 0, & \text{if } n \geq 2, \\ \mathbb{Z}^4, & \text{if } n = 1, \\ \mathbb{Z}, & \text{if } n = 0. \end{cases}$$

by the Singular-Cellular-H_*-Proposition 47.3

by the above chain complex and some elementary linear algebra

□

In the following proposition we give the calculation for the isomorphism types of the homology groups of all surfaces $\Sigma_{g,n}$ and $N_{k,m}$:

Proposition 47.9. (Surface-H_*-Proposition) For any $g \in \mathbb{N}_0$ we have

$$\mathrm{H}_n(\Sigma_g) \cong \begin{cases} 0, & \text{if } n \geq 3, \\ \mathbb{Z}, & \text{if } n = 2, \\ \mathbb{Z}^{2g}, & \text{if } n = 1, \\ \mathbb{Z}, & \text{if } n = 0. \end{cases} \quad \text{Furthermore for } m \geq 1 \text{ we have} \quad \mathrm{H}_n(\Sigma_{g,m}) \cong \begin{cases} 0, & \text{if } n \geq 3, \\ 0, & \text{if } n = 2, \\ \mathbb{Z}^{2g+m-1}, & \text{if } n = 1, \\ \mathbb{Z}, & \text{if } n = 0. \end{cases}$$

For any $k \in \mathbb{N}$ we have

$$\mathrm{H}_n(N_k) \cong \begin{cases} 0, & \text{if } n \geq 3, \\ 0, & \text{if } n = 2, \\ \mathbb{Z}^{k-1} \oplus \mathbb{Z}_2, & \text{if } n = 1, \\ \mathbb{Z}, & \text{if } n = 0. \end{cases} \quad \text{Furthermore for } m \geq 1 \text{ we have} \quad \mathrm{H}_n(N_{k,m}) \cong \begin{cases} 0, & \text{if } n \geq 3, \\ 0, & \text{if } n = 2, \\ \mathbb{Z}^{k+m-1}, & \text{if } n = 1, \\ \mathbb{Z}, & \text{if } n = 0. \end{cases}$$

Proof. The calculation of the homology groups Σ_g is a slight generalization of the arguments that we provided on page 598. Furthermore the case N_k is dealt with in the same way as we dealt with the Klein bottle $K = N_2$ on page 599. Finally note that for $m \in \mathbb{N}$ the calculation of the homology groups of $\Sigma_{g,m}$ and $N_{k,m}$ is a slight generalization of the argument preceding the proposition. We leave the details to the reader. ■

47.6. **The homology groups of real projective spaces.** Recall that in the $\mathbb{C}\mathrm{P}^n$-H_*-Proposition 47.4 we calculated the homology groups of the complex projective spaces $\mathbb{C}\mathrm{P}^n$. In this section we will finally compute the homology groups of the real projective spaces $\mathbb{R}\mathrm{P}^n$. This calculation is significantly trickier than the calculation for $\mathbb{C}\mathrm{P}^n$, since this time we will need the theory of local degrees, which we developed in Section 44.2, to determine cellular boundary maps.

Proposition 47.10. ($\mathbb{R}\mathrm{P}^n$-H_*-Proposition)
(1) For any $n \in \mathbb{N}_0$ we have[386]

$$\mathrm{H}_k(\mathbb{R}\mathrm{P}^n) \cong \begin{cases} \mathbb{Z}, & \text{if } k = 0, \\ \mathbb{Z}_2, & \text{if } k \text{ is odd and } k < n, \\ 0, & \text{if } k \text{ is even and } 0 < k, \\ 0, & \text{if } k > n, \\ \mathbb{Z}, & \text{if } k = n \text{ and } n \text{ is odd.} \end{cases}$$

(2) We have

$$\mathrm{H}_k(\mathbb{R}\mathrm{P}^\infty) \cong \begin{cases} \mathbb{Z}, & \text{if } k = 0, \\ \mathbb{Z}_2, & \text{if } k \text{ is odd,} \\ 0, & \text{if } k \text{ is even and } k > 0. \end{cases}$$

Proof.

(1) Let $n \in \mathbb{N}_0$. On page 137 we showed that for any $k \leq n$ the map $\mathbb{R}\mathrm{P}^k \to \mathbb{R}\mathrm{P}^n$ that is given by $[x] \mapsto [(x, 0)]$ is an embedding. For the duration of the proof we use this embedding as an excuse to view $\mathbb{R}\mathrm{P}^k$ as a subspace of $\mathbb{R}\mathrm{P}^n$.

Next recall we recall the CW-structure on $\mathbb{R}\mathrm{P}^n$ which we introduced on page 410: For each $k = 0, \ldots, n$ we define

$$\begin{aligned} \Phi_k \colon \overline{B}^k &\to \mathbb{R}\mathrm{P}^k \\ x &\mapsto [(x, \sqrt{1 - \|x\|^2})]. \end{aligned}$$

We saw in the proof of the Projective Space–Cell Attachment Lemma 29.2 that the map

$$\begin{aligned} \mathbb{R}\mathrm{P}^{k-1} \cup_{\mathbb{R}\mathrm{P}^{k-1} \xleftarrow{\Phi_k|_{S^{k-1}}} S^{k-1}} \overline{B}^k &\xrightarrow{\cong} \mathbb{R}\mathrm{P}^k \\ [x] &\mapsto \begin{cases} [(x, 0)], & \text{if } x \in \mathbb{R}\mathrm{P}^{k-1} \\ \Phi_k(x), & \text{if } x \in \overline{B}^k \end{cases} \end{aligned}$$

is a homeomorphism. Thus we see inductively that we can view $\mathbb{R}\mathrm{P}^n$ as a charted CW-complex with a single cell in the dimensions $0, \ldots, n$ where the characteristic maps are given precisely by the maps $\Phi_0, \ldots, \Phi_n$.

Claim. For each $k = 1, \ldots, n$ the cellular boundary map

$$d_k \colon \mathbb{Z} \cong \mathrm{C}_k^{\mathrm{CW}}(\mathbb{R}\mathrm{P}^n) \to \mathrm{C}_{k-1}^{\mathrm{CW}}(\mathbb{R}\mathrm{P}^n) \cong \mathbb{Z}$$

is given by multiplication by $\pm(1 + (-1)^k)$.

Proof. Let $k \in \{1, \ldots, n\}$. We denote by $q \colon \mathbb{R}\mathrm{P}^{k-1} \to \mathbb{R}\mathrm{P}^{k-1}/\mathbb{R}\mathrm{P}^{k-2}$ the projection and we denote by $g \colon \overline{B}^{k-1}/S^{k-2} \to S^{k-1}$ the homeomorphism from the Ball Quotient-Sphere Lemma 6.5. By the Cellular Boundary–Degree Proposition 47.7 it suffices to show that the degree of the map

$$S^{k-1} \xrightarrow{\Phi_k|_{S^{k-1}}} \mathbb{R}\mathrm{P}^{k-1} \xrightarrow{q} \mathbb{R}\mathrm{P}^{k-1}/\mathbb{R}\mathrm{P}^{k-2} \xrightarrow{(\overline{\Phi}_{k-1})^{-1}} \overline{B}^{k-1}/S^{k-2} \xrightarrow{g} S^{k-1}$$

$$=: \Omega$$

[386] The calculation looks more complicated than it is, in fact the most efficient way to remember the calculation is to remember the description of the cellular chain complex given in the proof.

equals $\pm(1+(-1)^k)$. We will do so using the Degree-via-Local Orientations Proposition 44.7. To apply this proposition we recall a few definitions and we make a few preparations:

(a) If $\Omega\colon S^{k-1}\to S^{k-1}$ is a local diffeomorphism at a point x, then as on page 552 we write
$$\deg(\Omega,x) \coloneqq \begin{cases} +1, & \text{if } \Omega \text{ is orientation-preserving at } x \\ -1, & \text{if } \Omega \text{ is orientation-reversing at } x. \end{cases}$$

(b) If $f\colon S^{k-1}\to S^{k-1}$ is a diffeomorphism, then we write
$$\deg(f) \coloneqq \begin{cases} +1, & \text{if } f \text{ is orientation-preserving} \\ -1, & \text{if } f \text{ is orientation-reversing.} \end{cases}$$

(c) If $\Omega\colon S^{k-1}\to S^{k-1}$ is a local diffeomorphism at a point x and if $f\colon S^{k-1}\to S^{k-1}$ is a diffeomorphism, then it follows immediately from $\mathrm{D}(\Omega\circ f)_{f^{-1}(x)} = \mathrm{D}\Omega_x \circ \mathrm{D}\Omega_{f^{-1}(x)}$ that $\deg(\Omega\circ f, f^{-1}(x)) = \deg(\Omega,x)\cdot\deg(f)$.

Now we turn to our specific situation:

(d) We pick the point $y=(0,\dots,0,-1)\in S^{k-1}$. The preimage of y under the map g is $[(0)]$ and the preimages of y under the map Ω are $x_+=(0,\dots,0,1)$ and $x_-=(0,\dots,0,-1)$. One can easily verify that the restrictions of Ω to the open hemispheres $S^n_{>0}$ and $S^n_{<0}$ are smooth embeddings. Since $x_+\in S^n_{>0}$ and $x_-\in S^n_{<0}$ we see that Ω is a local diffeomorphism around x_+ and x_-.[387]

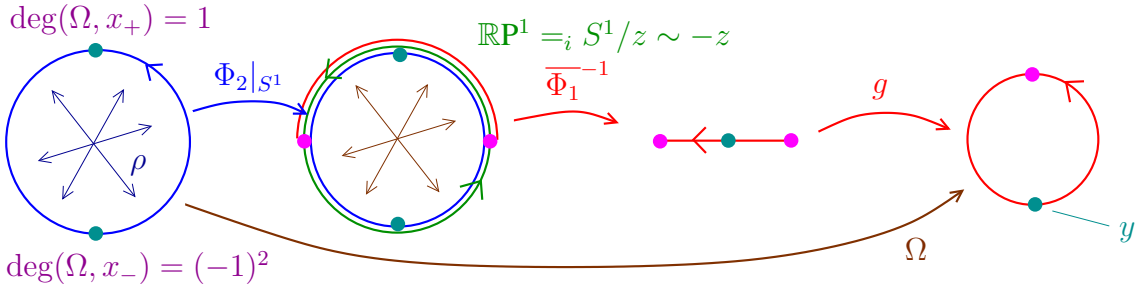
(e) Let $\rho\colon S^{k-1}\to S^{k-1}$ be the reflection in the origin, i.e. ρ is given by $z\mapsto -z$.

(f) It follows from the Sphere-Degree-Matrix Lemma 43.4 that $\deg(\rho)=\det(-\mathrm{id}_k)=(-1)^k$.

(g) Note that $(\Phi_k|_{S^{k-1}})\circ\rho = \Phi_k|_{S^{k-1}}\colon S^{k-1}\to \mathbb{R}\mathrm{P}^{k-1}$. This implies that $\Omega\circ\rho=\Omega$.

We now see that

$$\begin{array}{rcl} \deg(\Omega) & \overset{\text{by the Degree-via-Local Orientations Proposition 44.7 and by (d)}}{=} & \deg(\Omega,x_+)+\deg(\Omega,x_-) \\ & = & \deg(\Omega,x_+)+\deg(\Omega,x_+)\cdot\deg(\rho) \\ & \underset{\text{follows from (c), since } \rho(x_+)=x_-}{=} & \end{array} \quad \begin{array}{rcl} & \overset{\text{since by (g) we know that } \Omega\circ\rho=\Omega}{=} & \deg(\Omega,x_+)+\deg(\Omega\circ\rho,x_-) \\ & = & \underbrace{\deg(\Omega,x_+)}_{=\pm 1}\cdot(1+\underbrace{\deg(\rho)}_{=(-1)^k \text{ by (f)}}). \end{array} \qquad \boxplus$$



[387] If one struggles through all the orientation conventions and all the maps, then one can show that $\deg(\Omega,x_+)=1$ and that $\deg(\Omega,x_-)=(-1)^k$. We will not go down that route, instead we will use a more indirect, but hopefully more robust and more convincing argument.

It follows from the claim that the cellular chain complex of $\mathbb{R}\mathrm{P}^n$ is of the form

$$0 \to \underset{\substack{\uparrow\\ \text{degree } n}}{\mathbb{Z}} \xrightarrow{\pm(1+(-1)^n)} \mathbb{Z} \to \ldots \xrightarrow{\cdot(\pm 2)} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cdot(\pm 2)} \mathbb{Z} \xrightarrow{0} \underset{\substack{\uparrow\\ \text{degree } 0}}{\mathbb{Z}} \to 0.$$

It is now straightforward to determine the cellular homology groups which, by the Singular-Cellular-H_*-Proposition 47.3, are isomorphic to the singular homology groups of $\mathbb{R}\mathrm{P}^n$.

(2) On page 410 we equipped $\mathbb{R}\mathrm{P}^\infty = \operatorname{colim} \mathbb{R}\mathrm{P}^n$ with a CW-structure with precisely one cell in each dimension. Basically the same calculation as in (1) gives the desired result. Alternatively, as in the proof of the $\mathbb{C}\mathrm{P}^n$-H_*-Proposition 47.4 (2) one can use the calculation of (1) and the Exhaustions-H_*-Proposition 46.2 to obtain the desired result. We leave it to the reader to work out the details for this second approach. ■

Remark. We have now seen that the homology groups of CW-complexes can be computed fairly easily using cellular chain complexes. One might ask, why did we introduce singular chain complexes, if it is so much easier to do calculations with cellular chain complexes? There are two related answers:

(1) We also want to study topological spaces, e.g. topological manifolds, that are not (known to be) CW-complexes.

(2) It is much harder to prove some statements using cellular chain complexes, for example it is very difficult to show (without making recourse to singular homology) that homeomorphic (let alone homotopy equivalent) CW-complexes have isomorphic homology groups.

A general slogan in topology (and many other fields of mathematics) is that for calculations it is usually best to work with "small objects" (in our case the cellular chain complex), whereas for proving general properties, it is often best to work with "big objects" (in our case the singular chain complex). □

CHAPTER 48

The Euler characteristic

In this chapter we introduce the Euler characteristic $\chi(X) \in \mathbb{N}_0$ of a finite CW-complex X. This invariant has two interesting features:

(1) The Euler characteristic is easy to define and it is fun to calculate Euler characteristics.
(2) Using singular homology we will see that the Euler characteristic depends only on the homotopy type of X.

In this and the next chapter we will see that the Euler characteristic can be used to prove many statements and answer many classical problems (e.g. why does a football have 12 pentagons?).

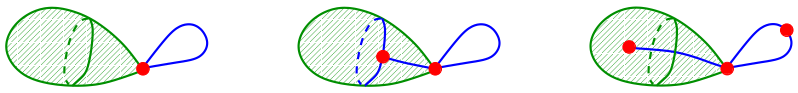
48.1. **The Euler characteristic of a CW-complex.** Let us start out with the straightforward definition of the Euler characteristic of a finite CW-complex.

Definition. Given a finite CW-complex X we define its Euler characteristic $\chi(X)$ as follows:

$$\chi(X) := \sum_{n \in \mathbb{N}_0} (-1)^n \cdot \text{number of } n\text{-cells}.$$

Examples.

(1) If G is a finite topological graph, then, according to the Graph–CW-Complex Lemma 30.1 we can view G as a 1-dimensional CW-complex where the number of 0-cells equals the number of vertices and the number of 1-cells equals the number of edges of G. Thus it follows immediately from the definitions that in this case the above Euler characteristic agrees with the definition of the Euler characteristic of a finite topological graph that we gave on page 155.
(2) In the figure below we show the topological space $S^2 \vee S^1$ with three different CW-structures.



number of 0-cells	1	2	3
number of 1-cells	1	3	3
number of 2-cells	1	2	1
Euler characteristic	1	1	1

We notice that even though the numbers of cells in the various dimensions differ, the Euler characteristic is always the same. □

The above example suggests that the Euler characteristic depends perhaps only on the homeomorphism type of the CW-complex. Our next result shows that this is indeed the case.

To formulate our next result we need the following basis definition.

Definition. If G is a finitely generated abelian group, then it follows from the Abelian Group-Classification Theorem 18.11 that G is isomorphic to $\mathbb{Z}^r \oplus T$, where T is finite. We refer to r as the rank of G.

Proposition 48.1. (Homology-χ-Proposition) Given any finite CW-complex X we have

$$\chi(X) \;=\; \sum_{n\in\mathbb{N}_0} (-1)^n \cdot \operatorname{rank}(\mathrm{H}_n(X)).$$

Proof. We will provide the proof of Homology-χ-Proposition 48.1 in the next section. ■

Before we do anything else let us record the following pretty obvious corollary.

Corollary 48.2. (Homotopy Type-χ-Corollary) Let X and Y be finite CW-complexes. If X and Y are homotopy equivalent, then $\chi(X) = \chi(Y)$.[388]

Proof of the Homotopy Type-χ-Corollary 48.2. The corollary is an immediate consequence of the Homology-χ-Proposition 48.1 together with the Homotopy Equivalence-H_*-Corollary 36.7 (2). ■

Now that we know that the Euler characteristic is a homotopy invariant (in particular a homeomorphism invariant) it is worth determining it for some of our favorite topological spaces.

Examples.

(1) Let $n \in \mathbb{N}_0$. On page 407 we showed that the n-dimensional sphere S^n has a CW-structure of S^n with one 0-cell and one extra cell in dimension n. It follows that[389]

$$\chi(S^n) \;=\; 1+(-1)^n \;=\; \begin{cases} 0, & \text{if } n \text{ is odd,} \\ 2, & \text{if } n \text{ is even.} \end{cases}$$

(2) Given $g \in \mathbb{N}_0$ we consider the surface Σ_g of genus g and given $k \in \mathbb{N}$ we consider the non-orientable surface N_k of genus k. On page 605 we equipped Σ_g and N_k with CW-structures that are also shown (for $g = 2$ and $k = 3$) in the figure below. One can now easily calculate that[390]

$$\chi(\Sigma_g) \;=\; 2-2g \qquad \text{and} \qquad \chi(N_k) \;=\; 2-k.$$

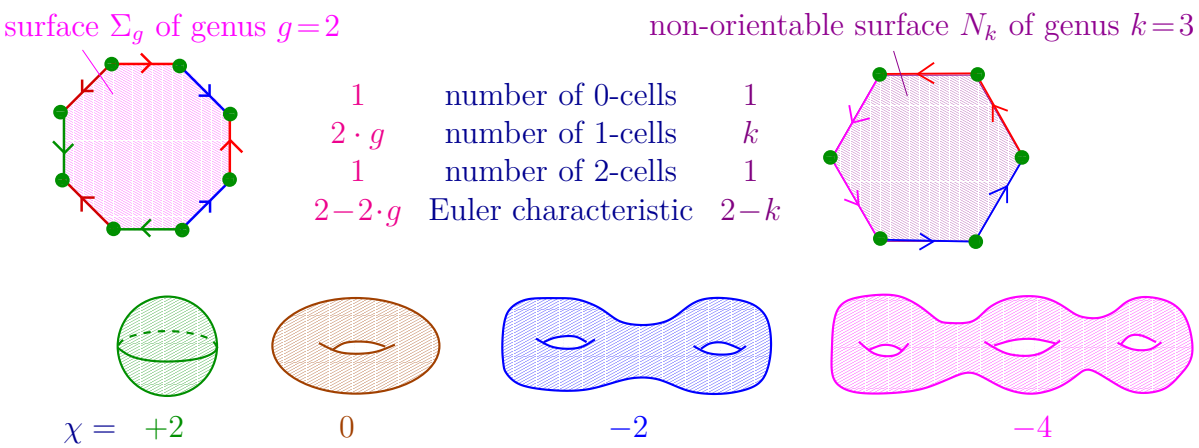
In the following figure we show the surfaces with increasing genus. The surface of genus 0, i.e. the sphere S^2, has Euler characteristic +2. When we increase the genus by one, the Euler characteristic drops by two.

(3) Let $n \in \mathbb{N}_0$. For $n = 0$ we have $\overline{B}^0 = \{0\}$, thus $\chi(\overline{B}^0) = 1$. For $n \geq 2$ a CW-structure for the ball $\overline{B}^n$ is given by the filtration $\varnothing \subset \{N\} \subset S^{n-1} \subset \overline{B}^n$, thus we have a

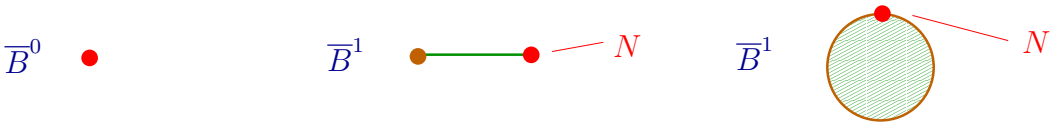
[388] This corollary implies in particular, that the Euler characteristic of a CW-complex does not depend on the choice of CW-structure.

[389] One can also calculate $\chi(S^n)$ using the Sphere-H_*-Proposition 41.2 and Homology-χ-Proposition 48.1.

[390] One can also calculate the Euler characteristics using the Surface-H_*-Proposition 47.9 together with the Homology-χ-Proposition 48.1.



CW-structure with one 0-cell $\{N\}$, one extra cell in dimension $n-1$ and one extra cell of dimension n. Thus again we obtain that $\chi(\overline{B}^n) = 1 + (-1)^{n-1} + (-1)^n = 1$.[391]



(4) On page 410 we showed that $\mathbb{R}\mathrm{P}^n$ admits a CW-structure with precisely one cell of dimensions $0, 1, \dots, n$ and no other cell. It follows that[392]

$$\chi(\mathbb{R}\mathrm{P}^n) \;=\; \sum_{i=0}^{n} (-1)^n \;=\; \begin{cases} 0, & \text{if } n \text{ is odd,} \\ 1, & \text{if } n \text{ is even.} \end{cases}$$

(5) On page 410 we showed that $\mathbb{C}\mathrm{P}^n$ admits a CW-structure with precisely one cell of dimensions $0, 2, \dots, 2n$ and no other cell. It follows that[393] $\chi(\mathbb{C}\mathrm{P}^n) = n+1$. □

48.2. **Proof of the Homology-χ-Proposition 48.1.** The following purely algebraic lemma is the key ingredient in the proof of the Homology-χ-Proposition 48.1.

Lemma 48.3. (Chain Complex–H_*–χ-Lemma) Given any chain complex

$$0 \;\to\; C_k \xrightarrow{\partial_k} C_{k-1} \xrightarrow{\partial_{k-1}} \dots\; C_1 \xrightarrow{\partial_1} C_0 \;\to\; 0$$

of abelian groups of finite rank the following equality holds:

$$\sum_{n=0}^{k} (-1)^n \cdot \operatorname{rank}(C_n) \;=\; \sum_{n=0}^{k} (-1)^n \cdot \operatorname{rank}(\mathrm{H}_n(C_*)).$$

Proof. For the above chain complex (C_*, ∂_*) we write for each $n \in \mathbb{N}_0$[394]

$$Z_n \;:=\; \ker(\partial_n), \quad B_n \;:=\; \operatorname{im}(\partial_{n+1}) \quad \text{and} \quad H_n \;=\; Z_n/B_n.$$

[391] Here is another approach to calculating the Euler characteristic: Note that $\overline{B}^n$ is contractible, i.e. $\overline{B}^n$ is homotopy equivalent to a singleton $\{*\}$. Thus it follows from the Homotopy Type-χ-Corollary 48.2 that $\chi(\overline{B}^n) = \chi(\{*\}) = 1$.

[392] We can also calculate $\chi(\mathbb{R}\mathrm{P}^n)$ using the $\mathbb{R}\mathrm{P}^n$-H_*-Proposition 47.10 and the Homology-χ-Proposition 48.1.

[393] We can also calculate $\chi(\mathbb{C}\mathrm{P}^n)$ using the $\mathbb{C}\mathrm{P}^n$-H_*-Proposition 47.4 and the Homology-χ-Proposition 48.1.

[394] It is a common notation to call these groups Z_n and B_n, the names go back to the German words "Zykel" and "Bild".

Note that these groups form the following two types of short exact sequences

$$\begin{array}{lccccccccc} \text{(a)} & 0 & \hookrightarrow & Z_n & \to & C_n & \xrightarrow{\partial_n} & B_{n-1} & \to & 0, \\ \text{(b)} & 0 & \hookrightarrow & B_n & \to & Z_n & \to & H_n & \to & 0. \end{array}$$

We see that

$$\begin{aligned} \sum_{n=0}^{k} (-1)^n \cdot \operatorname{rank} C_n &\overset{\text{by (a) we have } \operatorname{rank} C_n = \operatorname{rank} Z_n + \operatorname{rank} B_{n-1}}{\downarrow} \sum_{n=0}^{k} (-1)^n \cdot (\operatorname{rank} Z_n + \operatorname{rank} B_{n-1}) \\ &= \underset{\uparrow}{\sum_{n=0}^{k}} (-1)^n \cdot (\operatorname{rank} B_n + \operatorname{rank} H_n + \operatorname{rank} B_{n-1}) \\ &\overset{\text{by (b) we have } \operatorname{rank} Z_n = \operatorname{rank} B_n + \operatorname{rank} H_n}{=} \sum_{n=0}^{k} (-1)^n \cdot \operatorname{rank} H_n + \sum_{n=0}^{k} (-1)^n \cdot \operatorname{rank} B_n + \sum_{n=0}^{k} (-1)^n \cdot \operatorname{rank} B_{n-1} \\ &= \underset{\uparrow}{\sum_{n=0}^{k}} (-1)^n \cdot \operatorname{rank} H_n + \sum_{n=0}^{k-1} (-1)^n \cdot \operatorname{rank} B_n - \sum_{m=0}^{k-1} (-1)^m \cdot \operatorname{rank} B_m \\ &\overset{\text{we do the substitution } m = n-1 \text{ and we use that } B_k = 0 \text{ and } B_{-1} = 0}{=} \sum_{n=0}^{k} (-1)^n \cdot \operatorname{rank} H_n. \end{aligned}$$

■

Now we can easily provide the proof of the Homology-χ-Proposition 48.1.

Proof of the Homology-χ-Proposition 48.1. Let X be a finite k-dimensional CW-complex. We perform the following calculation:

$$\begin{aligned} \chi(X) &\overset{\text{by definition of } \chi(X)}{\downarrow} \sum_{n=0}^{k} (-1)^n \cdot \text{number of } n\text{-cells} &&\overset{\text{by the Skeleton-H}_*\text{-Lemma 47.1 (1)}}{\downarrow} \sum_{n=0}^{k} (-1)^n \cdot \operatorname{rank}(\mathrm{C}_n^{\mathrm{CW}}(X)) \\ &= \underset{\uparrow}{\sum_{n=0}^{k}} (-1)^n \cdot \operatorname{rank}(\mathrm{H}_n^{\mathrm{CW}}(X)) &&= \underset{\uparrow}{\sum_{n=0}^{k}} (-1)^n \cdot \operatorname{rank}(\mathrm{H}_n(X)). \end{aligned}$$

by the Chain Complex–H$_*$–χ-Lemma 48.3 — by the Singular-Cellular-H$_*$-Proposition 47.3

■

Using the Chain Complex–H$_*$–χ-Lemma 48.3 we can also easily prove the following lemma which be will useful on several occasions.

Lemma 48.4. (Exact Sequence-χ-Lemma) If

$$0 \to A_k \to A_{k-1} \to \ldots \to A_1 \to A_0 \to 0$$

is an exact sequence of finite-rank abelian groups, then

$$\sum_{n=0}^{k} (-1)^n \cdot \operatorname{rank}(A_n) = 0.$$

Proof. We have

$$\sum_{n=0}^{k} (-1)^n \cdot \operatorname{rank}(A_n) \underset{\uparrow}{=} \sum_{n=0}^{k} (-1)^n \cdot \operatorname{rank}(\mathrm{H}_n(A_*)) \underset{\uparrow}{=} \sum_{n=0}^{k} (-1)^n \cdot 0 = 0.$$

by the Chain Complex–H$_*$–χ-Lemma 48.3, which we can apply since an exact sequence is also a chain complex — since the sequence is exact

■

Finally note that the Homology-χ-Proposition 48.1 allows us to define the notion of a Euler characteristic in a somewhat more general setup:

Definition. Let X be a topological space and let $A \subset X$ be a subspace.

(1) We define the n-th Betti[401] number of (X, A) to be

$$b_n(X, A) \;:=\; \operatorname{rank}(\mathrm{H}_n(X, A)) \quad \in\; \mathbb{N}_0 \cup \{\infty\}$$

(2) If $\sum\limits_{n\in\mathbb{N}_0} b_n(X, A)$ is finite, then we define the Euler characteristic of (X, A) to be

$$\chi(X, A) \;:=\; \sum_{n\in\mathbb{N}_0} (-1)^n \cdot b_n(X, A).$$

Evidently we write $b_n(X) := b_n(X, \varnothing)$ and $\chi(X) := \chi(X, \varnothing)$.

Example. It now makes sense to say that $\chi(\mathbb{R}^n) = 1$, even though $\mathbb{R}^n$ is non-compact. □

Remark.

(1) It follows from the Homology-χ-Proposition 48.1 that for finite CW-complexes this new definition of the Euler characteristic agrees with the earlier definition from page 604.

(2) In the Smooth Manifold-CW-Theorem 30.2 we pointed out that compact smooth manifolds admit CW-structures. But it is actually unknown whether all compact *topological* manifolds admit CW-structures. Nonetheless we will see in the Topological Manifolds-Invariants Proposition 60.10 that the above definition of a Euler characteristic makes sense for compact topological manifolds. □

We conclude this section with the following elementary lemma:

Lemma 48.5. (Relative-χ-Lemma) Let (X, A) be a pair of topological spaces. If $\chi(X), \chi(A)$ and $\chi(X, A)$ are defined, then

$$\chi(X) \;=\; \chi(X, A) + \chi(A).$$

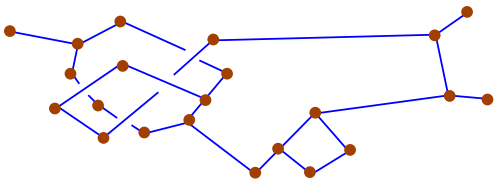
Proof. This lemma follows immediately from the LES-Proposition 37.4 and the Exact Sequence-χ-Lemma 48.4. ■

48.3. **Homology groups of topological graphs.** Using the Homology-χ-Proposition 48.1 we can now easily prove the following lemma:

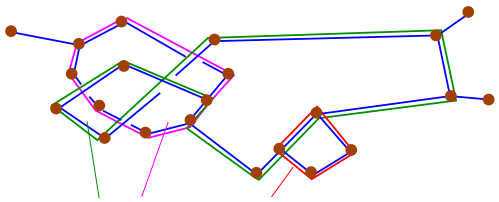
Lemma 48.6. (Graph-H_*-Lemma) Let G be a connected non-empty topological graph with v vertices and e edges. Then

$$\mathrm{H}_0(G) \;\cong\; \mathbb{Z}, \quad \mathrm{H}_1(G) \;\cong\; \mathbb{Z}^{e-v+1} \quad \text{and} \quad \mathrm{H}_i(G) \;=\; 0 \text{ for } i \geq 2.$$

Example. In the figure below we see a connected topological graph G with 22 vertices and 24 edges.



topological graph G with 22 vertices and 24 edges



basis for $\mathrm{H}_1(G) \cong \mathbb{Z}^3$

[401] These numbers are named after the Italian mathematician Enrico Betti (1823-1892) who studied in [**Bet70**] an early version of homology groups.

It follows from the Graph-H_*-Lemma 48.6 that its first homology group is isomorphic to $\mathbb{Z}^3$. Using the observation that an epimorphism of $\mathbb{Z}^3$ is an isomorphism one can show fairly easily that the three cycles which are shown in the figure below represent a basis for $H_1(G) \cong \mathbb{Z}^3$. □

Proof. Let G be a connected topological graph with v vertices and e edges. Recall that by the Graph–CW-Complex Lemma 30.1 we can view G as a 1-dimensional CW-complex with v 0-cells and e 1-cells. From the H_0-Proposition 35.11 together with the CW-Complex–Local Properties Proposition 31.7 (2) and the Locally Path-Connected Lemma 2.24 it follows immediately that $H_0(G) \cong \mathbb{Z}$. Furthermore from considering the cellular chain complex it follows that $H_i(G) = 0$ for $i \geq 2$ and that $H_1(G)$ is torsion-free. We now see that

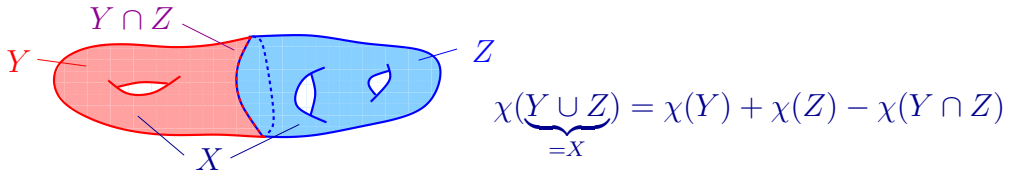
$$1 - \operatorname{rank}(H_1(G)) \underset{\text{by the above}}{\uparrow} = \sum_i (-1)^i \cdot \operatorname{rank}(H_i(X)) \underset{\text{by the Homology-}\chi\text{-Proposition 48.1}}{\uparrow} = \chi(X) = \#0\text{-cells} - \#1\text{-cells} = v - e.$$

Since $H_1(G)$ is torsion-free we obtain that $H_1(G) \cong \mathbb{Z}^{e-v+1}$. ■

48.4. **Properties of the Euler characteristic.** The Euler characteristic is an invariant which "sees less" than the homology groups. On the other hand we will see in this section that it behaves better than the homology groups under several natural operations. This makes the Euler characteristic an eminently computable invariant.

Proposition 48.7. (Union-χ-Proposition) Let $X = Y \cup Z$ be a decomposition of a finite CW-complex X into two subcomplexes Y and Z. The following equality holds [402]

$$\chi(X) = \chi(Y) + \chi(Z) - \chi(Y \cap Z).$$



Proof. Given $n \in \mathbb{N}_0$ and given a subcomplex W of X we denote by $n(W)$ the number of n-cells that are contained in W. In our setting we have

$$n(X) \underset{\text{since } X = Y \cup Z \text{ and since } Y \text{ and } Z \text{ are subcomplexes we obtain from the Subcomplex–Characterization Proposition 30.7 together with the Subcomplex–Closed Lemma 30.6 that every } n\text{-cell of } X \text{ lies in } Y \text{ or } Z, \text{ the } n\text{-cells that lie in the intersection } Y \cap Z \text{ get counted twice in } n(Y) + n(Z), \text{ so we have to subtract } n(Y \cap Z) \text{ to obtain } n(X)}{\uparrow} = n(Y) + n(Z) - n(Y \cap Z).$$

By performing the alternating sum over all $n \in \mathbb{N}_0$ we obtain immediately that

$$\chi(X) = \chi(Y) + \chi(Z) - \chi(Y \cap Z).$$

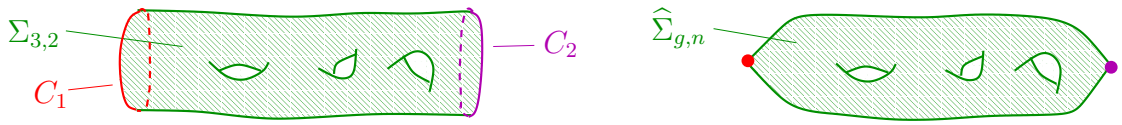
■

[402] It follows from the Subcomplex–Intersection-Union Lemma 30.8 that $Y \cap Z$ is also a CW-complex.

Proposition 48.8. (Quotient-χ-Proposition) Let X be a finite CW-complex and let A be a subcomplex of X. We equip the quotient X/A with the CW-structure given by the CW-Quotient Lemma 33.3. Then

$$\chi(X/A) \;=\; \chi(X) - \chi(A) + 1.$$

Example. Let $\Sigma_{g,n}$ be the surface of genus g minus n open disks. We equip $\Sigma_{g,n}$ with a CW-structure. Let $C_1, \ldots, C_n$ be the boundary components of $\Sigma_{g,n}$. We denote by $\widehat{\Sigma}_{g,n}$ the topological space that is obtained from $\Sigma_{g,n}$ by collapsing each boundary component to a point.



By the Surface Recovery Lemma 22.6 we know that $\widehat{\Sigma}_{g,n}$ is homeomorphic to the surface of genus g. We calculate that

$$2-2g \;\underset{\uparrow}{=}\; \chi(\widehat{\Sigma}_{g,n}) \;=\; \chi(\overbrace{(\ldots(\Sigma_{g,n}/C_1)/\ldots/C_k)}^{=\widehat{\Sigma}_{g,n}}) \;\underset{\uparrow}{=}\; \chi(\Sigma_{g,n}) - \sum_{i=1}^{n} \underbrace{\chi(C_i)}_{=0} + n \;=\; \chi(\Sigma_{g,n}) + n.$$

see page 605

by the Quotient-χ-Proposition 48.8, applied n times

Thus it follows that $\chi(\Sigma_{g,n}) = 2 - 2g - n$. □

Proof. Recall that in the CW-Quotient Lemma 33.3 we equipped X/A with a CW-structure such that for each $k \in \mathbb{N}_0$ we have the following equalities:

$$\#k\text{-cells of } X/A \;=\; \begin{cases} \#k\text{-cells of } X \text{ not in } A, & \text{if } k > 0, \\ \#k\text{-cells of } X \text{ not in } A \;+\; 1, & \text{if } k = 0. \end{cases}$$

Evidently these equalities imply the equality $\chi(X/A) = \chi(X) - \chi(A) + 1$. ■

Proposition 48.9. (Product-χ-Proposition) Let X and Y be two finite CW-complexes. If we equip $X \times Y$ with the CW-structure from the CW-Category–Product Proposition 33.7 and the CW-Complex Product Proposition 33.9, then

$$\chi(X \times Y) \;=\; \chi(X) \cdot \chi(Y).$$

Example. Let X be a finite CW-complex.

(1) It follows from the Product-χ-Proposition 48.9 that $\chi(S^1 \times X) = \chi(S^1) \cdot \chi(X) = 0$.
(2) We consider the suspension $\Sigma(X) = ((X \times [-1,1])/(X \times \{-1\}))/(X \times \{1\})$. We see that

$$\chi(\Sigma(X)) \;\underset{\downarrow}{=}\; \chi((X \times [-1,1]) - \chi(X \times \{-1\}) + 1 - \chi(X \times \{1\}) + 1$$
$$\;\underset{\uparrow}{=}\; 2 + \chi(X) - \chi(X) - \chi(X) \;=\; 2 - \chi(X).$$

follows from applying the Quotient-χ-Proposition 48.8 twice

follows from the Product-χ-Proposition 48.9 and the observation that $\chi([-1,1]) = 1$ □

Proof. Let X and Y be two finite CW-complexes. Given $k \in \mathbb{N}_0$ we denote by c_k the number of k-cells of X and we denote by d_k the number of k-cells of Y. We equip $X \times Y$

with the product CW-structure from the CW-Category–Product Proposition 33.7 and the CW-Complex Product Proposition 33.9. By construction we have

$$\#n\text{-cells of } X\times Y \;=\; \sum_{k+l=n} \#k\text{-cells of } X\cdot \#l\text{-cells of } Y \;=\; \sum_{k+l=n} c_k\cdot d_l.$$

It follows that

$$\chi(X)\cdot\chi(Y) \;=\; \Big(\sum_{k\in\mathbb{N}_0}(-1)^k\cdot c_k\Big)\cdot\Big(\sum_{l\in\mathbb{N}_0}(-1)^l\cdot d_l\Big) \overset{\text{by multiplying out}}{=} \sum_{n\in\mathbb{N}_0}\sum_{k+l=n}(-1)^{k+l}\cdot c_k\cdot d_l$$
$$= \sum_{n\in\mathbb{N}_0}(-1)^n\cdot\sum_{k+l=n} c_k\cdot d_l \;=\; \chi(X\times Y).$$
(the last equality: by the above discussion)

We have thus proved the desired equality $\chi(X)\cdot\chi(Y)=\chi(X\times Y)$. ∎

Next let $p\colon \widetilde{X}\to X$ be a covering of path-connected non-empty topological spaces. We know that the fundamental groups and the higher homotopy groups behave well under coverings:

(1) In the Covering-π_1-Monomorphism Proposition 14.11 we showed that the induced map $p_*\colon \pi_1(\widetilde{X})\to\pi_1(X)$ is a monomorphism and we saw in the Covering-Degree Proposition 14.13 that $[\pi_1(X) : p_*(\pi_1(\widetilde{X}))] = \deg(p)$.
(2) For $n\geq 2$ we obtain from the Covering-π_n-Isomorphism Proposition 28.10 that the induced map $p_*\colon \pi_n(\widetilde{X})\to\pi_n(X)$ is an isomorphism.

There are no analogous statements for homology groups. For example we saw on page 216 that there exists a 2-fold covering $p\colon S^1\times S^1\to K$ from the torus to the Klein bottle K. In this case we have by the Torus-H_*-Proposition 45.5 and the Klein Bottle-H_*-Lemma 45.6 that

$$\begin{array}{lcl} \mathrm{H}_0(S^1\times S^1) &\cong& \mathbb{Z},\\ \mathrm{H}_1(S^1\times S^1) &\cong& \mathbb{Z}^2,\\ \mathrm{H}_2(S^1\times S^1) &\cong& \mathbb{Z},\\ \mathrm{H}_i(S^1\times S^1) &\cong& 0,\ \text{for } i\geq 3, \end{array} \qquad \text{and} \qquad \begin{array}{lcl} \mathrm{H}_0(K) &\cong& \mathbb{Z},\\ \mathrm{H}_1(K) &\cong& \mathbb{Z}\oplus\mathbb{Z}_2,\\ \mathrm{H}_2(K) &\cong& 0,\\ \mathrm{H}_i(K) &\cong& 0,\ \text{for } i\geq 3. \end{array}$$

Thus we see that the homology groups are quite different and there is no clear connection between the n-th homology group of a covering and the n-th homology group of the base.

It is therefore perhaps surprising that we have the following proposition:

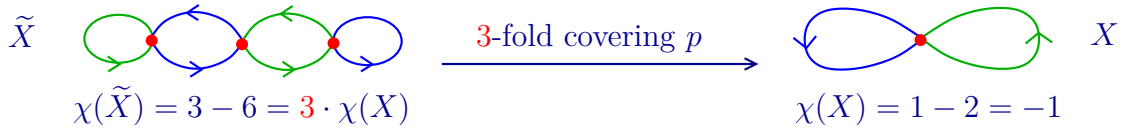
Proposition 48.10. (Covering-χ-Proposition) Let $p\colon \widetilde{X}\to X$ be a finite covering of a finite path-connected non-empty CW-complex X. We equip $\widetilde{X}$ with the structure of a CW-complex given by the CW-Complex Covering Proposition 32.1. Then

$$\chi(\widetilde{X}) \;=\; [\widetilde{X} : X]\cdot\chi(X).$$

Example.

(1) We consider again the 2-fold covering $p\colon S^1\times S^1\to K$. On page 605 we saw that $\chi(S^1\times S^1)=0$ and $\chi(K)=0$. This is consistent with the Covering-χ-Proposition 48.10.
(2) We consider the 2-fold covering $p\colon S^2\to\mathbb{R}\mathrm{P}^2$. In this case $\chi(S^2)=2$ is indeed twice $\chi(\mathbb{R}\mathrm{P}^2)=1$.
(3) In the Graph-π_1-Proposition 20.9 we showed that if X is a connected finite non-empty 1-dimensional CW-complex, then $\pi_1(X)$ is a free group of rank $1-\chi(X)$. As we implicitly already argued in the Free Subgroup Proposition 32.5, it follows fairly easily

from this observation and the Covering-χ-Proposition 48.10 that if F is a free group of finite rank and if G is a finite-index subgroup of F, then G is a free group with $\operatorname{rank}(G) - 1 = [F : G] \cdot (\operatorname{rank}(F) - 1)$.



Proof. As we mentioned in the CW-Complex Covering Proposition 32.1 (1), it follows immediately from the construction of the CW-structure on $\widetilde{X}$ that for each $k \in \mathbb{N}_0$ we have

$$\#k\text{-cells of } \widetilde{X} \;=\; \big[\widetilde{X} : X\big] \cdot \#k\text{-cells of } X.$$

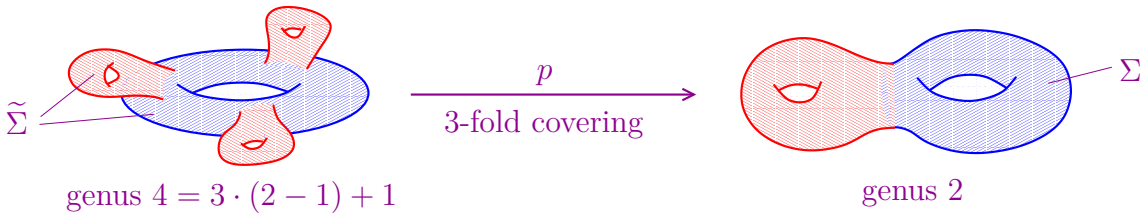
The proposition is an immediate consequence of this equality. ■

48.5. **Applications of Euler characteristics.** The following lemma shows how the genus changes under taking finite coverings:

Lemma 48.11. (Genus-of-Covering Lemma) Let Σ be the surface of genus g. For any connected finite covering $p\colon \widetilde{\Sigma} \to \Sigma$ we have[403]

$$\operatorname{genus}(\widetilde{\Sigma}) \;=\; \deg(p) \cdot (g-1) + 1.$$

Example. In the figure below we illustrate a 3-fold covering of the surface of genus 2 by the surface of genus $4 = 3 \cdot (2-1) + 1$.



Proof. We denote by $\widetilde{g}$ the genus of $\widetilde{\Sigma}$. The lemma follows immediately from the following two facts:

(1) By the discussion on page 605 we have $\chi(\Sigma) = 2 - 2g$ and $\chi(\widetilde{\Sigma}) = 2 - 2\widetilde{g}$.
(2) By the Covering-χ-Proposition 48.10 we have $\chi(\widetilde{\Sigma}) = \deg(p) \cdot \chi(\Sigma)$. ■

[403] Recall that by the Surface Classification Theorem 22.8 there exists precisely one $\widetilde{g}$ such that the closed, oriented 2-dimensional smooth manifold $\widetilde{\Sigma}$ is diffeomorphic to the surface of genus $\widetilde{g}$.

CHAPTER 49

Applications of the Euler characteristic

In this chapter we show that the fairly simple concept of the Euler characteristic, which we introduced in the previous chapter, can be used to address the following questions about 3-dimensional objects:

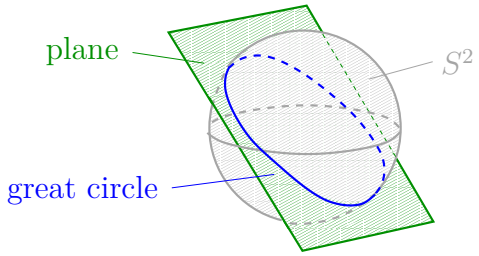
(1) Why is a classical football made using precisely 12 pentagons?
(2) What is Euler's famous formula?
(3) Why are there precisely five types of platonic solids?
(4) Can we embed all finite topological graphs into the plane?

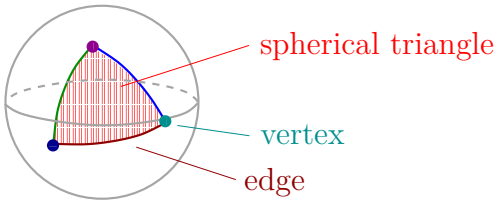
The reader is warned that in this chapter at times we will trade notational and mathematical rigor for readability.

49.1. **Building a leather football.** In this section we want to study the classic leather football from a mathematical point of view. This discussion will require the following definitions:

Definition.

(1) A great circle on the 2-dimensional sphere S^2 is the intersection of S^2 with a 2-dimensional vector subspace of $\mathbb{R}^3$.
(2) The angle between two great circles is defined as the smaller of the two angles given by the corresponding 2-dimensional vector subspaces.
(3) A spherical n-gon is a subset of S^2 such that the boundary is the union of n segments of n different great circles. These segments are called edges. The points where two edges intersect are called vertices.





In the following we are mostly interested in highly symmetric spherical n-gons. To define what we mean by "highly symmetric" we need to introduce the following general concept.

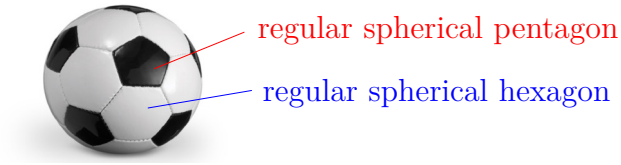
Definition. Let P be a subset of $\mathbb{R}^k$.

(1) A symmetry of P in $\mathbb{R}^k$ is a matrix $A \in \mathrm{O}(k)$ such that $A \cdot P = P$.
(2) The symmetry group $\mathrm{Sym}(P) \subset \mathrm{O}(k)$ is defined as the group of all symmetries of P.
(3) We set $\mathrm{Sym}^+(P) := \mathrm{Sym}(P) \cap \mathrm{SO}(k)$.

Using this general definition we can now introduce the main players of this section:

Definition. A spherical polygon P is called regular if the symmetry group $\mathrm{Sym}(P)$ acts transitively on the set of pairs (v, e) where v is a vertex of P and E is an edge of P with $v \in e$.

Example. The classical leather football is made out of 12 regular spherical pentagons and 20 regular spherical hexagons.



We want to determine how many regular spherical pentagons and hexagons are required to make a football. So suppose a football is made out of p regular spherical pentagons and h regular spherical hexagons such that any edge of a polygon is the edge of another polygon. This endows S^2 with a CW-structure such that the following holds:

- The 0-cells correspond to the vertices.
- The 1-cells correspond to the edges.
- The 2-cells correspond to the polygons.

We make the following observations:

(1) We first note that each 1-cell bounds two 2-cells.
(2) The interior angles of a regular n-gon are all the same.
(3) One can show, using elementary spherical geometry (see e.g. [**Bon09**, p. 53] or [**Fen01**, p. 260]), that the interior angle of a regular spherical n-gon is larger than the angle of the corresponding Euclidean regular n-gon.
(4) It follows from (2) and (3) that in our setting all the interior angles are larger than[404] $\pi - \frac{2\pi}{5} = \frac{3}{5}\pi$. Since the total angle at a vertex is 2π we see that at any vertex there are not more than three regular spherical hexagons and pentagons that meet. Of course this implies that at any vertex precisely three regular spherical hexagons and pentagons meet.

It follows that

$$\begin{aligned} \#\text{2-cells} &= p + h \\ \#\text{1-cells} &= \tfrac{1}{2}(5p + 6h) \quad \text{here we use Observation (1)} \\ \#\text{0-cells} &= \tfrac{1}{3}(5p + 6h) \quad \text{here we use Observation (4).} \end{aligned}$$

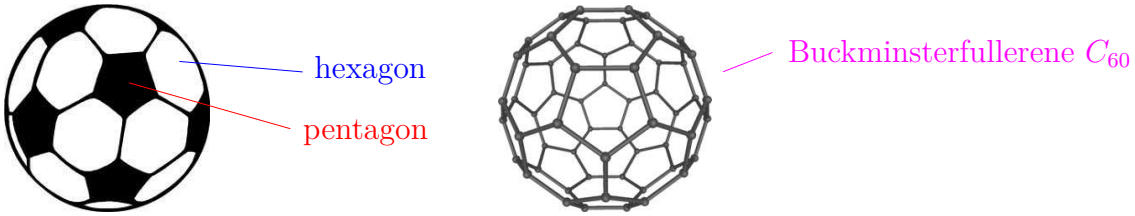
We now see that

$$2 \underset{\uparrow}{=} \chi(S^2) = \sum_{n=0}^{2} (-1)^n \cdot \#n\text{-cells} \underset{\uparrow}{=} \tfrac{1}{3}(5p + 6h) - \tfrac{1}{2}(5p + 6h) + (p + h) = \tfrac{1}{6}p.$$

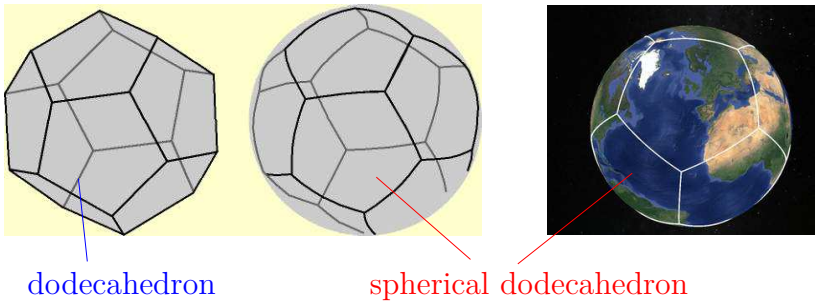
by the Homology-χ-Proposition 48.1 — by the above three equalities

Thus we see that a football has to have precisely 12 regular spherical pentagons. On the other hand, the number of regular spherical hexagons is not fixed. For example, as we mentioned above, the standard football has 20 hexagons.

[404] Where does that formula come from?



But in principle one could also build a football with zero hexagons, see the figure below on the right. This way one obtains a spherical dodecahedron. One could also build footballs with more hexagons, but they would be much less round since they would have bigger flat parts.



Finally note that the 1-skeleton of this CW-structure was in fact already "invented by nature", more precisely, there exists a molecule, called the Buckminsterfullerene C_{60}, built out of 60 carbon atoms, which has exactly the same shape as a football, except that it is a little smaller.

49.2. **Euler's Formula.** We start out by recalling the following definition from page 74:

Definition. Given a subset S of $\mathbb{R}^k$ the convex hull of S is defined as the intersection of all convex subsets of $\mathbb{R}^k$ that contain S. Since the intersection of convex sets is again convex we see that the convex hull of S is a convex subset of $\mathbb{R}^n$.

We move on to a new definition.

Definition.

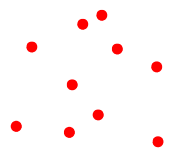
(1) A convex polyhedron is the convex hull of finitely many points in $\mathbb{R}^k$.
(2) The dimension of a convex polyhedron is defined as the minimal dimension of an affine subspace of $\mathbb{R}^k$ that contains the convex polyhedron.

Lemma 49.1. (Vertices-of-Polyhedron Lemma) If P is a convex polyhedron in $\mathbb{R}^k$, then there exists a unique finite set $V \subset \mathbb{R}^k$ such that P is the convex hull of V, but such that for any proper subset W of V, P is not the convex hull of W.

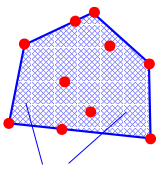
Proof. We leave the proof of this lemma to the reader. ■

Definition. Let P be a convex polyhedron in $\mathbb{R}^k$.

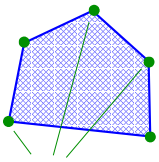
(1) We refer to the finite set of the Vertices-of-Polyhedron Lemma 49.1 as the vertices of the convex polyhedron P.



points in $\mathbb{R}^2$

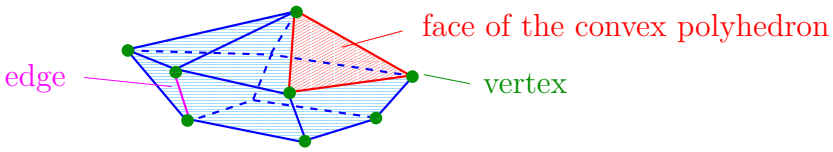


convex polyhedron
defined by these points



vertices of the
convex polyhedron

(2) (a) A closed affine half-space in $\mathbb{R}^k$ is a set of the form $\{v \in \mathbb{R}^k \mid \langle v, w\rangle \geq a\}$ for some fixed non-zero vector $w \in \mathbb{R}^k$ and some $a \in \mathbb{R}$.
(b) A subset $F \subset P$ is called a face of P if there exists a closed affine half-space H of $\mathbb{R}^k$ with $P \subset H$ such that $F = P \cap \partial H$.
Note that a face is a convex polyhedron in its own right, which implies in particular that it makes sense to talk of the dimension of a face.
(3) An edge is a 1-dimensional face. Sometimes we refer to a 2-dimensional face just as a face, without specifying the dimension.



Proposition 49.2. (Euler's Formula) Let P be a convex k-dimensional polyhedron in $\mathbb{R}^k$. The following statements hold:
(1) The interior $\mathring{P}$ of P is non-empty and all faces of P are of dimension $\leq k-1$.
(2) The boundary ∂P admits a CW-structure such that for each $j \in \mathbb{N}_0$ the j-cells are precisely the j-dimensional faces of P.
(3) The following equality hold:

$$\sum_{j=0}^{k-1}(-1)^j \cdot \#j\text{-dimensional faces} \;=\; 1+(-1)^{k-1}.$$

Remark. The 3-dimensional version of Proposition 49.2 gives us that for the boundary of a convex 3-dimensional polyhedron in $\mathbb{R}^3$ we have

$$\text{number of vertices} - \text{number of edges} + \text{number of faces} \;=\; 2.$$

This equality is known as Euler's Formula and was formulated by Leonhard Euler [**Eul**, p. 119] in 1758. This equality was already noted before by Francesco Maurolico [**Fri18**, p. 71] in 1537 and René Descartes [**Rab10**, p. 447] in 1628. Arguably Euler's Formula is historically the first result in topology. A famous book by Imre Lakatos [**Lak76**] is dedicated to the challenge of finding the correct statement of Euler's formula and of providing a rigorous proof of Euler's Formula without employing the technical machinery, namely singular homology, that we are about to use: □

Sketch of proof. We leave the pleasure of proving the first two statements to the reader. Now we calculate that

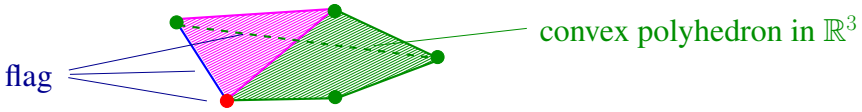
[404]Here ∂P is the boundary of P viewed as a subset of $\mathbb{R}^k$.

$$\sum_{j=0}^{k-1}(-1)^j \cdot \#j\text{-dimensional faces} \underset{\text{by (1) and (2)}}{=} \chi(\partial P) \underset{\text{by (1) together with the the Convex-to-Ball Proposition 2.12(2)}}{=} \chi(S^{k-1}) \underset{\text{see page 605}}{=} 1+(-1)^{k-1}.$$

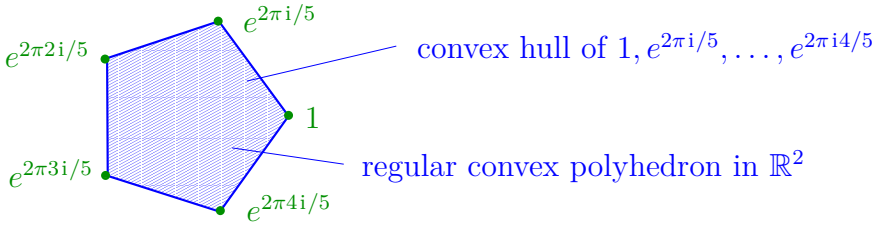
■

49.3. **Platonic solids.** In this section we will use Euler's Formula 49.2 to address the question: Why are there only five types of Platonic solids. Evidently we first need to introduce the notion of a Platonic solid. This requires some preparations:

Definition.
(1) Let P be a convex polyhedron in $\mathbb{R}^k$.
  (a) A flag of P is a sequence $F_0 \subset F_1 \subset \cdots \subset F_{k-1}$ of faces of P such that each F_i is i-dimensional.
  (b) We say that P is regular if the symmetry group $\mathrm{Sym}(P)$ acts transitively on the set of flags of P.
(2) Two convex polyhedra P and Q in $\mathbb{R}^k$ are called equivalent if there exists an $r > 0$, a $v \in \mathbb{R}^k$ and an orthogonal matrix $A \in \mathrm{O}(k)$ with $P = A \cdot (r \cdot Q) + v$.



Examples.
(1) Let $n \in \mathbb{N}_{\geq 3}$ and $r > 0$. The convex hull of the n points $r, r \cdot e^{2\pi \mathrm{i}/n}, \ldots, r \cdot e^{2\pi \mathrm{i}(n-1)/n}$ in $\mathbb{C} =_{\mathrm{i}} \mathbb{R}^2$ is a regular convex polyhedron with vertices $r, r \cdot e^{2\pi \mathrm{i}/n}, \ldots, r \cdot e^{2\pi \mathrm{i}(n-1)/n}$. We leave it to the reader to show that, up to equivalence, these are all regular convex 2-dimensional polyhedra in $\mathbb{R}^2$.



(2) The standard n-simplex Δ^n is a convex n-dimensional polyhedron in $\mathbb{R}^{n+1}$ with vertex set $\{(1,0,\ldots,0),(0,1,0,\ldots,0),\ldots,(0,\ldots,0,1)\}$. It is straightforward to see that Δ^n is regular. □

We now specialize to the "classical" setting of 3-dimensional objects. The next definition introduces one of the oldest concepts in the history of mathematics.

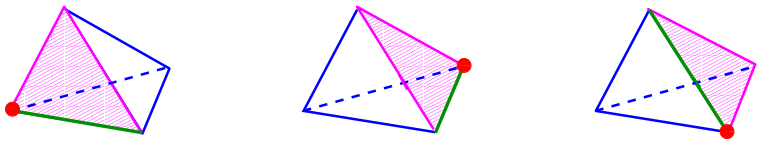
Definition. A Platonic solid is a regular convex 3-dimensional polyhedron in $\mathbb{R}^3$.

Before we discuss the five classical examples of Platonic solids let us introduce the following definition:

Definition. The valence of a vertex of a convex polyhedron is defined as the number of edges that contain the vertex.

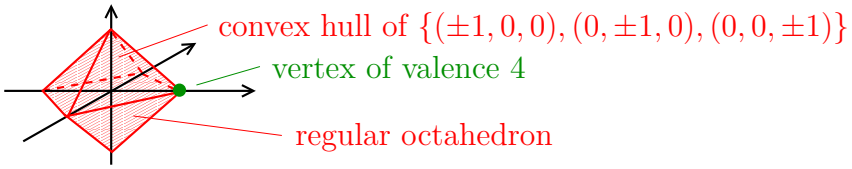
Examples.

(1) Let T be the 3-dimensional polyhedron in $\mathbb{R}^3$ that has four 2-dimensional faces, each of which is an equilateral triangle. In the figure below we show T together with three flags. Some classical geometry shows that T is, up to equivalence, uniquely defined and it is regular. We refer to T as *the regular tetrahedron*. Finally note that each vertex has valence 3.

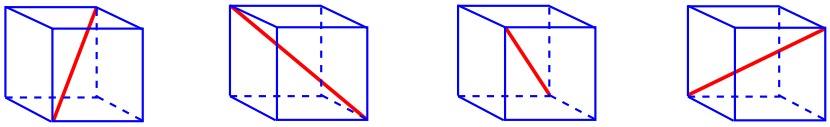


three flags in the regular tetrahedron T

(2) The convex polyhedron in $\mathbb{R}^3$ spanned by $\{(\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1)\}$ is a regular convex 3-dimensional polyhedron where each vertex has valence 4. This convex polyhedron is usually called *the regular octahedron*.



(3) We consider the cube $[-1, 1]^3 \subset \mathbb{R}^3$. In [**Arm88**, p. 38] it is shown that the group $\mathrm{Sym}^+([-1, 1]^3)$ is isomorphic to S_4. More precisely, we denote by D the set of the diagonals in the cube, i.e. the segments in $\mathbb{R}^3$ spanned by opposite vertices of the cube. Note that D has precisely four elements.



In [**Arm88**, p. 38] it is shown that the map

$$\begin{array}{rcl} \mathrm{Sym}^+([-1, 1]^3) & \to & S_4 \cong \text{permutation group of } D \\ A & \mapsto & \begin{pmatrix} D & \to & D \\ d & \mapsto & A \cdot d \end{pmatrix} \end{array}$$

is an isomorphism. It is clear that the cube is a regular convex polyhedron. Since it has six sides it is in our context also referred to as *the regular hexahedron*.

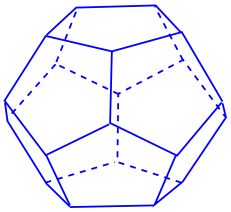
(4) Let $z \coloneqq 1 + \frac{\sqrt{5}}{2}$ be the golden ratio. We denote by D the polyhedron that is spanned by the 20 vertices

$$(\pm 1, \pm 1, \pm 1), \quad (0, \pm z, \pm \tfrac{1}{z}), \quad (\pm \tfrac{1}{z}, 0, \pm z), \quad \text{and} \quad (\pm z, \pm \tfrac{1}{z}, 0).$$

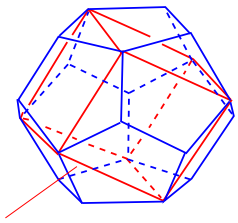
By [**Cox48**, p. 53] this is a regular polyhedron. We refer to it as *the regular dodecahedron*. Note that it has 12 faces and 20 vertices. We can inscribe five cubes $C_1, \ldots, C_5$ into the regular dodecahedron. In [**Arm88**, p. 40] it is shown that the map

$$\begin{array}{rcl} \mathrm{Sym}^+(D) & \to & A_5 \cong \text{positive permutations of } \{C_1, \ldots, C_5\} \\ A & \mapsto & (C_i \mapsto A \cdot C_i) \end{array}$$

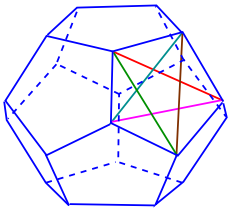
is an isomorphism.



regular dodecahedron



cube inscribed into the regular dodecahedron



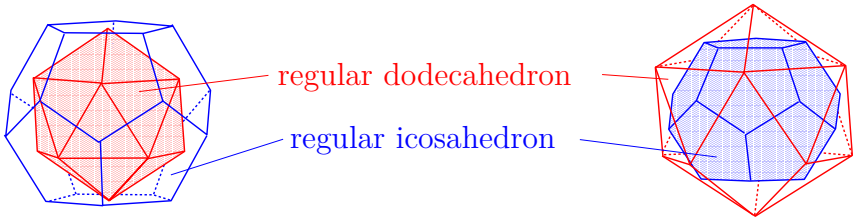
the five diagonals of a face correspond precisely to one inscribed cube

(5) Let $z := 1 + \frac{\sqrt{5}}{2}$ be the golden ratio. We denote by I the polyhedron that is spanned by the 12 vertices

$$(0, \pm 1, \pm z), \quad (\pm z, 0, \pm 1) \quad \text{and} \quad (\pm 1, \pm z, 0).$$

By [**Cox48**, p. 52] this is a regular polyhedron. We refer to it is as the *regular icosahedron*.

Note that the regular dodecahedron has 20 vertices and 12 faces whereas the regular icosahedron has 12 vertices and 20 faces. In fact the regular icosahedron is dual to the regular dodecahedron in the following sense: up to a scaling factor and a rotation, the regular icosahedron is spanned by the centers of the faces of the regular dodecahedron and vice-versa. We refer to the figure below for an illustration of this fact.

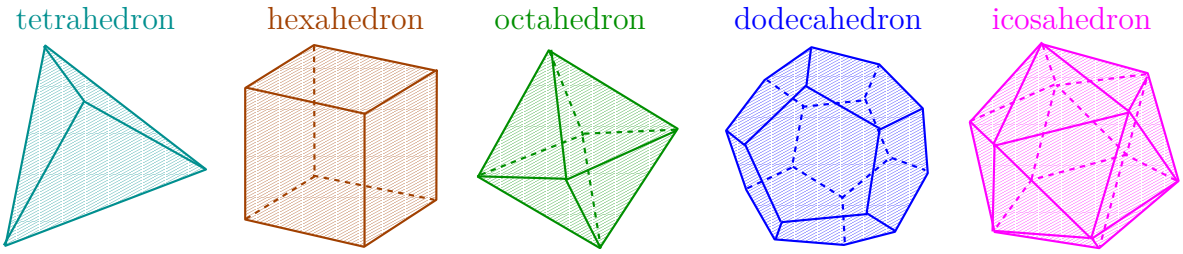


It follows that the symmetry group of the regular icosahedron I is in a suitable sense equal to the symmetry group of the regular dodecahedron D. In particular the group $\mathrm{Sym}^+(I)$ is isomorphic to A_5. □

Theorem 49.3. (Platonic Solid Theorem) Up to equivalence the only regular convex 3-dimensional polyhedra in $\mathbb{R}^3$ are the platonic solids discussed above and illustrated in the figure below. They have the following types of faces and valences and they have the following symmetry groups:

	tetrahedron	hexahedron	octahedron	dodecahedron	icosahedron
number of faces	4	6	8	12	20
shape of face	3-gon	4-gon	3-gon	5-gon	3-gon
valence of vertex	4	3	4	3	5
$\mathrm{Sym}^+(P)$	A_4	S_4	S_4	A_5	A_5

Sketch of proof. We will not provide a full proof for the theorem. Instead we refer to [**Cox48**, Section 6.7] for a full proof that the list of five platonic solids is indeed complete.



Furthermore we refer to [**Arm88**, p. 40] or alternatively to [**Aar08**, Chapter 5.5] for a proof of the statement regarding the symmetry groups.[405]

In the following we will show the weaker statement that any platonic solid fits into the table of the theorem. Put differently, we will show that given any platonic solid the number of faces, the shape of faces and the valence of a vertex corresponds to one of the five columns in the above table.

Thus let P be a regular convex 3-dimensional polyhedron in $\mathbb{R}^3$. We make the following observations:

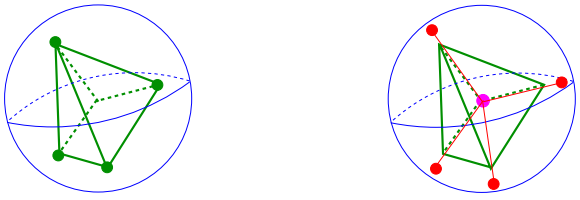
(1) It follows easily from the definition of a regular polyhedron that there exists an $m \in \mathbb{N}$ such that each face of P is a regular m-gon.
(2) The number of faces that meet at a vertex equals the valence of a vertex.
(3) The valences of the vertices are the same and it is at least three.

The following claim tells us the possible values for m and the number f of faces that meet at a vertex.

Claim.

possible values for m	3	4	5
f = number of faces that meet at a vertex	3, 4, 5	3	3

Proof. After a translation we can assume that the sum of all vertices is in the origin. Next we shrink the convex polyhedron so that it fits into the sphere S^2. Then we consider the projection of the vertices and edges of the convex polyhedron onto the sphere S^2.



embed polyhedron in the 3-ball

projection of vertices onto the sphere

projection of edges onto the sphere

Recall that on page 614 we noted that the interior angle of a regular spherical m-gon is greater than the angle of a regular Euclidean m-gon. So it follows that in our case the interior angle is greater than the angle of a regular m-gon in $\mathbb{R}^3$, i.e. it is greater than $\pi - \frac{2\pi}{m}$. Since at any vertex at least three faces meet we obtain the restriction that $f \geq 3$

[405] We sketched the proofs for the regular hexahedron, the regular dodecahedron and the regular icosahedron.

and that $f \cdot (\pi - \frac{2\pi}{m}) < 2\pi$. It is now straightforward to see that we obtain the stated table. ⊞

Now we treat the cases $m = 3, 4, 5$ separately. We start out with the case $m = 3$, i.e. we assume that all the faces of the convex polyhedron are equilateral triangles. Let us first consider the case that three triangles meet at a vertex. Then we have

$$\#\text{faces} =: k, \qquad \#\text{edges} \underset{\uparrow}{=} \tfrac{1}{2} \cdot 3k \qquad \text{and} \qquad \#\text{vertices} \underset{\uparrow}{=} \tfrac{1}{3} \cdot 3k.$$

since $m = 3$ and since each edge cobounds two faces

since $m = 3$ and since each vertex cobounds three faces

Now we see that

$$2 \underset{\uparrow}{=} \#\text{faces} - \#\text{edges} + \#\text{vertices} \underset{\uparrow}{=} k - \tfrac{3}{2}k + \tfrac{3}{3}k = \tfrac{1}{2}k.$$

Euler's Formula 49.2

by the above

It follows that $k = 4$. Note that this case is realized by the regular tetrahedron. Almost the same calculation shows that if four triangles meet at a vertex, then $2 = k - \frac{3}{2}k + \frac{3}{4}k$, i.e. $k = 8$. Note that this case is realized by the regular octahedron. Finally by the above claim it remains to consider the case that five triangles meet at a vertex. In this case we see that $2 = k - \frac{3}{2}k + \frac{3}{5}k$, i.e. $k = 20$. Note that this case is realized by the regular icosahedron.

Next we consider the case $m = 4$, i.e. each face is a square. The same argument as above shows that the valence of a vertex can only be three and the same calculation shows that there have to be precisely six faces. Note that this case is realized by the regular hexahedron (aka cube).

Finally if $m = 5$, i.e. if each face is a regular pentagon, then as above we see that the valence of a vertex can only be three and that there have to be precisely twelve faces. Note that this case is realized by the regular dodecahedron. ■

Remark. On page 617 we saw that there exist infinitely many equivalence classes of regular convex polyhedra in $\mathbb{R}^2$. On the other hand we have just seen in the Platonic Solid Theorem 49.3 that in $\mathbb{R}^3$ there are only finitely many, namely precisely five equivalence classes of regular convex polyhedra in $\mathbb{R}^3$. This begs the question, what is the number in higher dimensions? It turns out that in $\mathbb{R}^4$ there are six equivalence classes of regular convex polyhedra, whereas in all higher dimensions there are only three equivalence classes of convex polyhedra. A nice exposition of these facts is given on the following website:

`http://math.ucr.edu/home/baez/platonic.html` □

Part V

Singular Homology – Algebra and coefficients

In this part of the lecture notes we will develop our algebraic skills and bring them to bear on topology. In particular we will learn about homology with coefficients and we will determine the homology groups of a product of topological spaces.
This part of the lecture notes is organizes as follows:

- In Chapter 50 we will prove the Fundamental Theorem of Homological Algebra 50.1 which says that every abelian group has a free resolution which is unique up to chain homotopy. We will also prove that a chain map between chain complexes of free abelian groups, that induces isomorphisms on all homology groups, is in fact a chain homotopy equivalence.
- In the purely algebraic Chapter 51 we will introduce and study the tensor product of abelian groups.
- In Chapter 52 we will introduce the homology groups $\mathrm{H}_k(X, A; G)$ of a pair (X, A) of topological spaces with coefficients in an abelian group G. We will explore the main properties and we will see that most results on ordinary homology generalize in a straightforward way to homology with G-coefficients.
- In the short Chapter 53 we will introduce exact functors, we will study their effect on homology and we will give a few basic examples.
- In Chapter 54 we will prove the Universal Coefficient Theorems 54.8 and 54.9 which let us calculate the homology groups $\mathrm{H}_k(X, A; G)$ in terms of the ordinary homology groups $\mathrm{H}_k(X, A)$ and $\mathrm{H}_{k-1}(X, A)$.
- In Chapter 55 we introduce the tensor product $\mathcal{C}_* \otimes \mathcal{D}_*$ of two chain complexes $\mathcal{C}_*$ and $\mathcal{D}_*$. In the Algebraic–Künneth Theorem 55.5 we will compute the homology groups of a tensor product chain complex $\mathcal{C}_* \otimes \mathcal{D}_*$ in terms of homology groups of the chain complexes $\mathcal{C}_*$ and $\mathcal{D}_*$.
- In Chapter 56 we will state and prove the Eilenberg-Zilber Theorem 56.1 which says that for two topological spaces X and Y the singular chain complex $\mathrm{C}_*(X \times Y)$ is chain homotopic to the tensor chain complex $\mathrm{C}_*(X) \otimes \mathrm{C}_*(Y)$. We will use the Algebraic–Künneth Theorem 55.5 to prove the Künneth Theorem 56.9 which lets us compute the homology groups of a product $X \times Y$ of topological spaces in terms of the homology groups of X and Y.
- In Chapter 57 we will consider several applications of homology groups. In particular we will give a very very concise introduction to persistent homology and we will use homology with $\mathbb{F}_2$-coefficients to give one more proof of the Borsuk-Ulam Theorem 57.4.

CHAPTER 50

The Fundamental Theorem of Homological Algebra

We start out by recalling two results from earlier on:

(1) In the C_*–Functor Lemma 35.5 and the Relative-H_*–Functor Lemma 38.1 we showed that the maps

$$\begin{array}{rcl} \text{pair } (X,A) \text{ of topological spaces} & \mapsto & (\mathrm{C}_*(X,A),\partial_*) \\ (\text{continuous map } f\colon (X,A)\to(Y,B)) & \mapsto & (f_*\colon (\mathrm{C}_*(X,A),\partial_*)\to(\mathrm{C}_*(Y,B),\partial_*)) \end{array}$$

define a covariant functor from the category *PairTop* of pairs of topological spaces to the category *ChCplx* of chain complexes.

(2) In the Chain Complex-H_*-Functor Lemma 35.1 we showed that for every $n\in\mathbb{N}_0$ the maps

$$\begin{array}{rcl} (C_*,\partial_*) & \mapsto & \mathrm{H}_n(C_*) \\ (f\colon (C_*,\partial_*)\to(D_*,\partial_*)) & \mapsto & (f_*\colon \mathrm{H}_n(C_*)\to\mathrm{H}_n(D_*)) \end{array}$$

define a covariant functor from the category *ChCplx* of chain complexes to the category *AbGroup* of abelian groups.

In the previous chapters we mostly studied the composition of the two functors, i.e. for $n\in\mathbb{N}_0$ we studied the covariant functor

$$(X,A) \;\mapsto\; \mathrm{H}_n(X,A)$$

from the category *PairTop* of pairs of topological spaces to the category *AbGroup* of abelian groups. In this and the following chapters we want to consider the two functors more in their own right and we will see that it can be interesting to "intersperse" a functor from the category *ChCplx* of chain complexes to itself, between these two functors.

A typical question we want to answer in this chapter is the following: In the Excision Theorem 40.1 we showed that if X is a topological space and $Z\subset A\subset X$ are subsets such that the closure of Z is contained in the interior of A, then the inclusion $(X\setminus Z, A\setminus Z)\to(X,A)$ induces for each $n\in\mathbb{N}_0$ an isomorphism

$$\mathrm{H}_n(X\setminus Z, A\setminus Z) \;\to\; \mathrm{H}_n(X,A).$$

It is reasonable to ask whether the inclusion-induced map

$$(\mathrm{C}_*(X\setminus Z, A\setminus Z),\partial_*) \;\to\; (\mathrm{C}_*(X,A),\partial_*).$$

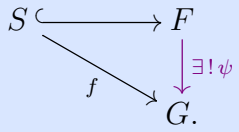
is in fact a chain homotopy equivalence.

In this chapter we will see that, for purely algebraic reasons, the above question can always be answered in the affirmative. More precisely, we will show the following completely algebraic statement: If a chain map $C_*\to D_*$ induces isomorphisms on all homology groups and if all chain groups are free abelian groups, then the chain map is in fact a chain homotopy equivalence. This algebraic result will be very useful when we study homology groups with coefficients and cohomology groups.

50.1. **The Fundamental Theorem of Homological Algebra.** We start out by recalling the following definitions from pages 256 and 257.

Definition.

(1) We say that a group F is a free abelian group on a subset $S \subset F$, if the following universal property holds: For any *map* $f\colon S \to G$ of the set S to an abelian group G there exists a unique *homomorphism* $\psi\colon F \to G$ that makes the following diagram commute:



We call S a basis of the free abelian group F.

(2) Let S be a set. By the Free Abelian Group–Existence Lemma 18.4 we know that

$$\mathbb{Z}^{(S)} := \text{all maps from } S \text{ to } \mathbb{Z} \text{ which are non-zero for only finitely many } s \in S$$

is a free abelian group generated by S.[406]

We move on to a new definition, which will turn out to be surprisingly important for our future discussions.

Definition. A (free) resolution of an abelian group H is an exact sequence

$$\ldots \xrightarrow{f_3} F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \to 0$$

where the F_i are (free) abelian groups.

Example. The following is a free resolution for the abelian group $H = \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_5$:

$$0 \to \mathbb{Z}^2 \xrightarrow{\begin{pmatrix} 0 & 0 \\ 2 & 0 \\ 0 & 5 \end{pmatrix}} \mathbb{Z}^3 \xrightarrow{\text{obvious projection}} \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_5 \to 0. \qquad \square$$

Definition. Let $\alpha\colon H \to H'$ be a homomorphism between abelian groups. Furthermore let F_* be a resolution of H and let F'_* be a resolution of H'. An extension of α to the resolutions F_* and F'_* is a sequence of homomorphisms $\alpha_i\colon F_i \to F'_i$, $i \in \mathbb{N}_0$ such that the following diagram commutes:

$$\begin{array}{ccccccccccc}
\ldots & \longrightarrow & F_2 & \xrightarrow{f_2} & F_1 & \xrightarrow{f_1} & F_0 & \xrightarrow{f_0} & H & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_0} & & \downarrow{\scriptstyle \alpha} & & \\
\ldots & \longrightarrow & F'_2 & \xrightarrow{f'_2} & F'_1 & \xrightarrow{f'_1} & F'_0 & \xrightarrow{f'_0} & H' & \longrightarrow & 0.
\end{array}$$

Note that the horizontal exact sequences on top and bottom are in particular chain complexes and that the maps α_* define a chain map from one chain complex to the other.

The following fundamental theorem underlies many arguments in homological algebra.

[406] Here we identify $s \in S$ with the homomorphism $S \to \mathbb{Z}$ which is given by $s \mapsto 1$ and $t \mapsto 0$ for $t \neq s$. This way we can view S as a subset of $\mathbb{Z}^{(S)}$.

Theorem 50.1. (Fundamental Theorem of Homological Algebra) Let $\alpha\colon H \to H'$ be a homomorphism between abelian groups. Furthermore let F_* be a free resolution of H and let F'_* be a resolution[407] of H'.

(1) There exists an extension of α to the resolutions F_* and F'_*.
(2) Any two extensions α_* and α'_*, viewed as chain maps, are chain homotopic.

Proof. Let F_* be a free resolution of H and let F'_* be a resolution of H'. Statements (1) and (2) are both a straightforward consequence of some mild diagram chasing and the ease with which one can find homomorphisms starting from a free abelian group. In the following we fill in all the details.

(1) We have to show that there exists a sequence $\{\alpha_i\}_{i\in\mathbb{N}_0}$ of homomorphisms such that the following diagram commutes:

$$\begin{array}{ccccccccccc}
\ldots & \longrightarrow & F_2 & \xrightarrow{f_2} & F_1 & \xrightarrow{f_1} & F_0 & \xrightarrow{f_0} & H & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_0} & & \downarrow{\scriptstyle \alpha} & & \\
\ldots & \longrightarrow & F'_2 & \xrightarrow{f'_2} & F'_1 & \xrightarrow{f'_1} & F'_0 & \xrightarrow{f'_0} & H' & \longrightarrow & 0.
\end{array}$$

First we define the homomorphism $\alpha_0\colon F_0 \to F'_0$. We choose a basis $\mathcal{B}_0$ of the free abelian group F_0. Since f'_0 is surjective we see that for each $x \in \mathcal{B}_0$ we can choose an $x' \in F'_0$ with $f'_0(x') = \alpha(f_0(x))$. By definition of a basis there exists a unique homomorphism $\alpha_0\colon F_0 \to F'_0$ with $\alpha_0(x) = x'$ for all $x \in \mathcal{B}_0$. It is now evident that the right square commutes.

Now we turn to the definition of the homomorphism $\alpha_1\colon F_1 \to F'_1$. We choose a basis $\mathcal{B}_1$ of the free abelian group F_1. Let $x \in \mathcal{B}_1$. It follows from the commutativity of the square on the right that

$$f'_0((\alpha_0 \circ f_1)(x)) \;=\; \alpha(\underbrace{(f_0 \circ f_1)}_{=0}(x)) \;=\; 0.$$

Since the lower sequence is exact there exists an $x' \in F'_1$ with $f'_1(x') = (\alpha_0 \circ f_1)(x)$. Since $\mathcal{B}_1$ is a basis of F_1 there exists a unique homomorphism $\alpha_1\colon F_1 \to F'_1$ such that $\alpha_1(x) = x'$ for all $x \in \mathcal{B}_1$. It is now evident that the "second square from the right" also commutes. It is now clear how inductively we can construct the homomorphisms $\alpha_2, \alpha_3, \ldots$.

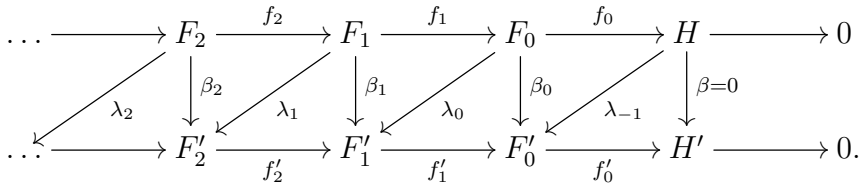
(2) Now let $\alpha_i, \alpha'_i\colon F_i \to F'_i, i \in \mathbb{N}_0$ be two extensions of α to the free resolutions F_* and F'_*. We have to find a chain homotopy from the chain map α_i to the chain map α'_i. Note that the maps $\beta_i \coloneqq \alpha_i - \alpha'_i$ are an extension of the zero map $\beta = \alpha - \alpha = 0\colon H \to H'$. Therefore it suffices to find a chain homotopy λ from $\beta_i = \alpha_i - \alpha'_i$ to the zero chain map. The construction of the chain homotopy is very similar to the construction of the homomorphisms α_i in the first part of the proof.

Recall that we have to find a sequence of homomorphisms λ_i, $i \in \mathbb{N}_0$ such that for every $i \in \mathbb{N}_0$ we have

$$f'_{i+1} \circ \lambda_i + \lambda_{i-1} \circ f_i \;=\; \beta_i.$$

[407] Note that we do not demand that F'_* is a *free* resolution.

In the following discussion it is convenient to keep an eye on the following diagram that contains all the homomorphisms that we are given and that we need to construct:



We put $\lambda_{-1} = 0$. Next we need to find a homomorphism $\lambda_0 \colon F_0 \to F'_1$ such that $\beta_0 = f'_1 \circ \lambda_0$. We pick a basis $\mathcal{B}_0$ of the free abelian group F_0. For each $x \in \mathcal{B}_0$ we choose an $x' \in F'_1$ with $f'_1(x') = \beta_0(x)$. The existence of such x' follows from the observation that $f'_0(\beta_0(x)) = \beta(f_0(x)) = 0$ and from the exactness of the lower sequence. Since $\mathcal{B}_0$ is a basis for F_0 we get an induced homomorphism $\lambda_0 \colon F_0 \to F'_1$.

Now we suppose that we have already found homomorphisms $\lambda_0, \dots, \lambda_{i-1}$ which have the desired property that

$$(*) \qquad f'_j \circ \lambda_{j-1} + \lambda_{j-2} \circ f_{j-1} \;=\; \beta_{j-1} \quad \text{for } j = 0, \dots, i.$$

We choose again a basis $\mathcal{B}_i$ of the free abelian group F_i. It suffices to show that for each $x \in \mathcal{B}_i$ we can find an element $x' \in F'_{i+1}$ which satisfies the equation

$$f'_{i+1}(x') \;=\; \beta_i(x) - \lambda_{i-1}(f_i(x)).$$

Thus let $x \in \mathcal{B}_i$. Note that it follows from the exactness of the lower sequence that $\operatorname{im}(f'_{i+1}) = \ker(f'_i)$. Thus it suffices to show that $f'_i \circ (\beta_i - \lambda_{i-1} \circ f_i) = 0$. It turns out that indeed we have

$$\begin{aligned} f'_i \circ (\beta_i - \lambda_{i-1} \circ f_i) \;&=\; f'_i \circ \beta_i - f'_i \circ \lambda_{i-1} \circ f_i \;\overset{\text{since the } \beta_i\text{'s are a chain map, i.e. } f'_i \circ \beta_i = \beta_{i-1} \circ f_i}{=}\; \beta_{i-1} \circ f_i - f'_i \circ \lambda_{i-1} \circ f_i \\ &=\; (\beta_{i-1} - f'_i \circ \lambda_{i-1}) \circ f_i \;\overset{\text{by } (*)}{=}\; \lambda_{i-2} \circ f_{i-1} \circ f_i \;\overset{\text{since } f_{i-1} \circ f_i = 0}{=}\; 0. \end{aligned}$$

■

We conclude this section with two applications of the Fundamental Theorem 50.1 of Homological Algebra. The following corollary says in particular that any two free resolutions of a given abelian group are chain homotopy equivalent.

Corollary 50.2. (Free Resolution–Chain Homotopic Corollary) Let

$$\dots \to F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \to 0$$

and

$$\dots \to F'_2 \xrightarrow{f'_2} F'_1 \xrightarrow{f'_1} F'_0 \xrightarrow{f'_0} H \to 0$$

be two free resolutions of the *same* abelian group H. Let $\{\alpha_i \colon F_i \to F'_i\}_{i \in \mathbb{N}}$ be an extension of the identity map $\mathrm{id} \colon H \to H$ to the free resolutions F_* and F'_*. The maps α_* form a chain homotopy equivalence between the two chain complexes F_* and F'_*.

Proof. Let $\{\alpha_i \colon F_i \to F'_i\}_{i \in \mathbb{N}}$ be an extension of the identity map $\mathrm{id} \colon H \to H$. We apply the Fundamental Theorem 50.1 of Homological Algebra to the free resolutions F'_* and F_* (i.e. we swap the roles of the free resolutions) and we obtain homomorphisms

$\{\beta_i\colon F_i' \to F_i\}_{i\in\mathbb{N}_0}$ that give us the lower half of the following commutative diagram:

$$\begin{array}{ccccccccccc}
\ldots & \longrightarrow & F_2 & \xrightarrow{f_2} & F_1 & \xrightarrow{f_1} & F_0 & \xrightarrow{f_0} & H & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_0} & & \downarrow{\scriptstyle \mathrm{id}} & & \\
\ldots & \longrightarrow & F_2' & \xrightarrow{f_2'} & F_1' & \xrightarrow{f_1'} & F_0' & \xrightarrow{f_0'} & H & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \beta_2} & & \downarrow{\scriptstyle \beta_1} & & \downarrow{\scriptstyle \beta_0} & & \downarrow{\scriptstyle \mathrm{id}} & & \\
\ldots & \longrightarrow & F_2 & \xrightarrow{f_2} & F_1 & \xrightarrow{f_1} & F_0 & \xrightarrow{f_0} & H & \longrightarrow & 0.
\end{array}$$

The maps $\beta_i \circ \alpha_i$ now provide us with the following commutative diagram:

$$\begin{array}{ccccccccccc}
\ldots & \longrightarrow & F_2 & \xrightarrow{f_2} & F_1 & \xrightarrow{f_1} & F_0 & \xrightarrow{f_0} & H & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \beta_2\circ\alpha_2} & & \downarrow{\scriptstyle \beta_1\circ\alpha_1} & & \downarrow{\scriptstyle \beta_0\circ\alpha_0} & & \downarrow{\scriptstyle \mathrm{id}} & & \\
\ldots & \longrightarrow & F_2 & \xrightarrow{f_2} & F_1 & \xrightarrow{f_1} & F_0 & \xrightarrow{f_0} & H & \longrightarrow & 0.
\end{array}$$

This shows that the maps $\beta_i \circ \alpha_i$ define an extension of $\mathrm{id}\colon H \to H$ to the free resolutions F_i and F_i. But the identity maps also define an extension. By the Fundamental Theorem 50.1 of Homological Algebra the extensions $\{\beta_i \circ \alpha_i\}_{i\in\mathbb{N}_0}$ and $\{\mathrm{id}\}_{i\in\mathbb{N}_0}$ are chain homotopic. The same way one shows that the extensions $\{\alpha_i \circ \beta_i\}_{i\in\mathbb{N}_0}$ and $\{\mathrm{id}\}_{i\in\mathbb{N}_0}$ are chain homotopic. This shows that the maps α_i form a chain homotopy equivalence. ■

Corollary 50.3. (Identity-of-Acyclic-is-Null Homotopic Corollary) Let (C_*, ∂_*) be a chain complex of free abelian groups. If (C_*, ∂_*) is acyclic, i.e. if $\mathrm{H}_n(C_*) = 0$ for all $n \in \mathbb{N}_0$, then the identity map and the zero map of C_* to itself are chain homotopic.

Proof. We consider the following pretty diagram:

$$\begin{array}{ccccccccccc}
\ldots & \longrightarrow & C_2 & \xrightarrow{\partial_2} & C_1 & \xrightarrow{\partial_1} & C_0 & \xrightarrow{0} & 0 & \longrightarrow & 0 \\
 & & {\scriptstyle 0}\downarrow\ \downarrow{\scriptstyle \mathrm{id}} & & {\scriptstyle 0}\downarrow\ \downarrow{\scriptstyle \mathrm{id}} & & {\scriptstyle 0}\downarrow\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle 0} & & \\
\ldots & \longrightarrow & C_2 & \xrightarrow{\partial_2} & C_1 & \xrightarrow{\partial_1} & C_0 & \xrightarrow{0} & 0 & \longrightarrow & 0.
\end{array}$$

The hypothesis that $\mathrm{H}_n(C_*) = 0$ for all $n \in \mathbb{N}_0$ means that C_* is a free resolution of the trivial group, i.e. the top and bottom are free resolutions. The zero map on the right now has two extensions:

(1) The identity maps $\mathrm{id}\colon C_i \to C_i$ define an extension.
(2) The zero maps $\mathrm{id}\colon C_i \to C_i$ define an extension.

By Statement (2) of the Fundamental Theorem 50.1 of Homological Algebra these two extensions are chain homotopic. But that is precisely what we wanted to show. ■

50.2. **Quasi-isomorphisms are chain homotopy equivalences.**

In this section we will state and discuss more consequences of the Fundamental Theorem 50.1 of Homological Algebra. We start out with the following definition:

Definition. Let $f\colon C_* \to D_*$ be a chain map between chain complexes. We say that f is a quasi-isomorphism if for each $k \in \mathbb{N}_0$ the induced map $f_*\colon \mathrm{H}_k(C_*) \to \mathrm{H}_k(D_*)$ is an isomorphism.

In the Chain Homotopy–H_*-Lemma 36.1 we saw that chain homotopy equivalences are quasi-isomorphisms. If we are dealing with chain complexes of free abelian groups, the following proposition gives us now the converse statement.

Proposition 50.4. (Quasi Iso–Homotopy Equivalence Proposition) Let C_* and D_* be two chain complexes of *free* abelian groups and let $f\colon C_* \to D_*$ be a chain map. If f is a quasi-isomorphism, then f is in fact a chain homotopy equivalence.

Example. As on page 625 we consider the following setting: Let X be a topological space and let $Z \subset A \subset X$ be subsets such that the closure of Z is contained in the interior of A. By the Excision Theorem 40.1 we know that the inclusion-induced chain map

$$\mathrm{C}_*(X \setminus Z, A \setminus Z) \;\to\; \mathrm{C}_*(X, A)$$

is a quasi-isomorphism. By the discussion on page 489 we know that the relative chain groups are free abelian groups. Thus it follows from the Quasi Iso–Homotopy Equivalence Proposition 50.4 that the above chain map is in fact a chain homotopy equivalence. □

Proof. We postpone the proof to the next section. ■

Before we turn to the proof of the Quasi Iso–Homotopy Equivalence Proposition 50.4, let us formulate the following corollary:

Corollary 50.5. (Two-Out-Of-Three-Homotopy Equivalence Corollary) Let

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_* & \longrightarrow & B_* & \longrightarrow & C_* & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A'_* & \longrightarrow & B'_* & \longrightarrow & C'_* & \longrightarrow & 0 \end{array}$$

be a commutative diagram of short exact sequences of chain complexes of free abelian groups. If two of the vertical maps are chain homotopy equivalences, then the third is also a chain homotopy equivalence.

Proof. We will prove this corollary in Exercise Sheet 1. ■

50.3. **Proof of the Quasi Iso–Homotopy Equivalence Proposition 50.4.** This section is completely dedicated to the proof of the Quasi Iso–Homotopy Equivalence Proposition 50.4. We start out with the following definition which associates to a chain map a chain complex.

Definition. Let (C_*, ∂^C_*) and (D_*, ∂^D_*) be two chain complexes and let $\varphi\colon C_* \to D_*$ be a chain map. The algebraic mapping cone of φ is the chain complex $(\mathrm{M}(\varphi), \partial_*)$ that is given by the chain groups

$$\mathrm{M}(\varphi)_n \;:=\; \begin{matrix} C_{n-1} \\ D_n \end{matrix} \;:=\; C_{n-1} \oplus D_n$$

and the boundary maps[408]

$$\partial_n\colon \begin{array}{c} \overbrace{\phantom{C_{n-1}}}^{\mathrm{M}(\varphi)_n} \\ C_{n-1} \\ D_n \\ x \\ y \end{array} \;\begin{array}{c} \to \\ \\ \\ \mapsto \\ \\ \end{array}\; \begin{array}{c} \overbrace{\phantom{C_{n-2}}}^{\mathrm{M}(\varphi)_{n-1}} \\ C_{n-2} \\ D_{n-1} \\ -\partial^C_{n-1}(x) \\ \varphi_{n-1}(x) + \partial^D_n(y) \end{array}$$

which we write as block matrices

$$\begin{array}{cc} C_{n-1} & D_n \\ \end{array}$$
$$\begin{pmatrix} -\partial^C_{n-1} & 0 \\ \varphi_{n-1} & \partial^D_n \end{pmatrix} \begin{array}{c} C_{n-2} \\ D_{n-1} \end{array}$$

Lemma 50.6. (Mapping Cone-LES Lemma) Let $\varphi\colon C_* \to D_*$ be a chain map between chain complexes (C_*, ∂_*^C) and (D_*, ∂_*^D). We denote by $i_n\colon D_n \to \mathrm{M}(\varphi)_n = C_{n-1} \oplus D_n$ the obvious natural inclusion map and we denote by $p_n\colon \mathrm{M}(\varphi)_n = C_{n-1} \oplus D_n \to C_{n-1}$ the obvious natural projection. These maps induce maps on homology groups and the following sequence is exact

$$\ldots \to \mathrm{H}_n(C_*) \xrightarrow{\varphi_*} \mathrm{H}_n(D_*) \xrightarrow{i_*} \mathrm{H}_n(\mathrm{M}(\varphi)) \xrightarrow{p_*} \mathrm{H}_{n-1}(C_*) \xrightarrow{\varphi_*} \mathrm{H}_{n-1}(D_*) \to \ldots$$

Proof. For each $n \in \mathbb{N}_0$ we consider the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & D_n & \xrightarrow{i_n} & \overbrace{\begin{matrix} C_{n-1} \\ D_n \end{matrix}}^{=\mathrm{M}(\varphi)_n} & \xrightarrow{p_n} & C_{n-1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \partial_n^D} & & \downarrow{\scriptstyle \partial_n \begin{pmatrix} -\partial_{n-1}^C & 0 \\ \varphi_{n-1} & \partial_n^D \end{pmatrix}} & & \downarrow{\scriptstyle \partial_{n-1}^C} & & \\
0 & \longrightarrow & D_{n-1} & \xrightarrow[i_{n-1}]{} & \underbrace{\begin{matrix} C_{n-2} \\ D_{n-1} \end{matrix}}_{=\mathrm{M}(\varphi)_{n-1}} & \xrightarrow[p_{n-1}]{} & C_{n-2} & \longrightarrow & 0.
\end{array}$$

We make the following observations:

(1) The horizontal sequences are exact.
(2) The left-hand square commutes, i.e. the maps i_* define a chain map.
(3) The right-hand square commutes up to a minus sign, i.e. we have

$$\partial_{n-1}^C \circ p_n \;=\; -\,p_{n-1} \circ \partial_n.$$

Fortunately the minus sign in the above equality does not change the fact that the maps p_n induce a map between homology groups.

Despite the minus sign in (3) we can use verbatim the same construction as on page 483 to obtain connecting homomorphisms $\partial_n\colon \mathrm{H}_n(D_*) \to \mathrm{H}_n(C_*)$, $n \in \mathbb{N}_0$, and verbatim the same proof as in the LES-Proposition 37.4 shows that these connecting homomorphisms give rise to a long exact sequence of the form

$$\ldots \to \mathrm{H}_n(C_*) \xrightarrow{\partial_n} \mathrm{H}_n(D_*) \xrightarrow{i_*} \mathrm{H}_n(\mathrm{M}(\varphi)) \xrightarrow{p_*} \mathrm{H}_{n-1}(C_*) \xrightarrow{\partial_{n-1}} \mathrm{H}_{n-1}(D_*) \to \ldots$$

Keeping the above diagram in mind one sees that it follows easily from the definition of the connecting homomorphism on page 483 that for each $n \in \mathbb{N}_0$ we have the equality[409]

$$\partial_n \;=\; \varphi_*\colon \mathrm{H}_n(C_*) \;\to\; \mathrm{H}_n(D_*).$$

We have thus shown that the sequence in the statement of the lemma is exact. ∎

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[408] Note that

$$\partial_{n-1}\circ\partial_n \;=\; \begin{pmatrix} -\partial_{n-2}^C & 0 \\ \varphi_{n-2} & \partial_{n-1}^D \end{pmatrix}\begin{pmatrix} -\partial_{n-1}^C & 0 \\ \varphi_{n-1} & \partial_n^D \end{pmatrix} = \begin{pmatrix} \partial_{n-2}^C\circ\partial_{n-1}^C & 0 \\ -\varphi_{n-2}\circ\partial_{n-1}^C + \partial_{n-1}^D\circ\varphi_{n-1} & \partial_{n-1}^D\circ\partial_n^D \end{pmatrix} \underset{\uparrow}{=} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

since φ is a chain map, since $\partial_{n-2}^C \circ \partial_{n-1}^C = 0$ and since $\partial_{n-1}^D \circ \partial_n^D = 0$

This shows that we have indeed defined a chain complex. This calculation also shows the necessity of the $-$-sign in the definition of the boundary map of the algebraic mapping cone.

[409] It is worth verifying that $\partial_n = \varphi_*$.

Now we are finally fully prepared to provide the proof of the Quasi Iso–Homotopy Equivalence Proposition 50.4.

Proof of the Quasi Iso–Homotopy Equivalence Proposition 50.4. Let $\varphi\colon C_* \to D_*$ be a chain map between chain complexes (C_*, ∂^C_*) and (D_*, ∂^D_*) of free abelian groups which is a quasi-isomorphism, i.e. which induces an isomorphism on all homology groups. We need to show that φ is in fact a chain homotopy equivalence.

As above we denote by $(\mathrm{M}(\varphi)_*, \partial_*)$ the algebraic mapping cone of φ. We consider the long exact sequence from the Mapping Cone-LES Lemma 50.6:

$$\ldots \to \mathrm{H}_n(C_*) \xrightarrow[\uparrow]{\varphi_*} \mathrm{H}_n(D_*) \xrightarrow{i_*} \mathrm{H}_n(\mathrm{M}(\varphi)) \xrightarrow{p_*} \mathrm{H}_{n-1}(C_*) \xrightarrow[\uparrow]{\varphi_*} \mathrm{H}_{n-1}(D_*) \to \ldots$$

isomorphism by hypothesis — isomorphism by hypothesis

By our hypothesis the maps φ induce isomorphisms of homology groups, thus we see that $\mathrm{H}_n(\mathrm{M}(\varphi)) = 0$ for all $n \geq 0$. By the Identity-of-Acyclic-is-Null Homotopic Corollary 50.3 this means that we have a chain homotopy $\{P_n\colon \mathrm{M}(\varphi)_n \to \mathrm{M}(\varphi)_{n+1}\}_{n\geq 0}$ from the identity map of $\mathrm{M}(\varphi)_*$ to the zero map. (Note that here we use that by our hypothesis on C_* and D_* each $\mathrm{M}(\varphi)_n = C_{n-1} \oplus D_n$ is a free abelian group.)

Recall that this means that for each $n \in \mathbb{N}_0$ we have the equality

$$(*) \qquad P_{n-1} \circ \partial_n + \partial_{n+1} \circ P_n \;=\; \mathrm{id}_{C_{n-1}\oplus D_n} - 0.$$

Using the obvious matrix notation we write

$$P_n \;=:\; \begin{array}{c} \\ C_n \\ D_{n+1} \end{array} \begin{array}{c} \begin{array}{cc} C_{n-1} & D_n \end{array} \\ \begin{pmatrix} P_n^{11} & P_n^{12} \\ P_n^{21} & P_n^{22} \end{pmatrix} \end{array} : \underbrace{\begin{array}{c} C_{n-1} \\ D_n \end{array}}_{=\mathrm{M}(\varphi)_n} \to \underbrace{\begin{array}{c} C_n \\ D_{n+1}. \end{array}}_{=\mathrm{M}(\varphi)_{n+1}}$$

Now we can rewrite the above equality $(*)$ and we obtain that

$$\begin{pmatrix} P_{n-1}^{11} & P_{n-1}^{12} \\ P_{n-1}^{21} & P_{n-1}^{22} \end{pmatrix} \circ \begin{pmatrix} -\partial^C_{n-1} & 0 \\ \varphi_{n-1} & \partial^D_n \end{pmatrix} + \begin{pmatrix} -\partial^C_n & 0 \\ \varphi_n & \partial^D_{n+1} \end{pmatrix} \circ \begin{pmatrix} P_n^{11} & P_n^{12} \\ P_n^{21} & P_n^{22} \end{pmatrix} = \begin{pmatrix} \mathrm{id}_{C_{n-1}} & 0 \\ 0 & \mathrm{id}_{D_n} \end{pmatrix}.$$

Looking at the four entries separately we obtain the following four equalities:

$$\begin{aligned}
&(a) && -P^{11}_{n-1} \circ \partial^C_{n-1} + P^{12}_{n-1} \circ \varphi_{n-1} - \partial^C_n \circ P^{11}_n &&= \mathrm{id}_{C_{n-1}},\\
&(b) && P^{12}_{n-1} \circ \partial^D_n - \partial^C_n \circ P^{12}_n &&= 0,\\
&(c) && -P^{21}_{n-1} \circ \partial^C_{n-1} + P^{22}_{n-1} \circ \varphi_{n-1} + \varphi_n \circ P^{11}_n + \partial^D_{n+1} \circ P^{21}_n &&= 0,\\
&(d) && P^{22}_{n-1} \circ \partial^D_n + \varphi_n \circ P^{12}_n + \partial^D_{n+1} \circ P^{22}_n &&= \mathrm{id}_{D_n}.
\end{aligned}$$

We make the following observation:

(1) Equality (b) implies that the homomorphisms $\{P^{12}_n\colon D_n \to C_n\}_{n\in\mathbb{N}_0}$ define a chain map $D \to C$.

Now we claim that the chain map $\{P^{12}_n\}_{n\in\mathbb{N}_0}$ is a chain homotopy inverse to the chain map φ. We do so with the following two observations:

(2) Equality (a) implies that $P^{12} \circ \varphi \simeq \mathrm{id}_{C_*}$ where the chain homotopy is given by the maps $P^{11}_n\colon C_{n-1} \to C_n$.

(3) Equality (d) implies that $\varphi \circ P^{12} \simeq \mathrm{id}_{D_*}$, where the chain homotopy is given by the maps $P^{22}_n\colon D_n \to D_{n+1}$,

The last two statements together imply that φ is indeed a chain homotopy equivalence. ■

50.4. **Singular and cellular chain complexes.** In Chapter 47 we introduced the cellular chain complex $\mathrm{C}_*^{\mathrm{CW}}(X)$ and the cellular homology groups $\mathrm{H}_k^{\mathrm{CW}}(X)$ of a CW-complex X. In the Singular-Cellular-H_*-Proposition 47.3 we showed that the cellular homology groups $\mathrm{H}_k^{\mathrm{CW}}(X)$ are naturally isomorphic to the singular homology groups $\mathrm{H}_k(X)$. At that point we did not say anything about the relationship between the cellular chain complex $\mathrm{C}_*^{\mathrm{CW}}(X)$ and the singular chain complex $\mathrm{C}_*(X)$. We will rectify this in this section. The extra information that we gain in this section will stand us in good stead when we explore the relationship between cellular and singular homology with coefficients, and again when we study the relationship between cellular and singular cohomology.

First we recall some definitions and some notation, mostly from page 588.

Definition. Let X be a CW-complex and let $n \in \mathbb{N}_0$.

(1) We denote by X^n the n-skeleton of X.

(2) We write $\mathrm{C}_n^{\mathrm{CW}}(X) := \mathrm{H}_n(X^n, X^{n-1})$ and we denote by $d_n \colon \mathrm{C}_n^{\mathrm{CW}}(X) \to \mathrm{C}_{n-1}^{\mathrm{CW}}(X)$ the cellular boundary map from page 588, i.e. d_n is the homomorphism that is given by the composition of the following two homomorphisms:

$$\mathrm{H}_n(X^n, X^{n-1}) \xrightarrow[\uparrow]{\partial_n} \mathrm{H}_{n-1}(X^{n-1}) \xrightarrow{\not\imath} \mathrm{H}_{n-1}(X^{n-1}, X^{n-2}).$$

connecting homomorphism of the long exact sequence of the pair (X^n, X^{n-1})

In the Cellular Boundary Map Lemma 47.2 we showed that for every $n \in \mathbb{N}_0$ we have $d_n \circ d_{n+1} = 0$.

(3) We refer to

$$\ldots \xrightarrow{d_{n+2}} \mathrm{C}_{n+1}^{\mathrm{CW}}(X) \xrightarrow{d_{n+1}} \mathrm{C}_n^{\mathrm{CW}}(X) \xrightarrow{d_n} \mathrm{C}_{n-1}^{\mathrm{CW}}(X) \xrightarrow{d_{n-1}} \ldots$$

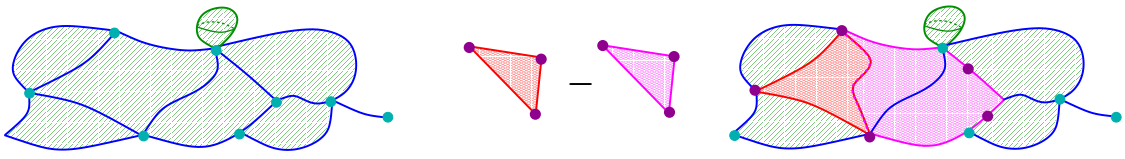
as the cellular chain complex. Its homology groups are denoted by $\mathrm{H}_n^{\mathrm{CW}}(X)$.

In the following we want to relate the cellular chain complex $\mathrm{C}_*^{\mathrm{CW}}(X)$ to the singular chain complex $\mathrm{C}_*(X)$. There is no obvious "interesting" chain map between these two chain complexes. But as we will see in a minute, it turns out that there is a nice "intermediate" chain complex which maps to both of the above chain complexes:

Definition. Given a CW-complex X and given $n \in \mathbb{N}_0$ we define the intermediate cellular chain complex[410]

$$\mathrm{C}_n^{\mathrm{int}}(X) := \ker\left(\mathrm{C}_n(X^n) \xrightarrow{\partial_n} \mathrm{C}_{n-1}(X^n) \xrightarrow{\pi_{n-1}} \mathrm{C}_{n-1}(X^n, X^{n-1})\right)$$

where $\pi_{n-1} \colon \mathrm{C}_{n-1}(X^n) \to \mathrm{C}_{n-1}(X^n, X^{n-1})$ is the natural projection.



CW-complex X element of $\mathrm{C}_2^{\mathrm{int}}(X)$

[410]Strictly speaking we only show in the Intermediate-Chain Complex-Lemma 50.7 that this is indeed a chain complex.

The following lemma summarizes a few basic properties of the intermediate cellular chain complexes.

Lemma 50.7. (Intermediate-Chain Complex-Lemma)

(1) Given any CW-complex X the intermediate cellular chain complex $\mathrm{C}_*^{\mathrm{int}}(X)$ is a subcomplex of the singular chain complex $\mathrm{C}_*(X)$.
(2) If $f\colon X \to Y$ is a cellular map between two CW-complexes, then the induced map $f_*\colon \mathrm{C}_*(X) \to \mathrm{C}_*(Y)$ restricts to a chain map $\mathrm{C}_*^{\mathrm{int}}(X) \to \mathrm{C}_*^{\mathrm{int}}(Y)$.
(3) For each $k \in \mathbb{N}_0$ the maps $X \mapsto \mathrm{H}_k(\mathrm{C}_*^{\mathrm{int}}(X))$ and $f \mapsto f_*$ define a covariant functor from the category $\mathcal{CW}$ of CW-complexes to the category $\mathcal{AbGroup}$ of abelian groups.

Proof.

(1) First note that by definition we have for any $k \in \mathbb{N}_0$ the equality

$$(*) \qquad \mathrm{C}_k^{\mathrm{int}}(X) \;=\; \{\sigma \in \mathrm{C}_k(X^k) \,|\, \partial_k(\sigma) \in \mathrm{C}_{k-1}(X^{k-1})\}.$$

Now let $\sigma \in \mathrm{C}_n^{\mathrm{int}}(X)$. We need to show that $\partial_n(\sigma) \in \mathrm{C}_{n-1}^{\mathrm{int}}(X)$.
(a) By $(*)$ we know that $\partial_n(\sigma) \in \mathrm{C}_{n-1}(X^{n-1})$.
(b) Note that $\partial_{n-1}(\partial_n(\sigma)) = (\partial_{n-1} \circ \partial_n)(\sigma) = 0$. It follows from this observation that $\partial_{n-1}(\partial_n(\sigma)) \in \mathrm{C}_{n-2}(X^{n-2})$.
(c) The combination of (a), (b) and $(*)$ shows that $\partial_n(\sigma) \in \mathrm{C}_{n-1}^{\mathrm{int}}(X)$.

(2) Let $f\colon X \to Y$ be a cellular map between two CW-complexes. We obtain the following diagram:

$$\begin{array}{ccccc}
\mathrm{C}_n(X^n) & \xrightarrow{\partial_n} & \mathrm{C}_{n-1}(X^n) & \xrightarrow{\pi_{n-1}} & \mathrm{C}_{n-1}(X^n, X^{n-1}) \\
\downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} \\
\mathrm{C}_n(Y^n) & \xrightarrow{\partial_n} & \mathrm{C}_{n-1}(Y^n) & \xrightarrow{\pi_{n-1}} & \mathrm{C}_{n-1}(Y^n, Y^{n-1}).
\end{array}$$

It follows from the C_*–Functor Lemma 35.5 that this diagram commutes. It follows immediately that $f_*\colon \mathrm{C}_*(X) \to \mathrm{C}_*(Y)$ restricts to a chain map $\mathrm{C}_*^{\mathrm{int}}(X) \to \mathrm{C}_*^{\mathrm{int}}(Y)$.

(3) This statement follows easily from the definitions. ∎

As a standalone invariant the intermediate chain complexes are not particular interesting. But as we will see in the following proposition, the intermediate chain complexes have interesting chain maps to the singular chain complex and to the cellular chain complex. The following proposition also gives us in particular a new proof of the main statements of the Singular-Cellular-H_*-Proposition 47.3.

Proposition 50.8. (Intermediate Chain Complex Proposition) Let X be a CW-complex.

- We denote by $i\colon \mathrm{C}_*^{\mathrm{int}}(X) \to \mathrm{C}_*(X)$ the natural inclusion map.
- We consider the natural map

$$\begin{array}{rcl} p\colon \mathrm{C}_*^{\mathrm{int}}(X) & \to & \mathrm{C}_*^{\mathrm{CW}}(X) = \mathrm{H}_n(X^n, X^{n-1}) \\ \sigma & \mapsto & [\pi_n(\sigma)]. \end{array}$$

For each $n \in \mathbb{N}_0$ the natural maps

(1) $i_*\colon \mathrm{H}_n(\mathrm{C}_*^{\mathrm{int}}(X)) \to \underbrace{\mathrm{H}_n(\mathrm{C}_*(X))}_{=\mathrm{H}_n(X)}$ and (2) $p_*\colon \mathrm{H}_n(\mathrm{C}_*^{\mathrm{int}}(X)) \to \underbrace{\mathrm{H}_n(\mathrm{C}_*^{\mathrm{CW}}(X))}_{=\mathrm{H}_n^{\mathrm{CW}}(X)}$

are isomorphisms.

Proof. We outsource the lengthy and dull proof of the proposition to the next section. ■

The following proposition gives us the promised relationship between the singular and cellular chain complex of a given CW-complex.

Proposition 50.9. (Singular-Cellular Chain Complex Proposition) Let X be a CW-complex.

(1) The two natural chain maps
$$i\colon \mathrm{C}_*^{\mathrm{int}}(X) \hookrightarrow \mathrm{C}_*(X) \quad \text{and} \quad p\colon \mathrm{C}_*^{\mathrm{int}}(X) \twoheadrightarrow \mathrm{C}_*^{\mathrm{CW}}(X)$$
from the Intermediate Chain Complex Proposition 50.8, that are given by the natural inclusion and the natural projection, are chain homotopy equivalences.

(2) There exists a[411] chain homotopy equivalence $\mathrm{C}_*(X) \to \mathrm{C}_*^{\mathrm{CW}}(X)$.

Proof. Let X be a CW-complex.

(1) By definition each $\mathrm{C}_k^{\mathrm{int}}(X)$ is a subgroup of the free abelian group $\mathrm{C}_k(X)$. It follows from the Free Abelian Subgroup Theorem 18.7 that each $\mathrm{C}_k^{\mathrm{int}}(X)$ itself is a free abelian group. In particular $\mathrm{C}_*^{\mathrm{int}}(X)$ is a chain complex of free abelian groups. By the Intermediate Chain Complex Proposition 50.8 we know that the chain maps $i\colon \mathrm{C}_*^{\mathrm{int}}(X) \hookrightarrow \mathrm{C}_*(X)$ and $p\colon \mathrm{C}_*^{\mathrm{int}}(X) \twoheadrightarrow \mathrm{C}_*^{\mathrm{CW}}(X)$ are quasi-isomorphisms. Since all chain complexes involved consist of free abelian groups we deduce from the Quasi Iso–Homotopy Equivalence Proposition 50.4 that the two chain maps are both chain homotopy equivalences.

(2) Recall that on page 472 we pointed out that it follows from the Chain Homotopy Equivalence-Reformulation Lemma 36.3 and the Isomorphism Lemma 3.3 that the relation "chain homotopy equivalence" of chain complexes is an equivalence relation on the class of all chain complexes. The desired statement follows immediately from this observation and Statement (1). ■

50.5. **Proof of the Intermediate Chain Complex Proposition 50.8.** We prove the two parts of the Intermediate Chain Complex Proposition 50.8 separately.

Proof of Proposition 50.8 (1). Let X be a CW-complex and let $n \in \mathbb{N}_0$. We need to show that the map
$$i_*\colon \mathrm{H}_n(\mathrm{C}_*^{\mathrm{int}}(X)) \to \mathrm{H}_n(\mathrm{C}_*(X))$$
is an isomorphism. Before we start, let us recall that in the Skeleton-H_*-Lemma 47.1 (3) and (4) we proved the following two convenient facts:

(a) The inclusion $k''\colon X^{n+1} \to X$ induces an isomorphism $k''_*\colon \mathrm{H}_n(X^{n+1}) \xrightarrow{\cong} \mathrm{H}_n(X)$.
(b) The inclusion $k'\colon X^n \to X$ induces an epimorphism $k'_*\colon \mathrm{H}_n(X^n) \twoheadrightarrow \mathrm{H}_n(X)$.

Claim. Let $n \in \mathbb{N}_0$.
(i) The map $i_*\colon \mathrm{H}_n(\mathrm{C}_*^{\mathrm{int}}(X)) \to \mathrm{H}_n(\mathrm{C}_*(X)) = \mathrm{H}_n(X)$ is a monomorphism.
(ii) The map $i_*\colon \mathrm{H}_n(\mathrm{C}_*^{\mathrm{int}}(X)) \to \mathrm{H}_n(\mathrm{C}_*(X)) = \mathrm{H}_n(X)$ is an epimorphism.

[411] Note that we do not claim that there exists a *natural* chain homotopy equivalence.

Proof. We consider the following two commutative diagrams

$$\begin{array}{ccccc} \mathrm{C}_{n+1}^{\mathrm{int}}(X) & \xhookrightarrow{j} & \mathrm{C}_{n+1}(X^{n+1}) & \xhookrightarrow{k''_*} & \mathrm{C}_{n+1}(X) \\ \downarrow{\scriptstyle \partial_{n+1}} & & \downarrow{\scriptstyle \partial_{n+1}} & & \downarrow{\scriptstyle \partial_{n+1}} \\ \mathrm{C}_n^{\mathrm{int}}(X) & \xhookrightarrow[i]{j} & \mathrm{C}_n(X^{n+1}) & \xhookrightarrow{k''_*} & \mathrm{C}_n(X) \end{array} \quad \text{and} \quad \begin{array}{ccccc} \mathrm{C}_n^{\mathrm{int}}(X) & \xhookrightarrow[j]{} & \mathrm{C}_n(X^n) & \xhookrightarrow[k'_*]{} & \mathrm{C}_n(X) \\ \downarrow{\scriptstyle \partial_n} & & \downarrow{\scriptstyle \partial_n} & & \downarrow{\scriptstyle \partial_n} \\ \mathrm{C}_{n-1}^{\mathrm{int}}(X) & \xhookrightarrow{j} & \mathrm{C}_{n-1}(X^n) & \xhookrightarrow{k'_*} & \mathrm{C}_{n-1}(X). \end{array}$$

Here in each diagram the maps to the left are the obvious inclusion maps; for better or worse we refer to all of them as j. The maps to the right are induced by the inclusions $k'' \colon X^{n+1} \to X$ and $k' \colon X^n \to X$. As we will now see, both statements of the claim follow easily from the diagrams together with (a) and (b) above:

(i) We consider the diagram to the left. Let $\sigma \in \mathrm{C}_n^{\mathrm{int}}(X)$ be a cycle with the property that $i(\sigma) \in \mathrm{C}_n(X)$ is null-homologous. It follows from (a) that $j(\sigma) \in \mathrm{C}_n(X^{n+1})$ is already null-homologous, i.e. there exists a $\tau \in \mathrm{C}_{n+1}(X^{n+1})$ with $\partial\tau = j(\sigma)$. By definition of $\mathrm{C}_n^{\mathrm{int}}(X)$ we know that $\sigma \in \mathrm{C}_n(X^n) \subset \mathrm{C}_n(X^{n+1})$. Thus we see that τ actually lies in
$$\mathrm{C}_{n+1}^{\mathrm{int}}(X) \;=\; \ker\big(\mathrm{C}_{n+1}(X^{n+1}) \xrightarrow{\partial} \mathrm{C}_n(X^{n+1}) \xrightarrow{\pi_n} \mathrm{C}_n(X^{n+1}, X^n)\big).$$
But this implies that σ is null-homologous in $\mathrm{C}_*^{\mathrm{int}}(X)$.

(ii) We consider the diagram to the right. Let $[\sigma] \in \mathrm{H}_n(X)$. By (b) we know that there exists a cycle $\tau \in \mathrm{C}_n(X^n)$ with $[k'_*(\tau)] = [\sigma]$. Since $\partial\tau = 0$ we see that $\tau \in \mathrm{C}_n^{\mathrm{int}}(X)$. Since the bottom-left horizontal map is a monomorphism we see that $\tau \in \mathrm{C}_n^{\mathrm{int}}(X)$ is also a cycle. But this shows that $[\sigma] = [i(\tau)]$. ∎

Proof of Proposition 50.8 (2). Let X be a CW-complex. For orientation purposes it is convenient to keep the following simple commutative diagram in mind:

$$\begin{array}{ccc} \mathrm{C}_n(X^n) & \xrightarrow{\pi_n} & \mathrm{C}_n(X^n, X^{n-1}) \\ \downarrow{\scriptstyle \partial_n} & & \downarrow{\scriptstyle \partial_n} \\ \mathrm{C}_{n-1}(X^n) & \xrightarrow{\pi_{n-1}} & \mathrm{C}_{n-1}(X^n, X^{n-1}). \end{array}$$

Furthermore it is convenient to recall the following definitions:

$$\begin{aligned} \mathrm{C}_n^{\mathrm{int}}(X) &= \ker(\pi_{n-1} \circ \partial_n \colon \mathrm{C}_n(X^n) \to \mathrm{C}_{n-1}(X^n, X^{n-1})) \\ \mathrm{C}_n^{\mathrm{CW}}(X) &= \mathrm{H}_n(X^n, X^{n-1}) = \mathrm{H}_n(\mathrm{C}_*(X^n, X^{n-1})). \end{aligned}$$

Claim 0. The maps
$$\begin{aligned} p \colon \mathrm{C}_*^{\mathrm{int}}(X) &\to \mathrm{C}_*^{\mathrm{CW}}(X) = \mathrm{H}_n(X^n, X^{n-1}) \\ \sigma &\mapsto [\pi(\sigma)] \end{aligned}$$
are well-defined, in the sense that $[\pi(\sigma)]$ is a cycle in $\mathrm{C}_*^{\mathrm{CW}}(X)$, and these maps form a chain map.

Proof. We consider the above commutative diagram. If $\sigma \in \mathrm{C}_n^{\mathrm{int}}(X)$, then the bottom-right map is by definition zero. Since the diagram commutes we see that the right-bottom map is also zero. But this means precisely that $\pi_n(\sigma)$ is a cycle in $\mathrm{C}_n(X^n, X^{n-1})$, thus $\pi_n(\sigma)$ defines an element in $\mathrm{C}_*^{\mathrm{CW}}(X) = \mathrm{H}_n(X^n, X^{n-1})$. We leave it to the reader to verify that the maps $p \colon \mathrm{C}_*^{\mathrm{int}}(X) \to \mathrm{C}_*^{\mathrm{CW}}(X) = \mathrm{H}_n(X^n, X^{n-1})$ form a chain map. ⊞

Now let $n \in \mathbb{N}_0$. We need to show that the map

$$\begin{array}{rcl} p_*\colon \mathrm{H}_n(\mathrm{C}_*^{\mathrm{int}}(X)) & \to & \mathrm{H}_n(\mathrm{C}_*^{\mathrm{CW}}(X)) \\ [\sigma] & \mapsto & [\pi_n(\sigma)] \end{array}$$

is an isomorphism.

Claim 1. For each $n \in \mathbb{N}_0$ the map $p\colon \mathrm{C}_n^{\mathrm{int}}(X) \to \mathrm{C}_n^{\mathrm{CW}}(X)$ is an epimorphism.

Proof. Let $[\sigma] \in \mathrm{C}_n^{\mathrm{CW}}(X)$, i.e. let $\sigma \in \mathrm{C}_n^{\mathrm{CW}}(X)$ such that $\partial_n(\sigma) = 0$. Since the projection $\pi_n\colon \mathrm{C}_n(X^n) \to \mathrm{C}_n(X^n, X^{n-1})$ is an epimorphism there exists a $\nu \in \mathrm{C}_n(X^n)$ with $\partial_n(\nu) = \sigma$. Using the above commutative diagram one sees easily that $\nu \in \mathrm{C}_n^{\mathrm{int}}(X)$. ⊞

Given $n \in \mathbb{N}_0$ we set

$$D_n \coloneqq \ker\left\{p\colon \mathrm{C}_n^{\mathrm{int}}(X) \xrightarrow{p} \mathrm{C}_n^{\mathrm{CW}}(X)\right\}.$$

Note that $\nu \in \mathrm{C}_n(X^n)$ lies in D_n if and only if the following two conditions are satisfied:

(a) $(\pi_{n-1} \circ \partial_n)(\nu) = 0$,

(b) $\pi_n(\nu) \in \mathrm{C}_n(X^n, X^{n-1})$ represents the zero element of $\mathrm{C}_n^{\mathrm{CW}}(X) = \mathrm{H}_n(X^n, X^{n-1})$.

Furthermore note that the groups D_* form a subcomplex of $\mathrm{C}_*^{\mathrm{int}}(X)$.

Claim 2. For each $n \in \mathbb{N}_0$ we have $\mathrm{C}_{n+1}(X^n) \subset D_{n+1}$.

Proof. For orientation purposes we consider the following commutative diagram:

$$\begin{array}{ccccc} \mathrm{C}_{n+1}(X^n) & \hookrightarrow & \mathrm{C}_{n+1}(X^{n+1}) & \xrightarrow{\pi_{n+1}} & \mathrm{C}_{n+1}(X^{n+1}, X^n) \\ \downarrow{\scriptstyle \partial_{n+1}} & & \downarrow{\scriptstyle \partial_{n+1}} & & \downarrow{\scriptstyle \partial_{n+1}} \\ \mathrm{C}_n(X^n) & \hookrightarrow & \mathrm{C}_n(X^{n+1}) & \xrightarrow{\pi_n} & \mathrm{C}_n(X^{n+1}, X^n). \end{array}$$

Let $\Theta \in \mathrm{C}_n(X^{n+1})$. Since the diagram commutes and since the horizontal sequences are exact we see that Θ satisfies (a) and we see that $\pi_{n+1}(\Theta) = 0 \in \mathrm{C}_{n+1}(X^{n+1}, X^n)$, the latter statement implies that (b) is also satisfied. ⊞

Note that by Claim 1 we have a short exact sequence

$$0 \to D_* \to \mathrm{C}_n^{\mathrm{int}}(X) \xrightarrow{p} \mathrm{C}_n^{\mathrm{CW}}(X) \to 0$$

of chain complexes. Considering the corresponding long exact sequence of homology groups, see the LES-Proposition 37.4, it suffices to prove the following claim:

Claim 3. The chain complex D_* is acyclic, i.e. all homology groups of D_* are zero.

Proof. For orientation purposes we consider the following commutative diagram:

$$\begin{array}{ccccc} \mathrm{C}_{n+1}(X^{n-1}) & \xrightarrow{i} & \mathrm{C}_{n+1}(X^n) & \xrightarrow{\pi_{n+1}} & \mathrm{C}_{n+1}(X^n, X^{n-1}) \\ \downarrow{\scriptstyle \partial_{n+1}} & & \downarrow{\scriptstyle \partial_{n+1}} & & \downarrow{\scriptstyle \partial_{n+1}} \\ \mathrm{C}_n(X^{n-1}) & \xrightarrow{i} & \mathrm{C}_n(X^n) & \xrightarrow{\pi_n} & \mathrm{C}_n(X^n, X^{n-1}) \\ \downarrow{\scriptstyle \partial_n} & & \downarrow{\scriptstyle \partial_n} & & \downarrow{\scriptstyle \partial_n} \\ \mathrm{C}_{n-1}(X^{n-1}) & \xrightarrow{i} & \mathrm{C}_{n-1}(X^n) & \xrightarrow{\pi_{n-1}} & \mathrm{C}_{n-1}(X^n, X^{n-1}). \end{array}$$

So let $\mu \in \ker(\partial_n\colon D_n \to D_{n-1}) \subset D_n \subset \mathrm{C}_n^{\mathrm{int}}(X) \subset \mathrm{C}_n(X)$. We need to show that there exists a $\Theta \in D_{n+1}$ with $\partial_{n+1}(\Theta) = \mu$. By (b) we know that $\pi_n(\mu) \in \mathrm{C}_n(X^n, X^{n-1})$ represents the zero element of $\mathrm{C}_n^{\mathrm{CW}}(X) = \mathrm{H}_n(X^n, X^{n-1})$. It follows that there exists a $\gamma \in \mathrm{C}_{n+1}(X^n, X^{n-1})$ with $\partial_{n+1}(\gamma) = \pi_n(\mu)$. Since the top horizontal sequence is exact we can pick $\widetilde{\gamma} \in \mathrm{C}_{n+1}(X^n)$ with $\pi_{n+1}(\widetilde{\gamma}) = \gamma$. Note that we have the following equality in

$\mathrm{C}_n(X^n, X^{n-1})$:

$$\pi_n(\partial_{n+1}(\widetilde{\gamma}) - \mu) \underset{\substack{\uparrow \\ \text{since } \partial_{n+1}\circ\pi_{n+1} = \pi_n\circ\partial_{n+1}}}{=} (\pi_n \circ \partial_{n+1})(\widetilde{\gamma}) - \pi(\mu) = \partial_{n+1}\underbrace{(\pi_{n+1}(\widetilde{\gamma}))}_{=\gamma} - \pi_n(\mu) \underset{\substack{\uparrow \\ \text{since } \partial_{n+1}(\gamma) = \pi_n(\mu)}}{=} 0.$$

Since the middle horizontal sequence is exact there exists an $\alpha \in \mathrm{C}_n(X^{n-1})$ such that $i(\alpha) = \partial_{n+1}(\widetilde{\gamma}) - \mu$. Note that

$$i(\partial_n(\alpha)) = \partial_n(i(\alpha)) = i(\partial_n(\partial_{n+1}(\widetilde{\gamma}) - \mu)) = i(\underbrace{(\partial_n \circ \partial_{n+1})}_{=0}(\widetilde{\gamma})) - i(\underbrace{\partial_n(\mu)}_{\substack{=0, \text{ since} \\ \mu \in \ker(\partial_n)}}) = 0.$$

Since i is a monomorphism we see that $\partial_n(\alpha) = 0$. As we mentioned above, by the Skeleton-H_*-Lemma 47.1 (2) we know that $\mathrm{H}_n(X^{n-1}) = 0$. This implies that there exists an $\widetilde{\alpha} \in \mathrm{C}_{n+1}(X^{n-1})$ with $\partial_{n+1}(\widetilde{\alpha}) = \alpha$. Then

$$\partial_{n+1}(\widetilde{\gamma} - i(\widetilde{\alpha})) \underset{\substack{\uparrow \\ \text{since } \partial_{n+1}\circ i = i\circ\partial_{n+1}}}{=} \partial_{n+1}(\widetilde{\gamma}) - (i \circ \partial_{n+1})(\widetilde{\alpha}) = \partial_{n+1}(\widetilde{\gamma}) - \underbrace{i(\alpha)}_{=\partial_{n+1}(\widetilde{\gamma})-\mu} = \partial_{n+1}(\widetilde{\gamma}) - \partial_{n+1}(\widetilde{\gamma}) + \mu = \mu.$$

We set $\Theta \coloneqq \widetilde{\gamma} - i(\widetilde{\alpha}) \in \mathrm{C}^{n+1}(X^n)$. We just showed that $\partial_{n+1}(\Theta) = \mu$. Since $\Theta \in \mathrm{C}^{n+1}(X^n)$ we obtain from Claim 2 that Θ actually lies in D_{n+1}. ∎

50.6. **Excisive triads.** In this section we introduce the notion of an "excisive triad". The name sounds mysterious. It turns out that the concept is a little technical, but it will appear on many occasions. The Quasi Iso–Homotopy Equivalence Proposition 50.4, which we just proved, will help us to prove a characterisation of excisive triads, which we will use on many occasions in the future.

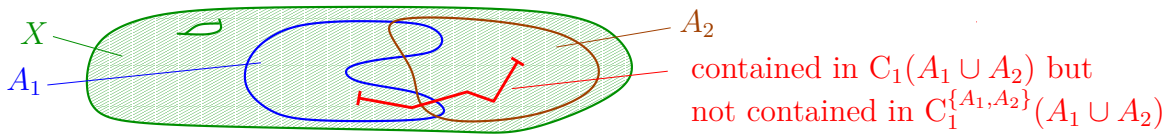
First let us recall the following notation from page 509.

Definition. Let again X be a topological space and let $A_1, A_2 \subset X$ be subsets of X. We consider the following subgroup of $\mathrm{C}_n(X)$:

$$\mathrm{C}_n^{\{A_1,A_2\}}(X) \coloneqq \left\{ \sum_{j=1}^{k} a_j \cdot \sigma_j \;\middle|\; \begin{array}{l}\text{for each } j \text{ there exists an } i \in \{1,2\} \text{ such} \\ \text{that the image of } \sigma_j \text{ lies in } A_i\end{array} \right\}.$$

On page 509 we saw that these groups define a subcomplex of $\mathrm{C}_*(X)$. Thus we can define

$$\mathrm{H}_n^{\{A_1,A_2\}}(X) \coloneqq \mathrm{H}_n\big(\mathrm{C}_*^{\{A_1,A_2\}}(X), \partial_*\big).$$



Now we can provide the definition of an (excisive) triad of topological spaces:

Definition.

(1) A triad of topological spaces is a triple (X, A_1, A_2) where X is a topological space and where both A_1 and A_2 are subspaces of X.

(2) A continuous map $f\colon (X, A_1, A_2) \to (Y, B_1, B_2)$ between two triads is a continuous map $f\colon X \to Y$ with $f(A_1) \subset B_1$ and $f(A_2) \subset B_2$.

(3) We say that a triad (X, A_1, A_2) is excisive if the inclusion map

$$\mathrm{C}_*^{\{A_1,A_2\}}(A_1 \cup A_2) \;\to\; \mathrm{C}_*(A_1 \cup A_2)$$

is a chain homotopy equivalence.

We will soon see that many of the triads that a working topologist comes across are excisive. But before we get to this point let us first formulate the following characterization of excisive triads.

Lemma 50.10. (Excisive Characterization Lemma) Let (X, A_1, A_2) be a triad of topological spaces. The following statements are equivalent:

(1) The triad is excisive.
(2) For every $n \in \mathbb{N}_0$ the inclusion-induced map

$$\mathrm{H}_n^{\{A_1,A_2\}}(A_1 \cup A_2) \;\to\; \mathrm{H}_n(A_1 \cup A_2)$$

is an isomorphism.
(3) For every $n \in \mathbb{N}_0$ the inclusion-induced map

$$\mathrm{H}_n(A_1, A_1 \cap A_2) \;\to\; \mathrm{H}_n(A_1 \cup A_2, A_2)$$

is an isomorphism.

Proof. Let (X, A_1, A_2) be a triad of topological spaces. The equivalence of (1) and (2) is an immediate consequence of the Chain Homotopy Equivalence-H_*-Corollary 36.4 and the Quasi Iso–Homotopy Equivalence Proposition 50.4.

Now we turn to the proof that (2) and (3) are equivalent. We write $Y = A_1 \cup A_2$. We consider the following sequences of chain complexes:

$$\text{(I)} \quad 0 \longrightarrow \mathrm{C}_*^{\{A_1,A_2\}}(Y) \longrightarrow \mathrm{C}_*(Y) \longrightarrow \mathrm{C}_*(Y)/\mathrm{C}_*^{\{A_1,A_2\}}(Y) \longrightarrow 0$$

$$\text{(II)} \quad 0 \longrightarrow \underbrace{\mathrm{C}_*(A_1)/\mathrm{C}_*(A_1 \cap A_2)}_{=\mathrm{C}_*(A_1, A_1\cap A_2)} \longrightarrow \underbrace{\mathrm{C}_*(Y)/\mathrm{C}_*(A_2)}_{=\mathrm{C}_*(Y, A_2)} \longrightarrow \mathrm{C}_*(Y)/\mathrm{C}_*^{\{A_1,A_2\}}(Y) \longrightarrow 0.$$

The top sequence is exact by definition. It is straightforward to see that the bottom sequence is also exact. We can thus apply the LES-Proposition 37.4 to get the corresponding long exact sequences of homology groups:

$$\text{(I)} \quad \to \mathrm{H}_n(\mathrm{C}_*^{\{A_1,A_2\}}(Y)) \longrightarrow \mathrm{H}_n(Y) \longrightarrow \mathrm{H}_n(\mathrm{C}_*(Y)/\mathrm{C}_*^{\{A_1,A_2\}}(Y)) \xrightarrow{\partial} \mathrm{H}_{n-1}(\mathrm{C}_*^{\{A_1,A_2\}}(Y)) \to$$

$$\text{(II)} \quad \to \mathrm{H}_n(A_1, A_1 \cap A_2) \to \mathrm{H}_n(Y, A_2) \to \mathrm{H}_n(\mathrm{C}_*(Y)/\mathrm{C}_*^{\{A_1,A_2\}}(Y)) \xrightarrow{\partial} \mathrm{H}_{n-1}(A_1, A_1 \cap A_2) \to$$

But using these long exact sequences we see that

$$\text{(2) holds} \underset{\text{long exact sequence (I)}}{\iff} \mathrm{H}_n(\mathrm{C}_*(Y)/\mathrm{C}_*^{\{A_1,A_2\}}(Y)) = 0 \text{ for all } n \in \mathbb{N}_0 \underset{\text{long exact sequence (II)}}{\iff} \text{(3) holds.}$$

■

The following proposition gives us a very long list of examples of excisive triads:

Proposition 50.11. (Excisive Triad-Proposition) Let (X, A_1, A_2) be a triad of topological spaces. The triad is excisive if one of the following conditions is satisfied:

(0) (a) $A_1 \subset A_2$ or $A_2 \subset A_1$.
(b) $A_1 = \varnothing$ or $A_2 = \varnothing$ or $A_1 = A_2$.

(c) A_1 and A_2 are both unions of path components of $A_1 \cup A_2$.

(1) (a) $A_1 \cup A_2 = \mathring{A}_1 \cup \mathring{A}_2$ where we denote by $\mathring{A}_1$ and $\mathring{A}_2$ the interiors of A_1 and A_2 in $A_1 \cup A_2$.

(b) A_1 and A_2 are both open subsets of $A_1 \cup A_2$.

(2) $A_1 \cup A_2$ is a topological manifold and A_1 and A_2 are codimension-zero submanifolds which are both closed subsets and such that each component of $A_1 \cap A_2$ is a boundary component of A_1 and a boundary component of A_2.

(3) X is a CW-complex and A_1 and A_2 are both subcomplexes.

(4) Let A and B be topological spaces and let $a_0 \in A$ and $b_0 \in B$ be good points.[412] Then[413] $((A, a_0) \vee (B, b_0), A, B)$ is an excisive triad.



Proof. Let (X, A_1, A_2) be a triad of topological spaces.

(0) (a) If $A_1 \subset A_2$, then $\mathrm{C}_*^{\{A_1,A_2\}}(A_1 \cup A_2) = \mathrm{C}_*(A_2) = \mathrm{C}_*(A_1 \cup A_2)$, so the triad is excisive. The same argument also applies if $A_2 \subset A_1$.

(b) This case is just a special case of (a).

(c) A_1 and A_2 are both unions of path components of $A_1 \cup A_2$, then it follows from the Path-Component–C_*–Lemma 35.6 that $\mathrm{C}_*^{\{A_1,A_2\}}(A_1 \cup A_2) = \mathrm{C}_*(A_2) = \mathrm{C}_*(A_1 \cup A_2)$.

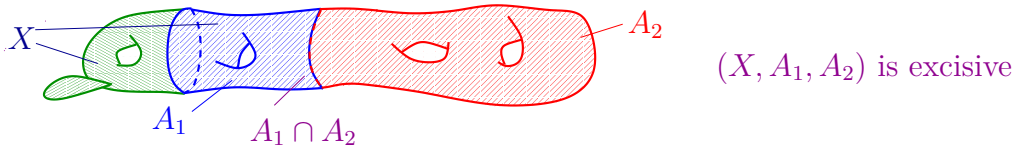
(1) (a) We suppose that $A_1 \cup A_2 = \mathring{A}_1 \cup \mathring{A}_2$. By the Comfortable Cover Proposition 40.7 (which we proved in the context of the Excision Theorem 40.1) we know that the natural chain maps $\mathrm{C}_k^{\{A_1,A_2\}}(A_1 \cup A_2) \to \mathrm{C}_k(A_1 \cup A_2)$ induce isomorphisms of all homology groups. It thus follows from the Excisive Characterization Lemma 50.10 (2)⇒(1) that the triad (X, A_1, A_2) is excisive.

(b) The case that A_1 and A_2 are both open subsets of $A_1 \cup A_2$ is a special case of (a).

(2) Suppose that $A_1 \cup A_2$ is a compact topological manifold and that A_1 and A_2 are submanifolds such that each component of $A_1 \cap A_2$ is a boundary component of A_1 and a boundary component of A_2. The Manifold–Excision Theorem 40.3 implies that for every $n \in \mathbb{N}_0$ the inclusion-induced map

$$\mathrm{H}_n(A_1, A_1 \cap A_2) \;\to\; \mathrm{H}_n(A_1 \cup A_2, A_2)$$

is an isomorphism. By the Excisive Characterization Lemma 50.10 (3)⇒(1) this implies that the triad (X, A_1, A_2) is excisive.



[411] Recall that according to the definition on page 292 we say that a point x in a topological space X is *good*, if $\{x\}$ is a closed subset and if there exists an open neighborhood U of x such that $\{x\}$ is a deformation retract of U.

[411] As we discussed on page 247, we can view X and Y as subspaces of the wedge.

(3) Suppose that X is a CW-complex and that A_1 and A_2 are both subcomplexes. The proof is identical to the proof of (2), except that we need to replace the Manifold–Excision Theorem 40.3 by the CW-Excision Theorem 40.4.

(4) We denote by z the point in $A \vee B$ that is given by identifying a_0 with b_0. For each $k \in \mathbb{N}_0$ we consider the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{H}_k(B, \{b_0\}) & \longrightarrow & \mathrm{H}_k(A, \{a_0\}) \oplus \mathrm{H}_k(B, \{b_0\}) & \longrightarrow & \mathrm{H}_k(A, \{a_0\}) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{H}_k(B, \{b_0\}) & \longrightarrow & \mathrm{H}_k(A \vee B, \{z\}) & \longrightarrow & \mathrm{H}_k(A \vee B, B) & \longrightarrow & 0 \end{array}$$

where all the maps are the obvious maps. The top horizontal sequence is evidently exact. The bottom horizontal sequence is an excerpt from the long exact sequence of the triple $(\{b_0\} = \{z\}, B, A \vee B)$, see the LES-of-Triple Proposition 38.2. The excerpt shown is also exact since $(B, \{b_0\})$ is a retract of $(A \vee B, \{z\})$, which implies that for any $l \in \mathbb{N}_0$ the inclusion-induced map $\mathrm{H}_l(B, \{b_0\}) \to \mathrm{H}_l(A \vee B, \{z\})$ is a monomorphism.

The middle vertical map in the above diagram is an isomorphism by the Wedge-H_*-Proposition 41.10 and the left vertical map is the identity. It follows from the Five Lemma 37.5 that the right vertical hand map is an isomorphism. We obtain from the Excisive Characterization Lemma 50.10 (3)⇒(1) that $(A \vee B, A, B)$ is an excisive triad. ■

CHAPTER 51

Algebraic interlude: The tensor product of abelian groups

In this purely algebraic chapter we introduce and study the tensor product of two abelian groups. It follows from the fact that an abelian group is essentially the same as a $\mathbb{Z}$-module that the tensor product of two abelian groups is essentially the same as the tensor product of two $\mathbb{Z}$-modules. Thus if the reader is familiar with the tensor product of abelian groups or of modules over a commutative ring, then the reader might as well move straight to the next chapter.

51.1. **The definition of the tensor product of abelian groups.** We start out with the definition of the tensor product of two abelian groups:

Definition. Let A and B be abelian groups.

(1) We consider the free abelian group $\mathbb{Z}^{(A\times B)}$ generated by the set $A \times B$ and we define the tensor product $A \otimes B$ to be the following abelian group:

$$A \otimes B \;:=\; \mathbb{Z}^{(A\times B)}/N(A,B)$$

where $N(A,B)$ is the subgroup of $\mathbb{Z}^{(A\times B)}$ generated by the elements of the form

$$((a+a'),b) - ((a,b)+(a',b)) \qquad \text{and} \qquad (a,(b+b')) - ((a,b)+(a,b'))$$

with $a, a' \in A$ and $b, b' \in B$.

(2) For $a \in A$ and $b \in B$ we write $a \otimes b := [(a,b)] \in \mathbb{Z}^{(A\times B)}/N(A,B) = A \otimes B$.

As it stands the definition of a tensor product is very ad hoc and seems to be rather random. Our first goal is thus to show that the tensor product is characterized by a universal property. To formulate this universal property we need the following definition:

Definition. Let A, B and C be abelian groups. We say that a map $f\colon A \times B \to C$ is bilinear, if if the following conditions hold:

(1) For all $a, a' \in A$, $b, b' \in B$ we have

$$f(a+a',b) \;=\; f(a,b)+f(a',b) \quad \text{and} \quad f(a,b+b') \;=\; f(a,b)+f(a,b')$$

(2) For all $a \in A$, $b \in B$ and $m \in \mathbb{Z}$ we have

$$f(m\cdot a,b) \;=\; f(a,m\cdot b) \;=\; m\cdot f(a,b).$$

Lemma 51.1. (Tensor Product–Universal Property Lemma) Let A and B be abelian groups.

(1) The map

$$\begin{array}{rcl} f\colon A\times B & \to & A\otimes B \\ (a,b) & \mapsto & a\otimes b \end{array}$$

is bilinear. In other words, the following two conditions hold:
(a) For any $a, a' \in A$ and $b, b' \in B$ we have

$$\begin{aligned}(a+a')\otimes b &= a\otimes b + a'\otimes b \in A\otimes B\\ a\otimes(b+b') &= a\otimes b + a\otimes b' \in A\otimes B.\end{aligned}$$

(b) For any $a \in A, b \in B$ and $m \in \mathbb{Z}$ we have

$$(m\cdot a)\otimes b = a\otimes(m\cdot b) = m\cdot(a\otimes b).$$

(2) (a) Any element in $A\otimes B$ is of the form $\sum_{i=1}^{n} a_i\otimes b_i$ with $a_1,\dots,a_n \in A$ and $b_1,\dots,b_n \in B$.
(b) Given any $a \in A$ and $b \in B$ we have $a\otimes 0 = 0 = 0\otimes b \in A\otimes B$.
(3) Given any abelian group G and given any bilinear map $g\colon A\times B\to G$ there exists a unique homomorphism $\varphi\colon A\otimes B\to G$ such that the following diagram commutes:

$$\begin{array}{ccc} A\times B & \xrightarrow[\text{bilinear}]{(a,b)\mapsto a\otimes b} & A\otimes B \\ & \searrow^{\text{bilinear } g} & \downarrow \exists!\,\varphi \text{ homomorphism} \\ & & G. \end{array}$$

In fact this homomorphism is given by

$$\begin{aligned}\varphi\colon A\otimes B &\to G\\ \sum_{i=1}^{n} a_i\otimes b_i &\mapsto \sum_{i=1}^{n} g(a_i,b_i).\end{aligned}$$

(4) Let H be an abelian group and let $g\colon A\times B\to H$ be a bilinear map. If this bilinear map satisfies the universal property of (3), then there exists a unique isomorphism $\varphi\colon A\otimes B\to H$ such that the following diagram commutes:

$$\begin{array}{ccc} A\times B & \xrightarrow[\text{bilinear}]{(a,b)\mapsto a\otimes b} & A\otimes B \\ & \searrow^{\text{bilinear } g} & \cong\downarrow \exists!\,\varphi \text{ isomorphism} \\ & & H. \end{array}$$

Remark.

(1) The tensor product of two abelian groups was first introduced by the topologist Hassler Whitney [**Whi34**, p. 497]. The characterization through a universal property was first given by Nicolas Bourbaki in 1942 [**Bou07**, Chapitre II.3.1].
(2) For the most part we will not work with the actual definition of the tensor product $A\otimes B = \mathbb{Z}^{(A\times B)}/N(A,B)$. Instead we will mostly work with the universal property formulated in the Tensor Product–Universal Property Lemma 51.1 (3). □

Proof. Let A and B be two abelian groups.

(1) (a) This statement is an immediate consequence of the definition of the tensor product.
(b) Let $a \in A$, $b \in B$ and let $m \in \mathbb{Z}$. We first consider the case that $m > 0$. We have

$$(m\cdot a)\otimes b = \Big(\sum_{i=1}^{m} a\Big)\otimes b \underset{\text{by (a)}}{=} \sum_{i=1}^{m} a\otimes b = m\cdot(a\otimes b) \underset{\text{same argument backwards}}{=} \dots = a\otimes(m\cdot b).$$

It follows easily from (a) that $0 \otimes b = a \otimes 0 = 0$ and that $(-a) \otimes b = a \otimes (-b) = -a \otimes b$. Using these observations we get Statement (b) for $m = 0$. Finally note that we can easily reduce the case $m < 0$ of (b) to the case $m > 0$ of (b).

(2) (a) It follows immediately from the definition of the tensor product that any element in $A \otimes B$ is of the form $\sum_{i=1}^{n} r_i \cdot (a_i \otimes b_i)$, with $r_1, \ldots, r_n \in \mathbb{Z}$, $a_1, \ldots, a_n \in A$ and $b_1, \ldots, b_n \in B$. But by (1b) we know that $r_i \cdot (a_i \otimes b_i) = (r_i \cdot a_i) \otimes b_i$, thus we get the desired result.

(b) Let $a \in A$ and $b \in B$. It follows from (1b) that we have the following equality in the abelian group $A \otimes B$: $a \otimes 0_B = a \otimes (0_{\mathbb{Z}} \cdot 0_B) = 0_{\mathbb{Z}} \cdot (a \otimes 0_B) = 0_{A \otimes B}$. The same way we see that $0_A \otimes b = 0_{A \otimes B}$.

(3) Let G be an abelian group and let $g\colon A \times B \to G$ be a bilinear map.

Claim. The map

$$\begin{array}{rcl} A \otimes B & \to & G \\ \sum_{i=1}^{n} a_i \otimes b_i & \mapsto & \sum_{i=1}^{n} g(a_i, b_i) \end{array}$$

is a well-defined homomorphism and it is the unique homomorphism $\varphi\colon A \otimes B \to G$ such that the following diagram commutes:

$$\begin{array}{ccc} A \times B & \xrightarrow{(a,b) \mapsto a \otimes b} & A \otimes B \\ & \searrow^{g} & \downarrow \varphi \\ & & G. \end{array}$$

Proof. First note that it follows from the fact that $A \times B$ is a basis for the free abelian group $\mathbb{Z}^{(A \times B)}$ that there exists a *unique* homomorphism

$$\mathbb{Z}^{(A \times B)} \to G$$

which sends each $(a, b) \in A \times B$ to $g(a, b)$. Since g is bilinear we see that the subgroup $N(A, B)$ lies in the kernel of the above homomorphism. It follows that the above homomorphism descends to a well-defined homomorphism

$$\begin{array}{rcl} \varphi\colon A \otimes B \coloneqq \mathbb{Z}^{(A \times B)}/N(A, B) & \to & G \\ \left[\sum_{i=1}^{n} (a_i, b_i)\right] = \sum_{i=1}^{n} a_i \otimes b_i & \mapsto & \sum_{i=1}^{n} g(a_i, b_i). \end{array}$$

It is clear that the following diagram commutes:

$$\begin{array}{ccc} A \times B & \xrightarrow{(a,b) \mapsto a \otimes b} & A \otimes B \\ & \searrow^{g} & \downarrow \varphi \\ & & G. \end{array}$$

Finally note that it follows easily from the fact that $\mathbb{Z}^{(A \times B)} \to A \otimes B$ is an epimorphism that φ is uniquely determined. ⊞

(4) This statement follows from the usual argument involving universal properties. We leave it to the reader to fill in the details. ■

51.2. **Homomorphisms between tensor products.** In this section we will study homomorphisms between tensor products. Let us start out with the following definition:

Definition. Let $f\colon A \to A'$ and $g\colon B \to B'$ be homomorphisms between abelian groups. It follows from the Tensor Product–Universal Property Lemma 51.1 (1) that the map

$$\begin{array}{rcl} A \times B & \to & A' \otimes B' \\ (a_i, b_i) & \mapsto & f(a_i) \otimes g(b_i) \end{array}$$

is bilinear. Thus it follows from the Tensor Product–Universal Property Lemma 51.1 (3) that the map

$$\begin{array}{rcl} f \otimes g\colon A \otimes B & \to & A' \otimes B' \\ \sum_{i=1}^{n} a_i \otimes b_i & \mapsto & \sum_{i=1}^{n} f(a_i) \otimes g(b_i) \end{array}$$

is a well-defined homomorphism. We refer to $f \otimes g$ as the tensor product of the homomorphisms f and g.

The following lemma summarizes three key properties of the tensor product of two homomorphisms.

Lemma 51.2. (Tensor–Map Composition Lemma)

(1) Given any abelian groups A and B we have $\mathrm{id}_A \otimes \mathrm{id}_B = \mathrm{id}_{A\otimes B}$.

(2) Let $f\colon A \to A'$, $f'\colon A' \to A''$, $g\colon B \to B'$ and $g'\colon B' \to B''$ be homomorphisms between abelian groups. Then we have

$$(f' \otimes g') \circ (f \otimes g) \;=\; (f' \circ f) \otimes (g' \circ g)\colon A \otimes B \;\to\; A'' \otimes B''.$$

(3) Let $f, g\colon A \to B$ be homomorphisms between abelian groups and let G be an abelian group. Then

$$(f + g) \otimes \mathrm{id}_G \;=\; f \otimes \mathrm{id}_G + g \otimes \mathrm{id}_G\colon A \otimes G \;\to\; B \otimes G.$$

The same statement also holds if we swap the roles of the two sides.

Proof. All the statements follow immediately from the above explicit description of the tensor product of homomorphisms. Alternatively one can also deduce the statements from the universal property stated in the Tensor Product–Universal Property Lemma 51.1 (3). ∎

The next lemma gives us an important, albeit slightly dull, calculation of a tensor product.

Lemma 51.3. (Tensor-with-$\mathbb{Z}$–Lemma) Let A be an abelian group. The maps

$$\begin{array}{rcl} \Theta\colon \mathbb{Z} \otimes A & \to & A \\ \sum_{i=1}^{m} n_i \otimes a_i & \mapsto & \sum_{i=1}^{m} n_i \cdot a_i \end{array} \qquad \text{and} \qquad \begin{array}{rcl} \Omega\colon A & \to & \mathbb{Z} \otimes A \\ a & \mapsto & 1 \otimes a \end{array}$$

are well-defined homomorphisms, they are natural and they are inverses to one another. In particular both are natural isomorphisms. Here "natural" means that for any homomorphism $\varphi\colon A \to B$ of abelian groups the following diagrams commute:

$$\begin{array}{ccc} \mathbb{Z} \otimes A & \xrightarrow[\cong]{\Theta} & A \\ {\scriptstyle \mathrm{id}_{\mathbb{Z}} \otimes \varphi}\downarrow & & \downarrow{\scriptstyle \varphi} \\ \mathbb{Z} \otimes B & \xrightarrow[\cong]{\Theta} & B \end{array} \qquad \text{and} \qquad \begin{array}{ccc} A & \xrightarrow[\cong]{a \mapsto 1\otimes a} & \mathbb{Z} \otimes A \\ {\scriptstyle \varphi}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_{\mathbb{Z}} \otimes \varphi} \\ B & \xrightarrow[\cong]{b \mapsto 1\otimes b} & \mathbb{Z} \otimes B. \end{array}$$

The analogous statement also hold if the $\mathbb{Z}$-term is on the right instead of the left.

Proof. Note that the map

$$\begin{array}{rcl} \mathbb{Z} \times A & \to & A \\ (n, a) & \mapsto & n \cdot a \end{array}$$

is bilinear. It thus follows from the Tensor Product–Universal Property Lemma 51.1 (3) that the given map

$$\begin{array}{rcl} \Theta\colon \mathbb{Z} \otimes A & \to & A \\ \sum_{i=1}^{m} n_i \otimes a_i & \mapsto & \sum_{i=1}^{m} n_i \cdot a_i \end{array}$$

is a well-defined homomorphism. It follows easily from the definitions that this map is natural.

It is clear that $\Omega\colon A \to \mathbb{Z} \otimes A$ is a well-defined natural homomorphism. Furthermore it is clear that $\Theta \circ \Omega\colon A \to A$ is the identity. It remains to prove that $\Omega \circ \Theta\colon \mathbb{Z} \otimes A \to \mathbb{Z} \otimes A$ is the identity. This follows easily from the definitions together with the following claim:

Claim. Any element in $\mathbb{Z} \otimes A$ is represented by an element of the form $1 \otimes a$.

Proof. By the Tensor Product–Universal Property Lemma 51.1 (2) any element of $\mathbb{Z} \otimes A$ is of the form $\sum_{i=1}^{m} n_i \otimes a_i \in \mathbb{Z} \otimes A$. We see that

$$\sum_{i=1}^{m} n_i \otimes a_i \;=\; \sum_{i=1}^{m} (n_i \cdot 1) \otimes a_i \;\underset{\uparrow}{=}\; \sum_{i=1}^{m} 1 \otimes (n_i \cdot a_i) \;\underset{\uparrow}{=}\; 1 \otimes \Big(\sum_{i=1}^{m} (n_i \cdot a_i) \Big).$$

by the Tensor Product–Universal Lemma 51.1 (1b) — by the Tensor Product–Universal Lemma 51.1 (1a)

The argument if the $\mathbb{Z}$-term is on the right is evidently almost identical.[412] ⊞ ■

We move on to a slightly more interesting calculation of a tensor product.

Lemma 51.4. (Tensor-with-$\mathbb{Z}_k$–Lemma) Let $k \in \mathbb{N}_0$ and let A be an abelian group. We write

$$k \cdot A \;:=\; \{k \cdot a \,|\, a \in A\}.$$

The maps

$$\begin{array}{rcl} \Theta\colon \mathbb{Z}_k \otimes A & \to & A/k \cdot A \\ \sum_{i=1}^{m} [n_i] \otimes a_i & \mapsto & \Big[\sum_{i=1}^{m} n_i \cdot a_i \Big] \end{array} \quad \text{and} \quad \begin{array}{rcl} \Omega\colon A/k \cdot A & \to & \mathbb{Z}_k \otimes A \\ [a] & \mapsto & [1] \otimes a \end{array}$$

are well-defined homomorphisms, they are natural and they are inverses to one another. In particular both are natural isomorphisms. Here "natural" means that for any homomorphism $\varphi\colon A \to B$ of abelian groups the following diagrams commute:

$$\begin{array}{ccc} \mathbb{Z}_k \otimes A & \xrightarrow[\cong]{\Theta} & A/k \cdot A \\ {\scriptstyle \mathrm{id}_{\mathbb{Z}_k} \otimes \varphi}\downarrow & & \downarrow{\scriptstyle \varphi} \\ \mathbb{Z}_k \otimes B & \xrightarrow[\cong]{\Theta} & B/k \cdot B \end{array} \quad \text{and} \quad \begin{array}{ccc} A/k \cdot A & \xrightarrow[\cong]{a \mapsto [1] \otimes a} & \mathbb{Z}_k \otimes A \\ {\scriptstyle \varphi}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_{\mathbb{Z}_k} \otimes \varphi} \\ B/k \cdot B & \xrightarrow[\cong]{b \mapsto [1] \otimes b} & \mathbb{Z}_k \otimes B. \end{array}$$

The analogous statement also hold if the $\mathbb{Z}$-term is on the right instead of the left.

[412] Alternatively one could use the elementary Tensor Commutative-Associative Lemma 51.5 (1) below.

Proof. [413][414] First note that one can easily show that the map

$$\begin{array}{rcl} \mathbb{Z}_k \times A & \to & A/k \cdot A \\ ([n], a) & \mapsto & [n \cdot a] \end{array}$$

is well-defined and bilinear. It thus follows from the Tensor Product–Universal Property Lemma 51.1 (3) that the given map

$$\begin{array}{rcl} \Theta\colon \mathbb{Z}_k \otimes A & \to & A/k \cdot A \\ \sum_{i=1}^{m} [n_i] \otimes a_i & \mapsto & \left[\sum_{i=1}^{m} n_i \cdot a_i\right] \end{array}$$

is a well-defined homomorphism. It follows easily from the definitions that this map is natural.

Using the Tensor Product–Universal Property Lemma 51.1 (1) one can easily verify that $\Omega\colon A/k \cdot A \to \mathbb{Z}_k \otimes A$ is a well-defined natural homomorphism. Furthermore it is clear that $\Theta \circ \Omega\colon A/k \cdot A \to A/k \cdot A$ is the identity. It remains to prove that $\Omega \circ \Theta\colon \mathbb{Z}_k \otimes A \to \mathbb{Z}_k \otimes A$ is the identity. This follows easily from the definitions together with the following claim:

Claim. Any element in $\mathbb{Z}_k \otimes A$ is represented by an element of the form $[1] \otimes a$.

Proof. By the Tensor Product–Universal Property Lemma 51.1 (2) any element of $\mathbb{Z}_k \otimes A$ is of the form $\sum_{i=1}^{m} [n_i] \otimes a_i \in \mathbb{Z}_k \otimes A$. We see that

$$\sum_{i=1}^{m} [n_i] \otimes a_i \;=\; \sum_{i=1}^{m} (n_i \cdot [1]) \otimes a_i \;\underset{\uparrow}{=}\; \sum_{i=1}^{m} [1] \otimes (n_i \cdot a_i) \;\underset{\uparrow}{=}\; [1] \otimes \left(\sum_{i=1}^{m} (n_i \cdot a_i)\right).$$

by the Tensor Product–Universal Lemma 51.1 (1b) — by the Tensor Product–Universal Lemma 51.1 (1a)

The argument if the $\mathbb{Z}_k$-term is on the right is evidently almost identical.[415] ⊞ ■

We continue with general results on tensor products.

Lemma 51.5. (Tensor Commutative-Associative Lemma)

(1) For any abelian group A we have $A \otimes 0 \cong 0 \cong 0 \otimes A$.

(2) Given any two abelian groups A and B the map

$$\begin{array}{rcl} A \otimes B & \to & B \otimes A \\ \sum_{i=1}^{n} a_i \otimes b_i & \mapsto & \sum_{i=1}^{n} b_i \otimes a_i \end{array}$$

is well-defined and it is a natural isomorphism of abelian groups. Here "natural" means that if $f\colon A \to A'$ and $g\colon B \to B'$ are two homomorphisms between abelian groups, then the following diagram commutes:

$$\begin{array}{ccc} A \otimes B & \xrightarrow{a\otimes b \mapsto b\otimes a} & B \otimes A \\ {\scriptstyle f\otimes g}\downarrow & & \downarrow{\scriptstyle g\otimes f} \\ A' \otimes B' & \xrightarrow{a'\otimes b' \mapsto b'\otimes a'} & B' \otimes A'. \end{array}$$

[413] The proof of the Tensor-with-$\mathbb{Z}_k$–Lemma 51.4 is very similar to the proof of the Tensor-with-$\mathbb{Z}$–Lemma 51.3. In fact the case $k = 0$ of the Tensor-with-$\mathbb{Z}_k$–Lemma 51.4 is essentially the content of the Tensor-with-$\mathbb{Z}$–Lemma 51.3.

[414] On page 670 we will give a different, more conceptual proof, of the lemma.

[415] Alternatively one could use the elementary Tensor Commutative-Associative Lemma 51.5 (1) below.

(3) Let A, B, C be abelian groups. The map[416]

$$\begin{array}{rcl} (A \otimes B) \otimes C & \to & A \otimes (B \otimes C) \\ \sum_{j=1}^{n} (a_j \otimes b_j) \otimes c_j & \mapsto & \sum_{j=1}^{n} a_j \otimes (b_j \otimes c_j) \end{array}$$

is well-defined and it is a natural isomorphism of abelian groups.

Proof.

(1) This statement follows immediately from the Tensor Product–Universal Property Lemma 51.1 (2b).

(2) First note that the maps

$$\begin{array}{rcl} A \times B & \to & B \otimes A \\ (a, b) & \mapsto & b \otimes a \end{array} \quad \text{and} \quad \begin{array}{rcl} B \times A & \to & A \otimes B \\ (b, a) & \mapsto & a \otimes b \end{array}$$

are bilinear. It follows from the Tensor Product–Universal Property Lemma 51.1 (3) that we get induced homomorphisms

$$\Phi \colon A \otimes B \to B \otimes A \qquad \text{and} \qquad \Psi \colon B \otimes A \to A \otimes B.$$

Since $\Phi \circ \Psi$ and $\Psi \circ \Phi$ are the identity on the generating sets $\{b \otimes a \,|\, b \in B, a \in A\}$ of $B \otimes A$ and $\{a \otimes b \,|\, a \in A, b \in B\}$ of $A \otimes B$ we see that $\Phi \circ \Psi = \mathrm{id}_{B \otimes A}$ and $\Psi \circ \Phi = \mathrm{id}_{A \otimes B}$. This shows that Φ and Ψ are isomorphisms. It follows easily from the definitions that Φ and Ψ are natural.

(3) Let $\widetilde{c} \in C$ and let $\widetilde{a} \in A$. One can easily verify that the maps

$$\begin{array}{rcl} A \times B & \to & A \otimes (B \otimes C) \\ (a, b) & \mapsto & a \otimes (b \otimes \widetilde{c}) \end{array} \quad \text{and} \quad \begin{array}{rcl} B \times C & \to & (A \otimes B) \otimes C \\ (b, c) & \mapsto & (\widetilde{a} \otimes b) \otimes c \end{array}$$

are bilinear. It follows from the Tensor Product–Universal Property Lemma 51.1 (3) that we get induced homomorphisms $A \otimes B \to A \otimes (B \otimes C)$ and $B \otimes C \to (A \otimes B) \otimes C$. Thus we have defined maps

$$\begin{array}{rcl} (A \otimes B) \times C & \to & A \otimes (B \otimes C) \\ \left(\sum_{i=1}^{n} a_i \otimes b_i, c \right) & \mapsto & \sum_{i=1}^{n} a_i \otimes (b_i \otimes c) \end{array} \quad \text{and} \quad \begin{array}{rcl} A \times (B \otimes C) & \to & (A \otimes B) \otimes C \\ \left(a, \sum_{i=1}^{n} (b_i \otimes c_i) \right) & \mapsto & \sum_{i=1}^{n} (a \otimes b_i) \otimes c_i. \end{array}$$

Again one can easily verify that these maps are bilinear. Thus it follows again from the Tensor Product–Universal Property Lemma 51.1 (3) that we get induced homomorphisms

$$\Phi \colon (A \otimes B) \otimes C \to A \otimes (B \otimes C) \qquad \text{and} \qquad \Psi \colon A \otimes (B \otimes C) \to (A \otimes B) \otimes C.$$

It is clear that the homomorphisms $\Phi \circ \Psi$ and $\Psi \circ \Phi$ are the identity on the generating sets $\{a \otimes (b \otimes c) \,|\, a \in A, b \in B, c \in C\}$ and $\{(a \otimes b) \otimes c) \,|\, a \in A, b \in B, c \in C\}$. we see that $\Phi \circ \Psi = \mathrm{id}_{A \otimes (B \otimes C)}$ and $\Psi \circ \Phi = \mathrm{id}_{(A \otimes B) \otimes C}$. This shows that Φ and Ψ are isomorphisms. Finally note that it follows easily from the definitions that Φ and Ψ are natural. ∎

[416] It follows easily from the Tensor Product–Universal Property Lemma 51.1 (1) and (2) that every element of $(A \otimes B) \otimes C$ is of that form.

Using the notion of the tensor product of two homomorphisms we can now formulate the following lemma which shows that the tensor product is "distributive" with respect to direct sums:

Lemma 51.6. (Tensor–Direct Sum Lemma) Let $\{A_i\}_{i\in I}$ be a family of abelian groups and let B be an abelian group. Given $j \in I$ let $\iota_j \colon A_j \to \bigoplus_{i\in I} A_i$ be the natural homomorphism. Note that it follows from the Coproduct-of-Abelian Groups Lemma 18.1 that the homomorphisms $\iota_j \otimes \mathrm{id}_B \colon A_j \otimes B \to \left(\bigoplus_{i\in I} A_i\right) \otimes B$ give rise to a natural homomorphism

$$\bigoplus_{i\in I} (A_i \otimes B) \;\to\; \left(\bigoplus_{i\in I} A_i\right) \otimes B.$$

This homomorphism is a natural isomorphism of abelian groups.

Sketch of proof. We need to show that the above homomorphism

$$\Phi \colon \bigoplus_{i\in I} (A_i \otimes B) \;\to\; \left(\bigoplus_{i\in I} A_i\right) \otimes B$$

has an inverse. Let $b \in B$. For each $j \in I$ we consider the homomorphism $A_j \to A_j \otimes B$ that is given by $a \mapsto a \otimes b$. We compose this homomorphism with the natural homomorphism $A_j \otimes B \to \bigoplus_{i\in I} (A_i \otimes B)$ to obtain a homomorphism $A_j \to \bigoplus_{i\in I} (A_i \otimes B)$. By the Coproduct-of-Abelian Groups Lemma 18.1 we get an induced homomorphism

$$\bigoplus_{j\in I} A_j \;\to\; \bigoplus_{i\in I} (A_i \otimes B).$$

In summary we have now constructed a map

$$\left(\bigoplus_{j\in I} A_j\right) \times B \;\to\; \bigoplus_{i\in I} (A_i \otimes B).$$

One can easily verify that this map is bilinear. It follows from the Tensor Product–Universal Property Lemma 51.1 (3) that we get an induced homomorphism

$$\Psi \colon \left(\bigoplus_{j\in I} A_j\right) \otimes B \;\to\; \bigoplus_{i\in I} (A_i \otimes B).$$

Note that the two homomorphisms $\Phi \circ \Psi$ and $\Psi \circ \Phi$ are both the identity on the generating set $\left\{ a \otimes b \;\middle|\; a \in \bigcup_{j\in I} A_j \text{ and } b \in B \right\}$. Thus we see that $\Phi \circ \Psi = \mathrm{id}$ and $\Psi \circ \Phi = \mathrm{id}$. This implies that Φ is indeed an isomorphism. Finally note that one can easily verify that Φ is natural. ∎

The combination of the Tensor-with-$\mathbb{Z}$–Lemma 51.3, the Tensor-with-$\mathbb{Z}_k$–Lemma 51.4, the Tensor Commutative-Associative Lemma 51.5 and the Tensor–Direct Sum Lemma 51.6 allows us to calculate most tensor products that arise naturally.

Examples.

(1) Let $m, n \in \mathbb{N}$. We write $r = \gcd(m,n)$ and $s = m/r$. Then

$$\mathbb{Z}_m \otimes \mathbb{Z}_n \underset{\text{by the Tensor-with-}\mathbb{Z}_k\text{–Lemma 51.4}}{\overset{\uparrow}{\cong}} \mathbb{Z}_n/m\mathbb{Z}_n = \mathbb{Z}_n/rs\mathbb{Z}_n = \mathbb{Z}_n/r\cdot(s\mathbb{Z}_n) \underset{\text{since } s \text{ is coprime to } n}{\overset{\uparrow}{=}} \mathbb{Z}_n/r\mathbb{Z}_n \xleftarrow[\cong]{\cdot n/r} \mathbb{Z}_r = \mathbb{Z}_{\gcd(m,n)}.$$

(2) Let A be a finitely generated abelian group of rank r. By the Abelian Group-Classification Theorem 18.11 we have $A \cong \mathbb{Z}^r \oplus \bigoplus_{i=1}^{n} \mathbb{Z}_{a_i}$ where $a_1, \ldots, a_n$ are non-zero natural

numbers. For any abelian group S we have

$$A \otimes S \cong \left(\mathbb{Z}^r \oplus \bigoplus_{i=1}^n \mathbb{Z}_{a_i}\right) \otimes S \quad \underset{\uparrow}{\cong} \quad (\mathbb{Z} \otimes S)^r \oplus \left(\bigoplus_{i=1}^n (\mathbb{Z}_{a_i} \otimes S)\right) \quad \underset{\uparrow}{\cong} \quad S^r \oplus \bigoplus_{i=1}^n S/a_i \cdot S.$$

by the Tensor Commutative-Associative Lemma 51.5 (2)

by the Tensor-with-$\mathbb{Z}$–Lemma 51.3 and the Tensor-with-$\mathbb{Z}_k$ Lemma 51.4

In particular for $S = \mathbb{Q}, S = \mathbb{R}$ or $S = \mathbb{C}$ we have

$$A \otimes S \cong \left(\mathbb{Z}^r \oplus \bigoplus_{i=1}^n \mathbb{Z}_{a_i}\right) \otimes \quad \underset{\uparrow}{\cong} \quad S^r \oplus \bigoplus_{i=1}^n S/a_i \cdot S \quad \underset{\uparrow}{=} \quad S^r.$$

by the above

since a_i are invertible in $S = \mathbb{Q}, \mathbb{R}, \mathbb{C}$

(3) Recall that given a set W we denote by $\mathbb{Z}^{(W)}$ the free abelian group generated by W. Let A and B be two sets. We then have

$$\mathbb{Z}^{(A)} \otimes \mathbb{Z}^{(B)} = \left(\bigoplus_{a \in A} \mathbb{Z} \cdot a\right) \otimes \left(\bigoplus_{b \in B} \mathbb{Z} \cdot b\right) \overset{\downarrow}{\cong} \bigoplus_{\substack{a \in A \\ b \in B}} \mathbb{Z} \cdot a \otimes \mathbb{Z} \cdot b \overset{\downarrow}{\cong} \bigoplus_{\substack{a \in A \\ b \in B}} \mathbb{Z} \cdot (a, b) \cong \mathbb{Z}^{(A \times B)}.$$

follows from applying the Tensor Commutative-Associative Lemma 51.5 (2) twice

Tensor-with-$\mathbb{Z}$–Lemma 51.3

In summary we have shown that the tensor product of two free abelian groups on sets A and B is the free abelian group on the set $A \times B$. □

51.3. **Tensoring with commutative rings.** For this section it is convenient to introduce the following notation:

Notation. Let R be a commutative ring. We denote by $\mathcal{M}od_R$ the category of R-modules.

The following elementary lemma will be used frequently, without actually referring to it.

Lemma 51.7. (Tensor-with-Ring Lemma) Let R be a commutative ring.

(1) Given any abelian group A the map

$$\begin{array}{rcl} (A \otimes R) \times R & \to & A \otimes R \\ \left(\sum_{i=1}^n a_i \otimes r_i, s\right) & \mapsto & \sum_{i=1}^n a_i \otimes \underbrace{(r_i \cdot s)}_{\in R} \end{array}$$

is well-defined and it defines an R-module structure on $A \otimes R$.

(2) If $f\colon A \to B$ is a homomorphism of abelian groups, then the map

$$\begin{array}{rcl} f \otimes \mathrm{id}_R\colon A \otimes R & \to & B \otimes R \\ \sum_{i=1}^n a_i \otimes r_i & \mapsto & \sum_{i=1}^n f(a_i) \otimes r_i \end{array}$$

is a homomorphism of R-modules.

(3) The maps

$$\begin{array}{rcl} A & \mapsto & A \otimes R \\ (f\colon A \to B) & \mapsto & (f \otimes \mathrm{id}_R\colon A \otimes R \to B \otimes R) \end{array}$$

define a covariant functor from the category $\mathcal{A}b\mathcal{G}roup$ of abelian groups to the category $\mathcal{M}od_R$ of R-modules.

Proof.

(1) Let $s \in R$. Using the fact that R is a *commutative* ring and using the Tensor Product–Universal Property Lemma 51.1 (1) one can easily verify that the map

$$\begin{array}{ccc} A \times R & \to & A \otimes R \\ (a, r) & \mapsto & a \otimes (r \cdot s) \end{array}$$

is bilinear. Thus it follows from the Tensor Product–Universal Property Lemma 51.1 (3) that the map descends to a well-defined homomorphism $A \otimes R \to A \otimes R$. This shows that the given map $(A \otimes R) \times R \to A \otimes R$ is well-defined. One can now easily verify that this defines an R-module structure on $A \otimes R$.

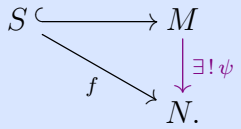
(2) This statement follows easily from the definitions.

(3) This statement also follows immediately from (2) together with the Tensor–Map Composition Lemma 51.2. ■

We continue with the following variation on the definition of a free abelian group (see page 256).

Definition. Let R be a commutative ring.

(1) We say that an R-module M is a free R-module on a subset $S \subset M$, if the following universal property holds: For any *map* $f\colon S \to N$ of the set S to an R-module N there exists a unique *homomorphism* $\psi\colon M \to N$ of R-modules that makes the following diagram commute:



We call S a basis of the R-module M.[417] [418]

(2) Given a set S we consider

$$R^{(S)} \coloneqq \text{ all maps from } S \text{ to } R \text{ which are non-zero for only finitely many } s \in S$$

with the obvious R-module structure. The same argument as in the Free Abelian Group–Existence Lemma 18.4 shows that the R-module $R^{(S)}$ is a free R-module on the subset $S \subset R^{(S)}$

(3) We identify elements in $R^{(S)}$ with finite formal linear combinations $r_1 \cdot s_1 + \cdots + r_k \cdot s_k$ where $r_1, \ldots, r_k \in R$ and $s_1, \ldots, s_k \in S$.

Lemma 51.8. (Matrix–Tensor-with-Ring Lemma) Let R be a commutative ring.

(1) Let I be a set. The composition

$$\mathbb{Z}^{(I)} \otimes R = \Big(\bigoplus_{i\in I} \mathbb{Z}\Big) \otimes R \xleftarrow[\uparrow]{\cong} \bigoplus_{i\in I} (\mathbb{Z} \otimes R) \xrightarrow[\uparrow]{\cong} \bigoplus_{i\in I} R = R^{(I)}$$

by the Tensor–Direct Sum Lemma 51.6 — by the Tensor-with-$\mathbb{Z}$–Lemma 51.3

defines an isomorphism

$$\Theta_I\colon \mathbb{Z}^{(I)} \otimes R \to R^{(I)}$$

of R-modules. This isomorphism is natural in R.

[418] It follows easily from the definitions that a $\mathbb{Z}$-module is essentially the same as an abelian group.

(2) Let A be an $(m \times n)$-matrix over $\mathbb{Z}$. We denote by $\varphi\colon \mathbb{Z} \to R$ the ring homomorphism that is given by $\varphi(k) = k \cdot 1_R$. With this notation the following diagram commutes:

$$\begin{array}{ccc} \mathbb{Z}^n \otimes R & \xrightarrow[\cong]{\Theta_n} & R^n \\ \downarrow{\scriptstyle (v \mapsto A\cdot v)\otimes \mathrm{id}_R} & & \downarrow{\scriptstyle w \mapsto \varphi(A)\cdot w} \\ \mathbb{Z}^m \otimes R & \xrightarrow[\cong]{\Theta_m} & R^m. \end{array}$$

Example. Let $m, n \in \mathbb{N}_0$ such that $\mathbb{Z}^m \cong \mathbb{Z}^n$. We want to show that $m = n$. Indeed, we have

$$m = \dim_{\mathbb{Q}}(\mathbb{Q}^m) \underset{\uparrow}{=} \dim_{\mathbb{Q}}(\mathbb{Z}^m \otimes \mathbb{Q}) \overset{\text{by the Tensor-with-Ring Lemma 51.7 (3)}}{\overset{\downarrow}{\cong}} \dim_{\mathbb{Q}}(\mathbb{Z}^n \otimes \mathbb{Q}) \underset{\uparrow}{\cong} \dim_{\mathbb{Q}}(\mathbb{Q}^n) = n.$$

follows from the Matrix–Tensor-with-Ring Lemma 51.8 (2)

□

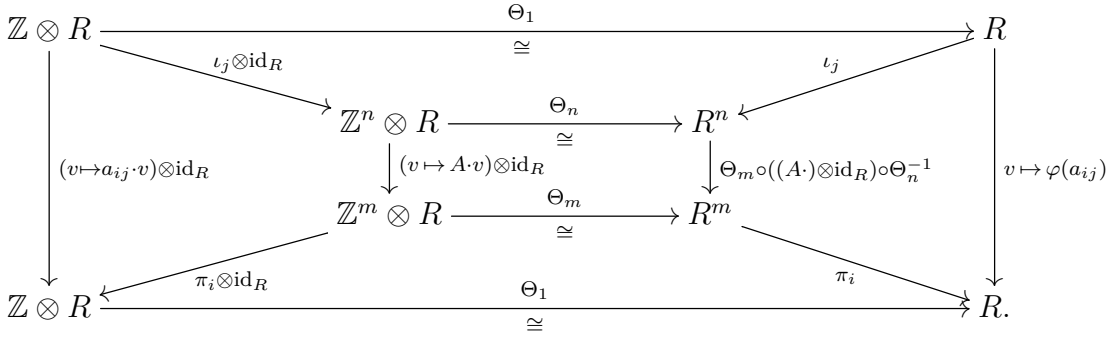
Proof.

(1) This statement follows almost immediately from the Tensor–Direct Sum Lemma 51.6 and the Tensor-with-$\mathbb{Z}$–Lemma 51.3.

(2) We start out with the following simple observation: It follows easily from the Tensor Product–Universal Property Lemma 51.1 (1b) that given any $k \in \mathbb{Z}$ the following diagram commutes:

$$\begin{array}{ccc} \mathbb{Z} \otimes R & \xrightarrow{\Theta_1} & R \\ \downarrow{\scriptstyle (v \mapsto k\cdot v)\otimes \mathrm{id}_R} & & \downarrow{\scriptstyle r \mapsto \varphi(k)\cdot r} \\ \mathbb{Z} \otimes R & \xrightarrow[\Theta_1]{} & R. \end{array}$$

Now let $A = (a_{ij}) \in \mathrm{M}(m \times n)$. Let $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$. We denote by $\pi_i\colon \mathbb{Z}^m \to \mathbb{Z}$ the projection onto the i-th term and we denote by $\iota_j\colon \mathbb{Z} \to \mathbb{Z}^n$ the inclusion into the j-th coordinate. We consider the following diagram:



We make the following observations:

(a) The outer rectangle commutes by the discussion at the beginning of the proof.

(b) The small inner rectangle commutes by definition.

(c) The quadrilateral to the left commutes by the Tensor–Map Composition Lemma 51.2 (2).

(d) The quadrilaterals to the top and bottom commute by definition of Θ_m and Θ_n.

It follows from these observations and the fact that the horizontal maps are isomorphisms that the quadrilateral to the right commutes. But this implies precisely that the vertical map from R^n to R^m is given by multiplication by the matrix $\varphi(A)$. ■

CHAPTER 52

Homology with coefficients

After the algebraic interlude from the last chapter we now return to topology. As we discussed in the beginning of Chapter 50, the functors we were so far mostly interested in are the covariant functor

$$(X, A) \;\mapsto\; (\mathrm{C}_*(X, A), \partial_*)$$

from the category *PairTop* of pairs of topological spaces to the category *ChCplx* of chain complexes and the covariant functors

$$(C_*, \partial_*) \;\to\; \mathrm{H}_k(C_*, \partial_*)$$

from the category *ChCplx* of chain complexes to the category *AbGroup* of abelian groups. The idea now is to "intersperse" a functor between these two functors.

In this chapter the functor of choice is the following: Given a fixed abelian group G we consider the functor $A \mapsto A \otimes G$ from the category *AbGroup* of abelian groups to itself. Combining this functor with the above two by now "classical" functors we will obtain the functor

$$(X, A) \;\mapsto\; \underbrace{\mathrm{H}_n(\mathrm{C}_*(X, A) \otimes G)}_{=:\mathrm{H}_n(X,A;G)}$$

from the category *PairTop* of pairs of topological spaces to the category *AbGroup* of abelian groups (or the category $\mathcal{M}od_G$ of G-modules.) We call $\mathrm{H}_n(X, A; G)$ the *n-th homology group of (X, A) with coefficients in an abelian group G* (or with coefficients in a commutative ring G). These homology groups will have many applications:

(1) If we take a "small" ring, like $G = \mathbb{F}_2$, then the homology groups with coefficients become much more computable.
(2) If we consider homology groups with coefficients in a field $\mathbb{F}$, then these homology groups are $\mathbb{F}$-vector spaces and thus easier to handle than abelian groups.
(3) Homology with coefficients will be essential in proving theoretical statements. For example we will need to work with $\mathbb{F}_2$-coefficients to prove the higher-dimensional analogue of the 2D-Borsuk-Ulam Theorem 12.2.

52.1. **Additive functors applied to chain complexes.** Our next goal is to figure out what properties on a self-functor of the category *AbGroup* of abelian groups we need to get a self-functor of the category *ChCplx* of chain complexes.

As topologists we also want our functors to preserve the property of being chain-homotopic. Some thought leads us to the following definition:

Definition. We say that a covariant functor $F\colon \mathcal{A}b\mathcal{G}roup \to \mathcal{A}b\mathcal{G}roup$ from the category *AbGroup* of abelian groups to itself is additive if for any two abelian groups A and B and any two homomorphisms $\varphi, \psi\colon A \to B$ we have $F(\varphi + \psi) = F(\varphi) + F(\psi)\colon F(A) \to F(B)$.

Lemma 52.1. (Additive Functor–Basics Lemma) Let $F\colon \mathcal{AbGroup} \to \mathcal{AbGroup}$ be a covariant functor from the category $\mathcal{AbGroup}$ of abelian groups to itself. If F is additive, then the following two statement holds:

(1) $F(\text{zero homomorphism}) = \text{zero homomorphism}$.
(2) $F(\text{zero group}) = \text{zero group}$.

Proof. Let $F\colon \mathcal{AbGroup} \to \mathcal{AbGroup}$ be an additive covariant functor.

(1) Let A, B be two abelian groups. Since F is an additive covariant functor we see that $F(0\colon A \to B) = F(0+0) = F(0) + F(0)\colon F(A) \to F(B)$. It follows that $F(0)$ is the zero homomorphism.
(2) Let G be a trivial abelian group. Then $\mathrm{id}_G = 0$. It follows from functoriality and (1) that $\mathrm{id}_{F(G)} = F(\mathrm{id}_G) = F(0) = 0$. This implies that $F(G)$ is a trivial group. ∎

We can now formulate the elementary but useful lemma:

Lemma 52.2. (Functor-applied-to-Chain Complex Lemma) Suppose we are given an additive covariant functor $F\colon \mathcal{AbGroup} \to \mathcal{AbGroup}$.

(1) (a) If (C_*, ∂_*) is a chain complex, then
$$\ldots \to F(C_2) \xrightarrow{F(\partial_2)} F(C_1) \xrightarrow{F(\partial_1)} F(C_0) \xrightarrow{F(\partial_0)} 0$$
is also a chain complex.
(b) If $\varphi_*\colon C_* \to D_*$ is a chain map, then $F(\varphi)_*\colon F(C)_*, \to F(D)_*$ is also a chain map.
(2) The maps
$$\begin{aligned} (C_*, \partial_*) &\mapsto (F(C_*), F(\partial_*)) \\ (\varphi_*\colon C_* \to D_*) &\mapsto (F(\varphi)_*\colon F(C)_*, \to F(D)_*) \end{aligned}$$
define a covariant functor from the category $\mathcal{ChCplx}$ of chain complexes to itself.
(3) If $\varphi, \psi\colon C_* \to D_*$ are chain homotopic, then $F(\varphi), F(\psi)\colon F(C_*) \to F(D_*)$ are also chain homotopic.

Proof. Let $F\colon \mathcal{AbGroup} \to \mathcal{AbGroup}$ be an additive covariant functor from the category $\mathcal{AbGroup}$ of abelian groups to itself.

(1) (a) Let (C_*, ∂_*) be a chain complex. Given any $n \in \mathbb{N}_0$ we see that
$$F(\partial_{n-1}) \circ F(\partial_n) \underset{\uparrow}{=} F(\partial_{n-1} \circ \partial_n) \underset{\uparrow}{=} F(0) \underset{\uparrow}{=} 0.$$
since F is a covariant functor; since (C_*, ∂_*) is a chain complex; by the Additive Functor–Basics Lemma 52.1 (1)

(b) Let $\varphi\colon (C_*, \partial_*^C) \to (D_*, \partial_*^D)$ be a chain map. Given any $n \in \mathbb{N}_0$ we have
$$F(\varphi_n) \circ F(\partial_n^C) \underset{\uparrow}{=} F(\varphi_n \circ \partial_n^C) \underset{\uparrow}{=} F(\partial_n^D \circ \varphi_n) \underset{\uparrow}{=} F(\partial_n^D) \circ F(\varphi_n).$$
since F is a covariant functor; since φ_* is a chain map; since F is a covariant functor

(2) This statement follows easily from the definitions.
(3) Let $\varphi, \psi\colon C_* \to D_*$ be chain maps between two chain complexes (C_*, ∂_*^C) and (D_*, ∂_*^D). Furthermore let $\{P_n\colon C_n \to D_{n+1}\}_{n \in \mathbb{N}_0}$ be a chain homotopy between the chain maps

φ and ψ. Given any $n \in \mathbb{N}_0$ we have

$$\begin{array}{rcll} F(\partial^D_{n+1}) \circ F(P_n) + F(P_{n-1}) \circ F(\partial^C_n) & \overset{\text{since } F \text{ is an additive functor}}{\overset{\downarrow}{=}} & F(\partial^D_{n+1} \circ P_n + P_{n-1} \circ \partial^C_n) & \\ & \underset{\text{since the } P_n \text{ form a chain homotopy between } \varphi \text{ and } \psi}{\overset{\uparrow}{=}} & F(\varphi_n - \psi_n) & \underset{\text{since } F \text{ is additive}}{\overset{\uparrow}{=}} F(\varphi_n) - F(\psi_n). \end{array}$$

We have thus shown that the homomorphisms $F(P_n)\colon F(C_n) \to F(D_{n+1})$ form a chain homotopy between the chain maps $F(\varphi)$ and $F(\psi)$. ■

Fortunately the tensor product from the last chapter gives us an interesting additive covariant functor from the category $\mathcal{AbGroup}$ of abelian groups to itself:

Lemma 52.3. (Tensor–Functor Lemma) Let G be an abelian group. The maps

$$\begin{array}{rcl} A & \mapsto & A \otimes G \\ (f\colon A \to B) & \mapsto & (f \otimes \mathrm{id}_G\colon A \otimes G \to B \otimes G) \end{array}$$

define an additive covariant functor from the category $\mathcal{AbGroup}$ of abelian groups to itself.

Proof.

(1) We verify the two properties of a covariant functor as defined on page 95:
   (a) Let A be an abelian group. By the Tensor–Map Composition Lemma 51.2 (1) we have $\mathrm{id}_A \otimes \mathrm{id}_G = \mathrm{id}_{A\otimes G}$.
   (b) Let $f\colon A \to B$ and $g\colon B \to C$ be two homomorphisms between two abelian groups. It follows immediately from the Tensor–Map Composition Lemma 51.2 (2) that we have $(g \otimes \mathrm{id}_G) \circ (f \otimes \mathrm{id}_G) = (g \circ f) \otimes \mathrm{id}_G$. ■

(2) It follows from the Tensor–Map Composition Lemma 51.2 (3) that the functor is additive.

52.2. **Tensoring chain complexes.** In this section we will apply the results from the previous section to tensor a chain complex with a fixed abelian group. In the following section we will apply this principle to the singular chain complexes of a topological space.

Definition. Let

$$\ldots \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \ldots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \to 0$$

be a chain complex in the sense of the definition on page 454.

(1) Let G be an abelian group. It follows from the Tensor–Functor Lemma 52.3 together with the Functor-applied-to-Chain Complex Lemma 52.2 (1a) that the following sequence of homomorphisms

$$\ldots \to C_n \otimes G \xrightarrow{\partial_n \otimes \mathrm{id}_G} C_{n-1} \otimes G \xrightarrow{\partial_{n-1} \otimes \mathrm{id}_G} \ldots \xrightarrow{\partial_2 \otimes \mathrm{id}_G} C_1 \otimes G \xrightarrow{\partial_1 \otimes \mathrm{id}_G} C_0 \otimes G \to 0.$$

is again a chain complex. We refer to $\mathrm{H}_n(C_* \otimes G) = \mathrm{H}_n(C_* \otimes G, \partial_* \otimes \mathrm{id}_G)$ as the homology groups of C_* with G-coefficients.

(2) If we tensor the chain complex C_* with a commutative ring R, then it follows from the Tensor-with-Ring Lemma 51.7 that we obtain a chain complex of R-modules, i.e. all chain groups are R-modules and all chain maps are R-homomorphisms. In particular also all homology groups are R-modules.

The following lemma collects some low-hanging mathematical fruit.

Lemma 52.4. (Homology-with-Coefficients Lemma) Let G be an abelian group.

(1) (a) If $f\colon C_* \to D_*$ is a chain map between two chain complexes C_* and D_*, then the maps $f \otimes \mathrm{id}_G\colon C_* \otimes G \to D_* \otimes G$ also form a chain map. Thus for each $n \in \mathbb{N}_0$ we get an induced homomorphism

$$(f \otimes \mathrm{id}_G)_*\colon \mathrm{H}_n(C_* \otimes G) \;\to\; \mathrm{H}_n(D_* \otimes G).$$

(b) Given $n \in \mathbb{N}_0$ the maps

$$\begin{array}{rcl} C_* & \mapsto & \mathrm{H}_n(C_* \otimes G) \\ (f\colon C_* \to D_*) & \mapsto & (\underbrace{(f \otimes \mathrm{id}_G)_*}_{=:f_*}\colon \mathrm{H}_n(C_* \otimes G) \to \mathrm{H}_n(D_* \otimes G)) \end{array}$$

define a covariant functor from the category $\mathcal{ChCplx}$ of chain complexes to the category $\mathcal{AbGroup}$ of abelian groups. Furthermore, if G is a commutative ring, then we get a functor to the category $\mathcal{Mod}_G$ of G-modules.

(2) (a) Let $f, g\colon C_* \to D_*$ be chain maps between two chain complexes (C_*, ∂_*) and (D_*, ∂_*). If $\{P_n\colon C_n \to D_{n+1}\}_{n\in\mathbb{N}_0}$ is a chain homotopy between f and g, then the maps $\{P_n \otimes \mathrm{id}_G\colon C_n \otimes G \to D_{n+1} \otimes G\}_{n\in\mathbb{N}_0}$ form a chain homotopy between $f \otimes \mathrm{id}_G$ and $g \otimes \mathrm{id}_G$. It follows from the Chain Homotopy–H_*-Lemma 36.1 that we have

$$(f \otimes \mathrm{id}_G)_* = (g \otimes \mathrm{id}_G)_*\colon \mathrm{H}_n(C_* \otimes G) \;\to\; \mathrm{H}_n(D_* \otimes G).$$

(b) Let C_* and D_* be chain complexes. If $f\colon C_* \to D_*$ is a chain homotopy equivalence, then $f \otimes \mathrm{id}_G\colon C_* \otimes G \to D_* \otimes G$ is also a chain homotopy equivalence. It follows immediately from the Chain Homotopy Equivalence-H_*-Corollary 36.4, that the induced map $(f \otimes \mathrm{id}_G)_*\colon \mathrm{H}_n(C_* \otimes G) \to \mathrm{H}_n(D_* \otimes G)$ is an isomorphism.

Proof. All statements follow almost immediately from the Tensor–Functor Lemma 52.3 together with the Functor-applied-to-Chain Complex Lemma 52.2. ∎

One of the most important tools in homology is the long exact sequence of homology groups corresponding to a short exact sequence of chain complexes. In the setting of homology with coefficients this requires a little bit of care.

Proposition 52.5. (LES-with-Coefficients Proposition) Let G be an abelian group and let $0 \to A_* \xrightarrow{i} B_* \xrightarrow{p} C_* \to 0$ be a short exact sequence of chain complexes such that all chain groups C_* are free abelian. We get natural connecting homomorphisms $\partial_k\colon \mathrm{H}_k(C_* \otimes G) \to \mathrm{H}_{k-1}(A_* \otimes G)$ and a long exact sequence

$$\ldots \to \mathrm{H}_k(A_* \otimes G) \xrightarrow{(i\otimes\mathrm{id}_G)_*} \mathrm{H}_k(B_* \otimes G) \xrightarrow{(p\otimes\mathrm{id}_G)_*} \mathrm{H}_k(C_* \otimes G) \xrightarrow{\partial_k} \mathrm{H}_{k-1}(A_* \otimes G) \to \ldots$$

in homology with G-coefficients. Furthermore, if G is a commutative ring, then the connecting homomorphisms are in fact G-module homomorphisms.

The proof of the proposition requires some preparations. First let us recall the following definition from page 480:

Definition. We say that a short exact sequence

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$$

of groups splits, if there exists a splitting $s\colon C \to B$, i.e. a homomorphism $s\colon C \to B$ such that $p \circ s = \mathrm{id}_C$.

Lemma 52.6. (Tensor-Split-SES Lemma) Let

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$$

be a short exact sequence of abelian groups and let G be an abelian group. If the short exact sequence splits[419], then

$$0 \to A \otimes G \xrightarrow{i\otimes \mathrm{id}} B \otimes G \xrightarrow{p\otimes \mathrm{id}} C \otimes G \to 0$$

is also a short exact sequence and it also splits.

Proof of the Tensor-Split-SES Lemma 52.6. We start out with the following claim:

Claim. Let $0 \to F \xrightarrow{\alpha} G \xrightarrow{\beta} H \to 0$ be a sequence of homomorphisms between abelian groups. The following two statements are equivalent:

(1) The sequence is exact and the sequence splits.
(2) We have $\beta \circ \alpha = 0$ and there exist homomorphisms $\sigma\colon G \to F$ and $\tau\colon H \to G$ such that $\sigma \circ \alpha = \mathrm{id}_F$, $\beta \circ \tau = \mathrm{id}_H$ and $\alpha \circ \sigma + \tau \circ \beta = \mathrm{id}_G$.

Proof. The (1)⇒(2) Statement follows almost immediately from the Splitting Lemma 37.2. Conversely the (2)⇒(1)-Statement can easily be verified by hand. ⊞

Let

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$$

be a short exact sequence of abelian groups that splits and let G be an abelian group. We pick homomorphisms $\sigma\colon B \to A$ and $\tau\colon C \to B$ as in the claim. It follows from the Tensor–Map Composition Lemma 51.2 (2) and (3) that $(p\otimes\mathrm{id}_G)\circ(\iota\otimes\mathrm{id}_G) = (p\circ\iota)\otimes\mathrm{id}_G = 0\otimes\mathrm{id}_G = 0$ and that the homomorphisms $\sigma\otimes\mathrm{id}_G\colon B\otimes G \to A\otimes G$ and $\tau\otimes\mathrm{id}_G\colon C\otimes G \to B\otimes G$ have the desired properties for the sequence

$$0 \to A \otimes G \xrightarrow{i\otimes \mathrm{id}_G} B \otimes G \xrightarrow{p\otimes \mathrm{id}_G} C \otimes G \to 0.$$

It follows from the claim that this sequence is exact and that it splits. ■

Proof of the LES-with-Coefficients Proposition 52.5. Let G be an abelian group and let $0 \to A_* \to B_* \to C_* \to 0$ be a short exact sequence of chain complexes such that all chain groups C_* are free abelian. Since each chain group C_* is free abelian we obtain from the Free Abelian Group–Splitting Lemma 18.5 that each of the above short exact sequences splits. It follows from this observation together with the Tensor-Split-SES Lemma 52.6 and Statement (1) that $0 \to A_* \otimes G \to B_* \otimes G \to C_* \otimes G \to 0$ is again a short exact sequence of chain complexes. Thus we see that the Connecting Homomorphism Lemma 37.3 and the LES Proposition 37.4 apply in the present context. Finally, if G is a commutative ring, then one can easily verify that the connecting homomorphisms are G-module homomorphisms. ■

The following lemma shows that, not surprisingly, tensoring with $\mathbb{Z}$ has no real effect on the homology groups.

[419] Recall that by the Free Abelian Group–Splitting Lemma 18.5 this happens if C is free abelian.

Lemma 52.7. (Homology-$\mathbb{Z}$-Coefficients Lemma) Let (C_*, ∂_*) be a chain complex. The natural isomorphisms $\Theta\colon C_k \otimes \mathbb{Z} \to C_k$ from the Tensor-with-$\mathbb{Z}$–Lemma 51.3 define a natural isomorphism of chain complexes, in particular for each $k \in \mathbb{N}_0$ we obtain natural isomorphisms

$$\Theta\colon \mathrm{H}_k(C_* \otimes \mathbb{Z}, \partial_* \otimes \mathrm{id}_{\mathbb{Z}}) \;\to\; \mathrm{H}_k(C_*, \partial_*).$$

Proof. It follows from the Tensor-with-$\mathbb{Z}$–Lemma 51.3 that for each $k \in \mathbb{N}_0$ we obtain the following *commutative* diagram with vertical isomorphisms:

$$\begin{array}{ccccccccc}
\ldots & \longrightarrow & C_{k+1} \otimes \mathbb{Z} & \xrightarrow{\partial_{k+1} \otimes \mathrm{id}_{\mathbb{Z}}} & C_k \otimes \mathbb{Z} & \xrightarrow{\partial_k \otimes \mathrm{id}_{\mathbb{Z}}} & C_{k-1} \otimes \mathbb{Z} & \longrightarrow & \ldots \\
 & & \Theta \downarrow \cong & & \Theta \downarrow \cong & & \Theta \downarrow \cong & & \\
\ldots & \longrightarrow & C_{k+1} & \xrightarrow{\partial_{k+1}} & C_k & \xrightarrow{\partial_k} & C_{k-1} & \longrightarrow & \ldots
\end{array}$$

But this observation gives us precisely the desired statement. ∎

The following elementary lemma shows that homology groups are "covariantly functorial in the coefficients".

Lemma 52.8. (Homology-Coefficients-Functor Lemma) Let C_* be a chain complex and let $\varphi\colon G \to \widetilde{G}$ be a homomorphism of abelian groups. The maps

$$\mathrm{id}_{C_*} \otimes \varphi\colon C_* \otimes G \;\to\; C_* \otimes \widetilde{G}$$

form a natural chain map. In particular for each $n \in \mathbb{N}_0$ we obtain a natural induced homomorphism

$$(\mathrm{id}_{C_*} \otimes \varphi)_*\colon \mathrm{H}_n(C_* \otimes G) \;\to\; \mathrm{H}_n(C_* \otimes \widetilde{G}).$$

Here "natural" means that for any chain map $f\colon C_* \to D_*$ we get a commutative diagram

$$\begin{array}{ccc}
\mathrm{H}_n(C_* \otimes G) & \xrightarrow{(\mathrm{id}_{C_*} \otimes \varphi)_*} & \mathrm{H}_n(C_* \otimes \widetilde{G}) \\
{\scriptstyle (f \otimes \mathrm{id}_G)_*}\downarrow & & \downarrow{\scriptstyle (f \otimes \mathrm{id}_{\widetilde{G}})_*} \\
\mathrm{H}_n(D_* \otimes G) & \xrightarrow{(\mathrm{id}_{C_*} \otimes \varphi)_*} & \mathrm{H}_n(D_* \otimes \widetilde{G}).
\end{array}$$

Example. Let R be a commutative ring with neutral element 1_R. We have a natural homomorphism $\varphi\colon \mathbb{Z} \to R$ that is given by $n \mapsto n \cdot 1_R$. We get the following natural homomorphisms

$$\mathrm{H}_k(C_*) \quad \xleftarrow[\uparrow]{\cong} \quad \mathrm{H}_k(C_* \otimes \mathbb{Z}) \quad \xrightarrow[\uparrow]{(\mathrm{id} \otimes \varphi)_*} \quad \mathrm{H}_k(C_* \otimes R).$$

by the Homology-$\mathbb{Z}$-Coefficients Lemma 52.7 — from the Homology-Coefficients-Functor Lemma 52.8 □

Proof. Given $k \in \mathbb{N}_0$ we consider the following diagram:

$$\begin{array}{ccc}
C_k \otimes G & \xrightarrow{\mathrm{id}_{C_k} \otimes \varphi} & C_k \otimes \widetilde{G} \\
{\scriptstyle \partial_k \otimes \mathrm{id}_G}\downarrow & & \downarrow{\scriptstyle \partial_k \otimes \mathrm{id}_{\widetilde{G}}} \\
C_{k-1} \otimes G & \xrightarrow{\mathrm{id}_{C_{k-1}} \otimes \varphi} & C_{k-1} \otimes \widetilde{G}.
\end{array}$$

It follows immediately from the Tensor–Map Composition Lemma 51.2 (2) that the diagram commutes. This means precisely that the horizontal maps are chain maps. The proof that the horizontal maps are natural follows by the same type of argument. ∎

52.3. **The homology groups of a topological space with coefficients.** Now we are ready to define the homology groups of a (pair of) topological spaces with coefficients in an abelian group G.

Definition. Let (X, A) be a pair of topological spaces. For an abelian group G we define

$$\begin{aligned} \mathrm{C}_*(X, A; G) &:= \mathrm{C}_*(X, A) \otimes G \\ \mathrm{H}_k(X, A; G) &:= \mathrm{H}_k(\mathrm{C}_*(X, A; G), \partial_* \otimes \mathrm{id}_G). \end{aligned}$$

We call $\mathrm{H}_k(X, A; G)$ the k-th homology group of (X, A) with G-coefficients. If $A = \varnothing$, then we just drop it from the notation.

We start out with the following three observations. Let (X, A) be a pair of topological spaces.

(1) If the coefficients are given by a commutative ring R, e.g. if $R = \mathbb{Z}_n$ or $R = \mathbb{Q}, \mathbb{R}, \mathbb{C}$, then, as we pointed out above, each $\mathrm{C}_n(X, A; R) = \mathrm{C}_n(X, A) \otimes R$ is an R-module, each boundary map is an R-module homomorphism, and each homology group $\mathrm{H}_n(X, A; R)$ is also an R-module.

(2) It follows from the Homology-$\mathbb{Z}$-Coefficients Lemma 52.7 that we have natural isomorphisms

$$\mathrm{C}_*(X, A; \mathbb{Z}) \xrightarrow{\cong} \mathrm{C}_*(X, A)$$

of chain complexes and natural isomorphisms

$$\mathrm{H}_*(X, A; \mathbb{Z}) \xrightarrow{\cong} \mathrm{H}_*(X, A)$$

of abelian groups. In the following we will identify these chain complexes and homology groups and we will go back and forth between these two points of view, depending on which one is more convenient.

(3) Let G be an abelian group. It follows easily from the Tensor–Direct Sum Lemma 51.6 together with the Tensor-with-$\mathbb{Z}$–Lemma 51.3 (1) and the remark on page 489 that any element in $\mathrm{C}_k(X, A; G)$ can be written uniquely as $\sum_{i=1}^{n} \sigma_i \otimes g_i$ where the following holds:
   (a) $\sigma_1, \ldots, \sigma_n \colon \Delta^k \to X$ are pairwise distinct singular k-simplices whose images are not entirely contained in A.
   (b) $g_1, \ldots, g_n \in G \setminus \{0\}$.

Most results for homology groups of (pairs of) topological spaces carry over, with the obvious modifications, to the theory of homology groups with G-coefficients. For example, given an abelian group G we obtain from the Homology-with-Coefficients Lemma 52.4 the following statements:

(1) (a) Let $f \colon (X, A) \to (Y, B)$ be a continuous map between two pairs of topological spaces. It follows from the C_*–Functor Lemma 35.5 (and the Relative-H_*–Functor Lemma 38.1) together with the Homology-with-Coefficients Lemma 52.4 (1a) that

$$f_* \otimes \mathrm{id} \colon \mathrm{C}_*(X, A; G) \to \mathrm{C}_*(Y, B; G)$$

is a chain map which in turn, for each $k \in \mathbb{N}_0$, induces a homomorphism

$$(f_* \otimes \mathrm{id})_* \colon \mathrm{H}_k(X, A; G) \to \mathrm{H}_k(Y, B; G).$$

To simplify the notation we usually denote this induced map on homology groups also by f_*.

(b) It follows from the C_*–Functor Lemma 35.5 (and the Relative-H_*–Functor Lemma 38.1) together with the Homology-with-Coefficients Lemma 52.4 (1b) that for any $k \in \mathbb{N}_0$ the maps

$$\begin{array}{rcl} (X,A) & \mapsto & \mathrm{H}_k(X,A;G) \\ (f\colon (X,A)\to(Y,B)) & \mapsto & (f_* = (f_*\otimes \mathrm{id}_G)_*\colon \mathrm{H}_k(X,A;G)\to \mathrm{H}_k(Y,B;G)) \end{array}$$

define a covariant functor from the category *PairTop* of pairs of topological spaces to the category *AbGroup* of abelian groups.

As we mentioned in the Homology-with-Coefficients Lemma 52.4 (1b), if G is a commutative ring, then the above induced maps are G-module homomorphisms and we get a covariant functor from the category *PairTop* of pairs of topological spaces to the category $\mathcal{M}od_G$ of G-modules.

(2) (a) In the Homotopy-H_*-Proposition 36.5 we showed that if $f, g\colon X\to Y$ are homotopic maps between topological spaces, then the induced maps $f_*, g_*\colon C_*(X)\to C_*(Y)$ are chain homotopic. It follows from the Homology-with-Coefficients Lemma 52.4 (2a) that for any $n\in\mathbb{N}_0$ we have

$$f_* = g_*\colon \mathrm{H}_n(X;G) \ \to\ \mathrm{H}_n(Y;G).$$

(b) It follows from (a) that if X and Y are homotopy equivalent topological spaces, then for any $n\in\mathbb{N}_0$ we have an isomorphism

$$\mathrm{H}_n(X;G) \ \xrightarrow{\cong}\ \mathrm{H}_n(Y;G).$$

(3) Let X be a topological space with path-components $\{X_a\}_{a\in A}$. Given $a\in A$ we denote by $i_a\colon X_a\to X$ the inclusion map. It follows from the Path-Component-C_*-Lemma 35.6, the Tensor–Direct Sum Lemma 51.6 and the Direct Sum-H_*-Lemma 35.2 that for each $n\in\mathbb{N}_0$ the map

$$\bigoplus_{a\in A} i_{a*}\colon \bigoplus_{a\in A}\mathrm{H}_n(X_a;G)\ \to\ \mathrm{H}_n(X;G)$$

is an isomorphism. This result is the analogue of the Path-Component-H_*-Lemma 35.8.



$i_{1*}\oplus i_{2*}\colon \mathrm{H}_n(X_1;G)\oplus \mathrm{H}_n(X_2;G)\xrightarrow{\cong}\mathrm{H}_n(X_1\sqcup X_2;G)$

(4) (a) Precisely as on page 497 we can use the natural augmentation map

$$\begin{array}{rcl} \epsilon_X\colon \mathrm{C}_0(X;G) & \to & G \\ \sum\limits_{i=1}^{k}\sigma_i\otimes g_i & \mapsto & \sum\limits_{i=1}^{k} g_i. \end{array}$$

to define reduced homology groups $\widetilde{\mathrm{H}}_k(X;G)$ with G-coefficients.

(b) One can easily verify that the augmentation homomorphism from (a) descends to a natural augmentation homomorphism

$$\begin{array}{rcl} \epsilon_X\colon \mathrm{H}_0(X;G) & \to & G \\ \left[\sum\limits_{i=1}^{k}\sigma_i\otimes g_i\right] & \mapsto & \sum\limits_{i=1}^{k} g_i. \end{array}$$

If X is path-connected and non-empty, then the argument of the H_0-Proposition 35.11 implies immediately that the augmentation map $\epsilon\colon \mathrm{H}_0(X;G) \to G$ is an isomorphism.

(c) If $f\colon X \to Y$ is a continuous map between path-connected non-empty topological spaces, then one can easily show, as in the H_0-Proposition 35.11, that the induced map $f_*\colon \mathrm{H}_0(X;G) \to \mathrm{H}_0(Y;G)$ is an isomorphism.

All the other earlier results on reduced homology groups also have obvious analogues for reduced homology with G-coefficients. There is surely no need to spell out all of these results. The earlier results on reduced homology groups have obvious analogues for reduced homology with G-coefficients.

(5) Let (X, B, A) be a triple of topological spaces. As we mentioned in the proof of the LES-of-Triple Proposition 38.2, it follows from the third isomorphism theorem that we have the following short exact sequence of chain complexes:

$$0 \;\to\; \mathrm{C}_n(B,A) \;\xrightarrow{i}\; \mathrm{C}_n(X,A) \;\xrightarrow{p}\; \mathrm{C}_n(X,B) \;\to\; 0.$$

As we pointed out on page 489, the relative chain groups $\mathrm{C}_n(X,B)$ are free abelian groups. Note that it follows immediately from these two observations and the LES-with-Coefficients Proposition 52.5 that we obtain natural connecting homomorphisms $\partial_n\colon \mathrm{H}_n(X,B;G) \to \mathrm{H}_{n-1}(B,A;G)$ which give rise to the following long exact sequence of homology groups with G-coefficients:

$$\ldots \to \mathrm{H}_n(B,A;G) \xrightarrow{i_*} \mathrm{H}_n(X,A;G) \xrightarrow{p_*} \mathrm{H}_n(X,B;G) \xrightarrow{\partial_n} \mathrm{H}_{n-1}(B,A;G) \to \ldots$$

If G is a commutative ring, then the connecting homomorphism is a G-module homomorphism.

(6) (a) Let X be a topological space and let $Z \subset A \subset X$ be subsets. We assume that the closure of Z is contained in the interior of A. It follows from the Excision Theorem 40.1, the fact that relative chain groups are free abelian groups, the Quasi Iso–Homotopy Equivalence Proposition 50.4 and the Homology-with-Coefficients Lemma 52.4 (2) that for each $n \in \mathbb{N}_0$ the inclusion $(X \setminus Z, A \setminus Z) \to (X, A)$ induces an isomorphism

$$\mathrm{H}_n(X \setminus Z, A \setminus Z; G) \;\xrightarrow{\cong}\; \mathrm{H}_n(X,A;G).$$



$$\mathrm{H}_n(X \setminus Z, A \setminus Z; G) \;\xrightarrow{\cong}\; \mathrm{H}_n(X,A;G)$$

(b) We have an obvious version of the Mayer–Vietoris Theorem 45.1 for homology groups with G-coefficients. We can either deduce this version from (a) and the argument of Section 45.1.1. Alternatively we can argue as in Section 45.1.2. For this argument we only have to note that when we consider the exact sequence

$$0 \;\to\; \mathrm{C}_*(A \cap B) \xrightarrow{(i^A_{A\cap B}, -i^B_{A\cap B})} \mathrm{C}_*(A) \oplus \mathrm{C}_*(B) \xrightarrow{i^X_A \oplus i^X_B} \underbrace{\mathrm{C}_*^{\{A,B\}}(X)}_{\text{free abelian}} \;\to\; 0,$$

then it is a consequence of the Tensor-Split-SES Lemma 52.6 together with the Free Abelian Group–Splitting Lemma 18.5 that the corresponding long exact sequence of homology groups with G-coefficients is also exact.

(7) For a CW-complex X we can also consider the corresponding cellular chain complex $\mathrm{C}_*^{\mathrm{CW}}(X;G) := \mathrm{C}_*^{\mathrm{CW}}(X) \otimes G$ and the corresponding cellular homology groups $\mathrm{H}_k^{\mathrm{CW}}(X;G) = \mathrm{H}_k(\mathrm{C}_*^{\mathrm{CW}}(X);G)$.

Recall that by the Singular-Cellular Chain Complex Proposition 50.9 we have natural chain homotopy equivalences

$$\mathrm{C}_*(X) \xleftarrow{i} \mathrm{C}_*^{\mathrm{int}}(X) \xrightarrow{p} \mathrm{C}_*^{\mathrm{CW}}(X).$$

It follows from the Homology-with-Coefficients Lemma 52.4 (2) that we obtain natural isomorphisms[420]

$$\mathrm{H}_k(X;G) \xleftarrow[\cong]{(i\otimes \mathrm{id}_G)_*} \mathrm{H}_k(\mathrm{C}_*^{\mathrm{int}}(X) \otimes G) \xrightarrow[\cong]{(p\otimes \mathrm{id}_G)_*} \mathrm{H}_*^{\mathrm{CW}}(X;G).$$

We can draw the following conclusions:

(a) If X is an n-dimensional CW-complex, then $\mathrm{H}_i(X;G) = 0$ for $i \geq n+1$.

(b) If X has only finitely many n-cells, then given any field $\mathbb{F}$ the vector space $\mathrm{H}_n(X;\mathbb{F})$ is finite-dimensional.

Next we record two important calculations of homology groups with coefficients as lemmas:

Proposition 52.9. (Sphere-H_*-Proposition) For any $n \in \mathbb{Z}_{\geq -1}$ and any abelian group G we have

$$\widetilde{\mathrm{H}}_k(S^n;G) \cong \begin{cases} G, & \text{if } k = n, \\ 0, & \text{if } k \neq n. \end{cases}$$

Proof. We can calculate these homology groups by exactly the same argument as on page 504 or as in the proof of the original Sphere-H_*-Proposition 41.2. Alternatively we can equip S^n with the CW-structure with one 0-cell and one n-cell, which we introduced on page 407. With this CW-structure we can easily calculate the cellular homology groups $\mathrm{H}_k^{\mathrm{CW}}(S^n;G)$. By the discussion on page 662 we obtain this way also the singular homology groups $\mathrm{H}_k(S^n;G)$. Using the obvious analogue of the Reduced Homology Lemma 39.2 we finally obtain the reduced homology groups $\widetilde{\mathrm{H}}_k(S^n;G)$. ∎

The next calculation is more interesting and more surprising.

Proposition 52.10. ($\mathbb{R}\mathrm{P}^n$-H_*-Proposition)

(1) Let $n \in \mathbb{N}_0$. Given $k \in \mathbb{N}_0$ we have

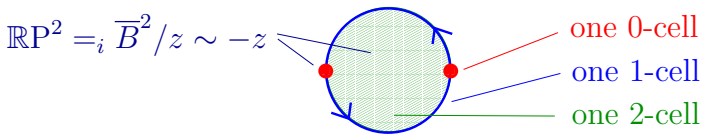
$$\mathrm{H}_k(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2) \cong \begin{cases} \mathbb{F}_2, & \text{if } k = 0,\dots,n \\ 0, & \text{if } k > n. \end{cases}$$

(2) For every $k \in \mathbb{N}_0$ we have $\mathrm{H}_k(\mathbb{R}\mathrm{P}^\infty;\mathbb{F}_2) \cong \mathbb{F}_2$.

Proof.

(1) Let $n \in \mathbb{N}_0$. As usual we equip $\mathbb{R}\mathrm{P}^n$ with the CW-structure that we introduced on page 410 and which has precisely one cell in dimensions $0,\dots,n$.

[420] Alternatively we could easily modify the proof of the Singular-Cellular-H_*-Proposition 47.3 to see that there exists a natural isomorphism $\mathrm{H}_*(X;G) \xrightarrow{\cong} \mathrm{H}_*^{\mathrm{CW}}(X;G)$.



In the proof of the $\mathbb{R}\mathrm{P}^n$-H_*-Proposition 47.10 we saw that the cellular chain complex is given by

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot(1+(-1)^n)} \mathbb{Z} \longrightarrow \ldots \longrightarrow \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0$$

We tensor the above cellular chain complex with $\mathbb{F}_2$. Using the Matrix–Tensor-with-Ring Lemma 51.8 we obtain the following commutative diagram with vertical isomorphisms:

$$\begin{array}{ccccccccccccc}
0 & \to & \mathbb{Z}\otimes\mathbb{F}_2 & \xrightarrow{(\cdot(1+(-1)^n))\otimes\mathrm{id}_{\mathbb{F}_2}} & \mathbb{Z}\otimes\mathbb{F}_2 & \to \ldots \to & \mathbb{Z}\otimes\mathbb{F}_2 & \xrightarrow{(\cdot 2)\otimes\mathrm{id}_{\mathbb{F}_2}} & \mathbb{Z}\otimes\mathbb{F}_2 & \xrightarrow{0\otimes\mathrm{id}_{\mathbb{F}_2}} & \mathbb{Z}\otimes\mathbb{F}_2 & \to & 0 \\
 & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \\
0 & \longrightarrow & \mathbb{F}_2 & \xrightarrow{0} & \mathbb{F}_2 & \xrightarrow{\cdot 0} \ldots \longrightarrow & \mathbb{F}_2 & \xrightarrow{\cdot 0} & \mathbb{F}_2 & \xrightarrow{\cdot 0} & \mathbb{F}_2 & \longrightarrow & 0.
\end{array}$$

We now see that

$$\mathrm{H}_k(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2) \underset{\uparrow}{\cong} \mathrm{H}_k^{\mathrm{CW}}(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2) \underset{\uparrow}{\cong} \begin{cases} \mathbb{F}_2, & \text{if } k = 0,\ldots,n, \\ 0, & \text{otherwise.} \end{cases}$$

since by the discussion on page 662 we know that singular and cellular homology groups are isomorphic

by the above commutative diagram

(2) For $\mathbb{R}\mathrm{P}^\infty$ we use the CW-structure from page 410 which has precisely one n-cell in each dimension. The same argument as in (1) gives us the desired result. ■

We conclude this chapter with the following analogue of the Homology-χ-Proposition 48.1:

Proposition 52.11. (Homology-χ-Proposition) Let X be a finite CW-complex. For any field $\mathbb{F}$ we have

$$\chi(X) \;=\; \sum_i (-1)^i \cdot \dim_{\mathbb{F}}(\mathrm{H}_i(X;\mathbb{F})).$$

Example. Let us consider the real projective plane $\mathbb{R}\mathrm{P}^2$. By the $\mathbb{R}\mathrm{P}^n$-H_*-Propositions 47.10 and 52.10 we have the following table:

	$k=3$	$k=2$	$k=1$	$k=0$
$\mathrm{H}_k(\mathbb{R}\mathrm{P}^2)$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}$
$\mathrm{H}_k(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2)$	0	$\mathbb{F}_2$	$\mathbb{F}_2$	$\mathbb{F}_2$

But even though the "classical homology groups" of $\mathbb{R}\mathrm{P}^2$ and the homology groups with $\mathbb{F}_2$-coefficients are quite different, in both cases the alternating sums of the ranks and dimensions give, as promised in the Homology-χ-Propositions 48.1 and 52.11, the Euler

characteristic. Indeed, we have

$$\operatorname{rank}(\mathrm{H}_2(\mathbb{R}\mathrm{P}^2)) - \operatorname{rank}(\mathrm{H}_1(\mathbb{R}\mathrm{P}^2)) + \operatorname{rank}(\mathrm{H}_0(\mathbb{R}\mathrm{P}^2)) \;=\; 0 - 0 + 1 \;=\; 1,$$

and

$$\dim_{\mathbb{F}_2}(\mathrm{H}_2(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2)) - \dim_{\mathbb{F}_2}(\mathrm{H}_1(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2)) + \dim_{\mathbb{F}_2}(\mathrm{H}_0(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2)) \;=\; 1 - 1 + 1 \;=\; 1.$$

□

Proof. [421] Our proof builds on the following claim, which is a variation on the Chain Complex–H_*–χ-Lemma 48.3.

Claim. Given any chain complex

$$0 \;\to\; V_k \xrightarrow{\partial_k} V_{k-1} \xrightarrow{\partial_{k-1}} \dots\; V_1 \xrightarrow{\partial_1} V_0 \;\to\; 0$$

of finite-dimensional $\mathbb{F}$-modules the following equality holds:

$$\sum_{n=0}^{k} (-1)^n \cdot \dim_{\mathbb{F}}(V_n) \;=\; \sum_{n=0}^{k} (-1)^n \cdot \dim_{\mathbb{F}}(\mathrm{H}_n(V_*)).$$

Proof. The proof builds is almost identical to the proof of the Chain Complex–H_*–χ-Lemma 48.3. All we need to do is to replace the additivity of ranks under short exact sequences by the well-known fact that for any short exact sequence $0 \to A \to B \to C \to 0$ of finite dimensional $\mathbb{F}$-vector spaces we have

$$\dim_{\mathbb{F}}(B) \;=\; \dim_{\mathbb{F}}(A) + \dim_{\mathbb{F}}(C).$$

⊞

Now let X be a finite k-dimensional CW-complex. As on page 607 we perform the following calculation:

$$\begin{array}{ccccc}
 & & \text{by definition of } \chi(X) & & \text{by the Skeleton-H}_*\text{-Lemma 47.1 (1) and the Matrix–Tensor-with-Ring Lemma 51.8 (1)} \\
 & & \downarrow & & \downarrow \\
\chi(X) & = & \sum_{n=0}^{k} (-1)^n \cdot \text{number of } n\text{-cells} & = & \sum_{n=0}^{k} (-1)^n \cdot \dim_{\mathbb{F}}(\mathrm{C}_n^{\mathrm{CW}}(X) \otimes \mathbb{F}) \\
 & = & \sum_{n=0}^{k} (-1)^n \cdot \dim_{\mathbb{F}}(\mathrm{H}_n^{\mathrm{CW}}(X;\mathbb{F})) & = & \sum_{n=0}^{k} (-1)^n \cdot \dim_{\mathbb{F}}(\mathrm{H}_n(X;\mathbb{F})). \\
 & & \uparrow & & \uparrow \\
 & & \text{by the claim} & & \text{by the discussion on page 662}
\end{array}$$

■

Remark. Why should we study the homology groups of a topological space with G-coefficients? As we mentioned in the beginning of the chapter, there are several answers one can give:

(1) If we work over the field $\mathbb{F}_2$ then many calculations become particularly simple, since we do not need to worry about signs. Even in the previous example we already saw that working with $\mathbb{F}_2$-coefficients can simplify calculations.
(2) The theory of (finite dimensional) vector spaces is much easier than the theory of (finitely generated) abelian groups. Thus on many occasions it is easier to use homology with coefficients in a field, then to work with the "tradition" homology groups.
(3) We will use homology groups with coefficients for more theoretic purposes, e.g. in the proof of the Borsuk–Ulam Theorem.

□

Exercise 52.1. Let M and M' be two oriented connected n-dimensional smooth manifolds with non-empty boundary. We consider the boundary connected sum $M \#_{\mathrm{b}} M'$ that we

[421] The proof of this proposition is of course very similar to the proof of the Homology-χ-Proposition 48.1.

introduced on page **??**. Show that for any $i \geq 1$ and any commutative ring R the inclusion maps induce an isomorphism

$$\mathrm{H}_i(M;R) \oplus \mathrm{H}_i(M';R) \xrightarrow{\cong} \mathrm{H}_i(M \#_{\mathrm{b}} M';R).$$

Remark. Use the Boundary Connected Sum Proposition **??** (3) and collars or tubular neighborhoods.

CHAPTER 53

Exact functors and homology

The following question naturally arises from the discussion in the previous chapter:

Question 53.1. Given a topological space X and an abelian group G, what is the relationship between the usual homology groups of X and the homology groups with G-coefficients?

We will discuss this question in great detail in this and in the following chapter.

53.1. **Exact functors.** Before we can hope to answer Question 53.1 we have to introduce many more definitions and algebraic tools. We start out with the following definition from category theory.

Definition. Let $F\colon \mathcal{AbGroup} \to \mathcal{AbGroup}$ be an additive covariant functor from the category $\mathcal{AbGroup}$ of abelian groups to itself.

(1) We say that F is left-exact, if for every exact sequence[422]

$$0 \longrightarrow A \xrightarrow{\varphi} B \xrightarrow{\psi} C$$

of abelian groups the sequence

$$0 \longrightarrow F(A) \xrightarrow{F(\varphi)} F(B) \xrightarrow{F(\psi)} F(C)$$

is also exact.

(2) We say that F is right-exact, if for every exact sequence

$$A \xrightarrow{\varphi} B \xrightarrow{\psi} C \longrightarrow 0$$

of abelian groups the sequence

$$F(A) \xrightarrow{F(\varphi)} F(B) \xrightarrow{F(\psi)} F(C) \longrightarrow 0$$

is also exact.

(3) We say that F is exact if it is left-exact and right-exact.

Remark. Let $F\colon \mathcal{AbGroup} \to \mathcal{AbGroup}$ be an additive covariant functor and let $\varphi\colon A \to B$ be a homomorphism between abelian groups.

(1) If F is left-exact, then "taking F" commutes with "taking kernels". More precisely, if we apply the left-exact functor F to the exact sequence $0 \to \ker(\varphi) \xrightarrow{\iota} A \xrightarrow{\varphi} B$, then we obtain the *exact* sequence $0 \to F(\ker(\varphi)) \xrightarrow{F(\iota)} F(A) \xrightarrow{F(\varphi)} F(B)$. From the Exactness Lemma 37.1 (3) we obtain that

$$F(\ker(\varphi\colon A \to B)) \xrightarrow{F(\iota)} \ker(F(\varphi)\colon F(A) \to F(B))$$

[422] Note that we know by the Additive Functor–Basics Lemma 52.1 (2) that $F(0) \cong 0$.

is an isomorphism.

(2) If F is right-exact, then "taking F" commutes with "taking cokernels". More precisely, if we apply the right-exact functor F to the exact sequence $A \xrightarrow{\varphi} B \xrightarrow{\pi} \operatorname{coker}(\varphi) \to 0$, then we obtain the *exact* sequence $F(A) \xrightarrow{F(\varphi)} F(B) \xrightarrow{F(\pi)} F(\operatorname{coker}(\varphi)) \to 0$. From the Exactness Lemma 37.1 (3) we obtain that $F(\pi)$ induces an isomorphism

$$\operatorname{coker}(F(\varphi)\colon F(A) \to F(B)) \xrightarrow{\text{induced by } F(\pi)} F(\operatorname{coker}(\varphi)). \qquad \square$$

Lemma 53.2. (Exact Functor Lemma) Let $F\colon \mathcal{AbGroup} \to \mathcal{AbGroup}$ be an additive covariant functor. The following two statements are equivalent:

(1) F is exact.

(2) For every short exact sequence

$$0 \longrightarrow A \xrightarrow{\varphi} B \xrightarrow{\psi} C \longrightarrow 0$$

of abelian groups the sequence

$$0 \longrightarrow F(A) \xrightarrow{F(\varphi)} F(B) \xrightarrow{F(\psi)} F(C) \longrightarrow 0$$

is also exact.

Proof. We will prove this lemma in Exercise Sheet 2. ■

The following lemma shows why we are suddenly interested in *exact* additive covariant functors.

Lemma 53.3. (Exact Functor–Homology Lemma) Let $F\colon \mathcal{AbGroup} \to \mathcal{AbGroup}$ be an additive covariant functor and let (C_*, ∂_*) be a chain complex. If F is exact, then for any $k \in \mathbb{N}_0$ we have a natural isomorphism

$$F(\mathrm{H}_k(C_*, \partial_*)) \;\to\; \mathrm{H}_k(F(C_*), F(\partial_*)).$$

Proof. Let $F\colon \mathcal{AbGroup} \to \mathcal{AbGroup}$ be an *exact* additive covariant functor and let (C_*, ∂_*) be a chain complex. Let $k \in \mathbb{N}_0$. We denote by $\iota\colon \ker(\partial_k\colon C_k \to C_{k-1}) \to C_k$ the inclusion. Since $\operatorname{im}(\partial_{k+1}\colon C_{k+1} \to C_k) \subset \ker(\partial_k\colon C_k \to C_{k-1})$ we see that there exists a homomorphism $\widetilde{\partial}\colon C_{k+1} \to \ker(\partial_k\colon C_k \to C_{k-1})$ with $\partial_{k+1} = \iota \circ \widetilde{\partial}\colon C_{k+1} \to C_k$.

Now we consider the following diagram:

$$\begin{array}{ccccccccc}
 & & & & C_{k+1} & & & & \\
 & & & & \downarrow{\scriptstyle \widetilde{\partial}} & \searrow^{\partial_{k+1}} & & & \\
0 & \longrightarrow & \ker(\partial_k\colon C_k \to C_{k-1}) & \xrightarrow[\iota]{} & C_k & \xrightarrow[\partial_k]{} & C_{k-1} & & \\
 & & \downarrow{\scriptstyle \pi} & & & & & & \\
 & & \mathrm{H}_k(C_*) & & & & & & \\
 & & \downarrow & & & & & & \\
 & & 0. & & & & & &
\end{array}$$

Note that the vertical and the horizontal sequence are exact. Since F is an additive covariant functor we get the following commutative diagram:

$$\begin{array}{ccccccccc} & & & & F(C_{k+1}) & & & & \\ & & & & \downarrow{\scriptstyle F(\widetilde{\partial})} & \searrow^{F(\partial_{k+1})} & & & \\ 0 & \longrightarrow & F(\ker(\partial_k\colon C_k \to C_{k-1})) & \xrightarrow[F(\iota)]{} & F(C_k) & \xrightarrow[F(\partial_k)]{} & F(C_{k-1}) & & \\ & & \downarrow{\scriptstyle F(\pi)} & & & & & & \\ & & F(\mathrm{H}_k(C_*)) & & & & & & \\ & & \downarrow & & & & & & \\ & & 0. & & & & & & \end{array}$$

Since F is exact we see that the vertical and the horizontal sequence are still exact. It follows from the exactness that the following maps: are isomorphisms:

$$F(\mathrm{H}_k(C_*)) \xleftarrow{F(\pi)} \frac{F\Big(\ker(C_k \xrightarrow{\partial_k} C_{k-1})\Big)}{\mathrm{im}(F(\widetilde{\partial}))} \xrightarrow{F(\iota)} \frac{\ker\Big(F(C_k) \xrightarrow{F(\partial_k)} F(C_{k-1})\Big)}{\mathrm{im}\Big(F(C_{k+1}) \xrightarrow{F(\partial_{k+1})} F(C_k)\Big)} = \mathrm{H}_k(F(C_*)).$$

Since all the isomorphisms involved are natural we see also that the isomorphism from left to right is natural. ■

Example. Given a fixed abelian group G we have studied in great detail the covariant functor $A \mapsto A \otimes G$. For better or worse we will now see that in general this is not a *left-exact* functor. For example for any $k \in \mathbb{N}_{\geq 2}$ the sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot k} \mathbb{Z} \xrightarrow{n \mapsto [n]} \mathbb{Z}_k$$

is exact. But after tensoring with $G = \mathbb{Z}_k$ we obtain, by the Matrix–Tensor-with-Ring Lemma 51.8 and the Tensor-with-$\mathbb{Z}_k$–Lemma 51.4, the following diagram with vertical isomorphisms:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}_k & \xrightarrow[0]{\cdot k} & \mathbb{Z}_k & \xrightarrow{\mathrm{id}_{\mathbb{Z}_k}} & \mathbb{Z}_k & & \\ & & \cong\downarrow{\scriptstyle [n]\mapsto 1\otimes[n]} & & \cong\downarrow{\scriptstyle [n]\mapsto 1\otimes[n]} & & \cong\downarrow{\scriptstyle [n]\mapsto [1]\otimes[n]} & & \\ 0 & \longrightarrow & \mathbb{Z}\otimes\mathbb{Z}_k & \xrightarrow[(\cdot k)\otimes\mathrm{id}_{\mathbb{Z}_k}]{} & \mathbb{Z}\otimes\mathbb{Z}_k & \xrightarrow[(n\mapsto[n])\otimes\mathrm{id}_{\mathbb{Z}_k}]{} & \mathbb{Z}_k\otimes\mathbb{Z}_k & & \end{array}$$

It follows from $(k \cdot a) \otimes b = a \otimes (k \cdot b)$ that the diagram commutes. Note that the top left horizontal homomorphism $\mathbb{Z}_k \to \mathbb{Z}_k$ is not a monomorphism, thus the top horizontal sequence is not exact. But this implies that the bottom horizontal sequence is not exact either. □

In contrast to the previous example we will now see that tensoring with an abelian group G is *right-exact*.

Lemma 53.4. (Tensor–Right Exact Lemma) For every abelian group G the functor

$$\begin{array}{ccc} \mathcal{A}b\mathcal{G}roup & \to & \mathcal{A}b\mathcal{G}roup \\ A & \mapsto & A \otimes G \end{array}$$

is right-exact.

Proof. Let

$$A \xrightarrow{i} B \xrightarrow{j} C \to 0$$

be an exact sequence of abelian groups and let G be an abelian group. We have to show that the sequence

$$A \otimes G \xrightarrow{i\otimes \mathrm{id}_G} B \otimes G \xrightarrow{j\otimes \mathrm{id}_G} C \otimes G \to 0$$

is also exact.

(a) Since $j\colon B \to C$ is an epimorphism it follows immediately from the Tensor Product–Universal Property Lemma 51.1 (2) that the homomorphism

$$\begin{aligned} j \otimes \mathrm{id}_G\colon B \otimes G &\to C \otimes G \\ \sum_{i=1}^{n} b_i \otimes g_i &\mapsto \sum_{i=1}^{n} j(b_i) \otimes g_i \end{aligned}$$

is also an epimorphism.

(b) As on so many other occasions we obtain from the Tensor–Map Composition Lemma 51.2 that we have $(j \otimes \mathrm{id}_G) \circ (i \otimes \mathrm{id}_G) = (j \circ i) \otimes \mathrm{id}_G = 0$, i.e. we have

$$\mathrm{im}(i \otimes \mathrm{id}_G\colon A \otimes G \to B \otimes G) \subset \ker(j \otimes \mathrm{id}_G\colon B \otimes G \to C \otimes G).$$

Note that this implies that we get a well-defined homomorphism

$$\begin{aligned} \Psi\colon (B \otimes G)/\mathrm{im}(i \otimes \mathrm{id}_G\colon A \otimes G \to B \otimes G) &\to C \otimes G \\ \left[\sum_{i=1}^{k} b_i \otimes g_i\right] &\mapsto \sum_{i=1}^{k} j(b_i) \otimes g_i. \end{aligned}$$

(c) By (a) and (b) it remains to show that

$$\ker(j \otimes \mathrm{id}_G\colon B \otimes G \to C \otimes G) \subset \mathrm{im}(i \otimes \mathrm{id}_G\colon A \otimes G \to B \otimes G).$$

Put differently it remains to show that the above homomorphism

$$\begin{aligned} \Psi\colon (B \otimes G)/\mathrm{im}(i \otimes \mathrm{id}_G\colon A \otimes G \to B \otimes G) &\to C \otimes G \\ \left[\sum_{i=1}^{k} b_i \otimes g_i\right] &\mapsto \sum_{i=1}^{k} j(b_i) \otimes g_i \end{aligned}$$

is a monomorphism. We will do so by providing a left inverse for Ψ, i.e. we will provide a map $\Phi\colon C \otimes G \to (B \otimes G)/\mathrm{im}(i \otimes \mathrm{id}_G)$ with $\Phi \circ \Psi = \mathrm{id}_{(B\otimes G)/\mathrm{im}(i\otimes \mathrm{id}_G)}$. As a first step we prove the following claim

Claim 1. The map

$$\begin{aligned} \varphi\colon C \times G &\to (B \otimes G)/\mathrm{im}(i \otimes \mathrm{id}_G\colon A \otimes G \to B \otimes G) \\ (c, g) &\mapsto [\underset{\uparrow}{b} \otimes g] \end{aligned}$$

where $b \in B$ is an element with $j(b) = c$

is well-defined.

Proof. Let $(c, g) \in C \times G$. Since j is by hypothesis surjective we know that there exists a $b \in B$ with $j(b) = c$. We have to show that $\varphi(c, g)$ is independent of the choice of b. So let $b, b' \in B$ with $j(b) = j(b') = c$, i.e. with $j(b - b') = 0$. From the exactness of the original short exact sequence at B it follows that there exists an $a \in A$ with $b - b' = i(a)$. Then we obtain that

$$b \otimes g - b' \otimes g \underset{\uparrow}{=} (b - b') \otimes g = i(a) \otimes g \in \mathrm{im}(i \otimes \mathrm{id}_G\colon A \otimes G \to B \otimes G).$$

by the Tensor Product–Universal Property Lemma 51.1 (1)

Thus we have proved that φ is well-defined. ⊞

The map φ is easily seen to bilinear. It follows from the Tensor Product–Universal Property Lemma 51.1 (3) that φ induces a well-defined homomorphism

$$\begin{array}{rcl} \Phi\colon C\otimes G & \to & (B\otimes G)/\mathrm{im}(i\otimes \mathrm{id}_G\colon A\otimes G\to B\otimes G) \\ \sum_{i=1}^{k} c_i\otimes g_i & \mapsto & \sum_{i=1}^{k}\varphi(c_i,g_i). \end{array}$$

It remains to prove the following claim:

Claim 2. We have $\Phi\circ\Psi=\mathrm{id}_{(B\otimes G)/\mathrm{im}(i\otimes\mathrm{id}_G)}$.

Proof. We make two observations:

(a) It follows easily from the definitions that for all $b\in B$ and $g\in G$ we have

$$(\Phi\circ\Psi)([b\otimes g]) \;=\; \Phi(j(b)\otimes g) \;=\; [b\otimes g]\in (B\otimes G)/\mathrm{im}(i\otimes \mathrm{id}_G).$$

(b) By the Tensor Product–Universal Property Lemma 51.1 (2) we know that the elements $b\otimes g$ generate $B\otimes G$.

It follows from the two observations that $\Phi\circ\Psi=\mathrm{id}_{(B\otimes G)/\mathrm{im}(i\otimes\mathrm{id}_G)}$. ■

Using the Tensor–Right Exact Lemma 53.4 we can now provide a new, arguably more conceptual, proof of the Tensor-with-$\mathbb{Z}_k$–Lemma 51.4.

Lemma 51.4 (Tensor-with-$\mathbb{Z}_k$–Lemma) Let $k\in\mathbb{N}$ and let A be an abelian group. The maps

$$\begin{array}{rcl} \Theta\colon \mathbb{Z}_k\otimes A & \to & A/k\cdot A \\ \sum_{i=1}^{m}[n_i]\otimes a_i & \mapsto & \left[\sum_{i=1}^{m} n_i\cdot a_i\right] \end{array} \quad\text{and}\quad \begin{array}{rcl} \Omega\colon A/k\cdot A & \to & \mathbb{Z}_k\otimes A \\ [a] & \mapsto & [1]\otimes a \end{array}$$

are well-defined, they are inverses of one another and they are natural isomorphisms.

Proof. We already discussed in our original proof of the Tensor-with-$\mathbb{Z}_k$–Lemma 51.4 that the two maps are well-defined natural homomorphisms. It is clear that $\Theta\circ\Omega=\mathrm{id}_{A/k\cdot A}$. So let us concentrate on the most interesting aspect of the proof, namely let us prove that $\Omega\colon A/k\cdot A\to\mathbb{Z}_k\otimes A$ is an isomorphism. We consider the exact sequence

$$\mathbb{Z} \xrightarrow{a\,\mapsto\, k\cdot a} \mathbb{Z} \xrightarrow{n\,\mapsto\,[n]} \mathbb{Z}_k \to 0.$$

Tensoring this exact sequence with A gives use the following diagram:

$$\begin{array}{ccccccc} \mathbb{Z}\otimes A & \xrightarrow{(a\mapsto a\cdot k)\otimes\mathrm{id}_A} & \mathbb{Z}\otimes A & \xrightarrow{n\otimes a\mapsto [n]\otimes a} & \mathbb{Z}_k\otimes A & \longrightarrow & 0 \\ {\scriptstyle a\mapsto 1\otimes a}\uparrow{\scriptstyle\cong} & & {\scriptstyle a\mapsto 1\otimes a}\uparrow{\scriptstyle\cong} & & {\scriptstyle [a]\mapsto [1]\otimes a}\uparrow{\scriptstyle\Omega} & & \\ A & \xrightarrow{a\mapsto k\cdot a} & A & \xrightarrow{a\mapsto [a]} & A/k\cdot A & \longrightarrow & 0. \end{array}$$

We make the following observations:

(1) It follows from the Tensor–Right Exact Lemma 53.4 that the top sequence is exact.

(2) The two vertical maps to the left are the natural isomorphisms given by the Tensor-with-$\mathbb{Z}$–Lemma 51.3. It follows from the naturality that the square to the left commutes.

(3) It follows immediately from the definitions that the square to the right commutes.

It follows from the above and the Five Lemma 37.5 that the vertical map to the right is also an isomorphism. ■

On page 668 we saw that tensoring with $\mathbb{Z}_k$ is not exact. The following proposition shows that in contrast tensoring with torsion-free abelian groups is in fact exact:

Proposition 53.5. (Tensor-with-Torsion Free Proposition) For any torsion-free abelian group G the functor $A \mapsto A \otimes G$ from the category $\mathcal{A}b\mathcal{G}roup$ of abelian groups to itself is exact.

Proof. Let G be a torsion-free abelian group. We want to show that tensoring with G is exact. By the Exact Functor Lemma 53.2 it suffices to show that for every short exact sequence of abelian groups

$$0 \to A \xrightarrow{\varphi} B \xrightarrow{\psi} C \to 0,$$

the "tensored" sequence is still exact. First we consider the special case that G is a finitely generated torsion-free abelian group:

Claim 1. If G is in fact a finitely generated torsion-free abelian group, then the sequence $0 \to A \otimes G \xrightarrow{\varphi\otimes \mathrm{id}_G} B \otimes G \xrightarrow{\psi\otimes G} C \otimes G \to 0$ is exact.

Proof. Since G is a *finitely generated* torsion-free abelian group we obtain from the Abelian Group-Classification Theorem 18.11 that there exists an $m \in \mathbb{N}_0$ and an isomorphism $\Xi\colon \bigoplus_{i=1}^{m} \mathbb{Z} \to G$. We consider the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \bigoplus_{i=1}^{m} A & \xrightarrow{\bigoplus_{i=1}^{m} \varphi} & \bigoplus_{i=1}^{m} B & \xrightarrow{\bigoplus_{i=1}^{m} \psi} & \bigoplus_{i=1}^{m} C & \longrightarrow & 0 \\
 & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \\
0 & \longrightarrow & \bigoplus_{i=1}^{m} (A \otimes \mathbb{Z}) & \xrightarrow{\bigoplus_{i=1}^{m} (\varphi\otimes \mathrm{id}_{\mathbb{Z}})} & \bigoplus_{i=1}^{m} (B \otimes \mathbb{Z}) & \xrightarrow{\bigoplus_{i=1}^{m} (\psi\otimes \mathrm{id}_{\mathbb{Z}})} & \bigoplus_{i=1}^{m} (C \otimes \mathbb{Z}) & \longrightarrow & 0 \\
 & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \\
0 & \longrightarrow & A \otimes \left(\bigoplus_{i=1}^{m} \mathbb{Z}\right) & \xrightarrow{\varphi\otimes \mathrm{id}_{\mathbb{Z}^m}} & B \otimes \left(\bigoplus_{i=1}^{m} \mathbb{Z}\right) & \xrightarrow{\psi\otimes \mathrm{id}_{\mathbb{Z}^m}} & C \otimes \left(\bigoplus_{i=1}^{m} \mathbb{Z}\right) & \longrightarrow & 0 \\
 & & \mathrm{id}_A\otimes\Xi \downarrow\cong & & \mathrm{id}_B\otimes\Xi \downarrow\cong & & \mathrm{id}_C\otimes\Xi \downarrow\cong & & \\
0 & \longrightarrow & A \otimes G & \xrightarrow{\varphi\otimes \mathrm{id}_G} & B \otimes G & \xrightarrow{\psi\otimes \mathrm{id}_G} & C \otimes G & \longrightarrow & 0.
\end{array}$$

The top vertical maps are the natural isomorphisms from the Tensor-with-$\mathbb{Z}$–Lemma 51.3. Note that naturality implies that the top diagram commutes. The middle vertical maps are the natural isomorphisms from the Tensor–Direct Sum Lemma 51.6. Again naturality implies that the middle part of the diagram commutes. The bottom vertical maps are induced by the isomorphism Ξ, thus they are isomorphisms. It follows from the Tensor–Map Composition Lemma 51.2 that the bottom part of the diagram commutes.

Evidently the top sequence is exact. It follows from the above discussion that the bottom sequence is also exact. ⊞

Now we turn to the general case:

Claim 2. For every torsion-free abelian group G the corresponding "tensored" sequence $0 \to A \otimes G \xrightarrow{\varphi\otimes \mathrm{id}_G} B \otimes G \xrightarrow{\psi\otimes G} C \otimes G \to 0$ is exact.

Proof. By the Tensor–Right Exact Lemma 53.4 we only need to worry about the left part of the sequence. In other words, it suffices to prove that for any monomorphism $\varphi\colon A \to B$ of abelian groups the induced homomorphism $\varphi \otimes \mathrm{id}_G\colon A \otimes G \to B \otimes G$ is also injective.

We introduce and recall the following notation:

- We denote by $R(B,G)$ the set of elements of $\mathbb{Z}^{(B\times G)}$ that are of the form
$$((b+b'),g)-((b,g)+(b',g)) \qquad \text{or} \qquad (b,(g+g'))-((b,g)+(b,g'))$$
with $b,b'\in B$ and $g,g'\in G$.
- Let $N(B,G)$ be the subgroup of $\mathbb{Z}^{(B\times G)}$ that is generated by $R(B,G)$.
- By definition $B\otimes G=\mathbb{Z}^{(B\times G)}/N(B,G)$.
- Given $b\in B$ and $g\in G$ we denote by $b\otimes g$ the image of (b,g) in $B\otimes G=\mathbb{Z}^{(B\times G)}/N(B,G)$.

Now we turn to the actual proof.

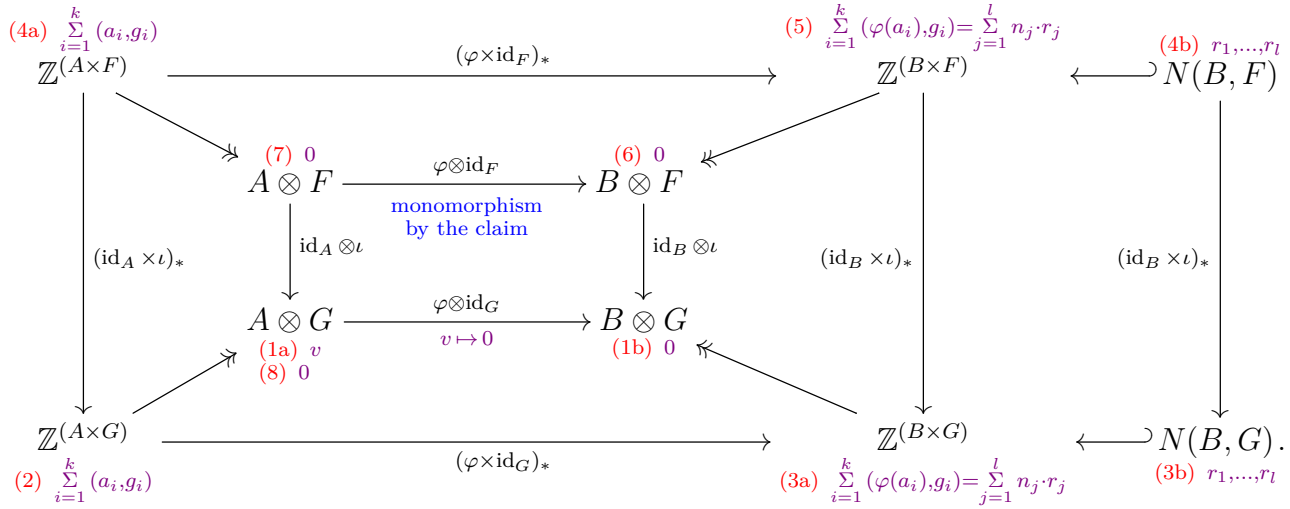
(1) Let $v\in\ker(\varphi\otimes\mathrm{id}_G\colon A\otimes G\to B\otimes B)$. We want to show that $v=0$.

(2) We write $v=\sum_{i=1}^{k} a_i\otimes g_i\in A\otimes G$ and we consider $\sum_{i=1}^{k}(a_i,g_i)\in\mathbb{Z}^{(A\times G)}$.

(3) We have the following equality in $B\otimes G=\mathbb{Z}^{(B\times G)}/N(B,G)$:
$$\Big[\sum_{i=1}^{k}(\varphi(a_i),g_i)\Big] \;=\; \sum_{i=1}^{k}\varphi(a_i)\otimes g_i \;=\; (\varphi\otimes\mathrm{id}_G)\Big(\sum_{i=1}^{k}a_i\otimes g_i\Big) \;=\; (\varphi\otimes\mathrm{id}_G)(v) \;=\; 0.$$
By definition of $N(B,G)$ we see that there exist $r_1,\dots,r_l\in R(B,G)$ and $n_1,\dots,n_l\in\mathbb{Z}$ such that
$$\sum_{i=1}^{k}(\varphi(a_i),g_i) \;=\; \sum_{j=1}^{l} n_j\cdot r_j \;\in\; \mathbb{Z}^{(B\times G)}.$$

Let F be the subgroup of G that is generated by $g_1,\dots,g_k$ and by all the second entries appearing in the summands of $r_1,\dots,r_l\in R(B,G)$. By construction F is finitely generated. Furthermore, as the subgroup of the torsion-free group G we see that F is also torsion-free. In summary, F is a finitely generated torsion-free group. We denote by $\iota\colon F\to G$ the inclusion homomorphism.

We consider the following diagram:



Note that it follows from the Tensor–Map Composition Lemma 51.2 that the small inner rectangle commutes. All other quadrilaterals commute basically by definition.

We continue with the following observations:

(4) (a) By definition of F we have $\sum_{i=1}^{k}(a_i,g_i)\in\mathbb{Z}^{(A\times F)}\subset\mathbb{Z}^{(A\times G)}$.

(b) By definition of F we have $r_1,\dots,r_l\in R(B,F)\subset R(B,G)$.

(5) (a) It follows from (4a) that $\sum_{i=1}^{k}(\varphi(a_i), g_i) \in \mathbb{Z}^{(B\times F)} \subset \mathbb{Z}^{(B\times G)}$.

(b) It follows from (4b) that $\sum_{j=1}^{l} n_j \cdot r_j \in N(B,F) \subset N(B,G)$.

It follows from (3) that these two terms are equal in $\mathbb{Z}^{(B\times F)}$.

(6) By (5) we know that the image of $\sum_{i=1}^{k}(a_i, g_i)$ in $\mathbb{Z}^{(B\times F)}$ is a linear combination of $r_1, \dots, r_l$. Since $r_1, \dots, r_l$ lie in the image of $N(B,F) \to N(B,G)$ we see that the image of $\sum_{i=1}^{k}(a_i, g_i)$ in $\mathbb{Z}^{(B\times F)}$ lies in the image of $N(B,F)$. It follows that the image of $\sum_{i=1}^{k}(a_i, g_i)$ in $B \otimes F = \mathbb{Z}^{(B\times F)}/N(B,F)$ is zero.

(7) Since F is a free abelian group we know by Claim 1 that the horizontal map $\varphi \otimes \mathrm{id}_F \colon A \otimes F \to B \otimes F$ is a monomorphism. It follows from (6) that the image of $\sum_{i=1}^{k}(a_i, g_i)$ in $A \otimes F$ is zero.

(8) It follows from (7) that the image of $\sum_{i=1}^{k}(a_i, g_i)$ in $A \otimes G$ is zero. But the image of $\sum_{i=1}^{k}(a_i, g_i)$ in $A \otimes G$ is precisely v. Thus we have shown that $v = 0$. ∎

The combination of the Exact Functor–Homology Lemma 53.3 and the Tensor-with-Torsion Free Proposition 53.5 shows that for torsion-free abelian groups G we have isomorphisms $\mathrm{H}_n(C_*) \otimes G \to \mathrm{H}_n(C_* \otimes G)$. The following lemma allows us to specify an explicit isomorphism:

Lemma 53.6. (Homology-Multiplication Lemma) Let (C_*, ∂_*) be a chain complex let G be an abelian group and let $n \in \mathbb{N}_0$. The map

$$\begin{array}{rcl} \mu \colon \mathrm{H}_n(C_*) \otimes G & \to & \mathrm{H}_n(C_* \otimes G) \\ \sum_{i=1}^{k} [\sigma_i] \otimes g_i & \mapsto & \sum_{i=1}^{k} [\sigma_i \otimes g_i] \end{array}$$

is a well-defined natural[423] homomorphism. If G is a commutative ring, then this is a homomorphism of G-modules.

Proof. It is straightforward to show that the map

$$\begin{array}{rcl} \mathrm{H}_n(C_*) \times G & \to & \mathrm{H}_n(C_*; G) \\ ([\sigma], g) & \mapsto & [\sigma \otimes g] \end{array}$$

is well-defined and bilinear. Thus it follows from the Tensor Product–Universal Property Lemma 51.1 (1) that the given map is well-defined. It follows easily from the definitions that the resulting homomorphism μ is natural. If G is a commutative ring, then one can easily verify that the given map is an isomorphism of G-modules. ∎

We can now formulate the following convenient proposition.

[423]Here "natural" means that for a chain map $f_* \colon C_* \to D_*$ and a homomorphism $\varphi \colon G \to G'$ between abelian groups the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}_n(C_*) \otimes G & \xrightarrow{\mu} & \mathrm{H}_n(C_* \otimes G) \\ \downarrow{\scriptstyle f_* \otimes \varphi} & & \downarrow{\scriptstyle (f \otimes \varphi)_*} \\ \mathrm{H}_n(D_*) \otimes G' & \xrightarrow{\mu} & \mathrm{H}_n(D_* \otimes G') \end{array}$$

Proposition 53.7. (Torsion–Free Coefficients Proposition) Let (C_*, ∂_*) be a chain complex let G be an abelian group and let $n \in \mathbb{N}_0$. If G is a torsion-free abelian group, then the map

$$\mu\colon \mathrm{H}_n(C_*) \otimes G \;\to\; \mathrm{H}_n(C_*; G)$$

from the Homology-Multiplication Lemma 53.6 is a natural isomorphism. In particular, if (X, A) is a pair of topological spaces, then the map

$$\mu\colon \mathrm{H}_n(X, A) \otimes G \;\to\; \mathrm{H}_n(X, A; G)$$

is a natural isomorphism.

Example. We have the following calculation:

$$\mathrm{H}_k(\mathbb{R}\mathrm{P}^m; \mathbb{R}) \;\overset{\cong}{\underset{\uparrow}{\leftarrow}}\; \mathrm{H}_k(\mathbb{R}\mathrm{P}^m) \otimes \mathbb{R} \;\underset{\uparrow}{\cong}\; \begin{cases} \mathbb{R}, & \text{if } k = 0 \text{ or if } k = m \text{ and } m \text{ is odd,} \\ 0, & \text{otherwise.} \end{cases}$$

by the Torsion–Free Coefficients Proposition 53.7

follows from the $\mathbb{R}\mathrm{P}^n$-H_*-Proposition 47.10 together with the Tensor-with-$\mathbb{Z}$–Lemma 51.3 and the Tensor-with-$\mathbb{Z}_k$–Lemma 51.4

□

Proof. It follows from the Tensor-with-Torsion Free Proposition 53.5 that we can apply the Exact Functor–Homology Lemma 53.3 in our context. This implies that we have a natural isomorphism

$$\mathrm{H}_n(C_*) \otimes G \;\to\; \mathrm{H}_n(C_*; G).$$

In fact, in the Exact Functor–Homology Lemma 53.3 this natural isomorphism is given explicitly: For an exact additive covariant functor $F\colon \mathcal{A}\mathcal{b}\mathcal{G}\mathcal{r}\mathcal{o}\mathcal{u}\mathcal{p} \to \mathcal{A}\mathcal{b}\mathcal{G}\mathcal{r}\mathcal{o}\mathcal{u}\mathcal{p}$ and for the chain complex C_* we showed that we get the following two isomorphisms:

$$F(\mathrm{H}_k(C_*)) \underset{\cong}{\overset{F(\pi)}{\leftarrow}} \frac{F\Big(\ker(C_k \xrightarrow{\partial_k} C_{k-1})\Big)}{\mathrm{im}(F(\widetilde{\partial}))} \xrightarrow[\cong]{F(\iota)} \frac{\ker\Big(F(C_k) \xrightarrow{F(\partial_k)} F(C_{k-1})\Big)}{\mathrm{im}\Big(F(C_{k+1}) \xrightarrow{F(\partial_{k+1})} F(C_k)\Big)} = \mathrm{H}_k(F(C_*)).$$

It follows immediately from the definitions that in our context this isomorphism is precisely the homomorphism μ from the Homology-Multiplication Lemma 53.6. ■

53.2. Appendix: The rank of an abelian group.

We recall the following definition from page 605:

Definition. If G is a finitely generated abelian group, then it follows from the Abelian Group-Classification Theorem 18.11 that G is isomorphic to $\mathbb{Z}^r \oplus T$, where T is finite. We refer to r as the rank of G.

Proposition 53.8. (Rank-via-Dimensions Proposition) For any finitely generated abelian group G and any field $\mathbb{F}$ of characteristic zero we have

$$\mathrm{rank}(G) \;=\; \dim_{\mathbb{F}}(G \otimes \mathbb{F}) \;\in\; \mathbb{N}_0.$$

Proof. [424] Let $\mathbb{F}$ be a field of characteristic zero and let G be a finitely generated abelian group of rank r. By the Abelian Group-Classification Theorem 18.11 we have an isomorphism $G \cong \mathbb{Z}^r \oplus \bigoplus_{i=1}^{n} \mathbb{Z}_{a_i}$ where $a_1, \ldots, a_n$ are non-zero natural numbers. We calculate that

$$G \otimes \mathbb{F} \cong \left(\mathbb{Z}^r \oplus \bigoplus_{i=1}^{n} \mathbb{Z}_{a_i}\right) \otimes \mathbb{F} \underset{\uparrow}{\cong} (\mathbb{Z} \otimes \mathbb{F})^r \oplus \left(\bigoplus_{i=1}^{n} (\mathbb{Z}_{a_i} \otimes \mathbb{F})\right) \underset{\uparrow}{\cong} \mathbb{F}^r \oplus \bigoplus_{i=1}^{n} \mathbb{F}/a_i \cdot \mathbb{F} \underset{\uparrow}{\cong} \mathbb{F}^r.$$

by the Tensor Commutative-Associative Lemma 51.5 (2); by the Tensor-with-$\mathbb{Z}$–Lemma 51.3 and the Tensor-with-$\mathbb{Z}_k$ Lemma 51.4; since $\mathbb{F}$ has characteristic zero

Thus we have shown that $G \otimes \mathbb{F} \cong \mathbb{F}^r$. Thus $\dim_{\mathbb{F}}(G \otimes \mathbb{F}) = \dim_{\mathbb{F}}(\mathbb{F}^r) = r = \operatorname{rank}(G)$. ■

We can use the previous proposition to prove the following result, which we used already on a few occasions.

Proposition 53.9. (Rank Additivity Proposition) . Given any short exact sequence of abelian groups

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$$

we have the following equality in $\mathbb{N}_0 \cup \{\infty\}$:

$$\operatorname{rank}(B) = \operatorname{rank}(A) + \operatorname{rank}(C).$$

Proof. Let

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$$

be a short exact sequence of abelian groups. It follows from the Tensor-with-Torsion Free Proposition 53.5 and the Tensor-with-Ring Lemma 51.7 that

$$0 \to A \otimes \mathbb{Q} \xrightarrow{\alpha \otimes \mathrm{id}_{\mathbb{Q}}} B \otimes \mathbb{Q} \xrightarrow{\beta \otimes \mathrm{id}_{\mathbb{Q}}} C \otimes \mathbb{Q} \to 0$$

is a short exact sequence of $\mathbb{Q}$-vector spaces. We obtain the following equalities in $\mathbb{N}_0 \cup \{\infty\}$:

$$\operatorname{rank}(B) \underset{\uparrow}{=} \dim_{\mathbb{Q}}(B \otimes \mathbb{Q}) \underset{\uparrow}{=} \dim_{\mathbb{Q}}(A \otimes \mathbb{Q}) + \dim_{\mathbb{Q}}(C \otimes \mathbb{Q}) \underset{\uparrow}{=} \operatorname{rank}(A) + \operatorname{rank}(C).$$

by the Rank-via-Dimensions Proposition 53.8; since dimensions are additive under short exact sequences; by the Rank-via-Dimensions Proposition 53.8 ■

We conclude this appendix with a short discussion of Betti numbers.

Definition. Given a pair (X, A) of topological spaces and $n \in \mathbb{N}_0$ we refer to

$$b_n(X, A) := \operatorname{rank}(\mathrm{H}_n(X, A))$$

as the n-th Betti number of (X, A). For $A = \varnothing$ we write of course $b_n(X) := b_n(X, \varnothing)$.

Lemma 53.10. (Betti-via-Dimension Lemma) Let (X, A) be a pair of topological spaces. For any $n \in \mathbb{N}_0$ and any field $\mathbb{F}$ of characteristic zero we have the following equality in $\mathbb{N}_0 \cup \{\infty\}$:

$$\operatorname{rank}(\mathrm{H}_n(X, A)) = \dim_{\mathbb{F}}(\mathrm{H}_n(X, A; \mathbb{F})).$$

Proof. Let (X, A) be a pair of topological spaces and let $\mathbb{F}$ be a field of characteristic zero. We have

$$\operatorname{rank}(\mathrm{H}_n(X, A)) \underset{\uparrow}{=} \dim_{\mathbb{F}}(\mathrm{H}_n(X, A) \otimes \mathbb{F}) \underset{\uparrow}{=} \dim_{\mathbb{F}}(\mathrm{H}_n(X, A; \mathbb{F}).$$

by the Rank-via-Dimensions Proposition 53.8; by the Torsion–Free Coefficients Proposition 53.7 ■

[424] The proof consists mostly of repeating the calculation from page 650.

CHAPTER 54

The Universal Coefficient Theorem

We are still trying to answer the following question:

Question 53.1. Given a topological space X and an abelian group G, what is the relationship between the usual homology groups of X and the homology groups with G-coefficients?

In the Torsion–Free Coefficients Proposition 53.7 we showed that if G is a torsion-free abelian group, then the natural map $\mu\colon \mathrm{H}_k(X)\otimes G \to \mathrm{H}_k(X;G)$ is an isomorphism.

Nonetheless, in general the situation is more complex. For example, if we compare the $\mathbb{R}\mathrm{P}^n$-H_*-Proposition 47.10 with the $\mathbb{R}\mathrm{P}^n$-H_*-Proposition 52.10 we see that

$$\mathrm{H}_2(\mathbb{R}\mathrm{P}^2)\otimes\mathbb{F}_2 \;=\; 0 \;\ncong\; \mathbb{F}_2 \;=\; \mathrm{H}_2(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2).$$

This shows that, for better or worse, in general it is not true that $\mathrm{H}_k(X)\otimes G \cong \mathrm{H}_k(X;G)$.

In this chapter will see that, despite this sobering example, one can always deduce the homology groups $\mathrm{H}_*(X;G)$ of a topological space X from the usual homology groups $\mathrm{H}_*(X)$. But the connection between these homology groups will turn out to be less naive than one might have thought initially.

54.1. The G-torsion of an abelian group.

Our goal in this chapter is to relate the homology groups $\mathrm{H}_*(C_*)$ of a chain complex C_* to the homology groups $\mathrm{H}_*(C\otimes G)$ of the same chain complex with G-coefficients. To state the theorem relating these groups we need to introduce the G-torsion of an abelian group.

We start our discussion with recalling the definition of a (free) resolution, which we had introduced on page 626.

Definition. A (free) resolution of an abelian group H is an exact sequence

$$\ldots \xrightarrow{f_3} F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \;\to\; 0$$

where the F_i are (free) abelian groups. The length of the resolution is the maximal $n\in\mathbb{Z}_{\geq -1}\cup\{\infty\}$ such that $F_n\neq 0$.

Examples.

(1) Let $m\in\mathbb{N}$. A free resolution of length 1 for $H=\mathbb{Z}_m$ is given by

$$0 \;\to\; \mathbb{Z} \xrightarrow{\cdot m} \mathbb{Z} \;\to \mathbb{Z}_m \;\to\; 0.$$

(2) Let G be a *finitely generated* abelian group. By the Abelian Group-Classification Theorem 18.11 there exists an $r\in\mathbb{N}_0$, non-zero natural numbers $a_1,\ldots,a_k$ and an isomorphism

$$\Phi\colon G \xrightarrow{\cong} \mathbb{Z}^r \oplus \bigoplus_{i=1}^{k}\mathbb{Z}_{a_i}.$$

Thus we obtain a free resolution[425]

$$0 \to \mathbb{Z}^k \xrightarrow{0 \oplus \begin{pmatrix} a_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & a_k \end{pmatrix}} \mathbb{Z}^r \oplus \mathbb{Z}^k \to \underbrace{\mathbb{Z}^r \oplus \bigoplus_{i=1}^{k} \mathbb{Z}_{a_i}}_{\cong G} \to 0.$$

□

The final example is so important that it becomes a definition.

Definition.

(1) We recall the following basic notation: If M is an abelian group, then we know by the Free Abelian Group–Existence Lemma 18.4 that there exists a unique homomorphism $\alpha(M)\colon \mathbb{Z}^{(M)} \to M$ such that for each $m \in M$ we have $\alpha(M)(m) = m$. Evidently this map is an epimorphism.

(2) Let H be an abelian group. We consider the sequence

$$\mathbb{Z}^{(H)} \xrightarrow{\alpha(H)} H \to 0.$$

The group $\mathbb{Z}^{(H)}$ is free abelian and $\alpha(H)$ is an epimorphism. In particular the sequence is exact. We set $H_1 := \ker(\alpha(H)\colon \mathbb{Z}^{(H)} \to H)$. Then

$$H_1 \hookrightarrow \mathbb{Z}^{(H)} \xrightarrow{\alpha(H)} H \to 0$$

is exact. Now we consider

$$\mathbb{Z}^{(H_1)} \xrightarrow{\alpha(H_1)} \mathbb{Z}^{(H)} \xrightarrow{\alpha(H)} H \to 0.$$

This sequence is exact and both the groups $\mathbb{Z}^{(H_1)}$ and $\mathbb{Z}^{(H)}$ are free abelian. Now we set $H_2 := \ker(\alpha(H_1)\colon \mathbb{Z}^{(H_1)} \to H_1)$ and we iterate this procedure. This way we obtain a free resolution of H.[426] Note that this free resolution is unambiguously defined, we did not make any choices along the way. Therefore we refer to this free resolution of H as the natural[433] free resolution of H.

Lemma 54.1. (Natural Free Resolution Lemma) Every abelian group admits a natural free resolution.

Proof. The lemma follows from the definition preceding the lemma. ■

The following definition is one of the key definitions in homological algebra.

Definition. Let G and H be abelian groups. We denote by

$$\ldots \to F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \to 0$$

[425]Note that the homomorphism to the left is a monomorphism since $a_1, \ldots, a_k$ are non-zero.

[432]Note that this procedure does not stop, i.e. it produces an infinite free resolution. If H is non-trivial, then basically all groups involved are free groups of infinite rank.

[432]This resolution is "natural" in the weakest sense, namely only under isomorphisms.

the natural free resolution from above. We define the n-th G-torsion group of H as

$$\mathrm{Tor}_n(H,G) := \mathrm{H}_n(\underbrace{F_* \otimes G}_{\text{chain complex}}) = \mathrm{H}_n(\ldots \to F_2 \otimes G \xrightarrow{f_2 \otimes \mathrm{id}_G} F_1 \otimes G \xrightarrow{f_1 \otimes \mathrm{id}_G} \underbrace{F_0 \otimes G}_{\text{degree } 0.} \xrightarrow{0} 0).$$

We will now see that the torsion groups $\mathrm{Tor}_n(H,G)$ are "covariantly functorial in both entries":

Definition.

(1) We fix an abelian group G. Let $\beta\colon H \to H'$ be a homomorphism between two abelian groups. It follows from the Fundamental Theorem 50.1 of Homological Algebra applied to the natural free resolutions F_* and F'_* of H and H', that we get an extension $F_* \to F'_*$ of β to the free resolutions, and this extension is unique up to chain homotopy. It follows from the Chain Homotopy–H_*-Lemma 36.1, that β induces for each $n \in \mathbb{N}_0$ a well-defined homomorphism

$$\beta_*\colon \underbrace{\mathrm{Tor}_n(H,G)}_{=\mathrm{H}_n(F_* \otimes G)} \to \underbrace{\mathrm{Tor}_n(H',G)}_{=\mathrm{H}_n(F'_* \otimes G)}.$$

One can now easily show that the maps given by

$$\begin{array}{rcl} H & \mapsto & \mathrm{Tor}_n(H,G) \\ (\beta\colon H \to H') & \mapsto & (\beta_*\colon \mathrm{Tor}_n(H,G) \to \mathrm{Tor}_n(H',G)) \end{array}$$

define a covariant functor from the category $\mathcal{AbGroup}$ of abelian groups $\mathcal{AbGroup}$ to itself.

(2) We fix an abelian group H. Let $\alpha\colon G \to G'$ be a homomorphism between two abelian groups. Let F_* be the natural free resolution of H. Then $\mathrm{id} \otimes \alpha$ defines a chain map from $F_* \otimes G$ to $F_* \otimes G'$ and, as in the Homology-Coefficients-Functor Lemma 52.8, we get an induced homomorphism

$$\alpha_*\colon \mathrm{Tor}_n(H,G) \to \mathrm{Tor}_n(H,G').$$

One can now easily show that the maps given by

$$\begin{array}{rcl} G & \mapsto & \mathrm{Tor}_n(H,G) \\ (\alpha\colon G \to G') & \mapsto & (\alpha_*\colon \mathrm{Tor}_n(H,G) \to \mathrm{Tor}_n(H,G')) \end{array}$$

define a covariant functor from the category $\mathcal{AbGroup}$ of abelian groups to itself.

The following proposition shows that we can use any free resolution to determine the G-torsion groups. This fact is the key to calculating G-torsion groups.

Proposition 54.2. (Tor-Free Resolution Proposition) Let G and H be abelian groups and let

$$\ldots \to F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \to 0$$

be a free resolution of H. For every $n \in \mathbb{N}_0$ there exists an isomorphism

$$\mathrm{Tor}_n(H,G) \xrightarrow{\cong} \mathrm{H}_n(F_* \otimes G)$$

which is natural in G.

Proof. Let G and H be abelian groups and let

$$\ldots \to F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \to 0$$

be a free resolution of H. We want to show that there exists a natural isomorphism

$$\mathrm{Tor}_n(H,G) \xrightarrow{\cong} \mathrm{H}_n(F_* \otimes G).$$

We denote by (F'_*, f'_*) the natural free resolution of H. Since F'_* is a free resolution we obtain from the Fundamental Theorem 50.1 of Homological Algebra that there exists an extension $\alpha_i, i \in \mathbb{N}$ of the identity map $\mathrm{id}_H \colon H \to H$ from the free resolutions F'_* to the free resolution F_*. In other words, we obtain we obtain a commutative diagram of the following form:

$$\begin{array}{ccccccccccc}
\ldots & \longrightarrow & F'_2 & \xrightarrow{f'_2} & F'_1 & \xrightarrow{f'_1} & F'_0 & \xrightarrow{f'_0} & H & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_0} & & \downarrow{\scriptstyle \mathrm{id}_H} & & \\
\ldots & \longrightarrow & F_2 & \xrightarrow{f_2} & F_1 & \xrightarrow{f_1} & F_0 & \xrightarrow{f_0} & H & \longrightarrow & 0.
\end{array}$$

Since both resolutions are free we obtain from the Free Resolution–Chain Homotopic Corollary 50.2 that the maps α_* form a chain homotopy equivalence between the two chain complexes F'_* and F_*. It follows from the Homology-with-Coefficients Lemma 52.4 (2) that $\alpha_* \otimes \mathrm{id}_G \colon F'_* \otimes G \to F_* \otimes G$ is also a chain homotopy equivalence. It follows from this observation together with the Chain Homotopy Equivalence-H_*-Corollary 36.4 that for every $n \in \mathbb{N}_0$ the map

$$\begin{array}{rccc}
\alpha_n \otimes \mathrm{id}_G \colon & \overbrace{\mathrm{H}_n(F'_* \otimes G)}^{=:\mathrm{Tor}_n(H,G)} & \to & \mathrm{H}_n(F_* \otimes G) \\
 & \left[\sum_{i=1}^{k} c_i \otimes g_i\right] & \mapsto & \left[\sum_{i=1}^{k} \alpha_n(c_i) \otimes g_i\right]
\end{array}$$

is an isomorphism. Using Statement (2) of the Fundamental Theorem 50.1 of Homological Algebra one can easily show that this map does not depend on the choice of the extension α_*. Finally note that it follows easily from the Tensor–Map Composition Lemma 51.2 (2) that this isomorphism is natural in G. ∎

We now turn to the calculation of the G-torsion groups. The following lemma shows that the 0-th G-torsion groups are objects that we are already familiar with.

Lemma 54.3. (Tor_0–Lemma) For any choice of abelian groups G and H there exists a natural isomorphism [433]

$$\mathrm{Tor}_0(H,G) \xrightarrow{\cong} H \otimes G.$$

Identification. In the remainder of this course we will use the natural isomorphisms from the Tor_0–Lemma 54.3 to make the identification $\mathrm{Tor}_0(H,G) =_{\mathrm{i}} H \otimes G$.

Proof. Let G and H be abelian groups. We consider the natural free resolution of H:

$$\cdots \to F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \to 0$$

By the Tensor–Right Exact Lemma 53.4 we know that the sequence

$$(*) \qquad F_1 \otimes G \xrightarrow{f_1 \otimes \mathrm{id}_G} F_0 \otimes G \xrightarrow{f_0 \otimes \mathrm{id}_G} H \otimes G \to 0$$

[433] Note that "natural isomorphism" means in this context that given an abelian group G we have natural isomorphisms from the functor $H \mapsto \mathrm{Tor}_0(H,G)$ to the functor $H \mapsto H \otimes G$ and also from the functor $G \mapsto \mathrm{Tor}_0(H,G)$ to the functor $G \mapsto H \otimes G$.

is exact. Now we obtain the following isomorphism

$$\begin{array}{rcl} \operatorname{Tor}_0(H,G) & \overset{\text{by definition}}{=} & \mathrm{H}_0\big(\cdots \to F_1 \otimes G \xrightarrow{f_1\otimes \mathrm{id}_G} F_0 \otimes G \to 0\big) \\ & = & \operatorname{coker}\big(F_1 \otimes G \xrightarrow{f_1\otimes \mathrm{id}_G} F_0 \otimes G\big) \xrightarrow{f_0\otimes \mathrm{id}_G} H \otimes G. \end{array}$$

since $(*)$ is exact we obtain from the Exactness Lemma 37.1 (3) that this map is an isomorphism

It follows easily from the Tensor–Map Composition Lemma 51.2 (2) that this isomorphism $\operatorname{Tor}_0(H,G) \to H \otimes G$ is natural in G. Furthermore, using the Fundamental Theorem 50.1 of Homological Algebra one can easily show that this isomorphism is natural in H. ■

The following two lemmas show that we do not need to worry about G-torsion groups $\operatorname{Tor}_n(H,G)$ of degree $n \geq 2$.

Lemma 54.4. (Length 1–Free Resolution Lemma) Every abelian group H admits a free resolution of length ≤ 1.

Proof. Let H be an abelian group. Recall that we have the exact sequence

$$0 \to \ker(\alpha(H)\colon \mathbb{Z}^{(H)} \to H) \to \mathbb{Z}^{(H)} \xrightarrow{\alpha(H)} H \to 0$$

where $\alpha(H)$ is defined as in the proof of the Natural Free Resolution Lemma 54.1. The lemma follows from the observation that it follows from the Free Abelian Subgroup Theorem 18.7 that the group $\ker(\alpha(H)\colon \mathbb{Z}^{(H)} \to H)$, as the subgroup of the free abelian group $\mathbb{Z}^{(H)}$, is itself free abelian. ■

Lemma 54.5. (Higher Tor Lemma) For any two abelian groups H and G we have $\operatorname{Tor}_n(H,G) = 0$ for $n \geq 2$.

Proof. Let G and H be abelian groups. By (1) there exists a free resolution of H of length 1, i.e. of the form

$$0 \to F_1 \to F_0 \to H \to 0.$$

It now follows that for $n \geq 2$ we have

$$\operatorname{Tor}_n(H,G) \overset{\text{by the Tor-Free Resolution Proposition 54.2}}{\cong} \mathrm{H}_n(F_* \otimes G) \overset{\text{since } n \geq 2 \text{ implies that } F_n = 0}{=} 0.$$

■

The Tor_0–Lemma 54.3 together with the Higher Tor Lemma 54.5 motivates the following simplification of our notation:

Definition. Given two abelian groups H and G we write $\operatorname{Tor}(H,G) := \operatorname{Tor}_1(H,G)$ and we refer to this group as as the G-torsion of H.

The following is one of the most important calculations of G-torsion groups.

Lemma 54.6. ($\mathbb{Z}_k$-Tor-Lemma) Let $k \in \mathbb{N}$.

(1) For any abelian group G we have a natural isomorphism

$$\ker\big(G \xrightarrow{g \mapsto g\cdot k} G\big) \xrightarrow{\cong} \operatorname{Tor}(\mathbb{Z}_k, G).$$

(2) Given any $m \in \mathbb{N}$ we have $\operatorname{Tor}(\mathbb{Z}_k, \mathbb{Z}_m) \cong \mathbb{Z}_{\gcd(k,m)}$.

Proof. [434]

(1) We consider the free resolution $0 \to \mathbb{Z} \xrightarrow{n\mapsto n\cdot k} \mathbb{Z} \to \mathbb{Z}_k \to 0$ of $\mathbb{Z}_k$. Given any abelian group G this gives rise to the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \overbrace{\ker\left(\mathbb{Z}\otimes G \xrightarrow{(n\mapsto n\cdot k)\otimes \mathrm{id}_G} \mathbb{Z}\otimes G\right)}^{\cong \mathrm{Tor}(\mathbb{Z}_k, G)} & \longrightarrow & \mathbb{Z}\otimes G & \xrightarrow{(n\mapsto n\cdot k)\otimes \mathrm{id}_G} & \mathbb{Z}\otimes G \\ & & \uparrow & & \uparrow{\scriptstyle g\mapsto 1\otimes g}\ \cong & & \uparrow{\scriptstyle g\mapsto 1\otimes g}\ \cong \\ 0 & \longrightarrow & \ker(G \xrightarrow{g\mapsto g\cdot k} G) & \longrightarrow & G & \xrightarrow{g\mapsto g\cdot k} & G. \end{array}$$

We make the following observations:

(a) Since $0 \to \mathbb{Z} \xrightarrow{n\mapsto n\cdot k} \mathbb{Z} \to \mathbb{Z}_k \to 0$ is a free resolution of $\mathbb{Z}_k$ we obtain from the Tor-Free Resolution Proposition 54.2 the isomorphism in the top left corner.

(b) The top horizontal sequence is exact by definition.

(c) The two vertical maps to the right are the natural isomorphisms given by the Tensor-with-$\mathbb{Z}$–Lemma 51.3. It follows immediately from the Tensor Product–Universal Property Lemma 51.1 (1) that the square to the right commutes.

It now follows from a mild diagram chase that from the above we get an induced isomorphism $\ker\left(G \xrightarrow{g\mapsto g\cdot k} G\right) \xrightarrow{\cong} \mathrm{Tor}(\mathbb{Z}_k, G)$. One can easily verify that this isomorphism is natural.

(2) We write $r = \gcd(m,k)$ and $s = m/r$. We see that

$$\mathrm{Tor}(\mathbb{Z}_k, \mathbb{Z}_m) \underset{\text{by (1)}}{\cong} \ker(\mathbb{Z}_m \xrightarrow{h\mapsto h\cdot k} \mathbb{Z}_m) = s\cdot\mathbb{Z}_m \underset{\text{elementary number theory}}{\cong} \mathbb{Z}_r.$$

■

The following lemma contains many useful properties of the torsion groups $\mathrm{Tor}(H,G)$:

Lemma 54.7. (Tor-Properties Lemma) Let G, H, $\{H_i\}_{i\in I}$ and $\{G_j\}_{j\in J}$ be abelian groups. Then the following holds:

(1) There exists a natural isomorphism $\mathrm{Tor}\left(\bigoplus_i H_i, G\right) \cong \bigoplus_i \mathrm{Tor}(H_i, G)$.

(2) There exists a natural isomorphism $\mathrm{Tor}\left(H, \bigoplus_j G_j\right) \cong \bigoplus_j \mathrm{Tor}(H, G_j)$.

(3) If G is a torsion-free abelian group, then $\mathrm{Tor}(H,G) = 0$.

(4) If H is free abelian group, then $\mathrm{Tor}(H,G) = 0$.

Example. Using this lemma one can usually determine the G-torsion of an abelian group H without any problems. For example the groups $(\mathbb{Q},+)$, $(\mathbb{R},+)$ and $(\mathbb{C},+)$ are torsion-free abelian groups. Thus it follows immediately from the Tor-Properties Lemma 54.7 (3) that for every abelian group H we have

$$\mathrm{Tor}(H,\mathbb{Q}) = \mathrm{Tor}(H,\mathbb{R}) = \mathrm{Tor}(H,\mathbb{C}) = 0.$$

□

Proof.

[434] The following proof shares some features with the proof of the Tensor-with-$\mathbb{Z}_k$–Lemma 51.4 on page 670.

(1) For each $i \in I$ let F_*^i be the natural free resolution of H_i. One can easily verify that $\bigoplus_{i\in I} F_*^i$ is a free resolution of $\bigoplus_{i\in I} H_i$. We have the following natural isomorphisms:

$$\begin{array}{ccccc}
\operatorname{Tor}\Big(\bigoplus_{i\in I} H_i, G\Big) & \overset{\text{by the Tor-Free Resolution Proposition 54.2}}{\cong} & \mathrm{H}_1\Big(\Big(\bigoplus_{i\in I} F_*^i\Big)\otimes G\Big) & \overset{\text{by the Tensor–Direct Sum Lemma 51.6}}{\xleftarrow{\cong}} & \mathrm{H}_1\Big(\bigoplus_{i\in I}(F_*^i\otimes G)\Big) \\
 & & & \overset{\text{by the Direct Sum-H}_*\text{-Lemma 35.2}}{\xleftarrow{\cong}} & \bigoplus_{i\in I}\mathrm{H}_1(F_*^i\otimes G) \quad \overset{\text{by definition}}{\cong} \quad \bigoplus_{i\in I}\operatorname{Tor}(H_i,G).
\end{array}$$

(2) Let F_* be the natural resolution for H. We have the following natural isomorphisms:

$$\begin{array}{ccccc}
\operatorname{Tor}\Big(H, \bigoplus_{j\in J} G_j\Big) & \overset{\text{by definition}}{=} & \mathrm{H}_1\Big(F_*\otimes \bigoplus_{j\in J} G_j\Big) & \overset{\text{by the Tensor–Direct Sum Lemma 51.6}}{\xleftarrow{\cong}} & \mathrm{H}_1\Big(\bigoplus_{j\in J}(F_*\otimes G_j)\Big) \\
 & & & \overset{\text{by the Direct Sum-H}_*\text{-Lemma 35.2}}{\xleftarrow{\cong}} & \bigoplus_{j\in J}\mathrm{H}_1(F_*\otimes G_j) \quad \overset{\text{by definition}}{=} \quad \bigoplus_{j\in J}\operatorname{Tor}(H,G_j).
\end{array}$$

(3) We need to show that if G is a torsion-free abelian group, then for any abelian group H we have $\operatorname{Tor}(H,G)=0$. First recall that it follows from the Length 1–Free Resolution Lemma 54.4 that we can pick a free resolution

$$0 \to F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \to 0$$

of length ≤ 1 for H. Since G is a torsion-free abelian group we know by the Tensor-with-Torsion Free Proposition 53.5 that tensoring with G is exact, in particular it is left-exact. It follows that

$$0 \to F_1\otimes G \xrightarrow{f_1\otimes \mathrm{id}_G} F_0\otimes G \xrightarrow{f_0\otimes \mathrm{id}_G} H\otimes G$$

is also exact. In particular $f_1\otimes \mathrm{id}_G\colon F_1\otimes G\to F_0\otimes G$ is injective. It now follows that

$$\begin{array}{ccccc}
\operatorname{Tor}(H,G) & \overset{\text{by the Tor-Free Resolution Proposition 54.2}}{\cong} & \mathrm{H}_1\big(0\to F_1\otimes G \xrightarrow{f_1\otimes \mathrm{id}_G} F_0\otimes G\to 0\big) & \overset{\text{since we just saw that } f_1\otimes \mathrm{id}_G \text{ is injective}}{=} & 0.
\end{array}$$

(4) Let H be a *free* abelian group. Note that in this case $0\to F_0 := H \xrightarrow{\mathrm{id}} H\to 0$ is already a free resolution, and it follows immediately from the Tor-Free Resolution Proposition 54.2 that $\operatorname{Tor}(H,G)=0$. ∎

54.2. **The Universal Coefficient Theorem.** Before we turn to the Algebraic Universal Coefficient Theorem let us recall the following a concept that we first introduced in the Homology-Multiplication Lemma 53.6:

Definition. Let (C_*,∂_*) be a chain complex and let G be an abelian group. Throughout this chapter, given $n\in\mathbb{N}_0$, we denote by μ the natural homomorphism

$$\begin{array}{rcl}
\mu\colon \mathrm{H}_n(C_*)\otimes G & \to & \mathrm{H}_n(C_*\otimes G) \\
\sum_{i=1}^m [c_i]\otimes g_i & \mapsto & \Big[\sum_{i=1}^m c_i\otimes g_i\Big].
\end{array}$$

If G is a commutative ring, then we saw that this is a homomorphism of G-modules.

Now we can formulate and prove the following theorem which is the key result of the whole chapter.

Theorem 54.8. (Algebraic Universal Coefficient Theorem) Let (C_*, ∂_*) be a chain complex of free abelian groups and let G be an abelian group. For each $n \in \mathbb{N}_0$ there exists a natural homomorphism $\nu\colon \mathrm{H}_n(C_* \otimes G) \to \mathrm{Tor}(\mathrm{H}_{n-1}(C_*), G)$ such that the following sequence is exact:

$$0 \to \mathrm{H}_n(C_*) \otimes G \xrightarrow{\mu} \mathrm{H}_n(C_* \otimes G) \xrightarrow{\nu} \mathrm{Tor}(\mathrm{H}_{n-1}(C_*), G) \to 0.$$

If G is a commutative ring, then ν is a G-module homomorphism.

Proof. Let (C_*, ∂_*) be a chain complex of free abelian groups. For each $n \in \mathbb{N}_0$ we write as always

$$\begin{aligned} Z_n &:= \ker(\partial_n\colon C_n \to C_{n-1}) \\ B_n &:= \mathrm{im}(\partial_{n+1}\colon C_{n+1} \to C_n). \end{aligned}$$

Since C_n is by hypothesis free abelian we obtain from the Free Abelian Subgroup Theorem 18.7 that B_n and Z_n are free abelian groups.

We consider the following diagram:

$$\begin{array}{ccccccccc} & & \vdots & & \vdots & & \vdots & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & Z_n & \xhookrightarrow{\iota_n} & C_n & \xrightarrow{\partial_n} & B_{n-1} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \partial_n} & & \downarrow{\scriptstyle 0} & & \\ 0 & \longrightarrow & Z_{n-1} & \xhookrightarrow{\iota_{n-1}} & C_{n-1} & \xrightarrow{\partial_{n-1}} & B_{n-2} & \longrightarrow & 0. \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \vdots & & \vdots & & \vdots & & \end{array}$$

One sees easily that this diagram is commutative and that the horizontal sequences are exact. We can and will view the left and the right vertical columns as chain complexes. In other words, the above diagram is a short exact sequence of chain complexes.

Now let G be an abelian group. We tensor the above short exact sequence of chain complexes with G. This leads us to the following claim:

Claim. The following maps define a short exact sequence of "vertical" chain complexes:

$$\begin{array}{ccccccccc} & & \vdots & & \vdots & & \vdots & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & Z_n \otimes G & \xrightarrow{\iota_n \otimes \mathrm{id}_G} & C_n \otimes G & \xrightarrow{\partial_n \otimes \mathrm{id}_G} & B_{n-1} \otimes G & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \partial_n \otimes \mathrm{id}_G} & & \downarrow{\scriptstyle 0} & & \\ 0 & \longrightarrow & Z_{n-1} \otimes G & \xrightarrow{\iota_{n-1} \otimes \mathrm{id}_G} & C_{n-1} \otimes G & \xrightarrow{\partial_{n-1} \otimes \mathrm{id}_G} & B_{n-2} \otimes G & \longrightarrow & 0. \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \vdots & & \vdots & & \vdots & & \end{array}$$

Proof.

(1) It follows from the Tensor–Functor Lemma 52.3 together with the Functor-applied-to-Chain Complex Lemma 52.2 (1a) that the vertical sequences are again chain complexes. Thus the above is a sequence of "vertical" chain complexes.

(2) Recall that the groups B_n are free abelian groups. Thus it follows from the Free Abelian Group–Splitting Lemma 18.5 together with the Splitting Lemma 37.2 (1)⇒(3) that each of the horizontal sequences in the original diagram splits. It follows from the

Tensor-Split-SES Lemma 52.6 that the horizontal sequences obtained from tensoring with G are still exact. ⊞

We apply the LES-Proposition 37.4 to the short exact sequence of chain complexes from the claim. We obtain a natural connecting homomorphism

$$\underbrace{B_n \otimes G}_{=\mathrm{H}_n(B_* \otimes G)} \quad \to \quad \underbrace{Z_n \otimes G}_{=\mathrm{H}_n(Z_* \otimes G)} .$$

In fact, it follows rather easily from the definition of connecting homomorphisms (see page 483) that the connecting homomorphism $B_n \otimes G \to Z_n \otimes G$ is the map that is given by $i_n \otimes \mathrm{id}_G \colon B_n \otimes G \to Z_n \otimes G$ where $i_n \colon B_n \to Z_n$ denotes the obvious inclusion map. By the LES-Proposition 37.4 this connecting homomorphism gives us the following long exact sequence

$$\ldots \to B_n \otimes G \xrightarrow{i_n \otimes \mathrm{id}_G} Z_n \otimes G \to \mathrm{H}_n(C_* \otimes G) \xrightarrow{\partial_n} B_{n-1} \otimes G \xrightarrow{i_{n-1} \otimes \mathrm{id}_G} Z_{n-1} \otimes G \to \ldots$$

The above discussion implies that we obtain the following short exact sequence

$$0 \to \underbrace{\operatorname{coker}\left(B_n \otimes G \xrightarrow{i_n \otimes \mathrm{id}_G} Z_n \otimes G\right)}_{\text{need to show that } = \mathrm{H}_n(C_*) \otimes G} \to \mathrm{H}_n(C_* \otimes G) \to \underbrace{\ker\left(B_{n-1} \otimes G \xrightarrow{i_{n-1} \otimes \mathrm{id}_G} Z_{n-1} \otimes G\right)}_{\text{need to show that } = \operatorname{Tor}(\mathrm{H}_{n-1}(C_*), G)} \to 0.$$

We still have to show that the expressions left and right coincide with the desired expressions:

(1) We have natural isomorphisms

$$\operatorname{coker}\left(B_n \otimes G \xrightarrow{i_n \otimes \mathrm{id}_G} Z_n \otimes G\right) \xleftarrow[\uparrow]{\cong} \operatorname{Tor}_0(\mathrm{H}_n(C_*), G) \xrightarrow[\uparrow]{\cong} \mathrm{H}_n(C_*) \otimes G$$

by the Tor-Free Resolution Proposition 54.2 since $0 \to B_n \xrightarrow{i_n} Z_n \to \mathrm{H}_n(C_*)$ is a free resolution of $\mathrm{H}_n(C_*)$

by the Tor_0–Lemma 54.3

One can easily verify that the resulting homomorphism $\mathrm{H}_n(C_*) \otimes G \to \mathrm{H}_n(C_* \otimes G)$ is the given homomorphism μ.

(2) Similarly to (1) we have a natural isomorphism

$$\ker\left(B_{n-1} \otimes G \xrightarrow{i_{n-1} \otimes \mathrm{id}_G} Z_{n-1} \otimes G\right) \xrightarrow[\uparrow]{\cong} \operatorname{Tor}(\mathrm{H}_{n-1}(C_*), G).$$

by the Tor-Free Resolution Proposition 54.2 since $0 \to B_{n-1} \xrightarrow{i_n} Z_{n-1} \to \mathrm{H}_{n-1}(C_*)$ is a free resolution of $\mathrm{H}_{n-1}(C_*)$

In summary we have now finally shown the existence of the promised homomorphism $\nu \colon \mathrm{H}_n(C_* \otimes G) \to \operatorname{Tor}(\mathrm{H}_{n-1}(C_*), G)$ and we have shown that μ and ν give rise to a short exact sequence. The statement that the homomorphism ν is natural follows easily from the definitions and Statement (2) of the Fundamental Theorem 50.1 of Homological Algebra. We leave the verification of the details to the reader. ■

The following theorem is for us the main application of the Algebraic Universal Coefficient Theorem 54.8.

Theorem 54.9. (Universal Coefficient Theorem) Let (X, A) be a pair of topological spaces and let G be an abelian group. For each $n \in \mathbb{N}_0$ there exists a natural homomorphism $\nu \colon \mathrm{H}_n(X, A; G) \to \operatorname{Tor}(\mathrm{H}_{n-1}(X, A), G)$ such that the following sequence is exact:

$$0 \to \mathrm{H}_n(X, A) \otimes G \xrightarrow{\mu} \mathrm{H}_n(X, A; G) \xrightarrow{\nu} \operatorname{Tor}(\mathrm{H}_{n-1}(X, A), G) \to 0.$$

If G is a commutative ring, then the natural homomorphism ν is in fact a G-module homomorphism.

Remark. Note that the naturality of the maps μ and ν implies that for every continuous map $f\colon (X,A) \to (Y,B)$ between pairs of topological spaces and any homomorphism $\varphi\colon G \to G'$ between abelian groups we obtain the following *commutative* diagram of short exact sequences:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{H}_n(X,A) \otimes G & \xrightarrow{\mu} & \mathrm{H}_n(X,A;G) & \xrightarrow{\nu} & \mathrm{Tor}(\mathrm{H}_{n-1}(X,A),G) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{H}_n(Y,B) \otimes G' & \xrightarrow{\mu} & \mathrm{H}_n(Y,B;G') & \xrightarrow{\nu} & \mathrm{Tor}(\mathrm{H}_{n-1}(Y,B),G') & \longrightarrow & 0,
\end{array}$$

where the vertical maps are the obvious (?) ones induced by f and φ. □

Proof. Let (X,A) be a pair of topological spaces and let G be an abelian group. As we pointed out on page 489, the chain groups $\mathrm{C}_n(X,A)$, $n \in \mathbb{N}_0$, are free abelian groups. Thus we can apply the Algebraic Universal Coefficient Theorem 54.8 to the chain complex $\mathrm{C}_*(X,A)$ and we immediately obtain the desired natural homomorphism $\nu\colon \mathrm{H}_n(X,A;G) \to \mathrm{Tor}(\mathrm{H}_{n-1}(X,A),G)$ and the desired short exact sequence. ■

Example.

(1) Let (X,A) be a pair of topological spaces and let G be a torsion-free abelian group. From the Universal Coefficient Theorem 54.9 we obtain the short exact sequence

$$0 \;\to\; \mathrm{H}_n(X,A) \otimes G \;\xrightarrow{\mu}\; \mathrm{H}_n(X,A;G) \;\xrightarrow{\nu}\; \underbrace{\mathrm{Tor}(\mathrm{H}_{n-1}(X,A),G)}_{\substack{=0 \text{ by the Tor-Properties} \\ \text{Lemma 54.7 (3)}}} \;\to\; 0.$$

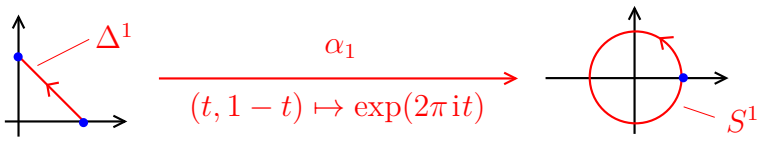
Thus we see that the map to the left is an isomorphism. In other words, we have reproved the Torsion–Free Coefficients Proposition 53.7.

(2) Let $n \in \mathbb{N}$ and let G be an abelian group. We consider the sphere S^n. By the Universal Coefficient Theorem 54.9 we know that the following sequence is exact:

$$\begin{array}{ccccccc}
0 & \to & \mathrm{H}_n(S^n) \otimes G & \xrightarrow{\mu} & \mathrm{H}_n(S^n;G) & \xrightarrow{\nu} & \underbrace{\mathrm{Tor}(\underbrace{\mathrm{H}_{n-1}(S^n)}_{=0 \text{ or } \cong \mathbb{Z}},G)}_{\substack{=0 \text{ by the Tor-Properties} \\ \text{Lemma 54.7 (4)}}} \;\to\; 0. \\
 & & \sum_{i=1}^{k} [\sigma_i] \otimes g_i & \mapsto & \sum_{i=1}^{k} [\sigma_i \otimes g_i] & &
\end{array}$$

Since the term on the right is zero, we see that the map μ to the left is an isomorphism. Thus we see that $G \cong \mathbb{Z} \otimes G \cong \mathrm{H}_n(S^n) \otimes G \cong \mathrm{H}_n(S^n;G)$. This gives us a new proof of the Sphere-H_*-Proposition 52.9. But this approach gives us more: In the Standard Generator-$\mathrm{H}_n(S^n)$-Proposition 42.4 we gave an explicit singular cycle $\alpha_n - \beta_n$ in $\mathrm{C}_n(S^n)$ that represents a generator of $\mathrm{H}_n(S^n)$. If G is a commutative ring, then it follows easily from the above that $(\alpha_n - \beta_n) \otimes 1_G \in \mathrm{C}_n(S^n;G)$ is a cycle that represents a generator of the G-module $\mathrm{H}_n(S^n;G) \cong G$. Furthermore, if n is odd, then we know by the Standard Generator-$\mathrm{H}_n(S^n)$-Proposition 42.4 that we can drop β_n, i.e. $\alpha_n \otimes 1_G$ represents already a generator.

(3) We consider again the real projective space $\mathbb{R}\mathrm{P}^n$. In the $\mathbb{R}\mathrm{P}^n$-H_*-Proposition 47.10 we saw that



$[\alpha_1] \otimes 1_{\mathbb{F}}$ is a generator of $\mathrm{H}_1(S^1;\mathbb{F})$

$$\mathrm{H}_k(\mathbb{R}\mathrm{P}^n) \cong \begin{cases} \mathbb{Z}, & \text{if } k = 0, \\ \mathbb{Z}_2, & \text{if } k \text{ is odd and } k < n, \\ 0, & \text{if } k \text{ is even and } 0 < k \leq n, \\ 0, & \text{if } k > n, \\ \mathbb{Z}, & \text{if } k = n \text{ and } n \text{ is odd.} \end{cases}$$

It follows from the Universal Coefficient Theorem 54.9, the discussion on page 649, the $\mathbb{Z}_k$-Tor-Lemma 54.6 and the Tor-Properties Lemma 54.7 (4) that for any even number $2k \in \{0, \dots, n\}$ we have a short exact sequence

$$0 \to \underbrace{\mathrm{H}_{2k}(\mathbb{R}\mathrm{P}^n) \otimes \mathbb{F}_2}_{\cong 0 \otimes \mathbb{F}_2 = 0} \xrightarrow{\mu} \mathrm{H}_{2k}(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2) \to \underbrace{\mathrm{Tor}(\mathrm{H}_{2k-1}(\mathbb{R}\mathrm{P}^n),\mathbb{F}_2)}_{\cong \mathrm{Tor}(\mathbb{F}_2,\mathbb{F}_2) \cong \mathbb{F}_2} \to 0$$

whereas for an odd number $2k+1 \in \{0, \dots, n\}$ we have a short exact sequence

$$0 \to \underbrace{\mathrm{H}_{2k+1}(\mathbb{R}\mathrm{P}^n) \otimes \mathbb{F}_2}_{\substack{\cong \mathbb{F}_2 \otimes \mathbb{F}_2 \cong \mathbb{F}_2 \\ \text{or } \cong \mathbb{Z} \otimes \mathbb{F}_2 \cong \mathbb{F}_2}} \xrightarrow{\mu} \mathrm{H}_{2k+1}(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2) \to \underbrace{\mathrm{Tor}(\mathrm{H}_{2k}(\mathbb{R}\mathrm{P}^n),\mathbb{F}_2)}_{\substack{\cong \mathrm{Tor}(0,\mathbb{F}_2) = 0 \\ \text{or } \cong \mathrm{Tor}(\mathbb{Z},\mathbb{F}_2) = 0}} \to 0.$$

Also it follows easily that $\mathrm{H}_k(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2) = 0$ for $k > n$. Summarizing we obtain the following result:

$$\mathrm{H}_k(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2) \cong \begin{cases} \mathbb{F}_2, & \text{if } k = 0, \dots, n, \\ 0, & \text{otherwise.} \end{cases}$$

Fortunately this calculation agrees with the calculation in the $\mathbb{R}\mathrm{P}^n$-H_*-Proposition 52.10.

□

54.3. **Splittings of the Universal Coefficient Theorem.** Our goal, first formulated in Question 53.1, has been to determine the homology groups $\mathrm{H}_n(C_* \otimes G)$ only in terms of the original groups $\mathrm{H}_k(C_*)$ and the abelian group G. The Algebraic Universal Coefficient Theorem 54.8 gives us a short exact sequence

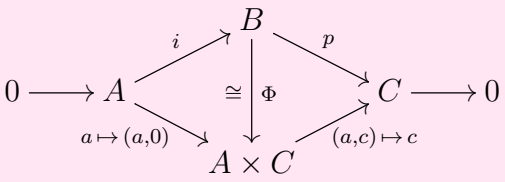
$$0 \to \mathrm{H}_n(C_*) \otimes G \xrightarrow{\mu} \mathrm{H}_n(C_* \otimes G) \xrightarrow{\nu} \mathrm{Tor}(\mathrm{H}_{n-1}(C_*), G) \to 0.$$

But in general, as we already saw on page 480, given a short exact sequence, it is not possible to determine the middle group of a short exact sequence from the groups to the left and the right. At this stage it is worth recalling the following lemma:

Lemma 37.2. (Splitting Lemma) Let $0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$ be a short exact sequence of abelian groups. The following three statements are equivalent:

(1) The short exact sequence splits, i.e. there exists a homomorphism $s\colon C \to B$ with $p \circ s = \mathrm{id}_C$.
(2) There exists a homomorphism $t\colon B \to A$ such that $t \circ i = \mathrm{id}_A$.

(3) There exists an isomorphism $\Phi\colon B \to A\times C$ such that the following diagram commutes:



.

The above discussion raises the question whether the short exact sequence in the Algebraic Universal Coefficient Theorem 54.8 splits. This is fortunately the case:

Proposition 54.10. (Algebraic UCT-Splitting Proposition) Let (C_*, ∂_*) be a chain complex of free abelian groups and let G be an abelian group. For any $n \in \mathbb{N}_0$ the short exact sequence

$$0 \;\to\; \mathrm{H}_n(C_*) \otimes G \;\xrightarrow{\mu}\; \mathrm{H}_n(C_* \otimes G) \;\xrightarrow{\nu}\; \mathrm{Tor}(\mathrm{H}_{n-1}(C_*), G) \;\to\; 0$$

of the Algebraic Universal Coefficient Theorem 54.8 splits. In particular there exists an isomorphism

$$\mathrm{H}_n(C_* \otimes G) \;\cong\; \mathrm{H}_n(C_*) \otimes G \oplus \mathrm{Tor}(\mathrm{H}_{n-1}(C_*), G).$$

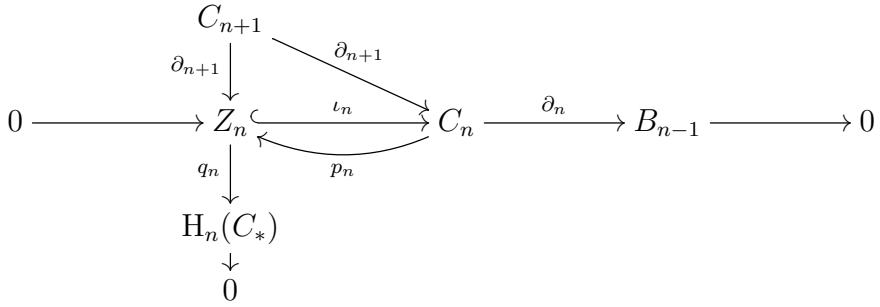
Proof. Let (C_*, ∂_*) be a chain complex of free abelian groups and let G be an abelian group. According to the Splitting Lemma 37.2 (3)⇒(1) it suffices to construct a homomorphism $\eta\colon \mathrm{H}_n(C_*\otimes G) \to \mathrm{H}_n(C_*)\otimes G$ such that $\eta\circ\mu$ is the identity on $\mathrm{H}_n(C_*)\otimes G$. We summarize this problem as follows:

$$0 \longrightarrow \mathrm{H}_n(C_*) \otimes G \underset{\overset{\longleftarrow}{?\,\eta\,?}}{\xrightarrow{\mu}} \mathrm{H}_n(C_* \otimes G) \xrightarrow{\nu} \mathrm{Tor}(\mathrm{H}_{n-1}(C_*), G) \longrightarrow 0.$$

For each $n \in \mathbb{N}_0$ we consider, as on page 683, $Z_n := \ker(\partial_n)$ and $B_n := \mathrm{im}(\partial_{n+1})$ and we consider the short exact sequence

$$0 \longrightarrow Z_n \xhookrightarrow{\iota_n} C_n \xrightarrow{\partial_n} B_{n-1} \longrightarrow 0.$$

The group $B_{n-1} \subset C_{n-1}$ is a subgroup of a free abelian group, thus it is itself a free abelian group by the Free Abelian Subgroup Theorem 18.7. Therefore we can deduce from the Free Abelian Group–Splitting Lemma 18.5 together with the Splitting Lemma 37.2 (1)⇒(3) that the above short exact sequence splits. In particular, according to the Splitting Lemma 37.2 (1)⇒(3), for each $n \in \mathbb{N}_0$ there exists a homomorphism $p_n\colon C_n \to Z_n$ such that $p_n \circ \iota_n = \mathrm{id}_{Z_n}$. Again we summarize the setting as follows:



where q_n is just the natural projection.

Claim 1. The following diagram commutes:

$$\begin{array}{ccccccccc}
\ldots & \longrightarrow & C_{n+1} & \xrightarrow{\partial_{n+1}} & C_n & \xrightarrow{\partial_n} & C_{n-1} & \longrightarrow & \ldots \\
 & & \downarrow{\scriptstyle q_{n+1}\circ p_{n+1}} & & \downarrow{\scriptstyle q_n\circ p_n} & & \downarrow{\scriptstyle q_{n-1}\circ p_{n-1}} & & \\
\ldots & \longrightarrow & \mathrm{H}_{n+1}(C_*) & \xrightarrow{0} & \mathrm{H}_n(C_*) & \xrightarrow{0} & \mathrm{H}_{n-1}(C_*) & \longrightarrow & \ldots
\end{array}$$

Proof. The map down-right is evidently zero. On the other hand we also have

$$(q_n \circ p_n) \circ \partial_{n+1} \underset{\substack{\uparrow \\ \text{since } \iota_n \circ \partial_{n+1} = \partial_{n+1}}}{=} q_n \circ p_n \circ \iota_n \circ \partial_{n+1} \underset{\substack{\uparrow \\ \text{since } p_n \circ \iota_n = \mathrm{id}_{Z_n}}}{=} q_n \circ \partial_{n+1} \underset{\substack{\uparrow \\ \text{since } \mathrm{H}_n(C_*) = Z_n/\operatorname{im}(\partial_{n+1})}}{=} 0.$$

⊞

The top and the bottom horizontal sequences of the claim are chain complexes. Since the diagram commutes we see that the vertical maps from top to bottom are chain maps. Now we tensor the top and the bottom horizontal sequences with G. From the Functor-applied-to-Chain Complex Lemma 52.2 together with the Tensor–Functor Lemma 52.3 we obtain the following diagram of chain complexes:

$$\begin{array}{ccccccccc}
\ldots & \longrightarrow & C_{n+1} \otimes G & \xrightarrow{\partial_{n+1}\otimes \mathrm{id}_G} & C_n \otimes G & \xrightarrow{\partial_n \otimes \mathrm{id}_G} & C_{n-1} \otimes G & \longrightarrow & \ldots \\
 & & \downarrow{\scriptstyle (q_{n+1}\circ p_{n+1})\otimes \mathrm{id}_G} & & \downarrow{\scriptstyle (q_n\circ p_n)\otimes \mathrm{id}_G} & & \downarrow{\scriptstyle (q_{n-1}\circ p_{n-1})\otimes \mathrm{id}_G} & & \\
\ldots & \longrightarrow & \mathrm{H}_{n+1}(C_*) \otimes G & \xrightarrow{0} & \mathrm{H}_n(C_*) \otimes G & \xrightarrow{0} & \mathrm{H}_{n-1}(C_*) \otimes G & \longrightarrow & \ldots
\end{array}$$

The induced maps on the homology of these two chain complexes give us a homomorphism

$$\eta\colon \mathrm{H}_n(C_* \otimes G) \;\to\; \mathrm{H}_n(\text{lower sequence}) = \mathrm{H}_n(C_*) \otimes G.$$

It remains to prove the following claim:

Claim 2. The composition $\eta \circ \mu$ is the identity on $\mathrm{H}_n(C_*) \otimes G$.

Proof. First let $[\sigma] \otimes g \in \mathrm{H}_n(C_*) \otimes G$ where $\sigma \in Z_n$. Then

$$(\eta \circ \mu)([\sigma] \otimes g) \underset{\substack{\uparrow \\ \text{definition of } \mu}}{=} \eta([\sigma \otimes g]) \underset{\substack{\uparrow \\ \text{definition of } \eta}}{=} (q_n(p_n(\sigma))) \otimes g \underset{\substack{\uparrow \\ \text{since } p_n = \mathrm{id} \text{ on } Z_n}}{=} q_n(\sigma) \otimes g \underset{\substack{\uparrow \\ \text{definition of } q_n}}{=} [\sigma] \otimes g.$$

The claim now follows from the observation that the Tensor Product–Universal Property Lemma 51.1 (2) tells us that the $[\sigma] \otimes g$ generate $\mathrm{H}_n(C_*) \otimes G$. ■

The following immediate corollary to the Algebraic UCT-Splitting Proposition 54.10 shows in particular that the isomorphism types of the homology groups with G-coefficients of a topological space X are completely determined by the usual homology groups of X and the abelian group G.

Theorem 54.11. (UCT-Splitting Theorem) Let (X, A) be a pair of topological spaces and let G be an abelian group. For each $n \in \mathbb{N}_0$ the short exact sequence

$$0 \;\to\; \mathrm{H}_n(X, A) \otimes G \;\xrightarrow{\mu}\; \mathrm{H}_n(X, A; G) \;\to\; \operatorname{Tor}(\mathrm{H}_{n-1}(X, A), G) \;\to\; 0$$

from the Universal Coefficient Theorem 54.9 splits. In particular there exists an isomorphism

$$\mathrm{H}_n(X, A; G) \;\cong\; \mathrm{H}_n(X, A) \otimes G \;\oplus\; \operatorname{Tor}(\mathrm{H}_{n-1}(X, A), G).$$

In particular the isomorphism types of the G-homology groups $\mathrm{H}_n(X, A; G)$ of (X, A) are completely determined by the isomorphism types of the "classical" homology groups $\mathrm{H}_n(X, A)$.

Proof. This theorem follows basically immediately from the Algebraic UCT-Splitting Proposition 54.10 applied to the chain complex $\mathrm{C}_*(X, A)$. ■

The careful reader might have noticed that the keyword in the Algebraic UCT-Splitting Proposition 54.10 and the UCT-Splitting Theorem 54.11 that did not bark is the adjective "natural". Put differently, the question arises, whether given an abelian group G there exist *natural* isomorphisms

$$\begin{array}{rcl} \mathrm{H}_n(C_* \otimes G) & \cong & \mathrm{H}_n(C_*) \otimes G \quad \oplus \quad \mathrm{Tor}(\mathrm{H}_{n-1}(C_*), G) \\ \mathrm{H}_n(X, A; G) & \cong & \mathrm{H}_n(X, A) \otimes G \quad \oplus \quad \mathrm{Tor}(\mathrm{H}_{n-1}(X, A), G). \end{array}$$

The following two proposition says that, given any cyclic group $G = \mathbb{Z}_k$, for chain complexes of free abelian groups such a natural isomorphism does not exist:

Proposition 54.12. (Algebraic No-Natural Splitting Proposition) Let $k \in \mathbb{N}_{\geq 2}$. It is not possible to assign to each chain complex C_* of free abelian groups an isomorphism

$$\Phi_{C_*} \colon \mathrm{H}_n(C_* \otimes \mathbb{Z}_k) \ \to \ \mathrm{H}_n(C_*) \otimes \mathbb{Z}_k \ \oplus \ \mathrm{Tor}(\mathrm{H}_{n-1}(C_*), \mathbb{Z}_k)$$

such that for every chain map $\psi \colon C_* \to D_*$ between chain complexes of free abelian groups the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}_n(C_* \otimes \mathbb{Z}_k) & \xrightarrow[\cong]{\Phi_{C_*}} & \mathrm{H}_n(C_*) \otimes \mathbb{Z}_k \ \oplus \ \mathrm{Tor}(\mathrm{H}_{n-1}(C_*), \mathbb{Z}_k) \\ \downarrow{\scriptstyle \psi_*} & & \downarrow{\scriptstyle \psi_*} \\ \mathrm{H}_n(D_* \otimes \mathbb{Z}_k) & \xrightarrow[\cong]{\Phi_{D_*}} & \mathrm{H}_n(D_*) \otimes \mathbb{Z}_k \ \oplus \ \mathrm{Tor}(\mathrm{H}_{n-1}(D_*), \mathbb{Z}_k). \end{array}$$

Proof. So suppose that such a natural isomorphism Φ_{C_*} exists for every chain complex C_* of free abelian groups. We consider the following "vertical" chain map ψ between "horizontal" chain complexes C_* and D_*:

$$\begin{array}{lccccccc} C_* := & 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\cdot k} & \mathbb{Z} & \longrightarrow & 0 \\ \downarrow{\scriptstyle \psi} & & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \\ D_* := & 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & 0. \end{array}$$

From the Tensor-with-$\mathbb{Z}_k$–Lemma 51.4 we obtain the following commutative diagram:

$$\begin{array}{lccccccc} C_* \otimes \mathbb{Z}_k \ \cong & 0 & \longrightarrow & \mathbb{Z}_k & \xrightarrow{(\cdot k)=0} & \mathbb{Z}_k & \longrightarrow & 0 \\ \downarrow{\scriptstyle \psi \otimes \mathrm{id}_{\mathbb{Z}_k}} & & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \\ D_* \otimes \mathbb{Z}_k \ \cong & 0 & \longrightarrow & \mathbb{Z}_k & \longrightarrow & 0 & \longrightarrow & 0. \end{array}$$

Using the above information, using the Tensor-with-$\mathbb{Z}_k$–Lemma 51.4 and using the $\mathbb{Z}_k$-Tor-Lemma 54.6 we can calculate the various homology groups in the following diagram:

$$\begin{array}{ccccc}
\overbrace{\mathrm{H}_1(C_*;\mathbb{Z}_k)}^{\cong\mathbb{Z}_k} & \xrightarrow[\cong]{\Phi_{C_*}} & \overbrace{\mathrm{H}_1(C_*)\otimes\mathbb{Z}_k}^{=0} & \oplus & \overbrace{\mathrm{Tor}(\mathrm{H}_0(C_*),\mathbb{Z}_k)}^{\cong\mathbb{Z}_k} \\
\Big\downarrow{\scriptstyle \psi_*} & & \Big\downarrow{\scriptstyle =0\ \psi_*} & & \Big\downarrow{\scriptstyle =0\ \psi_*} \\
\underbrace{\mathrm{H}_1(D_*;\mathbb{Z}_k)}_{\cong\mathbb{Z}_k} & \xrightarrow[\cong]{\Phi_{D_*}} & \underbrace{\mathrm{H}_1(D_*)\otimes\mathbb{Z}_k}_{\cong\mathbb{Z}_k} & \oplus & \underbrace{\mathrm{Tor}(\mathrm{H}_0(D_*),\mathbb{Z}_k)}_{=0}.
\end{array}$$

Since the left vertical map is an isomorphism of the non-trivial group $\mathbb{Z}_k$ (here we use that $k \geq 2$) and since the right vertical map is the zero map we see that this diagram just cannot be commutative. We have thus obtained a contradiction to our assumption on the existence of natural isomorphisms. ■

Proposition 54.13. (No-Natural Splitting Proposition) Let $k \in \mathbb{N}_{\geq 2}$. It is not possible to assign to each topological space X an isomorphism

$$\Phi_X \colon \mathrm{H}_n(X;\mathbb{Z}_k) \ \to\ \mathrm{H}_n(X)\otimes\mathbb{Z}_k \ \oplus\ \mathrm{Tor}(\mathrm{H}_{n-1}(X),\mathbb{Z}_k)$$

such that for every continuous map $f\colon X \to Y$ between topological spaces the following diagram commutes:

$$\begin{array}{ccccc}
\mathrm{H}_n(X;\mathbb{Z}_k) & \xrightarrow[\cong]{\Phi_X} & \mathrm{H}_n(X)\otimes\mathbb{Z}_k & \oplus & \mathrm{Tor}(\mathrm{H}_{n-1}(X),\mathbb{Z}_k) \\
\Big\downarrow{\scriptstyle f_*} & & \Big\downarrow{\scriptstyle f_*} & & \Big\downarrow{\scriptstyle f_*} \\
\mathrm{H}_n(Y;\mathbb{Z}_k) & \xrightarrow[\cong]{\Phi_Y} & \mathrm{H}_n(Y)\otimes\mathbb{Z}_k & \oplus & \mathrm{Tor}(\mathrm{H}_{n-1}(Y),\mathbb{Z}_k).
\end{array}$$

Proof. In the following we deal with the case $k = 2$. We leave it to the reader to take care of the case $k \geq 3$. So suppose that such a natural isomorphism Φ_X exists for every topological space X and for $\mathbb{Z}_2$.

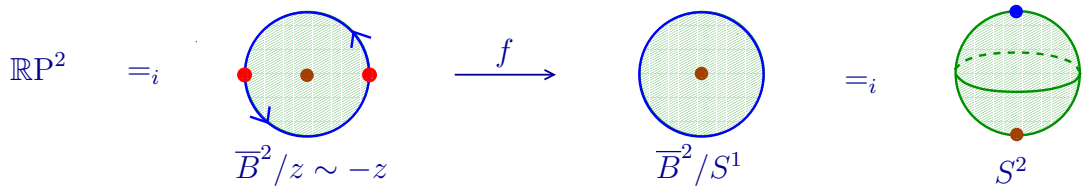
The idea of the proof is quite simple: We want to realize (a slight variation) on the chain complexes and the chain map of the Algebraic No-Natural Splitting Proposition 54.12 by the cellular chain complexes of CW-complexes and by a suitable cellular map.

We make the following preparations:

(1) As in the Ball Quotient-Sphere Lemma 6.5 and as on page 139 we make the usual identifications $S^2 =_{\mathrm{i}} \overline{B}^2/S^1$ and $\mathbb{R}\mathrm{P}^2 =_{\mathrm{i}} \overline{B}^2/z \sim -z$ where $z \in S^1$.

(2) We consider the continuous map

$$\begin{array}{rcl}
f\colon \mathbb{R}\mathrm{P}^2 =_{\mathrm{i}} \overline{B}^2/z \sim -z & \to & \overline{B}^2/S^1 =_{\mathrm{i}} S^2 \\
[P] & \mapsto & [P].
\end{array}$$



(3) As in on page 410 we equip $\mathbb{R}\mathrm{P}^2$ with the CW-structure with one 0-cell, one 1-cell and one 2-cell and as on page 407 we equip S^2 with the CW-structure with one 0-cell and one 2-cell. Note that with this CW-structure the map f is cellular.

Claim. With suitable bases of the cellular chain complexes we get the following description of the cellular chain complexes and the chain map induced by the cellular map f:

$$\begin{array}{ccccccccccc}
\mathrm{C}_*^{\mathrm{CW}}(\mathbb{R}\mathrm{P}^2) & \cong & 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\cdot 2} & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & \longrightarrow & 0 \\
\downarrow{\scriptstyle f_*} & & & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \\
\mathrm{C}_*^{\mathrm{CW}}(S^2) & \cong & 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & 0.
\end{array}$$

Proof. It follows from the discussion on page 601 that the horizontal cellular chain complexes are of the stated form. The fact that the right vertical map is an isomorphism is almost trivial and also irrelevant for what comes next. So let us concentrate on showing that the left vertical map is an isomorphism.

We consider the following commutative diagram:

$$\begin{array}{ccccc}
\mathrm{H}_2(\mathbb{R}\mathrm{P}^2/\mathbb{R}\mathrm{P}^1) & \xleftarrow[\cong]{\text{ю}} & \mathrm{H}_2(\mathbb{R}\mathrm{P}^2, \mathbb{R}\mathrm{P}^1) & \xrightarrow{\mathrm{id}} & \mathrm{C}_2^{\mathrm{CW}}(\mathbb{R}\mathrm{P}^2) \\
\downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} \\
\mathrm{H}_2(S^2/\{N\}) & \xleftarrow[\cong]{\text{ю}} & \mathrm{H}_2(S^2, \{N\}) & \xrightarrow{\mathrm{id}} & \mathrm{C}_2^{\mathrm{CW}}(S^2).
\end{array}$$

We make the following clarifications and observations:

(1) The left horizontal maps ю are the natural isomorphisms from the Relative-to-Quotient-H_*-Proposition 41.8.
(2) It follows from our choice of CW-structures and the definition of the cellular chain groups $\mathrm{C}_n^{\mathrm{CW}}(X) := \mathrm{H}_n(X^n, X^{n-1})$ that the second horizontal maps are the identity.
(3) It follows quite easily from the Compact-Hausdorff Proposition 2.10 (3) that the induced map $f\colon \mathbb{R}\mathrm{P}^2/\mathbb{R}\mathrm{P}^1 \to S^2/\{N\}$ is a homeomorphism. In particular we see that the left vertical map is an isomorphism. But this implies that the right vertical map is also an isomorphism. But that is precisely what we needed to show. ⊞

Now we consider the following diagram:

$$\begin{array}{ccccc}
\overbrace{\mathrm{H}_2(\mathbb{R}\mathrm{P}^2; \mathbb{F}_2)}^{\cong \mathbb{F}_2} & \xrightarrow[\cong]{\Phi_{\mathbb{R}\mathrm{P}^2}} & \overbrace{\mathrm{H}_2(\mathbb{R}\mathrm{P}^2) \otimes \mathbb{F}_2}^{=0} & \oplus & \overbrace{\mathrm{Tor}(\mathrm{H}_1(\mathbb{R}\mathrm{P}^2), \mathbb{F}_2)}^{\cong \mathbb{F}_2} \\
\downarrow{\scriptstyle f_*} & & \overset{=0}{\downarrow} f_* & & \overset{=0}{\downarrow} f_* \\
\underbrace{\mathrm{H}_2(S^2; \mathbb{F}_2)}_{\cong \mathbb{F}_2} & \xrightarrow[\cong]{\Phi_{S^2}} & \underbrace{\mathrm{H}_2(S^2) \otimes \mathbb{F}_2}_{\cong \mathbb{F}_2} & \oplus & \underbrace{\mathrm{Tor}(\mathrm{H}_1(S^2), \mathbb{F}_2)}_{=0}.
\end{array}$$

We make the following observations:

(1) By hypothesis the horizontal maps are isomorphisms and by hypothesis the diagram commutes.
(2) The various homology groups involved can be calculated easily using the claim, the Singular-Cellular-H_*-Proposition 47.3 and the discussion on page 662. The resulting tensor products and Tor-groups can be calculated using the Tensor-with-$\mathbb{Z}_k$–Lemma 51.4 and using the $\mathbb{Z}_k$-Tor-Lemma 54.6.

(3) It follows from the claim and the discussion on page 662 that the vertical map on the left is an isomorphism.
(4) The two vertical maps on the right are zero since either the domain or the target is zero. Thus this diagram cannot possibly commute.

We have thus found a contradiction to our assumption about the existence of the natural maps Φ_X. ■

CHAPTER 55

The tensor product of chain complexes

In this and the following chapter we want to study the following question:

Question 55.1. Let X and Y be two topological spaces. What is the connection between the homology groups of X and Y on the one hand and the homology groups of the product topological space $X \times Y$ on the other hand?

The idea for answering Question 55.1 is as follows:

(1) In this chapter we will introduce the completely algebraic notion of the tensor product $\mathcal{C} \otimes \mathcal{D}$ of two chain complexes $\mathcal{C}$ and $\mathcal{D}$.
(2) In this chapter we will prove the Algebraic–Künneth Theorem 55.5 which relates the homology groups of the chain complex $\mathcal{C} \otimes \mathcal{D}$ to the homology groups of the original chain complexes $\mathcal{C}$ and $\mathcal{D}$.
(3) In the next chapter we will prove the Eilenberg-Zilber Theorem 56.1 which says that, given two topological spaces X and Y, the singular chain complex $\mathrm{C}_*(X \times Y)$ is chain homotopy equivalent to the tensor product of the singular chain complexes $\mathrm{C}_*(X)$ and $\mathrm{C}_*(Y)$.

The combination of (2) and (3) then gives us an answer to Question 55.1, see the Künneth Theorem 56.3 for details.

55.1. **The tensor product of chain complexes.** In this section we will introduce the tensor product of two chain complexes. Initially this definition might come across as being rather ad hoc. Historically the concepts from the next, topological chapter, motivates the definition. But instead of going through a long-winded motivation we just provide the definition and hope that the reader enjoys the corresponding algebra.

Without further ado, here is the definition:

Definition. Let $\mathcal{C} = (C_*, \partial_*)$ and $\mathcal{C}' = (C'_*, \partial'_*)$ be two chain complexes. Given $n \in \mathbb{N}_0$ we define

$$(\mathcal{C} \otimes \mathcal{C}')_n \coloneqq \bigoplus_{p+q=n} C_p \otimes C'_q.$$

We consider the homomorphisms

$$\partial^{\otimes} \colon (\mathcal{C} \otimes \mathcal{C}')_n \to (\mathcal{C} \otimes \mathcal{C}')_{n-1}$$

which are uniquely determined by[435]

$$c_p \otimes c'_q \mapsto \underbrace{\partial_p(c_p) \otimes c'_q + (-1)^p \cdot c_p \otimes \partial'_q(c'_q)}_{=(\partial_p \otimes \mathrm{id} + (-1)^p \cdot \mathrm{id} \otimes \partial'_q)(c_p \otimes c'_q)}$$

for $c_p \in C_p$ and $c'_q \in C'_q$. By the Tensor–Boundary Lemma 55.2 below these maps $\partial^\otimes$ turn $\mathcal{C} \otimes \mathcal{C}'$ into a chain complex, called the tensor product[442] $\mathcal{C} \otimes \mathcal{C}'$ of the two chain complexes $\mathcal{C}$ and $\mathcal{C}'$.

Visualisation. We illustrate the definition of the tensor product of chain complexes in the following diagram:

$$\begin{array}{ccccccc}
C_0 \otimes C'_3 & \xleftarrow{\partial_1 \otimes \mathrm{id}} & C_1 \otimes C'_3 & \xleftarrow{\partial_2 \otimes \mathrm{id}} & C_2 \otimes C'_3 & \xleftarrow{\partial_3 \otimes \mathrm{id}} & C_3 \otimes C'_3 \\
\downarrow{\scriptstyle \mathrm{id} \otimes \partial'_3} & & \downarrow{\scriptstyle -\mathrm{id} \otimes \partial'_3} & & \downarrow{\scriptstyle \mathrm{id} \otimes \partial'_3} & & \downarrow{\scriptstyle -\mathrm{id} \otimes \partial'_3} \\
C_0 \otimes C'_2 & \xleftarrow{\partial_1 \otimes \mathrm{id}} & C_1 \otimes C'_2 & \xleftarrow{\partial_2 \otimes \mathrm{id}} & C_2 \otimes C'_2 & \xleftarrow{\partial_3 \otimes \mathrm{id}} & C_3 \otimes C'_2 \\
\downarrow{\scriptstyle \mathrm{id} \otimes \partial'_2} & & \downarrow{\scriptstyle -\mathrm{id} \otimes \partial'_2} & & \downarrow{\scriptstyle \mathrm{id} \otimes \partial'_2} & & \downarrow{\scriptstyle -\mathrm{id} \otimes \partial'_2} \\
C_0 \otimes C'_1 & \xleftarrow{\partial_1 \otimes \mathrm{id}} & C_1 \otimes C'_1 & \xleftarrow{\partial_2 \otimes \mathrm{id}} & C_2 \otimes C'_1 & \xleftarrow{\partial_3 \otimes \mathrm{id}} & C_3 \otimes C'_1 \\
\downarrow{\scriptstyle \mathrm{id} \otimes \partial'_1} & & \downarrow{\scriptstyle -\mathrm{id} \otimes \partial'_1} & & \downarrow{\scriptstyle \mathrm{id} \otimes \partial'_1} & & \downarrow{\scriptstyle -\mathrm{id} \otimes \partial'_1} \\
C_0 \otimes C'_0 & \xleftarrow{\partial_1 \otimes \mathrm{id}} & C_1 \otimes C'_0 & \xleftarrow{\partial_2 \otimes \mathrm{id}} & C_2 \otimes C'_0 & \xleftarrow{\partial_3 \otimes \mathrm{id}} & C_3 \otimes C'_0.
\end{array}$$

Here the "antidiagonal" groups of the same color "add up" to the chain groups of $\mathcal{C} \otimes \mathcal{C}'$.

In the above definition we already made use of the following little lemma:

Lemma 55.2. (Tensor–Boundary Lemma) Let $\mathcal{C} = (C_*, \partial_*)$ and $\mathcal{C}' = (C'_*, \partial'_*)$ be two chain complexes. The maps $\partial^\otimes$ introduced above turn $\mathcal{C} \otimes \mathcal{C}'$ into a chain complex.

Proof. Let $\mathcal{C} = (C_*, \partial_*)$ and $\mathcal{C}' = (C'_*, \partial'_*)$ be two chain complexes. We have to show that the maps $\partial^\otimes$ are indeed boundary maps of a chain complex, i.e. we have to verify that $\partial^\otimes \circ \partial^\otimes = 0$. This is the case since for any $c_p \in C_p$ and $c'_q \in C'_q$ we calculate that

$$\begin{aligned}
\partial^\otimes(\partial^\otimes(c_p \otimes c'_q)) &= \partial^\otimes(\overbrace{\partial_p(c_p)}^{\in C_{p-1}} \otimes c'_q) + \partial^\otimes((-1)^p \cdot c_p \otimes \overbrace{\partial'_q(c'_q)}^{\in C'_{q-1}}) \\
&= \underbrace{\partial_{p-1}(\partial_p(c_p))}_{=0} \otimes c'_q + (-1)^{p-1} \cdot \partial_p(c_p) \otimes \partial'_q(c'_q) \\
&\qquad + (-1)^p \cdot \partial_p(c_p) \otimes \partial'_q(c'_q) + (-1)^p \cdot c_p \otimes \underbrace{\partial'_{q-1}(\partial'_q(c'_q))}_{=0} \\
&= (-1)^{p-1} \cdot \partial_p(c_p) \otimes \partial'_q(c'_q) - (-1)^{p-1} \cdot \partial_p(c_p) \otimes \partial'_q(c'_q) \;=\; 0.
\end{aligned}$$

Thus we have shown that $(\mathcal{C} \otimes \mathcal{C}', \partial^\otimes)$ is indeed a chain complex.[443] ∎

Example. Let

$$\mathcal{C} \;=\; \cdots \to C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} C_{n-2} \to \ldots \qquad \text{and} \qquad \mathcal{C}' \;=\; 0 \;\to\; \overset{\text{in degree } k}{\overset{\downarrow}{G}} \;\to\; 0$$

[441] Secretly we are using here the Tensor Product–Universal Property Lemma 51.1 (3) and the Coproduct-of-Abelian Groups Lemma 18.1 (2).

[442] The author is not aware of any universal properties of the tensor product of chain complexes.

[443] Note that in the calculation we made use of the extra-term $(-1)^p$ in the definition of the boundary map. If we had defined the boundary map "naively" as $\partial^\otimes(c_p \otimes c'_q) = \partial_p(c_p) \otimes c'_q + c_p \otimes \partial'_q(c'_q)$, then the result would in general not have been a chain complex.

be two chain complexes where the second consists of a single abelian group G. In this case we have for each $n \in \mathbb{N}_0$ the equality

$$(\mathcal{C}_* \otimes \mathcal{C}'_*)_n \;=\; \mathcal{C}_{n-k} \otimes G$$

and the boundary map is given by $\partial_{n-k} \otimes \mathrm{id}_G$. Thus we see that in this case the tensor product of chain complexes is just given by tensoring the chain groups of $\mathcal{C}$ by G and shifting the degrees "upwards" by k. □

We conclude this section with the following lemma, which we will use subconsciously on several occasions:

Lemma 55.3. (Tensoring Chain Maps Lemma) Let $\mathcal{C}, \mathcal{D}, \mathcal{C}', \mathcal{D}'$ be chain complexes.

(1) Given two chain maps $f\colon \mathcal{C} \to \mathcal{D}$ and $f'\colon \mathcal{C}' \to \mathcal{D}'$ the induced map

$$\begin{aligned} f \otimes f'\colon \mathcal{C} \otimes \mathcal{C}' &\to \mathcal{D} \otimes \mathcal{D}' \\ \sum_{i=1}^{n} c_i \otimes c'_i &\mapsto \sum_{i=1}^{n} f(c_i) \otimes f'(c'_i) \end{aligned}$$

is also a chain map.

(2) If $f, g\colon \mathcal{C} \to \mathcal{D}$ and $f', g'\colon \mathcal{C}' \to \mathcal{D}'$ are chain homotopic maps, then the induced maps

$$f \otimes f', g \otimes g'\colon \mathcal{C} \otimes \mathcal{C}' \;\to\; \mathcal{D} \otimes \mathcal{D}'$$

are also chain homotopic.

(3) If $f\colon \mathcal{C} \to \mathcal{D}$ and $f'\colon \mathcal{C}' \to \mathcal{D}'$ are chain homotopy equivalences, then the induced map

$$f \otimes f'\colon \mathcal{C} \otimes \mathcal{C}' \;\to\; \mathcal{D} \otimes \mathcal{D}'$$

is also a chain homotopy equivalence.

Proof.

(1) Let $f\colon \mathcal{C} \to \mathcal{D}$ and $f'\colon \mathcal{C}' \to \mathcal{D}'$ be chain maps. Given any $c \in C_p$ and $c' \in C'_q$ we calculate that

$$\begin{aligned} \partial^{\otimes}(f(c) \otimes f'(c')) &\overset{\text{since } \deg(f(c)) = \deg(c) = p}{=} \partial_p(f(c)) \otimes f'(c') + (-1)^p \cdot f(c) \otimes \partial_q(f'(c')) \\ &= f(\partial_p(c)) \otimes f'(c') + (-1)^p \cdot f(c) \otimes f'(\partial_q(c')) \\ &\overset{\text{since } f \text{ and } f' \text{ are chain maps}}{=} (f \otimes f')\big(\partial_p(c) \otimes c' + (-1)^p \cdot c \otimes \partial_q(c')\big) \;=\; (f \otimes f')(\partial^{\otimes}(c \otimes c')). \end{aligned}$$

(2) Let $f, g\colon \mathcal{C} \to \mathcal{D}$ and $f', g'\colon \mathcal{C}' \to \mathcal{D}'$ be chain maps.

Claim 1. If $P\colon \mathcal{C} \to \mathcal{D}$ is a chain homotopy between the chain maps f and g, then the maps $P_n \otimes f'\colon C_n \otimes \mathcal{C}' \to D_{n+1} \otimes \mathcal{D}'$ are a chain homotopy between $f \otimes f'$ and $g \otimes f'$.

Proof. We need to show that $\partial^{\otimes} \circ (P \otimes f') + (P \otimes f') \circ \partial^{\otimes} = f \otimes f' - g \otimes f'$. We only have to verify that this statement holds on each $C_p \otimes C'_q$. As maps $C_p \otimes C'_q \to D_p \otimes D'_q$

we have

$$\begin{aligned}
&(\partial^{\otimes}_{p+q+1}) \circ (P \otimes f') + (P \otimes f') \circ \partial^{\otimes}_{p+q} \\
&= \underbrace{(\partial_{p+1} \otimes \mathrm{id} + (-1)^{p+1} \cdot \mathrm{id} \otimes \partial'_q)}_{=\partial^{\otimes}_{p+q+1}\colon C_{p+1}\otimes D_q \to C_p \otimes D_q \oplus C_{p+1} \otimes D_{q-1}} \circ (P \otimes f') + (P \otimes f') \circ \underbrace{(\partial_p \otimes \mathrm{id} + (-1)^p \cdot \mathrm{id} \otimes \partial'_q)}_{=\partial^{\otimes}_{p+q}\colon C_p \otimes D_q \to C_{p-1} \otimes D_q \oplus D_p \otimes D_{q-1}} \\
&= (\partial_{p+1} \circ P) \otimes f' + (-1)^{p+1} \cdot P \otimes (\partial'_q \circ f') + (P \circ \partial_p) \otimes f' + (-1)^p \cdot P \otimes \underbrace{(f' \circ \partial'_q)}_{=\partial'_q \circ f'} \\
&= (\partial_{p+1} \circ P) \otimes f' + (P \circ \partial_p) \otimes f' \;=\; (\partial_{p+1} \circ P + P \circ \partial_p) \otimes f' \;=\; (f_p - g_p) \otimes f' \\
&= f_p \otimes f' - g_p \otimes f'.
\end{aligned}$$

⊞

Since the situation is not entirely symmetric between the two factors let us also consider explicitly the situation with the two entries reversed:

Claim 2. Let $P'\colon \mathcal{C}' \to \mathcal{D}'$ be a chain homotopy between the chain maps f' and g'. Then the maps $(-1)^p \cdot f_p \otimes P'_q\colon C_p \otimes C'_q \to C_p \otimes D'_{q+1}$ define a chain homotopy between $f \otimes f'$ and $f \otimes g'$.

Proof. We need to show that $\partial^{\otimes} \circ (f \otimes P') + (f \otimes P') \circ \partial^{\otimes} = f \otimes f' - f \otimes g'$. We only have to verify that this statement holds on each $C_p \otimes C'_q$. We set $Q \coloneqq (-1)^p \cdot f_p \otimes P'_q$. As maps $C_p \otimes C'_q \to D_p \otimes D'_q$ we have

$$\begin{aligned}
&\partial^{\otimes}_{p+q+1} \circ Q + Q \circ \partial^{\otimes}_{p+q} \\
&= \underbrace{(\partial_p \otimes \mathrm{id} + (-1)^p \cdot \mathrm{id} \otimes \partial'_{q+1})}_{=\partial^{\otimes}_{p+q+1}\colon C_p \otimes D_{q+1} \to C_{p-1} \otimes D_{q+1} \oplus C_p \otimes D_q} \circ Q + Q \circ \underbrace{(\partial_p \otimes \mathrm{id} + (-1)^p \cdot \mathrm{id} \otimes \partial'_q)}_{=\partial^{\otimes}\colon C_p \otimes D_q \to C_{p-1} \otimes D_q \oplus C_p \otimes D_{q-1}} \\
&= (-1)^p \cdot (\partial_p \circ f_p) \otimes P' + f_p \otimes (\partial'_{q+1} \circ P') + (-1)^{p-1} \cdot \underbrace{(f_p \circ \partial_p)}_{=\partial_p \circ f_p} \otimes P' + f_p \otimes (P' \circ \partial'_q) \\
&= f_p \otimes (\partial'_{q+1} \circ P' + P' \circ \partial'_q) \;=\; f_p \otimes (f'_q - g'_q) \;=\; f_p \otimes f'_q - f_p \otimes g'_q.
\end{aligned}$$

⊞

Now we finally turn to the actual proof of Statement (2) of the lemma. Thus let $f, g\colon \mathcal{C} \to \mathcal{D}$ and $f', g'\colon \mathcal{C}' \to \mathcal{D}'$ be chain homotopic chain maps. It follows from Claim 1, Claim 2 and the Chain Homotopy-Lemma 36.2 that $f \otimes f', g \otimes g'\colon \mathcal{C} \otimes \mathcal{C}' \to \mathcal{D} \otimes \mathcal{D}'$ are chain homotopic.

(3) This statement follows easily from the definitions and (2). ■

55.2. **The Künneth Theorem for chain complexes.** Let $\mathcal{C}$ and $\mathcal{D}$ be two chain complexes of free abelian groups. In this section we want to determine the homology groups of $\mathcal{C} \otimes \mathcal{D}$ in terms of the homology groups of $\mathcal{C}$ and $\mathcal{D}$. We start out with the following elementary lemma:

Lemma 55.4. (Cycle Tensor Lemma) Let $\mathcal{C}$ and $\mathcal{C}'$ be chain complexes. The map

$$\begin{array}{rcl}
\mu\colon \mathrm{H}_p(\mathcal{C}) \otimes \mathrm{H}_q(\mathcal{C}') & \to & \mathrm{H}_{p+q}(\mathcal{C} \otimes \mathcal{C}') \\
\sum_{i=1}^{r} [c_i] \otimes [c'_i] & \mapsto & \sum_{i=1}^{r} [c_i \otimes c'_i]
\end{array}$$

is a well-defined homomorphism and it is natural with respect to the chain complexes $\mathcal{C}$ and $\mathcal{C}'$.

Proof. Let $\mathcal{C} = (C_*, \partial_*)$ and $\mathcal{C}' = (C'_*, \partial'_*)$ be two chain complexes. We start out with the following claim:

Claim. The map
$$\begin{array}{rcl} \mu\colon \mathrm{H}_p(\mathcal{C}) \times \mathrm{H}_q(\mathcal{C}') & \to & \mathrm{H}_{p+q}(\mathcal{C} \otimes \mathcal{C}') \\ ([c],[c']) & \mapsto & [c \otimes c'] \end{array}$$
is well-defined and bilinear.

Proof. First note that if $c \in C_p$ and $c' \in C_q$ are cycles, then
$$\partial^{\otimes}(c \otimes c') \;=\; \underbrace{\partial_p(c)}_{=0} \otimes c' + (-1)^p \cdot c \otimes \underbrace{\partial_q(c')}_{=0} \;=\; 0$$
i.e. $c \otimes c' \in \mathcal{C} \otimes \mathcal{C}'$ is indeed a cycle. Second, if $c + \partial_{p+1}(e)$ is another representative of $[c]$ then we have the following equality in $\mathrm{H}_{p+q}(\mathcal{C} \otimes \mathcal{C}')$:
$$[(c + \partial_{p+1}(e)) \otimes c'] \;=\; [c \otimes c' + \partial_{p+1}(e) \otimes c'] \underset{\substack{\uparrow \\ \text{since } \partial'_q(c') = 0}}{=} [c \otimes c' + \partial^{\otimes}(e \otimes c')] \;=\; [c \otimes c'].$$
The same way one can show that the value of μ does not depend on the choice of representative of the class in $\mathrm{H}_q(\mathcal{C}')$. We have thus shown that the map given in the claim is well-defined. It is clear that it is bilinear. ⊞

It follows from the claim and the Tensor Product–Universal Property Lemma 51.1 (3) that the map μ of the lemma is a well-defined homomorphism. Finally note that it is straightforward to verify that μ is in fact a natural map. ■

Example. In some instances the homomorphism μ from the Cycle Tensor Lemma 55.4 is an isomorphism. For example let $\mathcal{C} = (C_*, \partial_*)$ be a chain complex of free abelian groups such that all boundary maps $C_n \to C_{n-1}$ in $\mathcal{C}$ are the zero map. In this case we have for any chain complex $\mathcal{C}'$ that we have natural isomorphisms
$$\bigoplus_{n\in\mathbb{N}_0} \mathrm{H}_p(\mathcal{C}) \otimes \mathrm{H}_{n-p}(\mathcal{C}'_*) \;\underset{\substack{\uparrow \\ \text{since all boundary maps in } \mathcal{C} \text{ are zero}}}{=}\; \bigoplus_{n\in\mathbb{N}_0} C_p \otimes \mathrm{H}_n(\mathcal{C}'_{*-p}) \;\underset{\substack{\uparrow \\ \text{by the Algebraic Universal Coefficient Theorem 54.8 and the Tor-Properties Lemma 54.7 (4) (here we use that all } C_p \text{ are free abelian)}}}{\overset{\cong}{\to}}\; \bigoplus_{n\in\mathbb{N}_0} \mathrm{H}_n(C_p \otimes \mathcal{C}'_{*-p}) \;\underset{\substack{\uparrow \\ \text{by the example on page 694 since } \mathcal{C} \text{ is a direct sum of trivial chain complexes}}}{\overset{\cong}{\to}}\; \mathrm{H}_n(\mathcal{C}_* \otimes \mathcal{C}'_*).$$
It follows easily from the definitions that the maps from left to right are induced by the maps μ from the previous lemma. We will shortly make good use of this example. □

Now we can formulate the following promised theorem which is the key result of this chapter.

Theorem 55.5. (Algebraic–Künneth Theorem)[444] Let $\mathcal{C}$ and $\mathcal{C}'$ be two chain complexes. If all chain groups of the chain complex $\mathcal{C}$ are free abelian groups, then there exists a natural homomorphism
$$\nu\colon \mathrm{H}_n(\mathcal{C} \otimes \mathcal{C}') \;\to\; \bigoplus_{p+q=n-1} \mathrm{Tor}(\mathrm{H}_p(\mathcal{C}), \mathrm{H}_q(\mathcal{C}'))$$
such that the following sequence is exact:
$$0 \;\to\; \bigoplus_{p+q=n} \mathrm{H}_p(\mathcal{C}) \otimes \mathrm{H}_q(\mathcal{C}') \;\xrightarrow{\mu}\; \mathrm{H}_n(\mathcal{C} \otimes \mathcal{C}') \;\xrightarrow{\nu}\; \bigoplus_{p+q=n-1} \mathrm{Tor}(\mathrm{H}_p(\mathcal{C}), \mathrm{H}_q(\mathcal{C}')) \;\to\; 0.$$
If the chain groups of $\mathcal{C}'$ are also free abelian, then this short exact sequence splits.[445]

Remark.

(1) The adjective "natural" means in this context that if $f\colon \mathcal{C} \to \mathcal{D}$ and $f'\colon \mathcal{C}' \to \mathcal{D}'$ are chain maps, then we obtain a commutative diagram of short exact sequences:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \bigoplus_{p+q=n} \mathrm{H}_p(\mathcal{C}) \otimes \mathrm{H}_q(\mathcal{C}') & \xrightarrow{\mu} & \mathrm{H}_n(\mathcal{C} \otimes \mathcal{C}') & \xrightarrow{\nu} & \bigoplus_{p+q=n} \mathrm{Tor}(\mathrm{H}_p(\mathcal{C}), \mathrm{H}_q(\mathcal{C}')) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f_*\otimes f'_*} & & \downarrow{\scriptstyle (f\otimes f')_*} & & \downarrow{\scriptstyle \mathrm{Tor}(f_*, f'_*)} & & \\
0 & \longrightarrow & \bigoplus_{p+q=n} \mathrm{H}_p(\mathcal{D}) \otimes \mathrm{H}_q(\mathcal{D}') & \xrightarrow{\mu} & \mathrm{H}_n(\mathcal{D} \otimes \mathcal{D}') & \xrightarrow{\nu} & \bigoplus_{p+q=n} \mathrm{Tor}(\mathrm{H}_p(\mathcal{D}), \mathrm{H}_q(\mathcal{D}')) & \longrightarrow & 0.
\end{array}$$

Here the middle map is given by the Tensoring Chain Maps Lemma 55.3 (1) and the map on the right is given by the functoriality of the torsion-groups that we had discussed on page 678.

(2) If $\mathcal{C}'$ is a chain complex which consists of a single abelian group G, i.e. if it is of the form $0 \to G \to 0$ with G in degree 0, then $\mathrm{H}_0(\mathcal{C}') = G$ and all other homology groups are zero. As on page 694 we see that $(\mathcal{C} \otimes \mathcal{C}')_n = C_n \otimes G$. Thus the Algebraic–Künneth Theorem 55.5 gives us the short exact sequence

$$0 \;\to\; \mathrm{H}_n(\mathcal{C}) \otimes G \;\to \mathrm{H}_n(\underbrace{\mathcal{C} \otimes G}_{=\mathcal{C}\otimes\mathcal{C}'}) \;\to\; \mathrm{Tor}(\mathrm{H}_{n-1}(\mathcal{C}), G) \;\to\; 0,$$

i.e. we recover the Algebraic Universal Coefficient Theorem 54.8 as a special case of the Algebraic–Künneth Theorem 55.5. In fact, as we will see, the proof of the Künneth Theorem is quite similar to the proof of the Algebraic Universal Coefficient Theorem 54.8.

□

Now we turn to the proof of the Algebraic–Künneth Theorem 55.5. To improve readability we break the proof into two parts. First we define ν and we show that we obtain a short exact sequence. Afterwards we will show that the short exact sequence splits if all chain groups of $\mathcal{C}'$ are also free abelian.

Proof that the short exact sequence exists. Let $\mathcal{C} = (C_*, \partial_*)$ and $\mathcal{C}' = (C'_*, \partial'_*)$ be two chain complexes such that all C_n are free abelian groups. First we introduce the following notation and objects:

(1) (a) For $n \in \mathbb{N}_0$ we set $Z_n := \ker(\partial_n\colon C_n \to C_{n-1})$.
(b) We denote by $\mathcal{Z}$ the chain complex which is given by the groups Z_n where all the boundary maps are the zero map.
(c) We denote by $\iota\colon Z_n \to C_n$ the inclusion map. Note that these maps form a chain map $\iota\colon \mathcal{Z} \to \mathcal{C}$.

(2) (a) For $n \in \mathbb{N}_0$ we set $B_n := \mathrm{im}(\partial_{n+1}\colon C_{n+1} \to C_n)$.
(b) We denote by $\mathcal{B}$ the chain complex which is given by the groups B_n where all the boundary maps are the zero map.

[444] Hermann Künneth (1892-1975) was a German mathematician. In [**Kün23a, Kün24**] he proved early versions of the above theorem. In [**Kün23b**] he also gave one of the first proofs of the (much easier) Product-π_1-Proposition 13.7.

[445] If the Künneth Theorem splits, then it follows from the Splitting Lemma 37.2 that the middle group is isomorphic to the direct sum of the two groups to the left and right. One can show, analogous to the Algebraic No-Natural Splitting Proposition 54.12, that there is no natural isomorphism.

(c) We denote by $\mathcal{B}[1]$ the chain complex with $\mathcal{B}[1]_n = \mathcal{B}_{n-1}$, i.e. it is the chain complex $\mathcal{B}$ but all groups got shifted by 1 to the left.
(d) Note that the boundary maps $\partial_n \colon C_n \to B_{n-1}$ define a chain map $\partial \colon \mathcal{C} \to \mathcal{B}[1]$.

Claim. The maps

$$0 \;\to\; \mathcal{Z} \otimes \mathcal{C}' \;\xrightarrow{\iota\otimes\mathrm{id}}\; \mathcal{C} \otimes \mathcal{C}' \;\xrightarrow{\partial\otimes\mathrm{id}}\; \mathcal{B}[1] \otimes \mathcal{C}' \;\to\; 0$$

form a short exact sequence of chain complexes.

Proof. It follows from the Tensoring Chain Maps Lemma 55.3 (1) that the homomorphisms $\iota \otimes \mathrm{id} \colon \mathcal{Z} \otimes \mathcal{C}' \to \mathcal{C} \otimes \mathcal{C}'$ and $\partial \otimes \mathrm{id} \colon \mathcal{C} \otimes \mathcal{C}' \to \mathcal{B}[1] \otimes \mathcal{C}$ are chain maps. It remains to show that the sequence is exact at each degree. The argument is exactly the same as on page 683. For the reader's convenience we recall the argument. First note that by definition the sequences

$$0 \;\to\; Z_p \;\xrightarrow{\iota}\; C_p \;\xrightarrow{\partial_p}\; \underbrace{B_{p-1}}_{=\mathcal{B}[1]_p} \;\to\; 0$$

are exact. Also recall that by hypothesis each C_n is a free abelian group. It follows from the Free Abelian Subgroup Theorem 18.7 that each B_n is also free abelian. By the Free Abelian Group–Splitting Lemma 18.5 together with the Splitting Lemma 37.2 (1)⇒(3) this means that the above short exact sequence splits. But then it follows from the Tensor-Split-SES Lemma 52.6 that the sequence stays exact even after tensoring it with any C'_q. The claim is an immediate consequence of these observations. ⊞

From the LES-Proposition 37.4 and the claim we obtain a natural connecting homomorphism

$$\underbrace{\mathrm{H}_{n+1}(\mathcal{B}[1] \otimes \mathcal{C}')}_{=\mathrm{H}_n(\mathcal{B}\otimes\mathcal{C}')} \;\to\; \mathrm{H}_n(\mathcal{Z} \otimes \mathcal{C}').$$

As in the proof of the Algebraic Universal Coefficient Theorem 54.8 it follows easily from the definition that the connecting homomorphism $\mathrm{H}_n(\mathcal{B} \otimes \mathcal{C}') \to \mathrm{H}_n(\mathcal{Z} \otimes \mathcal{C}')$ is induced by the map $i \otimes \mathrm{id}$ where i denotes the inclusion $\mathcal{B} \to \mathcal{Z}$. From the LES-Proposition 37.4 we obtain the following long exact sequence[446]

$$\cdots \to \mathrm{H}_n(\mathcal{B}\otimes\mathcal{C}') \xrightarrow{i\otimes\mathrm{id}} \mathrm{H}_n(\mathcal{Z}\otimes\mathcal{C}') \to \mathrm{H}_n(\mathcal{C} \otimes \mathcal{C}') \to \mathrm{H}_{n-1}(\mathcal{B}\otimes\mathcal{C}') \xrightarrow{i\otimes\mathrm{id}} \mathrm{H}_{n-1}(\mathcal{Z}\otimes\mathcal{C}') \to \dots$$

Since the boundary maps in the chain complexes $\mathcal{B}$ and $\mathcal{Z}$ are the zero maps it follows from the example on page 697 that we obtain the following long exact sequence:

$$\dots \xrightarrow{i\otimes\mathrm{id}} \bigoplus_{p+q=n} Z_p \otimes \mathrm{H}_q(\mathcal{C}') \to \mathrm{H}_n(\mathcal{C} \otimes \mathcal{C}') \to \bigoplus_{p+q=n-1} B_p \otimes \mathrm{H}_q(\mathcal{C}') \xrightarrow{i\otimes\mathrm{id}} \bigoplus_{p+q=n-1} Z_p \otimes \mathrm{H}_q(\mathcal{C}') \to \dots$$

[446]The index goes down by one in the map $\mathrm{H}_n(\mathcal{C} \otimes \mathcal{C}') \to \mathrm{H}_{n-1}(\mathcal{B} \otimes \mathcal{C}')$, since we replace $\mathcal{B}[1]$ by $\mathcal{B}$.

We obtain a short exact sequence[447] [448]

$$0 \to \bigoplus_{p+q=n} \overbrace{\operatorname{coker}\left(B_p \otimes \mathrm{H}_q(\mathcal{C}') \xrightarrow{i\otimes \mathrm{id}} Z_p \otimes \mathrm{H}_q(\mathcal{C}')\right)}^{\cong \mathrm{H}_p(\mathcal{C}) \otimes \mathrm{H}_q(\mathcal{C}'),\ \text{since } 0 \to B_p \to Z_p \to \mathrm{H}_p(\mathcal{C}) \to 0 \text{ is a free resolution}} \to \mathrm{H}_n(\mathcal{C} \otimes \mathcal{C}') \to$$
$$\to \bigoplus_{p+q=n-1} \underbrace{\ker\left(B_p \otimes \mathrm{H}_q(\mathcal{C}') \xrightarrow{i\otimes \mathrm{id}} Z_p \otimes \mathrm{H}_q(\mathcal{C}')\right)}_{\cong \operatorname{Tor}(\mathrm{H}_p(\mathcal{C}), \mathrm{H}_q(\mathcal{C}')),\ \text{since } 0 \to B_p \to Z_p \to \mathrm{H}_p(\mathcal{C}) \to 0 \text{ is a free resolution}} \to 0.$$

This proves the existence of the homomorphism ν and the fact that μ and ν give us a short exact sequence. Using Statement (2) of the Fundamental Theorem 50.1 of Homological Algebra one can show that ν is natural. We leave it to the reader to verify the details. ■

Proof that the short exact sequence splits. [449] Let $\mathcal{C} = (C_*, \partial_*)$ and $\mathcal{C}' = (C'_*, \partial'_*)$ be two chain complexes such that all C_n and all C'_n are free abelian groups. Recall that according to the Splitting Lemma 37.2 (3)⇒(1) it suffices to construct a homomorphism $\eta\colon \mathrm{H}_n(\mathcal{C}\otimes\mathcal{C}') \to \bigoplus_{p+q=n} \mathrm{H}_p(\mathcal{C})\otimes\mathrm{H}_q(\mathcal{C}')$ such that $\eta\circ\mu$ is the identity on $\bigoplus_{p+q=n} \mathrm{H}_p(\mathcal{C})\otimes\mathrm{H}_q(\mathcal{C}')$. We summarize this problem as follows:

$$0 \longrightarrow \bigoplus_{p+q=n} \mathrm{H}_p(\mathcal{C}) \otimes \mathrm{H}_q(\mathcal{C}') \xrightarrow[\leftarrow\ -\ ?\ \eta\ ?\ -\ ]{\mu} \mathrm{H}_n(\mathcal{C} \otimes \mathcal{C}') \xrightarrow{\nu} \bigoplus_{p+q=n-1} \operatorname{Tor}(\mathrm{H}_p(\mathcal{C}), \mathrm{H}_q(\mathcal{C}')) \longrightarrow 0.$$

For each $n \in \mathbb{N}_0$ we consider $Z_n := \ker(\partial_n)$ and $B_n := \operatorname{im}(\partial_{n+1})$ and we consider the short exact sequence

$$0 \longrightarrow Z_n \xhookrightarrow{\iota_n} C_n \xrightarrow{\partial_n} B_{n-1} \longrightarrow 0.$$

Since the chain groups C_n are free abelian we see, as in the proof of the Algebraic UCT-Splitting Proposition 54.10, that for each $n \in \mathbb{N}_0$ there exists a homomorphism $p_n\colon C_n \to Z_n$ such that $p_n \circ \iota_n = \mathrm{id}_{Z_n}$. We denote by $q_n\colon Z_n \to \mathrm{H}_n(\mathcal{C})$ the natural projection and we set $f_n := q_n \circ p_n\colon C_n \to \mathrm{H}_n(\mathcal{C})$.

Next recall that in the Algebraic UCT-Splitting Proposition 54.10 we showed that the following diagram commutes:

$$\begin{array}{ccccccccc}
\ldots & \longrightarrow & C_{n+1} & \xrightarrow{\partial_{n+1}} & C_n & \xrightarrow{\partial_n} & C_{n-1} & \longrightarrow & \ldots \\
 & & \downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & \\
\ldots & \longrightarrow & \mathrm{H}_{n+1}(\mathcal{C}) & \xrightarrow{0} & \mathrm{H}_n(\mathcal{C}) & \xrightarrow{0} & \mathrm{H}_{n-1}(\mathcal{C}) & \longrightarrow & \ldots
\end{array}$$

Since each C'_n is also a free abelian group we can do all the definitions and constructions for the chain complex $\mathcal{C}'$. In particular we obtain homomorphisms $f'_n\colon C'_n \to \mathrm{H}_n(\mathcal{C}')$ which have the same properties as the homomorphisms f_n.

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[447] We are using that for homomorphisms $\{\varphi_i\colon V_i \to W_i\}_{i\in I}$ we have $\ker\left(\bigoplus_{i\in I}\varphi_i\right) = \bigoplus_{i\in I}\ker(\varphi_i)$ and similarly for cokernels.

[448] As before we also make use of the Tor-Free Resolution Proposition 54.2 and the Tor_0–Lemma 54.3.

[449] As we will see, the proof of this statement is quite similar to the proof of the Algebraic UCT-Splitting Proposition 54.10.

Using the above commutative diagrams involving f_n and f'_n one can easily show that the following diagram of two horizontal chain complexes commutes:

$$\begin{array}{ccccccccc}
\ldots & \longrightarrow & \overbrace{\bigoplus_{p+q=n+1} \mathcal{C}_p \otimes \mathcal{C}'_q}^{=(\mathcal{C}\otimes\mathcal{C}')_{n+1}} & \xrightarrow{\partial^{\otimes}_{n+1}} & \overbrace{\bigoplus_{p+q=n} \mathcal{C}_p \otimes \mathcal{C}'_q}^{=(\mathcal{C}\otimes\mathcal{C}')_{n}} & \xrightarrow{\partial^{\otimes}_{n}} & \overbrace{\bigoplus_{p+q=n-1} \mathcal{C}_p \otimes \mathcal{C}'_q}^{=(\mathcal{C}\otimes\mathcal{C}')_{n-1}} & \longrightarrow & \ldots \\
 & & \downarrow{\scriptstyle \bigoplus_{p+q=n+1} f_p\otimes f'_q} & & \downarrow{\scriptstyle \bigoplus_{p+q=n} f_p\otimes f'_q} & & \downarrow{\scriptstyle \bigoplus_{p+q=n-1} f_p\otimes f'_q} & & \\
\ldots & \to & \bigoplus_{p+q=n+1} \mathrm{H}_p(\mathcal{C}) \otimes \mathrm{H}_q(\mathcal{C}') & \xrightarrow{0} & \bigoplus_{p+q=n} \mathrm{H}_p(\mathcal{C}) \otimes \mathrm{H}_q(\mathcal{C}') & \xrightarrow{0} & \bigoplus_{p+q=n-1} \mathrm{H}_p(\mathcal{C}) \otimes \mathrm{H}_q(\mathcal{C}') & \to & \ldots
\end{array}$$

Since the diagram commutes we obtain induced maps on the homology of these two chain complexes. These give us a homomorphism

$$\eta\colon \mathrm{H}_n(\mathcal{C}\otimes\mathcal{C}') \;\to\; \mathrm{H}_n(\text{lower sequence}) = \bigoplus_{p+q=n} \mathrm{H}_p(\mathcal{C}) \otimes \mathrm{H}_q(\mathcal{C}').$$

As in the proof of the Algebraic UCT-Splitting Proposition 54.10 it follows easily from the definitions that the composition $\eta \circ \mu$ is the identity on $\bigoplus_{p+q=n} \mathrm{H}_p(\mathcal{C}) \otimes \mathrm{H}_q(\mathcal{C}')$. ■

CHAPTER 56

The Eilenberg-Zilber Theorem and the Künneth Theorem

As we already mentioned in the beginning of the last chapter, the goal of this chapter is to formulate and prove the Eilenberg-Zilber Theorem 56.1 which says that for topological spaces X and Y there exists a natural chain homotopy equivalence between $\mathrm{C}_*(X \times Y)$ and the tensor product $\mathrm{C}_*(X) \otimes \mathrm{C}_*(Y)$ of the chain complexes $\mathrm{C}_*(X)$ and $\mathrm{C}_*(Y)$. Together with the Algebraic–Künneth Theorem 55.5 this allows us to calculate the homology groups of products of topological spaces.

In fact the Eilenberg-Zilber Theorem 56.1 gives us much more, namely it allows us to define the cross product on homology and later it will be at the heart of the all important definition of the cup product on cohomology.

56.1. **The Eilenberg-Zilber theorem.** We start out by recalling and introducing some definitions:

Definition.

(1) Let $\mathcal{C} = (C_*, \partial_*)$ and $\mathcal{C} = (C'_*, \partial'_*)$ be two chain complexes. Recall that on page 693 we defined the tensor product $\mathcal{C} \otimes \mathcal{C}'$ of the chain complexes where the n-th chain group is defined as

$$(\mathcal{C} \otimes \mathcal{C}')_n \quad := \quad \bigoplus_{p+q=n} C_p \otimes C'_q$$

and where the boundary homomorphisms are given by

$$\begin{array}{rccl} \partial^{\otimes}\colon & (\mathcal{C} \otimes \mathcal{C}')_n & \to & (\mathcal{C} \otimes \mathcal{C}')_{n-1} \\ & c_p \otimes c'_q & \mapsto & \partial_p(c_p) \otimes c'_q + (-1)^p \cdot c_p \otimes \partial'_q(c'_q). \end{array}$$

(2) Given a point x in a topological space X we denote by x also the singular 0-simplex given by sending the unique point of Δ^0 to x.

(3) Let X be a topological space and let $x \in X$. If $\sigma\colon \Delta^n \to Y$ is a singular n-simplex in a topological space Y, then we denote by $x \times \sigma\colon \Delta^n \to X \times Y$ the singular n-simplex given by $v \mapsto (x, \sigma(v))$. We use the analogous notation for the roles of X and Y swapped.

(4) Given two topological spaces X and Y we refer to the maps

$$\begin{array}{rcl} \mathrm{C}_0(X) \otimes \mathrm{C}_0(Y) & \to & \mathrm{C}_0(X \times Y) \\ x \otimes y & \mapsto & (x, y) \end{array} \quad \text{and} \quad \begin{array}{rcl} \mathrm{C}_0(X \times Y) & \to & \mathrm{C}_0(X) \otimes \mathrm{C}_0(Y) \\ (x, y) & \mapsto & x \otimes y \end{array}$$

as the canonical maps. We refer to the identity maps on $\mathrm{C}_0(X) \otimes \mathrm{C}_0(Y)$ and on $\mathrm{C}_0(X \times Y)$ also as the canonical maps.

The following theorem, which was first proved in 1953 by Samuel Eilenberg and Joseph Zilber [**EZ53**], relates the singular chain complex $\mathrm{C}_*(X \times Y)$ of the product of two topological spaces X and Y to the tensor product $\mathrm{C}_*(X) \otimes \mathrm{C}_*(Y)$ of the two individual chain complexes.

Theorem 56.1. (Eilenberg-Zilber Theorem)
(1) Given any topological spaces X and Y there exist natural[450] chain maps

$$\Upsilon = \Upsilon_{X,Y} \colon \mathrm{C}_*(X) \otimes \mathrm{C}_*(Y) \quad \to \quad \mathrm{C}_*(X \times Y)$$

and

$$\Theta = \Theta_{X,Y} \colon \mathrm{C}_*(X \times Y) \quad \to \quad \mathrm{C}_*(X) \otimes \mathrm{C}_*(Y),$$

which on the 0-level are the above canonical maps.
(2) The maps Υ and Θ are unique up to natural chain homotopies.[451]
(3) The maps Υ and Θ are naturally[452] chain homotopy inverses of one another.
In fact, we can find $\Upsilon \colon \mathrm{C}_*(X) \otimes \mathrm{C}_*(Y) \to \mathrm{C}_*(X \times Y)$ with the following extra property:
(∗) For any $x \in X$ and any singular simplex $\sigma \colon \Delta^q \to Y$ we have $\Upsilon(x \otimes \sigma) = x \times \sigma$ and the same statement also holds with the roles of X and Y swapped.

Definition. We refer to the maps Υ and Θ from the Eilenberg-Zilber Theorem 56.1 as Eilenberg-Zilber maps.

Proof. We postpone the elegant, but technically slightly intricate, proof of the Eilenberg-Zilber Theorem 56.1 to Section 56.3.[453] ∎

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[450]Here "natural" requires some explanation. Let $\mathcal{TupTop}$ be the category of tuples of topological spaces with

$$\begin{array}{rcl} \mathrm{Ob}(\mathcal{TupTop}) & = & \text{all tuples } (X,Y) \text{ of topological spaces,} \\ \mathrm{Mor}((X,Y),(X',Y')) & = & \text{all tuples } (f,g) \text{ of continuous maps } f \colon X \to X' \text{ and } g \colon Y \to Y'. \end{array}$$

Now "natural" means that the chain maps $\Upsilon_{X,Y}$ form a natural transformation between the functors

$$\begin{array}{rcl} \mathcal{TupTop} & \to & \text{category } \mathcal{ChCplx} \text{ of chain complexes} \\ (X,Y) & \mapsto & \mathrm{C}_*(X) \otimes \mathrm{C}_*(Y) \\ (X,Y) & \mapsto & \mathrm{C}_*(X \times Y) \end{array}$$

and similarly for Θ. In other words, this means that for continuous maps $f \colon X \to X'$ and $g \colon Y \to Y'$ of topological spaces we obtain commutative diagrams

$$\begin{array}{ccc} \mathrm{C}_*(X) \otimes \mathrm{C}_*(Y) & \xrightarrow{\Upsilon_{X,Y}} & \mathrm{C}_*(X \times Y) \\ \downarrow{\scriptstyle f_* \otimes f'_*} & & \downarrow{\scriptstyle (f \times g)_*} \\ \mathrm{C}_*(X') \otimes \mathrm{C}_*(Y') & \xrightarrow{\Upsilon_{X',Y'}} & \mathrm{C}_*(X' \times Y') \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathrm{C}_*(X \times Y) & \xrightarrow{\Theta_{X,Y}} & \mathrm{C}_*(X) \otimes \mathrm{C}_*(Y) \\ \downarrow{\scriptstyle (f \times g)_*} & & \downarrow{\scriptstyle f_* \otimes f'_*} \\ \mathrm{C}_*(X' \times Y') & \xrightarrow{\Theta_{X',Y'}} & \mathrm{C}_*(X') \otimes \mathrm{C}_*(Y'). \end{array}$$

[451]Here "natural chain homotopy" means that given two such maps Υ and Υ' there exists a choice of chain homotopies $(\mathrm{C}_*(X) \otimes \mathrm{C}_*(Y))_k \to \mathrm{C}_{k+1}(X \times Y)$, which is natural in X and Y, between Υ and Υ'. An analogous statement holds for maps Θ and Θ'.
[452]Here "naturally" means that $\Upsilon \circ \Theta$ and $\Theta \circ \Upsilon$ are naturally homotopic to the identity maps, i.e. there exist natural chain homotopies from $\Upsilon \circ \Theta$ to the identity map and from $\Theta \circ \Upsilon$ to the identity map.
[453]The existence proof for the two maps Υ and Θ which we will provide in the following sections is non-explicit. A little later, in the Shuffle Proposition 56.11 we will give an explicit example of an Eilenberg-Zilber map Υ. For the most part the existence is enough, but on a few occasions it will be helpful to have an explicit example of an Eilenberg-Zilber map Υ at our disposal.

56.2. **The Künneth Theorem for topological spaces.** Before we turn to the proof of the Eilenberg-Zilber Theorem 56.1 let us first discuss a major application. To formulate the application it is convenient to introduce the following definition:

Definition. Let X and Y be topological spaces and let $\Upsilon\colon \mathrm{C}_*(X)\otimes\mathrm{C}_*(Y)\to\mathrm{C}_*(X\times Y)$ be an Eilenberg-Zilber map. We refer to the map[454]

$$\times\colon \mathrm{H}_p(X)\otimes\mathrm{H}_q(Y) \xrightarrow{\mu} \mathrm{H}_{p+q}(\mathrm{C}_*(X)\otimes\mathrm{C}_*(Y)) \xrightarrow{\Upsilon_*} \mathrm{H}_{p+q}(X\times Y).$$

(μ: map from the Cycle Tensor Lemma 55.4)

as the cross product.

In the following lemma we summarize some elementary properties of the cross product.

Lemma 56.2. (Cross Product Lemma)

(0) The cross product does not depend on the choice of the Eilenberg-Zilber map Υ.

(1) Let X and Y be two topological spaces. The map

$$\times\colon \mathrm{H}_0(X)\otimes\mathrm{H}_n(Y) \to \mathrm{H}_n(X\times Y)$$

is given by

$$[x]\otimes\Big[\sum_{i=1}^{k} r_i\cdot\sigma_i\Big] \mapsto \Big[\sum_{i=1}^{k} r_i\cdot(x\times\sigma_i)\Big].$$

The statement also holds with the roles of X and Y reversed.

(2) The cross product is natural, i.e. if $f\colon X\to X'$ and $g\colon Y\to Y'$ are continuous maps between topological spaces, then for any $p,q\in\mathbb{N}_0$ the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}_p(X)\otimes\mathrm{H}_q(Y) & \xrightarrow{\times} & \mathrm{H}_{p+q}(X\times Y) \\ \downarrow{\scriptstyle f_*} & \downarrow{\scriptstyle g_*} & \downarrow{\scriptstyle (f\times g)_*} \\ \mathrm{H}_p(X')\otimes\mathrm{H}_q(Y') & \xrightarrow{\times} & \mathrm{H}_{p+q}(X'\times Y'). \end{array}$$

Proof.

(0) This statement follows from the Eilenberg-Zilber Theorem 56.1 (2) together with the Chain Homotopy–H_*-Lemma 36.1.

(1) This statement follows easily from Statement $(*)$ of the Eilenberg-Zilber Theorem 56.1.

(2) This statement follows immediately from the naturality of μ and Υ. ∎

With the Eilenberg-Zilber Theorem 56.1, the results from the previous chapter and the cross product we have now all the tools to deal with Question 55.1. More precisely, the following theorem tells us what is the relationship between the homology groups of the topological spaces X, Y and $X\times Y$.

Theorem 56.3. (Künneth Theorem) For all topological spaces X and Y there exists a natural homomorphism

$$\widetilde{\nu}\colon \mathrm{H}_n(X\times Y) \to \bigoplus_{p+q=n-1} \mathrm{Tor}(\mathrm{H}_p(X),\mathrm{H}_q(Y))$$

[454] As we just mentioned in footnote 453, in the Shuffle Proposition 56.11 we will give an explicit example of an Eilenberg-Zilber map Υ. This makes it possible to write down the cross product explicitly.

such that the following sequence

$$0 \to \bigoplus_{p+q=n} \mathrm{H}_p(X) \otimes \mathrm{H}_q(Y) \xrightarrow{\times} \mathrm{H}_n(X \times Y) \xrightarrow{\widetilde{\nu}} \bigoplus_{p+q=n-1} \mathrm{Tor}(\mathrm{H}_p(X), \mathrm{H}_q(Y)) \to 0$$

is exact and such that it splits.

Proof. Let X and Y be topological spaces. Let

$$\Upsilon \colon \mathrm{C}_*(X) \otimes \mathrm{C}_*(Y) \to \mathrm{C}_*(X \times Y)$$

be an Eilenberg-Zilber map. We consider the following maps:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \displaystyle\bigoplus_{p+q=n} \mathrm{H}_p(X) \otimes \mathrm{H}_q(Y) & \xrightarrow{\mu} & \mathrm{H}_n(\mathrm{C}_*(X) \otimes \mathrm{C}_*(Y)) & \xrightarrow{\nu} & \displaystyle\bigoplus_{p+q=n-1} \mathrm{Tor}(\mathrm{H}_p(X), \mathrm{H}_q(Y)) & \longrightarrow & 0 \\ & & & \searrow^{\times} & \cong \downarrow \Upsilon_* & \nearrow_{=:\widetilde{\nu}} & & & \\ & & & & \underbrace{\mathrm{H}_n(\mathrm{C}_*(X \times Y))}_{=\mathrm{H}_n(X \times Y)}. & & & & \end{array}$$

We make the following observations and clarifications:

(a) The homomorphisms μ and ν are the natural homomorphisms from the Cycle Tensor Lemma 55.4 and the Algebraic–Künneth Theorem 55.5.
(b) By definition we have $\Upsilon_* \circ \mu = \times$. In other words, the left triangle commutes.
(c) Since the chain groups $\mathrm{C}_*(X)$ and $\mathrm{C}_*(Y)$ are free abelian groups it follows from the Algebraic–Künneth Theorem 55.5 that the top sequence is exact and that it splits.
(d) The vertical map is a natural isomorphism by the Eilenberg-Zilber Theorem 56.1 together with the Chain Homotopy Equivalence-H_*-Corollary 36.4.

It follows from the above that the homomorphism $\widetilde{\nu} \coloneqq \nu \circ (\Upsilon_*)^{-1}$ has all the desired properties. ■

Examples.

(1) Let X be a topological space and let $k \in \mathbb{N}$. We want to determine the homology groups of $X \times S^k$. By the Künneth Theorem 56.3 we obtain a short exact sequence[455]

$$0 \to \bigoplus_{p+q=n} \underbrace{\mathrm{H}_p(X) \otimes \mathrm{H}_q(S^k)}_{\substack{= \mathrm{H}_n(X), \quad \text{for } p=n, q=0, \\ = \mathrm{H}_{n-k}(X), \quad \text{for } p=n-k, q=k, \\ = 0, \quad \text{else}}} \xrightarrow{\times} \mathrm{H}_n(X \times S^k) \xrightarrow{\widetilde{\nu}} \underbrace{\bigoplus_{p+q=n-1} \mathrm{Tor}(\mathrm{H}_p(X), \mathrm{H}_q(S^k))}_{=0} \to 0$$

where all maps are natural. Thus we see that we have a natural isomorphism

$$\mathrm{H}_n(X) \oplus \mathrm{H}_{n-k}(X) \xrightarrow{\cong} \mathrm{H}_n(X \times S^k).$$

For $k = 1$ this is basically the same result as in the Product-with-S^1-H_*-Proposition 45.4.

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[454] As in the case of the Algebraic Universal Coefficient Theorem 54.8, see the No-Natural Splitting Proposition 54.13, one can show that the splitting is not natural.

[455] Here we use the following:

- The homology groups of S^k are isomorphic to $\mathbb{Z}$ in the dimensions 0 and k and 0 otherwise. Furthermore the standard generator $[S^k]$ gives us a preferred isomorphism $\mathbb{Z} \to \mathrm{H}_k(S^k)$.
- For any abelian group A we have by the Tensor-with-$\mathbb{Z}$–Lemma 51.3 a natural isomorphism $A \otimes \mathbb{Z} \xrightarrow{\cong} A$ and by the Tor-Properties Lemma 54.7 (3) we have $\mathrm{Tor}(A, \mathbb{Z}) = 0$.

(2) Let $n \in \mathbb{N}$ and let $m \in \mathbb{N}$. Using (1) and using an induction argument one can easily show that for any $k \in \mathbb{N}_0$ there exists an isomorphism

$$\mathrm{H}_i((S^m)^n) \cong \begin{cases} \mathbb{Z}^{\binom{n}{k}} = \mathbb{Z}^{\frac{n!}{k!\cdot(n-k)!}} & \text{if } i = k \cdot m \text{ with } k \in \{0, \dots, n\}, \\ 0, & \text{otherwise.} \end{cases}$$

(3) Finally let us discuss the product $S^2 \times S^4$ in slightly more detail. It follows easily from (1) and the Sphere-H_*-Proposition 41.2 that

$$\mathrm{H}_k(S^2 \times S^4) \cong \begin{cases} \mathbb{Z}, & \text{if } k = 0, 2, 4, 6, \\ 0, & \text{otherwise.} \end{cases}$$

But by the $\mathbb{C}\mathrm{P}^n$-H_*-Proposition 47.4 these are precisely the homology groups of the complex projective space $\mathbb{C}\mathrm{P}^3$. Furthermore it follows from the Sphere-π_1-Proposition 11.15 and the Product-π_1-Proposition 13.7 that $S^2 \times S^4$ is simply connected and it follows from the Projective Space–π_1-Proposition 29.3 that $\mathbb{C}\mathrm{P}^3$ is also simply connected. So all the invariants we are aware of, and that we know how to compute[456], agree for $S^2 \times S^4$ and $\mathbb{C}\mathrm{P}^3$. □

The last example raises the following question.

Question 56.4. Are the topological spaces $S^2 \times S^4$ and $\mathbb{C}\mathrm{P}^3$ homeomorphic?

56.3. **Proof of the Eilenberg-Zilber Theorem 56.1.** The proof of the Eilenberg-Zilber Theorem 56.1 breaks up into three parts:

(1) In the EZ-Construction Proposition 56.5 we build $\Upsilon\colon \mathrm{C}_*(X) \otimes \mathrm{C}_*(Y) \to \mathrm{C}_*(X \times Y)$.
(2) In the EZ-Inverse Proposition 56.6 we construct $\Theta\colon \mathrm{C}_*(X \times Y) \to \mathrm{C}_*(X) \otimes \mathrm{C}_*(Y)$.
(3) Using the EZ–Natural Chain Homotopy Proposition 56.8 we show that these maps Υ and Θ are unique in a suitable sense and that they have the desired properties.

We start out with the following definition:

Definition. A topological space X is called acyclic if $\mathrm{H}_0(X; \mathbb{Z}) \cong \mathbb{Z}$ and if all higher homology groups are zero.

Example. In the Contractible-H_*-Corollary 36.8 we showed that every contractible topological space is acyclic. In particular every standard simplex Δ^n is contractible and thus acyclic. Furthermore, all products $\Delta^p \times \Delta^q$ of standard simplices are contractible and thus acyclic.[457] □

Now we can formulate the first of the three promised propositions.

Proposition 56.5. (EZ-Construction Proposition) For all topological spaces X and Y and all $p, q \in \mathbb{N}_0$ there exists a homomorphism

$$\Upsilon_{p,q}\colon \mathrm{C}_p(X) \otimes \mathrm{C}_q(Y) \to \mathrm{C}_{p+q}(X \times Y)$$

[456] As of right now we do not know the higher homotopy groups of these two smooth manifolds.

[457] This can be seen in many ways, for example pick points $x \in \Delta^p$ and $y \in \Delta^q$, it is straightforward to show that there exists a deformation retraction r from Δ^p to $\{x\}$ and that there exists a deformation retraction s from Δ^q to $\{y\}$. But these two deformation retractions also define a deformation retraction from $\Delta^p \times \Delta^q$ to (x, y).

such that the following conditions are satisfied:

(1) For any $x \in X$ and any singular simplex $\sigma\colon \Delta^q \to Y$ we have $\Upsilon_{0,q}(x \otimes \sigma) = x \times \sigma$. The same statement also holds with the roles of X and Y swapped.

(2) The maps $\Upsilon_{p,q}$ form a chain map, i.e. for any $a \in \mathrm{C}_p(X)$ and any $b \in \mathrm{C}_q(Y)$ we have the following equality in $\mathrm{C}_{p+q-1}(X \times Y)$:

$$\partial_{p+q}(\Upsilon_{p,q}(a \otimes b)) \;=\; \underbrace{\Upsilon_{p-1,q}(\partial_p(a) \otimes b) + (-1)^p \cdot \Upsilon_{p,q-1}(a \otimes \partial_q(b))}_{=\Upsilon(\partial^{\otimes}(a \otimes b))}.$$

(3) The map $\Upsilon_{p,q}$ is natural, i.e. if $f\colon X' \to X$ and $g\colon Y' \to Y$ are continuous maps between topological spaces, then for any $a \in \mathrm{C}_p(X')$ and any $b \in \mathrm{C}_q(Y')$ we have

$$\underbrace{\Upsilon_{p,q}(f_*(a) \otimes g_*(b))}_{\in \mathrm{C}_{p+q}(X \times Y)} \;=\; (f \times g)_*(\underbrace{\Upsilon_{p,q}(a \otimes b)}_{\in \mathrm{C}_{p+q}(X' \times Y')}).$$

Proof. We start out with the outline of proof:

(a) We use (1) to define $\Upsilon_{p,q}$ for $p = 0$ and $q = 0$.
(b) We assume that $\Upsilon_{p',q'}$ is defined for $p' \cdot q' < p \cdot q$. We then use (2) to find a suitable value for $\Upsilon_{p,q}(\mathrm{id}_{\Delta^p}, \mathrm{id}_{\Delta^q})$.
(c) We then use the naturality statement of (3) to define $\Upsilon_{p,q}$ for all topological spaces X and Y and all singular simplices in $\mathrm{C}_p(X) \otimes \mathrm{C}_q(Y)$.

Now we carry out the above program. We prove the existence of the maps $\Upsilon_{p,q}$ by induction on $p \cdot q$. It is quite elementary to show that the maps from (1) have the properties (2) and (3).[458] So now suppose that we are given $p, q \in \mathbb{N}_0$ with $p \cdot q > 0$ such that the maps $\Upsilon_{p',q'}$ have been defined for all p', q' with $p' \cdot q' < p \cdot q$.

Given any $n \in \mathbb{N}_0$ we denote by $\mathrm{id}_n\colon \Delta^n \to \Delta^n$ the identity map. We first want to come up with a reasonable definition of $\Upsilon_{p,q}(\mathrm{id}_p \otimes \mathrm{id}_q) \in \mathrm{C}_{p+q}(\Delta^p \times \Delta^q)$. If we want (2) to be satisfied we have to define $\Upsilon_{p,q}(\mathrm{id}_p \otimes \mathrm{id}_q) =: z_{p,q}$ where $z_{p,q} \in \mathrm{C}_{p+q}(\Delta^p \times \Delta^q)$ satisfies

$$\partial_{p+q}(z_{p,q}) \;=\; \Upsilon_{p-1,q}(\partial_p(\mathrm{id}_p) \otimes \mathrm{id}_q) + (-1)^p \cdot \Upsilon_{p,q-1}(\mathrm{id}_p \otimes \partial_q(\mathrm{id}_q)) \;\in\; \mathrm{C}_{p+q-1}(\Delta^p \times \Delta^q).$$

In other words, we have to prove the following claim:

Claim 1. There exists a $z_{p,q} \in \mathrm{C}_{p+q}(\Delta^p \times \Delta^q)$ with

$$\partial_{p+q}(z_{p,q}) \;=\; \Upsilon_{p-1,q}(\partial_p(\mathrm{id}_p) \otimes \mathrm{id}_q) + (-1)^p \cdot \Upsilon_{p,q-1}(\mathrm{id}_p \otimes \partial_q(\mathrm{id}_q)) \;\in\; \mathrm{C}_{p+q-1}(\Delta^p \times \Delta^q).$$

Proof. As we pointed out on page 706, the product $\Delta^p \times \Delta^q$ is acyclic. Since $p, q \geq 1$ this implies that $\mathrm{H}_{p+q-1}(\Delta^p \times \Delta^q) = 0$. This in turn implies that every singular $(p+q-1)$-chain *that is a cycle* is in fact the boundary of some $(p+q)$-chain $z_{p,q}$. So all we have to do is to

[458] Note though that the proof of property (3) is not completely content-free.

show that the right-hand side of the claim is indeed a cycle. We compute that

$$\begin{aligned}
&\partial_{p+q-1}(\Upsilon_{p-1,q}(\partial_p(\mathrm{id}_p)\otimes \mathrm{id}_q)+(-1)^p\cdot\Upsilon_{p,q-1}(\mathrm{id}_p\otimes\partial_q(\mathrm{id}_q))) \overset{\text{by the linearity of } \partial_{p+q}}{=}\\
&=\partial_{p+q}(\Upsilon_{p-1,q}(\partial_p(\mathrm{id}_p)\otimes \mathrm{id}_q))+(-1)^p\cdot\partial_{p+q}(\Upsilon_{p,q-1}(\mathrm{id}_p\otimes\partial_q(\mathrm{id}_q)))\\
&=\Upsilon_{p-2,q}(\underbrace{\partial_{p-1}\partial_p\,\mathrm{id}_p}_{=0}\otimes \mathrm{id}_q)+(-1)^{p-1}\cdot\Upsilon_{p-1,q-1}(\partial_p(\mathrm{id}_p)\otimes\partial(\mathrm{id}_q))\\
&\quad+(-1)^p\cdot\Upsilon_{p-1,q-1}(\partial_p(\mathrm{id}_p)\otimes\partial_q(\mathrm{id}_q))+(-1)^{2p}\cdot\Upsilon_{p,q-2}(\mathrm{id}_p\otimes\underbrace{\partial_{q-1}\partial_q\,\mathrm{id}_q}_{=0})=0.
\end{aligned}$$

by our induction hypothesis we can apply the boundary formula (2) to $\Upsilon_{p-1,q}(\partial\,\mathrm{id}_p\otimes\mathrm{id}_q)$ and $\Upsilon_{p,q-1}(\mathrm{id}_p\otimes\partial\,\mathrm{id}_q)$ ⊞

This discussion gives us the definition of $\Upsilon(\mathrm{id}_p\otimes\mathrm{id}_q):=z_{p,q}\in \mathrm{C}_{p+q}(\Delta^p\times\Delta^q)$. Now let X and Y be topological spaces and let $\varphi\colon\Delta^p\to X$ and $\psi\colon\Delta^q\to Y$ be singular simplices. If we want (2) to be satisfied then we have to define

$$\Upsilon_{p,q}(\underbrace{\varphi\otimes\psi}_{\in \mathrm{C}_p(X)\otimes\mathrm{C}_q(Y)}):=(\varphi\times\psi)_*(\underbrace{\Upsilon_{p,q}(\mathrm{id}_p\otimes\mathrm{id}_q)}_{\in \mathrm{C}_{p+q}(\Delta^p\times\Delta^q)})=(\varphi\times\psi)_*(z_{p,q})\in \mathrm{C}_{p+q}(X\times Y).$$

It follows from the discussion on page 650 that the $\varphi\otimes\psi$ form a basis of $\mathrm{C}_p(X)\otimes\mathrm{C}_q(Y)$. Thus we can extend this map Θ linearly to all of $\mathrm{C}_p(X)\otimes\mathrm{C}_q(Y)$.

Claim 2. The map $\Upsilon_{p,q}\colon \mathrm{C}_p(X)\otimes\mathrm{C}_q(Y)\to\mathrm{C}_{p+q}(X\times Y)$ has all the desired properties.

Proof. First note that it follows almost immediately from the definition of Υ that the map Υ is natural in X and Y.

Thus it remains to prove (2). This property holds basically by definition. But for completeness' sake we carry out the proof. So let $\varphi\colon\Delta^p\to X$ and $\psi\colon\Delta^q\to Y$ be singular simplices. We then have the following equality in $\mathrm{C}_{p+q-1}(X\times Y)$:

$$\begin{aligned}
\partial_{p+q}(\Upsilon_{p,q}(\varphi\otimes\psi))&\overset{\text{definition of }\Upsilon_{p,q}}{=}\partial_{p+q}((\varphi\times\psi)_*(\Upsilon_{p,q}(\mathrm{id}_p\otimes\mathrm{id}_q)))\overset{\text{since } (\varphi\times\psi)_* \text{ is a chain map}}{=}(\varphi\times\psi)_*(\partial_{p+q}(\overbrace{\Upsilon_{p,q}(\mathrm{id}_p\otimes\mathrm{id}_q)}^{=z_{p,q}}))\\
&\overset{\uparrow}{=}(\varphi\times\psi)_*(\Upsilon_{p-1,q}(\partial\,\mathrm{id}_p\otimes\mathrm{id}_q))+(-1)^p\cdot(\varphi\times\psi)_*(\Upsilon_{p,q-1}(\mathrm{id}_p\otimes\partial\,\mathrm{id}_q))\\
&\overset{\text{by the choice of } z_{p,q} \text{ and the linearity of } (\varphi\times\psi)_*}{=}\Upsilon_{p-1,q}(\varphi_*(\partial\,\mathrm{id}_p)\otimes\psi_*(\mathrm{id}_q)))+(-1)^p\cdot\Upsilon_{p,q-1}(\varphi_*(\mathrm{id}_p)\otimes\psi_*(\partial\,\mathrm{id}_q))\\
&\overset{\uparrow}{=}\Upsilon_{p-1,q}(\partial(\varphi)\otimes\psi)+(-1)^p\cdot\Upsilon_{p,q-1}(\varphi\otimes\partial_q(\psi))).\\
&\qquad\text{since } \varphi_* \text{ and } \psi_* \text{ are chain maps and since } \varphi_*(\mathrm{id}_p)=\varphi \text{ and } \psi_*(\mathrm{id}_q)=\psi
\end{aligned}$$

■

Now we turn to the construction of the second map in the Eilenberg-Zilber Theorem 56.1.

Proposition 56.6. (EZ-Inverse Proposition) For all topological spaces X and Y and all $k\in\mathbb{N}_0$ there exists a homomorphism

$$\Theta_k\colon \mathrm{C}_k(X\times Y)\to(\mathrm{C}_*(X)\otimes\mathrm{C}_*(Y))_k=\bigoplus_{p+q=k}\mathrm{C}_p(X)\otimes\mathrm{C}_q(Y)$$

such that the following conditions are satisfied:

(1) The map Θ_0 is the canonical map on the 0-level, i.e. for every $x\in X$ and $y\in Y$ we have $\Theta_0((x,y))=x\otimes y$.

(2) The maps Θ_* form a chain map, i.e. for any $k \in \mathbb{N}_0$ we have
$$\partial_k^{\otimes} \circ \Theta_k \;=\; \Theta_{k-1} \circ \partial_k.$$
(3) The map Θ_k is natural in X and Y, i.e. for any two continuous maps $f\colon X' \to X$ and $g\colon Y' \to Y$ of topological spaces and any $c \in \mathrm{C}_k(X' \times Y')$ we have
$$\Theta_k((f \times g)_*(c)) \;=\; (f_* \otimes g_*)(\Theta_k(c)).$$

In the proof of the EZ-Inverse Proposition 56.6 we will need the following lemma:

Lemma 56.7. (Acyclic Tensor Product Lemma) Let X and Y be two acyclic topological spaces. For every $k \geq 1$ we have
$$\mathrm{H}_k(\mathrm{C}_*(X) \otimes \mathrm{C}_*(Y)) \;=\; 0.$$
Furthermore we have an isomorphism
$$\begin{array}{rcl} \epsilon\colon \mathrm{H}_0(\mathrm{C}_*(X) \otimes \mathrm{C}_*(Y)) & \xrightarrow{\cong} & \mathbb{Z} \\ \Big[\sum\limits_{i=1}^{k} \underbrace{a_i}_{\in\mathbb{Z}} \cdot \underbrace{x_i}_{\in X} \otimes \underbrace{y_i}_{\in Y}\Big] & \mapsto & \sum\limits_{i=1}^{k} a_i. \end{array}$$

Proof. The lemma is a straightforward consequence of our hypotheses, the H_0-Proposition 35.11 and the Algebraic–Künneth Theorem 55.5. ∎

Proof of the EZ-Inverse Proposition 56.6. [459] We start out with an outline of the proof:

(a) We use (1) to define Θ_0.
(b) We suppose that the maps $\Theta_0, \ldots, \Theta_{k-1}$ are defined. We use (2) to find a suitable value for $\Theta_k(d_k)$ where we consider
  (i) the topological spaces $X = Y = \Delta^k$ and
  (ii) the singular k-simplex $d_k\colon \Delta^k \to \Delta^k \times \Delta^k$ that is given by the diagonal map $d_k(x) = (x, x)$.
(c) We then use the naturality statement of (3) to define Θ_k for all topological spaces X and Y and all singular simplices in $\mathrm{C}_*(X \times Y)$.

Now we carry out the above program. First of all, given a 0-simplex (x, y) in $X \times Y$ we define $\Theta_0((x, y)) = x \otimes y$. Then we extend this definition linearly to get the desired map $\Theta_0\colon \mathrm{C}_0(X \times Y) \to \mathrm{C}_0(X) \otimes \mathrm{C}_0(Y)$.

Now suppose we have already constructed the maps $\Theta_0, \ldots, \Theta_{k-1}$. We want to define Θ_k and we start out by defining $\Theta_k(d_k\colon \Delta^k \to \Delta^k \times \Delta^k)$. Since we need (2) to be satisfied we need to define $\Theta_k(d_k) := c$ where $c \in (\mathrm{C}_*(\Delta^k) \otimes \mathrm{C}_*(\Delta^k))_k$ satisfies $\partial_k^{\otimes}(c) = \Theta_{k-1}(\partial_k(d_k))$.

Claim 1. There exists a $c \in (\mathrm{C}_*(\Delta^k) \otimes \mathrm{C}_*(\Delta^k))_k$ with $\partial_k^{\otimes}(c) = \Theta_{k-1}(\partial_k(d_k))$.

Proof. We first consider the case $k \geq 2$. In the chain complex $\mathrm{C}_*(\Delta^k) \otimes \mathrm{C}_*(\Delta^k)$ we calculate that
$$\partial^{\otimes}(\Theta_{k-1}(\partial_k(d_k))) \underset{\substack{\uparrow \\ \text{by induction we can apply (2)}}}{=} (\Theta_{k-2} \circ \partial_{k-1})(\partial_k(d_k)) \;=\; \Theta_{k-2}(\underbrace{(\partial_{k-1} \circ \partial_k)}_{=0}(d_k)) \;=\; 0,$$

[459] As we will see the idea behind the proof of the EZ-Inverse Proposition 56.6 is very similar to the idea of the proof of the EZ-Construction Proposition 56.5.

i.e. the chain $\Theta_{k-1}(\partial_k(d_k)) \in (\mathrm{C}_*(\Delta^k) \otimes \mathrm{C}_*(\Delta^k))_{k-1}$ is in fact a cycle. Since Δ^k is acyclic it follows immediately from the Acyclic Tensor Product Lemma 56.7 that this cycle is null-homologous. Thus there exists a $c \in (\mathrm{C}_*(\Delta^k) \otimes \mathrm{C}_*(\Delta^k))_k$ with $\partial_k^{\otimes}(c) = \Theta_{k-1}(\partial_k(d_k))$.

Now we consider the case $k = 1$. In this case one verifies easily, using the explicit description of the map $\Theta_0 \colon \mathrm{C}_0(\Delta^k \times \Delta^k) \to \mathrm{C}_0(\Delta^k) \otimes \mathrm{C}_0(\Delta^k)$, that $\Theta_0(\partial_1(d_1))$ represents an element in the kernel of the isomorphism $\epsilon \colon \mathrm{H}_0(\mathrm{C}_*(\Delta^k) \otimes \mathrm{C}_*(\Delta^k)) \to \mathbb{Z}$. Again this means that there exists a singular chain $c \in (\mathrm{C}_*(\Delta^k) \otimes \mathrm{C}_*(\Delta^k))_k$ with $\partial^{\otimes}(c) = \Theta_{k-1}(\partial_k(d_k))$. ⊞

We set $\Theta_k(d_k) \coloneqq c$. Next we define Θ_k for arbitrary topological spaces X and Y and an arbitrary singular k-simplex $\sigma \colon \Delta^k \to X \times Y$. We denote by $p \colon X \times Y \to X$ and $q \colon X \times Y \to Y$ the natural projections. Note that we have a commutative diagram

$$\begin{array}{ccc} & & \Delta^k \times \Delta^k \\ & \overset{d_k}{\nearrow} & \big\downarrow {\scriptstyle (p\circ\sigma)\times(q\circ\sigma)} \\ \Delta^k & & \\ & \underset{\sigma}{\searrow} & X \times Y, \end{array}$$

i.e. we have $\sigma = ((p \circ \sigma) \times (q \circ \sigma)) \circ d_k = ((p \circ \sigma) \times (q \circ \sigma))_*(d_k)$. For the naturality condition (2) to be satisfied we need to define

$$\Theta_k(\sigma) \;\coloneqq\; ((p \circ \sigma)_* \otimes (q \circ \sigma)_*)(\underbrace{\Theta_k(d_k)}_{=c}) \;\in\; (\mathrm{C}_*(X) \otimes \mathrm{C}_*(Y))_k.$$

We extend Θ_k linearly to $\Theta_k \colon \mathrm{C}_k(X \times Y) \;\to\; (\mathrm{C}_*(X) \otimes \mathrm{C}_*(Y))_k$.

Claim 2. The map $\Theta_k \colon \mathrm{C}_k(X \times Y) \to (\mathrm{C}_*(X) \otimes \mathrm{C}_*(Y))_k$ has all the desired properties.

Proof. First note that it follows easily from the definitions that the map Θ_k is natural in X and Y. Thus it remains to show that Θ_k satisfies the boundary formula (2). So as before let $\sigma \colon \Delta^k \to X \times Y$ be a singular k-simplex. Then

$$\begin{array}{lll} & \text{by definition of } \Theta_k(\sigma) & \text{by the Tensoring Chain Maps Lemma 55.3 we know that } (p\circ\sigma)_* \otimes (q\circ\sigma)_* \colon \mathrm{C}_*(\Delta^k) \otimes \mathrm{C}_*(\Delta^k) \to \mathrm{C}_*(X) \otimes \mathrm{C}_*(Y) \text{ is in fact a chain map} \\ \partial_k^{\otimes}(\Theta_k(\sigma)) & \overset{\downarrow}{=} \partial_k^{\otimes}((p \circ \sigma)_* \otimes (q \circ \sigma)_*)(\Theta_k(d_k)) & \overset{\downarrow}{=} ((p \circ \sigma)_* \otimes (q \circ \sigma)_*)(\partial_k^{\otimes}(\Theta_k(d_k))) \\ & = ((p \circ \sigma)_* \otimes (q \circ \sigma)_*)(\Theta_{k-1}(\partial_k(d_k))) & = \Theta_{k-1}(((p \circ \sigma) \times (q \circ \sigma))_*(\partial_k(d_k))) \\ & \overset{\uparrow}{=} \text{by definition of } \Theta_k(d_k) & \overset{\uparrow}{=} \text{by naturality of } \Theta_{k-1} \\ & \Theta_{k-1}(\partial_k(((p \circ \sigma) \times (q \circ \sigma))_*(d_k))) & \Theta_{k-1}(\partial_k(\sigma)). \\ & \overset{\uparrow}{=} \text{since } ((p \circ \sigma) \times (q \circ \sigma))_* \text{ is a chain map} & \overset{\uparrow}{=} \text{since } \sigma = ((p \circ \sigma) \times (q \circ \sigma)) \circ d_k \end{array}$$

We have now verified that Θ_k has all the desired properties. ■

In the proofs of the EZ-Construction Proposition 56.5 and the EZ-Inverse Proposition 56.6 we saw that the constructions of the natural chain maps $\Upsilon \colon \mathrm{C}_*(X) \otimes \mathrm{C}_*(Y) \to \mathrm{C}_*(X \times Y)$ and $\Theta \colon \mathrm{C}_*(X \times Y) \to \mathrm{C}_*(X) \otimes \mathrm{C}_*(Y)$ depend on many choices. The next theorem says that different choices in the proofs lead to natural chain maps that are naturally homotopy equivalent to one another.

Proposition 56.8. (EZ–Natural Chain Homotopy Proposition) Let X and Y be topological spaces.

(a) Any two natural chain maps

$$\mathrm{C}_*(X \times Y) \;\to\; \mathrm{C}_*(X) \otimes \mathrm{C}_*(Y),$$

that are the canonical maps on the 0-level, are naturally chain homotopic.

The same conclusion also holds for any two natural chain maps that are the canonical maps on the 0-level in the following three setups:

(b) $\mathrm{C}_*(X) \otimes \mathrm{C}_*(Y) \;\to\; \mathrm{C}_*(X) \otimes \mathrm{C}_*(Y)$.
(c) $\mathrm{C}_*(X \times Y) \;\to\; \mathrm{C}_*(X \times Y)$.
(d) $\mathrm{C}_*(X) \otimes \mathrm{C}_*(Y) \;\to\; \mathrm{C}_*(X \times Y)$.

Proof. [460] In the following we prove Statement (1). The proofs of the other three statements are very similar and they are left to the reader. So let

$$F, G\colon \mathrm{C}_*(X \times Y) \;\to\; \mathrm{C}_*(X) \otimes \mathrm{C}_*(Y)$$

be two natural chain maps that are the canonical maps on the 0-chains, i.e. which satisfy $F((x,y)) = x \otimes y = G((x,y))$ for all $x \in X, y \in Y$. For every $k \in \mathbb{Z}_{\geq -1}$ we have to define a homomorphism

$$P_k\colon \mathrm{C}_k(X \times Y) \;\to\; (\mathrm{C}_*(X) \otimes \mathrm{C}_*(Y))_{k+1}$$

that satisfies the following three properties:

(1) The map P_{-1} is the zero map.
(2) We have $\partial^{\otimes}_{k+1} \circ P_k + P_{k-1} \circ \partial_k = F_k - G_k$.
(3) The map P_k is natural in X and Y.

So suppose we have already defined the maps $P_{-1}, \ldots, P_{k-1}$ with $k \geq 0$. As before we denote by $d_k\colon \Delta^k \to \Delta^k \times \Delta^k$ the singular k-simplex that is given by the diagonal map $d_k(x) \coloneqq (x,x)$. We first want to define $P_k(d_k) \in (\mathrm{C}_*(\Delta^k) \otimes \mathrm{C}_*(\Delta^k)_{k+1}$.

Claim 1. There exists a $c \in (\mathrm{C}_*(\Delta^k) \otimes \mathrm{C}_*(\Delta^k))_{k+1}$ such that

$$\partial^{\otimes}_{k+1}(c) \;=\; (F_k - G_k - P_{k-1} \circ \partial_k)(d_k).$$

Proof. In the chain complex $\mathrm{C}_*(\Delta^k) \otimes \mathrm{C}_*(\Delta^k)$ we calculate that

$$\begin{aligned}
\partial^{\otimes}_k((F_k - G_k - P_{k-1} \circ \partial_k)(d_k)) \;&=\; \partial^{\otimes}_k(F_k(d_k)) - \partial^{\otimes}_k(G_k(d_k)) - (\partial^{\otimes}_k \circ P_{k-1})(\partial_k(d_k)) \\
&\underset{\uparrow}{=}\; F_{k-1}(\partial_k(d_k)) - G_{k-1}(\partial_k(d_k)) - (F_{k-1} - G_{k-1} - (P_{k-2} \circ \partial_{k-1}))(\partial_k(d_k)) \\
&\qquad \text{since } F \text{ and } G \text{ are chain maps and since inductively} \\
&\qquad \text{we know by (2) that } \partial^{\otimes}_k \circ P_{k-1} + P_{k-2} \circ \partial_{k-1} = F_{k-1} - G_{k-1} \\
&=\; P_{k-2}(\underbrace{(\partial_{k-1} \circ \partial_k)}_{=0}(d_k)) \;=\; 0.
\end{aligned}$$

We first assume that $k \geq 1$. It follows in this case from the above calculation and the Acyclic Tensor Product Lemma 56.7 that there exists a $(k+1)$-chain $c \in (\mathrm{C}_*(\Delta^k) \otimes \mathrm{C}_*(\Delta^k))_{k+1}$ with $\partial^{\otimes}_{k+1}(c) = (F_k - G_k - P_{k-1} \circ \partial_k)(d_k)$. Now we consider the case $k = 0$. Using the fact that F and G agree on 0-simplices one can easily verify that for the augmentation homomorphism $\epsilon\colon \mathrm{C}_0(X) \otimes \mathrm{C}_0(Y) \to \mathbb{Z}$ from the Acyclic Tensor Product Lemma 56.7 we

[460] The proof of this theorem is in many ways quite similar to the proofs of the EZ-Construction Proposition 56.5 and the EZ-Inverse Proposition 56.6.

have $\epsilon((F_0 - G_0)(d_0)) = 0$. It follows again from the Acyclic Tensor Product Lemma 56.7 that there exists a 1-chain $c \in (\mathrm{C}_*(\Delta^0) \otimes \mathrm{C}_*(\Delta^0))_1$ with $\partial^{\otimes}(c) = (F_0 - G_0)(d_0)$. ⊞

We set $P_k(d_k) := c$. Next let X and Y be two topological spaces and furthermore let $\sigma \in \Delta^k \to X \times Y$ be a continuous map. As in the proof of the EZ-Construction Proposition 56.5 we denote by $p\colon X \times Y \to X$ and $q\colon X \times Y \to Y$ the natural projections and we set

$$P_k(\sigma) := ((p \circ \sigma)_* \otimes (q \circ \sigma)_*)(\underbrace{P_k(d_k)}_{=c}) \in (\mathrm{C}_*(X) \otimes \mathrm{C}_*(Y))_{k+1}.$$

Claim 2. The map $P_k\colon \mathrm{C}_k(X \times Y) \to (\mathrm{C}_*(X) \otimes \mathrm{C}_*(Y))_k$ has all the desired properties.

Proof. First note that basically by definition this map is natural in X and Y. Finally note that, as in the proof of the EZ-Construction Proposition 56.5, one easily verifies that condition (2) is satisfied. We leave the details to the reader. ■

Now we can easily provide the proof of the Eilenberg-Zilber Theorem 56.1.

Proof of the Eilenberg-Zilber Theorem 56.1. We pick natural chain maps

$$\Upsilon\colon \mathrm{C}_*(X) \otimes \mathrm{C}_*(Y) \to \mathrm{C}_*(X \times Y)$$

and

$$\Theta\colon \mathrm{C}_*(X \times Y) \to \mathrm{C}_*(X) \otimes \mathrm{C}_*(Y)$$

as provided by the EZ-Construction Proposition 56.5 and the EZ-Inverse Proposition 56.6. These maps are the canonical maps on the 0-level. It follows from the EZ–Natural Chain Homotopy Proposition 56.8 (d) and (a) that the maps Υ and Θ are unique up to a natural chain homotopy.

It remains to show that the maps Υ and Θ are naturally chain homotopy inverses of one another. The map $\Theta \circ \Upsilon\colon \mathrm{C}_*(X) \otimes \mathrm{C}_*(Y) \to \mathrm{C}_*(X) \otimes \mathrm{C}_*(Y)$ is a natural chain map that is the canonical map on the 0-level. But this is also the case for the identity map. It follows from the EZ–Natural Chain Homotopy Proposition 56.8 (b) that $\Theta \circ \Upsilon$ is naturally chain homotopic to the identity. The same argument, using the EZ–Natural Chain Homotopy Proposition 56.8 (c), shows that $\Upsilon \circ \Theta$ is naturally chain homotopic to the identity. ■

56.4. The relative Eilenberg–Zilber map Υ.

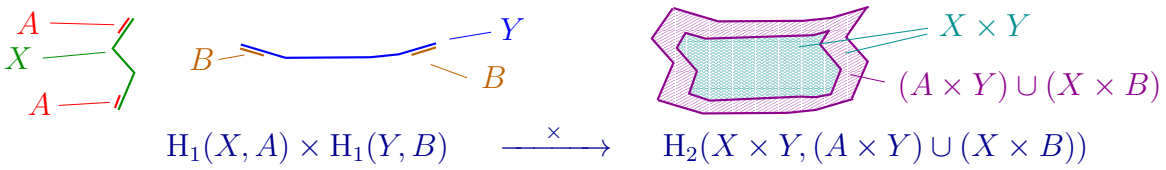
In this section we introduce a relative version of the Eilenberg–Zilber map Υ and we prove a relative version of the Künneth Theorem 56.3. Most of the discussion is straightforward, but at some point some care needs to be taken.

The following definition is a mild generalization of the definition on page 704:

Definition. Let (X, A) and (Y, B) be two pairs of topological spaces. Furthermore let $\Upsilon\colon \mathrm{C}_*(X) \otimes \mathrm{C}_*(Y) \to \mathrm{C}_*(X \times Y)$ be an Eilenberg-Zilber map. We refer to the homomorphism[461] [462]

$$\begin{array}{rcl} \times\colon \mathrm{H}_p(X, A) \otimes \mathrm{H}_q(Y, B) & \xrightarrow{\times} & \mathrm{H}_{p+q}(X \times Y, (A \times Y) \cup (X \times B)) \\ [c] \otimes [d] & \mapsto & [c] \times [d] := [\Upsilon(c \otimes d)] \end{array}$$

as the cross product.



We have the following slightly subtle generalization of the earlier Künneth Theorem 56.3.

Theorem 56.9. (Künneth Theorem) Let (X, A) and (Y, B) be pairs of topological spaces. We assume that one of the following three statements holds:

(1) $A = \varnothing$ or $B = \varnothing$.
(2) $A \subset X$ and $B \subset Y$ are open subsets.
(3) (X, A) and (Y, B) are pairs of CW-complexes and $X \times Y$ has the topology from page 439 which turns $X \times Y$ into a CW-complex.[463]

For each $n \in \mathbb{N}$ there exists a natural homomorphism

$$\widetilde{\nu}\colon \mathrm{H}_n(X \times Y, (A \times Y) \cup (X \times B)) \to \bigoplus_{p+q=n-1} \mathrm{Tor}(\mathrm{H}_p(X, A), \mathrm{H}_q(Y, B))$$

such that the following sequence

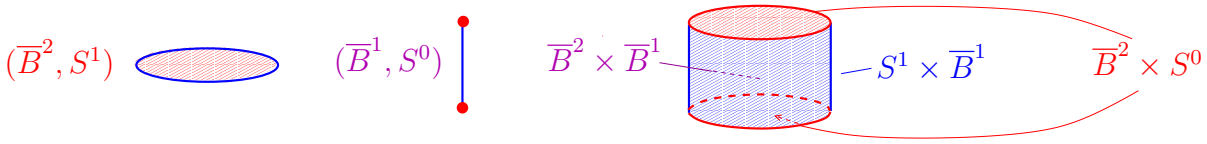
$$0 \to \bigoplus_{p+q=n} \mathrm{H}_p(X, A) \otimes \mathrm{H}_q(Y, B) \xrightarrow{\times} \mathrm{H}_n(X \times Y, (A \times Y) \cup (X \times B)) \xrightarrow{\widetilde{\nu}} \bigoplus_{p+q=n-1} \mathrm{Tor}(\mathrm{H}_p(X, A), \mathrm{H}_q(Y, B)) \to 0.$$

is exact and such that it splits.

Example. We consider the pairs $(\overline{B}^p, S^{p-1})$ and $(\overline{B}^q, S^{q-1})$. In this context the Künneth Theorem 56.9 (3) simplifies to the statement that the cross product[464]

$$\underbrace{\mathrm{H}_p(\overline{B}^p, S^{p-1})}_{\cong \mathbb{Z}} \otimes \underbrace{\mathrm{H}_q(\overline{B}^q, S^{q-1})}_{\cong \mathbb{Z}} \xrightarrow{\times} \mathrm{H}_{p+q}\underbrace{(\overline{B}^p \times \overline{B}^q, (S^{p-1} \times \overline{B}^q) \cup (\overline{B}^p \times S^{q-1}))}_{\cong (\overline{B}^{p+q}, S^{p+q-1})}$$

is an isomorphism.



The extra requirements in the Künneth Theorem 56.9 already indicate that the proof of the Künneth Theorem 56.9 is a little less straightforward than one might have hoped. As a first step to proving the Künneth Theorem 56.9 let us formulate and prove the following lemma:

[461] It follows easily from the naturality of Υ that this map on relative homology is well-defined.
[462] As in the Cross Product Lemma 56.2 we see that this map is natural and that it is independent of the choice of the Eilenberg-Zilber map Υ.
[463] If X and Y are finite CW-complexes, then it follows from the CW-Complex Product Proposition 33.9 that this topology is the usual product topology.
[464] Recall that it follows from the long exact sequence in reduced homology of the pair $(\overline{B}^n, S^{n-1})$ that $\mathrm{H}_k(\overline{B}^n, S^{n-1}) \cong \widetilde{\mathrm{H}}_{k-1}(S^{n-1})$ which is $\cong \mathbb{Z}$ for $k = n$ and $= 0$ otherwise.

Lemma 56.10. (Relative–Υ-Chain Equivalence Lemma) Let X and Y be topological spaces and let $A \subset X$ and $B \subset Y$ be subspaces. Any choice of an Eilenberg-Zilber map $\Upsilon\colon \mathrm{C}_*(X) \otimes \mathrm{C}_*(Y) \to \mathrm{C}_*(X \times Y)$ induces a natural chain homotopy equivalence[465]

$$\begin{array}{rcl} \mathrm{C}_*(X, A) \otimes \mathrm{C}_*(Y, B) & \to & \mathrm{C}_*(X \times Y)/(\mathrm{C}_*(A \times Y) + \mathrm{C}_*(X \times B)) \\ \sum_{i=1}^{n} [c_i] \otimes [d_i] & \mapsto & \sum_{i=1}^{n} [\Upsilon(c_i \otimes d_i)] \end{array}$$

which we also denote by Υ.

Proof of the Relative–Υ-Chain Equivalence Lemma 56.10. Let X and Y be topological spaces and let $A \subset X$ and $B \subset Y$ be subspaces. We start out with the following claim:

Claim. For any topological space Z the Eilenberg-Zilber map

$$\Upsilon\colon \mathrm{C}_*(X) \otimes \mathrm{C}_*(Z) \to \mathrm{C}_*(X \times Z)$$

induces a chain homotopy equivalence

$$\overline{\Upsilon}\colon \mathrm{C}_*(X, A) \otimes \mathrm{C}_*(Z) \to \mathrm{C}_*(X \times Z, A \times Z).$$

Proof. Let $i\colon A \to Z$ be the inclusion. We consider the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{C}_*(A) \otimes \mathrm{C}_*(Z) & \xrightarrow{i_* \otimes \mathrm{id}} & \mathrm{C}_*(X) \otimes \mathrm{C}_*(Z) & \longrightarrow & \mathrm{C}_*(X, A) \otimes \mathrm{C}_*(Z) & \longrightarrow & 0 \\ & & \simeq \downarrow \Upsilon & & \simeq \downarrow \Upsilon & & \overline{\Upsilon} \downarrow [\sigma] \otimes \tau \mapsto [\Upsilon(\sigma \otimes \tau)] & & \\ 0 & \longrightarrow & \mathrm{C}_*(A \times Z) & \xrightarrow{(i \times \mathrm{id})_*} & \mathrm{C}_*(X \times Z) & \longrightarrow & \mathrm{C}_*(X \times Z)/\mathrm{C}_*(A \times Z) & \longrightarrow & 0. \end{array}$$

We make the following observations:

(1) Since each chain group $\mathrm{C}_k(Z)$ is free abelian we obtain from the Tensor-with-Torsion Free Proposition 53.5 that the top horizontal sequence is exact.
(2) The bottom sequence is exact by definition.
(3) By the Tensoring Chain Maps Lemma 55.3 we know that $i_* \otimes \mathrm{id}$ is a chain map.
(4) It follows from the Eilenberg-Zilber Theorem 56.1 that two vertical maps to the left are chain homotopy equivalences.
(5) It follows from the naturality of Υ that the square to the left commutes.
(6) It follows immediately from (5) that $\overline{\Upsilon}$ is well-defined. It is clear that the right square also commutes.

It follows from the above and the Two-Out-Of-Three-Homotopy Equivalence Corollary 50.5 that the right vertical map is also a chain homotopy equivalence. ⊞



[465]Note that $\mathrm{C}_*(A \times Y) + \mathrm{C}_*(X \times B)$ is in fact a subcomplex of the chain complex $\mathrm{C}_*(X \times Y)$, so the right-hand side is indeed a chain complex.

Next we consider the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{C}_*(X,A)\otimes\mathrm{C}_*(B) & \longrightarrow & \mathrm{C}_*(X,A)\otimes\mathrm{C}_*(Y) & \longrightarrow & \mathrm{C}_*(X,A)\otimes\mathrm{C}_*(Y,B) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \overline{\Upsilon}} & & \downarrow{\scriptstyle \overline{\Upsilon}} & & \downarrow{\scriptstyle \overline{\Upsilon}} & & \\ 0 & \longrightarrow & \mathrm{C}_*(X\times B)/\mathrm{C}_*(A\times B) & \longrightarrow & \mathrm{C}_*(X\times Y)/\mathrm{C}_*(A\times Y) & \longrightarrow & \dfrac{\mathrm{C}_*(X\times Y)}{\mathrm{C}_*(A\times Y)+\mathrm{C}_*(X\times B)} & \longrightarrow & 0. \end{array}$$

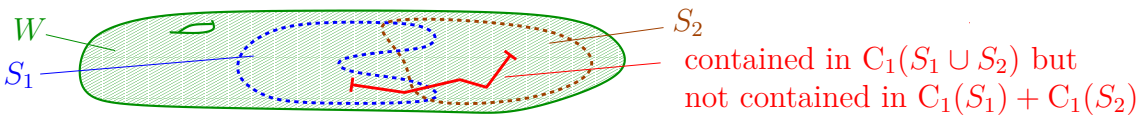
We make the following observations:

(1) It follows from the fact that each $\mathrm{C}_k(X,A)$ is a free abelian group and the Tensor-with-Free Abelian Group Proposition 53.5 that the top sequence is exact.
(2) It follows easily from
$$\mathrm{C}_*(X\times B)\cap\mathrm{C}_*(A\times Y) \;=\; \mathrm{C}_*((X\times B)\cap(A\times Y)) \;=\; \mathrm{C}_*(A\times B)$$
that the bottom sequence is also exact.
(3) It follows from the above claim that the two vertical maps to the left are chain homotopy equivalences.
(4) It follows easily from the naturality of Υ that the square to the left commutes. The square to the right commutes by definition of the third vertical map.

It follows again from the Two-Out-Of-Three-Homotopy Equivalence Corollary 50.5 that the right vertical map is also a homotopy equivalence. ■

For the reader's convenience we recall some statements from the Excisive Triad-Proposition 50.11:

Proposition 50.11. (Excisive Triad-Proposition) Let W be a topological space and let $S_1, S_2 \subset W$ be subspaces. We assume that one of the following conditions is satisfied:
(1) $S_1 = \varnothing$ or $S_2 = \varnothing$.
(2) S_1 and S_2 are open subsets of W.
(4) W is a CW-complex and S_1 and S_2 are both subcomplexes.
Then the inclusion maps
$$\underbrace{\mathrm{C}_k(S_1)+\mathrm{C}_k(S_2)}_{\text{subgroup of } \mathrm{C}_k(S_1\cup S_2)} \;\to\; \mathrm{C}_k(S_1\cup S_2)$$
form a chain homotopy equivalence.



Proof of the Künneth Theorem 56.9. We consider the following maps:

$$\begin{array}{ccccccccc} 0 \to \bigoplus_{p+q=n} \mathrm{H}_p(X,A)\otimes \mathrm{H}_q(Y,B) & \xrightarrow[(a)]{\mu} & \mathrm{H}_n(\mathrm{C}_*(X,A)\otimes \mathrm{C}_*(Y,B)) & \xrightarrow[(a)]{\nu} & \bigoplus_{p+q=n-1} \mathrm{Tor}(\mathrm{H}_p(X,A),\mathrm{H}_q(Y,B)) \to 0. \\ & & \text{(b)}\ \downarrow \Upsilon_* & & \\ & & \mathrm{H}_n(\mathrm{C}_*(X\times Y)/(\mathrm{C}_*(A\times Y)+\mathrm{C}_*(X\times B))) & & \\ & & \text{(c)}\ \downarrow & & \\ & & \mathrm{H}_n(\mathrm{C}_*(X\times Y)/\mathrm{C}_*((A\times Y)\cup(X\times B))) & & \\ & & \downarrow \mathrm{id} & & \\ & & \mathrm{H}_n(X\times Y,(A\times Y)\cup(X\times B)) & & \end{array}$$

The theorem is a consequence of the following observations and clarifications:

(a) The top horizontal maps μ and ν are the natural maps from the Cycle Tensor Lemma 55.4 and the Algebraic–Künneth Theorem 55.5. The Algebraic–Künneth Theorem 55.5 implies that the top horizontal sequence is exact and that it splits.[466]

(b) It follows from the Relative–Υ-Chain Equivalence Lemma 56.10 that the top vertical map is an isomorphism.

(c) Now we consider the middle vertical map:

Claim. The middle vertical map is an isomorphism.

Proof. Recall that we assume that one of the following three statements holds:
(1) $A = \varnothing$ or $B = \varnothing$.
(2) $A \subset X$ and $B \subset Y$ are open subsets.
(3) (X, A) and (Y, B) are pairs of CW-complexes.
In the three cases we get the following three corresponding consequences:
(1) $A \times Y = \varnothing$ or $X \times B = \varnothing$.
(2) $A \times Y$ and $X \times B$ are open subsets of $(A \times Y) \cup (X \times B)$.
(3) It follows easily from the CW-Category–Product Proposition 33.7 that $A \times Y$ and $X \times B$ are subcomplexes of the CW-complex $X \times Y$.
It follows from these two observations and the Excisive Triad-Proposition 50.11 (0), (1) respectively (3) that the inclusion map

$$\mathrm{C}_*(A\times Y)+\mathrm{C}_*(X\times B) \ \to\ \mathrm{C}_*((A\times Y)\cup(X\times B))$$

is a chain homotopy equivalence. We now consider the following commutative diagram:

$$\begin{array}{ccccccccc} 0 \to & \mathrm{C}_*(A\times Y)+\mathrm{C}_*(X\times B) & \to & \mathrm{C}_*(X\times Y) & \to & \mathrm{C}_*(X\times Y)/(\mathrm{C}_*(A\times Y)+\mathrm{C}_*(X\times B)) & \to 0 \\ & \downarrow & & \downarrow \mathrm{id} & & \downarrow & \\ 0 \to & \mathrm{C}_*((A\times Y)\cup(X\times B)) & \to & \mathrm{C}_*(X\times Y) & \longrightarrow & \mathrm{C}_*(X\times Y,((A\times Y)\cup(X\times B)) & \longrightarrow 0. \end{array}$$

By the above discussion the vertical maps to the left induce isomorphisms on homology. If follows from the long exact sequences in homology and the Five Lemma 37.5 that the right vertical map also induces an isomorphism on homology. But that is exactly what we had to show.

(d) It follows immediately from the definition of the cross product that the induced "diagonal map"

[466] Here we use again the fact (shown on page 489) that relative chain groups are free abelian groups.

$$\bigoplus_{p+q=n} \mathrm{H}_p(X, A) \otimes \mathrm{H}_q(Y, B) \;\to\; \mathrm{H}_n(X \times Y, (A \times Y) \cup (X \times B))$$

is precisely the cross product. ■

56.5. **Appendix: The shuffle product.** In the EZ-Construction Proposition 56.5 we showed that there exist homomorphisms

$$\Upsilon_{p,q} \colon \mathrm{C}_p(X) \otimes \mathrm{C}_q(Y) \;\to\; \mathrm{C}_{p+q}(X \times Y)$$

which have the following properties:

(1) On the 0-level they equal the canonical maps that we introduced on page 702. In other words, for all $x \in X$ and $y \in Y$ we have $\nabla_{0,0}(x \otimes y) = (x, y) \in \mathrm{C}_0(X \times Y)$.
(2) They form a chain map $\mathrm{C}_*(X) \otimes \mathrm{C}_*(Y) \to \mathrm{C}_*(X \times Y)$.
(3) They are natural.

We gave a non-constructive proof of the existence of the map Υ. For the most part the non-explicit version of the Eilenberg-Zilber map Υ is enough. But on a few occasions, e.g. if one wants to make the Künneth Theorem 56.3 explicit, or in the proof of Product Orientation Proposition 62.7 (2), it will be essential to have an explicit Eilenberg-Zilber map.

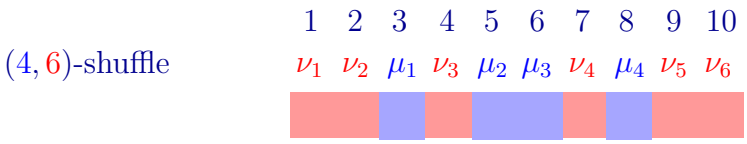
In this appendix we will give an *explicit* description of a map ∇ which has all the properties that we stated above. We start out with several definitions.

Definition. Let $m, n \in \mathbb{N}_0$.

(1) An (m, n)-shuffle is a pair (μ, ν) of disjoint sets of integers

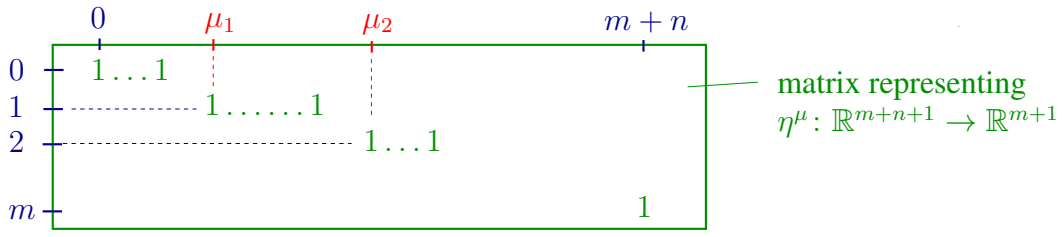
$$1 \le \mu_1 < \mu_2 < \cdots < \mu_m \le m + n \quad \text{and} \quad 1 \le \nu_1 < \nu_2 < \cdots < \nu_n \le m + n.$$

Note that an (m, n)-shuffle is the same data as picking m numbers out of $\{1, \ldots, m+n\}$.
(2) We denote by $S_{m,n}$ the set of all (m, n)-shuffles.
(3) We denote by $\operatorname{sign}(\mu, \nu)$ the sign of the permutation $(\mu_1, \ldots, \mu_m, \nu_1, \ldots, \nu_n)$.



Definition.

(4) Given a sequence $1 \le \mu_1 < \mu_2 < \cdots < \mu_m \le m + n$ we denote by $\eta^\mu \colon \mathbb{R}^{m+n+1} \to \mathbb{R}^{m+1}$ the unique linear map which satisfies $\eta^\mu(\mathfrak{v}_i) = \mathfrak{v}_j$ for $\mu_j \le i < \mu_{j+1}$.[467] One can easily verify that η^μ restricts to a map $\Delta^{m+n} \to \Delta^m$ which we also denote by η^μ.

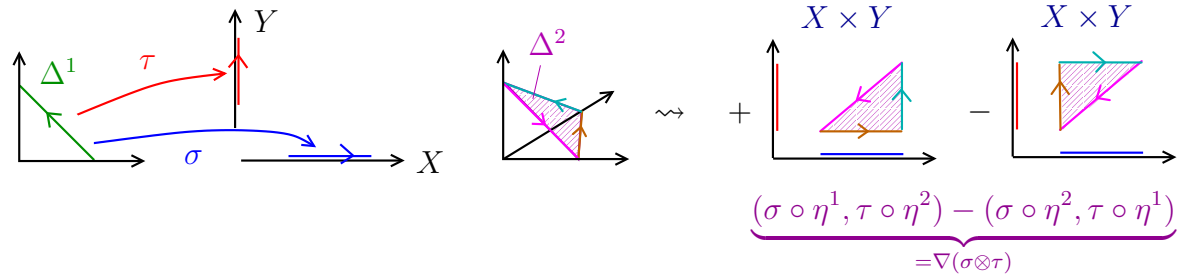


Definition.

(5) Given two topological spaces X and Y we consider the homomorphism[468]

$$\begin{aligned}\nabla := \nabla_{m,n}\colon \mathrm{C}_m(X)\otimes\mathrm{C}_n(Y) &\to \mathrm{C}_{m+n}(X\times Y)\\ (\sigma\otimes\tau) &\mapsto \sum_{(\mu,\nu)\in S_{m,n}} \operatorname{sign}(\mu,\nu)\cdot\Big(\underbrace{\overbrace{(\sigma\times\tau)}^{\Delta^m\times\Delta^n\to X\times Y}\circ\overbrace{(\eta^\mu,\eta^\nu)}^{\Delta^{m+n}\to\Delta^m\times\Delta^n}}_{\text{map }\Delta^{m+n}\to X\times Y}\Big).\end{aligned}$$

Example. We consider the case $X = Y = \mathbb{R}$ and $m = n = 1$. There exist two $(1,1)$-shuffles, namely $\{\mu_1 = 1, \nu_1 = 2\}$ and $\{\mu_1 = 2, \nu_1 = 1\}$. Note that $\eta^1\colon \Delta^2 \to \Delta^1$ equals the map[469] $[\mathfrak{v}_0, \mathfrak{v}_1, \mathfrak{v}_1]$ and that $\eta^2\colon \Delta^2 \to \Delta^1$ equals the map $[\mathfrak{v}_0, \mathfrak{v}_0, \mathfrak{v}_1]$. In the figure below we illustrate $\nabla_{1,1}(\sigma\otimes\tau)$ for two affine linear singular 1-simplices $\sigma\colon \Delta^1 \to X = \mathbb{R}$ and $\tau\colon \Delta^1 \to Y = \mathbb{R}$.



Now we can formulate the main result of this appendix:

Proposition 56.11. (Shuffle Proposition) The maps $\nabla_{m,n}$ that we had just defined have all the properties that we had mentioned in the EZ-Construction Proposition 56.5, in other words, the maps $\nabla_{m,n}\colon \mathrm{C}_m(X)\otimes\mathrm{C}_n(Y) \to \mathrm{C}_{m+n}(X\times Y)$ have the following properties:

(1) On the 0-level they equal the canonical maps that we introduced on page 702. In other words, for all $x \in X$ and $y \in Y$ we have $\nabla_{0,0}(x\otimes y) = (x,y) \in \mathrm{C}_0(X\times Y)$.
(2) They form a chain map $\mathrm{C}_*(X)\otimes\mathrm{C}_*(Y) \to \mathrm{C}_*(X\times Y)$.
(3) They are natural.
(∗) For any $x \in X$ and any singular simplex $\sigma\colon \Delta^q \to Y$ we have $\nabla(x\otimes\sigma) = x\times\sigma$ and the same statement also holds with the roles of X and Y swapped.[470]

The key to the proof of the Shuffle Proposition 56.11 is the following lemma:

Lemma 56.12. (∇-Chain Map Lemma) For any choice of $m, n \in \mathbb{N}_0$ we have the following equality in $\mathrm{C}_{m+n-1}(\Delta^m\times\Delta^n)$:

$$\nabla(\partial^{\otimes}_{m+n}(\underbrace{\mathrm{id}_{\Delta^m}\otimes\mathrm{id}_{\Delta^n}}_{\in\mathrm{C}_m(\Delta^m)\otimes\mathrm{C}_n(\Delta^n)})) \;=\; \partial_{m+n}(\nabla(\underbrace{\mathrm{id}_{\Delta^m}\otimes\mathrm{id}_{\Delta^n}}_{\in\mathrm{C}_m(\Delta^m)\otimes\mathrm{C}_n(\Delta^n)})).$$

[467] Let $X = \Delta^m$, $Y = \Delta^n$ and $\sigma = \mathrm{id}_{\Delta^m}$ and $\tau = \mathrm{id}_{\Delta^n}$. If we compare the above definitions with definition on page **??** of the product of two ordered simplicial complexes we see that the summands correspond precisely to the $(m+n)$-simplices of the product of the ordered simplicial complexes Δ^m and $Delta^n$.
[468] Here we implicitly use that $\mu_0 = 0$ and $\mu_{m+1} = m+n+1$.
[469] As on page 457, given points $a_0, \ldots, a_k \in \mathbb{R}^n$ we denote by $[a_0, \ldots, a_k]\colon \Delta^k \to \mathbb{R}^n$ the unique affine linear map that sends the i-th standard basis vector e_i to a_i.
[470] This shows that ∇ has the extra property (∗) which we formulated in the Eilenberg-Zilber Theorem 56.1.

Proof of the ∇-Chain Map Lemma 56.12. The lemma follows from an elementary, albeit slightly tricky calculation. We leave the details to [**Fri24**]. ■

Proof of Proposition 56.11 assuming Lemma 56.12. It follows immediately from the definitions that the maps ∇ satisfy (1), (3) and $(*)$. Thus it remains to show that the maps form a chain map. In other words, we need to prove the following claim:

Claim. Let X and Y be topological spaces. For any $\sigma \in \mathrm{C}_m(X)$ and $\tau \in \mathrm{C}_n(Y)$ we have

$$\nabla(\partial^{\otimes}_{m+n}(\sigma \otimes \tau)) \;=\; \partial_{m+n}(\nabla(\sigma \otimes \tau)).$$

Proof. As we will see this claim follows easily from the naturality of ∇, the naturality of boundary maps and from the ∇-Chain Map Lemma 56.12. More precisely, we have the following equalities:

$$\begin{aligned}
\nabla(\partial^{\otimes}_{m+n}(\sigma \otimes \tau)) &= \nabla(\partial^{\otimes}_{m+n}(\sigma_*(\mathrm{id}_{\Delta^m}) \otimes \tau_*(\mathrm{id}_{\Delta^n}))) && = \nabla(\partial^{\otimes}_{m+n}((\sigma \otimes \tau)_*(\mathrm{id}_{\Delta^m} \otimes \mathrm{id}_{\Delta^n})))\\
&= \nabla((\sigma \otimes \tau)_*(\partial^{\otimes}_{m+n}(\mathrm{id}_{\Delta^m} \otimes \mathrm{id}_{\Delta^n}))) && = (\sigma \times \tau)_*(\nabla(\partial^{\otimes}_{m+n}(\mathrm{id}_{\Delta^m} \otimes \mathrm{id}_{\Delta^n})))\\
&\quad \uparrow \text{ since } \sigma_* \text{ and } \tau_* \text{ are chain maps} && \quad \uparrow \text{ by the naturality of } \nabla\\
&= (\sigma \times \tau)_*(\partial_{m+n}(\nabla(\mathrm{id}_{\Delta^m} \otimes \mathrm{id}_{\Delta^n}))) && = \partial_{m+n}((\sigma \times \tau)_*(\nabla(\mathrm{id}_{\Delta^m} \otimes \mathrm{id}_{\Delta^n})))\\
&\quad \uparrow \text{ by the } \nabla\text{-Chain Map Lemma 56.12} && \quad \uparrow \text{ since } (\sigma \times \tau)_* \text{ is a chain map}\\
&= \partial_{m+n}(\nabla((\sigma \otimes \tau)_*(\mathrm{id}_{\Delta^m} \otimes \mathrm{id}_{\Delta^n}))) && = \partial_{m+n}(\nabla(\sigma \otimes \tau)).\\
&\quad \uparrow \text{ by the naturality of } \nabla
\end{aligned}$$

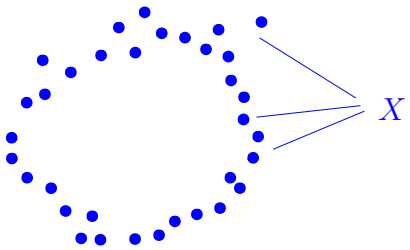
■

CHAPTER 57

Applications of homology groups

57.1. **Persistent homology.** In this section we outline the idea behind "persistent homology". Persistent homology was introduced by Gunnar Carlsson [**Car09**] in 2009. More information and a much more careful treatment of persistent homology can for example be found in [**EH10**]. Some extra information can also be found in [**Ghr14**].

We consider the figure below. What do we see? If we forget about topology for a second, then we would say that we see a set X of dots that form a loop.

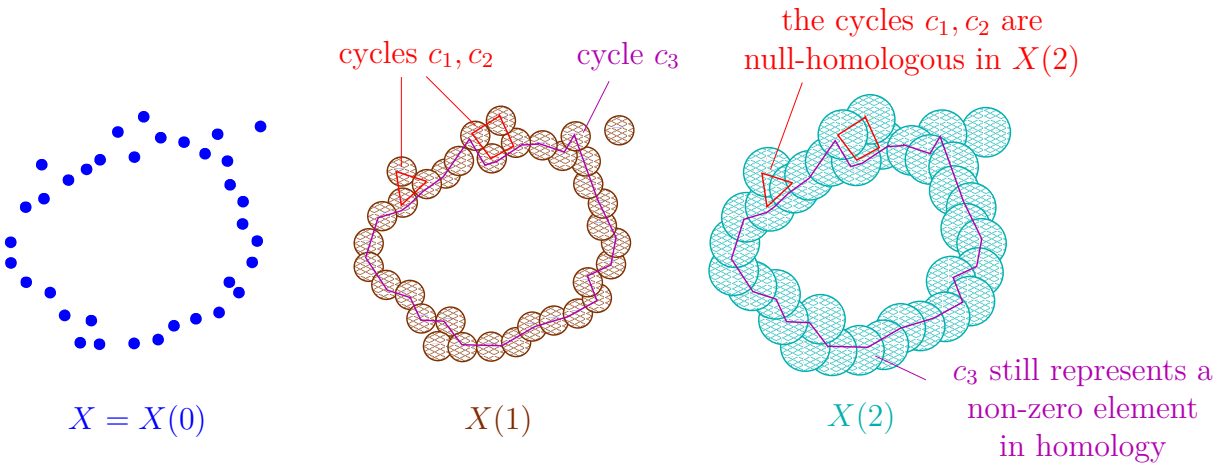


dots that appear to the eye to form a loop

If we view X as a topological space, then it is just a finite discrete subset of $\mathbb{R}^2$. In particular its homology groups are non-zero only in dimension 0. So how can we use the mathematical machine that we have developed over the last two semesters to "see the loop"?

Given $r \in [0, \infty)$ we denote by $X(r)$ the subset of $\mathbb{R}^2$ that is given by replacing each point $x \in X$ by the closed ball $\overline{B}_r(x)$ of radius r around x. The homeomorphism type of $X(r)$ changes with the radius r. For example in the figure below we see that at $r = 1$ all but one ball are joined up and we see that $\mathrm{H}_1(X(1)) \cong \mathbb{Z}^3$. We show three cycles c_1, c_2, c_3 in $X(1)$ which form a basis for $\mathrm{H}_1(X(1))$. Two of the generators, namely c_1, c_2 are formed "by accident", namely they arise from balls which are close by. This gets captured by the fact that they become null-homologous in $\mathrm{H}_1(X(2))$ whereas c_3 represents a non-trivial element for "much longer", i.e. for much larger values of r. Put differently, c_3 is a persistent element in homology, hence it captures global information.
Summarizing, given any subset X of $\mathbb{R}^2$ we can do the following: For any $r \in [0, \infty)$ we can study the topological spaces $X(r)$ and furthermore for any $r < s$ we can study the inclusion-induced map $\mathrm{H}_n(X(r)) \to \mathrm{H}_n(X(s))$. If there is a non-zero element in $\mathrm{H}_n(X(r))$ which stays non-zero for a "long time", then we say that it is persistent and we hope that it captures something interesting. For computational convenience it might also be interesting to consider homology with say $\mathbb{F}_2$-coefficients.

The same approach can be applied to any finite subset of some $\mathbb{R}^n$. Pretty much any data one has in practice, e.g. blue pixels of a picture, measurements in an experiment, statistical data and so on, can be viewed as a finite subset of some $\mathbb{R}^n$, so in principle one



can study its persistent homology. The real question is, whether one captures something significant or not. For example persistent homology was applied in [**SHP16**] to study the "The topological shape of Brexit". It is perhaps debatable whether one can really obtain anything meaningful about the politics of Brexit using persistent homology.

57.2. **The transfer map.** Before we turn to the Borsuk-Ulam Theorem let us introduce the transfer map, which is interesting in its own right.[471]

Definition. Let $p\colon \widetilde{X} \to X$ be a k-fold covering of a topological space X. Let $\sigma\colon \Delta^n \to X$ be a singular simplex. Since Δ^n is simply connected it follows from the Map Lifting Criterion 27.1 together with the Lift Uniqueness Lemma 11.10 that there exist precisely k different lifts of σ to $\widetilde{X}$, i.e. there exist precisely k different singular simplices $\widetilde{\sigma}_1, \dots, \widetilde{\sigma}_k\colon \Delta^n \to \widetilde{X}$ with $p \circ \widetilde{\sigma}_i = \sigma$ for $i = 1, \dots, k$. We write

$$p^*(\sigma) \;:=\; \sum_{i=1}^{k} \widetilde{\sigma}_i \;\in\; \mathrm{C}_n(\widetilde{X}).$$

Given any commutative ring R we refer to the R-module homomorphism[472]

$$\begin{array}{rcl} p^*\colon \mathrm{C}_n(X;R) & \to & \mathrm{C}_n(\widetilde{X};R) \\ \sum_{i=1}^{m} \sigma_i \otimes r_i & \mapsto & \sum_{i=1}^{m} p^*(\sigma_i) \otimes r_i \end{array}$$

as the transfer map of the covering.

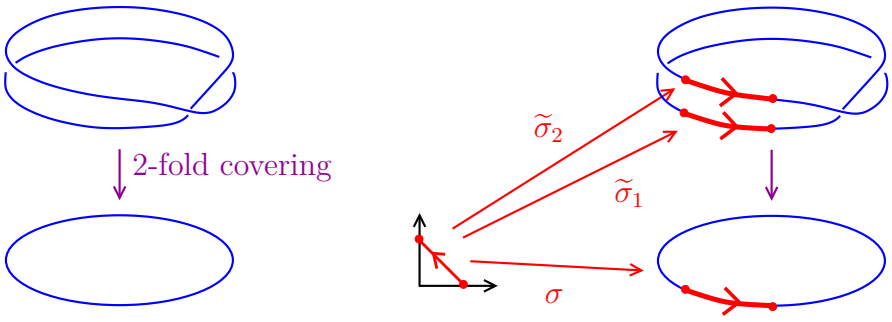
The following lemma summarizes two essential properties of transfer maps:

Lemma 57.1. (Transfer Map Lemma) Let $p\colon \widetilde{X} \to X$ be a k-fold covering of a topological space X and let R be a commutative ring.

(1) Each transfer map $p^*\colon \mathrm{C}_n(X;R) \to \mathrm{C}_n(\widetilde{X};R)$ is a monomorphism.

[471] The following definition of the transfer map is the analogue of the definition on page **??** of the transfer map for smooth forms on manifold.

[472] Here we use that it follows easily from the Matrix–Tensor-with-Ring Lemma 51.8 that for any $k \in \mathbb{N}_0$ the set $\{\sigma \otimes 1_R\}_{\sigma\colon \Delta^k \to X}$ forms a basis for the free R-module $\mathrm{C}_k(X;R)$.



(2) The transfer map $p^*\colon \mathrm{C}_*(X;R) \to \mathrm{C}_*(\widetilde{X};R)$ is a chain map.
(3) The transfer map is natural in the following sense: Suppose we are given a diagram

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{p} & X \\ {\scriptstyle \widetilde{f}}\downarrow & & \downarrow{\scriptstyle f} \\ \widetilde{Y} & \xrightarrow{q} & Y \end{array}$$

of continuous maps such that the following statements hold:
(a) The diagram commutes.
(b) $p\colon \widetilde{X} \to X$ and $q\colon \widetilde{Y} \to Y$ are k-fold coverings.
(c) For each $x \in X$ the map $\widetilde{f}|_{p^{-1}(x)}\colon p^{-1}(x) \to q^{-1}(f(x))$ is a bijection.
Then for every $n \in \mathbb{N}_0$ the following diagram commutes:

$$\begin{array}{ccc} \mathrm{C}_n(X;R) & \xrightarrow{p^*} & \mathrm{C}_n(\widetilde{X};R) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle \widetilde{f}_*} \\ \mathrm{C}_n(Y;R) & \xrightarrow{q^*} & \mathrm{C}_n(\widetilde{Y};R). \end{array}$$

Proof. Let $p\colon \widetilde{X} \to X$ be a k-fold covering of a topological space X.
(1) Let $c = \sum_{i=1}^{m} \sigma_i \cdot r_i \in \mathrm{C}_n(X) \otimes R$ be a non-zero element. We can assume that the σ_i are pairwise distinct and that all the r_i are non-zero. Let $\widetilde{\sigma}_i\colon \Delta^n \to \widetilde{X}$ be a lift of σ_i. Let $\pi\colon \mathrm{C}_n(\widetilde{X}) \to \mathbb{Z}$ be the epimorphism with $\widetilde{\sigma}_i \mapsto 1$ and that sends all other singular simplices to 0. Clearly $((\pi \otimes \mathrm{id}_R) \circ p^*)(c) = 1 \otimes r_i \in \mathbb{Z} \otimes R \cong R$ is non-zero. But this shows that p^* is a monomorphism.

For the proof of (2) and (3) let $\sigma\colon \Delta^n \to X$ be a singular simplex. Furthermore let $\widetilde{\sigma}_1, \dots, \widetilde{\sigma}_k\colon \Delta^n \to \widetilde{X}$ be the k different continuous maps with $p \circ \widetilde{\sigma}_j = \sigma$ for $j = 1, \dots, k$.
(2) This statement follows immediately from the observation that for any $r = 0, \dots, n$ the maps $\widetilde{\sigma}_j \circ i_r^n\colon \Delta^{n-1} \to \widetilde{X}$, $j = 1, \dots, k$ are k different lifts of $\sigma_j \circ i_r^n\colon \Delta^{n-1} \to X$.
(3) This statement follows immediately from the following observation: It is a consequence of (c) that for any $r = 0, \dots, n$ the maps $\widetilde{f} \circ \widetilde{\sigma}_j\colon \Delta^n \to \widetilde{Y}$, $j = 1, \dots, k$ are k different lifts of $f \circ \sigma\colon \Delta^n \to Y$. ■

Definition. Let $p\colon \widetilde{X} \to X$ be a k-fold covering of a topological space X. It follows from the Transfer Map Lemma 57.1 (2) that the natural transfer map $p^*\colon \mathrm{C}_*(X;R) \to \mathrm{C}_*(\widetilde{X};R)$

induces a natural R-module homomorphism

$$p^*\colon \mathrm{H}_*(X;R) \;\to\; \mathrm{H}_*(\widetilde{X};R)$$

which we also call transfer map. Finally suppose that we are given a subset $A \subset X$. We write $\widetilde{A} \coloneqq p^{-1}(A)$. One easily verifies that the transfer map $p^*\colon \mathrm{C}_n(X;R) \to \mathrm{C}_*(\widetilde{X};R)$ induces an R-module homomorphism

$$p^*\colon \mathrm{H}_*(X,A;R) \;\to\; \mathrm{H}_*(\widetilde{X},\widetilde{A};R).$$

Proposition 57.2. (Transfer–Projection Proposition) Let $p\colon \widetilde{X} \to X$ be a k-fold covering of a topological space X and let $n \in \mathbb{N}_0$. The following statements hold:

(1) For any commutative ring R the map $p_* \circ p^*\colon \mathrm{H}_n(X;R) \to \mathrm{H}_n(X;R)$ is given by multiplication by k.
(2) The projection induced map $p_*\colon \mathrm{H}_n(\widetilde{X};\mathbb{Q}) \to \mathrm{H}_n(X;\mathbb{Q})$ is an epimorphism.

Remark. The conclusion of the Transfer–Projection Proposition 57.2 (2) does not hold if we drop the condition that the covering is finite. For example, we have an infinite covering $p\colon \mathbb{R} \to S^1$ but $0 = \mathrm{H}_1(\mathbb{R};\mathbb{Q}) \xrightarrow{p_*} \mathrm{H}_1(S^1;\mathbb{Q}) = \mathbb{Q}$ is not an epimorphism. □

Proof. Let $p\colon \widetilde{X} \to X$ be a k-fold covering of a topological space X and let $n \in \mathbb{N}_0$.

(1) Let R be a commutative ring. We denote by $p^*\colon \mathrm{C}_n(X;R) \to \mathrm{C}_n(\widetilde{X};R)$ the transfer map. Let $\sigma\colon \Delta^n \to X$ be a singular simplex. We denote the corresponding lifts by $\widetilde{\sigma}_1, \dots, \widetilde{\sigma}_k$. Then

$$(p_* \circ p^*)(\sigma) \;=\; p_*(\widetilde{\sigma}_1 + \cdots + \widetilde{\sigma}_k) \;=\; \underbrace{p\circ\widetilde{\sigma}_1 + \cdots + p\circ\widetilde{\sigma}_k}_{k \text{ summands}} \overset{\text{since } \widetilde{\sigma}_i \text{ is a lift of } \sigma}{=} \underbrace{\sigma + \cdots + \sigma}_{k \text{ summands}} \;=\; k\cdot\sigma.$$

We thus see that $p_* \circ p^*\colon \mathrm{C}_n(X) \to \mathrm{C}_n(X)$ is multiplication by k. It follows easily that the map

$$\mathrm{H}_n(X;R) \xrightarrow{p^*} \mathrm{H}_n(\widetilde{X};R) \xrightarrow{p_*} \mathrm{H}_n(X;R)$$

is also given by multiplication by k.

(2) Since multiplication by $k \neq 0$ is an isomorphism of any rational vector space we obtain from (1) that $p_* \circ p^*\colon \mathrm{H}_n(X;\mathbb{Q}) \to \mathrm{H}_n(X;\mathbb{Q})$ is an isomorphism. In particular the second map, namely p_*, is an epimorphism. ■

For two-fold coverings we will now see that the transfer map fits into a long exact sequence:

Proposition 57.3. (Double Covering-LES-Proposition) Let $p\colon \widetilde{X} \to X$ be a 2-fold covering. There exist homomorphisms $\partial_n\colon \mathrm{H}_n(X;\mathbb{F}_2) \to \mathrm{H}_{n-1}(X;\mathbb{F}_2)$, which are natural in the sense of the Transfer Map Lemma 57.1, such that the following sequence is exact:

$$\dots \longrightarrow \mathrm{H}_n(X;\mathbb{F}_2) \xrightarrow{p^*} \mathrm{H}_n(\widetilde{X};\mathbb{F}_2) \xrightarrow{p_*} \mathrm{H}_n(X;\mathbb{F}_2) \xrightarrow{\partial_n} \mathrm{H}_{n-1}(X;\mathbb{F}_2) \longrightarrow \dots.$$

Example. Let $n \in \mathbb{N}$. We will see that we can use the Double Covering-LES-Proposition 57.3 to calculate the homology groups $\mathrm{H}_k(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2)$ just using the following basic facts:

(a) By the Sphere-H_*-Proposition 52.9 we know that

$$\mathrm{H}_k(S^n;\mathbb{F}_2) \;\cong\; \begin{cases} \mathbb{F}_2, & \text{if } k = 0 \text{ or } k = n, \\ 0, & \text{otherwise.} \end{cases}$$

(b) Since $\mathbb{R}\mathrm{P}^n$ is an n-dimensional CW-complex we know from the discussion on page 662 that $\mathrm{H}_k(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2) = 0$ for $k > n$.

In the following discussion we write $\mathbb{F} = \mathbb{F}_2$ for space reasons. Applying the Double Covering-LES-Proposition 57.3 to the obvious 2-fold covering $p\colon S^n \to \mathbb{R}\mathrm{P}^n$ we obtain the following long exact sequence:

$$\underbrace{\mathrm{H}_{n+1}(\mathbb{R}\mathrm{P}^n;\mathbb{F})}_{=0 \text{ by (b)}} \xrightarrow{\partial} \mathrm{H}_n(\mathbb{R}\mathrm{P}^n;\mathbb{F}) \xrightarrow{p^*} \underbrace{\mathrm{H}_n(S^n;\mathbb{F})}_{=\mathbb{F} \text{ by (a)}} \xrightarrow{p_*} \mathrm{H}_n(\mathbb{R}\mathrm{P}^n;\mathbb{F}) \xrightarrow{\partial} \mathrm{H}_{n-1}(\mathbb{R}\mathrm{P}^n;\mathbb{F}) \xrightarrow{p^*} \underbrace{\mathrm{H}_{n-1}(S^n;\mathbb{F})}_{=0 \text{ by (a)}} \to$$

$$\ldots \to \mathrm{H}_1(\mathbb{R}\mathrm{P}^n;\mathbb{F}) \xrightarrow{p^*} \underbrace{\mathrm{H}_1(S^n;\mathbb{F})}_{=0 \text{ by (a)}} \xrightarrow{p_*} \mathrm{H}_1(\mathbb{R}\mathrm{P}^n;\mathbb{F}) \xrightarrow{\partial} \mathrm{H}_0(\mathbb{R}\mathrm{P}^n;\mathbb{F}) \xrightarrow{p^*} \underbrace{\mathrm{H}_0(S^n;\mathbb{F})}_{=\mathbb{F} \text{ by (a)}} \xrightarrow{p_*} \mathrm{H}_0(\mathbb{R}\mathrm{P}^n;\mathbb{F}) \to 0.$$

Now we can make the following observations:

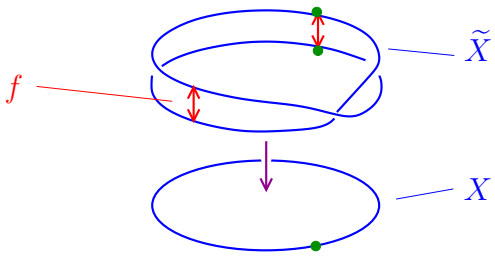
(1) Considering the bottom part of the long exact sequence we see that $\mathrm{H}_0(\mathbb{R}\mathrm{P}^n;\mathbb{F}) \cong \mathbb{F}$.
(2) If we work our way upward through the long exact sequence we see that the connecting homomorphisms give us isomorphisms
$$\mathrm{H}_{n-1}(\mathbb{R}\mathrm{P}^2;\mathbb{F}) \xrightarrow[\cong]{\partial} \mathrm{H}_{n-2}(\mathbb{R}\mathrm{P}^2;\mathbb{F}) \xrightarrow[\cong]{\partial} \ldots \xrightarrow[\cong]{\partial} \mathrm{H}_1(\mathbb{R}\mathrm{P}^2;\mathbb{F}) \xrightarrow[\cong]{\partial} \mathrm{H}_0(\mathbb{R}\mathrm{P}^2;\mathbb{F}) \cong \mathbb{F}.$$
(3) Now that we know that $\mathrm{H}_{n-1}(\mathbb{R}\mathrm{P}^n;\mathbb{F}) \cong \mathbb{F}$ let us look again at the upper end of the long exact sequence:
$$0 \longrightarrow \mathrm{H}_n(\mathbb{R}\mathrm{P}^n;\mathbb{F}) \xrightarrow{p^*} \underbrace{\mathrm{H}_n(S^n;\mathbb{F})}_{=\mathbb{F}} \xrightarrow{p_*} \mathrm{H}_n(\mathbb{R}\mathrm{P}^n;\mathbb{F}) \xrightarrow{\partial} \underbrace{\mathrm{H}_{n-1}(\mathbb{R}\mathrm{P}^n;\mathbb{F})}_{=\mathbb{F}} \xrightarrow{p^*} \underbrace{\mathrm{H}_{n-1}(S^n;\mathbb{F})}_{=0} \longrightarrow$$
It follows easily from staring at this data that the maps $\mathrm{H}_n(\mathbb{R}\mathrm{P}^n;\mathbb{F}) \xrightarrow{p^*} \mathrm{H}_n(S^n;\mathbb{F})$ and $\mathrm{H}_n(\mathbb{R}\mathrm{P}^n;\mathbb{F}) \xrightarrow{\partial} \mathrm{H}_{n-1}(\mathbb{R}\mathrm{P}^n;\mathbb{F})$ are also isomorphisms.

In particular we have shown that
$$\mathrm{H}_k(\mathbb{R}\mathrm{P}^n;\mathbb{F}) \cong \begin{cases} \mathbb{F}, & \text{if } k = 0,\ldots,n, \\ 0, & \text{otherwise.} \end{cases}$$
This agrees of course with the result that we had obtained in the $\mathbb{R}\mathrm{P}^n$-H_*-Proposition 52.10. □

Proof of the Double Covering-LES-Proposition 57.3. Let $p\colon \widetilde{X} \to X$ be a 2-fold covering.

Claim. The following sequence of chain maps
$$0 \to \mathrm{C}_*(X;\mathbb{F}_2) \xrightarrow{p^*} \mathrm{C}_*(\widetilde{X};\mathbb{F}_2) \xrightarrow{p_*} \mathrm{C}_*(X;\mathbb{F}_2) \to 0$$
is in fact exact.

Proof. Since $p\colon \widetilde{X} \to X$ is a 2-fold cover we can consider the continuous map
$$\begin{aligned} f\colon \widetilde{X} &\to \widetilde{X} \\ x &\mapsto \text{the unique other element of } p^{-1}(p(x)). \end{aligned}$$



Using the Map Lifting Criterion 27.1 we pick for each singular simplex $\sigma\colon \Delta^k \to X$ a lift $\widetilde{\sigma}\colon \Delta^k \to \widetilde{X}$. We make the following observations which follow easily from the definitions and the Map Lifting Criterion 27.1:

(a) Given any singular n-simplex σ we have $p_*(\widetilde{\sigma}) = \sigma$.
(b) Given any singular n-simplex σ we have $p^*(\sigma) = \widetilde{\sigma} + f_*(\widetilde{\sigma})$.
(c) Given any singular simplex μ of $\widetilde{X}$ we have $p_*(f_*(\mu)) = p_*(\mu)$.
(d) Every singular n-simplex μ of $\widetilde{X}$ is of the form $\mu = \widetilde{\sigma}$ or $\mu = f_*(\widetilde{\sigma})$ for some singular n-simplex σ of X.

Now we turn to the proof of the exactness of the above sequence.

(1) We first show that the map $p_*\colon \mathrm{C}_n(\widetilde{X};\mathbb{F}_2) \to \mathrm{C}_n(X;\mathbb{F}_2) = \mathrm{C}_n(X) \otimes \mathbb{F}_2$ is surjective. So let $c = \sigma_1 \otimes b_1 + \cdots + \sigma_k \otimes b_k \in \mathrm{C}_n(X;\mathbb{F}_2)$. It follows from (a) that
$$p_*(\widetilde{\sigma}_1 \otimes b_1 + \cdots + \widetilde{\sigma}_k \otimes b_k) \;=\; \sigma_1 \otimes b_1 + \cdots + \sigma_k \otimes b_k \;=\; c.$$
(2) Note that it follows from the Transfer Map Lemma 57.1 (1) that the transfer map $p^*\colon \mathrm{C}_*(X;\mathbb{F}_2) \to \mathrm{C}_*(\widetilde{X};\mathbb{F}_2)$ is a monomorphism.
(3) Next we show that $\mathrm{im}(p^*) \subset \ker(p_*)$. For any singular n-simplex σ of $\widetilde{X}$ we have[473]
$$(p_* \circ p^*)(\sigma \otimes 1) \;=\; p_*(\widetilde{\sigma} \otimes 1 + f_*(\widetilde{\sigma}) \otimes 1) \underset{\text{by (b) and (c)}}{=} 2\sigma \otimes 1 \;=\; \sigma \otimes 2 \;=\; 0 \;\in\; \mathrm{C}_*(X) \otimes \mathbb{F}_2.$$
(4) Finally we have to show that $\ker(p_*) \subset \mathrm{im}(p^*)$. So let $c \in \ker(p_*) \subset \mathrm{C}_*(\widetilde{X};\mathbb{F}_2)$. By (d) we can write
$$c \;=\; (\widetilde{\sigma}_1 \otimes a_1 + f_*(\widetilde{\sigma}_1) \otimes b_1) + \cdots + (\widetilde{\sigma}_k \otimes a_k + f_*(\widetilde{\sigma}_k) \otimes b_k)$$
where $\sigma_1, \ldots, \sigma_k$ are pairwise different and where $a_1, \ldots, a_k, b_1, \ldots, b_k \in \mathbb{F}_2$. By (c) we have
$$p(c) \;=\; \sigma_1 \otimes (a_1 + b_1) + \cdots + \sigma_k \otimes (a_k + b_k).$$
But since $c \in \ker(p_*)$ this implies that $a_i = -b_i$ for $i = 1, \ldots, k$. But since we work over $\mathbb{F}_2$ we actually have $a_i = b_i$ for $i = 1, \ldots, k$. It follows from (b) that
$$c \;=\; p^*(\sigma_1 \otimes a_1 + \cdots + \sigma_k \otimes a_k) \;\in\; \mathrm{im}(p^*)$$
as desired. ⊞

It follows from the claim and the LES-Proposition 37.4 that we get the desired connecting homomorphism $\partial_n\colon \mathrm{H}_n(X;\mathbb{F}_2) \to \mathrm{H}_{n-1}(X;\mathbb{F}_2)$. Finally note that the naturality of the connecting homomorphism follows from the naturality statement of the Transfer Map Lemma 57.1 and the Connecting Homomorphism Lemma 37.3 (2). ■

[473] That is of course basically the same argument as in the proof of the Transfer–Projection Proposition 57.2.

57.3. **The Borsuk-Ulam Theorem.** Our goal in this section is to give a proof of the following major theorem:

Theorem 57.4. (Borsuk-Ulam Theorem) Let $n \in \mathbb{N}_0$. For every continuous map $f\colon S^n \to \mathbb{R}^n$ there exists a pair of antipodal points x and $-x$ on S^n with $f(x) = f(-x)$.

Example. We consider the continuous map

$$\begin{array}{rcl} \text{earth} = S^2 & \to & \mathbb{R}^2 \\ x & \mapsto & (\text{temperature at } x, \text{distance to Kneitinger from } x). \end{array}$$

It follows from the Borsuk-Ulam Theorem 57.4, applied to the above function, that on earth there exists a pair of antipodal points where the temperature and the distance to the Kneitinger agree. □

Remark.

(1) In the 2D-Borsuk Ulam Theorem 12.2 we proved the Borsuk-Ulam Theorem 57.4 in the special case that $n = 2$.
(2) The Borsuk-Ulam Theorem 57.4 has many applications, in particular the Lusternik-Schnirelmann Theorem, the Ham-Sandwich Theorem, Pfister's Level Theorem and the Necklace Theorem. We refer to [**Fri24**] for details.

□

The key ingredient in the proof of the Borsuk-Ulam Theorem 57.4 is the following proposition:

Proposition 57.5. (Odd Map–Odd Degree Proposition) Let $f\colon S^n \to S^n$ be a continuous map. If $f(-x) = -f(x)$ for all $x \in S^n$, then $\deg(f)$ is odd.

Proof of the Odd Map–Odd Degree Proposition 57.5. We start out with a very general claim, which might be interesting in its own right.

Claim. Let $f\colon S^n \to S^n$ be a continuous map. For every $k \in \mathbb{N}$ the induced map

$$f_*\colon \mathrm{H}_n(S^n;\mathbb{Z}_k) \to \mathrm{H}_n(S^n;\mathbb{Z}_k)$$

is given by multiplication by $\deg(f)$.

Proof. Let $f\colon S^n \to S^n$ be a continuous map and let $k \in \mathbb{N}$. Since the two homomorphisms μ and ν of the Universal Coefficient Theorem 54.9 are natural we obtain the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{H}_n(S^n) \otimes \mathbb{Z}_k & \xrightarrow{\mu} & \mathrm{H}_n(S^n;\mathbb{Z}_k) & \xrightarrow{\nu} & \mathrm{Tor}(\mathrm{H}_{n-1}(S^n),\mathbb{Z}_k) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_* \otimes \mathrm{id}_{\mathbb{Z}_k}} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \\ 0 & \longrightarrow & \mathrm{H}_n(S^n) \otimes \mathbb{Z}_k & \xrightarrow{\mu} & \mathrm{H}_n(S^n;\mathbb{Z}_k) & \xrightarrow{\nu} & \mathrm{Tor}(\mathrm{H}_{n-1}(S^n),\mathbb{Z}_k) & \longrightarrow & 0 \end{array}$$

of short exact sequences. Note that $\mathrm{H}_{n-1}(S^n) = 0$ or $\cong \mathbb{Z}$, hence it follows from the Tor-Properties Lemma 54.7 (4) that $\mathrm{Tor}(\mathrm{H}_{n-1}(S^n),\mathbb{Z}_k) = 0$. It follows from this observation and the exactness of the horizontal sequences that the horizontal maps μ on the left are isomorphisms.

By definition the map $f_*\colon \mathrm{H}_n(S^n) \to \mathrm{H}_n(S^n)$ is given by multiplication by $\deg(f)$. It follows from the Tensor-with-$\mathbb{Z}_k$–Lemma 51.4 that $f_* \otimes \mathrm{id}_{\mathbb{Z}_k}$ is also given by multiplication

by $\deg(f)$. It now follows that the middle vertical map is also given by multiplication by $\deg(f)$. ⊞

Now we turn to the actual proof of the proposition. Thus let $f\colon S^n \to S^n$ be a continuous map such that $f(-x) = -f(x)$ for all $x \in S^n$. We need to show that $\deg(f)$ is odd. By the above claim it suffices to show that $f_*\colon \mathrm{H}_n(S^n;\mathbb{Z}_2) \to \mathrm{H}_n(S^n;\mathbb{Z}_2)$ is an isomorphism.

We start out with the observation that the map $f\colon S^n \to S^n$ evidently descends to a well-defined continuous map

$$\begin{array}{rccc} \overline{f}\colon & \mathbb{R}\mathrm{P}^n & \to & \mathbb{R}\mathrm{P}^n \\ & [x] & \mapsto & [f(x)]. \end{array}$$

Note that this map gives rise to the following diagram:

$$\begin{array}{ccc} S^n & \xrightarrow{\quad p \quad} & \mathbb{R}\mathrm{P}^n \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \overline{f}} \\ S^n & \xrightarrow{\quad p \quad} & \mathbb{R}\mathrm{P}^n \end{array}$$

of continuous maps. Note that the following statements hold:

(a) The diagram commutes.
(b) $p\colon S^n \to \mathbb{R}\mathrm{P}^n$ is a 2-fold covering.
(c) For each $x \in \mathbb{R}\mathrm{P}^n$ the map $f|_{p^{-1}(x)}\colon p^{-1}(x) \to p^{-1}(\overline{f}(x))$ is a bijection. (Note that here we use that $f(-x) = -f(x)$).

Now we consider the following diagram:

$$\begin{array}{ccccccccccc} \mathrm{H}_n(S^n;\mathbb{Z}_2) & \xleftarrow{p^*} & \mathrm{H}_n(\mathbb{R}\mathrm{P}^n;\mathbb{Z}_2) & \xrightarrow{\partial_n} & \mathrm{H}_{n-1}(\mathbb{R}\mathrm{P}^n;\mathbb{Z}_2) & \xrightarrow{\partial_{n-1}} & \dots & \xrightarrow{\partial_2} & \mathrm{H}_1(\mathbb{R}\mathrm{P}^n;\mathbb{Z}_2) & \xrightarrow{\partial_1} & \mathrm{H}_0(\mathbb{R}\mathrm{P}^n;\mathbb{Z}_2) \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle \overline{f}_*} & & \downarrow{\scriptstyle \overline{f}_*} & & & & \downarrow{\scriptstyle \overline{f}_*} & & \downarrow{\scriptstyle \overline{f}_*} \\ \mathrm{H}_n(S^n;\mathbb{Z}_2) & \xleftarrow{p^*} & \mathrm{H}_n(\mathbb{R}\mathrm{P}^n;\mathbb{Z}_2) & \xrightarrow{\partial_n} & \mathrm{H}_{n-1}(\mathbb{R}\mathrm{P}^n;\mathbb{Z}_2) & \xrightarrow{\partial_{n-1}} & \dots & \xrightarrow{\partial_2} & \mathrm{H}_1(\mathbb{R}\mathrm{P}^n;\mathbb{Z}_2) & \xrightarrow{\partial_1} & \mathrm{H}_0(\mathbb{R}\mathrm{P}^n;\mathbb{Z}_2). \end{array}$$

We make the following clarifications and observations:

(1) The map p^* is of course the natural transfer map from the Transfer Map Lemma 57.1. It follows from (a)–(c) above that the naturality statement applies, i.e. the square to the left commutes.
(2) The maps $\partial_n, \dots, \partial_1$ are the natural connecting homomorphisms from the Double Covering-LES-Proposition 57.3. It follows from (a)–(c) above that the naturality statement applies, i.e. the squares to the right commutes.
(3) All the horizontal maps are isomorphisms by the discussion on page 724.
(4) It follows from the obvious generalization of the H_0-Proposition 35.11, see also the discussion on page 661, that the right vertical map is an isomorphism.

It follows from the above that the left vertical map is an isomorphism. But that is exactly what we had wanted to prove. ■

Now we finally turn to the proof of the Borsuk-Ulam Theorem 57.4.

Proof of the Borsuk-Ulam Theorem 57.4. Let $f\colon S^n \to \mathbb{R}^n$ be a continuous map. We suppose that there exists no pair of antipodal points x and $-x$ in S^n with $f(x) = f(-x)$. Put differently, we suppose that for every $x \in S^n$ we have $f(x) \neq f(-x)$. We consider the

composition of the following two maps

$$\begin{array}{rcccl} \Phi\colon S^{n-1}\times[0,1] & \to & S^n_{\geq 0} & \to & S^{n-1}. \\ (y,t) & \mapsto & (y\cdot\sqrt{1-t^2},t) & & \\ & & x & \mapsto & \underbrace{\frac{f(x)-f(-x)}{\|f(x)-f(-x)\|}} \\ & & & & \text{defined since } f(x)\neq f(-x) \end{array}$$

Now we see that

$$\text{odd number} \overset{\text{by the Sphere–Degree Lemma 43.1 (3) since } \Phi_0 \text{ and } \Phi_1 \text{ are homotopic}}{=} \deg(\Phi_0) \overset{\text{by definition of } \Phi_1}{=} \deg(\Phi_1) = \deg(\text{constant map}) \overset{\text{by the Sphere–Degree Lemma 43.1 (2)}}{=} 0.$$

by definition we have $\Phi(-x,0)=-\Phi(x,0)$, thus it follows from the Odd Map–Odd Degree Proposition 57.5 that $\deg(\Phi_0)$ is odd

We have thus obtained a contradiction. ■

Part VI

Singular Homology – extra topics

In this last part of the lecture notes, which is e dedicated to homology groups, we prove a couple of loosely or even unconnected results. The main topic will be the proof and discussion of the Hurewicz Theorems 58.5 and 59.6.

This part of the lecture notes is organized as follows:

- In Chapter 58 we will prove the Hurewicz Theorem 58.5 which says that for a pointed connected topological space (X, x_0) there exists a natural isomorphism between the abelianization $\pi_1(X, x_0)_{\mathrm{ab}}$ of the fundamental group and the first homology group $\mathrm{H}_1(X)$.
- In Chapter 59 we will prove a high-dimensional analogue of the Hurewicz Theorem 58.5. More precisely, the Hurewicz Theorem 59.6 says that if (X, x_0) is a connected pointed topological space such that $\pi_i(X, x_0) = 0$ for $i = 1, \ldots, n-1$ with $n \geq 2$, then $\pi_n(X, x_0)$ is naturally isomorphic to $\mathrm{H}_n(X)$. This result allows us in particular to finally show that for any $n \in \mathbb{N}$ we have an isomorphism $\pi_n(S^n) \cong \mathbb{Z}$.
- In Chapter **??** we will use the calculation $\pi_n(S^n) \cong \mathbb{Z}$ to study the not entirely serious "social choice problem".
- In Chapter **??** we will introduce groups rings and we will see that for $n \geq 2$ we should view $\pi_n(X, x_0)$ as a module over the group ring $\mathbb{Z}[\pi_1(X, x_0)]$. Furthermore we will use the calculation $\pi_n(S^n) \cong \mathbb{Z}$ to calculate the $\mathbb{Z}[\pi_1(S^1 \vee S^2)]$-module $\pi_2(S^1 \vee S^2)$.
- In Chapter **??** we will prove the Sphere Complement-H_*-Theorem **??** which gives us the calculation of the homology groups of $S^n \setminus h(S^k)$ for any embedding $S^k \to S^n$. We will use this calculation to prove the Jordan-Brouwer Separation Theorem **??** and to prove the Closed Manifold–Injection Corollary **??** which says that an injective continuous map between closed connected topological manifolds of the same dimension is already a homeomorphism.
- In Chapter **??** we will show in the Commutative Division Algebra Theorem **??**, using the Closed Manifold–Injection Corollary **??** that any finite-dimensional division algebra over $\mathbb{R}$ which is *commutative* and which has a multiplicatively neutral element is isomorphic either to $\mathbb{R}$ or to $\mathbb{C}$.
- In Chapter **??** we will return to the study of the covering dimension of topological spaces, which we first introduced in Chapter **??**. Now that we have homology groups at our disposal we can finally show that the covering dimension of any non-empty n-dimensional manifold equals n.
- In Chapter **??** we will introduce for each abelian group π and each $k \in \mathbb{N}$ the corresponding Moore space $\mathrm{M}(\pi, k)$.

CHAPTER 58

The Hurewicz Theorem in dimension one

Given a pointed path-connected topological space (X, x_0) we can consider the fundamental group $\pi_1(X, x_0)$ and the first homology group $\mathrm{H}_1(X)$. Both groups are defined by mapping "intervals" into X. Hence it is reasonable to expect that there is a relationship between these two groups. But in general these two groups cannot be isomorphic, since the first homology group is by definition abelian, whereas we saw on many occasions that fundamental groups can be non-abelian.

In this chapter we will prove and discuss the Hurewicz Theorem 58.5 which explains the relationship between the fundamental group $\pi_1(X, x_0)$ and the first homology group $\mathrm{H}_1(X)$. In the next chapter we will explore the relationship between higher homology groups and higher homotopy groups.

58.1. **The Hurewicz Theorem in dimension one.** We start out by reminding the reader of the following definitions from page 179 and page 181:

Definition. Let X be a topological space.

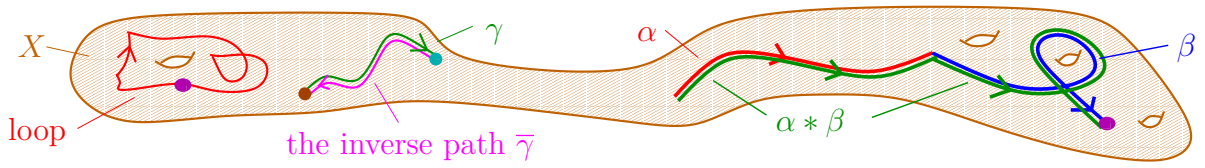
(1) A path in X is a continuous map $\gamma\colon [0,1] \to X$. We call $\gamma(0)$ the starting point of γ and we call $\gamma(1)$ the endpoint of γ. Often we say that γ is a path from $\gamma(0)$ to $\gamma(1)$.

(2) A loop is a path for which the starting point and the endpoint coincide.

(3) Given a path $\gamma\colon [0,1] \to X$ we refer to the path $\overline{\gamma}\colon [0,1] \to X$ that is defined by $\overline{\gamma}(t) = \gamma(1-t)$ as the inverse path of γ.

(4) Let $\alpha\colon [0,1] \to X$ and $\beta\colon [0,1] \to X$ be two paths with $\alpha(1) = \beta(0)$. We define the concatenation of α and β as the path $\alpha * \beta$ which is given by

$$\begin{array}{rcl} \alpha * \beta\colon [0,1] & \to & X \\ t & \mapsto & \begin{cases} \alpha(2t), & \text{if } t \in [0, \frac{1}{2}] \\ \beta(2t-1), & \text{if } t \in [\frac{1}{2}, 1]. \end{cases} \end{array}$$



Notation. Throughout this chapter we work with the homeomorphism

$$\begin{array}{rcl} \zeta\colon \Delta^1 & \to & [0,1] \\ (1-t, t) & \mapsto & t. \end{array}$$

The following proposition is the first step towards relating the fundamental group and the first homology group.

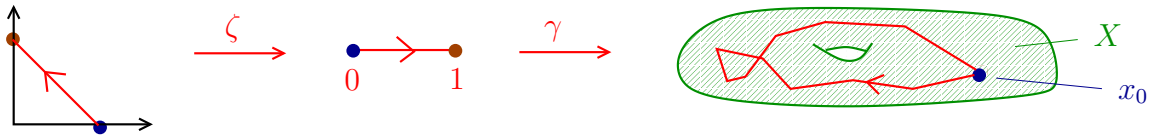
Proposition 58.1. (Hurewicz Homomorphism Proposition) Let (X, x_0) be a pointed topological space.

(1) The map
$$\begin{array}{rcl} \Phi_{(X,x_0)}\colon \pi_1(X,x_0) & \to & \mathrm{H}_1(X) \\ [\gamma\colon [0,1]\to X] & \mapsto & [\gamma\circ\zeta\colon \Delta^1\to X] \end{array}$$
is well-defined.

(2) The map $\Phi_{(X,x_0)}\colon \pi_1(X,x_0)\to \mathrm{H}_1(X)$ is a group homomorphism.

(3) Given any path $\gamma\colon [0,1]\to X$ from x_0 to some $x_1\in X$ the following diagram commutes:[474]
$$\begin{array}{ccc} \pi_1(X,x_1) & \xrightarrow{\Phi_{(X,x_1)}} & \\ \Big\downarrow{\scriptstyle \text{homomorphism } \gamma_* \text{ from the Change-of-Base Point Proposition 11.6 (1)}} & & \mathrm{H}_1(X). \\ \pi_1(X,x_0) & \xrightarrow{\Phi_{(X,x_0)}} & \end{array}$$

(4) Let $\mathcal{PTop}$ be the category of pointed topological spaces and let $\mathcal{Group}$ be the category $\mathcal{Group}$ of groups. The homomorphisms
$$\Phi_{(X,x_0)}\colon \pi_1(X,x_0) \to \mathrm{H}_1(X)$$
define a natural transformation from the functor[475]
$$\begin{array}{rcl} \mathcal{PTop} & \to & \mathcal{Group} \\ (X,x_0) & \mapsto & \pi_1(X,x_0) \end{array}$$
to the functor
$$\begin{array}{rcl} \mathcal{PTop} & \to & \mathcal{Group} \\ (X,x_0) & \mapsto & \mathrm{H}_1(X). \end{array}$$
In plain English this means the following: Given any continuous map $f\colon X\to Y$ between topological spaces and given any $x_0\in X$ the following diagram commutes:
$$\begin{array}{ccc} \pi_1(X,x_0) & \xrightarrow{f_*} & \pi_1(Y,f(x_0)) \\ \Big\downarrow{\scriptstyle \Phi_{(X,x_0)}} & & \Big\downarrow{\scriptstyle \Phi_{(Y,f(x_0))}} \\ \mathrm{H}_1(X) & \xrightarrow{f_*} & \mathrm{H}_1(Y). \end{array}$$



The proof of the Hurewicz Homomorphism Proposition 58.1 relies on the following elementary lemma:

Lemma 58.2. (Path-Chain–Homomorphism Lemma) Let X be a topological space.

(1) If γ is a loop in X, then $\gamma\circ\zeta\in \mathrm{C}_1(X)$ is a cycle.

(2) If γ is a path in X, then $-\gamma\circ\zeta$ and $\overline{\gamma}\circ\zeta$ are homologous.

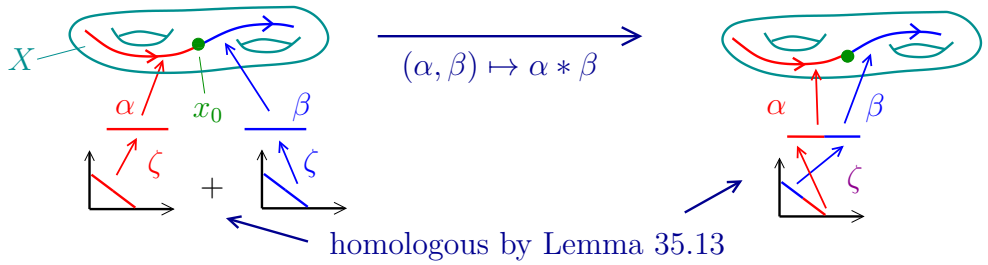
(3) If α and β are two paths in X such that the endpoint of α agrees with the starting point of β, then the singular 1-chains $\alpha\circ\zeta+\beta\circ\zeta$ and $(\alpha*\beta)\circ\zeta$ are homologous.

Proof of the Path-Chain–Homomorphism Lemma 58.2.

[474] This statement shows in particular that the question of whether the Hurewicz homomorphism is a monomorphism, epimorphism, or isomorphism does not depend on the choice of a base point in a path-component.

[475] Strictly speaking we viewed H_1 as a functor from the category $\mathcal{Top}$ of topological spaces to the category $\mathcal{AbGroup}$ of abelian groups. But of course it also defines a functor from the category $\mathcal{PTop}$ of pointed topological spaces to the category $\mathcal{Group}$ of groups.

(1) This statement is just the Singular 1-Simplex Lemma 35.13 (1).
(2) This statement follows easily from the Singular 1-Simplex Lemma 35.13 (2).
(3) Let α and β be two paths in X such that the endpoint of α agrees with the starting point of β.



It follows easily from the Singular 1-Simplex Lemma 35.13 (3) that the singular 1-chains $\alpha \circ \zeta + \beta \circ \zeta$ and $(\alpha * \beta) \circ \zeta$ are homologous. ■

Proof of the Hurewicz Homomorphism Proposition 58.1.

(1) First note that it follows from the Path-Chain–Homomorphism Lemma 58.2 (1) that if $\gamma\colon [0,1] \to X$ is a loop in (X, x_0), then $\gamma \circ \zeta \in \mathrm{C}_1(X)$ is a cycle. It remains to prove the following claim:

Claim. If $f, g\colon [0,1] \to X$ are two path-homotopic loops in x_0, then $f \circ \zeta$ and $g \circ \zeta$ are homologous.

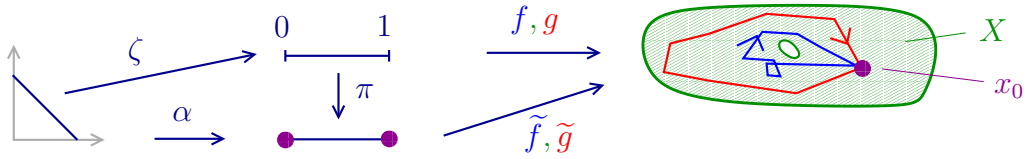
Proof. We denote by $\sim$ the equivalence relation on $[0,1]$ that is generated by $0 \sim 1$. We denote by $\pi\colon [0,1] \to [0,1]/\!\sim$ the projection. We set $\alpha := \pi \circ \zeta$. Note that α defines an element in $\mathrm{H}_1([0,1]/\!\sim)$. Now suppose that $f, g\colon [0,1] \to X$ are two path-homotopic loops in x_0. Recall that this means that there exists a homotopy $F\colon [0,1] \times [0,1] \to X$ with $F_0 = f$, $F_1 = g$ and such that for all $t \in [0,1]$ we have $F_t(0) = f(0) = g(0)$ and we have $F_t(1) = f(1) = g(1)$.

We denote by $\widetilde{f}, \widetilde{g}\colon [0,1]/\!\sim \to X$ and $\widetilde{F}\colon ([0,1]/\!\sim) \times [0,1] \to X$ the induced maps. It follows from the Topological-Quotient Proposition 6.1 (1b) and the Homotopy Quotient Theorem 8.5 that these induced maps are continuous. In particular we see that $\widetilde{f}$ and $\widetilde{g}$ are homotopic. We obtain the following equalities in $\mathrm{H}_1(X)$:

$$[f \circ \zeta] \underset{\uparrow}{=} [\widetilde{f} \circ \alpha] = \widetilde{f}_*([\alpha]) \underset{\uparrow}{=} \widetilde{g}_*([\alpha]) = [\widetilde{g} \circ \alpha] = [g \circ \zeta].$$

since $f \circ \zeta$ and $\widetilde{f} \circ \alpha$ are the same map $\Delta^1 \to X$

by the Homotopy-H_*-Proposition 36.5, since $\widetilde{f}$ and $\widetilde{g}$ are homotopic maps $[0,1]/\!\sim \to X$



⊞

(2) This statement follows from the Path-Chain–Homomorphism Lemma 58.2 (3).
(3) This statement follows from the Path-Chain–Homomorphism Lemma 58.2 (2) and (3).
(4) The last statement follows immediately from the definitions. ■

The Hurewicz Homomorphism Proposition 58.1 allows us to introduce the following definition:

Definition. Given a pointed topological space (X, x_0) we refer to the homomorphism

$$\begin{array}{rcl} \Phi_{(X,x_0)}\colon \pi_1(X,x_0) & \to & \mathrm{H}_1(X) \\ [\gamma\colon [0,1] \to X] & \mapsto & [\gamma \circ \zeta\colon \Delta^1 \to X] \end{array}$$

as the Hurewicz homomorphism.

Example. We consider the pointed topological space $(X, x_0) = (S^1, 1)$. It follows from the discussion on page 540 that $\Phi_{(S^1,1)}([t \mapsto \exp(2\pi\,\mathrm{i}t)]) = [S^1] \in \mathrm{H}_1(S^1)$. □

It is clear that the Hurewicz homomorphism is in general neither injective nor surjective:

(1) The Hurewicz homomorphism is in general not injective since $\pi_1(X, x_0)$ is in many cases a non-abelian group, whereas $\mathrm{H}_1(X)$ is an abelian group, and it is not possible that a non-abelian group is a subgroup of an abelian group.

(2) The Hurewicz homomorphism is in general not surjective, since $\pi_1(X, x_0)$ depends only on the path-component of x_0, while $\mathrm{H}_1(X)$ "sees" all path-components of X. But as we will see in the next proposition, this turns out to be the only obstacle to Φ being surjective.

Proposition 58.3. (Hurewicz Epimorphism Proposition) Let X be a topological space and let $x_0 \in X$. If X is path-connected, then the Hurewicz homomorphism

$$\Phi_{(X,x_0)}\colon \pi_1(X,x_0) \;\to\; \mathrm{H}_1(X)$$

is surjective.

In the proof of the Hurewicz Epimorphism Proposition 58.3 we will need the following elementary lemma:

Lemma 58.4. (Matching Lemma) Let X be a set and let $x_1, \ldots, x_k$ and $y_1, \ldots, y_l$ be elements in X. If

$$\sum_{i=1}^{k} x_i \;-\; \sum_{i=1}^{l} y_i \;=\; 0 \;\in\; \mathbb{Z}^{(X)},$$

then $k = l$ and there exists a permutation $s \in S_k$ such that $y_i = x_{s(i)}$ for $i = 1, \ldots, k$.

Proof. Let X be a set and let $x_1, \ldots, x_k$ and $y_1, \ldots, y_l$ be elements in X such that

$$\underbrace{\sum_{i=1}^{k} x_i}_{=:a} \;=\; \underbrace{\sum_{i=1}^{l} y_i}_{=:b} \quad \in\; \mathbb{Z}^{(X)}.$$

It follows from the Free Abelian Group–Existence Lemma 18.4 that there exists a unique homomorphism $\psi\colon \mathbb{Z}^{(X)} \to \mathbb{Z}$ with the property that $\psi(x) = 1$ for all $x \in X$. By definition we have $\psi(a) = k$ and $\psi(b) = l$. So from $a = b$ we deduce that $k = l$.

Note that it follows from applying the Free Abelian Group–Existence Lemma 18.4 once again that there exists a unique homomorphism $\psi\colon \mathbb{Z}^{(X)} \to \mathbb{Z}$ with the property that $\psi(x_k) = 1$ and $\psi(x) = 0$ for all $x \in X \setminus \{x_k\}$. By definition we have $\psi(a) > 0$. From $\psi(b) = \psi(a) > 0$ it follows that there exists an $n \in \{1, \ldots, k\}$ such that $y_n = x_k$. We define $s(k) = n$. We obtain that

$$\sum_{i=1}^{k-1} x_i \;=\; \sum_{i \in \{1,\ldots,k\} \setminus \{s(k)\}} y_i \;\in\; \mathbb{Z}^{(X)}$$

and we iterate this procedure to define $s(k-1), \ldots, s(1)$. By construction the map $s\colon \{1, \ldots, k\} \to \{1, \ldots, k\}$ is a bijection. ■

Proof of Proposition 58.3. The homeomorphism $\zeta\colon \Delta^1 \to [0,1]$ from page 731 gives us a bijection

$$\begin{array}{rcl} \Phi\colon \{\text{paths } [0,1] \to X\} & \to & \{\text{singular 1-simplices in } X\} \\ (\gamma\colon [0,1] \to X) & \mapsto & (\gamma \circ \zeta\colon \Delta^1 \to X). \end{array}$$

We denote by Ψ the inverse of the above bijection.

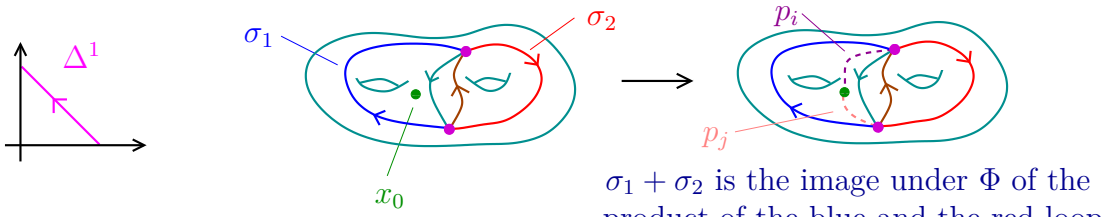
We pick a homology class $z \in \mathrm{H}_1(X)$. Let $c = \sum_{i=1}^{k} a_i \cdot \sigma_i$ be a cycle in $\mathrm{C}_1(X)$ that represents z. It follows from the Singular 1-Simplex Lemma 35.13 (2) that, after possibly replacing a singular chain σ_i by $\overline{\sigma}_i$, we can assume that all the coefficients a_i are positive. Furthermore, since we do not require that the σ_i's are different, we can in fact suppose that $a_1 = \cdots = a_k = 1$. Next note that we have

$$0 \underset{\uparrow}{=} \partial_1(c) = \partial_1\Big(\sum_{i=1}^{k} \sigma_i\Big) \underset{\uparrow}{=} \sum_{i=1}^{k} \sigma_i(0,1) - \sigma_i(1,0) = \sum_{i=1}^{k} \sigma_i(0,1) - \sum_{i=1}^{k} \sigma_i(1,0).$$

since c is a cycle — see the discussion on page 459

On the right hand side each "positive summand" needs to cancel with a "negative summand". This means, see also the Matching Lemma 58.4, that there exists a permutation $\theta \in S_k$ such that for each $i \in \{1,\dots,k\}$ the "endpoint" $\sigma_i(0,1)$ of σ_i is just the "starting point" $\sigma_{\theta(i)}(1,0)$ of $\sigma_{\theta(i)}$.

Since X is path-connected we can choose for each $i \in \{1,\dots,k\}$ a path p_i from the base point x_0 to the starting point $\sigma_i(1,0)$ of σ_i.



$\sigma_1 + \sigma_2$ is the image under Φ of the product of the blue and the red loop

We obtain the following equality in $\mathrm{H}_1(X)$:

$$\begin{array}{ccccccc} c = \sum_{i=1}^{k} \sigma_i & \underset{\downarrow}{=} & \sum_{i=1}^{k} \sigma_i + \sum_{i=1}^{k} \Phi(p_i) + \sum_{i=1}^{k} \Phi(\overline{p}_i) & \underset{\downarrow}{=} & \sum_{i=1}^{k} \big(\Phi(p_i) + \Phi(\Psi(\sigma_i)) + \Phi(\overline{p}_{\theta(i)})\big) \\ & = & \sum_{i=1}^{k} \Phi\big(\underbrace{p_i * \Psi(\sigma_i) * \overline{p}_{\theta(i)}}_{\text{loop in } (X,x_0), \text{ by choice of } \theta}\big) & = & \Phi\Big(\underbrace{\prod_{i=1}^{k} p_i * \Psi(\sigma_i) * \overline{p}_{\theta(i)}}_{\text{loop in } (X,x_0)}\Big). \\ & \underset{\uparrow}{} & & \underset{\uparrow}{} & \end{array}$$

by the Singular 1-Simplex Lemma 35.13: the singular chain $\Phi(p_i) + \Phi(\overline{p_i})$ is a null-homologous cycle

since $\Phi \circ \Psi = \mathrm{id}$ and since permutation of the summands does not change the sum

by the Path-Chain–Homomorphism Lemma 58.2

by the Path-Chain–Homomorphism Lemma 58.2

Thus we have shown that $z = [c] \in \Phi_{(X,x_0)}(\pi_1(X,x_0))$. ∎

In the following we want to study to what degree the Hurewicz homomorphism fails to be injective. Before we can do so, we need to recall some definitions and results from Section 19.3.

Definition. Let π be a group.

(1) For $x, y \in \pi$ we denote by $[x,y] := xyx^{-1}y^{-1}$ the commutator of x and y.

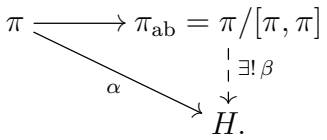
(2) We refer to

$$[\pi,\pi] := \left\{ \prod_{i=1}^{n} [x_i, y_i] \,\middle|\, x_1, y_1, \ldots, x_n, y_n \in \pi \right\}$$

as the commutator subgroup of π.[476]

(3) We refer to $\pi_{\mathrm{ab}} := \pi/[\pi,\pi]$ as the abelianization of π.

In Section 19.3 we saw that the abelianization is "the largest abelian quotient" of π. More precisely, if $\alpha\colon \pi \to H$ is a homomorphism from π to an abelian group H, then the Commutator Subgroup Lemma 19.5 (3) says that there exists a unique homomorphism $\beta\colon \pi_{\mathrm{ab}} = \pi/[\pi,\pi] \to H$ such that the following diagram commutes:



This discussion leads us to the following definition:

Definition. Given a pointed topological space (X, x_0), the corresponding Hurewicz homomorphism $\Phi_{(X,x_0)}\colon \pi_1(X,x_0) \to \mathrm{H}_1(X)$ induces a homomorphism $\pi_1(X,x_0)_{\mathrm{ab}} \to \mathrm{H}_1(X)$ that we also denote by $\Phi_{(X,x_0)}$ and that we also refer to as the Hurewicz homomorphism.

Now we can formulate the main result of this chapter:[477]

Theorem 58.5. (Hurewicz Theorem) Let (X, x_0) be a pointed topological space. If X is path-connected, then the Hurewicz homomorphism

$$\Phi_{(X,x_0)}\colon \pi_1(X,x_0)_{\mathrm{ab}} \;\to\; \mathrm{H}_1(X)$$

is a natural isomorphism[478].

Example. Using our earlier calculations of fundamental groups and homology groups in the Torus-π_1-Corollary 13.8, the Real Projective Space–π_1-Proposition 15.3, the Surface-π_1-Proposition 22.1, the Torus-H_*-Proposition 45.5 (1), the $\mathbb{R}\mathrm{P}^n$-H_*-Proposition 47.10, the Klein Bottle-H_*-Lemma 45.6 and the Surface-H_*-Proposition 47.9 we obtain the following table:

topological space	$(S^1)^n$	$\mathbb{R}\mathrm{P}^n$	Klein bottle	surface of genus g
π_1	$\mathbb{Z}^n$	$\mathbb{Z}_2$	$\langle x, y \mid yxyx^{-1}\rangle$	$\langle x_1, y_1, \ldots, x_g, y_g \mid [x_1,y_1]\cdot\ldots\cdot[x_g,y_g]\rangle$
H_1	$\mathbb{Z}^n$	$\mathbb{Z}_2$	$\mathbb{Z} \oplus \mathbb{Z}_2$	$\mathbb{Z}^{2g}$

Using the Abelianization Calculation Proposition 19.8 (1) and the Abelianization-of-Presentation Proposition 21.5 one can verify that these calculations are indeed consistent with the Hurewicz Theorem 58.5. □

[476] By the Commutator Subgroup Lemma 19.5 (1) we know that $[\pi,\pi]$ is a normal subgroup of π.

[477] This theorem is called Hurewicz Theorem since Witold Hurewicz [**Hur35b**] proved the high-dimensional generalization, namely the Hurewicz Theorem 59.6. This 1-dimensional statement was known before the work of Witold Hurewicz, see e.g. [**ST34**, Satz 48.3].

[478] If $\pi_1(X,x_0)$ is abelian, then it follows from the elementary Abelianization Calculation Proposition 19.8 (1) that we obtain an isomorphism $\pi_1(X,x_0) \xrightarrow{\cong} \mathrm{H}_1(X)$.

The proof of the Hurewicz Theorem 58.5 requires the remainder of this section. We continue in Section 58.2 with one application.

Proof of the Hurewicz Theorem 58.5. Throughout this proof we consider the following two homeomorphisms:

$$\begin{array}{rcl} \zeta\colon \Delta^1 & \to & [0,1] \\ (1-t,t) & \mapsto & t \end{array} \qquad \text{and} \qquad \begin{array}{rcl} \psi\colon [0,1] & \to & \Delta^1 \\ t & \mapsto & (1-t,t). \end{array}$$

Let (X,x_0) be a path-connected pointed topological space. By the Hurewicz Homomorphism Proposition 58.1 we know that the Hurewicz homomorphism

$$\begin{array}{rcl} \Phi_{(X,x_0)}\colon \pi_1(X,x_0)_{\mathrm{ab}} & \to & \mathrm{H}_1(X) \\ [f\colon [0,1]\to X] & \mapsto & [f\circ\zeta] \end{array}$$

is natural. Furthermore it follows from the Hurewicz Epimorphism Proposition 58.3 that the Hurewicz homomorphism $\Phi_{(X,x_0)}\colon \pi_1(X,x_0)_{\mathrm{ab}} \to \mathrm{H}_1(X)$ is surjective. It remains to show that $\Phi_{(X,x_0)}$ is injective.

So let $f\colon [0,1]\to X$ be a loop in (X,x_0) such that $\Phi_{(X,x_0)}([f]) = [f\circ\zeta] = 0 \in \mathrm{H}_1(X)$. By definition of $\mathrm{H}_1(X)$ this means that there exists a singular 2-chain $\sum_{j=1}^{s} n_j\cdot\sigma_j$ in X whose boundary is the singular 1-simplex $f\circ\zeta$. Note that, since we do not demand that the σ_j are distinct, we can assume that all n_j are -1 or 1. Given $j\in\{1,\dots,s\}$ and $k\in\{0,1,2\}$ we write[479] $\mu_{(j,k)} \coloneqq \sigma_j\circ i_k^2\circ\psi\colon [0,1]\to X$ and $n_{(j,k)} \coloneqq (-1)^k\cdot n_j$. We have the following equality in $\mathrm{C}_1(X)$:

$$f\circ\zeta \;=\; \partial_1\Big(\sum_{j=1}^{s} n_j\cdot\sigma_j\Big) \;\underset{\substack{\uparrow\\ \text{by definition of } \partial_1 \text{ and since } \psi\circ\zeta=\mathrm{id}_{\Delta^1}}}{=}\; \sum_{j=1}^{s}\sum_{k=0}^{2} n_{(j,k)}\cdot(\mu_{j,k}\circ\zeta) \;=\; \sum_{a\in\{1,\dots,s\}\times\{0,1,2\}} \underbrace{n_a}_{=\pm1}\cdot(\mu_a\circ\zeta).$$

We make the following choices and we introduce the following definitions:

(1) We set $A \coloneqq \{1,\dots,s\}\times\{0,1,2\}$.

(2) By comparing the left and the right-hand side of the above equality we see that there exists a $b\in A$ with $1\cdot(f\circ\zeta) = n_b\cdot(\mu_b\circ\zeta)$.

(3) We set $A_{\pm} \coloneqq \{a\in A\setminus\{b\} \mid n_a = \pm 1\}$.

(4) We obtain from the above equality that

$$\sum_{a\in A_+}(\mu_a\circ\zeta) - \sum_{a\in A_-}(\mu_a\circ\zeta) \;=\; 0.$$

It follows from the Matching Lemma 58.4 that there exists a bijection $\theta\colon A_+\to A_-$ such that for each $a\in A_+$ we have $\mu_{\theta(a)} = \mu_a$.

(5) For each point P in X that is a starting point or an endpoint of some μ_a we pick a path β_P from x_0 to P. For $P = x_0$ we just pick the constant path const_{x_0}.

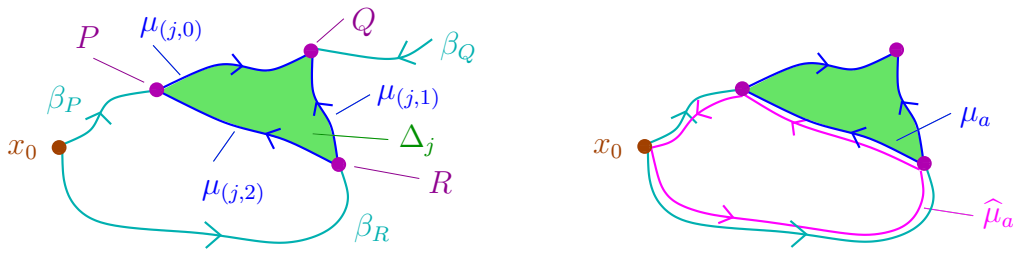
(6) Given $(i,j)\in A$ we set

$$\widehat{\mu}_{(i,j)} \;\coloneqq\; \beta_{\text{starting point of } \mu_{(i,j)}} * \mu_{(i,j)} * \overline{\beta}_{\text{endpoint of } \mu_{(i,j)}}.$$

[479] Recall that for $j\in\{0,\dots,n\}$ we denote by i_j^n the j-th face map

$$\begin{array}{rcl} i_j^n\colon \Delta^{n-1} & \to & \Delta^n \\ (t_0,\dots,t_{n-1}) & \mapsto & (t_0,\dots,t_{j-1},0,t_j,\dots,t_{n-1}). \end{array}$$

Note that $\widehat{\mu}_{(i,j)}$ is a loop in (X, x_0), in particular it defines an element in the group $\pi_1(X, x_0)$.



Next note that it follows easily from the description of inverses in the π_1-Group Proposition 11.5 that for any $(i,j) \in \{1,\dots,s\} \times \{0,1,2\}$ we have the following equality in $\pi_1(X, x_0)$:

$$(*) \qquad [\widehat{\mu}_{(i,j)}]^{-1} \coloneqq [\beta_{\text{starting point of } \overline{\mu}_{(i,j)}} * \overline{\mu}_{(i,j)} * \overline{\beta}_{\text{endpoint of } \overline{\mu}_{(i,j)}}].$$

Finally note that in $\pi_1(X, x_0)_{\text{ab}}$ we have the following equalities:

$$0 = [e] \underset{\uparrow}{=} \prod_{j=1}^{s} [\beta_{\text{starting point of } \mu_{(j,0)}} * (\overbrace{\mu_{(j,0)} * \overline{\mu}_{(j,1)} * \mu_{(j,2)}}^{\text{loop in the starting point of } \mu_{(j,0)}})^{n_j} * \overline{\beta}_{\text{starting point of } \mu_{(j,0)}}]$$

since $\mu_{(j,0)} * \overline{\mu}_{(j,1)} * \mu_{(j,2)}$ is null-homotopic

$$\underset{\uparrow}{=} \prod_{j=1}^{s} [\widehat{\mu}_{(j,0)}]^{n_{(j,0)}} \cdot [\widehat{\mu}_{(j,1)}]^{n_{(j,1)}} \cdot [\widehat{\mu}_{(j,2)}]^{n_{(j,2)}} \underset{\uparrow}{=} \prod_{a \in A = \{1,\dots,s\} \times \{0,1,2\}} [\widehat{\mu}_a]^{n_a}$$

follows from inserting null-homotopic paths $\overline{\beta}_P * \beta_P$ together with the above equality $(*)$ and the fact that $n_{(j,k)} = (-1)^k \cdot n_j$

since $\pi_1(X, x_0)_{\text{ab}}$ is abelian

$$\underset{\uparrow}{=} [f] \cdot \prod_{a \in A_+} [\widehat{\mu}_a]^{n_a} \cdot [\widehat{\mu}_{\theta(a)}]^{n_{\theta(a)}} \underset{\uparrow}{=} [f] \cdot \prod_{a \in A_+} [\widehat{\mu}_a]^{n_a} \cdot [\widehat{\mu}_a]^{-n_a} = [f].$$

since $\{1,\dots,k\} \times \{0,1,2\} = \{b\} \sqcup A_+ \sqcup \theta(A_+)$, since $\mu_b = f$, since $n_b = 1$ and since $\beta_{x_0} = \text{const}_{x_0}$

since for $a \in A_+$ we have $\mu_{\theta(a)} = \mu_a$ and we have $n_{\theta(a)} = -n_a$ ■

58.2. **Lens spaces.** For most topological spaces for which we computed the fundamental group we also determined the (first) homology groups. So we do not really gain much new information from the Hurewicz Theorem 58.5. But there is one rather interesting nugget, namely the lens spaces $L(p,q) = S^3/\sim$ which we introduced on page 212:

Corollary 58.6. (Lens Space-H$_1$-Corollary) For any coprime $p, q \in \mathbb{N}$ we have

$$\mathrm{H}_1(\text{lens space } L(p,q)) \cong \mathbb{Z}_p.$$

Proof. We have

$$\mathrm{H}_1(L(p,q)) \underset{\uparrow}{\cong} \pi_1(L(p,q)) \underset{\uparrow}{\cong} \mathbb{Z}_p.$$

by the Hurewicz Theorem 58.5 — by the Lens Space-π_1-Proposition 15.2 ■

CHAPTER 59

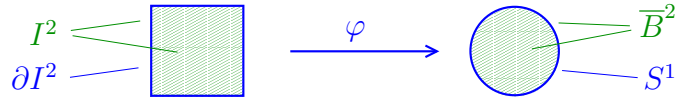
The Hurewicz Theorem in higher dimensions

The main goal of this chapter is to formulate and prove a suitable high-dimensional analogue of the Hurewicz Theorem 58.5 from Chapter 58.

59.1. **The Hurewicz homomorphism in higher dimensions.** The Hurewicz Theorem 58.5 evidently raises the question whether there is a connection between higher homotopy groups and higher homology groups. As in the 1-dimensional case we first try to find a promising map from homotopy groups to homology groups.

We start out with introducing the following notation:

Notation.
(1) We write $I := [0,1]$.
(2) Given $n \in \mathbb{N}_0$ we denote by $I^n \subset \mathbb{R}^n$ the cube in $\mathbb{R}^n$.
(3) We denote by ∂I^n the boundary of I^n viewed as a subset of $\mathbb{R}^n$.
(4) Let $\varphi\colon (I^n, \partial I^n) \to (\overline{B}^n, S^{n-1})$ be the homeomorphism from page 381.
(5) Let $[\overline{B}^n] \in \mathrm{H}_n(\overline{B}^n, S^{n-1})$ be the standard generator as defined on page 593.[480] We refer to $[I^n] := (\varphi^{-1})_*([\overline{B}^n]) \in \mathrm{H}_n(I^n, \partial I^n)$ as the standard generator of $\mathrm{H}_n(I^n, \partial I^n)$.



Now comes the actual definition we are interested in.

Definition. Let X be a topological space, let $x_0 \in X$ and let $n \geq 1$. Note that it follows from the LES-of-Pair Proposition 38.3 together with the Singleton-H_*-Lemma 35.12 that the natural map $\text{я}\colon \mathrm{H}_n(X) \to \mathrm{H}_n(X, \{x_0\})$ is an isomorphism. We refer to the map

$$\begin{array}{rcccc} \Phi_{(X,x_0)}\colon \pi_n(X,x_0) & \to & \mathrm{H}_n(X,\{x_0\}) & \xrightarrow[\cong]{\text{я}^{-1}} & \mathrm{H}_n(X) \\ [f\colon (I^n,\partial I^n) \to (X,x_0)] & \mapsto & f_*([I^n]) & & \end{array}$$

as the Hurewicz homomorphism $\Phi_{(X,x_0)}\colon \pi_n(X,x_0) \to \mathrm{H}_n(X)$.[481]

Remark. It follows rather easily from the explicit description of the standard generator of $\mathrm{H}_1(\overline{B}^1, S^0)$ on page 593 and the fact that the homeomorphism $\varphi\colon I^1 \to \overline{B}^1$ from page 381 is the map $t \mapsto 2t-1$, that for $n=1$ the above definition of the Hurewicz homomorphism agrees with the definition given on page 734. □

[480] For the most part the precise choice of the generator of $\mathrm{H}_n(\overline{B}^n, S^{n-1})$ is irrelevant.
[481] It follows from the Homotopy-H_*-Proposition 36.5 that the Hurewicz homomorphism is well-defined.

In the next proposition, which can be viewed as an analogue of the Hurewicz Homomorphism Proposition 58.1, we show in particular that the Hurewicz homomorphism is in fact a homomorphism:

Proposition 59.1. (Hurewicz Homomorphism Proposition) Let $n \in \mathbb{N}$.

(1) For any pointed topological space (X, x_0) the Hurewicz homomorphism

$$\Phi_{(X,x_0)} \colon \pi_n(X, x_0) \;\to\; \mathrm{H}_n(X)$$

is indeed a homomorphism.

(2) Let X be a topological space and let $\gamma \colon I \to X$ be a path from a point x_0 to some point x_1. The following diagram commutes:

$$\begin{array}{ccc} \pi_n(X, x_1) & \xrightarrow{\Phi_{(X,x_1)}} & \mathrm{H}_n(X) \\ {\scriptstyle \gamma_*}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ \pi_n(X, x_0) & \xrightarrow{\Phi_{(X,x_0)}} & \mathrm{H}_n(X). \end{array}$$

Here the vertical map to the left is defined in the Change-of-Base Point Propositions 11.6 (1) and 28.4.

(3) Let $\mathcal{PTop}$ be the category of pointed topological spaces and let $\mathcal{Group}$ be the category $\mathcal{Group}$ of groups. The Hurewicz homomorphisms define a natural transformation from the functor

$$\begin{array}{ccc} \mathcal{PTop} & \to & \mathcal{Group} \\ (X, x_0) & \mapsto & \pi_n(X, x_0) \end{array} \quad \text{to the functor} \quad \begin{array}{ccc} \mathcal{PTop} & \to & \mathcal{Group} \\ (X, x_0) & \mapsto & \mathrm{H}_n(X). \end{array}$$

(4) Let (X, x_0) be a pointed topological space. If we take the point of view from page 381, i.e. if we view elements in the n-th homotopy group $\pi_n(X, x_0)$ as homotopy classes of continuous maps $(S^n, *) \to (X, x_0)$, then the Hurewicz homomorphism is given by the map

$$\begin{array}{ccc} \pi_n(X, x_0) & \to & \mathrm{H}_n(X) \\ [f \colon (S^n, *) \to (X, x_0)] & \mapsto & f_*([S^n]) \end{array}$$

where $[S^n] \in \mathrm{H}_n(S^n)$ is the standard generator as defined on page 540.

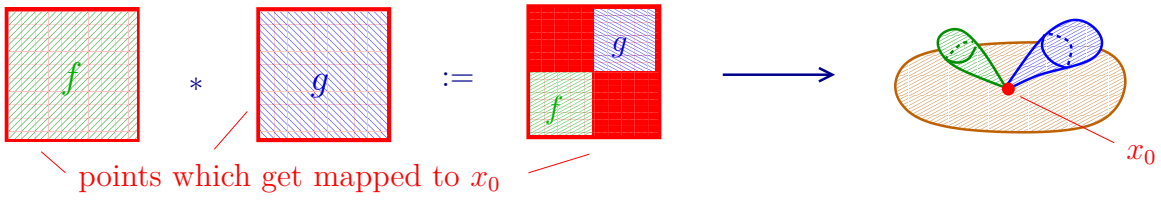
The proof of the Hurewicz Homomorphism Proposition 59.1 will require the remainder of this section. The reader who instead wants to see first applications can jump to Section 59.2.

The key ingredient in the proof of the Hurewicz Homomorphism Proposition 59.1 (1) is the following technical lemma:

Lemma 59.2. (Multiplication–Addition Lemma) Let (X, x_0) be a pointed topological space, let $n \in \mathbb{N}$, let $\omega \in \mathrm{H}_n(I^n, \partial I^n)$ and let $f, g \colon (I^n, \partial I^n) \to (X, x_0)$ be two continuous maps. Then

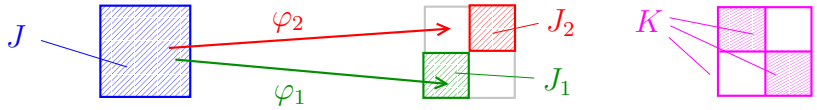
$$f_*(\omega) + g_*(\omega) \;=\; (f \underset{\uparrow}{*} g)_*(\omega) \;\in\; \mathrm{H}_n(X, x_0).$$

product of f and g as defined on page 378



Proof of the Multiplication–Addition Lemma 59.2. We introduce the following notation:

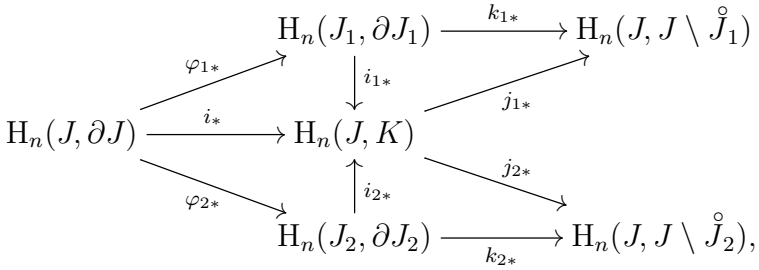
(1) We write $J \coloneqq I^n$, $J_1 \coloneqq [0, \frac{1}{2}]^n$ and $J_2 \coloneqq [\frac{1}{2}, 1]^n$.
(2) We write $K \coloneqq J \setminus (\mathring{J}_1 \cup \mathring{J}_2)$.
(3) For $k = 1, 2$ we denote by $i_k \colon (J_k, \partial J_k) \to (J, K)$ the inclusion map.
(4) We denote by $\varphi_k \colon (J, \partial J) \to (J_k, \partial J_k)$ the obvious homeomorphism given by reparametrizing the coordinates.



Claim. Given any $\omega \in \mathrm{H}_n(I^n, \partial I^n)$ we have

$$i_*(\omega) \;=\; i_{1*}(\varphi_{1*}(\omega)) + i_{2*}(\varphi_{2*}(\omega)) \;\in\; \mathrm{H}_n(J, K).$$

Proof. Let $\omega \in \mathrm{H}_n(I^n, \partial I^n)$. For $s = 1, 2$ we write $\omega_s = \varphi_{s*}(\omega) \in \mathrm{H}_n(J_s, \partial J_s)$. To prove the desired equality we consider the following diagram:



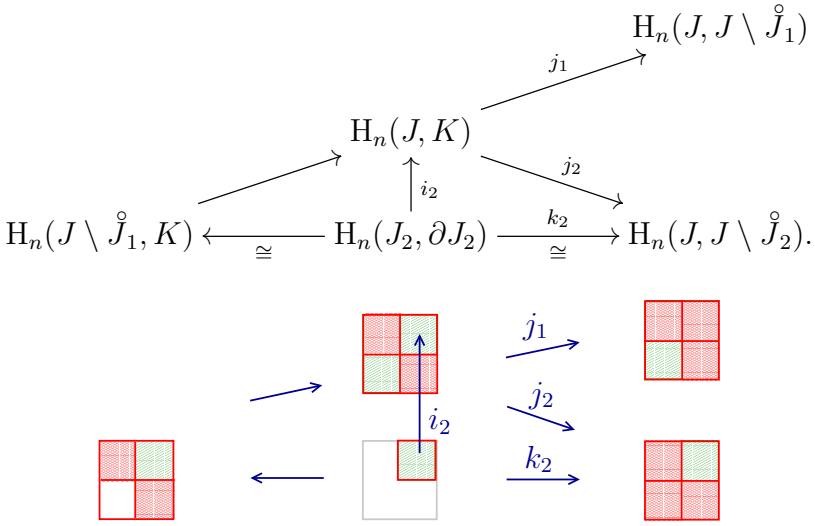
where the five maps i, j_1, j_2, k_1, k_2 are the obvious inclusion maps of pairs of topological spaces. Note that the two triangles to the right commute by the functoriality of relative homology groups. Furthermore note that one can easily verify that the two maps $j_s \circ i$ and $k_s \circ \varphi_s \colon (J, \partial J) \to (J, J \setminus \mathring{J}_s)$ are homotopic as maps of pairs of topological spaces. Thus it follows from the Homotopy-H$_*$-Proposition 38.6 that the top and bottom parallelogram commute.[482] Next note that

$$\begin{aligned} j_{1*}(i_*(\omega)) &\overset{\text{since the top parallelogram commutes}}{=} (k_{1*} \circ \varphi_{1*})(\omega) = k_{1*}(\omega_1) \\ &\overset{\text{since the top right triangle commutes}}{=} j_{1*}(i_{1*}(\omega_1)) \\ &\overset{\text{since } (j_1 \circ i_2)_* \colon \mathrm{H}_n(J_2, \partial J_2) \to \mathrm{H}_n(J, J \setminus \mathring{J}_1) \text{ is zero we have } j_{1*} \circ i_{2*} = (j_1 \circ i_2)_* = 0}{=} j_{1*}(i_{1*}(\omega_1)) + j_{1*}(i_{2*}(\omega_2)) \\ &= j_{1*}(i_{1*}(\omega_1) + i_{2*}(\omega_2)). \end{aligned}$$

and similarly we show that $j_{2*}(i_*(\omega)) = j_{2*}(i_{1*}(\omega_1)) + j_{2*}(i_{2*}(\omega_2)) = j_{2*}(i_{1*}(\omega_1) + i_{2*}(\omega_2))$.

[482] The triangles to the left do not commute, but this is irrelevant.

To prove the desired equality $i_*(\omega) = i_{1*}(\varphi_{1*}(\omega)) + i_{2*}(\varphi_{2*}(\omega))$ it now remains to show that $\ker(j_{1*}) \cap \ker(j_{2*}) = 0$. We consider the following diagram:



We make the following clarifications and observations:

(1) The unnamed maps are induced by the obvious inclusions of pairs of topological spaces.
(2) The diagram commutes by the functoriality of relative homology.
(3) The diagonal sequence is exact by the LES-of-Triple Proposition 38.2.
(4) It follows from the CW-Excision Theorem 40.4 that the two bottom horizontal maps are isomorphisms.
(5) It now follows from some mild diagram chasing that $\ker(j_{1*}) \cap \ker(j_{2*}) = 0$. ⊞

Now we can prove the actual statement of the lemma.

Claim 2. For any $\omega \in \mathrm{H}_n(I^n, \partial I^n)$ we have the following equality:

$$f_*(\omega) + g_*(\omega) \;=\; (f * g)_*(\omega) \;\in\; \mathrm{H}_n(X, x_0).$$

Proof. The claim follows from the following equalities:

$$\begin{array}{rcl} (f*g)_*(\omega) & \overset{\text{since } (f*g)(K)=\{x_0\}}{\underset{\downarrow}{=}} & (f*g)_*(i_*(\omega)) \\ & \overset{\text{by Claim 1}}{\underset{\downarrow}{=}} & (f*g)_*(i_{1*}(\varphi_{1*}(\omega)) + i_{2*}(\varphi_{2*}(\omega))) \\ & = & (f*g)_*(i_{1*}(\varphi_{1*}(\omega)) + (f*g)_*(i_{2*}(\varphi_{2*}(\omega))) \\ & = & f_*(\omega) + g_*(\omega). \\ & & \underset{\uparrow}{} \\ & & \text{since } (f*g)\circ i_1 \circ \varphi_1 = f \text{ and since } (f*g)\circ i_2 \circ \varphi_2 = g \end{array}$$

■

The proof of the Hurewicz Homomorphism Proposition 59.1 is now, except for one technicality, almost a breeze:

Proof of the Hurewicz Homomorphism Proposition 59.1.

(1) Let $[I^n] \in \mathrm{H}_n(I^n, \partial I^n)$ be the standard generator. It follows from the Multiplication–Addition Lemma 59.2 that for all continuous maps $f, g\colon (I^n, \partial I^n) \to (X, x_0)$ we have the desired equality $f_*([I^n]) + g_*([I^n]) = (f * g)_*([I^n]) \in \mathrm{H}_n(X, x_0)$.
(2) We start out with the following technical claim, which later will allow us to describe the Hurewicz homomorphism in an alternative way:

Claim. Let $g\colon (I^n, \partial I^n) \to (X, x)$ be a continuous map. Note that g descends to a continuous map $\overline{g}\colon I^n/\partial I^n \to X$. We denote by $p\colon X \to X/\{x\}$ the natural projection. With this notation we obtain the following diagram:

$$\begin{array}{ccccc}
\mathrm{H}_n(I^n, \partial I^n) & \xrightarrow{g_*} & \mathrm{H}_n(X, \{x\}) & \xleftarrow{\text{я}} & \mathrm{H}_n(X) \\
\downarrow{\scriptstyle \text{ю}} & & \downarrow{\scriptstyle \text{ю}} & \swarrow{\scriptstyle \text{ж}_{(X,\{x\})}} & \uparrow{\scriptstyle \text{ж}} \\
\widetilde{\mathrm{H}}_n(I^n/\partial I^n) & \xrightarrow{(p\circ\overline{g})_*} & \widetilde{\mathrm{H}}_n(X/\{x\}) & \xleftarrow{p_*} & \widetilde{\mathrm{H}}_n(X).
\end{array}$$

$$\overline{g}_*$$

We claim that this diagram commutes.

Proof. It follows easily from the definitions that the square to the left commutes and that the triangle to the bottom right commutes. The triangle to the top right commutes by definition, see page 500. The bottom part commutes by the functoriality of reduced homology. ⊞

Now let $f\colon (I^n, \partial I^n) \to (X, x_1)$ be a continuous map and let $\gamma\colon I \to X$ be a path from a point x_0 to some point x_1. As we mentioned above, f descends to a continuous map $\overline{f}\colon I^n/\partial I^n \to X$ and f^γ descends to a continuous map $\overline{f^\gamma}\colon I^n/\partial I^n \to X$.



One can easily write down an explicit homotopy $F\colon I^n \times [0,1] \to X$ from $F_0 = f$ to $F_1 = f^\gamma$ such that for each $t \in [0,1]$ the image $F_t(\partial I^n)$ consists of a single point. It follows from the Homotopy Quotient Theorem 8.5 that F descends to a homotopy $\overline{F}\colon (I^n/\partial I^n) \times [0,1] \to X$ from $\overline{f}$ to $\overline{f^\gamma}$.

We now obtain the following equalities in $\mathrm{H}_n(X, \{x_1\})$:

$$\begin{array}{lllll}
\text{я}(\Phi_{(X,x_1)}([f])) & \overset{\text{by definition of } \Phi_{(X,x_1)}}{\underset{\uparrow}{=}} & f_*([I^n]) & \overset{\text{since the diagram commutes}}{\underset{\uparrow}{=}} & \text{ж}_{(X,\{x\})}(\overline{f}_*(\text{ю}([I^n]))) \\
& = & f^\gamma_*([I^n]) & = & \text{я}(\Phi_{(X,x_0)}([f^\gamma])) \\
& & \text{since the diagram commutes} & & \text{by definition of } \Phi_{(X,x_0)}
\end{array}$$

$$\begin{array}{ll}
\overset{\text{by the Homotopy-H}_*\text{-Proposition 36.5 since } \overline{f} \text{ and } \overline{f^\gamma} \text{ are homotopic}}{\underset{\uparrow}{=}} & \text{ж}_{(X,\{x\})}(\overline{f}^\gamma_*(\text{ю}([I^n]))) \\
= & \text{я}(\Phi_{(X,x_0)}(\gamma_*([f]))). \\
& \text{by definition of } \gamma_*
\end{array}$$

Since $n \geq 1$ we obtain from the LES-of-Pair Proposition 38.3 together with the Singleton-H$_*$-Lemma 35.12 that я is an isomorphism. It follows that the diagram given in the statement is commutative.

(3) This statement follows immediately from the functoriality of relative homology groups.

(4) This statement is a straightforward consequence of the Standard Generator-of-$\overline{B}^n$-Lemma 47.8. ■

59.2. **First applications of the Hurewicz homomorphism.** The following lemma gives in particular an affirmative answer to Question 28.11 (1).

Lemma 59.3. (Hurewicz Sphere Epimorphism Lemma) For any $n \in \mathbb{N}$ the Hurewicz homomorphism $\Phi_{(S^n,*)}\colon \pi_n(S^n, *) \to \mathrm{H}_n(S^n) \cong \mathbb{Z}$

sends $[\mathrm{id}_{S^n}\colon S^n \to S^n] \in \pi_n(S^n)$ to the standard generator of $\mathrm{H}_n(S^n)$, in particular this Hurewicz homomorphism is an epimorphism.

Proof. We denote by $[S^n] \in \mathrm{H}_n(S^n)$ the standard generator. Note that

$$\Phi_{(S^n,*)}(\underbrace{[\mathrm{id}_{S^n}\colon S^n \to S^n]}_{\in \pi_n(S^n,*)}) \underset{\text{by the Hurewicz Homomorphism Proposition 59.1 (4)}}{=} (\mathrm{id}_{S^n})_*([S^n]) = [S^n].$$

Thus we see that the standard generator of $\mathrm{H}_n(S^n) \cong \mathbb{Z}$ lies in the image of the Hurewicz homomorphism. But this implies that the Hurewicz homomorphism $\Phi\colon \pi_n(S^n,*) \to \mathrm{H}_n(S^n)$ is an epimorphism. ∎

The fairly simple-minded Hurewicz Sphere Epimorphism Lemma 59.3 actually already turns out to be quite useful. As an example, recall that on page 137 we showed that we can view the real projective space $\mathbb{R}\mathrm{P}^m$ as a subspace of the projective space $\mathbb{R}\mathrm{P}^n$. Now we can formulate the following application of the Hurewicz Sphere Epimorphism Lemma 59.3.

Proposition 59.4. ($\mathbb{R}\mathbf{P}^n$-Non-Retraction Proposition) If $1 \leq m < n$, then the subspace $\mathbb{R}\mathrm{P}^m \subset \mathbb{R}\mathrm{P}^n$ is not a retract of $\mathbb{R}\mathrm{P}^n$.

Proof. Note that

$$\pi_m(\mathbb{R}\mathrm{P}^m) \underset{\text{by the Covering-}\pi_n\text{-Isomorphism Proposition 28.10}}{\cong} \pi_m(S^m) \underset{\text{by the Hurewicz Epimorphism Sphere Lemma 59.3}}{\neq} 0 \quad \text{and} \quad \pi_m(\mathbb{R}\mathrm{P}^n) \underset{\text{by the Covering-}\pi_n\text{-Isomorphism Proposition 28.10}}{\cong} \pi_m(S^n) \underset{\text{follows from } m \neq n \text{ and the Sphere–Low-}\pi_n\text{-Proposition 28.8}}{=} 0.$$

With this calculation, and the fact that π_m is functorial, the argument of the Circle–Non Retract Proposition 13.3 can be easily modified to show that $\mathbb{R}\mathrm{P}^m$ is not a retract of $\mathbb{R}\mathrm{P}^n$. ∎

The $\mathbb{R}\mathrm{P}^n$-Non-Retraction Proposition 59.4 raises the following question:

Question 59.5. Let $m < n$. Is the complex projective space $\mathbb{C}\mathrm{P}^m$ a retract of $\mathbb{C}\mathrm{P}^n$?

59.3. **The Hurewicz Theorem.** It is natural to ask whether, given a path-connected topological space X and $n \geq 2$, the Hurewicz homomorphism

$$\Phi_{(X,x_0)}\colon \pi_n(X,x_0) \to \mathrm{H}_n(X)$$

is necessarily an isomorphism. Since $n \geq 2$ we know by the π_n-Abelian Proposition 28.3 that the group $\pi_n(X,x_0)$ is abelian. So this time, in contrast to the situation for π_1 which we discussed on page 734, there is no obvious obstruction to this map being a monomorphism.

Nonetheless, for better or worse, browsing through our examples shows that the Hurewicz homomorphism is in general neither an epimorphism nor a monomorphism:

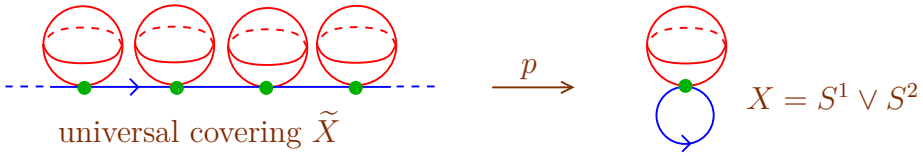
(1) In the Torus-H_*-Proposition 45.5 (1) we showed that $\mathrm{H}_2(S^1 \times S^1) \cong \mathbb{Z}$ is non-zero, whereas on page 387 we showed that $\pi_2(S^1 \times S^1) = 0$. This shows that the Hurewicz homomorphism $\pi_2(S^1 \times S^1) \to \mathrm{H}_2(S^1 \times S^1)$ is not an epimorphism.

(2) Here are two examples that show that in general the Hurewicz homomorphism is not a monomorphism:

(a) We consider $X = \mathbb{R}\mathrm{P}^2$. By the Covering-$\pi_n$-Isomorphism Proposition 28.10 and the Hurewicz Sphere Epimorphism Lemma 59.3 we know that $\pi_2(\mathbb{R}\mathrm{P}^2) \cong \pi_2(S^2)$

is non-trivial. On the other hand we saw in the $\mathbb{R}\mathrm{P}^n$-H_*-Proposition 47.10 that $\mathrm{H}_2(\mathbb{R}\mathrm{P}^2) = 0$.

(b) We consider $X = S^1 \vee S^2$. It follows immediately from the Wedge-H_*-Proposition 41.10 and the Sphere-H_*-Proposition 41.2 that $\mathrm{H}_2(S^1 \vee S^2) \cong \mathbb{Z}$. On the other hand we know by the Covering-π_n-Isomorphism Proposition 28.10 and the discussion on page 374 that $\pi_2(S^1 \vee S^2)$ is isomorphic to $\pi_2(\widetilde{X})$, where $\widetilde{X}$ is the real line "with infinitely many 2-spheres attached".



Using the projections onto the various spheres and using the Hurewicz Sphere Epimorphism Lemma 59.3 one can now show that $\pi_2(\widetilde{X})$ admits an epimorphism onto $\mathbb{Z}^\infty$. It follows that there is no monomorphism from $\pi_2(X) = \pi_2(\widetilde{X})$ into $\mathrm{H}_2(X) \cong \mathbb{Z}$. In particular the Hurewicz homomorphism $\pi_2(S^1 \vee S^2) \to \mathrm{H}_2(S^1 \vee S^2)$, whatever it is precisely, cannot be a monomorphism.

The above examples show that we need to put some hypotheses on X for a higher Hurewicz homomorphism to be an isomorphism. Some quick thinking leads us to the following definition:

Definition. Let $k \in \mathbb{N}_0$. We say that a topological space X is k-connected if $\pi_i(X) = 0$ for $i = 0, \dots, k$.[483] [484]

The main goal of this chapter is to prove the following theorem which was first proved by Witold Hurewicz [**Hur35b**]. It can be viewed as the high-dimensional analogue of the original Hurewicz Theorem 58.5.

Theorem 59.6. (Hurewicz Theorem) Let (X, x_0) be a pointed topological space. Furthermore let $n \in \mathbb{N}_{\geq 2}$. If X is $(n-1)$-connected, then the following two statements hold:

(1) We have $\widetilde{\mathrm{H}}_k(X) = 0$ for $k = 0, \dots, n-1$.

(2) The Hurewicz homomorphism $\Phi_{(X,x_0)} \colon \pi_n(X, x_0) \to \mathrm{H}_n(X)$ is an isomorphism.

Using the Hurewicz Theorem 59.6 we can now determine some homotopy groups that had been eluding us for a long time:

Corollary 59.7. ($\pi_n(S^n)$–Corollary) For any $n \in \mathbb{N}$ we have

$$\pi_n(S^n) \;=\; \mathbb{Z} \cdot [\mathrm{id}_{S^n}].$$

Proof. Let $n \in \mathbb{N}$. By the Sphere–Low-π_n-Proposition 28.8 we know that S^n is $(n-1)$-connected. Thus the corollary follows immediately from the Hurewicz Theorems 58.5

[483] Recall that according to our definition on page 82 we write $\pi_0(X) = 0$ if and only if X is path-connected and *non-empty*. Furthermore note that by the definition on page 186 a topological space X is 1-connected if and only if it is simply connected.

[484] Note that a k-connected space is in particular 0-connected and thus path-connected. It follows from the Change-of-Base Point Proposition 28.4 that we can ignore base points.

[484] Here, following the discussion on page 381, we view $\pi_n(X, x_0)$ as the set of homotopy classes of continuous maps $(S^n, *) \to (X, x_0)$.

and 59.6, the discussion of the Hurewicz homomorphism in the Hurewicz Sphere Epimorphism Lemma 59.3, and the fact that by the Sphere-H_*-Proposition 41.2 we know that $\mathrm{H}_n(S^n) = \mathbb{Z} \cdot [S^n]$. ■

Corollary 59.8. ($\pi_n(\mathbb{R}\mathbf{P}^n)$–Corollary) For any $n \in \mathbb{N}$ we have

$$\pi_n(\mathbb{R}\mathrm{P}^n, *) \;=\; \mathbb{Z} \cdot \big[p\colon S^n \xrightarrow{x \mapsto [x]} \mathbb{R}\mathrm{P}^n\big].$$

Proof. This corollary follows immediately from the fact, established on page 217, that the projection $p\colon S^n \to \mathbb{R}\mathrm{P}^n = (\mathbb{R}^{n+1} \setminus \{0\})/(\mathbb{R} \setminus \{0\})$ is a covering, the Covering-π_n-Isomorphism Proposition 28.10 and the $\pi_n(S^n)$–Corollary 59.7. ■

Corollary 59.9. ($\pi_2(\mathbb{C}\mathbf{P}^n)$–Corollary) Let $n \in \mathbb{N} \cup \{\infty\}$. Let $f\colon S^2 \to \mathbb{C}\mathrm{P}^1$ be the standard homeomorphism from page 139 and let $i\colon \mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^n$ be the continuous map that is given by $[x] \mapsto [(x, 0)]$. With this notation we have

$$\pi_2(\mathbb{C}\mathrm{P}^n) \;=\; \mathbb{Z} \cdot [i \circ f\colon S^2 \to \mathbb{C}\mathrm{P}^n].$$

Proof. Let $n \in \mathbb{N} \cup \{\infty\}$. We consider the following diagram:

$$\begin{array}{ccccc}
\pi_2(S^2, *) & \xrightarrow[\cong]{f_*} & \pi_2(\mathbb{C}\mathrm{P}^1, *) & \xrightarrow{i_*} & \pi_2(\mathbb{C}\mathrm{P}^n, *) \\
\cong \downarrow (\Phi_{(S^2,*)} & & \cong \downarrow \Phi_{(\mathbb{C}\mathrm{P}^1,*)} & & \cong \downarrow \Phi_{(\mathbb{C}\mathrm{P}^n,*)} \\
\mathrm{H}_2(S^2) & \xrightarrow[\cong]{f_*} & \mathrm{H}_2(\mathbb{C}\mathrm{P}^1) & \xrightarrow[\cong]{i_*} & \mathrm{H}_2(\mathbb{C}\mathrm{P}^n).
\end{array}$$

We make the following observations:

(1) It follows from the Hurewicz Homomorphism Proposition 59.1 that the diagram commutes.
(2) The left horizontal maps are isomorphisms since f is a homeomorphism.
(3) By the Projective Space-π_1-Proposition 29.3 and the discussion on page 419 we know that $\pi_1(S^2) = \pi_1(\mathbb{C}\mathrm{P}^1) = \pi_1(\mathbb{C}\mathrm{P}^n) = 0$.
(4) It follows from (3) and the Hurewicz Theorem 59.6 that the vertical maps are isomorphisms.
(5) It follows from the discussion on page 592 that the bottom right horizontal map is an isomorphism.
(6) It follows from (1), (4) and (5) that the top right horizontal map is an isomorphism.
(7) By the $\pi_n(S^n)$–Corollary 59.7 we know that $\pi_2(S^2, *) = \mathbb{Z} \cdot [\mathrm{id}_{S^n}]$.

It follows from the above discussion that $\pi_2(\mathbb{C}\mathrm{P}^n, *) = \mathbb{Z} \cdot [i \circ f\colon S^2 \to \mathbb{C}\mathrm{P}^n]$. ■

On several occasions we will also need the following straightforward corollary to the Hurewicz Theorem 59.6.

Corollary 59.10. (Hurewicz Corollary) Let (X, x_0) be a pointed topological space and let $n \in \mathbb{N}_{\geq 2}$. We suppose that X is *simply connected*. If given every $k \in \{2, \ldots, n-1\}$ we have $\pi_k(X, x_0) = 0$ or $\mathrm{H}_k(X) = 0$, then the following two statements hold:
(1) For $k = 1, \ldots, n-1$ we have $\pi_k(X, x_0) = \mathrm{H}_k(X) = 0$.
(2) The Hurewicz homomorphism $\pi_n(X, x_0) \to \mathrm{H}_n(X)$ is an isomorphism.

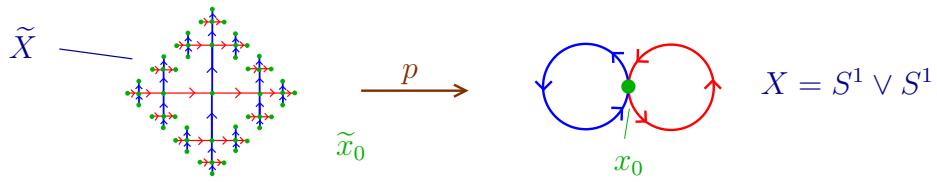
Proof. We prove the corollary by induction on n. First consider the case $n = 2$. Since X is simply connected we obtain from the Hurewicz Theorem 58.5 that $\mathrm{H}_1(X) = 0$. Furthermore it follows from the Hurewicz Theorem 59.6 that $\Phi_{(X,x_0)} \colon \pi_2(X, x_0) \to \mathrm{H}_2(X)$ is an isomorphism.

Now suppose the statement holds for some n with $n \geq 2$. We need to show that the statement holds for $n + 1$. Thus we now suppose that for every $k \in \{2, \ldots, n\}$ we have $\pi_k(X, x_0) = 0$ or $\mathrm{H}_k(X) = 0$. By induction we know that the homotopy and homology groups vanish up to degree $n - 1$. Thus we can apply the Hurewicz Theorem 59.6 which says that $\pi_n(X)$ and $\mathrm{H}_n(X)$ are isomorphic. Since one is zero, both are zero. Hence we can apply the Hurewicz Theorem 59.6 once more to see that the Hurewicz homomorphism $\pi_{n+1}(X, x_0) \to \mathrm{H}_{n+1}(X)$ is an isomorphism. ∎

As a nice application to the Hurewicz Corollary 59.10 we can prove the following lemma.[485]

Lemma 59.11. (1-Complex-π_n-Lemma) Let X be a 1-dimensional CW-complex and let $x_0 \in X$ be base point. Then $\pi_n(X, x_0) = 0$ for every $n \geq 2$.

Proof. Let X be a 1-dimensional CW-complex, let $n \geq 2$ and let $x_0 \in X$ be a base point. Without loss of generality we can assume that X is path-connected. We denote by $\widetilde{X}$ the universal covering of X and we pick $\widetilde{x}_0 \in p^{-1}(x_0)$.



Let $n \geq 2$. We calculate that

$$0 \underset{\uparrow}{=} \mathrm{H}_n(\widetilde{X}) \xleftarrow[\text{isomorphism by the Hurewicz Corollary 59.10, since } \widetilde{X} \text{ is simply connected}]{\Phi_{(\widetilde{X},\widetilde{x}_0)}} \pi_n(\widetilde{X}, \widetilde{x}_0) \xleftarrow[\text{isomorphism by the Covering-}\pi_n\text{-Isomorphism}]{p_*} \pi_n(X, x_0).$$

by the CW-Complex Covering Proposition 32.1 we know that $\widetilde{X}$ is a 1-dimensional CW-complex, it follows that $\mathrm{H}_n(\widetilde{X}) = 0$ for all $n \in \mathbb{N}_{\geq 2}$

∎

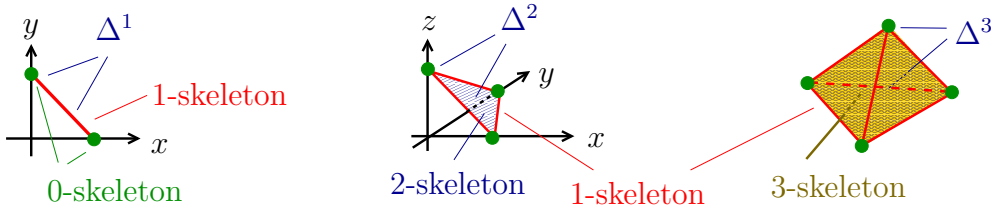
59.4. The proof of the Hurewicz Theorem 59.6.

The proof of the Hurewicz Theorem 59.6 requires some more preparations.

Convention. Let $k \in \mathbb{N}_0$. We view Δ^k as a CW-complex where the i-skeleton is given by

$$(\Delta^k)^i := \{(t_0, \ldots, t_k) \in \Delta^k \mid \text{at least } k - i \text{ coordinates are zero}\}.$$

Note that the $(k-1)$-skeleton of Δ^k is given by $\partial\Delta^k$.



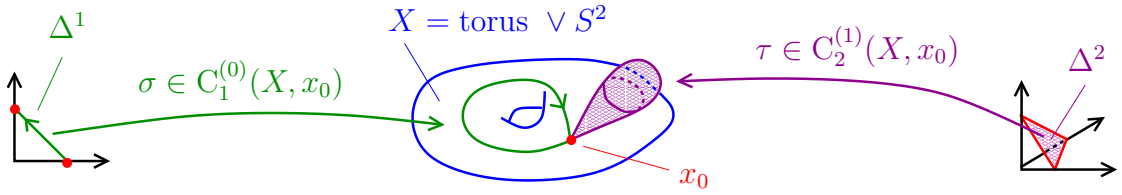
[485] The content of this lemma is the same as for the 1-Complex-π_n-Lemma 32.2, but the proof is quite different.

Definition. Let (X, x_0) be a pointed topological space and let $n \in \mathbb{N}_0$. Given $k \in \mathbb{N}_0$ we define[486]

$$\mathrm{C}_k^{(n)}(X, x_0) \;:=\; \begin{array}{l}\text{the subgroup of the relative chain group } \mathrm{C}_k(X, \{x_0\}) \text{ generated by} \\ \text{all singular simplices } \sigma\colon \Delta^k \to X \text{ with } \sigma(n\text{-skeleton of } \Delta^k) \subset \{x_0\}.\end{array}$$

It follows almost immediately from the definitions that for fixed $n \in \mathbb{N}_0$ these groups define a subcomplex of $\mathrm{C}_*(X, \{x_0\})$. In particular we obtain corresponding homology groups $\mathrm{H}_k^{(n)}(X, x_0)$. The inclusions $\mathrm{C}_*^{(n)}(X, x_0) \to \mathrm{C}_*(X, \{x_0\})$ are evidently chain maps and thus they define natural maps $\mathrm{H}_k^{(n)}(X, x_0) \to \mathrm{H}_k(X, \{x_0\})$.

Example. In the figure below we show a pointed topological space (X, x_0) and we consider an element $\sigma \in \mathrm{C}_1^{(0)}(X, x_0)$ and an element $\tau \in \mathrm{C}_2^{(1)}(X, x_0)$.



From the pictures it becomes apparent that elements in $\mathrm{C}_n^{(n-1)}(X, x_0)$ are "close" to elements in $\pi_n(X, x_0)$. In particular it seems reasonable to expect that the groups $\mathrm{H}_n^{(k)}(X, x_0)$ with $k = 0, \ldots, n-1$ "sit between" $\mathrm{H}_n(X)$ and $\pi_n(X, x_0)$. □

The following proposition is one of the two key ingredients in the proof of the Hurewicz Theorem 59.6.

Proposition 59.12. (n-Connected–Chain Group Lemma) Let (X, x_0) be a pointed topological space and let $n \in \mathbb{N}$. If X is n-connected, then for any $k \in \mathbb{N}_0$ the induced map

$$\mathrm{H}_k^{(n)}(X, x_0) \;\to\; \mathrm{H}_k(X, \{x_0\})$$

is an isomorphism.

Proof. We postpone the proof to Section 59.5. ■

Before we continue let us introduce some notations and conventions.

Notation.

(1) We write $* = (0, \ldots, 0, 1) \in S^n$.
(2) Let $\varphi\colon (I^n, \partial I^n) \to (\overline{B}^n, S^{n-1})$ be the homeomorphism from page 381.
(3) As on page 381 we use φ to make the identifications

$$\pi_n(X, x_0) \;:=\; \begin{array}{c}\text{set of homotopy classes} \\ \text{of continuous maps} \\ (I^n, \partial I^n) \to (X, x_0)\end{array} \;\overset{\text{via } \varphi}{=}\; \begin{array}{c}\text{set of homotopy classes} \\ \text{of continuous maps} \\ (\overline{B}^n, S^{n-1}) \to (X, x_0).\end{array}$$

In the remainder of this chapter we will mostly work with the description on the right.

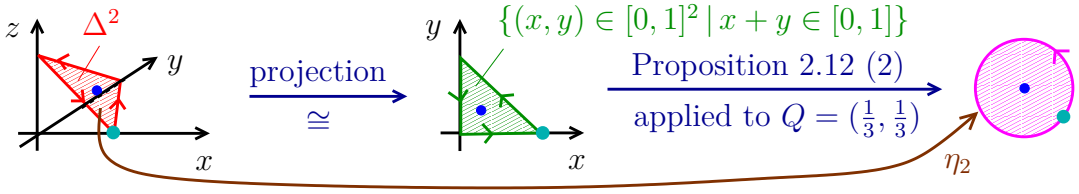
Next we need to relate the standard balls (used in the definition of homotopy groups) to the standard simplex (used in the above definition of singular homology). We do so via the following definition:

[486] Note that the notation $\mathrm{C}_k^{(n)}(X, x_0)$ instead of $\mathrm{C}_k^{(n)}(X, \{x_0\})$ is on purpose.

Notation. Let $n \in \mathbb{N}$. We consider the homeomorphism $\eta_n \colon (\Delta^n, \partial\Delta^n) \to (\overline{B}^n, S^{n-1})$ that is given by the composition of the following two maps:[487]

$$\begin{array}{rcl} \eta_n \colon \Delta^n & \xrightarrow{\cong} & \left\{(x_1, \ldots, x_n) \in [0,1]^n \;\middle|\; \sum_{i=1}^{n} x_i \in [0,1]\right\} \xrightarrow[\text{with } Q=(\frac{1}{n+1}, \ldots, \frac{1}{n+1})]{\text{Proposition 2.12 (2)}} \overline{B}^n. \\ (t_0, \ldots, t_n) & \mapsto & (t_0, \ldots, t_{n-1}) \end{array}$$

Note that η_n restricts to a homeomorphism $\eta_n|_{\partial\Delta^n} \colon \partial\Delta^n \to S^{n-1}$.



The following proposition is the other key ingredient in the proof of the Hurewicz Theorem 59.6.

Proposition 59.13. (Homotopy-via-Homology Proposition) Let $k \in \mathbb{N}$ and let (X, x_0) be a pointed $(k-1)$-connected[488] topological space.

(i) The map

$$\begin{array}{rcl} \widetilde{\Phi}_{(X,x_0)} \colon \pi_k(X, x_0) & \to & \mathrm{H}_k^{(k-1)}(X, x_0) \\ [f \colon (\overline{B}^k, S^{k-1}) \to (X, x_0)] & \mapsto & [\underbrace{f \circ \eta_k \colon \Delta^k \to X}_{\in \mathrm{C}_k^{(k-1)}(X,x_0)}] \end{array}$$

is well-defined and it is an isomorphism.

(ii) The following diagram commutes up to the sign $(-1)^k$:

$$\begin{array}{ccc} \pi_k(X, x_0) & \xrightarrow{\Phi_{(X,x_0)}} & \mathrm{H}_k(X) \\ {\scriptstyle \widetilde{\Phi}_{(X,x_0)}}\downarrow & & {\scriptstyle \text{я}}\downarrow{\scriptstyle \cong} \\ \mathrm{H}_k^{(k-1)}(X, x_0) & \xrightarrow{\iota} & \mathrm{H}_k(X, \{x_0\}), \end{array}$$

here ι is induced by the inclusion $\mathrm{C}_*^{(k-1)}(X, x_0) \to \mathrm{C}_k(X, \{x_0\})$.

(iii) The right vertical map of the diagram in (ii) is an isomorphism.

Proof. We postpone the proof of the proposition to Section 59.6. ∎

After cleverly hiding two proofs in future sections we can now turn to the proof of the Hurewicz Theorem 59.6.

Proof of the Hurewicz Theorem 59.6. Let $n \in \mathbb{N}_{\geq 2}$. Furthermore let (X, x_0) be a pointed $(n-1)$-connected topological space. Note that it suffices to prove the following claim:

Claim. For $k = 1, \ldots, n$ the Hurewicz homomorphism $\Phi_{(X,x_0)} \colon \pi_k(X, x_0) \to \mathrm{H}_k(X)$ is an isomorphism.

[487] This is precisely the homeomorphism from the Δ^n-Homeomorphism Lemma 35.3 (2).

[488] It is not clear to the author whether this hypothesis is strictly necessary.

Proof. Let $k \in \{1, \dots, n\}$. As in the Homotopy-via-Homology Proposition 59.13 we consider the following diagram:

$$\begin{array}{ccc} \pi_k(X, x_0) & \xrightarrow{\Phi_{(X,x_0)}} & \mathrm{H}_k(X) \\ {\scriptstyle \widetilde{\Phi}_{(X,x_0)}}\downarrow & & {\scriptstyle я}\downarrow{\scriptstyle \cong} \\ \mathrm{H}_k^{(k-1)}(X, x_0) & \xrightarrow[\iota]{\cong} & \mathrm{H}_k(X, \{x_0\}) \end{array}$$

We make the following observations:

(a) By the Homotopy-via-Homology Proposition 59.13 we know that the diagram commutes up to the sign $(-1)^k$ and that the two vertical maps are isomorphisms.
(b) Since X is $(n-1)$-connected and since $1 \leq k \leq n$ we obtain from the n-Connected–Chain Group Proposition 59.12 that the bottom horizontal map is an isomorphism.

It follows from (a) and (b) that the top horizontal map is an isomorphism. ■

The remainder of this chapter is taken up by the proofs of the n-Connected–Chain Group Proposition 59.12 and the Homotopy-via-Homology Proposition 59.13.

59.5. **Proof of the n-Connected–Chain Group Proposition 59.12.** For the reader's convenience we recall the statement we need to prove:

Proposition 59.12. (n-Connected–Chain Group Lemma) Let (X, x_0) be a pointed topological space and let $n \in \mathbb{N}$. If X is n-connected, then the inclusion map

$$\mathrm{C}_*^{(n)}(X, x_0) \;\to\; \mathrm{C}_*(X, \{x_0\})$$

is a chain homotopy equivalence. In particular for any $k \in \mathbb{N}_0$ the induced map

$$\mathrm{H}_k^{(n)}(X, x_0) \;\to\; \mathrm{H}_k(X, \{x_0\})$$

is an isomorphism.

Proof. Let (X, x_0) be a pointed topological space and let $n \in \mathbb{N}$. We suppose that X is n-connected. We recall the following notation from the Prism Boundary Lemma 36.9: Given $t = 0$ and $t = 1$ and $k \in \mathbb{N}_0$ we denote by

$$\begin{array}{rccc} \eta_t \colon & \Delta^k & \to & \Delta^k \times [0,1] \\ & x & \mapsto & (x, t) \end{array}$$

the obvious embedding. We start out with the following claim which basically says that singular simplices $\Delta^k \to X$ can be homotoped consistently to singular simplices that lie in $\mathrm{C}_k^{(n)}(X, x_0)$.

Claim 1. Given any $k \in \mathbb{N}_0$ and given any singular simplex $\sigma \colon \Delta^k \to X$ we can choose a continuous map $P_k(\sigma) \colon \Delta^k \times [0,1] \to X$ such that the following four statements are satisfied:

(a) We have $P_k(\sigma) \circ \eta_0 = \sigma \colon \Delta^k \to X$.
(b) The singular k-simplex $P_k(\sigma) \circ \eta_1 \colon \Delta^k \to X$ lies in $\mathrm{C}_k^{(n)}(X, x_0)$.
(c) If $\sigma \in \mathrm{C}_k^{(n)}(X, x_0)$, then $P_k(\sigma) = \sigma \circ \pi$ where $\pi \colon \Delta^k \times [0,1] \to \Delta^k$ is the projection.

(d) For any $j \in \{0, \dots, k\}$ we have

$$P_k(\sigma) \circ \underbrace{(i_j^k \times \mathrm{id}_{[0,1]})}_{\Delta^{k-1}\times[0,1]\to\Delta^k\times[0,1]} = P_{k-1}(\underbrace{\sigma \circ i_j^k}_{\Delta^{k-1}\to X}) \colon \Delta^{k-1} \times [0,1] \to X.$$

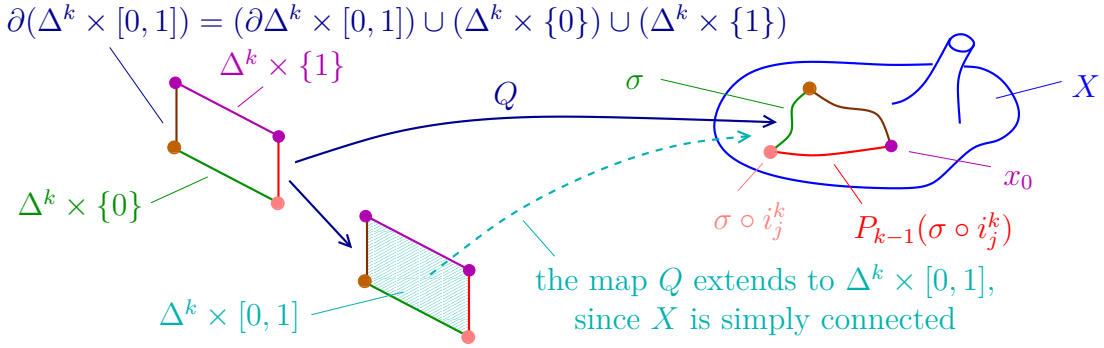
Proof. Property (d) already suggests that we should prove the claim by induction on k. It is convenient to view $k = -1$ as the (trivial) base case. Now let $k \in \mathbb{N}_0$ and suppose that we have proved the claim for all singular i-simplices with $i < k$. Now let $\sigma\colon \Delta^k \to X$ be a singular simplex. If $\sigma \in \mathrm{C}_k^{(n)}(X, x_0)$, then we set $P_k(\sigma) = \sigma \circ \pi$. In this case it is straightforward to verify that the conditions (a), (b) and (d) are satisfied.

Now consider the case that $\sigma \notin \mathrm{C}_k^{(n)}(X, x_0)$. We distinguish two cases.

(1) First we consider the case that $k \le n$. By induction the maps $P_{k-1}(\sigma \circ i_j^k)$ have been defined for all j. We set $\partial(\Delta^k \times [0,1]) \coloneqq (\partial\Delta^k \times [0,1]) \cup (\Delta^k \times \{0,1\})$ and we consider the map

$$Q\colon \partial(\Delta^k \times [0,1]) \to X$$

that is defined via σ on $\Delta^k \times \{0\}$, via the maps $P_{k-1}(\sigma \circ i_j^k)$ on $\partial\Delta^k \times [0,1]$[489] and which sends all points in $\Delta^k \times \{1\}$ to x_0.[490] (We refer to the figure below for an illustration of this map.) It follows almost immediately from the Convex-to-Ball Proposition 2.12 (2) that the pair $(\Delta^k \times [0,1], \partial(\Delta^k \times [0,1]))$ is homeomorphic to $(\overline{B}^{k+1}, S^k)$. Since $k \le n$ and since X is n-connected we see that there exists an extension of Q to a continuous map $\Delta^k \times [0,1] \to X$. We define $P_k(\sigma)$ to be any such extension. We obtain a map $P_k(\sigma)$ which is easily seen to have all the desired properties.

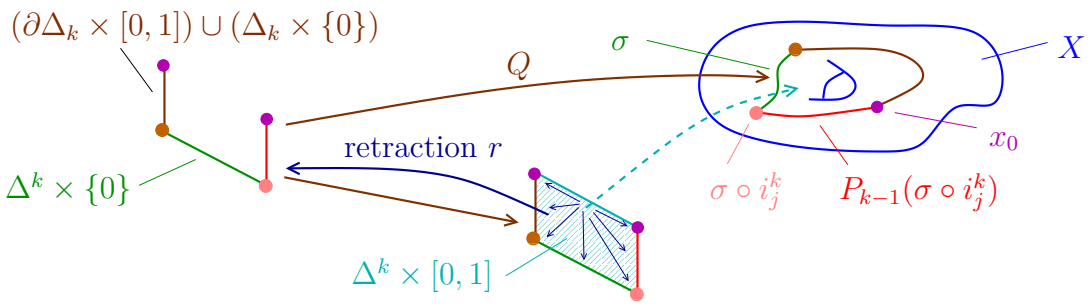


(2) Now we consider the case that $k > n$. Note that in this case condition (b) is satisfied "for free" by any continuous map $\Delta^k \times [0,1] \to X$ that satisfies (d). Thus we only have to find a map $P_k(\sigma)$ that satisfies (a), (c) and (d). This is much easier than in (1). Indeed, similar to (1) we consider the map

$$Q\colon (\Delta^k \times \{0\}) \cup (\partial\Delta^k \times [0,1]) \to X$$

that is defined via σ on $\Delta^k \times \{0\}$ and via the maps $P_{k-1}(\sigma \circ i_j^k)$ on $\partial\Delta^k \times [0,1]$.
It is elementary to show that there exists a retraction r from $\Delta^k \times [0,1]$ to the domain of Q. Thus combining this retraction r with Q we obtain an extension of Q to a map $\Delta^k \times [0,1] \to X$. We define $P_k(\sigma)$ to be this extension. It follows basically by construction that all the statements are satisfied. ⊞

[489] Here we use that we assume that the maps $P(\mu)$ are defined for simplices of dimension $< k$.
[490] It follows from the Pasting Proposition 2.4 (2′) that this map is indeed continuous.



Now we turn to the actual proof of the statement that the inclusion map

$$\iota\colon \mathrm{C}^{(n)}_*(X, x_0) \;\to\; \mathrm{C}_*(X, \{x_0\})$$

is a chain homotopy equivalence. To do so we consider the homomorphisms

$$\phi\colon \mathrm{C}_k(X, \{x_0\}) \;\to\; \mathrm{C}^{(n)}_k(X, x_0)$$

that are determined by

$$[\sigma\colon \Delta^k \to X)] \;\mapsto\; \phi([\sigma]) \coloneqq \underbrace{[P_k(\sigma) \circ \eta_1]}_{\Delta^k \to \Delta^k \times [0,1] \to X}.$$

It follows from Claim 1 (b) that ϕ does indeed take values in $\mathrm{C}^{(n)}_*(X, x_0)$ and it follows easily from Claim 1 (d) that ϕ is in fact a chain map. We want to show that ϕ is a chain homotopy inverse to ι. By Claim 1 (c) we have $\phi \circ \iota = \mathrm{id}$. Thus it remains to prove the following claim:

Claim 2. There exists a chain homotopy from $\iota \circ \phi$ to the identity of $\mathrm{C}_*(X, \{x_0\})$.

Proof. Let $k \in \mathbb{N}_0$. We recall that on page 475 we introduced the singular $(k+1)$-chain

$$\Omega_k \;\coloneqq\; \sum_{j=0}^{k} (-1)^j \cdot \underbrace{[v_0, \dots, v_j, w_j, \dots, w_k]}_{\Delta^{k+1} \to \Delta^k \times [0,1]} \;\in\; \mathrm{C}_{k+1}(\Delta^k \times [0,1])$$

where $v_0, \dots, v_k, w_0, \dots, w_k$ denote the vertices of $\Delta^k \times [0,1]$, see page 475 for details. The precise definition of Ω_k is in fact irrelevant to us, all we need to know is that the Prism Boundary Lemma 36.9 says that for any $k \in \mathbb{N}_0$ we have

$$(*) \qquad \partial_{k+1}(\Omega_k) + \sum_{j=0}^{k} (-1)^j \cdot \underbrace{(i^k_j \times \mathrm{id}_{[0,1]})_*}_{\mathrm{C}_*(\Delta^{k-1} \times [0,1]) \to \mathrm{C}_*(\Delta^k \times [0,1])} (\Omega_{k-1}) \;=\; \eta_1 - \eta_0 \;\in\; \mathrm{C}_k(\Delta^k \times [0,1]).$$

For any $k \in \mathbb{N}_0$ we consider the homomorphism

$$D_k\colon \mathrm{C}_k(X, \{x_0\}) \;\to\; \mathrm{C}_{k+1}(X, \{x_0\})$$

that is determined by

$$[\sigma\colon \Delta^k \to X] \;\mapsto\; \underbrace{[P_k(\sigma)_*}_{\mathrm{C}_*(\Delta^k \times [0,1]) \to \mathrm{C}_*(X)} (\overbrace{\Omega_k}^{\in \mathrm{C}_{k+1}(\Delta^k \times [0,1])})]$$

We claim that these homomorphisms define a chain homotopy from $\iota \circ \phi$ to the identity of $\mathrm{C}_*(X, \{x_0\})$. Thus let $\sigma\colon \Delta^k \to X$ be a singular k-simplex. We need to show that $\partial_{k+1}(D_k(\sigma)) + D_{k-1}(\partial_k(\sigma)) = \phi(\sigma) - \sigma$. Indeed we have the following equalities in

$\mathrm{C}_k(X, \{x_0\})$:[491]

$$\begin{aligned}
\partial_{k+1}(D_k(\sigma)) + D_{k-1}(\partial_k(\sigma)) &\overset{\text{definition of } \partial_k(\sigma)}{=} \partial_{k+1}(D_k(\sigma)) + D_{k-1}\Big(\sum_{j=0}^{k}(-1)^j \cdot \sigma \circ i_j^k\Big) \overset{\text{definition of } D_k \text{ and } D_{k-1}}{=} \\
&= \partial_{k+1}(P_k(\sigma)_*(\Omega_k)) + \sum_{j=0}^{k}(-1)^j \cdot P_{k-1}(\sigma \circ i_j^k)_*(\Omega_{k-1}) \\
&= P_k(\sigma)_*\Big(\partial_{k+1}(\Omega_k) + \sum_{j=0}^{k}(-1)^j \cdot (i_j^k \times \mathrm{id}_{[0,1]})_*(\Omega_{k-1})\Big)
\end{aligned}$$

in the first summand we used that by the C_*-Functor Lemma 35.5 the induced map $P_k(\sigma)_*$ is a chain map and in the second summand we used that by Claim 1 (d) we have $P_{k-1}(\sigma \circ i_j^k) = P_k(\sigma) \circ (i_j^k \times \mathrm{id}_{[0,1]})$ and we used the covariant functoriality of chain groups

$$= P_k(\sigma)_*(\eta_1 - \eta_0) = P_k(\sigma) \circ \eta_1 - P_k(\sigma) \circ \eta_0 \overset{\text{by definition of } \phi \text{ and by Claim 1 (a)}}{=} \phi(\sigma) - \sigma.$$

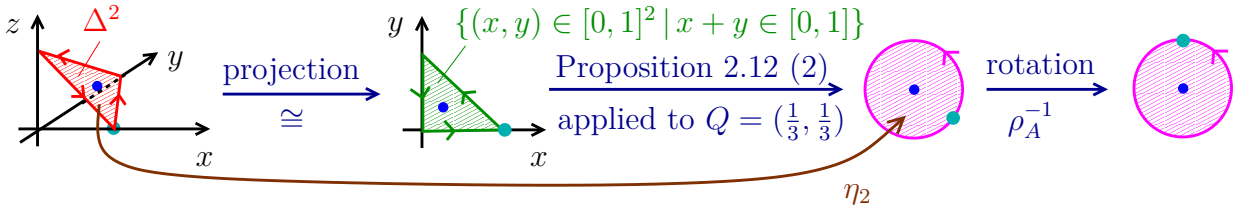
by the above equality (∗) ■

59.6. **Proof of the Homotopy-via-Homology Proposition 59.13.** The proof of the Homotopy-via-Homology Proposition 59.13 requires some preparation. First we introduce the following notation:

Notation.

(1) We write $* = (0, \dots, 0, 1) \in S^n$.

(2) We pick a matrix $A \in \mathrm{SO}(n+1)$ such that $A \cdot * \in \eta_{n+1}((n-1)\text{-skeleton of } \partial\Delta^{n+1})$. We denote by $\rho_A \colon \overline{B}^{n+1} \to \overline{B}^{n+1}$ the map that is given by $x \mapsto A \cdot x$.

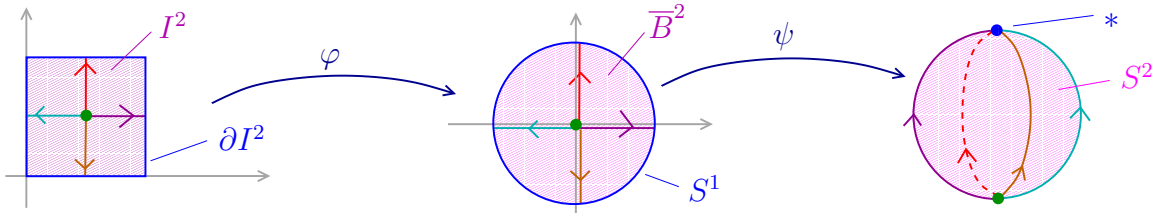


We also recall and introduce more notation:

Notation.

(1) Let $\varphi \colon (I^n, \partial I^n) \to (\overline{B}^n, S^{n-1})$ be the homeomorphism from page 381.

(2) Let $\psi \colon (\overline{B}^n, S^{n-1}) \to (S^n, *)$ be the map from the Ball Quotient-Sphere Lemma 6.5 that induces the standard homeomorphism $\overline{B}^n / S^{n-1} \to S^n$.



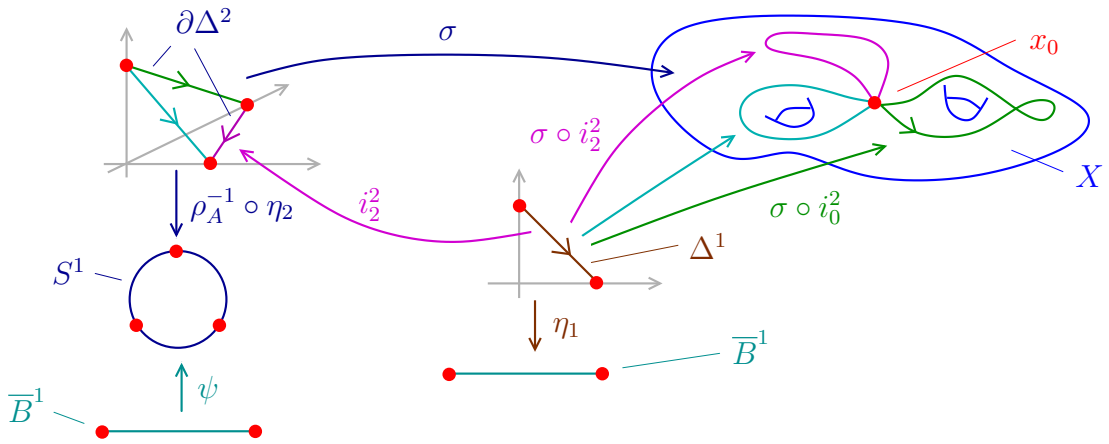
The following lemma is a key technical ingredient in the proof of the Homotopy-via-Homology Proposition 59.13:

[491] In this calculation we do not distinguish in the notation between a singular chain in $\mathrm{C}_k(X)$ and the element it represents in $\mathrm{C}_k(X, \{x_0\})$.

Lemma 59.14. (Faces–Addition Lemma) Let (X, x_0) be a pointed topological space, let $n \in \mathbb{N}_{\geq 2}$ and let $\sigma\colon \partial\Delta^{n+1} \to X$ be a continuous map that sends the $(n-1)$-skeleton of $\partial\Delta^{n+1}$ to x_0. Then we have the following equality in $\pi_n(X, x_0)$:[492] [493]

$$[\underbrace{\sigma \circ (\eta_{n+1}|_{\partial\Delta^{n+1}})^{-1} \circ \rho_A|_{S^n} \circ \psi}_{(\overline{B}^n, S^{n-1}) \to (X, x_0)}] \;=\; -\sum_{j=0}^{n+1} (-1)^j \cdot [\underbrace{\sigma \circ i_j^{n+1} \circ \eta_n^{-1}}_{(\overline{B}^n, S^{n-1}) \to (X, x_0)}].$$

We refer to the figure below for an illustration for $n = 1$, even though strictly speaking this case is not allowed.



Proof. We outsource the technically intricate proof to [**Fri24**]. ∎

For the reader's convenience we recall the proposition we are supposed to prove:

Proposition 59.13. (Homotopy-via-Homology Proposition) Let $n \in \mathbb{N}$ and let (X, x_0) be a pointed $(n-1)$-connected topological space.

(i) The map

$$\begin{array}{rcl} \widetilde{\Phi}_{(X,x_0)}\colon \pi_n(X, x_0) & \to & \mathrm{H}_n^{(n-1)}(X, x_0) \\ [f\colon (\overline{B}^n, S^{n-1}) \to (X, x_0)] & \mapsto & [\underbrace{f \circ \eta_n\colon \Delta^n \to X}_{\in \mathrm{C}_n^{(n-1)}(X, x_0)}] \end{array}$$

is well-defined and it is an isomorphism.

(ii) The following diagram commutes up to the sign $(-1)^n$:

$$\begin{array}{ccc} \pi_n(X, x_0) & \xrightarrow{\Phi_{(X,x_0)}} & \mathrm{H}_n(X) \\ {\scriptstyle \widetilde{\Phi}_{(X,x_0)}}\downarrow & & {\scriptstyle \text{я}}\downarrow{\scriptstyle \cong} \\ \mathrm{H}_n^{(n-1)}(X, x_0) & \xrightarrow{\iota} & \mathrm{H}_n(X, \{x_0\}), \end{array}$$

here ι is induced by the inclusion $\mathrm{C}_*^{(n-1)}(X, x_0) \to \mathrm{C}_n(X, \{x_0\})$.

(iii) The right vertical map of the diagram in (ii) is an isomorphism.

Proof. We start out with the following claim:

[492] Note that A is chosen in such a way that $(\sigma \circ (\eta_{n+1}|_{\partial\Delta^{n+1}})^{-1} \circ \rho_A|_{S^n} \circ \psi)(S^{n-1}) = \{x_0\}$.

[493] Note that our hypothesis that σ sends the $(n-1)$-skeleton of $\partial\Delta^{n+1}$ to x_0 implies in particular that for all $j \in \{0, \ldots, n+1\}$ we have $(\sigma \circ i_j^{n+1} \circ \eta_n^{-1})(S^{n-1}) = \{x_0\}$.

Claim 1.

(i) The map

$$\begin{array}{rcl} \widetilde{\Phi}_{(X,x_0)}\colon \pi_n(X,x_0) & \to & \mathrm{H}_n^{(n-1)}(X,x_0) \\ [f\colon (\overline{B}^n, S^{n-1}) \to (X,x_0)] & \mapsto & \big[\underbrace{f\circ\eta_n\colon \Delta^n \to X}_{\in \mathrm{C}_n^{(n-1)}(X,x_0)}\big] \end{array}$$

is well-defined and it is a homomorphism.

(ii) The following diagram commutes up to the sign $(-1)^n$:

$$\begin{array}{ccc} \pi_n(X,x_0) & \xrightarrow{\Phi_{(X,x_0)}} & \mathrm{H}_n(X) \\ {\scriptstyle \widetilde{\Phi}_{(X,x_0)}}\downarrow & & {\scriptstyle я}\downarrow{\scriptstyle \cong} \\ \mathrm{H}_n^{(n-1)}(X,x_0) & \xrightarrow{\iota} & \mathrm{H}_n(X,\{x_0\}) \end{array}$$

where ι is induced by the inclusion $\mathrm{C}_*^{(n-1)}(X,x_0) \to \mathrm{C}_n(X,\{x_0\})$.

(iii) The right vertical map of the diagram in (ii) is an isomorphism.

Proof. We prove the claim using a somewhat convoluted logic.[494] We consider the following diagram:

$$\begin{array}{ccc} \{\text{set of continuous maps } (\Delta^n,\partial\Delta^n)\to(X,x_0)\} & & \\ \Big\downarrow{\scriptstyle f\mapsto[f\circ\eta_n^{-1}]} & & \\ \pi_n(X,x_0) & \xrightarrow{\Phi_{(X,x_0)}} & \mathrm{H}_n(X) \\ {\scriptstyle f\mapsto[f]}\big(\big\downarrow{\scriptstyle \widetilde{\Phi}_{(X,x_0)}} & \searrow^{[g]\mapsto g_*([\overline{B}^n])} & {\scriptstyle я}\downarrow{\scriptstyle \cong} \\ \mathrm{H}_n^{(n-1)}(X,x_0) & \xrightarrow[\iota]{\cong} & \mathrm{H}_n(X,\{x_0\}) \end{array}$$

We make the following observations:

(a) It follows from $n \geq 2$, from the long exact sequence of homology groups of the pair $(X,\{x_0\})$ (see the LES-of-Pair Proposition 38.3) and the Singleton-H_*-Lemma 35.12 that the right-hand vertical map $я$ of the diagram is an isomorphism.

(b) Since X is $(n-1)$-connected we obtain from the n-Connected–Chain Group Proposition 59.12 that the bottom horizontal map is an isomorphism.

(c) Let $f\colon (\Delta^n,\partial\Delta^n)\to(X,x_0)$ be a continuous map. The $(n-1)$-skeleton of Δ^n equals $\partial\Delta^n$, thus the map $f\colon \Delta^n \to X$ does indeed define an element in $\mathrm{C}_n^{(n-1)}(X,x_0)$. It follows easily from the definitions that every element in $\mathrm{C}_n^{(n-1)}(X,x_0) \subset \mathrm{C}_n(X,\{x_0\})$ is a cycle, thus f defines an element in $\mathrm{H}_n^{(n-1)}(X,x_0)$. This shows that in the above diagram the curved left vertical map is well-defined.

(d) In this proof we think of $\pi_n(X,x_0)$ as the set of homotopy classes of continuous maps $(\overline{B}^n, S^{n-1}) \to (X,x_0)$. It follows immediately from the definitions that the right triangle commutes.

[494] The convoluted logic is forced upon us since proving (i) as a stand-alone statement is actually slightly tricky.

(e) For any continuous map $f\colon (\Delta^n, \partial\Delta^n) \to (X, \{x_0\})$ of pairs of topological spaces we have the following equality in $\mathrm{H}_n(X, \{x_0\})$:

$$(f \circ \eta_n^{-1})_*([\overline{B}^n]) \underset{\uparrow}{=} (-1)^n \cdot (f \circ \eta_n^{-1})_*([\eta_n]) = (-1)^n \cdot [f \circ \eta_n^{-1} \circ \eta_n] = (-1)^n \cdot [f].$$

it is clear that the equality holds up to a sign which only depends on n, the sign (which is irrelevant for the purpose of the proof of the Hurewicz Theorem), is determined in [**Fri24**]

This shows that the two possibilities of going from the upper left corner to the lower right corner differ by the sign $(-1)^n$.

(f) We want to show that $\widetilde{\Phi}_{(X,x_0)}\colon \pi_n(X, x_0) \to \mathrm{H}_n^{(n-1)}(X, x_0)$ is well-defined and that it is a homomorphism. By (b) we know that $\iota\colon \mathrm{H}_n^{(n-1)}(X, x_0) \xrightarrow{\sim} \mathrm{H}_n(X, \{x_0\})$ is an isomorphism. Thus it suffices to show these statements for $\iota \circ \widetilde{\Phi}_{(X,x_0)}$. We make the following observations:

(α) It follows from (e) and the observation that the top vertical map is a surjection that the rectangle commutes up to the sign $(-1)^n$. In particular the bottom-right map $\iota \circ \widetilde{\Phi}_{(X,x_0)}\colon \pi_n(X, x_0) \to \mathrm{H}_n(X, \{\widetilde{x}_0\})$ is well-defined.

(β) By (d) we know that the right triangle commutes.

(γ) By the Hurewicz Homomorphism Proposition 59.1 we know that $\Phi_{(X,x_0)}$ is a homomorphism. Evidently я is also a homomorphism.

(δ) It follows from (α), (β) and (γ) that $\iota \circ \widetilde{\Phi}_{(X,x_0)}\colon \pi_n(X, x_0) \to \mathrm{H}_n(X, \{\widetilde{x}_0\})$ is a homomorphism. ⊞

By Claim 1 it suffices to show that the homomorphism

$$\begin{array}{rcl} \widetilde{\Phi}_{(X,x_0)}\colon \pi_n(X, x_0) & \to & \mathrm{H}_n^{(n-1)}(X, x_0) \\ [f\colon (\overline{B}^n, S^{n-1}) \to (X, x_0)] & \mapsto & [f \circ \eta_n\colon \Delta^n \to X] \end{array}$$

is in fact an isomorphism. We will do so by defining an explicit inverse.

Claim 2.

(i) The map

$$\begin{array}{rcl} \Xi_{(X,x_0)}\colon \mathrm{C}_n^{(n-1)}(X, x_0) & \to & \pi_n(X, x_0) \\ \sum_{i=1}^{s} a_i \cdot (\sigma_i\colon (\Delta^n, \partial\Delta^n) \to (X, x_0)) & \mapsto & \sum_{i=1}^{s} a_i \cdot [\sigma_i \circ \eta_n^{-1}\colon (\overline{B}^n, S^{n-1}) \to (X, x_0)] \end{array}$$

is a well-defined homomorphism.

(ii) The map $\Xi_{(X,x_0)}$ from (i) descends to a well-defined homomorphism

$$\mathrm{H}_n^{(n-1)}(X, x_0) \to \pi_n(X, x_0).$$

(iii) The map $\Xi_{(X,x_0)}$ is an inverse to $\widetilde{\Phi}_{(X,x_0)}$.

Proof.

(i) Recall that we know from the π_n-Abelian Proposition 28.3 and our hypothesis $n \geq 2$ that the group $\pi_n(X, x_0)$ on the right is abelian. Therefore it follows from the Free Abelian Group–Existence Lemma 18.4 that the map is well-defined.[495]

(ii) Let $\sigma\colon \Delta^{n+1} \to X$ be a singular $(n+1)$-simplex that defines an element in the group $\mathrm{C}_{n+1}^{(n-1)}(X, x_0)$. We need to show that $\Xi_{(X,X_0)}(\partial_{n+1}(\sigma)) = 0$. We calculate that we have

[495]Put differently, it would be a serious dereliction of duty to use the additive notation for a group structure, if the group was not known to be abelian.

the following equality in $\pi_n(X, x_0)$:

$$\begin{aligned}\Xi_{(X,X_0)}(\partial_{n+1}(\sigma)) &= \Xi_{(X,X_0)}\Big(\sum_{j=0}^{n+1}(-1)^j\cdot\sigma\circ i_j^n\Big)\\ &= \sum_{j=0}^{n+1}(-1)^j\cdot[\sigma\circ i_j^{n+1}\circ\eta_n^{-1}] \;=\; -[\sigma|_{\partial\Delta^{n+1}}\circ\eta_{n+1}^{-1}\circ\rho_A|_{S^n}] \;=\; 0.\end{aligned}$$

by definition of $\Xi_{(X,x_0)}$ — follows from the Faces–Addition Lemma 59.14 — this follows from the fact that $\sigma|_{\partial\Delta^{n+1}}\circ\eta_{n+1}^{-1}\circ\rho_A|_{S^n}\colon S^n\to X$ extends to $\sigma\circ\eta_{n+1}^{-1}\circ\rho_A\colon B^{n+1}\to X$

(iii) It follows immediately from the definitions that $\Xi_{(X,x_0)}\circ\widetilde{\Phi}_{(X,x_0)}$ is the identity on $\pi_n(X, x_0)$. Now let us consider the converse. As on page 755 we see that every element of $\mathrm{C}_n^{(n-1)}(X, x_0)$ is a cycle. It follows from this observation and the definitions that $\widetilde{\Phi}_{(X,x_0)}\circ\Xi_{(X,x_0)}$ is the identity on a generating set for the group $\mathrm{H}_n^{(n-1)}(X, x_0)$. But by Claim 1 and Claim 2 we know that $\widetilde{\Phi}_{(X,x_0)}$ and $\Xi_{(X,x_0)}$ are both homomorphisms. Thus we see that $\widetilde{\Phi}_{(X,x_0)}\circ\Xi_{(X,x_0)}$ is actually the identity on all of $\mathrm{H}_n^{(n-1)}(X, x_0)$.

This concludes the proof of the claim and thus, modulo the proof of the Faces–Addition Lemma 59.14, we are done with the proof of the Hurewicz Theorem 59.6. ■

Part VII

Homology groups of topological manifolds

At least this particular author is very interested in topological and smooth manifolds. The theory of smooth manifolds is often times easier than the theory of topological manifolds. For example, using smooth structures, we already introduced and proved the following notions and results:

- In Chapter **??** we introduced the tangent spaces of smooth manifolds.
- In Chapter **??** we used tangent spaces to introduce the notion of an orientation of a smooth manifold.
- In the Smooth Manifold-Simplicial Structure Theorem **??** we showed that every smooth manifold admits a (finite) smooth simplicial structure. This result allowed us to prove the Smooth Manifolds-Invariants Proposition **??** which says in particular, that for any compact connected smooth manifold M the fundamental group $\pi_1(M)$ is finitely presented and that the homology groups $\mathrm{H}_k(M)$ are finitely generated.
- Let M be an n-dimensional smooth manifold. In the Manifold-Top-H_*-Theorems **??** and **??** we used the Smooth Manifold-Simplicial Structure Theorem **??** to calculate $\mathrm{H}_n(M)$ and $\mathrm{H}_n(M, \partial M)$ in terms of the topology of M.
- In Chapter **??** we used the Manifold-Top-H_*-Theorems **??** and **??** to introduce the fundamental class $[M] \in \mathrm{H}_n(M, \partial M)$ of a compact oriented n-dimensional smooth manifold M.
- In Chapter **??** we used fundamental classes to introduce and study the degree of a continuous map between compact oriented smooth manifolds.

In this part of the lecture notes we now turn to the study of the homology groups of *topological* manifolds. We will manage to extend most of the earlier notions and results from smooth manifolds to topological manifolds.

This part of the lecture notes is organized as follows:

- In Chapter 60 we will formulate and state Borsuk's Theorem 60.4. Among many other statements this theorem is essential in the proof of the Topological Manifold-Simplicial Complex Retract Theorem 60.9 which says that every compact topological manifold is the retract of a finite simplicial complex. This results allows us to prove a generalization of the Smooth Manifolds-Invariants Proposition **??**. More precisely, in the Topological Manifolds-Invariants Proposition 60.10 we will show that for any compact connected topological manifold X the fundamental group $\pi_1(X)$ is finitely presented and that the homology groups $\mathrm{H}_k(X)$ are finitely generated.
- In Chapter **??** we will give a survey on further results on topological manifolds. In particular we will discuss the question whether topological manifolds are actually homotopy equivalent to CW-complexes, we will introduce the Euler characteristic of compact topological manifolds and we will introduce the notion of the tangent micro bundle.
- In Chapter 61 we will introduce the notion of an orientation of a topological manifold. We will see that for smooth manifolds this new notion of an orientation is consistent with the earlier notion.
- In Chapter 62 we will study induced orientations. For example we will see, that for an oriented topological manifold, the boundary, any covering and any codimension-zero submanifold inherit a natural orientation. Furthermore we will use the cross product on homology to introduce the product orientation.

- In Chapter 63 we will study the homology groups of topological manifolds. More precisely, given an n-dimensional topological manifold X we will determine in the Manifold-Top-H_*-Theorem 63.1 the homology groups $\mathrm{H}_n(X)$ and $\mathrm{H}_n(X, \partial X)$ in terms of the topology of X.
- In Chapter 64 we will use the Manifold-Top-H_*-Theorem 63.1 to introduce the fundamental class $[X] \in \mathrm{H}_n(X, \partial X)$ of a compact oriented n-dimensional topological manifold.
- In Chapter 65 we will use fundamental classes to introduce and study the degree of a continuous map between compact oriented topological manifolds. We will see that this notion is consistent with the earlier notions of degrees.

CHAPTER 60

Neighborhood retracts and Borsuk's Theorem

In this chapter we will state and prove Borsuk's Theorem 60.4 which will give us many goodies:

(1) Using Borsuk's Theorem we can get some control over the invariants of compact topological manifolds. For example we will be able to show that the homology groups of a compact topological manifold are finitely generated and that the fundamental group is finitely presented.
(2) We will use Borsuk's Theorem in the proof of the Generalized Sphere–Alexander Duality Theorem 86.1 which allows us in particular to calculate the homology groups of topological spaces of the form $S^n \setminus X$ where X is a compact topological manifold.

60.1. **Neighborhood retracts.** We start out by recalling the following definition from pages 164 and 201:

Definition. Let Y be a topological space. We say that a subspace $A \subset Y$ is a retract of Y if there exists a retraction from Y to A, i.e. if there exists a continuous map $r\colon Y \to A$ with $r(a) = a$ for all $a \in A$.



retraction r from Y to A

The following lemma gives the most basic necessary condition for a subspace to be a retract:

Lemma 60.1. (Retract Closed Lemma) Let Y be a topological space and let $A \subset Y$ be a retract. If Y is Hausdorff, then $A \subset Y$ is closed.

Proof. Let Y be a Hausdorff space, let $A \subset Y$ be a retract and let $r\colon Y \to A$ be a retraction. We consider the map

$$\begin{array}{rccc} f\colon & Y & \to & Y \times Y \\ & x & \mapsto & (x, r(x)). \end{array}$$

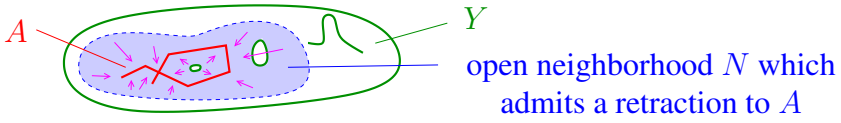
We make the following observations:

(1) Since r is continuous it follows easily from the Topological-Product Proposition 5.1 (1∗) that f is continuous.
(2) We consider the diagonal $\Delta = \{(y, y) \mid y \in Y\} \subset Y \times Y$. Since Y is Hausdorff we obtain from the Hausdorff-Diagonal Lemma 5.8 that Δ is a closed subset of $Y \times Y$.
(3) It follows easily from the definition of a retraction that $f^{-1}(\Delta) = A$.
(4) Since f is continuous and since Δ is closed we obtain from the Continuity-via-Closed Subsets Lemma 2.3 that $A = f^{-1}(\Delta)$ is indeed a closed subset of Y. ■

The functoriality of the fundamental group, of higher homotopy groups and of homology groups gave us many tools for showing that a given subspace is *not* a retract. For example in the Circle–Non Retract Proposition 13.3 we used the functoriality of the fundamental group to show that S^1 is not a retract of $\overline{B}^2$.

In the following we will not be interested in "global retracts" but in more "local retracts". This leads us to the following definition:

Definition. Let Y be a topological space and let A be a subset. We denote by $\overline{A}$ the closure of A in Y. We say that A is a neighborhood retract in Y if there exists an open neighborhood N of $\overline{A}$ in Y such that A is a retract of N.[496]



Examples.

(1) Let M be a smooth manifold and let $A \subset M$ be a compact proper smooth submanifold of M. It follows from the Tubular Neighborhood Theorem [**Fri24**] that A is a neighborhood retract in M.



(2) One can easily show by hand that $A = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\}$, viewed as a subset of $Y = \mathbb{R}$, is not a neighborhood retract. □

Here is a moderately exciting question:

Question 60.2. Which subsets of topological manifolds are neighborhood retracts?

Recall that every topological manifold is by definition Hausdorff and recall that a subspace of a Hausdorff space is again Hausdorff. It follows easily from these observations and the Retract Closed Lemma 60.1 that every neighborhood retract A of a topological manifold X is necessarily a closed subset of X.

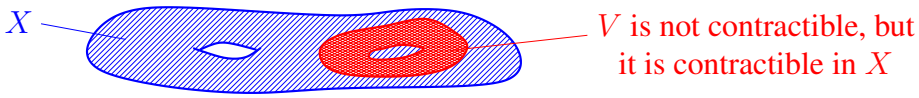
The above example of $\{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\} \subset \mathbb{R}$ shows that being closed is not a sufficient condition for being a neighborhood retract. We will discuss further conditions in the next section.

60.2. **Borsuk's Theorem.** Our attempt to answer Question 60.2 leads us, after some serious pondering (which we skip for time reasons), to the notion of a "weakly locally contractible" topological space. To introduce this new notion we first need to recall two definitions from pages 85 and 171 and we need to introduce two related definitions.

[496]Note that if there exists a neighborhood N of $\overline{A}$ that admits a retraction to A, then the restriction to the interior of N is also a retraction. This shows that our definition of a neighborhood retract is equivalent to demanding that there exists any neighborhood (not necessarily open) of $\overline{A}$ that admits a retract to A. This formulation is often the definition of a neighborhood retract that is used in the literature. For our discussion it is convenient to already assume that the neighborhood is open.

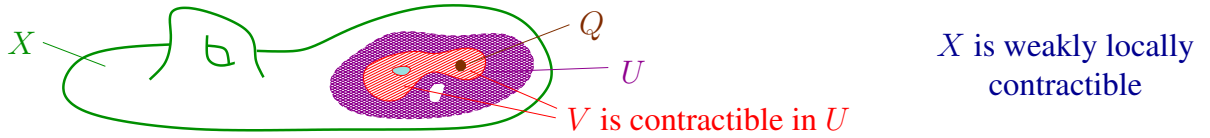
Definition. Let X be a topological space.

(1) We say that X is contractible if X is homotopy equivalent to a point.
(2) It follows from the definition of a local property of a topological space, see page 85, that X is locally contractible if given any $Q \in X$ and given any open neighborhood U of Q there exists an open neighborhood V of Q that is contained in U and that is contractible.
(3) We say that a subspace V of X is contractible in X if the inclusion map $V \to X$ is homotopic to a constant map.[497]



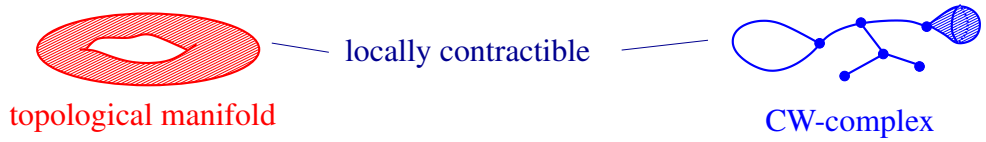
Definition.

(4) We say that X is weakly locally contractible if given any $Q \in X$ and given any open neighborhood U of Q there exists an open neighborhood V of Q that is contained in U and that is contractible in U.



Examples.

(1) (a) It follows almost immediately from the definitions that a topological space that is locally contractible is also weakly locally contractible.
(b) One can easily show that every topological manifold is locally contractible and thus weakly locally contractible.
(c) In the CW-Complex–Local Properties Proposition 31.7 (2) we showed that CW-complexes are locally contractible and thus weakly locally contractible. In particular topological graphs are (weakly) locally contractible.



(2) (a) It is easy to show that $\mathbb{R}^2 \setminus \{(\frac{1}{n}, 0) \,|\, n \in \mathbb{N}\}$ is not weakly locally contractible.
(b) It is an amusing little exercise to show that the Topologist's Sine Curve

$$\{(0, y) \,|\, y \in [-1, 1]\} \cup \{(x, \sin(\tfrac{1}{x})) \,|\, x \in (0, \tfrac{1}{\pi}]\} \subset \mathbb{R}^2,$$

which we had initially introduced on page 80, is not weakly locally contractible.
(c) It is an equally amusing (but not entirely trivial) exercise to come up with a topological space that is weakly locally contractible but not locally contractible. □

The following lemma gives a partial explanation why we introduced the notion of a weakly locally contractible topological space:

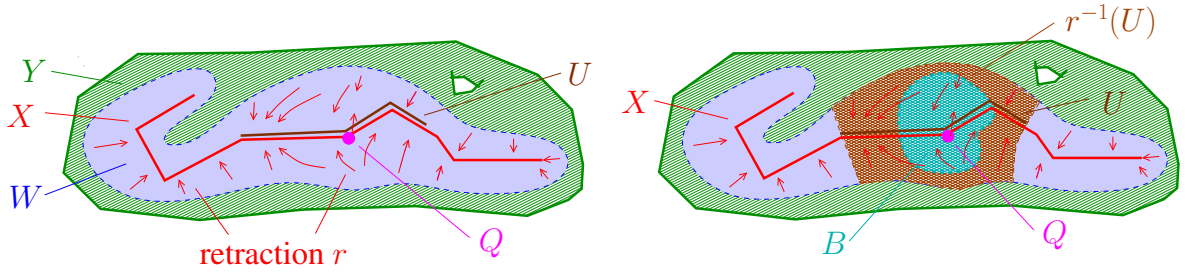
[497]Note that a topological space X is contractible if and only if the identity map is homotopic to a constant map. Thus we see that the subspace X is contractible in X if and only if X itself is contractible.



Lemma 60.3. (Manifold Neighborhood Retract–Contractibility Lemma) Let Y be a topological manifold and let X be a subspace of Y. If X is a neighborhood retract in Y, then the topological space X is weakly locally contractible.

Proof. Let Y be a topological manifold and let X be a subspace of Y. Furthermore let W be an open neighborhood of X that admits a retraction $r\colon W \to X$. Let $Q \in X$ and let U be a neighborhood of Q in X.

Claim. There exists an open neighborhood B of $Q \in Y$ with $B \subset r^{-1}(U)$ and a homotopy $F\colon B \times [0,1] \to B$ from the identity to a constant map.



Proof. One can easily show that Y is locally contractible. It follows from this information, applied to the open neighborhood $r^{-1}(U)$ of Q, that there exists an open contractible neighborhood B of $Q \in Y$ with $B \subset r^{-1}(U)$. The fact that B is contractible gives us the desired homotopy. ⊞

Next we consider the map

$$\begin{array}{rcl} (B \cap U) \times [0,1] & \to & U \\ (x,t) & \mapsto & r(\underbrace{F(x,t)}_{\in B \subset r^{-1}(U)}). \end{array}$$

Evidently this map is a homotopy from the identity to a constant map. In particular $V := B \cap U$ is contractible in U. ■

As we will see in a minute, rather stunningly for subsets of topological manifolds the converse to the Retract Closed Lemma 60.1 and the Manifold Neighborhood Retract–Contractibility Lemma 60.3 holds.

We start out with the important special case of subsets of $\mathbb{R}^n$. The following theorem was first proved by Karol Borsuk [**Bor32**] in 1932:

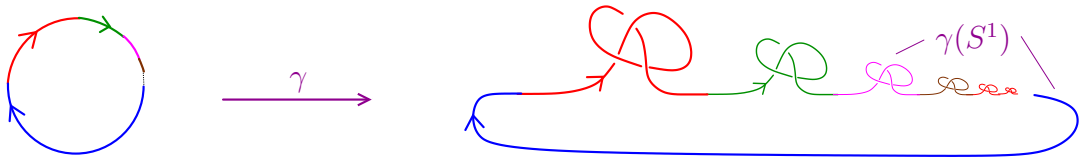
Theorem 60.4. (Borsuk's Theorem) Let $n \in \mathbb{N}_0$ and let X be a closed subset of $\mathbb{R}^n$. If the topological space X is weakly locally contractible, then X is a neighborhood retract in $\mathbb{R}^n$.

We now proceed as follows:

(1) First we discuss an example.

(2) In the Manifold–Neighborhood Retract Theorem 60.5 we use Borsuk's Theorem 60.4 to also deal with subspaces of arbitrary topological manifolds.
(3) In Section 60.3 we provide the tricky proof of Borsuk's Theorem 60.4.
(4) In Section 60.4 and 60.5 we will discuss applications of Borsuk's Theorem 60.4.

Example. We consider the map $\gamma\colon S^1 \to \mathbb{R}^3$ that is shown in the figure below:

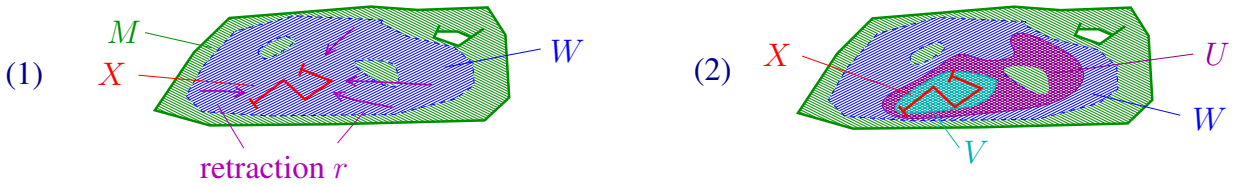


Using the Compact-Hausdorff Proposition 2.10 (3) it is not difficult to show that γ is an embedding. Evidently S^1 is locally contractible and by the Compact Image Lemma 2.8 together with the Compact-Closed Lemma 1.9 we know that $\gamma(S^1)$ is a closed subset of $\mathbb{R}^3$. Thus it follows from Borsuk's Theorem 60.4 that $\gamma(S^1)$ is a neighborhood retract in $\mathbb{R}^3$. It is a mindboggling exercise to try to visualize an (open) neighborhood of $\gamma(S^1) \subset \mathbb{R}^3$ together with a retraction to $\gamma(S^1)$. □

We conclude this section with a theorem which generalizes the statement of Borsuk's Theorem 60.4 from $\mathbb{R}^n$ to topological manifolds and that furthermore gives another rather technical conclusion. This second conclusion will play a significant role later on in our proof of the General Alexander Duality Theorem 86.1.

Theorem 60.5. (Manifold–Neighborhood Retract Theorem) Let M be a topological manifold and let $X \subset M$ be a closed subset that is weakly locally contractible.

(1) X is a neighborhood retract, i.e. there exists an open neighborhood $W \subset M$ of X for which there exists a retraction $r\colon W \to X$.[498]
(2) We can find W and r such that furthermore the following condition is satisfied: Given any open neighborhood U of X there exists an open neighborhood V with $X \subset V \subset U \cap W$ such that the inclusion $i\colon V \to U$ and the retraction $r\colon V \to X \subset U$ are homotopic maps from V to U.



The proof of the Manifold–Neighborhood Retract Theorem 60.5 relies on the following theorem:

Theorem 60.6. (Euclidean Embedding Theorem) Given any topological manifold M (not necessarily compact) there exists an $n \in \mathbb{N}$ and an embedding $f\colon M \to \mathbb{R}^n$ such that $f(M)$ is a closed subset of $\mathbb{R}^n$.

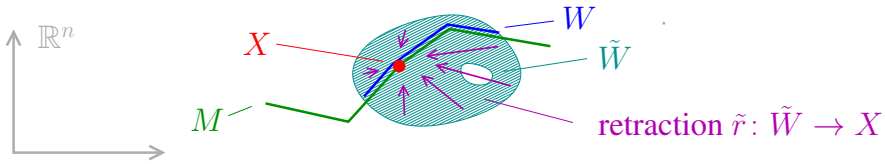
[498] This statement together with the Manifold Neighborhood Retract–Contractibility Lemma 60.3 and the discussion on page 763 shows that a subspace X of a topological manifold is a neighborhood retract if and only if X is a closed subset and if X is weakly locally contractible.

Proof of the Euclidean Embedding Theorem 60.6. This theorem is proved in [**Fri24**]. The case that M is compact can be done fairly easily, using partitions of unity. The non-compact case requires more of an effort. ■

Proof of the Manifold–Neighborhood Retract Theorem 60.5. Let M be a topological manifold and let $X \subset M$ be a closed subset that is weakly locally contractible. By the Euclidean Embedding Theorem 60.6 there exists an $n \in \mathbb{N}$ and an embedding $f\colon M \to \mathbb{R}^n$ of topological manifolds such that $f(M)$ is a closed subset of $\mathbb{R}^n$. Since $X \subset M$ is by hypothesis closed it follows from the elementary Open-Closed Inclusion Lemma 2.7 that $f(X)$ is a closed subset of $\mathbb{R}^n$.

The above discussion shows that we only need to consider the case that M is a closed subset of $\mathbb{R}^n$ and that $X \subset M$ is a closed subset of $\mathbb{R}^n$.

(1) Since X is by hypothesis weakly locally contractible we know by Borsuk's Theorem 60.4 that there exists an open neighborhood $\widetilde{W} \subset \mathbb{R}^n$ of $X \subset \mathbb{R}^n$ for which there exists a retraction $\widetilde{r}\colon \widetilde{W} \to X$. We set $W \coloneqq \widetilde{W} \cap M$ and $r \coloneqq \widetilde{r}|_W\colon W \to X$. By the definition of the subspace topology we know that W is indeed an open neighborhood of $X \subset M$. Furthermore clearly $r\colon W \to X$ is a retraction. Thus we have shown that X is a neighborhood retract in M.

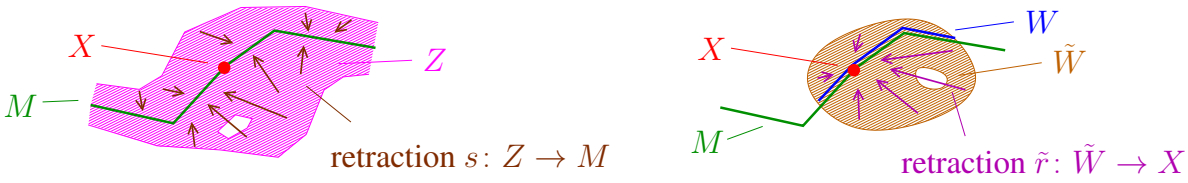


(2) To get the desired more refined statement we now apply Borsuk's Theorem 60.4 not once, but twice:

(a) Since M is a topological manifold we know that M is weakly locally contractible. Since M is a closed subset of $\mathbb{R}^n$ we know by Borsuk's Theorem 60.4 that there exists an open neighborhood $Z \subset \mathbb{R}^n$ of $M \subset \mathbb{R}^n$ for which there exists a retraction $s\colon Z \to M$.

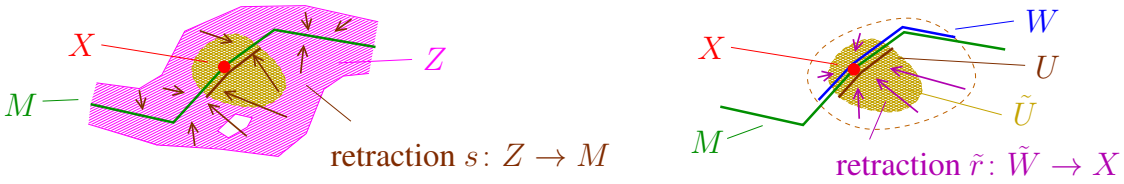
(b) Since X is weakly locally contractible we know again by Borsuk's Theorem 60.4 that there exists an open neighborhood $\widetilde{W} \subset \mathbb{R}^n$ of $X \subset \mathbb{R}^n$ for which there exists a retraction $\widetilde{r}\colon \widetilde{W} \to X$. After possibly replacing $\widetilde{W}$ by $\widetilde{W} \cap Z$ we can assume that $\widetilde{W} \subset Z$.

We set $W \coloneqq \widetilde{W} \cap M$ and $r \coloneqq \widetilde{r}|_W\colon W \to X$. By the definition of the subspace topology we know that $\widetilde{W}$ is indeed an open neighborhood of $X \subset M$.



Now let us show that the above W and r actually satisfy Statement (2). Thus let U be an open neighborhood of $X \subset M$. We need to show that there exists an open neighborhood V with $X \subset V \subset U \cap W$ such that the inclusion $i\colon V \to U$ and the retraction $r\colon V \to X \subset U$ are homotopic maps from V to U.

Claim 1. There exists an open subset $\widetilde{U}$ of $\mathbb{R}^n$ with $X \subset \widetilde{U} \cap \widetilde{W}$ and with $s(\widetilde{U}) \subset U$.



Proof. By definition of the subspace topology there exists an open subset U' of $\mathbb{R}^n$ with $U' = \widetilde{U} \cap M$. We set $\widetilde{U} := U' \cap \widetilde{W} \cap s^{-1}(U)$. Since clearly $U \subset \widetilde{W}$ and $U \subset s^{-1}(U)$ we see that $\widetilde{U}$ has all the desired properties. ⊞

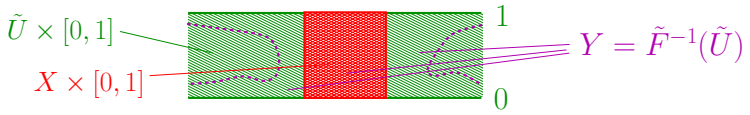
We consider the homotopy

$$\begin{array}{rcl} \widetilde{F}\colon \widetilde{U} \times [0,1] & \to & \mathbb{R}^n \\ (x,t) & \mapsto & (1-t)\cdot x + t \cdot \widetilde{r}(x) \end{array}$$

from the identity $\mathrm{id}_{\widetilde{U}}$ to the retraction $\widetilde{r}\colon \widetilde{U} \to X$. Note that this is a homotopy rel X.

Claim 2. There exists an open neighborhood $\widetilde{V}$ of $X \subset \mathbb{R}^n$, that is contained in $\widetilde{U}$, such that $\widetilde{F}(\widetilde{V} \times [0,1]) \subset \widetilde{U}$.

Proof. We consider the open subset $Y := \widetilde{F}^{-1}(\widetilde{U}) \subset \widetilde{U} \times [0,1]$. Since $X \subset \widetilde{U}$ and since $\widetilde{F}$ is a homotopy rel X we see that Y contains $X \times [0,1]$. (For the record we note that Y contains $\widetilde{U} \times \{0,1\}$.)



Since $[0,1]$ is compact and since Y is open see that for every $x \in X$ there exists an open neighborhood R_x of $x \in \widetilde{U}$ such that $R_x \times [0,1] \subset Y$. We set $\widetilde{V} := \bigcup_{x \in X} R_x$. This is an open neighborhood of $X \subset \mathbb{R}^n$ that is contained in $\widetilde{U}$ such that $\widetilde{F}(\widetilde{V} \times [0,1]) \subset \widetilde{U}$. ⊞

We set $V := \widetilde{V} \cap M \cap W$. By definition of the subspace topology this is an open neighborhood of $X \subset M$. Evidently $X \subset V \subset U \cap W$. Finally we consider the map

$$\begin{array}{rcl} F\colon V \times [0,1] & \to & M \\ (x,t) & \mapsto & \underbrace{s(\underbrace{\widetilde{F}(x,t)}_{\in \widetilde{U} \subset Z})}_{\in U}. \end{array}$$

Note that this map takes values in $s(\widetilde{U}) = U$. Furthermore note that this is a homotopy from $F_0 = i\colon V \to U$ to $F_1 = r\colon V \to U$. We have thus shown that W and r satisfy the desired condition. ■

60.3. **Proof of Borsuk's Theorem 60.4.** Before we provide the proof of Borsuk's Theorem 60.4 it is convenient to introduce one definition and to prove one lemma.

Definition. Let $k \in \mathbb{N}_0$ and let $n \in \mathbb{N}$. A $\frac{1}{2^k}$-cube is a subset C of $\mathbb{R}^n$ of the form

$$C = \left[\tfrac{a_1}{2^k}, \tfrac{a_1+1}{2^k}\right] \times \cdots \times \left[\tfrac{a_n}{2^k}, \tfrac{a_n+1}{2^k}\right] \quad \text{with } a_1, \ldots, a_n \in \mathbb{Z}.$$

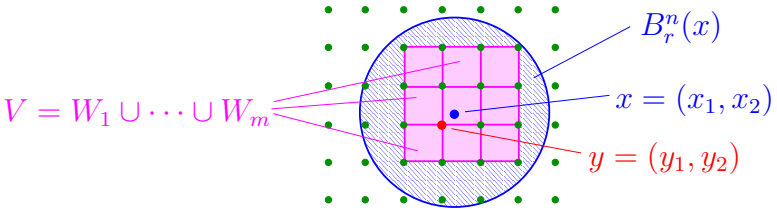
Put differently, a closed $\frac{1}{2^k}$-cube is a cube in $\mathbb{R}^n$ with side length $\frac{1}{2^k}$ and where the coordinates of all vertices lie in $\frac{1}{2^k} \cdot \mathbb{Z}$.

Lemma 60.7. (Point-in-Cubes-in-Ball Lemma) Let $x \in \mathbb{R}^n$ and let $r > 0$. If $k \in \mathbb{N}_0$ is chosen such that $\frac{1}{2^k} < \frac{1}{3\sqrt{n}} \cdot r$, then there exist finitely many $\frac{1}{2^k}$-cubes $W_1, \ldots, W_m$ such that

$$x \in \text{interior of } (W_1 \cup \cdots \cup W_m) \subset W_1 \cup \cdots \cup W_m \subset B_r^n(x).$$

Proof. Let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and let $r > 0$. Furthermore let $k \in \mathbb{N}_0$ such that $\frac{1}{2^k} < \frac{1}{3\sqrt{n}} \cdot r$. We write $\epsilon := \frac{1}{2^k}$. For $i = 1, \ldots, n$ we denote by y_i the result of rounding x_i down to the closest number in $\mathbb{Z} \cdot \epsilon$. Clearly we have

$$\begin{aligned}(x_1, \ldots, x_n) \in [y_1, y_1+\epsilon] \times \cdots \times [y_n, y_n+\epsilon] &\subset (y_1-\epsilon, y_1+2\epsilon) \times \ldots \times (y_n-\epsilon, y_n+2\epsilon) \\ &= \text{interior of } \underbrace{[y_1-\epsilon, y_1+2\epsilon] \times \ldots \times [y_n-\epsilon, y_n+2\epsilon]}_{=:V}.\end{aligned}$$



Since V is the union of finitely many ϵ-cubes it suffices to show that V is contained in $B_r^n(x)$. Thus let $y \in V$. We have

$$d(y,x) \underset{\substack{\uparrow \\ \text{since } x \in V}}{\leq} \text{diameter of } V = \sqrt{n} \cdot 3\epsilon = \sqrt{n} \cdot \frac{3}{2^k} \underset{\substack{\uparrow \\ \text{choice of } k}}{<} r$$

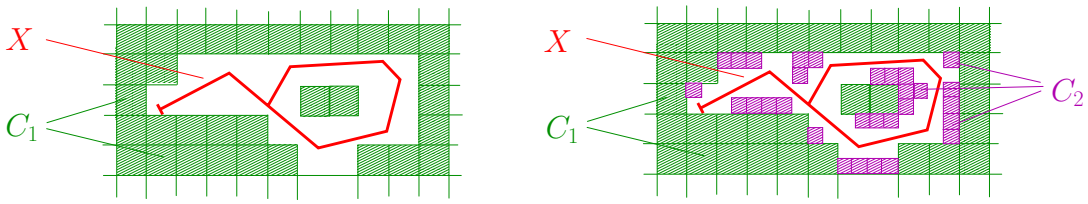
which means by definition that $y \in B_r^n(x)$. Thus we have shown that $V \subset B_r^n(x)$. ∎

Now we turn to the actual proof of Borsuk's Theorem 60.4:

Proof of Borsuk's Theorem 60.4. Let X be a closed, weakly locally contractible subset of $\mathbb{R}^n$. It follows from the discussion in Footnote 496 that it suffices to show that there exists a neighborhood (not necessarily open) W of X in $\mathbb{R}^n$ that admits a retraction $W \to X$. Evidently we only have to consider the case that X is non-empty. We introduce the following notation:

(1) We define C_1 to be the union of all 1-cubes that do not intersect X.
(2) We define C_2 to be the union of all $\frac{1}{2}$-cubes that do not intersect $\overset{\circ}{C}_1$ and that do not intersect X.
(3) We iterate this process, i.e. for any $k \geq 3$ we define C_k to be the union of all $\frac{1}{2^k}$-cubes that do not intersect the interior of $C_1 \cup \cdots \cup C_{k-1}$ and that do not intersect X.

Claim 1.
(1) We have $\bigcup_{k \in \mathbb{N}} C_k = \mathbb{R}^n \setminus X$.



(2) Given any $x_0 \in \mathbb{R}^n \setminus X$ there exists an open neighborhood U that intersects only finitely many cubes of $C_1, C_2, \ldots$.

Proof. Let $x_0 \in \mathbb{R}^n \setminus X$. Since X is closed we know that $\mathbb{R}^n \setminus X$ is open. This means that there exists an $r > 0$ such that $B_r^n(x_0) \subset \mathbb{R}^n \setminus X$. We pick $k \in \mathbb{N}_0$ such that $\frac{3 \cdot \sqrt{n}}{2^k} < r$. By the Point-in-Cubes-in-Ball Lemma 60.7 we can find finitely many $\frac{1}{2^k}$-cubes $W_1, \ldots, W_d$ such that

$$x_0 \in \underbrace{\text{interior of } W_1 \cup \cdots \cup W_d}_{:=U} \subset B_r^n(x).$$

This implies in particular that x_0 is contained in C_k or it is contained already in one of $C_1, \ldots, C_{k-1}$.

Subclaim. If there exists an $m \geq k$ and a $\frac{1}{2^m}$-cube in $\mathbb{R}^n \setminus X$ which intersects U, there exists already a $\frac{1}{2^k}$-cube in $\mathbb{R}^n \setminus X$ which intersects U.

Proof of subclaim. First note that an elementary argument shows that

$$U = (W_1 \cup \cdots \cup W_d) \setminus \text{the union of all other } \tfrac{1}{2^k}\text{-cubes}.$$

Now let Y be a $\frac{1}{2^m}$-cube in $\mathbb{R}^n \setminus X$ which intersects U, where $m \geq k$. Note that there exists a unique $\frac{1}{2^k}$-cube Z such that $Y \subset \mathring{Z}$. Note that this implies that $Y \subset Z$.



It follows from this discussion and the above description of U that Z is one of the W_i. The subclaim now follows from the observation that each W_i intersects U.

Finally note that follows from this subclaim and the iterative definition of the C_i that U intersects only finitely many cubes of $C_1, C_2, \ldots$. ⊞

Claim 2. The topological space $\mathbb{R}^n \setminus X = C := \bigcup_{k \in \mathbb{N}} C_k$ admits a CW-structure that restricts to the standard CW-structure on each cube.

Proof. We view each C_i as a CW-complex in the obvious way: The j-skeleton of C_i is given by all points of C_i such that at most j coordinates lie outside $\mathbb{Z} \cdot \frac{1}{2^k}$. One can easily verify that with this CW-structure the n-cells correspond precisely to the $\frac{1}{2^i}$-cubes and the lower dimensional cells correspond precisely to the faces of the cubes. Furthermore one can easily show, say using the the CW-Gluing Proposition 33.2 that for each k the union $C_1 \cup \cdots \cup C_k$ is also a CW-complex in an obvious way and finally we can also view the union $C := \bigcup_{k \in \mathbb{N}} C_k = \mathbb{R}^n \setminus X$ as a CW-complex with the obvious filtration.[499] ⊞

[499] We leave it to the reader to verify, using Statement (2) of the claim (which says that the collection of cubes involved is locally finite) and using the Cover-Open Subset Lemma 1.5 (2), that condition (CW1) of the definition of a CW-complex, see page 405, is satisfied.

We continue with the following definition:

- Given $x, y \in \mathbb{R}^n$ we write $d(x,y) \coloneqq \|x - y\|$.
- Given $x \in X$ and given a subset $\varnothing \neq Y \subset \mathbb{R}^n$ we write $d(x,Y) \coloneqq \inf\{d(x,y) \,|\, y \in Y\}$.
- Given a cell σ of C and a continuous map $f\colon \sigma \to X$ we set[500] [501]

$$\rho(f) \;\coloneqq\; \max\{d(z, f(z)) \,|\, z \in \sigma\}.$$

We think of $\rho(f)$ as the maximal displacement of f.

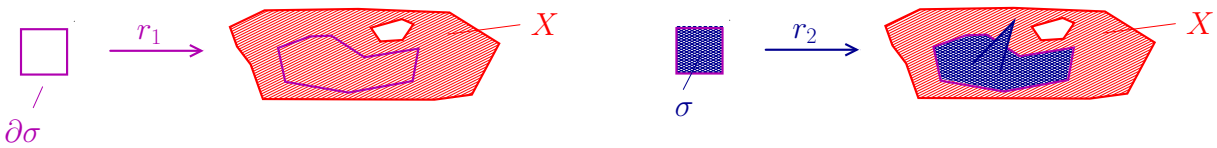
We start with the iterative definition of a subset of $\mathbb{R}^n$ together with a map to X.

(a) We denote by A_0 the 0-skeleton of C. We define a map $r_0\colon A_0 \to X$. Let $a \in A_0$. Since X is a closed subset of $\mathbb{R}^n$ it follows froman elementary argument that we can find a point $r_0(a) \in X$ with $d(a, r_0(a)) = d(a, X)$.

(b) Now suppose inductively that we have defined a map $r_i\colon A_i \to X$ on a union A_i of some of the i-cells of C. We define the next set A_{i+1} and the corresponding map $r_{i+1}\colon A_{i+1} \to X$ as follows:

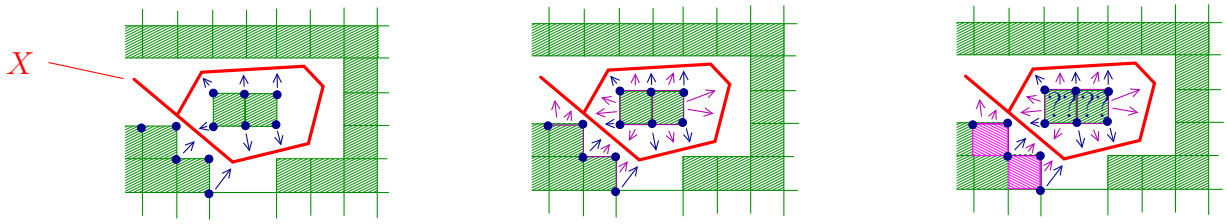
(b_1) Let σ be an $(i+1)$-cell of C for which r_i is defined on $\partial\sigma$. Suppose that r_i can be extended to a continuous map $g\colon \sigma \to X$ that agrees with r_i on $\partial\sigma$. Then we pick a continuous map $r_{i+1}\colon \sigma \to X$ with $r_{i+1}|_{\partial\sigma} = r_i$ such that r_{i+1} has "small displacement", more precisely, such that

$$\rho(r_{i+1}) \;\leq\; 2 \cdot \underbrace{\inf\{\rho(g) \,|\, g\colon \sigma \to X \text{ is a map with } g|_{\partial\sigma} = r_i|_{\partial\sigma}\}}_{\geq d(\sigma, X) \text{ which in turn is } > 0 \text{ since } X \subset \mathbb{R}^n \text{ is closed}}.$$



(b_2) We denote by A_{i+1} the union of all $(i+1)$-cells as in (b_1) and we define r_{i+1} on A_{i+1} as in (b_1).

(c) We set $A \coloneqq \bigcup_{i \in \mathbb{N}} A_i \cup X$ and we define $r\colon A \to X$ to be the unique map that equals r_i on each A_i and that is the identity on X.



It remains to show that the set A is a neighborhood of X and that the map $r\colon A \to X$ is continuous.

---

[500] Given $a, b \in \mathbb{R}^n$ we denote by $d(a,b) = \|a - b\|$ the usual Euclidean distance.

[501] Note that σ is compact, hence the maximum exists by the Compact Image Lemma 2.8 (2).

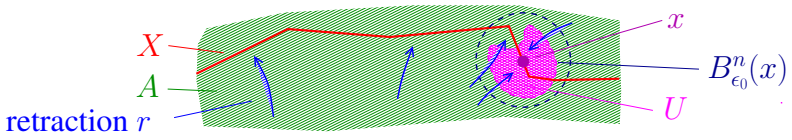
Let us first discuss the idea behind the proof. We start out with two observations:

- We can extend the map r_i over an $(i+1)$-cube σ if the image $r_i|_{\partial\sigma}$ is contractible.
- The extension we picked (assuming it exists) is "close" to the original map if $r_i|_{\partial\sigma}$ is contractible in a small neighborhood of $r_i(\partial\sigma)$.

To get a suitable extension and to get an extension that is close to the original map we will use that X is weakly locally contractible.

We now have to make this idea work. We start out with the following claim, which implies already that A is indeed a neighborhood of X and which will be essential in showing that the map $r\colon A \to X$ is indeed continuous:

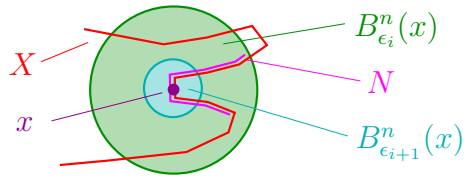
Claim 3. Given any $x \in X$ and any $\epsilon_0 > 0$ there exists an open neighborhood U of x that is contained in A and such that $\|\underbrace{r(x)}_{=x} - r(y)\| < \epsilon_0$ for all $y \in U$, i.e. such that $r(U) \subset B^n_{\epsilon_0}(x)$.



Proof. So let $x \in X$ and let $\epsilon_0 > 0$. We adopt the following notation: Given $\mu > 0$ we write $K(\mu) \coloneqq X \cap B^n_\mu(x)$.

Subclaim 1. We can find $\epsilon_0, \dots, \epsilon_{2n} \in \mathbb{R}_{>0}$[502] with the following two properties:
(1) For each $i \in \{0, \dots, 2n-1\}$ we have $\epsilon_{i+1} < \frac{1}{3} \cdot \epsilon_i$.
(2) For every $i = 1, \dots, 2n$ the subset $K(\epsilon_i)$ is contractible in $K(\epsilon_{i-1})$.

Proof of subclaim. First we set $\epsilon_0 \coloneqq 1$. We now suppose that we have already found $\epsilon_0, \dots, \epsilon_i$ for some $i \in \{0, \dots, 2n-1\}$ with the desired properties. Since X is weakly locally contractible we can find an open neighborhood N of x in X that is contained in $K(\epsilon_i)$ and that is contractible in $K(\epsilon_i)$. Since N is open there exists an $\epsilon_{i+1} < \frac{1}{3}\epsilon_i$ such that $K(\epsilon_{i+1}) \subset N$.



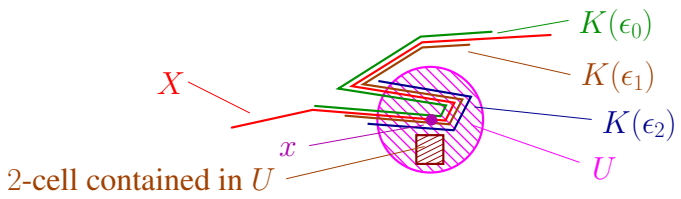
Since N is contractible in $K(\epsilon_i)$ the subset $K(\epsilon_{i+1})$ is also contractible in $K(\epsilon_i)$. □

We set $U \coloneqq B^n_{\frac{1}{4}\epsilon_{2n}}(x)$. It suffices to prove the following two subclaims:

Subclaim 2. For each $i = 0, \dots, n$ the map r is defined for each i-cell σ of the CW-complex C that lies in U and for each such σ we have $r(\sigma) \subset K(\epsilon_{2n-2i})$.

Subclaim 3. The union of X with all cells of C that lie in U is a neighborhood of x.

[502] We only stop at the index $2n$ since that is what we need later on. One could also find an infinite sequence of ϵ_i with the stated properties.



Proof of subclaim 2. We prove Subclaim 2 iteratively for $i = 0, 1, \dots, n$. We start out with $i = 0$. Thus we have to show that $r(U \cap A_0) \subset K(\epsilon_{2n})$. Therefore let $y \in U \cap A_0$. Then we have

$$d(x, r(y)) \underset{\substack{\uparrow \\ \text{by the triangle inequality} \\ \text{and since } r(y) = r_0(y)}}{\leq} d(x,y) + d(y, r_0(y)) \underset{\substack{\uparrow \\ \text{by (a) since } x \in X}}{\leq} d(x,y) + d(y,x) = 2d(x,y) \underset{\substack{\uparrow \\ \text{since } y \in U = B^n_{\frac{1}{4}\epsilon_{2n}}(x)}}{<} 2 \cdot \tfrac{1}{4} \cdot \epsilon_{2n} = \tfrac{1}{2} \cdot \epsilon_{2n}.$$

But since $r(y) \in X$ this means precisely that $r(y) \in K(\epsilon_{2n}) = X \cap B^n_{\epsilon_{2n}}(x)$. This shows that Subclaim 1 holds for $i = 0$.

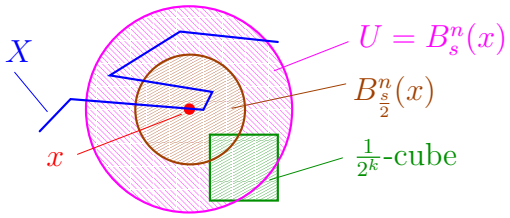
Now suppose that Subclaim 1 holds for some $i \in \{0, \dots, n-1\}$. Let σ be an $(i+1)$-cell of the CW-complex C that is contained in U. Since $r(\partial\sigma) \subset K(\epsilon_{2n-2i})$ and since $K(\epsilon_{2n-2i})$ is contractible in $K(2n-2i-1)$ it follows from the Convex-to-Ball Proposition 2.12 and the Homotopy-π_n-Proposition 28.6 (3) that we can extend $r|_{\partial\sigma} = r_i|_{\partial\sigma}$ to a continuous map $s\colon \sigma \to K(\epsilon_{2n-2i-1})$. By definition of A_{i+1} this implies that $\sigma \subset A_{i+1}$. In particular r is defined on σ. It remains to show that $r(\sigma) \subset K(\epsilon_{2n-2i-2})$. Thus let $y \in \sigma$.

We have

$$\begin{aligned}
d(x, r(y)) &\underset{\substack{\uparrow \\ \text{by the triangle inequality and since } r(y) = r_{i+1}(y)}}{\leq} d(x,y) + d(y, r_{i+1}(y)) \underset{\substack{\downarrow \\ \text{since } y \in \sigma \subset U = B^n_{\frac{1}{4}\epsilon_{2n}}(x) \text{ and by definition of } \rho(r_{i+1})}}{\leq} \tfrac{1}{4}\epsilon_{2n} + \rho(r_{i+1}) \underset{\substack{\uparrow \\ \text{by } (b_1)}}{\leq} \tfrac{1}{4}\epsilon_{2n} + 2 \cdot \inf\left\{ \rho(g) \,\middle|\, \begin{matrix} g\colon \sigma \to X \text{ with} \\ g|_{\partial\sigma} = r_i|_{\partial\sigma} \end{matrix} \right\}. \\
&\underset{\substack{\uparrow \\ \text{since } s \text{ is such a map}}}{\leq} \tfrac{1}{4}\epsilon_{2n} + 2 \cdot \underbrace{\max\{d(z, s(z)) \mid z \in \sigma\}}_{=\rho(s)} \underset{\substack{\uparrow \\ \text{by triangle inequality}}}{\leq} \tfrac{1}{4}\epsilon_{2n} + 2 \cdot \max\{d(z,x) + d(x, s(z)) \mid z \in \sigma\} \\
&\underset{\substack{\uparrow \\ \text{since } \sigma \subset U = B^n_{\frac{1}{4}\epsilon_{2n}}(x) \text{ and since for each} \\ z \in \sigma \text{ we have } s(z) \in K(\epsilon_{2n-2i-1}) \subset B^n_{\epsilon_{2n-2i-1}}(x)}}{\leq} \tfrac{1}{4}\epsilon_{2n} + 2\left(\tfrac{1}{4}\epsilon_{2n} + \epsilon_{2n-2i-1}\right) = \tfrac{3}{4}\epsilon_{2n} + 2\epsilon_{2n-2i-1} \underset{\substack{\uparrow \\ \text{since for each } j \text{ we have } \epsilon_{j+1} \leq \frac{1}{3}\epsilon_j}}{\leq} \tfrac{1}{12}\epsilon_{2n-2i-2} + \tfrac{2}{3}\epsilon_{2n-2i-2} \leq \epsilon_{2n-2i-2}.
\end{aligned}$$

This concludes the proof that $r(\sigma) \subset K(\epsilon_{2n-2i-2})$. □

Proof of subclaim 3. We set $s = \frac{1}{4}\epsilon_{2n}$. As a reminder, we have to show that the union of X with all cells of C that lie in $B^n_s(x)$ is a neighborhood of x. Let W be a $\frac{1}{2^k}$-cube with $W \cap B^n_{\frac{s}{2}}(x) \neq \varnothing$ and $W \cap (\mathbb{R}^n \setminus B^n_s(x)) \neq \varnothing$.



It follows that the diameter of W is at least $\frac{s}{2}$ which in turn implies that $\frac{\sqrt{n}}{2^k} \geq \frac{s}{2}$. We set $k_0 = \lfloor \ln_2(\frac{2\sqrt{n}}{s}) \rfloor$. We see that

$$
\begin{aligned}
&\text{union of } X \text{ with all cells of } C \text{ that lie in } B^n_s(x) \\
&\quad \supset \text{ union of } X \text{ with all cubes of } C \text{ that lie in } B^n_s(x) \\
&\quad \supset B^n_s(x) \setminus \text{ union of all cubes of } C \text{ that are not completely contained in } B^n_s(x) \\
&\qquad \uparrow \\
&\qquad \text{since } \mathbb{R}^n \setminus X = C \\
&\quad \supset B^n_s(x) \setminus \text{ union of all cubes of } C \text{ that are not completely contained in } B^n_s(x) \\
&\qquad\qquad \text{which do not contain the point } x \text{ but which intersect } B^n_{\frac{s}{2}}(x) \\
&\quad \supset B^n_s(x) \setminus \underbrace{\text{union of all } \tfrac{1}{2^k}\text{-cubes with } k \leq k_0 \text{ of } C \text{ that are not contained in } B^n_s(x)}_{\text{finite union of cubes, hence a closed subset}} \\
&\qquad \uparrow \qquad\qquad \underbrace{\text{which do not contain the original point } x \text{ but which do intersect } B^n_{\frac{s}{2}}(x).}_{\text{finite union of cubes, hence a closed subset}} \\
&\qquad \text{definition of } k_0
\end{aligned}
$$

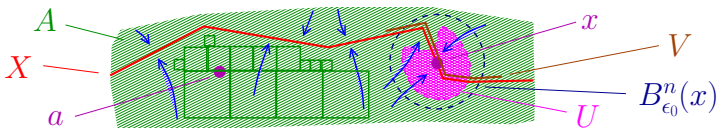
The last set is clearly an (open) neighborhood of x. But then the first set is also a neighborhood of x. This concludes the proof of Subclaim 3 and thus the proof of Claim 3. ⊞

Recall that we need to show that A is indeed a neighborhood of X and we that we need to show that the map $r\colon A \to X$ is indeed continuous. Fortunately it follows immediately from Claim 3 that A is a neighborhood of X. So it remains to prove the following claim:

Claim 4. The map $r\colon A \to X$ is continuous.

Proof. Let V be an open subset of X. We need to show that $r^{-1}(V) \subset A$ is open. It follows from the JH-Openness Criterion 1.1 that it suffices to show that for each $a \in A$ there exists an open neighborhood U_a of $a \in A$ with $r(U_a) \subset W$.

First we consider the case that $a \notin X$. In this case a is in the interior of the union C of finitely many cubes. By construction the restriction of r to each cube is continuous. It follows from the Pasting Proposition 2.4 that $r|_C\colon C \to X$ is continuous. In particular $r^{-1}(W) \subset C$ is open. It follows that $U_a := r^{-1}(W) \cap \mathring{C}$ has the desired properties.



Finally we need to deal with the case $a \in X$. Since V is an open neighborhood of a we see that there exists an $\epsilon_0 > 0$ such that $B_{\epsilon_0}(a) \cap V \subset V$. We now obtain the desired open neighborhood U_a of a by appealing to Claim 3. ■

60.4. **Applications to topological manifolds.** In the following we want to study applications of Borsuk's Theorem 60.4. The key to many applications is the following proposition:

Proposition 60.8. (Simplicial Complex Retract Proposition) Let $K \subset \mathbb{R}^n$ be a non-empty subspace.

(1) If K is a neighborhood retract, then K is also the retract of a countable n-dimensional (possibly infinite) simplicial complex.[503]
(2) If $K \subset \mathbb{R}^n$ is compact and if it is a neighborhood retract, then K is the retract of a finite n-dimensional simplicial complex.

Proof. We prove the proposition in the reverse order:

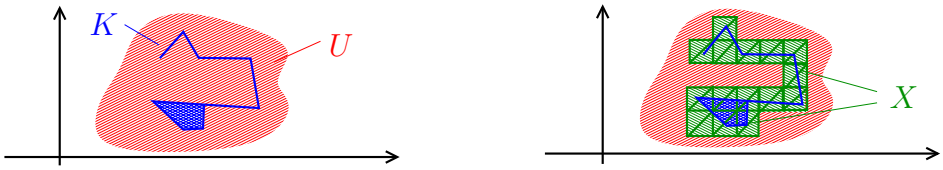
(2) Let K be a compact non-empty subspace of $\mathbb{R}^n$ which is a retract of some open neighborhood $U \subset \mathbb{R}^n$. Clearly it suffices to prove the following claim:

Claim. There exists a finite n-dimensional simplicial complex X with $K \subset X \subset U$.

Proof. The idea is to construct a suitable simplicial complex $X \subset \mathbb{R}^n$ as the union of finitely many sufficiently small cubes. To do so we set

$$r \;:=\; d(K, \mathbb{R}^n \setminus U) \;=\; \inf\{d(x,y) \,|\, x \in K, y \in \mathbb{R}^n \setminus U\}.$$

Since K is compact and since $\mathbb{R}^n \setminus U$ is closed it follows froman elementary argument that $r > 0$. We pick $k \in \mathbb{N}_0$ such that $\frac{1}{2^k} < \frac{1}{\sqrt{n}} \cdot r$. We denote by X the union of all $\frac{1}{2^k}$-cubes that intersect K. (See page 769 for the definition of a $\frac{1}{2^k}$-cube.) It follows easily from the hypothesis that K is compact that X is the union of finitely many $\frac{1}{2^k}$-cubes. The same argument as in the proof of the Point-in-Cubes-in-Ball Lemma 60.7 shows, by our choice of k, that each $\frac{1}{2^k}$-cube is contained in U.



We equip each cube with the standard simplicial structure. It follows easily from the Compact-Hausdorff Proposition 2.10 (3) that these simplicial structures on the cubes define an n-dimensional simplicial structure on X. ⊞

(1) Let $K \subset \mathbb{R}^n$ be a non-empty neighborhood retract. By definition K is a retract of an open non-empty neighborhood U. Thus it remains to prove the following claim:

Claim. Every open non-empty subset U of $\mathbb{R}^n$ admits the structure of a countable n-dimensional simplicial complex.

Proof. We provide a sketch of the proof:

(a) We introduce the following subsets of $\mathbb{R}^n$:
- We define C_1 to be the union of all 1-cubes that are contained in U.
- We define C_2 to be the union of all $\frac{1}{2}$-cubes that do not intersect $\mathring{C}_1$ and that are contained in U.
- We iterate this process, i.e. for any $k \geq 3$ we define C_k to be the union of all $\frac{1}{2^k}$-cubes that do not intersect the interior of $C_1 \cup \cdots \cup C_{k-1}$ and that are contained in U.

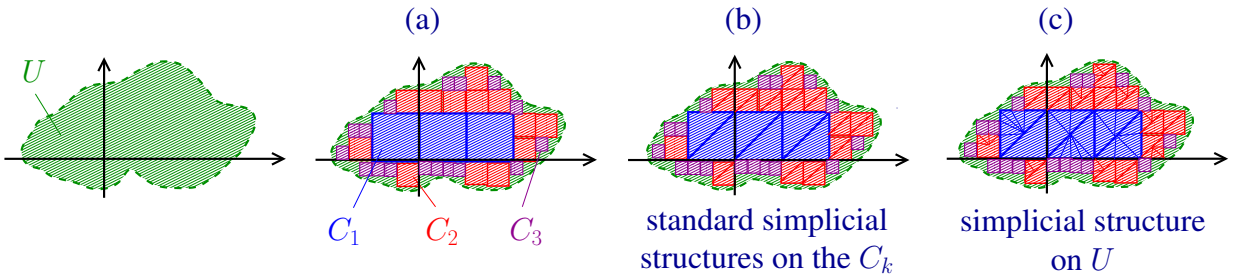
One can easily show (using the same logic as in the proof of Claim 1 on page 769) that $U = \bigcup_{k \in \mathbb{N}_0} C_k$.

(b) We equip each cube with the standard simplicial structure.

(c) We iteratively subdivide the simplicial structures on $C_1, C_2, \ldots$ so that the simplicial structures match.

This construction defines an abstract simplicial complex K together with a bijective map $f\colon |K| \to U$. It follows relatively easily from the Pasting Proposition 2.4 that the

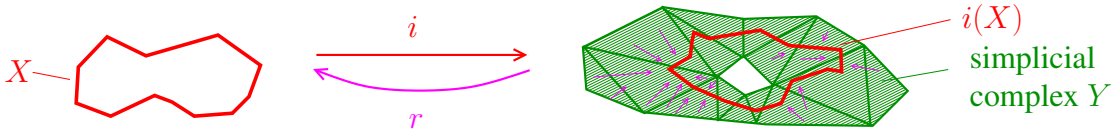
[503] We did not define the notion of a simplicial complex. For the purpose of this course it suffices to think of a CW-complex made out of embedded simplices which intersect only in faces.



inverse $f^{-1}\colon U \to |K|$ is also continuous. Thus we have shown that $f\colon |K| \to U$ is a homeomorphism. Finally note that it follows easily from the construction that K is in fact a countable simplicial complex and that each simplex has dimension $\leq n$. ■

Before we state the next theorem we need to introduce the following elementary definition:

Definition. Let X be a topological space and let $\mathcal{C}$ be a class of topological spaces (e.g. $\mathcal{C}$ could be the class of simplicial complexes or CW-complexes). We say that X is a retract of an element in $\mathcal{C}$ if there exists a topological space $Y \in \mathcal{C}$, a continuous map $i\colon X \to Y$ and a continuous map $r\colon Y \to X$ with $r \circ i = \mathrm{id}_X$.



We can now formulate the following theorem:

Theorem 60.9. (Topological Manifold-Simplicial Complex Retract Theorem)
(1) Every topological manifold is a retract of a countable finite-dimensional simplicial complex.
(2) Every compact topological manifold is a retract of a finite simplicial complex.

Proof. Let M be a topological manifold. By the Euclidean Embedding Theorem 60.6 we know that there exists an embedding $i\colon M \to \mathbb{R}^n$ such that $i(M)$ is a closed subset of $\mathbb{R}^n$. As we discussed on page 764, every topological manifold is weakly locally contractible. Thus we can apply Borsuk's Theorem 60.4 to deduce that $i(M) \subset \mathbb{R}^n$ is a neighborhood retract. The desired statement now follows from the Simplicial Complex Retract Proposition 60.8. ■

60.5. **Invariants of topological manifolds.** The Topological Manifold-Simplicial Complex Retract Theorem 60.9 allows us to get a grip on the size of invariants of topological manifolds:

Proposition 60.10. (Topological Manifolds-Invariants Proposition) Let M be a connected non-empty topological manifold. The following statement holds:
(1) (a) The fundamental group of M is countable.
    (b) If M is compact, then the fundamental group of M is finitely presented.
(2) (a) All homology groups $\mathrm{H}_k(M)$ are countably generated abelian groups.

(b) If M is compact, then all homology groups $\mathrm{H}_k(M)$ are finitely generated abelian groups.

Furthermore, given any field $\mathbb{F}$ the homology groups $\mathrm{H}_k(M;\mathbb{F})$ are of countable dimension and if M is compact, then they are finite-dimensional.

(3) There exists an $l \in \mathbb{N}_0$ such that for every $k > l$ and for every abelian group G we have $\mathrm{H}_k(M;G) = 0$.[504]

(4) The higher homotopy groups of M are countable.

The following lemma is the one extra input we need to prove the Topological Manifolds-Invariants Proposition 60.10.

Lemma 60.11. (Simplicial Complex-Retract-Invariants Lemma) Let X be a path-connected non-empty topological space that is a retract of a countable n-dimensional simplicial complex Y. The following statement holds:

(1) (a) The fundamental group of X is countable.

(b) If X is compact, then the fundamental group of X is finitely presented.

(2) (a) All homology groups $\mathrm{H}_k(X)$ are countably generated abelian groups.

(b) If X is compact, then all homology groups $\mathrm{H}_k(X)$ are finitely generated abelian groups.

Furthermore, given any field $\mathbb{F}$ the homology groups $\mathrm{H}_k(X;\mathbb{F})$ are of countable dimension and if X is compact, then they are finite-dimensional.

(3) For every $k > n$ and for every abelian group G we have $\mathrm{H}_k(X;G) = 0$.

(4) The higher homotopy groups of X are countable.

Proof of Proposition 60.10 assuming Lemma 60.11. Let M be a path-connected non-empty n-dimensional topological manifold. By the Topological Manifold-Simplicial Complex Retract Theorem 60.9 we know that M is the retract of a countable finite-dimensional simplicial complex Y. The theorem also tells us that if M is compact, then we can arrange that Y is finite. Thus the desired statements follow immediately from the Simplicial Complex-Retract-Invariants Lemma 60.11. ■

Proof of the Simplicial Complex-Retract-Invariants Lemma 60.11. Let X be a path-connected non-empty topological space that is a retract of a countable n-dimensional simplicial complex Y. Recall that this means that there exists a continuous map $i\colon X \to Y$ and a continuous map $r\colon Y \to X$ with $r \circ i = \mathrm{id}_X$. By considering the path-component of Y that contains $i(X)$ we can assume that Y is path-connected. Furthermore let $\mathbb{F}$ be a field and let G be an abelian group. We pick a base point $x_0 \in X$. We write $y_0 \coloneqq i(x_0)$.

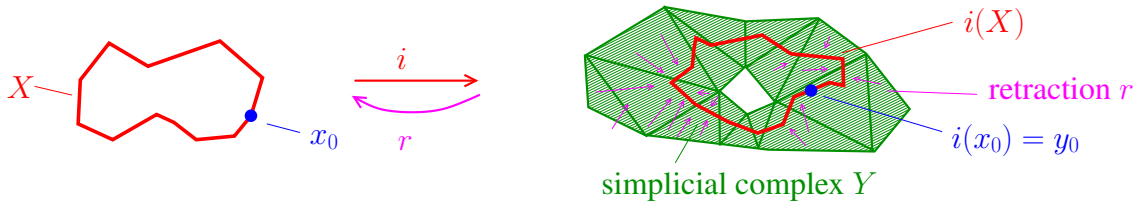
If X is compact, then it follows from the Compact Image Lemma 2.8 and the CW-Complex-Finiteness Theorem 30.10 that X is contained in a finite subcomplex. So if X is compact, we can arrange that Y is a finite simplicial complex.

We start out with the following claim:

Claim.

(1) (a) $\pi_1(Y)$ is countable.

[504] Let M be an n-dimensional topological manifold. In the Manifold-Top-H_*-Theorem 63.1 we will see that we can take $l = \dim(M)$ and we will in fact be able compute the homology group $\mathrm{H}_n(M)$ in terms of the topology of M.



(b) If Y is a finite simplicial complex, then $\pi_1(Y)$ is finitely presented.
(2) (a) All homology groups $\mathrm{H}_k(Y)$ are countable and for any field $\mathbb{F}$ the homology groups $\mathrm{H}_k(Y;\mathbb{F})$ are of countable dimension.
(b) If Y is finite, then all homology groups $\mathrm{H}_k(Y)$ are finitely generated and for any field $\mathbb{F}$ the homology groups $\mathrm{H}_k(Y;\mathbb{F})$ are finite-dimensional.
(3) For every $k > n$ we have $\mathrm{H}_k(Y;G) = 0$.
(4) The higher homotopy groups of Y are countable.

Proof.

(1) This statement follows immediately from the the CW-Skeleton-π_1-Proposition 30.14 and the 1-Complex-π_1-Theorem 31.5.
(2),(3) These two statements follow from theSingular-Cellular-H_*-Proposition 47.3 together with the Skeleton-H_*-Lemma 47.1.
(4)) This statement follows from the Simplicial Approximation Theorem [**Fri24**] which says that any continuous map from a finite simplicial complex to a simplicial complexes is, after barycentric subdivisions, homotopic to a simplicial (thus combinatorial) map. This implies that the homotopy groups of a countable simplicial complex are countable. ⊞

Now we turn to the invariants of X. Note that we deal with them in a slightly unexpected order:

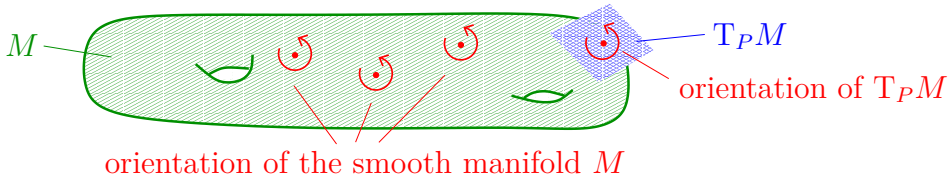
(2) Let $k \in \mathbb{N}_0$. Note that it follows from the functoriality of homology groups and the fact that $r \circ i = \mathrm{id}_X$ that the maps $r_*\colon \mathrm{H}_k(Y) \to \mathrm{H}_k(X)$ and $r_*\colon \mathrm{H}_k(Y;G) \to \mathrm{H}_k(X;G)$ are epimorphisms. In particular if one of the groups to the left is countable (finitely generated, finite-dimensional, of countable dimension), then so is the corresponding group to the right. This observation together with Claim (2) implies the desired Statement (2).
(3) In the proof of (2) we just saw that the induced maps $r_*\colon \mathrm{H}_k(Y) \to \mathrm{H}_k(X)$ and $r_*\colon \mathrm{H}_k(Y;\mathbb{F}) \to \mathrm{H}_k(X;\mathbb{F})$ are epimorphisms. In particular if the groups to the left vanish, then so do the groups to the right. This observation together with Claim (3) implies the desired Statement (3).
(4) As in (2) we see that $r_*\colon \pi_n(Y, y_0) \to \pi_n(X, x_0)$ is an epimorphism. It follows from Claim (4) and the Countability Lemma 0.6 that $\pi_n(X, x_0)$ is also countable.
(1) (a) The logic of (2), (3) and (4) together with Claim (1) shows immediately that $\pi_1(X, x_0)$ is countable.
(b) Now assume that X is compact. As we discussed above, this means that we can assume that Y is a finite simplicial complex. The logic of (2), (3) and (4) together with Claim (1) shows immediately that $\pi_1(X, x_0)$ is *finitely generated*. The

statement that $\pi_1(X, x_0)$ is *finitely presented* requires a little more thought. More precisely, by the functoriality of fundamental groups we know that $\pi_1(X, x_0)$ is a *retract* (in the sense of the definition on page 311) of $\pi_1(Y, y_0)$. By Claim (1) we know that the group $\pi_1(Y, y_0)$ is finitely presented. Thus we obtain from the purely group-theoretic Finite Presentation-Retract Lemma 21.8 that $\pi_1(X, x_0)$ is also finitely presented. ■

CHAPTER 61

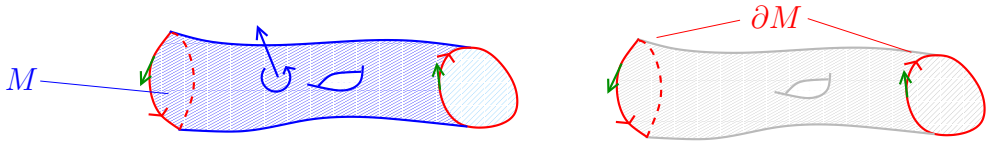
Orientations of topological manifolds

In an earlier course we introduced the notion of an orientation of a smooth manifold M. The basic idea is that an orientation of a smooth manifold M is an orientation of each tangent space $\mathrm{T}_P M$ such that the orientations are "continuous in P".



We showed that orientations have various "inheritance properties":

(1) Codimension-zero submanifolds and coverings of oriented smooth manifolds inherit a natural orientation.
(2) The boundary of an oriented smooth manifold inherits a natural orientation.



(3) The product of two oriented smooth manifolds admits a natural orientation.

But what is an orientation actually good for? There are several answers:

(a) Let M be a closed oriented non-empty n-dimensional smooth manifold. In an earlier course we showed that we can integrate any smooth n-form ω over M. Furthermore we showed, using Stokes' Theorem, that the map

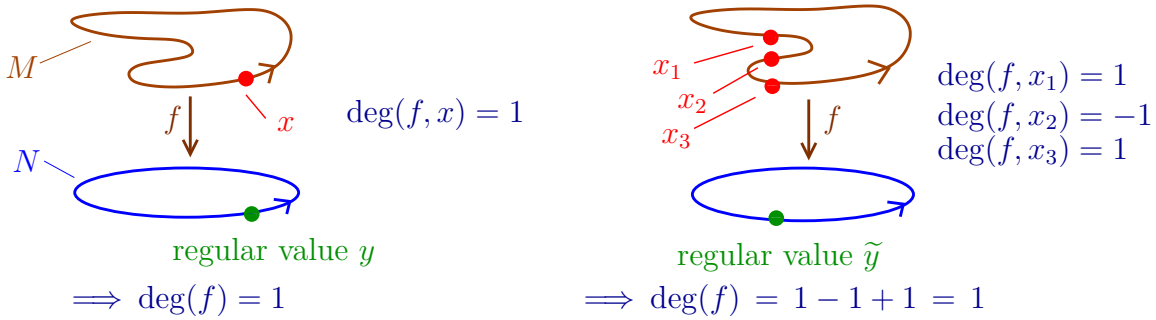
$$\begin{array}{ccc} \overbrace{\mathrm{H}^n_{\mathrm{dR}}(M)}^{\text{de Rham cohomology}} & \to & \mathbb{R} \\ [\omega] & \mapsto & \displaystyle\int_M \omega \\ \uparrow & & \\ \text{smooth } n\text{-form} & & \end{array}$$

is well-defined and that it is an epimorphism. Finally note than one can show that if M is furthermore connected, then the above map $\mathrm{H}^n_{\mathrm{dR}}(M) \to \mathbb{R}$ is even an isomorphism.
(b) To every smooth map $f\colon M \to N$ between two closed oriented connected non-empty n-dimensional smooth manifolds we can associate a degree $\deg(f) \in \mathbb{Z}$. Here is an outline of the definition: We pick a regular value $y \in N$ (which exists by Sard's Theorem

[**Fri24**]) and we define

$$\deg(f) \coloneqq \sum_{x\in f^{-1}(y)} \begin{cases} +1, & \text{if } \mathrm{D}f_x\colon \mathrm{T}_xM \to \mathrm{T}_yN \text{ is orientation-preserving,} \\ -1, & \text{if } \mathrm{D}f_x\colon \mathrm{T}_xM \to \mathrm{T}_yN \text{ is orientation-reversing.} \end{cases}$$

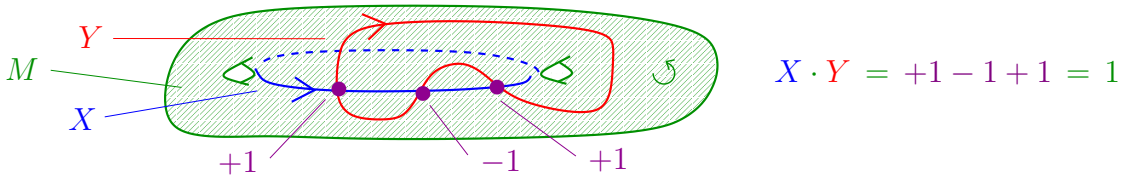
In [**Fri24**] it is shown that the degree is well-defined. We will not really make use of this concept, but it can be helpful to be aware of this concept.



(c) Let M be an oriented smooth manifold and let $X, Y \subset M$ be two closed oriented submanifolds of M that intersect transversally and with $\dim(X) + \dim(Y) = \dim(M)$. Given $P \in X \cap Y$ we define $\operatorname{sign}(P)$ as follows: If a positive basis of T_PX followed by a positive basis of T_PY is a positive basis of T_PM, then we set $\operatorname{sign}(P) = +1$. Otherwise we set $\operatorname{sign}(P) = -1$.

One can define the algebraic intersection number of X and Y as follows:

$$X \cdot Y \coloneqq \sum_{P\in X\cap Y} \operatorname{sign}(P).$$



Our goal in this chapter is to introduce the notion of an orientation of a *topological* manifold M which has the following features:

(0) It extends the previous notion in the sense that an oriented smooth manifold is naturally also an oriented topological manifold.
(1) Codimension-zero submanifolds and coverings of oriented topological manifolds have natural orientations.
(2) Boundaries of oriented topological manifolds have natural orientations.
(3) Products of oriented topological manifolds have natural orientations.

In the following chapters we will see that we also have the following analogues and generalizations of the above statements (a), (b) and (c):

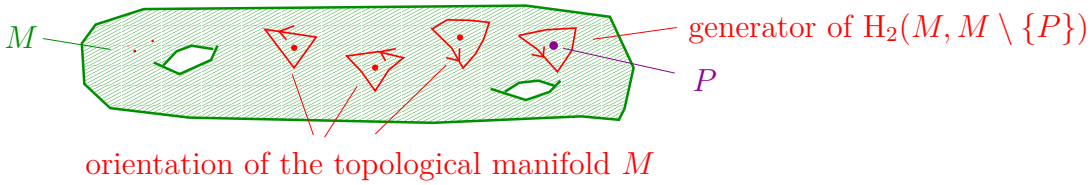
(a) We will see that every compact oriented connected non-empty n-dimensional topological manifold admits a natural isomorphism $\mathrm{H}_n(M, \partial M) \to \mathbb{Z}$.
(b) On page 545 we introduced the degree of any continuous map $S^n \to S^n$. We will use (a) to extend this concept. More precisely, we will assign to every continuous

map $f\colon M \to N$ between two closed oriented connected non-empty n-dimensional topological manifolds a degree $\deg(f) \in \mathbb{Z}$.

(c) We will introduce the intersection form of any compact oriented topological manifold. We will see that the intersection form is a generalization of the algebraic intersection number.

So how can we achieve this? As we just mentioned, an orientation of a smooth manifold is a "continuous" choice of orientations for each tangent space. Since topological manifolds do not come with tangent spaces this definition does not make sense for topological manifolds.

What we can do is the following: If M is an n-dimensional topological manifold, then given any $P \in M$ we can consider the local homology group $\mathrm{H}_n(M, M \setminus \{P\})$. If $P \notin \partial M$, then one can show easily that $\mathrm{H}_n(M, M \setminus \{P\}) \cong \mathbb{Z}$. Instead of picking one of the two orientations of the tangent space at P we now pick one of the two generators of the local homology group $\mathrm{H}_n(M, M \setminus \{P\})$. We then have to make sense of the notion that this choice is "continuous in P".



In the following sections we will develop this idea.

61.1. **Local homology of topological manifolds.** We recall the following definition from page 536:

Definition. Given a pointed topological space (X, x_0) and given a commutative ring R the homology groups $\mathrm{H}_n(X, X \setminus \{x_0\}; R)$ is called the n-th local homology group with R-coefficients of X at the point x_0.

In this section we will study the local homology of topological manifolds. We start out with the following basic lemma.

Lemma 61.1. (Manifold-Local R-Homology Lemma) Let M be an n-dimensional topological manifold and let R be a commutative ring. For every $k \in \mathbb{N}$ and every $x \in M$ the following statements hold:

(1) We have

$$\mathrm{H}_k(M, M \setminus \{x\}) \cong \begin{cases} \mathbb{Z}, & \text{if } x \in M \setminus \partial M \text{ and if } k = n, \\ 0, & \text{otherwise.} \end{cases}$$

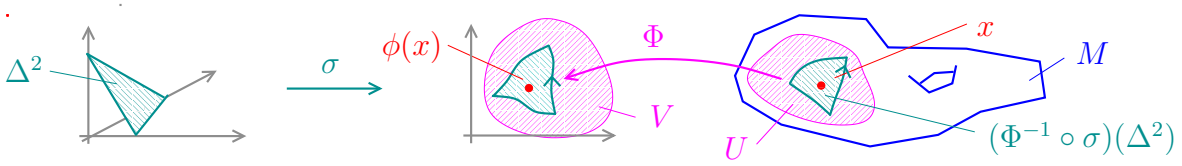
(2) The following map is an isomorphism of R-modules:

$$\begin{array}{rcl} \mathrm{H}_k(M, M \setminus \{x\}) \otimes R & \xrightarrow{\mu} & \mathrm{H}_k(M, M \setminus \{x\}; R) \\ \sum_{i=1}^{m} [\sigma_i] \otimes r_i & \mapsto & \sum_{i=1}^{m} [\sigma_i \otimes r_i]. \end{array}$$

(3) We have

$$\mathrm{H}_k(M, M \setminus \{x\}; R) \cong \begin{cases} R, & \text{if } x \in M \setminus \partial M \text{ and if } k = n, \\ 0, & \text{otherwise.} \end{cases}$$

Furthermore, if $x \in M \setminus \partial M$, if $\Phi\colon U \to V$ is a chart around x and if $\sigma\colon \Delta^n \to V$ represents a generator of $\mathrm{H}_n(V, V \setminus \{\Phi(x)\})$, then $(\Phi^{-1} \circ \sigma) \otimes 1_R$ represents a generator of $\mathrm{H}_n(M, M \setminus \{x\}; R)$.



Proof.

(1) First we consider the case that $x \in M \setminus \partial M$. Since $x \notin \partial M$ we can pick a chart $\Phi\colon U \to V$ around x such that V is an open subset of $\mathbb{R}^n$. We write $y = \Phi(x)$. We pick $r > 0$ with $\overline{B}^n_r(y) \subset V$. For any $k \in \mathbb{N}_0$ we then have

$$\mathrm{H}_k(M, M \setminus \{x\}) \xleftarrow[\uparrow]{\cong} \mathrm{H}_k(U, U \setminus \{x\}) \xrightarrow[\uparrow]{\Phi_*} \mathrm{H}_k(V, V \setminus \{y\}) \xleftarrow[\uparrow]{\cong} \mathrm{H}_k(\overline{B}^n_r(y), \overline{B}^n_r(y) \setminus \{y\})$$

by the Excision Theorem 40.2; isomorphism since Φ is a homeomorphism; by the Excision Theorem 40.2

$$\xrightarrow[\uparrow]{\partial_k} \widetilde{\mathrm{H}}_{k-1}(\overline{B}^n_r(y) \setminus \{y\}) \xrightarrow[\uparrow]{\cong} \widetilde{\mathrm{H}}_{k-1}(S^{n-1}_r(y)) \xrightarrow[\uparrow]{\cong} \begin{cases} \mathbb{Z}, & \text{if } k = n, \\ 0, & \text{otherwise.} \end{cases}$$

isomorphism since $\widetilde{\mathrm{H}}_*(\overline{B}^n_r(y)) = 0$; since $S^{n-1}_r(y)$ is a deformation retract of $\overline{B}^n_r(y) \setminus \{y\}$; by the Sphere-$\mathrm{H}_*$-Proposition 52.9

The case $x \in \partial M$ is dealt with by a similar argument which we leave to the reader.

(2) This statement follows easily from (1), the Universal Coefficient Theorem 54.9 and the Tor-Properties Lemma 54.7 (4).

(3) The calculation of $\mathrm{H}_k(M, M \setminus \{x\}; R)$ is an almost immediate consequence of (1), (2) and the Tensor-with-$\mathbb{Z}$-Lemma 51.3. The statement about the generator follows easily from the argument in (1). ∎

On a few occasions the following lemma will be useful.

Lemma 61.2. (Local-H_*-Reflection Lemma) Let $n \in \mathbb{N}_0$ and let $P \in \mathbb{R}^n$. Furthermore let $\rho\colon \mathbb{R}^n \to \mathbb{R}^n$ be the reflection in an affine hyperplane through P. For every commutative ring R the induced map

$$\rho_*\colon \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{P\}; R) \to \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{P\}; R)$$

is given by multiplication by minus -1.

Proof. Let $\rho_V\colon \mathbb{R}^n \to \mathbb{R}^n$ be the reflection in an affine hyperplane V through a point P. We denote by $f\colon \mathbb{R}^n \to \mathbb{R}^n$ the map that is given by addition by P and furthermore we write $W = f^{-1}(V)$. Note that W goes through the origin. We denote by $\rho_W\colon \mathbb{R}^n \to \mathbb{R}^n$ the reflection in the hyperplane W. We consider the following diagram:

$$\begin{array}{ccccccc}
\widetilde{\mathrm{H}}_{n-1}(S^{n-1}) \otimes R & \xrightarrow[\cong]{\mu} & \widetilde{\mathrm{H}}_{n-1}(S^{n-1}; R) & \xrightarrow{f_*} & \widetilde{\mathrm{H}}_{n-1}(\mathbb{R}^n \setminus \{P\}; R) & \xleftarrow[\cong]{\partial_n} & \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{P\}; R) \\
\downarrow{\scriptstyle \rho_{W*} \otimes \mathrm{id}} & & \downarrow{\scriptstyle \rho_{W*}} & & \downarrow{\scriptstyle \rho_{V*}} & & \downarrow{\scriptstyle \rho_{V*}} \\
\widetilde{\mathrm{H}}_{n-1}(S^{n-1}) \otimes R & \xrightarrow[\cong]{\mu} & \widetilde{\mathrm{H}}_{n-1}(S^{n-1}; R) & \xrightarrow{f_*} & \widetilde{\mathrm{H}}_{n-1}(\mathbb{R}^n \setminus \{P\}; R) & \xleftarrow[\cong]{\partial_n} & \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{P\}; R).
\end{array}$$

where μ is the natural map from page 673 and where the right-hand horizontal maps are the connecting homomorphisms from the Reduced LES-of-Pair Proposition 39.6.

It follows from the Sphere-Degree-Matrix Lemma 43.4 and the Matrix-Tensor-with-Ring Lemma 51.8 that the vertical map on the left is given by multiplication by -1. It follows from the naturality of μ and of the connecting homomorphism ∂_n, and the functoriality of homology, that the diagram commutes. Furthermore it follows from the Universal Coefficient Theorem 54.9 and other standard arguments that all horizontal maps are isomorphisms. Furthermore the horizontal maps on the top and bottom are identical. It thus follows that the vertical map on the right is also given by multiplication by -1. ■

Next we want to find explicit generators of local homology groups $\mathrm{H}_n(M, M \setminus \{x\})$ for n-dimensional topological manifolds M and ideally we would like to find "standard generators". To do so it is worth recalling the following notation from page 457.

Definition. Let $n \in \mathbb{N}$. We write

$$\partial\Delta^n \;:=\; \bigcup_{i=0}^{n} \{(t_0, \dots, t_n) \in \Delta^n \mid t_i = 0\}.$$

and we refer to $\mathring{\Delta}^n := \Delta^n \setminus \partial\Delta^n$ as the interior of Δ^n.

The following lemma can be viewed as a mild extension of the Identity-is-Generator Lemma 42.1.

Lemma 61.3. (Identity-is-Generator Lemma) For any $k \in \mathbb{N}_0$ we have

$$\mathrm{H}_k(\Delta^n, \partial\Delta^n; R) \;=\; \begin{cases} 0, & \text{if } k \neq n, \\ R \cdot [\mathrm{id}_{\Delta^n} \otimes 1_R], & \text{if } k = n. \end{cases}$$

Proof. This lemma follows easily from the following two observations:

(1) By the "original" Identity-is-Generator Lemma 42.1 we know that

$$\mathrm{H}_k(\Delta^n, \partial\Delta^n) \;=\; \begin{cases} 0, & \text{if } k \neq n, \\ \mathbb{Z} \cdot [\mathrm{id}_{\Delta^n}], & \text{if } k = n. \end{cases}$$

(2) By the Universal Coefficient Theorem 54.9 we have the following exact sequence:

$$\begin{array}{ccccccccc} 0 & \to & \mathrm{H}_k(\Delta^n, \partial\Delta^n) \otimes R & \xrightarrow{\mu} & \mathrm{H}_k(\Delta^n, \partial\Delta^n; R) & \xrightarrow{\nu} & \underbrace{\mathrm{Tor}(\mathrm{H}_{k-1}(\Delta^n, \partial\Delta^n), R)} & \to & 0. \\ & & \sum_{i=1}^{m} [\sigma_i] \otimes r_i & \mapsto & \sum_{i=1}^{m} [\sigma_i \otimes r_i]. & & \text{by (1) and the Tor-Properties Lemma 54.7 (4)} & & \end{array}$$

■

Next we recall more definitions from page 457.

Definition. Let $n \in \mathbb{N}$.

(1) We equip $\mathbb{R}^n$ with the orientation given by the standard basis $e_1, \dots, e_n$. With this orientation we can view any open subset of $\mathbb{R}^n$ as an oriented smooth manifold.

(2) We say that a basis $\{v_1, \dots, v_n\}$ for $\mathrm{T}_P\mathring{\Delta}^n$ is positive, if the ordered set of vectors $\{(1, \dots, 1), v_1, \dots, v_n\}$ is a positive basis for $\mathbb{R}^{n+1}$. This equips $\mathring{\Delta}^n$ with the structure of an oriented smooth manifold.

(3) Given a subset V of $\mathbb{R}^n$ we say that a map $f\colon V \to \mathbb{R}^k$ is affine linear if there exists a $P \in \mathbb{R}^k$ and a $(k \times n)$–matrix A such that $f(v) = P + A \cdot v$ for all $v \in V$.

Example. The projection

$$\begin{array}{rcl} p\colon \Delta^n & \to & \mathbb{R}^n \\ (x_1, \dots, x_{n+1}) & \mapsto & (x_1, \dots, x_n) \end{array}$$



orientations of the interiors of the standard simplices

is an injective affine linear map.



projection is orientation-reversing projection is orientation-preserving

One can easily verify that the restriction of this map to the interior $\mathring{\Delta}^n$ is orientation-preserving if and only if n is even. □

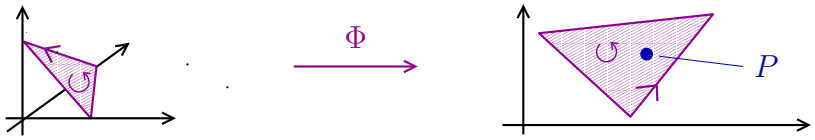
Lemma 61.4. (Local-H_*-Basics Lemma) Let R be a commutative ring, let $U \subset \mathbb{R}^n$ be an open subset and let $P \in \mathbb{R}^n$. Let $\Phi\colon \Delta^n \to U$ be an injective affine linear map such that P lies in the image of $\mathring{\Delta}^n$ and such that the restriction of Φ to $\mathring{\Delta}^n$ is *orientation-preserving*.

(1) For any $k \in \mathbb{N}_0$ the induced map

$$\Phi_*\colon \mathrm{H}_k(\Delta^n, \partial\Delta^n; R) \;\to\; \mathrm{H}_k(U, U \setminus \{P\}; R)$$

is an isomorphism

(2) The map in (1) does not depend on the choice of the orientation-preserving Φ.



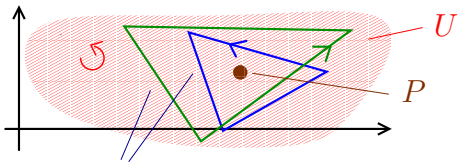
Proof. We delay the proof of the lemma to Section 61.2. ■

Definition. Let R be a commutative ring, let U be an open subset of $\mathbb{R}^n$ and let $P \in U$. Let $\Phi\colon \Delta^n \to U$ be an injective affine linear map such that P lies in the image of $\mathring{\Delta}^n$ and such that the restriction of Φ to $\mathring{\Delta}^n$ is *orientation-preserving*. By the Local-H_*-Basics Lemma 61.4 the induced map

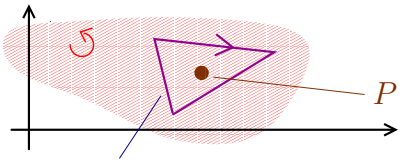
$$\Phi_*\colon \underbrace{\mathrm{H}_n(\Delta^n, \partial\Delta^n; R)}_{=[\mathrm{id}_{\Delta^n} \otimes 1_R]} \;\xrightarrow{\cong}\; \mathrm{H}_n(U, U \setminus \{P\}; R)$$

is an isomorphism and this isomorphism does not depend on the choice of Φ. By the Identity-is-Generator Lemma 61.3 we know that $\mathrm{H}_n(\Delta^n, \partial\Delta^n; R) = [\mathrm{id}_{\Delta^n} \otimes 1_R]$. We refer to the image of the generator $[\mathrm{id}_{\Delta^n} \otimes 1_R] \in \mathrm{H}_n(\Delta^n, \partial\Delta^n; R)$ as the standard generator of $\mathrm{H}_n(U, U \setminus \{P\}; R)$.

Example. We consider the following figure:



standard generator of $\mathrm{H}_2(U, U\setminus\{P\};\mathbb{Z})$



minus the standard generator

To the left we see representatives of the standard generators.[505] Here, for simplicity, given a singular 2-simplex $\Delta^2 \to \mathbb{R}^2$ we only sketch the image of $\partial\Delta^2$ with the given orientation. Also note that it follows from the Local-H_*-Reflection Lemma 61.2 that the singular simplex illustrated in the right of the figure below represents *minus* the standard generator. □

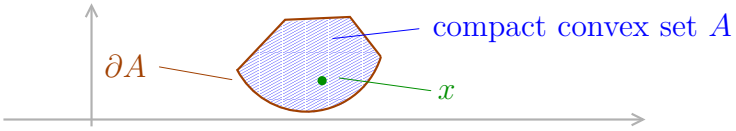
61.2. **Proof of the Local-H_*-Basics Lemma 61.4.** In this section we will provide the proof of the Local-H_*-Basics Lemma 61.4. The general discussion of this chapter continues in Section 61.3.

We start out with the following lemma:

Lemma 61.5. (Convex-Rel-Boundary Homology Lemma) Let $n \in \mathbb{N}$, let A be a compact convex subset of $\mathbb{R}^n$ and let $x \in \mathring{A}$. For each abelian group G the inclusion-induced maps

$$\mathrm{H}_i(A, \partial A; G) \;\to\; \mathrm{H}_i(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\}; G)$$

are isomorphisms.



Proof. Let $n \in \mathbb{N}$, let A be a compact convex subset of $\mathbb{R}^n$ and let $x \in \mathring{A}$. To simplify the notation we drop the coefficients G from the notation and we assume that $x = 0$.

Claim. The subspace ∂A is a deformation retract of $\mathbb{R}^n \setminus \{0\}$.

Proof. First note that since $0 \in \mathring{A}$ we can find $\epsilon > 0$ with $\overline{B}^n_\epsilon(0) \subset A$. Next note that it follows from the Convex-to-Ball Proposition 2.12 that the following map is continuous:

$$\begin{array}{rcl} \rho\colon S^{n-1}_\epsilon(0) & \to & \mathbb{R}_{\geq 0} \\ x & \mapsto & \sup\{\|r\cdot x\| \,\big|\, r \in \mathbb{R}_{>0} \text{ and } r\cdot x \in A\}. \end{array}$$

Using this fact and using the convexity of A one can easily verify that the map

$$\begin{array}{rcl} (\mathbb{R}^n \setminus \{0\}) \times [0,1] & \to & \mathbb{R}^n \setminus \{0\} \\ (x,t) & \mapsto & x \cdot ((1-t) + t \cdot \rho(\epsilon \cdot \frac{x}{\|x\|})) \end{array}$$

is a deformation retraction from $\mathbb{R}^n \setminus \{0\}$ to ∂A. ⊞

[505] At this occasion it is good to remember that if $\sigma\colon \Delta^k \to X$ is a continuous map with $\sigma(\partial\Delta^n) \subset A$, then σ is a cycle in $\mathrm{C}_k(X, A)$, i.e. it represents a class in $\mathrm{H}_k(X, A)$.

Now let $i \in \mathbb{N}_0$. From the Reduced LES-of-Pair Proposition 39.6 we obtain the following commutative diagram of long exact sequences:

$$\begin{array}{ccccccccccc}
\ldots & \longrightarrow & \widetilde{\mathrm{H}}_i(\partial A) & \longrightarrow & \widetilde{\mathrm{H}}_i(A) & \xrightarrow{\text{я}} & \mathrm{H}_i(A, \partial A) & \xrightarrow{\partial_i} & \widetilde{\mathrm{H}}_{i-1}(\partial A) & \longrightarrow & \widetilde{\mathrm{H}}_{i-1}(A) \longrightarrow \ldots \\
 & & \downarrow\cong & & \downarrow\cong & & \downarrow & & \downarrow\cong & & \downarrow\cong \\
\ldots & \longrightarrow & \widetilde{\mathrm{H}}_i(\mathbb{R}^n \setminus \{x\}) & \longrightarrow & \widetilde{\mathrm{H}}_i(\mathbb{R}^n) & \xrightarrow{\text{я}} & \mathrm{H}_i(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\}) & \xrightarrow{\partial_i} & \widetilde{\mathrm{H}}_{i-1}(\mathbb{R}^n \setminus \{x\}) & \longrightarrow & \widetilde{\mathrm{H}}_{i-1}(\mathbb{R}^n) \longrightarrow \ldots
\end{array}$$

We make the following observations:

(1) It follows from the claim and the Homotopy Equivalence-$\widetilde{\mathrm{H}}_*$-Corollary 39.4 (1b) that the first and fourth vertical maps are isomorphisms.
(2) Since A and $\mathbb{R}^n$ are contractible we see that the second and fifth vertical maps are isomorphisms.
(3) It follows from (1) and (2) together with the Five Lemma 37.5 that the middle vertical map is also an isomorphism. ∎

Definition. Let R be a commutative ring, let U be an open subset of $\mathbb{R}^n$ and let $P \in U$. We have isomorphisms

$$\overbrace{\mathrm{H}_n(\Delta^n, \partial\Delta^n; R)}^{=[\mathrm{id}_{\Delta^n} \otimes 1_R]} \xrightarrow[\uparrow]{\cong} \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{P\}; R) \xleftarrow[\uparrow]{\cong} \mathrm{H}_n(U, U \setminus \{P\}; R).$$

isomorphism from the Local-H_*-Basics Lemma 61.4

by the Excision Theorem 40.1, see also page 661

By the Identity-is-Generator Lemma 61.3 we know that $\mathrm{H}_n(\Delta^n, \partial\Delta^n; R) = [\mathrm{id}_{\Delta^n} \otimes 1_R]$. We refer to the image of the generator $[\mathrm{id}_{\Delta^n} \otimes 1_R] \in \mathrm{H}_n(\Delta^n, \partial\Delta^n; R)$ under the given isomorphisms as the standard generator of the various relative homology modules.

Proof of the Local-H_*-Basics Lemma 61.4. Let R be a commutative ring, let $V \subset \mathbb{R}^n$ be an open subset and let $P \in \mathbb{R}^n$. Note that it follows from the Excision Theorem 40.1 and the discussion on page 661 that the inclusion-induced map

$$\mathrm{H}_k(V, V \setminus \{P\}; R) \;\to\; \mathrm{H}_k(\mathbb{R}^n, \mathbb{R}^n \setminus \{P\}; R)$$

is an isomorphism. It follows almost immediately from this observation that we can restrict ourselves to the case that $V = \mathbb{R}^n$.

Let $\Phi\colon \Delta^n \to \mathbb{R}^n$ be an injective affine linear map such that P lies in the image of $\mathring{\Delta}^n$ and such that the restriction of Φ to $\mathring{\Delta}^n$ is orientation-preserving.

(1) We need to show that the induced map $\Phi_*\colon \mathrm{H}_k(\Delta^n, \partial\Delta^n; R) \to \mathrm{H}_k(\mathbb{R}^n, \mathbb{R}^n \setminus \{P\}; R)$ is an isomorphism. We have the following commutative diagram:

$$\begin{array}{ccc}
\mathrm{H}_k(\Delta^n, \partial\Delta^n; R) & \xrightarrow{\quad\Phi_*\quad} & \mathrm{H}_k(\mathbb{R}^n, \mathbb{R}^n \setminus \{P\}; R). \\
& {\scriptstyle\Phi_*}\searrow \quad \nearrow & \\
& \mathrm{H}_k(\Phi(\Delta^n), \Phi(\partial\Delta^n); R) &
\end{array}$$

where the right diagonal map is induced by the inclusions. Note that the left diagonal map is an isomorphism since Φ is an embedding[506]. Since Φ is an injective affine linear map we see that $\Phi(\Delta^n)$ is a convex subset of $\mathbb{R}^n$ with interior $\Phi(\mathring{\Delta}^n)$ and boundary

[506]Here we implicitly use the ever popular Compact-Hausdorff Proposition 2.10 (2).

$\Phi(\partial\Delta^n)$. By our hypothesis the point P lies in the interior of $\Phi(\Delta^n)$. It follows from the Convex-Rel-Boundary Homology Lemma 61.5 that the right diagonal map is an isomorphism. It follows from this discussion that the top horizontal map is an isomorphism.

(2) Now we need to show that the homomorphism of (1) does not depend on the choice of Φ. To simplify the notation a little bit we now assume that $P = 0$. Thus let $\Psi\colon \Delta^n \to \mathbb{R}^n$ be another such injective affine linear orientation-preserving map. It follows from elementary linear algebra that there exists a $P \in \mathbb{R}^n$ and a matrix A with *positive* determinant such that

$$\Phi \;=\; (v \mapsto P + A \cdot v) \circ \Psi\colon \Delta^n \to \mathbb{R}^n.$$

We write $\Omega = (v \mapsto A \cdot v) \circ \Psi$.



It suffices to show that $\Psi_* = \Omega_* = \Phi_*\colon \mathrm{H}_k(\Delta^n, \partial\Delta^n; R) \to \mathrm{H}_k(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}; R)$.

(a) Recall that it follows from the Matrix Group-π_0-Proposition 2.23 (1) that the group $\{A \in \mathrm{GL}(n,\mathbb{R}) \mid \det(A) > 0\}$ is path-connected. This implies in particular that there exists a path $\gamma\colon [0,1] \to \mathrm{GL}(n,\mathbb{R})$ with $\gamma(0) = \mathrm{id}$ and $\gamma(1) = A$. We consider the map

$$\begin{aligned} F\colon \Delta^n \times [0,1] &\to \mathbb{R}^n \\ (Q,t) &\mapsto \gamma(t) \cdot \Psi(Q). \end{aligned}$$

Since by hypothesis we have $0 \notin \Phi(\partial\Delta^n)$ and since $\gamma(t) \cdot 0 = 0$ we see that $\gamma(t)(\partial\Delta^n) \subset \mathbb{R}^n \setminus \{0\}$ for all t. In particular we deduce that F defines a homotopy between Ψ and Ω as maps $(\Delta^n, \partial\Delta^n) \to (\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$. But by the Homotopy-$\mathrm{H}_*$-Proposition 38.6 this means that $\Psi_* = \Omega_*$ as maps on relative homology.

(b) Now we compare Ω and $\Phi = (w \mapsto P + w) \circ \Omega$. We consider the map

$$\begin{aligned} G\colon \Delta^n \times [0,1] &\to \mathbb{R}^n \\ (w,t) &\mapsto (P \cdot t + \Omega(w)). \end{aligned}$$

Note that $0 \in \Omega(\mathring{\Delta}^n) = A(\Psi(\mathring{\Delta}^n))$ and that $0 \in \Phi(\mathring{\Delta}^n)$ which implies by $\Phi = P + \Omega$ that $-P \in \Omega(\mathring{\Delta}^n)$. Since $\Omega(\mathring{\Delta}^n)$ is convex we see that G defines a homotopy between Ω and Φ as maps $(\Delta^n, \partial\Delta^n) \to (\mathbb{R}^n, \mathbb{R}^n \setminus \{0\})$. By the Homotopy-$\mathrm{H}_*$-Proposition 38.6 this means that $\Omega_* = \Phi_*$ as maps on relative homology. ∎

61.3. ***R*-Sections on topological manifolds.** Recall the idea from page 782: We want to define an orientation of an n-dimensional topological manifold M to consist of a generator of each local homology group $\mathrm{H}_n(M, M \setminus \{x\})$ for $x \in M \setminus \partial M$, such that the generators change "continuously" in x.

In this section we will define more generally what it means for a choice of elements in the local homology groups $\mathrm{H}_n(M, M \setminus \{x\})$ to be "continuous in x". This discussion will have many benefits, in particular in the next section we will be able to finally define the notion of an orientation of a topological manifold.

We start out with the following, initially rather obscure definition.

Definition. Let M be an n-dimensional topological manifold and let R be a commutative ring.

(1) An R-section for M at $x \in M \setminus \partial M$ is an element $\mu_x \in \underbrace{\mathrm{H}_n(M, M \setminus \{x\}; R)}_{\cong R}$.

(2) Let $W \subset M$ and let $x \in W$. Given any $k \in \mathbb{N}_0$ we denote by ρ_x the natural map

$$\rho_x \colon \mathrm{H}_k(M, M \setminus W; R) \;\to\; \mathrm{H}_k(M, M \setminus \{x\}; R).$$

Given $\varphi \in \mathrm{H}_k(M, M \setminus W; R)$ we often write $\varphi_x := \rho_x(\varphi)$, i.e. the above map is also written as

$$\varphi \;\mapsto\; \varphi_x.$$

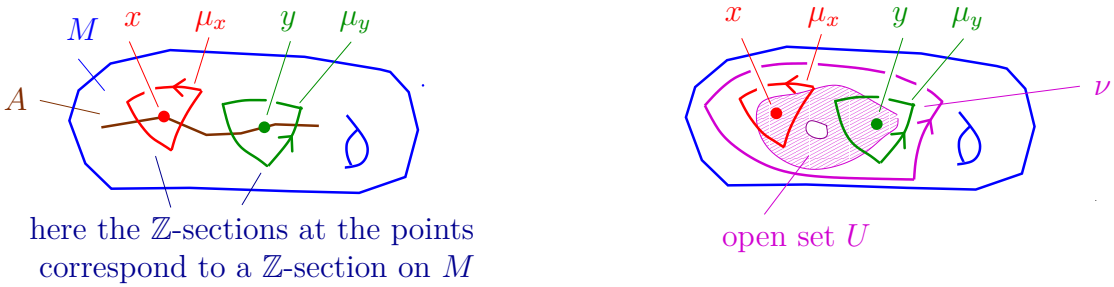
(3) Let A be a subset of $M \setminus \partial M$. (In the following we will mostly be interested in the case $A = M \setminus \partial M$). An R-section for M along A is a choice for each $x \in A$ of an R-section $\mu_x \in \mathrm{H}_n(M, M \setminus \{x\}; R)$ at x such that the following "continuity condition" is satisfied:

($*$) For each $x \in A$ there exists an open neighborhood U of x in A and a homology class $\nu \in \mathrm{H}_n(M, M \setminus U; R)$ such that for each $y \in U$ the image of ν under the natural map

$$\rho_y \colon \mathrm{H}_n(M, M \setminus U; R) \;\to\; \mathrm{H}_n(M, M \setminus \{y\}; R)$$

equals μ_y.

(4) We denote by $\Gamma_R^A(M)$ the set of all R-sections of M along A.



Lemma 61.6. (Homology-defines-Section Lemma) Let M be an n-dimensional topological manifold and let R be a commutative ring.

(1) Given any subset $A \subset M \setminus \partial M$ the set $\Gamma_R^A(M)$ admits a unique R-module structure such that for each $y \in A$ the map $\Gamma_R^A(M) \to \mathrm{H}_n(M, M \setminus \{y\}; R)$ that is given by $\{\alpha_x\}_{x \in A} \mapsto \alpha_y$ is an R-module homomorphism.

(2) For every open subset $A \subset M \setminus \partial M$ (often we will take $A = M \setminus \partial M$) the map

$$\begin{aligned} \mathrm{H}_n(M, M \setminus A; R) &\;\to\; \Gamma_R^A(M) \\ \alpha &\;\mapsto\; \{\alpha_x\}_{x \in A} \end{aligned}$$

is a well-defined homomorphism of R-modules.

Proof.

(1) This statement follows easily from the definitions and a few basic considerations.

(2) Let $A \subset M \setminus \partial M$ be an open subset and let $\alpha \in \mathrm{H}_n(M, M \setminus A; R)$. Let $x \in A$. We need to show that there exists an open neighborhood U of x in A and a homology class

$\nu \in \mathrm{H}_n(M, M \setminus U; R)$ such that for each $y \in U$ we have $\rho_x(\nu) = \alpha_x$. But this is trivial, we just take $U = A$ and $\nu = \alpha$. ■

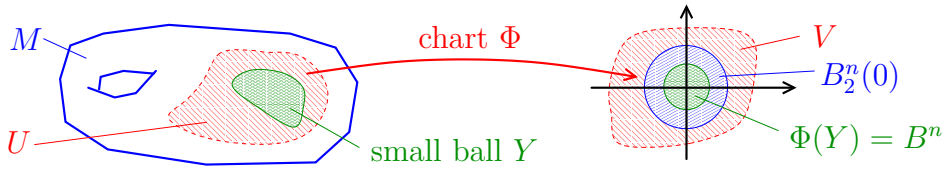
The following proposition contains much more interesting results on $\Gamma_R^A(M)$.

Proposition 61.7. (Sections-on-Connected Proposition) Let M be an n-dimensional topological manifold and let R be a commutative ring. Furthermore let A be a *connected* subset of $M \setminus \partial M$[507] and let $\{\alpha_x\}_{x\in A}$ and $\{\beta_x\}_{x\in A}$ be two R-sections for M along A. If there exists an $x \in A$ with $\alpha_x = \beta_x$, then the two R-sections agree.

Proof. We outsource the proof of the Sections-on-Connected Proposition 61.7 to the next section. The reader who wants to skip the proof can also go straight to Section 61.5. ■

61.4. **Proof of the Sections-on-Connected Proposition 61.7.** Before we provide the proof of the Sections-on-Connected Proposition 61.7 we need to prove one little lemma which in turn requires the following definition.

Definition. Let M be an n-dimensional topological manifold. We say that $Y \subset M$ is a small ball in M if there exists a chart $\Phi\colon U \to V$ such that $B_2^n(0) \subset V$ and such that $\Phi(Y) = B^n$.



Lemma 61.8. (Small Ball-Lemma) Let M be an n-dimensional topological manifold.

(0) Given any point $x \in M \setminus \partial M$ and any neighborhood U of x there exists a small ball Y with $x \in Y \subset U$.

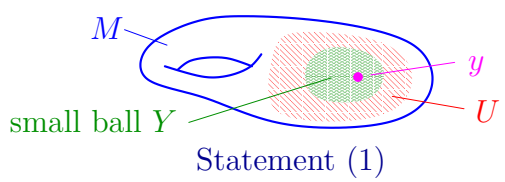
Let R be a commutative ring.

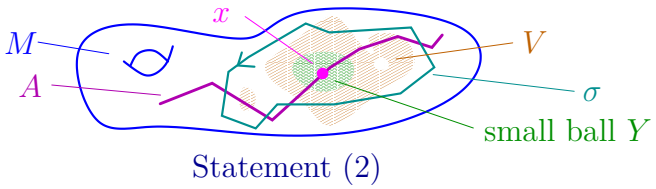
(1) For any point y in a small ball Y and any $k \in \mathbb{N}_0$ the map

$$\rho_y\colon \mathrm{H}_k(M, M \setminus Y; R) \;\to\; \mathrm{H}_k(M, M \setminus \{y\}; R)$$

is an isomorphism.

(2) Let A be a subset of $M \setminus \partial M$, let $\{\alpha_x\}_{x\in A}$ be an R-section along A and let $x \in A$. There exists a small ball Y in $M \setminus \partial M$ that contains x and a $\varphi \in \mathrm{H}_k(M, M \setminus Y; R)$ such that for every $y \in Y \cap A$ we have $\rho_y(\varphi) = \alpha_y \in \mathrm{H}_n(M, M \setminus \{y\}; R)$.





Proof. Let M be an n-dimensional topological manifold.

(0) The proof of Statement (1) is elementary and is thus left to the reader.

[507] In some applications we will use the fact that if M is a connected topological manifold, then $M \setminus \partial M$ is also connected.

(1) Let y be a point that is contained in a small ball Y. To simplify the notation we drop the commutative ring R from the notation. We consider the long exact sequence of the triple $(M, M\setminus\{y\}, M\setminus Y)$ given by the LES-of-Triple Proposition 38.2:

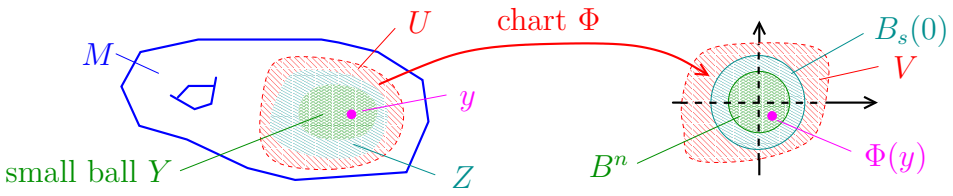
$$\to \mathrm{H}_k(M\setminus\{y\}, M\setminus Y) \to \mathrm{H}_k(M, M\setminus Y) \to \mathrm{H}_k(M, M\setminus\{y\}) \to \mathrm{H}_{k-1}(M\setminus\{y\}, M\setminus Y) \to$$

We see immediately that it suffices to show that all the relative homology groups $\mathrm{H}_k(M\setminus\{y\}, M\setminus Y)$ vanish. Since Y is a small ball we can pick a chart $\Phi\colon U \to V$ such that $B_2^n(0) \subset V$ and such that $\Phi(Y) = B^n$. We pick $s \in (1,2)$ and we write $Z = \Phi^{-1}(B_s^n(0))$. Then

$$\mathrm{H}_k(M\setminus\{y\}, M\setminus Y) \xleftarrow[\uparrow]{\cong} \mathrm{H}_k(Z\setminus\{y\}, Z\setminus Y) \xrightarrow[\cong]{\Phi_*} \mathrm{H}_k(B_s^n(0)\setminus\{\Phi(y)\}, B_s^n(0)\setminus B^n) = \underset{\uparrow}{0}.$$

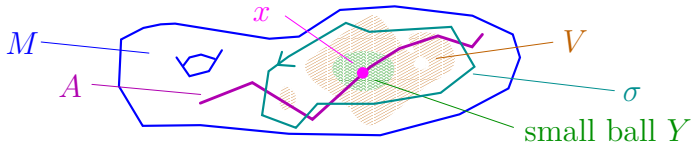
by the Excision Theorem 40.2

by the Deformation Retract-Relative H_*-Corollary 38.5, since $B_s^n(0)\setminus B^n$ is a deformation retract of $B_s^n(0)\setminus\{\Phi(y)\}$



(2) Let A be a subset of M, let $\{\alpha_x\}_{x\in A}$ be an R-section along A and let $x \in A$. By definition of an R-section along A and by definition of the subspace topology there exists an open neighborhood V of $x \in M\setminus\partial M$ and a singular chain $\sigma \in \mathrm{C}_n(M;R)$ that represents a cycle in $\mathrm{C}_n(M, M\setminus(V\cap A);R)$ such that for each $y \in V\cap A$ we have $\alpha_y = \rho_y([\sigma]) \in \mathrm{H}_n(M, M\setminus\{y\};R)$.

Since σ represents a cycle in the relative chain complex $\mathrm{C}_*(M, M\setminus(V\cap A);R)$ we know that $\partial\sigma \in \mathrm{C}_{n-1}(M\setminus(V\cap A);R)$. It follows pretty easily from the fact that Δ^{n-1} is compact together with the Compact Image Lemma 2.8 and the Compact-Closed Lemma 1.9 that there exists an open neighborhood $U \subset M$ of x with $U \subset V$ such that $\partial\sigma \in \mathrm{C}_{n-1}(M\setminus U;R)$. In other words, σ represents a cycle in $\mathrm{C}_n(M, M\setminus U;R)$.



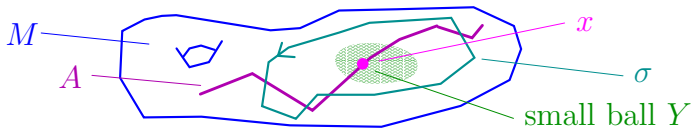
By (0) there exists a small ball Y with $x \in Y \subset U$. It follows immediately from the functoriality of homology groups that the homology class $[\sigma] \in \mathrm{H}_n(M, M\setminus Y;R)$ has all the desired properties. ∎

Proof of the Sections-on-Connected Proposition 61.7. Let M be an n-dimensional topological manifold, let A be a connected subset of $M\setminus\partial M$ and let R be a commutative ring. Let $\{\alpha_x\}_{x\in A}$ and $\{\beta_x\}_{x\in A}$ be two R-sections for M along A. We assume that there exists a $z \in A$ with $\alpha_z = \beta_z$. By the Homology-defines-Section Lemma 61.6 (1) we can consider the R-section along A that is given by $\gamma_x = \alpha_x - \beta_x$ for $x \in A$. Next we consider the subsets $V_{=0} := \{y \in A \,|\, \gamma_y = 0\}$ and $V_{\neq 0} := \{y \in A \,|\, \gamma_y \neq 0\}$. We need to show that

$V_{=0} = A$. Since $V_{=0}$ is by hypothesis non-empty and since A is connected it suffices to show that $V_{=0}$ and $V_{\neq 0}$ are both open. In other words, it suffices to prove the following claim:

Claim. Let $x \in A$. There exists an open neighborhood U of x in A such that either γ_y is zero for every $z \in U$ or γ_y is non-zero for every $z \in U$.

Proof. Let $x \in A$. It follows immediately from the Small Ball-Lemma 61.8 (0) and (2) that there exists a small ball Y in $M \setminus \partial M$ that contains x and a $\varphi \in \mathrm{H}_k(M, M \setminus Y; R)$ such that for every $y \in Y \cap A$ we have $\rho_y(\varphi) = \alpha_y \in \mathrm{H}_n(M, M \setminus \{y\}; R)$.



Note that by the Small Ball-Lemma 61.8 (1) we know that for every $y \in Y$ the map $\rho_y \colon \mathrm{H}_n(M, M \setminus Y; R) \to \mathrm{H}_n(M, M \setminus \{y\}; R)$ is an isomorphism. But this implies that $\gamma_x = 0$ if and only if $\gamma_y = 0$ for all $y \in Y$. Thus $U := Y \cap A$ has the desired properties. ⊞ ■

61.5. **Orientations of topological manifolds.** We start out with the following addendum to the definition from page 789 of a section along a subspace A.

Definition. Let M be an n-dimensional topological manifold and let R be a commutative ring. For $A = M \setminus \partial M$ we say "section" instead of "section along A" and we write $\Gamma_R(M)$ instead of $\Gamma_R^{M \setminus \partial M}(M)$.

After the technical preparations from the last section we can now finally introduce the notion of an orientation of a topological manifold.

Definition. Let M be an n-dimensional topological manifold and let R be a commutative ring.

(1) An R-orientation for M at a point $x \in M \setminus \partial M$ is a generator of the R-module $\underbrace{\mathrm{H}_n(M, M \setminus \{x\}; R)}_{\cong R}$.
(2) An R-section for M that is an R-orientation at each point $x \in M \setminus \partial M$ is called an R-orientation of M.
(3) The topological manifold M is called R-orientable if it admits an R-orientation.
(4) An R-oriented topological manifold is a topological manifold that is equipped with an R-orientation.

If we work with homology groups $\mathrm{H}_n(M, M \setminus \{x\})$ "without coefficients", then we say orientation, section and orientable. Note that it follows from the discussion on page 659 that a $\mathbb{Z}$-orientation is naturally the same as an orientation.



two different orientations for the torus

Example. Let M be a 0-dimensional topological manifold. For each $x \in M$ we have a natural isomorphism

$$\mathrm{H}_0(M, M \setminus \{x\}) \xleftarrow[\uparrow]{\cong} \mathrm{H}_0(\{x\}) \xleftarrow[\uparrow]{n \mapsto n \cdot [x]} \mathbb{Z}.$$

this is an isomorphism since M has the discrete topology — by the H_0-Proposition 35.11 this is an isomorphism

Since M has the discrete topology we see that any assignment of generators satisfies the continuity condition of a section. Thus we see that an orientation of a 0-dimensional topological manifold is basically the same as a map $M \to \{\pm 1\}$. We refer to the orientation which assigns $1 = 1 \cdot [x]$ to each point $x \in M$ as the standard orientation of the 0-dimensional topological manifold M. □

Before we move on to other examples let us get the following basic lemma out of the way:

Lemma 61.9. (Number-of-Orientations Lemma) Let M be a topological manifold.

(1) For any point $x \in M \setminus \partial M$ there exist precisely two orientations for M at x.
(2) If M is orientable, connected and non-empty, then M admits precisely two orientations.

Proof.

(1) This statement is an immediate consequence of the fact, proved in the Manifold-Local R-Homology Lemma 61.1 (1), that for any $x \in M \setminus \partial M$ we have $\mathrm{H}_n(M, M \setminus \{x\}) \cong \mathbb{Z}$.
(2) This statement follows easily from the Sections-on-Connected Proposition 61.7 and the various definitions. ■

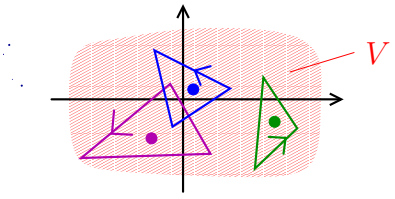
We formulate our first interesting example of an orientation of a topological manifold as a lemma:

Lemma 61.10. (Standard Euclidean Orientation Lemma) Let V be an open subset of $\mathbb{R}^n$.

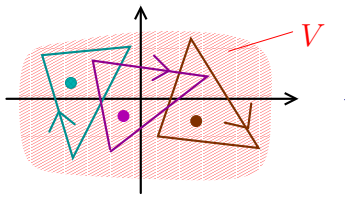
(1) For any $x \in V$ we set

$$\mu_x \;:=\; \text{the standard generator of } \mathrm{H}_n(V, V \setminus \{x\}) \text{ as defined on page 785.}$$

These homological classes μ_x define an orientation for V. We refer to this orientation as the standard orientation of V.
(2) If V is connected and non-empty, then V admits precisely two orientations, namely the one from (1) and the one where we take minus the standard generator.[508]



standard generators for $\mathrm{H}_2(V, V \setminus \{x\})$, these define the standard orientation of V



the negatives of the standard generators, these define the opposite of the standard orientation

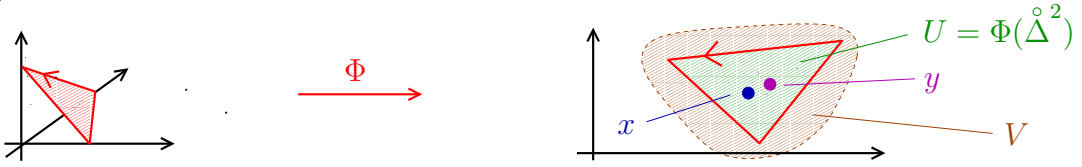
[508] By the Local-H_*-Reflection Lemma 61.2 the other one is given by a reflection of the standard generators in hyperplanes.

Proof.

(1) Let V be an open subset of $\mathbb{R}^n$ and let $x \in \mathbb{R}^n$. We pick an orientation-preserving injective affine linear map $\Phi\colon \Delta^n \to V$ such that $x \in \Phi(\mathring{\Delta}^n)$. We set $U := \Phi(\mathring{\Delta}^n)$ and we set $\nu := [\Phi] \in \mathrm{H}_n(V, V \setminus U)$. It remains to prove the following claim:

Claim. For every $y \in U$ the class $\nu_y \in \mathrm{H}_n(V, V \setminus \{y\})$ is the standard generator.

Proof. It follows immediately from the Local-H_*-Basics Lemma 61.4 (2), the definition of the standard generator of $\mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{y\})$ and the fact that y is contained in $\Phi(\mathring{\Delta}^n)$ that $\rho_y([\Phi]) = [\Phi] \in \mathrm{H}_n(V, V \setminus \{y\})$ represents the standard generator of $\mathrm{H}_n(V, V \setminus \{y\})$. ⊞



(2) This statement follows immediately from (1) and the Number-of-Orientations Lemma 61.9 (2). ■

The following lemma gives another essential example of an orientable topological manifold.

Lemma 61.11. (Sphere Orientation Lemma) Let $n \in \mathbb{N}$. We denote by $[S^n] \in \mathrm{H}_n(S^n)$ the standard generator that we introduced on page 540. For each $x \in S^n$ we consider the natural map

$$\rho_x\colon \mathrm{H}_n(S^n) \;\to\; \mathrm{H}_n(S^n, S^n \setminus \{x\}).$$

The images of $[S^n]$ in the local homology groups $\mathrm{H}_n(S^n, S^n \setminus \{x\})$ define an orientation of S^n. We refer to it as the standard orientation of S^n.

Proof. First note that it follows easily from the long exact sequence of homology groups of the pair $(S^n, S^n \setminus \{x\})$ and the fact that $S^n \setminus \{x\} \cong \mathbb{R}^n$ is contractible, that each map ρ_x is an isomorphism. Since $[S^n]$ is a generator of $\mathrm{H}_n(S^n)$ we see that each $\rho_x([S^n])$ is a generator of $\mathrm{H}_n(S^n, S^n \setminus \{x\})$.

Now we show that the homology classes $\rho_x([S^n])$ define a section. Thus let $x \in S^n$. In the definition on page 789 we just need to set $U = S^n$ and $\nu = [S^n] \in \mathrm{H}_n(S^n, \underbrace{S^n \setminus S^n}_{=\varnothing})$. ■

Lemma 61.12. (Orientable-for-all-Rings Lemma) Let M be a topological manifold and let R be a commutative ring. Every orientation on M defines naturally an R-orientation on M.

Proof. Let M be an orientable n-dimensional topological manifold and let R be a commutative ring. Furthermore let $\{\alpha_x\}_{x \in M \setminus \partial M}$ be an orientation for M. By the Manifold-Local R-Homology Lemma 61.1 (2) we know that for each $x \in M \setminus \partial M$ the natural map

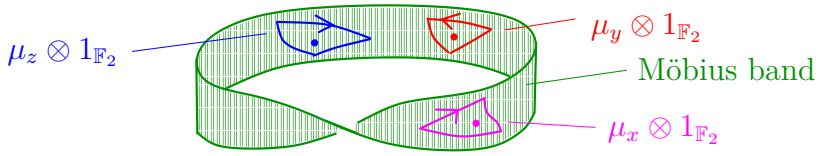
$$\begin{aligned} \mu\colon \mathrm{H}_n(M, M \setminus \{x\}) \otimes R \;&\to\; \mathrm{H}_n(M, M \setminus \{x\}; R) \\ \sum_{i=1}^{k} [\sigma_i] \otimes r_i \;&\mapsto\; \Big[\sum_{i=1}^{k} \sigma_i \otimes r_i\Big] \end{aligned}$$

is an isomorphism. For each $x \in M \setminus \partial M$ we denote by $\widetilde{\alpha}_x \in \mathrm{H}_n(M, M \setminus \{x\}; R)$ the image of $\alpha_x \otimes 1_R$ under the above isomorphism. It is straightforward to verify that these generators $\{\widetilde{\alpha}_x\}_{x \in M \setminus \partial M}$ define an R-orientation for the topological manifold M. ■

The following proposition shows that the notion of R-orientability is much more flexible than the previous notion of orientability of smooth manifolds.

Proposition 61.13. ($\mathbb{F}_2$-Orientability Proposition) Every topological manifold is $\mathbb{F}_2$-orientable and it admits a unique $\mathbb{F}_2$-orientation.

Example. The $\mathbb{F}_2$-Orientability Proposition 61.13 says in particular that the real projective plane, the Klein bottle and the Möbius band are $\mathbb{F}_2$-orientable.



This observation will turn out to be more useful than one might think right now. □

Proof. Let M be an n-dimensional topological manifold. Let $x \in M \setminus \partial M$. By the Manifold-Local R-Homology Lemma 61.1 we know that $\mathrm{H}_n(M, M \setminus \{x\}; \mathbb{F}_2) \cong \mathbb{F}_2$. We define φ_x to be the unique non-trivial element in $\mathrm{H}_n(M, M \setminus \{x\}; \mathbb{F}_2) \cong \mathbb{F}_2$. Using the Small Ball-Lemma 61.8 (0) and (1) it is straightforward to show that the continuity condition is satisfied, in particular we have defined an $\mathbb{F}_2$-orientation on M. ■

61.6. Orientation-preserving local homeomorphisms.

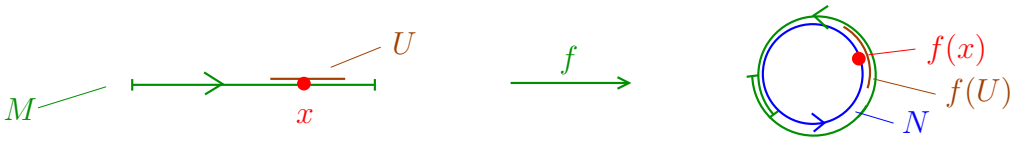
Definition. Let R be a commutative ring. Let M and N be two n-dimensional topological manifolds that are equipped with R-orientations $\{\alpha_x\}_{x \in M \setminus \partial M}$ and $\{\beta_x\}_{x \in N \setminus \partial N}$. Let $f\colon M \to N$ be a continuous map that is a local homeomorphism.

- We say that the map f is orientation-preserving at a point $x \in M \setminus \partial M$ if for every open neighborhood U of x, such that $f\colon U \to f(U)$ is a homeomorphism and such that $f(U)$ is open, the image of α_x under the map[509][510]

$$\mathrm{H}_n(M, M\setminus\{x\}) \underset{\uparrow}{\overset{\cong}{\leftarrow}} \mathrm{H}_n(U, U\setminus\{x\}) \underset{\cong}{\overset{f_*}{\longrightarrow}} \mathrm{H}_n(f(U), f(U)\setminus\{f(x)\}) \underset{\uparrow}{\overset{\cong}{\rightarrow}} \mathrm{H}_n(N, N\setminus\{f(x)\}).$$

it follows from the Excision Theorem 40.2 that these two maps are isomorphisms

  equals $\beta_{f(x)}$. Otherwise we say that f is orientation-reversing at x.
- We say that $f\colon M \to N$ is orientation-preserving (respectively orientation-reversing) if it is orientation-preserving (respectively orientation-reversing) at all $x \in M \setminus \partial M$.



Example. Let $n \in \mathbb{N}_0$, $v \in \mathbb{R}^n$, let $A \in \mathrm{GL}(n, \mathbb{R})$ and let $U \subset \mathbb{R}^n$ be an open subset. We consider the embedding

$$\begin{array}{ccc} U & \to & \mathbb{R}^n \\ x & \mapsto & A \cdot x + v. \end{array}$$

If $\det(A) > 0$, then it follows almost immediately from the definition of the standard orientation (see page 785) that the above map is orientation-preserving. If $\det(A) < 0$,

[509] Since $f\colon U \to f(U)$ is a homeomorphism we have $f(U \setminus \{x\}) = f(U) \setminus \{f(x)\}$.

[510] We suppress the R-coefficients from the notation for space reasons.

then it follows easily from the Local-H_*-Reflection Lemma 61.2 that the above map is orientation-reversing. □

Lemma 61.14. (Orientation Preserving Lemma) Let R be a commutative ring.

(1) Let $f\colon M \to N$ be a local homeomorphism between R-oriented n-dimensional manifolds.
   (a) To check whether f is orientation-preserving at a point $x \in M \setminus \partial M$ it suffices to check the above condition for a single open neighborhood of x.
   (b) If f is orientation-preserving at some point $x \in M \setminus \partial M$ and if M is connected, then $f\colon M \to N$ is orientation-preserving.

(2) The composition of k orientation-preserving and l orientation-reversing local homeomorphisms is orientation-preserving if and only if l is even.

Sketch of proof.

(1) (a) This statement follows easily from the functoriality of homology.
   (b) This statement follows from the Sections-on-Connected Proposition 61.7.

(2) This statement is basically clear. ■

61.7. **Smooth and topological orientations.** The following proposition shows in particular that for smooth manifolds the notion of orientability agrees with the above notion of orientability.

Proposition 61.15. (Smooth-Topological Orientation Proposition)

(1) Give any smooth manifold M there exists a unique natural bijection

$$\left\{\begin{array}{c}\text{orientations of the}\\ \text{smooth manifold } M\end{array}\right\} \xrightarrow{\cong} \left\{\begin{array}{c}\text{orientations of the}\\ \text{topological manifold } M \text{ in}\\ \text{the sense of page 792}\end{array}\right\}$$

such that the following two statements hold:
   (a) For any open subset of some $\mathbb{R}^n$ the above bijection sends the smooth standard orientation to the above topological standard orientation.
   (b) If $f\colon M \to N$ is a local diffeomorphism between oriented connected smooth manifolds, then the following two statements are equivalent:
   - f preserves the orientations on M and N as smooth manifolds,
   - f preserves the orientations on M and N in the sense of page 792.

(2) Given an n-dimensional smooth manifold M we have

$$M \text{ is orientable as a smooth manifold} \iff M \text{ is orientable.}$$

In the proof of the Smooth-Topological Orientation Proposition 61.15 we will use the following lemma:

Lemma 61.16. (Orientations–Preserved Lemma) Let $\Psi\colon V \to V'$ be a diffeomorphism between two open subsets of $\mathbb{R}^n$. For any $x \in V$ we have

$$\begin{array}{c}\text{at the point } x \text{ the map } \Psi \text{ preserves}\\ \text{the standard orientation of the}\\ \text{topological manifolds } V \text{ and } V'\end{array} \iff \begin{array}{c}\text{at the point } x \text{ the map } \Psi \text{ preserves}\\ \text{the standard orientation of the}\\ \text{smooth manifolds } V \text{ and } V'\end{array}$$

Proof. Let $\Psi\colon V \to V'$ be a diffeomorphism between two open subsets of $\mathbb{R}^n$ and let $x \in V$. We write $x' = \Psi(x)$. Given $w \in \mathbb{R}^n$ we denote by t_w the translation $\mathbb{R}^n \to \mathbb{R}^n$ that is given by $v \mapsto v + w$. We set $\epsilon := 1$ if Ψ preserves the orientation of the smooth manifold V at the point x. Otherwise we set $\epsilon := -1$. We consider the following diagram:

$$\begin{array}{ccccc}
\mathrm{H}_n(V, V\setminus\{x\}) & \xleftarrow{(t_x)_*} & \mathrm{H}_n(t_{-x}(V), t_{-x}(V)\setminus\{0\}) & \xrightarrow{\cong} & \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n\setminus\{0\}) \\
\downarrow{\scriptstyle \Psi_*} & & \downarrow{\scriptstyle (t_x\circ\Psi\circ t_{-x'})_*} & & \downarrow{\scriptstyle \cdot\epsilon} \\
\mathrm{H}_n(V', V'\setminus\{x'\}) & \xleftarrow[\cong]{(t_{x'})_*} & \mathrm{H}_n(t_{-x}(V), t_{-x}(V)\setminus\{0\}) & \xrightarrow{\cong} & \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n\setminus\{0\}).
\end{array}$$

We make the following observations:

(1) Note that each translation t_v is an orientation-preserving diffeomorphism of smooth manifolds. Furthermore it follows from the Orientation Preserving Lemma 61.14 that each translation t_v is an orientation-preserving homeomorphism of topological manifolds.
(2) It follows from the Excision Theorem 40.2 that the horizontal homomorphisms to the right are isomorphisms.
(3) It follows from the Local Diffeomorphism-Degree Proposition 44.3 together with (1) that the square at the right commutes.
(4) It is clear that the square at the left commutes.
(5) The statement of the lemma follows from (2)–(4) together with the Orientation Preserving Lemma 61.14 (2) and the the smooth analogue of the Orientation Preserving Lemma 61.14 (2).. ∎

Sketch of the proof of Proposition 61.15. First, as we saw on page 793, the case of 0-dimensional smooth manifolds is basically a tautology, at least as long as all statements are interpreted in a reasonable way. Thus in the following we assume that we are dealing with smooth manifolds of dimension ≥ 1.

Next note that Statement (2) is just a reader-friendly condensation of Statement (1). Thus we need to prove Statement (1). We define the promised natural bijections. Thus let M be an n-dimensional smooth manifold where $n \geq 1$. We first suppose that M is oriented as a smooth manifold. Let $x \in M \setminus \partial M$. We pick a chart $\Phi\colon U \to V$ around x that is orientation-preserving as a diffeomorphism between smooth manifolds. We define

$$\mu_x \;:=\; \Phi_*^{-1}(\text{standard generator of } \mathrm{H}_n(V, V \setminus \{\Phi(x)\}) \text{ as defined on page 785}).$$

It follows from the Orientations–Preserved Lemma 61.16 that this definition does not depend on the choice of the orientation-preserving chart Φ. It now follows easily from the Standard Euclidean Orientation Lemma 61.10 that the classes $\{\mu_x\}_{x\in M\setminus\partial M}$ form an orientation for the topological manifold M.

Now we define the inverse map. So suppose that we are given an orientation $\{\mu_x\}_{x\in M\setminus\partial M}$ for the topological manifold M. One can easily show that we only need to find an orientation for $M \setminus \partial M$. Thus let $x \in M \setminus \partial M$. We pick a chart $\Phi\colon U \to V$ around x such that V is connected. It follows from the Orientation Preserving Lemma 61.14 that, after possibly reflecting V in a hyperplane, we can assume that Φ preserves the orientation at x, i.e. it sends the given orientation μ_x to the standard orientation of the open subset V of $\mathbb{R}^n$ that

we introduced on page 793. We define

$$\text{orientation of } \mathrm{T}_x M \coloneqq \Phi^*(\text{standard orientation of } \mathrm{T}_{\Phi(x)} V = \mathbb{R}^n).$$

It follows from the Orientations–Preserved Lemma 61.16 that this definition does not depend on the choice of Φ. It is now straightforward to verify that these orientations define an orientation of the smooth manifold M.

It is basically clear, that by construction, the bijections have the desired properties (a) and (b). It also straightforward to see that the map between the two types of orientations is determined uniquely by properties (a) and (b). This concludes the proof of Statement (1). ■

We formulate one essential example as a lemma:

Lemma 61.17. (Sphere Orientations Agree Lemma) Let $n \in \mathbb{N}$. We consider the smooth manifold S^n. Under the natural bijection from the Smooth-Topological Orientation Proposition 61.15 the smooth standard orientation of S^n gets matched to the topological standard orientation that we introduced on page 794.

Proof. We outsource the proof to [**Fri24**]. ■

CHAPTER 62

Induced orientations

In the last chapter we introduced the notion of an orientation of a topological manifold. In the Smooth-Topological Orientation Proposition 61.15 we showed that an orientation on a smooth manifold naturally defines an orientation of the underlying topological manifold.

For smooth manifolds we know that if we start with oriented smooth manifolds, then various constructions of smooth manifolds can naturally be endowed with an orientation.

Similar to the setting of smooth manifolds we will now see that various constructions of topological manifolds out of given given oriented topological manifolds can also be endowed with natural orientations. For the most part the statements sound reasonable and familiar from the setting of smooth manifolds. Nonetheless, the constructions of the induced representations for topological manifolds are at times quite different from the constructions of induced orientations for smooth manifolds.

62.1. Induced orientations.

Lemma 62.1. (Topological Manifold-Induced Orientation Lemma) Let R be a commutative ring and let M be a topological manifold that is equipped with an R-orientation.

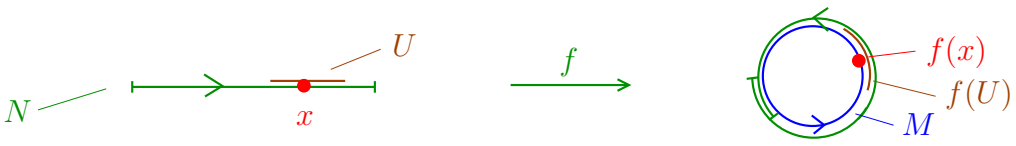
(1) Let N be a topological manifold and let $f\colon N \to M$ be a local homeomorphism, then there exists a unique R-orientation on N such that f is orientation-preserving.
(2) If N is a submanifold of codimension zero (for example N could be an open subset of M), then N admits a unique R-orientation such that the inclusion map $N \to M$ is orientation-preserving.
(3) Let $p\colon \widetilde{M} \to M$ be a covering of topological manifolds. There exists a unique R-orientation on $\widetilde{M}$ that turns p into an orientation-preserving map.

Proof. To simplify the notation we suppress the ring R from the notation. Thus let M be an n-dimensional topological manifold that is equipped with an orientation $\{\alpha_x\}_{x\in M\setminus\partial M}$.

(1) Let N be a topological manifold and let $f\colon N \to M$ be a local homeomorphism. Let $x \in N \setminus \partial N$. We pick an open neighborhood U of x such that $f\colon U \to f(U)$ is a homeomorphism and such that the image $f(U)$ is an open subset of M. We define $\beta_x \in \mathrm{H}_n(N, N\setminus\{x\}; R)$ to be the image of α_x under the map

$$\mathrm{H}_n(M, M\setminus\{f(x)\}) \xleftarrow[\uparrow]{\cong} \mathrm{H}_n(f(U), f(U)\setminus\{f(x)\}) \xrightarrow[\cong]{(f_*)^{-1}} \mathrm{H}_n(U, U\setminus\{x\}) \xrightarrow[\uparrow]{\cong} \mathrm{H}_n(N, N\setminus\{x\}).$$

it follows from the Excision Theorem 40.2 that these two maps are isomorphisms



We leave it to the reader to verify that these classes β_x define an orientation on N. By definition the map f is orientation-preserving with respect to the orientations on M and N. It is elementary to verify that the orientation on N is uniquely determined.

(2) This statement follows from (1) since the inclusion map of a codimension-zero submanifold is, basically by definition, a local homeomorphism.

(3) This statement also follows from (1) since a covering map is a local homeomorphism. ■

Proposition 62.2. (Topological Orientation-Action Proposition) Let M be a topological manifold. Furthermore let G be a group that acts freely properly and continuously on M. Note that the quotient M/G is a topological manifold and the projection $p\colon M \to M/G$ is a local homeomorphism.

(1) If M is oriented and if the G-action is orientation-preserving, then the topological manifold M/G admits a unique orientation such that the projection $p\colon M \to M/G$ is orientation-preserving.

(2) Conversely, if the topological manifold M/G admits an orientation, then there exists a unique orientation on M such that the projection $p\colon M \to M/G$ is orientation-preserving. Furthermore this orientation has the property that the G-action on M is orientation-preserving.

Example. We consider the action of $\mathbb{Z}^n$ on $\mathbb{R}^n$ that is given by addition. It follows from the discussion on page 796 that this action is orientation-preserving. It follows from the Orientation-Action Proposition 62.2 that $\mathbb{R}^n/\mathbb{Z}^n$ is an orientable topological manifold. □

Proof. The proof is similar to the proof for the case of smooth manifolds. We leave the details to the reader. ■

Note that the boundary of an oriented smooth manifold can be equipped with a natural orientation. The following proposition is the topological analogue:

Proposition 62.3. (Boundary Orientation Proposition)

(1) Let R be a commutative ring. Given any topological manifold M there exists a natural map

$$\{R\text{-orientations of } M\} \;\to\; \{R\text{-orientations of } \partial M\}.$$

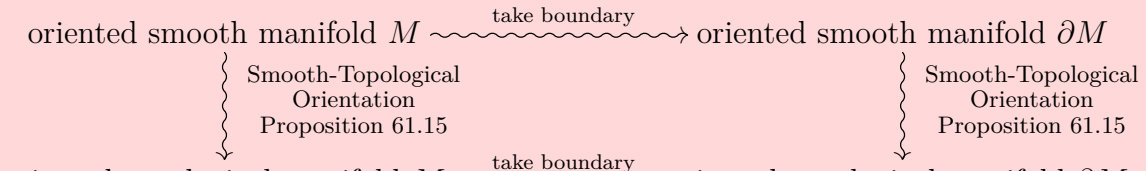
Here natural means that if $f\colon M \to N$ is a local homeomorphism, then the following diagram commutes:

$$\begin{array}{ccc} \{R\text{-orientations of } M\} & \longrightarrow & \{R\text{-orientations of } \partial M\} \\ \downarrow f_* & & \downarrow f_* \\ \{R\text{-orientations of } N\} & \longrightarrow & \{R\text{-orientations of } \partial N\}. \end{array}$$

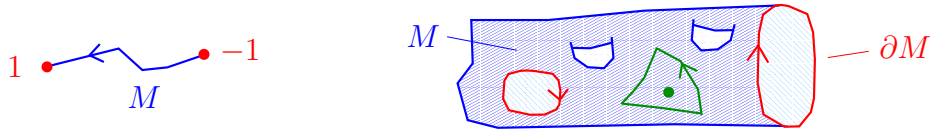
This orientation has the property that $\partial(-M) = -\partial M$. The same results also hold for homology without coefficients.

(2) For homology without coefficients this boundary orientation can be chosen to be compatible with the boundary orientation in the smooth setting. More precisely: Let M be an n-dimensional smooth manifold that is equipped with an orientation as a smooth

manifold. We have the following commutative "diagram"

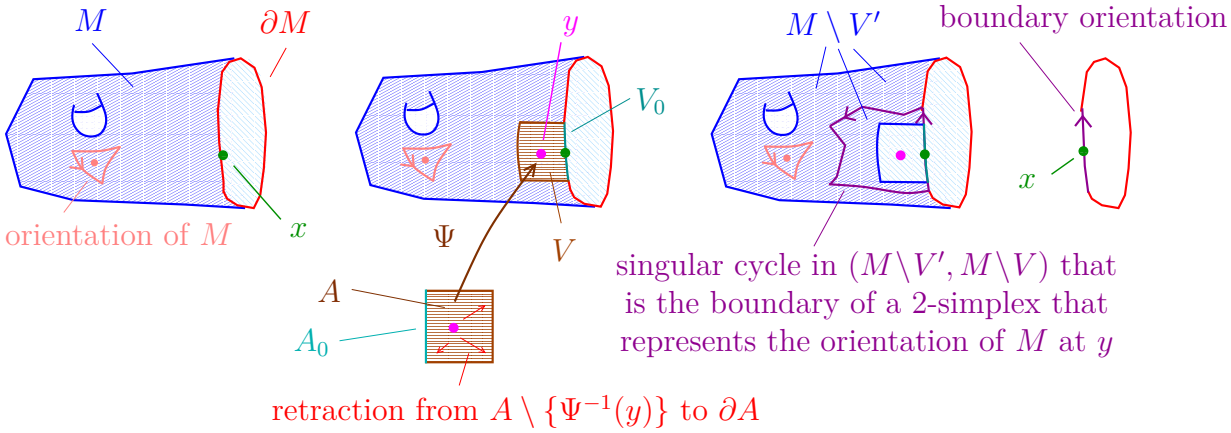


Convention. If M is an R-oriented topological manifold, then invariably we will equip ∂M with the R-orientation provided by the construction in the proof of the Boundary Orientation Proposition 62.3.



Proof of Statement (1). To simplify the notation we will only deal with case that we have no coefficients. That is anyway the only case we care about. We suppose that we are given an orientation $\{\mu_y\}_{y\in M\setminus\partial M}$ for M. In the following we will define a natural orientation for the topological manifold ∂M.

We set $n := \dim(M)$ and we write $A = [0,1]\times[-1,1]^{n-1}$, we write $A_0 := \{0\}\times[-1,1]^{n-1}$ and we write $A' = (0,1)\times(-1,1)^{n-1}$. Now let $x \in \partial M$. Since M is a topological manifold and since $x \in \partial M$ we can pick a chart $\Phi\colon U \to [0,2)\times(-2,2)^{n-1}$ with $\Phi(x) = (0,0)$. We write $\Psi := \Phi^{-1}$. Furthermore we write $V = \Psi(A)$, $V_0 = \Psi(A_0)$ and we write $V' = \Psi(A')$. Finally we pick a point $y \in V'$.



We consider the following sequence of maps:

$$\begin{array}{ccccc} \mathrm{H}_n(M, M\setminus\{y\}) & \xleftarrow{(1)} & \mathrm{H}_n\big(M, M\setminus V'\big) & \xrightarrow[(2)]{\partial_n} & \mathrm{H}_{n-1}(M\setminus V', (M\setminus V')\setminus\{x\}) \\ & & & & \uparrow{\scriptstyle (3)} \\ & & & & \mathrm{H}_{n-1}(V_0, V_0\setminus\{x\}) \xrightarrow[(4)]{} \mathrm{H}_{n-1}(\partial M, \partial M\setminus\{x\}). \end{array}$$

(5)

Here all the maps are induced by the obvious inclusions of pairs of topological spaces, except for the second map which is the connecting homomorphism of the long exact sequence of the triple $(M, M\setminus V', (M\setminus V')\setminus\{x\})$ that is provided by the LES-of-Triple Proposition 38.2.

Claim 1. All the maps in the above diagram are isomorphisms.

Proof. In the following we treat the five maps separately.

(1) It is elementary to see that $A \setminus \Psi^{-1}(y)$ is a deformation retract of ∂A. It follows that $M \setminus V'$ is a deformation retract of $M \setminus \{y\}$.[511] We obtain from the Homotopy Equivalence-Relative-H_*-Corollary 38.4 that (1) an isomorphism.

(2) We consider the long exact sequence of the triple $(M, M \setminus V', (M \setminus V') \setminus \{x\})$ that is provided by the LES-of-Triple Proposition 38.2:

$$\begin{aligned}\mathrm{H}_n(M, (M \setminus V') \setminus \{x\}) \to \mathrm{H}_n(M, M \setminus V') \xrightarrow{\partial_n} \mathrm{H}_{n-1}(M \setminus V', (M \setminus V') \setminus \{x\}) \to \\ \to \mathrm{H}_{n-1}(M, (M \setminus V') \setminus \{x\}).\end{aligned}$$

Similarly to (1) one sees that $(M \setminus V') \setminus \{x\}$ is a deformation retract of M. Therefore it follows from the Deformation Retract-Relative H_*-Corollary 38.5 that the groups at the two extremes are zero. In other words, the connecting homomorphism is an isomorphism.

(3) It follows quite easily from the Excision Theorem 40.2 that (3) is an isomorphism.

(4) It follows equally easily from the Excision Theorem 40.2 that (4) is an isomorphism.

(5) The triangle is given by inclusion-induced maps, hence it commutes. It now follows from (3) and (4) that (5) is also an isomorphism. ⊞

Given $x \in \partial M$ we now define $\nu_x \in \mathrm{H}_{n-1}(\partial M, \partial M \setminus \{x\})$ to be the image of the generator $\mu_y \in \mathrm{H}_n(M, M \setminus \{y\})$ under the above sequence of isomorphisms. We continue with our next claim.

Claim 2.

(1) The definition of ν_x is independent of the choice of y.

(2) The definition of ν_x is independent of the choice of Φ.

Proof.

(1) We need to show that for every $z \in \Psi(V')$ we have

$$\begin{array}{c}\text{preimage of } \mu_y \text{ under the map} \\ \mathrm{H}_n(M, M\setminus V') \xrightarrow{\cong} \mathrm{H}_n(M, M\setminus\{y\})\end{array} = \begin{array}{c}\text{preimage of } \mu_z \text{ under the map} \\ \mathrm{H}_n(M, M\setminus V') \xrightarrow{\cong} \mathrm{H}_n(M, M\setminus\{z\}).\end{array}$$

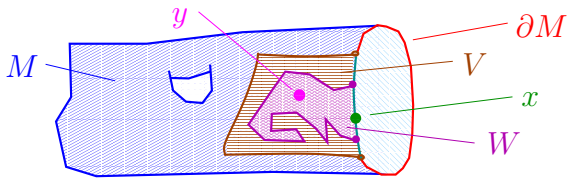
But this statement can be shown in a similar fashion as the Sections-on-Connected Proposition 61.7. We leave it to the reader to fill in the details.

[511] Looking at the figure on page 801 this statement sounds reasonable. But the statement nonetheless requires a proof. So let r be a deformation retraction from $A \setminus \Psi^{-1}(y)$ to ∂A. We consider the map

$$\begin{array}{rcl} s\colon (M \setminus \{y\}) \times [0,1] & \to & M \setminus \{y\} \\ (P,t) & \mapsto & \begin{cases} P, & \text{if } P \notin V \setminus \{y\}, \\ \Psi(r(\Psi^{-1}(P), t)), & \text{if } P \in V \setminus \{y\}. \end{cases} \end{array}$$

Note that $V \setminus \{y\}$ and $M \setminus \Psi(A')$ are closed subsets of $M \setminus \{y\}$. (Hmm, why is that?). It follows from the Homotopy Combination Lemma 8.2 (1) that the map s is continuous.

(2) Suppose we are given a different chart than Φ. We define W, W_0, W' in the obvious way. By possibly taking a third chart whose image lies in the intersection of V and W we can assume, without loss of generality, that $W \subset V$.



Furthermore by (1) we can work with a point $y \in W$ for both charts. We consider the following diagram

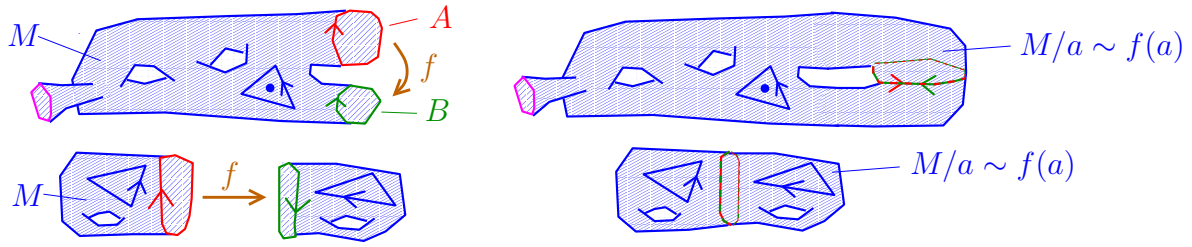
$$\begin{array}{ccc}
\mathrm{H}_n(M, M\backslash\{y\}) & \xleftarrow[(1)]{\cong} & \mathrm{H}_n\big(M, M\backslash V'\big) \\
\cong\uparrow{\scriptstyle (1)} & & {\scriptstyle (2)}\downarrow\partial_n \\
\mathrm{H}_n\big(M, M\backslash W'\big) & \leftarrow\!\!\!

Proposition 62.4. (Gluing-Topological Manifolds-Proposition) Let M be an n-dimensional topological manifold (possibly disconnected), let A and B be disjoint unions of boundary components of M and let $f\colon A \to B$ be a homeomorphism. Then the following statements hold:

(1) The topological space
$$M/a \sim f(a) \quad \text{for } a \in A$$
is an n-dimensional topological manifold.

(2) The boundary of $M/a \sim f(a)$ is given by $\partial M \setminus (A \cup B)$.
   (a) The image of A in $M/a \sim f(a)$ is a codimension-one submanifold of $M/a \sim f(a)$.
   (b) If M consists of two components X and Y with $A \subset X$ and $B \subset Y$, then the following statements hold:
      (i) The obvious maps $i\colon X \to M/a \sim f(a)$ and $j\colon Y \to M/a \sim f(a)$ are embeddings.
      (ii) The images $i(X)$ and $j(Y)$ are closed subsets of $M/a \sim f(a)$ and they are codimension-zero submanifolds of $M/a \sim f(a)$.

(3) Let R be a commutative ring. Assume that M is R-oriented. We equip A and B with the boundary orientations, as defined in the Boundary Orientation Proposition 62.3.
   (a) If f is orientation-reversing, then $M/a \sim f(a)$ admits an R-orientation such that the inclusion $M \setminus (A \cup B) \to M/a \sim f(a)$ is orientation-preserving.
   (b) If M is connected and if f is not orientation-reversing, then $M/a \sim f(a)$ is not R-orientable.



Proof. The proofs for (1) and (2) can be found in [**Fri24**]. We still need to prove Statement (3). We will not make use of Statement (3b), thus we only provide a proof of Statement (3a), and even that proof will only be sketch.

We now assume that M is equipped with an R-orientation. We set $\widetilde{M} = M/a \sim f(a)$ and we denote by $p\colon M \to \widetilde{M}$ the projection. We need to show that $\widetilde{M}$ admits an orientation such that the inclusion $M \setminus (A \cup B) \to M/a \sim f(a)$ is orientation-preserving.

Now let $x \in \widetilde{M} \setminus \partial\widetilde{M}$. We need to distinguish two cases:

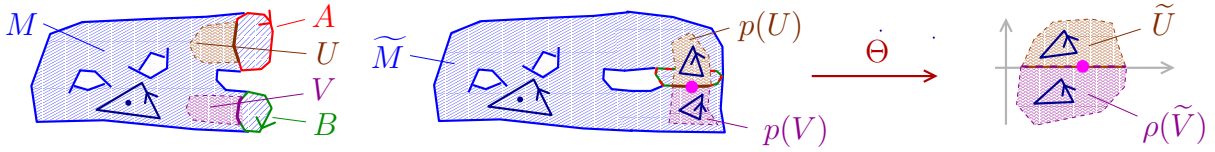
(α) We assume that $x = p(y)$ for some $y \in M \setminus \partial M$. In this case we have natural isomorphisms $\mathrm{H}_n(\widetilde{M}, \widetilde{M}\setminus\{y\}; R) \xleftarrow{\cong} \mathrm{H}_n(M\setminus\partial M, (M\setminus\partial M)\setminus\{y\}; R) \xrightarrow{\cong} \mathrm{H}_n(M, M\setminus\{y\}; R)$ and we equip the local homology group of $\widetilde{M}$ with the generator given by the R-orientation of M.

(β) If (α) does not hold, then there exist $a \in A$ and $b \in B$ with $x = p(a) = p(b)$. We pick an orientation-preserving chart $\Phi\colon U \to \widetilde{U}$ of type (ii) for $a \in M$ and we pick an orientation-reversing chart $\Psi\colon V \to \widetilde{V}$ of type (ii) for $b \in M$. After possibly shrinking U and V we can assume that $f(U \cap A) = V \cap B$ and we can assume that $U \cap V = \varnothing$. We denote by $\rho\colon \mathbb{R}^n \to \mathbb{R}^n$ the reflection in the $(x_n = 0)$-hyperplane. As we discussed

in the proof of the Gluing-Topological Manifolds-Proposition 62.4 the map

$$\begin{aligned}\Theta\colon p(U)\cup p(V) &\to \widetilde{U}\cup\rho(\widetilde{V})\\ x &\mapsto \begin{cases}\Phi(u), & \text{if } x=p(u) \text{ for some } u\in U,\\ ((f|_{U\cap A},\mathrm{id}_{\mathbb{R}})\circ\rho)(\Psi(v)), & \text{if } x=p(v) \text{ for some } v\in V\end{cases}\end{aligned}$$

is a chart of type (i) for x.



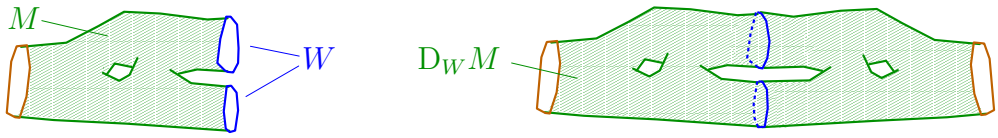
It follows from our choice of Φ and Ψ, the example on page 796 and the Orientation Preserving Lemma 61.14 that this map Θ is orientation-preserving on $p(U\backslash A)\cup p(V\backslash B)$. But then we can use this chart to also equip the points on $p(U\cap A)=p(V\cap B)$ with a coherent R-orientation.

We leave it to the reader to verify that these choices do indeed define an R-orientation on $\widetilde{M}$ with the desired properties. ■

Definition. Let M be a topological manifold and let W be the union of some components of ∂M. We refer to

$$\mathrm{D}_W M \;:=\; (M\times\{1\})\cup_{W\times\{1\}=W\times\{2\}}(M\times\{2\})$$

as the double of M along W. We refer to $\mathrm{D}M := \mathrm{D}_{\partial M}M$ as the double of M.



Lemma 62.5. (Double-Orientability Lemma) Let M be a connected topological manifold and let W be the union of some components of ∂M. Furthermore let R be a commutative ring. Then

$$M \text{ is } R\text{-orientable} \iff \text{the double } \mathrm{D}_W M \text{ is } R\text{-orientable.}$$

Furthermore, if M is equipped with an R-orientation, then $\mathrm{D}_W M$ admits a unique R-orientation such that the "first" inclusion map $M\to \mathrm{D}_W M$ is orientation-preserving and such that the "second" inclusion map $M\to \mathrm{D}_W M$ is orientation-reversing.

Proof. The"⇒"–direction follows from the Gluing-Topological Manifolds-Proposition 62.4 applied to the following data:

- We consider the topological manifold $(M\times\{1\})\sqcup(M\times\{2\})$. We equip it with an R-orientation on $M\times\{1\}$ and with the opposite R-orientation on $M\times\{2\}$.
- We set $A=W\times\{1\}$ and $B=W\times\{2\}$.
- We let $f\colon A\to B$ be the obvious homeomorphism. Note that f is orientation-reversing.

The"⇐"–direction is an immediate consequence of the Topological Manifold-Induced Orientation Lemma 62.1. ■

62.3. **Product orientations of topological manifolds.** In this last section we want to consider products of topological manifolds.

Proposition 62.6. (Topological Manifold Product Proposition) Let M be an m-dimensional topological manifold and let N be an n-dimensional topological manifold. Then the following hold:

(1) The product $M \times N$ is an $(m+n)$-dimensional topological manifold.
(2) We have $\partial(M \times N) = (\partial M \times N) \cup (M \times \partial N)$.

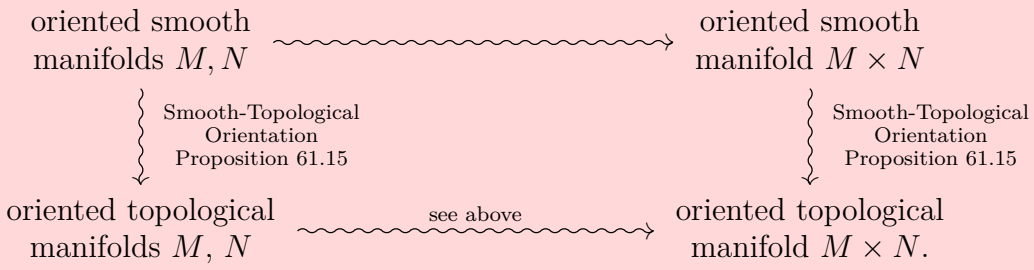


Proof. The proof is fairly elementary. We leave the details to [**Fri24**]. ■

The orientation of the product of two oriented smooth manifolds is a well-known concept. The following proposition gives us the topological analogue.

Proposition 62.7. (Product Orientation Proposition)

(1) Let M be an m-dimensional topological manifold and let N be an n-dimensional topological manifold.
   (a) For any $x \in M$ and any $y \in N$ the cross product
   $$\mathrm{H}_m(M, M \setminus \{x\}) \otimes \mathrm{H}_n(N, N \setminus \{y\}) \xrightarrow{\times} \mathrm{H}_{m+n}(M \times N, (M \times N) \setminus \{(x,y)\})$$
   is an isomorphism.
   (b) If $\{\mu_x\}_{x \in M \setminus \partial M}$ is an orientation of M and if $\{\nu_y\}_{y \in N \setminus \partial N}$ is an orientation on N, then the cross products $\mu_x \times \nu_y$ define a natural[512] orientation on $M \times N$.
(2) The product orientation of (1) is compatible with the product orientation of smooth manifolds. More precisely, for any two smooth manifolds M and N (where at most one has non-empty boundary) we have the following commutative diagram:



Example. Let $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ be open non-empty subsets. We equip U and V with the smooth and topological standard orientations. In the smooth setting the product orientation on $U \times V$ is clearly the orientation coming from viewing $U \times V$ as a subspace of $\mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n}$. It follows from the Product Orientation Proposition 62.7 that the same conclusion also holds in the context of orientations of topological manifolds. □

[512] Here "natural" refers to local homeomorphisms between topological manifolds.

Definition. Let M and N be oriented topological manifolds. We refer to the orientation on $M \times N$ defined in the Product Orientation Proposition 62.7 (1) as the product orientation.

The remainder of this section is occupied by the proof of the Product Orientation Proposition 62.7. The reader who is comfortable without seeing the full details of the proof can also jump straight to Chapter 63, where we consider homology groups of topological manifolds.

The proof of the Product Orientation Proposition 62.7 (2) requires that we work with an explicit Eilenberg-Zilber map. Fortunately in the Shuffle Proposition 56.11 we already gave an explicit example of an Eilenberg-Zilber map. It is worthwhile to recall the statement of the Shuffle Proposition 56.11. To do so it is necessary to recall some definitions from page 717.

Definition. Let $m, n \in \mathbb{N}_0$.

(1) An (m,n)-shuffle is a pair (μ, ν) of disjoint sets of integers

$$1 \leq \mu_1 < \mu_2 < \cdots < \mu_m \leq m+n \quad \text{and} \quad 1 \leq \nu_1 < \nu_2 < \cdots < \nu_n \leq m+n.$$

(2) We denote by $S_{m,n}$ the set of all (m,n)-shuffles.

(3) We denote by $\operatorname{sign}(\mu, \nu)$ the sign of the permutation $(\mu_1, \ldots, \mu_m, \nu_1, \ldots, \nu_n)$.

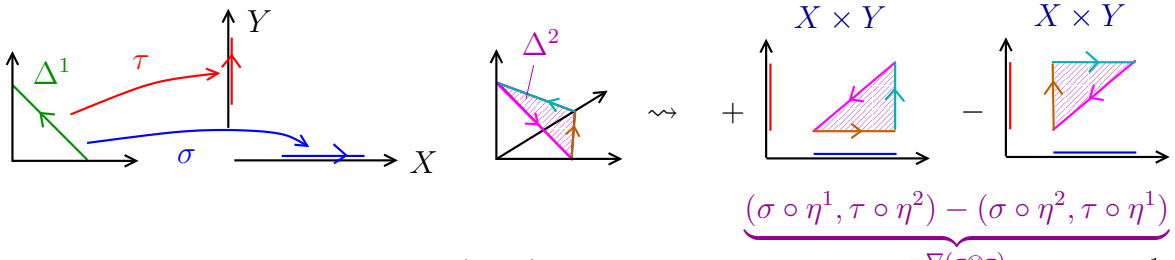
(4) Given a sequence $1 \leq \mu_1 < \mu_2 < \cdots < \mu_m \leq m+n$ we denote by $\eta^\mu \colon \Delta^{m+n} \to \Delta^m$ the unique affine linear map which satisfies $\eta^\mu(\mathfrak{v}_i) = \mathfrak{v}_j$ for $\mu_j \leq i < \mu_{j+1}$.[513]

(5) Given two topological spaces X and Y we consider the homomorphism

$$\begin{aligned} \nabla := \nabla_{m,n} \colon \mathrm{C}_m(X) \otimes \mathrm{C}_n(Y) &\to \mathrm{C}_{m+n}(X \times Y) \\ (\sigma \otimes \tau) &\mapsto \sum_{(\mu,\nu) \in S_{m,n}} \operatorname{sign}(\mu,\nu) \cdot (\overbrace{(\sigma \circ \eta^\mu)}^{\Delta^{m+n} \to X}, \overbrace{(\tau \circ \eta^\nu)}^{\Delta^{m+n} \to Y}). \end{aligned}$$

$(\sigma \circ \eta^\mu, \tau \circ \eta^\nu)$: map $\Delta^{m+n} \to X \times Y$

Example. We consider the case $X = Y = \mathbb{R}$ and $m = n = 1$. There exist two $(1,1)$-shuffles, namely $\{\mu_1 = 1, \nu_1 = 2\}$ and $\{\mu_1 = 2, \nu_1 = 1\}$. Note that $\eta^1 \colon \Delta^2 \to \Delta^1$ equals the map[514] $[\mathfrak{v}_0, \mathfrak{v}_1, \mathfrak{v}_1]$ and that $\eta^2 \colon \Delta^2 \to \Delta^1$ equals the map $[\mathfrak{v}_0, \mathfrak{v}_0, \mathfrak{v}_1]$.



In the figure above we illustrate $\nabla_{1,1}(\sigma \otimes \tau)$ for two affine linear singular 1-simplices $\sigma \colon \Delta^1 \to X = \mathbb{R}$ and $\tau \colon \Delta^1 \to Y = \mathbb{R}$. □

With this notation we can now recall the following result:

[513] Here we implicitly use that $\mu_0 = 0$ and $\mu_{m+1} = m+n+1$.

[514] Given points $a_0, \ldots, a_k \in \mathbb{R}^n$ we denote by $[a_0, \ldots, a_k] \colon \Delta^k \to \mathbb{R}^n$ the unique affine linear map that sends the i-th standard basis vector e_i to a_i.

Proposition 56.11 (Shuffle Proposition) The maps $\nabla_{m,n}$ that we had just defined have all the properties that we had mentioned in the EZ-Construction Proposition 56.5, in other words, the maps $\nabla_{m,n}\colon \mathrm{C}_m(X)\otimes\mathrm{C}_n(Y)\to\mathrm{C}_{m+n}(X\times Y)$ have the following properties:

(1) On the 0-level they equal the canonical maps that we introduced on page 702. In other words, for all $x\in X$ and $y\in Y$ we have $\nabla_{0,0}(x\otimes y)=(x,y)\in\mathrm{C}_0(X\times Y)$.
(2) They form a chain map $\mathrm{C}_*(X)\otimes\mathrm{C}_*(Y)\to\mathrm{C}_*(X\times Y)$.
(3) They are natural.

Now we can provide the proof of the Product Orientation Proposition 62.7:

Proof of the Product Orientation Proposition 62.7.

(1) Let M be an m-dimensional topological manifold and let N be an n-dimensional topological manifold.
   (a) First note that we have $(M\times N)\setminus\{(x,y)\}=((M\setminus\{x\})\times N)\cup(M\times(N\setminus\{y\}))$. This shows that the cross product does indeed take values in the relative homology group $\mathrm{H}_{m+n}(M\times N,(M\times N)\setminus\{(x,y)\})$. It follows from the Künneth Theorem 56.9 (2), the Manifold-Local-H_*-Lemma 61.1 and the Tor-Properties Lemma 54.7 (4) that the given cross product is indeed an isomorphism.
   (b) Let $(x,y)\in(M\times N)\setminus\partial(M\times N)$. Recall that this means that $x\in M\setminus\partial M$ and $y\in N\setminus\partial N$. We pick an open neighborhood U of $x\in M\setminus\partial M$ and a class $\mu_U\in\mathrm{H}_m(M,M\setminus U)$ and an open neighborhood V of $y\in N\setminus\partial N$ and a class $\nu_V\in\mathrm{H}_n(N,N\setminus V)$ as given by the definition on page 789. By the naturality of the cross product, see the Cross Product Lemma 56.2 (2), we obtain the following commutative diagram:

$$\begin{array}{ccc}
\mathrm{H}_m(M,M\setminus U)\times\mathrm{H}_n(N,N\setminus V) & \xrightarrow{\quad\times\quad} & \mathrm{H}_{m+n}(M\times N,M\times N\setminus(U\times V)) \\
\Big\downarrow{\scriptstyle \mu_U\mapsto\mu_x,\ \nu_V\mapsto\nu_y} & & \Big\downarrow \\
\mathrm{H}_m(M,M\setminus\{x\})\times\mathrm{H}_n(N,N\setminus\{y\}) & \xrightarrow[(\mu_x,\nu_y)\mapsto\mu_x\times\nu_y]{\times} & \mathrm{H}_{m+n}(M\times N,M\times N\setminus\{(x,y)\}).
\end{array}$$

   It is now straightforward to verify that $\phi_{U\times V}\coloneqq\mu_U\times\nu_V$ has the properties desired by the definition on page 789. Finally note that it follows again from the naturality of the cross product that the product orientation is natural.
(2) It follows easily from the naturality of the product orientations of smooth and topological statements that it suffices to prove the statement for the smooth manifolds $M=\mathbb{R}^m$ and for $N=\mathbb{R}^n$ and only for a suitable point $(x,y)\in\mathbb{R}^m\times\mathbb{R}^n=\mathbb{R}^{m+n}$. Given any $k\in\mathbb{N}_0$ we consider the singular chain $\sigma_k=(-1)^k\cdot\pi_k$ where[515] $\pi_k=[\mathfrak{v}_0,\dots,\mathfrak{v}_k-1,0]$. (In plain English the singular simplex $\pi_k=[\mathfrak{v}_0,\dots,\mathfrak{v}_k-1,0]\colon\Delta^k\to\mathbb{R}^k$ is just given by the obvious projection onto the first n coordinates.) We make the following observation:
   (∗) Let $z\in\mathbb{R}^k$ be any point that lies in the interior of the image of σ_k. It follows from the discussion on page 785 that the singular chain $\sigma_k=(-1)^k\cdot\pi_k$ represents the standard generator of $\mathrm{H}_k(\mathbb{R}^k,\mathbb{R}^k\setminus\{z\})$.

[515]Given points $a_0,\dots,a_k\in\mathbb{R}^n$ we denote by $[a_0,\dots,a_k]\colon\Delta^k\to\mathbb{R}^n$ the unique affine linear map that sends the i-th standard basis vector e_i to a_i.

Next we consider the (m,n)-shuffle that is given by the sets $\mu_0 = \{1,\dots,m\}$ and by $\nu_0 = \{m+1,\dots,m+n\}$. It follows easily from the definitions that

$$\eta^{\mu_0} = [\mathfrak{v}_0,\dots,\mathfrak{v}_m,\dots,\mathfrak{v}_m]\colon \Delta^{m+n} \to \Delta^m \quad \& \quad \eta^{\nu_0} = [\mathfrak{v}_m,\dots,\mathfrak{v}_m,\dots,\mathfrak{v}_{m+n}]\colon \Delta^{m+n} \to \Delta^n.$$

Also note that it is a straightforward consequence of the definitions that we have the following equality of singular $(m+n)$-chains in $\mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n}$:

$$(\pi_m \circ \eta^{\mu_0}, \pi_n \circ \eta^{\nu_0}) \;=\; [\mathfrak{v}_0,\dots,\mathfrak{v}_{m-1},\mathfrak{v}_m,\dots,\mathfrak{v}_{m+n-1},0] \;=\; \pi_{m+n}.$$

We pick a point $(x,y) \in \mathbb{R}^m \times \mathbb{R}^n$ that lies in the interior of the image of π_{m+n}. Note that x lies in the interior of the image of π_m and that y lies in the interior of the image of π_n.

It follows from the observation (∗), the definition of the product orientation on page 807 and the Sections-on-Connected Proposition 61.7 that it suffices to prove the following claim:

Claim. The following equality holds in $\mathrm{H}_{m+n}(\mathbb{R}^{m+n}, \mathbb{R}^{m+n} \setminus \{(x,y)\})$:

$$[\sigma_m] \times [\sigma_n] \;=\; [\sigma_{m+n}].$$

Proof. We have

$$[\sigma_m] \times [\sigma_n] \;\underset{\uparrow}{=}\; [\nabla_{m,n}(\sigma_m \otimes \sigma_n)] \;\underset{\uparrow}{=}\; \Big[\sum_{(\mu,\nu)} (-1)^m \cdot (-1)^n \cdot \mathrm{sign}(\mu,\nu) \cdot ((\pi_m \circ \eta^\mu), (\pi_n \circ \eta^\nu))\Big]$$

by the Shuffle Proposition 56.11 and the EZ–Natural Chain Homotopy Proposition 56.8 (d)

by definition of $\nabla_{m,n}$

$$\underset{\uparrow}{=}\; \big[(-1)^{m+n} \cdot \underbrace{\mathrm{sign}(\mu_0,\nu_0)}_{=1} \cdot \underbrace{(\pi_m \circ \eta^{\mu_0} \times \pi_n \circ \eta^{\nu_0})}_{=\pi_{m+n}}\big] \;=\; [\sigma_{m+n}].$$

for all other shuffles (μ,ν) the point (x,y) does not lie in the image of $(\pi_m \circ \eta^\mu, \pi_n \circ \eta^\nu)$

■

CHAPTER 63

The top homology groups of topological manifolds

In this chapter, given an n-dimensional topological manifold M, we will determine the top homology groups $\mathrm{H}_n(M)$ and $\mathrm{H}_n(M, \partial M)$ in terms of basic data (like compactness, orientability) of M.

63.1. **Homology and orientability.** The following is the main theorem of this chapter:

Theorem 63.1. (Manifold-Top-H_*-Theorem) Let M be a connected non-empty n-dimensional topological manifold and let R be a commutative ring. The following three statements hold:

(1) (a) For every $i > n$ we have $\mathrm{H}_i(M; R) = 0$.
(b) For every $i > n$ we have $\mathrm{H}_i(M, \partial M; R) = 0$.

(2) (a)
$$\mathrm{H}_n(M; R) \cong \begin{cases} R, & \text{if } M \text{ is closed and } R\text{-orientable,} \\ \{r \in R \,|\, 2r = 0\}, & \text{if } M \text{ is closed and not } R\text{-orientable,} \\ 0, & \text{if } M \text{ is not closed.} \end{cases}$$
(b)
$$\mathrm{H}_n(M, \partial M; R) \cong \begin{cases} R, & \text{if } M \text{ is compact and } R\text{-orientable,} \\ \{r \in R \,|\, 2r = 0\}, & \text{if } M \text{ is compact and not } R\text{-orientable,} \\ 0, & \text{if } M \text{ is non-compact.} \end{cases}$$

(3) If M is compact and R-orientable, then for every $x \in M \setminus \partial M$ the natural map
$$\mathrm{H}_n(M, \partial M; R) \to \mathrm{H}_n(M, M \setminus \{x\}; R)$$
is an isomorphism.

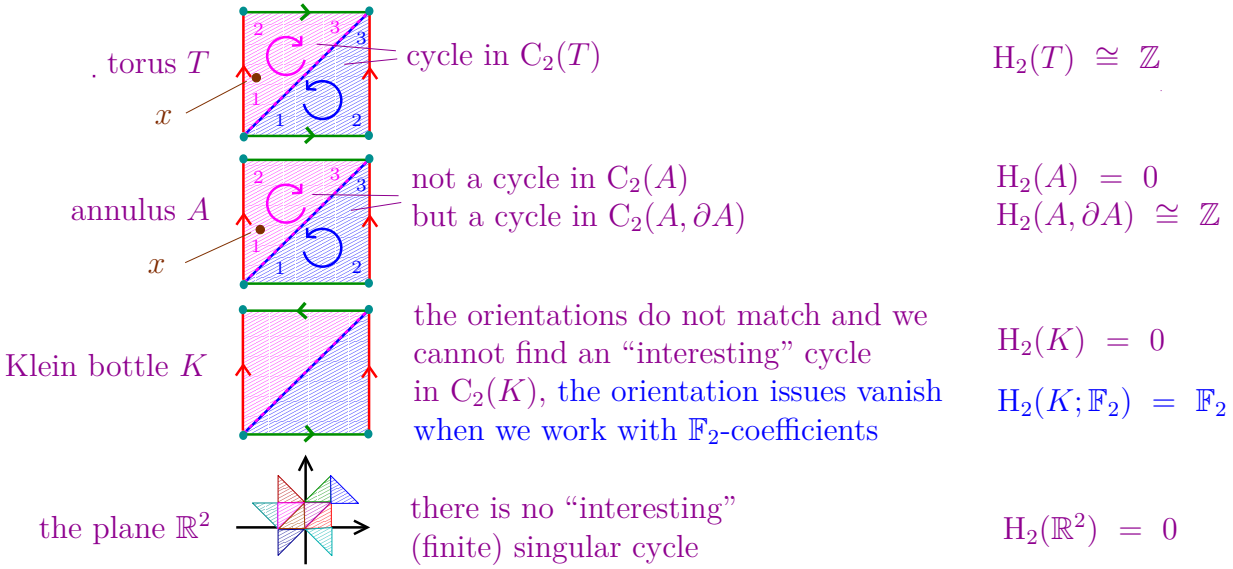
Proof. We will prove the theorem in Section 63.4 after long and arduous preparations. ■

Example. In the following figure we show some examples which hopefully illustrate the statement of the theorem and which work as mnemonic devices to remember the many statements.

For the most part we will be interested in the rings $R = \mathbb{Z}$ and $R = \mathbb{F}_2$. In these two cases we obtain two specialized theorems, which are worth recording:

Theorem 63.2. (Top-$\mathbb{F}_2$-Homology Theorem) Let M be a compact connected non-empty n-dimensional topological manifold M. The following two statements hold:

(1) $\mathrm{H}_n(M, \partial M; \mathbb{F}_2) \cong \mathbb{F}_2$.
(2) For every $x \in M \setminus \partial M$ the natural map $\mathrm{H}_n(M, \partial M; \mathbb{F}_2) \to \mathrm{H}_n(M, M \setminus \{x\}; \mathbb{F}_2)$ is an isomorphism.



Proof. Recall that in the $\mathbb{F}_2$-Orientability Proposition 61.13 we showed that every topological manifold admits an $\mathbb{F}_2$-orientation. This theorem is thus an immediate consequence of the Manifold-Top-H_*-Theorem 63.1 (2b) and (3) applied to $R = \mathbb{F}_2$. ■

Not every topological manifold is orientable. The following theorem gives us several criteria for orientability.

Theorem 63.3. (Orientation-Top Homology-Theorem) Let M be a compact connected non-empty n-dimensional topological manifold. The following statements are equivalent:

(1) The topological manifold M is orientable.
(2) $H_n(M, \partial M; \mathbb{Z}) \neq 0$.
(3) $H_n(M, \partial M; \mathbb{Z}) \cong \mathbb{Z}$.
(4) For every $x \in M \setminus \partial M$ the natural map $H_n(M, \partial M; \mathbb{Z}) \to H_n(M, M \setminus \{x\}; \mathbb{Z})$ is an isomorphism.

Proof of the Orientation-Top Homology-Theorem 63.3. Let M be a compact connected non-empty n-dimensional topological manifold. We have (1)⇒(4) by the Manifold-Top-H_*-Theorem 63.1 (3) applied with $R = \mathbb{Z}$. Since M is non-empty there exists a point $x \in M \setminus \partial M$. Therefore we have (4)⇒(3) by the Manifold-Local R-Homology Lemma 61.1. Evidently we have (3)⇒(2). Furthermore we have (2)⇒(1) by the Manifold-Top-H_*-Theorem 63.1 (2b) and the little observation that $\{r \in \mathbb{Z} \mid 2r = 0\} = 0$. ■

Let M be a compact n-dimensional topological manifold. With the Manifold-Top-H_*-Theorem 63.1 we now completely understand the n-dimensional homology of M for all rings. We will use these results to obtain partial information on the $(n-1)$-st homology groups with $\mathbb{Z}$-coefficients:

Theorem 63.4. (Manifold-H_{n-1}-Theorem) Let M be a compact connected n-dimensional topological manifold.

(1) $$H_{n-1}(M;\mathbb{Z}) \cong \begin{cases} \text{free abelian group}, & \text{if } M \text{ is orientable or if } \partial M \neq \varnothing, \\ \text{free abelian group} \oplus \mathbb{Z}_2, & \text{if } M \text{ is non-orientable and closed.} \end{cases}$$

(2) $$H_{n-1}(M,\partial M;\mathbb{Z}) \cong \begin{cases} \text{free abelian group}, & \text{if } M \text{ is orientable,} \\ \text{free abelian group} \oplus \mathbb{Z}_2, & \text{if } M \text{ is non-orientable.} \end{cases}$$

Proof. Much later in Section 63.5 we will provide a proof of the Manifold-H_{n-1}-Theorem 63.4. As we will see, for the most part it is a fairly formal consequence of the Manifold-Top-H_*-Theorem 63.1 applied to $R = \mathbb{Z}$ and $R = \mathbb{F}_p$ for all primes p together with the Universal Coefficient Theorem 54.9. ■

Example. As a reality check it is good to recall the calculation of the homology groups of closed 2-dimensional topological manifolds that we gave in the Surface-H_*-Proposition 47.9 and of the real projective spaces $\mathbb{R}P^n$ in the $\mathbb{R}P^n$-H_*-Proposition 47.10:

topological manifold M	orientable	$H_{\dim(M)}(M)$	$H_{\dim(M)-1}(M)$
surface Σ_g of genus g	yes	$\mathbb{Z}$	$\mathbb{Z}^{2g}$
non-orientable surface N_g of genus g	no	0	$\mathbb{Z}_2 \oplus \mathbb{Z}^{g-1}$
real projective space $\mathbb{R}P^n$ with n odd	yes	$\mathbb{Z}$	0
real projective space $\mathbb{R}P^n$ with n even	no	0	$\mathbb{Z}_2$.

Fortunately these calculations are coherent with the Manifold-Top-H_*-Theorem 63.1 and the Manifold-H_{n-1}-Theorem 63.4. □

The structure of the chapter is as follows:

(1) In Section 63.2 we will use the above results to calculate the homology groups of lens spaces.
(2) In Section 63.3 we will formulate the all important Homology-Section Theorem 63.6.
(3) The intricate proof of the Homology-Section Theorem 63.6 will take place in Subsections 63.3.1, 63.3.2 and 63.3.3.
(4) In Section 63.4 we will use the Homology-Section Theorem 63.6 to provide the proof of the Manifold-Top-H_*-Theorem 63.1.
(5) In Section 63.5 we will deduce the Manifold-H_{n-1}-Theorem 63.4 from the Manifold-Top-H_*-Theorem 63.1.

Readers who are not all that interested in lens spaces and the proofs of the Manifold-Top-H_*-Theorem 63.1 can jump straight for Chapter 64 where we apply the above theorems.

63.2. Homology groups of lens spaces.

In this section we will apply the results from the previous section to calculate the homology groups of lens spaces.

Lemma 63.5. (Lens Space-H_*-Lemma) Let $p \in \mathbb{N}$ and let $q \in \mathbb{Z}$ be coprime to p. For the corresponding lens space $L(p,q) = S^3/\mathbb{Z}_p$ we have

$$H_n(L(p,q);\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & \text{if } n = 0, 3, \\ 0, & \text{if } n = 2 \text{ or } n > 3, \\ \mathbb{Z}_p, & \text{if } n = 1. \end{cases}$$

Proof. We start out with the following observations:

(1) We already know from the H_0-Proposition 35.11 that $H_0(L(p,q)) \cong \mathbb{Z}$.
(2) By the Lens Space-H_1-Corollary 58.6 we know that $H_1(L(p,q)) \cong \mathbb{Z}_p$.
(3) On page 212 we pointed out that every lens space $L(p,q)$ is a closed orientable connected 3-dimensional smooth manifold.
(4) By the Manifold-Top-H_*-Theorem 63.1 (1) we know that $H_n(L(p,q)) = 0$ for $n \geq 4$.
(5) By the Orientation-Top Homology-Theorem 63.3 we know that $H_3(L(p,q)) \cong \mathbb{Z}$.

Thus at the moment we are still missing $H_2(L(p,q))$. To obtain this homology group we make two more observations:

(6) By the Smooth Manifold-CW-Theorem 30.2 we know that $L(p,q)$ admits the structure of a finite CW-complex.
(7) It follows from the Action–Covering Proposition 14.5 and the definition of a lens space that $L(p,q)$ is finitely covered by S^3. It follows from $\chi(S^3) = 0$ together with the Covering-χ-Proposition 48.10 that $\chi(L(p,q)) = 0$.

Now note that

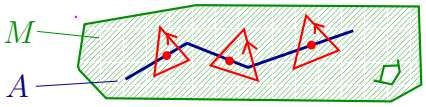
$$\operatorname{rank} H_2(L(p,q)) \overset{\text{by the Homology-}\chi\text{-Proposition 48.1}}{=} \underbrace{\chi(L(p,q))}_{=0} - \underbrace{\operatorname{rank} H_0(L(p,q))}_{=1} + \underbrace{\operatorname{rank} H_1(L(p,q))}_{=0} + \underbrace{\operatorname{rank} H_3(L(p,q))}_{=1} = 0.$$

We know by the Manifold-H_{n-1}-Theorem 63.4 that $H_2(L(p,q))$ is a finitely generated free abelian group. Therefore we see that $H_2(L(p,q)) = 0$. ∎

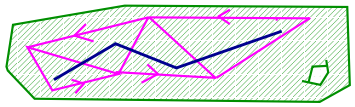
63.3. **Lifts of sections on topological manifolds.** In this section we will state and prove a fundamental theorem of topological manifolds which lies at the heart of the proof of the Manifold-Top-H_*-Theorem 63.1. Before we can state the theorem we introduce the following helpful notation.

Notation. Let M be a topological manifold with $\partial M = \varnothing$, let R be a commutative ring and let $A \subset M$ be a subset.

(1) We write $H_n(M|A;R) := H_n(M, M \setminus A; R)$. Given a point $x \in M$ we use the obvious abbreviation $H_n(M|x;R) := H_n(M|\{x\};R)$.
(2) Given a homology class $\alpha \in H_n(M|A;R)$ and a point $x \in A$ we denote by α_x the image of α under the obvious map $H_n(M|A;R) \to H_n(M|x;R)$.
(3) Let $A \subset M$ be a subset and let $\{\alpha_x\}_{x\in A}$ be an R-section of M along A. We call a class $\beta \in H_n(M|A;R)$ a lift to A if for all $x \in A$ we have $\beta_x = \alpha_x$.
(4) As on page 789 we denote by $\Gamma_R^A(M)$ the set of all R-sections of M along A. Recall that by the Homology-defines-Section Lemma 61.6 we know that $\Gamma_R^A(M)$ is naturally an R-module.



section $\{\alpha_x\}_{x\in A}$ along A


lift β of the section to $H_2(X, A)$

The following theorem is the main technical result of this chapter.

Theorem 63.6. (Homology-Section Theorem) Let M be an n-dimensional topological manifold with $\partial M = \varnothing$ and let $A \subset M$ be a compact subset. For every commutative ring R the following statements hold:

(i) The map

$$\begin{array}{ccc} \mathrm{H}_n(M|A;R) & \to & \Gamma_R^A(M) \\ \varphi & \mapsto & \{\varphi_x\}_{x\in A} \end{array}$$

is an isomorphism of R-modules. In other words, if $\{\alpha_x\}_{x\in A}$ is an R-section for M along A, then there exists a unique lift $\alpha_A^M = \alpha_A$ to A, i.e. there exists a unique homology class $\alpha_A^M = \alpha_A \in \mathrm{H}_n(M|A;R)$ such that for each $x \in A$ we have $\alpha_{A,x} = \alpha_x$.

(ii) For $i > n$ we have $\mathrm{H}_i(M|A;R) = 0$.

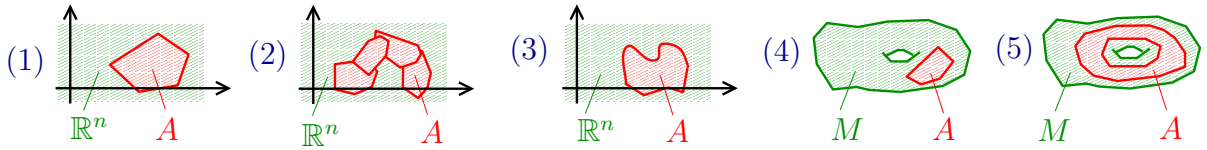
Remark. To avoid any nervousness later on we remark that the statements (i) and (ii) hold trivially for $A = \varnothing$ since $\mathrm{H}_n(M|\varnothing;R) = \mathrm{H}_n(M,M;R) = 0$.

How can we prove the Homology-Section Theorem 63.6? We need to show two statements for all topological manifolds M with $\partial M = \varnothing$ and all compact subsets $A \subset M$. We proceed as follows:

(∗) First we show in the Homology Section–Union Lemma 63.7 the following: If we are given a topological manifold M and two compact subsets A and B such that the statements (i) and (ii) hold for M and compact subsets A, B and $A \cap B$, then the statement holds for $A \cup B$.

We then work our way up from simple cases to the general cases:

(1) We show the statements for $M = \mathbb{R}^n$ and A a convex compact subset.
(2) We use (1) and (∗) to show the statements for $M = \mathbb{R}^n$ and A a finite union of convex compact subsets.
(3) We use (2) to show the statements for $M = \mathbb{R}^n$ and any compact subset A.
(4) We use (3) to show the statements for any M and any compact subset $A \subset M$ contained in a chart.
(5) We use (4) and (∗) to show the general case.



63.3.1. **Proof of the Homology-Section Theorem 63.6 I.** We start out with the following lemma which corresponds to the statement (∗) above.

Lemma 63.7. (Homology Section–Union Lemma) Let M be an n-dimensional topological manifold with $\partial M = \varnothing$, let R be a commutative ring and let A and B be two compact subsets of M. If the statements (i) and (ii) from the Homology-Section Theorem 63.6 hold for A, B and $A \cap B$, then they also hold for $A \cup B$.

The proof of the Homology Section–Union Lemma 63.7 requires the following variation on the Mayer–Vietoris Theorem 45.1.

Theorem 63.8. (Inverted Mayer–Vietoris Theorem) Let X be a topological space and let U and V be open subsets. Then for all $n \in \mathbb{N}_0$ there exists a natural homomorphism $\partial_n \colon \mathrm{H}_n(X, U \cup V) \to \mathrm{H}_{n-1}(X, U \cap V)$ such that

$$.. \to \mathrm{H}_n(X, U \cap V) \xrightarrow{(i,-i)} \mathrm{H}_n(X,U) \oplus \mathrm{H}_n(X,V) \xrightarrow{i \oplus i} \mathrm{H}_n(X, U \cup V) \xrightarrow{\partial_n} \mathrm{H}_{n-1}(X, U \cap V) \to ..$$

is an exact sequence. The same statement also holds for coefficients in any commutative ring R.

Proof of the Inverted Mayer–Vietoris Theorem 63.8. Let X be a topological space and let U and V be open subsets. We recall the following notation from page 509:

$$\mathrm{C}_n^{\{U,V\}}(X) \;\coloneqq\; \{\sigma + \tau \,|\, \sigma \in \mathrm{C}_n(U) \text{ and } \tau \in \mathrm{C}_n(V)\}.$$

As in the second proof of the Mayer–Vietoris Theorem 45.1 on page 565 we have a short exact sequence

$$0 \;\to\; \mathrm{C}_n(X, U \cap V) \;\xrightarrow{(i,-i)}\; \mathrm{C}_n(X,U) \oplus \mathrm{C}_n(X,V) \;\xrightarrow{i \oplus i}\; \mathrm{C}_n(X)/\mathrm{C}_n^{\{U,V\}}(U \cup V) \;\to\; 0.$$

Thus we get a corresponding long exact sequence of homology groups. It suffices to prove the following claim:

Claim. There exists an isomorphism $\mathrm{H}_n(\mathrm{C}_*(X)/\mathrm{C}_*^{\{U,V\}}(U \cup V)) \to \mathrm{H}_n(X, U \cup V)$.

Proof. We consider the following commutative pair of short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{C}_*^{\{U,V\}}(U \cup V) & \longrightarrow & \mathrm{C}_*(X) & \longrightarrow & \mathrm{C}_*(X)/\mathrm{C}_*^{\{U,V\}}(U \cup V) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{C}_*(U \cup V) & \longrightarrow & \mathrm{C}_*(X) & \longrightarrow & \mathrm{C}_*(X, U \cup V) & \longrightarrow & 0.
\end{array}$$

We obtain the corresponding long exact sequence of homology groups:

$$\begin{array}{ccccccccccc}
\ldots & \longrightarrow & \mathrm{H}_n^{\{U,V\}}(U \cup V) & \longrightarrow & \mathrm{H}_n(X) & \longrightarrow & \mathrm{H}_n\big(\mathrm{C}_*(X)/\mathrm{C}_*^{\{U,V\}}(U \cup V)\big) & \longrightarrow & \mathrm{H}_{n-1}^{\{U,V\}}(U \cup V) & \longrightarrow & \ldots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\ldots & \longrightarrow & \mathrm{H}_n(U \cup V) & \longrightarrow & \mathrm{H}_n(X) & \longrightarrow & \mathrm{H}_n(X, U \cup V) & \longrightarrow & \mathrm{H}_{n-1}(U \cup V) & \longrightarrow & \ldots
\end{array}$$

It follows from the Comfortable Cover Proposition 40.7 that the first and fourth vertical map are isomorphisms. The second map and the fifth map (which for space reasons is not shown) are the identity. Hence it follows from the Five Lemma 37.5 that the third vertical map is also an isomorphism. ∎

Proof of the Homology Section–Union Lemma 63.7. Let M be an n-dimensional topological manifold with $\partial M = \varnothing$ and let R be a commutative ring. We start out with a little observation:

Observation. Let $X \subset Y \subset M$ be two subsets. Let $\{\alpha_x\}_{x \in Y}$ be an R-section on M along Y. If $\mu_Y \in \mathrm{H}_n(M|Y; R)$ is a lift of this R-section to Y, then it follows immediately from the functoriality of relative homology groups that the image of μ_Y under the natural map $\mathrm{H}_n(M|Y; R) \to \mathrm{H}_n(M|X; R)$ is a lift of the given R-section to the subset X.

Now we start with the actual proof of the lemma. Thus let A and B be two compact subsets of M such that A, B and $A \cap B$ satisfy statements (i) and (ii). To simplify the notation we drop R from the notation. We consider the inverted Mayer–Vietoris Sequence

from the Inverted Mayer–Vietoris Theorem 63.8

$$\cdots \to \overbrace{\mathrm{H}_{n+1}(M|A\cap B)}^{=0,\text{ by hypothesis on } A\cap B} \to \mathrm{H}_n(M|A\cup B) \xrightarrow{\Phi} \mathrm{H}_n(M|A)\oplus \mathrm{H}_n(M|B) \xrightarrow{\Psi} \mathrm{H}_n(M|A\cap B)\to \dots$$
$$\alpha \mapsto (\alpha,-\alpha) \qquad (\alpha,\beta)\mapsto \alpha+\beta$$

Now we can prove that $A\cup B$ also satisfies statements (i) and (ii).

(ii) For $i>n$ the group $\mathrm{H}_i(M|A\cup B)$ is stuck, by our hypothesis on A, B and $A\cap B$, between groups that are zero, thus we see that $\mathrm{H}_i(M|A\cup B)=0$ for $i>n$.

(i) Now suppose that we are given a section $\{\alpha_x\}_{x\in A\cup B}$ on M along $A\cup B$. We denote by α_A, α_B and $\alpha_{A\cap B}$ the unique lifts to A, B and $A\cap B$ given by our hypothesis. By the above observation the images of α_A and α_B in $\mathrm{H}_n(M|A\cap B)$ are lifts to $A\cap B$. By the uniqueness of the lift $\alpha_{A\cap B}$ these images thus equal $\alpha_{A\cap B}$. By the exactness of the above sequence we see that $(\alpha_A,-\alpha_B)=\Phi(\alpha_{A\cup B})$ for some $\alpha_{A\cup B}\in \mathrm{H}_n(M|A\cup B)$. We claim that $\alpha_{A\cup B}$ is indeed a lift to $A\cup B$. So let $x\in A\cup B$. First we consider the case that $x\in A$. We have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{H}_n(M|A\cup B) & \xrightarrow{[\alpha_{A\cup B}\mapsto \alpha_A]} & \mathrm{H}_n(M|A) \\ & \searrow & \downarrow \\ & & \mathrm{H}_n(M|x). \end{array}$$

We see that the image of $\alpha_{A\cup B}$ in $\mathrm{H}_n(M|x)$ is the image of α_A in $\mathrm{H}_n(M|x)$, hence it equals α_x. The same argument applies to the case that $x\in B$.

It remains to prove the uniqueness of the lift $\alpha_{A\cup B}$. If $\beta\in \mathrm{H}_n(M|A\cup B)$ is another lift, then it follows immediately from the above observation and the uniqueness of α_A and α_B that $\Phi(\beta)=(\alpha_A,-\alpha_B)$. By the exactness of the sequence we see that we already have the equality $\alpha_{A\cup B}=\beta$ in $\mathrm{H}_n(M|A\cup B)$. ■

Later on we will need the following variation on the Homology Section–Union Lemma 63.7.

Lemma 63.9. (Homology Section–Finite Union Lemma) Let M be an n-dimensional topological manifold and let R be a commutative ring. Furthermore let $\mathcal{A}$ be a family of compact subsets of M with the following two properties:

(1) Statements (i) and (ii) from the Homology-Section Theorem 63.6 hold for each $A\in\mathcal{A}$.
(2) The family $\mathcal{A}$ is closed under intersections, i.e. whenever $A, B\in\mathcal{A}$, then $A\cap B\in\mathcal{A}$.

Then the statements hold for all finite unions of elements in $\mathcal{A}$.

Proof. Given $k\in\mathbb{N}$ we write

$$\mathcal{U}_k \coloneqq \{\text{all subsets of } M \text{ that are the union of at most } k \text{ subsets in } \mathcal{A}\}.$$

We claim that for every $k\in\mathbb{N}$ statements (i) and (ii) hold for all elements in $\mathcal{U}_k$. We prove this claim by induction on k.

(1) If $k=1$, then the claim holds by hypothesis (1).

(2) We now suppose that the claim holds for some $k\in\mathbb{N}$. Let $A_1,\dots,A_k,A_{k+1}\in\mathcal{A}$. We write $B=A_2\cup\cdots\cup A_{k+1}$. Note that

$$A_1\cap B \;=\; A_1\cap(A_2\cup\cdots\cup A_{k+1}) \;=\; \bigcup_{i=2}^{k+1} \underbrace{(A_1\cap A_i)}_{\in\mathcal{A} \text{ by (2)}} \;\in \mathcal{U}_k.$$

Thus by induction hypothesis we see that the statements hold for A_1, B and also for $A_1 \cap B$. So the statements also hold for $A_1 \cup B = A_1 \cup \cdots \cup A_{k+1}$ by the Homology Section–Union Lemma 63.7. ■

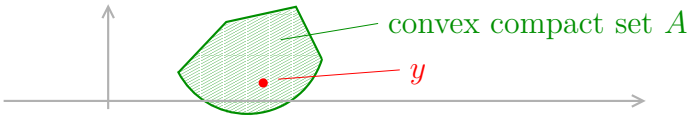
63.3.2. **Proof of the Homology-Section Theorem 63.6 II: $\mathbb{R}^n$.** We now turn to the five steps in the proof of the Homology-Section Theorem 63.6 that we had formulated on page 814. In this section we provide the proofs of the first three steps, namely the ones, which cover the case $M = \mathbb{R}^n$.

Lemma 63.10. (Homology Section–Convex Compact Lemma) Statements (i) and (ii) from the Homology-Section Theorem 63.6 hold for any commutative ring R, $M = \mathbb{R}^n$ and any convex compact subset A of $\mathbb{R}^n$.

Proof. We start out with the following general claim:

Claim. Let A be a convex compact subset of $\mathbb{R}^n$. For every $y \in A$ and every $i \in \mathbb{N}_0$ the following inclusion-induced map is an isomorphism:

$$\mathrm{H}_i(\mathbb{R}^n, \mathbb{R}^n \setminus A; R) \;\to\; \mathrm{H}_i(\mathbb{R}^n, \mathbb{R}^n \setminus \{y\}; R).$$



Proof. To simplify the notation we drop the coefficients R from the notation and we assume that $x = 0$. Now let $i \in \mathbb{N}_0$. In the following we will prove that the following inclusion-induced maps are isomorphisms:

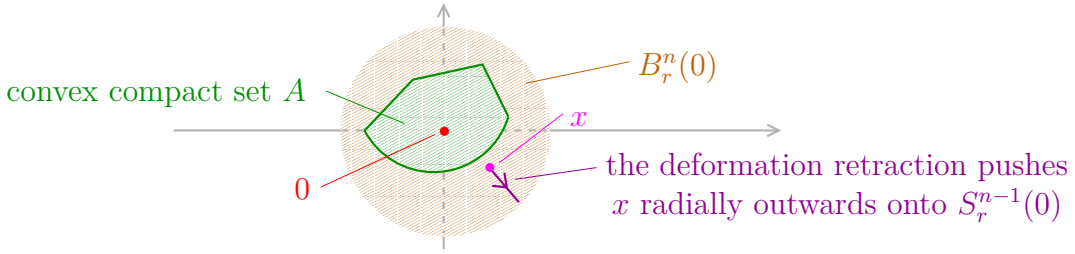
$$\begin{array}{lrcl} \text{(a)} & \mathrm{H}_i(\mathbb{R}^n \setminus A) & \to & \mathrm{H}_i(\mathbb{R}^n \setminus \{y\}) \\ \text{(b)} & \mathrm{H}_i(\mathbb{R}^n, \mathbb{R}^n \setminus A) & \to & \mathrm{H}_i(\mathbb{R}^n, \mathbb{R}^n \setminus \{y\}). \end{array}$$

So let us turn to the proofs:

(a) Since A is compact we know by the Heine-Borel Theorem 1.12 that A is bounded. Thus we can pick an $r > 0$ with $A \subset B_r^n(0)$. We consider the following map:

$$\begin{array}{rcl} F\colon (\mathbb{R}^n \setminus A) \times [0,1] & \to & \mathbb{R}^n \\ (x,t) & \mapsto & \begin{cases} x, & \text{if } x \in \mathbb{R}^n \setminus B_r^n(0), \\ x \cdot \underbrace{((1-t) + t \cdot \frac{r}{\|x\|})}_{\geq 1 \text{ since } r > \|x\|}, & \text{if } x \in \overline{B}_r^n(0) \setminus A. \end{cases} \end{array}$$

Note that if $x \notin A$, then it follows from the convexity of A and the fact that $0 \in A$, that for any $s \geq 1$ the point $x \cdot s$ cannot lie in A either. This shows that the above map actually takes values in $\mathbb{R}^n \setminus A$.



Now it is pretty obvious that the map F defines a deformation retraction from $\mathbb{R}^n \setminus A$ to $\mathbb{R}^n \setminus B_r^n(0)$. [516] Clearly $\mathbb{R}^n \setminus B_r^n(0)$ is also a deformation retract of $\mathbb{R}^n \setminus \{0\}$. Next we consider the following commutative diagram of inclusion-induced maps:

$$\begin{array}{ccc} \mathrm{H}_i(\mathbb{R}^n \setminus A) & \longrightarrow & \mathrm{H}_i(\mathbb{R}^n \setminus \{0\}) \\ & \nwarrow \quad \nearrow & \\ & \mathrm{H}_i(\mathbb{R}^n \setminus B_r^n(0)). & \end{array}$$

It follows from the Homotopy Equivalence-H_*-Corollary 36.7 (1a) and the above discussion that the two diagonal maps are isomorphisms. It follows that the horizontal map is also an isomorphism.

(b) From the LES-of-Pair Proposition 38.3 we obtain the following commutative diagram of long exact sequences:

$$\begin{array}{ccccccccccc} \ldots \longrightarrow & \mathrm{H}_i(\mathbb{R}^n \setminus A) & \longrightarrow & \mathrm{H}_i(\mathbb{R}^n) & \xrightarrow{\text{я}} & \mathrm{H}_i(\mathbb{R}^n, \mathbb{R}^n \setminus A) & \xrightarrow{\partial_i} & \mathrm{H}_{i-1}(\mathbb{R}^n \setminus A) & \longrightarrow & \mathrm{H}_{i-1}(\mathbb{R}^n) & \longrightarrow \ldots \\ & \downarrow \cong & & \downarrow \mathrm{id} & & \downarrow & & \downarrow \cong & & \downarrow \mathrm{id} & \\ \ldots \longrightarrow & \mathrm{H}_i(\mathbb{R}^n \setminus \{y\}) & \longrightarrow & \mathrm{H}_i(\mathbb{R}^n) & \xrightarrow{\text{я}} & \mathrm{H}_i(\mathbb{R}^n, \mathbb{R}^n \setminus \{y\}) & \xrightarrow{\partial_i} & \mathrm{H}_{i-1}(\mathbb{R}^n \setminus \{y\}) & \longrightarrow & \mathrm{H}_{i-1}(\mathbb{R}^n) & \longrightarrow \ldots \end{array}$$

By (a) the first and the fourth vertical maps are isomorphisms. The second and the fifth vertical map are the identity. It follows from the Five Lemma 37.5 that the middle vertical map is also an isomorphism. ⊞

Now we turn to the actual proof of the lemma. Let A be a convex compact subset of $\mathbb{R}^n$ and let R be a commutative ring. As pointed out above, if $A = \varnothing$, then there is nothing to prove. Thus we can assume that there exists a $y \in A$. Now let us prove that A satisfies the statements (i) and (ii) of the Homology-Section Theorem 63.6:

(i) Let $\{\alpha_x\}_{x \in \mathbb{R}^n}$ be a section for $\mathbb{R}^n$. We denote by $\alpha_A \in \mathrm{H}_n(\mathbb{R}^n | A)$ the preimage of α_y under the isomorphism

$$\mathrm{H}_n(\mathbb{R}^n | A; R) \;\to\; \mathrm{H}_n(\mathbb{R}^n | \{y\}; R)$$

given by the claim. Since A is in particular connected it follows from the Sections-on-Connected Proposition 61.7 (applied to the sections $\{\alpha_{A,x}\}_{x \in A}$ and $\{\alpha_x\}_{x \in A}$) that $\alpha_{A,x} = \alpha_x$ for all $x \in A$, i.e. we have shown that α_A is a lift to A. The uniqueness of the lift α_A follows again from the above isomorphism.

(ii) For every $i > n$ we have

$$\mathrm{H}_i(\mathbb{R}^n | A; R) \underset{\text{by the claim}}{\cong} \mathrm{H}_i(\mathbb{R}^n | \{y\}; R) = \mathrm{H}_i(\mathbb{R}^n, \mathbb{R}^n \setminus \{y\}) \underset{\text{by the Manifold-Local-H}_*\text{-Lemma 61.1}}{=} 0.$$

■

Lemma 63.11. (Homology Section–Finite Union-of-Convex Compact Lemma) Statements (i) and (ii) from the Homology-Section Theorem 63.6 hold for any commutative ring R, $M = \mathbb{R}^n$ and any compact subset A of $\mathbb{R}^n$ that is the union of finitely many convex compact sets.

Proof. We consider

$$\mathcal{A} \coloneqq \{\text{all convex compact subsets of } \mathbb{R}^n\}.$$

[516] As always, checking that a map is a deformation retraction also requires showing that the map is continuous. In this case the map is continuous by the Pasting Proposition 2.4 and the fact that $B_r^n(0) \setminus A$ and $\mathbb{R}^n \setminus B_r^n(0)$ are closed subsets of $\mathbb{R}^n \setminus A$.

This family of subsets of $\mathbb{R}^n$ is closed under intersections.[517] By Lemma 63.10 the two statements hold for each $A \in \mathcal{A}$. The lemma is now an immediate consequence of the Homology Section–Finite Union Lemma 63.9. ■

The following lemma is arguably the hardest step towards proving the fundamental theorem of topological manifolds.

Lemma 63.12. (Homology Section–Compact-in-$\mathbb{R}^n$-Lemma) Statements (i) and (ii) from the Homology-Section Theorem 63.6 hold for any commutative ring R, $M = \mathbb{R}^n$ and any compact subset A of $\mathbb{R}^n$.

Proof. To simplify the notation we again drop the R from the notation. Let $A \subset \mathbb{R}^n$ be a convex compact subset and let $\{\alpha_x\}_{x\in A}$ be a section for $\mathbb{R}^n$. If $A = \varnothing$, then there is nothing really to show. So we assume that $A \neq \varnothing$.

(i) Recall that we need to show that the map

$$\mathrm{H}_n(M|A;R) \;\to\; \Gamma^A_R(M)$$

is an isomorphism.

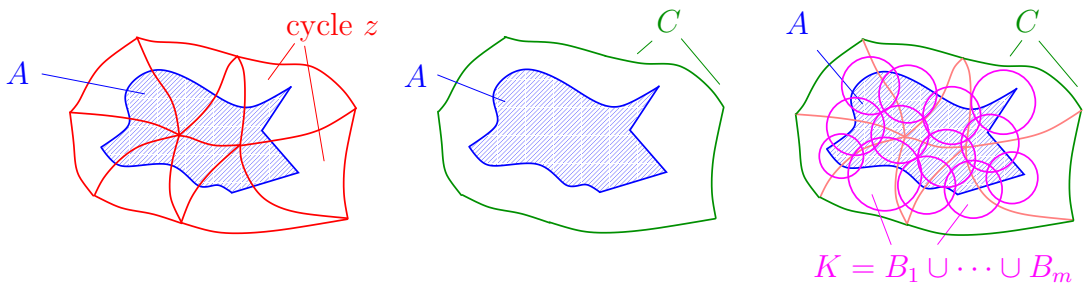
We first show that the map is an epimorphism. Since $A \neq \varnothing$ we can pick a $y \in A$. Since A is compact we can pick $r > 0$ with $A \subset \overline{B}^n_r(y)$. It follows from the claim on page 817 that $i_*\colon \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \overline{B}^n_r(y)) \to \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{y\})$ is an isomorphism. We set $\beta := (i_*)^{-1}(\alpha_y)$. Note that the classes $\{\beta_x\}_{x\in A}$ define an R-section on A which agrees with the section $\{\alpha_x\}_{x\in A}$ on y. Since A is in particular connected we obtain from the Sections-on-Connected Proposition 61.7 that $\beta_x = \alpha_x$ for all $x \in A$. The image of β in $\mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus A)$ is thus a lift of $\{\alpha_x\}_{x\in A}$.

Now we show that the above map is a monomorphism. Thus we need to show that if $\beta \in \mathrm{H}_n(\mathbb{R}^n|A)$ is a class such that $\beta_x = 0$ for all $x \in A$, then β is already zero. So let $\beta \in \mathrm{H}_n(\mathbb{R}^n|A)$. We pick a singular n-chain $z \in \mathrm{C}_n(\mathbb{R}^n)$ that represents β. We denote by $C \subset \mathbb{R}^n \setminus A$ the union of the images of the singular simplices appearing in $\partial z \in \mathrm{C}_{n-1}(\mathbb{R}^n \setminus A)$. Note that C is the union of finitely many compact sets, hence C itself is compact.

Claim. There exist finitely many closed balls $B_1, \ldots, B_m$ with the following three properties:
(1) each B_i is contained in $\mathbb{R}^n \setminus C$,
(2) each B_i contains a point $x_i \in A$,
(3) we have $A \subset B_1 \cup \cdots \cup B_m$.

Proof. Since C is compact and since $A \cap C = \varnothing$ we can find for each $x \in A$ an $r_x > 0$ with $B_{2r_x}(x) \subset \mathbb{R}^n \setminus C$. Since A is compact there exist finitely many points $x_1, \ldots, x_m$ in A such that A is contained in $B_{r_{x_1}}(x_1) \cup \cdots \cup B_{r_{x_m}}(x_m)$.
It is now clear that $B_i := \overline{B}_{r_{x_i}}(x_i)$, $i = 1, \ldots, m$ have the desired property. ⊞

[517] Here we use one of the most convenient properties of convex sets: the intersection of two convex sets is again convex.



We write $K := B_1 \cup \cdots \cup B_m$. By (1) the singular n-chain $z \in \mathrm{C}_n(\mathbb{R}^n)$ also represents an element $\gamma \in \mathrm{H}_n(\mathbb{R}^n|K)$. Note that for each $x \in A$ we have a commutative diagram

$$\begin{array}{ccc} \mathrm{H}_n(\mathbb{R}^n|K) & \xrightarrow{\gamma=[z]\mapsto\beta=[z]} & \mathrm{H}_n(\mathbb{R}^n|A) \\ & {\scriptstyle \gamma\mapsto\gamma_x} \searrow \quad \swarrow {\scriptstyle \beta\mapsto\beta_x} & \\ & \mathrm{H}_n(\mathbb{R}^n|x). & \end{array}$$

It follows in particular from the commutative diagram that it suffices to show that we have $\gamma = 0 \in \mathrm{H}_n(\mathbb{R}^n|K)$. We do so as follows:

(1) By our hypothesis on β and by the above commutative diagram we know that for each $i \in \{1, \ldots, m\}$ we have $\gamma_{x_i} = \beta_{x_i} = 0$.

(2) Let $i \in \{1, \ldots, m\}$. Since $\gamma_{x_i} = 0$ and since B_i is connected it follows from the Sections-on-Connected Proposition 61.7 (applied to the 0-section and the section γ_x) that $\gamma_x = 0$ for all $x \in B_i$.

(3) By (2) we know that $\gamma_x = 0$ for all $x \in K = B_1 \cup \cdots \cup B_m$.

(4) But now it follows from the Homology Section–Finite Union-of-Convex Compact Lemma 63.11 that $\gamma = 0$.

(ii) We need to show that $\mathrm{H}_i(\mathbb{R}^n|A) = 0$ for $i > n$. So let $\beta \in \mathrm{H}_i(\mathbb{R}^n|A)$. As above we pick a singular n-chain $z \in \mathrm{C}_n(\mathbb{R}^n)$ that represents β. We define C, K and γ as above. By the Homology Section–Finite Union-of-Convex Compact Lemma 63.11 we have $\gamma = 0 \in \mathrm{H}_i(\mathbb{R}^n|K)$. But as before this implies that also $\beta = 0 \in \mathrm{H}_i(\mathbb{R}^n|A)$. ■

Lemma 63.13. (Homology Section–Compact-in-Chart Lemma) Statements (i) and (ii) from the Homology-Section Theorem 63.6 hold for any commutative ring R, any n-dimensional topological manifold with empty boundary and any compact subset A of M such that A is contained in the domain of a chart.

Proof. So let $A \subset M$ be a compact subset that is contained in the domain of a chart $\Phi\colon U \to V$. The desired statements follow fairly easily from the Homology Section–Compact-in-$\mathbb{R}^n$-Lemma 63.12 and the fact that we have isomorphisms

$$\mathrm{H}_n(M, M \setminus A; R) \xleftarrow[\uparrow]{\cong} \mathrm{H}_n(U, U \setminus A; R) \xrightarrow[\uparrow]{\Phi} \mathrm{H}_n(\mathbb{R}^n, \mathbb{R}^n \setminus \Phi(A); R)$$

by the Excision Theorem 40.2 — isomorphism since Φ is a homeomorphism

We leave it to the reader to fill in the details. ■

63.3.3. **Proof of the Homology-Section Theorem 63.6 III.** Now we can finally complete the proof of the Homology-Section Theorem 63.6.

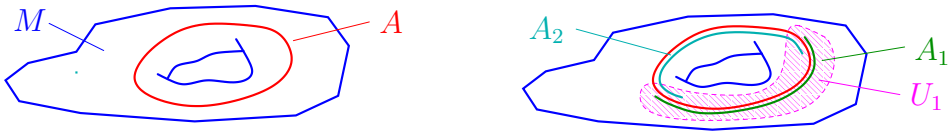
Proof of the Homology-Section Theorem 63.6. Let M be an n-dimensional topological manifold with empty boundary. We consider

$\mathcal{A} :=$ all compact subsets A of M for which there exists a chart $\Phi\colon U \to \mathbb{R}^n$ with $A \subset U$.

By the Homology Section–Compact-in-Chart Lemma 63.13 statements (i) and (ii) hold for all subsets in $\mathcal{A}$. It is clear that $\mathcal{A}$ is closed under intersections. Thus it follows from the Homology Section–Finite Union Lemma 63.9 that the statements (i) and (ii) hold for all finite unions of sets in $\mathcal{A}$. Therefore it suffices to prove the following claim:

Claim. Every compact subset A of M is the union of finitely many sets in $\mathcal{A}$.

Proof. Let A be a compact subset of M. It is clear that given any point $x \in A$ there exists a chart $\Phi_x\colon U_x \to V_x$ around x such that the image is some open ball in $\mathbb{R}^n$. But since each open ball in $\mathbb{R}^n$ is homeomorphic to $\mathbb{R}^n$ we can find around each $x \in A$ a chart around x such that the image V_x equals $\mathbb{R}^n$. Now we write $A_x := A \cap \Phi_x^{-1}(\overline{B}_1^n(0))$ and $U'_x := \Phi_x^{-1}(B_1^n(0))$. Since A is compact we can find finitely many $x_1, \dots, x_m$ in A such that A is contained in $U'_{x_1} \cup \dots \cup U'_{x_m}$. By construction the corresponding charts $\Phi_i := \Phi_{x_i}\colon U_{x_i} \to \mathbb{R}^n$ and the sets $A_i := A_{x_i}$ have the desired properties. ⊞



It follows from the Homology Section–Compact-in-Chart Lemma 63.13 that statements (i) and (ii) hold for all the A_i and for all arbitrary intersections of the A_i. But then the Homology Section–Union Lemma 63.7 implies that they also hold for $A = A_1 \cup \dots \cup A_m$. ■

63.4. **Proof of the Manifold-Top-H$_*$-Theorem 63.1.** In this section we will finally provide the long-delayed proof of Manifold-Top-H$_*$-Theorem 63.1. We break the proof into three parts:

(1) The case that M is closed.
(2) The case that M is non-compact.
(3) The case that M is compact but not closed.

These three cases are dealt with in the following three subsections.

63.4.1. **Proof of the Manifold-Top-H$_*$-Theorem 63.1: Closed case.** The following is the statement of the Manifold-Top-H$_*$-Theorem 63.1 for closed topological manifolds:

Theorem 63.1 (Manifold-Top-H$_*$-Theorem) Let M be a closed connected non-empty n-dimensional topological manifold and let R be a commutative ring. The following statements hold:

(1) For every $i > n$ we have $\mathrm{H}_i(M; R) = 0$.
(2)
$$\mathrm{H}_n(M; R) \;\cong\; \begin{cases} R, & \text{if } M \text{ is } R\text{-orientable,} \\ \{r \in R \mid 2r = 0\}, & \text{if } M \text{ is not } R\text{-orientable.} \end{cases}$$
(3) (a) If M is R-orientable, then for every $x \in M$ the map
$$\mathrm{H}_n(M; R) \;\to\; \mathrm{H}_n(M, M \setminus \{x\}; R)$$

is an isomorphism.

(b) If M is not R-orientable, then for every $x \in M$ the natural map

$$\mathrm{H}_n(M;R) \;\to\; \mathrm{H}_n(M, M\setminus\{x\};R)$$

is a monomorphism and the image equals $\{r \in \mathrm{H}_n(M, M\setminus\{x\};R) \,|\, 2\cdot r = 0\}$.

Proof. Let M be a closed connected non-empty n-dimensional topological manifold and let R be a commutative ring.

(1) Let $i > n$. We have

$$\mathrm{H}_i(M;R) \;=\; \mathrm{H}_i(M, \overbrace{M\setminus M}^{=\varnothing};R) \;=\; \mathrm{H}_i(M|M;R) \;\underset{\uparrow}{\cong}\; 0.$$

by the Homology-Section Theorem 63.6 (ii) since M is compact and since M has no boundary

(3) Let $x \in M$.

(a) We assume that M is R-orientable. We start out with the following claim:

Claim. The map

$$\begin{array}{rcl} \Phi\colon \Gamma_R(M) & \to & \overbrace{\mathrm{H}_n(M, M\setminus\{x\};R)}^{\cong R} \\ \{\alpha_y\}_{y\in M} & \mapsto & \alpha_x \end{array}$$

is an isomorphism of R-modules.

Proof. Since M is R-orientable we can pick an R-orientation $\{\mu_y\}_{y\in M}$ for M. By definition μ_x is a generator of the R-module $\mathrm{H}_n(M, M\setminus\{x\};R) \cong R$. This means that every element in $\mathrm{H}_n(M, M\setminus\{x\};R)$ is of the form $r\cdot\mu_x$ for some unique $r \in R$. We consider the map[518]

$$\begin{array}{rcl} \Psi\colon \mathrm{H}_n(M, M\setminus\{x\};R) & \to & \Gamma_R(M) \\ r\cdot\mu_x & \mapsto & \{r\cdot\mu_y\}_{y\in M}. \end{array}$$

Evidently we have $\Phi\circ\Psi = \mathrm{id}$. It follows from our hypothesis that M is connected and from the Sections-on-Connected Proposition 61.7 that $\Psi\circ\Phi$ is also the identity. ⊞

We consider the following diagram:

$$\begin{array}{ccc} \mathrm{H}_n(M;R) & \xrightarrow{\quad\varphi\mapsto\varphi_x\quad} & \mathrm{H}_n(M, M\setminus\{x\};R) \\ & {\scriptstyle \varphi\mapsto\{\varphi_x\}_{x\in M}}\searrow \quad \nearrow{\scriptstyle \{\varphi_x\}_{x\in M}\mapsto\varphi_x} & \\ & \Gamma_R(M) & \end{array}$$

The diagram commutes by definition. Since M is compact, we know by the Homology-Section Theorem 63.6 (i) that the left diagonal map is an isomorphism. Furthermore, since M is connected and since M is R-orientable we obtain from the claim that the right diagonal map is an isomorphism. Thus we see that the horizontal map is an isomorphism, as promised.

(b) We now assume that M is not R-orientable. We start out with the following claim:

Claim. The map

$$\begin{array}{rcl} \Phi_x\colon \Gamma_R(M) & \to & \mathrm{H}_n(M, M\setminus\{x\};R) \\ \{\alpha_y\}_{y\in M} & \mapsto & \alpha_x \end{array}$$

[518] It follows easily from the Homology-defines-Section Lemma 61.6 (1) that this map is well-defined.

is a monomorphism of R-modules with image $\{\gamma \in \mathrm{H}_n(M, M \setminus \{x\}; R) \,|\, 2 \cdot \gamma = 0\}$.

Proof. It follows from our hypothesis that M is connected and from the Sections-on-Connected Proposition 61.7 that Φ_x is a monomorphism. It remains to prove that the following equality holds:

$$\mathrm{im}(\Phi_x) \;=\; \{\gamma \in \mathrm{H}_n(M, M \setminus \{x\}; R) \,|\, 2 \cdot \gamma = 0\}.$$

Before we prove equality we make two preparations:

- For each $x \in M \setminus \partial M$ we choose a generator $\varphi_x \in \mathrm{H}_n(M, M \setminus \{x\})$. Note that each φ_x is well-defined up to sign. Also note that we do not make any claims about continuity.
- Given any $x \in M \setminus \partial M$ we consider the isomorphisms

$$\begin{array}{l} R \to \mathrm{H}_n(M, M \setminus \{x\}) \otimes R \\ r \mapsto \varphi_x \otimes r \end{array} \quad \& \quad \begin{array}{l} \mathrm{H}_n(M, M \setminus \{x\}) \otimes R \xrightarrow{\mu} \mathrm{H}_n(M, M \setminus \{x\}; R) \\ [\varphi] \otimes r \mapsto [\varphi \otimes r] \end{array}$$

of R-modules which are given by the Manifold-Local R-Homology Lemma 61.1.

We want to prove the description of $\mathrm{im}(\Phi_x)$. We will show the two inclusions "$\subset$" and "$\supset$" separately. First we show "$\supset$". Let $\gamma \in \mathrm{H}_n(M, M \setminus \{x\}; R)$ with $2 \cdot \gamma = 0$. We pick $r \in R$ with $\mu(\varphi_x \otimes r) = \gamma$. Since $2 \cdot \gamma = 0$ we see that $2 \cdot r = 0$. For each $x \in M \setminus \partial M$ we set $\alpha_x := \mu(\varphi_x \otimes r)$. Since $2 \cdot r = 0$ we have $r = -r$. We see that $\mu((-\varphi_x) \otimes r) = \mu(\varphi_x \otimes (-r)) = \mu(\varphi_x \otimes r)$. Thus we see that α_x does not depend on the choice of φ_x. Using charts one can now easily verify that $\{\alpha_x\}_{x \in M \setminus \partial M}$ is an R-section. We leave it to the reader to fill in the details.

Now we show "$\subset$". We show the following equivalent statement: If $\{\alpha_x\}_{x \in M \setminus \partial M}$ is an R-section such that $2 \cdot \alpha_y \neq 0$, then M is orientable (which by the Orientable-for-all-Rings Lemma 61.12 implies that M is also R-orientable). So suppose that $2 \cdot \alpha_y \neq 0$. We pick $r \in R$ such that $[\varphi_y \otimes r] = \alpha_y$. Note that $2 \cdot r \neq 0$. Using the fact that $\{\alpha_x\}_{x \in M \setminus \partial M}$ is an R-section one can show fairly easily that for each $x \in M \setminus \partial M$ there exists an $\epsilon_x \in \{\pm 1\}$ with $[\epsilon_x \cdot \varphi_x \otimes r] = \alpha_x$. Since $2 \cdot r \neq 0$ we see that ϵ_x is uniquely determined and one can now show without any real problems that $\{\epsilon_x \cdot \varphi_x\}_{x \in M \setminus \partial M}$ is a section. Since each $\epsilon_x \cdot \varphi_x$ is a generator we see that this is in fact an orientation. ⊞

As in (a) we consider the following diagram:

$$\begin{array}{ccc} \mathrm{H}_n(M; R) & \xrightarrow{\quad \varphi \mapsto \varphi_x \quad} & \mathrm{H}_n(M, M \setminus \{x\}; R) \\ & {\scriptstyle \varphi \mapsto \{\varphi_x\}_{x \in M}} \searrow \quad \nearrow {\scriptstyle \{\varphi_x\}_{x \in M} \mapsto \varphi_x} & \\ & \Gamma_R(M) & \end{array}$$

As in (a) we note that the diagram commutes and that the left diagonal map is an isomorphism. Furthermore, since M is connected and since M is R-orientable we obtain from the claim that the right diagonal map is a monomorphism with image $\{\gamma \in \mathrm{H}_n(M, M \setminus \{x\}, R) \,|\, 2 \cdot \gamma = 0\}$. Thus we see that the horizontal map has the promised properties.

(2) This statement follows immediately from (3) and the Manifold-Local R-Homology Lemma 61.1 which tells us that we have an isomorphism $\mathrm{H}_n(M, M \setminus \{x\}; R) \cong R$ of R-modules. ■

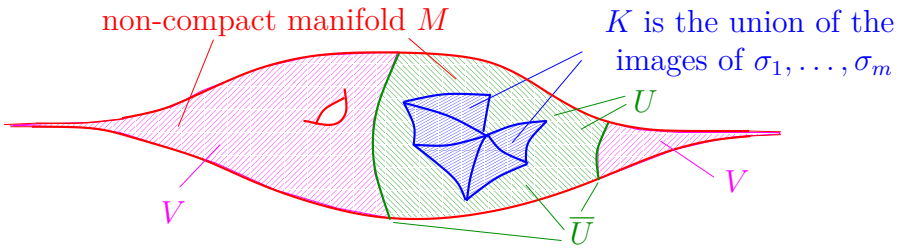
63.4.2. **Proof of the Manifold-Top-H$_*$-Theorem 63.1: Non-compact case.** The key extra ingredient in the proof of the Manifold-Top-H$_*$-Theorem 63.1 is the following proposition.

Proposition 63.14. (Non Compact–Top Homology Proposition) Let M be a connected n-dimensional topological manifold and let R be a commutative ring. If M is non-compact and if $\partial M = \varnothing$, then $\mathrm{H}_k(M;R) = 0$ for every $k \geq n$.

Proof. Let M be a connected non-compact n-dimensional topological manifold such that $\partial M = \varnothing$, let R be a commutative ring and let $k \geq n$. Let $\alpha \in \mathrm{H}_k(M;R)$. We need to show that $\alpha = 0$. We represent α by a cycle $z = \sum_{i=1}^{m} \sigma_i \otimes r_i \in \mathrm{C}_k(M;R) = \mathrm{C}_k(M) \otimes R$. The union of the images of $\sigma_1, \ldots, \sigma_m$ is a compact subset K of M. Since K is compact it follows that there exists an open subset $U \subset M$ that contains K and such that the closure $\overline{U}$ is compact. We consider the open subset $V = M \setminus \overline{U}$ and we consider the long exact sequence of the triple $(M, U \cup V, V)$, see the LES-of-Triple Proposition 38.2. It fits into the following commutative diagram:

$$\begin{array}{ccccccc} \ldots \longrightarrow \mathrm{H}_{k+1}(M, U\cup V;R) & \xrightarrow{\partial_{k+1}} & \mathrm{H}_k(U\cup V, V;R) & \longrightarrow & \mathrm{H}_k(M,V;R) & \longrightarrow \ldots \\ & & \cong\uparrow & & \uparrow & \\ & & \mathrm{H}_k(U;R) & \xrightarrow{[z]\mapsto[z]=\alpha} & \mathrm{H}_k(M;R). & \end{array}$$

Here the left vertical map is an isomorphism since $U \cup V$ is the disjoint union of the open subsets U and V.[519]



Now we consider the modules to the top-left and top-right of the diagram. We have

$$\mathrm{H}_{k+1}(M, U\cup V;R) = \mathrm{H}_{k+1}(M|\overbrace{M\setminus(U\cup V)}^{=\overline{U}\setminus U};R) \underset{\uparrow}{=} 0 \quad \text{if } k > n-1$$

by the Homology-Section Theorem 63.6 (ii) since M has no boundary and since $\overline{U}$ and $\overline{U} \setminus U$ are compact

$$\mathrm{H}_k(M,V;R) = \mathrm{H}_k(M|\underbrace{M\setminus V}_{=\overline{U}};R) \underset{\downarrow}{=} 0 \quad \text{if } k > n.$$

After these preparations we come to the actual proof that $\alpha = [z] = 0 \in \mathrm{H}_k(M;R)$. We first consider the case $k > n$. We deduce from the above exact sequence and the above calculations that $\mathrm{H}_k(U;R) = 0$. But z represents an element in $\mathrm{H}_k(U;R)$. It follows that $[z] = 0$ in $\mathrm{H}_k(M;R)$.

[519] One could at this point refer to the Excision Theorem 40.1 but this isomorphism follows already from the observation that $\mathrm{C}_*(U\cup V) = \mathrm{C}_*(U) \oplus \mathrm{C}_*(V)$ as chain complexes, since U and V are both open subsets of $U \cup V$.

Now we consider the case $k = n$. It follows from the above discussion that we have a commutative diagram

$$\begin{array}{ccc} \mathrm{H}_n(U \cup V, V; R) & \hookrightarrow & \mathrm{H}_n(M, V; R) \\ \cong\uparrow & & \uparrow \\ \mathrm{H}_n(U; R) & \xrightarrow{[z]\mapsto[z]=\alpha} & \mathrm{H}_n(M; R). \end{array}$$

Thus it suffices to show that the image of α in $\mathrm{H}_n(M, V; R)$ is zero. We showed in the Homology-defines-Section Lemma 61.6 (2) that $\{\alpha_x\}_{x\in M}$ is an R-section of M.[520] Since M is non-compact we can find a point x outside of the compact set K. Note that the class $\alpha_x \in \mathrm{H}_n(M, M \setminus \{x\}; R)$ is again represented by z which lies in $K \subset M \setminus \{x\}$. It is thus clear that $\alpha_x = 0 \in \mathrm{H}_n(M, M \setminus \{x\}; R)$. But since M is connected it follows from the Sections-on-Connected Proposition 61.7 that $\{\alpha_x\}_{x\in M}$ is the zero section. Both the image of α in $\mathrm{H}_n(M, V; R)$ and the zero-class are lifts of the zero section to $\mathrm{H}_n(M, V; R)$. By the Homology-Section Theorem 63.6 (i) both lifts agree, i.e. the image of α in $\mathrm{H}_n(M, V; R)$ is zero. ■

In the proof of the non-compact case of the Manifold-Top-H_*-Theorem 63.1 we also need the following two little lemmas:

Lemma 63.15. (Manifold-minus-Boundary Lemma) Let M be an n-dimensional topological manifold.

(1) M is homotopy equivalent to the topological manifold $M \setminus \partial M$.
(2) The topological manifold $M \setminus \partial M$ has no boundary.
(3) If $\partial M \neq \varnothing$, then $M \setminus \partial M$ is non-compact.

Proof. Statement (1) follows from the existence of collar neighborhoods. The other two statements follow from elementary arguments. We refer to [**Fri24**] for details. ■

Lemma 63.16. (Double Homology Lemma) Let M be an n-dimensional topological manifold. If $\partial M \neq \varnothing$, then $\mathrm{H}_n(\mathrm{D}M; R)$ and $\mathrm{H}_n(M, \partial M; R)$ are isomorphic R-modules.

Proof. As before we think of the double as $\mathrm{D}M = (M \sqcup M')/\sim$ where M' is another copy of M. We consider the following diagram:

$$\begin{array}{ccccccccc} \underbrace{\mathrm{H}_n(M'; R)}_{=0} & \longrightarrow & \mathrm{H}_n(\mathrm{D}M; R) & \longrightarrow & \mathrm{H}_n(\mathrm{D}M, M'; R) & \longrightarrow & \mathrm{H}_{n-1}(M'; R) & \hookrightarrow & \mathrm{H}_{n-1}(\mathrm{D}M; R) \\ & & & & \uparrow\cong & & & & \\ & & & & \mathrm{H}_n(M, \partial M; R), & & & & \end{array}$$

We make the following clarifications and observations:

- On the top we have written down the long exact sequence of the pair $(\mathrm{D}M, M')$.
- Since $\partial M' \neq \varnothing$ we obtain from the Manifold-minus-Boundary Lemma 63.15 and the Non Compact–Top Homology Proposition 63.14 that $\mathrm{H}_n(M'; R) = 0$.
- The vertical map is an isomorphism by the Manifold–Excision Theorem 40.3.
- Let $i\colon M' \to \mathrm{D}M$ the natural embedding. The map $r\colon \mathrm{D}M \to M'$ given by $[(P, n)] \mapsto P$ is a retraction from $\mathrm{D}M$ to M'. Since $r \circ i = \mathrm{id}_{M'}$ we obtain from a functoriality argument that the homomorphism $i_*\colon \mathrm{H}_{n-1}(M'; R) \to \mathrm{H}_{n-1}(\mathrm{D}M; R)$ is a monomorphism.

[520] Here we used a second time that M has no boundary.

It follows immediately from the above information that $\mathrm{H}_n(\mathrm{D}M;R)$ and $\mathrm{H}_n(M,\partial M;R)$ are isomorphic R-modules. ■

The following is the statement of the Manifold-Top-H_*-Theorem 63.1 for non-compact topological manifolds:

Theorem 63.1 (Manifold-Top-H_*-Theorem) Let M be a non-compact connected n-dimensional topological manifold and let R be a commutative ring. The following statements hold:

(1) (a) For every $i > n$ we have $\mathrm{H}_i(M;R) = 0$.
(b) For every $i > n$ we have $\mathrm{H}_i(M,\partial M;R) = 0$.
(2) (a) $\mathrm{H}_n(M;R) = 0$.
(b) $\mathrm{H}_n(M,\partial M;R) = 0$.

Proof. Let M be a non-compact connected n-dimensional topological manifold and let R be a commutative ring.

(1) (a) Let $i > n$. If $\partial M = \varnothing$, then this statement follows immediately from the Non Compact–Top Homology Proposition 63.14. If $\partial M \neq \varnothing$, then the statement follows via the extra detour given by the Manifold-minus-Boundary Lemma 63.15.
(b) By (a) we know that for $i > n$ we have $\mathrm{H}_i(M;R) = 0$. Note that ∂M is an $(n-1)$-dimensional topological manifold with empty boundary. Thus it follows from (a) and the discussion on page 821 of the closed case that $\mathrm{H}_i(\partial M;R) = 0$ for all $i > n-1$. It now follows from the long exact sequence of the pair $(M,\partial M)$ that for every $i > n$ we have $\mathrm{H}_i(M,\partial M;R) = 0$.
(2) (a) It follows from the Manifold-minus-Boundary Lemma 63.15 and the Non Compact–Top Homology Proposition 63.14 that $\mathrm{H}_n(M;R) = 0$.
(b) Note that $\mathrm{D}M$ is the union of two copies of M. It follows that if M is not compact, then $\mathrm{D}M$ is not compact. It now follows from (a) and the Double Homology Lemma 63.16 that $\mathrm{H}_n(M,\partial M;R) = 0$. ■

63.4.3. Proof of the Manifold-Top-H_*-Theorem 63.1: Compact but not closed.

The following theorem give us the remaining statements of the Manifold-Top-H_*-Theorem 63.1, namely it gives us the statements for compact topological manifolds that are not closed:

Theorem 63.1 (Manifold-Top-H_*-Theorem) Let M be a connected non-empty n-dimensional topological manifold and let R be a commutative ring. We assume that M is compact but not closed. The following statements hold:

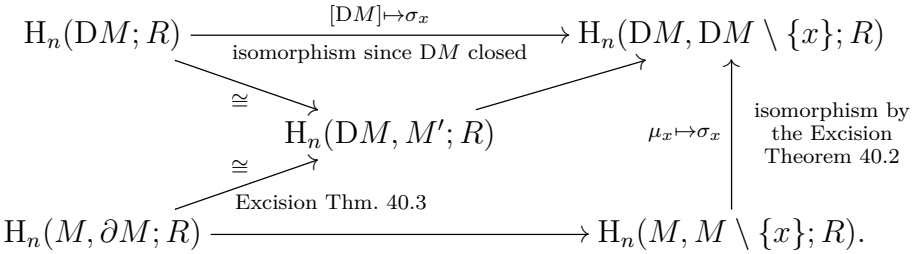
(1) (a) For every $i > n$ we have $\mathrm{H}_i(M;R) = 0$.
(b) For every $i > n$ we have $\mathrm{H}_i(M,\partial M;R) = 0$.
(2) (a) $\mathrm{H}_n(M;R) = 0$.
(b)
$$\mathrm{H}_n(M,\partial M;R) \cong \begin{cases} R, & \text{if } M \text{ is compact and } R\text{-orientable,} \\ \{r \in R \,|\, 2\cdot r = 0\}, & \text{if } M \text{ is compact and not } R\text{-orientable.} \end{cases}$$
(3) If M is R-orientable, then for every $x \in M \setminus \partial M$ the natural map
$$\mathrm{H}_n(M,\partial M;R) \to \mathrm{H}_n(M, M\setminus\{x\};R)$$
is an isomorphism.

Proof. Let M be a connected non-empty n-dimensional topological manifold and let R be a commutative ring. We assume that M is compact but not closed.

(1) Let $i > n$.
   (a) It follows from the Manifold-minus-Boundary Lemma 63.15 and the non-compact case that $\mathrm{H}_i(M;R) = 0$ for $i > n$.
   (b) Note that (b) is an immediate consequence of (a), applied to M and ∂M, and the long exact sequence in homology of the pair $(M, \partial M)$.

(2) (a) This statement follows from the Manifold-minus-Boundary Lemma 63.15 and the Non Compact–Top Homology Proposition 63.14.
   (b) We consider the double $\mathrm{D}M$. It follows from the Double-Orientability Lemma 62.5 and some elementary arguments that M is compact and R-orientable if and only if $\mathrm{D}M$ is closed and R-orientable. Thus the promised result follows from the closed case.

(3) We suppose that M is R-orientable. Let $x \in M \setminus \partial M$. We consider the following diagram:



Here all maps are the obvious inclusion-induced maps. This implies that the diagram commutes. Since $\mathrm{D}M$ is closed and since $\mathrm{D}M$ is R-orientable by the Double-Orientability Lemma 62.5 we know from the first part of the proof, i.e. the from the discussion on page 822, that the top horizontal map is an isomorphism. Next note that in the proof of the Double Homology Lemma 63.16 we showed that the top-left diagonal map is an isomorphism. Finally two of the maps are isomorphisms by the Manifold–Excision Theorem 40.3. It now follows immediately that the bottom horizontal map is also an isomorphism. But that is exactly what we were supposed to show. ■

63.5. **Proof of the Manifold-H_{n-1}-Theorem 63.4.** As we discussed on page 659, homology with $\mathbb{Z}$-coefficients is naturally isomorphic to homology without coefficients. Thus the theorem that we want to prove can be formulated as follows:

Theorem 63.4 (Manifold-H_{n-1}-Theorem) Let M be a compact connected n-dimensional topological manifold.

(1) $$\mathrm{H}_{n-1}(M) \cong \begin{cases} \text{free abelian group}, & \text{if } M \text{ is orientable or if } \partial M \neq \varnothing, \\ \text{free abelian group} \oplus \mathbb{Z}_2, & \text{if } M \text{ is non-orientable and closed.} \end{cases}$$

(2) $$\mathrm{H}_{n-1}(M, \partial M) \cong \begin{cases} \text{free abelian group}, & \text{if } M \text{ is orientable,} \\ \text{free abelian group} \oplus \mathbb{Z}_2, & \text{if } M \text{ is non-orientable.} \end{cases}$$

In the proof of the Manifold-H_{n-1}-Theorem 63.4 we will use the following lemma:

Lemma 63.17. (Free Abelian–Characterization Lemma) Let A be a finitely generated abelian group. Then the following holds:

$$A \text{ is a free abelian group} \iff \operatorname{Tor}(A, \mathbb{Z}_p) = 0 \text{ for all primes } p.$$

Proof. The lemma is an immediate consequence of the Abelian Group-Classification Theorem 18.11, the $\mathbb{Z}_k$-Tor-Lemma 54.6 and the Tor-Properties Lemma 54.7. ■

Now we can provide the proof of the Manifold-H_{n-1}-Theorem 63.4. First we consider Statement (1). For that statement we consider the two cases that the topological manifold is closed and that it has non-empty boundary separately.

Proof of the Manifold-H_{n-1}-Theorem 63.4 (1) I. Let M be a closed connected n-dimensional topological manifold. In the proof we constantly use that by the Topological Manifolds-Invariants Proposition 60.10 we know that $\mathrm{H}_{n-1}(M)$ is finitely generated.

We first consider the case that M is orientable. We want to show that $\mathrm{H}_{n-1}(M)$ is a free abelian group. It follows from the Free Abelian–Characterization Lemma 63.17 that it suffices to prove the following claim:

Claim 1. For every prime p we have $\operatorname{Tor}(\mathrm{H}_{n-1}(M), \mathbb{F}_p) = 0$.

Proof. So let p be a prime. We have

$$\mathbb{F}_p \underset{\text{by the Manifold-Top-H}_*\text{-Theorem 63.1 (2a), since by the Orientable-for-all-Rings Lemma 61.12 we know that } M \text{ is also } \mathbb{F}_p\text{-orientable}}{\overset{\downarrow}{\cong}} \mathrm{H}_n(M; \mathbb{F}_p) \underset{\text{by the UCT-Splitting Theorem 54.11}}{\overset{\uparrow}{\cong}} \mathrm{H}_n(M) \otimes \mathbb{F}_p \oplus \operatorname{Tor}(\mathrm{H}_{n-1}(M), \mathbb{F}_p) \underset{\text{by the Orientation-Top Homology-Theorem 63.3 (3) we have } \mathrm{H}_n(M) \cong \mathbb{Z}}{\overset{\uparrow}{\cong}} \mathbb{F}_p \oplus \operatorname{Tor}(\mathrm{H}_{n-1}(M), \mathbb{F}_p).$$

This implies that $\operatorname{Tor}(\mathrm{H}_{n-1}(M), \mathbb{F}_p) = 0$. ⊞

Now we turn to the case that M is non-orientable. As we explain in [**Fri24**], there exists a connected 2-fold covering $q\colon \widetilde{M} \to M$ such that $\widetilde{M}$ is orientable. We consider the maps

$$\mathrm{H}_{n-1}(M) \xrightarrow{q^*} \mathrm{H}_{n-1}(\widetilde{M}) \xrightarrow{q_*} \mathrm{H}_{n-1}(M), \qquad \mathrm{H}_{n-1}(M) \xrightarrow{\cdot 2} \mathrm{H}_{n-1}(M).$$

Here q^* denotes the transfer map that we introduced on page 721 and q_* denotes of course the map induced by the projection q. By the Transfer–Projection Proposition 57.2 the composition of q^* and q_* is given by multiplication by 2. Put differently, the above diagram commutes.

Since $\widetilde{M}$ is orientable we obtain from the case that we had just proved that $\mathrm{H}_{n-1}(\widetilde{M})$ is torsion-free. Now we see that

$$\text{torsion subgroup of } \mathrm{H}_{n-1}(M) \underset{\text{since } \mathrm{H}_{n-1}(\widetilde{M}) \text{ is torsion-free}}{\overset{\uparrow}{\subset}} \ker(q^*) \subset \ker(q_* \circ q^*) \underset{\text{by the above commutative diagram}}{\overset{\uparrow}{\subset}} \{a \in \mathrm{H}_{n-1}(M) \mid 2a = 0\}.$$

From these inclusions and the classification of finitely generated abelian groups we obtain easily that $\mathrm{H}_{n-1}(M) \cong (\mathbb{Z}_2)^r \oplus \mathbb{Z}^k$ for some $r \in \mathbb{N}_0$ and some $k \in \mathbb{N}_0$.

It remains to show that $r = 1$. In light of the $\mathbb{Z}_k$-Tor-Lemma 54.6 and the Tor-Properties Lemma 54.7 it suffices to prove the following claim:

Claim 2. We have $\operatorname{Tor}(\mathrm{H}_{n-1}(M), \mathbb{F}_2) \cong \mathbb{F}_2$.

Proof. The argument is very similar to the proof of Claim 1. More precisely, we have

$$\mathbb{F}_2 \underset{\text{by the } \mathbb{F}_2\text{-Orientability Proposition 61.13 and the Top-}\mathbb{F}_2\text{-Homology Theorem 63.2}}{\cong} \mathrm{H}_n(M;\mathbb{F}_2) \overset{\text{by the UCT-Splitting Theorem 54.11}}{\cong} \mathrm{H}_n(M) \otimes \mathbb{F}_2 \oplus \operatorname{Tor}(\mathrm{H}_{n-1}(M), \mathbb{F}_2) \underset{\text{by the Orientation-Top Homology-Theorem 63.3 (3) we have } \mathrm{H}_n(M) = 0}{\cong} 0 \oplus \operatorname{Tor}(\mathrm{H}_{n-1}(M), \mathbb{F}_2).$$

■

Now we deal with topological manifolds with non-empty boundary.

Proof of the Manifold-H_{n-1}-Theorem 63.4 (1) II. Let M be a connected, compact n-dimensional topological manifold with non-empty boundary. We need to show that $\mathrm{H}_{n-1}(M)$ is a free abelian group. There are two approaches to the proof of this statement:

(1) We can imitate the proof of the closed case, or
(2) we reduce the proof of the proposition to the case of closed topological manifolds that we had just dealt with.

We will pursue the second approach. In the proof we again constantly use that by the Topological Manifolds-Invariants Proposition 60.10 we know that $\mathrm{H}_{n-1}(M)$ is finitely generated. First recall that by the Double-Orientability Lemma 62.5 the double $\mathrm{D}M$ is orientable if and only if M is orientable.

We first consider the case that M is orientable. By the Manifold-H_{n-1}-Theorem 63.4 we know that $\mathrm{H}_{n-1}(\mathrm{D}M)$ is a finitely generated free abelian group. One can easily show that M is a retract of $\mathrm{D}M$. It follows that the group $\mathrm{H}_{n-1}(M)$ is isomorphic to a subgroup of $\mathrm{H}_{n-1}(\mathrm{D}M)$. It follows from the Free Abelian Subgroup Theorem 18.7 that $\mathrm{H}_{n-1}(M)$ is also a finitely generated free abelian group.

Now we consider the case that M is non-orientable. We denote by M' a copy of M and as usual we denote by $\mathrm{D}M = M \cup_{\partial M = \partial M'} M'$ the double. From the Mayer–Vietoris Theorem 45.1 we obtain the following long exact sequence:

$$\dots \to \mathrm{H}_n(\mathrm{D}M) \to \mathrm{H}_{n-1}(\partial M) \to \mathrm{H}_{n-1}(M) \oplus \mathrm{H}_{n-1}(M') \to \mathrm{H}_{n-1}(\mathrm{D}M) \to \dots$$

We make the following observations:

(1) Note that ∂M is a closed $(n-1)$-dimensional topological manifold. Thus we obtain from the Orientation-Top Homology-Theorem 63.3 that $\mathrm{H}_{n-1}(\partial M)$ is isomorphic to a free abelian group $\mathbb{Z}^r$ for some $r \in \mathbb{N}_0$. (We do not have any control over the orientability of ∂M and the number of components of ∂M but this does not matter.)
(2) Since $\mathrm{D}M$ is non-orientable we know from the Orientation-Top Homology-Theorem 63.3 that $\mathrm{H}_n(\mathrm{D}M) = 0$. Since $\mathrm{D}M$ is closed we know from the closed case of the Manifold-H_{n-1}-Theorem 63.4 (1) that $\mathrm{H}_{n-1}(\mathrm{D}M) \cong \mathbb{Z}^s \oplus \mathbb{Z}_2$ for some $s \in \mathbb{N}_0$.
(3) Since $M = M'$ we have $\mathrm{H}_{n-1}(M) = \mathrm{H}_{n-1}(M')$. By the classification of finitely generated abelian groups we can pick an isomorphism $\mathrm{H}_{n-1}(M) \cong F \oplus T$ where F is a free abelian group and T is a torsion group.

We can thus rewrite the above sequence as follows:

$$0 \to \mathbb{Z}^r \xrightarrow{\varphi} F \oplus F \oplus T \oplus T \to \mathbb{Z}^s \oplus \mathbb{Z}_2 \to \dots$$

The map $\varphi\colon \mathbb{Z}^r \to F \oplus F \oplus T \oplus T$ is a monomorphism, but since $\mathbb{Z}^r$ is torsion-free and since T is finite we see that $\operatorname{im}(\varphi) \cap (T \oplus T)$ is trivial. But this implies that $T \oplus T$ is isomorphic

to a subgroup of $\mathbb{Z}^s \oplus \mathbb{Z}_2$. But this is only possible if $T = 0$. This concludes the proof that $\mathrm{H}_{n-1}(M)$ is a free abelian group. ■

Proof of the Manifold-H_{n-1}-Theorem 63.4 (2). First we assume that M is orientable. Note that by the above Free Abelian–Characterization Lemma 63.17 it suffices to show that $\mathrm{Tor}(\mathrm{H}_{n-1}(M, \partial M), \mathbb{F}_p) = 0$ for all primes p. So let p be a prime. We have

$$\begin{array}{ccccc} & & \text{by the UCT-Splitting Theorem 54.11} & & \\ & & \downarrow & & \\ \mathbb{F}_p & \cong & \mathrm{H}_n(M, \partial M; \mathbb{F}_p) & \cong & \mathrm{H}_n(M, \partial M) \otimes \mathbb{F}_p \oplus \mathrm{Tor}(\mathrm{H}_{n-1}(M, \partial M), \mathbb{F}_p) \\ & & \uparrow & & \cong \mathbb{F}_p \oplus \mathrm{Tor}(\mathrm{H}_{n-1}(M, \partial M), \mathbb{F}_p). \\ & & \text{by the Manifold-Top-H}_*\text{-Theorem 63.1 (2a),} & & \uparrow \\ & & \text{since by the Orientable-for-all-Rings Lemma 61.12} & & \text{by the Orientation-Top Homology} \\ & & M \text{ is also } \mathbb{F}_p\text{-orientable} & & \text{Theorem 63.3 (3) we have } \mathrm{H}_n(M, \partial M) \cong \mathbb{Z} \end{array}$$

This implies that $\mathrm{Tor}(\mathrm{H}_{n-1}(M, \partial M), \mathbb{F}_p) = 0$. The case that M is non-orientable is very similar, we leave it to the reader to make the necessary modifications. ■

CHAPTER 64

The fundamental class of topological manifolds

In this chapter we introduce the fundamental class of a compact oriented topological manifold. This concept is extremely useful for calculations of homology groups and it will play an essential role in the formulation of the Poincaré Duality Theorem 79.1.

64.1. **The definition of the fundamental class.** First we recall the following notation from page 789.

Notation. Let M be an n-dimensional topological manifold and let R be a commutative ring. Given any $k \in \mathbb{N}_0$ and given any $x \in M \setminus \partial M$ we denote by ρ_x the natural map

$$\rho_x \colon \mathrm{H}_k(M, M \setminus \partial M; R) \;\to\; \mathrm{H}_k(M, M \setminus \{x\}; R).$$

Given $\varphi \in \mathrm{H}_k(M, M \setminus \partial M; R)$ we often write $\varphi_x := \rho_x(\varphi)$, i.e. the above map is also written as

$$\varphi \;\mapsto\; \varphi_x.$$

The following proposition is the starting point for this chapter:

Proposition 64.1. (Fundamental Class-Proposition) Let M be a compact n-dimensional topological manifold, let R be a commutative ring and let $\{\alpha_x\}_{x \in M \setminus \partial M}$ be an R-orientation for M.

(1) There exists a unique $[M]_R \in \mathrm{H}_n(M, \partial M; R)$ with $\rho_x([M]_R) = \alpha_x$ for all $x \in M \setminus \partial M$.
(2) We now suppose that M is connected, R-orientable and non-empty.
   (a) We have $\mathrm{H}_n(M, \partial M; R) = R \cdot [M]_R$.[521]
   (b) Let $\Theta \in \mathrm{H}_n(M, \partial M; R)$. If there exists a single $x \in M \setminus \partial M$ with $\rho_x(\Theta) = \alpha_x$, then $\Theta = [M]_R$.

The analogous statements hold for homology without coefficients.

Proof. Let M be a compact n-dimensional topological manifold, let R be a commutative ring and let $\{\alpha_x\}_{x \in M \setminus \partial M}$ be an R-orientation for M.

(1) We first consider the case that M is connected and non-empty. In this setting it follows from the Manifold-Top-H_*-Theorem 63.1 (3) that for every $x \in M \setminus \partial M$ the map

$$\rho_x \colon \mathrm{H}_n(M, \partial M; R) \;\to\; \mathrm{H}_n(M, M \setminus \{x\}; R)$$

is an isomorphism of R-modules.

We pick a $y \in M \setminus \partial M$ and we set $[M]_R := \rho_y^{-1}(\alpha_y) \in \mathrm{H}_n(M, \partial M; R)$. Since ρ_y is an isomorphism the class $[M]_R$ is uniquely determined. Thus it remains to show

[521] It follows from the Homology-defines-Section Lemma 61.6 (2) together with the Manifold-Top-H_*-Theorem 63.1 that given any generator Θ of $\mathrm{H}_n(M, \partial M; R)$ the homology classes $\{\Theta_x\}_{x \in M \setminus \partial M}$ form an R-orientation of M. Thus we see that in the connected cases there is a bijection between generators of $\mathrm{H}_n(M, \partial M; R)$ and R-orientations on M.

that $\rho_x([M]_R) = \alpha_x$ for all $x \in M \setminus \partial M$. By the Homology-defines-Section Lemma 61.6 (2) the classes $\{\rho_x([M]_R)\}_{x\in M\setminus\partial M}$ form an R-section. By construction we have $\rho_y([M]_R) = \alpha_y$. Since M is connected it follows from the Sections-on-Connected Proposition 61.7 that $\rho_x([M]_R) = \alpha_x$ for all $x \in M \setminus \partial M$.

Now we consider the general case. Let $\{M_i\}_{i=1,\dots,k}$ be the components of M. By the Path-Component-H_*-Lemma 35.8 we have a natural isomorphism

$$\bigoplus_{i=1}^{k} \mathrm{H}_n(M_i, \partial M_i; R) \;\to\; \mathrm{H}_n(M, \partial M; R).$$

One can easily verify that the image of $([M_1]_R, \dots, [M_k]_R)$ under this map has all the desired properties and that it is unique.

(2) We now assume that M is connected. Let $x \in M \setminus \partial M$. Since M is connected and orientable we know by the Manifold-Top-H_*-Theorem 63.1 (3) that the map

$$\rho_x \colon \mathrm{H}_n(M, \partial M; R) \;\to\; \underbrace{\mathrm{H}_n(M, M \setminus \{x\}; R)}_{=R\cdot\alpha_x}$$

is an isomorphism of R-modules. The two promised statements follow immediately from this observation and the definition of $[M]_R$. ∎

The Fundamental Class-Proposition 64.1 motivates the following definition:

Definition. Let M be a compact n-dimensional topological manifold.

(1) Let R be a commutative ring and let $\{\alpha_x\}_{x\in M\setminus\partial M}$ be an R-orientation for M. Recall that we know by the Fundamental Class-Proposition 64.1 that there exists a unique $[M]_R \in \mathrm{H}_n(M, \partial M; R)$ with $\rho_x([M]_R) = \alpha_x$ for all $x \in M \setminus \partial M$. We refer to $[M]_R$ as the fundamental class of the R-oriented topological manifold M.

(2) If we consider homology without coefficients, then we write $[M] \in \mathrm{H}_n(M, \partial M)$ for this uniquely determined element.

Example. Let $n \in \mathbb{N}$. We equip the topological manifold S^n with the standard orientation that we introduced in the Sphere Orientation Lemma 61.11. It follows immediately from the definition of this orientation that the corresponding fundamental class $[S^n] \in \mathrm{H}_n(S^n)$ equals the standard generator that we introduced on page 540.

The case $R = \mathbb{F}_2$ is particularly interesting:

Example. Let M be a compact n-dimensional topological manifold. In the $\mathbb{F}_2$-Orientability Proposition 61.13 we showed that M admits a unique $\mathbb{F}_2$-orientation. This implies that M has a natural $\mathbb{F}_2$-fundamental class $[M]_{\mathbb{F}_2} \in \mathrm{H}_n(M, \partial M; \mathbb{F}_2)$. If M is connected and non-empty, then it follows immediately from the Top-$\mathbb{F}_2$-Homology Theorem 63.2 (2) that

$$[M]_{\mathbb{F}_2} \;=\; \text{the unique non-zero element of } \mathrm{H}_n(M, \partial M; \mathbb{F}_2) \cong \mathbb{F}_2. \qquad \square$$

64.2. **Finding explicit fundamental classes.** In this section we will state and prove two propositions that can be useful for finding explicit fundamental classes.

Before we can state these two propositions we need to talk about Δ^n viewed as an oriented topological manifold:

Convention. We equip the topological manifold Δ^n with the orientation that is given by the images of[522] $[\mathrm{id}_{\Delta^n}] \in \mathrm{H}_n(\Delta^n, \partial\Delta^n)$ in $\mathrm{H}_n(\Delta^n, \Delta^n \setminus \{x\})$, $x \in \Delta^n \setminus \partial\Delta^n$.

Lemma 64.2. (Simplex-Orientation Lemma) Let $n \in \mathbb{N}_0$. We consider the oriented topological manifold Δ^n. We equip the codimension-zero submanifold $\mathring{\Delta}^n$ with the corresponding orientation coming from the Topological Manifold-Induced Orientation Lemma 62.1. Under the bijection from the Smooth-Topological Orientation Proposition 61.15 this orientation matches the orientation of the oriented smooth manifold $\mathring{\Delta}^n$ that is given by the outward pointing normal vector $(1, \ldots, 1)$.



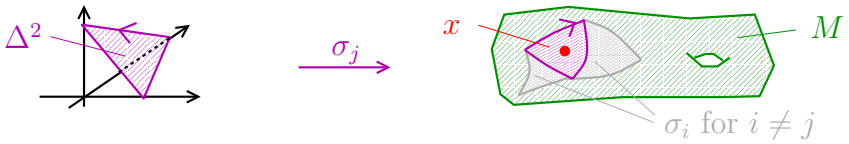
Sketch of proof. The lemma follows easily from the following observations:

(1) Let $\pi\colon \Delta^n \to \mathbb{R}^n$ be the projection onto the first n coordinates. This map π is given by an affine linear map that is orientation-preserving if and only if n is even.
(2) It follows fairly easily from (1) and the definition of the standard orientation of the topological manifold $\mathbb{R}^n$ that the map $\pi\colon \mathring{\Delta}^n \to \mathbb{R}^n$ is orientation-preserving, as a local homeomorphism between oriented topological manifolds, if and only if n is even.
(3) It follows from the definition of the orientation of the smooth manifold $\mathring{\Delta}^n$ that the map $\pi\colon \mathring{\Delta}^n \to \mathbb{R}^n$ is orientation-preserving, as a local diffeomorphism between oriented smooth manifolds, if and only if n is even.
(4) By the Smooth-Topological Orientation Proposition 61.15 (1a) the bijection matches the standard topological and smooth orientations of $\mathbb{R}^n$. ■

Proposition 64.3. (Fundamental Class-Criterion)[523] Let M be a compact oriented connected n-dimensional topological manifold. Let $r_1 \cdot \sigma_1 + \cdots + r_m \cdot \sigma_m \in \mathrm{C}_n(M, \partial M)$ be a cycle. Suppose that there exists a $j \in \{1, \ldots, m\}$ such that the following three conditions are satisfied:

(1) The map $\sigma_j\colon \mathring{\Delta}^n \to M$ is an orientation-preserving embedding.
(2) There exists an $x \in \sigma_j(\mathring{\Delta}^n)$ which is not contained in any $\sigma_i(\Delta^n)$, $i \neq j$.
(3) $r_j = 1$.

Then the cycle $r_1 \cdot \sigma_1 + \cdots + r_m \cdot \sigma_m$ represents the fundamental class $[M] \in \mathrm{H}_n(M, \partial M)$ of the oriented topological manifold M.



Proof. Let $\{\alpha_y\}_{y \in M \setminus \partial M}$ be the given orientation of M, let $\sum_{i=1}^{m} r_i \cdot \sigma_i \in \mathrm{C}_n(M, \partial M)$ be a cycle and let $j \in \{1, \ldots, m\}$ such that conditions (1)–(3) above are satisfied. Since M is

[522] Recall that by the Identity-is-Generator Lemma 42.1 we know that $\mathbb{Z} \cdot [\mathrm{id}_{\Delta^n}] = \mathrm{H}_n(\Delta^n, \partial\Delta^n)$.
[523] This statement and its proof are closely related to the Detect-Cycle Proposition 42.3.

connected it follows from the Fundamental Class-Proposition 64.1 (2b) that it suffices to prove the following claim:

Claim. There exists an $x \in M \setminus \partial M$ such that $\rho_x([r_1 \cdot \sigma_1 + \cdots + r_m \cdot \sigma_m]) = \alpha_x$.

Proof. We make a few preparations:

- We pick $x \in \sigma_j(\mathring{\Delta}^n)$ as in hypothesis (2).
- Let $y \in \mathring{\Delta}^n$ be the unique point with $\sigma_j(y) = x$.
- We denote by $i\colon \mathring{\Delta}^n \to \Delta^n$ the inclusion.
- It follows from the Excision Theorem 40.2 that $i_*\colon \mathrm{H}_n(\mathring{\Delta}^n, \mathring{\Delta}^n \setminus \{y\}) \to \mathrm{H}_n(\Delta^n, \Delta^n \setminus \{y\})$ is an isomorphism.
- We denote by $\gamma_y \in \mathrm{H}_n(\mathring{\Delta}^n, \mathring{\Delta}^n \setminus \{y\})$ the generator corresponding to the orientation of $\mathring{\Delta}^n$. By definition of the orientations of the topological manifolds Δ^n and $\mathring{\Delta}^n$ we have $i_*(\gamma_y) = \rho_y([\mathrm{id}_{\Delta^n}])) \in \mathrm{H}_n(\Delta^n, \Delta^n \setminus \{y\})$.
- Since $\sigma_j\colon \mathring{\Delta}^n \to M$ is orientation-preserving we have $\sigma_{j*}(\gamma_y) = \alpha_x$.

We summarize some of the maps in the following basic commutative diagram:

$$\begin{array}{ccccc}
\mathrm{H}_n(\Delta^n, \partial\Delta^n) & \xrightarrow{\rho_y} & \mathrm{H}_n(\Delta^n, \Delta^n \setminus \{y\}) & \xleftarrow[\cong]{i_*} & \mathrm{H}_n(\mathring{\Delta}^n, \mathring{\Delta}^n \setminus \{y\}) \\
{\scriptstyle [\mathrm{id}_{\Delta^n}]} & & {\scriptstyle \rho_y([\mathrm{id}_{\Delta^n}])} & & {\scriptstyle \gamma_y} \\
& {\scriptstyle \sigma_{j*}} \searrow & \downarrow {\scriptstyle \sigma_{j*}} & \swarrow {\scriptstyle \sigma_{j*}} & \\
& & \mathrm{H}_n(M, M \setminus \{x\}). & & \\
& & {\scriptstyle \alpha_x} & &
\end{array}$$

We obtain the following equalities in $\mathrm{H}_n(M, M \setminus \{x\})$:

$$\begin{array}{lcccc}
\rho_x([r_1 \cdot \sigma_1 + \cdots + r_m \cdot \sigma_m]) & = & [\overbrace{r_1 \cdot \sigma_1 + \cdots + r_m \cdot \sigma_m}^{\in \mathrm{C}_n(M, M\setminus\{x\})}] & = & [r_j \cdot \sigma_j] \\
& & & & \uparrow \text{ since } r_j = 1 \\
& & & & = [\sigma_j\colon \Delta^n \to M]. \\
& & \uparrow \text{ since (2) implies that for } i \neq j \text{ we have } \sigma_i(\Delta^n) \subset M \setminus \{x\} & & \\
& = & \sigma_{j*}([\mathrm{id}_{\Delta^n}]) & = & \sigma_{j*}(\gamma_y) \\
& & \uparrow \text{ by definition of } \sigma_{j*} & & \uparrow \text{ by the above diagram} \\
& & & & = \alpha_x. \\
& & & & \uparrow \text{ since } \sigma_j\colon \mathring{\Delta}^n \to M \text{ is orientation-preserving}
\end{array}$$

■

Proposition 64.4. ($\mathbb{F}_2$-Fundamental Class-Criterion) Let M be a compact connected n-dimensional topological manifold. Let $(\sigma_1 + \cdots + \sigma_m) \otimes 1 \in \mathrm{C}_n(M, \partial M) \otimes \mathbb{F}_2$ be a cycle. Suppose that there exists a $j \in \{1, \ldots, m\}$ such that the following two conditions are satisfied:

(1) The map $\sigma_j\colon \mathring{\Delta}^n \to M$ is an embedding.

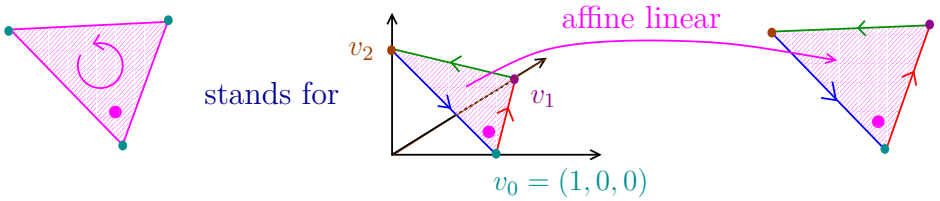
(2) There exists an $x \in \sigma_j(\mathring{\Delta}^n)$ which is not contained in any $\sigma_i(\Delta^n)$, $i \neq j$.

Then the cycle $(\sigma_1 + \cdots + \sigma_m) \otimes 1 \in \mathrm{C}_n(M, \partial M) \otimes \mathbb{F}_2 = \mathrm{C}_n(M, \partial M; \mathbb{F}_2)$ represents the $\mathbb{F}_2$-fundamental class in $\mathrm{H}_n(M, \partial M; \mathbb{F}_2)$ of M.

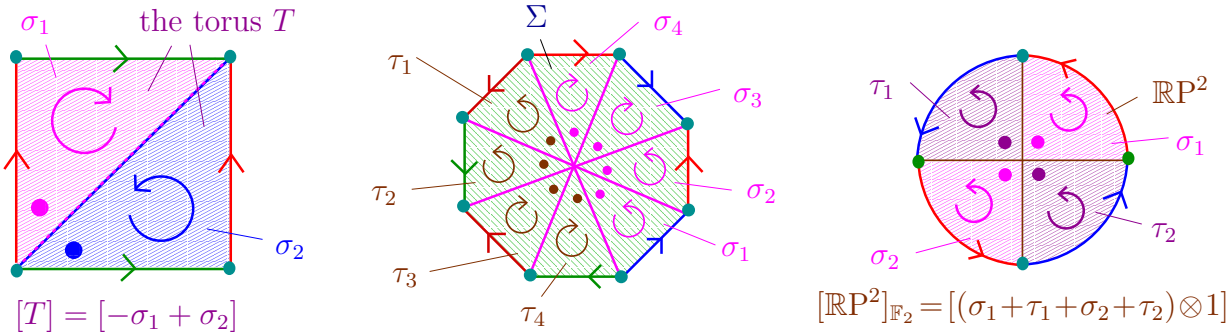
Proof. We leave it to the reader to simplify the proof of the Fundamental Class-Criterion 64.3 to obtain the desired result. ■

Example. We consider the topological manifold S^1 with the standard orientation. It follows quite easily from the Fundamental Class-Criterion 64.3 that the singular simplex $\mu\colon \Delta^1 \to S^1$ that is given by $(1-t, t) \mapsto \exp(2\pi\,\mathrm{i}t)$ represents the fundamental class of S^1. □

Example. We will give some examples of fundamental classes of 2-dimensional topological manifolds. We use the notation from the figure below to indicate a singular simplex.



In the figure below we now show various cycles.



It follows from the Fundamental Class-Criterion 64.3 and the $\mathbb{F}_2$-Fundamental Class-Criterion 64.4 that these cycles represent the fundamental classes of the torus and the surface of genus 2 and the $\mathbb{F}_2$-fundamental class of $\mathbb{R}\mathrm{P}^2 =_{\mathrm{i}} \overline{B}^2/z \sim -z$. □

64.3. **The fundamental class and induced orientations.** In Section 62.1 we saw the following:

(1) An orientation on a topological manifold induces orientations on codimension-zero submanifolds and on coverings.
(2) An orientation on a topological manifold induces an orientation on the boundary.
(3) The product of two oriented topological manifolds has a natural orientation.

If the topological manifolds are compact, then the orientations give rise to fundamental classes. In the following three subsections we will see how the fundamental classes of the original compact oriented topological manifolds are related to the fundamental classes of the resulting compact oriented topological manifolds.

64.3.1. **The fundamental class and induced orientations: Local homeomorphisms.** In the Topological Manifold-Induced Orientation Lemma 62.1 we saw that a codimension-zero submanifold W of an R-oriented topological manifold M inherits an R-orientation. For compact topological manifolds the following lemma shows the relationship between the corresponding fundamental classes:

Lemma 64.5. (Codimension Zero-Fundamental Class Lemma) Let M be a compact n-dimensional topological manifold and let R be a commutative ring. Furthermore

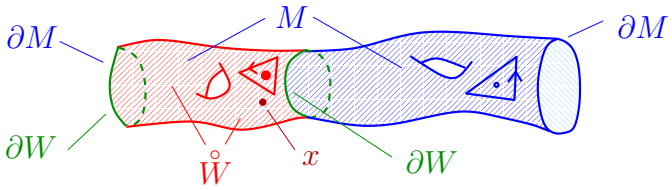
let $W \subset M$ be a compact codimension-zero submanifold. We write $\mathring{W} = W \setminus \partial W$. The following two statements hold:

(1) The inclusion-induced map $i_*\colon \mathrm{H}_n(W, \partial W; R) \to \mathrm{H}_n(M, M \setminus \mathring{W}; R)$ is an isomorphism.
(2) We assume that M is R-oriented. We equip W with the R-orientation defined in the Topological Manifold-Induced Orientation Lemma 62.1 (2). The image of the fundamental classes $[M]_R \in \mathrm{H}_n(M, \partial M; R)$ and $[W]_R \in \mathrm{H}_n(W, \partial W; R)$ under the maps

$$\mathrm{H}_n(M, \partial M; R) \xrightarrow{[M]_R} \xrightarrow{j_*} \mathrm{H}_n(M, M \setminus \mathring{W}; R) \xleftarrow{i_*} \xleftarrow{[W]_R} \mathrm{H}_n(W, \partial W; R)$$

agree.

The analogous statement holds for homology without coefficients.



Proof.

(1) This statement is an immediate consequence of the Manifold–Excision Theorem 40.3.
(2) We have to show that $(i_*)^{-1}(j_*([M]_R))$ is the fundamental class for W. Thus let $x \in W \setminus \partial W$. We consider the following diagram:

$$\begin{array}{ccccc}
\mathrm{H}_n(M, \partial M; R) & \xrightarrow{j_*} & \mathrm{H}_n(M, M \setminus \mathring{W}; R) & \xleftarrow[\cong]{i_*} & \mathrm{H}_n(W, \partial W; R) \\
\downarrow{\scriptstyle \rho_x^M} & \swarrow & & & \downarrow{\scriptstyle \rho_x^W} \\
\mathrm{H}_n(M, M \setminus \{x\}; R) & \xleftarrow[\text{isomorphism by the Excision Theorem 40.2}]{\cong} & & & \mathrm{H}_n(W, W \setminus \{x\}; R).
\end{array}$$

Since all maps are induced by inclusions we see that the diagram commutes. Since $[M]_R$ is the fundamental class we see that $\rho_x^M([M]_R)$ equals the generator given by the R-orientation of M. It follows from the above diagram that ρ_x^W sends $(i_*)^{-1}(j_*([M]_R))$ to the generator corresponding to the induced R-orientation of W. ∎

In the Topological Manifold-Induced Orientation Lemma 62.1 (3) we showed that a covering space $\widetilde{M}$ of an R-oriented topological manifold M inherits an R-orientation from M. The following proposition relates the fundamental classes of M and of any finite covering $\widetilde{M}$.

Proposition 64.6. (Fundamental Class-Covering Proposition) Let R be a commutative ring, let M be a compact R-oriented n-dimensional topological manifold and let $p\colon \widetilde{M} \to M$ be a finite covering. We equip $\widetilde{M}$ with the induced R-orientation coming from the Topological Manifold-Induced Orientation Lemma 62.1 (3). We denote by $[M]_R$ the fundamental class of M and we denote by $p^*\colon \mathrm{H}_n(M, \partial M; R) \to \mathrm{H}_n(\widetilde{M}, \partial \widetilde{M}; R)$ the transfer map that we introduced on page 721. Then

$$p^*([M]_R) \;=\; [\widetilde{M}]_R \;\in\; \mathrm{H}_n(\widetilde{M}, \partial \widetilde{M}; R).$$

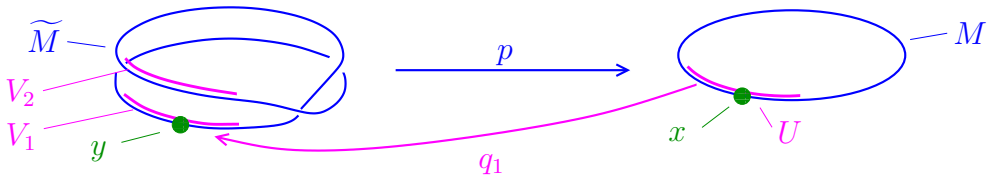
The analogous statement holds for homology without coefficients.

Proof. To simplify the notation we work with homology without coefficients. Let M be a compact n-dimensional topological manifold and let $p\colon \widetilde{M} \to M$ be a finite covering. Let $\{\nu_x\}_{x\in M\setminus\partial M}$ be an orientation on M. We equip $\widetilde{M}$ with the orientation $\{\widetilde{\nu}_y\}_{y\in\widetilde{M}\setminus\partial\widetilde{M}}$ that we defined in the Topological Manifold-Induced Orientation Lemma 62.1 (3).

We want to show that $p^*([M])$ equals the fundamental class $[\widetilde{M}]$. By definition of the fundamental class we need to prove the following claim:

Claim. For all $y \in \widetilde{M}$ we have $\rho_y(p^*([M])) = \widetilde{\nu}_y \in \mathrm{H}_n(\widetilde{M}, \widetilde{M} \setminus \{y\})$.

Proof. Let $y \in \widetilde{M}$. We set $x \coloneqq p(y)$. We pick an open neighborhood U around x such that U is uniformly covered. Recall that this means that we can write $p^{-1}(U) = V_1 \sqcup \cdots \sqcup V_d$ where $d = [\widetilde{M} : M]$, where each $V_i \subset \widetilde{M}$ is open and such that the restriction of p to each V_i is a homeomorphism. We denote by $q_i\colon U \to V_i$ the inverses. We can assume that $y \in V_1$.



We consider the following diagram:

$$\begin{array}{ccccc}
\mathrm{H}_n(M,\partial M) & \xrightarrow{\rho_x} & \mathrm{H}_n(M, M\setminus\{x\}) & \xleftarrow{\cong} & \mathrm{H}_n(U, U\setminus\{x\}) \\
\downarrow{\scriptstyle p^*} & & \downarrow{\scriptstyle p^*} & & \downarrow{\scriptstyle q_{1*}+\cdots+q_{d*}}\ {\scriptstyle p^*} \\
\mathrm{H}_n(\widetilde{M},\partial\widetilde{M}) & \longrightarrow & \mathrm{H}_n(\widetilde{M}, \widetilde{M}\setminus p^{-1}(\{x\})) & \xleftarrow{\cong} & \mathrm{H}_n(p^{-1}(U), p^{-1}(U)\setminus p^{-1}(\{x\})) \\
 & \searrow^{\rho_y} & \downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \pi_*} \\
 & & \mathrm{H}_n(\widetilde{M}, \widetilde{M}\setminus\{y\}) & \xleftarrow{\cong} & \mathrm{H}_n(p^{-1}(U), p^{-1}(U)\setminus\{y\}) \\
 & & & & \cong\uparrow{\scriptstyle i_*} \\
 & & & & \mathrm{H}_n(V_1, V_1\setminus\{y\})
\end{array}$$

(with q_{1*} from the top-right corner to the bottom-right corner)

We make the following clarifications and observations;

(1) The undecorated maps are induced by the obvious maps of pairs of topological spaces.
(2) The diagram commutes by the functoriality of homology and since it follows from the Transfer Map Lemma 57.1 that the transfer map is natural.
(3) It follows easily from the definition of the transfer map that the right hand map $p^*\colon \mathrm{H}_n(U, U\setminus\{x\}) \to \mathrm{H}_n(p^{-1}(U), p^{-1}(U)\setminus p^{-1}(\{x\}))$ is given by $q_{1*} + \cdots + q_{d*}$.
(4) We denote by $i\colon V_1 \to p^{-1}(U)$ the inclusion. Note that the corresponding bottom vertical map is an isomorphism by the Excision Theorem 40.2.
(5) We denote by $\pi\colon (p^{-1}(U), p^{-1}(U)\setminus p^{-1}(\{x\})) \to (p^{-1}(U), p^{-1}(U)\setminus\{y\})$ the obvious map. It follows easily from (3) that $\pi_* \circ p^* = i_* \circ q_{1*}$.
(6) If we apply the "outside route" from $\mathrm{H}_n(M, M\setminus\{x\})$ to $\mathrm{H}_n(\widetilde{M}, \widetilde{M}\setminus\{y\})$ via q_{1*} we send, by definition, ν_x to $\widetilde{\nu}_y$.

We leave it to the reader to verify that the above information implies the desired statement. ∎

64.3.2. **The fundamental class and induced orientations: Boundaries.** In the Boundary Orientation Proposition 62.3 we saw that the boundary of an oriented topological manifold inherits a natural orientation. The following proposition now gives us the expected result about the relationship between the fundamental classes of an oriented topological manifold and its boundary.

Proposition 64.7. (Fundamental Class-of-∂-Proposition) Let R be a commutative ring and let M be a compact R-oriented n-dimensional topological manifold. If we equip ∂M with the R-orientation coming from the Boundary Orientation Proposition 62.3, then

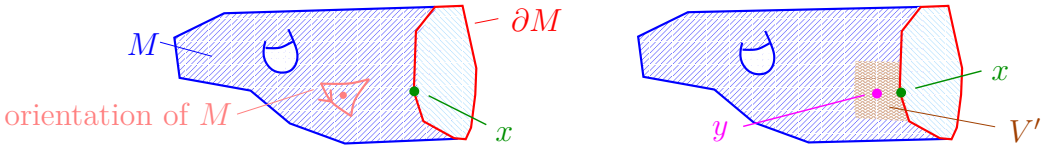
$$\underset{\uparrow}{\partial_n}([M]_R) \;=\; [\partial M]_R \quad \in\; \mathrm{H}_{n-1}(\partial M; R).$$

connecting homomorphism of the long exact sequence of the pair $(M, \partial M)$

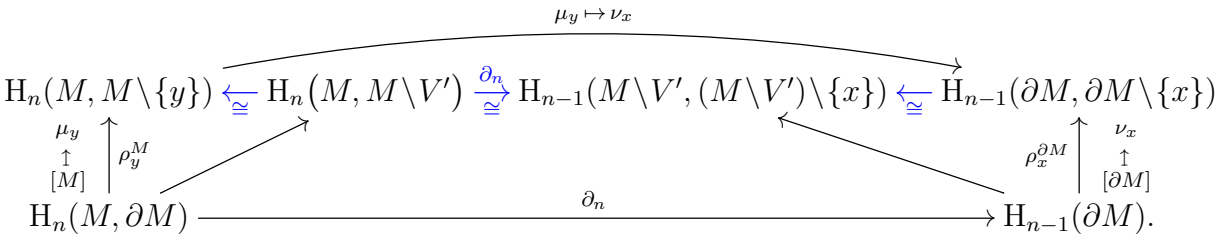
The analogous statement also holds if we consider homology without coefficients.

Proof. To simplify the notation we consider in the proof only homology without coefficients. As a health warning we point out that in this proof we make heavy use of the notation introduced in the Boundary Orientation Proposition 62.3.

Let $\{\mu_y\}_{y\in M\setminus\partial M}$ be the orientation of for M. Next let $x \in \partial M$. We pick a chart $\Phi\colon U \to [0,2)\times(-2,2)^{n-1}$ with $\Phi(x) = (0,0)$. We write $V' = \Phi^{-1}((0,1)\times(-1,1)^{n-1})$. Finally we pick a point $y \in V'$.



We consider the following diagram:



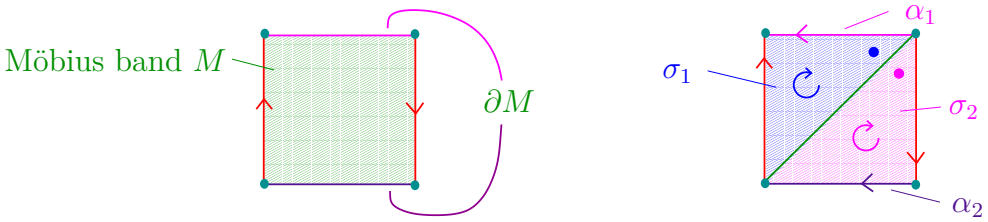
We make the following observations and clarifications:

(1) All the unmarked maps are induced by the obvious inclusions of pairs of topological space.
(2) Recall that in the proof of the Boundary Orientation Proposition 62.3 we showed that the three horizontal blue maps on the top, connecting $\mathrm{H}_n(M, M\setminus\{y\})$ to $\mathrm{H}_{n-1}(\partial M, \partial M\setminus\{x\})$, are isomorphisms.
(3) By definition of the orientation $\{\nu_x\}_{x\in\partial M}$ of ∂M the composition of the top horizontal maps sends μ_x to ν_y.
(4) The maps ∂_n are the connecting homomorphism of the long exact sequence of the triple $(M, M\setminus V', (M\setminus V')\setminus\{x\})$ respectively of the pair $(M, \partial M)$.
(5) The triangles are given by inclusion-induced maps, hence they both commute.

(6) Finally note that it follows from the naturality of the connecting homomorphisms, see the LES-of-Triple Proposition 38.2 (3), that the quadrilateral commutes.

It follows from the above and the defining property of $[M]$ that for any $x \in \partial M$ the image of $\partial_n([M]) \in \mathrm{H}_{n-1}(M)$ under the map $\mathrm{H}_{n-1}(\partial M) \to \mathrm{H}_{n-1}(\partial M, \partial M \setminus \{x\})$ equals ν_x. But that means that $\partial_n([M]) = [\partial M]$. ■

Example. In the figure below we consider the Möbius band M with two singular 1-simplices α_1, α_2 and two singular 2-simplices σ_1, σ_2.



In this case we have $$\partial_2(\underbrace{\sigma_1 + \sigma_2}_{=:\mu}) = \underbrace{\alpha_1 + \alpha_2}_{\in \mathrm{C}_1(\partial M)}.$$

In particular, $\mu \otimes 1$ is a cycle in $\mathrm{C}_2(M, \partial M; \mathbb{F}_2)$. It follows easily from the $\mathbb{F}_2$-Fundamental Class-Criterion 64.4 that $\mu \otimes 1$ represents the $\mathbb{F}_2$-fundamental class in $\mathrm{H}_2(M, \partial M; \mathbb{F}_2)$. We now see that

$$\partial_2([M]_{\mathbb{F}_2}) = \partial_2([\mu \otimes 1]) \underset{\text{by the LES-of-Triple Proposition 38.2}}{\uparrow} = [\partial\mu \otimes 1] \underset{\text{by the above}}{\uparrow} = [(\alpha_1 + \alpha_2) \otimes 1].$$

It follows again from $\mathbb{F}_2$-Fundamental Class-Criterion 64.4 that the last class is the $\mathbb{F}_2$-fundamental class for the circle ∂M. □

Corollary 64.8. (Kernel-of-Boundary Inclusion Corollary) Let R be a commutative ring and let M be a compact R-oriented connected n-dimensional topological manifold. We denote by $i\colon \partial M \to M$ the inclusion map. Then

$$\ker(i_*\colon \mathrm{H}_{n-1}(\partial M; R) \to \mathrm{H}_{n-1}(M; R)) = R \cdot [\partial M]_R \quad \in \mathrm{H}_{n-1}(\partial M; R).$$

Proof. It follows from the Fundamental Class-of-∂-Proposition 64.7 and the long exact sequence in homology of the pair $(M, \partial M)$ that the following sequence is exact:

$$\underbrace{\mathrm{H}_n(M, \partial M; R)}_{= R \cdot [M] \text{ by the Fundamental Class-Proposition 64.1}} \xrightarrow{[M]_R \mapsto [\partial M]_R} \mathrm{H}_{n-1}(\partial M; R) \xrightarrow{i_*} \mathrm{H}_{n-1}(M; R).$$

We are done. ■

The following proposition is the culmination of a long line of results, see the Circle–Non Retract Proposition 13.3 and the Sphere-Non Retract Proposition 41.5.

Proposition 64.9. (Boundary Non-Retraction Proposition) Let M be a *compact* non-empty topological manifold. There is no retraction from M to ∂M.

Example. It follows that the Möbius band does not retract to its boundary. What other proofs do you know for this statement? □

Proof. The proof of the proposition follows the same logic as the proof of the Circle–Non Retract Proposition 13.3: We denote by $i\colon \partial M \to M$ the inclusion. We now suppose that there exists a retraction $r\colon M \to \partial M$, i.e. a continuous map such that $r \circ i = \mathrm{id}_{\partial M}$. We assume that M is non-empty, which implies in our case that $\partial M \neq \varnothing$. For $n \coloneqq \dim(M)$ we consider the following commutative diagram:

$$\begin{array}{ccc} & \mathrm{H}_{n-1}(M;\mathbb{F}_2) & \\ {\scriptstyle i_*}\nearrow & & \searrow{\scriptstyle r_*} \\ \mathrm{H}_{n-1}(\partial M;\mathbb{F}_2) & \xrightarrow{(r\circ i)_*=\mathrm{id}} & \mathrm{H}_{n-1}(\partial M;\mathbb{F}_2). \end{array}$$

Next recall that we know by the $\mathbb{F}_2$-Orientability Proposition 61.13 that M is $\mathbb{F}_2$-orientable. It follows from this observation, the hypothesis that M is compact, the above observation that $\partial M \neq \varnothing$ and the Fundamental Class-of-∂-Proposition 64.7 that i_* is not injective. But this leads immediately to a contradiction. ∎

64.3.3. **The fundamental class and induced orientations: Products.** We conclude this section with the last construction of topological manifolds, namely products:

Proposition 64.10. (Fundamental Class-of-Product Proposition) Let M be a compact oriented m-dimensional topological manifold and let N be a compact oriented n-dimensional topological manifold. We consider the cross product

$$\mathrm{H}_m(M,\partial M) \otimes \mathrm{H}_n(N,\partial N) \xrightarrow{\times} \mathrm{H}_{m+n}(M\times N, \partial(M\times N))$$

which we introduced on page 712.

(1) We have
$$[M]\times[N] \;=\; [M\times N] \;\in\; \mathrm{H}_{m+n}(M\times N,\partial(M\times N))$$
where $[M\times N]$ denotes the fundamental class of the compact topological manifold $M\times N$ equipped with the orientation defined in the Product Orientation Proposition 62.7 (1).

(2) If M and N are connected, then the above cross product is an isomorphism.

Remark. In the Shuffle Proposition 56.11 we gave an explicit description of the cross product on the chain level. As an example we consider $M = N = [0,1]/0 \sim 1$ with the standard orientation and the cycles $\mu\colon \Delta^1 \to M$ and $\nu\colon \Delta^1 \to M$ that are both given by $(1-t,t) \mapsto t$. One can easily check that $\mu\times\nu \in \mathrm{C}_2(([0,1]\times[0,1])/\sim)$ is precisely the singular cycle that we considered on page 835. □

Proof. Let M be a compact oriented m-dimensional topological manifold and let N be a compact oriented n-dimensional topological manifold. First note that by the Product Topology-Properties Proposition 5.4 the product $M\times N$ is also compact. Furthermore note that by the Topological Manifold Product Proposition 62.6 we know that

$$\partial(M\times N) \;=\; (\partial M\times N)\cup(M\times\partial N).$$

(1) Let $x \in M\setminus\partial M$ and let $y \in N\setminus\partial N$. It follows from the above description of $\partial(M\times N)$ that $(x,y) \in (M\times N)\setminus\partial(M\times N)$. Note that, by the (relative version of the) Cross

Product Lemma 56.2 (2) we have the following *commutative* diagram

$$\begin{array}{ccc} \mathrm{H}_m(M,\partial M)\otimes \mathrm{H}_n(N,\partial N) & \xrightarrow{\times} & \mathrm{H}_{m+n}(M\times N,\partial(M\times N)) \\ \downarrow{\scriptstyle \rho_x\otimes\rho_y} & & \downarrow{\scriptstyle \rho_{(x,y)}} \\ \mathrm{H}_m(M,M\setminus\{x\})\otimes \mathrm{H}_n(N,N\setminus\{y\}) & \xrightarrow{\times} & \mathrm{H}_{m+n}(M\times N,(M\times N)\setminus\{(x,y)\}). \end{array}$$

It follows immediately from the fact that the diagram commutes and the definition of the product orientation that $\rho_{(x,y)}([M]\times[N])$ is the generator given by the product orientation.

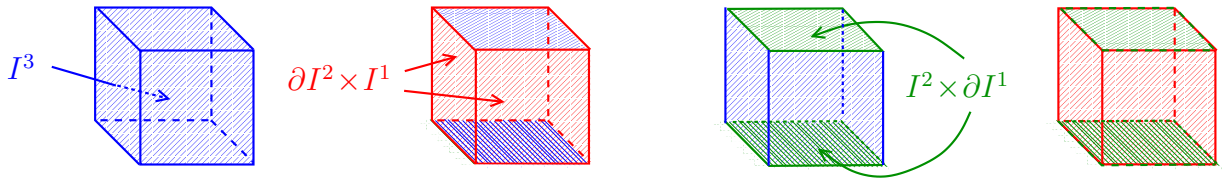
(2) We now assume that M and N are connected. If one of them is empty, then there is nothing to show. Otherwise we pick $x\in M\setminus\partial M$ and $y\in N\setminus\partial N$. We consider again the commutative diagram from (1). The vertical maps are isomorphisms by the Orientation-Top Homology-Theorem 63.3 (3). In the Product Orientation Proposition 62.7 (1) we showed that the bottom horizontal map is an isomorphism. It follows, as claimed, that the top horizontal map is also an isomorphism. ■

Example. We write $I=[-1,1]$. We view I as a 1-dimensional topological manifold and we equip I with the orientation coming from $\mathbb{R}$. Given $k\in\mathbb{N}_0$ we write

$$\partial I^k \;:=\; \{(x_1,\ldots,x_k)\,|\,\text{there exists an } i \text{ with } x_i=-1 \text{ or } x_i=1\}.$$

Given any $m,n\in\mathbb{N}$ one sees easily that

$$\partial(I^m\times I^n) \;=\; (\partial I^m\times I^n)\cup(I^m\times\partial I^n) \;=\; \partial(I^{m+n}).$$



It follows from the Topological Manifold Product Proposition 62.6 and the above discussion that we can view each $I^k=I\times\cdots\times I$ as a k-dimensional topological manifold with boundary given by the above subset ∂I^k. We equip I^k with the product orientation. It follows immediately from the Product Orientation Proposition 62.7 (2) that the orientation on the open subset $I^k\setminus\partial I^k\subset\mathbb{R}^k$ is the standard orientation.[524] As usual we denote by $[I^k]\in\mathrm{H}_k(I^k,\partial I^k;\mathbb{Z})$ the fundamental class.

Now let $m,n\in\mathbb{N}_0$. We get a cross product

$$\overbrace{\mathrm{H}_m(I^m,\partial I^m)}^{=\mathbb{Z}\cdot[I^m]}\otimes\overbrace{\mathrm{H}_n(I^n,\partial I^n)}^{=\mathbb{Z}\cdot[I^n]}\;\to\;\overbrace{\mathrm{H}_{m+n}(I^{m+n},\partial I^{m+n})}^{=\mathbb{Z}\cdot[I^{n+n}]}$$

which by the Fundamental Class-of-Product Proposition 64.10 has the property that

$$[I^m]\times[I^n] \;=\; [I^{m+n}].$$ □

[524]Note that the homeomorphism $f\colon(I^k,\partial I^k)\to(\overline{B}^k,S^{k-1})$ from the Convex-to-Ball Proposition 2.12 (2) has the property that the restriction to a dense open subset of $I^k\setminus\partial I^k\to B^k$ is an orientation-preserving diffeomorphism. In particular this homeomorphism is thus an orientation-preserving homeomorphism from the oriented topological manifold I^k to the oriented topological manifold $\overline{B}^k$.

CHAPTER 65

The degree of a map between oriented topological manifolds

In this chapter we introduce the notion of the degree of a continuous map between two compact oriented topological manifolds of the same dimension. This notion of degree generalizes the concept of the degree of a continuous self-map of a sphere which we first introduced on page 545.

65.1. **The degree of a map between oriented topological manifolds.** The following definition gives us the key concept of this chapter.

Definition. Let $f\colon M \to N$ be a continuous map between two compact oriented connected non-empty n-dimensional topological manifolds. We assume that $f(\partial M) \subset \partial N$. We denote by $\deg(f) \in \mathbb{Z}$ the unique integer with

$$\underset{\substack{\uparrow\\ \text{since } f(\partial M)\subset \partial N \text{ we have an}\\ \text{induced map } f_*\colon \mathrm{H}_n(M,\partial M)\to \mathrm{H}_n(N,\partial N)}}{f_*([M])} \;=\; \deg(f)\cdot[N] \;\in\; \mathrm{H}_n(N,\partial N) \;\underset{\substack{\uparrow\\ \text{by the Fundamental}\\ \text{Class-Proposition 64.1}}}{=}\; \mathbb{Z}\cdot[N].$$

We refer to $\deg(f)$ as the degree of f.

Remark. For a continuous map $f\colon S^n \to S^n$ the above degree is, by definition, exactly the same as the degree that we defined on page 545.

Example. Let $f\colon M \to N$ be a *homeomorphism* be tween two compact oriented connected non-empty n-dimensional topological manifolds. It follows easily from the defining properties of the fundamental class, see the Fundamental Class-Proposition 64.1, that

$$\deg(f) \;=\; \begin{cases} +1, & \text{if } f \text{ is orientation-preserving,}\\ -1, & \text{if } f \text{ is orientation-reversing.}\end{cases}$$

The following lemma generalizes several statements of the Sphere–Degree Lemma 43.1.

Lemma 65.1. (Manifold Degree Lemma) Let $n \in \mathbb{N}$ and let L, M and N be compact oriented connected non-empty n-dimensional topological manifolds.

(1) Let $f, g\colon M \to N$ be two continuous maps with $f(\partial M) \subset \partial N$ and $g(\partial M) \subset \partial N$. If f is homotopic to g, as a map of pairs $(M,\partial M) \to (N,\partial N)$, then $\deg(f) = \deg(g)$.

(2) For any two continuous maps $f\colon L \to M$ and $g\colon M \to N$ with $f(\partial L) \subset \partial M$ and $g(\partial M) \subset \partial N$ we have $\deg(f\circ g) = \deg(f)\cdot\deg(g)$.

Proof.

(1) This statement follows immediately from the Homotopy-H_*-Proposition 36.5.

(2) Exactly as in the proof of the Sphere–Degree Lemma 43.1 (4) we have

$$(g\circ f)_*([L]) \;=\; g_*(f_*([L])) \;=\; g_*(\deg(f)\cdot[M]) \;=\; \deg(g)\cdot\deg(f)\cdot[N].$$

It follows that $\deg(g\circ f) = \deg(g)\cdot\deg(f)$. ∎

65.2. **Local and global degrees.** In the S^n-Local-Global Degree Proposition 44.2 we already saw that in some settings we can calculate the "global" degree of a continuous map by "local" degrees. We now want to generalize these results to our setting.

Before we can formulate this result we need to define the notion of a local degree. At this point it is convenient to recall the following definition from page 553:

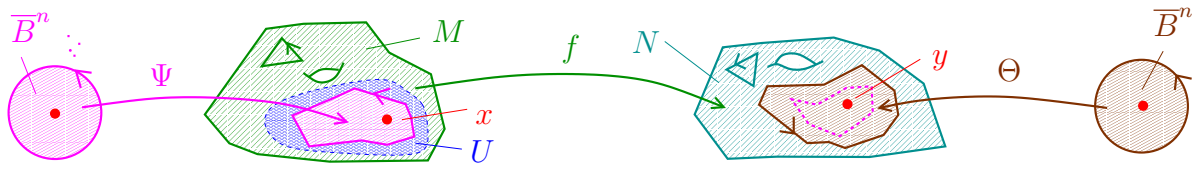
Definition. We say that a map $f\colon X \to Y$ between two topological spaces is nice at a point $x \in X$ if there exists a neighborhood U of x such that for all $x' \neq x$ in U we have $f(x') \neq f(x)$. We refer to such a neighborhood as a nice neighborhood of x.

With this definition we can now introduce our new notion of a local degree:

Definition. Let $f\colon M \to N$ be a continuous map between two oriented n-dimensional topological manifolds and let $x \in M \setminus \partial M$ such that $f(x) \in N \setminus \partial N$. We assume that x admits a nice neighborhood U. We set $y := f(x)$. We pick an orientation-preserving embedding $\Theta\colon \overline{B}^n \to N$ with $\Theta(0) = y$ and we pick an orientation-preserving embedding $\Psi\colon \overline{B}^n \to M$ with $\Psi(0) = x$ and such that $\Psi(\overline{B}^n) \subset U \cap f^{-1}(\Theta(\overline{B}^n))$. We define the local degree $\deg(f, x)$ as follows:

$$\deg(f,x) := \deg\left((\Theta^{-1} \circ f \circ \Psi)_*\colon \mathrm{H}_n(\overline{B}^n, \overline{B}^n \setminus \{0\}) \to \mathrm{H}_n(\overline{B}^n, \overline{B}^n \setminus \{0\})\right).$$

In the Local Degree Lemma 65.2 below we will see that $\deg(f, x)$ is well-defined.



Lemma 65.2. (Local Degree Lemma) Let $f\colon M \to N$ be a continuous map between two oriented n-dimensional topological manifolds such that $f(\partial M) \subset \partial N$.

(1) The local degree, as defined above, is well-defined, i.e. independent of the choice of Ψ and Θ.

(2) Let $x \in M \setminus \partial M$. If f is a local homeomorphism around x, then x is nice and

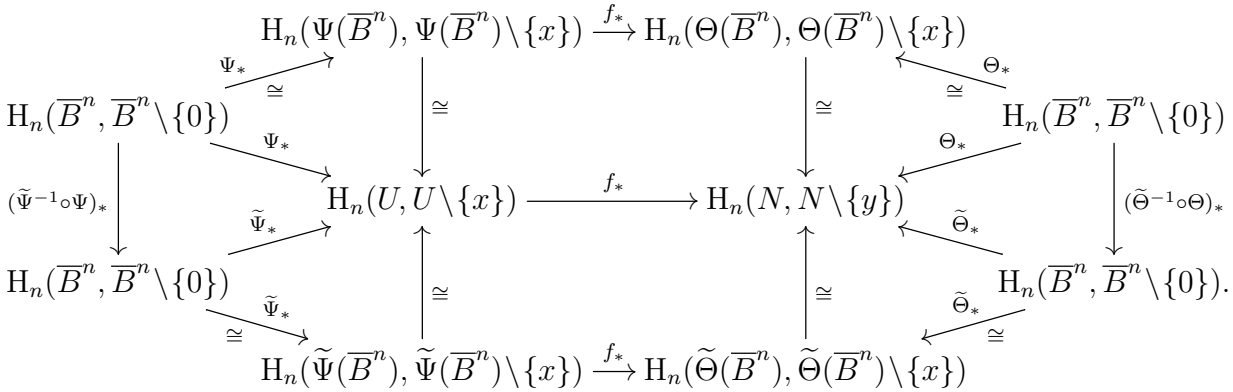
$$\deg(f,x) = \begin{cases} +1, & \text{if } f \text{ is orientation-preserving at } x \\ -1, & \text{if } f \text{ is orientation-reversing at } x. \end{cases}$$

(3) For $M = N = S^n$ the above local degree agrees with the definition given on page 554.

Proof.

(1) Let $f\colon M \to N$ be a continuous map between two oriented n-dimensional topological manifolds and let $x \in M \setminus \partial M$ such that $f(x) \in N \setminus \partial N$. We assume that x admits a nice neighborhood U. We set $y := f(x)$. Let $\Theta\colon \overline{B}^n \to N$ be an orientation-preserving embedding with $\Theta(0) = y$ and let $\Psi\colon \overline{B}^n \to M$ an orientation-preserving embedding with $\Psi(0) = x$ and such that $\Psi(\overline{B}^n) \subset U \cap f^{-1}(\Theta(\overline{B}^n))$. We now suppose that we have alternative orientation-preserving embeddings $\widetilde{\Psi}$ and $\widetilde{\Theta}$. By possibly comparing with embeddings with smaller images we might as well assume that $\Psi(\overline{B}^n) \subset \widetilde{\Psi}(\overline{B}^n)$ and that $\Theta(\overline{B}^n) \subset \widetilde{\Theta}(\overline{B}^n)$.

We now have the following commutative diagram:



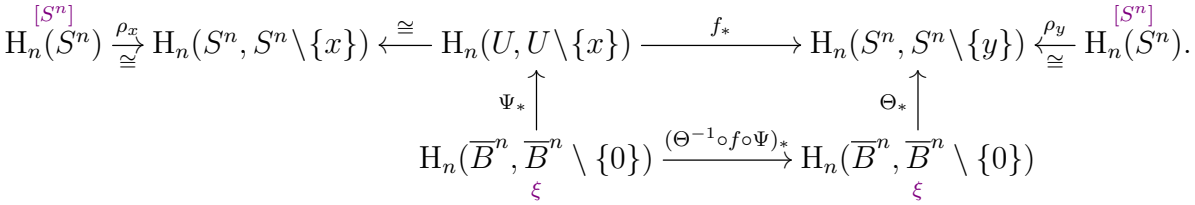
Since $\Psi, \widetilde{\Psi}, \Theta$ and $\widetilde{\Theta}$ are orientation-preserving we see that the vertical maps to the left and right are the identity. It now follows that the definitions of the local degree with the two different choices of orientation-preserving embeddings agree.

(2) This statement follows easily from the definitions.

(3) Let $n \geq 1$, let $f\colon S^n \to S^n$ be a continuous map and let $x \in S^n$ be a nice point. We make a few preparations:

- We pick a nice neighborhood U of x and we set $y = f(x)$.
- We pick an orientation-preserving embedding $\Theta\colon \overline{B}^n \to S^n$ with $\Theta(0) = y$ and we pick an orientation-preserving embedding $\Psi\colon \overline{B}^n \to S^n$ with $\Psi(0) = x$ and such that $\Psi(\overline{B}^n) \subset U \cap f^{-1}(\Theta(\overline{B}^n))$.
- We denote by $\xi \in \mathrm{H}_n(\overline{B}^n, \overline{B}^n \setminus \{0\})$ the generator corresponding to the standard orientation of $\overline{B}^n$.

We consider the following diagram:



The statement follows easily from the following observations:

- The composition of the top horizontal maps is, by definition, given by the multiplication of the local degree as defined on page 554.
- The bottom horizontal map is given by multiplication by the "new" local degree.
- It follows from the definition of the standard orientation of S^n (see the Sphere Orientation Lemma 61.11) and the fact that Ψ and Θ are orientation-preserving that the images of ξ and $[S^n]$ in $\mathrm{H}_n(S^n, S^n \setminus \{x\})$ and $\mathrm{H}_n(S^n, S^n \setminus \{y\})$ agree. From this observation it follows that the two definitions of local degree agree. ■

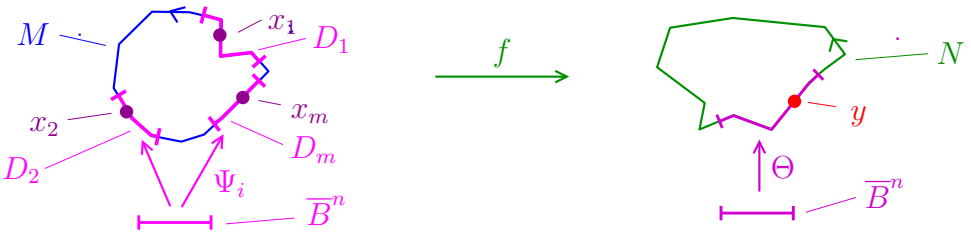
It follows from the Local Degree Lemma 65.2 (3) that the following proposition is a generalization of the S^n-Local-Global Degree Proposition 44.2

Proposition 65.3. (Local-Global Degree Proposition) Let $f\colon M \to N$ be a continuous map between two closed oriented connected n-dimensional topological manifolds M and N. Let $y \in N$ such that $f^{-1}(y)$ consists of finitely many points $x_1, \dots, x_m$. Then all the points $x_1, \dots, x_m$ are nice and we obtain the equality

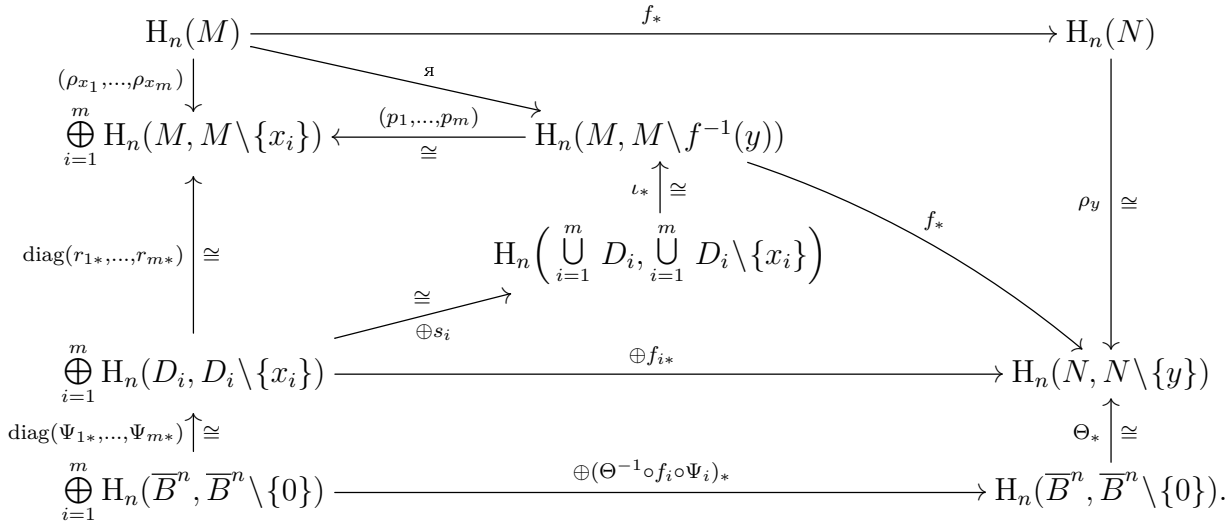
$$\deg(f\colon M \to N) \;=\; \sum_{i=1}^{m} \deg(f, x_i).$$

Proof. Let $n \in \mathbb{N}$, let $f\colon M \to N$ be a continuous map and let $y \in M$ such that $f^{-1}(y)$ consists of finitely many points $x_1, \dots, x_m$.

- It follows easily from the fact that M is Hausdorff that we can choose pairwise disjoint open neighborhoods $U_1, \dots, U_m$ for $x_1, \dots, x_m$. Note that this observation implies already that all the x_i are nice.
- We pick an orientation-preserving embedding $\Theta\colon \overline{B}^n \to N$ with $\Theta(0) = y$.
- For $i = 1, \dots, m$ we pick orientation-preserving embeddings $\Psi_i\colon \overline{B}^n \to M$ with $\Psi_i(0) = x_i$ and such that $\Psi_i(\overline{B}^n) \subset U_i \cap f^{-1}(\Theta(\overline{B}^n))$.
- For $i = 1, \dots, m$ we write $D_i := \Psi_i(\overline{B}^n)$.



We consider the following diagram:[525]



We make the following clarifications and observations:

(1) The maps r_i, s_i, p_i and ι are the maps induced by the obvious inclusions of pairs of topological spaces. Furthermore the maps $f_i\colon (D_i, D_i \setminus \{x_i\}) \to (N, N \setminus \{y\})$ are given by the restriction of f to D_i.

[525] We use some of the formal notation that we introduced on page 555 for maps out of and into direct sums.

(2) It follows pretty easily from the functoriality of relative homology groups and elementary considerations that the diagram commutes.
(3) It follows easily from the fact that the $\overline{B}^n$ are disjoint open subsets of M and the Path-Component–H_*–Lemma 35.8 that the diagonal map $\bigoplus_{i=1}^{m} s_i$ to the center left is an isomorphism.
(4) It follows from the Excision Theorem 40.2 that the vertical map ι_* and the maps r_{i*} are isomorphisms.
(5) It follows from the fact that the diagram commutes together with (3) and (4) that the horizontal map $(p_1, \ldots, p_m)$ is also an isomorphism.

By the Fundamental Class-Proposition 64.1 (2b) it suffices to prove the following claim:

Claim. We have $$\rho_y(f_*([M])) \;=\; \sum_{i=1}^{m} \deg(f, x_i) \cdot \rho_y([N]) \;\in\; \mathrm{H}_n(N, N \setminus \{y\}).$$

Proof. We verify the claim by performing the following calculation:

$$\begin{aligned}
\rho_y(f_*([M])) &\overset{\text{since the diagram commutes and since the map } \mathrm{diag}(r_{1*}, \ldots, r_{m*}) \text{ is an isomorphism}}{=} \left(\bigoplus_{i=1}^{m} f_{i*} \right) \circ \mathrm{diag}(r_{1*}, \ldots, r_{m*})^{-1} \circ (\rho_{x_1}, \ldots, \rho_{x_m})([M]) \overset{\text{since each } r_{i*} \text{ is an isomorphism}}{=} \\
&= \left(\bigoplus_{i=1}^{m} f_{i*} \right) \circ (r_{1*}^{-1} \circ \rho_{x_1}, \ldots, r_{m*}^{-1} \circ \rho_{x_m})([M]) \\
&\underset{\text{since we always have the equality } \left(\left(\bigoplus_{i=1}^{m} \varphi_i \right) \circ (\psi_1, \ldots, \psi_m) \right)(x) = \sum_{i=1}^{m} (\varphi_i \circ \psi_i)(x)}{=} \sum_{i=1}^{m} (f_i \circ r_i^{-1} \circ \rho_{x_i})([M]) \underset{\text{by the definition of the local degree and since } \Psi_i \text{ and } \Theta \text{ are orientation-preserving}}{=} \sum_{i=1}^{m} \deg(f, x_i) \cdot \rho_y([N]).
\end{aligned}$$

■

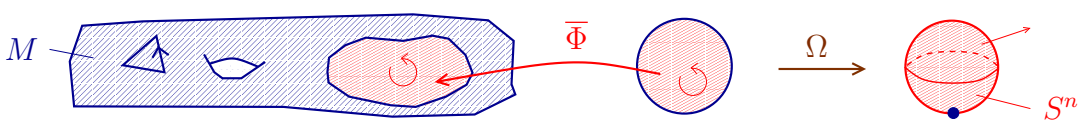
65.3. **Properties of the degree.** The following proposition gives us interesting maps of non-zero degree:

Proposition 65.4. (Degree Realization Proposition) Let $n \in \mathbb{N}$ and let M be a closed oriented connected non-empty n-dimensional topological manifold. Given any $k \in \mathbb{Z}$ there exists a continuous map $f\colon M \to S^n$ with $\deg(f) = k$.

Proof. Let M be a closed oriented connected non-empty n-dimensional topological manifold and let $k \in \mathbb{Z}$. It follows from the Degree-of-Power Lemma 43.2 together with the Suspension-Degree Lemma 43.3 that there exists a continuous map $g\colon S^n \to S^n$ of degree k. It follows from this fact and the Manifold Degree Lemma 65.1 that it suffices to prove the following claim:

Claim. There exists a continuous map $f\colon M \to S^n$ of degree $+1$.

Proof. We pick an orientation-preserving embedding $\Phi\colon \overline{B}^n \to M$. It follows fairly easily from the Compact-Hausdorff Proposition 2.10 (3) together with the Compact Image Lemma 2.8 and the Quotient Topology-Properties Proposition 6.2 (3), that the induced map $\overline{\Phi}\colon \overline{B}^n/S^{n-1} \to M/(M \setminus \Phi(B^n))$ is a homeomorphism. We denote by $\Omega\colon \overline{B}^n/S^{n-1} \to S^n$ the explicit homeomorphism from the Ball Quotient-Sphere Lemma 6.5.



We consider the map $f\colon M \to S^n$ that is given by

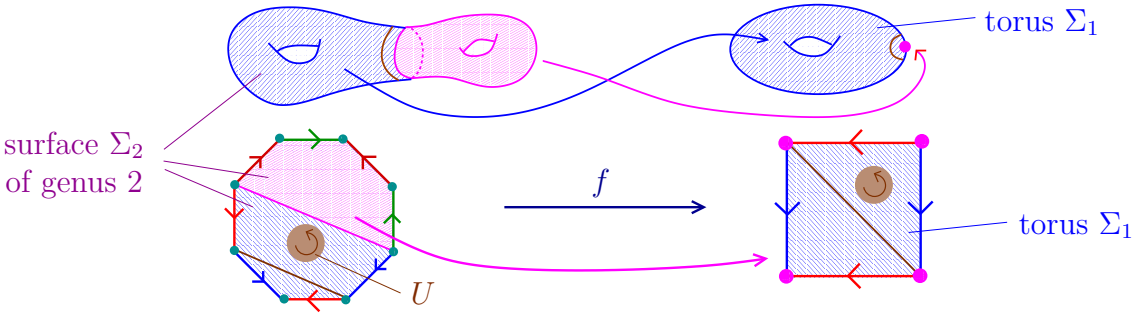
$$M \longrightarrow M/(M \setminus \Phi(B^n)) \xrightarrow{\overline{\Phi}^{-1}} \overline{B}/S^{n-1} \xrightarrow{\Omega} S^n.$$
$$\underbrace{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}_{=:f}$$

It follows almost immediately from the Local-Global Degree Proposition 65.3 together with the Local Degree Lemma 65.2 (2) that f is a map of degree $+1$. ■

In the following discussion, given $g \in \mathbb{N}_0$, we denote as usual by Σ_g the surface of genus g. We move on to our next list of interesting examples of continuous maps of non-zero degree.

Lemma 65.5. (Larger-to-Smaller Genus Lemma) Let $k \in \mathbb{N}_0$. For every $g \geq k$ there exists a degree-one map from Σ_g to Σ_k.

Proof. We prove the lemma for $g = 2$ and $k = 1$. It should be clear how to modify the discussion for other values of $g \geq k$. Let us consider the map from the surface Σ_2 of genus two to the torus Σ_1 that is shown in the figure below.



In fact in the figure we show the map twice, on top we show a slightly informal description of the map, at the bottom we give a more formal description. Note that the hypotheses of the Local-Global Degree Proposition 65.3 are satisfied. Thus we see that $\deg(f) = 1$. ■

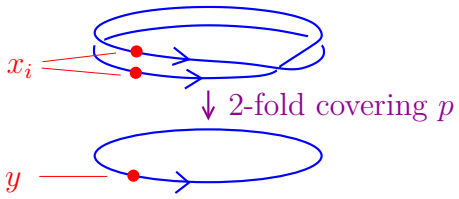
The following proposition shows that for finite coverings of compact oriented topological manifolds the two notions of degree agree.

Proposition 65.6. (Covering Map-Degree Proposition) Let $p\colon \widetilde{M} \to M$ be a finite covering of compact oriented connected non-empty topological manifolds. If p is orientation-preserving, then

$$\deg(p) \;=\; [\widetilde{M} : M].$$

Proof. It turns out that there are two quick proofs for the proposition:

(1) We pick $y \in M$. Note that $p^{-1}(y)$ consists of precisely $[\widetilde{M} : M]$ points and at each point in $p^{-1}(y)$ the projection is an orientation-preserving local homeomorphism. Thus each local degree is $+1$ and the desired statement follows from the Local-Global Degree Proposition 65.3.



(2) We denote by $p^*\colon \mathrm{H}_n(M,\partial M) \to \mathrm{H}_n(\widetilde{M},\partial\widetilde{M})$ the transfer map that we introduced on page 721, then we see that

$$p_*([\widetilde{M}]) \underset{\text{by the Fundamental Class-Covering Proposition 65.6}}{=} p_*(p^*([M])) \underset{\text{by the Transfer–Projection Proposition 57.2}}{=} [\widetilde{M} : M] \cdot [M].$$

■

We continue with the following rather remarkable proposition:

Proposition 65.7. (Degree One-Epimorphism Proposition) Let $f\colon M \to N$ be a continuous map between compact oriented connected non-empty n-dimensional topological manifolds such that $f(\partial M) \subset \partial N$. If $\deg(f) = \pm 1$, then $f_*\colon \pi_1(M) \to \pi_1(N)$ is an epimorphism.

The statement of the Degree One-Epimorphism Proposition 65.7 is somewhat surprising, in so far as it takes as input fairly simple information, namely the induced map on n-th homology, which is an abelian group, and gives as output information about fundamental groups, which are usually much harder to deal with. Also it is a priori not clear why the homology in the maximal dimension should have any impact on fundamental groups.

Proof. Let $f\colon M \to N$ be a continuous map between compact oriented connected non-empty n-dimensional topological manifolds such that $f(\partial M) \subset \partial N$. We fix a base point $P \in M$ and we write $Q = f(P)$. Furthermore we write $\pi = \pi_1(N,Q)$ and $\Gamma = f_*(\pi_1(M,P))$.

Note that by the Covering Existence Theorem 27.2[526] there exists a path-connected covering $p\colon (\widetilde{N},\widetilde{Q}) \to (N,Q)$ such that $p_*(\pi_1(\widetilde{N},\widetilde{Q})) = \Gamma$ and such that the degree of the covering is given by $[\widetilde{N} : N] = [\pi : \Gamma]$. In [**Fri24**] it is shown that $\widetilde{N}$ is a topological manifold and by the Topological Orientation-Action Proposition 62.2 (3) we know that we can equip $\widetilde{N}$ with an orientation such that the projection $p\colon \widetilde{N} \to N$ is orientation-preserving. Since p is a local homeomorphism we have $\partial\widetilde{N} = p^{-1}(\partial N)$. Since by design we know that $p_*(\pi_1(\widetilde{N},\widetilde{Q})) = \Gamma = f_*(\pi_1(M,P))$ we obtain from the Map Lifting Criterion 27.1 that the map $f\colon (M,P) \to (N,Q)$ lifts to a continuous map $\widetilde{f}\colon (M,P) \to (\widetilde{N},\widetilde{Q})$. We can summarize the situation in the following simple commutative diagram

$$\begin{array}{ccc} & & \widetilde{N} \\ & \overset{\widetilde{f}}{\nearrow} & \Big\downarrow{\scriptstyle p}\ \text{covering of degree } [\pi\colon\Gamma] \\ M & \xrightarrow{f} & N. \end{array}$$

First suppose that Γ is a subgroup of π of infinite index. It follows immediately from $[\widetilde{N} : N] = [\pi : \Gamma]$ and the Covering–Basics Lemma 14.6 (4) that $\widetilde{N}$ is non-compact. Therefore we obtain from the Manifold-Top-H_*-Theorem 63.1 that $\mathrm{H}_n(\widetilde{N}) = 0$. But this

[526]Here we use that according to the discussion on pages 360 any topological manifold is locally simply connected.

shows that $f_* = p_* \circ \widetilde{f}_* \colon \mathrm{H}_n(M) \to \mathrm{H}_n(N)$ factors through the trivial group. Hence $\deg(f) = 0$. But this is a contradiction to our hypothesis that $\deg(f) = \pm 1$.

So now we know that Γ is necessarily a finite-index subgroup of π. It follows from $[\widetilde{N} : N] = [\pi : \Gamma]$ and the Covering–Basics Lemma 14.6 (4) that $\widetilde{N}$ is compact. Therefore, by the above we now know that $\widetilde{N}$ is a compact oriented connected n-dimensional topological manifold. We deduce that

$$\pm 1 \;=\; \deg(f) \underset{\uparrow}{=} \deg(\widetilde{f}) \cdot \deg(p) \underset{\uparrow}{=} \deg(\widetilde{f}) \cdot [\widetilde{N} : N] \;=\; \deg(\widetilde{f}) \cdot [\Gamma \colon \pi].$$

by the Manifold Degree Lemma 65.1 — by the Covering Map-Degree Proposition 65.6

This shows that $[\Gamma \colon \pi] = 1$, i.e. $\Gamma = \pi$. ■

The following proposition is a satisfactory complement to the Larger-to-Smaller Genus Lemma 65.5.

Proposition 65.8. (Smaller-to-Larger Genus Proposition) Let $g, k \in \mathbb{N}_0$. If $g < k$, then there is no degree-one map from Σ_g to Σ_k.

Proof. Suppose that there exists a degree-one map from Σ_g to Σ_k. By the Degree One-Epimorphism Proposition 65.7 we obtain an epimorphism $\pi_1(\Sigma_g) \to \pi_1(\Sigma_k)$. In particular we obtain an epimorphism from the abelianization $\pi_1(\Sigma_g)$ onto the abelianization of $\pi_1(\Sigma_k)$. By the Surface-π_1-Proposition 22.1 the former is isomorphic to $\mathbb{Z}^{2g}$ and the latter is isomorphic to $\mathbb{Z}^{2k}$. It follows that $g \geq k$. ■

We conclude this chapter with several questions. Recall that it follows from the Degree Realization Proposition 65.4 that there exists a degree-one map $S^2 \times S^2 \to S^4$. It is natural to ask whether the converse holds. This leads us to the following questions which we will answer later on:

Question 65.9. Does there exist a degree-one map from the 4-dimensional sphere S^4 to $S^2 \times S^2$?

Question 65.10. Does there exist a degree-one map from $S^1 \times S^2$ to $\mathbb{R}\mathrm{P}^3$?

Finally we conclude this chapter with two conjectures that were formulated by Heinz Hopf in 1931. We refer to [**Hau87, Neu53**] for some more information and partial results.

Conjecture 65.11. (Hopf Conjecture) Let M be a closed oriented connected topological manifold and let $f \colon M \to M$ be a continuous map.

(1) If $\deg(f) = \pm 1$, then $f_* \colon \pi_1(M) \to \pi_1(M)$ is an isomorphism.
(2) If $\deg(f) = \pm 1$, then f is a homotopy equivalence.

Part VIII

Singular Cohomology

In this part of the lecture notes we introduce the singular cohomology groups $\mathrm{H}^i(X;G)$ of a topological space X with coefficients in an abelian group G. As groups they are only mildly interesting, since they are fully determined by the ordinary homology groups of X. What makes cohomology interesting is that if we work with cohomology groups with coefficients in a commutative ring R, then we obtain the cup product

$$\cup\colon \mathrm{H}^k(X;R) \times \mathrm{H}^l(X;R) \ \to\ \mathrm{H}^{k+l}(X;R),$$

which gives us the tools to handle many problems, which cannot be dealt with by homology.

This part of the lecture notes is organized as follows:

- In the purely algebraic Chapter 66 we will introduce the algebraic notions of cochain complexes and dual complexes.
- In Chapter 67 we will use the tools from Chapter 66 to introduce the cohomology groups $\mathrm{H}^i(X;G)$ of a topological space X with coefficients in an abelian group G.
- In Chapter 68 we introduce the Kronecker pairing which relates homology to cohomology. We will also introduce cellular cohomology which allows us to calculate cohomology groups of CW-complexes.
- In Chapter 69 we will state and prove the Excision-in-H^*-Theorem 69.1 and the Mayer–Vietoris-H^*-Theorem 69.5 which are the cohomological analogues of the earlier similar sounding statements in homology.
- In Chapter 70 we will introduce the Ext-groups $\mathrm{Ext}(G,H)$ of two abelian groups and we will use them to state and prove the Universal Coefficient-H^*-Theorem 70.14 which allows us to calculate the cohomology $\mathrm{H}^k(X;G)$ in terms of the homology groups of X.
- In Chapter 72, given a commutative ring R, we will introduce the all important cup product
  $$\cup\colon \mathrm{H}^k(M;R) \times \mathrm{H}^l(X;R) \ \to\ \mathrm{H}^{k+l}(X;R)$$
  on the cohomology groups of a topological space X. In fact we will give two definitions: The first definition using the Eilenberg-Zilber Theorem 56.1 is more systematic, the second one is more ad hoc, but more elementary and works well for most purposes. Of course we will show that the two definitions agree.
- In Chapter 73 we will introduce the relative cup product for suitable relative cohomology groups.
- In Chapter 74 we will apply the relative cup product to give lower bounds on the Lusternik-Schnirelmann category.
- In Chapter 75 we will introduce the cap product
  $$\cap\colon \mathrm{H}^k(M;R) \times \mathrm{H}_l(X;R) \ \to\ \mathrm{H}_{l-k}(X;R)$$
  which relates homology and cohomology groups.
- In Chapter 76 we will prove the Cross-Cup-Cap Theorem 76.1 which, as the name suggests, relates the cross product, the cup product and the cap product.
- In Chapter 77 we will prove the Cohomological Künneth Theorem 77.1 which calculates, under mild hypotheses, the cohomology groups of a product $X \times Y$ in terms of the cohomology groups of X and Y.

CHAPTER 66

Cochain complexes and dual complexes

In the following chapter we will introduce the cohomology groups $\mathrm{H}^k(X;G)$ of a topological space X together with a choice of an abelian group G. The idea is quite similar to the idea for homology with coefficients from Chapter 52:

(1) We consider the functor $X \mapsto (\mathrm{C}_*(X), \partial_*)$ from the category $\mathcal{T}op$ of topological spaces to the category $\mathcal{C}h\mathcal{C}pl\chi$ of chain complexes.
(2) We apply a suitable "intermediate'' algebraic functor.
(3) Finally we take homology.

For homology with coefficients the "intermediate" functor we chose was the *covariant* functor $A \mapsto A \otimes G$ for a fixed abelian group G. This idea lead to the homology groups $\mathrm{H}_k(X;G) = \mathrm{H}_k(\mathrm{C}_*(X) \otimes G)$ with coefficients. This time we will use the *contravariant* functor $A \mapsto \mathrm{Hom}(A,G)$ for a fixed abelian group G. This leads us to the cohomology groups $\mathrm{H}^k(X;G)$.

In this chapter we lay the algebraic foundations for carrying out this idea: We will study the Hom-functor $A \mapsto \mathrm{Hom}(A,G)$, we will study cochain complexes (which are basically chain complexes where the boundary maps go upwards instead of downwards) and dual chain complexes. The reader who is familiar with these notions can also move straight to Chapter 67.

66.1. **The Hom-functor.** We start out with the following obvious notation.

Notation. Given a group A and an abelian group G we write

$$\mathrm{Hom}(A,G) \;=\; \{\text{all group homomorphisms } A \to G\}.$$

We make two observations:

(1) The set of homomorphisms $\mathrm{Hom}(A,G)$ has the structure of an abelian group with the sum of two homomorphisms $\varphi, \psi \colon A \to G$ given by the group homomorphism that is defined by $(\varphi + \psi)(a) \coloneqq \varphi(a) + \psi(a)$.
(2) If R is in fact a commutative ring, then for any group A the set of homomorphisms $\mathrm{Hom}(A,R)$ forms an R-module in an obvious way. More precisely, given a homomorphism $\varphi \colon A \to R$ and $r \in R$ we define $r \cdot \varphi$ to be the homomorphism that is given by $a \mapsto r \cdot \varphi(a)$.

We move straight to induced maps:

Definition. Let G be an abelian group and let $f\colon A \to B$ be a group homomorphism. Then f induces a homomorphism

$$\begin{array}{rcl} f^*\colon \operatorname{Hom}(B,G) & \to & \operatorname{Hom}(A,G) \\ (\varphi\colon B \to G) & \mapsto & \left(\begin{array}{rcl} \varphi \circ f\colon A & \to & G \\ a & \mapsto & \varphi(f(a)) \end{array} \right). \end{array}$$

Example. Let G be an abelian group. If A is a subgroup of a group B and if $i\colon A \to B$ is the inclusion map, then for each homomorphism $\varphi\colon B \to G$ the homomorphism $i^*\varphi\colon A \to G$ is just the restriction of φ to $A \subset B$. □

On page 653 we introduced the notion of an additive covariant functor. The following definition gives us the obvious analogue for contravariant functors:

Definition. We say that a contravariant functor $F\colon \mathcal{AbGroup} \to \mathcal{AbGroup}$ from the category $\mathcal{AbGroup}$ of abelian groups to itself is additive if for any two abelian groups A and B and any two homomorphisms $\varphi, \psi\colon A \to B$ we have $F(\varphi+\psi) = F(\varphi) + F(\psi)\colon F(B) \to F(A)$.

With this definition we can now formulate the following analogue of the Tensor-Functor Lemma 52.3.

Lemma 66.1. (Hom–Contravariant Functor Lemma)

(1) Given an abelian group G the maps

$$\begin{array}{rcl} A & \mapsto & \operatorname{Hom}(A,G) \\ (f\colon A \to B) & \mapsto & (f^*\colon \operatorname{Hom}(B,G) \to \operatorname{Hom}(A,G)) \end{array}$$

define an additive *contravariant* functor from the category $\mathcal{Group}$ of groups to the category $\mathcal{AbGroup}$ of abelian groups. In particular we have

$$\begin{array}{rcll} (\mathrm{id}_A)_* & = & \mathrm{id}_{\operatorname{Hom}(A,G)}, & \text{for all groups } A \\ (g \circ f)^* & = & f^* \circ g^* & \text{for all homomorphisms } f\colon A \to B \text{ and } g\colon B \to C. \end{array}$$

(2) If R is a commutative ring, then the above maps $A \mapsto \operatorname{Hom}(A,R)$ and $f \mapsto f^*$ define a contravariant functor from the category $\mathcal{Group}$ of groups to the category $\mathcal{Mod}_R$ of R-modules.

Proof. Both statements follow easily from the definitions. ■

Lemma 66.2. (Abelian Group-Homomorphism Lemma) Let G be an abelian group.

(1) For any $n \in \mathbb{N}_0$ and any abelian group G the natural map

$$\begin{array}{rcl} \operatorname{Hom}(\mathbb{Z}_n, G) & \cong & \ker(G \xrightarrow{\cdot n} G) \\ \varphi & \mapsto & \varphi([1]) \end{array}$$

is an isomorphism of abelian groups.

(2) If F is a free abelian group with basis S, then the homomorphism[527]

$$\begin{array}{rcl} \operatorname{Hom}(F,G) & \to & \{\text{all maps from the set } S \text{ to } G\} \\ (\varphi\colon F \to G) & \mapsto & \left(\begin{array}{rcl} S & \to & G \\ s & \mapsto & \varphi(s) \end{array} \right) \end{array}$$

is an isomorphism of abelian groups.

(3) (a) For any family $\{A_i\}_{i\in I}$ of groups the natural map[528]

$$\begin{array}{rcl} \mathrm{Hom}\left(\bigoplus\limits_{i\in I} A_i, G\right) & \to & \prod\limits_{i\in I} \mathrm{Hom}(A_i, G) \\ f & \mapsto & \prod\limits_{i\in I} f|_{A_i} \end{array}$$

is an isomorphism of abelian groups.

(b) For any group A and any family $\{G_i\}_{i\in I}$ of abelian groups the natural map

$$\begin{array}{rcl} \mathrm{Hom}\left(A, \prod\limits_{i\in I} G_i\right) & \to & \prod\limits_{i\in I} \mathrm{Hom}(A, G_i) \\ f & \mapsto & \prod\limits_{i\in I} \text{the composition of } f \text{ with the natural projection} \end{array}$$

is an isomorphism of abelian groups.

Proof.

(1) We leave the elementary proof of this statement to the reader.

(2) This statement follows immediately from the definition of a free abelian group.

(3) (a) Note that for each $j \in I$ we have, by definition of a direct sum, a natural inclusion $\iota_j\colon A_j \to \bigoplus\limits_{i\in I} A_i$. We get an induced map $\iota_j^*\colon \mathrm{Hom}\left(\bigoplus\limits_{i\in I} A_i, G\right) \to \mathrm{Hom}(A_j, G)$. By definition of the product these homomorphisms give rise to a homomorphism

$$\mathrm{Hom}\left(\bigoplus_{i\in I} A_i, G\right) \to \prod_{j\in I} \mathrm{Hom}(A_j, G).$$

Using the Coproduct-of-Abelian Groups Lemma 18.1 one can easily verify that this map is an isomorphism.

(b) This statement follows easily from the definitions. ■

Example.

(1) Let $m, n \in \mathbb{N}$. It follows from the Abelian Group-Homomorphism Lemma 66.2 (1) and some elementary arguments that $\mathrm{Hom}(\mathbb{Z}_n, \mathbb{Z}_m) \cong \mathbb{Z}_{\gcd(n,m)}$.

(2) Let S be a set. In the Free Abelian Group–Existence Lemma 18.4 we showed that $\mathbb{Z}^{(S)}$ is a free abelian group on the set S. It follows from the Abelian Group-Homomorphism Lemma 66.2 (2) that the homomorphism

$$\begin{array}{rcl} \mathrm{Hom}(\mathbb{Z}^{(S)}, G) & \to & \{\text{all maps from the set } S \text{ to } G\} \\ (\varphi\colon \mathbb{Z}^{(S)} \to G) & \mapsto & \begin{pmatrix} S & \to & G \\ s & \mapsto & \varphi(s) \end{pmatrix} \end{array}$$

is an isomorphism.

(3) Let A be a finitely generated abelian group. Recall that it follows from the Abelian Group-Classification Theorem 18.11 that A is the direct sum of finitely many cyclic groups. Using the Abelian Group-Homomorphism Lemma 66.2 we now fully understand $\mathrm{Hom}(A, G)$.

(4) Let F be a free abelian group.

(a) If F is finitely generated, then it follows from the Abelian Group-Homomorphism Lemma 66.2 (1) and (3) that $\mathrm{Hom}(F, \mathbb{Z})$ is again a free abelian group.

[527] Note that the group structure on G turns both sides of the statement into a group, so the given map clearly respects this group structure, i.e. the given map is a homomorphism.

[528] This statement can be viewed as the analogue of the Tensor-Direct Sum Lemma 51.6.

(b) If $F = \mathbb{Z}^{(\mathbb{N})}$, then it follows from the Abelian Group-Homomorphism Lemma 66.2 (1) and (3) that $\mathrm{Hom}(\mathbb{Z}^{(\mathbb{N})}, \mathbb{Z}) \cong \prod_{i\in\mathbb{N}} \mathbb{Z}$. Note that this group is *not* a free abelian group. This surprising statement is known as Baer's Theorem and was first proved by Reinhold Baer [**Bae37**, Theorem 12.4]. We refer to [**Sch08**] and [**Rot09**, Theorem 4.17] for more recent accounts. □

The following lemma is an analogue of the Matrix–Tensor-with-Ring Lemma 51.8.

Lemma 66.3. (Matrix–Hom-to-Ring Lemma) Let R be a commutative ring.

(1) Let $k \in \mathbb{N}_0$. The map
$$\begin{array}{rccc} \Theta_k\colon & \mathrm{Hom}(\mathbb{Z}^k, R) & \to & R^k \\ & (f\colon \mathbb{Z}^k \to R) & \mapsto & (f(e_1), \ldots, f(e_k)) \end{array}$$
is an isomorphism of R-modules.

(2) Let A be an $(m \times n)$-matrix over $\mathbb{Z}$. We denote by $\varphi\colon \mathbb{Z} \to R$ the ring homomorphism that is given by $\varphi(k) = k \cdot 1_R$. With this notation the following diagram commutes:
$$\begin{array}{ccc} \mathrm{Hom}(\mathbb{Z}^m, R) & \xrightarrow[\cong]{\Theta_m} & R^m \\ \downarrow{\scriptstyle (v\mapsto A\cdot v)^*} & & \downarrow{\scriptstyle w\mapsto \varphi(A^T)\cdot w} \\ \mathrm{Hom}(\mathbb{Z}^n, R) & \xrightarrow[\cong]{\Theta_n} & R^n. \end{array}$$

Proof.

(1) As in the case of the Abelian Group-Homomorphism Lemma 66.2 this statement follows almost immediately from the fact that $\mathbb{Z}^k$ is a free abelian group on the standard basis $e_1, \ldots, e_k$.

(2) We introduce some notation:
- We denote by $\widetilde{e}_1, \ldots, \widetilde{e}_k$ the standard R-basis of R^k.
- We denote by $\pi_i\colon \mathbb{Z}^m \to \mathbb{Z}$ the projection onto the i-th coordinate.
- Let $\Psi\colon \mathbb{Z}^m \to \mathbb{Z}^n$ be the map that is given by multiplication by $A = (a_{ij})$.

For $i \in \{1, \ldots, m\}$ we calculate that we have the following equality in R^n:
$$\begin{array}{lllll} (\Theta_n \circ \Psi^* \circ \Theta_m^{-1})(\widetilde{e}_i) & \overset{\text{since } \Theta_m(\pi_i) = \widetilde{e}_i}{=} & \Theta_n(\Psi^*(\pi_i)) & \overset{\text{definition of } \Theta_n}{=} & \sum_{j=1}^n (\Psi^*\pi_i)(e_j)\cdot\widetilde{e}_j \\ & = \sum_{j=1}^n \pi_i\Big(\sum_{k=1}^n a_{kj}\cdot e_k\Big)\cdot\widetilde{e}_j & & = \sum_{j=1}^n \varphi(a_{ij})\cdot\widetilde{e}_j & \\ & \overset{\text{definition of } \Psi}{=} & & \overset{\substack{\text{by definition of } \pi_i \text{ and since for} \\ a\in\mathbb{Z},\ v\in R^n \text{ we have } a\cdot v = \varphi(a)\cdot v}}{=} & \end{array}$$
$$\begin{array}{l} \overset{\text{definition of } \Psi^*}{=} \sum_{j=1}^n \pi_i(\Psi(e_j))\cdot\widetilde{e}_j \\ = \varphi(A^T)\cdot\widetilde{e}_i. \\ \overset{\text{definition of } A^T}{=} \end{array}$$
■

The following lemma is similar to the Tensor-Split-SES Lemma 52.6.

Lemma 66.4. (Hom-Split-SES Lemma) Let
$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$$
be a short exact sequence of abelian groups and let G be an abelian group. If the short exact sequence splits[529], then the dual sequence
$$0 \to \mathrm{Hom}(C, G) \xrightarrow{p^*} \mathrm{Hom}(B, G) \xrightarrow{i^*} \mathrm{Hom}(A, G) \to 0$$

is also exact.

Proof. We start out with the following claim:

Claim. Let $0 \to F \xrightarrow{\alpha} G \xrightarrow{\beta} H \to 0$ be a sequence of homomorphisms between abelian groups. The following two statements are equivalent:

(1) The sequence is exact and the sequence splits.
(2) We have $\beta \circ \alpha = 0$ and there exist homomorphisms $\sigma\colon G \to F$ and $\tau\colon H \to G$ such that $\sigma \circ \alpha = \mathrm{id}_F$, $\beta \circ \tau = \mathrm{id}_H$ and $\alpha \circ \sigma + \tau \circ \beta = \mathrm{id}_G$.

Proof. The (1)⇒(2) Statement follows almost immediately from the Splitting Lemma 37.2. Conversely the (2)⇒(1)-Statement can easily be verified by hand. ⊞

Let

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$$

be a short exact sequence of abelian groups that splits and let G be an abelian group. We pick homomorphisms $\sigma\colon B \to A$ and $\tau\colon C \to B$ as in the claim. By the Hom–Contravariant Functor Lemma 66.1 we know that $D \mapsto \mathrm{Hom}(D, G)$ is an *additive contravariant* functor. It follows that $\alpha^* \circ \sigma^* = \mathrm{id}_{\mathrm{Hom}(A,G)}$, $\tau^* \circ \beta^* = \mathrm{id}_{\mathrm{Hom}(C,G)}$ and $\sigma^* \circ \alpha^* + \beta^* \circ \tau^* = \mathrm{id}_{\mathrm{Hom}(B,G)}$. Note that this shows that the two homomorphisms $\tau^*\colon \mathrm{Hom}(B, G) \to \mathrm{Hom}(C, G)$ and $\sigma^*\colon \mathrm{Hom}(B, G) \to \mathrm{Hom}(A, G)$ have the desired properties for the sequence

$$0 \to \mathrm{Hom}(G, C) \xrightarrow{p^*} \mathrm{Hom}(B, G) \xrightarrow{i^*} \mathrm{Hom}(A, G) \to 0.$$

It follows from the claim that this sequence is exact and that it splits. ■

66.2. **Cochain complexes.** Now we introduce the notion of a cochain complex and of its cohomology groups. A cochain complex is exactly the same as a chain complex except that the boundary map now increases the degree by one instead of lowering it by one.

Definition.

(1) A cochain complex is a sequence

$$\leftarrow D^n \xleftarrow{\delta_{n-1}} D^{n-1} \xleftarrow{\delta_{n-2}} \dots \xleftarrow{\delta_1} D^1 \xleftarrow{\delta_0} D^0 \leftarrow 0$$

of homomorphisms between abelian groups such that for all $i \in \mathbb{N}_0$ we have $\delta_{i+1} \circ \delta_i = 0$. We refer to the maps δ_* as coboundary maps.

(2) We define the n-th cohomology group of a cochain complex (D^*, δ_*) to be[530]

$$\mathrm{H}^n(D) \coloneqq \frac{\ker(\delta_n\colon D^n \to D^{n+1})}{\mathrm{im}(\delta_{n-1}\colon D^{n-1} \to D^n)}.$$

Now we will see that most of the definitions and algebraic results for chain complexes can be reinterpreted with minuscule changes for cochain complexes. We start out with the following definition which is modelled on the definition of a chain map on page 454.

Definition. A cochain map $f\colon C^* \to D^*$ between two cochain complexes consists of a family $\{f_n\colon \mathrm{C}^n \to D^n\}_{n \in \mathbb{N}_0}$ of homomorphisms that commute with the coboundary maps,

[529] Recall that by the Free Abelian Group–Splitting Lemma 18.5 this happens if C is free abelian.
[530] Evidently it follows from $\delta_n \circ \delta_{n-1} = 0$ that $\mathrm{im}(\delta_{n-1}\colon D^{n-1} \to D^n) \subset \ker(\delta_n\colon D^n \to D^{n+1})$.

i.e. we demand that the following diagram commutes:

$$\begin{array}{ccccccccc}
\xleftarrow{\delta_{n+1}} & C^{n+1} & \xleftarrow{\delta_n} & C^n & \xleftarrow{\delta_{n-1}} & C^{n-1} & \xleftarrow{\delta_{n-2}} & \\
& \downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & \\
\xleftarrow[\delta_{n+1}]{} & D^{n+1} & \xleftarrow[\delta_n]{} & D^n & \xleftarrow[\delta_{n-1}]{} & D^{n-1} & \xleftarrow[\delta_{n-2}]{} &
\end{array}$$

The above definition naturally leads us to the following definition:

Definition. We refer to category $\mathcal{CoChCplx}$ that is given by

$$\begin{aligned} \mathrm{Ob}(\mathcal{CoChCplx}) &:= \text{all cochain chain complexes,} \\ \mathrm{Mor}(C^*, D^*) &:= \text{all cochain maps from } C^* \text{ to } D^*. \end{aligned}$$

and with morphisms with the obvious composition of morphism as the category of cochain complexes.

The following lemma is the obvious analogue of the Chain Complex-H_*-Functor Lemma 35.1.

Lemma 66.5. (Cochain Complex-H^*-Functor Lemma)

(1) If $f\colon C^* \to D^*$ is a cochain map between cochain complexes C^* and D^*, then the map

$$\begin{array}{rcl} f^*\colon \mathrm{H}^n(C^*) & \to & \mathrm{H}^n(D^*) \\ [c] & \mapsto & [f_n(c)] \end{array}$$

is a well-defined homomorphism.

(2) Given $n \in \mathbb{N}_0$ the maps

$$\begin{array}{rcl} C^* & \mapsto & \mathrm{H}^n(C^*) \\ (f\colon C^* \to D^*) & \mapsto & (f^*\colon \mathrm{H}^n(C^*) \to \mathrm{H}^n(D^*)) \end{array}$$

define a covariant functor from the category $\mathcal{CoChCplx}$ of cochain complexes to the category $\mathcal{AbGroup}$ of abelian groups.

Proof. The proof is basically identical to the proof of the Cochain Complex-H^*-Lemma **??** and of the Chain Complex-H_*-Functor Lemma 35.1. ■

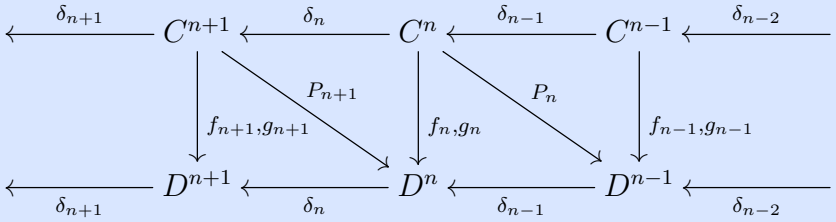
We continue with the following definition which is modelled on the definitions on pages 470 and 471.

Definition.

(1) Let $f, g\colon C^* \to D^*$ be two cochain maps between two cochain complexes C^* and D^*. A cochain homotopy $P = \{P_n\}_{n\in\mathbb{N}_0}$ between the cochain maps f and g consists of a family $\{P_n\colon \mathrm{C}^n \to D^{n-1}\}_{n\in\mathbb{N}_0}$ of homomorphisms such that for each $n \in \mathbb{N}_0$ we have

$$\delta_{n-1} \circ P_n + P_{n+1} \circ \delta_n \;=\; f_n - g_n.$$

The maps are illustrated in the following diagram:



If there exists a cochain homotopy between cochain maps f and g, then we say that f and g are cochain homotopic.[531]

(2) A cochain map $f\colon C^* \to D^*$ between two cochain complexes C^* and D^* is called a cochain homotopy equivalence if there exists a cochain map $g\colon D^* \to C^*$ such that $f \circ g$ is cochain homotopic to id_{C^*} and $g \circ f$ is cochain homotopic to id_{D^*}.

The following lemma is the cohomological analogue of the Chain Homotopy–H_*-Lemma 36.1 and the Chain Homotopy Equivalence-H_*-Corollary 36.4.

Lemma 66.6. (Cochain Homotopy-H^*-Lemma)

(1) If two cochain maps are cochain homotopic, the they induce the same map on cohomology groups.

(2) Every cochain homotopy equivalence induces an isomorphism of cohomology groups.

Proof. Statement (1) is proved the same way as the Chain Homotopy–H_*-Lemma 36.1. The proof of Statement (2) is basically the same as the proof of the Chain Homotopy Equivalence-H_*-Corollary 36.4. ■

One of the most important algebraic constructions in homology theory was that we could associate to a short exact sequence of chain complexes a long exact sequence of homology groups. We have an almost identical construction for short exact sequences of cochain complexes and their cohomology groups. We will discuss this construction in detail:

Definition. Let

$$0 \to C^* \xrightarrow{f} B^* \xrightarrow{g} A^* \to 0$$

be a short exact sequence of cochain complexes. This hypothesis implies that for each $n \in \mathbb{N}_0$ we have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C^{n+1} & \xrightarrow{f_{n+1}} & B^{n+1} & \xrightarrow{g_{n+1}} & A^{n+1} & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle \delta_n} & & \uparrow{\scriptstyle \delta_n} & & \uparrow{\scriptstyle \delta_n} & & \\
0 & \longrightarrow & C^n & \xrightarrow{f_n} & B^n & \xrightarrow{g_n} & A^n & \longrightarrow & 0
\end{array}$$

where the horizontal sequences are exact. Now we will define a connecting homomorphism

$$\delta_n\colon \mathrm{H}^n(A) \to \mathrm{H}^{n+1}(C).$$

The definition is almost verbatim the same as the definition on page 483 of the connecting homomorphism in homology. More precisely, we perform the following steps:

(1) Let $z \in \mathrm{H}^n(A)$.

[531] The argument of the Chain Homotopy-Lemma 36.2 shows of course that being cochain homotopic is an equivalence relation on the set of all cochains $C^* \to D^*$.

(2) We pick $\alpha \in A^n$ with $z = [\alpha]$.
(3) By the surjectivity of g_n we have $\alpha = g_n(\beta)$ for some $\beta \in B^n$.
(4) From the commutativity of the diagram and the exactness of the horizontal sequences we see that $\delta_n(\beta) = f_{n+1}(\gamma)$ for some uniquely determined $\gamma \in C^{n+1}$.
(5) We put $\delta_n(z) \coloneqq [\gamma] \in \mathrm{H}^{n+1}(C)$.

The following proposition is now the obvious analogue of the Connecting Homomorphism Lemma 37.3 and the LES-Proposition 37.4:

Proposition 66.7. (Cohomology–LES Proposition)
(1) The connecting homomorphisms are well-defined and they are indeed homomorphisms.[532]
(2) The connecting homomorphisms are natural in the sense that commuting diagrams of short exact sequences of cochain complexes give rise to commuting diagrams of cohomology groups involving the connecting homomorphisms.
(3) If
$$0 \to C^* \xrightarrow{f} B^* \xrightarrow{g} A^* \to 0$$
is a short exact sequence of cochain complexes, then
$$\ldots \to \mathrm{H}^n(C) \xrightarrow{f^*} \mathrm{H}^n(B) \xrightarrow{g^*} \mathrm{H}^n(A) \xrightarrow{\delta_n} \mathrm{H}^{n+1}(C) \xrightarrow{f^*} \mathrm{H}^{n+1}(B) \xrightarrow{g^*} \ldots$$
forms an exact sequence.

Proof. The proof is almost identical to the proofs of the Connecting Homomorphism Lemma 37.3 and the LES-Proposition 37.4, we just have to flip the indices. We leave it to the reader to fill in the details. ∎

Finally the following lemma says that cohomology commutes with products. This lemma can be viewed as a cohomological analogue of the Direct Sum-H_*-Lemma 35.2:

Lemma 66.8. (Product–H^*-Lemma) If $\{C_a\}_{a\in A}$ is a family of cochain complexes, then the natural projections $\prod_{a\in A} C_a \to C_b$, $b \in A$ (which come from the definition of a product) induce for each $n \in \mathbb{N}_0$ an isomorphism[533]
$$\mathrm{H}^n\Big(\prod_{a\in A} C_a\Big) \xrightarrow{\cong} \prod_{a\in A} \mathrm{H}^n(C_a).$$

Proof. Let $n \in \mathbb{N}_0$. Note that for each $b \in A$ the projection $\prod_{a\in A} C_a \to C_b$ induces a homomorphism
$$\mathrm{H}^n\Big(\prod_{a\in A} C_a\Big) \to \mathrm{H}^n(C_b).$$
It follows from the definition of a product, see page 101, that we get a well-defined homomorphism
$$\mathrm{H}^n\Big(\prod_{a\in A} C_a\Big) \to \prod_{b\in A} \mathrm{H}^n(C_b).$$
We leave it to the reader to verify that this map is in fact an isomorphism. ∎

66.3. **Dual cochain complexes.** Now we will present the arguably most important source of cochain complexes:

Definition. Let
$$(C_*, \partial_*)\colon \quad \to C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} \ldots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \to 0$$

[532] Furthermore, if the cochain complexes are cochain complexes over a commutative ring R, then the connecting homomorphism is in fact an R-module homomorphism.
[533] The same statement also holds with direct sums instead of products. The reason we formulated it for direct products is that that is what we will study later on.

be a chain complex and let G be an abelian group. For each $n \in \mathbb{N}_0$ we write

$$\delta_n \coloneqq \partial_{n+1}^* \colon \operatorname{Hom}(C_n, G) \to \operatorname{Hom}(C_{n+1}, G).$$

We obtain the following cochain complex[534]

$$\leftarrow \operatorname{Hom}(C_{n+1}, G) \xleftarrow{\delta_n} \operatorname{Hom}(C_n, G) \xleftarrow{\delta_{n-1}} \dots \xleftarrow{\delta_1} \operatorname{Hom}(C_1, G) \xleftarrow{\delta_0} \operatorname{Hom}(C_0, G) \leftarrow 0.$$

We refer to it as cochain complex dual to (C_*, ∂_*) and we denote it by $\operatorname{Hom}(C_*, G)$. We define the n-th cohomology group of the chain complex (C_*, ∂_*) with G-coefficients as the cohomology of the cochain complex $(\operatorname{Hom}(C_*, G), \delta_*)$, i.e. we define

$$\mathrm{H}^n(C; G) \coloneqq \frac{\ker(\delta_n \colon \operatorname{Hom}(C_n, G) \to \operatorname{Hom}(C_{n+1}, G))}{\operatorname{im}(\delta_{n-1} \colon \operatorname{Hom}(C_{n-1}, G) \to \operatorname{Hom}(C_n, G))}.$$

Remark. If R is a commutative ring and if (C_*, ∂_*) is a chain complex, then it follows immediately from the Hom–Contravariant Functor Lemma 66.1 that the corresponding cohomology groups $\mathrm{H}^n(C; R)$ come naturally with the structure of R-modules. □

The following lemma is related in spirit to the Functor-applied-to-Chain Complex Lemma 52.2.

Lemma 66.9. (Dual Cochain Map Lemma) Let G be an abelian group.

(1) Let $f = \{f_n \colon C_n \to D_n\}_{n \in \mathbb{N}_0}$ be a chain map between chain complexes C_* and D_*, then the maps

$$\begin{aligned} f^* \colon \operatorname{Hom}(D_n, G) &\to \operatorname{Hom}(C_n, G) \\ (\varphi \colon D_n \to G) &\mapsto \underbrace{\left(C_n \xrightarrow{f_n} D_n \xrightarrow{\varphi} G\right)}_{=\varphi \circ f_n = f_n^*(\varphi)} \end{aligned}$$

form a cochain map.

(2) The maps

$$\begin{aligned} C_* &\mapsto \operatorname{Hom}(C_*, G) \\ f &\mapsto f^* \end{aligned}$$

define a *contravariant* functor from the category $\mathcal{ChCplx}$ of chain complexes to the category $\mathcal{CoChCplx}$ of cochain complex.

(3) (a) Let $f, g \colon C_* \to D_*$ be two chain maps. If they are chain homotopic, then the induced cochain maps $f^*, g^* \colon \operatorname{Hom}(D_*, G) \to \operatorname{Hom}(C_*, G)$ are cochain homotopic.

(b) Let $f \colon C_* \to D_*$ be a chain map. If f is a chain homotopy equivalence, then $f^* \colon \operatorname{Hom}(D_*, G) \to \operatorname{Hom}(C_*, G)$ is a cochain homotopy equivalence.

Proof.

(1) Let $f = \{f_n \colon C_n \to D_n\}_{n \in \mathbb{N}_0}$ be a chain map. This means that for each $n \in \mathbb{N}_0$ we have

$$f_{n-1} \circ \partial_n = \partial_n \circ f_n.$$

It follows from the Hom–Contravariant Functor Lemma 66.1 that

$$\underbrace{\partial_n^*}_{=\delta_{n-1}} \circ f_{n-1}^* = f_n^* \circ \underbrace{\partial_n^*}_{=\delta_{n-1}}.$$

We have thus shown that f^* is a cochain map.

(2) This statement follows easily from the definitions.

[534] It follows from the Hom–Contravariant Functor Lemma 66.1 that $\delta_n \circ \delta_{n-1} = \partial_{n+1}^* \circ \partial_n^* = (\partial_n \circ \partial_{n+1})^* = 0$, i.e. this is indeed a cochain complex.

(3) (a) Let $\{P_n\colon C_n \to D_{n+1}\}_{n\in\mathbb{N}_0}$ be a chain homotopy $P = \{P_n\}_{n\in\mathbb{N}_0}$ between f and g. Recall that this means that for each $n \in \mathbb{N}_0$ we have

$$\partial_{n+1} \circ P_n + P_{n-1} \circ \partial_n \;=\; g_n - f_n.$$

We set $Q_n \coloneqq P^*_{n-1}\colon \operatorname{Hom}(D_n, G) \to \operatorname{Hom}(C_{n-1}, G)$. By the Hom–Contravariant Functor Lemma 66.1 we know that $A \mapsto \operatorname{Hom}(A, G)$ is an *additive contravariant* functor. This implies that

$$\underbrace{P^*_n}_{=Q_{n+1}} \circ \underbrace{\partial^*_{n+1}}_{=\delta_n} + \underbrace{\partial^*_n}_{=\delta_{n-1}} \circ \underbrace{P^*_{n-1}}_{=Q_n} \;=\; g^*_n - f^*_n.$$

This shows that the maps $Q_n, n \in \mathbb{N}_0$ form a cochain homotopy.

(b) This statement follows easily from (a). ∎

The following lemma plays the role of the Homology-Coefficients-Functor Lemma 52.8. In a nutshell it says that "cohomology is covariant in the coefficients".

Lemma 66.10. (Cohomology-Coefficients-Functor Lemma) Let $f\colon A \to B$ be a homomorphism of abelian groups.

(1) Given a chain complex C_* and $n \in \mathbb{N}_0$ the maps

$$\begin{array}{rcl} A & \mapsto & \mathrm{H}^n(C_*; A) \\ (f\colon A \to B) & \mapsto & \left(\begin{array}{rcl} f_*\colon \mathrm{H}^n(C_*; A) & \to & \mathrm{H}^n(C_*; B) \\ [\varphi\colon C_n \to A] & \mapsto & [f \circ \varphi\colon C_n \to B] \end{array} \right) \end{array}$$

form a covariant functor from the category *AbGroup* of abelian groups to the category *AbGroup* of abelian groups.

(2) Given a group homomorphism $f\colon A \to B$ the induced maps f_* define a natural transformation from the contravariant functor $C_* \mapsto \mathrm{H}^n(C_*; A)$ to the contravariant functor $C_* \mapsto \mathrm{H}^n(C_*; B)$.

Proof. Let C_* be a chain complex and let $f\colon A \to B$ be a homomorphism of abelian groups. One can easily verify that the maps

$$\begin{array}{rcl} f_*\colon \operatorname{Hom}(C_n, A) & \to & \operatorname{Hom}(C_n, B) \\ \varphi & \mapsto & f \circ \varphi \end{array}$$

define a cochain map. The two statements of the lemma now follow basically immediately from the definitions. ∎

CHAPTER 67

The cohomology groups of a topological space

In this chapter we finally introduce the cohomology groups $\mathrm{H}^k(X;G)$ of a topological space X together with a choice of an abelian group G. As we mentioned in the beginning of Chapter 66, the idea is similar to the idea for homology with coefficients:

(1) We consider the functor $X \mapsto (\mathrm{C}_*(X), \partial_*)$ from the category $\mathcal{Top}$ of topological spaces to the category $\mathcal{ChCplx}$ of chain complexes.
(2) We fix an abelian group G. But instead of applying the covariant functor $A \mapsto A \otimes G$ we now apply the contravariant functor $A \mapsto \mathrm{Hom}(A, G)$.
(3) Finally we take (co-) homology of the resulting (co-) chain complex.

The effort to introduce homology with coefficients only produced moderately interesting results. Initially it might appear that this new idea is not particularly promising either. But we will eventually see that cohomology is much more exciting, since it has a product structure which turns cohomology into a ring.

The study of cohomology will keep us busy for a very long time. For many mathematicians it takes a long time till they feel fully comfortable with cohomology groups since elements of cohomology groups are much less visual than elements of homology groups. Perhaps, before we embark on our new adventure it is perhaps good to quote the great John von Neumann:



In mathematics you don't understand things.
You just get used to them.

67.1. **The definition of singular cohomology groups.** Now we apply the somewhat dry algebra from the last chapter to our favorite chain complex, namely the singular chain complex of a pair of topological spaces:

Definition. Let (X, A) be a pair of topological spaces and let G be an abelian group.

(1) For each $n \in \mathbb{N}_0$ we write

$$\mathrm{C}^n(X, A; G) := \mathrm{Hom}(\mathrm{C}_n(X, A), G) \text{ and } \delta_n := \partial_{n+1}^* \colon \mathrm{C}^n(X, A; G) \to \mathrm{C}^{n+1}(X, A; G).$$

We use the following notation:

(a) We refer to the elements of $\mathrm{C}^n(X, A; G)$ as singular n-cochains, or shorter, as n-cochain.
(b) We refer to the elements in the kernel of $\delta_n \colon \mathrm{C}^n(X, A; G) \to \mathrm{C}^{n+1}(X, A; G)$ as singular n-cocycles.

(c) We refer to the elements in the image of $\delta_{n-1}\colon \mathrm{C}^{n-1}(X,A;G) \to \mathrm{C}^n(X,A;G)$ as singular n-coboundaries.

When $n \in \mathbb{N}_0$ is understood then sometimes we just write δ instead of δ_n.

(2) By the discussion on page 861 we obtain a cochain complex

$$\ldots \leftarrow \mathrm{C}^{n+1}(X,A;G) \xleftarrow{\delta_n} \mathrm{C}^n(X,A;G) \xleftarrow{\delta_{n-1}} \mathrm{C}^{n-1}(X,A;G) \leftarrow \ldots \xleftarrow{\delta_0} C^0(X,A;G) \leftarrow 0.$$

We refer to it as the singular cochain complex of (X,A) with G-coefficients. We define the n-th singular cohomology group of (X,A) with G-coefficients[535] as follows:

$$\mathrm{H}^n(X,A;G) \coloneqq \mathrm{H}^n(\mathrm{C}_*(X,A);G) = \frac{\ker(\delta_n\colon \mathrm{C}^n(X,A;G) \to \mathrm{C}^{n+1}(X,A;G))}{\mathrm{im}(\delta_{n-1}\colon \mathrm{C}^{n-1}(X,A;G) \to \mathrm{C}^n(X,A;G))}.$$

(3) If $A = \varnothing$ then we just write[536] $\mathrm{C}^n(X;G) = \mathrm{C}^n(X,\varnothing;G)$ and similar for cohomology groups.

Before we try to get a feeling for these cohomology groups, let us first get an important formality out to the way: The following definition show that continuous maps between pairs of topological spaces induce cochain maps and homomorphisms on cohomology groups:

Definition. Let G be an abelian group and let $f\colon (X,A) \to (Y,B)$ be a continuous map between pairs of topological spaces. By the C_*–Functor Lemma 35.5 we know that f induces a chain map $f_*\colon \mathrm{C}_*(X,A) \to \mathrm{C}_*(Y,B)$. It follows from the Dual Cochain Map Lemma 66.9 (1) that we get an induced cochain map

$$f^*\colon \mathrm{C}^*(Y,B;G) \to \mathrm{C}^*(X,A;G)$$

and it follows from the Cochain Complex-H^*-Functor Lemma 66.5 (1) that for each $n \in \mathbb{N}_0$ we get an induced homomorphism

$$f^*\colon \mathrm{H}^n(Y,B;G) \to \mathrm{H}^n(X,A;G).$$

Example. Let X be a topological space, let A be a subspace, let G be an abelian group and let $n \in \mathbb{N}_0$.

(1) The inclusion $i\colon A \to X$ induces a homomorphism $i^*\colon \mathrm{H}^n(X;G) \to \mathrm{H}^n(A;G)$ which is sometimes called the *restriction map*.

(2) The natural map $(X,\varnothing) \to (X,A)$ induces a homomorphism $\mathrm{H}^n(X,A;G) \to \mathrm{H}^n(X;G)$ which is denoted by я. □

We have the following analogue of the C_*–Functor Lemma 35.5 and the H_n–Functor Lemma 35.7:

Lemma 67.1. (H^n–Functor Lemma)

(1) The maps

$$\begin{aligned} (X,A) &\mapsto \mathrm{C}^*(X,A;G) \\ (f\colon (X,A) \to (Y,B)) &\mapsto (f^*\colon \mathrm{C}^*(Y,B;G) \to \mathrm{C}^*(X,A;G)) \end{aligned}$$

[535] In most cases we drop the adjective "singular" and we just talk of the n-th cohomology group of (X,A) with coefficients in G. We only refer to it as singular cohomology when we want to distinguish it from de Rham cohomology or cellular cohomology.

[536] As we had already pointed out on page 861, if R is a commutative ring, e.g. if $R = \mathbb{Z}_n, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ or $R = \mathbb{C}$, then the cohomology groups $\mathrm{H}^n(X,A;R)$ can in fact be viewed as R-modules and the coboundary maps are R-module homomorphisms.

define a contravariant functor from the category *PairTop* of pairs of topological spaces to the category *CoChCplx* of cochain complexes.

(2) For any $n \in \mathbb{N}_0$ the maps

$$\begin{array}{rcl}(X,A) & \mapsto & \mathrm{H}^n(X,A;G)\\ (f\colon (X,A)\to(Y,B)) & \mapsto & (f^*\colon \mathrm{H}^n(Y,B;G)\to \mathrm{H}^n(X,A;G))\end{array}$$

define a contravariant functor from the category *PairTop* of pairs of topological spaces to the category *AbGroup* of abelian groups. (If G is a commutative ring, then this is in fact a contravariant functor to the category $\mathcal{M}od_G$ of G-modules.)

Proof.

(1) This statement follows immediately from the C_*–Functor Lemma 35.5 (2) and the Dual Cochain Map Lemma 66.9.

(2) This statement follows immediately from Statement (1) and the Cochain Complex-H^*-Functor Lemma 66.5 (2). ■

67.2. **Explicit cochains and cocycles.** In this section we will consider a few explicit examples of cochains and cocycles. These examples gives us hopefully some intuition what cohomology does and later they will be useful to exhibit various statements on cohomology groups. We will continue with developing the theory of cohomology groups in the next Section 67.3.

Let X be a topological space. Recall that a homology class in dimension k is represented by a singular chain of the form $\sum_{i=1}^{m} a_i\cdot(\sigma_i\colon \Delta^k\to X)$. Thus homology classes can be described using "finite" data and in explicit cases one can often visualize a homology class.

A cohomology class in dimension k is a homomorphism $\mathrm{C}_k(X)\to G$. In almost all cases it is very hard to explicitly write down a cohomology class. Nonetheless, in this section we will present a few explicit cocycles in dimension 1 which are interesting in their own right, and which will be our go-to examples throughout the following chapters.

First we discuss examples in "absolute" cohomology groups. We start this discussion with the following remark.

Identification. Let G be an abelian group.

(1) Let X be a topological space and let $n\in\mathbb{N}_0$. We denote by $\mathrm{S}_n(X)$ the set of singular n-simplices of X. We have the following identities and natural identification

$$\mathrm{C}^n(X;G) \;:=\; \mathrm{Hom}(\mathrm{C}_n(X),G) \underset{\substack{\uparrow\\ \text{by definition } \mathrm{C}_n(X)=\mathbb{Z}^{(\mathrm{S}_n(X))}}}{=} \mathrm{Hom}\left(\mathbb{Z}^{(\mathrm{S}_n(X))},G\right) \underset{\substack{\uparrow\\ \text{identification given by the Abelian}\\ \text{Group-Homomorphism Lemma 66.2 (2)}}}{=_{\mathrm{i}}} \{\text{maps from } \mathrm{S}_n(X) \text{ to } G\}.$$

Put differently, a singular n-cochain is precisely the same as assigning to each singular n-simplex an element in G. We will use this bijection throughout the course without explicitly referring to it.

(2) Let X be a topological space. We recall that, following the discussion on page 458, we can identify the singular 0-simplices in X with the points in X. According to the discussion in (1) we can now make the identification

$$\mathrm{C}^0(X;G) \;=_{\mathrm{i}}\; \{\text{all maps from } X \text{ to } G\}.$$

We now consider explicit cochains and cocycles on the topological spaces $\mathbb{R}$ and S^1:

Examples.

(1) We start out with the topological space $X = \mathbb{R}$. We consider the two maps

$$\begin{array}{rcl} \alpha_{\mathbb{R}}\colon \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & x \end{array} \quad \text{and} \quad \begin{array}{rcl} \alpha_{\mathbb{Z}}\colon \mathbb{R} & \to & \mathbb{Z} \\ x & \mapsto & \lfloor x \rfloor := \max\{n \in \mathbb{Z} \,|\, n \le x\}. \end{array}$$

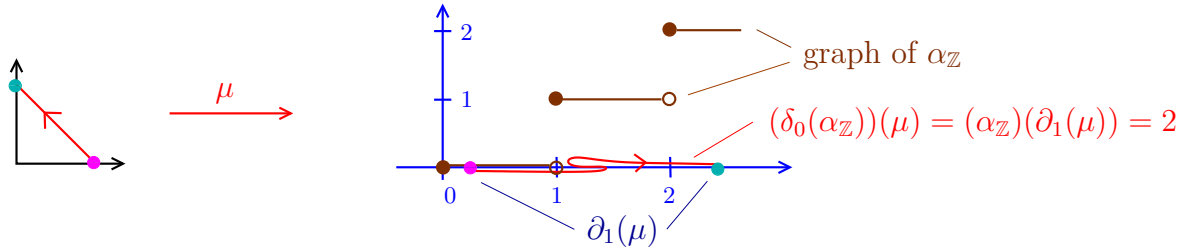
By the above identification, these two maps define singular 0-cochains on $\mathbb{R}$. We denote these 0-cochains again by $\alpha_{\mathbb{R}}$ and $\alpha_{\mathbb{Z}}$. Note that for any singular 1-simplex $\mu\colon \Delta^1 \to \mathbb{R}$ we have

$$(\delta_0(\alpha_{\mathbb{R}}))(\mu) \underset{\substack{\uparrow \\ \text{definition of } \delta_0 = (\partial_1)^*}}{=} \alpha_{\mathbb{R}}(\partial_1(\mu)) \underset{\substack{\uparrow \\ \text{see page 459 for the} \\ \text{description of } \partial_1}}{=} \alpha_{\mathbb{R}}(\mu(0,1) - \mu(1,0)) \underset{\substack{\uparrow \\ \text{definition of } \alpha_{\mathbb{R}}}}{=} \mu(0,1) - \mu(1,0).$$

Similarly we have[537]

$$(\delta_0(\alpha_{\mathbb{Z}}))(\mu) = \alpha_{\mathbb{Z}}(\partial_1(\mu)) = \alpha_{\mathbb{Z}}(\mu(0,1) - \mu(1,0)) = \lfloor \mu(0,1) \rfloor - \lfloor \mu(1,0) \rfloor.$$

In a somewhat informal way we can say that the cochain $\delta_0(\alpha_{\mathbb{Z}})\colon \mathrm{C}_1(\mathbb{R}) \to \mathbb{Z}$ applied to a singular 1-simplex μ measures how many integers the simplex μ "crosses from left to right". We refer to the figure below for an illustration of $\delta_0(\alpha_{\mathbb{Z}})$.



(2) We consider the topological space $X = S^1$. Let $p\colon \mathbb{R} \to S^1$, $t \mapsto \exp(2\pi i t)$ be the universal covering of S^1. Let $\sigma\colon \Delta^1 \to S^1$ be a singular 1-simplex. Since Δ^1 is simply connected it follows from the Covering Existence Theorem 27.2 that σ lifts to a continuous map $\widetilde{\sigma}\colon \Delta^1 \to \mathbb{R}$. We set

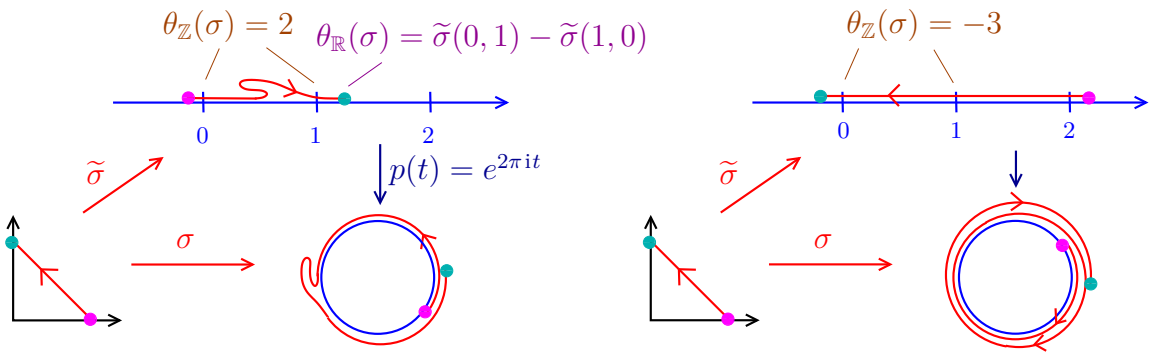
$$\begin{aligned} \theta_{\mathbb{R}}(\sigma) &:= (\delta_1(\alpha_{\mathbb{R}}))\underset{\substack{\uparrow \\ \in \mathrm{C}_1(\mathbb{R}) \\ \downarrow}}{(\widetilde{\sigma})} = \widetilde{\sigma}(0,1) - \widetilde{\sigma}(1,0) \quad \in \mathbb{R} \\ \theta_{\mathbb{Z}}(\sigma) &:= (\delta_1(\alpha_{\mathbb{Z}}))(\widetilde{\sigma}) = \lfloor \widetilde{\sigma}(0,1) \rfloor - \lfloor \widetilde{\sigma}(1,0) \rfloor \quad \in \mathbb{Z}. \end{aligned}$$

Evidently the lift $\widetilde{\sigma}$ is not unique, but it follows almost immediately from the Lift Uniqueness Lemma 14.7 that any two lifts differ by addition by an integer. Using this observation one can easily show that $\theta_{\mathbb{R}}(\sigma)$ and $\theta_{\mathbb{Z}}(\sigma)$ are both well-defined, i.e. independent of the choice of the lift $\widetilde{\sigma}$. The definitions of $\theta_{\mathbb{R}} \in \mathrm{C}_1(S^1, \mathbb{R})$ and $\theta_{\mathbb{Z}} \in \mathrm{C}_1(S^1, \mathbb{Z})$ are illustrated in the figure below.

It follows from the discussion on page 865 that these two maps $\theta_{\mathbb{R}}$ and $\theta_{\mathbb{Z}}$ actually define singular 1-cochains in $\mathrm{C}^1(S^1; \mathbb{R})$ and $\mathrm{C}^1(S^1; \mathbb{Z})$. As an example we consider the singular 1-simplex

[537] In other words:

$$(\delta_0(\alpha_{\mathbb{Z}})(\mu)) = \begin{cases} \#\{n \in \mathbb{Z} \,|\, \mu(1,0) < n \le \mu(0,1)\}, & \text{if } \mu(0,1) \ge \mu(1,0) \\ -\#\{n \in \mathbb{Z} \,|\, \mu(0,1) < n \le \mu(1,0)\}, & \text{if } \mu(1,0) \ge \mu(0,1). \end{cases}$$



$$\mu\colon \Delta^1 \to S^1 \quad \text{that is defined by} \quad \mu(1-t,t) = \exp(2\pi i t).$$

A lift of μ to the above universal covering $\mathbb{R} \to S^1$ is given by the singular 1-simplex

$$\widetilde{\mu}\colon \Delta^1 \to \mathbb{R} \quad \text{that is defined by} \quad \widetilde{\mu}(1-t,t) = t.$$

It follows immediately from the definitions that $\theta_{\mathbb{R}}(\mu) = \theta_{\mathbb{Z}}(\mu) = 1$. □

The following lemma shows that $\theta_{\mathbb{R}} \in \mathrm{C}^1(S^1;\mathbb{R})$ and $\theta_{\mathbb{Z}} \in \mathrm{C}^1(S^1;\mathbb{Z})$ are cocycles; in fact even better, it shows that these cocycles represent non-zero elements in cohomology:

Lemma 67.2. (θ-Cocycle Lemma) We continue with the notation from the previous example.

(1) The singular 1-cochains $\theta_{\mathbb{R}} \in \mathrm{C}^1(S^1;\mathbb{R})$ and $\theta_{\mathbb{Z}} \in \mathrm{C}^1(S^1;\mathbb{Z})$ are cocycles, in particular $\theta_{\mathbb{R}}$ defines an element in $\mathrm{H}^1(S^1;\mathbb{R})$ and $\theta_{\mathbb{Z}}$ defines an element in $\mathrm{H}^1(S^1;\mathbb{Z})$.

(2) The singular 1-cochains $\theta_{\mathbb{R}}$ and $\theta_{\mathbb{Z}}$ are not the coboundary of a singular 0-cochain, i.e. the cohomology classes $[\theta_{\mathbb{R}}] \in \mathrm{H}^1(S^1;\mathbb{R})$ and $[\theta_{\mathbb{Z}}] \in \mathrm{H}^1(S^1;\mathbb{Z})$ are non-trivial.

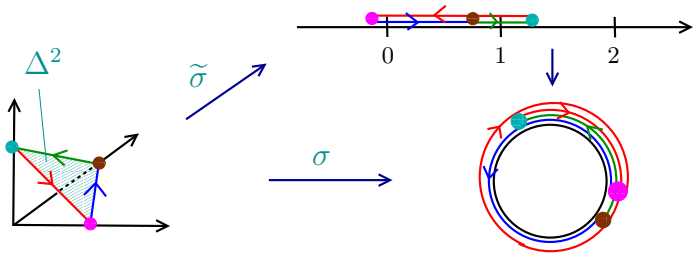
Remark. Note that in the θ-Cocycle Lemma 67.2 we have just given a mysteriously simple proof that $\mathrm{H}^1(S^1;\mathbb{R}) \neq 0$ and also that $\mathrm{H}^1(S^1;\mathbb{Z}) \neq 0$. □

Proof.

(1) We will first prove that the singular 1-cochain $\theta_{\mathbb{R}} \in \mathrm{C}^1(S^1;\mathbb{R})$ is a cocycle:

Claim. We have $\delta_1(\theta_{\mathbb{R}}) = 0 \in \mathrm{C}^2(S^1,\mathbb{R}) = \mathrm{Hom}(\mathrm{C}_2(S^1),\mathbb{R})$.

Proof. So let $\sigma\colon \Delta^2 \to S^1$ be a singular 2-simplex. Note that we have to show that $(\delta_1(\theta_{\mathbb{R}}))(\sigma) = \theta_{\mathbb{R}}(\partial_2(\sigma)) = 0$. As before we denote by $p\colon \mathbb{R} \to S^1$, $t \mapsto \exp(2\pi i t)$ the universal covering of S^1. Since Δ^2 is simply connected we can apply the Map Lifting Criterion 27.1 to obtain a lift of $\sigma\colon \Delta^2 \to S^1$ to a continuous map $\widetilde{\sigma}\colon \Delta^2 \to \mathbb{R}$.



Then

$$(\delta_1(\theta_{\mathbb{R}}))(\sigma) \underset{\text{definition of } \delta_1}{=} \theta_{\mathbb{R}}(\partial_2(\sigma)) \underset{\text{definition of } \partial_2}{=} \theta_{\mathbb{R}}\Big(\sum_{j=0}^{2}(-1)^j \cdot \sigma \circ i_j^2\Big) \underset{\text{by definition of } \theta_{\mathbb{R}} \text{ and since } \widetilde{\sigma}\circ i_j^2 \text{ is a lift of } \sigma\circ i_j^2}{=} (\delta_1(\alpha_{\mathbb{R}}))\Big(\overbrace{\sum_{j=0}^{2}(-1)^j \cdot \widetilde{\sigma} \circ i_j^2}^{=\partial_2(\widetilde{\sigma})}\Big)$$
$$\underset{\text{by definition of } \delta_1(\alpha_{\mathbb{R}}) \text{ and } \partial_2(\widetilde{\sigma})}{=} \alpha_{\mathbb{R}}(\partial_1(\partial_2\widetilde{\sigma})) \underset{\text{since } \partial_2\circ\partial_1 = 0}{=} 0. \qquad \boxplus$$

Exactly the same argument can be used to show that $\theta_{\mathbb{Z}} \in \mathrm{C}^1(S^1;\mathbb{Z})$ is a cocycle.

(2) We still need to show that the singular 1-cochain cochains $\theta_{\mathbb{R}}$ and $\theta_{\mathbb{Z}}$ are not the coboundary of a singular 0-cochain. So let β be a singular 0-cochain. As usual we view β as a function $S^1 \to \mathbb{R}$. We consider again the singular 1-simplex $\mu\colon \Delta^1 \to S^1$ that is defined by $\mu(1-t,t) = \exp(2\pi\,\mathrm{i}t)$. Then

$$(\delta_0(\beta))(\mu) \underset{\text{definition of } \delta_0}{=} \beta(\partial_1(\mu)) = \beta(\underbrace{\mu(0,1)}_{=1\in S^1} - \underbrace{\mu(1,0)}_{=1\in S^1}) = 0 \neq 1 \underset{\text{by the example preceding the lemma}}{=} \theta_{\mathbb{R}}(\mu) = \theta_{\mathbb{Z}}(\mu).$$ ■

We continue with another example of cochains.

Example. Let $I \subset \mathbb{R}$ be an interval, e.g. $I = [0,1]$ or $I = \mathbb{R}$. We fix a point $y \in I$. We consider the cochain

$$\varphi_y\colon \mathrm{C}_0(I) \to \mathbb{Z}$$

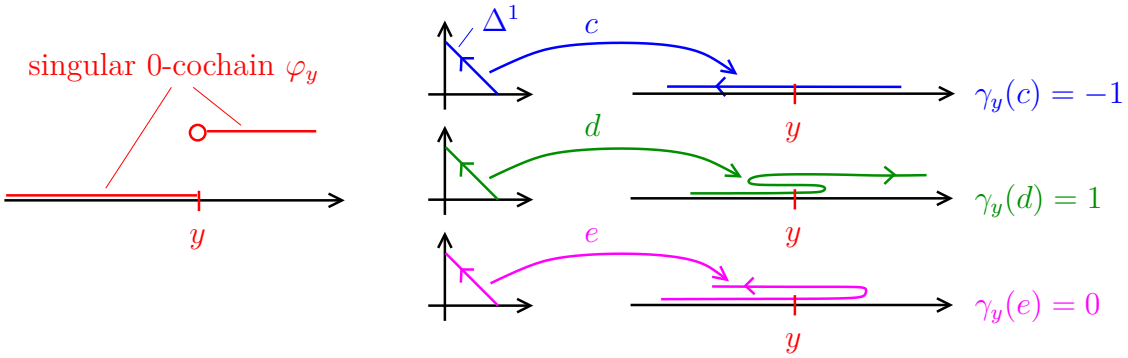
that is given by

$$\begin{aligned} I &\to \mathbb{Z} \\ x &\mapsto \begin{cases} 0, & \text{if } x \le y, \\ 1, & \text{else.} \end{cases} \end{aligned}$$

For any singular 1-simplex $d\colon \Delta^1 \to X = I$ we have

$$(\delta_0(\varphi_y))(d) = \varphi_y(\partial_1(d)) \underset{\substack{\text{since } \partial_1(d) = d(0,1) - d(1,0) \text{ and} \\ \text{since } \varphi_y\colon \mathrm{C}_0(I) \to \mathbb{Z} \text{ is linear}}}{=} \varphi_y(d(0,1)) - \varphi_y(d(1,0)) = \begin{cases} 0, & \text{if } d(0,1), d(1,0) \le y, \\ 0, & \text{if } d(0,1), d(1,0) \ge y, \\ 1, & \text{if } d(1,0) \le y < d(0,1) \\ -1, & \text{if } d(0,1) \le y < d(1,0). \end{cases}$$

In a slightly informal way we can say that the homomorphism $\delta_0(\varphi_y)\colon \mathrm{C}_1(I) \to \mathbb{Z}$ applied to a singular 1-simplex d equals $+1$ if the simplex "crosses y from right to left" and it equals -1 if the simplex "crosses y from left to right".



We will make use of this singular 1-cochain on several occasions, hence we give it its own notation, namely we write

$$\gamma_y \coloneqq \delta_0(\varphi_y) \in \mathrm{C}^1(I;\mathbb{Z}).$$ □

We want to conclude this section with an explicit cocycle in a relative cohomology group. Before we do so, it is again helpful to work with a few identifications:

Identification.

(1) Given an abelian group C, a subgroup D and an abelian group G we have a natural isomorphism

$$\begin{array}{rcl} \operatorname{Hom}(C/D,G) & \xrightarrow{\cong} & \text{all homomorphisms } C \to G \text{ that vanish on } D \\ (\varphi\colon C/D \to G) & \mapsto & (C \xrightarrow{c\mapsto[c]} C/D \xrightarrow{\varphi} G). \end{array}$$

We will use this isomorphism to identify the groups left and right.

(2) If S is a set and T is a subset, then for any abelian group G the identification

$$\operatorname{Hom}(\mathbb{Z}^{(S)},G) =_{\mathrm{i}} \{\text{all maps from } S \text{ to } G\}$$

from the Abelian Group-Homomorphism Lemma 66.2 (2) restricts to an identification

$$\left\{f \in \operatorname{Hom}(\mathbb{Z}^{(S)},G) \,\middle|\, f \text{ vanishes on } \mathbb{Z}^{(T)}\right\} =_{\mathrm{i}} \{\text{all maps } S \to G \text{ that are trivial on } T\}.$$

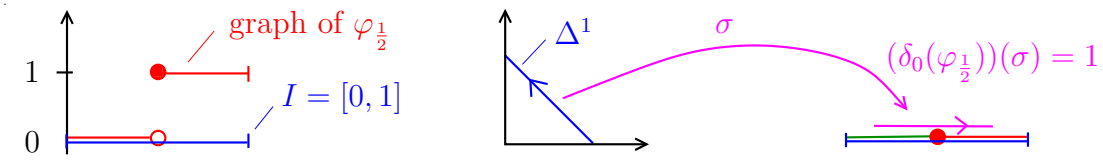
(3) It follows immediately from the Abelian Group-Homomorphism Lemma 66.2 (2) that given a pair of topological spaces (X,A) and an abelian group G we have for every $n \in \mathbb{N}_0$ an identification

$$\underbrace{\operatorname{Hom}(\mathrm{C}_n(X,A),G)}_{=\mathrm{C}^n(X,A;G)=_{\mathrm{i}}\operatorname{Hom}(\mathrm{C}_n(X)/\mathrm{C}_n(A),G)} = \text{all maps } \mathrm{S}_n(X) \to G \text{ that are trivial on } \mathrm{S}_n(A).$$

Example. We write $I=[0,1]$. We want to construct a potentially interesting element in $\mathrm{H}^1(I,\partial I;\mathbb{Z})$. As in the previous example we consider the map

$$\begin{array}{rcl} \varphi_{\frac12}\colon I & \to & \mathbb{Z} \\ x & \mapsto & \begin{cases} 0, & \text{if } x \in [0,\frac12) \\ 1, & \text{otherwise} \end{cases} \end{array}$$

which we use to define the singular 1-cochain $\gamma_{\frac12} := \delta_0(\varphi_{\frac12}) \in \mathrm{C}^1(I;\mathbb{Z})$.



Note that for any singular simplex $\sigma\colon \Delta^1 \to I$ we have

$$\gamma_{\frac12}(\sigma) = (\delta_0(\varphi_{\frac12}))(\sigma) = \varphi_{\frac12}(\partial_1(\sigma)) = \varphi_{\frac12}(\sigma(0,1)) - \varphi_{\frac12}(\sigma(0,1)).$$

We make two observations:

(1) Let $\sigma\colon \Delta^1 \to \partial I = \{0,1\}$ be a singular simplex. Since Δ^1 is connected and since $\{0,1\}$ is discrete it follows from the Connected-to-Discrete Lemma 2.18 that $\sigma\colon \Delta^1 \to \partial I = \{0,1\}$ is constant. It follows from the above calculation that $\gamma_{\frac12}(\sigma)=0$. This shows that $\gamma_{\frac12}$ defines a cochain $\overline{\gamma}_{\frac12}\colon \mathrm{C}_1(I,\partial I) \to \mathbb{Z}$.

(2) It is an amusing exercise to show that the cochain $\overline{\gamma}_{\frac12}\colon \mathrm{C}_1(I,\partial I) \to \mathbb{Z}$ is a cocycle. Finally on page 874 we will see that $\mathrm{H}^1(I,\partial I;\mathbb{Z}) \cong \mathbb{Z}$ and that the cocycle $\overline{\gamma}_{\frac12}\colon \mathrm{C}_1(I,\partial I) \to \mathbb{Z}$ represents a generator of $\mathrm{H}_1(I,\partial I;\mathbb{Z})$. □

67.3. **First calculations of cohomology.** After this long discussion of explicit cochains and cohomology classes we want to turn to more general results and calculations. Recall that the only long hanging fruit for calculating homology groups were the following:

(1) The Empty Space-H_*-Lemma 35.9.
(2) The Path-Component–H_*–Lemma 35.8.
(3) The calculation of the 0-th homology group of a path-connected space in the H_0-Proposition 35.11.
(4) The homology groups of a singleton in the Singleton-H_*-Lemma 35.12.

Not surprisingly we can also deal with the same situations with cohomology groups. First of all the following lemma is the analogue of the Empty Space-H_*-Lemma 35.9.

Lemma 67.3. (Empty Space-H^*-Lemma) For the empty topological space $\varnothing$ we have $\mathrm{H}^k(\varnothing; G) = 0$ for all abelian groups G and all $k \in \mathbb{N}_0$.

Proof. This lemma follows immediately from the observation that in this case we have $\mathrm{C}_k(\varnothing) = 0$ for all $k \in \mathbb{N}_0$. ∎

We continue with the somewhat more interesting analogue of the Path-Component–H_*–Lemma 35.8:

Lemma 67.4. (Path-Component-H^*-Lemma) Let X be a topological space and let G be an abelian group. We denote by $\{X_j\}_{j\in J}$ the path-components of X. The inclusion maps $\iota_j \colon X_j \to X$, $j \in J$, induce an isomorphism

$$\prod_{j\in J} \iota_j^* \colon \mathrm{H}^n(X; G) \xrightarrow{\cong} \prod_{j\in J} \mathrm{H}^n(X_j; G).$$

Proof. Let X be a topological space with path-components $\{X_j\}_{j\in J}$ and let G be an abelian group. We have the following isomorphisms of cochain complexes:

$$\mathrm{Hom}(\mathrm{C}_*(X), G) \xrightarrow{\left(\bigoplus_{j\in J} \iota_{j*}\right)^*} \mathrm{Hom}\left(\bigoplus_{j\in J} \mathrm{C}_*(X_j), G\right) \xrightarrow{f \mapsto \prod_{j\in J} f\circ i_j} \prod_{j\in J} \mathrm{Hom}(\mathrm{C}_*(X_j), G).$$

by the Path-Component–C_*–Lemma 35.6 this map is an isomorphism

by the Abelian Group-Homomorphism Lemma 66.2 (3) this is an isomorphism

It follows immediately from the definitions that the composition of these two maps is precisely the map $\prod_{j\in J} \iota_j^* \colon \mathrm{Hom}(\mathrm{C}_*(X), G) \to \prod_{j\in J} \mathrm{Hom}(\mathrm{C}_*(X_j), G)$. The lemma now follows from these observations together with the Product–H^*-Lemma 66.8, which says that cohomology commutes with products. ∎

We continue with the following analogue of the H_0-Proposition 35.11.

Proposition 67.5. (H^0-Proposition) Let G be an abelian group.

(1) Given any topological space the map

$$\begin{aligned} \zeta_X \colon G &\to \mathrm{H}^0(X; G) \\ g &\mapsto [\text{0-cochain that assigns to each } x \in X \text{ the value } g] \end{aligned}$$

is a well-defined homomorphism and it is natural. Here by "natural" we mean that for any continuous map $f \colon X \to Y$ between topological spaces the following diagram

commutes:

$$\begin{array}{ccc} G & \xrightarrow{\quad\mathrm{id}\quad} & G \\ \downarrow{\scriptstyle \zeta_Y} & & \downarrow{\scriptstyle \zeta_X} \\ \mathrm{H}^0(Y;G) & \xrightarrow{\quad f^*\quad} & \mathrm{H}^0(X;G). \end{array}$$

(2) If X is a path-connected non-empty topological space, then the map $\zeta_X\colon G \to \mathrm{H}^0(X;G)$ from (1) is an isomorphism. For any point $P \in X$ an inverse is given by the map

$$\begin{array}{rcl} \mathrm{H}^0(X;G) & \to & G \\ [f] & \mapsto & f(P). \end{array}$$

Proof. We prove this statement in Exercise Sheet 8. ∎

The following lemma is an analogue of the Singleton-H_*-Lemma 35.12.

Lemma 67.6. (Singleton-H^*-Lemma) Let $X = \{*\}$ be a topological space that consists of a single point $*$. For any abelian group G we have

$$\mathrm{H}^n(X;G) \cong \begin{cases} 0, & \text{if } n \geq 1, \\ G, & \text{if } n = 0. \end{cases}$$

Proof. We leave the proof as an exercise to the reader. ∎

67.4. **Basic properties of cohomology groups.** The following proposition is the analogue of the the Homotopy-H_*-Proposition 36.5, and the Homotopy-H_*-Proposition 38.6:

Proposition 67.7. (Homotopy-H^*-Proposition) Let G be an abelian group and let $n \in \mathbb{N}_0$. Let $f, g\colon (X,A) \to (Y,B)$ be continuous maps between pairs of topological spaces. If f and g are homotopic, then they induce the same maps

$$f^* = g^*\colon \mathrm{H}^n(Y,B;G) \to \mathrm{H}^n(X,A;G).$$

Proof. The statement is an immediate consequence of the Homotopy-H_*-Proposition 38.6 together with the Dual Cochain Map Lemma 66.9 (3a) and the Cochain Homotopy-H^*-Lemma 66.6 (1). ∎

The following proposition is the analogue of the the Homotopy Equivalence-H_*-Corollary 36.7 and the Homotopy Equivalence-Relative-H_*-Corollary 38.4:

Corollary 67.8. (Homotopy Equivalence-H^*-Corollary) Let G be an abelian group and let $n \in \mathbb{N}_0$.

(1) If $f\colon X \to Y$ is a homotopy equivalence between two topological spaces, then the induced map

$$f^*\colon \mathrm{H}^n(Y;G) \xrightarrow{\cong} \mathrm{H}^n(X;G)$$

is an isomorphism.

(2) The cohomology groups of homotopy equivalent topological spaces are isomorphic.

Proof. This statement follows immediately from the Homotopy-H^*-Proposition 67.7 and the H^n–Functor Lemma 67.1. ∎

Example. Let X be a contractible topological space (e.g. $X = \mathbb{R}^k, \overline{B}^k, B^k, \Delta^k$), then for any abelian group G and any $n \in \mathbb{N}_0$ we have

$$\begin{array}{ccccc} \mathrm{H}^n(X;G) & \cong & \mathrm{H}^n(\{*\};G) & \cong & \begin{cases} 0, & \text{if } n \geq 1, \\ G, & \text{if } n = 0. \end{cases} \\ \uparrow & & \uparrow & & \\ \text{by the Homotopy Equivalence} & & \text{by the Singleton-H}^*\text{-Lemma 67.6} & & \\ \text{H}^*\text{-Corollary 67.8 (2)} & & & & \end{array}$$

□

The following proposition is the analogue of the LES-of-Triple Proposition 38.2 in homology theory.

Proposition 67.9. (LES-of-Triple Proposition) Given a triple (X, B, A) of topological spaces and an abelian group G we obtain a long exact sequence

$$\ldots \xrightarrow{\delta_{n-1}} \mathrm{H}^n(X, B; G) \xrightarrow{p^*} \mathrm{H}^n(X, A; G) \xrightarrow{i^*} \mathrm{H}^n(B, A; G) \xrightarrow{\delta_n} \mathrm{H}^{n+1}(X, B; G) \xrightarrow{p^*} \ldots$$

hereby the maps are given as follows:

(1) p^* is the map induced by the map $p\colon (X, A) \to (X, B)$.
(2) i^* is the map induced by the map $i\colon (B, A) \to (X, A)$.
(3) The connecting homomorphism $\delta_n\colon \mathrm{H}^n(B, A; G) \to \mathrm{H}^{n+1}(X, B; G)$ is the natural homomorphism that is given by the map

$$\begin{array}{ccc} \mathrm{H}^n(B, A; G) & \to & \mathrm{H}^{n+1}(X, B; G) \\ [f\colon \mathrm{C}_n(B, A) \to G] & \mapsto & \left[\begin{array}{ccc} \mathrm{C}_{n+1}(X, B) & \to & G \\ \sigma & \mapsto & f(q(\partial_n(\sigma))) \\ \uparrow & & \\ \text{with } \sigma \in \mathrm{C}_{n+1}(X) & & \end{array}\right] \end{array}$$

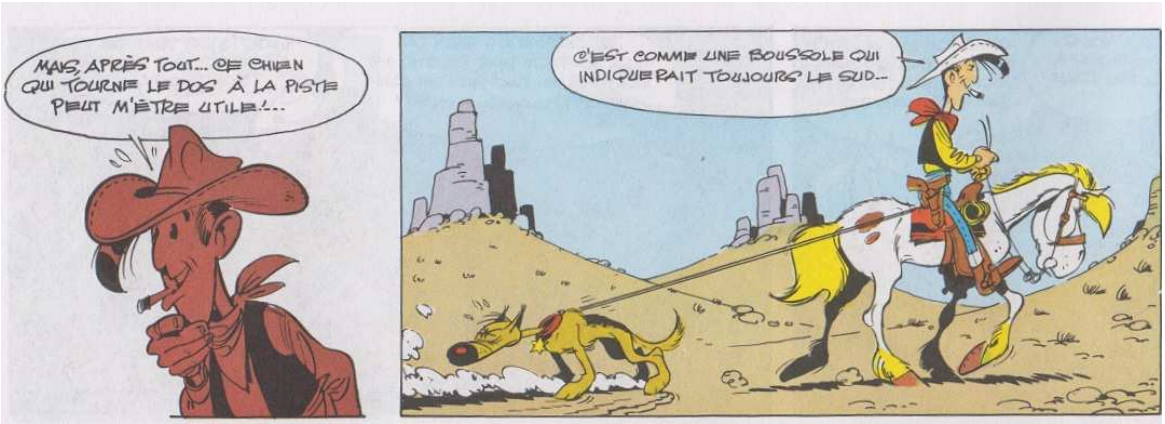
here $q\colon \mathrm{C}_n(X) \to \mathrm{C}_n(B)$ is the homomorphism given by sending all singular simplices that do not lie in B to zero.

Remark.

(1) One can easily verify that an equivalent approach to expressing the connecting homomorphism is given as follows:
   (a) take a cocycle $f \in \mathrm{C}^n(B, A; G) = \mathrm{Hom}(\mathrm{C}_n(B, A), G)$,
   (b) denote by $\pi\colon \mathrm{C}_n(B) \to \mathrm{C}_n(B, A)$ the projection ,
   (c) denote by $\widetilde{f}$ the result of extending $f \circ \pi\colon \mathrm{C}_n(B) \to G$ to a map $\mathrm{C}_n(X) \to G$ by sending all singular simplices that do not lie in B to zero (sometimes we call $\widetilde{f}$ the *trivial extension* of $f \circ \pi$),
   (d) then consider the coboundary $\delta_n(\widetilde{f}) \in \mathrm{C}^{n+1}(X; G)$, since f is a cocycle it follows that $\delta_n(\widetilde{f})$ actually defines an element in $\mathrm{C}^{n+1}(X, B; G)$ that we also denote by $\delta_n(\widetilde{f})$, and which turns out to be a cocycle,
   (e) then $\delta_n([f]) = [\delta_n(\widetilde{f})] \in \mathrm{H}^{n+1}(X, B; G)$.
(2) The most frequent application of the LES-of-Triple Proposition 67.9 is that given a pair (X, B) of topological spaces, i.e. given the triple $(X, B, \varnothing)$ and given an abelian group G we obtain a natural long exact sequence

$$\ldots \xrightarrow{\delta_{n-1}} \mathrm{H}^n(X, B; G) \xrightarrow{p^*} \mathrm{H}^n(X; G) \xrightarrow{i^*} \mathrm{H}^n(B; G) \xrightarrow{\delta_n} \mathrm{H}^{n+1}(X, B; G) \xrightarrow{p^*} \ldots$$

(3) One of the difficulties with the long exact sequence of cohomology groups is that all maps seem to point the wrong way.



You should use the Lucky Luke approach to cohomology: If Rantanplan (i.e. your intuition) points one way, you should take the opposite direction. □

Proof. Let (X, B, A) be a triple of topological spaces. As in the proof of the LES-of-Triple Proposition 38.2 we note that

$$0 \to \mathrm{C}_*(B, A) \xrightarrow{i_*} \mathrm{C}_*(X, A) \xrightarrow{p_*} \mathrm{C}_*(X, B) \to 0$$

is a short exact sequence of chain complexes. Now let G be an abelian group. Recall that we pointed out on page 489 that the chain groups $\mathrm{C}_*(X, B)$ are free abelian groups. Thus we deduce from the Hom-Split-SES Lemma 66.4 that the following induced sequence of cochain complexes is also exact:

$$0 \to \mathrm{Hom}(\mathrm{C}_*(X, B), G) \xrightarrow{p^*} \mathrm{Hom}(\mathrm{C}_*(X, A), G) \xrightarrow{i^*} \mathrm{Hom}(\mathrm{C}_*(B, A), G) \to 0.$$

The lemma now follows easily from the Cohomology–LES Proposition 66.7 and the construction of the connecting homomorphism. We leave the details to the reader. ■

Examples.

(1) Let X be a path-connected topological space and let A be any *non-empty* subset of X. We denote by $i\colon A \to X$ the inclusion map. By the LES-of-Triple Proposition 67.9 we have a long exact sequence

$$0 \to \mathrm{H}^0(X, A; G) \to \mathrm{H}^0(X; G) \xrightarrow{i^*} \mathrm{H}^0(A; G) \to \dots$$

it follows from the H^0-Proposition 67.5 and our hypothesis on X and A that this map is a monomorphism

Thus it follows that $\mathrm{H}^0(X, A; G) = 0$.

(2) We consider the beginning of the long exact sequence in cohomology with $\mathbb{Z}$-coefficients of the pair $([0,1], \{0,1\})$:

$$\dots \to \mathrm{H}^0([0,1]; \mathbb{Z}) \xrightarrow{i^*} \mathrm{H}^0(\{0,1\}; \mathbb{Z}) \xrightarrow{\delta_0} \mathrm{H}^1([0,1], \{0,1\}; \mathbb{Z}) \to \underbrace{\mathrm{H}^1([0,1]; \mathbb{Z})}_{=0}.$$

$\overline{\gamma}_{\frac{1}{2}}$

In the above sequence we already indicated the cocycle $\overline{\gamma}_{\frac{1}{2}} \in \mathrm{C}^1([0,1], \{0,1\})$ that we introduced on page 869. It follows easily from the Path-Component-H^*-Lemma 67.4 and the H^0-Proposition 67.5 that the cokernel of the map $i^*\colon \mathrm{H}^0([0,1]; \mathbb{Z}) \to \mathrm{H}^0(\{0,1\}; \mathbb{Z})$

is isomorphic to $\mathbb{Z}$ and that it is represented by the singular 0-cochain that is given by

$$\begin{array}{rccc} \psi\colon & \{0,1\} & \to & \mathbb{Z} \\ & x & \mapsto & x \end{array}$$

The following claim now shows that $\mathrm{H}^1([0,1],\{0,1\};\mathbb{Z}) = \mathbb{Z}\cdot[\overline{\gamma}_{\frac{1}{2}}]$:

Claim. The class $\delta_0([\psi]) \in \mathrm{H}^1([0,1],\{0,1\};\mathbb{Z})$ is represented by $[\overline{\gamma}_{\frac{1}{2}}]$.

Proof. We need to determine $\delta_0([\psi]) \in \mathrm{H}^1([0,1],\{0,1\};\mathbb{Z})$. It is perhaps helpful to consider the following commutative diagram of short exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Hom}(\mathrm{C}_1([0,1],\{0,1\}),\mathbb{Z}) & \xrightarrow{p^*} & \mathrm{Hom}(\mathrm{C}_1([0,1]),\mathbb{Z}) & \xrightarrow{i^*} & \mathrm{Hom}(\mathrm{C}_1(\{0,1\}),\mathbb{Z}) & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle \delta_0} & & \uparrow{\scriptstyle \delta_0} & & \uparrow{\scriptstyle \delta_0} & & \\ 0 & \longrightarrow & \mathrm{Hom}(\mathrm{C}_0([0,1],\{0,1\},\mathbb{Z}) & \xrightarrow{p^*} & \mathrm{Hom}(\mathrm{C}_0([0,1]),\mathbb{Z}) & \xrightarrow{i^*} & \underbrace{\mathrm{Hom}(\mathrm{C}_0(\{0,1\},\mathbb{Z})}_{\text{contains } \psi} & \longrightarrow & 0. \end{array}$$

We start with the map $\psi\colon \{0,1\} \to \mathbb{Z}$. We need to pick an extension to a map $[0,1] \to \mathbb{Z}$. We might as well take the map $\varphi_{\frac{1}{2}}\colon [0,1] \to \mathbb{Z}$ from page 868. Note that the coboundary $\gamma_{\frac{1}{2}} \coloneqq \delta_0(\varphi_{\frac{1}{2}})$ defines the cocycle $\overline{\gamma}_{\frac{1}{2}} \in \mathrm{C}^1([0,1],\{0,1\};\mathbb{Z})$. The claim now follows from the definition of the connecting homomorphism. ⊞

We conclude this section with the cohomological analogue of the popular Homotopy Equivalence-Relative-H_*-Corollary 38.4:

Corollary 67.10. (Homotopy Equivalence-Relative-H^*-Corollary) In the following let $f\colon (X,A) \to (Y,B)$ be a continuous map of pairs of topological spaces. If $f\colon X \to Y$ is a homotopy equivalence and if $f\colon A \to B$ is a homotopy equivalence, then for every $n \in \mathbb{N}_0$ and every abelian group G the induced map $f^*\colon \mathrm{H}^n(Y,B;G) \to \mathrm{H}^n(X,A;G)$ is an isomorphism.

Proof. We consider the following diagram:

$$\begin{array}{ccccccccccc} \longrightarrow & \mathrm{H}^n(Y;G) & \longrightarrow & \mathrm{H}^n(B;G) & \longrightarrow & \mathrm{H}^n(Y,B;G) & \xrightarrow{\delta_n} & \mathrm{H}^{n+1}(Y;G) & \longrightarrow & \mathrm{H}^{n+1}(B;G) & \longrightarrow \\ & \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} & \\ \longrightarrow & \mathrm{H}^n(X;G) & \longrightarrow & \mathrm{H}^n(A;G) & \longrightarrow & \mathrm{H}^n(X,A;G) & \xrightarrow{\delta_n} & \mathrm{H}^{n+1}(X;G) & \longrightarrow & \mathrm{H}^{n+1}(A;G) & \longrightarrow \end{array}$$

By the LES-of-Triple Proposition 67.9 we know that the diagram commutes and that the horizontal sequences are exact. Furthermore note that it follows from our hypothesis on f and the Homotopy Equivalence-H^*-Corollary 67.8 (1) that the two vertical maps to the left and the two vertical maps to the right are isomorphisms. It follows from the Five Lemma 37.5 that the middle vertical map is, as promised, an isomorphism. ■

67.5. **Reduced cohomology groups.** We saw on several occasions that the reduced homology groups $\widetilde{\mathrm{H}}_k(X)$ of a topological space can be convenient for expressing calculations and results. In this last section we quickly introduce the reduced cohomology groups. None of the concepts or results of this section are in anyway surprising. So the reader might as well move on to the next chapter.

Definition. Let X be a topological space. By the Augmentation Lemma 35.10 (1) we know that the following is a chain complex

$$\begin{array}{ccccccccccc} \dots & \xrightarrow{\partial_3} & \mathrm{C}_2(X) & \xrightarrow{\partial_2} & \mathrm{C}_1(X) & \xrightarrow{\partial_1} & \mathrm{C}_0(X) & \xrightarrow{\epsilon} & \mathbb{Z} & \to & 0. \\ & & & & & & \sum_{i=1}^{k} n_i \cdot \sigma_i & \mapsto & \sum_{i=1}^{k} n_i. & & \end{array}$$

We introduce the following two definitions.

(1) We refer to this chain complex as the augmented chain complex $\widetilde{\mathrm{C}}_*(X)$ of X.
(2) Let G be an abelian group. Given $n \in \mathbb{Z}_{\geq -1}$ we call $\widetilde{\mathrm{H}}^n(X;G) := \mathrm{H}^n(\mathrm{Hom}(\widetilde{\mathrm{C}}_*(X), G))$ the n-th reduced cohomology group of X with G-coefficients.

The following lemma is the analogue of the $\widetilde{\mathrm{H}}_n$–Functor Lemma 39.1 and the H^n–Functor Lemma 67.1:

Lemma 67.11. ($\widetilde{\mathbf{H}}^n$–Functor Lemma) Let G be an abelian group.

(1) If $f\colon X \to Y$ is a continuous map between two topological spaces, then we denote by $f_*\colon \widetilde{\mathrm{C}}_*(X) \to \widetilde{\mathrm{C}}_*(Y)$ the map that is the usual map on $\widetilde{\mathrm{C}}_n(X) = \mathrm{C}_n(X)$ for $n \geq 0$ and that is the identity on the final $\mathbb{Z}$-term. This map is a chain map and thus induces homomorphisms $f^*\colon \widetilde{\mathrm{H}}^n(Y;G) \to \widetilde{\mathrm{H}}^n(X;G)$ on the reduced cohomology groups.
(2) For any $n \in \mathbb{Z}_{\geq -1}$ the maps
$$\begin{array}{ccc} X & \mapsto & \widetilde{\mathrm{H}}^n(X;G) \\ (f\colon X \to Y) & \mapsto & (f^*\colon \widetilde{\mathrm{H}}^n(Y;G) \to \widetilde{\mathrm{H}}^n(X;G)) \end{array}$$
define a contravariant functor from the category *Top* of topological spaces to the category *AbGroup* of abelian groups.
(3) The maps $\mathrm{id}\colon \widetilde{\mathrm{C}}_n(X) \to \mathrm{C}_n(X)$ for $n \geq 0$ and the zero map $\widetilde{\mathrm{C}}_{-1}(X) \to \mathrm{C}_{-1}(X)$ define a natural chain map and for any $n \geq -1$ they induce a natural transformation
$$ж\colon \mathrm{H}^n(X;G) \to \widetilde{\mathrm{H}}^n(X;G)$$
from the functor $X \mapsto \mathrm{H}^n(X;G)$ to the functor $X \mapsto \widetilde{\mathrm{H}}^n(X;G)$.

Proof. The lemma follows easily from the $\widetilde{\mathrm{H}}_n$–Functor Lemma 39.1 together with the Dual Cochain Map Lemma 66.9. ∎

The following lemma is the analogue of the Reduced Homology Lemma 39.2:

Lemma 67.12. (Reduced Cohomology Lemma) Let G be an abelian group. For every topological space X the following statements hold:

(-1) We have
$$\widetilde{\mathrm{H}}^{-1}(X;G) = \begin{cases} 0, & \text{if } X \neq \varnothing, \\ \mathrm{Hom}(\mathbb{Z}, G) =_{\mathrm{i}} G, & \text{if } X = \varnothing. \end{cases}$$
(0) (a) If X is non-empty, then the following sequence is exact:
$$\begin{array}{ccccccccc} 0 & \to & G & \to & \mathrm{H}^0(X;G) & \xrightarrow{ж} & \widetilde{\mathrm{H}}^0(X;G) & \to & 0. \\ & & g & \mapsto & [0\text{-simplex} \mapsto g] & & & & \end{array}$$
(b) For any $P \in X$ the map
$$\begin{array}{ccc} \mathrm{H}^0(X;G) & \to & \widetilde{\mathrm{H}}^0(X;G) \oplus G \\ [\sigma] & \mapsto & ж([\sigma]) + \sigma(P) \end{array}$$

is an isomorphism.
(1) For every $n \geq 1$ the natural map $\text{ж}\colon \mathrm{H}^n(X;G) \to \widetilde{\mathrm{H}}^n(X;G)$ is the identity.

Proof. The proof is elementary and left to the reader. ■

We also have the following analogue of the Homotopy-$\widetilde{\mathrm{H}}_*$-Proposition 39.3 and the Homotopy-H^*-Proposition 67.7.

Proposition 67.13. (Homotopy-$\widetilde{\mathrm{H}}^*$-Proposition) Let X and Y be two topological spaces and let $f, g\colon X \to Y$ be two continuous maps. If f and g are homotopic, then for each $n \in \mathbb{N}_0$ and each abelian group G we have $f^* = g^*\colon \widetilde{\mathrm{H}}^n(Y;G) \to \widetilde{\mathrm{H}}^n(X;G)$.

Proof. This proposition follows from the Homotopy-$\widetilde{\mathrm{H}}_*$-Proposition 39.3 (1) together with the Dual Cochain Map Lemma 66.9 (3a) and the Cochain Homotopy-H^*-Lemma 66.6 (1). ■

Next the following proposition plays the role of the Reduced LES-of-Pair Proposition 39.6 and the LES-of-Triple Proposition 67.9.

Proposition 67.14. (Reduced LES-of-Pair Proposition) Let X be a topological space and let $B \subset X$ be a non-empty subset. Given any abelian group G there exist natural homomorphisms $\delta_n\colon \widetilde{\mathrm{H}}^n(B;G) \to \mathrm{H}^{n+1}(X,B;G)$ which give rise to a long exact sequence

$$\ldots \xrightarrow{\delta_{n-1}} \mathrm{H}^n(X,B;G) \xrightarrow{p^*} \widetilde{\mathrm{H}}^n(X;G) \xrightarrow{i^*} \widetilde{\mathrm{H}}^n(B;G) \xrightarrow{\delta_n} \mathrm{H}^{n+1}(X,B;G) \xrightarrow{p^*} \ldots$$

Proof. In the proof of the Reduced LES-of-Pair Proposition 39.6 we showed that we have a short exact sequence $0 \to \widetilde{\mathrm{C}}_*(B) \to \widetilde{\mathrm{C}}_*(X) \to \mathrm{C}_*(X,B) \to 0$ of chain complexes. By the discussion on page 489 we know that each $\mathrm{C}_n(X,B)$ is a free abelian group. The proposition now follows from the Hom-Split-SES Lemma 66.4 and the Cohomology–LES Proposition 66.7. ■

We conclude this section with the following analogue of the Reduced-to-Relative-H_*-Lemma 39.7:

Lemma 67.15. (Reduced-to-Relative-H^*-Lemma) Let X be a topological space and let $n \in \mathbb{N}_0$. For each $x_0 \in X$ there exists an explicit natural isomorphism

$$\text{ж}_{X,\{x_0\}}\colon \mathrm{H}^n(X,\{x_0\}) \to \widetilde{\mathrm{H}}^n(X).$$

Proof. Similar to the proof of the Reduced-to-Relative-H_*-Lemma 39.7 one can easily obtain a proof using the Reduced LES-of-Pair Proposition 67.14. We leave it to the reader to fill in the details. ■

CHAPTER 68

Cellular cohomology and the Kronecker pairing

After all this theory we now want to calculate cohomology groups. In Chapter 47 we saw that in many cases the most efficient way to calculate homology groups is to go through cellular homology. In this chapter we will repeat the trick and we will see that we can calculate singular cohomology via cellular cohomology.

This allows us to calculate the *isomorphism types* of the cohomology groups of many topological spaces, e.g. of surfaces and of the real and complex projective spaces. We also introduce the Kronecker pairing between homology and cohomology. This simple minded object will be surprisingly helpful at showing that explicit cycles and cocycles represent non-trivial elements and bases of (co-) homology.

68.1. **Cellular cohomology.** First we recall some definitions and some notation from the context of cellular chain complexes.

Definition. Let X be a CW-complex.

(1) Given $n \in \mathbb{N}_0$ we denote by X^n the n-skeleton of X.

(2) We write $\mathrm{C}_n^{\mathrm{CW}}(X) := \mathrm{H}_n(X^n, X^{n-1})$. We refer to elements of $\mathrm{C}_n^{\mathrm{CW}}(X)$ as a cellular chain.

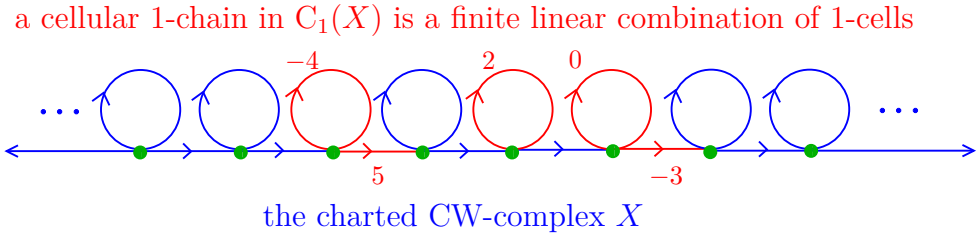
(3) (a) A charted CW-complex is a pair consisting of a CW-complex X together with a full set $\{\Phi_c \colon \overline{B}^{\dim(c)} \to X^{\dim(c)}\}_{c \in \mathrm{Cell}(X)}$ of characteristic maps of the cells of X.
(b) For an n-cell α we write $\Xi_\alpha := \Phi_{\alpha *}([\overline{B}^n]) \in \mathrm{H}_n(X^n, X^{n-1}) = \mathrm{C}_n^{\mathrm{CW}}(X)$.
By the Cellular Chain Complex–Basis Lemma 47.5 we know that for every $n \in \mathbb{N}_0$ the set $\{\Xi_c\}_{c \in \mathrm{Cell}_n(X)}$ is a basis of the free abelian group $\mathrm{C}_n^{\mathrm{CW}}(X) = \mathrm{H}_n(X^n, X^{n-1})$.

(4) We denote by $d_n \colon \mathrm{C}_n^{\mathrm{CW}}(X) \to \mathrm{C}_{n-1}^{\mathrm{CW}}(X)$ the cellular boundary map from page 588.

(5) We refer to
$$\ldots \xrightarrow{d_{n+2}} \mathrm{C}_{n+1}^{\mathrm{CW}}(X) \xrightarrow{d_{n+1}} \mathrm{C}_n^{\mathrm{CW}}(X) \xrightarrow{d_n} \mathrm{C}_{n-1}^{\mathrm{CW}}(X) \xrightarrow{d_{n-1}} \ldots$$
as the cellular chain complex.

(6) The homology groups of the cellular chain complex are the cellular homology groups $\mathrm{H}_n^{\mathrm{CW}}(X)$.

Example. Given a charted CW-complex X a cellular 1-chain in $\mathrm{C}_1^{\mathrm{CW}}(X)$ can be viewed as a finite linear combination of 1-cells.



Next we turn to a new concept, namely cellular cohomology:

Definition. Let X be a CW-complex and let G be an abelian group.

(1) Let $n \in \mathbb{N}_0$.

(a) We refer to the elements of $\mathrm{Hom}(\mathrm{C}_n^{\mathrm{CW}}(X), G)$ as cellular n-cochains.

(b) If X is charted, then we have

$$\mathrm{Hom}(\mathrm{C}_n^{\mathrm{CW}}(X), G) \underset{\uparrow}{=_{\mathrm{i}}} \text{maps from the set of } n\text{-cells to } G \;=\; G^{\#\, n\text{-cells}}.$$

by the Skeleton-H_*-Lemma 47.1 (1) and the Abelian Group-Homomorphism Lemma 66.2 (2)

(2) Given any $n \in \mathbb{N}_0$ we write $\delta_n := d_{n+1}^*$. We refer to

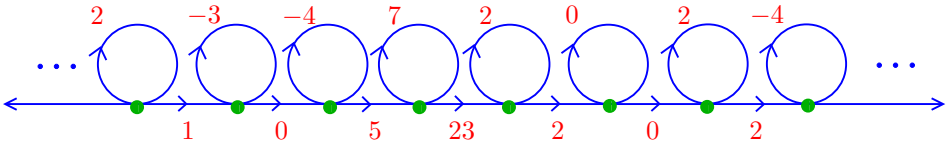
$$\ldots \leftarrow \mathrm{Hom}(\mathrm{C}_{n+1}^{\mathrm{CW}}(X), G) \xleftarrow{\delta_n} \mathrm{Hom}(\mathrm{C}_n^{\mathrm{CW}}(X), G) \xleftarrow{\delta_{n-1}} \mathrm{Hom}(\mathrm{C}_{n-1}^{\mathrm{CW}}(X), G) \leftarrow \ldots$$

as the cellular cochain complex. We define the n-th cellular cohomology group of X with G-coefficients as

$$\mathrm{H}_{\mathrm{CW}}^n(X; G) := \mathrm{H}^n(\mathrm{C}_*^{\mathrm{CW}}(X), G) = \frac{\ker(\delta_n \colon \mathrm{Hom}(\mathrm{C}_n^{\mathrm{CW}}(X), G) \to \mathrm{Hom}(\mathrm{C}_{n+1}^{\mathrm{CW}}(X), G))}{\mathrm{im}(\delta_{n-1} \colon \mathrm{Hom}(\mathrm{C}_{n-1}^{\mathrm{CW}}(X), G) \to \mathrm{Hom}(\mathrm{C}_n^{\mathrm{CW}}(X), G))}.$$

Example. As we discussed above, given a charted CW-complex X an integral cellular 1-cochain in $\mathrm{Hom}(\mathrm{C}_1^{\mathrm{CW}}(X), \mathbb{Z})$ assigns to *every* 1-cell an integer. So naively speaking a cellular 1-cochain can be viewed as an infinite linear combination of 1-cells.

a cellular 1-cochain in $\mathrm{C}^1(X; \mathbb{Z})$ assigns to *every* 1-cell an integer



the charted CW-complex X

Definition. Let $f \colon X \to Y$ be a cellular[538] map between two CW-complexes. We get an induced chain map $f_* \colon \mathrm{C}_*^{\mathrm{CW}}(X) \to \mathrm{C}_*^{\mathrm{CW}}(Y)$. Thus we obtain from the Dual Cochain Map Lemma 66.9 (1) and the Cochain Homotopy-H^*-Lemma 66.6 for any abelian group G and any $n \in \mathbb{N}_0$ an induced homomorphism

$$f^* \colon \mathrm{H}_{\mathrm{CW}}^n(Y; G) \;\to\; \mathrm{H}_{\mathrm{CW}}^n(X; G).$$

For completeness' sake we jot down the following fairly self-evident lemma.

Lemma 68.1. (Cellular Cohomology-Functor Lemma) Let G be an abelian group and let $n \in \mathbb{N}_0$. The maps

$$\begin{aligned} X &\mapsto \mathrm{H}_{\mathrm{CW}}^n(X; G) \\ (f \colon X \to Y) &\mapsto (f^* \colon \mathrm{H}_{\mathrm{CW}}^n(Y; G) \to \mathrm{H}_{\mathrm{CW}}^n(X; G)) \end{aligned}$$

define a contravariant functor from the category $\mathcal{CW}$ of CW-complexes to the category $\mathcal{AbGroup}$ of abelian groups. If G is a commutative ring, then this defines a contravariant functor to the category of $\mathcal{Mod}_G$ of G-modules.

[538] According to the definition on page 435 a map $f \colon X \to Y$ between two CW-complexes is called *cellular* if f is continuous and if for each $n \in \mathbb{N}_0$ we have $f(X^n) \subset Y^n$.

Proof. This lemma follows immediately from the Dual Cochain Map Lemma 66.9 (2) and the Cochain Complex-H*-Functor Lemma 66.5. ■

Perhaps not surprisingly we have the following theorem which plays a similar role as the Singular-Cellular-H$_*$-Proposition 47.3.

Proposition 68.2. (Singular-Cellular H*-Isomorphism Proposition) For any CW-complex X, any abelian group G and any $n \in \mathbb{N}_0$ there exists a natural isomorphism[539]

$$\mathrm{H}^n(X;G) \xrightarrow{\cong} \mathrm{H}^n_{\mathrm{CW}}(X;G).$$

Proof. Let X be a CW-complex. Recall that on page 633 we introduced the intermediate cellular chain complex that is given by

$$\mathrm{C}^{\mathrm{int}}_n(X) := \ker\big(\mathrm{C}_n(X^n) \xrightarrow{\partial_n} \mathrm{C}_{n-1}(X^n) \to \mathrm{C}_{n-1}(X^n, X^{n-1})\big) \quad \text{for } n \in \mathbb{N}_0.$$

We consider the two *natural* chain maps

$$\mathrm{C}_*(X) \xleftarrow{i} \mathrm{C}^{\mathrm{int}}_*(X) \xrightarrow{p} \mathrm{C}^{\mathrm{CW}}_*(X),$$

which are given by the obvious inclusion and the obvious projection. In the Singular-Cellular Chain Complex Proposition 50.9 we had seen that these two chain maps are chain homotopy equivalences. Thus it follows from the Dual Cochain Map Lemma 66.9 (3b) together with the Cochain Homotopy-H*-Lemma 66.6 that for any abelian group G we get isomorphisms

$$\mathrm{H}^n(X;G) = \mathrm{H}^n(\mathrm{C}_*(X);G) \underset{\cong}{\xrightarrow{i^*}} \mathrm{H}^n(\mathrm{C}^{\mathrm{int}}_*(X);G) \underset{\cong}{\xleftarrow{p^*}} \mathrm{H}^n(\mathrm{C}^{\mathrm{CW}}_*(X);G) = \mathrm{H}^n_{\mathrm{CW}}(X;G).$$

As pointed out above, the two chain maps i and p are natural. Therefore we see that the above isomorphism $\mathrm{H}^n(X;G) \to \mathrm{H}^n_{\mathrm{CW}}(X;G)$ is in fact natural. ■

68.2. **Calculations of (cellular) cohomology.** In this section we will use our expertise on cellular chain complexes to calculate many cellular cohomology groups. By the Singular-Cellular H*-Isomorphism Proposition 68.2 these calculations also give us the isomorphism types of singular cohomology.

Examples. Let G be an abelian group.

(1) Let $n \in \mathbb{N}$. On page 407 we showed that we can view S^n as a CW-complex with one 0-cell and one other cell of dimension n. Furthermore the cellular boundary maps are the zero maps.[540] Thus we see that the cellular chain complex is of the form

$$0 \to \mathbb{Z} \xrightarrow{0} \cdots \xrightarrow{0} \mathbb{Z} \to 0.$$

The dual cochain complex is then given by

$$0 \leftarrow \underbrace{\mathrm{Hom}(\mathbb{Z},G)}_{\cong G} \xleftarrow{0} \cdots \xleftarrow{0} \underbrace{\mathrm{Hom}(\mathbb{Z},G)}_{\cong G} \leftarrow 0.$$

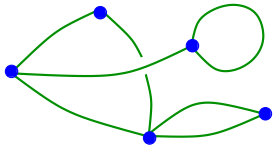
[539]Here "natural" means that there exists a natural transformation between the contravariant functors $X \mapsto \mathrm{H}^n(X;G)$ and $X \mapsto \mathrm{H}^n_{\mathrm{CW}}(X;G)$ from the category $\mathcal{CW}$ of CW-complexes to the category $\mathcal{AbGroup}$ of abelian groups.

[540]For $n \geq 2$ this follows by definition, for $n = 1$ this is a consequence of the Cellular-First Boundary Map Lemma 47.6.

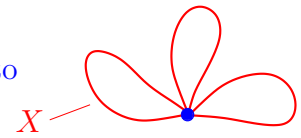
We obtain that
$$\mathrm{H}^k(S^n;G) \cong \begin{cases} G, & \text{if } k=0,n, \\ 0, & \text{otherwise.} \end{cases}$$
Similarly we see that
$$\mathrm{H}^k(S^0;G) \cong \begin{cases} G^2, & \text{if } k=0, \\ 0, & \text{if } k\neq 0. \end{cases}$$

(2) Let Γ be a connected finite topological graph. It follows from the Graph-Homotopy Type Corollary 17.6 (1) that Γ is homotopy equivalent to a CW-complex X with one 0-cell and $1-\chi(\Gamma)$ 1-cells. We. see that for any $n\in\mathbb{N}_0$ we have

the topological graph Γ



is homotopy equivalent to



by the Singular-Cellular H^*-Isomorphism Proposition 68.2

$$\mathrm{H}^n(\Gamma;G) \cong \mathrm{H}^n(X;G) \cong \mathrm{H}^n_{\mathrm{CW}}(X;G) \cong \mathrm{Hom}(\mathrm{C}^{\mathrm{CW}}_n(X);G) \cong \begin{cases} G, & \text{if } n=0, \\ G^{1-\chi(\Gamma)}, & \text{if } n=1, \\ 0, & \text{if } n\geq 2. \end{cases}$$

by the Homotopy Equivalence-H^* Corollary 67.8 (2)

it follows from the Cellular-First Boundary Map Lemma 47.6 that the cellular coboundary map is zero

(3) Given $g\in\mathbb{N}_0$ and $m\in\mathbb{N}_0$ we denote by $\Sigma_{g,m}$ the surface of genus g minus m open disks. Note that for $m\geq 1$ the topological space $\Sigma_{g,m}$ is homotopy equivalent to the wedge of $2g+m-1$ circles. The previous example, together with the Homotopy Equivalence-H^*-Corollary 67.8 (2), thus gives us the calculation of the cohomology groups of $\Sigma_{g,m}$ for $m\geq 1$.

(4) Let Σ_g be the surface of genus g. In the figure on page 408 we showed that we can view Σ_g as a CW-complex with one 0-cell, $2g$ 1-cells and one 2-cell. On page 599 we saw that the cellular boundary maps are all zero. Hence for any abelian group G and any $n\in\mathbb{N}_0$ we have
$$\mathrm{H}^n(\Sigma_g;G) \cong \mathrm{H}^n_{\mathrm{CW}}(\Sigma_g;G) = \mathrm{Hom}\left(\mathrm{C}^{\mathrm{CW}}_n(\Sigma_g);G\right) \cong \begin{cases} G, & \text{if } n=0 \text{ or } n=2, \\ G^{2g}, & \text{if } n=1, \text{ and} \\ 0, & \text{if } n\neq 0,1,2. \end{cases}$$

by the Singular-Cellular H^*-Isomorphism Proposition 68.2

□

Next we consider CW-complexes with interesting cellular boundary maps:

Examples.

(1) We consider the n-dimensional projective space $\mathbb{R}\mathrm{P}^n$. In the proof of the $\mathbb{R}\mathrm{P}^n$-H_*-Proposition 47.10 we saw that there exists a CW-structure on $\mathbb{R}\mathrm{P}^n$ such that the corresponding cellular chain complex is of the following form:
$$0 \to \underset{\substack{\uparrow \\ \text{degree } n}}{\mathbb{Z}} \xrightarrow{\cdot(1+(-1)^n)} \mathbb{Z} \to \ldots \xrightarrow{0} \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$$
which allowed us to compute the homology groups of $\mathbb{R}\mathrm{P}^n$. Now let G be an abelian group. We dualize this chain complex, i.e. we apply the functor $\mathrm{Hom}(-,G)$. It follows from the Matrix–Hom-to-Ring Lemma 66.3 that the resulting cochain complex is isomorphic to the following cochain complex:
$$0 \leftarrow G \xleftarrow{\cdot(1+(-1)^n)} G \leftarrow \ldots \xleftarrow{0} G \xleftarrow{\cdot 2} G \xleftarrow{0} G \leftarrow 0.$$

For $G = \mathbb{Z}$ we thus obtain from the Singular-Cellular H^*-Isomorphism Proposition 68.2 that for any $k \in \{0, \dots, n\}$ we have

$$\mathrm{H}^k(\mathbb{R}\mathrm{P}^n;\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & \text{if } k = 0, \\ 0, & \text{if } k \text{ is odd and } k \neq n, \\ \mathbb{Z}_2, & \text{if } k \text{ is even and } k \neq 0, \\ \mathbb{Z}, & \text{if } k = n \text{ and } n \text{ is odd} \end{cases} \quad \text{but } \mathrm{H}_k(\mathbb{R}\mathrm{P}^n) \cong \begin{cases} \mathbb{Z}, & \text{if } k = 0, \\ \mathbb{Z}_2, & \text{if } k \text{ is odd and } k \neq n, \\ 0, & \text{if } k \text{ is even and } k \neq 0, \\ \mathbb{Z}, & \text{if } k = n \text{ and } n \text{ is odd} \end{cases}$$

where on the right-hand side we recall the calculation of the homology groups using the above cellular chain complex, see also the $\mathbb{R}\mathrm{P}^n$-H_*-Proposition 47.10. Note that the calculations for homology and cohomology are mysteriously similar and different at the same time.

(2) To add to the confusion we also compute the cohomology of $\mathbb{R}\mathrm{P}^n$ with $\mathbb{F}_2$-coefficients. It follows from the Matrix–Hom-to-Ring Lemma 66.3 that the corresponding dual cellular cochain complex is then given by

$$0 \to \mathbb{F}_2 \xleftarrow{0} \mathbb{F}_2 \leftarrow \dots \leftarrow \mathbb{F}_2 \xleftarrow{\cdot 2 = 0} \mathbb{F}_2 \xleftarrow{0} \mathbb{F}_2 \to 0$$

and we see that

$$\mathrm{H}^k(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2) \cong \begin{cases} \mathbb{F}_2, & \text{if } k = 0, \dots, n, \\ 0, & \text{otherwise.} \end{cases}$$

One can easily modify the above argument and apply it to $\mathbb{R}\mathrm{P}^\infty$, which we can view as a CW-complex with precisely one cell in each dimension, see 352. We obtain that for each $k \in \mathbb{N}_0$ we have

$$\mathrm{H}^k(\mathbb{R}\mathrm{P}^\infty;\mathbb{F}_2) \cong \mathbb{F}_2$$

(3) We recall that on page 410 we showed that $\mathbb{C}\mathrm{P}^n$ admits a CW-structure with exactly one cell in dimensions $0, 2, \dots, 2n$ and with no odd-dimensional cells. It follows easily from the Singular-Cellular H^*-Isomorphism Proposition 68.2 that

$$\mathrm{H}^k(\mathbb{C}\mathrm{P}^n;\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & \text{if } k = 0, 2, \dots, 2n, \\ 0, & \text{otherwise} \end{cases} \quad \begin{matrix} \text{Similarly} \\ \text{we obtain} \end{matrix} \quad \mathrm{H}^k(\mathbb{C}\mathrm{P}^\infty;\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & \text{f } k \text{ even}, \\ 0, & \text{if } k \text{ odd}. \end{cases}$$

Also note the following: Given $k < l \in \mathbb{N} \cup \{\infty\}$ we have an inclusion $\mathbb{C}\mathrm{P}^k \to \mathbb{C}\mathrm{P}^l$. Since the isomorphism of the Singular-Cellular H^*-Isomorphism Proposition 68.2 is natural and since the cellular chain complexes of $\mathbb{C}\mathrm{P}^n$ and $\mathbb{C}\mathrm{P}^\infty$ agree up to the dimension $2n+1$ we obtain almost immediately that the inclusion-induced maps $\mathrm{H}^k(\mathbb{C}\mathrm{P}^\infty;\mathbb{Z}) \to \mathrm{H}^k(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$ are actually isomorphisms up to dimension $2n$. □

68.3. **The Kronecker pairing.** In the previous section we used cellular cohomology to compute the isomorphism types of the cohomology groups of many different topological spaces. But this calculation of the isomorphism types does not give us any explicit generators of the cohomology groups. For example on page 866 we introduced a singular 1-cochain $\theta_{\mathbb{Z}} \colon \mathrm{C}_1(S^1) \to \mathbb{Z}$ and we showed in the θ-Cocycle Lemma 67.2 that $\theta_{\mathbb{Z}}$ defines a non-zero cohomology class $[\theta_{\mathbb{Z}}] \in \mathrm{H}^1(S^1;\mathbb{Z})$. By the discussion on page 880 we now also know that $\mathrm{H}^1(S^1;\mathbb{Z}) \cong \mathbb{Z}$, but we do not know whether $[\theta_{\mathbb{Z}}] \in \mathrm{H}^1(S^1;\mathbb{Z})$ is a generator.

On the other hand in earlier chapters we had obtained a pretty good understanding of generators of homology groups. The idea now is to translate information on homology groups to information on cohomology groups. The key tool for relating homology and cohomology is given by the Kronecker pairing that we now introduce.

Lemma 68.3. (Kronecker Pairing Lemma) Let C_* be a chain complex and let G be an abelian group.

(1) (a) The map

$$\begin{array}{rcl} \langle\ ,\ \rangle\colon \mathrm{H}^n(C;G)\times \mathrm{H}_n(C) & \to & G \\ ([\varphi],[c]) & \mapsto & \langle[\varphi],[c]\rangle \coloneqq \underset{\uparrow}{\varphi(c)} \end{array}$$

here $\varphi\in\mathrm{Hom}(C_n,G)$ and $c\in C_n$, so $\varphi(c)\in G$ is defined

is well-defined and bilinear.[541]

(b) If (X,A) is a pair of topological spaces, then the map

$$\begin{array}{rcl} \langle\ ,\ \rangle_X \coloneqq \langle\ ,\ \rangle_{(X,A)}\colon \mathrm{H}^n(X,A;G)\times \mathrm{H}_n(X,A) & \to & G \\ ([\varphi],[c]) & \mapsto & \langle[\varphi],[c]\rangle_X \coloneqq \varphi(c) \end{array}$$

is well-defined and bilinear.

We refer to both pairings as the Kronecker pairing.[542]

(2) (a) Let $f\colon C_*\to D_*$ be a chain map between two chain complexes and let G be an abelian group. For any $c\in\mathrm{H}_n(C)$ and any $\varphi\in\mathrm{H}^n(D;G)$ we have

$$\langle f^*(\varphi),c\rangle \;=\; \langle\varphi,f_*(c)\rangle.$$

(b) Let $f\colon (X,A)\to(Y,B)$ be a continuous map between two pairs of topological spaces. For any $c\in\mathrm{H}_n(X,A)$ and any $\varphi\in\mathrm{H}^n(Y,B;G)$ we have

$$\langle f^*(\varphi),c\rangle_X \;=\; \langle\varphi,f_*(c)\rangle_Y.$$

Proof. Let C_* be a chain complex and let G be an abelian group.

(1) (a) Let $\varphi\in\mathrm{Hom}(C_n,G)$ be a cocycle and let $c\in C_n$ be a cycle. Furthermore let $\alpha\in\mathrm{Hom}(C_{n-1},G)$ and let $d\in C_{n+1}$. Note that the following equalities hold in G:

$$(\varphi+\delta_{n-1}(\alpha))(c) \underset{\substack{\uparrow\\ \text{definition of } \delta_{n-1}}}{=} \varphi(c)+\alpha(\partial_n(c)) \underset{\substack{\uparrow\\ \text{since } c \text{ is a cycle}}}{=} \varphi(c) \;\&\; \varphi(c+\partial_n(d)) \underset{\substack{\uparrow\\ \text{definition of } \delta_{n-1}}}{=} \varphi(c)+(\delta_{n-1}(\varphi))(d) \underset{\substack{\uparrow\\ \text{since } \varphi \text{ is a cocycle}}}{=} \varphi(c).$$

This calculation shows that the definition of the Kronecker pairing does not depend on the choices of the representatives of the cohomology and the homology classes. Finally note that it is clear that the Kronecker pairing is bilinear.

(b) This is just a special case of (a).

(2) (a) The statement also follows immediately from the definitions: Indeed, if $[c]\in\mathrm{H}_n(C)$ and $[\varphi]\in\mathrm{H}^n(D;G)$, then

$$\langle f^*([\varphi]),[c]\rangle \;=\; \langle[f^*\varphi],[c]\rangle \;=\; (f^*\varphi)(c) \underset{\substack{\uparrow\\ \text{definition of } f^*\varphi}}{=} \varphi(f_*(c)) \;=\; \langle[\varphi],[f_*(c)]\rangle \;=\; \langle[\varphi],f_*([c])\rangle.$$

(b) This statement is just a special case of (a). ∎

[541] In general, given a commutative ring R a map $\Theta\colon U\times V\to W$ between R-modules is called *bilinear*, if for all $u,u'\in U$, $v,v'\in V$ and $r\in R$ we have

$$\begin{array}{lllllll} \Theta(u+u',v) & = & \Theta(u,v)+\Theta(u',v) & \text{and} & \Theta(r\cdot u,v) & = & r\cdot\Theta(u,v), \quad \text{and} \\ \Theta(u,v+v') & = & \Theta(u,v)+\Theta(u,v') & \text{and} & \Theta(u,r\cdot v) & = & r\cdot\Theta(u,v). \end{array}$$

In our special case we consider the case of $\mathbb{Z}$-modules.

[542] Leopold Kronecker (1823–1891) was a German mathematician who worked on number theory. I don't know how his name got attached to this pairing.

Now we state the lemma which will allow us in many situations to show that a given collection of cocycles forms a basis for a given cohomology group.

Lemma 68.4. (Kronecker-Basis Lemma)

(1) Let A and B be free abelian groups of rank k, let $a_1, \dots, a_k \in A$ and let $b_1, \dots, b_k \in B$. Furthermore let $\langle\ ,\ \rangle\colon A \times B \to \mathbb{Z}$ be a bilinear pairing. If

$$\det\Big(\underbrace{\langle a_i, b_j\rangle_{i,j=1,\dots,k}}_{=(k\times k)\text{–matrix}}\Big) \;=\; \pm 1,$$

then both $a_1, \dots, a_k$ and $b_1, \dots, b_k$ are bases.

(2) Let X be a topological space and let $n \in \mathbb{N}_0$. Suppose that $\mathrm{H}^n(X;\mathbb{Z})$ and $\mathrm{H}_n(X;\mathbb{Z})$ are free abelian groups of rank k and suppose that we are given $\varphi_1, \dots, \varphi_k \in \mathrm{H}^n(X;\mathbb{Z})$ and $c_1, \dots, c_k \in \mathrm{H}_n(X)$. If

$$\det\Big(\underbrace{\langle \varphi_i, c_j\rangle_{i,j=1,\dots,k}}_{=(k\times k)\text{–matrix}}\Big) \;=\; \pm 1,$$

then $\varphi_1, \dots, \varphi_k \in \mathrm{H}^n(X;\mathbb{Z})$ and $c_1, \dots, c_k \in \mathrm{H}_n(X)$ are both bases.

Remark. The Kronecker-Basis Lemma 68.4 (2) is really quite generous, it gives us as output that both sets are bases. In particular to apply the lemma we do not even have to verify that the c_i form a basis. □

Example. Let $X = S^1$. We denote by $\theta_{\mathbb{Z}}\colon \mathrm{C}_1(S^1) \to \mathbb{Z}$ the singular 1-cocycle that we introduced on page 866. We consider the singular cycle $\mu\colon \Delta^1 \to S^1$ that is given by $(1-t, t) \mapsto \exp(2\pi\,\mathrm{i}t)$. We now see that

$$\langle [\theta_{\mathbb{Z}}], [\mu]\rangle \;\underset{\text{definition of } \langle\ ,\ \rangle_{S^1}}{=}\; \theta_{\mathbb{Z}}(\mu) \;\underset{\text{definition of } \theta_{\mathbb{Z}}}{=}\; \alpha_{\mathbb{Z}}\Big(\text{boundary of the lift } \begin{array}{ccc}\widetilde{\mu}\colon \Delta^1 & \to & \mathbb{R}\\ (1-t,t) & \mapsto & t\end{array} \text{ of } \mu\Big) \;\underset{\text{see the discussion on page 866}}{=}\; 1.$$

From the Sphere-H_*-Proposition 41.2 we know that $\mathrm{H}_1(S^1) \cong \mathbb{Z}$ and by the discussion on page 880 we know that $\mathrm{H}^1(S^1;\mathbb{Z}) \cong \mathbb{Z}$. Therefore it follows from the Kronecker-Basis Lemma 68.4 (2) that $[\theta_{\mathbb{Z}}] \in \mathrm{H}^1(S^1;\mathbb{Z})$ is in fact a generator.[543] In particular we have an equality $\mathrm{H}^1(S^1;\mathbb{Z}) = \mathbb{Z} \cdot [\theta_{\mathbb{Z}}]$. □

Proof.

(1) We denote by $i\colon \mathbb{Z}^k \to A$ and by $j\colon \mathbb{Z}^k \to B$ the homomorphisms that are given by $e_i \mapsto a_i$ and by $e_i \mapsto b_i$. We denote by $\Phi\colon A \to \mathrm{Hom}(B, \mathbb{Z})$ the homomorphism that is given by $a \mapsto (b \mapsto \langle a, b\rangle)$.[544] The desired statement follows from the following three observations:

(a) Let $\gamma\colon G \to H$ be an epimorphism between two finitely generated free abelian groups of the same rank. An elementary argument shows that this implies that γ is an isomorphism.

[543] It also follows from the Kronecker-Basis Lemma 68.4 (2) that the singular 1-simplex $\mu\colon \Delta^1 \to S^1$ represents a generator for $\mathrm{H}_1(S^1) \cong \mathbb{Z}$. We had obtained that result on page 540.

[544] The fact that $\langle\ ,\ \rangle$ is bilinear implies that $b \mapsto \langle a, b\rangle$ is a homomorphism and that $\Phi\colon A \to \mathrm{Hom}(B, \mathbb{Z})$ is a homomorphism.

(b) We consider the following sequence of homomorphisms

$$\mathbb{Z}^k \xrightarrow{i} A \xrightarrow{\Phi} \operatorname{Hom}(B,\mathbb{Z}) \xrightarrow{j^*} \operatorname{Hom}(\mathbb{Z}^k,\mathbb{Z}).$$

With respect to the basis $e_1,\dots,e_k$ of $\mathbb{Z}^k$ and the dual basis $e_1^*,\dots,e_k^*$ of $\operatorname{Hom}(\mathbb{Z}^k,\mathbb{Z})$ this linear map is represented by the matrix $\langle a_i, b_j\rangle_{i,j=1,\dots,k}$. Thus by our hypotheses this map is an isomorphism. Furthermore by our hypotheses each of the four individual groups is a free abelian group of rank k.

(c) Since the map in (b) is an isomorphism it is in particular an epimorphism. Starting from the right and using (a) three times in a row we see that all maps are in fact isomorphisms. In particular we see that $i\colon \mathbb{Z}^k \to A$ is an isomorphism. But this implies that $a_1,\dots,a_k$ is a basis for A. The same argument turned around shows that $b_1,\dots,b_k$ is a basis for B.

(2) This statement follows immediately from (1) and the fact, proved in the Kronecker Pairing Lemma 68.3 (1), that the Kronecker pairing is bilinear. ■

Now we will use the Kronecker-Basis Lemma 68.4 (2) to determine explicit bases for the first cohomology groups of surfaces. We start out with the torus.

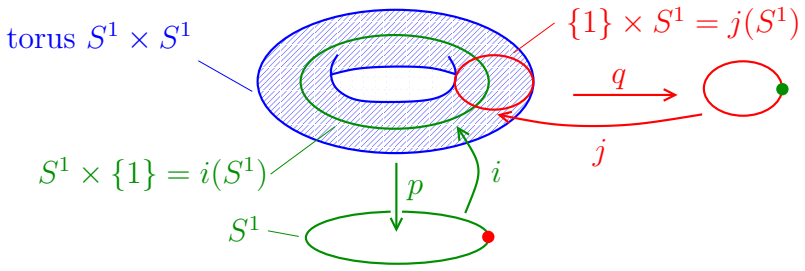
Lemma 68.5. (Torus-H^1-Lemma) We denote by $\theta_{\mathbb{Z}}\colon \mathrm{C}_1(S^1) \to \mathbb{Z}$ the singular 1-cocycle that we introduced on page 866. Furthermore we consider the two projections

$$\begin{array}{rcl} p\colon S^1\times S^1 & \to & S^1 \\ (v,w) & \mapsto & v \end{array} \quad \text{and} \quad \begin{array}{rcl} q\colon S^1\times S^1 & \to & S^1 \\ (v,w) & \mapsto & w. \end{array}$$

Then $p^*([\theta_{\mathbb{Z}}])$ and $q^*([\theta_{\mathbb{Z}}])$ form a basis for $\mathrm{H}^1(S^1\times S^1;\mathbb{Z}) \cong \mathbb{Z}^2$.

Proof. First we introduce some notation:

- Let $\mu\colon \Delta^1 \to S^1$ be the singular 1-cycle that is given by $\mu(1-t,t) = \exp(2\pi i t)$.
- Let $c\colon \Delta^1 \to S^1$ be the constant singular 1-cycle that is given by $(1-t,t) \mapsto 1$.
- We write $\theta = \theta_{\mathbb{Z}}$.
- We denote by $\begin{array}{rcl} i\colon S^1 & \to & S^1\times S^1 \\ z & \mapsto & (z,1) \end{array}$ and $\begin{array}{rcl} j\colon S^1 & \to & S^1\times S^1 \\ z & \mapsto & (1,z) \end{array}$ the two obvious inclusion maps.



We calculate that

it follows from the definition of the Kronecker pairing that $\langle [p^*\theta],[i_*(\mu)]\rangle = (p^*\theta)(i_*(\mu)) = \theta(p_*(i_*(\mu)))$, the other terms are dealt with in a similar fashion

$$\begin{pmatrix} \langle [p^*\theta],[i_*(\mu)]\rangle & \langle [p^*\theta],[j_*(\mu)]\rangle \\ \langle [q^*\theta],[i_*(\mu)]\rangle & \langle [q^*\theta],[j_*(\mu)]\rangle \end{pmatrix} \overset{\downarrow}{=} \begin{pmatrix} \theta(p_*(i_*(\mu))) & \theta(p_*(j_*(\mu))) \\ \theta(q_*(i_*(\mu))) & \theta(q_*(j_*(\mu))) \end{pmatrix} = \begin{pmatrix} \theta(\mu) & \theta(c) \\ \theta(c) & \theta(\mu) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

since $p\circ i = \mathrm{id}_{S^1}$ and $q\circ j = \mathrm{id}_{S^1}$ and since $p\circ j = c\colon \Delta^1 \to S^1$ and $q\circ i = c\colon \Delta^1 \to S^1$ $\uparrow$

We know from the Torus-H_*-Proposition 45.5 (1) that $H_1(S^1 \times S^1) \cong \mathbb{Z}^2$. Recall that we saw on page 880 that $H^1(S^1 \times S^1; \mathbb{Z}) \cong \mathbb{Z}^2$. The above calculation, together with the Kronecker-Basis Lemma 68.4 (2) shows that $p^*([\theta_\mathbb{Z}])$ and $q^*([\theta_\mathbb{Z}])$ form a basis for $H^1(S^1 \times S^1; \mathbb{Z})$. ■

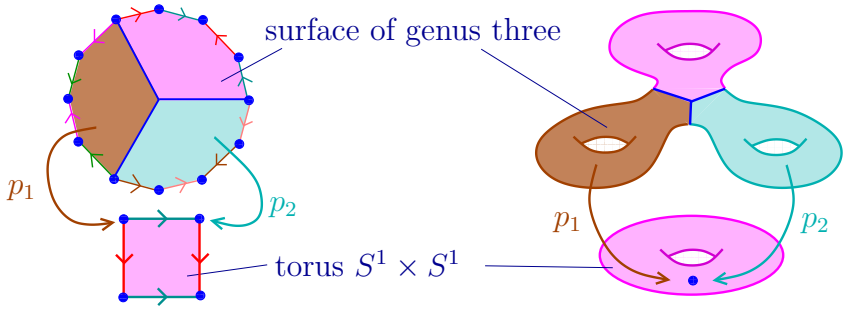
We will answer the question in the Torus–Cup Product Lemma 72.11.

Now we turn to surfaces of higher genus. As usual, given $g \in \mathbb{N}$ we denote by Σ_g the surface of genus g. The following lemma is a generalization of the Torus-H^1-Lemma 68.5.

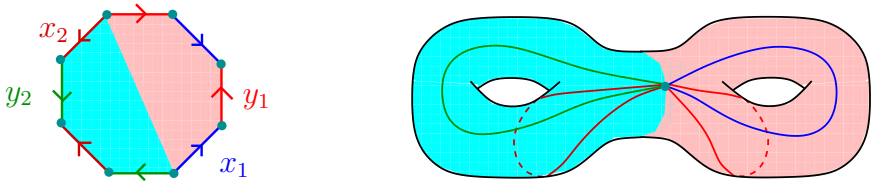
Lemma 68.6. (Surface-H^1-Lemma) Let $g \in \mathbb{N}_{\geq 2}$. We introduce the following notation:

- In the figure below we show projections $p_i \colon \Sigma_g \to S^1 \times S^1$, $i = 1, \dots, g$.
- Let $\alpha, \beta \in H^1(S^1 \times S^1; \mathbb{Z})$ be the basis that is given by the Torus-H^1-Lemma 68.5.

With this notation a basis for $H^1(\Sigma_g; \mathbb{Z}) \cong \mathbb{Z}^{2g}$ is given by $p_i^*(\alpha), p_i^*(\beta)$, $i = 1, \dots, g$.



Proof. As we will see, the proof of the lemma is very similar to the proof of the Torus-H^1-Lemma 68.5. For $i = 1, \dots, g$ we write $\varphi_i = p_i^*(\alpha)$ and $\psi_i := p_i^*(\beta)$. We consider the singular 1-cycles $x_1, \dots, x_g$ and $y_1, \dots, y_g$ that are indicated in the figure below.



As in the proof of the Torus-H^1-Lemma 68.5 it follows easily from the definitions that (for $g = 2$) we have

$$\begin{pmatrix} \langle \varphi_1, x_1 \rangle & \langle \varphi_1, y_1 \rangle & \langle \varphi_1, x_2 \rangle & \langle \varphi_1, y_2 \rangle \\ \langle \psi_1, x_1 \rangle & \langle \psi_1, y_1 \rangle & \langle \psi_1, x_2 \rangle & \langle \psi_1, y_2 \rangle \\ \langle \varphi_2, x_1 \rangle & \langle \varphi_2, y_1 \rangle & \langle \varphi_2, x_2 \rangle & \langle \varphi_2, y_2 \rangle \\ \langle \psi_2, x_1 \rangle & \langle \psi_2, y_1 \rangle & \langle \psi_2, x_2 \rangle & \langle \psi_2, y_2 \rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The lemma now follows immediately from the Kronecker-Basis Lemma 68.4 (2). ■

CHAPTER 69

Excision and Mayer–Vietoris for cohomology

In this chapter we want to extend the following deep theorems from the homological setup to the cohomological setting:

(1) The Excision Theorem 40.1.
(2) The Mayer-Vietoris Theorem 45.1.

These theorems will allow us in particular to calculate the cohomology groups of wedges.

69.1. **Excision theorems for cohomology groups.** We start out with the following analogue of the Excision Theorems 40.1 and 40.2, the Manifold–Excision Theorem 40.3 and the CW-Excision Theorem 40.4.

Theorem 69.1. (Excision-in-H*-Theorem) Let X be a topological space, let G be an abelian group and let $n \in \mathbb{N}_0$.

(1) (a) Let $Z \subset A \subset X$ be subsets such that the closure of Z is contained in the interior of A. The inclusion $i\colon (X \setminus Z, A \setminus Z) \to (X, A)$ induces an isomorphism

$$i^*\colon \mathrm{H}^n(X, A; G) \xrightarrow{\cong} \mathrm{H}^n(X \setminus Z, A \setminus Z; G).$$

(b) Let K be a subset of X and let U be a neighborhood of the closure of K. The inclusion $i\colon (U, U \setminus K) \to (X, X \setminus K)$ induces an isomorphism

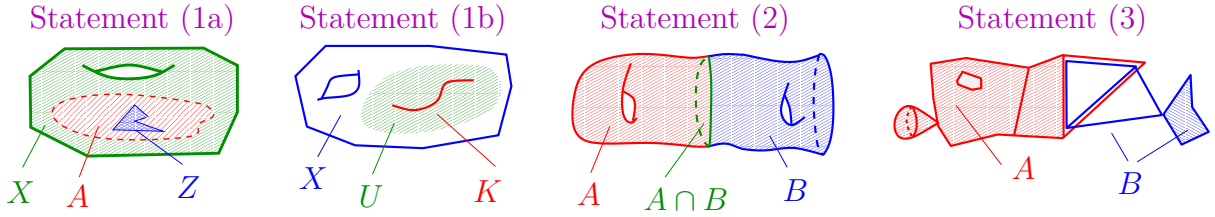
$$i^*\colon \mathrm{H}^n(X, X \setminus K; G) \xrightarrow{\cong} \mathrm{H}^n(U, U \setminus K; G).$$

(2) We suppose that X is a topological manifold. Let $A, B \subset X$ be two codimension-zero submanifolds such that the following statements hold:
(a) $X = A \cup B$.
(b) $A \cap B$ is a union of components of ∂A.
(c) A and B are closed subsets of X.
The following inclusion-induced map is an isomorphism:

$$\mathrm{H}^n(X, A; G) \to \mathrm{H}^n(B, A \cap B; G).$$

(3) If X is a CW-complex and if A and B are two subcomplexes of X with $X = A \cup B$, then the following inclusion-induced map is an isomorphism:

$$\mathrm{H}^n(X, A; G) \to \mathrm{H}.(B, A \cap B; G).$$



The proof of the Excision-in-H*-Theorem 69.1 relies on the following wonderful proposition:

Proposition 69.2. (Also-Iso-on-Cohomology Proposition) Let $f\colon (X,A) \to (Y,B)$ be a continuous map between pairs of topological spaces. If for all $n \in \mathbb{N}_0$ the induced map

$$f_*\colon \mathrm{H}_n(X,A) \;\to\; \mathrm{H}_n(Y,B)$$

is an isomorphism, then for all abelian groups G and all $n \in \mathbb{N}_0$ the induced map

$$f^*\colon \mathrm{H}^n(Y,B;G) \;\to\; \mathrm{H}^n(X,A;G)$$

is also an isomorphism for all $n \in \mathbb{N}_0$.

Proof of the Also-Iso-on-Cohomology Proposition 69.2. Let $f\colon (X,A) \to (Y,B)$ be a continuous map between pairs of topological spaces and let G be an abelian group. We suppose that $f_*\colon \mathrm{H}_n(X,A) \to \mathrm{H}_n(Y,B)$ is an isomorphism for all $n \in \mathbb{N}_0$. By the discussion on page 489 we know that all chain groups $\mathrm{C}_n(X,A)$ and $\mathrm{C}_n(Y,B)$ are free abelian groups. It thus follows from the Quasi Iso–Homotopy Equivalence Proposition 50.4 that the induced map $f_*\colon \mathrm{C}_*(X,A) \to \mathrm{C}_*(Y,B)$ is a chain homotopy equivalence. Now we get the desired conclusion from the Dual Cochain Map Lemma 66.9 (3b) together with the Cochain Homotopy-H*-Lemma 66.6 (2). ■

Proof of the Excision-in-H*-Theorem 69.1. It follows from Excision Theorems 40.1 and 40.2, the Manifold–Excision Theorem 40.3 and the CW-Excision Theorem 40.4 that in each of the four situations under consideration the inclusion-induced map on homology is an isomorphism for all $n \in \mathbb{N}_0$. It follows from the Also-Iso-on-Cohomology Proposition 69.2 that in all cases for all abelian groups G all inclusion-induced maps on cohomology groups with G-coefficients are isomorphisms. ■

69.2. **Good subsets and relative cohomology.** Psychologically it is often helpful to replace relative (co-) homology groups by the (co-) homology groups of a quotient space. In the Relative-to-Quotient-H$_*$-Proposition 41.8 we had seen that in the homological setting we can perform this switch for reasonable subsets. In this section we will consider the cohomological setting. We start out with the following cohomological analogue of the definition on page 527.

Definition. Let X be a topological space and let $A \subset X$ be a subset. We denote by $p\colon (X,A) \to (X/A, A/A)$ the natural projection. For each $n \in \mathbb{N}_0$ and any abelian group G we define

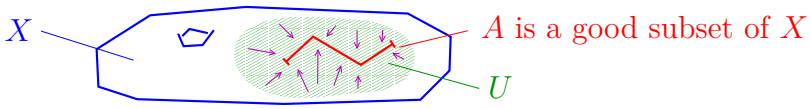
$$\widetilde{\mathrm{H}}^n(X/A;G) \xleftarrow[\cong]{\text{isomorphism } \text{ж} \text{ from the Reduced-to-Relative Lemma 67.15}} \mathrm{H}^n(X/A, A/A;G) \xrightarrow{p^*} \mathrm{H}^n(X,A;G)$$

$$\text{=: } \text{ю}$$

Next we recall the following definition from page 527:

Definition. Let X be a topological space.

(1) We say that a subset $A \subset X$ is good if A is closed and if there exists an open neighborhood U of A in X such that A is a deformation retract of U.
(2) We say that a point $a \in A$ is good, if $\{a\} \subset X$ is good.



Example. It follows from the Subcomplex–Closed Lemma 30.6 and the Subcomplex–Neighborhood Proposition 31.8 (1) that every subcomplex A of a CW-complex X is good. □

The following proposition is the cohomological analogue of the Relative-to-Quotient-H_*-Proposition 41.8.

Proposition 69.3. (Quotient-to-Relative-H^*-Proposition) Let X be a topological space, let $A \subset X$ be a subset and let $n \in \mathbb{N}_0$. If A is a good subset, then for any abelian group G the natural homomorphism $\imath\jmath\colon \widetilde{\mathrm{H}}^n(X/A;G) \to \mathrm{H}^n(X,A;G)$ is an isomorphism.

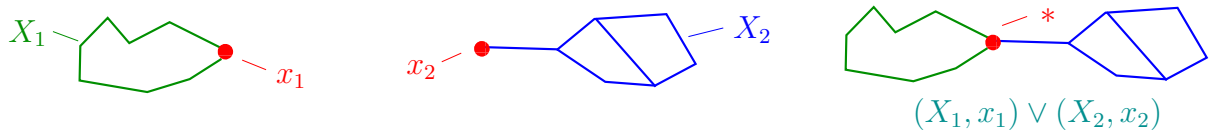
Proof. At this stage we have all the tools to rewrite the proof of the Relative-to-Quotient-H_*-Proposition 41.8. We just need to replace the Reduced-to-Relative-H_*-Lemma 39.7 by the Reduced-to-Relative-H^*-Lemma 67.15 and (more importantly) we need to replace the Excision Theorem 40.1 by the Excision-in-H^*-Theorem 69.1 (1a). We leave it to the reader to verify that nothing goes wrong. ■

69.3. **The cohomology groups of wedges.** We recall the following definition from page 245:

Definition. Let $\{(X_k,x_k)\}_{k\in K}$ be a family of pointed topological spaces. We refer to

$$\bigvee_{k\in K}(X_k,x_k) \;:=\; \Big(\bigsqcup_{k\in K} X_k\Big)\Big/\bigsqcup_{k\in k}\{x_k\}$$

as the wedge of the pointed topological spaces (X_k,x_k).



The following proposition is the analogue of the Wedge-H_*-Proposition 41.10.

Proposition 69.4. (Wedge-H^*-Proposition) Let $\{(X_k,x_k)\}_{k\in K}$ be a family of pointed topological spaces such that each $x_k \in X_k$ is good. We denote the wedge point of $\bigvee_{k\in K}(X_k,x_k)$ by $*$. Let $n \in \mathbb{N}_0$.

(1) Given $j \in K$ we denote by $i_j\colon X_j \to \bigvee_{k\in K}(X_k,x_k)$ the natural inclusion that we introduced on page 245. The natural map

$$\prod_{k\in K} i_k^*\colon \mathrm{H}^n\Big(\bigvee_{k\in K}(X_k,x_k),\{*\};G\Big) \;\to\; \prod_{k\in K}\mathrm{H}^n(X_k,\{x_k\};G)$$

is an isomorphism.

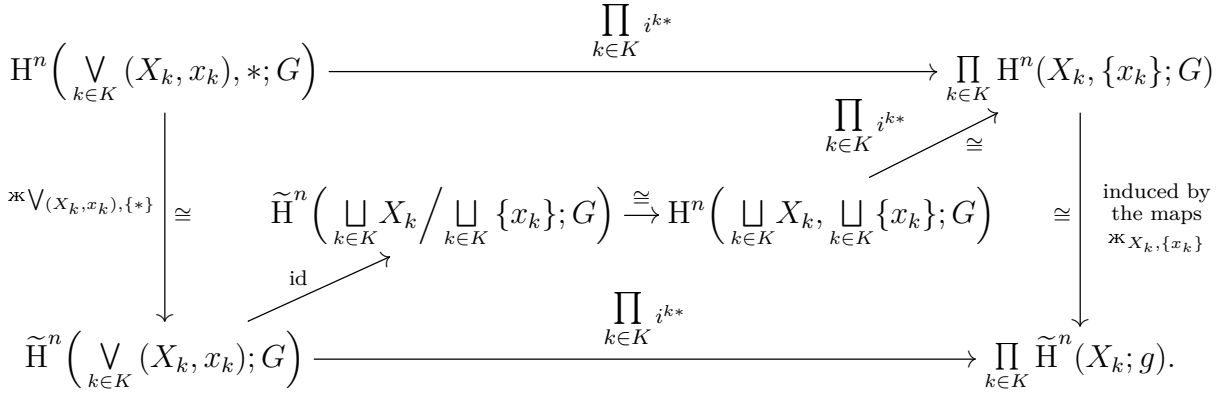
(2) For $j \in K$ let $p_j \colon \bigvee_{k\in K} X_k \to X_j$ be the natural map that we introduced on page 245. If K is finite, then the map

$$\bigoplus_{k\in K} p_k^* \colon \bigoplus_{k\in K} \mathrm{H}^n(X_k, \{x_k\}; G) \underset{\text{since } K \text{ is finite}}{=} \prod_{k\in K} \mathrm{H}^n(X_k, \{x_k\}; G) \to \mathrm{H}^n\Big(\bigvee_{k\in K} (X_k, x_k), \{*\}; G\Big)$$

is defined and it is the inverse to the isomorphism from (1).

Both statements also hold for reduced cohomology instead of relative cohomology.

Proof of the Wedge-H*-Proposition 69.4 (1). [545] We consider the following diagram:



The proposition follows from the following clarifications and observations:

(a) It follows from the Reduced-to-Relative-H*-Lemma 67.15 that the vertical maps to the left and right are isomorphisms.

(b) It follows from the obvious generalization of the Path-Component-H*-Lemma 67.4 that the top right diagonal map is an isomorphism.

(c) By hypothesis each $x_k \in X_k$ is good. It follows easily from the definitions that $\bigsqcup_{k\in K}\{x_k\}$ is a good subset of $\bigsqcup_{k\in K} X_k$. Thus we can apply the Quotient-to-Relative-H*-Proposition 69.3 to see that the middle horizontal map is an isomorphism.

(d) The bottom left diagonal map is by definition of the wedge indeed the identity.

(e) One can easily verify that the upper triangle commutes. This implies the desired statement for relative homology groups.

(f) One can also easily verify that the lower triangle commutes. Thus we also get the desired statement on reduced homology groups. ∎

Proof of the Wedge-H*-Proposition 69.4 (2). By (1) it suffices to prove the following claim:

Claim. The composition

$$\prod_{k\in K} \mathrm{H}^n(X_k;\{x_k\};G) \xrightarrow{\bigoplus_{k\in K} p_k^*} \mathrm{H}^n\Big(\bigvee_{k\in K}(X_k,x_k),\{*\};G\Big) \xrightarrow{\prod_{k\in K} i_k^*} \prod_{k\in K} \mathrm{H}^n(X_k,\{x_k\};G)$$

[545] The proof we provide is obtained from the proof of the Wedge-H$_*$-Proposition 41.10 by making all the obvious modifications. But we figured, that every now and then it does not hurt to write down the modified proof.

is the identity.

Proof. This claim follows immediately from the observation that $p_k \circ i_k = \mathrm{id}_{X_k}$ and that for $k \neq l$ the map $p_k \circ i_l$ is the constant map $x \mapsto x_k$ which induces the zero map on relative cohomology. ■

69.4. **The Mayer–Vietoris Theorem for cohomology groups.** We conclude this chapter with the following analogue of the the Mayer–Vietoris Theorem 45.1:

Theorem 69.5. (Mayer–Vietoris-H*-Theorem) Let X be a topological space and let $A, B \subset X$ be subspaces such that one of the following three conditions holds:

(1) X is the union of the interiors of A and B, i.e. $X = \mathring{A} \cup \mathring{B}$.
(2) X is a CW-complex and A and B are subcomplexes with $X = A \cup B$.
(3) X is a topological manifold and $A, B \subset X$ are two codimension-zero submanifolds such that the following conditions are satisfied:
   - $X = A \cup B$.
   - $A \cap B$ is a union of components of ∂A and it is a union of components of ∂B.
   - A and B are closed subsets of X.

It follows from the Excision-in-H*-Theorem 69.1 that in each of the three cases the inclusion-induced map $i^{(X,B)}_{(A,A\cap B)} \colon \mathrm{H}^{n+1}(X, B; G) \to \mathrm{H}^{n+1}(A, A \cap B; G)$ is an isomorphism. We refer to the homomorphism

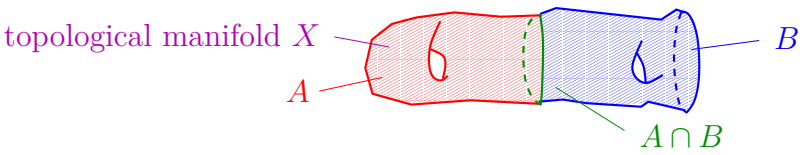
$$\mathrm{H}^n(A \cap B; G) \xrightarrow[\uparrow]{\delta_n} \mathrm{H}^{n+1}(A, A \cap B) \xrightarrow{\left(i^{(X,B)}_{(A,A\cap B)}\right)^{-1}} \mathrm{H}^{n+1}(X, B; G) \xrightarrow{\text{я}} \mathrm{H}^{n+1}(X; G)$$

connecting homomorphism of the long exact sequence corresponding to the pair $(A, A \cap B)$, provided by the LES-of-Triple Proposition 67.9

as the connecting homomorphism $\delta_n \colon \mathrm{H}^n(A \cap B; G) \to \mathrm{H}^{n+1}(X; G)$ in cohomology corresponding to (X, A, B). The connecting homomorphism is natural in a suitable sense and w with this notation the following sequence is exact:

$$\to \mathrm{H}^n(X; G) \xrightarrow{i^X_A \oplus i^X_B} \mathrm{H}^n(A; G) \oplus \mathrm{H}^n(B; G) \xrightarrow{(i^A_{A\cap B}, -i^B_{A\cap B})} \mathrm{H}^n(A \cap B; G) \xrightarrow{\delta_n} \mathrm{H}^{n+1}(X; G) \to$$

Totally analogous statements also hold if we replace the cohomology groups of $A \cap B, A, B$ and X by the corresponding reduced cohomology groups.



Proof. We gave two slightly different proofs for the Mayer–Vietoris Theorem 45.1. Both proofs can be rewritten easily for the current context:

(1) In the first proof of the Mayer–Vietoris Theorem 45.1, which we gave in Subsection 45.1.1, we just need to replace the LES-of-Pair Proposition 38.3 by the LES-of-Triple Proposition 67.9 and (more importantly), we need to replace the Excision Theorem 40.1, the Manifold–Excision Theorem 40.3 and the CW-Excision Theorem 40.4 by the Excision-in-H*-Theorem 69.1.
(2) Using the Hom-Split-SES Lemma 66.4, the Quasi Iso–Homotopy Equivalence Proposition 50.4 and the Dual Cochain Map Lemma 66.9 one can easily modify the second

proof of the Mayer–Vietoris Theorem 45.1, which we gave in Section 45.1.2, to obtain the desired result.

We leave it to the reader to fill in the details. ■

CHAPTER 70

The Universal Coefficient Theorem for Cohomology Groups

One of the questions that arises from the last chapter is, how can we explain the erratic relationship between homology groups and cohomology groups that we saw on page 881. Furthermore it would be interesting to know whether given a chain complex one can determine the cohomology groups with G-coefficients from the usual homology groups.

These questions are purely in the realm of algebra. More precisely, let (C_*, ∂_*) be a chain complex of free abelian groups and let G an abelian group. We want to find a connection between the cohomology groups $\mathrm{H}^*(C;G) = \mathrm{H}^*(\mathrm{Hom}(C_*, G))$ and the homology groups $\mathrm{H}_*(C)$. This problem, and later on its solution, is very similar to the problem we faced in Chapter 52, where we showed that we can determine the homology groups $\mathrm{H}_*(C_* \otimes G)$ of a chain complex with G-coefficients from the usual homology groups $\mathrm{H}_*(C_*)$ of the chain complex.

70.1. **Contravariant functors and exactness.** In the following we denote, as usual, by *AbGroup* the category *AbGroup* of abelian groups. Recall that on page 666 we defined what it means for an additive *covariant* functor $F\colon$ *AbGroup* $\to$ *AbGroup* to be left-exact, right-exact and exact. Basically the same way we can introduce these notions for additive *contravariant* functors $F\colon$ *AbGroup* $\to$ *AbGroup*:

Definition. Let $F\colon$ *AbGroup* $\to$ *AbGroup* be an additive contravariant functor.

(1) We say that F is left-exact[546] if for every exact sequence[547]

$$A \xrightarrow{i} B \xrightarrow{j} C \longrightarrow 0$$

of abelian groups the sequence

$$0 \longrightarrow F(C) \xrightarrow{F(j)} F(B) \xrightarrow{F(i)} F(A)$$

is also exact.

(2) We say that F is right-exact, if for every exact sequence

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{j} C$$

of abelian groups the sequence

$$F(C) \xrightarrow{F(j)} F(B) \xrightarrow{F(i)} F(A) \longrightarrow 0$$

is also exact.

(3) We say that F is exact if it is left-exact and right-exact.

The following lemma is the analogue of the Exact Functor–Homology Lemma 53.3:

Lemma 70.1. (Exact Functor–Cohomology Lemma) Let $F\colon \mathcal{A}b\mathcal{G}roup \to \mathcal{A}b\mathcal{G}roup$ be an additive contravariant functor and let (C_*, ∂_*) be a chain complex. If F is exact, then for any $k \in \mathbb{N}_0$ we have a natural isomorphism

$$\mathrm{H}^k(\underbrace{F(C_*), F(\partial_*)}_{\text{cochain complex}}) \to F(\mathrm{H}_k(\underbrace{C_*, \partial_*}_{\text{chain complex}})).$$

Proof. We leave it to the reader to modify the proof of the Exact Functor–Homology Lemma 53.3 to obtain our new lemma. ■

Recall that in the Tensor–Right Exact Lemma 53.4 we saw that the covariant functor $A \mapsto A \otimes G$ is right-exact but that in general it is not left-exact. With the contravariant functor $A \mapsto \mathrm{Hom}(A, G)$ the situation is exactly the opposite. More precisely, first we give an example that shows that the contravariant functor $A \mapsto \mathrm{Hom}(A, G)$ is in general not right-exact and then we prove a lemma that shows that this functor is left-exact.

Example. We consider the chain complex

$$0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{p} \mathbb{Z}_2$$

where $p\colon \mathbb{Z} \to \mathbb{Z}_2$ denotes the natural projection. We apply the contravariant functor $A \mapsto \mathrm{Hom}(A, \mathbb{Z})$. By the Matrix–Hom-to-Ring Lemma 66.3 the resulting cochain complex is of the form

$$\underbrace{\mathrm{Hom}(\mathbb{Z}_2, \mathbb{Z})}_{=0} \xrightarrow{p^*} \underbrace{\mathrm{Hom}(\mathbb{Z}, \mathbb{Z})}_{\cong \mathbb{Z}} \xrightarrow[\cdot 2]{(\cdot 2)^*} \underbrace{\mathrm{Hom}(\mathbb{Z}, \mathbb{Z})}_{\cong \mathbb{Z}} \to 0.$$

We see that the rightmost map is not surjective. Thus this cochain complex is not exact. This shows that the functor $A \mapsto \mathrm{Hom}(A, \mathbb{Z})$ is not right-exact. □

The following lemma is the promised analogue of the Tensor–Right Exact Lemma 53.4.

Lemma 70.2. (Hom-Left Exact Lemma) For any abelian group G the contravariant functor $A \mapsto \mathrm{Hom}(A, G)$ is left-exact.

Proof. Let G be an abelian group and let

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

be an exact sequence of abelian groups. We need to show that the sequence

$$0 \longrightarrow \mathrm{Hom}(C, G) \xrightarrow{\beta^*} \mathrm{Hom}(B, G) \xrightarrow{\alpha^*} \mathrm{Hom}(A, G)$$

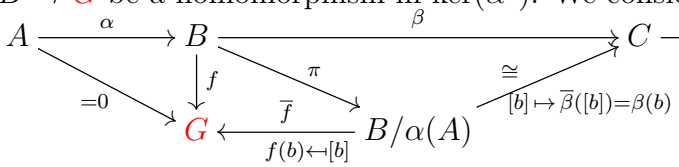
is also exact. It is clear that β^* is injective. Next note that $\alpha^* \circ \beta^* = (\beta \circ \alpha)^* = 0$, i.e. we have $\mathrm{im}(\beta^*) \subset \ker(\alpha^*)$. Thus it remains to prove the following claim:

Claim. We have $\ker(\alpha^*) \subset \mathrm{im}(\beta^*)$.

[546] Admittedly one could also refer to this definition as "right"-exact, since the direction changes. But the definition we give is the one which is commonly used in the literature.

[547] As in the Additive Functor–Basics Lemma 52.1 we see that for an additive contravariant functor we have $F(0) = 0$.

Proof. Let $f\colon B \to G$ be a homomorphism in $\ker(\alpha^*)$. We consider the following diagram:



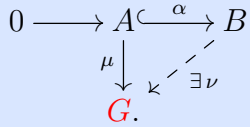
We make the following observations:

(1) Since $f \in \ker(\alpha^*)$ we know that $\alpha^*(f) = 0$ which in turn implies that f vanishes on $\alpha(A)$. We denote by $\overline{f}\colon B/\alpha(A) \to G$ the resulting homomorphism given by $[b] \mapsto f(b)$.
(2) We denote $\pi\colon B \to B/\alpha(A)$ the projection.
(3) It follows from the exactness of the original sequence and the Exactness Lemma 37.1 (3) that the homomorphism β induces an isomorphism $\overline{\beta}\colon B/\alpha(A) \to C$.
(4) By construction the above diagram commutes.
(5) Now we see that $f = \overline{f} \circ \overline{\beta}^{-1} \circ \beta$, i.e. we have $f = \beta^*(\overline{f} \circ \overline{\beta}^{-1})$. ∎

In the Tensor-with-Torsion Free Proposition 53.5 we showed that for any torsion-free abelian group G the covariant functor $A \mapsto A \otimes G$ from the category $\mathcal{AbGroup}$ of abelian groups to itself is exact. The question is, what condition on G do we have to impose so that $A \mapsto \mathrm{Hom}(A, G)$ is exact?

The following definition gives us the answer:

Definition. An abelian group G is called injective[548] if for all abelian groups A and B, every injective homomorphism $\alpha\colon A \to B$ and every homomorphism $\mu\colon A \to G$ there exists a homomorphism $\nu\colon B \to G$ such that the following diagram commutes:



The following lemma shows that we found the correct adjective:

Lemma 70.3. (Hom-to-Injective Lemma) If G is an injective abelian group, then the contravariant functor $A \mapsto \mathrm{Hom}(A, G)$ from the category $\mathcal{AbGroup}$ of abelian groups to itself is exact.

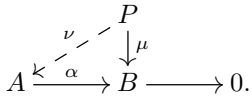
Proof. As in the Exact Functor Lemma 53.2 it suffices to show that for every short exact sequence

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$$

of abelian groups the resulting sequence

$$0 \to \mathrm{Hom}(C, G) \xrightarrow{\beta^*} \mathrm{Hom}(B, G) \xrightarrow{\alpha^*} \mathrm{Hom}(A, G) \to 0$$

[548]The notion of begin "injective" also makes sense for modules over a commutative ring R. In this setting the notion "injective" comes with the twin notion of being "projective": An R-module P is called *projective* if for all R-modules A and B, every epimorphism $\alpha\colon A \to B$ and every homomorphism $\mu\colon P \to B$ there exists a homomorphism $\nu\colon P \to A$ such that the following diagram commutes:



is also exact. Note that by the Hom-Left Exact Lemma 70.2 it remains to prove that $\alpha^*\colon \mathrm{Hom}(B, G) \to \mathrm{Hom}(A, G)$ is an epimorphism. But this follows immediately from the fact that α is a monomorphism and the hypothesis that G is injective. ■

Of course the Hom-to-Injective Lemma 70.3 is not all that interesting without examples of injective abelian groups. The key to finding interesting examples of injective abelian groups is the following definition:

Definition. An abelian group G is called divisible if for every $g \in G$ and $n \in \mathbb{N}$ there exists an $h \in G$ with $n \cdot h = g$.

Examples.

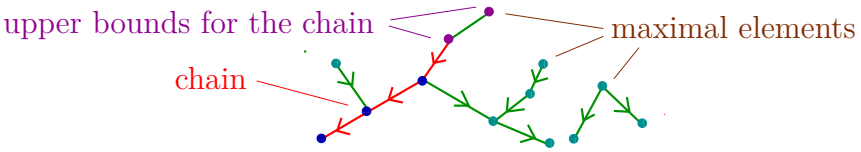
(1) The abelian groups $(\mathbb{Q}, +), (\mathbb{R}, +)$ and $(\mathbb{C}, +)$ are clearly divisible.
(2) Every quotient of a divisible group is easily seen to be divisible again. For example the quotient group $\mathbb{Q}/\mathbb{Z}$ is also divisible. This example also shows, using the notation of the definition, that h is not necessarily unique. For example, for $g = [0]$ and $n = 3$ we could take $h = [0]$ or $h = [\frac{1}{3}]$.
(3) The group $\mathbb{Z}$ is not divisible, and the only finite group that is divisible is the trivial group. □

The previous examples together with the following proposition give us many examples of injective abelian groups.

Proposition 70.4. (Divisible-implies Injective Proposition) Every abelian group that is divisible is also injective.

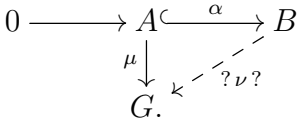
In the proof of the Divisible-implies Injective Proposition 70.4 we will need Zorn's Lemma which we now recall for the reader's convenience:

Lemma 70.5. (Zorn's Lemma) [549] Suppose a partially ordered set $(P, \leq)$ has the property that every chain has an upper bound in P. Then the set P contains at least one maximal element.



Now we can give the proof of the Divisible-implies Injective Proposition 70.4.

Proof of Proposition 70.4. Let G be an abelian group that is divisible. Suppose we are given a diagram

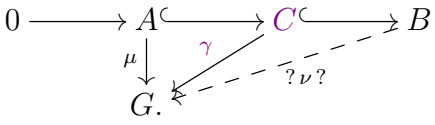


[549] We recall the definitions in the statement of Zorn's Lemma:

(1) A *partially ordered set* $(P, \leq)$ is a set together with a relation $\leq$ that is reflexive, transitive and antisymmetric, i.e. that has the property that $a \leq b$ and $b \leq a$ implies that $a = b$.
(2) A *chain* in P is a subset Q such that for any two elements $q, q' \in Q$ we have $q \leq q'$ or $q' \leq q$.
(3) An *upper bound* of a subset Q is an element $p \in P$ such that $q \leq p$ for all $q \in Q$.
(4) A *maximal element* of a subset Q is an element $q \in Q$ such that there is no $r \in Q$ with $q < r$.

Since α is a monomorphism we can assume that A is in fact a subgroup of B and that α is the inclusion map.

Let P be the set of all pairs (C, γ) where C is a subgroup of B which contains A and where $\gamma\colon C \to G$ is a homomorphism such that $\gamma|_A = \mu$. Note that such pairs fit into the following diagram:



We define $(C_i, \gamma_i) \leq (C_j, \gamma_j)$ if $C_i \subset C_j$ and if $\gamma_j|_{C_i} = \gamma_i$. One can easily verify that this defines a partial order on P.

Claim 1. Every chain $(C_j, \gamma_j)_{j\in J}$ in P admits an upper bound.

Proof. Let $(C_j, \gamma_j)_{j\in J}$ be a chain in P. If we are dealing with the empty chain, then an upper bound is given by the pair $(A, \mu) \in P$. Now suppose that the chain $(C_j, \gamma_j)_{j\in J}$ in P is non-empty. We get an upper bound by setting

$$C := \bigcup_{j\in J} C_j \qquad \text{and} \qquad \begin{aligned} \gamma\colon C &\to G \\ c &\mapsto \gamma_j(c) \end{aligned}$$

where we pick $j \in J$ with $c \in C_j$

Since $(C_j, \gamma_j)_{j\in J}$ is a non-empty chain one can easily verify that C is in fact a subgroup of B which contains A. Furthermore it follows easily from the definitions that γ is a homomorphism with $\gamma|_A = \mu$. ⊞

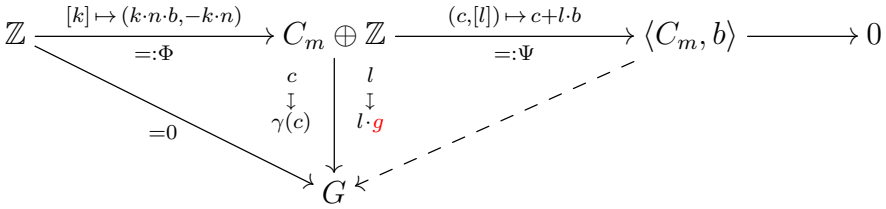
It follows from Claim 1 that we can apply Zorn's Lemma 70.5 to this partially ordered set $(P, \leq)$ and we obtain a maximal pair (C_m, γ_m). It remains to prove the following claim:

Claim 2. We have $C_m = B$.

Proof. Suppose that this is not the case. Then there exists a $b \in B \setminus C_m$. We denote by $\langle C_m, b\rangle$ the subgroup of B generated by C_m and b, i.e. $\langle C_m, b\rangle$ is the smallest subgroup of B that contains C_m and b. To obtain a contradiction it suffices to show that there exists a homomorphism $\delta\colon \langle C_m, b\rangle \to G$ that extends γ_m. We distinguish two cases:

(1) If there exists no $n \in \mathbb{N}$ such that $n \cdot b \in C_m$, then one can show easily that we have $C_m \oplus \langle b\rangle = \langle C_m, b\rangle$. We extend γ_m to $C_m \oplus \langle b\rangle = \langle C_m, b\rangle$ by setting $\delta(b) = 0$.

(2) Otherwise there exists a minimal $n \in \mathbb{N}$ such that $n \cdot b \in C_m$. Since G is divisible there exists a $g \in G$ such that $n \cdot g = \gamma(n \cdot b)$. Next we consider the following little diagram:



We make the following observations:

(a) The horizontal sequence is exact: Indeed, it is clear that $\Psi \circ \Phi = 0$. Next note that the image of Ψ lies a priori in B, but it is straightforward to see that the image of Ψ is precisely $\langle C_m, b\rangle$. Finally note that it follows easily from the definition of n that we have $\ker(\Psi) \subset \operatorname{im}(\Phi)$.

(b) It follows from the Coproduct-of-Abelian Groups Lemma 18.1 that the vertical homomorphism is well-defined.
(c) One can easily verify that $\operatorname{im}(\Phi)$ is contained in the kernel of the vertical map.
(d) It follows from (a) and (c) that we get the desired extension $\langle C_m, b\rangle \to G$ of $\gamma\colon C_m \to G$.

This concludes the proof of the claim and thus of the lemma. ■

It follows from the Exact Functor–Cohomology Lemma 70.1 and the Hom-to-Injective Lemma 70.3 that for any chain complex C_* and any injective abelian group G we have an isomorphism $\mathrm{H}^k(\operatorname{Hom}(C_*), G) \to \operatorname{Hom}(\mathrm{H}_k(C_*), G)$. Our next goal is to give an explicit isomorphism. This leads us to the following lemma, which is an analogue of the Homology-Multiplication Lemma 53.6.

Lemma 70.6. (Evaluation Map Lemma) Let (C_*, ∂_*) be a chain complex and let G be an abelian group. The map

$$\begin{array}{rcl} \mathrm{ev}\colon \mathrm{H}^n(C_*; G) & \to & \operatorname{Hom}(\mathrm{H}_n(C_*), G) \\ [\varphi\colon C_n \to G] & \mapsto & \left(\begin{array}{rcl} \mathrm{H}_n(C_*) & \to & G \\ [c] & \mapsto & \underset{\text{Kronecker Pairing}}{\langle [\varphi], [c]\rangle} = \varphi(c) \end{array} \right) \end{array}$$

is a well-defined natural[550] homomorphism. It is called evaluation homomorphism.

Proof. It follows from the Kronecker Pairing Lemma 68.3 (1) that this map is well-defined, that it is a homomorphism and that it is natural. ■

The following proposition is the cohomological analogue of the Torsion–Free Coefficients Proposition 53.7.

Proposition 70.7. (Injective Coefficients Proposition) Let (C_*, ∂_*) be a chain complex, let G be an abelian group and let $n \in \mathbb{N}_0$. If G is an injective abelian group, then the map

$$\mathrm{ev}\colon \mathrm{H}^n(C_*; G) \to \operatorname{Hom}(\mathrm{H}_n(C_*), G)$$

from the Evaluation Map Lemma 70.6 is a natural isomorphism. In particular for any pair (X, A) of topological spaces the map

$$\mathrm{ev}\colon \mathrm{H}^n(X, A; G) \to \operatorname{Hom}(\mathrm{H}_n(X, A); G)$$

[550]Here "natural" means two statements. If $f\colon C_* \to D_*$ is a chain map, then for any $n \in \mathbb{N}_0$ and any abelian group G the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}^n(D_*; G) & \xrightarrow{\mathrm{ev}} & \operatorname{Hom}(\mathrm{H}_n(D_*), G) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle (f_*)^*} \\ \mathrm{H}^n(C_*; G) & \xrightarrow{\mathrm{ev}} & \operatorname{Hom}(\mathrm{H}_n(C_*), G). \end{array}$$

Furthermore, if $\varphi\colon G_1 \to G_2$ is a homomorphism of abelian groups, then for any $n \in \mathbb{N}_0$ and any chain complex C_* the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}^n(C_*; G_1) & \xrightarrow{\mathrm{ev}} & \operatorname{Hom}(\mathrm{H}_n(C_*), G_1) \\ {\scriptstyle \varphi_*}\downarrow & & \downarrow{\scriptstyle \varphi_*} \\ \mathrm{H}^n(C_*; G_2) & \xrightarrow{\mathrm{ev}} & \operatorname{Hom}(\mathrm{H}_n(C_*), G_2). \end{array}$$

Here the map on the left is the one we had introduced in the Cohomology-Coefficients-Functor Lemma 66.10.

is an isomorphism.

Proof. Similar to the proof of the Torsion–Free Coefficients Proposition 53.7 this proposition follows quite easily from the Exact Functor–Cohomology Lemma 70.1 and the Hom-to-Injective Lemma 70.3. We leave it to the reader to fill in the details. ■

70.2. **The Ext group.** In Chapter 54.1 we introduced the torsion groups $\mathrm{Tor}(H, G)$ which "measure" to what degree the functor $A \mapsto A \otimes G$ fails to be a left-exact functor. The same way we now introduce the Ext-groups $\mathrm{Ext}(H, G)$ which "measure" to what degree the functor $A \mapsto \mathrm{Hom}(A, G)$ fails to be right-exact:

Definition. Let G and H be abelian groups. We denote by

$$\ldots \to F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \to 0$$

the natural free resolution from page 677. We define the n-th G-Ext-group of H as

$$\mathrm{Ext}^n(H, G) := \mathrm{H}^n(\mathrm{Hom}(F_*, G)).$$

On page 678 we saw that the groups $\mathrm{Tor}_n(H, G)$ are covariantly functorial in both G and H. Now we will see that the groups $\mathrm{Ext}^n(H, G)$ are contravariantly functorial in H and covariantly functorial in G:

Definition.

(1) We fix an abelian group G. Let $\alpha\colon H \to H'$ be a homomorphism between abelian groups. It follows from the Fundamental Theorem 50.1 of Homological Algebra together with the definition of the G-Ext groups that α induces a homomorphism

$$\alpha^*\colon \mathrm{Ext}^n(H', G) \to \mathrm{Ext}^n(H, G).$$

It is now straightforward to show that the maps

$$\begin{aligned} H &\mapsto \mathrm{Ext}^n(H, G) \\ (\alpha\colon H \to H') &\mapsto (\alpha^*\colon \mathrm{Ext}^n(H', G) \to \mathrm{Ext}^n(H, G)) \end{aligned}$$

define a *contravariant* functor from the category $\mathcal{AbGroup}$ of abelian groups to itself.

(2) We fix an abelian group H. If $\beta\colon G \to G'$ is a homomorphism between abelian groups then one sees easily, using the Cohomology-Coefficients-Functor Lemma 66.10, that β induces a homomorphism

$$\beta_*\colon \mathrm{Ext}^n(H, G) \to \mathrm{Ext}^n(H, G')$$

Furthermore one can easily prove that the maps

$$\begin{aligned} G &\mapsto \mathrm{Ext}^n(H, G) \\ (\beta\colon G \to G') &\mapsto (\beta_*\colon \mathrm{Ext}^n(H, G) \to \mathrm{Ext}^n(H, G')) \end{aligned}$$

define a *covariant* functor from the category $\mathcal{AbGroup}$ of abelian groups to itself.

The following proposition is the analogue of the Tor-Free Resolution Proposition 54.2.

Proposition 70.8. (Ext-Free Resolution Proposition) Let G and H be abelian groups and let

$$\ldots \to F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \to 0$$

be a free resolution of H. For every $n \in \mathbb{N}_0$ there exists an isomorphism

$$\mathrm{Ext}^n(H,G) \xrightarrow{\cong} \mathrm{H}^n(F_*,G)$$

which is natural in G.

Proof. The proof is almost the same as the proof of the Tor-Free Resolution Proposition 54.2. For the most part we just need to replace the Homology-with-Coefficients Lemma 52.4 (2) by the Dual Cochain Map Lemma 66.9 (3b) and at some point we need to make use of the Cochain Homotopy-H*-Lemma 66.6. We leave it to the reader to fill in the details. ■

In the Tor_0–Lemma 54.3 we saw that the 0-th Torsion functor is an old acquaintance. More precisely we saw that for all abelian groups G and H there exists a natural isomorphism $\mathrm{Tor}_0(H,G) \xrightarrow{\cong} H \otimes G$. The following lemma gives us the analogous result for the 0-th Ext functor:

Lemma 70.9. (Ext^0–Lemma) For any choice of abelian groups G and H there exists a natural[551] isomorphism $\mathrm{Ext}^0(H,G) \xrightarrow{\cong} \mathrm{Hom}(H,G)$.

Identification. In the remainder of this course we will use the natural isomorphisms from the Ext^0–Lemma 70.9 to make the identification $\mathrm{Ext}^0(H,G) =_{\mathrm{i}} \mathrm{Hom}(H,G)$.

Proof. Let G and H be abelian groups. We consider the natural free resolution of H:

$$\cdots \to F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \to 0$$

We know from the Hom-Left Exact Lemma 70.2 that the sequence

$$(*) \qquad 0 \to \mathrm{Hom}(H,G) \xrightarrow{f_0^*} \mathrm{Hom}(F_0,G) \xrightarrow{f_1^*} \mathrm{Hom}(F_1,G)$$

is exact. Now we obtain the following isomorphisms

$$\begin{aligned} \mathrm{Ext}^0(H,G) &\overset{\text{by definition}}{=} \mathrm{H}^0\big(\cdots \leftarrow \mathrm{Hom}(F_1,G) \xleftarrow{f_1^*} \mathrm{Hom}(F_0,G) \leftarrow 0\big) \\ &= \ker\big(\mathrm{Hom}(F_1,G) \xleftarrow{f_1^*} \mathrm{Hom}(F_0,G)\big) \xrightarrow[\text{isomorphism since } (*) \text{ is exact}]{f_0^*} \mathrm{Hom}(H,G). \end{aligned}$$

It is straightforward to verify that these isomorphisms are in fact natural. ■

In the Higher Tor Lemma 54.5 we saw that the higher torsion groups of abelian groups vanish. The same statement, with basically the same proof, also holds for Ext groups:

Lemma 70.10. (Higher Ext Lemma) For every choice of abelian groups G and H we have $\mathrm{Ext}^n(H,G) = 0$ for $n \geq 2$.

Proof. By the Length 1–Free Resolution Lemma 54.4 we know that the abelian group H admits a free resolution of length 2:

$$0 \to F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \to 0.$$

[551] Here by "natural" we mean that these isomorphisms define a natural isomorphism from the contravariant functor $H \mapsto \mathrm{Ext}^0(H,G)$ to the contravariant functor $H \mapsto \mathrm{Hom}(H,G)$ and that they also define a natural transformation from the covariant functor $G \mapsto \mathrm{Ext}^0(H,G)$ to the covariant functor $G \mapsto \mathrm{Hom}(H,G)$.

It now follows that for $n \geq 2$ we have

$$\underset{\text{by the Ext-Free Resolution Proposition 70.8}}{\operatorname{Ext}_n(H,G) \cong \operatorname{H}^n(\operatorname{Hom}(F_*,G))} \underset{\text{since } F_n = 0 \text{ for } n \geq 2}{= 0.}$$

■

The Ext^0–Lemma 70.9 and the Higher Ext Lemma 70.10 justify the following definition:

Definition. Given abelian groups G and H we refer to

$$\operatorname{Ext}(H,G) := \operatorname{Ext}^1(H,G)$$

as *the* G-Ext-group of H.

In light of the Ext^0–Lemma 70.9 and the Higher Ext Lemma 70.10 the remaining task is to compute the Ext-group $\operatorname{Ext}(H,G) = \operatorname{Ext}^1(H,G)$. We start out with the following lemma which is the analogue of the $\mathbb{Z}_k$-Tor-Lemma 54.6.

Lemma 70.11. ($\mathbb{Z}_k$-Ext-Lemma) Let $k \in \mathbb{N}$.

(1) For any abelian group G we have a natural isomorphism

$$\operatorname{Ext}(\mathbb{Z}_k, G) \xrightarrow{\cong} G/k \cdot G.$$

(2) Given any $m \in \mathbb{N}$ we have $\operatorname{Ext}(\mathbb{Z}_k, \mathbb{Z}_m) \cong \mathbb{Z}_{\gcd(k,m)}$.

Proof.

(1) We consider the free resolution $0 \to \mathbb{Z} \xrightarrow{n \mapsto n \cdot k} \mathbb{Z} \to \mathbb{Z}_k \to 0$ of $\mathbb{Z}_k$. Given any abelian group G this gives rise to the following diagram:

$$\begin{array}{ccccccccc}
 & & & & & & \overbrace{\qquad\qquad\qquad\qquad}^{\cong \operatorname{Ext}(\mathbb{Z}_k,G)} & & \\
\operatorname{Hom}(\mathbb{Z},G) & \xrightarrow{(n\mapsto n\cdot k)^*} & \operatorname{Hom}(\mathbb{Z},G) & \longrightarrow & \operatorname{coker}\big(\operatorname{Hom}(\mathbb{Z},G) & \xrightarrow{(n\mapsto n\cdot k)^*} & \operatorname{Hom}(\mathbb{Z},G)\big) & \longrightarrow & 0 \\
\downarrow{\scriptstyle \varphi\mapsto\varphi(1)}\,{\scriptstyle\cong} & & \downarrow{\scriptstyle \varphi\mapsto\varphi(1)}\,{\scriptstyle\cong} & & & & \vdots & & \\
G & \xrightarrow{g\mapsto g\cdot k} & G & \longrightarrow & \operatorname{coker}(G & \xrightarrow{g\mapsto g\cdot k} & G) & \longrightarrow & 0.
\end{array}$$

We make the following observations:

(a) Since $0 \to \mathbb{Z} \xrightarrow{n \mapsto n \cdot k} \mathbb{Z} \to \mathbb{Z}_k \to 0$ is a free resolution of $\mathbb{Z}_k$ we obtain from the Ext-Free Resolution Proposition 70.8 the isomorphism in the top right corner.

(b) The top horizontal sequence is exact by definition.

(c) The two vertical maps to the left are the natural isomorphisms given by the Abelian Group-Homomorphism Lemma 66.2. It follows immediately from the definitions that the square to the left commutes.

It now follows from a mild diagram chase that from the above we get an induced isomorphism $\operatorname{coker}\big(G \xrightarrow{g \mapsto g \cdot k} G\big) \xrightarrow{\cong} \operatorname{Ext}(\mathbb{Z}_k, G)$. One can easily verify that this isomorphism is natural.

(2) We write $r = \gcd(m,k)$ and $s = m/r$. We see that

$$\operatorname{Ext}(\mathbb{Z}_k,\mathbb{Z}_m) \underset{\text{by (1)}}{\cong} \operatorname{coker}(\mathbb{Z}_m \xrightarrow{h \mapsto h \cdot k} \mathbb{Z}_m) \underset{\text{elementary number theory}}{=} \mathbb{Z}_m/s \cdot \mathbb{Z}_m \underset{\text{elementary number theory}}{\cong} \mathbb{Z}_r.$$

■

The following lemma plays the role of the Tor-Properties Lemma 54.7:

Lemma 70.12. (Ext-Properties Lemma) Let G, H and $\{G_j\}_{j\in J}$ and $\{H_i\}_{i\in I}$ be abelian groups. The following statements hold:

(1) There exists a natural isomorphism $\operatorname{Ext}\left(\bigoplus_{i\in I} H_i, G\right) \cong \prod_{i\in I} \operatorname{Ext}(H_i, G)$.

(2) There exists a natural isomorphism $\operatorname{Ext}\left(H, \prod_{j\in J} G_j\right) \cong \prod_{j\in J} \operatorname{Ext}(H, G_j)$.

(3) If G is injective (e.g. if G is divisible), then $\operatorname{Ext}(H, G) = 0$.

(4) If H is a free abelian group, then $\operatorname{Ext}(H, G) = 0$.

Example. Let H be a finitely generated abelian group. By the Abelian Group-Classification Theorem 18.11 we know that H is isomorphic to a direct sum of cyclic groups. It follows easily from this observation, the $\mathbb{Z}_k$-Ext-Lemma 70.11 and the Ext-Properties Lemma 70.12 (1) and (4) that we have an isomorphism

$$\operatorname{Ext}(H, \mathbb{Z}) \cong \text{ torsion subgroup of } H := \{h \in H \mid \text{there exists an } n \in \mathbb{N} \text{ with } n \cdot h = 0\}.$$

In particular $\operatorname{Ext}(H, \mathbb{Z})$ is a finite group. □

Proof.

(1) For each $i \in I$ we let F^i_* be the natural free resolution of H_i. One can easily verify that $\bigoplus_{i\in I} F^i_*$ is a free resolution of $\bigoplus_{i\in I} H_i$. We have the following natural isomorphisms:

$$\begin{array}{ccccc}
\operatorname{Ext}\left(\bigoplus_{i\in I} H_i, G\right) & \overset{\text{by the Ext-Free Resolution Proposition 70.8}}{\cong} & \mathrm{H}^1\left(\operatorname{Hom}\left(\bigoplus_{i\in I} F^i_*, G\right)\right) & \overset{\text{by the Abelian Group-Homomorphism Lemma 66.2 (3a)}}{\underset{\cong}{\leftarrow}} & \mathrm{H}^1\left(\prod_{i\in I} \operatorname{Hom}(F^i_*, G)\right) \\
& \underset{\text{by the Product–H}^*\text{-Lemma 66.8}}{\overset{\cong}{\leftarrow}} & \prod_{i\in I} \mathrm{H}^1(\operatorname{Hom}(F^i_*, G)) & \underset{\text{by definition}}{=} & \prod_{i\in I} \operatorname{Ext}(H_i, G).
\end{array}$$

(2) Let F_* be the natural resolution for H. We have the following natural isomorphisms:

$$\begin{array}{ccccc}
\operatorname{Ext}\left(H, \prod_{j\in J} G_j\right) & \overset{\text{by definition}}{=} & \mathrm{H}^1\left(\operatorname{Hom}\left(F_*, \prod_{j\in J} G_j\right)\right) & \overset{\text{by the Abelian Group-Homomorphism Lemma 66.2 (3b)}}{\underset{\cong}{\leftarrow}} & \mathrm{H}^1\left(\prod_{j\in J} \operatorname{Hom}(F_*, G_j)\right) \\
& \underset{\text{by the Product–H}^*\text{-Lemma 66.8}}{\overset{\cong}{\leftarrow}} & \prod_{j\in J} \mathrm{H}^1(\operatorname{Hom}(F_*, G_j)) & \underset{\text{by definition}}{=} & \prod_{j\in J} \operatorname{Ext}(H, G_j).
\end{array}$$

(3) We need to show that if G is a divisible abelian group, then for any abelian group H we have $\operatorname{Ext}(H, G) = 0$. First recall that it follows from the Length 1–Free Resolution Lemma 54.4 that we can pick a free resolution

$$0 \to F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} H \to 0$$

of length ≤ 1 for H. Since G is divisible we know by the Hom-to-Injective Lemma 70.3 that the functor $A \mapsto \operatorname{Hom}(A, G)$ is exact, in particular it is right-exact. It follows that

$$\operatorname{Hom}(H, G) \xrightarrow{f_0^*} \operatorname{Hom}(F_0, G) \xrightarrow{f_1^*} \operatorname{Hom}(F_1, G) \to 0$$

is exact. In particular $f_1^*\colon \operatorname{Hom}(F_0, G) \to \operatorname{Hom}(F_1, G)$ is surjective. It now follows that

$$\operatorname{Ext}(H, G) \underset{\text{by the Ext-Free Resolution Proposition 70.8}}{\cong} \operatorname{H}^1\big(0 \leftarrow \operatorname{Hom}(F_1, G) \xleftarrow{f_1^*} \operatorname{Hom}(F_0, G) \leftarrow 0\big) \underset{\text{since we just saw that } f_1^* \text{ is surjective}}{=} 0.$$

(4) Let H be a *free* abelian group. Note that in this case $0 \to F_0 := H \xrightarrow{\mathrm{id}} H \to 0$ is already a free resolution. It follows immediately from the Ext-Free Resolution Proposition 70.8 that $\operatorname{Ext}(H, G) = 0$. ■

70.3. **The Universal Coefficient Theorem for Cohomology Groups.** In this section we want to explore the connection between the cohomology groups of a chain complex (C_*, ∂_*) and the usual homology groups of the chain complex. The key to relating homology groups to cohomology groups is given by the evaluation homomorphism which we introduced in the Evaluation Map Lemma 70.6:

Definition. Let (C_*, ∂_*) be a chain complex and let G be an abelian group. We refer to the map

$$\begin{array}{rcl} \mathrm{ev}\colon \operatorname{H}^n(C_*; G) & \to & \operatorname{Hom}(\operatorname{H}_n(C_*), G) \\ [\varphi\colon C_n \to G] & \mapsto & \left(\begin{array}{rcl} \operatorname{H}_n(C_*) & \to & G \\ [c] & \mapsto & \langle [\varphi], [c] \rangle = \varphi(c) \end{array} \right) \end{array}$$

as the evaluation homomorphism.

The following theorem relates homology groups and cohomology groups via the evaluation homomorphism and the Ext-groups.

Theorem 70.13. (Algebraic Universal Coefficient-H*-Theorem) Let (C_*, ∂_*) be a chain complex of free abelian groups and let G be an abelian group. For each $n \in \mathbb{N}_0$ there exists a homomorphism

$$\epsilon\colon \operatorname{Ext}(\operatorname{H}_{n-1}(C_*), G) \to \operatorname{H}^n(C_*; G)$$

which is natural in the chain complex C_* and natural in the abelian group G and which has the property that the following sequence is exact:

$$0 \to \operatorname{Ext}(\operatorname{H}_{n-1}(C_*), G) \xrightarrow{\epsilon} \operatorname{H}^n(C_*; G) \xrightarrow{\mathrm{ev}} \operatorname{Hom}(\operatorname{H}_n(C_*), G) \to 0.$$

This short exact sequence splits and there exists therefore an isomorphism[552]

$$\operatorname{H}^n(C_*; G) \cong \operatorname{Ext}(\operatorname{H}_{n-1}(C_*), G) \oplus \operatorname{Hom}(\operatorname{H}_n(C_*), G).$$

Remark. The Algebraic Universal Coefficient-H*-Theorem 70.13 is the analogue of the Algebraic Universal Coefficient Theorem 54.8 together with the UCT-Splitting Theorem 54.11. □

Before we provide the proof of the above Algebraic Universal Coefficient-H*-Theorem 70.13 let us formulate the most important application, namely let us apply it to the singular chain complex of a pair of topological spaces:

[552] Once again the two maps in the short exact sequence are natural, whereas the isomorphism is not natural. More precisely, the discussion on page 691 can easily be modified to show that there exists no *natural* isomorphism $\operatorname{H}^n(C_*; G) \cong \operatorname{Ext}(\operatorname{H}_{n-1}(C_*), G) \oplus \operatorname{Hom}(\operatorname{H}_n(C_*), G)$.

Theorem 70.14. (Universal Coefficient-H*-Theorem) Let (X, A) be a pair of topological spaces and let G be an abelian group. For each $n \in \mathbb{N}_0$ there exists a homomorphism

$$\epsilon\colon \operatorname{Ext}(\mathrm{H}_{n-1}(X, A), G) \to \mathrm{H}^n(X, A; G)$$

which is natural in (X, A), which is natural in the abelian group G and which has the property that the following sequence is exact:

$$0 \to \operatorname{Ext}(\mathrm{H}_{n-1}(X, A), G) \xrightarrow{\epsilon} \mathrm{H}^n(X, A; G) \xrightarrow{\mathrm{ev}} \operatorname{Hom}(\mathrm{H}_n(X, A), G) \to 0.$$

This short exact sequence splits and there exists therefore an isomorphism

$$\mathrm{H}^n(X, A; G) \cong \operatorname{Ext}(\mathrm{H}_{n-1}(X, A), G) \oplus \operatorname{Hom}(\mathrm{H}_n(X, A), G).$$

Proof of Theorem 70.14 assuming Theorem 70.13. As the reader will have noticed there is really not much to say, except that we recall that we pointed out on page 489 that the chain groups $\mathrm{C}_n(X, A)$, $n \in \mathbb{N}_0$ are free abelian groups. ■

Remark. The Universal Coefficient-H*-Theorem 70.14 implies in particular that topological spaces with isomorphic homology groups also have isomorphic cohomology groups. In other words, if our only goal is to distinguish homeomorphism types of topological spaces, then cohomology groups will not be of much help.

In the next Section 70.4 we will study many examples and applications. The remainder of this section is occupied with the proof of the Algebraic Universal Coefficient-H*-Theorem 70.13. We break the proofs into two parts:

(1) First we show that the short exact sequences exist.
(2) Afterwards we show that the short exact sequences split.

The proof of the two parts are, perhaps not surprisingly, rather similar to the proofs of the Algebraic Universal Coefficient Theorem 54.8 and the Algebraic UCT-Splitting Proposition 54.10.

Proof of the existence of short exact sequence. Let (C_*, ∂_*) be a chain complex of free abelian groups. For each $n \in \mathbb{N}_0$ we write as always $Z_n := \ker(\partial_n)$ and $B_n := \operatorname{im}(\partial_{n+1})$. Note that it follows from the Free Abelian Subgroup Theorem 18.7 that B_n and Z_n are free abelian groups.

As in the proof of the Universal Coefficient Theorem 54.9 we consider the following short exact sequence of vertical chain complexes:

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Z_n & \longrightarrow & C_n & \xrightarrow{\partial_n} & B_{n-1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \partial_n} & & \downarrow{\scriptstyle 0} & & \\
0 & \longrightarrow & Z_{n-1} & \longrightarrow & C_{n-1} & \xrightarrow{\partial_{n-1}} & B_{n-2} & \longrightarrow & 0. \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

Now let G be an abelian group. We apply the functor $A \mapsto \operatorname{Hom}(A, G)$ to the above short exact sequence of chain complexes. This leads us to the following claim:

Claim. The following maps define a short exact sequence of "vertical" cochain complexes:

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathrm{Hom}(B_{n-1},G) & \xrightarrow{\partial_n^*} & \mathrm{Hom}(C_n,G) & \longrightarrow & \mathrm{Hom}(Z_n,G) & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle 0} & & \uparrow{\scriptstyle \partial_n^*} & & \uparrow{\scriptstyle 0} & & \\
0 & \longrightarrow & \mathrm{Hom}(B_{n-2},G) & \xrightarrow{\partial_{n-1}^*} & \mathrm{Hom}(C_{n-1},G) & \longrightarrow & \mathrm{Hom}(Z_{n-1},G) & \longrightarrow & 0. \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

Proof.

(1) It follows from the Dual Cochain Map Lemma 66.9 that this is a sequence of cochain complexes.

(2) Recall that the groups B_n are free abelian groups. Thus it follows from the Free Abelian Group–Splitting Lemma 18.5 together with the Splitting Lemma 37.2 (1)⇒(3) that each of the horizontal sequences in the original diagram splits. It follows from the Hom-Split-SES Lemma 66.4 that the above horizontal sequences of cochain complexes are still exact. ⊞

We apply the Cohomology–LES Proposition 66.7 to the short exact sequence of cochain complexes from the claim. We obtain a natural connecting homomorphism

$$\underbrace{\mathrm{Hom}(Z_n,G)}_{=\mathrm{H}^n(Z_*;G)} \to \underbrace{\mathrm{Hom}(B_n,G)}_{=\mathrm{H}^n(B_*;G)}.$$

In fact, it follows easily from the definition of connecting homomorphisms (see page 859) that the connecting homomorphism $\mathrm{Hom}(Z_n,G) \to \mathrm{Hom}(B_n,G)$ is the map that is induced by the inclusion $i_n\colon B_n \to Z_n$. By the Cohomology–LES Proposition 66.7 this connecting homomorphism gives us the following long exact sequence:

$$\to \mathrm{Hom}(Z_{n-1},G) \xrightarrow{i_n^*} \mathrm{Hom}(B_{n-1},G) \to \mathrm{H}^n(C;G) \to \mathrm{Hom}(Z_n,G) \xrightarrow{d_{n-1}} \mathrm{Hom}(B_n,G) \to \dots$$

We obtain the following short exact sequence

$$\begin{aligned} 0 \to \mathrm{coker}(\mathrm{Hom}(Z_{n-1},G) \to \mathrm{Hom}(B_{n-1},G)) &\to \mathrm{H}^n(C;G) \\ &\to \ker(\mathrm{Hom}(Z_n,G) \to \mathrm{Hom}(B_n,G)) \to 0. \end{aligned}$$

We have to show that the expressions left and right coincide with the desired expressions. Both equalities follow immediately from the Ext-Free Resolution Proposition 70.8 (together with the Ext_0–Lemma 70.9 for the right-hand term) and the observation that for each $l \in \mathbb{N}_0$ a free resolution of $\mathrm{H}_l(C)$ is given by $0 \to B_l \xrightarrow{i_l} Z_l \to \mathrm{H}_l(C_*)$. We have thus shown the existence of the desired short exact sequence

$$0 \to \mathrm{Ext}(\mathrm{H}_{n-1}(C_*),G) \xrightarrow{\epsilon} \mathrm{H}^n(C_*;G) \xrightarrow{\mathrm{ev}} \mathrm{Hom}(\mathrm{H}_n(C_*),G) \to 0.$$

The statement that the homomorphism ϵ to the left is natural follows easily from the definitions and the Fundamental Theorem 50.1 (2) of Homological Algebra. We leave the verification of the details to the reader. ■

Proof that the short exact sequence splits. Let (C_*,∂_*) be a chain complex of free abelian groups and let G be an abelian group. According to the Splitting Lemma 37.2 (3)⇒(1) it suffices to construct a homomorphism $\eta\colon \mathrm{Hom}(\mathrm{H}_n(C_*),G) \to \mathrm{H}^n(C_*;G)$ such

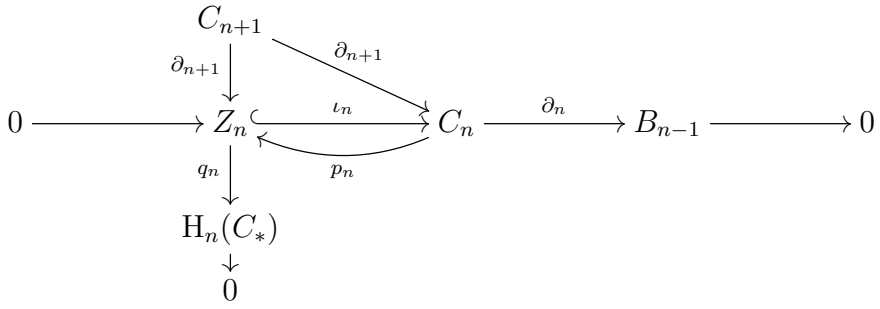
that $\eta \circ \mathrm{ev}$ is the identity on $\mathrm{H}_n(C_*) \otimes G$. We summarize this problem as follows:

$$0 \longrightarrow \mathrm{Ext}(\mathrm{H}_{n-1}(C_*), G) \longrightarrow \mathrm{H}^n(\mathrm{Hom}(C_*, G)) \xrightarrow{\mathrm{ev}} \mathrm{Hom}(\mathrm{H}_n(C_*), G) \longrightarrow 0.$$
$$\overset{\longleftarrow}{\underset{?\,\eta\,?}{- - - - -}}$$

For each $n \in \mathbb{N}_0$ we consider, as on page 683, $Z_n := \ker(\partial_n)$ and $B_n := \mathrm{im}(\partial_{n+1})$ and we consider the short exact sequence

$$0 \longrightarrow Z_n \overset{\iota_n}{\hookrightarrow} C_n \xrightarrow{\partial_n} B_{n-1} \longrightarrow 0.$$

The group $B_{n-1} \subset C_{n-1}$ is a subgroup of a free abelian group, thus it is itself a free abelian group by the Free Abelian Subgroup Theorem 18.7. Therefore we can deduce from the Free Abelian Group–Splitting Lemma 18.5 together with the Splitting Lemma 37.2 (1)⇒(3) that the above short exact sequence splits. In particular, according to the Splitting Lemma 37.2 (1)⇒(3), for each $n \in \mathbb{N}_0$ there exists a homomorphism $p_n \colon C_n \to Z_n$ such that $p_n \circ \iota_n = \mathrm{id}_{Z_n}$. Again we summarize the setting as follows:



where q_n is just the natural projection.

Claim 1. The following diagram commutes:

$$\begin{array}{ccccccccc}
\ldots & \longrightarrow & C_{n+1} & \xrightarrow{\partial_{n+1}} & C_n & \xrightarrow{\partial_n} & C_{n-1} & \longrightarrow & \ldots \\
 & & \downarrow{\scriptstyle q_{n+1}\circ p_{n+1}} & & \downarrow{\scriptstyle q_n \circ p_n} & & \downarrow{\scriptstyle q_{n-1}\circ p_{n-1}} & & \\
\ldots & \longrightarrow & \mathrm{H}_{n+1}(C_*) & \xrightarrow{0} & \mathrm{H}_n(C_*) & \xrightarrow{0} & \mathrm{H}_{n-1}(C_*) & \longrightarrow & \ldots
\end{array}$$

Proof. The map down-right is evidently zero. On the other hand we also have

$$\begin{array}{ccccccc}
(q_n \circ p_n) \circ \partial_{n+1} & = & q_n \circ p_n \circ \iota_n \circ \partial_{n+1} & = & q_n \circ \partial_{n+1} & = & 0. \\
 & & \uparrow & & \uparrow & & \uparrow \\
 & & \text{since } \iota_n \circ \partial_{n+1} = \partial_{n+1} & & \text{since } p_n \circ \iota_n = \mathrm{id}_{Z_n} & & \text{since } \mathrm{H}_n(C_*) = Z_n / \mathrm{im}(\partial_{n+1})
\end{array}$$

⊞

The top and the bottom horizontal sequences of the claim are chain complexes. Since the diagram commutes we see that the vertical maps from top to bottom are chain maps. Now we apply the $\mathrm{Hom}(-, G)$ functor to the top and the bottom horizontal sequences. From the Dual Cochain Map Lemma 66.9 we obtain the following diagram of cochain complexes:

$$\begin{array}{ccccccccc}
\ldots & \longleftarrow & \mathrm{Hom}(C_{n+1}, G) & \xleftarrow{\partial_{n+1}^*} & \mathrm{Hom}(C_n, G) & \xleftarrow{\partial_n^*} & \mathrm{Hom}(C_{n-1}, G) & \longleftarrow & \ldots \\
 & & \uparrow{\scriptstyle (q_{n+1}\circ p_{n+1})^*} & & \uparrow{\scriptstyle (q_n \circ p_n)^*} & & \uparrow{\scriptstyle (q_{n-1}\circ p_{n-1})^*} & & \\
\ldots & \longleftarrow & \mathrm{Hom}(\mathrm{H}_{n+1}(C_*), G) & \xleftarrow{0} & \mathrm{Hom}(\mathrm{H}_n(C_*), G) & \xleftarrow{0} & \mathrm{Hom}(\mathrm{H}_{n-1}(C_*), G) & \longleftarrow & \ldots
\end{array}$$

The induced maps on the cohomology of these two cochain complexes give us a homomorphism

$$\eta\colon \mathrm{H}^n(\text{lower sequence}) = \mathrm{Hom}(\mathrm{H}_n(C_*), G) \;\to\; \mathrm{H}_n(\mathrm{Hom}(C_*, G)).$$

It remains to prove the following claim:

Claim 2. The composition $\mathrm{ev} \circ \eta$ is the identity on $\mathrm{Hom}(\mathrm{H}_n(C_*), G)$.

Proof. Let $\varphi\colon \mathrm{H}_n(C_*) \to G$ be homomorphism. We need to show that $(\mathrm{ev} \circ \eta)(\varphi) = \varphi$ in $\mathrm{Hom}(\mathrm{H}_n(C_*), G)$. So let $[\sigma] \in \mathrm{H}_n(C_*)$ with $\sigma \in Z_n$. Then

$$\big((\mathrm{ev} \circ \eta)(\varphi)\big)([\sigma]) \underset{\text{definition of ev}}{=} \langle \overbrace{\eta(\varphi)}^{\in \mathrm{H}^n(C;G)}, [\sigma] \rangle \underset{\text{definition of } \eta \text{ and } \langle\,,\,\rangle}{=} (\overbrace{\varphi \circ q_n \circ p_n}^{\in \mathrm{Hom}(C_n, G)})(\sigma) \underset{\text{since } p_n = \mathrm{id} \text{ on } Z_n}{=} (\varphi \circ q_n)(\sigma) \underset{\text{since } q_n(\sigma) = [\sigma]}{=} \varphi([\sigma]).$$

■

70.4. **Applications of the Universal Coefficient-H*-Theorem 70.14.** We now apply the Universal Coefficient-H*-Theorem 70.14 to some of our favorite topological spaces:

Example. We consider the real projective space $P := \mathbb{R}\mathrm{P}^2$. As a reminder, in the $\mathbb{R}\mathrm{P}^n$-H$_*$-Proposition 47.10 we showed that $\mathrm{H}_2(P) = 0$, $\mathrm{H}_1(P) \cong \mathbb{Z}_2$ and $\mathrm{H}_0(P) \cong \mathbb{Z}$. It follows from the Universal Coefficient-H*-Theorem 70.14 and from the $\mathbb{Z}_k$-Ext-Lemma 70.11 and the Ext-Properties Lemma 70.12 (4) that

$$\begin{aligned}
\mathrm{H}^2(P;\mathbb{Z}) &\cong \mathrm{Hom}(\mathrm{H}_2(P),\mathbb{Z}) \oplus \mathrm{Ext}(\mathrm{H}_1(P),\mathbb{Z}) \cong \mathrm{Hom}(0,\mathbb{Z}) \oplus \mathrm{Ext}(\mathbb{Z}_2,\mathbb{Z}) = 0 \oplus \mathbb{Z}_2 = \mathbb{Z}_2\\
\mathrm{H}^1(P;\mathbb{Z}) &\cong \mathrm{Hom}(\mathrm{H}_1(P),\mathbb{Z}) \oplus \mathrm{Ext}(\mathrm{H}_0(P),\mathbb{Z}) \cong \mathrm{Hom}(\mathbb{Z}_2,\mathbb{Z}) \oplus \mathrm{Ext}(\mathbb{Z},\mathbb{Z}) = 0 \oplus 0 = 0\\
\mathrm{H}^0(P;\mathbb{Z}) &\cong \mathrm{Hom}(\mathrm{H}_0(P),\mathbb{Z}) \cong \mathrm{Hom}(\mathbb{Z},\mathbb{Z}) \cong \mathbb{Z}
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{H}^2(P;\mathbb{Z}_2) &\cong \mathrm{Hom}(\mathrm{H}_2(P),\mathbb{Z}_2) \oplus \mathrm{Ext}(\mathrm{H}_1(P),\mathbb{Z}_2) \cong \mathrm{Hom}(0,\mathbb{Z}_2) \oplus \mathrm{Ext}(\mathbb{Z}_2,\mathbb{Z}_2) = 0 \oplus \mathbb{Z}_2 = \mathbb{Z}_2\\
\mathrm{H}^1(P;\mathbb{Z}_2) &\cong \mathrm{Hom}(\mathrm{H}_1(P),\mathbb{Z}_2) \oplus \mathrm{Ext}(\mathrm{H}_0(P),\mathbb{Z}_2) \cong \mathrm{Hom}(\mathbb{Z}_2,\mathbb{Z}_2) \oplus \mathrm{Ext}(\mathbb{Z},\mathbb{Z}_2) = \mathbb{Z}_2 \oplus 0 = \mathbb{Z}_2\\
\mathrm{H}^0(P;\mathbb{Z}_2) &\cong \mathrm{Hom}(\mathrm{H}_0(P),\mathbb{Z}_2) \cong \mathrm{Hom}(\mathbb{Z},\mathbb{Z}_2) = \mathbb{Z}_2.
\end{aligned}$$

Fortunately this coincides with the calculations from page 881. □ □

We move on to the ever mysterious lens spaces:

Example. Let $p, q \in \mathbb{N}$ be coprime. We consider the corresponding lens space $L(p,q)$. In the Lens Space-H$_*$-Lemma 63.5 we saw that

$$\mathrm{H}_n(L(p,q);\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & \text{if } n = 0, 3,\\ 0, & \text{if } n = 2 \text{ or } n > 3,\\ \mathbb{Z}_p, & \text{if } n = 1. \end{cases}$$

We can now use the Universal Coefficient-H*-Theorem 70.14, the $\mathbb{Z}_k$-Ext-Lemma 70.11 and the Ext-Properties Lemma 70.12 (4) to compute the cohomology groups of $L(p,q)$. We leave it to the reader to verify that[553]

$$\mathrm{H}^n(L(p,q);\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & \text{if } n = 0, 3\\ 0, & \text{if } n = 1 \text{ or } n > 3,\\ \mathbb{Z}_p, & \text{if } n = 2. \end{cases}$$

This shows that cohomology groups also cannot distinguish between two lens spaces of the form $L(p,q)$ and $L(p,r)$. □

The following proposition gives us a reinterpretation of the first cohomology group:

[553]Note that the $\mathbb{Z}_p$ shifted from $n = 1$ in homology to $n = 2$ in cohomology.

Proposition 70.15. (H^1-Hom-Proposition) Let X be a path-connected topological space, let $x_0 \in X$ and let G be an abelian group. We have natural isomorphisms

$$\mathrm{H}^1(X;G) \underset{\uparrow}{\overset{\cong}{\longrightarrow}} \mathrm{Hom}(\mathrm{H}_1(X);G) \underset{\uparrow}{\overset{\cong}{\longrightarrow}} \mathrm{Hom}(\pi_1(X,x_0),G).$$

given by the evaluation homomorphism ev

induced by the Hurewicz homomorphism $\pi_1(X,x_0) \to \mathrm{H}_1(X)$ from page 734

Here, "natural" means that for any continuous map $(X,x_0) \to (Y,y_0)$ of pointed topological spaces the following diagram commutes:

$$\begin{array}{ccccc} \mathrm{H}^1(Y;G) & \xrightarrow{\cong} & \mathrm{Hom}(\mathrm{H}_1(Y);G) & \xrightarrow{\cong} & \mathrm{Hom}(\pi_1(Y,y_0),G) \\ \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} \\ \mathrm{H}^1(X;G) & \xrightarrow{\cong} & \mathrm{Hom}(\mathrm{H}_1(X);G) & \xrightarrow{\cong} & \mathrm{Hom}(\pi_1(X,x_0),G). \end{array}$$

Proof. As we pointed out in the Evaluation Map Lemma 70.6 and the Hurewicz Homomorphism Proposition 58.1 (4), both maps are natural. So it remains to show that both maps are isomorphisms.

We insert two extra maps into the above sequence of maps to see that the maps are indeed isomorphisms:

$$\mathrm{H}^1(X;G) \underset{\uparrow}{\cong} \overbrace{\mathrm{Ext}(\mathrm{H}_0(X),G)}^{=0} \oplus \mathrm{Hom}(\mathrm{H}_1(X),G) \underset{\uparrow}{=} \mathrm{Hom}(\mathrm{H}_1(X),G).$$

by the Universal Coefficient Theorem 70.14

by the H_0-Proposition 35.11 we know that $\mathrm{H}_0(X) \cong \mathbb{Z}$, it follows from the Ext-Properties Lemma 70.12 that the Ext-Term is zero

$$\underset{\uparrow}{\overset{\cong}{\longrightarrow}} \mathrm{Hom}(\pi_1(X,x_0)_{\mathrm{ab}},G) \underset{\uparrow}{\overset{\cong}{\longrightarrow}} \mathrm{Hom}(\pi_1(X,x_0),G).$$

by the Hurewicz Theorem 58.5

by the Commutator Subgroup Lemma 19.5 (3), since G is abelian ∎

70.5. **Cohomology groups of topological manifolds.** In this section we consider cohomology groups of compact topological manifolds. Before we delve into the topic it is convenient to recall the following definition from page 832:

Definition. Let M be a compact n-dimensional topological manifold and let $\{\alpha_x\}_{x \in M \setminus \partial M}$ be an orientation for M. We know by the Fundamental Class-Proposition 64.1 that there exists a unique $[M] \in \mathrm{H}_n(M,\partial M)$ with $\rho_x([M]) = \alpha_x$ for all $x \in M \setminus \partial M$. We refer to $[M]$ as the fundamental class of the oriented topological manifold M. If M is non-empty and connected, then $\mathrm{H}_n(M,\partial M) = \mathbb{Z} \cdot [M]$.

The following theorem is the cohomological analogue of the Manifold-Top-H_*-Theorem 63.1:

Theorem 70.16. (Top Cohomology Theorem) Let M be a compact[554] connected non-empty n-dimensional topological manifold and let G be an abelian group. The following statements hold:

(1) (a) For every $i > n$ we have $\mathrm{H}^i(M;G) = 0$.
(b) For every $i > n$ we have $\mathrm{H}^i(M,\partial M;G) = 0$.
(2) (a)

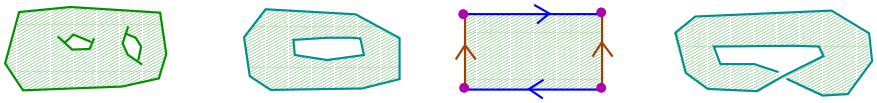
$$\mathrm{H}^n(M;G) \cong \begin{cases} G, & \text{if } M \text{ is closed and orientable,} \\ G/2\cdot G, & \text{if } M \text{ is closed and not orientable,} \\ 0, & \text{if } M \text{ is not closed.} \end{cases}$$

(b)
$$\mathrm{H}^n(M,\partial M;G) \cong \begin{cases} G, & \text{if } M \text{ is orientable,} \\ G/2\cdot G, & \text{if } M \text{ is not orientable.} \end{cases}$$

Furthermore, if M is orientable, then the evaluation homomorphism

$$\mathrm{ev}\colon \mathrm{H}^n(M,\partial M;G) \to \mathrm{Hom}(\underbrace{\mathrm{H}_n(M,\partial M)}_{=\mathbb{Z}\cdot[M] \text{ if oriented}}, G)$$

is an isomorphism.



$\mathrm{H}^2(M;G)$	G	0	$G/2\cdot G$	0
$\mathrm{H}^2(M,\partial M;G)$	G	G	$G/2\cdot G$	$G/2\cdot G$

Proof. Let M be a compact connected non-empty n-dimensional topological manifold. By the Manifold-Top-H_*-Theorem 63.1 we know that the following statements hold:

(1) (a) For every $i > n$ we have $\mathrm{H}_i(M) = 0$.
(b) For every $i > n$ we have $\mathrm{H}_i(M,\partial M) = 0$.

(2) (a)
$$\mathrm{H}_n(M) \cong \begin{cases} \mathbb{Z}, & \text{if } M \text{ is closed and orientable,} \\ 0, & \text{if } M \text{ is closed and not orientable,} \\ 0, & \text{if } M \text{ is not closed.} \end{cases}$$

(b)
$$\mathrm{H}_n(M,\partial M) \cong \begin{cases} \mathbb{Z}, & \text{if } M \text{ is orientable,} \\ 0, & \text{if } M \text{ is not orientable} \end{cases}$$

By the Manifold-H_{n-1}-Theorem 63.4 we also know the following:

(3) (a)
$$\mathrm{H}_{n-1}(M) \cong \begin{cases} \text{free abelian group}, & \text{if } M \text{ is orientable or if } \partial M \neq \varnothing, \\ \text{free abelian group} \oplus \mathbb{Z}_2, & \text{if } M \text{ is non-orientable and closed.} \end{cases}$$

(b)
$$\mathrm{H}_{n-1}(M,\partial M) \cong \begin{cases} \text{free abelian group}, & \text{if } M \text{ is orientable,} \\ \text{free abelian group} \oplus \mathbb{Z}_2, & \text{if } M \text{ is non-orientable.} \end{cases}$$

All the statements of the theorem now follow easily from the above facts together with the Universal Coefficient-H^*-Theorem 70.14, the $\mathbb{Z}_k$-Ext-Lemma 70.11 and the Ext-Properties Lemma 70.12. For example, if M is closed and not orientable, then we have

$$\mathrm{H}^n(M;G) \overset{\text{by the Universal Coefficient Theorem 70.14}}{\cong} \mathrm{Hom}(\mathrm{H}_n(M),G) \oplus \mathrm{Ext}(\mathrm{H}_{n-1}(M),G) \underset{\text{by the above discussion in (2a) we have } \mathrm{H}_n(M)=0 \text{ and by (3a) we have } \mathrm{H}_{n-1}(M)\cong\mathbb{Z}^k\oplus\mathbb{Z}_2 \text{ for some } k\in\mathbb{N}_0}{\cong} \mathrm{Ext}(\mathbb{Z}^k\oplus\mathbb{Z}_2,G) \overset{\text{by the Ext-Properties Lemma 70.12 and the } \mathbb{Z}_k\text{-Ext-Lemma 70.11}}{\cong} G/2\cdot G.$$

[554]Note that we restrict ourselves to the compact case. One would expect that for a non-compact connected n-dimensional topological manifold M we have $\mathrm{H}^n(M;G) = 0$ for any abelian group G. With the methods at our disposal it is not possible to prove such a statement.

All the other statements of the theorem can be deduced in a similar fashion. ■

The Manifold-Top-H*-Theorem 70.16 (2b) now allows us to introduce the dual fundamental class of a compact oriented topological manifold:

Definition.

(1) Let M be a compact oriented connected non-empty n-dimensional topological manifold M. The Manifold-Top-H*-Theorem 70.16 (2b) says that the evaluation homomorphism

$$\begin{array}{rcl} \mathrm{ev}\colon \mathrm{H}^n(M,\partial M;\mathbb{Z}) & \to & \mathrm{Hom}(\overbrace{\mathrm{H}_n(M,\partial M)}^{=\mathbb{Z}\cdot[M]},\mathbb{Z}) \\ \varphi & \mapsto & (\sigma \mapsto \langle \varphi,\sigma\rangle) \end{array}$$

is an isomorphism. This implies that there exists a generator $[M]^*$ of $\mathrm{H}^n(M,\partial M;\mathbb{Z})$ with $\langle [M]^*,[M]\rangle = 1$. We refer to $[M]^*$ as the dual fundamental class of M.

(2) Given a compact oriented non-empty n-dimensional topological manifold M with finitely many components $M_1,\dots,M_k$ we define its dual fundamental class to be

$$[M]^* := [M_1]^* + \cdots + [M_k]^* \in \underbrace{\mathrm{H}^n(M_1,\partial M_1;\mathbb{Z}) \oplus \cdots \oplus \mathrm{H}^n(M_k,\partial M_k;\mathbb{Z})}_{=\mathrm{H}^n(M,\partial M;\mathbb{Z}),\text{ by the Path-Component-H}^*\text{-Lemma 67.4}}.$$

Example. We consider the topological manifold S^1 with the standard orientation. Note that the singular simplex $\mu\colon \Delta^1 \to S^1$ that is given by $(1-t,t) \mapsto \exp(2\pi\mathrm{i}t)$ represents the fundamental class of S^1. On page 866 we introduced a cocycle $\theta_{\mathbb{Z}} \in \mathrm{C}^1(S^1;\mathbb{Z})$ and we had explicitly calculated that $\langle [\theta_{\mathbb{Z}}],[S^1]\rangle = \theta_{\mathbb{Z}}(\mu) = 1$. This shows immediately that $[\theta_{\mathbb{Z}}] \in \mathrm{H}^1(S^1;\mathbb{Z})$ is the dual fundamental class of S^1. □

Later on we will need the following lemma:

Lemma 70.17. (Dual Fundamental Class–Degree Lemma) Let $f\colon M \to N$ be a continuous map between closed oriented connected non-empty n-dimensional topological manifolds. Then

$$f^*([N]^*) = \deg(f)\cdot[M]^*.$$

Proof. Let $f\colon M \to N$ be a continuous map between closed oriented connected non-empty n-dimensional topological manifolds. It follows from the discussion on page 909 that any class in $\varphi = \mathrm{H}^n(M;\mathbb{Z})$ is of the form $\varphi = k\cdot[M]^*$ for some $k \in \mathbb{Z}$ and it follows from $\langle [M]^*,[M]\rangle = 1$ and the bilinearity of the Kronecker pairing that $k = \langle \varphi,[M]\rangle_M$. In our case we have

$$\begin{array}{rcl} \langle f^*([N]^*),[M]\rangle_M & \overset{\text{by the Kronecker Pairing Lemma 68.3}}{=} & \langle [N]^*, f_*([M])\rangle_N \\ & = & \deg(f)\cdot \underbrace{\langle [N]^*,[N]\rangle_N}_{=1} \\ & \overset{\text{since } \langle\,,\,\rangle_N \text{ is bilinear}}{=} & \end{array} \quad \begin{array}{rcl} & \overset{\text{by definition of } \deg(f)}{=} & \langle [N]^*, \deg(f)\cdot[N]\rangle_N \\ & = & \deg(f). \end{array}$$

It follows from the above discussion that $f^*([N]^*) = \deg(f)\cdot[M]^*$. ■

CHAPTER 71

The Universal Coefficient Theorem for Cohomology Groups II

In this chapter we prove a variation on the Algebraic Universal Coefficient-H^*-Theorem 70.13, namely, in the Universal Coefficient-H^*–Field Theorem 71.2 we will relate homology with coefficients in a field $\mathbb{F}$ to cohomology in the same field $\mathbb{F}$.

71.1. **Cohomology with field coefficients.** We start out by introducing a slight variation on the Kronecker pairing defined in the Kronecker Pairing Lemma 68.3:

Lemma 71.1. (Ring–Kronecker Pairing Lemma) Let C_* be a chain complex and let R be a commutative ring.

(1) The map

$$\begin{array}{rccccc} \langle\ ,\ \rangle\colon & \overbrace{\mathrm{H}^n(C;R)}^{=\mathrm{H}^n(\mathrm{Hom}(C_*,R))} & \times & \overbrace{\mathrm{H}_n(C;R)}^{=\mathrm{H}_n(C_*\otimes R)} & \to & R \\ & & & \Big([\varphi],\Big[\sum_{i=1}^k c_i\otimes a_i\Big]\Big) & \mapsto & \sum_{i=1}^k \varphi(c_i)\cdot a_i. \end{array}$$

is well-defined, bilinear and natural. We refer to this pairing also as the Kronecker pairing.

(2) For any $\varphi\in\mathrm{H}^n(C;R)$ and $\sigma\in\mathrm{H}_n(C)$ we have

$$\underbrace{\langle\varphi,[\sigma\otimes 1]\rangle}_{\text{as defined in (1)}} = \underbrace{\langle\varphi,\sigma\rangle}_{\substack{\text{as defined in the}\\ \text{Kronecker Pairing}\\ \text{Lemma 68.3}}}.$$

Proof. [555] In the following we show that the Kronecker pairing is well-defined. Once this is out of the way it is basically clear that the Kronecker pairing is bilinear and natural and that statement (2) holds.

Let $\varphi\in\mathrm{Hom}(C_n,R)$ be a cocycle. We consider the map

$$\begin{array}{ccc} C_n\times R & \to & R \\ \sum_{i=1}^k (c_i,r_i) & \to & \sum_{i=1}^k \varphi(c_i)\cdot r_i. \end{array}$$

One can easily verify that this map is bilinear. It follows from the Tensor Product–Universal Property Lemma 51.1 (3) that we get an induced map $C_n\otimes R\to R$. This shows that the map

$$\begin{array}{rccc} \langle\ ,\ \rangle\colon & \mathrm{Hom}(C_n,R)\times(C_n\otimes R) & \to & R \\ & \Big(\varphi,\sum_{i=1}^k c_i\otimes r_i\Big) & \to & \sum_{i=1}^k \varphi(c_i)\cdot r_i \end{array}$$

is well-defined. One can easily verify that this map is bilinear. It remains to show that this map defines a map on (co-) homology.

[555] As we will see, the proof of the lemma is elementary. Nonetheless, the calculations involved are much longer than the calculations in the proof of the original Kronecker Pairing Lemma 68.3.

First let $\varphi \in \operatorname{Hom}(C_n, G)$, let $\sigma \in C_n \otimes R$ and let $\sum_{i=1}^{k} \widetilde{c}_i \otimes \widetilde{r}_i \in C_{n+1} \otimes R$. We see that

$$\Big\langle \varphi, \sigma + \partial_{n+1}\Big(\sum_{i=1}^{k} \widetilde{c}_i \otimes \widetilde{r}_i\Big)\Big\rangle \underset{\text{by definition of } \partial_* \text{ on } C_* \otimes R}{=} \Big\langle \varphi, \sigma + \sum_{i=1}^{k} \partial_{n+1}(\widetilde{c}_i) \otimes \widetilde{r}_i\Big\rangle \underset{\text{by definition of } \langle\ ,\ \rangle}{=} \langle \varphi, \sigma\rangle + \sum_{i=1}^{k} \underbrace{\langle \varphi, \partial_{n+1}(\widetilde{c}_i)\rangle}_{=0, \text{ since } \varphi \text{ cocycle}} \cdot \widetilde{r}_i.$$

Next let $\varphi \in \operatorname{Hom}(C_n, G)$ be a cochain, let $\mu \in \operatorname{Hom}(C_{n-1}, G)$ and let $\sigma = \sum_{i=1}^{k} c_i \otimes r_i \in C_n \otimes R$ be a cycle. Then

$$\begin{aligned}
\langle \varphi + \delta_{n-1}(\mu), \sigma\rangle &\underset{\text{since } \langle\ ,\ \rangle \text{ is bilinear}}{=} \langle \varphi, \sigma\rangle + \Big\langle \delta_{n-1}(\mu), \sum_{i=1}^{k} c_i \otimes r_i\Big\rangle \\
&\underset{\text{definition of } \delta_{n-1}}{=} \langle \varphi, \sigma\rangle + \sum_{i=1}^{k} \mu(\partial_n(c_i)) \cdot r_i \\
&\underset{\text{definition of } \partial_* \text{ on } C_* \otimes R}{=} \langle \varphi, \sigma\rangle + \Big\langle \mu, \partial_n\Big(\sum_{i=1}^{k} c_i \otimes r_i\Big)\Big\rangle \\
&\underset{\text{definition of } \langle\ ,\ \rangle}{=} \langle \varphi, \sigma\rangle + \sum_{i=1}^{k} (\delta_{n-1}(\mu))(c_i) \cdot r_i \\
&\underset{\text{definition of } \langle\ ,\ \rangle}{=} \langle \varphi, \sigma\rangle + \Big\langle \mu, \sum_{i=1}^{k} \partial_n(c_i) \otimes r_i\Big\rangle. \\
&= \langle \varphi, \sigma\rangle + \langle \mu, \partial_n(\sigma)\rangle \underset{\text{since } \sigma \text{ is a cycle}}{=} \langle \varphi, \sigma\rangle + 0.
\end{aligned}$$

■

In the Universal Coefficient-H*-Theorem 70.14 we determined, given an abelian group G, the relationship between the ordinary homology groups $\mathrm{H}_*(X, A)$ and the cohomology groups $\mathrm{H}^*(X, A; G)$. In the following theorem we prove a similar statement: Given a field $\mathbb{F}$ we determine the relationship between the homology groups $\mathrm{H}_*(X, A; \mathbb{F})$ and the cohomology groups $\mathrm{H}^*(X, A; \mathbb{F})$:

Theorem 71.2. (Universal Coefficient-H*–Field Theorem) Let $\mathbb{F}$ be a field.

(1) For any chain complex C_* and any $n \in \mathbb{N}_0$ the evaluation homomorphism

$$\begin{array}{rcl} \mathrm{ev}\colon \mathrm{H}^n(\operatorname{Hom}(C_*, \mathbb{F})) & \to & \operatorname{Hom}_{\mathbb{F}}(\mathrm{H}_n(C_* \otimes \mathbb{F}), \mathbb{F}) \\ \varphi & \mapsto & \left(\begin{array}{rcl} \mathrm{H}_n(C_* \otimes \mathbb{F}) & \to & \mathbb{F} \\ \sigma & \mapsto & \langle \varphi, \sigma\rangle \end{array}\right) \end{array}$$

Kronecker pairing from the Ring–Kronecker Pairing Lemma 71.1

is a natural isomorphism of $\mathbb{F}$-vector spaces.

(2) Let (X, A) be a pair of topological spaces. For every $n \in \mathbb{N}_0$ the evaluation map

$$\mathrm{ev}\colon \mathrm{H}^n(X, A; \mathbb{F}) \to \operatorname{Hom}_{\mathbb{F}}(\mathrm{H}_n(X, A; \mathbb{F}), \mathbb{F})$$

from (1) is a natural isomorphism of $\mathbb{F}$-vector spaces.

The proof of the Universal Coefficient-H*–Field Theorem 71.2 consists of two steps:

(1) In the Evaluation-Isomorphism Theorem 71.3 we formulate and prove a "Universal Coefficient Type"-Theorem for chain complexes of $\mathbb{F}$-vector spaces.

(2) We need to massage the setting from the Universal Coefficient-H*–Field Theorem 71.2 to be able to apply (1).

We carry out this program in the next section.

71.2. **Proof of the Universal Coefficient-H*–Field Theorem 71.2.** In this section we give the proof of the Universal Coefficient-H*–Field Theorem 71.2. Before we can provide the proof we need to prepare the ground with some almost self-explanatory definitions:

Definition. Let $\mathbb{F}$ be a field.

(1) We define a chain complex over $\mathbb{F}$ to be a chain complex (V_*, v_*) where all chain groups are $\mathbb{F}$-vector spaces and all boundary maps are $\mathbb{F}$-homomorphisms.

(2) Given a chain complex (V_*, v_*) over the field $\mathbb{F}$ and given $n \in \mathbb{N}_0$ its n-th homology

$$\mathrm{H}_n(V_*) \coloneqq \frac{\ker(v_n \colon V_n \to V_{n-1})}{\mathrm{im}(v_{n+1} \colon V_{n+1} \to V_n)}.$$

is an $\mathbb{F}$-vector space in a natural way.

(3) We define cochain complexes and cochain maps over $\mathbb{F}$ in the obvious way.

(4) Given a chain complex (V_*, v_*) over $\mathbb{F}$ we obtain a cochain complex by considering $(\mathrm{Hom}_{\mathbb{F}}(V_*; \mathbb{F}), v^*)$ and we define its n-th cohomology group to be the $\mathbb{F}$-vector space

$$\mathrm{H}^n(V; \mathbb{F}) \coloneqq \frac{\ker(v_{n+1}^* \colon \mathrm{Hom}_{\mathbb{F}}(V_n, \mathbb{F}) \to \mathrm{Hom}_{\mathbb{F}}(V_{n+1}, \mathbb{F}))}{\mathrm{im}(v_n^* \colon \mathrm{Hom}_{\mathbb{F}}(V_{n-1}, \mathbb{F}) \to \mathrm{Hom}_{\mathbb{F}}(V_n, \mathbb{F}))}.$$

Theorem 71.3 (Evaluation-Isomorphism Theorem) Let $\mathbb{F}$ be a field and let (V_*, v_*) be a chain complex over $\mathbb{F}$. The map

$$\begin{array}{rcl} \mathrm{ev}_{\mathbb{F}} \colon \mathrm{H}^k(V_*; \mathbb{F}) & \to & \mathrm{Hom}_{\mathbb{F}}(\mathrm{H}_k(V_*), \mathbb{F}). \\ (\varphi \colon V_k \to \mathbb{F}) & \mapsto & \begin{pmatrix} \mathrm{H}_k(V_*) & \to & \mathbb{F} \\ [c] & \mapsto & \varphi(c) \end{pmatrix} \end{array}$$

is well-defined and it is a natural isomorphism of $\mathbb{F}$-vector spaces.

The following theorem can be viewed as an analogue of the Algebraic Universal Coefficient-H^*-Theorem 70.13 for chain complexes over fields:

Theorem 71.3. (Evaluation-Isomorphism Theorem) Let $\mathbb{F}$ be a field and let (V_*, v_*) be a chain complex over $\mathbb{F}$. The map

$$\begin{array}{rcl} \mathrm{ev}_{\mathbb{F}} \colon \mathrm{H}^k(V_*; \mathbb{F}) & \to & \mathrm{Hom}_{\mathbb{F}}(\mathrm{H}_k(V_*), \mathbb{F}). \\ [\varphi \colon V_k \to \mathbb{F}] & \mapsto & \begin{pmatrix} \mathrm{H}_k(V_*) & \to & \mathbb{F} \\ [v] & \mapsto & \varphi(v) \end{pmatrix} \end{array}$$

is well-defined and it is a natural isomorphism of $\mathbb{F}$-vector spaces.

Auxiliary Lemma 71.4. Let $\mathbb{F}$ be a field and let $0 \to B \xrightarrow{i} Z \xrightarrow{p} H \to 0$ be a short exact sequence of $\mathbb{F}$-vector spaces. Then the sequence

$$0 \to \mathrm{Hom}_{\mathbb{F}}(H, \mathbb{F}) \xrightarrow{p^*} \mathrm{Hom}_{\mathbb{F}}(Z, \mathbb{F}) \xrightarrow{i^*} \mathrm{Hom}_{\mathbb{F}}(B, \mathbb{F}) \to 0$$

is also exact.

Proof of Lemma 71.4. As in the Free Abelian Group–Splitting Lemma 18.5 we see that the original short exact sequence splits. The desired statement then follows from the analogue of the Hom-Split-SES Lemma 66.4 for chain complexes over a field. ■

Proof of the Evaluation-Isomorphism Theorem 71.3. [556] Let $\mathbb{F}$ be a field and let (V_*, ∂_*) be a chain complex of $\mathbb{F}$-vector spaces. First note that it follows, as in the Evaluation Map Lemma 70.6, from an elementary argument that the evaluation homomorphism $\mathrm{ev}_{\mathbb{F}} \colon \mathrm{H}^k(V_*; \mathbb{F}) \to \mathrm{Hom}_{\mathbb{F}}(\mathrm{H}_k(V_*), \mathbb{F})$ is actually well-defined and that it is natural. So it remains to show that it is an isomorphism.

[556] The following proof is very similar to the proof of the Algebraic Universal Coefficient-H^*-Theorem 70.13, which in turn is very similar to the proof of the Algebraic Universal Coefficient Theorem 54.8.

For each $n \in \mathbb{N}_0$ we write as always $Z_n := \ker(\partial_n)$ and $B_n := \operatorname{im}(\partial_{n+1})$. We consider the following short exact sequence of vertical chain complexes:

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Z_n & \longrightarrow & V_n & \xrightarrow{\partial_n} & B_{n-1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \partial_n} & & \downarrow{\scriptstyle 0} & & \\
0 & \longrightarrow & Z_{n-1} & \longrightarrow & V_{n-1} & \xrightarrow{\partial_{n-1}} & B_{n-2} & \longrightarrow & 0. \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

We apply the contravariant functor $A \mapsto \operatorname{Hom}_{\mathbb{F}}(A, \mathbb{F})$ to the above short exact sequence of chain complexes. This leads us to the following claim:

Claim. The following maps define a short exact sequence of "vertical" cochain complexes:

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \operatorname{Hom}_{\mathbb{F}}(B_{n-1}, \mathbb{F}) & \xrightarrow{\partial_n^*} & \operatorname{Hom}_{\mathbb{F}}(V_n, \mathbb{F}) & \longrightarrow & \operatorname{Hom}_{\mathbb{F}}(Z_n, \mathbb{F}) & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle 0} & & \uparrow{\scriptstyle \partial_n^*} & & \uparrow{\scriptstyle 0} & & \\
0 & \longrightarrow & \operatorname{Hom}_{\mathbb{F}}(B_{n-2}, \mathbb{F}) & \xrightarrow{\partial_{n-1}^*} & \operatorname{Hom}_{\mathbb{F}}(V_{n-1}, \mathbb{F}) & \longrightarrow & \operatorname{Hom}_{\mathbb{F}}(Z_{n-1}, \mathbb{F}) & \longrightarrow & 0. \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

Proof.

(1) It follows from the Dual Cochain Map Lemma 66.9 that this is a sequence of cochain complexes.
(2) Since we work over a field we obtain from Lemma 71.4 that the above horizontal sequences of cochain complexes are still exact. ⊞

We apply the Cohomology–LES Proposition 66.7 to the short exact sequence of cochain complexes from the claim. We obtain a natural connecting homomorphism

$$\underbrace{\operatorname{Hom}_{\mathbb{F}}(Z_n, \mathbb{F})}_{=\mathrm{H}^n(Z_*;\mathbb{F})} \to \underbrace{\operatorname{Hom}_{\mathbb{F}}(B_n, \mathbb{F})}_{=\mathrm{H}^n(B_*;\mathbb{F})}.$$

In fact, it follows easily from the definition of connecting homomorphisms (see page 859) that the connecting homomorphism $\operatorname{Hom}_{\mathbb{F}}(Z_n, \mathbb{F}) \to \operatorname{Hom}_{\mathbb{F}}(B_n, \mathbb{F})$ is the map that is induced by the inclusion $i_n \colon B_n \to Z_n$. By the Cohomology–LES Proposition 66.7 this connecting homomorphism gives us the following long exact sequence:

$$\begin{aligned}
\to \operatorname{Hom}_{\mathbb{F}}(Z_{n-1}, \mathbb{F}) \xrightarrow{i_n^*} \operatorname{Hom}_{\mathbb{F}}(B_{n-1}, \mathbb{F}) &\to \mathrm{H}^n(\operatorname{Hom}(V_*, \mathbb{F})) \\
&\to \operatorname{Hom}_{\mathbb{F}}(Z_n, \mathbb{F}) \xrightarrow{i_{n-1}^*} \operatorname{Hom}_{\mathbb{F}}(B_n, \mathbb{F}) \to \dots
\end{aligned}$$

We obtain the following short exact sequence

$$0 \to \overbrace{\operatorname{coker}(\operatorname{Hom}_{\mathbb{F}}(Z_{n-1},\mathbb{F}) \to \operatorname{Hom}_{\mathbb{F}}(B_{n-1},\mathbb{F}))}^{=0 \text{ by Lemma 71.4 applied to the short exact sequence } 0 \to B_l \to Z_l \to \mathrm{H}_l(V_*) \to 0} \to \mathrm{H}^n(\operatorname{Hom}_{\mathbb{F}}(V_*,\mathbb{F})) \to \underbrace{\ker(\operatorname{Hom}_{\mathbb{F}}(Z_n,\mathbb{F}) \to \operatorname{Hom}_{\mathbb{F}}(B_n,\mathbb{F}))}_{\cong \operatorname{Hom}(\mathrm{H}_n(V_*),\mathbb{F}) \text{ by Lemma 71.4 applied to the short exact sequence } 0 \to B_l \to Z_l \to \mathrm{H}_l(V_*) \to 0} \to 0.$$

■

Now we can finally provide the proof of the Universal Coefficient-H*–Field Theorem 71.2.

Proof of the Universal Coefficient-H*–Field Theorem 71.2. Statement (2) of the theorem is of course just a special case of Statement (1). So let us provide a proof of Statement (1). Let C_* be a chain complex and let $\mathbb{F}$ be a field. We need to show that for any $n \in \mathbb{N}_0$ the evaluation homomorphism

$$\begin{array}{rcl} \mathrm{ev}\colon \mathrm{H}^n(\operatorname{Hom}(C_*,\mathbb{F})) & \to & \operatorname{Hom}_{\mathbb{F}}(\mathrm{H}_n(C_*\otimes\mathbb{F}),\mathbb{F}) \\ \varphi & \mapsto & \left(\begin{array}{rcl} \mathrm{H}_n(C_*\otimes\mathbb{F}) & \to & \mathbb{F} \\ \sigma & \mapsto & \langle\varphi,\sigma\rangle \end{array}\right) \end{array}$$

from the Ring–Kronecker Pairing Lemma 71.1 is a natural isomorphism of $\mathbb{F}$-vector spaces. Naturality follows from the naturality of the Kronecker pairing which we discussed in the Ring–Kronecker Pairing Lemma 71.1. So it remains to show that the map is an isomorphism.

One might think that the desired statement follows immediately from the Evaluation-Isomorphism Theorem 71.3, but that is not quite the case since we first have to translate our setting into a problem about a chain complex over the field $\mathbb{F}$. To do so need the following elementary claim.

Claim.

(1) Let C be an abelian group. The map[557] [558]

$$\begin{array}{rcl} \Phi_C\colon \operatorname{Hom}(C,\mathbb{F}) & \to & \operatorname{Hom}_{\mathbb{F}}(C\otimes\mathbb{F},\mathbb{F}) \\ (\varphi\colon C\to\mathbb{F}) & \mapsto & \left(\begin{array}{rcl} C\otimes\mathbb{F} & \to & \mathbb{F} \\ \sum_{i=1}^{k} c_i\otimes a_i & \mapsto & \sum_{i=1}^{k}\varphi(c_i)\cdot a_i \end{array}\right) \end{array}$$

is a natural isomorphism of $\mathbb{F}$-vector spaces.

(2) The maps

$$\Phi_{C_k}\colon \operatorname{Hom}(C_k,\mathbb{F}) \to \operatorname{Hom}_{\mathbb{F}}(C_k\otimes\mathbb{F},\mathbb{F})$$

from (1) define an isomorphism of cochain complexes of $\mathbb{F}$-vector spaces.

Proof.

(1) First note that one can verify using the Tensor Product–Universal Property Lemma 51.1 (3) that the map Φ_C is well-defined. Next note that one can easily check that the map is natural. Finally note that it is straightforward to verify that an inverse map to Φ_C

[557] It follows from the Tensor-with-Ring Lemma 51.7 that we can view $C\otimes F$ as an $\mathbb{F}$-vector space.

[558] Note that to the left we consider homomorphisms of abelian groups whereas to the right we consider homomorphisms of $\mathbb{F}$-vector spaces.

is given by the following homomorphism of $\mathbb{F}$-vector spaces:

$$\begin{aligned} \Psi_C\colon \operatorname{Hom}_{\mathbb{F}}(C\otimes\mathbb{F},\mathbb{F}) &\to \operatorname{Hom}(C,\mathbb{F}) \\ (\varphi\colon C\otimes\mathbb{F}\to\mathbb{F}) &\mapsto \big(C \xrightarrow{c\mapsto c\otimes 1} C\otimes\mathbb{F} \xrightarrow{\varphi} \mathbb{F}\big). \end{aligned}$$

(2) It follows from the naturality of Φ_C that the maps in (1) actually form a cochain map. By (1) the maps Φ_{C_k}, $k\in\mathbb{N}_0$, are isomorphisms of the cochain complexes of $\mathbb{F}$-vector spaces. ⊞

Given $n\in\mathbb{N}_0$ we consider the diagram

$$\begin{array}{ccc} & \mathrm{H}^n(\operatorname{Hom}_{\mathbb{F}}(C\otimes\mathbb{F},\mathbb{F})) & \\ {\scriptstyle\Phi}\nearrow & & \searrow{\scriptstyle \mathrm{ev}_{\mathbb{F}}} \\ \mathrm{H}^n(\operatorname{Hom}(C_*,\mathbb{F})) & \xrightarrow[\mathrm{ev}]{} & \operatorname{Hom}_{\mathbb{F}}(\mathrm{H}_n(C\otimes\mathbb{F}),\mathbb{F}). \end{array}$$

We make the following observations:

- It follows easily from the definitions that for any representative in $\mathrm{H}^n(\operatorname{Hom}(C_*,\mathbb{F}))$ the image under the composition of the two diagonal maps agrees with the image under the map ev. This shows that the diagram commutes.[559]
- It follows from the claim that the left diagonal map is an isomorphism.
- It follows from the Evaluation-Isomorphism Theorem 71.3 that the right diagonal map is an isomorphism.

It follows from the above observations that the bottom map is an isomorphism. But that is exactly what we had wanted to show. ■

[559] In fact this also gives a new proof that ev is well-defined.

CHAPTER 72

The cup product

The previous chapters on cohomology produced mostly disappointing results. For example we showed in the Universal Coefficient-H*-Theorem 70.14 that the cohomology groups are completely determined by the usual humble singular homology groups. In this chapter we will finally get to see something new: We will define the cup product on cohomology, which has no real analogue for homology.

72.1. **A first definition of the cup product.** We start out by recalling some definitions from Chapters 55 and 56. First of all we have the following definition from page 693:

Definition. Let $\mathcal{C} = (C_*, \partial_*)$ and $\mathcal{D} = (D_*, \partial_*)$ be two chain complexes. Given $n \in \mathbb{N}_0$ we set

$$(\mathcal{C} \otimes \mathcal{D})_n \coloneqq \bigoplus_{p+q=n} C_p \otimes D_q.$$

The homomorphisms

$$\begin{aligned} \partial^{\otimes}\colon (\mathcal{C} \otimes \mathcal{D})_n &\to (\mathcal{C} \otimes \mathcal{D})_{n-1} \\ c_p \otimes d_q &\mapsto \underbrace{\partial_p(c_p) \otimes d_q + (-1)^p \cdot c_p \otimes \partial_q(d_q)}_{=(\partial_p \otimes \mathrm{id} \,+ (-1)^p \cdot \mathrm{id} \,\otimes \partial_q)(c_p \otimes d_q)} \end{aligned}$$

which are given by

turn the groups $(\mathcal{C} \otimes \mathcal{D})_n$ into a chain complex, called the tensor product $\mathcal{C} \otimes \mathcal{D}$.

Visualisation. During the following discussions it might be convenient to keep the following visualization of $\mathcal{C} \otimes \mathcal{D}$ from page 694 in mind:

$$\begin{array}{ccccccc}
C_0 \otimes D_3 & \xleftarrow{\partial_1 \otimes \mathrm{id}} & C_1 \otimes D_3 & \xleftarrow{\partial_2 \otimes \mathrm{id}} & C_2 \otimes D_3 & \xleftarrow{\partial_3 \otimes \mathrm{id}} & C_3 \otimes D_3 \\
\downarrow{\scriptstyle \mathrm{id} \otimes \partial_3} & & \downarrow{\scriptstyle -\mathrm{id} \otimes \partial_3} & & \downarrow{\scriptstyle \mathrm{id} \otimes \partial_3} & & \downarrow{\scriptstyle -\mathrm{id} \otimes \partial_3} \\
C_0 \otimes D_2 & \xleftarrow{\partial_1 \otimes \mathrm{id}} & C_1 \otimes D_2 & \xleftarrow{\partial_2 \otimes \mathrm{id}} & C_2 \otimes D_2 & \xleftarrow{\partial_3 \otimes \mathrm{id}} & C_3 \otimes D_2 \\
\downarrow{\scriptstyle \mathrm{id} \otimes \partial_2} & & \downarrow{\scriptstyle -\mathrm{id} \otimes \partial_2} & & \downarrow{\scriptstyle \mathrm{id} \otimes \partial_2} & & \downarrow{\scriptstyle -\mathrm{id} \otimes \partial_2} \\
C_0 \otimes D_1 & \xleftarrow{\partial_1 \otimes \mathrm{id}} & C_1 \otimes D_1 & \xleftarrow{\partial_2 \otimes \mathrm{id}} & C_2 \otimes D_1 & \xleftarrow{\partial_3 \otimes \mathrm{id}} & C_3 \otimes D_1 \\
\downarrow{\scriptstyle \mathrm{id} \otimes \partial_1} & & \downarrow{\scriptstyle -\mathrm{id} \otimes \partial_1} & & \downarrow{\scriptstyle \mathrm{id} \otimes \partial_1} & & \downarrow{\scriptstyle -\mathrm{id} \otimes \partial_1} \\
C_0 \otimes D_0 & \xleftarrow{\partial_1 \otimes \mathrm{id}} & C_1 \otimes D_0 & \xleftarrow{\partial_2 \otimes \mathrm{id}} & C_2 \otimes D_0 & \xleftarrow{\partial_3 \otimes \mathrm{id}} & C_3 \otimes D_0.
\end{array}$$

Here the "antidiagonal" groups of the same color "add up" to the chain groups of $\mathcal{C} \otimes \mathcal{D}$.

Next we recall the following purely algebraic lemma:

Lemma 55.4. (Cycle Tensor Lemma) Let $\mathcal{C}$ and $\mathcal{D}$ be chain complexes. The map

$$\begin{aligned} \mu\colon \mathrm{H}_p(\mathcal{C}) \otimes \mathrm{H}_q(\mathcal{D}) &\to \mathrm{H}_{p+q}(\mathcal{C} \otimes \mathcal{D}) \\ \sum_{i=1}^{r} [c_i] \otimes [d_i] &\mapsto \sum_{i=1}^{r} [c_i \otimes d_i] \end{aligned}$$

is a well-defined natural homomorphism.

We turn to recalling topological concepts:

Notation. Given topological spaces X and Y we denote by

$$\begin{aligned} \Upsilon := \Upsilon_{X,Y} \colon \mathrm{C}_*(X) \otimes \mathrm{C}_*(Y) &\to \mathrm{C}_*(X \times Y) \\ \Theta := \Theta_{X,Y} \colon \mathrm{C}_*(X \times Y) &\to \mathrm{C}_*(X) \otimes \mathrm{C}_*(Y) \end{aligned}$$

the natural chain homotopy equivalences from the Eilenberg-Zilber Theorem 56.1.[560]

As on page 704 the above concepts allow us to define the cross product on homology:

Definition. Let X and Y be topological spaces. We refer to the homomorphism[561]

$$\times \colon \overbrace{\mathrm{H}_p(\mathrm{C}_*(X)) \otimes \mathrm{H}_q(\mathrm{C}_*(X))}^{=\mathrm{H}_p(X) \otimes \mathrm{H}_q(Y)} \xrightarrow{\mu} \mathrm{H}_{p+q}(\mathrm{C}_*(X) \otimes \mathrm{C}_*(Y)) \xrightarrow{\Upsilon_*} \overbrace{\mathrm{H}_{p+q}(\mathrm{C}_*(X \times Y))}^{=\mathrm{H}_{p+q}(X \times Y)}$$

as the cross product.

The cross product applied to the same topological space $X = Y$ gives us in particular the cross product

$$\times \colon \mathrm{H}_p(X) \otimes \mathrm{H}_q(X) \to \mathrm{H}_{p+q}(X \times X).$$

This is very charming, but this construction does not give us a product on the homology groups of X itself, since we end up in the homology of a different topological space.

Now we try our luck with cohomology groups. Our first goal is to formulate a cohomological version of the Cycle Tensor Lemma 55.4. This requires a little bit of preparation:

Definition. Let $\mathcal{C}_*$ and $\mathcal{D}_*$ be chain complexes.

(1) Given $p, q \in \mathbb{N}_0$ we denote by

$$\Pi_{p,q} \colon (\mathcal{C}_* \otimes \mathcal{D}_*)_{p+q} = \bigoplus_{i+j=p+q} \mathcal{C}_i \otimes \mathcal{D}_j \to \mathcal{C}_p \otimes \mathcal{D}_q$$

the natural projection.

(2) For any commutative ring R we consider the multiplication map[562]

$$\begin{aligned} \mu_R \colon R \otimes R &\to R \\ \sum_{i=1}^{m} r_i \otimes s_i &\mapsto \sum_{i=1}^{m} r_i \cdot s_i. \end{aligned}$$

The following lemma plays the role of the Cycle Tensor Lemma 55.4:

Lemma 72.1. (Cocycle Tensor Lemma) Let $\mathcal{C}_*$ and $\mathcal{D}_*$ be chain complexes. For any commutative ring R the maps

$$\begin{aligned} \mathrm{Hom}(\mathcal{C}_p, R) \times \mathrm{Hom}(\mathcal{D}_q, R) &\to \mathrm{Hom}((\mathcal{C} \otimes \mathcal{D})_{p+q}, R) \\ (\varphi, \psi) &\mapsto ((\mathcal{C}_* \otimes \mathcal{D}_*)_{p+q} \xrightarrow{\Pi_{p,q}} \mathcal{C}_p \otimes \mathcal{D}_q \xrightarrow{\varphi \otimes \psi} R \otimes R \xrightarrow{\mu_R} R) \end{aligned}$$

[560] In the Eilenberg-Zilber Theorem 56.1 we saw that these maps are unique up to a natural chain homotopy equivalence. So for the most part the precise choice of these maps is irrelevant.

[561] As we just mentioned in Footnote 560, it follows from the Eilenberg-Zilber Theorem 56.1 (2) together with the Chain Homotopy–H_*-Lemma 36.1 that this map is independent of the choice of Υ.

[562] Using the Tensor Product–Universal Property Lemma 51.1 (3) one can easily verify that μ_R is well-defined.

descend to a well-defined homomorphism

$$\begin{array}{rcl} \Xi\colon \mathrm{H}^p(\mathcal{C}_*;R)\otimes \mathrm{H}^q(\mathcal{D}_*;R) & \to & \mathrm{H}^{p+q}(\mathcal{C}_*\otimes\mathcal{D}_*;R) \\ [\varphi]\otimes[\psi] & \mapsto & [\mu_R\circ(\varphi\otimes\psi)\circ\Pi_{p,q}]. \end{array}$$

Proof. Let $\mathcal{C}_*$ and $\mathcal{D}_*$ be chain complexes. We start out with the following claim:

Claim 1. If $\varphi\colon C_p\to R$ and $\psi\colon D_q\to R$ are cocycles, then

$$\mu_R\circ(\varphi\otimes\psi)\circ\Pi_{p,q}\colon (\mathcal{C}_*\otimes\mathcal{D}_*)_{p+q} \to R$$

is a cocycle.

Proof. We need to show that $\delta_{p+q}(\mu_R\circ(\varphi\otimes\psi)\circ\Pi_{p,q})=0\in\mathrm{Hom}((\mathcal{C}\otimes\mathcal{D})_{p+q+1},R)$. In other words, we need to show that $(\mu_R\circ(\varphi\otimes\psi)\circ\Pi_{p,q})\circ\partial^{\otimes}_{p+q+1}\colon(\mathcal{C}_*\otimes\mathcal{D}_*)_{p+q+1}\to R$ is the zero map. This is indeed the case:

$$\begin{aligned} (\mu_R\circ(\varphi\otimes\psi)\circ\Pi_{p,q})\circ\partial^{\otimes}_{p+q+1} &\overset{\text{definition of }\partial^{\otimes}}{=} (\mu_R\circ(\varphi\otimes\psi)\circ\Pi_{p,q})\Big(\bigoplus_{r+s=p+q+1}(\partial_r\otimes\mathrm{id}_{D_s}+(-1)^r\cdot\mathrm{id}_{C_r}\otimes\partial_s)\Big) \\ &= (\mu_R\circ(\varphi\otimes\psi))(\partial_{p+1}\otimes\mathrm{id}_{D_q}+(-1)^p\cdot\mathrm{id}_{C_p}\otimes\partial_{q+1}) \\ &= (\mu_R\circ(\underbrace{(\varphi\circ\partial_{p+1})}_{=0,\text{ since }\varphi\text{ is a cocycle}}\otimes\psi))+(-1)^p\cdot(\mu_R\circ(\varphi\otimes\underbrace{(\psi\circ\partial_{q+1})}_{=0,\text{ since }\psi\text{ is a cocycle}}))\ =\ 0. \end{aligned}$$

⊞

Claim 2. The map

$$\begin{array}{rcl} \mathrm{H}^p(\mathcal{C}_*;R)\otimes \mathrm{H}^q(\mathcal{D}_*;R) & \to & \mathrm{H}^{p+q}(\mathcal{C}_*\otimes\mathcal{D}_*;R) \\ [\varphi]\otimes[\psi] & \mapsto & [\mu_R\circ(\varphi\otimes\psi)\circ\Pi_{p,q}] \end{array}$$

is well-defined, i.e. it does not depend on the choice of representatives of the cohomology classes.

Proof. Let $\varphi\colon C_p\to R$ and $\psi\colon D_q\to R$ be cocycles and let $\alpha\colon C_{p-1}\to R$ be a cochain. Note that in $\mathrm{H}^{p+q}(\mathcal{C}_*\otimes\mathcal{D}_*;R)$ we have

$$\begin{aligned} [(\mu_R\circ(\varphi\otimes\psi)\circ\Pi_{p,q})]-[(\mu_R\circ((\varphi+\delta_{p-1}(\alpha))\otimes\psi)\circ\Pi_{p,q}] &= [\mu_R\circ(\delta_{p-1}(\alpha)\otimes\psi)\circ\Pi_{p,q}] \\ &= [\mu_R\circ(\underbrace{(\alpha\circ\partial_p)}_{=\delta_{p-1}(\alpha)}\otimes\psi)\circ\Pi_{p,q}]+(-1)^p\cdot[\mu_R\circ(\alpha\otimes\underbrace{(\psi\circ\partial_{q+1})}_{=0,\text{ since }\psi\text{ is a cocycle}})\circ\Pi_{p-1,q+1}] \\ &= [(\mu_R\circ(\alpha\otimes\psi)\circ(\Pi_{p,q}+\Pi_{p-1,q+1}))\circ(\partial_p\otimes\mathrm{id}_{D_q}+(-1)^p\cdot\mathrm{id}_{C_p}\otimes\partial_{q+1})] \\ &= [(\mu_R\circ(\alpha\otimes\psi)\circ(\Pi_{p,q}+\Pi_{p-1,q+1}))\circ\partial^{\otimes}_{p+q}]\ \overset{\text{since cocycles represent the zero class in cohomology}}{=}\ 0. \\ &\uparrow \\ &\text{definition of }\partial^{\otimes}_{p+q} \end{aligned}$$

The same argument also shows that the definition does not depend on the choice of the representative of the element in $\mathrm{H}^q(\mathcal{D}_*;R)$. ⊞

It follows easily from the definitions that the map in Claim 2 is bilinear. Finally note that it follows from the Tensor Product–Universal Property Lemma 51.1 (3) that we get the promised homomorphism Ξ. ■

We return to topology. The following definition is the cohomological analogue of the above cross product:

Definition. Let X and Y be topological spaces and let R be a commutative ring. We refer to the homomorphism[563] [564]

$$\overbrace{\mathrm{H}^p(X;R)\otimes\mathrm{H}^q(Y;R)}^{=\mathrm{H}^p(X;R)\otimes\mathrm{H}^q(Y;R)} \xrightarrow{\Xi} \mathrm{H}^{p+q}(\mathrm{C}_*(X)\otimes\mathrm{C}_*(Y);R) \xrightarrow{\Theta^*_{X,Y}} \overbrace{\mathrm{H}^{p+q}(\mathrm{C}_*(X\times Y);R)}^{=\mathrm{H}^{p+q}(X\times Y;R)}.$$
$$[\varphi]\otimes[\psi] \quad\mapsto\quad [\mu_R\circ(\varphi\otimes\psi)\circ\Pi_{p,q}].$$

as the cross product on cohomology.

The cross product on cohomology gives us in particular a homomorphism

$$\times\colon \mathrm{H}^p(X;R)\otimes\mathrm{H}^q(X;R) \;\to\; \mathrm{H}^{p+q}(X\times X;R).$$

Again, this is nice, but exactly as in the case of the cross product on homology, we end up in the cohomology of $X\times X$, but we really would like to end up in the cohomology of X. Now we have to find a way how to go back from the topological space $X\times X$ to the original space X. In general there does not exist an "interesting"[565] map $X\times X\to X$. But, as we saw before, we still have one ace up in our sleeves: There does exist an interesting map in the opposite direction, namely the diagonal map:

Definition. Given a topological space X we refer to[566]

$$\begin{array}{rcl} d_X\colon X & \to & X\times X \\ x & \mapsto & (x,x) \end{array}$$

as the diagonal map d_X.

Note that the diagonal map induces a map

$$d_{X*}\colon \mathrm{H}_*(X;R) \;\to\; \mathrm{H}_*(X\times X;R)$$

on homology, and, much more interesting in our context, it also induces a map

$$d_X^*\colon \mathrm{H}^*(X\times X;R) \;\to\; \mathrm{H}^*(X;R).$$

on cohomology groups that goes in the "opposite direction". This observation shows that we can "marry" the cohomological cross product with d_X^*, and as an offspring we obtain the following wonderful definition:

Definition. Let X be a topological space and let R be a commutative ring. We define the cup product[567] on the cohomology groups of X as the composition of the following maps:

$$\mathrm{H}^p(X;R)\times\mathrm{H}^q(X;R) \to \mathrm{H}^p(X;R)\otimes\mathrm{H}^q(X;R) \xrightarrow{\Theta^*_{X,X}\circ\Xi} \mathrm{H}^{p+q}(X\times X;R) \xrightarrow{d_X^*} \mathrm{H}^{p+q}(X;R).$$

---

[563] The first map is provided by the Cocycle Tensor Lemma 72.1.

[564] It follows from the Eilenberg-Zilber Theorem 56.1 (2), the Dual Cochain Map Lemma 66.9 (3) and the Cochain Homotopy-H^*-Lemma 66.6 that the definition below does not depend on the choice of Θ.

[565] The projection from $\pi\colon X\times X\to X$ onto say the first factor does not count as "interesting", since it does not "mix" the two factors. In fact one can show that the map

$$\mathrm{H}_p(X)\otimes\mathrm{H}_q(X) \xrightarrow{\times} \mathrm{H}_{p+q}(X\times X) \xrightarrow{\pi_*} \mathrm{H}_*(X)$$

is the zero map unless $q=0$.

[566] It follows from the Topological-Product Proposition 5.1 (1a) and (1b) that the diagonal map d_X is continuous.

For the resulting pairing we use the notation

$$\begin{array}{rcl} \mathrm{H}^p(X;R) \times \mathrm{H}^q(X;R) & \to & \mathrm{H}^{p+q}(X;R) \\ (a,b) & \mapsto & a \smile b \end{array}$$

and we refer to[568] $a \smile b$ as the cup product of a and b.

The following lemma shows that the cup product is natural:

Lemma 72.2. (Cup Product-Naturality Lemma) Let R be a commutative ring and let $p, q \in \mathbb{N}_0$. For any continuous map $f\colon X \to Y$ between topological spaces the following diagram commutes:

$$\begin{array}{ccccc} \mathrm{H}^p(Y;R) \times \mathrm{H}^q(Y;R) & & \xrightarrow{\smile} & & \mathrm{H}^{p+q}(Y;R) \\ \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} \\ \mathrm{H}^p(X;R) \times \mathrm{H}^q(X;R) & & \xrightarrow{\smile} & & \mathrm{H}^{p+q}(X;R). \end{array}$$

In other words, for any $c \in \mathrm{H}^p(Y;R)$ and any $d \in \mathrm{H}^q(Y;R)$ we have

$$f^*(c) \smile f^*(d) \;=\; f^*(c \smile d).$$

Proof. Let $f\colon X \to Y$ be a continuous map between topological spaces. We consider the following diagram:

$$\begin{array}{ccccccccc} \mathrm{H}^p(Y;R) \times \mathrm{H}^q(Y;R) & \longrightarrow & \mathrm{H}^p(Y;R) \otimes \mathrm{H}^q(Y;R) & \xrightarrow{\Theta^*_{Y,Y} \circ \Xi} & \mathrm{H}^{p+q}(Y \times Y;R) & \xrightarrow{d_Y^*} & \mathrm{H}^{p+q}(Y;R) \\ \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^* \otimes f^*} & & \downarrow{\scriptstyle (f,f)^*} & & \downarrow{\scriptstyle f^*} \\ \mathrm{H}^p(X;R) \times \mathrm{H}^q(X;R) & \longrightarrow & \mathrm{H}^p(X;R) \otimes \mathrm{H}^q(X;R) & \xrightarrow{\Theta^*_{X,X} \circ \Xi} & \mathrm{H}^{p+q}(X \times X;R) & \xrightarrow{d_X^*} & \mathrm{H}^{p+q}(X;R). \end{array}$$

It follows easily from the Tensor–Map Composition Lemma 51.2 and the naturality of $\Theta_{X,X}$, Ξ and d_X that the diagram commutes. ∎

The above definition of the cup product is very natural (no pun intended). But the problem with the above definition is that it is not explicit at all. In the next section we will give an alternative and utterly explicit description of the cup product. This definition is much more suitable for calculations of the cup product.

Later, when we study properties of the cup product and when we develop the theory we will make use of the fact that we have two alternative approaches, and we will choose whatever definition is more practical for the given task.

72.2. **An alternative definition of the cup product.** In this section we give the promised alternative definition of the cup product which is more ad hoc, but which works surprisingly well in practice. We will see in the Cup Product–Definitions Proposition 72.5 that both approaches lead to the same concept.

Notation.

(1) For $i = 0, \ldots, n$ we write $\mathfrak{v}_i := (0, \ldots, 1, 0, \ldots, 0) \in \mathbb{R}^{n+1}$.
(2) For $a_0, \ldots, a_k \in \Delta^n$ we denote by $[a_0, \ldots, a_k]\colon \Delta^k \to \Delta^n$ the unique affine linear map that sends $\mathfrak{v}_i$ to a_i.

[567] The name "cup product" comes from the notation "$a \smile b$" for the composition since $\cup$ looks like a cup.
[568] The weird looking "$\sim$" will be removed shortly, once we show that this definition is equivalent to an alternative definition of the cup product.

With this notation we can now give our second definition of the cup product:

Definition. Let X be a topological space and let R be a commutative ring. Furthermore let $\varphi \in \mathrm{C}^p(X;R)$ and $\psi \in \mathrm{C}^q(X;R)$ be cochains. We define the cup product of φ and ψ to be the $(p+q)$-cochain

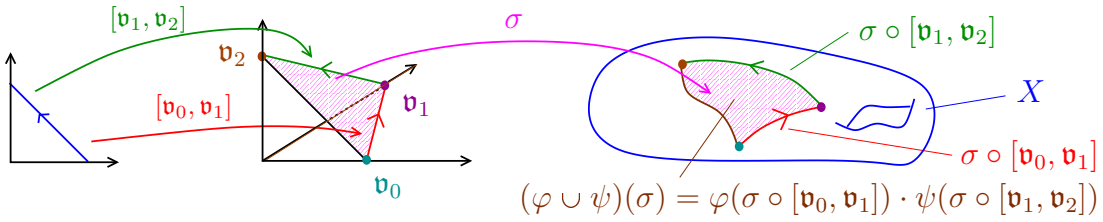
$$\varphi \cup \psi \colon \mathrm{C}_{p+q}(X) \ \to \ R$$

that is given by

$$(\sigma \colon \Delta^{p+q} \to X) \ \mapsto \ \underbrace{\varphi(\underbrace{\sigma \circ [\mathfrak{v}_0, \ldots, \mathfrak{v}_p]}_{\text{map } \Delta^p \to X})}_{\in R} \cdot \underbrace{\psi(\underbrace{\sigma \circ [\mathfrak{v}_p, \ldots, \mathfrak{v}_{p+q}]}_{\text{map } \Delta^q \to X})}_{\in R}.$$

The cup product thus defines a map

$$\cup \colon \mathrm{C}^p(X;R) \times \mathrm{C}^q(X;R) \ \to \ \mathrm{C}^{p+q}(X;R).$$



The definition of the cup product on cochains might look strange and arbitrary. But the following lemma shows that it behaves well with respect to coboundary maps.

Lemma 72.3. (Cup Product–Coboundary Lemma) Let X be a topological space and let R be a commutative ring. Given any $\varphi \in \mathrm{C}^p(X;R)$ and $\psi \in \mathrm{C}^q(X;R)$ we have

$$\delta_{p+q}(\varphi \cup \psi) \ = \ \delta_p(\varphi) \cup \psi \ + (-1)^p \cdot \varphi \cup \delta_q(\psi) \quad \in \ \mathrm{C}^{p+q+1}(X;R).$$

Proof. The lemma follows from an elementary, albeit slightly confusing, calculation. Indeed, let $\varphi \in \mathrm{C}^p(X;R)$ and $\psi \in \mathrm{C}^q(X;R)$ be cochains. We have to prove the following equality:

$$\delta_{p+q}(\varphi \cup \psi) \ = \ \delta_p(\varphi) \cup \psi \ + (-1)^p \cdot \varphi \cup \delta_q(\psi) \ \in \ \mathrm{C}^{p+q+1}(X;R) = \mathrm{Hom}(\mathrm{C}_{p+q+1}(X), R).$$

So let $\sigma \colon \Delta^{p+q+1} \to X$ be a singular $(p+q+1)$-simplex. We have the following three equalities in R:[569]

$$\begin{aligned}
\text{(a)} \quad (\delta_{p+q}(\varphi \cup \psi))(\sigma) &= (\varphi \cup \psi)(\partial_{p+q+1}(\sigma)) \ = \ (\varphi \cup \psi)\Big(\sum_{j=0}^{p+q+1} (-1)^j \cdot \sigma \circ i_j^{p+q+1}\Big) \\
&= (\varphi \cup \psi)\Big(\sum_{j=0}^{p} (-1)^j \cdot \sigma \circ i_j^{p+q+1}\Big) + (\varphi \cup \psi)\Big(\sum_{j=p+1}^{p+q+1} (-1)^j \cdot \sigma \circ i_j^{p+q+1}\Big) \\
&= \sum_{j=0}^{p} (-1)^j \cdot (\varphi \cup \psi)(\sigma \circ i_j^{p+q+1}) + (-1)^p \cdot \sum_{k=1}^{q+1} (-1)^k \cdot (\varphi \cup \psi)(\sigma \circ i_{k+p}^{p+q+1})
\end{aligned}$$

[569] On several occasions we use that for a singular n-cochain ϕ and a singular $(n+1)$-simplex μ_R we have by definition

$$(\delta_n(\phi))(\mu_R) \ = \ \phi(\partial_{n+1}(\mu_R)) \ = \ \phi\Big(\sum_{l=0}^{n+1} (-1)^l \cdot \mu_R \circ i_l^{n+1}\Big).$$

$$= \sum_{j=0}^{p}(-1)^j\cdot\varphi(\underbrace{\sigma\circ[\mathfrak{v}_0,\dots,\mathfrak{v}_{p+1}]\circ i_j^{p+1}}_{\text{map }\Delta^p\to X})\cdot\psi(\underbrace{\sigma\circ[\mathfrak{v}_{p+1},\dots,\mathfrak{v}_{p+q+1}]}_{\text{map }\Delta^q\to X})$$
$$+(-1)^p\cdot\sum_{k=1}^{q+1}(-1)^k\cdot\varphi(\underbrace{\sigma\circ[\mathfrak{v}_0,\dots,\mathfrak{v}_p]}_{\text{map }\Delta^p\to X})\cdot\psi(\underbrace{\sigma\circ[\mathfrak{v}_p,\dots,\mathfrak{v}_{p+q+1}]\circ i_k^{q+1}}_{\text{map }\Delta^q\to X}).$$

(b) $$(\delta_p(\varphi)\cup\psi)(\sigma) = (\delta_p(\varphi))(\sigma\circ[\mathfrak{v}_0,\dots,\mathfrak{v}_{p+1}])\cdot\psi(\sigma\circ[\mathfrak{v}_{p+1},\dots,\mathfrak{v}_{p+q+1}])$$
$$= \varphi\Big(\sum_{j=0}^{p+1}(-1)^j\cdot\underbrace{\sigma\circ[\mathfrak{v}_0,\dots,\mathfrak{v}_{p+1}]\circ i_j^{p+1}}_{\text{map }\Delta^p\to X}\Big)\cdot\psi(\underbrace{\sigma\circ[\mathfrak{v}_{p+1},\dots,\mathfrak{v}_{p+q+1}]}_{\text{map }\Delta^q\to X})$$
$$= \sum_{j=0}^{p+1}(-1)^j\cdot\varphi(\sigma\circ[\mathfrak{v}_0,\dots,\mathfrak{v}_{p+1}]\circ i_j^{p+1})\cdot\psi(\sigma\circ[\mathfrak{v}_{p+1},\dots,\mathfrak{v}_{p+q+1}])$$

and

(c) $$(-1)^p\cdot(\varphi\cup\delta_q(\psi))(\sigma) = (-1)^p\cdot\varphi(\sigma\circ[\mathfrak{v}_0,\dots,\mathfrak{v}_p])\cdot(\delta_q(\psi))(\sigma\circ[\mathfrak{v}_p,\dots,\mathfrak{v}_{p+q+1}])$$
$$= (-1)^p\cdot\varphi(\underbrace{\sigma\circ[\mathfrak{v}_0,\dots,\mathfrak{v}_p]}_{\text{map }\Delta^p\to X})\cdot\psi\Big(\sum_{k=0}^{q+1}(-1)^k\cdot\underbrace{\sigma\circ[\mathfrak{v}_p,\dots,\mathfrak{v}_{p+q+1}]\circ i_k^{q+1}}_{\text{map }\Delta^q\to X}\Big)$$
$$= (-1)^p\cdot\sum_{k=0}^{q+1}(-1)^k\cdot\varphi(\sigma\circ[\mathfrak{v}_0,\dots,\mathfrak{v}_p])\cdot\psi(\sigma\circ[\mathfrak{v}_p,\dots,\mathfrak{v}_{p+q+1}]\circ i_k^{q+1}).$$

Now we observe that if we take the sum of (b) and (c), then the $(j=p+1)$-summand in (b) cancels with the $(k=0)$-summand in (c). But after this observation it is obvious that the desired equality (b)+(c)=(a) holds. ∎

The Cup Product–Coboundary Lemma 72.3 leads us to the following lemma:

Lemma 72.4. (Cup Product–Definition Lemma) Let X be a topological space. For any commutative ring R the map

$$\begin{aligned}\cup\colon \mathrm{H}^p(X;R)\times\mathrm{H}^q(X;R) &\to \mathrm{H}^{p+q}(X;R)\\ ([\varphi],[\psi])] &\mapsto [\varphi]\cup[\psi]\coloneqq[\varphi\cup\psi]\end{aligned}$$

is well-defined.

Definition. We refer to the pairing of the Cup Product–Definition Lemma 72.4 also as the cup product.

Proof. Let $\varphi\in\mathrm{C}^p(X;R)$ and $\psi\in\mathrm{C}^q(X;R)$ be cocycles.

(1) It follows immediately from the Cup Product–Coboundary Lemma 72.3 that $\varphi\cup\psi$ is again a cocycle.

(2) If $\varphi'=\varphi+\delta_{p-1}(\tau)$ is another representative of $[\varphi]$, then

$$\varphi'\cup\psi-\varphi\cup\psi \underset{\uparrow}{=} (\varphi'-\varphi)\cup\psi = \delta_{p-1}(\tau)\cup\psi \underset{\uparrow}{=} \delta_{p+q-1}(\tau\cup\psi).$$

since "$\cup$" is clearly bilinear — by the Cup Product–Coboundary Lemma 72.3 and since $\delta_q(\psi)=0$

This shows that $\varphi\cup\psi$ and $\varphi'\cup\psi$ represent the same cohomology class, i.e. the cup product does not depend on the choice of the representative of the cohomology class of the first entry.

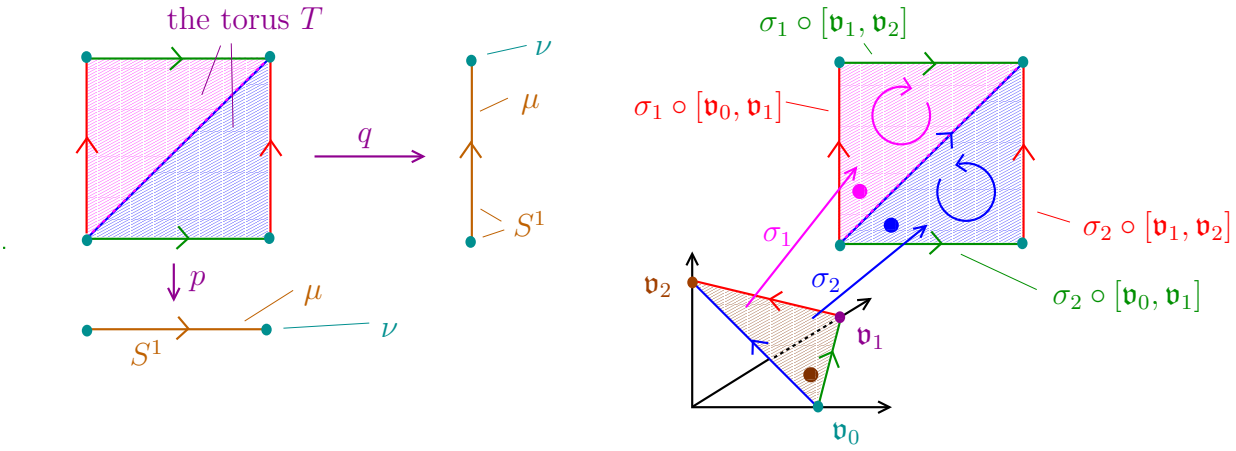
(3) The proof that the cup product does not depend on the choice of the representative of the second cohomology class is basically the same as the proof of (2). ∎

Example. In the following we consider the torus $T = S^1 \times S^1 = ([0,1] \times [0,1])/\sim$. We recall the following notation:

(1) We denote by p respectively q the projection $S^1 \times S^1 \to S^1$ onto the first respectively second factor.
(2) On page 866 we introduced an explicit cochain $\theta_{\mathbb{Z}} \in \mathrm{C}^1(S^1;\mathbb{Z})$. In the θ-Cocycle Lemma 67.2 we showed that $\theta_{\mathbb{Z}}$ is actually a cocycle.
(3) We consider the cohomology classes $p^*([\theta_{\mathbb{Z}}]), q^*([\theta_{\mathbb{Z}}]) \in \mathrm{H}^2(T;\mathbb{Z})$.

We want to show that the cup product $p^*([\theta_{\mathbb{Z}}]) \cup q^*([\theta_{\mathbb{Z}}]) \neq 0 \in \mathrm{H}^2(T;\mathbb{Z})$. To do so we need to introduce one more piece of notation:

(4) Let $\sigma_1, \sigma_2 \colon \Delta^2 \to T$ be the singular 2-simplices that are shown in the figure below.



First we want to determine the values of $p^*(\theta_{\mathbb{Z}}) \cup q^*(\theta_{\mathbb{Z}}) \in \mathrm{C}^2(T;\mathbb{Z}) = \mathrm{Hom}(\mathrm{C}_2(T),\mathbb{Z})$ on the singular 2-simplices σ_1 and σ_2. We calculate that

$$\begin{aligned}(p^*(\theta_{\mathbb{Z}}) \cup q^*(\theta_{\mathbb{Z}}))(\sigma_1) &= p^*(\theta_{\mathbb{Z}})(\sigma_1 \circ [\mathfrak{v}_0,\mathfrak{v}_1]) \cdot q^*(\theta_{\mathbb{Z}})(\sigma_1 \circ [\mathfrak{v}_1,\mathfrak{v}_2]) \\ &= \theta_{\mathbb{Z}}(\underbrace{p \circ \sigma_1 \circ [\mathfrak{v}_0,\mathfrak{v}_1]}_{\text{constant simplex } \nu}) \cdot \theta_{\mathbb{Z}}(\underbrace{q \circ \sigma_1 \circ [\mathfrak{v}_1,\mathfrak{v}_2]}_{\text{constant simplex } \nu}) \overset{\uparrow}{=} 0 \cdot 0 = 0.\end{aligned}$$

clearly $\theta_{\mathbb{Z}}(\nu) = 0$

Much more interestingly we see that

$$\begin{aligned}(p^*(\theta_{\mathbb{Z}}) \cup q^*(\theta_{\mathbb{Z}}))(\sigma_2) &= p^*(\theta_{\mathbb{Z}})(\sigma_2 \circ [\mathfrak{v}_0,\mathfrak{v}_1]) \cdot q^*(\theta_{\mathbb{Z}})(\sigma_2 \circ [\mathfrak{v}_1,\mathfrak{v}_2]) \\ &= \theta_{\mathbb{Z}}(\underbrace{p \circ \sigma_2 \circ [\mathfrak{v}_0,\mathfrak{v}_1]}_{\substack{\text{the simplex } \mu\colon \Delta^1 \to S^1 \text{ that} \\ \text{is given by } (1\text{–}t,t) \mapsto \exp(2\pi i t)}}) \cdot \theta_{\mathbb{Z}}(\underbrace{q \circ \sigma_2 \circ [\mathfrak{v}_1,\mathfrak{v}_2]}_{=\mu}) \overset{\uparrow}{=} 1 \cdot 1 = 1.\end{aligned}$$

on page 866 we saw that $\theta_{\mathbb{Z}}(\mu) = 1$

Now we consider the Kronecker pairing $\langle\ ,\ \rangle \colon \mathrm{H}^2(T;\mathbb{Z}) \times \mathrm{H}_2(T) \to \mathbb{Z}$ and we see that

$$\begin{aligned}\langle p^*([\theta_{\mathbb{Z}}]) \cup q^*([\theta_{\mathbb{Z}}]), [T]\rangle &\overset{\downarrow}{=} \langle [p^*(\theta_{\mathbb{Z}}) \cup q^*(\theta_{\mathbb{Z}})], [\sigma_1 - \sigma_2]\rangle \\ &\overset{\uparrow}{=} (p^*(\theta_{\mathbb{Z}}) \cup q^*(\theta_{\mathbb{Z}}))(\sigma_2) - (p^*(\theta_{\mathbb{Z}}) \cup q^*(\theta_{\mathbb{Z}}))(\sigma_1) \overset{\uparrow}{=} 1 - 0.\end{aligned}$$

on page 835 we showed that $\sigma_2 - \sigma_1$ is a cycle that represents $[T] \in \mathrm{H}_2(T)$

definition of the Kronecker pairing

see the above calculation

This calculation shows that the cup product $p^*([\theta_{\mathbb{Z}}]) \cup q^*([\theta_{\mathbb{Z}}]) \in \mathrm{H}^2(T;\mathbb{Z})$ is non-zero, in fact we have shown that it is a generator of $\mathrm{H}^2(T;\mathbb{Z})$. □

72.3. **The two definitions of the cup product.** The following proposition says that the original definition of the cup product and the new definition actually agree.

Proposition 72.5. (Cup Product–Definitions Proposition) Let X be a topological space and let R be a commutative ring. For any $\varphi \in \mathrm{H}^p(X;R)$ and any $\psi \in \mathrm{H}^q(X;R)$ we have

$$\underbrace{\varphi \mathbin{\widetilde{\cup}} \psi}_{\substack{\text{defined on}\\ \text{page 920}}} = \underbrace{\varphi \cup \psi}_{\substack{\text{defined on}\\ \text{page 921}}} \in \mathrm{H}^{p+q}(X;R).$$

We now know that our two definitions of the cup product agree. Beginning with the next section we will go back and forth between the two definitions, at each given moment we will pick the definition which is easier to work with.

The proof of the Cup Product–Definitions Proposition 72.5 will require the remainder of this section. In the next Section 72.4 we will then study the question whether or not the cup product is commutative.

The proof of the Cup Product–Definitions Proposition 72.5 rests on the notion of a "diagonal approximation":

Definition. A diagonal approximation consists of a natural chain map

$$\Phi\colon \mathrm{C}_*(X) \to \mathrm{C}_*(X) \otimes \mathrm{C}_*(X)$$

for each topological space X, such that for each $x \in X$, viewed as a singular 0-simplex, the following equality holds:

$$\Phi(x) = x \otimes x \in (\mathrm{C}_*(X) \otimes \mathrm{C}_*(X))_0 = \mathrm{C}_0(X) \otimes \mathrm{C}_0(X).$$

Example. It follows easily from the definitions that for any topological space X the maps

$$\Theta_{X,X} \circ d_{X*}\colon \mathrm{C}_*(X) \xrightarrow[\uparrow]{d_{X*}} \mathrm{C}_*(X \times X) \xrightarrow[\uparrow]{\Theta_{X,X}} \mathrm{C}_*(X) \otimes \mathrm{C}_*(X)$$

(d_X is the diagonal map; Eilenberg-Zilber map)

form a diagonal approximation in the above sense. □

The plan for the remainder of this section is as follows:

(1) In the Alexander-Whitney Lemma 72.6 we provide an explicit diagonal approximation.
(2) In the Diagonal Approximation–Uniqueness Proposition 72.7 we show that diagonal approximations are essentially unique.
(3) We use (1), (2) and the above example to prove the Cup Product–Definitions Proposition 72.5.

The following lemma gives a completely explicit diagonal approximation which will be very useful for doing actual calculations:

Lemma 72.6. (Alexander-Whitney Lemma) Given any topological space X we consider the map

$$\Delta\colon \mathrm{C}_*(X) \to \mathrm{C}_*(X) \otimes \mathrm{C}_*(X)$$

which is given by

$$(\sigma\colon \Delta^n \to X) \mapsto \sum_{p=0}^{n} \underbrace{\sigma \circ [\mathfrak{v}_0, \ldots, \mathfrak{v}_p]}_{\text{map } \Delta^p \to X} \otimes \underbrace{\sigma \circ [\mathfrak{v}_p, \ldots, \mathfrak{v}_n]}_{\text{map } \Delta^{n-p} \to X}.$$

These maps form a diagonal approximation.

Definition. The diagonal approximation from the Alexander-Whitney Lemma 72.6 is called the Alexander-Whitney diagonal approximation.

Proof. It follows almost immediately from the definition that the maps Δ are natural. Furthermore it is obvious that on the 0-chains the maps are of the desired form. It remains to prove the following claim:

Claim. For any topological space X the map

$$\begin{aligned} \Delta\colon \mathrm{C}_*(X) &\to \mathrm{C}_*(X) \otimes \mathrm{C}_*(X) \\ (\sigma\colon \Delta^n \to X) &\mapsto \sum_{p=0}^{n} \sigma \circ [\mathfrak{v}_0, \ldots, \mathfrak{v}_p] \otimes \sigma \circ [\mathfrak{v}_p, \ldots, \mathfrak{v}_n] \end{aligned}$$

which is given by

is a chain map.

Proof. Let $\sigma\colon \Delta^n \to X$ be a singular n-simplex. Given $i_0, i_1, \ldots, i_r$ we use the shorthand $[i_0, \ldots, i_r] \coloneqq \sigma \circ [\mathfrak{v}_{i_0}, \mathfrak{v}_{i_1}, \ldots, \mathfrak{v}_{i_r}]\colon \Delta^r \to X$. Furthermore we use the convention that the empty singular simplex $[\,]$ is 0. We calculate that

$$\begin{aligned} \Delta(\partial_n([0,\ldots,n])) &\overset{\text{definition of } \partial_n}{=} \Delta\Big(\sum_{i=0}^{n}(-1)^i \cdot [0,\ldots,\widehat{i},\ldots n]\Big) \\ &= \sum_{0\le p<i\le n}(-1)^i\cdot[0,\ldots,p]\otimes[p,\ldots,\widehat{i},\ldots,n] + \sum_{0\le i<q\le n}(-1)^i\cdot[0,\ldots,\widehat{i},\ldots,q]\otimes[q,\ldots,n] \\ &\uparrow \text{ in the second sum we secretly used the substitution } q = p+1 \end{aligned}$$

and

$$\begin{aligned} \partial_n^{\otimes}(\Delta([0,\ldots,n])) &= \partial_n^{\otimes}\Big(\sum_{p=0}^{n}[0,\ldots,p]\otimes[p,\ldots,n]\Big) \\ &= \sum_{p=0}^{n}\partial_p([0,\ldots,p])\otimes[p,\ldots,n] + \sum_{p=0}^{n}(-1)^p\cdot[0,\ldots,p]\otimes\partial_{n-q}([p,\ldots,n]) \\ &= \sum_{0\le i\le p\le n}(-1)^i\cdot[0,\ldots,\widehat{i},\ldots,p]\otimes[p,\ldots,n] + \sum_{0\le p\le i\le n}(-1)^i\cdot[0,\ldots,p]\otimes[p,\ldots,\widehat{i},\ldots,n]. \end{aligned}$$

We see that the difference between the two terms equals

$$\begin{aligned} &\sum_{i=p=0}^{n}(-1)^i\cdot[0,\ldots,\widehat{i},\ldots,p]\otimes[p,\ldots,n] + \sum_{i=p=0}^{n}(-1)^i\cdot[0,\ldots,p]\otimes[p,\ldots,\widehat{i},\ldots,n] \\ &= \sum_{i=1}^{n}(-1)^i\cdot[0,\ldots,i-1]\otimes[i,\ldots,n] + \sum_{i=0}^{n-1}(-1)^i\cdot[0,\ldots,i]\otimes[i+1,\ldots,n] \\ &\uparrow \text{ here we use that } [\,] = 0 \\ &= \sum_{i=1}^{n}(-1)^i\cdot[0,\ldots,i-1]\otimes[i,\ldots,n] + \sum_{j=1}^{n}(-1)^{j-1}\cdot[0,\ldots,j-1]\otimes[j,\ldots,n] \;=\; 0. \\ &\uparrow \text{ substitution } j = i+1 \end{aligned}$$

We have thus shown that Δ is indeed a chain map. ∎

The following proposition shows that diagonal approximations are essentially unique:

Proposition 72.7. (Diagonal Approximation–Uniqueness Proposition) Given any two diagonal approximations Φ and Ψ there exists a natural chain homotopy between Φ and Ψ.

Proof. [570] Let $\Phi, \Psi\colon \mathrm{C}_*(X) \to \mathrm{C}_*(X) \otimes \mathrm{C}_*(X)$ be two natural chain maps such that for each $x \in X$ we have

$$\Phi(x) \;=\; \Psi(x) \;=\; x \otimes x \quad \in (\mathrm{C}_*(X) \otimes \mathrm{C}_*(X))_0 = \mathrm{C}_0(X) \otimes \mathrm{C}_0(X).$$

For all $k \in \mathbb{Z}_{\geq -1}$ we have to find a homomorphism

$$P_k\colon \mathrm{C}_k(X) \;\to\; (\mathrm{C}_*(X) \otimes \mathrm{C}_*(X))_{k+1}$$

that satisfies the following three properties:

(1) The map P_{-1} is the zero map.
(2) We have $\partial^{\otimes}_{k+1} \circ P_k + P_{k-1} \circ \partial_k = \Phi_k - \Psi_k\colon \mathrm{C}_k(X) \to (\mathrm{C}_*(X) \otimes \mathrm{C}_*(X))_k$.
(3) The map P_k is natural in X.

So suppose we have already defined the maps $P_{-1}, \ldots, P_{k-1}$ with $k \geq 0$. Next we have to define the map P_k. We first consider a special case, namely we first want to define $P_k(\mathrm{id}_{\Delta^k}) \in (\mathrm{C}_*(\Delta^k) \otimes \mathrm{C}_*(\Delta^k)_{k+1}$.

Claim 1. There exists a $c \in (\mathrm{C}_*(\Delta^k) \otimes \mathrm{C}_*(\Delta^k))_{k+1}$ with

$$\partial^{\otimes}_{k+1}(c) \;=\; (\Phi_k - \Psi_k - P_{k-1} \circ \partial_k)(\mathrm{id}_{\Delta^k}).$$

Proof. In the proof of the claim we use the abbreviation $\mathrm{id}_k := \mathrm{id}_{\Delta^k}$. In the chain complex $\mathrm{C}_*(\Delta^k) \otimes \mathrm{C}_*(\Delta^k)$ we calculate that

$$\begin{aligned}
\partial^{\otimes}_k((\Phi_k - \Psi_k - P_{k-1} \circ \partial_k)(\mathrm{id}_k)) \;&=\; \partial^{\otimes}_k(\Phi_k(\mathrm{id}_k)) - \partial^{\otimes}_k(\Psi_k(\mathrm{id}_k)) - (\partial^{\otimes}_k \circ P_{k-1})(\partial_k(\mathrm{id}_k)) \\
&=\; \Phi_{k-1}(\partial_k(\mathrm{id}_k)) - \Psi_{k-1}(\partial_k(\mathrm{id}_k)) - (\Phi_{k-1} - \Psi_{k-1} - (P_{k-2} \circ \partial_{k-1}))(\partial_k(\mathrm{id}_k)) \\
&\quad \uparrow \\
&\quad \text{since } \Phi \text{ and } \Psi \text{ are chain maps and since inductively} \\
&\quad \text{we know by (2) that } \partial^{\otimes}_k \circ P_{k-1} + P_{k-2} \circ \partial_{k-1} = \Phi_{k-1} - \Psi_{k-1} \\
&=\; P_{k-2}(\underbrace{(\partial_{k-1} \circ \partial_k)}_{=0}(\mathrm{id}_k)) \;=\; 0.
\end{aligned}$$

In the following we first consider the case that $k \geq 1$. It follows in this case from the above calculation and the Acyclic Tensor Product Lemma 56.7 that there exists a $(k+1)$-chain $c \in (\mathrm{C}_*(\Delta^k) \otimes \mathrm{C}_*(\Delta^k))_{k+1}$ with $\partial^{\otimes}_{k+1}(c) = (\Phi_k - \Psi_k - P_{k-1} \circ \partial_k)(\mathrm{id}_k)$. Now we consider the case $k = 0$. Using the fact that Φ and Ψ agree on 0-simplices one can easily verify that for the augmentation homomorphism $\epsilon\colon \mathrm{C}_0(X) \otimes \mathrm{C}_0(X) \to \mathbb{Z}$ from the Acyclic Tensor Product Lemma 56.7 we have $\epsilon((\Phi_0 - \Psi_0)(\mathrm{id}_0)) = 0$. It follows again from the Acyclic Tensor Product Lemma 56.7 that there exists a 1-chain $c \in (\mathrm{C}_*(\Delta^0) \otimes \mathrm{C}_*(\Delta^0))_1$ with $\partial^{\otimes}_{k+1}(c) = (\Phi_0 - \Psi_0)(\mathrm{id}_0)$. ⊞

We set $P_k(\mathrm{id}_k) := c$. Next let X be a topological space and let $\sigma\colon \Delta^k \to X$. We set

$$P_k(\sigma) \;:=\; (\sigma_* \otimes \sigma_*)(\underbrace{P_k(\mathrm{id}_{\Delta^k})}_{=c}) \;\in\; (\mathrm{C}_*(X) \otimes \mathrm{C}_*(X))_{k+1}.$$

Claim 2. The map $P_k\colon \mathrm{C}_k(X) \to (\mathrm{C}_*(X) \otimes \mathrm{C}_*(X))_k$ has all the desired properties.

Proof. First note that basically by definition this map P_k is natural in X. Also note that, as in the proof of the EZ-Construction Proposition 56.5, one easily verifies that condition (2) is satisfied. We leave the details to the reader. ■

[570] The proof is a slight variation on the proof of the EZ–Natural Chain Homotopy Proposition 56.8.

Now we are ready to provide the proof of the Cup Product–Definitions Proposition 72.5.

Proof of Proposition 72.5. Let X be a topological space and let R be a commutative ring. We need to show that for any $\varphi \in \mathrm{H}^p(X;R)$ and any $\psi \in \mathrm{H}^q(X;R)$ we have

$$\underbrace{\varphi \mathbin{\widetilde{\cup}} \psi}_{\substack{\text{defined on}\\ \text{page 920}}} \;=\; \underbrace{\varphi \cup \psi}_{\substack{\text{defined on}\\ \text{page 921}}} \;\in\; \mathrm{H}^{p+q}(X;R).$$

In the following prove we use the following standard notation:

(1) We denote by $\Pi_{p,q}\colon (\mathrm{C}_*(X)\otimes \mathrm{C}_*(X))_{p+q} \to \mathrm{C}_p(X)\otimes \mathrm{C}_q(X)$ the projection.
(2) We denote by $\mu_R\colon R\otimes R \to R$ the multiplication map.
(3) We denote by $\Delta\colon \mathrm{C}_{p+q}(X) \to (\mathrm{C}_*(X)\otimes \mathrm{C}_*(X))_{p+q}$ the above Alexander-Whitney diagonalization.
(4) We denote by $d_X\colon X \to X\times X$ the diagonal map.

Now we can formulate the following claim:

Claim. For any two cochains $\varphi \in \mathrm{C}^p(X;R)$ and $\psi \in \mathrm{C}^q(X;R)$ we have

$$\varphi \cup \psi \;=\; \mu_R \circ (\varphi\otimes\psi)\circ \Pi_{p,q}\circ \Delta \;\in\; \mathrm{C}^{p+q}(X;R) = \mathrm{Hom}(\mathrm{C}_{p+q}(X),R).$$

Proof. We have to show that the two sides are the same maps $\mathrm{C}_{p+q}(X)\to R$. Therefore let $\sigma\colon \Delta^{p+q}\to X$ be a singular $(p+q)$-simplex. We calculate that

$$\begin{aligned}
(\mu_R\circ(\varphi\otimes\psi)\circ\Pi_{p,q}\circ\Delta)(\sigma) &\overset{\text{definition of the Alexander-Whitney diagonalization}}{\downarrow} \\
&= (\mu_R\circ(\varphi\otimes\psi))\Big(\Pi_{p,q}\Big(\sum_{i+j=p+q} \overbrace{\sigma\circ[\mathfrak{v}_0,\dots,\mathfrak{v}_i]}^{\in \mathrm{C}_i(X)} \otimes \overbrace{\sigma\circ[\mathfrak{v}_i,\dots,\mathfrak{v}_{p+q}]}^{\in \mathrm{C}_j(X)}\Big)\Big) \\
&= (\mu_R\circ(\varphi\otimes\psi))(\sigma\circ[\mathfrak{v}_0,\dots,\mathfrak{v}_p]\otimes\sigma\circ[\mathfrak{v}_p,\dots,\mathfrak{v}_{p+q}]) \\
&= \varphi(\sigma\circ[\mathfrak{v}_0,\dots,\mathfrak{v}_p])\cdot\psi(\sigma\circ[\mathfrak{v}_p,\dots,\mathfrak{v}_{p+q}]) \;=\; (\varphi\cup\psi)(\sigma).
\end{aligned}$$

(definition of $\mu_R\circ(\varphi\otimes\psi)$ ↑; definition of $\cup$ ↑) ⊞

Now suppose that we are given cocycles $\varphi \in \mathrm{C}^p(X;R)$ and $\psi\in\mathrm{C}^q(X;R)$. We obtain the following equality in $\mathrm{H}^{p+q}(X;R)$:

$$\begin{aligned}
[\varphi \mathbin{\widetilde{\cup}} \psi] &\overset{\text{by the definition on page 920}}{\downarrow} \\
&= [\mu_R\circ(\varphi\otimes\psi)\circ\Pi_{p,q}\circ(\Theta\circ d_{X*})] \;=\; (\Theta\circ d_{X*})^*\big(\overbrace{[\mu_R\circ(\varphi\otimes\psi)\circ\Pi_{p,q}]}^{\in \mathrm{H}^{p+q}(\mathrm{C}_*(X)\otimes\mathrm{C}_*(X);R)}\big) \\
&= \Delta^*([\mu_R\circ(\varphi\otimes\psi)\circ\Pi_{p,q}]) \;=\; [\mu_R\circ(\varphi\otimes\psi)\circ\Pi_{p,q}\circ\Delta] \;=\; [\varphi\cup\psi].
\end{aligned}$$

(by the Diagonal Approximation–Uniqueness Proposition 72.7 the maps $\Theta\circ d_{X*}, \Delta\colon \mathrm{C}_*(X)\to \mathrm{C}_*(X)\otimes\mathrm{C}_*(X)$ are chain homotopic, so by the Dual Cochain Map Lemma 66.9 (3a) together with the Cochain Homotopy-H*-Lemma 66.6 they induce the same map on cohomology; by the claim)

We have thus obtained the desired equality of cohomology classes. ■

72.4. **Graded commutativity of the cup product.** The following proposition shows that the cup product is (up to a sign) commutative:

Proposition 72.8. (Cup Product-Commutativity Proposition) Let X be a topological space and let R be a commutative ring. For any $a\in \mathrm{H}^k(X;R)$ and $b\in \mathrm{H}^l(X;R)$ we have

$$b\cup a \;=\; (-1)^{kl}\cdot a\cup b.$$

Remark. If $R=\mathbb{F}_2$, then we are in the pretty situation that $-1=1$, i.e. the cup product is commutative. □

Proof. We start out with the following claim:

Claim 1. Let A_* and B_* be chain complexes. The unique map

$$\tau\colon A_* \otimes B_* \quad \to \quad B_* \otimes A_*$$

that for $a \in A_k$ and $b \in B_l$ is given by

$$a \otimes b \quad \mapsto \quad (-1)^{kl} \cdot b \otimes a$$

is a chain map.

Proof. Given $a \in A_k$ and $b \in B_l$ we see that

$$\begin{aligned}
\tau(\partial^{\otimes}_{k+l}(a \otimes b)) &= \tau(\partial_k(a) \otimes b + (-1)^k \cdot a \otimes \partial_l(b)) \\
&= \tau(\partial_k(a) \otimes b) + (-1)^k \cdot \tau(a \otimes \partial_l(b)) \\
&= (-1)^{(k-1)l} \cdot b \otimes \partial_k(a) + (-1)^{k+k(l-1)} \cdot \partial_l(b) \otimes a \\
&= (-1)^{kl} \cdot \partial_l(b) \otimes a + (-1)^{l+kl} \cdot b \otimes \partial_k(a) \\
&= \partial^{\otimes}((-1)^{kl} \cdot b \otimes a) \quad = \quad \partial^{\otimes}(\tau(a \otimes b)).
\end{aligned}$$

⊞

Now we turn to topology. Let X be a topological space and let R be a commutative ring. It now remains to prove the following claim:

Claim 2. The following diagram commutes:

$$\begin{array}{ccccc}
\mathrm{H}^p(X;R) \times \mathrm{H}^q(X;R) & \xrightarrow{([a],[b]) \mapsto [\mu_R \circ (a \otimes b) \circ \Pi_{p,q}]} & \mathrm{H}^{p+q}(\mathrm{C}_*(X) \otimes \mathrm{C}_*(X);R) & \xrightarrow{d_X^* \circ \Theta^*} & \mathrm{H}^{p+q}(X;R) \\
\downarrow{\scriptstyle (a,b) \mapsto (-1)^{kl} \cdot (b,a)} & & \downarrow{\scriptstyle [a \otimes b] \mapsto [(-1)^{kl} \cdot b \otimes a]\ \tau} & & \downarrow{\scriptstyle \mathrm{id}} \\
\mathrm{H}^q(X;R) \times \mathrm{H}^p(X;R) & \xrightarrow{([c],[d]) \mapsto [\mu_R \circ (c \otimes d) \circ \Pi_{p,q}]} & \mathrm{H}^{p+q}(\mathrm{C}_*(X) \otimes \mathrm{C}_*(X);R) & \xrightarrow{d_X^* \circ \Theta^*} & \mathrm{H}^{p+q}(X;R).
\end{array}$$

$$(a,b) \mapsto a \cup b \qquad (c,d) \mapsto c \cup d$$

Proof. We make the following clarifications and observations:

(1) The maps $\mu_R\colon R \otimes R \to R$, $\Pi_{p,q}$, $d_X\colon X \to X \times X$ and $\Theta\colon \mathrm{C}_*(X \times X) \to \mathrm{C}_*(X) \otimes \mathrm{C}_*(X)$ are the maps appearing in the first definition of the cup product. In fact the upper and lower horizontal sequence give us the first definition of the cup product.

(2) It follows from Claim 1 together with the Dual Cochain Map Lemma 66.9 (1) that the second vertical map is defined.

(3) It is clear that the square to the left commutes.

(4) Note that the maps $\mathrm{id} \circ \Theta \circ d_X, \tau \circ \Theta \circ d_X\colon \mathrm{C}_*(X) \otimes \mathrm{C}_*(X) \to \mathrm{C}_*(X) \otimes \mathrm{C}_*(X)$ are both diagonal approximations. It follows from the Diagonal Approximation–Uniqueness Proposition 72.7 that these two chain maps are actually chain homotopic. It follows from this observation, the Dual Cochain Map Lemma 66.9 (3a) and the Cochain Homotopy-H*-Lemma 66.6 (2) that both maps induce the same map on cohomology. This implies that the square to the right commutes. ■

72.5. **The ring structure on cohomology.** In this section we will see that the cup product defines a ring structure on cohomology. To formulate this result it is helpful to introduce the following algebraic concepts:

Definition.

(1) We define a ring to be an abelian group together with a multiplicative structure which is associative, distributive and that admits a multiplicatively neutral element. We do not demand that a ring is commutative.

(2) A graded ring is a ring R together with a direct sum decomposition $R = \bigoplus_{n\in\mathbb{N}_0} R_n$ into subgroups such that for any $m, n \in \mathbb{N}_0$ the multiplication map restricts to a map

$$R_m \times R_n \to R_{m+n}.$$

(3) Let $R = \bigoplus_{n\in\mathbb{N}_0} R_n$ and $S = \bigoplus_{n\in\mathbb{N}_0} S_n$ be two graded rings. A morphism between the two graded rings is a ring homomorphism $\varphi\colon R \to S$ such that for any $n \in \mathbb{N}_0$ we have $\varphi(R_n) \subset S_n$.

(4) We define the category $\mathcal{GrRing}$ of graded rings to be the category where the objects are graded rings and the morphisms are morphisms of graded rings.

We mostly care about a special type of graded ring:

Definition. A superalgebra is a graded ring $R = \bigoplus_{n\in\mathbb{N}_0} R_n$ such that for any $a \in R_k$ and $b \in R_l$ we have[571]

$$b \cdot a = (-1)^{k\cdot l} \cdot a \cdot b.$$

We now return to topology:

Notation. Given a topological space X and an abelian group G we write

$$\mathrm{H}^*(X; G) := \bigoplus_{k\in\mathbb{N}_0} \mathrm{H}^k(X; G).$$

Now we can formulate the promised result:

Proposition 72.9. (Cup Product-Ring Proposition) Let X be a topological space and let R be a commutative ring.

(1) The cup product on $\mathrm{H}^*(X; R)$ is R-bilinear.

(2) The abelian group $\mathrm{H}^*(X; R) = \bigoplus_{n\in\mathbb{N}_0} \mathrm{H}^n(X; R)$ together with the map

$$\cup\colon \mathrm{H}^*(X; R) \times \mathrm{H}^*(X; R) \to \mathrm{H}^*(X; R)$$

is a superalgebra where the multiplicatively neutral element is given by

$$1_X := \text{constant map } X \to R \text{ given by } x \mapsto 1 \in \mathrm{H}^0(X; R).$$

(3) The maps

$$\begin{aligned} X &\mapsto (\mathrm{H}^*(X; R), \cup) \\ (f\colon X \to Y) &\mapsto (f^*\colon \mathrm{H}^*(Y; R) \to \mathrm{H}^*(X; R)) \end{aligned}$$

define a contravariant functor from the category $\mathcal{Top}$ of topological spaces to the category $\mathcal{GrRing}$ of graded rings.

Definition. Given a topological space X and given a commutative ring R we refer to $(\mathrm{H}^*(X; R), \cup)$ as the R-cohomology ring of X.

Remark. Let R be a commutative ring and let $f\colon X \to Y$ be a continuous map between topological spaces. If f is in fact homotopy equivalence, then it follows from the Homotopy

[571] The word "algebra" in our context is justified by the fact that any ring is a $\mathbb{Z}$-algebra. We could also have referred to such an object as a superring, but this terminology seems to be uncommon.

Equivalence-H^*-Corollary 67.8 (1) that the induced maps $f^*\colon \mathrm{H}^n(Y;R) \to \mathrm{H}^n(X;R)$ on cohomology *groups* are isomorphisms. It follows from the Cup Product-Ring Proposition 72.9 (2) that these isomorphisms preserve the ring structure, i.e. the map $f^*\colon \mathrm{H}^*(Y;R) \to \mathrm{H}^*(X;R)$ is in fact an isomorphism of graded rings.

Proof. Let X be a topological space and let R be a commutative ring.

(1) It is clear from both definitions of the cup product that it is R-bilinear.

(2) By (1) and by the Cup Product-Commutativity Proposition 72.8 it remains to show the following claim:

Claim.
(a) The cup product is associative.
(b) The 0-cochain 1_X is the multiplicatively neutral element.

Proof. Both statements follow easily from the second definition of the cup product:

(a) Let $\alpha \in \mathrm{C}^p(X;R)$, $\beta \in \mathrm{C}^q(X;R)$ and $\gamma \in \mathrm{C}^r(X;R)$ be cochains. Given any singular simplex $\sigma\colon \Delta^{p+q+r} \to X$ we have

$$\begin{aligned}((\alpha \cup \beta) \cup \gamma)(\sigma) &= (\alpha \cup \beta)(\sigma \circ [\mathfrak{v}_0, \dots, \mathfrak{v}_{p+q}]) \cdot \gamma(\sigma \circ [\mathfrak{v}_{p+q}, \dots, \mathfrak{v}_{p+q+r}]) \\ &= \alpha(\sigma \circ [\mathfrak{v}_0, \dots, \mathfrak{v}_p]) \cdot \beta(\sigma \circ [\mathfrak{v}_p, \dots, \mathfrak{v}_{p+q}]) \cdot \gamma(\sigma \circ [\mathfrak{v}_{p+q}, \dots, \mathfrak{v}_{p+q+r}]) \\ &= (\alpha \cup (\beta \cup \gamma))(\sigma). \\ &\quad\uparrow \\ &\quad\text{same argument backwards}\end{aligned}$$

Thus we see that the cup product is already associative for cochains, so it is in particular associative for cohomology classes.

(b) Let $\eta\colon X \to R$ be the map that assigns to each point in X the value $1 \in R$. As usual we view η as a singular 0-cochain. It follows from the H^0-Proposition 67.5 that η is a 0-cocycle and that we obtain an element $1_X \coloneqq \eta \in \mathrm{H}^0(X;R)$. It is clear from the second definition of the cup product that 1_X is the neutral element with respect to the cup product. ⊞

(3) By the H^n–Functor Lemma 67.1 it remains to prove the following claim:

Claim. Given any continuous map $f\colon X \to Y$ between topological spaces the induced map $f^*\colon \mathrm{H}^*(Y;R) \to \mathrm{H}^*(X;R)$ is a ring homomorphism.

Proof. The claim follows from the following observations:

(a) In the Cup Product-Naturality Lemma 72.2 we showed that for any $c \in \mathrm{H}^p(Y;R)$ and any $d \in \mathrm{H}^q(Y;R)$ we have $f^*(c) \cup f^*(d) = f^*(c \cup d)$.

Purely for fun we now also verify this equality using the second definition of the cup product. Thus let $c = [\varphi] \in \mathrm{H}^p(Y;R)$ and let $d = [\psi] \in \mathrm{H}^q(Y;R)$. Furthermore let $\sigma\colon \Delta^{p+q} \to X$ be a singular $(p+q)$-chain. Then we see that

$$\begin{aligned}(f^*\varphi \cup f^*\psi)(\sigma) &= (f^*\varphi)(\sigma \circ [\mathfrak{v}_0, \dots, \mathfrak{v}_p]) \cdot (f^*\psi)(\sigma \circ [\mathfrak{v}_p, \dots, \mathfrak{v}_{p+q}]) \\ &= \varphi(f \circ (\sigma \circ [\mathfrak{v}_0, \dots, \mathfrak{v}_p])) \cdot \psi(f \circ (\sigma \circ [\mathfrak{v}_p, \dots, \mathfrak{v}_{p+q}])) \\ &= \varphi((f \circ \sigma) \circ [\mathfrak{v}_0, \dots, \mathfrak{v}_p)) \cdot \psi((f \circ \sigma) \circ [\mathfrak{v}_p, \dots, \mathfrak{v}_{p+q}]) \\ &= (\varphi \cup \psi)(f \circ \sigma) \;=\; (f^*(\varphi \cup \psi))(\sigma).\end{aligned}$$

(b) It follows immediately from the definitions of 1_X, 1_Y and f^* that we have the equality $f^*(1_Y) = 1_X$. ■

72.6. **Calculations of the cup product.** In this section we calculate the cup product for the following topological spaces:

(1)	The torus	(with $\mathbb{Z}$-coefficients).
(2)	Any finite wedge of spheres	(with any coefficients).
(3)	The surface of genus 2	(with $\mathbb{Z}$-coefficients).
(4)	The real projective plane $\mathbb{R}\mathrm{P}^2$	(with $\mathbb{F}_2$-coefficients).

In the following calculations we will make use of the following elementary observations regarding the cup product:

Observation 72.10. Let R be a commutative ring.

(1) Let X be a path-connected non-empty topological space. It follows from the H^0-Proposition 67.5 that $\mathrm{H}^0(X;R) = R \cdot 1_X$. For any $r \cdot 1_X \in \mathrm{H}^0(X;R)$ and $\varphi \in \mathrm{H}^k(X;R)$ we have
$$(r \cdot 1_X) \cup \varphi \underset{\text{since } \cup \text{ is } R\text{-bilinear}}{=} r \cdot (1_X \cup \varphi) \underset{\substack{\text{Cup Product-Ring} \\ \text{Proposition 72.9}}}{=} r \cdot \varphi \underset{\text{same argument backwards}}{=} \varphi \cup (r \cdot 1_X).$$

(2) By the Cup Product-Ring Proposition 72.9 the cup product is R-bilinear, so it suffices to determine the cup product on a generating set.

(3) Let M be an n-dimensional topological manifold.

(a) For any p, q with $p+q > n$ the cup product on $\mathrm{H}^p(M;R) \times \mathrm{H}^q(M;R)$ is zero, since it takes values in the group $\mathrm{H}^{p+q}(M;R)$, which is zero by the Manifold-Top-H*-Theorem 70.16 (1).

(b) We assume that M is closed connected and non-empty.

- If M is oriented, then we know by the Top Cohomology Theorem 70.16 (2b) that the map
$$\begin{array}{rcl} \mathrm{H}^n(M;\mathbb{Z}) & \to & \mathbb{Z} \\ [\varphi] & \mapsto & \langle \varphi, [M] \rangle \end{array}$$
is an isomorphism.
- We know by the $\mathbb{F}_2$-Orientability Proposition 61.13, the Manifold-Top-H$_*$-Theorem 63.1 and the Universal Coefficient-H*–Field Theorem 71.2 that the map
$$\begin{array}{rcl} \mathrm{H}^n(M;\mathbb{F}_2) & \to & \mathbb{F}_2 \\ [\varphi] & \mapsto & \langle \varphi, [M]_{\mathbb{F}_2} \rangle \end{array}$$
is an isomorphism.

72.6.1. **The cup product of the torus.** In this section we want to determine the cup product on the torus $T = S^1 \times S^1$. In fact the following lemma, together with Observation 72.10, gives us the complete calculation of cup products on the torus:

Lemma 72.11. (Torus–Cup Product Lemma) Let $T = S^1 \times S^1$ be the 2-dimensional torus.

(1) We recall some notations from before:

- We denote by p respectively q the projection $S^1 \times S^1 \to S^1$ onto the first respectively second factor.
- On page 866 we introduced a cochain $\theta_{\mathbb{Z}} \in \mathrm{C}^1(S^1;\mathbb{Z})$. In the $\theta_{\mathbb{Z}}$-Cocycle Lemma 67.2 we showed that $\theta_{\mathbb{Z}}$ is a cocycle.

The cohomology classes $\alpha := p^*([\theta_{\mathbb{Z}}])$ and $\beta := q^*([\theta_{\mathbb{Z}}])$ form a basis for $\mathrm{H}^1(S^1 \times S^1; \mathbb{Z})$.

(2) We consider the pairing

$$\begin{array}{rcl} \mathrm{H}^1(T;\mathbb{Z}) \times \mathrm{H}^1(T;\mathbb{Z}) & \to & \mathbb{Z} \\ (\varphi, \psi) & \mapsto & \langle \varphi \cup \psi, [T] \rangle. \end{array}$$

With respect to the basis α, β of $\mathrm{H}^1(T;\mathbb{Z})$ this pairing is given by the matrix

$$\begin{pmatrix} \langle \alpha \cup \alpha, [T] \rangle & \langle \alpha \cup \beta, [T] \rangle \\ \langle \beta \cup \alpha, [T] \rangle & \langle \beta \cup \beta, [T] \rangle \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Proof.

(1) We proved this statement in the Torus-H^1-Lemma 68.5.

(2) We determine these four entries separately:

(a) We first consider the entry $\langle \alpha \cup \beta, [T] \rangle$. Fortunately we already did all the work on page 923: More precisely, we have

$$\begin{array}{rcl} \langle \alpha \cup \beta, [T] \rangle & = & (p^*(\theta_{\mathbb{Z}}) \cup q^*(\theta_{\mathbb{Z}}))(-\sigma_1 + \sigma_2) \\ & = & (p^*(\theta_{\mathbb{Z}}) \cup q^*(\theta_{\mathbb{Z}}))(-\sigma_1) + (p^*(\theta_{\mathbb{Z}}) \cup q^*(\theta_{\mathbb{Z}}))(\sigma_2) \underset{\uparrow}{=} 0 + 1 = 1. \end{array}$$

see the calculations on page 923

(b) We could do the same argument as in (a) with α and β reversed. But it is much easier to apply the Cup Product-Commutativity Proposition 72.8 which gives us that $\beta \cup \alpha = (-1)^{1 \cdot 1} \cdot \alpha \cup \beta = -\alpha \cup \beta$.

(c) There are many different ways for showing that $\alpha \cup \alpha = 0$:

- We could do the calculation as in (a), but that is the least efficient approach.
- We could use the symmetry from the Cup Product-Commutativity Proposition 72.8 to argue that $\alpha \cup \alpha = -\alpha \cup \alpha$, and hence $\alpha \cup \alpha = 0$ since $\mathrm{H}^2(T;\mathbb{Z})$ is torsion-free.
- We can also argue that

$$\alpha \cup \alpha = [p^*(\theta_{\mathbb{Z}})] \cup [p^*(\theta_{\mathbb{Z}})] \underset{\uparrow}{=} p^*([\theta_{\mathbb{Z}}] \cup [\theta_{\mathbb{Z}}]) \underset{\uparrow}{=} 0.$$

by the Cup Product-Naturality Lemma 72.2 — since $[\theta_{\mathbb{Z}}] \cup [\theta_{\mathbb{Z}}] \in \mathrm{H}^2(S^1;\mathbb{Z}) = 0$

(d) The same way as in (c) we see that $\beta \cup \beta = 0$. ∎

72.6.2. **The cup product of a wedge of spheres.** Next we calculate the cup product of the wedge of finitely many spheres:

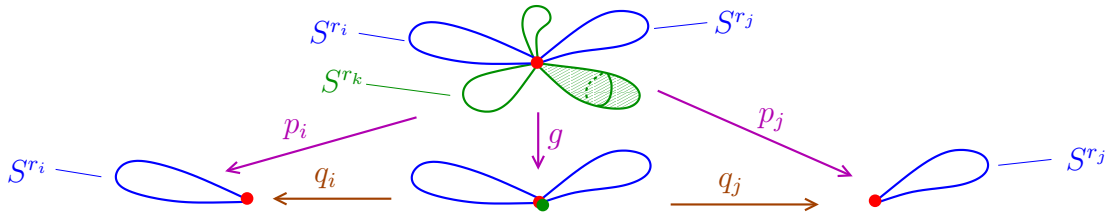
Lemma 72.12. (Wedge-of-Spheres–Cup Product Lemma) For any wedge of finitely many spheres all cup products (with respect to any commutative ring) of cohomology classes in degrees ≥ 1 are zero.

Proof. Let $S^{r_1} \vee \cdots \vee S^{r_k}$ be a wedge of finitely many spheres and let R be a commutative ring. For each $i \in \{1, \ldots, k\}$ we denote by $p_i \colon S^{r_1} \vee \cdots \vee S^{r_k} \to S^{r_i}$ the natural projection that we introduced on page 245. By the Wedge-H^*-Proposition 69.4 we know that for every $n \in \mathbb{N}$ the map

$$p_1^* \oplus \cdots \oplus p_k^* \colon \mathrm{H}^n(S^{r_1};R) \oplus \cdots \oplus \mathrm{H}^n(S^{r_k};R) \xrightarrow{\cong} \mathrm{H}^n(S^{r_1} \vee \cdots \vee S^{r_k};R)$$

is an isomorphism. It follows from the calculations of the cohomology groups of spheres on page 880 and from Observation 72.10 that it remains to show that for any $i, j \in \{1 \ldots, k\}$

with $r_i > 0$ and $r_j > 0$ we have $p_i^*([S^{r_i}]^*) \cup p_j^*([S^{r_j}]^*) = 0$. We prove the statement for $i \neq j$. The proof of the statement for $i = j$ is basically the same. So suppose that $i \neq j$. We denote by $g\colon S^{r_1} \vee \cdots \vee S^{r_k} \to S^{r_i} \vee S^{r_j}$ the natural projection and we denote by $q_i\colon S^{r_i} \vee S^{r_j} \to S^{r_i}$ and $q_j\colon S^{r_j} \vee S^{r_j} \to S^{r_j}$ the natural projections.



Then

$$p_i^*([S^{r_i}]^*) \cup p_j^*([S^{r_j}]^*) \overset{\text{since } q_i \circ g = p_i \text{ and } q_j \circ g = p_j}{\underset{\downarrow}{=}} g^*(q_i^*([S^{r_i}]^*)) \cup g^*(q_j^*([S^{r_j}]^*)) \overset{\text{by the Cup Product-Naturality Lemma 72.2}}{\underset{\downarrow}{=}} g^*(\underbrace{q_i^*([S^{r_i}]^*) \cup q_j^*([S^{r_j}]^*)}_{\substack{\in \mathrm{H}^{r_i+r_j}(S^{r_i} \vee S^{r_j}; R) \\ \text{which is } = 0 \text{ since } r_i, r_j > 0}}) = 0.$$

We have thus proved that all cup products in degrees ≥ 1 are zero. ■

Remark. We have now performed enough calculations to see that the cohomology rings contain more information than the homology or the cohomology groups. More precisely, it follows from the Torus-H_*-Proposition 45.5 (1), the Sphere-H_*-Proposition 41.2 and the Wedge-H_*-Proposition 41.10 that for the 2-dimensional torus $S^1 \times S^1$ and for $S^1 \vee S^1 \vee S^2$ we have

$$\mathrm{H}_i(S^1 \times S^1) \cong \mathrm{H}_i(S^1 \vee S^1 \vee S^2) \cong \begin{cases} \mathbb{Z}, & \text{if } i = 0, 2, \\ \mathbb{Z}^2, & \text{if } i = 1, \\ 0, & \text{if } i \geq 3. \end{cases}$$

We deduce from the Universal Coefficient-H^*-Theorem 70.14 that $S^1 \times S^1$ and $S^1 \vee S^1 \vee S^2$ also have isomorphic cohomology groups. In the Torus–Cup Product Lemma 72.11 we saw that the cup product

$$\mathrm{H}^1(S^1 \times S^1; \mathbb{Z}) \times \mathrm{H}^1(S^1 \times S^1; \mathbb{Z}) \to \mathrm{H}^2(S^1 \times S^1; \mathbb{Z})$$

is non-zero, whereas we showed in the Wedge-of-Spheres–Cup Product Lemma 72.12 that the cup product

$$\mathrm{H}^1(S^1 \vee S^1 \vee S^2; \mathbb{Z}) \times \mathrm{H}^1(S^1 \vee S^1 \vee S^2; \mathbb{Z}) \to \mathrm{H}^2(S^1 \vee S^1 \vee S^2; \mathbb{Z})$$

is zero.[572] □

72.6.3. **The cup product of the surface of genus 2.** In this section we want to determine the cup product on the surface of genus 2. In fact the following lemma, together with Observation 72.10, gives us the complete calculation of cup products on the surface of genus 2:

[572] These calculations, together with the Homotopy Equivalence-H^*-Corollary 67.8 (2) and the Cup Product-Ring Proposition 72.9 (2), give us a new proof that the torus $S^1 \times S^1$ and the wedge $S^1 \vee S^1 \vee S^2$ are not homotopy equivalent. This fact can of course already be seen by considering fundamental groups. Nonetheless it is encouraging to see that we found a new way to prove this statement.

Lemma 72.13. (Surface–Cup Product Lemma) Let Σ be the surface of genus 2.

(1) We recall the following notation:
- On page 885 we introduced for $i = 1, 2$ a continuous map $p_i \colon \Sigma \to T = S^1 \times S^1$. We refer to the figure below for a reminder of the definition of the maps.
- As on page 932 let $\alpha, \beta \in \mathrm{H}^1(T; \mathbb{Z})$ be the basis from the Torus-H^1-Lemma 68.5.

A basis for $\mathrm{H}^1(\Sigma; \mathbb{Z}) \cong \mathbb{Z}^4$ is given by $\alpha_i := p_i^*(\alpha)$, $\beta_i := p_i^*(\beta)$, $i = 1, 2$.

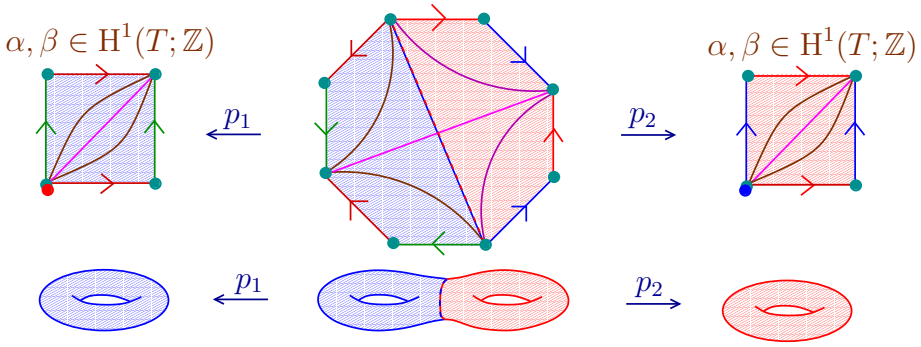
(2) We consider the pairing

$$\begin{array}{rcl} \mathrm{H}^1(\Sigma; \mathbb{Z}) \times \mathrm{H}^1(\Sigma; \mathbb{Z}) & \to & \mathbb{Z} \\ (\varphi, \psi) & \mapsto & \langle \varphi \cup \psi, [\Sigma] \rangle. \end{array}$$

With respect to the above basis $\alpha_1, \beta_1, \alpha_2, \beta_2$ of $\mathrm{H}^1(\Sigma; \mathbb{Z})$ this pairing is given by the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

An analogous statement holds for any surface of genus $g \geq 1$, i.e. the cup product on the first cohomology of the surface of genus $g \geq 1$ can be represented by a block diagonal matrix consisting of g copies of $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.



Proof.

(1) In the Surface-H^1-Lemma 68.6 we saw that the map

$$p_1^* \oplus p_2^* \colon \mathrm{H}^1(T; \mathbb{Z}) \oplus \mathrm{H}^1(T; \mathbb{Z}) \to \mathrm{H}^1(\Sigma; \mathbb{Z})$$

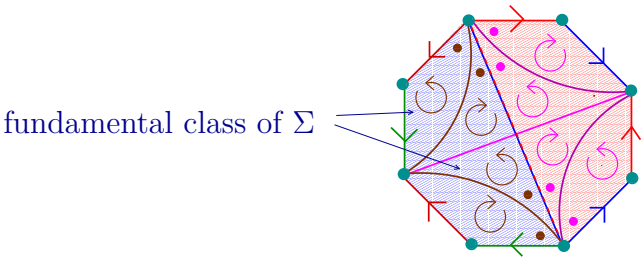
is an isomorphism. It follows from this fact together with the Torus-H^1-Lemma 68.5 that a basis for $\mathrm{H}^1(\Sigma; \mathbb{Z}) \cong \mathbb{Z}^4$ is indeed given by $\alpha_i := p_i^*(\alpha)$, $\beta_i := p_i^*(\beta)$, $i = 1, 2$.

(2) We need to calculate the 16 entries of the matrix. In fact there are several shortcuts we can make use of:
- (a) By the Cup Product-Commutativity Proposition 72.8 we know that the matrix is skew symmetric. In particular all diagonal entries are zero.
- (b) For $i = 1, 2$ we see that

$$\begin{array}{lll} \langle \alpha_i \cup \beta_i, [\Sigma]\rangle_\Sigma & = \langle p_i^*(\alpha) \cup p_i^*(\beta), [\Sigma]\rangle_\Sigma & \overset{\text{by the Cup Product-Naturality Lemma 72.2}}{=} \langle p_i^*(\alpha \cup \beta), [\Sigma]\rangle_\Sigma \\ & \overset{\text{by the Kronecker Pairing Lemma 68.3 (2b)}}{=} \langle \alpha \cup \beta, p_{i*}([\Sigma])\rangle_T & \overset{\text{definition of } \deg(p_i)}{=} \langle \alpha \cup \beta, \deg(p_i) \cdot [T]\rangle_T. \\ & & \overset{\text{it follows easily from the Local-Global Degree Proposition 65.3 that } \deg(p_i) = 1}{=} \langle \alpha \cup \beta, [T]\rangle_T. \end{array}$$

This shows that the top-left 2×2-submatrix and the bottom-right 2×2-submatrix are exactly given by the matrix from the Torus–Cup Product Lemma 72.11.

(c) It remain to show that the top-right 2×2-submatrix is zero. This we have to do "by hand". In the figure below we sketch the representative of a fundamental class $[\Sigma] \in \mathrm{H}_2(\Sigma)$.



A rather dull calculation now lets us calculate all the 4 entries of the matrix in the upper right corner. In fact it is fairly easy to see that all entries are zero. ■

72.6.4. **The cup product of the real projective plane.** In this subsection we want to understand the cup product of the real projective plane $\mathbb{R}\mathrm{P}^2$ in cohomology with $\mathbb{F}_2$-coefficients. Recall that on page 881 we showed that $\mathrm{H}^i(\mathbb{R}\mathrm{P}^2; \mathbb{F}_2) \cong \mathbb{F}_2$ for $i = 0, 1, 2$ and $= 0$ otherwise.

Similar to the case of the torus and of the surfaces of genus ≥ 2 we need an explicit 1-cocycle that represents the unique non-zero class in $\mathrm{H}^1(\mathbb{R}\mathrm{P}^2; \mathbb{F}_2)$. We will find such a representative in an analogous fashion to the construction of the popular singular 1-cocycle $\theta_{\mathbb{Z}} \in \mathrm{C}^1(S^1; \mathbb{Z})$ on page 867:

Lemma 72.14. (Generator-of-$\mathrm{H}^1(\mathbb{R}\mathrm{P}^2; \mathbb{F}_2)$-Lemma) Let $f\colon S^2 \to \mathbb{F}_2$ be any map such that for all $P \in S^2$ we have $f(-P) = 1 - f(P)$. For example we could consider the map

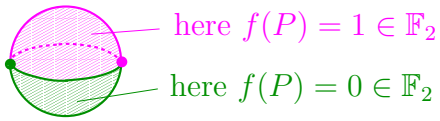
$$\begin{array}{rcl} f\colon S^2 & \to & \mathbb{F}_2 \\ (x,y,z) & \mapsto & \begin{cases} 1, & \text{if } (z > 0) \text{ or } (z = 0 \text{ and } y > 0) \text{ or } (z = y = 0 \text{ and } x = 1), \\ 0, & \text{otherwise.} \end{cases} \end{array}$$

We denote by $p\colon S^2 \to \mathbb{R}\mathrm{P}^2 = S^2/x \sim -x$ the projection. For a singular 1-simplex $\sigma\colon \Delta^1 \to \mathbb{R}\mathrm{P}^2$ we pick a lift $\widetilde{\sigma}\colon \Delta^1 \to S^2$ of σ and we set

$$\varphi(\sigma) \coloneqq \underbrace{f(\widetilde{\sigma}(0,1)) - f(\widetilde{\sigma}(1,0))}_{= f(\partial_1(\widetilde{\sigma}))} \in \mathbb{F}_2.$$

The map $\varphi\colon \mathrm{C}_1(\mathbb{R}\mathrm{P}^2) \to \mathbb{F}_2$ is a cocycle which represents the unique non-trivial element of $\mathrm{H}^1(\mathbb{R}\mathrm{P}^2; \mathbb{F}_2)$.

Proof. We start out with the following claim:



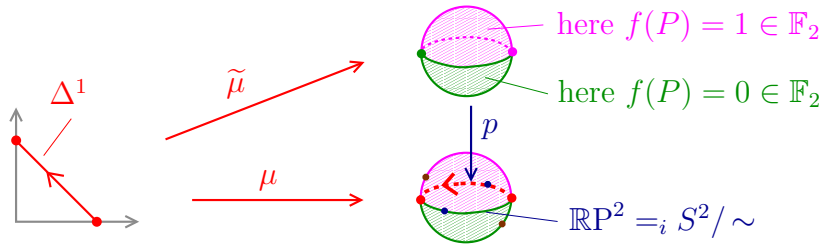
Claim. The map φ is well-defined.

Proof. Let $\sigma\colon \Delta^1 \to \mathbb{R}\mathrm{P}^2$ be a singular 1-simplex. Note that it follows from the Lift Uniqueness Lemma 14.7 that there are precisely two lifts, and these two lifts differ by reflection in the origin. It follows from $f(-P) = 1 - f(P) \in \mathbb{F}_2$ that the difference of the evaluations at the endpoints is unaffected by the reflection in the origin. (Here we use that we work with $\mathbb{F}_2$-coefficients instead of $\mathbb{Z}$-coefficients.) ⊞

Next note that the argument of the θ-Cocycle Lemma 67.2 can easily be adapted to show that the singular 1-cochain $\varphi\colon \mathrm{C}_1(\mathbb{R}\mathrm{P}^2) \to \mathbb{F}_2$ is a cocycle. It remains to show that $\varphi \neq 0 \in \mathrm{H}^1(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2)$. To do so we consider the Kronecker pairing

$$\langle\ ,\ \rangle\colon \mathrm{H}^1(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2) \times \mathrm{H}_1(\mathbb{R}\mathrm{P}^2) \ \to\ \mathbb{F}_2$$

from the Kronecker Pairing Lemma 68.3. We need a mate for φ: We consider the singular 1-simplex $\mu\colon \Delta^1 \to \mathbb{R}\mathrm{P}^2 =_{\mathrm{i}} S^2/\sim$ that is given by $\mu(1-t,t) = [(\cos(t),\sin(t),0)]$.



Note that μ is a cycle in $\mathrm{C}_1(\mathbb{R}\mathrm{P}^2)$. An explicit lift $\widetilde{\mu}\colon \Delta^1 \to S^2$ of μ is of course given by $(1-t,t) \mapsto (\cos(t),\sin(t),0)$. We now consider the above Kronecker pairing and we calculate that

$$\langle [\varphi],[\mu]\rangle \underset{\substack{\uparrow\\ \text{definition of } \langle\ ,\ \rangle}}{=} \varphi(\mu) \underset{\substack{\uparrow\\ \text{definition of } \varphi}}{=} f(\widetilde{\mu}(0,1)) - f(\widetilde{\mu}(1,0)) \underset{\substack{\uparrow\\ \text{since } \widetilde{\mu}(0,1) \text{ and } \widetilde{\mu}(1,0) \text{ are antipodal}}}{=} 1 \ \in\ \mathbb{F}_2.$$

This calculation implies that $\varphi \neq 0 \in \mathrm{H}^1(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2)$. ■

Observation 72.10 together with the following lemma gives us a complete understanding of the cup product on $\mathbb{R}\mathrm{P}^2$ with $\mathbb{F}_2$-coefficients:

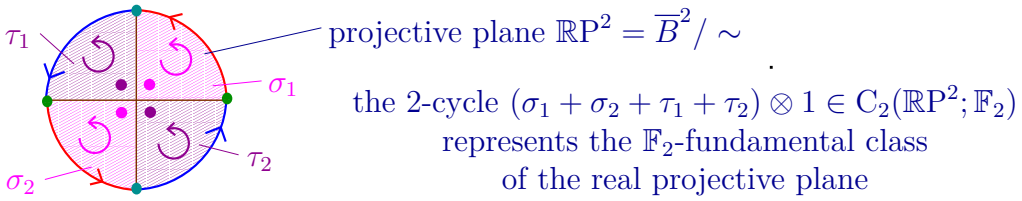
Lemma 72.15. (Projective Plane–Cup Product Lemma) Let $[\varphi] \in \mathrm{H}^1(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2)$ be the unique non-trivial element. We have

$$\underbrace{\langle [\varphi] \cup [\varphi], [\mathbb{R}\mathrm{P}^2]_{\mathbb{F}_2}\rangle}_{\substack{\text{pairing from the}\\ \text{Ring–Kronecker Pairing}\\ \text{Lemma 71.1}}} \ =\ 1 \ \in\ \mathbb{F}_2.$$

Proof. We introduce the following notation:

- Let $\varphi \in \mathrm{C}^1(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2)$ be the cocycle from the Generator-of-$\mathrm{H}^1(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2)$-Lemma 72.14.

- We consider the singular simplices $\sigma_1, \sigma_2, \tau_1, \tau_2$ that are shown in the figure below. It follows easily from the $\mathbb{F}_2$-Fundamental Class-Criterion 64.4 that $(\sigma_1+\sigma_2+\tau_1+\tau_2)\otimes 1 \in \mathrm{C}_2(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2)$ is a representative of the $\mathbb{F}_2$-fundamental class $[\mathbb{R}\mathrm{P}^2]_{\mathbb{F}_2}$ of $\mathbb{R}\mathrm{P}^2 =_{\mathrm{i}} \overline{B}^2/\sim$.



By the definition of the Kronecker pairing in the Ring–Kronecker Pairing 71.1 it remains to prove the following claim:

Claim. We have

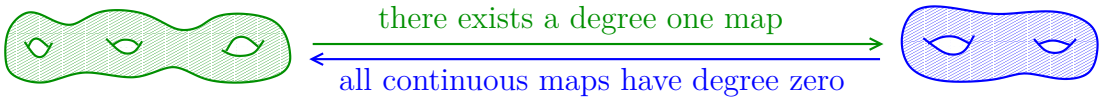
$$(\varphi\cup\varphi)(\sigma_1)+(\varphi\cup\varphi)(\sigma_2)+(\varphi\cup\varphi)(\tau_1)+(\varphi\cup\varphi)(\tau_2) \;=\; 1 \;\in\; \mathbb{F}_2.$$

Proof. We leave it to the reader to determine all four summands and to show that the sum is indeed 1. ∎

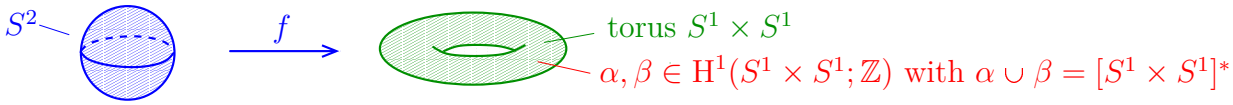
72.7. **Applications of the cup product.** Now that we had some fun with calculating the cup product, the question arises, what is it actually good for? Of course the cohomology together with the cup product is a stronger invariant, than the cohomology on its own. For example we saw that the torus $S^1\times S^1$ and the wedge $S^1\vee S^1\vee S^2$ have isomorphic cohomology groups, but the cohomology rings are not isomorphic. Once we have developed more tools for calculating cup products, this theme will become more important, for example we will be able to distinguish $S^2\times S^2$ from $\mathbb{C}\mathrm{P}^2\#\mathbb{C}\mathrm{P}^2$.

There are also many other questions which can be attacked using the cup product. For example we can give a new proof of the following proposition:

Proposition 65.8. (Smaller-to-Larger Genus Proposition) Let $g,k\in\mathbb{N}_0$. If $g<k$, then every continuous map $f\colon \Sigma_g\to\Sigma_k$ has degree zero.



Proof. In the following we will just deal with the case $g=0$ and $k=1$. In other words, we will show that every continuous map $f\colon S^2\to S^1\times S^1$ has degree zero. We leave the other cases as an entertaining exercise to the reader. So let $f\colon S^2\to S^1\times S^1$ be a continuous map. By the Torus–Cup Product Lemma 72.11 we know that there exist $\alpha,\beta\in \mathrm{H}^1(S^1\times S^1;\mathbb{Z})$ such that $[S^1\times S^1]^* = \alpha\cup\beta \in \mathrm{H}^2(S^1\times S^1;\mathbb{Z})$.



We now see that

$$\begin{array}{lcccl}
 & \overset{\text{by definition of } [S^2]^*}{\downarrow} & & \overset{\text{by the Dual Fundamental Class–Degree Lemma 70.17}}{\downarrow} & \overset{\text{choice of } \alpha \text{ and } \beta}{\downarrow} \\
\deg(f) & = & \langle \deg(f) \cdot [S^2]^*, [S^2] \rangle & = & \langle f^*([S^1 \times S^1]^*), [S^2] \rangle & = & \langle f^*(\alpha \cup \beta), [S^2] \rangle \\
 & = & \langle f^*(\alpha) \cup f^*(\beta), [S^2] \rangle & = & \langle 0 \cup 0, [S^2] \rangle & = & 0. \\
 & \underset{\text{by the Cup Product Naturality Lemma 72.2}}{\uparrow} & & \underset{\text{since } \mathrm{H}^1(S^2;\mathbb{Z}) = 0}{\uparrow} & &
\end{array}$$

■

CHAPTER 73

The relative cup product

Our goal in this chapter is to introduce a relative version of the cup product. More precisely, given an excisive triad (X, A, B) and a commutative ring R we will introduce a cup product

$$\mathrm{H}^p(X, A; R) \times \mathrm{H}^q(X, B; R) \xrightarrow{\cup} \mathrm{H}^{p+q}(X, A \cup B; R)$$

that is natural in an appropriate sense and that equals the previous definition of the cup product for the special case that $A = B = \varnothing$. This relative cup has many applications:

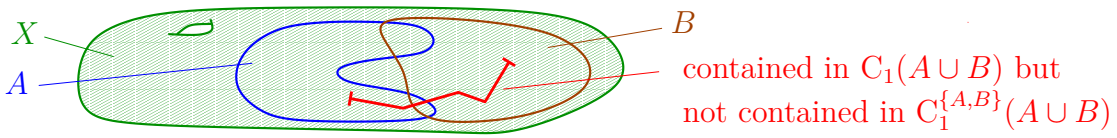
(1) It will help us in determining the absolute cup products of suspensions of topological spaces and of wedges of topological spaces.
(2) It can be used to define invariants of topological manifolds with boundary.
(3) In the next chapter we will see that it gives us lower bounds on the mysterious sounding Lusternik-Schnirelmann category.

73.1. **Excisive triads.** In the definition of the relative cup product, and later, in the definition of the relative cap product, we will have to work with the notion of an excisive triad which we introduced on page 639. We start out by recalling the following definition from page 509.

Definition. Let X be a topological space and let $A, B \subset X$ be subsets of X. We consider the following subgroup of $\mathrm{C}_n(X)$:

$$\mathrm{C}_n^{\{A,B\}}(X) \coloneqq \mathrm{C}_n(A) + \mathrm{C}_n(B) = \left\{ \sum_{j=1}^{k} a_j \cdot \sigma_j \;\middle|\; \begin{array}{l} \text{for each } j \text{ the image of } \sigma_j \\ \text{lies in } A \text{ or } B \end{array} \right\}.$$

On page 509 we saw that these groups define a subcomplex of $\mathrm{C}_*(X)$.



Next we recall the technical, but very useful, definition of an (excisive) triad of topological spaces from page 639.:

Definition.
(1) A triad of topological spaces is a triple (X, A, B) where X is a topological space and where both A and B are subspaces of X.
(2) We say that a triad (X, A, B) is excisive if the inclusion map

$$\mathrm{C}_*^{\{A,B\}}(A \cup B) \;\to\; \mathrm{C}_*(A \cup B)$$

is a chain homotopy equivalence.

Next we recall the following proposition which gives us many examples of excisive triads:

Proposition 50.11. (Excisive Triad-Proposition) Let (X, A, B) be a triad of topological spaces. The triad is excisive if one of the following conditions is satisfied:

(0) (a) $A \subset B$ or $B \subset A$.
(b) $A = \varnothing$ or $B = \varnothing$ or $A = B$.
(c) A_1 and A_2 are both unions of path components of $A_1 \cup A_2$.
(1) (a) $A \cup B = \mathring{A}_1 \cup \mathring{A}_2$ where we denote by $\mathring{A}_1$ and $\mathring{A}_2$ the interiors of A and B in $A \cup B$.
(b) A and B are both open subsets of $A \cup B$.
(2) $A \cup B$ is a topological manifold and A and B are codimension-zero submanifolds which are both closed subsets and such that each component of $A \cap B$ is a boundary component of A and a boundary component of B.
(3) X is a CW-complex and A and B are both subcomplexes.
(4) Let A and B be topological spaces and let $a_0 \in A$ and $b_0 \in B$ be good points in the sense of the definition on page 292. Then $((A, a_0) \vee (B, b_0), A, B)$ is an excisive triad.



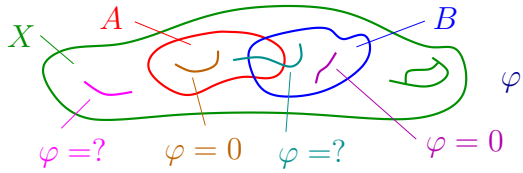
Definition. Let (X, A, B) be a triad of topological spaces. For each $k \in \mathbb{N}_0$ we write

$$\mathrm{C}_k(X, \{A, B\}) \coloneqq \frac{\mathrm{C}_k(X)}{\mathrm{C}_k^{\{A,B\}}(A \cup B)} = \frac{\mathrm{C}_k(X)}{\mathrm{C}_k(A) + \mathrm{C}_k(B)}.$$

Since $\mathrm{C}_*^{\{A,B\}}(A \cup B)$ is a subcomplex of $\mathrm{C}_*(X)$ we can view $\mathrm{C}_*(X, \{A, B\})$ also as a chain complex. For any abelian group G and any $k \in \mathbb{N}_0$ we define

$$\begin{aligned}
\text{the chain complex} \quad & \mathrm{C}_k(X, \{A, B\}; G) \coloneqq \mathrm{C}_k(X, \{A, B\}) \otimes G \\
\text{with homology groups} \quad & \mathrm{H}_k(X, \{A, B\}; G) \coloneqq \mathrm{H}_k(\mathrm{C}_*(X, \{A, B\}; G)) \\
\text{the cochain complex} \quad & \mathrm{C}^k(X, \{A, B\}; G) \coloneqq \mathrm{Hom}(\mathrm{C}_k(X, \{A, B\}), G) \\
\text{with cohomology groups} \quad & \mathrm{H}^k(X, \{A, B\}; G) \coloneqq \mathrm{H}^k(\mathrm{C}^*(X, \{A, B\}; G)).
\end{aligned}$$

Remark. Any element in $\mathrm{C}^n(X, \{A, B\}; G)$ corresponds to a cochain $\varphi\colon \mathrm{C}_n(X) \to G$ such that $\varphi(\sigma) = 0$ for all singular simplices $\sigma\colon \Delta^n \to X$ that take values in A or B.



$$\varphi \in \mathrm{C}^1(X, \{A, B\}; G) = \mathrm{Hom}\left(\frac{\mathrm{C}_1(X)}{\mathrm{C}_1(A) + \mathrm{C}_1(B)}, G\right)$$

The definition of the relative cup product (and later the relative cap product) relies very much on the following lemma:

Lemma 73.1. (Excisive Triad–Invariants Lemma) Let (X, A, B) be an excisive triad of topological spaces.

(1) The natural chain map

$$\iota\colon \mathrm{C}_*(X, \{A, B\}) \to \mathrm{C}_*(X, A \cup B)$$

is a chain homotopy equivalence.

(2) For any abelian group G and any $k \in \mathbb{N}_0$ the natural maps

$$\begin{array}{lrcl} \text{(a)} & \iota_* \colon \mathrm{H}_k(X, \{A, B\}; G) & \to & \mathrm{H}_k(X, A \cup B; G) \\ \text{(b)} & \iota^* \colon \mathrm{H}^k(X, A \cup B; G) & \to & \mathrm{H}^k(X, \{A, B\}; G) \end{array}$$

are isomorphisms.

Proof.

(1) Note that we have the following commutative diagram of short exact sequences of chain complexes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{C}_*^{\{A,B\}}(A \cup B) & \longrightarrow & \mathrm{C}_*(X) & \longrightarrow & \overbrace{\dfrac{\mathrm{C}_*(X)}{\mathrm{C}_*^{\{A,B\}}(A \cup B)}}^{=\mathrm{C}_*(X,\{A,B\})} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \mathrm{id} & & \downarrow \iota & & \\ 0 & \longrightarrow & \mathrm{C}_*(A \cup B) & \longrightarrow & \mathrm{C}_*(X) & \longrightarrow & \mathrm{C}_*(X, A \cup B) & \longrightarrow & 0. \end{array}$$

Claim. All groups involved are free abelian.

Proof. By definition each $\mathrm{C}_k(X)$ and $\mathrm{C}_k(A \cup B)$ is free abelian. Next note that $\mathrm{C}_k^{\{A,B\}}(A \cup B)$ is the free abelian group on the set of singular k-simplices that lie in A or B. On page 489 we already remarked that relative chain groups $\mathrm{C}_k(X, A \cup B)$ are free abelian. Finally note that the same same argument shows that the top right group is also free abelian, in fact a basis is given by all singular simplices of X that are not contained in A or B. ⊞

By our hypothesis the triad (X, A, B) is excisive. By definition this means that the left vertical map is a chain homotopy equivalence. The middle vertical map is the identity. It follows from this discussion together with the Two-Out-Of-Three-Homotopy Equivalence Corollary 50.5 that the right vertical map is also a chain homotopy equivalence.

(2) (a) This statement follows from (1) together with the Homology-with-Coefficients Lemma 52.4 (2b).

(b) This statement follows from (1) together with the Dual Cochain Map Lemma 66.9 (3b) and the Cochain Homotopy-H*-Lemma 66.6. ■

73.2. **Definition of the relative cup product.** In this section we will apply the rather dry discussion from the last section to introduce the relative cup product.

The relative cup product is defined on relative cohomology. In this section and the following sections we will thus work a lot with relative cochains. We will adopt the following sanity preserving notation:

Notation. Let (X, A) be a pair of topological spaces and let G be an abelian group. Given $\varphi \in \mathrm{C}^k(X, A; G) = \mathrm{Hom}(\mathrm{C}_k(X, A), G)$ we denote by φ also the corresponding homomorphism $\mathrm{C}_k(X) \to \mathrm{C}_k(X, A) \to G$.

The following lemma gives us a first version of a relative cup product:

Lemma 73.2. (Triad–Cup Product Lemma) Let (X, A, B) be a triad of topological spaces and let R be a commutative ring.

(1) The cup product

$$\begin{aligned} \cup\colon \mathrm{C}^p(X;R)\times\mathrm{C}^q(X;R) &\to \mathrm{C}^{p+q}(X;R) \\ (\varphi,\psi) &\mapsto \begin{pmatrix} \mathrm{C}_{p+q}(X) & \to & R \\ \sigma & \mapsto & \varphi(\sigma\circ[\mathfrak{v}_0,\dots,\mathfrak{v}_p])\cdot\psi(\sigma\circ[\mathfrak{v}_p,\dots,\mathfrak{v}_{p+q}]) \end{pmatrix} \end{aligned}$$

from page 921 descends to a well-defined R-bilinear map

$$\begin{aligned} \cup\colon \mathrm{C}^p(X,A;R)\times\mathrm{C}^q(X,B;R) &\to \mathrm{C}^{p+q}(X,\{A,B\};R) \\ (\varphi,\psi) &\mapsto \begin{pmatrix} \frac{\mathrm{C}_{p+q}(X)}{\mathrm{C}_{p+q}(A)+\mathrm{C}_{p+q}(B)} & \to & R \\ [\sigma] & \mapsto & (\varphi\cup\psi)(\sigma) \end{pmatrix}. \end{aligned}$$

(2) The map from (1) defines an R-bilinear map

$$\cup\colon \mathrm{H}^p(X,A;R)\times\mathrm{H}^q(X,B;R) \to \mathrm{H}^{p+q}(X,\{A,B\};R).$$

Proof.

(1) This statement follows easily from the following claim:

Claim. Let $\varphi\in\mathrm{C}^p(X,A;R)$ and $\psi\in\mathrm{C}^q(X,B;R)$. The homomorphism

$$\begin{aligned} \mathrm{C}_{p+q}(X) &\to R \\ (\sigma\colon\Delta^{p+q}\to X) &\mapsto \varphi(\sigma\circ[\mathfrak{v}_0,\dots,\mathfrak{v}_p])\cdot\psi(\sigma\circ[\mathfrak{v}_p,\dots,\mathfrak{v}_{p+q}]) \end{aligned}$$

vanishes on the subgroup $\mathrm{C}_{p+q}(A)+\mathrm{C}_{p+q}(B)$.

Proof. We have to show that the above cochain vanishes on all singular $(p+q)$-simplices that lie entirely in A or that lie entirely in B. But this is clear:

(a) Given $\sigma\colon\Delta^{p+q}\to A$ the first factor $\varphi(\sigma\circ[\mathfrak{v}_0,\dots,\mathfrak{v}_p])$ is necessarily zero since we have $\varphi\in\mathrm{C}^p(X,A;R)$.

(b) Given $\sigma\colon\Delta^{p+q}\to B$ the second factor $\psi(\sigma\circ[\mathfrak{v}_p,\dots,\mathfrak{v}_{p+q}])$ is necessarily zero since we have $\psi\in\mathrm{C}^q(X,B;R)$. ⊞

(2) Using the Cup Product–Coboundary Lemma 72.3 one can easily show that we get an induced cup product on cohomology groups. ■

The relative cup product from the Triad–Cup Product Lemma 73.2 is not all that interesting, since it takes values in the weird cohomology group $\mathrm{H}^{p+q}(X,\{A,B\};R)$. Fortunately, if we are dealing with an excisive triad, then the Excisive Triad–Invariants Lemma 73.1 (2b) lets us move on to a much more reasonable relative cohomology group. More precisely, the Triad–Cup Product Lemma 73.2 together with the Excisive Triad–Invariants Lemma 73.1 (2b) allows us to introduce the following definition:

Definition. Let (X, A, B) be an excisive triad of topological spaces and let R be a commutative ring. We refer to the composition

$$\mathrm{H}^p(X,A;R)\times\mathrm{H}^q(X,B;R) \xrightarrow[\uparrow]{(c,d)\mapsto c\cup d} \mathrm{H}^{p+q}(X,\{A,B\};R) \xleftarrow[\uparrow]{\cong} \mathrm{H}^{p+q}(X,A\cup B;R)$$

well-defined by the Triad–Cup Product Lemma 73.2

isomorphism by the Excisive Triad Invariants Lemma 73.1 (2b), here we use that (X, A, B) is excisive

again as a cup product (or sometimes as a relative cup product) and we denote it as usual by the "$\cup$"-symbol.

Remark. Let X be a topological space, let $A, B \subset X$ be subsets and let R be a commutative ring. Suppose that $\mathrm{C}_*^{\{A,B\}}(A \cup B) = \mathrm{C}_*(A \cup B)$. (This is for example the case if we are in the setting of Excisive Triad-Proposition 50.11 (0), e.g. if $A = \varnothing$, or $B = \varnothing$ or $A = B$.) Then in the above definition the horizontal map to the right is the identity and we see that the cup product is just given by the "naive" definition

$$\begin{aligned} \mathrm{C}^p(X, A; R) \times \mathrm{C}^q(X, B; R) &\to \mathrm{C}^{p+q}(X, A \cup B; R) \\ (\varphi, \psi) &\mapsto \Big(\begin{aligned} \mathrm{C}_{p+q}(X, A \cup B) &\to R \\ \sigma &\mapsto \underbrace{\varphi(\sigma \circ [\mathfrak{v}_0, \dots, \mathfrak{v}_p])}_{\in R} \cdot \underbrace{\psi(\sigma \circ [\mathfrak{v}_p, \dots, \mathfrak{v}_{p+q}])}_{\in R} \end{aligned} \Big). \end{aligned}$$

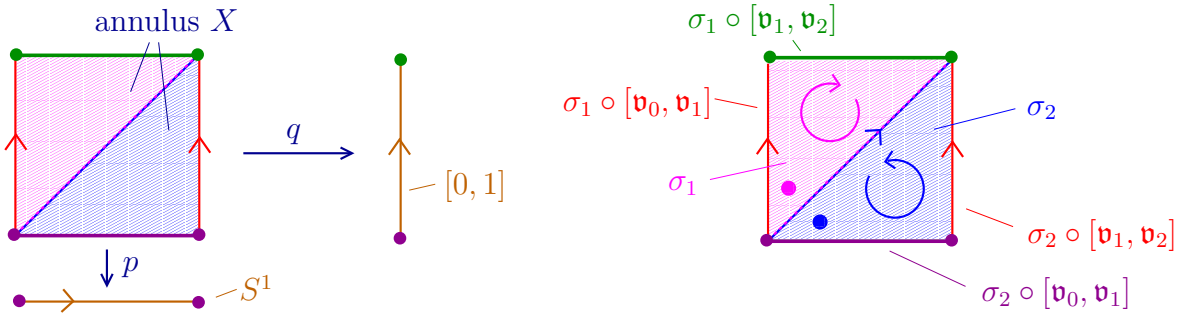
□

Example. We consider the annulus $X = S^1 \times [0,1]$, the subspace $A = \partial X = S^1 \times \{0,1\}$ and the subspace $B = \varnothing$. We want to understand the relative cup product

$$\begin{array}{ccccc} \mathrm{H}^1(X; \mathbb{Z}) \times \mathrm{H}^1(X, \partial X; \mathbb{Z}) & \to & \mathrm{H}^2(X, \partial X; \mathbb{Z}) & \xrightarrow{\cong} & \mathbb{Z} \\ (\varphi, \psi) & \mapsto & \varphi \cup \psi & & \\ & & \theta & \mapsto & \langle \theta, [X] \rangle. \end{array}$$

To perform this calculation we need explicit cocycles and an explicit representative of the fundamental class $[X] \in \mathrm{H}_2(X, \partial X)$. We introduce and recall the following notation:

(1) We denote by $p\colon X \to S^1$ and $q\colon X \to [0,1]$ the projections.
(2) Let $\theta_{\mathbb{Z}} \in \mathrm{C}^1(S^1, \mathbb{Z})$ the 1-cocycle from page 866.
(3) Let $\overline{\gamma}_{\frac{1}{2}} \in \mathrm{C}^1([0,1], \{0,1\}, \mathbb{Z})$ be the relative 1-cocycle from pages 869 and 874.
(4) Let $\sigma_1, \sigma_2\colon \Delta^2 \to X$ be the singular simplices shown in the figure below.



An explicit calculation, as on page 923, shows that

$$\begin{aligned} (p^*(\theta_{\mathbb{Z}}) \cup q^*(\overline{\gamma}_{\frac{1}{2}}))(\sigma_1) &= p^*(\theta_{\mathbb{Z}})(\sigma_1 \circ [\mathfrak{v}_0, \mathfrak{v}_1]) \cdot q^*(\overline{\gamma}_{\frac{1}{2}})(\sigma_1 \circ [\mathfrak{v}_1, \mathfrak{v}_2]) \\ &= \theta_{\mathbb{Z}}(\underbrace{p \circ \sigma_1 \circ [\mathfrak{v}_0, \mathfrak{v}_1]}_{\text{constant simplex}}) \cdot \overline{\gamma}_{\frac{1}{2}}(\underbrace{q \circ \sigma_1 \circ [\mathfrak{v}_1, \mathfrak{v}_2]}_{\text{constant simplex}}) \quad = \quad 0 \cdot 0 \;=\; 0. \end{aligned}$$

$$\begin{aligned} (p^*(\theta_{\mathbb{Z}}) \cup q^*(\overline{\gamma}_{\frac{1}{2}}))(\sigma_2) &= p^*(\theta_{\mathbb{Z}})(\sigma_2 \circ [\mathfrak{v}_0, \mathfrak{v}_1]) \cdot q^*(\overline{\gamma}_{\frac{1}{2}})(\sigma_2 \circ [\mathfrak{v}_1, \mathfrak{v}_2]) \\ &= \theta_{\mathbb{Z}}(\underbrace{p \circ \sigma_2 \circ [\mathfrak{v}_0, \mathfrak{v}_1]}_{\substack{\text{the simplex } \Delta^1 \to S^1 \\ \text{that is given} \\ \text{by } (1\text{–}t, t) \mapsto \exp(2\pi i t)}}) \cdot \overline{\gamma}_{\frac{1}{2}}(\underbrace{q \circ \sigma_2 \circ [\mathfrak{v}_1, \mathfrak{v}_2]}_{\substack{\text{the simplex } \Delta^1 \to [0,1] \\ \text{that is given} \\ \text{by } (1\text{–}t, t) \mapsto t}}) \quad = \quad 1 \cdot 1 \;=\; 1. \end{aligned}$$

↑ by the explicit definitions of $\theta_{\mathbb{Z}}$ and $\overline{\gamma}_{\frac{1}{2}}$

We now see that

$$\langle p^*([\theta_{\mathbb{Z}}]) \cup q^*([\overline{\gamma}_{\frac{1}{2}}]), [X]\rangle \;=\; \langle [p^*(\theta_{\mathbb{Z}}) \cup q^*(\overline{\gamma}_{\frac{1}{2}})], [-\sigma_1 + \sigma_2]\rangle \;=\; -0 + 1 \;=\; 1.$$

it follows easily from the Fundamental Class-Criterion 64.3 that $[-\sigma_1 + \sigma_2] = [X] \in \mathrm{H}_2(X, \partial X)$

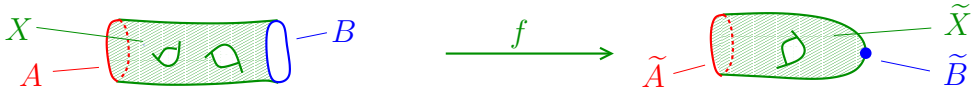
by the above calculations and the above remark on relative cup products

□

73.3. **Properties of the relative cup product.** In the following we summarize a few properties of the relative cup product. We start out with the following lemma, which is a generalization of the Cup Product-Naturality Lemma 72.2.

Lemma 73.3. (Cup Product-Naturality Lemma) Let (X, A, B) and $(\widetilde{X}, \widetilde{A}, \widetilde{B})$ be two excisive triads and let R be a commutative ring. Let $f\colon (X, A, B) \to (\widetilde{X}, \widetilde{A}, \widetilde{B})$ be a continuous map with $f(A) \subset \widetilde{A}$ and $f(B) \subset \widetilde{B}$. For all $p, q \in \mathbb{N}_0$ the following diagram commutes:

$$\begin{array}{ccc}
\mathrm{H}^p(\widetilde{X}, \widetilde{A}; R) \times \mathrm{H}^q(\widetilde{X}, \widetilde{B}; R) & \xrightarrow{\cup} & \mathrm{H}^{p+q}(\widetilde{X}, \widetilde{A} \cup \widetilde{B}; R) \\
\downarrow{\scriptstyle f^*} \quad \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} \\
\mathrm{H}^p(X, A; R) \times \mathrm{H}^q(X, B; R) & \xrightarrow{\cup} & \mathrm{H}^{p+q}(X, A \cup B; R).
\end{array}$$



Proof. We have to prove the following claim:

Claim. The following diagram commutes:

$$\begin{array}{ccccc}
\mathrm{H}^p(\widetilde{X}, \widetilde{A}; R) \times \mathrm{H}^q(\widetilde{X}, \widetilde{B}; R) & \xrightarrow{\cup} & \mathrm{H}^{p+q}(\widetilde{X}, \{\widetilde{A}, \widetilde{B}\}; R) & \xleftarrow{\cong} & \mathrm{H}^{p+q}(\widetilde{X}, \widetilde{A} \cup \widetilde{B}; R) \\
\downarrow{\scriptstyle f^*} \quad \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} \\
\mathrm{H}^p(X, A; R) \times \mathrm{H}^q(X, B; R) & \xrightarrow{\cup} & \mathrm{H}^{p+q}(X, \{A, B\}; R) & \xleftarrow{\cong} & \mathrm{H}^{p+q}(X, A \cup B; R).
\end{array}$$

Proof. The second argument in the proof of the Cup Product-Ring Proposition 72.9 (3) shows that the square to the left commutes. It follows immediately from the functoriality of cohomology that the square to the right commutes. ■

We continue with the obvious generalization of the Cup Product-Commutativity Proposition 72.8:

Proposition 73.4. (Cup Product-Commutativity Proposition) Let (X, A, B) be an excisive triad of topological spaces and let R be a commutative ring. For any cohomological classes $\varphi \in \mathrm{H}^p(X, A; R)$ and $\psi \in \mathrm{H}^q(X, B; R)$ we have

$$\varphi \cup \psi \;=\; (-1)^{pq} \cdot \psi \cup \varphi \;\in\; \mathrm{H}^{p+q}(X, A \cup B; R).$$

Proof. The proposition can be proved analogously to the Cup Product-Commutativity Proposition 72.8. For the sake of brevity we will not carry out the argument. We refer to [**Fri24**] for details. ■

73.4. **Calculations using the relative cup product.** So what is the relative cup product good for? In this section we will give a surprising answer, namely we will see that we can use it to calculate "absolute" cup products. The key to doing so is the Cup Product-Naturality

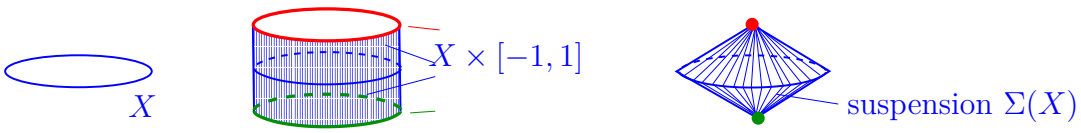
Lemma 73.3. To be more precise, in this section we will use relative cup products to calculate the cup products of the following types of topological spaces:

(1) Suspensions.
(2) Wedges of topological spaces.

73.4.1. **The cup product of suspensions.** In this subsection we study the cup product of the suspension $\Sigma(X)$ of a topological space X. For the reader's convenience we recall the definition of a suspension from page 282:

Definition. Given a topological space X we define its suspension $\Sigma(X)$ to be

$$\Sigma(X) \coloneqq ((X \times [-1,1])/(X \times \{-1\}))/(X \times \{1\}).$$



The following lemma shows that the cup product of the suspension of a topological space is quite dull.

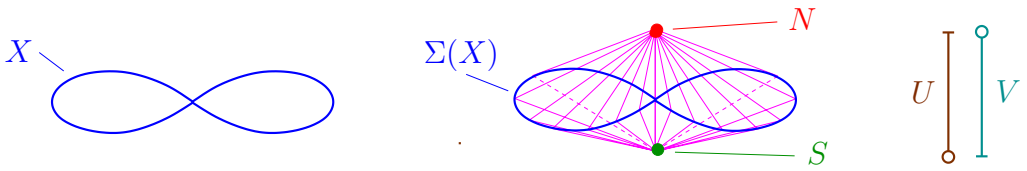
Lemma 73.5. (Suspension-Cup Product Lemma) Let X be a topological space and let R be a commutative ring. For any $k, l \geq 1$ the cup product

$$\mathrm{H}^k(\Sigma(X); R) \times \mathrm{H}^l(\Sigma(X); R) \ \to \ \mathrm{H}^{k+l}(\Sigma(X); R)$$

is the zero map.

Proof. We introduce the following notation:

(1) We denote by $N \coloneqq [X \times \{1\}]$ the "North Pole" and we denote by $S \coloneqq [X \times \{-1\}]$ the "South Pole" of $\Sigma(X)$.
(2) We write $U = \Sigma(X) \setminus \{S\}$ and $V = \Sigma(X) \setminus \{N\}$. Note that U admits deformation retraction to the North Pole N. This shows that U is contractible. The same argument shows that V is contractible.



Now let $p, q \geq 1$. We consider the following diagram:

$$\begin{array}{ccc} \mathrm{H}^p(\Sigma(X), U; R) \times \mathrm{H}^q(\Sigma(X), V; R) & \xrightarrow{\cup} & \mathrm{H}^{p+q}(\Sigma(X), \overbrace{U \cup V}^{=\Sigma(X)}; R) \\ \downarrow \cong & \cong \downarrow & \downarrow \\ \mathrm{H}^p(\Sigma(X); R) \times \mathrm{H}^q(\Sigma(X); R) & \xrightarrow{\cup} & \mathrm{H}^{p+q}(\Sigma(X); R). \end{array}$$

We make the following observations and clarifications:

(1) It follows from the Excisive Triad-Proposition 50.11 (1) that the triad $(\Sigma(X), U, V)$ is excisive. It follows that the relative cup product on top is defined and it follows from the Cup Product-Naturality Lemma 73.3 that the diagram commutes.

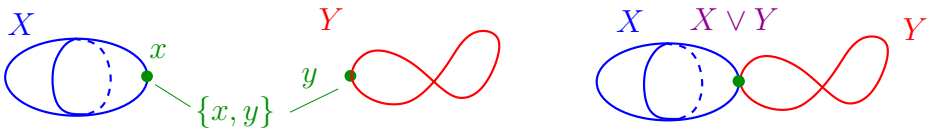
(2) It follows immediately from the long exact sequence of cohomology groups corresponding to the pair $(\Sigma(X), U)$, the fact that U is homotopy equivalent to a point (together with the Singleton-H*-Lemma 67.6 and the Homotopy Equivalence-H*-Corollary 67.8 (1)) and the hypothesis that $p \geq 1$ that the inclusion-induced map $\mathrm{H}^p(\Sigma(X), U; R) \to \mathrm{H}^p(\Sigma(X); R)$ is an isomorphism. Evidently the same also holds if we replace U by V.
(3) Since $U \cup V = \Sigma(X)$ we see that the cohomology group on the top-right is zero.

It follows from the above discussion that the cup product on the bottom is zero. ■

73.4.2. **The cup product of wedges.** In this subsection we want to calculate the cup product of wedges. For the reader's convenience we recall the relevant definition from page 245:

Definition. Let X and Y be two topological spaces and let $x \in X$ and $y \in Y$. We define the wedge

$$(X, x) \vee (Y, y) \coloneqq (X \sqcup Y)/\underbrace{\{x, y\}}_{\subset X \sqcup Y}.$$



The following lemma determines the cup product of $X \vee Y$ in terms of the cup products of X and Y. It can be viewed as a generalization of the Wedge-of-Spheres–Cup Product Lemma 72.12.

Lemma 73.6. (Wedge-Cup Product Lemma) Let X and Y be two topological spaces, let $x \in X$ and $y \in Y$ be good[573] points and let R be a commutative ring. We denote by $p\colon X \vee Y \to X$ and $q\colon X \vee Y \to Y$ the natural projections which we introduced on page 245.

(1) By the Wedge-H*-Proposition 69.4 we know that for every $n \in \mathbb{N}$ the map

$$p^* \oplus q^*\colon \mathrm{H}^n(X; R) \oplus \mathrm{H}^n(Y; R) \xrightarrow{\cong} \mathrm{H}^n(X \vee Y; R)$$

is an isomorphism.
(2) For any $\alpha_1, \alpha_2 \in \mathrm{H}^*(X; R)$ and $\beta_1, \beta_2 \in \mathrm{H}^*(Y; R)$ in degrees ≥ 1 the following statements hold in $\mathrm{H}^*(X \vee Y; R)$:

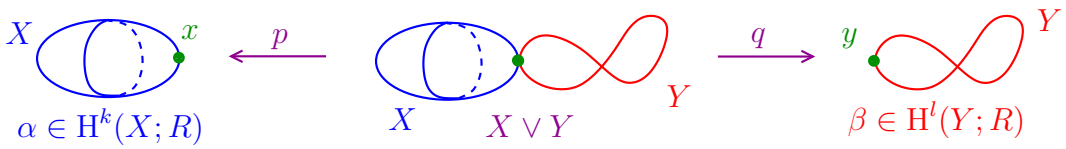
$\cup$	$p^*(\alpha_2)$	$q^*(\beta_2)$
$p^*(\alpha_1)$	$p^*(\alpha_1 \cup \alpha_2)$	0
$q^*(\beta_1)$	0	$q^*(\beta_1 \cup \beta_2)$

Proof. Note that we proved Statement (1) already in the Wedge-H*-Proposition 69.4. So it remains to prove Statement (2).

The statement regarding the "diagonal" terms of the table follows immediately from the Cup Product-Naturality Lemma 72.2. It remains to prove that the off-diagonal terms are zero. We show that the top-right entry is zero. The proof that the bottom-left entry is zero is basically identical.

Thus let $\alpha \in \mathrm{H}^k(X; R)$ and $\beta \in \mathrm{H}^l(Y; R)$ with $k, l \geq 1$. We want to show that $p^*(\alpha) \cup q^*(\beta) = 0$.

[573] We refer to page 292 for the definition of a good point.



We consider the following diagram:

$$\begin{array}{ccc} \mathrm{H}^k(X \vee Y, Y; R) \times \mathrm{H}^l(X \vee Y, X; R) & \xrightarrow{\cup} & \mathrm{H}^{k+l}(X \vee Y, \overbrace{X \cup Y}^{=X\vee Y}; R) \\ \downarrow{\scriptstyle g^*} \quad \downarrow{\scriptstyle f^*} & & \downarrow \\ \mathrm{H}^k(X \vee Y; R) \times \mathrm{H}^l(X \vee Y; R) & \xrightarrow{\cup} & \mathrm{H}^{k+l}(X \vee Y; R). \\ {\scriptstyle p^*(\alpha)} \quad {\scriptstyle q^*(\beta)} & & \end{array}$$

We make the following observations and clarifications:

(1) As we discussed on page 247, we can view X and Y as subspaces of $X \vee Y$.
(2) We denote by $f\colon (X \vee Y, \varnothing) \to (X \vee Y, X)$ and $g\colon (X \vee Y, \varnothing) \to (X \vee Y, Y)$ the two inclusion maps of pairs of topological spaces.
(3) It follows from the hypothesis that $x \in X$ and $y \in Y$ are good and the Excisive Triad-Proposition 50.11 (4) that the triad $(X \vee Y, X, Y)$ is excisive. It follows that the relative cup product on top is defined and it follows from the Cup Product-Naturality Lemma 73.3 that the diagram commutes.

Since the top right group is clearly zero it remains to prove the following claim:

Claim.
(a) There exists a $\widetilde{\alpha} \in \mathrm{H}^k(X \vee Y, Y; R)$ with $g^*(\widetilde{\alpha}) = p^*(\alpha)$.
(b) There exists a $\widetilde{\beta} \in \mathrm{H}^l(X \vee Y, X; R)$ with $f^*(\widetilde{\beta}) = q^*(\beta)$.

Proof. We provide a proof of (a). The proof of (b) is of course basically identical. To prove (a) we consider the following diagrams[574]

$$\begin{array}{ccc} \mathrm{H}^k(X, \{x_0\}; R) & \xrightarrow{p^*} & \mathrm{H}^k(X \vee Y, Y; R) \\ {\scriptstyle \cong}\downarrow{\scriptstyle \text{я}} & & \downarrow{\scriptstyle g^*} \\ \mathrm{H}^k(X; R) & \xrightarrow{p^*} & \mathrm{H}^k(X \vee Y; R). \\ {\scriptstyle \alpha} & & {\scriptstyle p^*(\alpha)} \end{array}$$

The claim is an immediate consequence of the following two observations:

- It follows from the functoriality of cohomology that the diagram commutes.
- It follows easily from the long exact sequence in cohomology of a pair of topological spaces and the fact that $k \geq 1$ that the left vertical map is an isomorphism. ∎

73.5. **The cup product and connecting homomorphisms.** In this last section we will study the connection between cup products and connecting homomorphisms. It is

[574]Recall that in $X \vee Y$ the points x and y get identified, so we can view x as a point in Y and we can view y as a point in X.

perhaps best to look up the lemmas, when they actually get used, and to move straight to Chapter 74.

Before we can state the relevant lemmas let us introduce the following convention:

Convention. Let A, B, C, D, X and Y be sets, let $f\colon C \to A$, $g\colon B \to D$ and $h\colon X \to Y$ be maps and let $\varphi\colon A \times B \to X$ and $\psi\colon C \times D \to Y$ be maps. We say that the diagram

$$\begin{array}{ccc} A \times \overset{b}{B} & \xrightarrow{\varphi} & X \\ {\scriptstyle f}\uparrow \quad \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} \\ \underset{c}{C} \times D & \xrightarrow{\psi} & Y \end{array}$$

commutes, if for all $c \in C$ and $b \in B$ we have

$$\psi(c, g(b)) \;=\; h(\varphi(f(c), b)).$$

A similar convention holds with the directions of f and g reversed. Furthermore, if we are dealing with homomorphisms of abelian groups, then there is an obvious notion of "commutes up to a given sign".

Using this convention we can now formulate the first lemma of this section:

Lemma 73.7. (Cup Product-Connecting Homomorphism Lemma) Let (X, A) be a pair of topological spaces and let R be a commutative ring. We denote by $i\colon A \to X$ the inclusion map and we denote by δ the connecting homomorphisms in the long exact sequence in cohomology corresponding to the pair (X, A). For any $p, q \in \mathbb{N}_0$ the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}^p(A; R) \times \mathrm{H}^q(A; R) & \xrightarrow{\cup} & \mathrm{H}^{p+q}(A; R) \\ {\scriptstyle i^*}\uparrow \qquad \downarrow{\scriptstyle \delta_q} & & \downarrow{\scriptstyle \delta_{p+q}} \\ \mathrm{H}^p(X; R) \times \mathrm{H}^{q+1}(X, A; R) & \xrightarrow{\cup} & \mathrm{H}^{p+q+1}(X, A; R) \end{array}$$

up to the sign $(-1)^p$. In other words, for any $\varphi \in \mathrm{H}^p(X; R)$ and any $\psi \in \mathrm{H}^q(A; R)$ we have

$$\varphi \cup \delta_q(\psi) \;=\; (-1)^p \cdot \delta_{p+q}(i^*(\varphi) \cup \psi) \;\in\; \mathrm{H}^{p+q+1}(X, A; R).$$

Proof. We pick cocycles $\varphi \in \mathrm{C}^p(X; R)$ and $\psi \in \mathrm{C}^q(A; R)$. In the following we denote by $\widetilde{\psi}\colon \mathrm{C}^q(X) \to R$ the extension of ψ that is defined to be zero on all singular p-simplices that do not lie in the subset A. We have the following equalities:

$$\begin{aligned} \delta_{p+q}([i^*\varphi] \cup [\psi]) \;&=\; \delta_{p+q}([i^*\varphi \cup \psi]) \;\overset{\downarrow}{=}\; [\delta_{p+q}(\varphi \cup \widetilde{\psi})] \;\underset{\uparrow}{=}\; [\delta_p(\varphi) \cup \widetilde{\psi} + (-1)^p \cdot (\varphi \cup \delta_q(\widetilde{\psi}))] \\ &\underset{\uparrow}{=}\; (-1)^p \cdot [\varphi \cup \delta_q(\widetilde{\psi})] \;=\; (-1)^p \cdot ([\varphi] \cup [\delta_q(\widetilde{\psi})]) \;\underset{\uparrow}{=}\; (-1)^p \cdot ([\varphi] \cup \delta_q([\psi])). \end{aligned}$$

by the discussion of the connecting homomorphism in the remark on page 872 and since $\varphi \cup \widetilde{\psi}$ is the trivial extension of $i^*\varphi \cup \psi\colon \mathrm{C}_{p+q}(A) \to R$ to a homomorphism $\mathrm{C}_{p+q}(X) \to R$

by the Cup Product–Coboundary Lemma 72.3

since φ is a cocycle

again by the remark on page 872

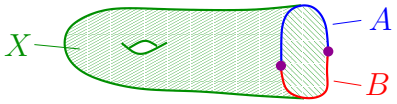
This equality that the given diagram commutes up to the sign $(-1)^p$. ∎

The Cup Product-Connecting Homomorphism Lemma 73.7 is in fact just the reader-friendly special case of the following lemma with $B = \varnothing$:

Lemma 73.8. (Cup Product-Connecting Homomorphism Lemma) Let (X, A, B) be an excisive triad of topological spaces and let R be a commutative ring. We denote by $i\colon (A, A\cap B) \to (X, B)$ and $j\colon (A, A\cap B) \to (A\cup B, B)$ the inclusion maps and we denote by δ the connecting homomorphisms in the long exact sequence in cohomology corresponding to the pair (X, A) and the triple $(X, A\cup B, B)$. Note that it follows from the Excisive Characterization Lemma 50.10 (1)⇒(3) together with the Also-Iso-on-Cohomology Proposition 69.2 that $j^*\colon \mathrm{H}^{p+q}(A\cup B, B; R) \to \mathrm{H}^{p+q}(A, A\cap B; R)$ is an isomorphism. The following diagram commutes:

$$\begin{array}{ccc}
\mathrm{H}^p(A, A\cap B; R) \times \mathrm{H}^q(A; R) & \xrightarrow{\cup} & \mathrm{H}^{p+q}(A, A\cap B; R) \\
\uparrow{\scriptstyle i^*} \quad \downarrow{\scriptstyle \delta_q} & & \cong\downarrow{\scriptstyle (j^*)^{-1}} \\
 & & \mathrm{H}^{p+q}(A\cup B, B; R) \\
 & & \downarrow{\scriptstyle \delta_{p+q}} \\
\mathrm{H}^p(X, B; R) \times \mathrm{H}^{q+1}(X, A; R) & \xrightarrow{\cup} & \mathrm{H}^{p+q+1}(X, A\cup B; R)
\end{array}$$

up to the sign $(-1)^p$.



excisive triad to which we will apply the Cup Product-Connecting Homomorphism Lemma 73.8

Sketch of proof. We consider the following diagram:

$$\begin{array}{ccccc}
\mathrm{H}^p(A, A\cap B; R) \times \mathrm{H}^q(A; R) & \xrightarrow{\cup} & \mathrm{H}^{p+q}(A, A\cap B; R) & \xleftarrow{\mathrm{id}} & \mathrm{H}^{p+q}(A, A\cap B; R) \\
\uparrow{\scriptstyle i^*} \quad \downarrow{\scriptstyle \delta_q} & & \cong\downarrow{\scriptstyle (j^*)^{-1}} & & \cong\downarrow{\scriptstyle (j^*)^{-1}} \\
 & & \mathrm{H}^{p+q}(\{A, B\}, B; R) & \xleftarrow{\cong} & \mathrm{H}^{p+q}(A\cup B, B; R) \\
 & & \downarrow{\scriptstyle \delta_{p+q}} & & \downarrow{\scriptstyle \delta_{p+q}} \\
\mathrm{H}^p(X, B; R) \times \mathrm{H}^{q+1}(X, A; R) & \xrightarrow{\cup} & \mathrm{H}^{p+q+1}(X, \{A, B\}; R) & \xleftarrow{\cong} & \mathrm{H}^{p+q+1}(X, A\cup B; R).
\end{array}$$

We make the following observations and clarifications:

(1) (a) We denote by $\mathrm{H}^k(\{A, B\}, B; R)$ the cohomology of the dual cochain complex corresponding to the chain complex $\mathrm{C}_*^{\{A,B\}}(A\cup B)/\mathrm{C}_*(B)$.
   (b) Clearly the obvious map $j_*\colon \mathrm{C}_k(A, A\cap B) \to \mathrm{C}_*^{\{A,B\}}(A\cup B)/\mathrm{C}_*(B)$ is an isomorphism of chain complexes. It follows that on cohomology we have $(j^*)^{-1} = ((j_*)^{-1})^*$.

(2) The cup products on the left are given by the "naive" formula on the cochain level.

(3) It follows easily from (1), (2) and the argument of the proof of the Cup Product-Connecting Homomorphism Lemma 73.7 that the diagram to the left commutes up to the sign $(-1)^p$.

(4) The right part of the diagram commutes by functoriality of cohomology and the fact, shown in the Cohomology–LES Proposition 66.7 (2) that connecting homomorphisms are natural.

It follows from the above observations that the diagram of the lemma commutes. ■

CHAPTER 74

The Lusternik-Schnirelmann category

In this chapter we introduce and study the Lusternik-Schnirelmann category of a topological space. This is a fun topic, but it is also a slight detour from developing the general theory of cup products and cohomology. In the next Chapter 75 we will continue the development of the general theory with the introduction of the cap product.

74.1. **The parametrization number.** We start out with the following definition:

Definition. Let M be an m-dimensional smooth manifold.

(1) A parametrization for M is a smooth map $\Psi\colon V \to M$ with the following properties:
   (a) The domain V is either an open m-dimensional rectangle or all of $\mathbb{R}^m$.[575]
   (b) The map Ψ is a diffeomorphism onto its image.

(2) We say that a family $\{\Psi_i\colon V_i \to M\}_{i\in I}$ of parametrizations covers M if $\bigcup_{i\in I} \Psi_i(V_i) = M$.

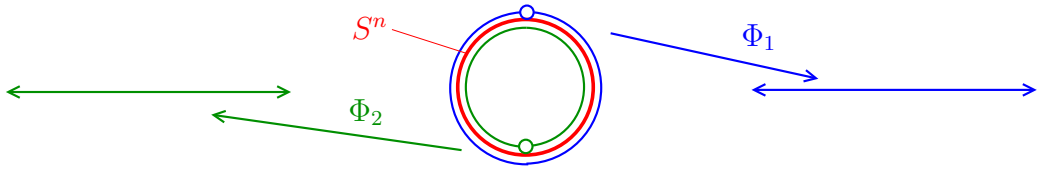
If M is a smooth manifold with empty boundary, then M can clearly be covered by a family of parametrizations. This leads us to the following definition:

Definition. Given a smooth manifold M with empty boundary we define its parametrization number as

$$\operatorname{param}(M) \;:=\; \text{minimal cardinality of a family of parametrizations that cover } M.$$

Examples.

(1) Let $N = (0,\ldots,0,1)$ be the North Pole and let $S = (0,\ldots,0,-1)$ be the South Pole of the n-dimensional sphere S^n. We can cover S^n by the two open subsets $S^n \setminus \{N\}$ and $S^n\setminus\{S\}$. Stereographic projection gives rise to two diffeomorphisms $\Phi_1\colon S^n\setminus\{N\} \to \mathbb{R}^n$ and $\Phi_2\colon S^n \setminus \{S\} \to \mathbb{R}^n$. The inverses of these two maps define parametrizations that cover S^n.



Since S^n is not diffeomorphic to any of the admissible domains of a parametrization we see that the above collection of parametrizations is optimal in the sense that we cannot parametrize S^n with fewer parametrizations. Thus we have shown that $\operatorname{param}(S^n) = 2$.

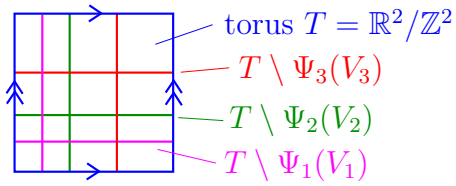
(2) We consider the n-torus $(S^1)^n =_{\text{i}} \mathbb{R}^n/\mathbb{Z}^n$. We denote by $p\colon \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$ the projection. We pick real numbers $0 < s_1 < s_2 < \cdots < s_{n+1} < 1$. For $i = 1,\ldots,n+1$ we consider

[575] The condition on V sounds slightly arbitrary, and it is, but it is reasonable that for practical purposes one prefers domains that are "easy to describe".

the map

$$\begin{aligned}\Psi_i\colon \overbrace{(s_i, s_i+1)\times\ldots\times(s_i, s_i+1)}^{=:V_i} &\to \mathbb{R}^n/\mathbb{Z}^n\\ (t_1,\ldots,t_n) &\mapsto p(t_1,\ldots,t_n).\end{aligned}$$

It is straightforward to see that the maps $\Psi_1,\ldots,\Psi_{n+1}$ are parametrizations that cover $(S^1)^n =_{\mathrm{i}} \mathbb{R}^n/\mathbb{Z}^n$. Note that this discussion shows that $\operatorname{param}((S^1)^n) \leq n+1$.



(3) For $i = 1,\ldots,n+1$ we consider the map

$$\begin{aligned}\Psi_i\colon \mathbb{R}^n &\to \mathbb{RP}^n = (\mathbb{R}^{n+1}\setminus\{0\})/\sim\\ (x_1,\ldots,x_n) &\mapsto \underbrace{[x_1:\ldots:x_{i-1}:1:x_i:\ldots:x_n]}_{\text{point in } \mathbb{RP}^n \text{ represented by } (x_1,\ldots,x_{i-1},1,x_i,\ldots,x_n)}.\end{aligned}$$

One can easily verify that the maps $\Psi_1,\ldots,\Psi_{n+1}$ are parametrizations that cover the real projective space $\mathbb{RP}^n$. This discussion shows that $\operatorname{param}(\mathbb{RP}^n) \leq n+1$.

(4) Almost the same construction as in (3) shows that we can cover the complex projective space $\mathbb{CP}^n = (\mathbb{C}^{n+1}\setminus\{0\})/\sim$ by $n+1$ parametrizations such that the domain of each parametrization is diffeomorphic to $\mathbb{C}^n \cong \mathbb{R}^{2n}$. This discussion shows that $\operatorname{param}(\mathbb{CP}^n) \leq n+1$. □

The above discussion leads us to the following question:

Question 74.1.

(1) Can we cover $(S^1)^n$ by fewer than $n+1$ parametrizations?
(2) Can we cover $\mathbb{RP}^n$ by fewer than $n+1$ parametrizations?
(3) Can we cover $\mathbb{CP}^n$ by fewer than $n+1$ parametrizations?

In the next section we will introduce a variation on the parametrization number which is defined for all topological spaces. We will then see how relative cup products can be used to give lower bounds on the parametrization number and its more general sibling.

On page 954 we will then use this approach to answer Question 74.1 for the 2-dimensional torus $S^1\times S^1$ and the real projective plane $\mathbb{RP}^2$. The other cases will have to wait till the LS Category–Torus Corollary 77.10 and the LS Category–Projective Space Proposition 81.10.

74.2. **The definition of the Lusternik-Schnirelmann category.** In the previous section we introduced the parametrization number of a smooth manifold. We now introduce a variation on this concept, which makes sense for arbitrary topological spaces:

Definition.

(1) Recall that according to the definition on page 158 a map $f\colon U \to X$ is called null-homotopic if it is homotopic to a constant map.
(2) Let X be a topological space.
  (a) A Lusternik-Schnirelmann covering of X is an open cover $\{U_i\}_{i\in I}$ of X such that each inclusion map $U_i \to X$ is null-homotopic.

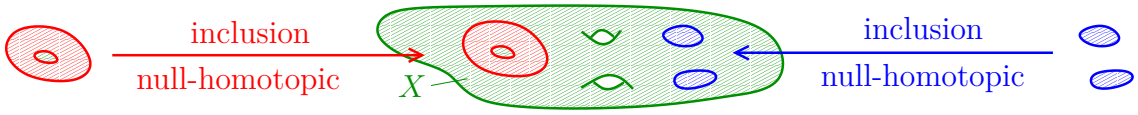
(b) If X admits a finite Lusternik-Schnirelmann covering, then we define the Lusternik-Schnirelmann category[582] $\operatorname{cat}(X) \in \mathbb{Z}_{\geq -1}$ as[583]

$$\operatorname{cat}(X) \coloneqq \min \left\{ n \in \mathbb{Z}_{\geq -1} \;\middle|\; \begin{array}{c} \text{there exists a Lusternik-Schnirelmann} \\ \text{covering with } n+1 \text{ sets} \end{array} \right\}.$$

Otherwise we set $\operatorname{cat}(X) \coloneqq \infty$.

Remark.

(1) The sets U_i in the definition of the Lusternik-Schnirelmann category do not have to be contractible. For example if X itself is contractible, then for any subset $U \subset X$ the inclusion map $U \to X$ is null-homotopic.



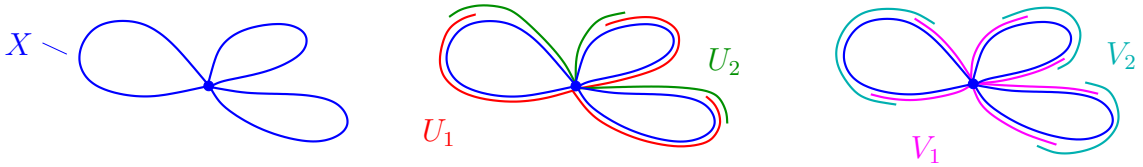
(2) The Lusternik-Schnirelmann category plays an important role in topological robotics, see e.g. [**Far08**] for more details. □

Examples.

(1) We have $\operatorname{cat}(X) = -1$ if and only if $X = \varnothing$.

(2) For a non-empty topological space X we have $\operatorname{cat}(X) = 0$ if and only if $\mathrm{id}\colon X \to X$ is null-homotopic. This holds precisely if X is contractible.

(3) (a) In the figure below we consider the wedge X of $n \geq 1$ circles together with two open covers $X = U_1 \cup U_2$ and also $X = V_1 \cup V_2$. Note that all the inclusion maps $U_i \to X$ and also $V_i \to X$ are null-homotopic. So this gives us two separate proofs that $\operatorname{cat}(X) \leq 1$. Together with Statement (2) this shows that we have $\operatorname{cat}(X) = 1$.



(b) Using the obvious generalization of the sets V_1 and V_2 shown in the figure above one can easily show that the Lusternik-Schnirelmann category of the wedge of a non-empty family of spheres of dimension ≥ 1 has Lusternik-Schnirelmann category equal to 1. □

The following lemma relates the notions of parametrization number and Lusternik-Schnirelmann category:

Lemma 74.2. (Parametrization–LS Category Lemma) For any smooth manifold M we have $\operatorname{cat}(M) \leq \operatorname{param}(M) - 1$.

---

[582]The word "category" with its modern meaning was first used by Eilenberg-Maclane [**EL45**] in 1945. On the other hand the term "Lusternik-Schnirelmann category" was already established by 1941, see [**Fox41**]. The concept goes back to the work of Lazar Lusternik and Lev Schnirelmann [**LS30, LS34**].

[583]Note the term "+1" in the definition. Sometimes, especially in the older literature the Lusternik-Schnirelmann category is defined without the adjustment given by the "+1".

Proof. Let $\Psi_i \colon V_i \to M$, $i = 1, \dots, n+1$ be a family of parametrizations that cover M. Note that each $U_i := \Psi(V_i)$ is an open subset of M. In particular $U_1, \dots, U_{n+1}$ is an open cover of M.

Since the domain of a parametrization is contractible and since a parametrization is a diffeomorphism onto its image we see that each U_i is contractible. It follows easily that each inclusion $U_i \to M$ is null-homotopic.

We have thus shown that $\operatorname{cat}(M) \leq (n+1) - 1 = n$. ■

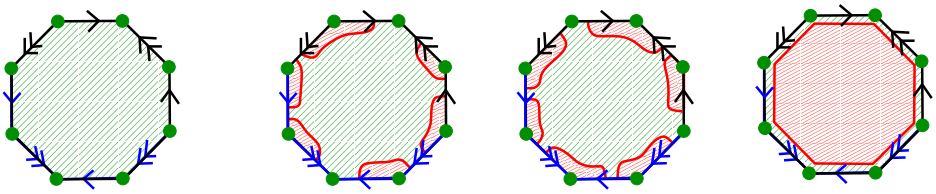
The Parametrization–LS Category Lemma 74.2 allows us to bring our expertise on parametrizations to bear:

Examples.

(4) The discussion of parametrizations of S^n on page 950 together with the above Example (2) and the Parametrization–LS Category Lemma 74.2 shows that $\operatorname{cat}(S^n) = 1$.

(5) As in (4) it follows from the above discussion of parametrizations that $\operatorname{cat}((S^1)^n) \leq n$, $\operatorname{cat}(\mathbb{R}\mathrm{P}^n) \leq n$ and also $\operatorname{cat}(\mathbb{C}\mathrm{P}^n) \leq n$.

(6) Let $g \geq 2$. The figure below convincingly shows that $\operatorname{cat}(\Sigma) \leq 2$.



Similarly one can show that for any non-orientable surface N_k of some genus $k \in \mathbb{N}$ we have $\operatorname{cat}(N_k) \leq 2$.

(7) Rather surprisingly one can generalize all of the above results on surfaces: For every connected n-dimensional smooth manifold M we have $\operatorname{cat}(M) \leq n$. We refer to [**Fri24**] for a proof of this statement. □

74.3. **Lower bounds on the Lusternik-Schnirelmann category.** As we saw in the above examples, one can obtain an upper bound on the Lusternik-Schnirelmann category of a given topological space by writing down explicit open subsets. To obtain lower bounds on the Lusternik-Schnirelmann category we introduce the following definition:

Definition. Given a non-empty topological space X we define its cup length $\operatorname{cl}(X)$ in $\mathbb{N}_0 \cup \{\infty\}$ as follows:[584]

$$\operatorname{cl}(X) := \max\left\{ n \in \mathbb{N}_0 \;\middle|\; \begin{array}{c}\text{there exists a commutative ring } R \text{ and } R\text{-cohomology} \\ \text{classes } \varphi_1, \dots, \varphi_n \text{ in degrees} \geq 1 \text{ with } \varphi_1 \cup \dots \cup \varphi_n \neq 0\end{array} \right\}.$$

Examples.

(1) It follows easily from the Wedge-of-Spheres–Cup Product Lemma 72.12 that for any wedge $X = S^{r_1} \vee \dots \vee S^{r_k}$ of finitely many spheres, where at least one sphere has dimension ≥ 1, the cup length equals 1.

[584] As usual we follow the convention that in a ring (in our case the cohomology ring), the product of the empty set of elements is the multiplicative neutral element. If X is non-empty we have $1_X \neq 0$. This shows that for a non-empty topological space X the definition actually makes sense.

(2) If X is an n-dimensional CW-complex, then it follows easily from the Singular-Cellular H^*-Isomorphism Proposition 68.2 that for any commutative ring R and any $i > n$ we have $\mathrm{H}^i(X;R) = 0$. This implies that the cup product of $\geq n+1$ cohomology classes of degree ≥ 1 is necessarily zero. In other words, this implies that $\mathrm{cl}(X) \leq n$.
(3) By the same logic as in (1) it follows from the Manifold-Top-H^*-Theorem 70.16 that for any n-dimensional topological manifold X we have $\mathrm{cl}(X) \leq n$.
(4) In the Torus–Cup Product Lemma 72.11 and the Surface–Cup Product Lemma 72.13 we saw that for any surface Σ of genus $g \geq 1$ there exist $\varphi, \psi \in \mathrm{H}^1(\Sigma;\mathbb{Z})$ with $\varphi \cup \psi \neq 0$, which together with (2) shows that $\mathrm{cl}(\Sigma) = 2$.
(5) Recall that in the Projective Plane–Cup Product Lemma 72.15 we showed that there exists a $\varphi \in \mathrm{H}_1(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2)$ with $\varphi \cup \varphi \neq 0$, which shows by (2) that $\mathrm{cl}(\mathbb{R}\mathrm{P}^2) = 2$. Similarlyone can easily show that for any $k \in \mathbb{N}$ we have for the non-orientable surface of genus k that $\mathrm{cl}(N_k) = 2$. □

The following proposition shows that the cup length is a lower bound on the Lusternik-Schnirelmann category.

Proposition 74.3. (Cup Length–LS Category Proposition) For every topological space X we have

$$\mathrm{cl}(X) \;\leq\; \mathrm{cat}(X).$$

Example. It follows immediately from the Cup Length–LS Category Proposition 74.3, the above calculations of cup lengths and the discussion on page 953 that

$$\mathrm{cat}(S^1 \times S^1) \;=\; \mathrm{cat}(\text{surface of genus } g \geq 2) \;=\; \mathrm{cat}(\mathbb{R}\mathrm{P}^2) \;=\; \mathrm{cat}(N_k) \;=\; 2.$$

Furthermore, it follows from the Parametrization–LS Category Lemma 74.2 and the examples on page 951 that $\mathrm{param}(S^1 \times S^1) = 3$ and $\mathrm{param}(\mathbb{R}\mathrm{P}^2) = 3$. □

Proof. Let X be a topological space, let R be a commutative ring and let $\varphi_j \in \mathrm{H}^{s_j}(X;R)$, $j = 1, \ldots, n$ be cohomology classes in degrees $s_j \geq 1$ such that $\varphi_1 \cup \varphi_2 \cup \cdots \cup \varphi_n \neq 0$. We have to show that $\mathrm{cat}(X) \geq n$.

Thus let $k \leq n$ and let $U_1, \ldots, U_k$ be open subsets in X such that each inclusion map $\iota_i \colon U_i \to X$ is null-homotopic. We have to show that $U_1 \cup \cdots \cup U_k$ is a *proper* subset of X.

Claim. For any $j \in \{1, \ldots, k\}$ the natural map $\text{я}_j \colon \mathrm{H}^{s_j}(X, U_j; R) \to \mathrm{H}^{s_j}(X;R)$ is an epimorphism.

Proof. We consider the long exact sequence in cohomology:

$$\mathrm{H}^{s_j}(X, U_j; R) \xrightarrow{\text{я}_j} \mathrm{H}^{s_j}(X;R) \xrightarrow[\uparrow]{\iota_j^*} \mathrm{H}^{s_j}(U_j;R).$$

since ι_j is homotopic to a constant map and since $s_j \geq 1$ it follows that this map is the zero map

Since the right map is zero, the map to the left has to be an epimorphism. ⊞

It follows from the claim that for each $j \in \{1 \ldots, k\}$ we can pick $\psi_j \in \mathrm{H}^{s_j}(X, U_j; R)$ with $\text{я}_j(\psi_j) = \varphi_j$. It follows from the Excisive Triad-Proposition 50.11 (1) and the Cup Product-Naturality Lemma 73.3 (1) (applied iteratively) that we have the following commutative

diagram:

$$\begin{array}{ccc} \mathrm{H}^{s_1}(X, U_1; R) \times \cdots \times \mathrm{H}^{s_k}(X, U_k; R) & \xrightarrow{\cup} & \mathrm{H}^{s_1+\cdots+s_k}(X, U_1 \cup \cdots \cup U_k; R) \\ \downarrow (\psi_1, \ldots, \psi_k) \\ \downarrow (\varphi_1, \ldots, \varphi_k) & & \downarrow \\ \mathrm{H}^{s_1}(X; R) \times \cdots \times \mathrm{H}^{s_k}(X; R) & \xrightarrow{\cup} & \mathrm{H}^{s_1+\cdots+s_k}(X; R). \end{array}$$

By our hypotheses the cup product $\varphi_1 \cup \cdots \cup \varphi_k$ is non-zero. This implies, by the commutativity of the diagram, that the cup product $\psi_1 \cup \cdots \cup \psi_k$ is also non-zero. But this implies that the group $\mathrm{H}^{s_1+\cdots+s_k}(X, U_1 \cup \cdots \cup U_k; R)$ is non-zero, which in turn implies that $U_1 \cup \cdots \cup U_k$ is a proper subset of X. ■

CHAPTER 75

The cap product

In this chapter we introduce the cap product that relates homology and cohomology groups. It can be viewed as a generalization of the Kronecker pairing $\mathrm{H}^k(X;R) \times \mathrm{H}_k(X;R) \to R$. Initially it might not be clear what the cap product is good for, but later on we will see that it plays an essential role in many calculations and in the formulation of the Poincaré Duality Theorem.

75.1. **The definition of the cap product.** First we recall some notation from pages 516 and 457:

Notation.
(1) For $i = 0, \dots, n$ we write $\mathfrak{v}_i := (0, \dots, 1, 0, \dots, 0) \in \mathbb{R}^{n+1}$.
(2) For $a_0, \dots, a_k \in \Delta^n$ we denote by $[a_0, \dots, a_k] \colon \Delta^k \to \Delta^n$ the unique affine linear map that sends $\mathfrak{v}_i$ to a_i.

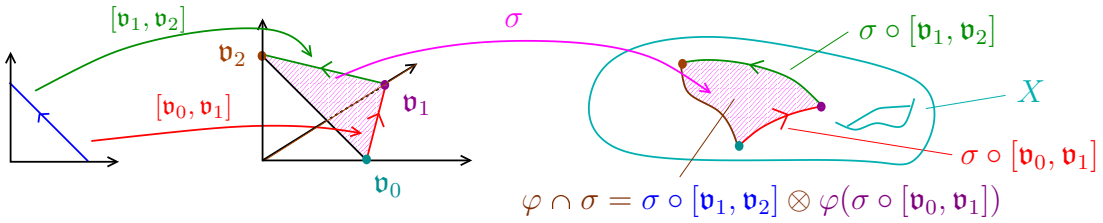
The following definition is inspired by the second definition of the cup product that we gave on page 921.

Definition. Let X be a topological space and let R be a commutative ring. Given a cochain $\varphi \in \mathrm{C}^k(X;R)$ and a singular simplex $\sigma \colon \Delta^l \to X$ with $k \le l$ we define the cap product as[585]

$$\varphi \cap \sigma := \underbrace{\sigma \circ [\mathfrak{v}_k, \dots, \mathfrak{v}_l]}_{\text{map } \Delta^{l-k} \to X} \otimes \underbrace{\varphi(\sigma \circ [\mathfrak{v}_0, \dots, \mathfrak{v}_k])}_{\in R} \in \mathrm{C}_{l-k}(X;R) = \mathrm{C}_{l-k}(X) \otimes R.$$

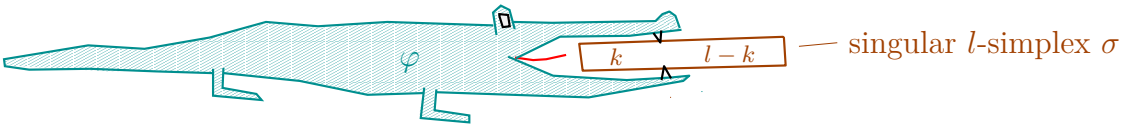
For $k > l$ we define the cap product to be the zero map. Since $\mathrm{C}^l(X)$ is the free abelian group on the set of singular l-simplices we get an induced map

$$\begin{aligned} \cap \colon \mathrm{C}^k(X;R) \times \mathrm{C}_l(X) &\to \mathrm{C}_{l-k}(X;R) \\ \Big(\varphi, \sum_{i=1}^d a_i \cdot \sigma_i\Big) &\mapsto \sum_{i=1}^d a_i \cdot \varphi \cap \sigma_i. \end{aligned}$$



[585] The notation is unfortunately slightly awkward: In the notation of the cap product $\varphi \cap \sigma$ in the literature the cochain is always to the left and the chain is always to the right. But in the tensor product $\mathrm{C}_{l-k}(X) \otimes R$ the chain is to the left and the result of the cochain is to the right.

Remark. The definition of the cap product can also be visualized by the cap product crocodile, which has the following eating habit: It bites the first k coordinates off a singular simplex, spits out $\varphi(\sigma \circ [\mathfrak{v}_0, \dots, \mathfrak{v}_k])$ and combines it with the "remaining" simplex $\sigma \circ [\mathfrak{v}_k, \dots, \mathfrak{v}_l]$.



As for the cup product, the first thing we want to verify is that this map on the cochain and chain level defines a map on the level of cohomology groups and homology groups. The key to doing so is the following elementary lemma, which can be viewed as an analogue of the Cup Product–Coboundary Lemma 72.3.

Lemma 75.1. (Cap Product–Boundary Lemma) Let X be a topological space and let R be a commutative ring. Furthermore let $\varphi \in \mathrm{C}^k(X;R)$ and let $\sigma\colon \Delta^l \to X$ be a singular simplex. Then

$$\partial_{l-k}(\varphi \cap \sigma) \;=\; (-1)^k \cdot (-\delta_k(\varphi) \cap \sigma \,+\, \varphi \cap \partial_l(\sigma)) \;\in\; \mathrm{C}_{l-k-1}(X;R).$$

Proof. For $k > l$ the statement is trivial. So assume that $k \le l$. Let $\varphi \in \mathrm{C}^k(X;R)$ and let $\sigma\colon \Delta^l \to X$ be a singular simplex. Note that

$$\begin{aligned}
\text{(a)} \qquad \varphi \cap \partial_l(\sigma) &= \sum_{i=0}^{l} (-1)^i \cdot \varphi \cap \sigma \circ [\mathfrak{v}_0, \dots, \widehat{\mathfrak{v}}_i, \dots, \mathfrak{v}_l] \\
&= \sum_{i=0}^{k} (-1)^i \cdot \sigma \circ [\mathfrak{v}_{k+1}, \dots, \mathfrak{v}_l] \otimes \varphi(\sigma \circ [\mathfrak{v}_0, \dots, \widehat{\mathfrak{v}}_i, \dots, \mathfrak{v}_{k+1}]) \\
&\qquad + \sum_{i=k+1}^{l} (-1)^i \cdot \sigma \circ [\mathfrak{v}_k, \dots, \widehat{\mathfrak{v}}_i, \dots, \mathfrak{v}_l] \otimes \varphi(\sigma \circ [\mathfrak{v}_0, \dots, \mathfrak{v}_k]) \\
\text{(b)} \qquad \delta_k(\varphi) \cap \sigma &= \sum_{i=0}^{k+1} (-1)^i \cdot \sigma \circ [\mathfrak{v}_{k+1}, \dots, \mathfrak{v}_l] \otimes \varphi(\sigma \circ [\mathfrak{v}_0, \dots, \widehat{\mathfrak{v}}_i, \dots, \mathfrak{v}_{k+1}]) \\
\text{(c)} \qquad (-1)^k \cdot \partial_{l-k}(\varphi \cap \sigma) &= \sum_{i=k}^{l} (-1)^i \cdot \sigma \circ [\mathfrak{v}_k, \dots, \widehat{\mathfrak{v}}_i, \dots, \mathfrak{v}_l] \otimes \varphi(\sigma \circ [\mathfrak{v}_0, \dots, \mathfrak{v}_k]).
\end{aligned}$$

Note that the summand of (b) corresponding to $i = k+1$ equals *minus* the summand of (c) corresponding to $i = k$. Adding (b) and (c) and comparing it to (a) we see that

$$\varphi \cap \partial_l(\sigma) \;=\; \delta_k(\varphi) \cap \sigma + (-1)^k \cdot \partial_{l-k}(\varphi \cap \sigma)$$

which gives the desired equality. ∎

The following lemma, which is the analogue of the Cup Product–Definition Lemma 72.4, says that "$\cap$" descends to a map on homology and cohomology.

Lemma 75.2. (Cap Product–Definition Lemma) Let X be a topological space and let R be a commutative ring. For any $k, l \in \mathbb{N}_0$ the map

$$\begin{aligned}
\mathrm{H}^k(X;R) \times \mathrm{H}_l(X;R) &\to \mathrm{H}_{l-k}(X;R) \\
\Big([\varphi], \Big[\sum_{i=1}^{m} \sigma_i \otimes r_i\Big]\Big) &\mapsto \Big[\sum_{i=1}^{m} \underbrace{(\varphi \cap \sigma_i)}_{\in \mathrm{C}_{l-k}(X;R)} \cdot r_i\Big]
\end{aligned}$$

is well-defined and R-bilinear.

Definition. We refer to the pairing

$$\cap\colon \mathrm{H}^k(X;R) \times \mathrm{H}_l(X;R) \ \to\ \mathrm{H}_{l-k}(X;R)$$

from the Cap Product–Definition Lemma 75.2 as the cap product.[586]

Proof. For $k > l$ the statement is trivial. So assume that $k \leq l$. As we will see, this case follows from the Cap Product–Boundary Lemma 75.1 together with routine arguments. But for peace of mind, let us go through the steps. First note that using the Tensor Product–Universal Property Lemma 51.1 (3) one can easily verify that the map

$$\begin{aligned} \cap\colon \mathrm{C}^k(X;R) \times \mathrm{C}_l(X;R) &\to \mathrm{C}_{l-k}(X;R) \\ \Big(\varphi, \sum_{i=1}^m \sigma_i \otimes r_i\Big) &\mapsto \sum_{i=1}^m \underbrace{(\varphi \cap \sigma_i)}_{\in \mathrm{C}_{l-k}(X;R)} \cdot r_i \end{aligned}$$

is well-defined. It remains to show that the above map descends to a well-defined map on (co-) homology. We start out with the following familiar looking claim.

Claim. For any $\varphi \in \mathrm{C}^k(X;R)$ and any $b \in \mathrm{C}_k(X;R)$ we have

$$\partial_{l-k}(\varphi \cap b) \ =\ (-1)^k \cdot (-\delta_k(\varphi) \cap b \ +\ \varphi \cap \partial_l(b)) \ \in\ \mathrm{C}_{l-k-1}(X;R).$$

Proof. Since "$\cap$" is evidently bilinear it suffices to show the desired equality for any $\varphi \in \mathrm{C}^k(X;R)$ and any element of $\mathrm{C}_k(X;R)$ of the form $b = \sigma \otimes r$ where $\sigma \in \mathrm{C}_k(X)$ and $r \in R$. For such element we have the following equality in $\mathrm{C}_{l-k-1}(X;R)$:

$$\begin{aligned} \varphi \cap (\sigma \otimes r) &\overset{\text{by the above definition of } \cap}{\downarrow =} (\varphi \cap \sigma) \cdot r \\ &\overset{\text{by the Cap Product–Boundary Lemma 75.1}}{\downarrow =} ((-1)^k \cdot (-\delta_k(\varphi) \cap \sigma \ +\ \varphi \cap \partial_l(\sigma))) \cdot r \\ &= (-1)^k \cdot (-\delta_k(\varphi) \cap (\sigma \otimes r) \ +\ \varphi \cap \underbrace{\partial_l(\sigma \otimes r)}_{=\partial_l(\sigma)\otimes r}). \\ &\overset{\text{by the above definition of } \cap}{\uparrow =} \end{aligned}$$

⊞

Now let $\varphi \in \mathrm{C}^k(X;R)$ be a cocycle and let $b \in \mathrm{C}_l(X;R)$ be a cycle.

(1) It follows from the claim that $\varphi \cap b$ is a cycle, i.e. it represents a homology class.

(2) If $b' = b + \partial_{l+1}(\tau)$ is another representative of $[b]$, then

$$\varphi \cap b - \varphi \cap b' \ =\ \varphi \cap (b - b') \ =\ \varphi \cap \partial_{l+1}(\tau) \underset{\text{by the claim}}{\uparrow =} (-1)^k \cdot \partial_{l+1-k}(\varphi \cap \tau).$$

This shows that $\varphi \cap b$ and $\varphi \cap b'$ represent the same homology class, i.e. the cap product does not depend on the choice of the representative of the homology class.

(3) The proof that the cap product does not depend on the choice of representative of the cohomology class is basically the same as the proof of (2).

This shows that the map is well-defined. It is clear that the map is indeed R-bilinear. ■

[586] The name "cap product" comes from the notation "$a \cap b$" for the composition since $\cap$ looks like, just a little bit, like the "a" in cap. The other reason is that the cup product was first introduced, and mathematicians needed a different name for a somewhat related object. Also mathematicians are well-known for their great sense of humor.

75.2. **Calculations of cap products.** In this section we consider the cap product on the following topological spaces:

(1) The torus with $\mathbb{Z}$-coefficients.
(2) The real projective plane with $\mathbb{F}_2$-coefficients.

75.2.1. **Cap product on the torus.** In this subsection we will calculate the cap product on the torus:

Lemma 75.3. (Torus–Cap Product Lemma) Let $T = S^1 \times S^1$ be the torus. We recall the following notations and results:

- Let p and q be the projections $S^1 \times S^1 \to S^1$ onto the first respectively second factor.
- We denote by $\theta_{\mathbb{Z}} \in \mathrm{C}^1(S^1;\mathbb{Z})$ the singular 1-cocycle that we introduced on page 866.
- We denote by $[\{1\} \times S^1]$ the image of the fundamental class of $\{1\} \times S^1$ under the monomorphism $\mathrm{H}_1(\{1\} \times S^1;\mathbb{Z}) \to \mathrm{H}_1(S^1 \times S^1;\mathbb{Z})$.

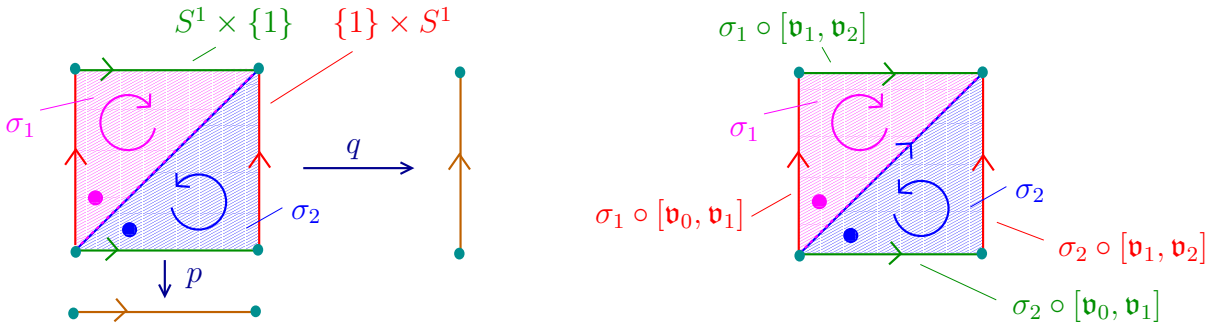
We have
$$\begin{array}{rcl} [p^*(\theta_{\mathbb{Z}})] \cap [T] & = & [\{1\} \times S^1] \\ [q^*(\theta_{\mathbb{Z}})] \cap [T] & = & -[S^1 \times \{1\}] \end{array}$$

In particular the map
$$\begin{array}{rcl} \mathrm{H}^1(T;\mathbb{Z}) & \to & \mathrm{H}_1(T;\mathbb{Z}) \\ \varphi & \mapsto & \varphi \cap [T] \end{array}$$

is an isomorphism.

Proof. We consider $T = S^1 \times S^1 =_{\mathrm{i}} ([0,1] \times [0,1])/\sim$. We introduce some more notation:

(1) We denote by σ_1, σ_2 the singular 2-simplices from the figure below. As we pointed out on page 835, the singular 2-chain $-\sigma_1 + \sigma_2$ is a representative of the fundamental class $[T]$.
(2) We denote by $\mu\colon \Delta^1 \to S^1$ the singular 1-simplex that is given by $\mu(1-t,t) = \exp(2\pi\,\mathrm{i}t)$. One can easily verify that μ represents the fundamental class $[S^1]$ of S^1 equipped with the standard orientation.



We calculate that
$$p^*(\theta_{\mathbb{Z}}) \cap \sigma_1 \;=\; \sigma_1 \circ [\mathfrak{v}_1, \mathfrak{v}_2] \otimes \overbrace{p^*(\theta_{\mathbb{Z}})(\sigma_1 \circ [\mathfrak{v}_0, \mathfrak{v}_1])}^{=\theta_{\mathbb{Z}}(p \circ \sigma_1 \circ [\mathfrak{v}_0, \mathfrak{v}_1]) = 0} \;=\; 0$$
and
$$p^*(\theta_{\mathbb{Z}}) \cap \sigma_2 \;=\; \sigma_2 \circ [\mathfrak{v}_1, \mathfrak{v}_2] \otimes \underbrace{p^*(\theta_{\mathbb{Z}})(\sigma_2 \circ [\mathfrak{v}_0, \mathfrak{v}_1])}_{=\theta_{\mathbb{Z}}(p \circ \sigma_2 \circ [\mathfrak{v}_0, \mathfrak{v}_1]) = 1} \;=\; \sigma_2 \circ [\mathfrak{v}_1, \mathfrak{v}_2] \;=\; 1 \times \mu.$$

Putting these calculations together we obtain that

$$[p^*(\theta_{\mathbb{Z}})] \cap [T] \;=\; [p^*(\theta_{\mathbb{Z}}) \cap (-\sigma_1 + \sigma_2)] \;=\; [\{1\} \times S^1].$$

Basically the same calculation shows that

$$[q^*(\theta_{\mathbb{Z}})] \cap [T] \;=\; [q^*(\theta_{\mathbb{Z}}) \cap (-\sigma_1 + \sigma_2)] \;=\; -[S^1 \times \{1\}].$$

Since $[p^*(\theta_{\mathbb{Z}})]$ and $[q^*(\theta_{\mathbb{Z}})]$ form a basis of $\mathrm{H}^1(T;\mathbb{Z})$ and since $[\{1\} \times S^1]$ and $[S^1 \times \{1\}]$ form a basis of $\mathrm{H}_1(T;\mathbb{Z})$ we have thus shown that the map $\varphi \mapsto \varphi \cap [T]$ is indeed an isomorphism. ■

Remark. Let Σ_2 be the surface of genus 2. We leave it to the reader to show that, similar to the Torus–Cap Product Lemma 75.3, the map

$$\begin{array}{rcl} \mathrm{H}^1(\Sigma_2;\mathbb{Z}) & \to & \mathrm{H}_1(\Sigma_2;\mathbb{Z}) \\ \varphi & \mapsto & \varphi \cap [\Sigma_2] \end{array}$$

is an isomorphism.

75.2.2. **The cap product on the real projective plane.** In this subsection we consider the real projective plane $\mathbb{R}\mathrm{P}^2$:

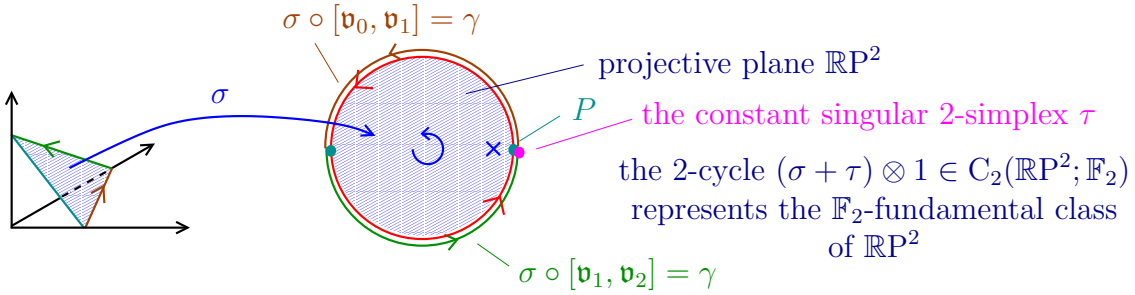
Lemma 75.4. (Projective Plane–Cap Product Lemma) The map

$$\begin{array}{rcl} \mathrm{H}^1(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2) & \to & \mathrm{H}_1(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2) \\ \varphi & \mapsto & \varphi \cap [\mathbb{R}\mathrm{P}^2]_{\mathbb{F}_2} \end{array}$$

is an isomorphism.

Proof. We consider $\mathbb{R}\mathrm{P}^2 =_{\mathrm{i}} \overline{B}^2/\!\sim$. We introduce the following notation:

- We denote by $\varphi \in \mathrm{C}^1(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2)$ the cocycle that we introduced in the Generator-of-$\mathrm{H}^1(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2)$-Lemma 72.14.
- Let $\gamma\colon \Delta^1 \to \mathbb{R}\mathrm{P}^2 =_{\mathrm{i}} \overline{B}^2/\!\sim$ be the singular 1-simplex that is given by $\gamma(t,1-t) = [e^{\pi i t}]$. Note that γ is a cycle in $\mathrm{C}_1(\mathbb{R}\mathrm{P}^2)$ and note that $\varphi(\gamma) = 1$.
- In the figure below we see the singular 2-simplex σ that sends the face $[\mathfrak{v}_0,\mathfrak{v}_2]$ to a point P and the singular 2-simplex τ that sends all points on Δ^2 to P. It is straightforward to see that $(\sigma+\tau)\otimes 1 \in \mathrm{C}_2(\mathbb{R}\mathrm{P}^2;\mathbb{F}_2)$ is a cycle. Using the $\mathbb{F}_2$-Fundamental Class-Criterion 64.4 we see that it represents the $\mathbb{F}_2$-fundamental class of $\mathbb{R}\mathrm{P}^2$.



It is straightforward to see that

$$[\varphi] \cap [\mathbb{R}\mathrm{P}^2]_{\mathbb{F}_2} \;=\; [\varphi] \cap [(\sigma+\tau)\otimes 1] \;=\; [\underbrace{(\varphi\cap\sigma)}_{=\gamma}\otimes 1 + \underbrace{(\varphi\cap\tau)}_{=0}\otimes 1] \;=\; [\gamma\otimes 1].$$

It follows from $\varphi(\gamma) = 1$ that $[\gamma \otimes 1]$ is the (unique) non-trivial element of $\mathrm{H}_1(\mathbb{R}\mathrm{P}^2; \mathbb{F}_2)$. ■

Remark. In the three examples above we saw that for the closed 2-dimensional smooth manifolds $F = S^1 \times S^1$, $F = \Sigma_2$, $F = \mathbb{R}\mathrm{P}^2$ and for $R = \mathbb{Z}$ respectively $R = \mathbb{F}_2$ the map

$$\begin{array}{rcl} \mathrm{H}^1(F; R) & \to & \mathrm{H}_{2-1}(F; R) \\ \varphi & \mapsto & \varphi \cap [F]_R \end{array}$$

is an isomorphism. Later we will see that this is not a coincidence. More precisely we will see that this observation can be generalized to all closed topological manifolds. □

75.3. **The relative cap product.** After considering the various examples of the cap product we continue with developing the theory.

Similar to the case of the cup product, we intend to extend the notion of cap product to relative homology and cohomology groups. As in the case of the cup product, this requires some delicate footwork to get to the right notion of a relative cap product. We start out with the following rather technical definition.

Definition. Let (X, A, B) be a triad of topological spaces and let R be a commutative ring.

(1) We write $\mathrm{C}_n^{\{A,B\}}(A \cup B) := \mathrm{C}_n(A) + \mathrm{C}_n(B) \subset \mathrm{C}_n(X)$.

(2) We write $\mathrm{C}_n(X, \{A, B\}; R) := \mathrm{C}_n(X, \{A, B\}) \otimes R = \left(\frac{\mathrm{C}_n(X)}{\mathrm{C}_n^{\{A,B\}}(A \cup B)}\right) \otimes R$.

It follows from the Functor-applied-to-Chain Complex Lemma 52.2 and the Tensor–Functor Lemma 52.3 that the usual boundary map on the chain complex $\mathrm{C}_*(X; R)$ defines a boundary map on $\mathrm{C}_*(X, \{A, B\}; R)$. We denote the corresponding homology groups by $\mathrm{H}_*(X, \{A, B\}; R)$.

This technical definition allows us to formulate the following lemma, which is a generalization of the Cap Product–Definition Lemma 75.2 and which can be viewed as an analogue of the Triad–Cup Product Lemma 73.2:

Lemma 75.5. (Triad–Cap Product Lemma) Let (X, A, B) be a triad of topological spaces, let $k \leq l$ and let R be a commutative ring.

(1) The map

$$\begin{array}{rcl} \cap \colon \mathrm{C}^k(X, A; R) \times \mathrm{C}_l(X, \{A, B\}) & \to & \mathrm{C}_{l-k}(X, B; R) \\ (\varphi, [\sigma \colon \Delta^l \to X]) & \mapsto & \underbrace{\sigma \circ [\mathfrak{v}_k, \ldots, \mathfrak{v}_l]}_{\text{map } \Delta^{l-k} \to X} \otimes \underbrace{\varphi(\sigma \circ [\mathfrak{v}_0, \ldots, \mathfrak{v}_k])}_{\in R} \end{array}$$

is well-defined.

(2) The map

$$\begin{array}{rcl} \mathrm{H}^k(X, A; R) \times \mathrm{H}_l(X, \{A, B\}; R) & \to & \mathrm{H}_{l-k}(X, B; R) \\ \left([\varphi], \left[\sum_{i=1}^m \sigma_i \otimes r_i\right]\right) & \mapsto & \left[\sum_{i=1}^m \underbrace{(\varphi \cap \sigma_i)}_{\in \mathrm{C}_{l-k}(X;R)} \cdot r_i\right] \end{array}$$

is well-defined and R-bilinear.

Proof.

(1) By definition of $\mathrm{C}_l(X, \{A, B\}) := \frac{\mathrm{C}_l(X)}{\mathrm{C}_l^{\{A,B\}}(A \cup B)} = \frac{\mathrm{C}_l(X)}{\mathrm{C}_l(A) + \mathrm{C}_l(B)}$ it remains to prove the following claim:

Claim. If we are given a singular l-simplex $\sigma\colon \Delta^l \to X$ such that
(a) the image of σ lies in A, or
(b) the image of σ lies in B,
then $\sigma \circ [\mathfrak{v}_0, \dots, \mathfrak{v}_k] \otimes \varphi(\sigma \circ [\mathfrak{v}_0, \dots, \mathfrak{v}_k])$ is zero in $\mathrm{C}_{l-k}(X, B; R) = \mathrm{C}_{l-k}(X, B) \otimes R$.

Proof. But this is indeed the case:
(a) If the image σ lies in A, then it follows from the hypothesis that $\varphi \in \mathrm{C}^k(X, A; R)$, that $\varphi(\sigma \circ [\mathfrak{v}_0, \dots, \mathfrak{v}_k]) = 0$.
(b) Now we assume that the image of σ lies in B. This implies that the image of $\sigma \circ [\mathfrak{v}_0, \dots, \mathfrak{v}_k]$Flies in B. Thus the factor $\sigma \circ [\mathfrak{v}_0, \dots, \mathfrak{v}_k]$ is zero in $\mathrm{C}_{l-k}(X, B)$. ⊞
(2) The argument that the map on (co-) chain complexes descends to a map on the level of homology and cohomology groups is the same as in the proof of the Cap Product–Definition Lemma 75.2. Finally note that it is obvious that the map $\cap$ is R-bilinear. ■

The Excisive Triad–Invariants Lemma 73.1 (2a) leads to the following definition:

Definition. Let (X, A, B) be an excisive triad of topological spaces, let $k \leq l$ and let R again be a commutative ring. We refer to the map

$$\mathrm{H}^k(X, A; R) \times \mathrm{H}_l(X, A \cup B; R) \xleftarrow[\uparrow]{\cong} \mathrm{H}^k(X, A; R) \times \mathrm{H}_l(X, \{A, B\}; R) \xrightarrow[\uparrow]{\cap} \mathrm{H}_{l-k}(X, B; R)$$

isomorphism given by the Excisive Triad–Invariants Lemma 73.1 (2a), here we use that (X, A, B) is excisive

map from the Triad–Cap Product Lemma 75.5

as the cap product (sometimes we refer to it as the relative cap product) that we also denote by "$\cap$". For $k > l$ we define the cap product to be the zero map.

Remark. [587] Let X be a topological space, let $A, B \subset X$ be subsets and let R be a commutative ring. Suppose that $\mathrm{C}_*^{\{A,B\}}(A \cup B) = \mathrm{C}_*(A \cup B)$. (This is for example the case if we are in the setting of the Excisive Triad-Proposition 50.11 (0), e.g. if $A = \varnothing$, or $B = \varnothing$ or $A = B$.) Then in the above definition the horizontal map to the left is the identity and we see that the relative cap product is just given by the "naive" definition

$$\begin{array}{rcl} \mathrm{C}^k(X, A; R) \times \mathrm{C}_l(X, A \cup B; R) & \to & \mathrm{C}^{l-k}(X, B; R) \\ (\varphi, \sigma\colon \Delta^l \to X) & \mapsto & \sigma \circ [\mathfrak{v}_k, \dots, \mathfrak{v}_l] \otimes \varphi(\sigma \circ [\mathfrak{v}_0, \dots, \mathfrak{v}_k]). \end{array}$$

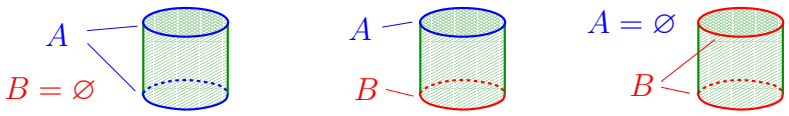
□

Again we formulate our main example as a lemma:

Lemma 75.6. (Annulus–Cap Product Lemma) Let $X = S^1 \times [0, 1]$ be the annulus. Let $A, B \subset \partial X$ be unions of boundary components such that $A \cup B = \partial X$ and $A \cap B = \varnothing$. Then the map

$$\begin{array}{rcl} \mathrm{H}^1(X, A; \mathbb{Z}) & \to & \mathrm{H}_1(X, B; \mathbb{Z}) \\ \varphi & \mapsto & \varphi \cap [X] \end{array}$$

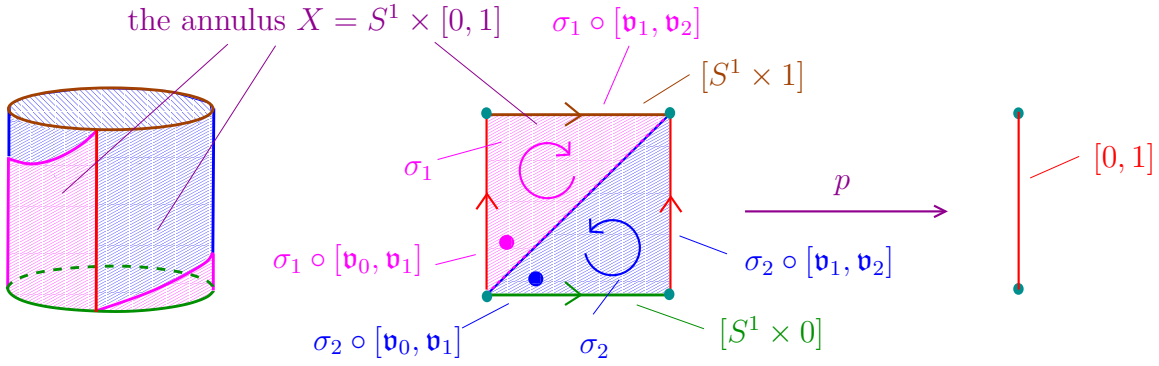
is an isomorphism.



[587] This remark is the analogue of the remark on page 943.

Proof. Let $X = S^1 \times [0,1]$ be the annulus. There are several options for A and B:

(1) First we assume that $A = \partial X$ and $B = \varnothing$. We will use the following notation:
- In the figure below we show two singular 2-simplices σ_1, σ_2. It follows easily from the Fundamental Class-Criterion 64.3 that $(-\sigma_1 + \sigma_2) \otimes 1$ represents the fundamental class $[X] \in \mathrm{H}_2(X, \partial X; \mathbb{Z})$ of the annulus X with the usual orientation.
- We denote by $p\colon X \to [0,1]$ the projection.
- Let $\overline{\gamma}_{\frac{1}{2}} \in \mathrm{C}^1([0,1], \{0,1\}, \mathbb{Z})$ be the relative 1-cocycle from pages 869 and 874.



We obtain that

$$\begin{aligned} [p^*(\overline{\gamma}_{\frac{1}{2}})] \cap \sigma_1 &= \overbrace{\sigma_1 \circ [\mathfrak{v}_1, \mathfrak{v}_2]}^{=[S^1 \times 1]} \otimes \overbrace{\overline{\gamma}_{\frac{1}{2}}(p \circ \sigma_1 \circ [\mathfrak{v}_0, \mathfrak{v}_1])}^{=1} &= [S^1 \times 1] \\ [p^*(\overline{\gamma}_{\frac{1}{2}})] \cap \sigma_2 &= \sigma_2 \circ [\mathfrak{v}_1, \mathfrak{v}_2] \otimes \underbrace{\overline{\gamma}_{\frac{1}{2}}(p \circ \sigma_2 \circ [\mathfrak{v}_0, \mathfrak{v}_1])}_{=0} &= 0. \end{aligned}$$

Since $[X] = (-\sigma + \tau) \otimes 1$ we obtain from these calculations that

$$[p^*(\overline{\gamma}_{\frac{1}{2}})] \cap [X] = [p^*(\overline{\gamma}_{\frac{1}{2}}) \cap (-\sigma_1 + \sigma_2)] = -[S^1 \times 1].$$

Since the inclusion $S^1 \times 1 \to X$ is a homotopy equivalence we see that $\mathrm{H}_1(X; \mathbb{Z}) \cong \mathbb{Z}$ and that $[S^1 \times 1]$ is a generator. A straightforward argument, e.g. using the long exact sequence of the pair $(X, \partial X)$, shows that $\mathrm{H}^1(X, \partial X; \mathbb{Z}) \cong \mathbb{Z}$. The above calculation thus shows that the map

$$\begin{aligned} \mathrm{H}^1(X, \partial X; \mathbb{Z}) &\to \mathrm{H}_1(X; \mathbb{Z}) \\ \varphi &\mapsto \varphi \cap [X] \end{aligned}$$

is an epimorphism. Since both groups are isomorphic to $\mathbb{Z}$ we see that the map is in fact an isomorphism and as a bonus we obtain that $[p^*(\overline{\gamma}_{\frac{1}{2}})] \in \mathrm{H}^1(X, \partial X; \mathbb{Z})$ is a generator.

(2) The case $A = \varnothing$ and $B = \partial X$ follows from a similar argument as in (1).

(3) The case that A and B both consist of a single boundary component is also left as a little exercise to the reader.. ∎

75.4. **Properties of the cap product.** The following lemma shows that for $k = l$ the cap product is essentially the same as the Kronecker pairing.

Lemma 75.7. (Cap Product-Kronecker Lemma) Let X be a topological space, let $A \subset X$ and let R be a commutative ring. The composition

$$\mathrm{H}^k(X, A; R) \times \mathrm{H}_k(X, A; R) \xrightarrow{\cap} \mathrm{H}_0(X; R) \xrightarrow[\text{from page 660}]{\text{augmentation } \epsilon_X} R$$

is precisely the Kronecker pairing $\langle\ ,\ \rangle_{(X,A)}$ from pages 882 and 910. In other words, for all $\varphi \in \mathrm{H}^k(X,A;R)$ and $\sigma \in \mathrm{H}_k(X,A;R)$ we have

$$\epsilon_X(\varphi \cap \sigma) \ = \ \langle \varphi, \sigma \rangle_{(X,A)}.$$

Proof. The lemma follows almost immediately from the definitions. For peace of mind, let u provide the details. Let $\varphi \in \mathrm{C}^k(X,A;R)$ by a cocycle and let $b = \sum_{i=1}^{m} \sigma_i \otimes r_i \in \mathrm{C}_k(X,A;R)$ be a cycle. We see that

$$\begin{array}{rcccl} \epsilon_X([\varphi] \cap [b]) & \overset{\text{follows from the discussion on page 962}}{=} & \epsilon_X\Big(\sum_{i=1}^{k} (\varphi \cap \sigma_i) \cdot r_i \Big) & = & \epsilon_X\Big(\sum_{i=1}^{k} \sigma([\mathfrak{v}_0]) \otimes \varphi(\sigma_i \circ \underbrace{[\mathfrak{v}_0, \ldots, \mathfrak{v}_k]}_{=\mathrm{id}_{\Delta^k}}) \cdot r_i \Big) \\ & = & \sum_{i=1}^{k} \varphi(\sigma_i) \cdot r_i & = & \langle [\varphi], [b] \rangle_{(X,A)}. \end{array}$$

■

Example. Let M be a compact oriented path-connected non-empty n-dimensional topological manifold. We calculate that

$$\epsilon_M([M]^* \cap [M]) \ \underset{\substack{\uparrow \\ \text{by the Cap Product-Kronecker Lemma 75.7}}}{=} \ \langle [M]^*, [M] \rangle \ \underset{\substack{\uparrow \\ \text{by the definition of } [M]^* \text{ on page 909}}}{=} \ 1.$$

Since M is path-connected we know that $\epsilon_M \colon \mathrm{H}_0(M;\mathbb{Z}) \to \mathbb{Z}$ is an isomorphism. Since for any $P \in M$ we have $\epsilon_M(\{P\} \otimes 1) = 1$ we now see that $[M]^* \cap [M] = [\{P\} \otimes 1]$. Running the argument backwards we see that $[M]^*$ is the only element in $\mathrm{H}^n(M, \partial M; \mathbb{Z}) \cong \mathbb{Z}$ that has that property. □

The following lemma can be viewed as an analogue of the Cup Product-Naturality Lemmas 72.2 and 73.3:

Lemma 75.8. (Cap Product-Naturality Lemma) Let R be a commutative ring. If $f \colon (X,A,B) \to (\widetilde{X}, \widetilde{A}, \widetilde{B})$ is a continuous map between two excisive triads, then for any $k, l \in \mathbb{N}_0$ the following diagram commutes:[588]

$$\begin{array}{ccccc} \mathrm{H}^k(X,A;R) & \times & \mathrm{H}_l(X, A \cup B; R) & \xrightarrow{\ \cap\ } & \mathrm{H}_{l-k}(X,B;R) \\ {\scriptstyle f^*}\uparrow & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} \\ \mathrm{H}^k(\widetilde{X}, \widetilde{A}; R) & \times & \mathrm{H}_l(\widetilde{X}, \widetilde{A} \cup \widetilde{B}; R) & \xrightarrow{\ \cap\ } & \mathrm{H}_{l-k}(\widetilde{X}, \widetilde{B}; R), \end{array}$$

i.e. for any $\varphi \in \mathrm{H}^k(\widetilde{X}, \widetilde{A}; R)$ and any $\sigma \in \mathrm{H}_l(X, A \cup B; R)$ we have

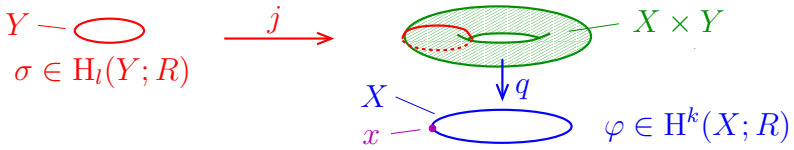
$$\varphi \cap f_*(\sigma) \ = \ f_*(\underbrace{f^*(\varphi) \cap \sigma}_{\in \mathrm{H}_{l-k}(X,B;R)}) \ \in \ \mathrm{H}_{l-k}(\widetilde{X}, \widetilde{B}; R).$$

Proof. Similar to the proof of the Cup Product-Naturality Lemma 73.3 the statement follows easily from the definitions. We leave it to the reader to fill in the details. ■

In Section 73.4 we saw that if we apply the Cup Product-Naturality Lemmas 72.2 and 73.3 judiciously, then it can be quite useful for calculating cup products. In a similar vein the Cap Product-Naturality Lemma 75.8 can also be helpful in the calculation of cap products:

[588] Here "the diagram commutes" is understood in the sense of the convention from page 948.

Example. Let X, Y be topological spaces, let $x \in X$ and let R be a commutative ring. We denote by $p\colon X \times Y \to X$ the natural projection and we denote by $j\colon Y \to X \times Y$ the embedding that is given by $y \mapsto (x, y)$. Let $\varphi \in \mathrm{H}^k(X; R)$ and let $\sigma \in \mathrm{H}_l(Y; R)$.



By the Cap Product-Naturality Lemma 75.8 we have the following commutative diagram:

$$\begin{array}{ccccc} \mathrm{H}^k(Y;R) & \times & \overset{\sigma}{\mathrm{H}_l(Y;R)} & \xrightarrow{\quad\cap\quad} & \mathrm{H}_{l-k}(Y;R) \\ {\scriptstyle j^*}\uparrow & & \downarrow{\scriptstyle j_*} & & \downarrow{\scriptstyle j_*} \\ \mathrm{H}^k(X\times Y;R) & \times & \mathrm{H}_l(X\times Y;R) & \xrightarrow{\quad\cap\quad} & \mathrm{H}_{l-k}(X\times Y;R). \\ {\scriptstyle p^*(\varphi)} & & & & \end{array}$$

It follows that for $k \geq 1$ we have

$$p^*(\varphi) \cap j_*(\sigma) \underset{\substack{\uparrow \\ \text{by the above} \\ \text{commutative diagram}}}{=} j_*(j^*(p^*(\varphi)) \cap \sigma) = j_*(\underbrace{(\underbrace{p \circ j}_{\text{constant map}})^*}_{\text{zero map since } k \geq 1}(\varphi) \cap \sigma) = 0.$$

□

75.5. **Homology as a module over the cohomology ring.** The following lemma relates the cup product to the cap product.

Lemma 75.9. (Cup-Cap Lemma) Let X be a topological space and let R be a commutative ring. Let $A \subset X$ be a subset and let $C, D \in \{\varnothing, A\}$. For any $\varphi \in \mathrm{H}^k(X, C; R)$, any $\psi \in \mathrm{H}^l(X, D; R)$ and any $\sigma \in \mathrm{H}_n(X, C \cup D; R)$ we have

$$\varphi \cap (\underbrace{\psi \cap \sigma}_{\in \mathrm{H}_{n-l}(X,C;R)}) = (\underbrace{\psi \cup \varphi}_{\in \mathrm{H}^{k+l}(X,C\cup D;R)}) \cap \sigma \quad \in \mathrm{H}_{n-k-l}(X; R).$$

Proof. Let X be a topological space and let R be a commutative ring. We start out with a claim on the (co-) chain level:

Claim. Let $\varphi \in \mathrm{C}^k(X; R)$ and let $\psi \in \mathrm{C}^l(X; R)$. Given any singular n-simplex $\sigma\colon \Delta^n \to X$ we have

$$\varphi \cap (\psi \cap \sigma) = (\psi \cup \varphi) \cap \sigma \in \mathrm{C}_{n-k-l}(X; R).$$

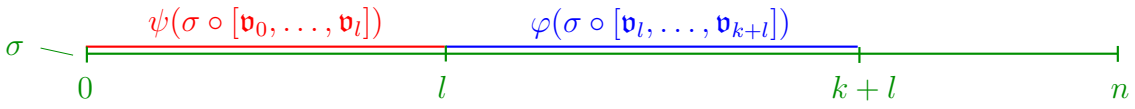
Proof. If $k + l > n$, then both sides are by definition 0. So suppose that $k + l \leq n$. We then see that

$$\begin{aligned} \varphi \cap (\psi \cap \sigma) &= \varphi \cap (\sigma \circ [\mathfrak{v}_l, \ldots, \mathfrak{v}_n] \otimes \psi(\sigma \circ [\mathfrak{v}_0, \ldots, \mathfrak{v}_l])) \\ &= (\sigma \circ [\mathfrak{v}_{l+k}, \ldots, \mathfrak{v}_n] \otimes \varphi(\sigma \circ [\mathfrak{v}_l, \ldots, \mathfrak{v}_{l+k}])) \cdot \psi(\sigma \circ [\mathfrak{v}_0, \ldots, \mathfrak{v}_l]) \\ &= F\sigma \circ [\mathfrak{v}_{l+k}, \ldots, \mathfrak{v}_n] \otimes \underbrace{\psi(\sigma \circ [\mathfrak{v}_0, \ldots, \mathfrak{v}_l]) \cdot \varphi(\sigma \circ [\mathfrak{v}_l, \ldots, \mathfrak{v}_{l+k}])}_{=(\psi\cup\varphi)(\sigma\circ[\mathfrak{v}_0,\ldots,\mathfrak{v}_{l+k}])} \\ &\underset{\substack{\uparrow \\ \text{since } R \text{ is commutative}}}{=} (\psi \cup \varphi) \cap \sigma. \end{aligned}$$

⊞

The above argument gets summarized in the following figure:

Now we turn to the proof of the desired statement. Let $A \subset X$ be a subset and let $C, D \in \{\varnothing, A\}$. Furthermore let $[\varphi] \in \mathrm{H}^k(X, C; R)$, $[\psi] \in \mathrm{H}^l(X, D; R)$ and finally let



$\left[\sum_{i=1}^{d} a_i \cdot \sigma_i\right] \in \mathrm{H}_n(X, C \cup D; R)$. Since $C, D \in \{\varnothing, A\}$ we obtain from the discussions on pages 962 and 943 that in all four calculations of cup and cap products we can deal with the "naive" definition on the (co-) chain level. The desired then follows immediately from the claim. ■

Lemma 75.10. (Cap-with-1-Lemma) Let X be a topological space, let $A \subset X$ be a subspace and let R be a commutative ring. We denote by $1_X \in \mathrm{H}^0(X; R)$ the multiplicatively neutral element from the Cup Product-Ring Proposition 72.9. For any $\sigma \in \mathrm{H}_k(X; R)$ we have

$$1_X \cap \sigma \;=\; \sigma.$$

Proof. The lemma follows immediately from the definitions. ■

One can summarize the above two lemmas in slightly more fancy language. To do so we need to introduce the notion of an S-right module:

Definition. Let S be a ring. An S-right module is an abelian group A together with a bilinear map

$$\begin{array}{rcl} S \times A & \to & A \\ (s, a) & \mapsto & a \cdot s \end{array}$$

such that the following conditions are satisfied:

$$\begin{array}{lrcll} \text{(a)} & (a \cdot s) \cdot \widetilde{s} & = & a \cdot (s \cdot \widetilde{s}) & \text{for all } s, \widetilde{s} \in S \text{ and } a \in A, \\ \text{(b)} & a \cdot 1 & = & a & \text{for all } a \in A. \end{array}$$

Lemma 75.11. (Homology-as-Module Lemma) Let X be a topological space, let $A \subset X$ be a subspace and let R be a commutative ring. The map

$$\begin{array}{rcl} \mathrm{H}^*(X; R) \times \mathrm{H}_*(X, A; R) & \to & \mathrm{H}_*(X, A; R) \\ (\varphi, \sigma) & \mapsto & \varphi \cap \sigma \end{array}$$

turns $\mathrm{H}_*(X, A; R) := \bigoplus_{n \in \mathbb{N}_0} \mathrm{H}_*(X, A; R)$ into a right module over the ring $(\mathrm{H}^*(X; R), \cup)$.

Proof. This statement follows immediately from the bilinearity of the cap product, the Cup-Cap Lemma 75.9 and the Cap-with-1-Lemma 75.10. ■

75.6. **Cap product and connecting homomorphisms.** In this last section we will relate in two lemmas the relative cap product to (co-) boundary maps on (co-) homology. These results are rather technical, and perhaps it is best to look them up, once they get actually used.

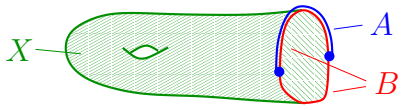
We start out with the first of the two lemmas:

Lemma 75.12. (Cap Product-Triple-Connecting Homomorphism Lemma) Let R be a commutative ring and let (X, B, A) be a triple of topological spaces. For any

$n, k \in \mathbb{N}_0$ the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}^k(B, A; R) \times \mathrm{H}_{n-1}(B, A; R) & \xrightarrow{\cap} & \mathrm{H}_{n-k-1}(B; R) \\ \downarrow{\scriptstyle \delta_k} & & \downarrow{\scriptstyle i_*} \\ \mathrm{H}^{k+1}(X, B; R) \times \mathrm{H}_n(X, B; R) & \xrightarrow{\cap} & \mathrm{H}_{n-k-1}(X; R). \end{array}$$

Here $i\colon B \to X$ denotes the inclusion map and δ_k and ∂_n denote the connecting homomorphisms in the long exact sequences in (co-) homology corresponding to the triple (X, B, A).



triple to which we will apply the Cap Product-Triple-Connecting Homomorphism Lemma 75.12

Proof. Let $\psi \in \mathrm{H}^k(B, A; R)$ and $\tau \in \mathrm{H}_n(B, A; R)$. We make the following preparations:

- We pick $\sigma \in \mathrm{C}_n(B; R)$ such that $[\sigma] \in \mathrm{C}_n(B, A; R)$ is a cycle that represents τ.
- We pick $\varphi \in \mathrm{C}^k(B; R) = \mathrm{Hom}(\mathrm{C}_k(B), R)$ that vanishes on $\mathrm{C}_k(A)$ and such that the corresponding element in $\mathrm{C}^k(B, A; R)$ represents ψ. We denote by $\widetilde{\varphi}\colon \mathrm{C}_k(X) \to R$ the extension of φ that is defined to be zero on all singular k-simplices whose image does not lie in B.

We obtain that (follows from the discussion of relative cap products on page 962)

$$\delta\psi \cap \tau \underset{\uparrow}{=} [\delta\widetilde{\varphi}] \cap [\sigma] \overset{\downarrow}{=} [\delta\widetilde{\varphi} \cap \sigma] \underset{\uparrow}{=} [\widetilde{\varphi} \cap \partial\sigma] = [\varphi \cap \partial\sigma] \overset{\downarrow}{=} [\varphi] \cap [\partial\sigma] \underset{\uparrow}{=} \psi \cap \partial\tau.$$

by the LES-of-Triple Proposition 67.9 and the discussion on page 872

by the Cap Product–Boundary Lemma 75.1 the classes $\delta\widetilde{\varphi} \cap \sigma$ and $\widetilde{\varphi} \cap \partial\sigma$ are homologous

by the Topological LES Proposition 38.2

This equality implies that the diagram commutes.[589] ∎

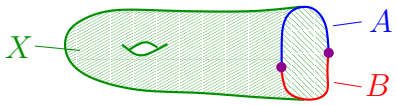
The following lemma is similar in spirit to the Cup Product-Connecting Homomorphism Lemma 73.7:

Lemma 75.13. (Cap Product-Triad-Connecting Homomorphism Lemma) Let R be a commutative ring, let (X, A, B) be an excisive triad and let $n \in \mathbb{N}_0$. We denote by $i\colon (A \cup B, A) \to (X, A)$ and $j\colon (B, A \cap B) \to (A \cup B, A)$ the inclusion maps. It follows from the Excisive Characterization Lemma 50.10 (1)⇒(3), the Quasi Iso–Homotopy Equivalence Proposition 50.4 and the Homology-with-Coefficients Lemma 52.4 (2b) that the map $j_*\colon \mathrm{H}_{n-1}(B, A \cap B; R) \to \mathrm{H}_{n-1}(A \cup B, A; R)$ is an isomorphism. For any $k \in \mathbb{N}_0$ the following diagram commutes up to the sign $(-1)^k$:

$$\begin{array}{ccc} \mathrm{H}^k(X, A; R) \times \mathrm{H}_n(X, A \cup B; R) & \xrightarrow{\cap} & \mathrm{H}_{n-k}(X, B; R) \\ \downarrow{\scriptstyle i^*} \quad \downarrow{\scriptstyle \partial_n} & & \\ \mathrm{H}^k(A \cup B, A; R) \times \mathrm{H}_{n-1}(A \cup B, A; R) & & \downarrow{\scriptstyle \partial_{n-k}} \\ \downarrow{\scriptstyle j^* \cong} \quad \downarrow{\scriptstyle (j_*)^{-1} \cong} & & \\ \mathrm{H}^k(B, A \cap B; R) \times \mathrm{H}_{n-1}(B, A \cap B; R) & \xrightarrow{\cap} & \mathrm{H}_{n-k-1}(B; R). \end{array}$$

[589] With the various quotient groups, lifts, extensions, inclusions and so on it is admittedly quite tricky to figure out in between where any of the equalities hold.

Here the ∂_i denote the connecting homomorphisms in the long exact sequences in homology corresponding to the triple $(X, A \cup B, A)$ respectively the pair (X, B).



excisive triad to which we will apply the Cap Product-Triad-Connecting Homomorphism Lemma 75.13

Proof. Let $\psi \in \mathrm{H}^k(X, A; R)$ and $\tau \in \mathrm{H}_n(X, A \cup B; R)$. We make the following observations:

(a) It follows immediately from our hypothesis that the triad (X, A, B) is excisive together with the Excisive Triad–Invariants Lemma 73.1 (2a) that we can find $\sigma \in \mathrm{C}_n(X; R)$ such that $\partial_n(\sigma) = \alpha + \beta$ with $\alpha \in \mathrm{C}_{n-1}(A; R)$ and $\beta \in \mathrm{C}_{n-1}(B; R)$ and such that the image of σ in $\mathrm{C}_n(X, A \cup B; R)$ is a cycle that represents τ.

(b) We pick a cocycle $\varphi \in \mathrm{C}^k(X, A; R) = \mathrm{Hom}(\mathrm{C}_k(X, A), R)$ that represents ψ. We denote by φ also the induced homomorphism $\mathrm{C}_k(X) \to \mathrm{C}_k(X, A) \to R$. Note that φ vanishes on $\mathrm{C}_k(A)$.

We have the following equalities:

$$
\begin{array}{lcccl}
 & & \text{choice of } \sigma \text{ and definition of relative cap product} & & \\
 & & \downarrow & & \\
\partial_{n-k}(\psi \cap \tau) & = & \partial_{n-k}([\varphi \cap \sigma]) & = & [\partial_{n-k}(\varphi \cap \sigma)] \\
 & & & & \downarrow \text{ by the Topological LES-Proposition 38.2} \\
 & & & & \text{by the Cap Product–Boundary Lemma 75.1} \\
 & & & & \downarrow \\
 & & = (-1)^k \cdot [\varphi \cap \alpha + \varphi \cap \beta]) & = & (-1)^k \cdot [\varphi \cap \beta] \\
 & & \uparrow & & \uparrow \\
 & & \text{since } \delta_k(\varphi) = 0 \text{ and } \partial_n(\sigma) = \alpha + \beta & & \text{since } \varphi \text{ vanishes on } \mathrm{C}_k(A) \text{ and since } \alpha \circ [\mathfrak{v}_0, \ldots, \mathfrak{v}_k] \in \mathrm{C}_k(A) \\
 & & = (-1)^k \cdot ([j^* i^* \varphi] \cap j_*^{-1}(\partial_n([\sigma]))). & & \\
 & & \uparrow & & \\
 & & \text{by the LES-of-Triple Proposition 38.2 we have } j_*([\beta]) = \partial_n([\sigma]) & &
\end{array}
$$

This equality implies that the diagram commutes up to the sign $(-1)^k$. ∎

CHAPTER 76

The interaction of cross-, cup- and cap-products

In this chapter we will study the interaction of cross-, cup- and cap-products which we introduced on pages 712, 942 and 962. It is perhaps worth recalling, that these products are (for suitable subspaces) of the form

$$\begin{aligned} \times\colon \mathrm{H}_k(X,A)\times\mathrm{H}_l(Y,B) &\to \mathrm{H}_{k+l}(X\times Y,(A\times Y)\cup(X\times B)) \\ \cup\colon \mathrm{H}^k(X,A;\mathbb{Z})\times\mathrm{H}^l(X,B;\mathbb{Z}) &\to \mathrm{H}^{k+l}(X,A\cup B;\mathbb{Z}) \\ \cap\colon \mathrm{H}^k(X,A;\mathbb{Z})\times\mathrm{H}_l(X,A\cup B;\mathbb{Z}) &\to \mathrm{H}_{l-k}(X,B;\mathbb{Z}). \end{aligned}$$

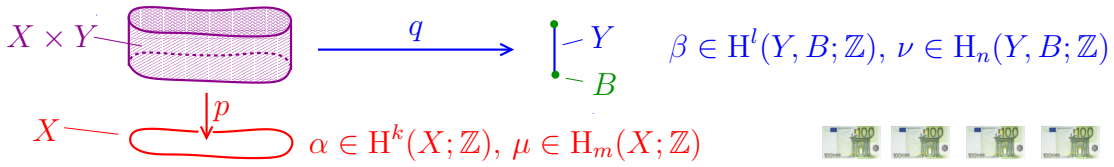
The main result of this chapter will allow us to determine the cup product of the product $S^m\times S^n$ of two spheres. The results of this chapter will also be essential in the next chapter when we calculate the cup product of the n-dimensional torus $(S^1)^n$.

76.1. **The formulation of the main theorem.** The following theorem relates the cross product to the cap product and the cup product:

Theorem 76.1. (Cross-Cup-Cap Theorem) Let (X,A) and (Y,B) be two pairs of topological spaces such that the triad $(X\times Y, A\times Y, X\times B)$ is excisive. We denote by $p\colon X\times Y\to X$ and $q\colon X\times Y\to Y$ the natural projections. Let[590] $A_1,A_2\in\{\varnothing,A\}$ and let $B_1,B_2\in\{\varnothing,B\}$. For any $\alpha\in\mathrm{H}^k(X,A_1;\mathbb{Z})$, $\mu\in\mathrm{H}_m(X,A_1\cup A_2;\mathbb{Z})$, $\beta\in\mathrm{H}^l(Y,B_1;\mathbb{Z})$ and $\nu\in\mathrm{H}_n(Y,B_1\cup B_2;\mathbb{Z})$ we have the equality[591]

$$(p^*(\alpha)\cup q^*(\beta))\cap(\mu\times\nu) \;=\; (-1)^{\deg(\alpha)\cdot\deg(\beta)+\deg(\beta)\cdot\deg(\mu)}\cdot(\alpha\cap\mu)\times(\beta\cap\nu)$$

in $\mathrm{H}_{m+n-k-l}(X\times Y,(X\times B_2)\cup(A_2\times Y);\mathbb{Z})$.



Remark. Even though the equality in the Product Theorem 76.1 is easy to write down, it is much harder to figure out in which (co-) homology groups the various objects actually

[590] By $A_1,A_2\in\{\varnothing,A\}$ we mean that $A_1=\varnothing$ or $A_1=A$ and that $A_2=\varnothing$ or $A_2=A$.

[591] Here, and throughout this section, we denote by p also the map of pairs $(X\times Y,A\times Y)\to(X,A)$ given by projection onto the first factor and we denote by q the map of pairs $(X\times Y,X\times B)\to(Y,B)$ given by projection onto the second factor.

live. In fact the equality is understood as follows:

$$\underbrace{\Big(\underbrace{p^*(\alpha)}_{\in \mathrm{H}^k(X\times Y, A_1\times Y)} \cup \underbrace{q^*(\beta)}_{\in \mathrm{H}^l(X\times Y, X\times B_1)}\Big)}_{\in \mathrm{H}^{k+l}(X\times Y,(A_1\times Y)\cup(X\times B_2))} \cap \overbrace{(\mu\times\nu)}^{\in \mathrm{lies\ in\ H}_{m+n}(X\times Y,((A_1\cup A_2)\times Y)\cup(X\times(B_1\cup B_2)))} = (-1)^{\deg(\alpha)\cdot\deg(\beta)+\deg(\beta)\cdot\deg(\mu)} \cdot \underbrace{\underbrace{(\alpha\cap\mu)}_{\in \mathrm{H}_{m-k}(X,A_2)} \times \underbrace{(\beta\cap\nu)}_{\in \mathrm{H}_{n-l}(Y,B_2)}}_{\in \mathrm{H}_{m+n-k-l}(X\times Y,(X\times B_2)\cup(A_2\times Y))}.$$

In this formula we have to ensure that all products are in fact defined. This follows easily from the Excisive Triad-Proposition 50.11, our hypothesis that $A_1, A_2 \in \{\varnothing, A\}$ and $B_1, B_2 \in \{\varnothing, B\}$ and our hypothesis that the triad $(X \times Y, A \times Y, X \times B)$ is excisive. □

Remark. The condition in the Cross-Cup-Cap Theorem 76.1 that the triad $(X \times Y, A \times Y, X \times B)$ needs to be excisive is arguably a little awkward. By the Excisive Triad-Proposition 50.11 this condition is satisfied if we are in one of the following situations:

(a) $A = \varnothing$ or $B = \varnothing$,
(b) (X, A) and (Y, B) are pairs of CW-complexes,
(c) $(X, A) = (X, \partial X)$ and $(Y, B) = (Y, \partial Y)$ where X and Y are topological manifolds,[592]
(d) A and B are open subsets of X respectively Y.

The proof of the Cross-Cup-Cap Theorem 76.1 is somewhat lengthy and technical. Therefore we first discuss some consequences of the Cross-Cup-Cap Theorem 76.1 which will then hopefully convince us that it is worth the effort to prove the Cross-Cup-Cap Theorem 76.1.

The following is one of the key calculations of the cup product and the cap product. As we will see later, many other calculations can be reduced to this calculation.

Proposition 76.2. (Fundamental Class-Product Proposition) Let X be a compact oriented connected m-dimensional topological manifold and let Y be a compact oriented connected n-dimensional topological manifold. Let $p\colon X \times Y \to X$ and $q\colon X \times Y \to Y$ be the natural projections. The following two statements hold:

(1)
$$\underbrace{p^*([X]^*)}_{\in \mathrm{H}^m(X\times Y,\partial X\times Y)} \cup \underbrace{q^*([Y]^*)}_{\in \mathrm{H}^n(X\times Y, X\times\partial Y)} = \underbrace{[X\times Y]^*}_{\in \mathrm{H}^{m+n}(X\times Y,\partial(X\times Y))}.$$

[592] Let X be an m-dimensional topological manifold and let Y be an n-dimensional smooth manifold. Note that $W := \partial(X \times Y) = (\partial X \times Y) \cup (X \times \partial Y)$ is an $(m + n - 1)$-dimensional topological manifold. It is straightforward to verify that $\partial X \times Y$ and $X \times \partial Y$ are submanifolds of W such that

$$(\partial X \times Y) \cap (X \times \partial Y) \;=\; \partial X \times \partial Y = \partial(\partial X \times Y) \;=\; \partial(X \times \partial Y).$$

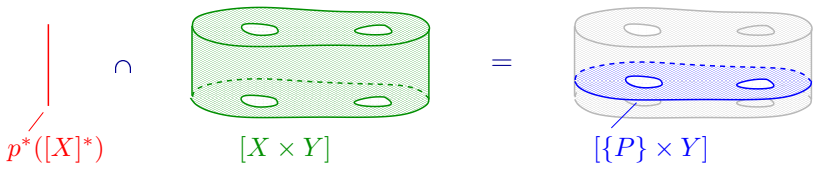
Therefore it follows from the Excisive Triad-Proposition 50.11 (2) that the triad $(X \times Y, \partial X \times Y, X \times \partial Y)$ is excisive.

(2) If $P \in X$ and $Q \in Y$ are points, then[593]

$$\overbrace{p^*([X]^*)}^{\in \mathrm{H}^m(X\times Y,\partial X\times Y)} \cap \overbrace{[X\times Y]}^{\in \mathrm{H}_{m+n}(X\times Y,\partial(X\times Y))} = \overbrace{[\{P\}\times Y]}^{\in \mathrm{H}_n(X\times Y,X\times\partial Y)}$$

and similarly

$$\underbrace{q^*([Y]^*)}_{\in \mathrm{H}^n(X\times Y,X\times\partial Y)} \cap \underbrace{[X\times Y]}_{\in \mathrm{H}_{m+n}(X\times Y,\partial(X\times Y))} = (-1)^{mn}\cdot \underbrace{[X\times\{Q\}]}_{\in \mathrm{H}_m(X\times Y,\partial X\times Y)}.$$



Example. If we apply the Fundamental Class-Product Proposition 76.2 (2) to $X = S^1$ and $Y = S^1$ we see that

$$p^*([S^1]^*) \cap [S^1\times S^1] = [\{1\}\times S^1] \quad \text{and} \quad q^*([S^1]^*) \cap [S^1\times S^1] = -[S^1\times\{1\}].$$

This matches our down-to-earth calculation from the Torus–Cap Product Lemma 75.3. □

Proof. Let X be a compact oriented connected m-dimensional topological manifold and let Y be a compact oriented connected n-dimensional topological manifold. Furthermore let P be a point in X and let Q be a point in Y. We start out with the following two observations:

(a) For any compact oriented connected k-dimensional topological manifold W and any $w_0 \in W$ we obtain from the discussion on page 964 that $[W]^* \cap [W] = [w_0]$ and that $[W]^*$ is the unique element in $\mathrm{H}^k(W,\partial W;\mathbb{Z}) \cong \mathbb{Z}$ with $[W]^* \cap [W] = [w_0]$ in $\mathrm{H}_0(W;\mathbb{Z}) = \mathbb{Z}\cdot[w_0]$.

(b) By the Fundamental Class-of-Product Proposition 64.10 we have $[X\times Y] = [X]\times[Y]$.

Now we turn to the actual proof of the two statements.

(1) We perform the following calculation in $\mathrm{H}_0(X\times Y;\mathbb{Z})$:

$$\begin{array}{ccccc} & & & \overset{\text{observation (b)}}{\downarrow} & \\ (p^*([X]^*)\cup q^*([Y]^*))\cap([X\times Y]) & \overset{\uparrow}{=} & (p^*([X]^*)\cup q^*([Y]^*))\cap([X]\times[Y]) & & \\ \underset{\text{to } (X,A_1,A_2)=(X,\partial X,\partial X)}{\underset{\text{and } (Y,B_1,B_2)=(Y,\partial Y,\partial Y)}{\underset{\uparrow}{=}}} (-1)^{mn+mn}\cdot([X]^*\cap[X])\times([Y]^*\cap[Y]) & \overset{\uparrow}{=} & [P]\times[Q] & \overset{\uparrow}{=} & [(P,Q)]. \\ \text{by the Cross-Cup-Cap Theorem 76.1 applied} & & \text{observation (a)} & & \text{by the Cross Product Lemma 56.2 (1)} \end{array}$$

This implies, by observation (a), that $p^*([X]^*)\cup q^*([Y]^*) = [X\times Y]^*$.

[593]Here we use the following suggestive notation: We can view $\{P\}\times Y$ as a submanifold of $X\times Y$. It is naturally homeomorphic to Y, thus can view $\{P\}\times Y$ as a compact oriented topological manifold. We denote by $[\{P\}\times Y]$ its fundamental class, and by a slight abuse of notation we also denote by $[\{P\}\times Y]$ the image of the fundamental class under the inclusion $\{P\}\times Y \to X\times Y$.

(2) Let $P \in X$. We have the following equality in $\mathrm{H}_n(X \times Y, X \times \partial Y; \mathbb{Z})$:

$$p^*([X]^*) \cap ([X \times Y]) \underset{\uparrow}{=} (p^*([X]^*) \cup q^*(1_Y)) \cap ([X] \times [Y])$$

by the Cup Product-Ring Proposition 72.9 (3) we have $q^*(1_Y) = 1_{X\times Y}$ and by observation (b) we have $[X\times Y] = [X]\times[Y]$

$$\underset{\uparrow}{=} ([X]^* \cap [X]) \times (1_Y \cap [Y]) \underset{\uparrow}{=} [P] \times [Y] \underset{\uparrow}{=} [\{P\} \times Y].$$

by the Cross-Cup-Cap Theorem 76.1 applied to $(Y, B_1, B_2) = (Y, \varnothing, \partial Y)$, note that here we use that 1_Y lies in $\mathrm{H}^0(Y;\mathbb{Z})$

this equality follows from the Cap-with-1-Lemma 75.10 and observation (a)

by the obvious relative version of the Cross Product Lemma 56.2 (1)

This proves the first equality of (2). Now let $Q \in Y$. Then, very similar to the above, we have

$$\begin{aligned} q^*([Y]^*) \cap ([X \times Y]) &= (p^*(1_X) \cup q^*([Y]^*)) \cap ([X] \times [Y]) \\ &= (-1)^{mn} \cdot (1_X \cap [X]) \times ([Y]^* \cap [Y]) \\ &= (-1)^{mn} \cdot [X] \times [Q] = (-1)^{mn} \cdot [X \times \{Q\}]. \end{aligned}$$

■

Using the Fundamental Class-Product Proposition 76.2 (1) we can now easily prove the following lemma, which can be viewed as a generalization of the Torus–Cup Product Lemma 72.11.

Lemma 76.3. (Product-of-Spheres–Cup Product Lemma) Let $m, n \in \mathbb{N}$. We denote by $p\colon S^m \times S^n \to S^m$ and by $q\colon S^m \times S^n \to S^n$ the two natural projections. Let $1 \in \mathrm{H}^0(S^m \times S^n;\mathbb{Z})$ be the neutral element from the Cup Product-Ring Proposition 72.9.

(1) The cohomology of $S^m \times S^n$ is given as follows:

$$\mathrm{H}^*(S^m \times S^n;\mathbb{Z}) = \mathbb{Z}\cdot 1 \oplus \mathbb{Z}\cdot p^*([S^m]^*) \oplus \mathbb{Z}\cdot q^*([S^n]^*) \oplus \mathbb{Z}\cdot [S^m \times S^n]^*.$$

(2) The multiplication table for the cup product is given by

$\cup$	1	$p^*([S^m]^*)$	$q^*([S^n]^*)$	$[S^m \times S^n]^*$
1	1	$p^*([S^m]^*)$	$q^*([S^n]^*)$	$[S^m \times S^n]^*$
$p^*([S^m]^*)$	$p^*([S^m]^*)$	0	$[S^m\times S^n]^*$	0
$q^*([S^n]^*)$	$q^*([S^n]^*)$	$(-1)^{mn}\cdot[S^m\times S^n]^*$	0	0
$[S^m \times S^n]^*$	$[S^m \times S^n]^*$	0	0	0.

Here the table shows the cup product of the term in the first column with the term in the first row.

Proof.

(1) Using the Sphere-H_*-Proposition 41.2, the Künneth Theorem 56.3 and the Universal Coefficient-H^*-Theorem 70.14 one can easily determine the isomorphism type of $\mathrm{H}^*(S^m \times S^n;\mathbb{Z})$. Furthermore, using the Kronecker-Basis Lemma 68.4 (2) and the homology classes given by the fundamental classes of S^m, S^n and $S^m \times S^n$ it is straightforward to verify that the given cohomology classes do indeed form a basis for $\mathrm{H}^*(S^m \times S^n;\mathbb{Z})$.

(2) By Observation 72.10 it remains to show that the red-colored and the blue-colored entries are correct.

(a) First we consider the red-colored entries. We have

$$p^*([S^m]^*) \cup p^*([S^m]^*) \underset{\uparrow}{=} p^*([S^m]^* \cup [S^m]^*) \underset{\uparrow}{=} p^*(0) = 0.$$

by the Cup Product-Naturality Lemma 72.2

since $[S^m]^* \cup [S^m]^* \in \mathrm{H}^{2m}(S^m;\mathbb{Z}) = 0$

The same argument also shows that $q^*([S^n]^*) \cup q^*([S^n]^*) = 0$.

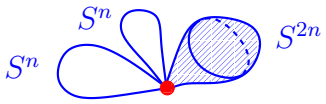
(b) We turn to the blue-colored entries. We calculate that

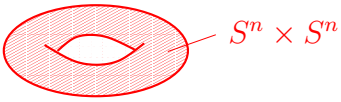
$$(-1)^{mn} \cdot q^*([S^n]^*) \cup p^*([S^m]^*) \underset{\uparrow}{=} p^*([S^m]^*) \cup q^*([S^n]^*) \underset{\uparrow}{=} [S^m \times S^n]^*.$$

by the Cup Product-Commutativity Proposition 72.8 — by the Fundamental Class Product Proposition 76.2 (1)

■

In the following we consider the wedge $S^n \vee S^n \vee S^{2n}$ and we consider the product $S^n \times S^n$. We point out that it follows from the Torus-H_*-Proposition 45.5 (1), the Sphere-H_*-Proposition 41.2, the Wedge-H_*-Proposition 41.10 and the Universal Coefficient-H^*-Theorem 70.14 that $S^n \vee S^n \vee S^{2n}$ and $S^n \times S^n$ have isomorphic homology and cohomology groups. Furthermore, if $n \geq 2$, then one obtains easily from the Sphere-π_1-Proposition 11.15, the Product-π_1-Proposition 13.7 and the Wedge-π_1-Proposition 20.6 that both $S^n \vee S^n \vee S^{2n}$ and $S^n \times S^n$ are simply connected.





It is quite easy to show that $S^n \vee S^n \vee S^{2n}$ and $S^n \times S^n$ are not homeomorphic.[594] The following lemma gives us the much more interesting result, that these two topological spaces are not even homotopy equivalent:

Lemma 76.4. (Wedge–Product-of-Spheres Lemma) The wedge $S^n \vee S^n \vee S^{2n}$ and the product $S^n \times S^n$ are not homotopy equivalent.

Proof. We had just seen in the Product-of-Spheres–Cup Product Lemma 76.3 that there exist cohomology classes α and β in $\mathrm{H}^n(S^n \times S^n; \mathbb{Z})$ with $\alpha \cup \beta \neq 0$. On the other hand in the Wedge-of-Spheres–Cup Product Lemma 72.12 we saw that the cup product

$$\cup \colon \mathrm{H}^n(S^n \vee S^n \vee S^{2n}; \mathbb{Z}) \times \mathrm{H}^n(S^n \vee S^n \vee S^{2n}; \mathbb{Z}) \to \mathrm{H}^{2n}(S^n \vee S^n \vee S^{2n}; \mathbb{Z})$$

is the zero-map. It follows from the Homotopy Equivalence-H^*-Corollary 67.8 (2) together with the Cup Product-Ring Proposition 72.9 (2) that the topological spaces $S^n \vee S^n \vee S^{2n}$ and $S^n \times S^n$ are not homotopy equivalent. ■

Let $n \in \mathbb{N}$. In the Degree Realization Proposition 65.4 we saw that there exists a degree-one map from $S^n \times S^n$ onto the $2n$-dimensional sphere S^{2n}. The following lemma is a generalization of some parts of the Smaller-to-Larger Genus Proposition 65.8.

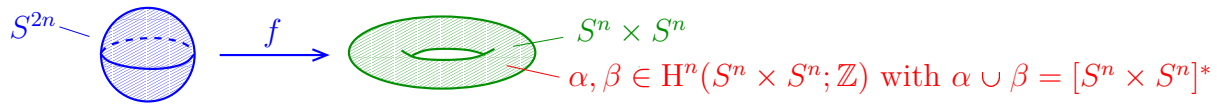
Lemma 76.5. (Sphere-to-Product-of-Spheres Lemma) Let $n \in \mathbb{N}$. There is no continuous map $f \colon S^{2n} \to S^n \times S^n$ of non-zero degree.

Proof. [595] Let $f \colon S^{2n} \to S^n \times S^n$ be a continuous map. By the Product-of-Spheres–Cup Product Lemma 76.3 we know that there exist $\alpha, \beta \in \mathrm{H}^n(S^n \times S^n; \mathbb{Z})$ such that $[S^n \times S^n]^* = \alpha \cup \beta \in \mathrm{H}^{2n}(S^n \times S^n; \mathbb{Z})$.

---

[594] Indeed, suppose that there exists a homeomorphism $\phi \colon S^n \vee S^n \vee S^{2n} \to S^n \times S^n$. We denote by P the wedge point of $S^n \vee S^n \vee S^{2n}$ and we write $Q = f(P)$. Then $(S^n \vee S^n \vee S^{2n}) \setminus \{P\}$ and $(S^n \times S^n) \setminus \{Q\}$ would also be homeomorphic. But the former topological space has three path-components whereas the latter has, since it is a 4-dimensional topological manifold, just one path-component.

[595] The proof is basically identical to the proof of the corresponding statement in the Smaller-to-Larger Genus Proposition 65.8 that we gave on page 937. We just need to replace the Torus–Cup Product Lemma 72.11 by the Product-of-Spheres–Cup Product Lemma 76.3.



We now see that

$$\begin{array}{ccccccc} & \text{by definition of } [S^{2n}]^* & & \text{by the Dual Fundamental Class–Degree Lemma 70.17} & & \text{choice of } \alpha \text{ and } \beta & \\ \deg(f) & \downarrow = & \langle \deg(f) \cdot [S^{2n}]^*, [S^{2n}] \rangle & \downarrow = & \langle f^*([S^n \times S^n]^*), [S^{2n}] \rangle & \downarrow = & \langle f^*(\alpha \cup \beta), [S^{2n}] \rangle \\ & = & \langle f^*(\alpha) \cup f^*(\beta), [S^{2n}] \rangle & = & \langle 0 \cup 0, [S^{2n}] \rangle & = & 0. \\ & \text{by the Cup Product Naturality Lemma 72.2} \uparrow & & \text{since } \mathrm{H}^n(S^{2n}; \mathbb{Z}) = 0 \uparrow & & & \end{array}$$

■

76.2. **Proof of the Cross-Cup-Cap Theorem 76.1.** Now that we are fully convinced of the value of the Cross-Cup-Cap Theorem 76.1 it is time to actually provide a proof thereof. This, very lengthy, proof will require the remainder of this chapter. The impatient reader can also move to Chapter 77 for the Künneth Theorem in cohomology.

76.2.1. **Proof of the Cross-Cup-Cap Theorem 76.1: Algebraic preparations.** Initially we introduced only the tensor product $C_* \otimes D_*$ of two chain complexes C_* and D_*. The definition of the boundary map in the tensor product $C_* \otimes D_*$ is *not* symmetric in the two terms. But it is straightforward to see that the tensor product of chain complexes is associative. This leads us to the following definition:

Definition. Given chain complexes $C_*^1, \ldots, C_*^k$ the boundary map on the tensor product $C_*^1 \otimes \cdots \otimes C_*^k$ is always determined by

$$\partial(c_1 \otimes c_2 \otimes \cdots \otimes c_k) \;=\; \sum_{i=1}^{k} (-1)^{\deg(c_1)+\cdots+\deg(c_{i-1})} \cdot c_1 \otimes \cdots \otimes \partial c_i \otimes \cdots \otimes c_k,$$

regardless of the way how one breaks the tensor product into the tensor product of pairs of two chain complexes. This observation lets us define the chain complex $C_*^1 \otimes \cdots \otimes C_*^k$ (regardless of parentheses).

The following lemma is basically the same as the Tensoring Chain Maps Lemma 55.3, but we state it here for the convenience of the reader.

Lemma 76.6. (Tensoring Chain Maps Lemma) Let $f_i \colon C_*^i \to D_*^i$, $i = 1, \ldots, k$ be chain maps between chain complexes. The map

$$f_1 \otimes \cdots \otimes f_k \colon C_*^1 \otimes \cdots \otimes C_*^k \;\to\; D_*^1 \otimes \cdots \otimes D_*^k$$

that is determined by

$$c_1 \otimes \cdots \otimes c_k \;\mapsto\; f(c_1) \otimes \cdots \otimes f(c_k)$$

is also a chain map.

Proof. Since later on, in the proof of the Cross-Cup-Cap Theorem 76.1, there are lots of steps which can make us nervous, let us convince ourselves that the statement is indeed

correct. We have

$$\begin{aligned}\partial((f_1\otimes\ldots\otimes f_k)(c_1\otimes\ldots\otimes c_k)) &= \partial(f_1(c_1)\otimes\ldots\otimes f_k(c_k)) \\ &= \sum_{i=1}^{k}(-1)^{\deg(f_1(c_1))+\cdots+\deg(f_{i-1}(c_{i-1}))}\cdot f_1(c_1)\otimes\ldots\otimes \overbrace{f_i(\partial c_i)}^{=\partial f_i(c_i)}\otimes\ldots\otimes f_k(c_k) \\ &= (f_1\otimes\ldots\otimes f_k)\Big(\sum_{i=1}^{k}(-1)^{\deg(c_1)+\cdots+\deg(c_{i-1})}\cdot c_1\otimes\ldots\partial c_i\otimes\ldots\otimes c_k\Big) \\ &\quad\uparrow \\ &\quad\text{since } \deg(f_i(c_i))=\deg(c_i) \\ &= (f_1\otimes\ldots\otimes f_k)(\partial(c_1\otimes\ldots\otimes c_k)).\end{aligned}$$

■

Note though that the naive swapping of two factors in the tensor product of two chain complexes is not necessarily a chain map. But we have the following lemma:

Lemma 76.7. (Swap Factors–Chain Map Lemma) Let A_* and B_* be chain complexes. The unique map

$$\tau\colon A_*\otimes B_* \;\to\; B_*\otimes A_*$$

that for $a\in A_k$ and $b\in B_l$ is given by

$$a\otimes b \;\mapsto\; (-1)^{kl}\cdot b\otimes a$$

is a chain map.

Proof. [596] The difficult bit about the lemma is coming up with the statement. The verification is now almost trivial. Indeed, given $a\in A_k$ and $b\in B_l$ we see that

$$\begin{aligned}\tau(\partial^{\otimes}_{k+l}(a\otimes b)) &= \tau(\partial_k(a)\otimes b+(-1)^k\cdot a\otimes\partial_l(b)) \\ &= \tau(\partial_k(a)\otimes b)+(-1)^k\cdot\tau(a\otimes\partial_l(b)) \\ &= (-1)^{(k-1)l}\cdot b\otimes\partial_k(a)+(-1)^{k+k(l-1)}\cdot\partial_l(b)\otimes a \\ &= (-1)^{kl}\cdot\partial_l(b)\otimes a+(-1)^{l+kl}\cdot b\otimes\partial_k(a) \\ &= \partial^{\otimes}((-1)^{kl}\cdot b\otimes a) \;=\; \partial^{\otimes}(\tau(a\otimes b)).\end{aligned}$$

■

Now we have to introduce a plethora of new definitions and notations.

Notation.

(1) We denote by $\mathbb{Z}$ also the chain complex given by the group $\mathbb{Z}$ in degree zero and where all other chain groups are zero.

(2) Given a chain complex C_* and $k\in\mathbb{N}_0$ we denote by $C_*[k]$ the chain complex which is the result of "shifting C_* by k to the left", i.e. it is the chain complex for which the m-th chain groups $(C_*[k])_m$ is given by C_{m-k}. [597]

(3) If $\alpha\colon C_*\to D_*$ is a chain map, then for any $k\in\mathbb{N}_0$ we denote by $\alpha[k]\colon C_*[k]\to D_*[k]$ the obvious shift of the chain map. Evidently this is again a chain map.

(4) Let C_* be a chain complex and let $k\in\mathbb{N}_0$.
   (a) Given a cocycle $\varphi\colon C_k\to\mathbb{Z}$ we denote by $\Pi_k(\varphi)\colon C_*\to\mathbb{Z}[k]$ the map that is given by φ on C_k and zero on all other chain groups.
   (b) Conversely, given a chain map $\psi\colon C_*\to\mathbb{Z}[k]$ we denote by $\psi_k\colon C_k\to\mathbb{Z}$ the map that is the k-th map of the chain map.

[596] We proved this statement already on page 928. We repeat the proof for the reader's convenience.

[597] For example, $\mathbb{Z}[3]$ is the following chain complex:

$$\ldots \;\to\; 0 \;\to\; \underset{\substack{\uparrow\\ \text{degree } 3}}{\mathbb{Z}} \;\to\; 0 \;\to\; 0 \;\to\; 0.$$

Lemma 76.8. (Projection–Cocycle Lemma) Let C_* be a chain complex. Given any $k \in \mathbb{N}_0$ the following statements hold:

(1) Let $\varphi\colon C_k \to \mathbb{Z}$ be a cocycle. The map $\Pi_k(\varphi)\colon C_* \to \mathbb{Z}[k]$ is a chain map.
(2) Let $\psi\colon C_* \to \mathbb{Z}[k]$ be a chain map. The map $\psi_k\colon C_k \to \mathbb{Z}$ is a cocycle.
(3) Let $\alpha, \beta\colon C_* \to \mathbb{Z}[k]$ be chain maps. If α and β are chain homotopic, then α_k and β_k are cohomologous.

Proof.

(1) Let $\varphi\colon C_k \to \mathbb{Z}$ be a cocycle. We consider the following diagram:

$$\begin{array}{ccccccccc} \dots & \longrightarrow & C_{k+1} & \xrightarrow{\partial_{k+1}} & C_k & \xrightarrow{\partial_k} & C_{k-1} & \longrightarrow & \dots \\ & & \downarrow & & \downarrow{\scriptstyle \varphi} & & \downarrow & & \\ \dots & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \dots . \end{array}$$

The hypothesis that φ is a cocycle means that $\delta_k(\varphi) = \varphi \circ \partial_{k+1} = 0$, which implies immediately that $\Pi_k(\varphi)$ is a chain map.

(2) The proof of Statement (2) is almost the same as the proof of Statement (1).

(3) Let $\alpha, \beta\colon C_* \to \mathbb{Z}[k]$ be chain maps and let $\{P_m\colon C_m \to \mathbb{Z}[k]_{m+1}\}_{m\in\mathbb{N}_0}$ be a chain homotopy between α and β. We summarize all relevant maps in the following diagram:

$$\begin{array}{ccccccccc} \dots & \longrightarrow & C_{k+1} & \xrightarrow{\partial_{k+1}} & C_k & \xrightarrow{\partial_k} & C_{k-1} & \longrightarrow & \dots \\ & & \downarrow & \swarrow{\scriptstyle P_k} & \downarrow{\scriptstyle \alpha_k-\beta_k} & \swarrow{\scriptstyle P_{k-1}} & \downarrow & & \\ \dots & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \dots . \end{array}$$

By definition of a chain homotopy we obtain that $\alpha_k - \beta_k = P_{k-1} \circ \partial_k$. But this means that $\alpha_k - \beta_k = \delta_{k-1}(P_{k-1})$, i.e. α_k and β_k are cohomologous. ■

Lemma 76.9. (Chain Complex–Shift Lemma) Let C_* be a chain complex and let $k \in \mathbb{Z}$. The maps

$$\begin{array}{rccc} \eta_m\colon (\mathbb{Z}[k] \otimes C_*)_m = \mathbb{Z} \otimes C_{m-k} & \to & (C_*[k])_m = C_{m-k} \\ a \otimes c_{m-k} & \mapsto & (-1)^{k(m-k)} \cdot a \cdot c_{m-k} \end{array}$$

form an isomorphism

$$\eta\colon \mathbb{Z}[k] \otimes C_* \xrightarrow{\cong} C_*[k]$$

of chain complexes.

Example. Let $k, l \in \mathbb{N}_0$. The chain map

$$\eta\colon \mathbb{Z}[k] \otimes \mathbb{Z}[l] \to \mathbb{Z}[k+l]$$

is determined by the fact that for $a \in (\mathbb{Z}[k])_k = \mathbb{Z}$ and $b \in (\mathbb{Z}[l])_l = \mathbb{Z}$ we have

$$\eta(a \otimes b) = (-1)^{kl} \cdot a \cdot b.$$ □

Proof. It follows from the Tensor-with-$\mathbb{Z}$–Lemma 51.3 that each η_m is an isomorphism. But we still need to verify that the maps η_m form a chain map $\eta\colon \mathbb{Z}[k] \otimes C_* \to C_*[k]$. Thus

let $k \in \mathbb{N}_0$, let $a \in (\mathbb{Z}[k])_k = \mathbb{Z}$ and let $c_{m-k} \in C_{m-k}$. Then

$$\begin{array}{rcl} & \overset{\text{since } a \text{ is of degree } k \text{ and since } \partial_k(a)=0}{\downarrow} & \overset{\text{since } \partial c_{m-k} \text{ is of degree } m-k-1}{\downarrow} \\ \eta_{m-1}(\partial_m^{\otimes}(a \otimes c_{m-k})) & = \eta_{m-1}((-1)^k \cdot a \otimes \partial_{m-k}(c_{m-k})) & = (-1)^k \cdot (-1)^{k(m-1-k)} \cdot a \cdot \partial_{m-k}(c_{m-k}) \\ & = \partial_{m-k}((-1)^{k(m-k)} \cdot a \cdot c_{m-k}) & = \partial_m(\eta_m(a \otimes c_{m-k})). \end{array}$$

It follows immediately that the maps η_m form a chain map. ■

76.2.2. **Proof of the Cross-Cup-Cap Theorem 76.1: Topological preparations.** After the purely algebraic notions from the last subsection we now return to topological concepts:

Notation.

(1) Given topological spaces X and Y we denote by

$$\begin{array}{rcl} \Upsilon\colon \mathrm{C}_*(X) \otimes \mathrm{C}_*(Y) & \to & \mathrm{C}_*(X \times Y) \\ \Theta\colon \mathrm{C}_*(X \times Y) & \to & \mathrm{C}_*(X) \otimes \mathrm{C}_*(Y) \end{array}$$

the chain maps from the Eilenberg-Zilber Theorem 56.1.

(2) Given a topological space W we denote by $\Delta_W\colon \mathrm{C}_*(W) \to \mathrm{C}_*(W) \otimes \mathrm{C}_*(W)$ the Alexander-Whitney diagonalization from page 925. Recall that Δ_W is also a chain map.

In the following two lemmas we will reinterpret the cup product and the cap product as chain maps.

Lemma 76.10. (Cup Product–Reinterpretation Lemma) Let W be a topological space and let $\alpha \in \mathrm{C}^k(W;\mathbb{Z})$ and $\beta \in \mathrm{C}^l(W;\mathbb{Z})$ be cocycles. Then we have the following equality of chain maps:

$$(-1)^{kl} \cdot \Pi_{k+l}\big(\underbrace{\alpha \cup \beta}_{\text{cocycle in } \mathrm{C}^{k+l}(W;\mathbb{Z})}\big) = \eta \circ (\Pi_k(\alpha) \otimes \Pi_l(\beta)) \circ \Delta_W\colon \mathrm{C}_*(W) \to \mathbb{Z}[k+l].$$

Proof. We introduce the following notation:

(1) We denote by $\mu_{\mathbb{Z}}\colon \mathbb{Z} \otimes \mathbb{Z} \to \mathbb{Z}$ the obvious isomorphism of abelian groups given by $\mu_{\mathbb{Z}}(a \otimes b) = a \cdot b$.

(2) We denote by $\Pi_{k,l}\colon (\mathrm{C}_*(W) \otimes \mathrm{C}_*(W))_{k+l} \to \mathrm{C}_k(W) \otimes \mathrm{C}_l(W)$ the natural projection.

In the proof of the Cup Product–Definitions Proposition 72.5 on page 927 we showed that

$$\alpha \cup \beta = \mu_{\mathbb{Z}} \circ (\alpha \otimes \beta) \circ \Pi_{k,l} \circ \Delta_W\colon \mathrm{C}_{k+l}(W) \to \mathbb{Z}.$$

The lemma follows immediately from this equality and the following claim:

Claim.

$$(-1)^{kl} \cdot \mu_{\mathbb{Z}} \circ (\alpha \otimes \beta) \circ \Pi_{k,l} = \underbrace{(\eta \circ (\Pi_k(\alpha) \otimes \Pi_l(\beta)))_{k+l}}_{\substack{\text{the restriction to the } (k+l)\text{-th chain group of the} \\ \text{chain map } \mathrm{C}_*(W) \otimes \mathrm{C}_*(W) \to \mathbb{Z}[k] \otimes \mathbb{Z}[l] \to \mathbb{Z}[k+l] \\ a \otimes b \mapsto (-1)^{kl} \cdot ab}} : (\mathrm{C}_*(W) \otimes \mathrm{C}_*(W))_{k+l} \to \mathbb{Z}.$$

Proof. Let $\sigma_i \in \mathrm{C}_i(W)$ and $\sigma_j \in \mathrm{C}_j(W)$ with $i+j = k+l$. If $(i,j) \neq (k,l)$, then both maps are zero. So now suppose that $(i,j) = (k,l)$. Then it is clear that both maps send

$\sigma_k \otimes \sigma_l$ to $(-1)^{kl} \cdot \alpha(\sigma_k) \cdot \beta(\sigma_l)$. (Note that the sign $(-1)^{kl}$ for the map on the right-hand side stems from the definition of the chain map η.) ■

Given a topological space W and given $\varphi \in \mathrm{C}^k(W;\mathbb{Z})$ we obtain for each $m \in \mathbb{N}_0$ a map[598]

$$\begin{array}{rcl} \varphi \cap -\colon \mathrm{C}_m(W) & \to & \mathrm{C}_{m-k}(W) = (\mathrm{C}_*(W)[k])_m \\ \sigma & \mapsto & \varphi \cap \sigma. \end{array}$$

Unfortunately in general these maps do *not* form a chain map from the chain complex $\mathrm{C}_*(W)$ to the chain complex $\mathrm{C}_*(W)[k]$. This motivates the introduction of the following slight variation on the cap product:

Definition. Let W be a topological space. Given $\varphi \in \mathrm{C}^m(W;\mathbb{Z})$ and given $\sigma \in \mathrm{C}_n(W)$ we define

$$\varphi \mathbin{\widetilde{\cap}} \sigma \;:=\; (-1)^{m(n-m)} \cdot \varphi \cap \sigma.$$

In particular for $m = n$ we have $\varphi \mathbin{\widetilde{\cap}} \sigma = \varphi \cap \sigma$.

Lemma 76.11. (Cap-as-Chain Map Lemma) Let W be a topological space.

(1) If $\varphi \in \mathrm{C}^m(W;\mathbb{Z})$ is a cocycle, then the map

$$\begin{array}{rcl} \varphi \mathbin{\widetilde{\cap}} -\colon \mathrm{C}_*(W) & \to & \mathrm{C}_*(W)[m] \\ \sigma & \mapsto & \varphi \mathbin{\widetilde{\cap}} \sigma \end{array}$$

is a chain map.

(2) If $\varphi, \psi \in \mathrm{C}^m(W;\mathbb{Z})$ are cohomologous cocycles, then the chain maps $\varphi \mathbin{\widetilde{\cap}} -$ and $\psi \mathbin{\widetilde{\cap}} -$ are chain homotopic.

Proof. Let W be a topological space. We start out with the following claim:

Claim. For any cochain $\tau \in \mathrm{C}^l(W;\mathbb{Z})$ and any chain $\mu \in \mathrm{C}_n(W;\mathbb{Z})$ we have

$$\partial_{n-l}(\tau \mathbin{\widetilde{\cap}} \mu) \;=\; (-1)^{l+n} \cdot \delta_l(\tau) \mathbin{\widetilde{\cap}} \mu + \tau \mathbin{\widetilde{\cap}} \partial_n(\mu).$$

Proof. The claim is an immediate consequence of the Cap Product–Boundary Lemma 75.1. Since we are worried about signs we write down the detailed calculation. We have

$$\begin{array}{rcl} \partial_{n-l}(\tau \mathbin{\widetilde{\cap}} \mu) & \overset{\text{definition of } \widetilde{\cap}}{\downarrow} & \\ & = & (-1)^{l(n-l)} \cdot \partial_{n-l}(\tau \cap \mu) \\ & \overset{\text{by the Cap Product–Boundary Lemma 75.1}}{\downarrow} & \\ & = & (-1)^{l(n-l)} \cdot (-1)^l \cdot (-\delta_l(\tau) \cap \mu + \tau \cap \partial_n(\mu)) \\ & = & (-1)^{l(n-l)+l} \cdot (-(-1)^{(l+1)\cdot(n-l-1)} \cdot \delta_l(\tau) \mathbin{\widetilde{\cap}} \mu + (-1)^{l\cdot(n-l-1)} \cdot \tau \mathbin{\widetilde{\cap}} \partial_n(\mu)) \\ & = & (-1)^{l+n} \cdot \delta_l(\tau) \mathbin{\widetilde{\cap}} \mu + \tau \mathbin{\widetilde{\cap}} \partial_n(\mu). \end{array}$$

⊞

Now we turn to the actual proof of the two statements.

(1) Let $\varphi \in \mathrm{C}^m(W;\mathbb{Z})$ be a cocycle. Furthermore let $\sigma \in \mathrm{C}_n(W)$. Note that by the definition of the cap product and by definition of the shifting of chain complexes we have $\varphi \cap \sigma \in \mathrm{C}_{n-m}(W) = (\mathrm{C}_*(W)[m])_n$. This shows that $\varphi \mathbin{\widetilde{\cap}} -$ preserves degrees. It remains to show that the map $\varphi \mathbin{\widetilde{\cap}} -$ commutes with the boundary maps. But this is an immediate consequence of the above claim and the fact that φ is a cocycle.

(2) So let $\varphi, \psi \in \mathrm{C}^m(W;\mathbb{Z})$ be cocycles. We assume they are cohomologous which means that there exists a $\tau \in \mathrm{C}^{m-1}(W;\mathbb{Z})$ such that $\varphi = \psi + \delta_{m-1}(\tau)$. For any $k \in \mathbb{N}_0$ we consider the map

$$\begin{array}{rcl} P_k\colon \mathrm{C}_k(W) & \to & (\mathrm{C}_*(W)[m])_{k+1} = \mathrm{C}_{k-m+1}(W) \\ \sigma & \mapsto & (-1)^{m+k-1} \cdot \tau \mathbin{\widetilde{\cap}} \sigma. \end{array}$$

[598] Here we use the natural isomorphism $\mathrm{C}_n(W) \otimes \mathbb{Z} =_{\mathrm{i}} \mathrm{C}_n(W)$. This way we can arrange that the cap product takes values in $\mathrm{C}_n(W)$ instead of $\mathrm{C}_n(W) \otimes \mathbb{Z}$.

For $\sigma \in \mathrm{C}_k(W)$ we see that

$$\begin{aligned}\varphi \mathbin{\widetilde{\cap}} \sigma - \psi \mathbin{\widetilde{\cap}} \sigma \;=\; \delta_{m-1}(\tau) \mathbin{\widetilde{\cap}} \sigma \;&\overset{\text{by the claim with } l=m-1 \text{ and } n=k \text{ we have } \partial_{k-m+1}(\tau \mathbin{\widetilde{\cap}} \sigma) = (-1)^{k+m-1} \cdot \delta_{m-1}(\tau) \mathbin{\widetilde{\cap}} \sigma + \tau \mathbin{\widetilde{\cap}} \partial_k(\sigma)}{=} (-1)^{m+k-1} \cdot \partial_{k-m+1}(\tau \mathbin{\widetilde{\cap}} \sigma) + (-1)^{m+k} \cdot \tau \mathbin{\widetilde{\cap}} \partial_k(\sigma) \\ &= (\partial_{k+1} \circ P_k)(\sigma) + (P_{k-1} \circ \partial_k)(\sigma).\end{aligned}$$

This shows that the maps $\{P_k\}_{k\in\mathbb{N}_0}$ form a chain homotopy between the chain maps $\varphi \mathbin{\widetilde{\cap}} -$ and $\psi \mathbin{\widetilde{\cap}} -$. ■

In the proof of the Cross-Cup-Cap Theorem 76.1 we will also need the following reformulation of the cap product.

Proposition 76.12. (Cap Product–Reinterpretation Lemma) Let W be a topological space and let $\varphi \in \mathrm{C}^m(W;\mathbb{Z})$ be a cocycle. We have the following equality of chain maps $\varphi \mathbin{\widetilde{\cap}} - \;=\; \eta \circ (\Pi_m(\varphi) \otimes \mathrm{id}) \circ \Delta_W \colon \mathrm{C}_*(W) \;\to\; \mathrm{C}_*(W)[m]$.

Proof. Let W be a topological space, let $\varphi \in \mathrm{C}^m(W)$ be a cocycle and let $\sigma\colon \Delta^n \to W$ be a singular n-simplex. We have the following equalities in $\mathrm{C}_*(W)[m]_n = \mathrm{C}_{n-m}(W)$:

$$\begin{aligned}\big(\eta \circ (\Pi_m(\varphi) \otimes \mathrm{id}) \circ \Delta_W\big)(\sigma) &\overset{\text{definition of the Alexander-Whitney diagonalization}}{=} \eta\Big((\Pi_m(\varphi) \otimes \mathrm{id})\Big(\sum_{j=0}^{n} \sigma \circ [\mathfrak{v}_0,\dots,\mathfrak{v}_j] \otimes \sigma \circ [\mathfrak{v}_j,\dots,\mathfrak{v}_n]\Big)\Big) \\ &= \eta\big(\underbrace{\varphi(\sigma \circ [\mathfrak{v}_0,\dots,\mathfrak{v}_m])}_{\in\mathbb{Z}[m]_m} \otimes \underbrace{\sigma \circ [\mathfrak{v}_m,\dots,\mathfrak{v}_n]}_{\in \mathrm{C}_{n-m}}\big) \\ &\overset{\text{by definition of the chain map } \Pi_m(\varphi)\otimes\mathrm{id}\colon \mathrm{C}_*(W)\otimes\mathrm{C}_*(W) \to \mathbb{Z}[m]\otimes\mathrm{C}_*(W)}{=} (-1)^{m(n-m)} \cdot \underbrace{\varphi(\sigma\circ[\mathfrak{v}_0,\dots,\mathfrak{v}_m]) \cdot \sigma \circ [\mathfrak{v}_m,\dots,\mathfrak{v}_n]}_{=\varphi\cap\sigma} \;=\; \varphi \mathbin{\widetilde{\cap}} \sigma.\end{aligned}$$

(definition of η in the Chain Complex–Shift Lemma 76.9; definition of $\widetilde{\cap}$ on page 978)

We have thus verified that the desired equality holds. ■

The following statement is just the definition of the cross product.

Tautology 76.13. Let X and Y two topological spaces. For any $[\mu] \in \mathrm{H}_m(X;\mathbb{Z})$ and $[\nu] \in \mathrm{H}_n(Y;\mathbb{Z})$ we have $[\mu] \times [\nu] \;=\; [\Upsilon(\mu \otimes \nu)]$.

We need one more slightly delicate chain map before we can finally turn to the proof of the Product Theorem 76.1.

Lemma 76.14. (Tensor-and-Shift Lemma) Let C_* and D_* be chain complexes and let $k, l \in \mathbb{N}_0$. The map

$$\Psi_{k,l}\colon C_*[k] \otimes D_*[l] \;\to\; (C_* \otimes D_*)[k+l]$$

which is defined by the property that for $c_m \in C_m$ and $d_n \in D_n$ we have

$$c_m \otimes d_n \;\mapsto\; (-1)^{k(l+n)} \cdot c_m \otimes d_n.$$

is an isomorphism of chain complexes.

Proof. First note that it is clear that $\Psi_{k,l}$ preserves degrees and that on each degree it is an isomorphism. It remains to show that the map $\Psi_{k,l}$ is a chain map. Let $c_m \in C_m$ and

$d_n \in D_n$. We calculate that

$$\begin{aligned}
\Psi_{k,l}(\partial^{\otimes}_{m+n}(c_m \otimes d_n)) &\overset{\text{since } c_m \text{ has degree } m+k \text{ in } C_*[k]}{=} \Psi_{k,l}(\partial_m(c_m) \otimes d_n + (-1)^{m+k} \cdot c_m \otimes \partial_n(d_n)) \\
&= (-1)^{k(l+n)} \cdot \partial_m(c_m) \otimes d_n + (-1)^{m+k} \cdot (-1)^{k(l+n-1)} \cdot c_m \otimes \partial_n(d_n) \\
&= (-1)^{k(l+n)} \cdot (\partial_m(c_m) \otimes d_n + (-1)^m \cdot c_m \otimes \partial_n(d_n)) \\
&= (-1)^{k(l+n)} \cdot \partial^{\otimes}_{m+n}(c_m \otimes d_n) \\
&\overset{\text{since } c_m \text{ has degree } m \text{ in } C_*}{=} \partial^{\otimes}_{m+n}((-1)^{k(l+n)} \cdot c_m \otimes d_n)) \;=\; \partial^{\otimes}_{m+n}(\Psi_{k,l}(c_m \otimes d_n)).
\end{aligned}$$

This calculation implies that $\Psi_{k,l}$ is a chain map. ■

76.2.3. **Proof of the Cross-Cup-Cap Theorem 76.1: The main step.** The following theorem is the main technical result of this section. As we will see in the next subsection, the Product Theorem 76.1 is a straightforward consequence of the following Cross-Cup-Cap-on-Chain Level Theorem 76.15.

Theorem 76.15. (Cross-Cup-Cap-on-Chain Level Theorem) Let X and Y be topological spaces. We denote by $p\colon X \times Y \to X$ and $q\colon X \times Y \to Y$ the projections. Let $\alpha \in \mathrm{C}^k(X;\mathbb{Z})$ and $\beta \in \mathrm{C}^l(Y;\mathbb{Z})$ be cocycles. Then the chain maps

$$(-1)^{kl} \cdot ((p^*\alpha \cup q^*\beta) \frown -) \circ \Upsilon \quad \text{and} \quad \Upsilon[k+l] \circ \Psi_{k,l} \circ (\alpha \frown - \otimes \beta \frown -)$$

from $\mathrm{C}_*(X) \otimes \mathrm{C}_*(Y)$ to $\mathrm{C}_*(X \times Y)[k+l]$ are chain homotopic.

Proof. Let X and Y be topological spaces and let $\alpha \in \mathrm{C}^k(X;\mathbb{Z})$ and $\beta \in \mathrm{C}^l(Y;\mathbb{Z})$ be cocycles. We consider the following diagram of chain maps

$$\begin{array}{ccccc}
\mathrm{C}_*X \otimes \mathrm{C}_*Y & \xrightarrow{\Upsilon} & \mathrm{C}_*(X\times Y) & \xrightarrow{\Delta_{X\times Y}} & \mathrm{C}_*(X\times Y)\otimes\mathrm{C}_*(X\times Y) \\
\downarrow{\scriptstyle \Delta_X\otimes\Delta_Y} & & & & \downarrow{\scriptstyle \Theta\otimes\mathrm{id}} \\
\mathrm{C}_*X\otimes\mathrm{C}_*X\otimes\mathrm{C}_*Y\otimes\mathrm{C}_*Y & \xrightarrow{\mathrm{id}\otimes\tau\otimes\mathrm{id}} & \mathrm{C}_*X\otimes\mathrm{C}_*Y\otimes\mathrm{C}_*X\otimes\mathrm{C}_*Y & \xrightarrow{\mathrm{id}\otimes\mathrm{id}\otimes\Upsilon} & \mathrm{C}_*X\otimes\mathrm{C}_*Y\otimes\mathrm{C}_*(X\times Y) \\
\downarrow{\scriptstyle \Pi_k(\alpha)\otimes\mathrm{id}\otimes\Pi_l(\beta)\otimes\mathrm{id}} & & \downarrow{\scriptstyle \Pi_k(\alpha)\otimes\Pi_l(\beta)\otimes\mathrm{id}\otimes\mathrm{id}} & & \downarrow{\scriptstyle \Pi_k(\alpha)\otimes\Pi_l(\beta)\otimes\mathrm{id}} \\
\mathbb{Z}[k]\otimes\mathrm{C}_*X\otimes\mathbb{Z}[l]\otimes\mathrm{C}_*Y & \xrightarrow{\mathrm{id}\otimes\tau\otimes\mathrm{id}} & \mathbb{Z}[k]\otimes\mathbb{Z}[l]\otimes\mathrm{C}_*X\otimes\mathrm{C}_*Y & \xrightarrow{\mathrm{id}\otimes\mathrm{id}\otimes\Upsilon} & \mathbb{Z}[k]\otimes\mathbb{Z}[l]\otimes\mathrm{C}_*(X\times Y) \\
\downarrow{\scriptstyle \eta\otimes\eta} & & \downarrow{\scriptstyle \eta\otimes\mathrm{id}\otimes\mathrm{id}} & & \downarrow{\scriptstyle \eta\otimes\mathrm{id}} \\
 & & \mathbb{Z}[k+l]\otimes(\mathrm{C}_*X\otimes\mathrm{C}_*Y) & \xrightarrow{\mathrm{id}\otimes\Upsilon} & \mathbb{Z}[k+l]\otimes\mathrm{C}_*(X\times Y) \\
 & & \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \eta} \\
\mathrm{C}_*X[k]\otimes\mathrm{C}_*Y[l] & \xrightarrow{\Psi_{k,l}} & (\mathrm{C}_*X\otimes\mathrm{C}_*Y)[k+l] & \xrightarrow{\Upsilon[k+l]} & \mathrm{C}_*(X\times Y)[k+l].
\end{array}$$

Claim 1. The above diagram commutes up to chain homotopies.

Proof. We consider the various smaller squares and rectangles.

(1) The two paths along the top rectangle from the top-left corner to the bottom-right corner are both natural chain maps. Furthermore, both have the property that for $x \in X$ and $y \in Y$ the element $x \otimes y \in \mathrm{C}_*(X) \otimes \mathrm{C}_*(Y)$ gets sent to $x \otimes y \otimes (x,y)$ in $\mathrm{C}_*(X) \otimes \mathrm{C}_*(Y) \otimes \mathrm{C}_*(X \times Y)$. In particular both paths agree on the 0-level.

Almost the same argument as in the proof of the EZ–Natural Chain Homotopy Proposition 56.8 (a) (which is essentially the same argument as in the proof of the Diagonal Approximation–Uniqueness Proposition 72.7) shows that the rectangle commutes up to a chain homotopy.

As we will see the other squares commute "on the nose" for formal reasons:

(2) It is clear that the two squares between the second and the third row commute.
(3) It is straightforward to verify that the two squares at the bottom right commute.
(4) It remains to consider the big square to the bottom left of the diagram. Starting with $a \in (\mathbb{Z}[k])_k = \mathbb{Z}$, $b \in \mathrm{C}_m(X)$, $c \in (\mathbb{Z}[l])_l = \mathbb{Z}$ and $d \in \mathrm{C}_n(Y)$ and keeping in mind the sign conventions of the Swap Factors–Chain Map Lemma 76.7, the Chain Complex–Shift Lemma 76.9 and the Tensor-and-Shift Lemma 76.14 we get the following results for $a \otimes b \otimes c \otimes d$:
   (a) going down-right gives $(-1)^{km} \cdot (-1)^{ln} \cdot (-1)^{k(l+n)} \cdot ac \cdot (b \otimes d)$,
   (b) going right-down-down gives $(-1)^{ml} \cdot (-1)^{kl} \cdot (-1)^{(k+l)(m+n)} \cdot ac \cdot (b \otimes d)$.
   One easily verifies that the signs agree. So we see that the bottom left square also commutes. ⊞

In light of the previous claim it suffices to prove the following claim:

Claim 2.
(1) The "left-bottom route" equals the chain map $\Upsilon[k+l] \circ \Psi_{k,l} \circ ((\alpha \mathbin{\widetilde{\cap}} -) \otimes (\beta \mathbin{\widetilde{\cap}} -))$.
(2) The "top-right route" is chain homotopic to $(-1)^{kl} \cdot ((p^*\alpha \cup q^*\beta) \mathbin{\widetilde{\cap}} -) \circ \Upsilon$.

Proof. We start out with the proof of Statement (1). First we note that

$$(\alpha \mathbin{\widetilde{\cap}} -) \otimes (\beta \mathbin{\widetilde{\cap}} -) \overset{\text{by the Cap Product–Reinterpretation Proposition 76.12}}{=} (\eta \circ (\Pi_k(\alpha) \otimes \mathrm{id}) \circ \Delta_X) \otimes (\eta \circ (\Pi_l(\beta) \otimes \mathrm{id}) \circ \Delta_Y) = (\eta \otimes \eta) \circ ((\Pi_k(\alpha) \otimes \mathrm{id} \otimes \Pi_l(\beta) \otimes \mathrm{id})) \circ (\Delta_X \otimes \Delta_Y).$$

(the second equality follows from the observation that for chain maps a, b, c and d we have $(a \circ b) \otimes (c \circ d) = (a \otimes b) \circ (c \otimes d)$)

If we compose this equality with $\Upsilon[k+l] \circ \Psi_{k,l}$ then we obtain the "bottom left route" in the above diagram. This concludes the proof of Statement (1).

We continue with the proof of Statement (2). In that proof we will make use of the following subclaim.

Subclaim. The two cocycles[599]

$$(\eta \circ (\Pi_k(\alpha) \otimes \Pi_l(\beta)) \circ \Theta)_{k+l} \quad \text{and} \quad (-1)^{kl} \cdot p^*(\alpha) \cup q^*(\beta)$$

in $\mathrm{C}^{k+l}(X \times Y; \mathbb{Z})$ are cohomologous.

Proof of subclaim. We consider the following diagram:

$$\begin{array}{ccccc} \mathrm{C}_{k+l}(X \times Y) & \hookrightarrow & \mathrm{C}_*(X \times Y) & \xrightarrow{\Theta} & \mathrm{C}_*(X) \otimes \mathrm{C}_*(Y) \\ & & \downarrow{\scriptstyle \Delta_{X\times Y}} & \nearrow{\scriptstyle p_* \otimes q_*} & \downarrow{\scriptstyle \Pi_k(\alpha) \otimes \Pi_l(\beta)} \\ & & \mathrm{C}_*(X \times Y) \otimes \mathrm{C}_*(X \times Y) & \xrightarrow[\Pi_k(p^*(\alpha)) \otimes \Pi_l(q^*(\beta))]{} & \mathbb{Z}[k] \otimes \mathbb{Z}[l] \xrightarrow[\eta]{} \mathbb{Z}[k+l]. \end{array}$$

[599]Recall that $\eta \circ (\Pi_k(\alpha) \otimes \Pi_l(\beta)) \circ \Theta$ is a chain map $\mathrm{C}_*(X \times Y) \to \mathbb{Z}[k+l]$. Furthermore recall that the map $(\eta \circ (\Pi_k(\alpha) \otimes \Pi_l(\beta)) \circ \Theta)_{k+l} \colon \mathrm{C}_{k+l}(X \times Y) \to (\mathbb{Z}[k+l])_{k+l} = \mathbb{Z}$ is the $(k+l)$-th map of the chain map.

Note that by the above discussion all maps except for the top left map are chain maps. It follows immediately from the definitions that the bottom triangle of chain maps commutes "on the nose". The top triangle of chain maps commutes, up to a chain homotopy, by the EZ–Natural Chain Homotopy Proposition 56.8 (a).[600] Note that it follows from the Projection–Cocycle Lemma 76.8 (3) that the restrictions of these two chain maps to maps $\mathrm{C}_{k+l}(X \times Y) \to \mathbb{Z} = \mathbb{Z}[k+l]$ are cohomologous cochains. But the restriction of the "top route" equals evidently the left-hand side of the subclaim. On the other hand, by the Cup Product–Reinterpretation Lemma 76.10 the restriction of the "bottom route" equals the right-hand side of the subclaim. □

Now we see that

$$
\begin{aligned}
(-1)^{kl} \cdot ((p^*\alpha \cup q^*\beta) \mathbin{\widetilde{\cap}} -) \circ \Upsilon \quad &\overset{\text{above subclaim and the Cap-as-Chain Map Lemma 76.11 (2)}}{\simeq} \quad \big((\eta \circ (\Pi_k(\alpha) \otimes \Pi_l(\beta)) \circ \Theta)_{k+l} \mathbin{\widetilde{\cap}} -\big) \circ \Upsilon \\
&= \quad \eta \circ \big(\Pi_{k+l}((\eta \circ (\Pi_k(\alpha) \otimes \Pi_l(\beta)) \circ \Theta)_{k+l}) \otimes \mathrm{id}\big) \circ \Delta_{X\times Y} \circ \Upsilon \\
&\qquad \uparrow \text{ by the Cap Product–Reinterpretation Proposition 76.12 applied to the } (k+l)\text{-cocycle } (\eta \circ (\Pi_k(\alpha) \otimes \Pi_l(\beta)) \circ \Theta)_{k+l} \\
&= \quad \eta \circ \big(\Pi_{k+l}((\eta \circ (\Pi_k(\alpha) \otimes \Pi_l(\beta)))_{k+l}) \circ \Theta \otimes \mathrm{id}\big) \circ \Delta_{X\times Y} \circ \Upsilon \\
&\qquad \uparrow \text{ since for every chain map } \Omega\colon \mathrm{C}_*(X) \times \mathrm{C}_*(Y) \to \mathbb{Z}[k+l] \text{ we have } \Pi_{k+l}((\Omega \circ \Theta)_{k+l}) = \Pi_{k+l}(\Omega_{k+l}) \circ \Theta \text{ as maps } \mathrm{C}_*(X\times Y) \to \mathbb{Z}[k+l], \\
&\qquad \text{applies this observation to } \Omega = \eta \circ (\Pi_k(\alpha) \otimes \Pi_l(\beta)) \\
&= \quad \eta \circ ((\eta \circ (\Pi_k(\alpha) \otimes \Pi_l(\beta)) \circ \Theta) \otimes \mathrm{id}) \circ \Delta_{X\times Y} \circ \Upsilon \\
&\qquad \uparrow \text{ here we use the equality of chain maps} \\
&\qquad \Pi_{k+l}((\eta \circ (\Pi_k(\alpha) \otimes \Pi_l(\beta)))_{k+l}) = \eta \circ (\Pi_k(\alpha) \otimes \Pi_l(\beta))\colon \mathrm{C}_*(X) \otimes \mathrm{C}_*(Y) \to \mathbb{Z}[k+l] \\
&\qquad \text{which is obvious once one unravels the definitions} \\
&= \quad \text{the "top right route" in the above diagram.} \\
&\qquad \uparrow \text{ follows from } (f \otimes \mathrm{id}) \circ (g \otimes \mathrm{id}) = (g \circ f) \otimes \mathrm{id}
\end{aligned}
$$

This concludes the proof of the claim and thus also of the theorem. ■

76.2.4. **Proof of the Cross-Cup-Cap Theorem 76.1: Conclusion.** Now we can finally provide the proof of the Cross-Cup-Cap Theorem 76.1:

Proof of the Cross-Cup-Cap Theorem 76.1. To simplify the notation we only consider the special case that $A = B = \varnothing$. For the general case one needs to take more care, we refer to [**Dol56**, p. 240 and p. 244] for details. So let X and Y be two topological spaces. We denote by $p\colon X \times Y \to X$ and $q\colon X \times Y \to Y$ the projections. Let $[\alpha] \in \mathrm{H}^k(X;\mathbb{Z})$, $[\beta] \in \mathrm{H}^l(Y;\mathbb{Z})$ and let $[\mu] \in \mathrm{H}_m(X;\mathbb{Z})$ and $[\nu] \in \mathrm{H}_n(Y;\mathbb{Z})$. We have to show that

$$
(\underbrace{[p^*(\alpha)]}_{\in \mathrm{H}^k(X\times Y)} \cup \underbrace{[q^*(\beta)]}_{\in \mathrm{H}^l(X\times Y)}) \cap \underbrace{([\mu] \times [\nu])}_{\in \mathrm{H}_{m+n}(X\times Y)} = (-1)^{kl+lm} \cdot \underbrace{([\alpha] \cap [\mu])}_{\in \mathrm{H}_{m-k}(X)} \times \underbrace{([\beta] \cap [\nu])}_{\in \mathrm{H}_{n-l}(Y)}
$$

[600] More precisely, given $x \in X$ and $y \in Y$ both routes along the triangle send (x, y) to $x \otimes y$. Furthermore both routes are natural chain maps. Thus we can apply the EZ–Natural Chain Homotopy Proposition 56.8 (a).

We have the following equality in $\mathrm{H}_{m+n-k-l}(X \times Y)$:

$$
\begin{aligned}
(p^*([\alpha]) \cup q^*([\beta])) \cap ([\mu] \times [\nu]) &\overset{\text{Tautology 76.13 and the definition of } \widetilde{\cap}\text{, see page 978}}{=} (-1)^{(k+l)(m+n-k-l)} \cdot ([p^*(\alpha) \cup q^*(\beta)]) \ \widetilde{\cap}\ [\Upsilon(\mu \otimes \nu)] \\
&= (-1)^{(k+l)(m+n-k-l)} \cdot (((p^*(\alpha) \cup q^*(\beta)) \ \widetilde{\cap}\ -) \circ \Upsilon)_*([\mu \otimes \nu]) \\
&\overset{\text{by the Cross-Cup-Cap-on-Chain Level Theorem 76.15 and the fact that chain homotopic maps induce the same map on homology groups}}{=} (-1)^{(k+l)(m+n-k-l)} \cdot ((-1)^{kl} \cdot \Upsilon[k+l] \circ \Psi_{k,l} \circ (\alpha \ \widetilde{\cap}\ - \otimes \beta \ \widetilde{\cap}\ -))_*([\mu \otimes \nu]) \\
&= (-1)^{(k+l)(m+n-k-l)} \cdot (-1)^{kl} \cdot [\Upsilon[k+l] \circ \Psi_{k,l} \circ (\underbrace{\alpha \ \widetilde{\cap}\ \mu}_{\substack{=(-1)^{k(m-k)} \cdot \alpha \cap \mu \\ \in \mathrm{C}_{m-k}(X)[k]}} \otimes \underbrace{\beta \ \widetilde{\cap}\ \nu}_{\substack{=(-1)^{l(n-l)} \cdot \beta \cap \nu \\ \in \mathrm{C}_{n-l}(Y)[l]}})] \\
&\overset{\text{definition of } \Psi_{k,l}\text{, see page 979, and of the shifting from } \Upsilon \text{ to } \Upsilon[k+l]}{=} (-1)^{(k+l)(m+n-k-l)} \cdot (-1)^{kl} \cdot (-1)^{k(l+n-l)} \cdot (-1)^{k(m-k)} \cdot (-1)^{l(n-l)} \cdot [\Upsilon(\alpha \cap \mu \otimes \beta \cap \nu)] \\
&= (-1)^{kl+lm} \cdot [\alpha \cap \mu \times \beta \cap \nu] = (-1)^{kl+lm} \cdot ([\alpha] \cap [\mu] \times [\beta] \cap [\nu]). \\
&\overset{\text{cleaning up signs and Tautology 76.13}}{}
\end{aligned}
$$

■

CHAPTER 77

The Künneth Theorem for cohomology

Given two topological spaces X and Y the Künneth Theorem 56.9 determines the homology groups of $X \times Y$ in terms of the homology groups of X and Y. In this chapter we prove a cohomological analogue of this result.

Together with results from the last chapter we will be able to fully calculate the cup product of the n-dimensional torus $(S^1)^n$. In particular this will allow us to determine the Lusternik-Schnirelmann category of the n-dimensional torus.

77.1. **The Künneth Theorem for cohomology groups.** The following theorem can be viewed as the cohomological analogue of the Künneth Theorem 56.9.

Theorem 77.1. (Cohomological Künneth Theorem) Let X and Y be two topological spaces. We denote by $p\colon X \times Y \to X$ and $q\colon X \times Y \to Y$ the natural projections. We suppose that X satisfies the following technical hypothesis:

(⋇) The singular chain complex of X is chain homotopy equivalent to a chain complex where each chain group is a finitely generated free abelian group.

Then given any $n \in \mathbb{N}_0$ there exists a[601] homomorphism

$$\nu\colon \mathrm{H}^n(X \times Y; \mathbb{Z}) \to \bigoplus_{k+l=n+1} \mathrm{Tor}(\mathrm{H}^k(X;\mathbb{Z}), \mathrm{H}^l(Y;\mathbb{Z}))$$

such that the following sequence is exact:[602]

$$0 \to \bigoplus_{k+l=n} \mathrm{H}^k(X;\mathbb{Z}) \otimes \mathrm{H}^l(Y;\mathbb{Z}) \to \mathrm{H}^n(X \times Y;\mathbb{Z}) \xrightarrow{\nu} \bigoplus_{k+l=n+1} \mathrm{Tor}(\mathrm{H}^k(X;\mathbb{Z}), \mathrm{H}^l(Y;\mathbb{Z})) \to 0.$$
$$\varphi \otimes \psi \mapsto p^*(\varphi) \cup q^*(\psi)$$

If Y also satisfies (⋇), then the short exact sequence splits (albeit not naturally).

The obvious question is, when is (⋇) satisfied? We will now give two answers, one is topological, one is purely algebraic:

Lemma 77.2. (CW–Chain Complex Condition Lemma) Let X be a topological space. If X is homotopy equivalent to a CW-complex $\widetilde{X}$ such that in each dimension $\widetilde{X}$ has only finitely many cells, then X satisfies Condition (⋇) from the Cohomological Künneth Theorem 77.1.

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[601] The dog that did not bark is that here we do not claim that the map is natural. It is conceivable that with more effort one can arrange this. But our proof does not give this straightaway and we will not need such a statement anyway.

[602] The map

$$\mathrm{H}^k(X;\mathbb{Z}) \times \mathrm{H}^l(Y;\mathbb{Z}) \to \mathrm{H}^{k+l}(X \times Y;\mathbb{Z}), \quad (\varphi, \psi) \mapsto p^*(\varphi) \cup q^*(\psi)$$

is often referred to as the *cohomology cross product* and it is written as $\varphi \times \psi \coloneqq p^*(\varphi) \cup q^*(\psi)$. We will not make use of this notation.

Proof. Let X be a topological space that is homotopy equivalent to a CW-complex $\widetilde{X}$ such that in each dimension $\widetilde{X}$ has only finitely many cells. The lemma follows immediately from the following observations:

(1) By the Chain Homotopy-Lemma 36.2 being "chain homotopy equivalent" is indeed an equivalence relation.
(2) It follows easily from the Homotopy-H_*-Proposition 36.5 (1) that the singular chain complex $\mathrm{C}_*(X)$ is chain homotopy equivalent to the singular chain complex $\mathrm{C}_*(\widetilde{X})$.
(3) By the Singular-Cellular Chain Complex Proposition 50.9 we know that the singular chain complex $\mathrm{C}_*(\widetilde{X})$ is chain homotopy equivalent to the cellular chain complex $\mathrm{C}_*^{\mathrm{CW}}(\widetilde{X})$.
(4) It follows from our hypothesis on $\widetilde{X}$ together with the Skeleton-H_*-Lemma 47.1 that each cellular chain group $\mathrm{C}_k^{\mathrm{CW}}(X)$ is a finitely generated free abelian group. ■

Lemma 77.3. (Algebraic–Chain Complex Condition Lemma) Let X be a topological space. If all the homology groups $\mathrm{H}_k(X)$ are finitely generated abelian groups, then X satisfies condition (⁕) from the Cohomological Künneth Theorem 77.1.

Sketch of proof. It is straightforward to show that there exists a chain complex D_*, such that all chain groups are finitely generated free abelian groups and such that for all $k \in \mathbb{N}_0$ we have $\mathrm{H}_k(D_*) \cong \mathrm{H}_k(X)$. With a little bit of work one can then show that there exists a chain map $D_* \to \mathrm{C}_*(X)$ which induces an isomorphism of homology groups. It then follows from the Quasi Iso–Homotopy Equivalence Proposition 50.4 that this chain map is actually a chain homotopy equivalence. The details are carried out in [**Fri24**]. ■

77.2. **Proof of the Cohomological Künneth Theorem 77.1.** This section is completely dedicated to the proof of the Cohomological Künneth Theorem 77.1. In the next Section 77.3 we will study applications.

77.2.1. **The Algebraic–Cohomological Künneth Theorem.** The proof of the Cohomological Künneth Theorem 77.1 rests on a purely algebraic theorem that, given two chain complexes $\mathcal{C}$ and $\mathcal{C}'$, relates the cohomology groups of $\mathcal{C}$ and $\mathcal{C}'$ to the cohomology groups of the tensor product[603] $\mathcal{C} \otimes \mathcal{C}'$. Before we can state the theorem we need to recall some definitions that we introduced on page 917:

Definition. Let $\mathcal{C}$ and $\mathcal{C}'$ be two chain complexes and let $p, q \in \mathbb{N}_0$.

(1) We denote by
$$\Pi_{p,q} \colon (\mathcal{C}_* \otimes \mathcal{C}'_*)_{p+q} = \bigoplus_{i+j=p+q} C_i \otimes C'_j \to C_p \otimes C'_q$$
the natural projection.
(2) We denote by $\mu_{\mathbb{Z}} \colon \mathbb{Z} \otimes \mathbb{Z} \to \mathbb{Z}$ the obvious isomorphism given by $\mu_{\mathbb{Z}}(a \otimes b) = ab$.
(3) It follows from the Cocycle Tensor Lemma 72.1 that the following map is well-defined:[604]
$$\begin{aligned} \Xi_{p,q} \colon \mathrm{H}^p(\mathcal{C};\mathbb{Z}) \otimes \mathrm{H}^q(\mathcal{C}';\mathbb{Z}) &\to \mathrm{H}^{p+q}(\mathcal{C} \otimes \mathcal{C}';\mathbb{Z}) \\ ([\varphi_p] \otimes [\varphi'_q]) &\mapsto [\mu_{\mathbb{Z}} \circ (\varphi_p \otimes \varphi'_q) \circ \Pi_{p,q}]. \end{aligned}$$
Furthermore it follows immediately from the definition that $\Xi_{p,q}$ is natural with respect to $\mathcal{C}$ and $\mathcal{C}'$.

[603] We refer to page 693 for the definition of the tensor product of two chain complexes.

Now we can state the promised purely algebraic theorem, which can be viewed as the cohomological analogue of the Algebraic–Künneth Theorem 55.5.

Theorem 77.4. (Algebraic–Cohomological Künneth Theorem) Let $\mathcal{C}$ and $\mathcal{C}'$ be chain complexes. We suppose that $\mathcal{C}$ satisfies the following hypothesis:

(⋇) The chain complex $\mathcal{C}$ is chain homotopy equivalent to a chain complex where each chain group is a finitely generated free abelian group.

Then for every $n \in \mathbb{N}_0$ there exists a[605] homomorphism

$$\nu\colon \mathrm{H}^n(\mathcal{C}\otimes\mathcal{C}';\mathbb{Z}) \;\to\; \bigoplus_{p+q=n+1} \mathrm{Tor}(\mathrm{H}^p(\mathcal{C};\mathbb{Z}),\mathrm{H}^q(\mathcal{C}';\mathbb{Z}))$$

such that the following sequence is exact

$$0 \to \bigoplus_{p+q=n} \mathrm{H}^p(\mathcal{C};\mathbb{Z})\otimes\mathrm{H}^q(\mathcal{C}';\mathbb{Z}) \xrightarrow{\Xi_{p,q}} \mathrm{H}^n(\mathcal{C}\otimes\mathcal{C}';\mathbb{Z}) \xrightarrow{\nu} \bigoplus_{p+q=n+1} \mathrm{Tor}(\mathrm{H}^p(\mathcal{C};\mathbb{Z}),\mathrm{H}^q(\mathcal{C}';\mathbb{Z})) \to 0.$$

If the second chain complex $\mathcal{C}'$ also satisfies (⋇), then the short exact sequence splits (albeit not naturally).

The idea behind the proof of the Algebraic–Cohomological Künneth Theorem 77.4 is quite simple: Since a cochain complex is essentially the same as a chain complex, just with opposite directions for boundary maps, it seems realistic that we can reduce the above theorem, with some massaging, to the original Algebraic–Künneth Theorem 55.5.

To make this idea work we need the following harmless definition:

Definition. A generalized chain complex (C_*,∂_*) is a sequence

$$\ldots \to C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} C_{-1} \xrightarrow{\partial_{-1}} C_{-2} \to \ldots$$

of homomorphisms between abelian groups such that for every $i \in \mathbb{Z}$ we have the equality $\partial_{i-1}\circ\partial_i = 0$.

Example. Let (C_*,∂_*) be a chain complex. Given $n \in \mathbb{Z}_{\leq 0}$ we write $E_n \coloneqq \mathrm{Hom}(C_{-n},\mathbb{Z})$ and $e_n \coloneqq \partial^*_{-n+1}\colon E_n \to E_{n-1}$. Then (E_*,e_*) is clearly a generalized chain complex.

Theorem 55.5. (Algebraic–Künneth Theorem) Let $\mathcal{E}$ and $\mathcal{E}'$ be two generalized chain complexes. If all chain groups of the chain complex $\mathcal{E}$ are free abelian groups, then there exists a natural homomorphism

$$\nu\colon \mathrm{H}_n(\mathcal{E}\otimes\mathcal{E}') \;\to\; \bigoplus_{p+q=n-1} \mathrm{Tor}(\mathrm{H}_p(\mathcal{E}),\mathrm{H}_q(\mathcal{E}'))$$

such that the following sequence is exact:

$$\begin{array}{ccccccccc} 0 & \to & \displaystyle\bigoplus_{p+q=m} \mathrm{H}_p(\mathcal{E})\otimes\mathrm{H}_q(\mathcal{E}') & \xrightarrow{\mu} & \mathrm{H}_m(\mathcal{E}\otimes\mathcal{E}') & \xrightarrow{\nu} & \displaystyle\bigoplus_{p+q=m-1} \mathrm{Tor}(\mathrm{H}_p(\mathcal{E}),\mathrm{H}_q(\mathcal{E}')) & \to & 0. \\ & & [\sigma_p]\otimes[\tau_q] & \mapsto & [\sigma_p\otimes\tau_q] & & & & \end{array}$$

If the chain groups of $\mathcal{E}'$ are also free abelian, then this short exact sequence splits.

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[604] Here we use the notation from page 645: Given two homomorphism $f\colon A\to A'$ and $g\colon B\to B'$ be tween abelian groups we consider the induced homomorphism $f\otimes g\colon A\otimes B\to A'\otimes B'$. Thus in this setting we have the homomorphism $\varphi_p\otimes\varphi'_q\colon C_p\otimes C'_q\to\mathbb{Z}\otimes\mathbb{Z}$.

[605] Again we do not claim that this homomorphism is natural.

Proof. We proved the Algebraic–Künneth Theorem 55.5 for "ordinary" chain complexes. The proof for generalized chain complexes is basically the same. ∎

Now let us try to prove the Algebraic–Cohomological Künneth Theorem 77.4. Thus let $\mathcal{C}$ and $\mathcal{C}'$ be chain complexes and let $n \in \mathbb{N}_0$. The idea is to apply the Algebraic–Künneth Theorem 55.5 to the generalized chain complexes $\mathcal{E}_* := \operatorname{Hom}(\mathcal{C}_{-*}, \mathbb{Z})$ and $\mathcal{E}' := \operatorname{Hom}(\mathcal{C}'_{-*}, \mathbb{Z})$ and to $m = -n$. If we compare the short exact sequence from the Algebraic–Künneth Theorem 55.5 with the desired short exact sequence from the Algebraic–Cohomological Künneth Theorem 77.4 we see that the groups to the left and right look about the same. But the middle group is different. In fact the following question arises: Under what circumstances do we have an isomorphism

$$\operatorname{Hom}(\mathcal{C}_*, \mathbb{Z}) \otimes \operatorname{Hom}(\mathcal{C}'_*, \mathbb{Z}) \xrightarrow{\cong} \operatorname{Hom}(\mathcal{C}_* \otimes \mathcal{C}'_*, \mathbb{Z})\,?$$

The following lemma is the key to answering this question:

Lemma 77.5. (Tensor-of-Hom Lemma) Let A and B be abelian groups. We consider the homomorphism[606]

$$\begin{array}{rcl} \Phi\colon \operatorname{Hom}(A,\mathbb{Z}) \otimes \operatorname{Hom}(B,\mathbb{Z}) & \to & \operatorname{Hom}(A \otimes B, \mathbb{Z}) \\ \alpha \otimes \beta & \mapsto & \mu_{\mathbb{Z}} \circ \left(\begin{array}{rcl} \alpha \otimes \beta\colon A \otimes B & \to & \mathbb{Z} \otimes \mathbb{Z} \\ \sum\limits_{i=1}^{m} a_i \otimes b_i & \mapsto & \sum\limits_{i=1}^{m} \alpha(a_i) \otimes \beta(b_i) \end{array} \right). \end{array}$$

If A is a finitely generated free abelian group, then above map is an isomorphism.

Proof. If A is a finitely generated free abelian group, then we can find a basis $a_1, \ldots, a_k$ for A. We denote by $a_1^*, \ldots, a_k^*$ the dual basis of $\operatorname{Hom}(A, \mathbb{Z})$. We consider the homomorphism

$$\begin{array}{rcl} \Psi\colon \operatorname{Hom}(A \otimes B, \mathbb{Z}) & \to & \operatorname{Hom}(A,\mathbb{Z}) \otimes \operatorname{Hom}(B,\mathbb{Z}) \\ \theta & \mapsto & \sum\limits_{i=1}^{k} a_i^* \otimes (b \mapsto \theta(a_i \otimes b)). \end{array}$$

It remains to prove the following claim:

Claim. The map Φ and Ψ are inverses of one another.

Proof. First we show that $\Psi \circ \Phi = \mathrm{id}_{\operatorname{Hom}(A,\mathbb{Z}) \otimes \operatorname{Hom}(B,\mathbb{Z})}$. Note that by the Tensor Product–Universal Property Lemma 51.1 (2a) it suffices to prove that $\Psi \circ \Phi$ is the identity on any element of the form $a_j^* \otimes \beta \in \operatorname{Hom}(A,\mathbb{Z}) \otimes \operatorname{Hom}(B,\mathbb{Z})$. But for those elements we have indeed

$$(\Psi \circ \Phi)(a_j^* \otimes \beta) \underset{\text{definition of } \Phi}{\overset{\uparrow}{=}} \Psi(\mu_{\mathbb{Z}} \circ (a_j^* \otimes \beta)) \underset{\text{definition of } \Psi}{\overset{\uparrow}{=}} \sum_{i=1}^{k} a_i^* \otimes (b \mapsto \underbrace{(\mu_{\mathbb{Z}} \circ (a_j^* \otimes \beta))(a_i \otimes b)}_{= a_j^*(a_i) \cdot \beta(b) = \delta_{ij} \cdot \beta(b)}) = a_j^* \otimes \beta.$$

It remains to show that $\Psi \circ \Phi = \mathrm{id}_{\operatorname{Hom}(A \otimes B, \mathbb{Z})}$. So let $\theta \in \operatorname{Hom}(A \otimes B, \mathbb{Z})$. We need to show that $(\Phi \circ \Psi)(\theta) = \theta\colon A \otimes B \to \mathbb{Z}$. Again by the Tensor Product–Universal Property Lemma 51.1 (2a) it suffices to show that these two homomorphisms agree on any element

[606] It follows easily from the Tensor Product–Universal Property Lemma 51.1 (3) that this map Φ is a well-defined homomorphism.

of the form $a_j \otimes \widetilde{b} \in A \otimes B$. For such elements we calculate

$$\begin{aligned}((\Phi \circ \Psi)(\theta))(a_j \otimes \widetilde{b}) &\overset{\text{definition of } \Psi}{=} \Phi\Big(\sum_{i=1}^{k} a_i^* \otimes (b \mapsto \theta(a_i \otimes b))\Big)(a_j \otimes \widetilde{b}) \overset{\text{definition of } \Phi}{=} \\ &= \sum_{i=1}^{k} \underbrace{(\mu_{\mathbb{Z}} \circ (a_i^* \otimes (b \mapsto \theta(a_i \otimes b))))(a_j \otimes \widetilde{b})}_{= \mu_{\mathbb{Z}}(a_i^*(a_j) \otimes \theta(a_i \otimes \widetilde{b})) = \delta_{ij} \cdot \theta(a_i \otimes \widetilde{b})} = \theta(a_j \otimes \widetilde{b}).\end{aligned}$$

■

Lemma 77.6. (Cochain–Tensor Lemma) Let $\mathcal{C}_*$ and $\mathcal{C}'_*$ be two chain complexes. If all chain groups of $\mathcal{C}$ are finitely generated free abelian groups, then the map

$$\begin{array}{rcl}\Omega\colon \operatorname{Hom}(\mathcal{C}_*, \mathbb{Z}) \otimes \operatorname{Hom}(\mathcal{C}'_*, \mathbb{Z}) & \to & \operatorname{Hom}(\mathcal{C}_* \otimes \mathcal{C}'_*, \mathbb{Z}) \\ (\varphi_p\colon C_p \to \mathbb{Z}) \otimes (\varphi'_q\colon C'_q \to \mathbb{Z}) & \mapsto & (\mu_{\mathbb{Z}} \circ (\varphi_p \otimes \varphi'_q) \circ \Pi_{p,q}\colon (\mathcal{C}_* \otimes \mathcal{C}'_*)_{p+q} \to \mathbb{Z})\end{array}$$

is an isomorphism of cochain complexes.

Proof. We start out with the following claim:

Claim 1. The map Ω is a cochain map.

Proof. This statement follows easily from the definitions. In fact, we basically proved this statement in the proof of the Cocycle Tensor Lemma 72.1. We leave it to the reader to fill in the details. ⊞

It follows from Claim 1 that it remains to show that for every degree $n \in \mathbb{N}_0$ the given map Ω is an isomorphism. In other words, we need to prove the following claim:

Claim 2. For every $n \in \mathbb{N}_0$ the following map is an isomorphism:

$$\Omega\colon \bigoplus_{p+q=n} \operatorname{Hom}(C_p, \mathbb{Z}) \otimes \operatorname{Hom}(C'_q, \mathbb{Z}) \to \operatorname{Hom}\Big(\bigoplus_{p+q=n} C_p \otimes C'_q, \mathbb{Z}\Big).$$

Proof. We consider the following two maps:

$$\begin{array}{ccccc}\displaystyle\bigoplus_{p+q=n} \operatorname{Hom}(C_p, \mathbb{Z}) \otimes \operatorname{Hom}(C'_q, \mathbb{Z}) & \overset{\Omega}{\to} & \operatorname{Hom}\Big(\displaystyle\bigoplus_{p+q=n} C_p \otimes C'_q, \mathbb{Z}\Big) & \to & \displaystyle\bigoplus_{p+q=n} \operatorname{Hom}(C_p \otimes C'_q, \mathbb{Z}) \\ \varphi_p \otimes \varphi'_q & \mapsto & \mu_{\mathbb{Z}} \circ (\varphi_p \otimes \varphi'_q) \circ \Pi_{p,q} & & \\ & & \theta & \mapsto & \displaystyle\sum_{p+q=n} \theta \circ \iota_{p,q}.\end{array}$$

We make the following clarifications and observations:

(1) The map to the left is precisely the map Ω for which we want to show that it is an isomorphism.

(2) For $p, q \in \mathbb{N}_0$ with $p + q = n$ let $\iota_{p,q}\colon C_p \otimes C'_q \to \displaystyle\bigoplus_{r+s=n} C_r \otimes C'_s$ the natural inclusion.

(3) It follows pretty easily from the Coproduct-of-Sets-and-Abelian Groups Lemma 4.4 that the map to the right is an isomorphism.

(4) The composition of the two maps is, for each $p, q \in \mathbb{N}_0$, given by the map

$$\begin{array}{rcl}\operatorname{Hom}(C_p, \mathbb{Z}) \otimes \operatorname{Hom}(C'_q, \mathbb{Z}) & \to & \operatorname{Hom}(C_p \otimes C'_q, \mathbb{Z}) \\ \varphi \otimes \varphi' & \mapsto & \mu_{\mathbb{Z}} \circ (\varphi \otimes \varphi').\end{array}$$

from the Tensor-of-Hom Lemma 77.5. By hypothesis all groups C_p are finitely generated free abelian groups. Thus it follows from the Cochain–Tensor Lemma 77.6 that the above map is an isomorphism.

In summary we have shown that the right map and the composition of the two maps is an isomorphism. It follows that the left map Ω is an isomorphism. ■

Now we can finally prove the Algebraic–Cohomological Künneth Theorem 77.4.

Proof of the Algebraic–Cohomological Künneth Theorem 77.4. Let $\mathcal{C} = (C_*, \partial_*)$ and $\mathcal{C}' = (C'_*, \partial'_*)$ be chain complexes. We assume that $\mathcal{C}$ satisfies (⁕), i.e. we assume that $\mathcal{C}$ is chain homotopy equivalent to a chain complex such that all chain groups are finitely generated abelian groups.

The proof proceeds in three steps:

(1) It follows from the Tensoring Chain Maps Lemma 55.3 (3), the Cochain Homotopy-H*-Lemma 66.6 and the Dual Cochain Map Lemma 66.9 (3b) that we can replace $\mathcal{C}$ by the chain complex where all chain groups are finitely generated free abelian groups. (Note though that at this point we loose naturality.)

(2) Given $n \in \mathbb{Z}_{\leq 0}$ we write

$$\begin{aligned} E_n &:= \operatorname{Hom}(C_{-n}, \mathbb{Z}), & e_n &:= \partial^*_{-n+1} \colon E_n \to E_{n-1}, \\ E'_n &:= \operatorname{Hom}(C'_{-n}, \mathbb{Z}), & e'_n &:= \partial^*_{-n+1} \colon E'_n \to E'_{n-1}. \end{aligned}$$

It follows immediately from the definitions that $\mathcal{E} := (E_*, e_*)$ and $\mathcal{E}' := (E'_*, e'_*)$ form chain complexes in negative degrees. Since the groups C_* are finitely generated free abelian groups we see that the groups E_* are free abelian.[607] Therefore we can apply the Algebraic–Künneth Theorem 55.5 and we obtain for each $m \in \mathbb{Z}_{\leq 0}$ a natural homomorphism

$$\nu \colon \mathrm{H}_n(\mathcal{E} \otimes \mathcal{E}') \to \bigoplus_{p+q=m-1} \operatorname{Tor}(\mathrm{H}_p(\mathcal{E}), \mathrm{H}_q(\mathcal{E}'))$$

such that the following sequence is exact:

$$\begin{array}{ccccccccc} 0 & \to & \displaystyle\bigoplus_{p+q=m} \mathrm{H}_p(\mathcal{E}) \otimes \mathrm{H}_q(\mathcal{E}') & \to & \mathrm{H}_m(\mathcal{E} \otimes \mathcal{E}') & \xrightarrow{\nu} & \displaystyle\bigoplus_{p+q=m-1} \operatorname{Tor}(\mathrm{H}_p(\mathcal{E}), \mathrm{H}_q(\mathcal{E}')) & \to & 0. \\ & & [\varphi] \otimes [\psi] & \mapsto & [\varphi \otimes \psi] & & & & \end{array}$$

Put differently, for each $n \in \mathbb{N}_0$ we can apply the above discussion to $m = -n$ and we obtain the following exact sequence:

$$0 \to \bigoplus_{-p-q=-n} \underbrace{\mathrm{H}_{-p}(\mathcal{E})}_{=\mathrm{H}^p(\mathcal{C};\mathbb{Z})} \otimes \underbrace{\mathrm{H}_{-q}(E')}_{=\mathrm{H}^q(\mathcal{C}';\mathbb{Z})} \to \mathrm{H}_{-n}(E \otimes E') \xrightarrow{\nu} \bigoplus_{-p-q=-n-1} \operatorname{Tor}(\underbrace{\mathrm{H}_{-p}(E)}_{=\mathrm{H}^p(\mathcal{C};\mathbb{Z})}, \underbrace{\mathrm{H}_{-q}(E')}_{=\mathrm{H}^p(\mathcal{C}';\mathbb{Z})}) \to 0.$$

(3) For each $n \in \mathbb{N}_0$ we have isomorphisms

$$\mathrm{H}_{-n}(E_* \otimes E'_*) \underset{\text{by definition}}{=} \mathrm{H}^n(E_{-*} \otimes E'_{-*}) \underset{\text{by definition}}{=} \mathrm{H}^n(\overbrace{\operatorname{Hom}(C_*, \mathbb{Z})}^{=\mathcal{E}_{-*}} \otimes \overbrace{\operatorname{Hom}(C'_*, \mathbb{Z})}^{=\mathcal{E}'_{-*}}) \underset{\substack{\text{by the Cochain–Tensor Lemma 77.6,} \\ \text{since all chain groups of } \mathcal{C}_* \text{ are} \\ \text{finitely generated free abelian groups}}}{\xrightarrow{\Omega}} \mathrm{H}^n(C_* \otimes C'_*; \mathbb{Z}).$$

We use this isomorphism to replace the middle term of the second short exact sequence in (2) and we obtain the desired short exact sequence.

[607] Here we use the following observation: If D is a finitely generated free abelian group, then it follows from the Abelian Group-Homomorphism Lemma 66.2 (1) and (3) that $\operatorname{Hom}(D, \mathbb{Z})$ is again a finitely generated free abelian group. As we discussed on page 856, the analogous statement for arbitrary free abelian groups does *not* hold.

Finally suppose that $\mathcal{C}'$ also satisfies (⋇). In other words, we assume that $\mathcal{C}'$ is chain homotopy equivalent to a chain complex such that all chain groups are finitely generated abelian groups. By the same discussion as above this allows us to replace $\mathcal{C}'$ by a chain complex such that all chain groups are finitely generated free abelian groups. This observation implies that $\mathcal{E}'_*$ is a chain complex consisting of free abelian groups. This then implies, by the last statement of the Algebraic–Künneth Theorem 55.5, that the short exact sequence splits. ■

77.2.2. **Proof of the Cohomological Künneth Theorem 77.1.** Now we are in a position to prove the Cohomological Künneth Theorem 77.1:

Proof of the Cohomological Künneth Theorem 77.1. Let X and Y be two topological spaces. We assume that X satisfies the following hypothesis:

(⋇) The singular chain complex of X is chain homotopy equivalent to a chain complex where each chain group is a finitely generated free abelian group.

We denote by $p\colon X \times Y \to X$ and $q\colon X \times Y \to Y$ the natural projections. Throughout the proof we use all the definitions that we introduced in Section 76.2. We consider the following diagram:

$$\begin{array}{ccccccccc} & & & & \mathrm{H}^n(\mathrm{C}_*(X)\otimes\mathrm{C}_*(Y);\mathbb{Z}) & & & & \\ & & & \overset{[\alpha]\otimes[\beta]\mapsto[(\alpha\otimes\beta)\circ\Pi_{k,l}]}{\nearrow} & \cong\big\downarrow\Theta^* & \overset{\nu}{\searrow} & & & \\ 0 \to & \bigoplus_{k+l=n} \mathrm{H}^k(X;\mathbb{Z})\otimes\mathrm{H}^l(Y;\mathbb{Z}) & \xrightarrow{[\alpha]\otimes[\beta]\mapsto[p^*\alpha\cup q^*\beta]} & \underbrace{\mathrm{H}^n(X\times Y;\mathbb{Z})}_{=\mathrm{H}^n(\mathrm{C}_*(X\times Y),\mathbb{Z})} & \to & \bigoplus_{k+l=n+1} \mathrm{Tor}(\mathrm{H}^k(X;\mathbb{Z}),\mathrm{H}^l(Y;\mathbb{Z})) & \to & 0. \end{array}$$

We make the following clarifications:

(1) The goal is to show that the "lower" sequence is exact.
(2) The "upper" sequence is the short exact sequence which we obtain from the Algebraic–Cohomological Künneth Theorem 77.4, where we use that X satisfies (⋇).
(3) The map $\Theta\colon \mathrm{C}_*(X\times Y) \to \mathrm{C}_*(X)\otimes\mathrm{C}_*(Y)$ is one of the chain homotopy equivalences provided by the Eilenberg-Zilber Theorem 56.1. It follows from the Dual Cochain Map Lemma 66.9 together with the Cochain Homotopy-H^*-Lemma 66.6 that the vertical map is an isomorphism.
(4) We define the bottom right map to be $\nu \circ (\Theta^*)^{-1}$.

We see that it remains to prove that the triangle to the left commutes. In other words, we need to prove the following claim:

Claim. For every $k, l \in \mathbb{N}_0$ the map

$$\begin{array}{rcl} \mathrm{H}^k(X;\mathbb{Z})\otimes\mathrm{H}^l(Y;\mathbb{Z}) & \to & \mathrm{H}^{k+l}(X\times Y;\mathbb{Z}) \\ [\alpha]\otimes[\beta] & \mapsto & [p^*(\alpha)\cup q^*(\beta)] \end{array}$$

agrees with the map that is given by

$$[\alpha]\otimes[\beta] \mapsto [(\alpha\otimes\beta)\circ\Pi_{k,l}\circ\Theta].$$

Proof. Let $\alpha \in \mathrm{C}^k(X;\mathbb{Z})$ and $\beta \in \mathrm{C}^l(Y;\mathbb{Z})$ be cocycles. We have to show that the two cocycles

$$\underbrace{p^*(\alpha) \cup q^*(\beta)}_{=:\aleph} \quad \text{and} \quad \underbrace{(\alpha \otimes \beta) \circ \Pi_{k,l}}_{=:\beth} \circ \Theta$$

are cohomologous.

As usual, given a topological space W we denote by $\epsilon_W \colon \mathrm{C}_0(W) \to \mathbb{Z}$ the augmentation map. Clearly ϵ_W is a cocycle, so by the Projection–Cocycle Lemma 76.8 (1) we can extend ϵ_W to a chain map $\mathrm{C}_*(W) \to \mathbb{Z}[0]$ which, by a slight abuse of notation, we also denote by ϵ_W. It follows from the Eilenberg-Zilber Theorem 56.1, the Cross-Cup-Cap-on-Chain Level Theorem 76.15 and the Projection–Cocycle Lemma 76.8 (2) that the maps

$$\underbrace{(-1)^{kl} \cdot \epsilon_{X\times Y}[k+l] \circ ((p^*\alpha \cup q^*\beta) \mathbin{\widetilde{\cap}} -)}_{=:\aleph'} \colon \mathrm{C}_{k+l}(X \times Y) \to (\mathbb{Z}[k+l])_{k+l} = \mathbb{Z}$$

and

$$\underbrace{\epsilon_{X\times Y}[k+l] \circ \Upsilon[k+l] \circ \Psi_{k,l} \circ (\alpha \mathbin{\widetilde{\cap}} - \otimes \beta \mathbin{\widetilde{\cap}} -)}_{=:\beth'} \circ \Theta \colon \mathrm{C}_{k+l}(X \times Y) \to (\mathbb{Z}[k+l])_{k+l} = \mathbb{Z}$$

are cocycles that are cohomologous. It remains to prove the following two subclaims.

Subclaim. We have $\aleph' = (-1)^{kl} \cdot \aleph \colon \mathrm{C}_{k+l}(X \times Y) \to \mathbb{Z}$.

Proof of subclaim. Let $\sigma \in \mathrm{C}_{k+l}(X \times Y)$. Then

$$\aleph'(\sigma) \underset{\uparrow}{=} (-1)^{kl} \cdot \epsilon_{X\times Y}((p^*\alpha \cup q^*\beta) \mathbin{\widetilde{\cap}} \sigma) \underset{\uparrow}{=} (-1)^{kl} \cdot (p^*\alpha \cup q^*\beta)(\sigma) = (-1)^{kl} \cdot \aleph(\sigma).$$

since we have $(\mathbb{Z}[k+l])_{k+l} = \mathbb{Z}$ we have $\epsilon_{X\times Y}[k+l] = \epsilon_{X\times Y}$

by definition of $\widetilde{\cap}$, see page 978, together with the Cap Product-Kronecker Lemma 75.7 □

Subclaim. We have $\beth' = (-1)^{kl} \cdot \beth \colon (\mathrm{C}_*(X) \otimes \mathrm{C}_*(Y))_{k+l} \to \mathbb{Z}$.

Proof of subclaim. We have to show that both sides take the same value on any tensor product of the form $\sigma_i \otimes \sigma_j$ with $i + j = k + l$ and $\sigma_i \in \mathrm{C}_i(X)$ and let $\sigma_j \in \mathrm{C}_j(Y)$. It follows easily from the definition that both sides are zero for $\sigma_i \otimes \sigma_j$ with $(i, j) \neq (k, l)$. So let $\sigma_k \in \mathrm{C}_k(X)$ and let $\sigma_l \in \mathrm{C}_l(Y)$. Then

$$\begin{aligned}
\beth'(\sigma_k \otimes \sigma_l) &\underset{\uparrow}{=} (\epsilon_{X\times Y} \circ \Upsilon)[k+l] \circ \Psi_{k,l} \circ (\alpha \mathbin{\widetilde{\cap}} \sigma_k \otimes \beta \mathbin{\widetilde{\cap}} \sigma_l) \\
&\qquad \text{since } \epsilon_{X\times Y}[k+l] \circ \Upsilon[k+l] = (\epsilon_{X\times Y} \circ \Upsilon)[k+l] \\
&\underset{\uparrow}{=} (-1)^{kl} \cdot \mu_{\mathbb{Z}} \circ (\epsilon_X \otimes \epsilon_Y) \circ (\alpha \cap \sigma_k \otimes \beta \cap \sigma_l) \underset{\uparrow}{=} (-1)^{kl} \cdot \alpha(\sigma_k) \cdot \beta(\sigma_l) \\
&\qquad \text{we use } \epsilon_{X\times Y} \circ \Upsilon = \mu_{\mathbb{Z}} \circ \epsilon_X \otimes \epsilon_Y \colon (\mathrm{C}_*(X) \otimes \mathrm{C}_*(Y))_0 \to \mathbb{Z}, \text{ definition of } \Psi_{k,l} \text{ in the Tensor-and-Shift Lemma 76.14 and the definition of } \widetilde{\cap} \text{ on page 978} \qquad \text{by the Cap Product-Kronecker Lemma 75.7} \\
&= (-1)^{kl} \cdot ((\alpha \otimes \beta) \circ \Pi_{k,l})(\sigma_k \otimes \sigma_l) = (-1)^{kl} \cdot \beth(\sigma_k \otimes \sigma_l).
\end{aligned}$$

This concludes the proof of the second subclaim, thus also of the claim and therefore also of the theorem. ■

77.3. **The cohomology ring of product topological spaces.** Let X and Y be two topological spaces. In the Cohomological Künneth Theorem 77.1 we just showed that (under some mild hypothesis) we have a short exact sequence

$$0 \to \bigoplus_{k+l=n} \mathrm{H}^k(X;\mathbb{Z}) \otimes \mathrm{H}^l(Y;\mathbb{Z}) \to \mathrm{H}^n(X \times Y;\mathbb{Z}) \xrightarrow{\nu} \bigoplus_{k+l=n+1} \mathrm{Tor}(\mathrm{H}^k(X;\mathbb{Z}), \mathrm{H}^l(Y;\mathbb{Z})) \to 0.$$

$$\varphi \otimes \psi \mapsto p^*(\varphi) \cup q^*(\psi)$$

In many cases this result allows us to calculate the cohomology *groups* of a product. But our real goal is to calculate the cohomology *rings* of a product. To get to this point, we

need to make some algebraic preparations. First we recall the following definition from page 929:

Definition.

(1) A graded ring is a ring R together with a decomposition $R = \bigoplus_{n\in\mathbb{N}_0} R_n$ such that for any $m, n \in \mathbb{N}_0$ the multiplication map restricts to a map $R_m \times R_n \to R_{m+n}$.
(2) A homomorphism $\varphi\colon A \to B$ between graded rings is a ring homomorphism that preserves the gradings, i.e. that satisfies $\varphi(A_n) \subset B_n$ for all $n \in \mathbb{N}_0$.
(3) We say that an element r of R that lies in some R_n is homogeneous and we refer to $\deg(r) := n$ as the degree of r.
(4) A graded ring $R = \bigoplus_{n\in\mathbb{N}_0} R_n$ such that for any two homogeneous elements a and b the equality $a \cdot b = (-1)^{\deg(a)\cdot\deg(b)} \cdot b \cdot a$ holds is called a superalgebra.

Examples.

(1) In the Cup Product-Ring Proposition 72.9 we saw that for any topological space X the cohomology ring $(\mathrm{H}^*(X;\mathbb{Z}), \cup)$ is a superalgebra. Furthermore the Cup Product-Ring Proposition 72.9 (3) implies that for any continuous map $f\colon X \to Y$ the induced map $f^*\colon \mathrm{H}^*(Y;\mathbb{Z}) \to \mathrm{H}^*(X;\mathbb{Z})$ is a homomorphism of superalgebras.
(2) The polynomial ring $\mathbb{Z}[x] = \bigoplus_{i\in\mathbb{N}_0} \mathbb{Z} \cdot x^i$ is a superalgebra if we equip the ring with the grading that is determined by setting $\deg(x^i) := 2 \cdot i$.
(3) More generally, the polynomial ring $\mathbb{Z}[x_1, \ldots, x_k]$ in k variables $x_1, \ldots, x_k$ can be viewed as a superalgebra by setting $\deg(x_1^{r_1} \cdot \ldots \cdot x_k^{r_k}) = 2 \cdot (r_1 + \cdots + r_k)$.
(4) The ring $\mathbb{Z}[x]/(x^2) = \mathbb{Z} \oplus \mathbb{Z} \cdot x$ is a superalgebra by viewing elements in $\mathbb{Z}$ as elements of degree 0 and by viewing elements in $\mathbb{Z} \cdot x$ as elements of degree 1. Note that the map

$$\begin{array}{rcl} \mathbb{Z}[x]/(x^2) & \to & \mathrm{H}^*(S^1;\mathbb{Z}) \\ a \cdot 1 + b \cdot x & \mapsto & a \cdot 1_{S^1} + b \cdot [S^1]^* \end{array}$$

is an isomorphism of superalgebras.
(5) We consider the ring

$$\Lambda[x_1, \ldots, x_k] := \mathbb{Z}[x_1, \ldots, x_k]/\text{ideal generated by } x_1^2, \ldots, x_k^2, \{x_i x_j + x_j x_i\}_{i,j=1,\ldots,k}.$$

In this ring we use the symbol "$\wedge$" for multiplication instead of "$\cdot$". For example for any $i, j \in \{1, \ldots, k\}$ we have $x_i \wedge x_i = 0$ and $x_i \wedge x_j = -x_j \wedge x_i$. This ring can be viewed in a unique way as a superalgebra such that $\deg(x_j) = 1$ for $j = 1, \ldots, k$. We leave it as an amusing exercise to verify that $\Lambda[x_1, \ldots, x_k]$ is a free abelian group of rank 2^k where a basis is given by the set $\{x_{n_1} \wedge \cdots \wedge x_{n_l} \mid 1 \le n_1 < \cdots < n_l \le k\}$. □

Definition. The tensor product of two superalgebras $A = \bigoplus_{n\in\mathbb{N}_0} A_n$ and $B = \bigoplus_{n\in\mathbb{N}_0} B_n$ is defined as the graded abelian group

$$A \otimes B := \bigoplus_{n\in\mathbb{N}_0} \underbrace{\bigoplus_{p+q=n} A_p \otimes B_q}_{=:(A\otimes B)_n}$$

where for $a_k \in A_k, a_l \in A_l, b_m \in B_m$ and $b_n \in B_n$ the multiplication is given by[608]

$$(a_k \otimes b_l) \cdot (a_m \otimes b_n) \;:=\; (-1)^{lm} \cdot a_k \cdot a_m \otimes b_l \cdot b_n.$$

Lemma 77.7. (Super Algebra–Tensor Product Lemma)

(1) The tensor product of two superalgebras is again a superalgebra.

(2) The tensor product of superalgebras is associative, i.e. for any three superalgebras A, B and C the map[609]

$$\begin{array}{rcl}(A \otimes B) \otimes C & \to & A \otimes (B \otimes C)\end{array}$$

that is induced by

$$\begin{array}{rcl}(a \otimes b) \otimes c & \mapsto & a \otimes (b \otimes c)\end{array}$$

is an isomorphism of superalgebras.

Proof. Let A, B and C be superalgebras and let $a_k \in A_k, a_l \in A_l, b_m \in B_m$ and $b_n \in B_n$.

(1) We have the following equalities:

$$\begin{array}{rcl}(a_k \otimes b_l) \cdot (a_m \otimes b_n) & = & (-1)^{lm} \cdot a_k \cdot a_m \otimes b_l \cdot b_n \;=\; (-1)^{lm+km+ln} \cdot a_m \cdot a_k \otimes b_n \cdot b_l \\ & = & (-1)^{lm+km+ln+kn} \cdot (a_m \otimes b_n) \cdot (a_k \otimes b_l) \\ & = & (-1)^{(m+n)(k+l)} \cdot (a_m \otimes b_n) \cdot (a_k \otimes b_l) \\ & = & (-1)^{\deg(a_m \otimes b_n) \cdot \deg(a_k \otimes b_l)} \cdot (a_m \otimes b_n) \cdot (a_k \otimes b_l).\end{array}$$

It follows immediately that $A \otimes B$ is again a superalgebra.

(2) It suffices to show that the indicated map is a homomorphism of superalgebras. Thus let $c_r \in C_r$ and $c_s \in C_s$. Then

$$\begin{array}{rcl}((a_k \otimes b_l) \otimes c_r) \cdot ((a_m \otimes b_n) \otimes c_s) & = & (-1)^{r(m+n)} \cdot ((a_k \otimes b_l) \cdot (a_m \otimes b_n)) \otimes (c_r \cdot c_s) \\ & = & (-1)^{r(m+n)+lm} \cdot ((a_k \cdot a_m) \otimes (b_l \cdot b_n) \otimes (c_r \cdot c_s)) \\ & = & (-1)^{m(l+r)} \cdot (a_k \cdot a_m) \otimes ((b_l \otimes c_r) \cdot (b_n \otimes c_s)) \\ & = & (a_k \otimes (b_l \otimes c_r)) \cdot (a_m \otimes (b_n \otimes c_s)).\end{array}$$

■

Examples.

(1) As before we view the polynomial rings $\mathbb{Z}[x_j]$, $j = 1, \ldots, k$ as superalgebras with $\deg(x_j^i) = 2i$. We consider the map

$$\begin{array}{rcl}\mathbb{Z}[x_1] \otimes \cdots \otimes \mathbb{Z}[x_k] & \to & \mathbb{Z}[x_1, \ldots, x_k] \\ x_1^{r_1} \otimes \cdots \otimes x_k^{r_k} & \mapsto & x_1^{r_1} \cdot \ldots \cdot x_k^{r_k}.\end{array}$$

One can easily verify that this map is an isomorphism of superalgebras.

(2) As above we view the rings $\mathbb{Z}[x_j]/(x_j^2)$ as superalgebras with $\deg(x_j) = 1$. We leave it as an amusing exercise to verify that the map

$$\begin{array}{rcl}\mathbb{Z}[x_1]/(x_1^2) \otimes \cdots \otimes \mathbb{Z}[x_k]/(x_k^2) & \to & \Lambda[x_1, \ldots, x_k] \\ x_1^{r_1} \otimes \cdots \otimes x_k^{r_k} & \mapsto & x_1^{r_1} \wedge \cdots \wedge x_k^{r_k}\end{array}$$

is an isomorphism of superalgebras. □

The following proposition explains our sudden interest in tensor products of superalgebras.

[608] The mnemonic for the multiplication rule is that swapping b_l and a_m introduces the (by now familiar) sign $(-1)^{lm}$.

[609] It follows from this statement that henceforth we can ignore parentheses when we consider the tensor product of several superalgebras.

Proposition 77.8. (Cohomology Ring-of-Product Proposition) Let X and Y be two topological spaces. We denote by $p\colon X \times Y \to X$ and $q\colon X \times Y \to Y$ the natural projections.

(1) The map
$$\begin{aligned} \Phi\colon \mathrm{H}^*(X;\mathbb{Z}) \otimes \mathrm{H}^*(Y;\mathbb{Z}) &\to \mathrm{H}^*(X \times Y;\mathbb{Z}) \\ \varphi \otimes \psi &\mapsto p^*(\varphi) \cup q^*(\psi) \end{aligned}$$
from the Cohomological Künneth Theorem 77.1 is a homomorphism of superalgebras.

(2) If X satisfies the condition (⁕) from the Cohomological Künneth Theorem 77.1, and if all homology groups of X are torsion-free, then the homomorphism from (1) is in fact an isomorphism.

Proof. Let X and Y be two topological spaces.

(1) Let $\varphi_k \in \mathrm{H}^k(X;\mathbb{Z})$, $\psi_l \in \mathrm{H}^l(Y;\mathbb{Z})$, $\varphi_m \in \mathrm{H}^m(X;\mathbb{Z})$ and $\psi_n \in \mathrm{H}^n(Y;\mathbb{Z})$. Then
$$\begin{aligned} & \overbrace{\Phi((-1)^{lm} \cdot (\varphi_k \cup \varphi_m) \otimes (\psi_l \cup \psi_n))}^{= \Phi((\varphi_k \otimes \psi_l) \cdot (\varphi_m \otimes \psi_n))} \\ &= (-1)^{lm} \cdot (p^*(\varphi_k) \cup p^*(\varphi_m) \cup q^*(\psi_l) \cup q^*(\psi_n)) \\ &\quad \uparrow \\ &\quad \text{by the Cup Product-Naturality Proposition 72.8} \end{aligned} \qquad \begin{aligned} &= (-1)^{lm} \cdot p^*(\varphi_k \cup \varphi_m) \cup q^*(\psi_l \cup \psi_n) \\ &= \underbrace{p^*(\varphi_k) \cup q^*(\psi_l) \cup p^*(\varphi_m) \cup q^*(\psi_n)}_{= \Phi(\varphi_k \otimes \psi_l) \cup \Phi(\varphi_m \otimes \psi_n))}. \\ &\quad \uparrow \\ &\quad \text{by the Cup Product-Commutativity Lemma 72.2} \end{aligned}$$
It follows immediately that Φ is a homomorphism of superalgebras.

(2) By our hypothesis on X we can apply the Cohomological Künneth Theorem 77.1 and we obtain the following short exact sequence:
$$\begin{aligned} 0 \to \bigoplus_{k+l=n} \mathrm{H}^k(X;\mathbb{Z}) \otimes \mathrm{H}^l(Y;\mathbb{Z}) &\to \mathrm{H}^n(X \times Y;\mathbb{Z}) \to \bigoplus_{k+l=n+1} \underbrace{\mathrm{Tor}(\mathrm{H}^k(X;\mathbb{Z}), \mathrm{H}^l(Y;\mathbb{Z}))}_{= 0, \text{ see below}} \to 0. \\ \varphi \otimes \psi &\mapsto p^*(\varphi) \cup q^*(\psi) \end{aligned}$$
Since the homology groups of X are torsion-free we obtain from the Universal Coefficient-H^*-Theorem 70.14 together with the Ext-Properties Lemma 70.12 (4) that the cohomology groups $\mathrm{H}^k(X;\mathbb{Z})$ are also torsion-free. But by the Tor-Properties Lemma 54.7 (3) this implies that all the Tor-terms on the right-hand side of the short exact sequence vanish. ∎

Proposition 77.9. (Torus–Cup Product Proposition) Let $n \in \mathbb{N}_0$. We denote by $(S^1)^n$ the n-dimensional torus. For $i = 1, \ldots, n$ we denote by $p_i\colon (S^1)^n \to S^1$ the projection onto the i-th factor. There exists a unique isomorphism
$$\psi\colon \Lambda[x_1, \ldots, x_n] \to \mathrm{H}^*((S^1)^n;\mathbb{Z})$$
of superalgebras which has the property that for each $i \in \{1, \ldots, n\}$ we have
$$\psi(x_i) = p_i^*([S^1]^*).$$

Proof. We consider the following isomorphism of superalgebras

$$\Lambda[x_1, \ldots, x_n] \overset{\cong}{\underset{\uparrow}{\leftarrow}} \mathbb{Z}[x_1]/(x_1^2) \otimes \ldots \otimes \mathbb{Z}[x_n]/(x_n^2) \overset{\cong}{\underset{\uparrow}{\rightarrow}} \mathrm{H}^*(S^1;\mathbb{Z}) \otimes \ldots \otimes \mathrm{H}^*(S^1;\mathbb{Z}) \overset{\cong}{\underset{\uparrow}{\rightarrow}} \mathrm{H}^*((S^1)^n;\mathbb{Z}).$$

isomorphism from page 993

isomorphism from page 992

by the Cohomology Ring-of-Product Proposition 77.8, which we can apply by the CW–Chain Complex Condition Lemma 77.2

It is clear from the definitions of the various maps that the map from left to right sends each x_i to $p_i^*([S^1]^*)$. The composition of the isomorphisms thus gives us the desired isomorphism. The uniqueness follows immediately from the fact that the x_i generate $\Lambda[x_1, \ldots, x_n]$ as a ring. ■

The following corollary gives in particular a negative answer to Question 74.1.

Corollary 77.10. (LS Category–Torus Corollary) Let $n \in \mathbb{N}$. For the n-dimensional torus $(S^1)^n$ the Lusternik-Schnirelmann category $\mathrm{cat}((S^1)^n)$ equals n.

Proof. Given any $n \in \mathbb{N}_0$ we have

see page 953

by the Cup Length–LS Category Proposition 74.3

$$n \overset{\downarrow}{\geq} \text{Lusternik-Schnirelmann category } \mathrm{cat}((S^1)^n) \overset{\downarrow}{\geq} \text{cup length } \mathrm{cl}((S^1)^n) \underset{\uparrow}{\geq} n.$$

by the Torus–Cup Product Proposition 77.9 there exists an isomorphism $\psi\colon \Lambda[x_1, \ldots, x_n] \to \mathrm{H}^*((S^1)^n;\mathbb{Z})$ of graded rings, furthermore one can easily show that $x_1 \wedge \cdots \wedge x_n \neq 0$, since ψ is an isomorphism we deduce that $\psi(x_1) \cup \cdots \cup \psi(x_n) \neq 0$, i.e. $\mathrm{cl}((S^1)^n) \geq n$ ■

Part IX

Poincaré Duality

In this part of the lecture notes we will state and prove the Poincaré Duality Theorem 79.1 which states (in simplified form) that given any closed oriented n-dimensional topological manifold M and given any $k \in \mathbb{N}_0$ the map

$$\begin{array}{rcl} \mathrm{H}^k(M;\mathbb{Z}) & \to & \mathrm{H}_{n-k}(M) \\ \varphi & \mapsto & \varphi \cap [M], \end{array}$$

which is given by capping with the fundamental class $[M] \in \mathrm{H}_n(M)$ of M, is an isomorphism. This part of the lecture notes contains the following chapters:

- In Chapter 78 we introduce cohomology with compact support of any topological space. This invariant is interesting in its own right, but for us the most interesting aspect is that it plays an essential role in the proof of the Poincaré Duality Theorem 79.1.
- In Chapter 79 we will state the Poincaré Duality Theorem 79.1 in full generality and we discuss many of the pretty immediate consequences.
- In Chapter 80 we provide the proof of the Poincaré Duality Theorem 79.1. The proof is long, but the main idea, is at the end not that hard.

CHAPTER 78

Cohomology with compact support

The Poincaré Duality Theorem 79.1 will say, in a slightly simplified version, that if M is a closed oriented n-dimensional topological manifold, then for each $k \in \mathbb{N}_0$ the map

$$\begin{array}{rcl} -\cap[M]\colon \mathrm{H}^k(M;\mathbb{Z}) & \xrightarrow{\cong} & \mathrm{H}_{n-k}(M;\mathbb{Z}) \\ \varphi & \mapsto & \varphi\cap[M] \end{array}$$

that is given by capping with the fundamental class $[M] \in \mathrm{H}_n(M)$, is an isomorphism. So the good news is that we now have all the tools to formulate the most important theorem in (co-) homology theory.

The idea for the proof is somewhat similar to the proof of the Homology-Section Theorem 63.6, namely we will do a five step process where we work up our way from a trivial case, through cleverly arranged Mayer–Vietoris type arguments, to the general statement.

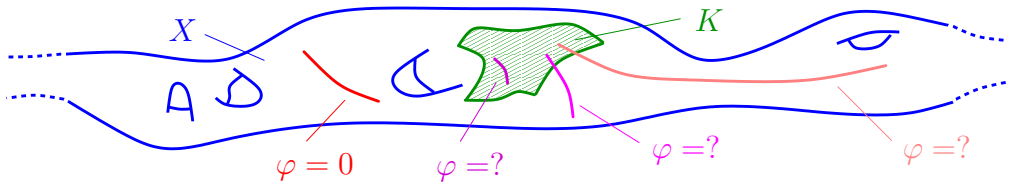
For better or worse this five step process involves non-compact topological manifolds and we will need to work with an analogue of the above map $-\cap[M]$ for non-compact topological manifolds. To get a sensible statement we will have to replace ordinary cohomology by "cohomology with compact support". In this chapter we will introduce and study "cohomology with compact support". As we will see, this is an interesting invariant in its own right.

78.1. **The definition of cohomology with compact support and basic properties.** In this section we introduce "cohomology with compact support". We start out with the following definition:

Definition. Let X be a topological space and let G be an abelian group.

(1) Given a subset $A \subset X$ we say that a cochain $\varphi \in \mathrm{C}^n(X;G) = \mathrm{Hom}(\mathrm{C}_n(X), G)$ vanishes on A if $\varphi(c) = 0$ for all singular n-simplices c which take values in A, i.e. if φ vanishes on the subgroup $\mathrm{C}_n(A) \subset \mathrm{C}_n(X)$.

(2) Given a subset $B \subset X$ we say that a cochain vanishes outside B if the cochain vanishes on $X \setminus B$.



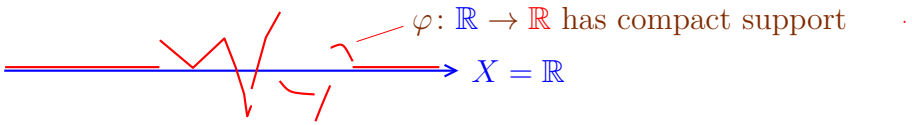
φ is a cohomology class that vanishes outside of the compact subset K

Definition.

(3) We say that a cochain $\varphi \in \mathrm{C}^n(X;G) = \mathrm{Hom}(\mathrm{C}_n(X), G)$ has compact support if there exists a compact set $K \subset X$ such that φ vanishes outside of K.
(4) We denote by $\mathrm{C}^n_{\mathrm{c}}(X;G)$ the set of all n-cochains with compact support.

Example. Let X be a topological space and let G be an abelian group. On page 865 we made the identification

$$\mathrm{C}^0(X;G) \;=_{\mathrm{i}}\; \{\text{maps from } X \text{ to } G\}.$$



With this identification we can now write

$$\mathrm{C}^0_{\mathrm{c}}(X;G) \;=\; \{\text{maps } f\colon X \to G \,|\, \text{there exists a compact subset } K \subset X \text{ with } f|_{X\setminus K} \equiv 0\}. \;\square$$

The first question that arises is whether cochains with compact support form a cochain complex. The following lemma shows in particular that this is indeed the case.

Lemma 78.1. (Cochain-with-Compact Support Lemma) Let X be a topological space and let G be an abelian group. The set of cochains that have compact support form a subcomplex $(\mathrm{C}^*_{\mathrm{c}}(X;G);\delta_*)$ of the cochain complex $(\mathrm{C}^*(X;G);\delta_*)$.

Proof. The lemma is an immediate consequence of the following claim and the observation that the union of two compact subspaces is again compact.

Claim. Let $A, B \subset X$ be two subsets.
(a) If $\varphi \in \mathrm{C}^n(X;G)$ vanishes outside A and if $\psi \in \mathrm{C}^n(X;G)$ vanishes outside B, then for any $r, s \in \mathbb{Z}$ the linear combination $r\cdot\varphi + s\cdot\psi \in \mathrm{C}^n(X;G)$ vanishes outside the union $A \cup B$.
(b) If $\varphi \in \mathrm{C}^n(X;G)$ vanishes outside A, then $\delta_n(\varphi) \in \mathrm{C}^{n+1}(X;G)$ also vanishes outside A.

Proof. Statement (1) is obvious. Now we turn to the proof of Statement (2). Suppose we are given a cochain $\varphi \in \mathrm{C}^n(X;G)$ that vanishes outside A. We need to show that $\delta_n(\varphi) \in \mathrm{C}^{n+1}(X;G)$ also vanishes outside A. So let c be a singular $(n+1)$-simplex in $X \setminus A$. Then

$$(\delta_n(\varphi))(c) \underset{\uparrow}{=} \varphi(\partial_{n+1}(c)) \underset{\uparrow}{=} \sum_{j=0}^{n+1} (-1)^n \cdot \varphi(c \circ i_j^{n+1}) \underset{\uparrow}{=} 0.$$

definition of δ_n — definition of ∂_{n+1} — since the image of each $c \circ i_j^{n+1}$ also lies in $X \setminus A$ and since φ vanishes outside A ■

The Cochain-with-Compact Support Lemma 78.1 allows us to introduce the following definition:

Definition. Let X be a topological space and let G be an abelian group. We define the n-th cohomology group with compact support $\mathrm{H}^n_{\mathrm{c}}(X;G)$ to be the cohomology of the cochain complex given by cochains with compact support, i.e. we define

$$\mathrm{H}^n_{\mathrm{c}}(X;G) \;:=\; \mathrm{H}^n(\mathrm{C}^*_{\mathrm{c}}(X;G)) \;:=\; \frac{\ker(\delta_n\colon \mathrm{C}^n_{\mathrm{c}}(X;G) \to \mathrm{C}^{n+1}_{\mathrm{c}}(X;G))}{\mathrm{im}(\delta_{n-1}\colon \mathrm{C}^{n-1}_{\mathrm{c}}(X;G) \to \mathrm{C}^n_{\mathrm{c}}(X;G))}.$$

Remark. If X is a compact space, then every cochain has compact support, in particular the cohomology groups with compact support are just the usual cohomology groups.

Example. Let X be a non-empty path-connected topological space. Then

$$\mathrm{H}^0_{\mathrm{c}}(X;G) \underset{\uparrow}{=} \left\{ X \xrightarrow{f} G \;\middle|\; \begin{array}{l} f \text{ is constant and there exists a} \\ \text{compact subset } K \text{ with } f|_{X\setminus K} \equiv 0 \end{array} \right\} \cong \begin{cases} 0, & \text{if } X \text{ is non-compact,} \\ G, & \text{if } X \text{ is compact.} \end{cases}$$

see the proof of the H^0-Proposition 67.5 and the description of $\mathrm{C}^0_{\mathrm{c}}(X;G)$ on page 1000.

This simple example already indicates that cohomology with compact support can, for non-compact topological spaces, be quite different from usual cohomology.

Example. We consider the topological space $X = \mathbb{R}$. As on page 868 we consider the singular 0-cochain

$$\varphi_0 \colon \mathrm{C}_0(X) \to \mathbb{Z}$$

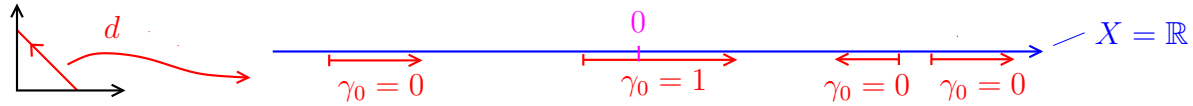
that is uniquely determined by the property that for any singular 0-simplex $x \in X$ we have

$$x \mapsto \begin{cases} 0, & \text{if } x \leq 0, \\ 1, & \text{if } x > 0. \end{cases}$$

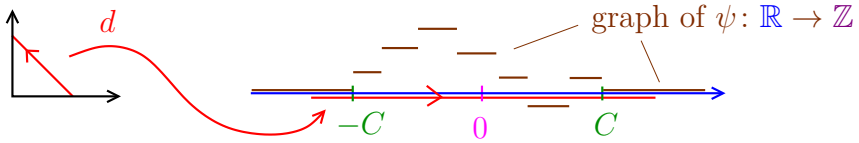
It is clear that this 0-cochain does *not have* compact support. As on page 868 we also consider the coboundary $\gamma_0 := \delta_0(\varphi_0) \in \mathrm{C}^1(\mathbb{R};\mathbb{Z})$. Since γ_0 is a coboundary, it is in particular a cocycle. By definition $\gamma_0 = \delta_0(\varphi_0)$ is uniquely determined by the property that for any singular 1-simplex $d \colon \Delta^1 \to \mathbb{R}$ we have

$$(\delta_0(\varphi_0))(d) = \varphi_0(\partial_1(d)) = \varphi_0(d(0,1)) - \varphi_0(d(1,0)).$$

One can easily verify that $\gamma_0 = \delta_0(\varphi_0)$ vanishes outside of the compact subset $K := \{0\}$.[610] Thus we see that $\gamma_0 = \delta_0(\varphi_0)$ defines an element in $\mathrm{H}^1_{\mathrm{c}}(\mathbb{R})$.



Now we claim that $[\gamma_0] \in \mathrm{H}^1_{\mathrm{c}}(\mathbb{R};\mathbb{Z})$ is non-zero. This means that we have to show that there is no $\psi \in \mathrm{C}^0_{\mathrm{c}}(\mathbb{R};\mathbb{Z})$ with $\delta_0(\psi) = \gamma_0$. So let $\psi \in \mathrm{C}^0_{\mathrm{c}}(\mathbb{R};\mathbb{Z})$. Following the discussion on page 1000 we view ψ as a function $\mathbb{R} \to \mathbb{Z}$ which is zero outside of a compact subset of $\mathbb{R}$. This implies that there exists a $C \geq 0$ such that $\psi(x) = 0$ for all $x \notin [-C, C]$. Now we consider the singular 1-chain $d \colon \Delta^1 \to \mathbb{R}$ that is given by $d(1-t,t) = -2C + 4C \cdot t$ with $t \in [0,1]$.



Then we have

[610] Indeed, if $d \colon \Delta^1 \to \mathbb{R}\setminus\{0\}$ is a singular 1-simplex, then it follows from the fact that Δ^1 is homeomorphic to the interval $[0,1]$ and the intermediate value theorem that either $d(0,1)$ and $d(1,0)$ both lie to the left of 0 or both lie to the right of 0. In both cases we obtain from the above formula that $\gamma_0(d) = (\delta_0(\varphi_0))(d) = 0$.

$$\gamma_0(d) \;=\; (\delta_0(\varphi_0))(d) \;=\; \varphi_0(d(0,1)) - \varphi_0(d(1,0)) \;=\; \varphi_0(-2C) - \varphi_0(2C) \;=\; 0 - 1 \;=\; -1$$

but on the other hand we have

$$(\delta_0(\psi))(d) \;=\; \psi(d(0,1)) - \psi(d(1,0)) \;=\; \psi(-2C) - \psi(2C) \underset{\uparrow}{=} 0 - 0 \;=\; 0.$$

since ψ is zero outside $[-C, C]$

This concludes the proof that $\gamma_0 = \delta_0(\varphi_0)$ is non-trivial in $\mathrm{H}^1_{\mathrm{c}}(\mathbb{R};\mathbb{Z})$. □

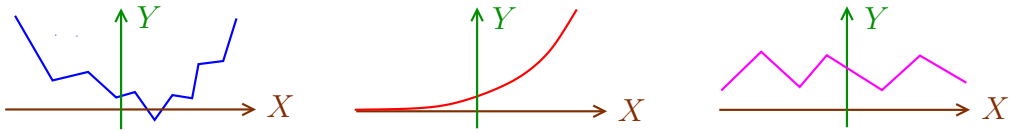
We conclude this section with the discussion of induced maps on cohomology groups with compact support. Let $f\colon X \to Y$ be a continuous map between topological spaces and let G be an abelian group. We saw that f induces a homomorphism $f^*\colon \mathrm{C}^*(Y;G) \to \mathrm{C}^*(X;G)$. We make the following simple observation:

$$\begin{aligned}
&\varphi \in \mathrm{C}^k(Y;G) \text{ vanishes outside of a subset } B \subset Y \\
\Longrightarrow\;& \varphi \in \mathrm{C}^k(Y;G) \text{ vanishes on } Y \setminus B \\
\Longrightarrow\;& f^*(\varphi) \in \mathrm{C}^k(X;G) \text{ vanishes on } f^{-1}(Y \setminus B) = X \setminus f^{-1}(B) \\
\Longrightarrow\;& f^*(\varphi) \in \mathrm{C}^k(X;G) \text{ vanishes outside of } f^{-1}(B) \subset X.
\end{aligned}$$

Since the preimage of a compact subspace is in general not compact we see that we have a problem with a "naive" approach to functoriality of cohomology with compact support.

This discussions leads us to the following definition.

Definition. A continuous map $f\colon X \to Y$ between topological spaces is called proper if the preimage of every compact subset of Y is a compact subset of X.



graph of proper map $\mathbb{R} \to \mathbb{R}$ | graph of a map that is *not* proper | graph of a map that is *not* proper

We can now formulate the following slightly subtle analogue of the H^*-Functor Lemma 67.1:

Lemma 78.2. ($\mathrm{H}^n_{\mathrm{c}}$-Functor Lemma) Let G be an abelian group.

(1) Let $f\colon X \to Y$ be a continuous map between topological spaces. If f is a *proper* map, then the cochain map $f^*\colon \mathrm{C}^*(Y;G) \to \mathrm{C}^*(X;G)$ restricts to a cochain map

$$f^*\colon \mathrm{C}^*_{\mathrm{c}}(Y;G) \;\to\; \mathrm{C}^*_{\mathrm{c}}(X;G).$$

In particular f^* induces a well-defined homomorphism

$$\mathrm{H}^*_{\mathrm{c}}(Y;G) \;\to\; \mathrm{H}^*_{\mathrm{c}}(X;G).$$

(2) For any $n \in \mathbb{N}_0$ the maps $X \mapsto \mathrm{H}^n_{\mathrm{c}}(X;G)$ and $f \mapsto f^*$ define a contravariant functor

category *TopPropMap* where objects are topological spaces and morphisms are proper maps[611] $\;\to\;$ category *AbGroup* of abelian groups.

In particular homeomorphic topological spaces have isomorphic cohomology groups with compact support.

Proof. Let G be an abelian group.

(1) Let $f\colon X \to Y$ be a proper map. Let $\varphi\colon \mathrm{C}_n(Y) \to G$ be a cocycle with compact support. We pick a compact subset K such that φ vanishes outside of K. It follows from the discussion above that $f^*(\varphi)$ vanishes outside of $f^{-1}(K)$. Since f is proper we see that $f^{-1}(K)$ is compact. This shows that $f^*\colon \mathrm{C}^*_{\mathrm{c}}(Y;G) \to \mathrm{C}^*(X;G)$ takes values in $\mathrm{C}^*_{\mathrm{c}}(X;G)$.

(2) Statement (2) follows easily from (1). ∎

78.2. **Cohomology with compact support and relative cohomology.** In this section we want to relate cohomology with compact support to relative cohomology groups. To do so it is worth introducing the following notation:

Notation. Let X be a topological space and let G be an abelian group. Given a subspace $A \subset X$ we denote by

$$я\colon \mathrm{C}^n(X,A;G) \to \mathrm{C}^n(X;G)$$

the natural chain map that is induced by the natural projection $\mathrm{C}_n(X) \to \mathrm{C}_n(X,A)$.

Lemma 78.3. (Compact Support–Relative Cochain Lemma) Let X be a topological space and let G be an abelian group.

(1) (a) A cochain $\varphi \in \mathrm{C}^n(X;G)$ vanishes on a subset $A \subset X$ if and only if φ lies in the image of the natural map $я\colon \mathrm{C}^n(X,A;G) \to \mathrm{C}^n(X;G)$.

(b) A cochain $\varphi \in \mathrm{C}^n(X;G)$ has compact support if and only if there exists a compact subset $K \subset X$ such that φ is contained in the image of the natural homomorphism $я\colon \mathrm{C}^n(X, X\setminus K;G) \to \mathrm{C}^n(X;G)$.

(2) (a) For any compact subset K of X the natural map $я\colon \mathrm{C}^n(X,X\setminus K;G) \to \mathrm{C}^n(X;G)$ restricts to a natural cochain map $я\colon \mathrm{C}^n(X,X\setminus K;G) \to \mathrm{C}^n_{\mathrm{c}}(X;G)$.

(b) Given any $\varphi \in \mathrm{H}^n_{\mathrm{c}}(X;G)$ there exists a compact subset K of X such that φ lies in the image of the natural map $я\colon \mathrm{H}^n(X,X\setminus K;G) \to \mathrm{H}^n_{\mathrm{c}}(X;G)$.

Proof. Let X be a topological space and let G be an abelian group.

(1) (a) We consider the following little diagram:

$$\begin{array}{ccccccc} \mathrm{C}_n(A) & \longrightarrow & \mathrm{C}_n(X) & \longrightarrow & \mathrm{C}_n(X,A) & \longrightarrow & 0 \\ & & \downarrow & \swarrow & & & \\ & & G. & & & & \end{array}$$

The desired conclusion is a consequence of the following observation: It follow immediately from the definition $\mathrm{C}_n(X,A) \coloneqq \mathrm{C}_n(X)/\mathrm{C}_n(A)$ that a homomorphism $\varphi\colon \mathrm{C}_n(X) \to G$ vanishes on $\mathrm{C}_n(A)$ if and only if φ descends to a homomorphism $\mathrm{C}_n(X,A) \to G$.

(b) This statement follows immediately from (a) and the definition of "compact support".

(2) (a) This statement is an immediate consequence of the definitions.

(b) Let $[\varphi] \in \mathrm{H}^n_{\mathrm{c}}(X;G)$. It follows from (1b) that there exists a compact subset K and a $\psi \in \mathrm{C}^n(X,X\setminus K;G)$ such that $я(\psi) = \varphi$. We consider the following little diagram:

[611] Clearly the composition of proper maps is again a proper map, thus we see that we do indeed have a category in our hands.

$$\begin{array}{ccc} \mathrm{C}^{n+1}(X, X\setminus K; G) & \xrightarrow{\text{я}} & \mathrm{C}_{\mathrm{c}}^{n+1}(X; G) \\ \uparrow{\scriptstyle \delta_n} & & \uparrow{\scriptstyle \delta_n} \\ \mathrm{C}^{n}(X, X\setminus K; G) & \xrightarrow{\text{я}} & \mathrm{C}_{\mathrm{c}}^{n}(X; G). \end{array}$$

Since the horizontal maps are monomorphisms we see that ψ is actually a cocycle in $\mathrm{C}^*(X, X\setminus K; G)$. This discussion implies that $[\varphi]$ lies in the image of the map $\mathrm{H}^n(X, X\setminus K; G) \to \mathrm{H}_{\mathrm{c}}^n(X; G)$. ■

The Compact Support–Relative Cochain Lemma 78.3 suggests that cohomology $\mathrm{H}_{\mathrm{c}}^n(X; G)$ with compact support is related to the relative cohomology groups $\mathrm{H}^n(X, X\setminus K; G)$ where $K \subset X$ is compact. In fact, it seems reasonable that for "larger and larger" compact $K \subset X$ one might be able to "approximate" cohomology with compact support. The notion of a direct colimit, which we discuss in the next section, will help us in turning this vague idea into proper mathematics.

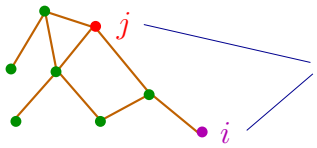
78.3. **Interlude: Direct colimits of direct systems.** In the next section we will give a reinterpretation of cohomology with compact support as a direct colimit of the relative cohomology groups $\mathrm{H}^n(X, X\setminus K; G)$ where K runs over all compact subsets of X.

In this section we recall the basic definition of direct systems and direct colimits from Chapter 25 and we generalize a few concepts. First we recall the following definition from page 338:

Definition. A directed set is a set I together with a relation "$\leq$" on I which satisfies the following three conditions:
(1) The relation is reflexive, i.e. for every $i \in I$ we have $i \leq i$.
(2) The relation is transitive, i.e. for every $i, j, k \in I$ with $i \leq j$ and $j \leq k$ we have $i \leq k$.
(3) For every $i, j \in I$ there exists a $k \in I$ with $i \leq k$ and $j \leq k$.

Examples.

(a) Evidently the set $\mathbb{N}$ together with the usual "$\leq$" relation forms a directed set $(\mathbb{N}, \leq)$.
(b) In the figure we show a directed set $(I, \leq)$:



we have $i \leq j$ if we can go from i to j via an upward pointing path

(c) Let X be a topological space. We denote by $\mathcal{K}(X)$ the set of all compact subsets of X. The set $\mathcal{K}(X)$, together with the relation "$\subset$" defined by inclusion, is a directed set since the union of two compact subsets is again compact.
(d) Given a metric space (X, d) the set of all closed balls in X, together with the relation "$\subset$" defined by inclusion, is a directed set since the union of two closed balls is clearly contained in a closed ball. □

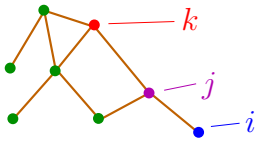
Next we recall the following definition from page 339:

Definition. Let $(I, \leq)$ be a directed set and let $\mathcal{C}$ be a category. A direct system over the directed set $(I, \leq)$ is a family of objects $\{X_i\}_{i \in I}$ in $\mathcal{C}$, together with a family of morphisms $\{f_{ji} \colon X_i \to X_j\}$ for all $i, j \in I$ with $i \leq j$ such that the following two conditions are satisfied:

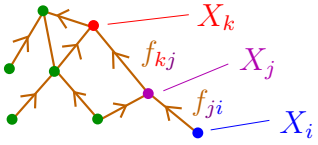
(1) $f_{ii} = \mathrm{id}_{X_i}$ for all $i \in I$,

(2) $f_{ki} = f_{kj} \circ f_{ji}$ for all $i, j, k \in I$ with $i \leq j \leq k$.

directed set:



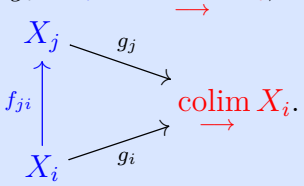
direct system:



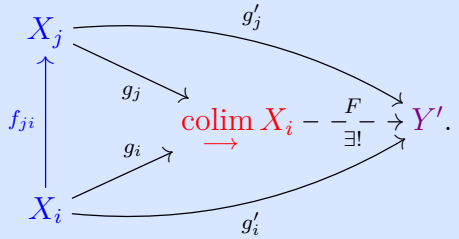
Next we recall the definition of a direct colimit from page 339.

Definition. Let $(I, \leq)$ be a directed set, let $\mathcal{C}$ be a category. Suppose we are given a direct system $(\{X_i\}_{i \in I}, \{f_{ji} \colon X_i \to X_j\}_{i \leq j})$ in the category $\mathcal{C}$. A direct colimit of the direct system is a colimit, in the sense of the definition on page 110, of the direct system viewed as a diagram. In other words, a direct colimit is an object $\varinjlim X_i$ in $\mathcal{C}$ together with a family of morphisms $g_i \colon X_i \to \varinjlim X_i$ in the category $\mathcal{C}$ such that the following two conditions are satisfied:

(1) For all $i \leq j$ we have $g_j \circ f_{ji} = g_i \colon X_i \to \varinjlim X_i$, i.e. for all $i \leq j$ the following diagram commutes:



(2) If we are given another object Y' and a family of morphisms $\{g'_i \colon X_i \to Y'\}_{i \in I}$ that satisfy (1), then there exists a unique morphism $F \colon \varinjlim X_i \to Y'$ such that for all $i \in I$ we have $g'_i = F \circ g_i \colon X_i \to Y'$. In particular for all $i \leq j$ the following diagram commutes:



By the Colimit Uniqueness Lemma 4.7 we know that if a direct colimit exists, then it is unique in the usual sense.

We recall the following existence result:

Proposition 25.3. (Algebraic–Direct Colimit Existence Proposition) Let $(I, \leq)$ be a directed set and let $(\{X_i\}_{i \in I}, \{f_{ji} \colon X_i \to X_j\}_{i \leq j})$ be a direct system in the category *AbGroup* of abelian groups. A direct colimit is given by

$$\varinjlim X_i = \left(\bigoplus_{i \in I} X_i \right) / \langle f_{ji}(x) - x \mid \ i \leq j \text{ and } x \in X_i \rangle.$$

We also recall the following lemma, which follows easily from the explicit description of the colimit in the Algebraic–Direct Colimit Existence Proposition 25.3.

Lemma 25.4. (Elements-in-Direct Colimit Lemma) Let $(I, \leq)$ be a directed set and let $(\{X_i\}_{i\in I}, \{f_{ji}\colon X_i \to X_j\}_{i\leq j})$ be a direct system over I in the category *AbGroup* of abelian groups. The following two statements hold:

(1) Every element in $\varinjlim X_i$ lies in the image of one of the maps $X_j \to \varinjlim X_i$.
(2) Let $a_j \in X_j$ and let $a_k \in X_k$. The images of a_j and a_k in $\varinjlim X_i$ agree if and only if there exists an $l \in I$ with $l \geq j$ and $l \geq k$ such that $f_{lj}(a_j) = f_{lk}(a_k) \in X_l$.

Definition. Let $(I, \leq)$ and $(J, \leq)$ be directed sets.

(1) A morphism between the directed sets $(I, \leq)$ and $(J, \leq)$ is a map $\varphi\colon I \to J$ such that for every $i_1 \leq i_2$ in I we have $\varphi(i_1) \leq \varphi(i_2)$.
(2) Let $(\{K_i\}_{i\in I}, \{\chi_{i_2,i_1}\}_{i_1\leq i_2})$ and $(\{L_j\}_{j\in J}, \{\lambda_{j_2,j_1}\}_{j_1\leq j_2})$ be direct systems in the category *AbGroup* of abelian groups. A morphism between the direct systems, modelled on a morphism $\varphi\colon I \to J$, is a family $\{f_i\colon K_i \to L_{\varphi(i)}\}_{i\in I}$ of homomorphisms that has the property that for any $i_1 \leq i_2$ the following diagram commutes:

$$\begin{array}{ccc} K_{i_2} & \xrightarrow{f_{i_2}} & L_{\varphi(i_2)} \\ {\scriptstyle \chi_{i_2,i_1}}\uparrow & & \uparrow{\scriptstyle \lambda_{\varphi(i_2),\varphi(i_1)}} \\ K_{i_1} & \xrightarrow{f_{i_1}} & L_{\varphi(i_1)}. \end{array}$$

(3) Suppose that we are in the setting of (2). The homomorphisms $K_i \xrightarrow{f_i} L_{\varphi(i)} \to \varinjlim L_j$ induce, by the universal property of a direct colimit, a homomorphism

$$\varinjlim_{i\in I} K_i \;\to\; \varinjlim_{j\in J} L_j$$

which is uniquely determined by the property that for any $i \in I$ the following diagram commutes:

$$\begin{array}{ccc} K_i & \xrightarrow{f_i} & L_{\varphi(i)} \\ \downarrow & & \downarrow \\ \varinjlim_{i\in I} K_i & \longrightarrow & \varinjlim_{j\in J} L_j. \end{array}$$

Next we want to give a criterion for an induced map of direct colimits to be an isomorphism. To formulate the lemma we need the following definition:

Definition. Let $(I, \leq)$ and $(J, \leq)$ be directed sets. A morphism $\varphi\colon I \to J$ is called cofinal if given any $j \in J$ there exists an $i \in I$ with $\varphi(i) \geq j$.



Lemma 78.4. (Cofinal-Isomorphism Lemma) Let $\{f_i \colon K_i \to L_{\varphi(i)}\}_{i \in I}$ be a morphism between two direct systems $(\{K_i\}_{i \in I}, \{\chi_{i_2,i_1}\}_{i_1 \leq i_2})$ and $(\{L_j\}_{j \in J}, \{\lambda_{j_j j_1}\}_{j_1 \leq j_2})$ in the category *AbGroup* of abelian groups. We suppose that the following two conditions are satisfied:

(1) The morphism $\varphi \colon I \to J$ is cofinal.

(2) For each $i \in I$ the homomorphism $f_i \colon K_i \to L_{\varphi(i)}$ is an isomorphism.

Then the induced homomorphism

$$\Phi \colon \operatorname*{colim}_{i \in I} K_i \to \operatorname*{colim}_{j \in J} L_j$$

is an isomorphism.

Proof. Our goal is to construct an inverse $\Psi \colon \operatorname{colim} L_j \to \operatorname{colim} K_i$ to the given map. First let $j \in J$. Since $\varphi \colon I \to J$ is cofinal we can pick $i \in I$ with $\varphi(i) \geq j$. We consider the homomorphism g_j that is given by

$$L_j \xrightarrow{\lambda_{\varphi(i),j}} L_{\varphi(i)} \xrightarrow[\cong]{(f_i)^{-1}} K_i \to \operatorname*{colim}_{i \in I} K_i.$$

It is easy to verify that this homomorphism does *not* depend on the choice of i.



Using this observation it is elementary to check that for any $j_1 \leq j_2$ the following diagram commutes:

$$\begin{array}{ccc} L_{j_2} & \xrightarrow{g_{j_2}} & \\ \uparrow{\scriptstyle \lambda_{j_2,j_1}} & & \operatorname*{colim}_{i \in I} K_i. \\ L_{j_1} & \xrightarrow{g_{j_1}} & \end{array}$$

By the universal property of the direct colimit this implies that we get an induced homomorphism

$$\operatorname*{colim}_{j \in J} L_j \to \operatorname*{colim}_{i \in I} K_i.$$

Using the fact that direct colimits are unique up to a unique isomorphism one can now easily show that $\Phi \circ \Psi$ and $\Psi \circ \Phi$ are the identity. We leave it to the reader to fill in the details. ∎

Example. Let $(J, \leq)$ be a directed set and let $(\{X_j\}_{j \in J}, \{f_{kj} \colon X_j \to X_k\}_{j \leq k})$ be a direct system over J in the category *AbGroup* of abelian groups. Let I be a subset of J such that $(I, \leq)$ is again a directed set. The inclusion $I \to J$ is obviously a morphism of directed sets. By the above definition there exists a corresponding natural homomorphism

$$\operatorname*{colim}_{j \in J} X_j \longrightarrow \operatorname*{colim}_{i \in I} X_i.$$

If given any $i \in I$ there exists a $j \in J$ with $i \leq j$, then the inclusion $I \to J$ is cofinal and it follows from the Cofinal-Isomorphism Lemma 78.4 that the above homomorphism is an isomorphism. □

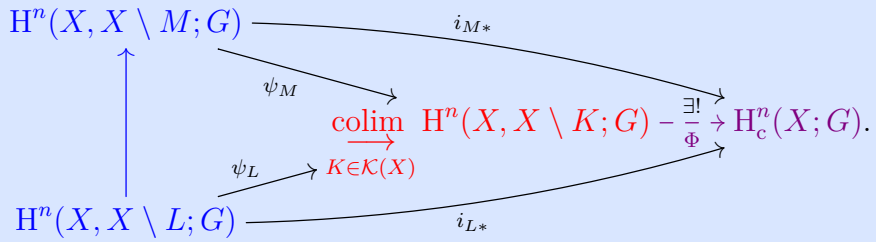
78.4. **Cohomology with compact support and direct colimits.** After this refresher on direct colimits we are now able to give the promised reinterpretation of cohomology with compact support.

Definition. Let X be a topological space and let G be an abelian group.

(1) We denote by $\mathcal{K}(X)$ the set of all compact subsets of X. As we discussed above, the set $\mathcal{K}(X)$ together with the relation "$\subset$" defined by inclusion is a directed set.

(2) Given any two compact subsets $K \subset L$ of X and given any $i \in \mathbb{N}_0$ we get an induced homomorphism[612]
$$\mathrm{H}^n(X, X \setminus K; G) \;\to\; \mathrm{H}^n(X, X \setminus L; G).$$
Since cohomology groups are contravariantly functorial we see that these cohomology groups form a *direct* system over the directed set $(\mathcal{K}(X), \subset)$. It follows from the Algebraic–Direct Colimit Existence Proposition 25.3 that we can form the direct colimit
$$\varinjlim_{K \in \mathcal{K}(X)} \mathrm{H}^n(X, X \setminus K; G).$$

(3) Given any compact subset $K \subset X$ we have by the Compact Support–Relative Cochain Lemma 78.3 (2a) a natural homomorphism $i_{K*} \colon \mathrm{H}^n(X, X \setminus K; G) \to \mathrm{H}^n_{\mathrm{c}}(X; G)$. Given compact subsets $L \subset M$ these homomorphisms give rise to the following diagram:



It follows immediately from the functoriality of cochain complexes that the outer triangle commutes for all compact subsets $L \subset M \subset X$. Thus we get the induced dotted arrow by the universal property of the direct colimit.

The following proposition says that the above dotted arrow is in fact an isomorphism. This gives in particular an often convenient reformulation of the concept of cohomology with compact support.

[612] The most confusing part about cohomology is to figure out which way the maps go. In this case $K \subset L$ implies that $X \setminus L \subset X \setminus K$ which means that we have a map of pairs $(X, X \setminus L) \to (X, X \setminus K)$ given by inclusion, which by the discussion on page 864 implies that we do indeed get an induced homomorphism $\mathrm{H}^n(X, X \setminus K; G) \to \mathrm{H}^n(X, X \setminus L; G)$.

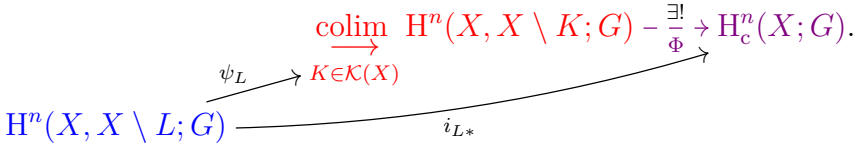
Proposition 78.5. ($\mathrm{H}^n_{\mathrm{c}}$-via-Direct Colimits Proposition) If X is a topological space and G is an abelian group, then the above map

$$\Phi\colon \varinjlim_{K\in\mathcal{K}(X)} \mathrm{H}^n(X, X\setminus K; G) \;\to\; \mathrm{H}^n_{\mathrm{c}}(X; G)$$

defines a natural isomorphism.

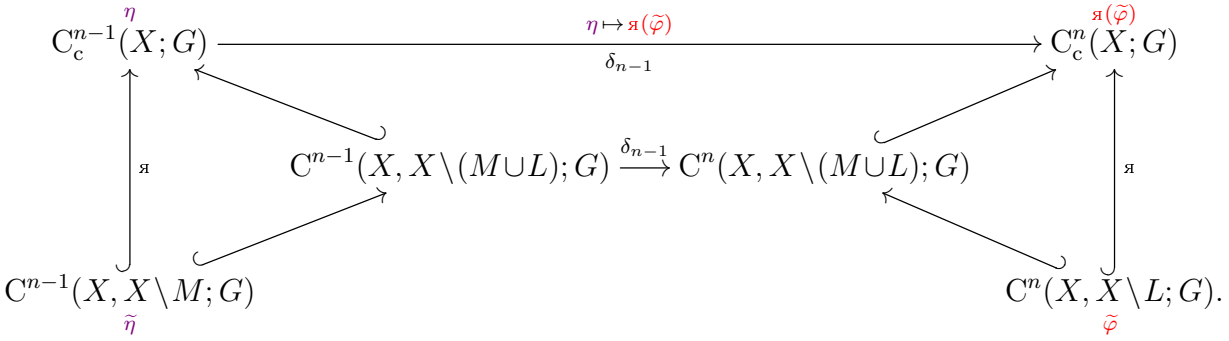
Proof. We need to show that the map Φ in the proposition is a bijection.

We first show that Φ is surjective. So let $\varphi \in \mathrm{H}^n_{\mathrm{c}}(X; G)$. By the Compact Support–Relative Cochain Lemma 78.3 (2b) there exists a compact subset $L \subset X$ such that φ lies in the image of $\mathrm{H}^n(X, X\setminus L; G) \to \mathrm{H}^n_{\mathrm{c}}(X; G)$. It follows immediately from the commutative diagram



that φ also lies in the image of $\Phi\colon \varinjlim \mathrm{H}^n(X, X\setminus K; G) \to \mathrm{H}^n_{\mathrm{c}}(X; G)$.

Now we show that the map Φ is injective. Suppose we are given an element c in the kernel of $\Phi\colon \varinjlim \mathrm{H}^n(X, X\setminus K; G) \to \mathrm{H}^n_{\mathrm{c}}(X; G)$. By the Elements-in-Direct Colimit Lemma 25.4 (1) we know that the element c lies in the image of some element in $\mathrm{H}^n(X, X\setminus L; G)$. We pick a representative $\widetilde{\varphi} \in \mathrm{C}^n(X, X\setminus L; G)$. Since $\text{я}(\widetilde{\varphi})$ represents the zero class in $\mathrm{H}^n_{\mathrm{c}}(X; G)$ there exists a compactly supported $(n-1)$-cocycle η with $\delta_{n-1}(\eta) = \text{я}(\widetilde{\varphi})$. We pick a compact subset M of X such that η vanishes on $X\setminus M$. It follows from the Compact Support–Relative Cochain Lemma 78.3 (1a) that there exists an $\widetilde{\eta} \in \mathrm{C}^{n-1}(X, X\setminus M; G)$ with $\text{я}(\widetilde{\eta}) = \eta$. We consider the following diagram:



Note that the diagram commutes by the functoriality of cochain complexes. It follows immediately from the diagram that in $\mathrm{C}^n(X, X\setminus(M\cup L); G)$ we have $\delta_{n-1}(\eta) = \varphi$, which shows that $[\varphi]$ represents the zero class in $\mathrm{H}^n(X, X\setminus(M\cup L); G)$. It follows from the Elements-in-Direct Colimit Lemma 25.4 (2) or alternatively from the commutative diagram from page 1008 that $\psi_L([\varphi]) = c$ is zero. ∎

Example. Now we turn to a key example, namely for each $k \in \mathbb{N}_0$ we consider the cohomology with compact support $\mathrm{H}^k_c(\mathbb{R}^n; G)$ of $\mathbb{R}^n$. We see that

$$\begin{array}{ccccc}
 & & & & \vdots \\
 & & & & \uparrow \\
 & \text{this follows from the Cofinal-Isomorphism Lemma 78.4 since every compact subspace } K \subset \mathbb{R}^n \text{ is contained in a disk } \overline{B}^n_s(0) \text{ with } s \in \mathbb{N}_0 & & & \mathrm{H}^k(\mathbb{R}^n, \mathbb{R}^n \setminus \overline{B}^n_2(0); G) \\
 & & & & \uparrow \\
\text{by the } \mathrm{H}^n_c\text{-via-Direct Limits Proposition 78.5} & & & & \mathrm{H}^k(\mathbb{R}^n, \mathbb{R}^n \setminus \overline{B}^n_1(0); G) \\
\downarrow & & \downarrow & & \uparrow \\
\mathrm{H}^k_c(\mathbb{R}^n; G) \xleftarrow{\cong} & \varinjlim_{K \in \mathcal{K}(\mathbb{R}^n)} \mathrm{H}^k(\mathbb{R}^n, \mathbb{R}^n \setminus K; G) & \xleftarrow{\cong} & \varinjlim_{s \in \mathbb{N}_0} & \mathrm{H}^k(\mathbb{R}^n, \mathbb{R}^n \setminus \overline{B}^n_0(0); G) \\
 & & & & \\
 & \xleftarrow{\cong} \mathrm{H}^k(\mathbb{R}^n, \mathbb{R}^n \setminus \overline{B}^n_0(0); G) & & \cong & \begin{cases} G, & \text{if } k = n, \\ 0, & \text{otherwise.} \end{cases} \\
 & \uparrow & & \uparrow & \\
 & \text{by the Homotopy Equivalence-Relative-H}_*\text{-Corollary 38.4 all vertical maps in the direct system are isomorphisms, hence we can apply the Direct Colimit–Stabilization Lemma 25.1} & & \text{follows from the Manifold-Local } R\text{-Homology Lemma 61.1} &
\end{array}$$

We draw three interesting conclusions from this example:

(1) This example shows that cohomology with compact support can be used to show that the topological spaces $\mathbb{R}^n$ and $\mathbb{R}^m$ are not homeomorphic for $n \neq m$.
(2) The example also shows that cohomology with compact support, in contrast to basically all other invariants we have studied so far, is not the same for homotopy equivalent topological spaces.
(3) Finally we consider the case $n = 1$ in more detail. We consider again the singular 1-cochain $\gamma_0 := \delta_0(\varphi_0)$ from page 868. The argument from page 868 shows that γ_0 defines an element in each $\mathrm{H}^1(\mathbb{R}, \mathbb{R} \setminus [-s, s]; \mathbb{Z})$. Furthermore the discussion on page 1001 shows that γ_0 also defines an element in $\mathrm{H}^1_c(\mathbb{R}; \mathbb{Z})$. Furthermore the argument from page 874 shows that for each $s \in N$ the class $[\gamma_0] \in \mathrm{H}^1(\mathbb{R}, \mathbb{R} \setminus [-s, s]; \mathbb{Z})$ is a generator. The above argument now implies that $[\gamma_0] \in \mathrm{H}^1_c(\mathbb{R}; \mathbb{Z})$ is also a generator. □

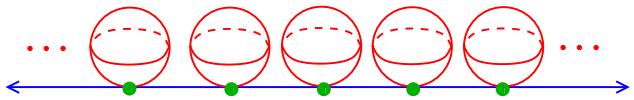
In the previous examples cohomology with compact support turns out to be "bigger" than ordinary cohomology. But there are also examples "in the other direction":

Example. We consider again the topological space

$$X \;:=\; (\mathbb{R} \cup (S^2 \times \mathbb{Z}))/\sim$$

where given $i \in \mathbb{Z}$ we identify $i \in \mathbb{Z} \subset \mathbb{R}$ with $((0, 0, -1), i) \in S^2 \times \mathbb{Z}$.

the topological space X



$\mathrm{H}^2(X; \mathbb{Z}) \cong \mathbb{Z}^{\mathbb{N}}$ but $\mathrm{H}^2_c(X; \mathbb{Z}) \cong \mathbb{Z}^{(\mathbb{N})}$

We leave it to the reader as an exercise to show

$$\mathrm{H}^2_c(X; \mathbb{Z}) \;\cong\; \mathbb{Z}^{(\mathbb{N})},$$

in particular $\mathrm{H}^2_c(X)$ is countable, whereas one can easily show that

$$\mathrm{H}^2(X; \mathbb{Z}) \;\cong\; \mathbb{Z}^{\mathbb{N}}$$

is uncountable. □

78.5. **Cohomology with compact support and inclusions.** In this section we will make the rather surprising observation that under fairly mild hypotheses an inclusion $U \to V$ of topological spaces induces for every $n \in \mathbb{N}_0$ a homomorphism $\mathrm{H}^n_c(U;G) \to \mathrm{H}^n_c(V;G)$ which "goes the same way".

Definition. Let U be an open subset of a Hausdorff space V and let G be an abelian group. We denote by $i_{VU*} = i_*\colon \mathrm{H}^n_c(U;G) \to \mathrm{H}^n_c(V;G)$ the composition of the following homomorphisms (for space reasons we ignore the coefficients)

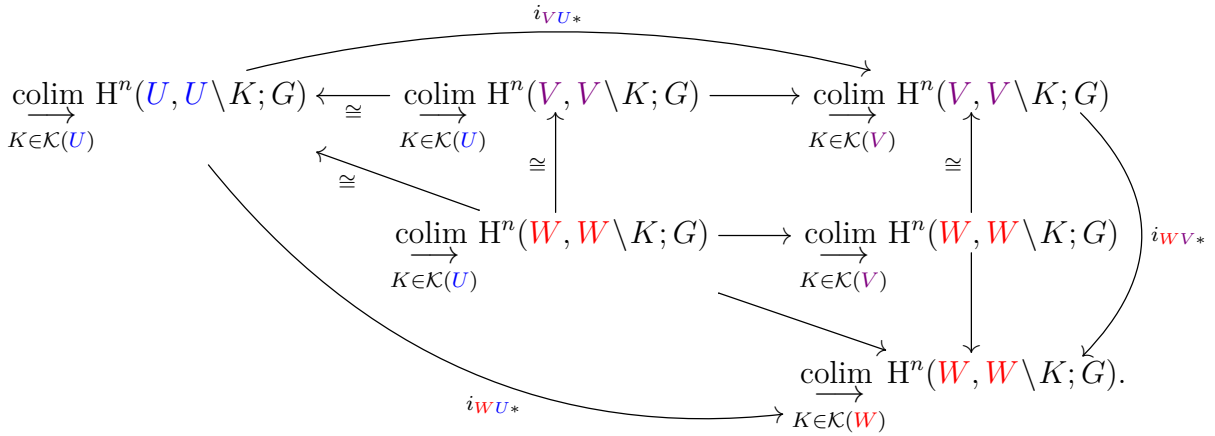
$$\mathrm{H}^n_c(U) \underset{\text{H}^n_c\text{-via-Direct Colimits Proposition 78.5}}{=} \operatorname*{colim}_{K\in\mathcal{K}(U)} \mathrm{H}^n(U, U\setminus K) \underset{\substack{\text{it follows from the Compact-Closed Lemma 1.9 together with the Excision-in-H}^*\text{-Theorem 69.1 (1b) that the map } \mathrm{H}^n(V,V\setminus K)\to\mathrm{H}^n(U,U\setminus K) \text{ is an isomorphism for every compact subset } K \text{ of } U}}{\overset{\cong}{\longleftarrow}} \operatorname*{colim}_{K\in\mathcal{K}(U)} \mathrm{H}^n(V, V\setminus K) \underset{\substack{\text{given by the functoriality of direct colimits, see page 344, and the fact that by the Compact Image Lemma 2.8 we have } \mathcal{K}(U)\subset\mathcal{K}(V)}}{\longrightarrow} \operatorname*{colim}_{K\in\mathcal{K}(V)} \mathrm{H}^n(V, V\setminus K) \underset{\text{H}^n_c\text{-via-Direct Colimits Proposition 78.5}}{=} \mathrm{H}^n_c(V).$$

Note that, confusingly, the induced map on cohomology with compact support "goes the same direction" as the inclusion map $i\colon U \to V$.

Lemma 78.6. (Covariant Cohomology Lemma) Let G be an abelian group. Given a topological space X that is Hausdorff the above construction defines for each $n \in \mathbb{N}_0$ a *covariant* functor

$$\begin{array}{c}\text{category where the objects are open subsets}\\ \text{of } X \text{ and where the morphisms are inclusions}\end{array} \longrightarrow \begin{array}{c}\text{category of}\\ \text{abelian groups.}\end{array}$$

Proof. Let X be a topological space that is Hausdorff and let $U \subset V \subset W$ be open subsets of X. Let $n \in \mathbb{N}_0$. We have to show that $i_{WU*} = i_{WV*} \circ i_{VU*}\colon \mathrm{H}^n_c(U;G) \to \mathrm{H}^n_c(W;G)$. We consider the following diagram:



It follows pretty easily from the universal property of direct colimits that the two smaller triangles and the rectangle commute. Hence the diagram commutes and we get the desired statement. ∎

We conclude this section with the following proposition.

Proposition 78.7. (Exhaustion-H_c^*-Proposition) Let X be a Hausdorff space, let G be an abelian group and let $\{X_i\}_{i\in\mathbb{N}}$ be a sequence of subsets of X such that the following holds:

(1) The sequence is nested, i.e. for each $i \in \mathbb{N}$ we have $X_i \subset X_{i+1}$.
(2) Each X_i is open in X.
(3) $X = \bigcup_{i\in\mathbb{N}} X_i$.

Then for any $n \in \mathbb{N}_0$ the above inclusion-induced maps $\mathrm{H}_c^n(X_i;G) \to \mathrm{H}_c^n(X;G)$ induce an isomorphism

$$\operatorname*{colim}_{\longrightarrow} \underbrace{\mathrm{H}_c^n(X_i;G)}_{\text{by the Covariant Cohomology Lemma 78.6 this is a direct system with the above inclusion-induced maps}} \xrightarrow{\cong} \mathrm{H}_c^n(X;G).$$

Proof. Using the H_c^n-via-Direct Colimits Proposition 78.5 we see that we have to show that the map

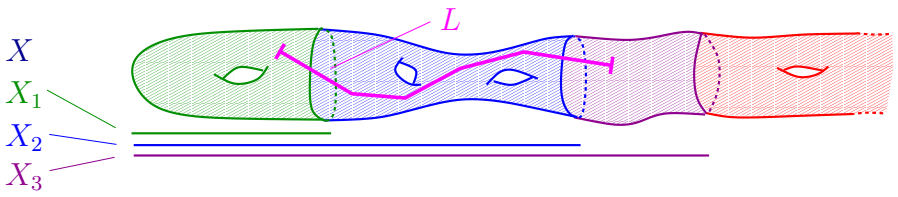
$$\Phi\colon \operatorname*{colim}_{\substack{\longrightarrow\\ i\in\mathbb{N}}} \operatorname*{colim}_{\substack{\longrightarrow\\ K\in\mathcal{K}(X_i)}} \mathrm{H}^n(X_i, X_i\setminus K;G) \xrightarrow{\cong} \operatorname*{colim}_{\substack{\longrightarrow\\ K\in\mathcal{K}(X)}} \mathrm{H}^n(X, X\setminus K;G)$$

is an isomorphism. We need to show that Φ is injective and surjective. The key to the proof of these two statements are the following observations:

(a) It follows from (1) and (2) that every compact subset L of X is already contained in some X_i.
(b) It follows from the Excision-in-H^*-Theorem 69.1 (1b) together with the Compact-Closed Lemma 1.9 (2) that for each compact subset $L \subset X_i$ we get an inclusion-induced isomorphism $\mathrm{H}^n(X, X\setminus L;G) \to \mathrm{H}^n(X_i, X_i\setminus L;G)$.

Now we prove that Φ is surjective, the proof that Φ is injective is fairly similar and we leave that part of the proof to the reader.

So let c be an element of the limit on the right-hand side. By the Elements-in-Direct Colimit Lemma 25.4 (1) there exists a compact subset L of X and some $\varphi \in \mathrm{H}^n(X, X\setminus L;G)$ such that c is the image of φ under the map $\mathrm{H}^n(X, X\setminus L;G) \to \operatorname*{colim}_{\longrightarrow} \mathrm{H}^n(X, X\setminus K;G)$.



By (a) there exists an $i \in \mathbb{N}$ with $L \subset X_i$. We consider the following commutative diagram:

$$\begin{array}{ccc}
\operatorname*{colim}_{\substack{\longrightarrow\\ i\in\mathbb{N}}} \operatorname*{colim}_{\substack{\longrightarrow\\ K\in\mathcal{K}(X_i)}} \mathrm{H}^n(X_i, X_i\setminus K;G) & \xrightarrow{\Phi} & \operatorname*{colim}_{\substack{\longrightarrow\\ K\in\mathcal{K}(X)}} \mathrm{H}^n(X, X\setminus K;G) \\
\uparrow{\scriptstyle =:\beta} & & \uparrow{\scriptstyle c} \\
\mathrm{H}^n(X_i, X_i\setminus L;G) & \xleftarrow[\cong]{=:\alpha} & \mathrm{H}^n(X, X\setminus L;G) \\
 & & \uparrow{\scriptstyle \varphi}
\end{array}$$

The bottom horizontal map is an isomorphism by (b). Since the diagram is commutative we now see that $\Phi(\beta(\alpha(\varphi))) = c$. ∎

CHAPTER 79

The Poincaré Duality Theorem

In this chapter we finally state and discuss the Poincaré Duality Theorem 79.1 which is undoubtedly one of the most influential results in topology. The proof of the theorem will be given in the next chapter.

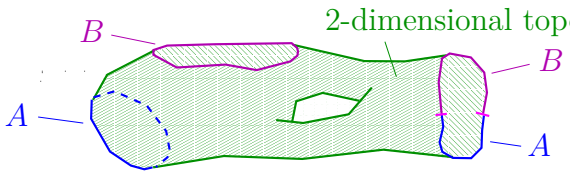
79.1. **The Poincaré Duality Theorem.** Here is the long awaited theorem:

Theorem 79.1. (Poincaré Duality Theorem)[613] Let R be a commutative ring, let M be a compact R-oriented[614] n-dimensional topological manifold and suppose that we are given a decomposition $\partial M = A \cup B$ where A and B are compact $(n-1)$-dimensional submanifolds of ∂M such that $A \cap B = \partial A = \partial B$. For each $k \in \mathbb{N}_0$ the map[615] [616] [617]

$$\begin{array}{ccc} \mathrm{H}^k(M, A; R) & \xrightarrow{\cong} & \mathrm{H}_{n-k}(M, B; R) \\ \varphi & \mapsto & \varphi \cap [M] \end{array}$$

$[M] \in \mathrm{H}_n(M, \partial M; R)$ is the R-fundamental class

is an isomorphism.



for all commutative rings R we have
$\mathrm{H}^k(M, A; R) \xrightarrow{\cong} \mathrm{H}_{2-k}(M, B; R)$

We proceed as follows:

(1) In the remainder of this chapter we discuss several special cases and applications of the Poincaré Duality Theorem 79.1. For the most part we will only make use of the fact that the theorem gives us an isomorphism between cohomology and homology groups.
(2) In the next Chapter 80 we will provide the proof of the Poincaré Duality Theorem 79.1.

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[613] For topological manifolds with non-empty boundary the Poincaré Duality Theorem is sometimes also called the "Poincaré-Lefschetz Duality Theorem".

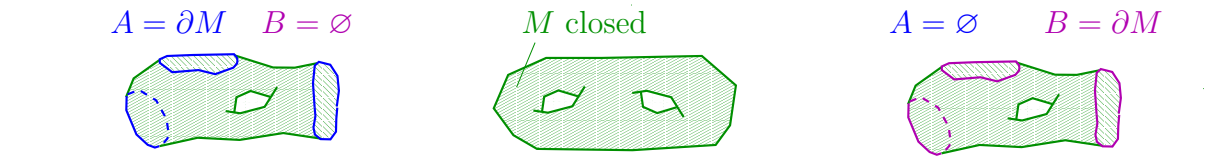
[614] The $\mathbb{F}_2$-Orientability Proposition 61.13 says that any topological manifold M is $\mathbb{F}_2$-orientable.

[615] If M is in fact orientable, then it follows from the Orientable-for-all-Rings Lemma 61.12 that M is R-orientable for any commutative ring R.

[616] The Excisive Triad-Proposition 50.11 (2) implies that (M, A, B) is an excisive triad. By the definition of the relative cap product on page 962, we thus obtain indeed a relative cap product

$$\mathrm{H}^k(M, A; R) \times \underbrace{\mathrm{H}_n(M, \overbrace{A \cup B}^{=\partial M}; R)}_{\text{contains } [M]} \xrightarrow{\cap} \mathrm{H}_{n-k}(X, B; R).$$

[616] In the Projective Plane–Cap Product Lemma 75.4 and the Annulus–Cap Product Lemma 75.6 we verified special case of this isomorphism "by hand".



(3) In Chapter 81 we will use the fact that the isomorphism is given by capping with the fundamental class to calculate the cohomology rings of $\mathbb{R}\mathrm{P}^n$ with $\mathbb{F}_2$-coefficients and of $\mathbb{C}\mathrm{P}^n$ with $\mathbb{Z}$-coefficients.

79.2. **The Poincaré Duality isomorphisms.** The Poincaré Duality Theorem 79.1 leads us to the following notation:

Notation. We continue with the notation from the Poincaré Duality Theorem 79.1. The isomorphism

$$\begin{array}{ccc} \mathrm{H}^k(M, A; R) & \xrightarrow{\cong} & \mathrm{H}_{n-k}(M, B; R) \\ \varphi & \mapsto & \varphi \cap [M] \end{array}$$

is called Poincaré Duality isomorphism. The corresponding inverse is denoted by

$$\mathrm{PD}_M \colon \mathrm{H}_{n-k}(M, B; R) \xrightarrow{\cong} \mathrm{H}^k(M, A; R)$$

and it will also be called Poincaré Duality isomorphism.

Example. Let M be a compact oriented connected n-dimensional topological manifold and let $P \in M$. As we pointed out on page 964, we have $[M]^* \cap [M] = [P] \in \mathrm{H}_0(M; \mathbb{Z})$. This implies that $\mathrm{PD}_M([P]) = [M]^* \in \mathrm{H}^n(M, \partial M; \mathbb{Z})$.

79.3. **Poincaré Duality Theorem 79.1 applied with $\mathbb{Z}$-coefficients.** In this and the following sections we give several more or less immediate corollaries to the Poincaré Duality Theorem 79.1.

We start out with the following proposition, which combines the Poincaré Duality Theorem 79.1 with the Universal Coefficient-H^*-Theorem 70.14.

Proposition 79.2. (Poincaré Duality–Non Natural Isomorphism Proposition) Let M be a compact orientable n-dimensional topological manifold and suppose that we are given a decomposition $\partial M = A \cup B$ where A and B are compact $(n-1)$-dimensional submanifolds of ∂M such that $A \cap B = \partial A = \partial B$. For every $k \in \mathbb{N}_0$ there exists an isomorphism

$$\mathrm{H}_{n-k}(M, A) \cong \underset{\uparrow}{\mathrm{FH}_k(M, B)} \oplus \underset{\uparrow}{\mathrm{Tor}}(\mathrm{H}_{k-1}(M, B)).$$

given an abelian group G we denote by $\mathrm{Tor}(G)$ its torsion-subgroup and we denote by $\mathrm{F}G \coloneqq G/\mathrm{Tor}(G)$ the maximal torsion-free quotient

Proof. We have the following isomorphisms:

$$\mathrm{H}_{n-k}(M, A) \xrightarrow{\cong} \mathrm{H}_{n-k}(M, A; \mathbb{Z}) \xleftarrow[\text{see page 659}]{\cong} \mathrm{H}^k(M, B; \mathbb{Z}) \underset{\text{Duality Theorem 79.1}}{\cong} \mathrm{Hom}(\mathrm{H}_k(M, B), \mathbb{Z}) \oplus \mathrm{Ext}(\mathrm{H}_{k-1}(M, B), \mathbb{Z}) \underset{\text{Universal Coefficient-H}^*\text{-Theorem 70.14}}{\cong} \mathrm{FH}_k(M, B) \oplus \mathrm{Tor}(\mathrm{H}_{k-1}(M, B)).$$

follows from the $\mathbb{Z}_k$-Ext-Lemma 70.11, the Ext-Properties Lemma 70.12 (4) and the observation that it follows from the Topological Manifolds-Invariants Proposition 60.10 (2) that the homology groups $\mathrm{H}_i(M, B)$ are finitely generated abelian groups

■

In the following corollary we apply the above proposition to the cases $k = 0$ and $k = 1$:

Corollary 79.3. (Homology-rel-Partial Boundary Corollary) Let M be a compact orientable connected non-empty n-dimensional topological manifold and let A be a compact codimension-zero submanifold of ∂M.

(1) We have[617]
$$\mathrm{H}_n(M, A) \cong \begin{cases} \mathbb{Z}, & \text{if } A = \partial M, \\ 0, & \text{otherwise.} \end{cases}$$
(2) The homology group $\mathrm{H}_{n-1}(M, A)$ is a free abelian group.[618]

Proof. We set $B := (\partial M) \setminus A$. Note that it follows from the Topological Submanifold Proposition 42.6 (3) that A and B satisfy the hypothesis of the Poincaré Duality–Non Natural Isomorphism Proposition 79.2.

(1) Note that
$$\mathrm{H}_n(M, A) \underset{\uparrow}{\cong} \mathrm{FH}_0(M, B) \underset{\uparrow}{\cong} \begin{cases} \mathbb{Z}, & \text{if } B = \varnothing, \text{ i.e. if } A = \partial M, \\ 0, & \text{otherwise.} \end{cases}$$
by the Poincaré Duality–Non Natural Isomorphism Proposition 79.2

follows from the H_0-Proposition 35.11 and the discussion on page 489, here we use that M is path-connected and non-empty

(2) Note that
$$\mathrm{H}_{n-1}(M, A) \underset{\text{by the Poincaré Duality–Non Natural Isomorphism Proposition 79.2}}{\cong} \underbrace{\mathrm{FH}_1(M, B)}_{\text{free abelian group}} \oplus \underbrace{\mathrm{Ext}(\overbrace{\mathrm{H}_0(M, B)}^{\text{free abelian group}}, \mathbb{Z})}_{= 0 \text{ by the Ext-Properties Lemma 70.12 (4)}}$$

Note that here we used that it follows from the Topological Manifolds-Invariants Proposition 60.10 that $\mathrm{H}_1(M, B)$ is finitely generated. ■

79.4. **Poincaré Duality Theorem 79.1 applied with field coefficients.** In this section we consider (co-) homology groups with field coefficients. The following proposition combines the Poincaré Duality Theorem 79.1 with the Universal Coefficient-H^*–Field Theorem 71.2:

Theorem 79.4. (Homology Symmetry Theorem) Let M be a compact n-dimensional topological manifold and let $k \in \mathbb{N}_0$. Furthermore suppose that we are given a decomposition $\partial M = A \cup B$ where A and B are compact $(n-1)$-dimensional submanifolds of ∂M such that $A \cap B = \partial A = \partial B$.

(1) We have[619]
$$\dim_{\mathbb{F}_2}(\mathrm{H}_k(M, A; \mathbb{F}_2)) = \dim_{\mathbb{F}_2}(\mathrm{H}_{n-k}(M, B; \mathbb{F}_2)).$$
(2) If M is orientable, then for any field $\mathbb{F}$ we have
$$\dim_{\mathbb{F}}(\mathrm{H}_k(M, A; \mathbb{F})) = \dim_{\mathbb{F}}(\mathrm{H}_{n-k}(M, B; \mathbb{F})).$$
By the Poincaré Duality Theorem 79.1 the analogous statements also hold for cohomology instead of homology.

[617] This statement is consistent with the statements of the Manifold-Top-H_*-Theorem 63.1.
[618] This statement gives a new proof of some statements of the Manifold-H_{n-1}-Theorem 63.4 (2).

Proof.

(1) We note that

$$\dim_{\mathbb{F}_2}(\mathrm{H}_k(M, A; \mathbb{F}_2)) \underset{\uparrow}{=} \dim_{\mathbb{F}_2}(\mathrm{H}^{n-k}(M, B; \mathbb{F}_2)) \underset{\uparrow}{=} \dim_{\mathbb{F}_2}(\mathrm{Hom}(\mathrm{H}_{n-k}(M, B; \mathbb{F}_2), \mathbb{F}_2))$$

by the Poincaré Duality Theorem 79.1, which we can apply since by the $\mathbb{F}_2$-Orientability Proposition 61.13 we know that M is $\mathbb{F}_2$-orientable

by the Universal Coefficient-H*–Field Theorem 71.2

$$= \dim_{\mathbb{F}_2}(\mathrm{H}_{n-k}(M, B; \mathbb{F}_2)).$$

since it follows from the Topological Manifolds-Invariants Proposition 60.10 that $\mathrm{H}_{n-k}(M, B; \mathbb{F}_2)$ is finite-dimensional

(2) Let $\mathbb{F}$ be any field. We now assume that M is orientable. It follows from the Orientable-for-all-Rings Lemma 61.12 that M is also $\mathbb{F}$-orientable. The proof of (2) is now basically identical to the proof of (1). ■

Example. We consider the real projective space $\mathbb{R}\mathrm{P}^n$. On page 685 we showed that

$$\mathrm{H}_k(\mathbb{R}\mathrm{P}^n; \mathbb{F}_2) \cong \begin{cases} \mathbb{F}_2, & \text{if } k = 0, \dots, n, \\ 0, & \text{otherwise.} \end{cases}$$

So the homology groups with $\mathbb{F}_2$-coefficients are in particular symmetric, which is exactly what the above Homology Symmetry Theorem 79.4 with $\mathbb{F}_2$-coefficients predicts. □

Next we recall the following definition from page 608:

Definition. Let X be a topological space and let $n \in \mathbb{N}_0$. We define the n-th Betti number of X as

$$b_n(X) := \mathrm{rank}(\mathrm{H}_n(X)) \underset{\uparrow}{=} \dim_{\mathbb{Q}}(\mathrm{H}_n(X; \mathbb{Q})) \in \mathbb{N}_0 \cup \{\infty\}.$$

by the Betti-via-Dimension Lemma 53.10

With this definition we can formulate the following straightforward corollary to the Homology Symmetry Theorem 79.4:

Corollary 79.5. (Betti Number Symmetry Corollary) Let M be a closed orientable n-dimensional topological manifold. Given any $k \in \mathbb{N}_0$ we have $b_{n-k}(M) = b_k(M)$.

Proof. Let M be a closed orientable n-dimensional topological manifold. We have

$$b_{n-k}(M) = \dim_{\mathbb{Q}}(\mathrm{H}_{n-k}(M; \mathbb{Q})) \underset{\uparrow}{=} \dim_{\mathbb{Q}}(\mathrm{H}_k(M; \mathbb{Q})) = b_k(M).$$

by the Homology Symmetry Theorem 79.4 ■

Remark. Henri Poincaré [**Poi95, Poi10, PGL53**, Chapter 9], formulated the conclusion of the Betti Number Symmetry Corollary 79.5 in a distinctly more elegant way: in his seminal paper "analysis situs", published in 1895, he wrote

> "[...] pour une variété fermé les nombres de Betti
> également distants des extrêmes sont égaux".

In fact when Henri Poincaré first developed what became known as Poincaré Duality he thought that for any closed orientable n-dimensional smooth manifold the groups $\mathrm{H}_k(M)$ and $\mathrm{H}_{n-k}(M)$ are isomorphic. A Danish mathematician, Poul Heegaard [**Hee98**], pointed

[619] The statement implies of course that $\mathrm{H}_k(M, A; \mathbb{F}_2)$ and $\mathrm{H}_{n-k}(M, B; \mathbb{F}_2)$ are isomorphic. We prefer to state the result in terms of dimensions to stress that the isomorphism is not natural.

out to him that this is not true in general, for example for the lens space $L(p,q)$ we have by the Lens Space-H_*-Lemma 63.5 that $\mathrm{H}_1(L(p,q)) \cong \mathbb{Z}_p$ but $\mathrm{H}_2(L(p,q)) = 0$. Shortly afterwards Henri Poincaré [**Poi00**] came up with the correct formulation of Poincaré Duality, namely the one given in the above Poincaré Duality–Non Natural Isomorphism Proposition 79.2. □

79.5. **Euler characteristics of topological manifolds.** Using the Homology Symmetry Theorem 79.4 we obtain the following pretty proposition:

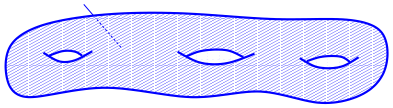
Proposition 79.6. (Odd-Dimensional Manifold-χ-Proposition) For every compact odd-dimensional topological manifold M we have

$$\chi(M) \;=\; \tfrac{1}{2}\cdot\chi(\partial M).$$

In particular if M is a closed odd-dimensional topological manifold, then $\chi(M) = 0$.

$$\begin{array}{rcl} \mathrm{H}_2(M) &=& 0 \\ \mathrm{H}_1(M) &\cong& \mathbb{Z}^3 \\ \mathrm{H}_0(M) &\cong& \mathbb{Z} \\ \Longrightarrow \chi(M) &=& -2 \end{array}$$

M is a 3-dimensional "handlebody"



∂M is a surface of genus 3

$$\begin{array}{rcl} \mathrm{H}_2(\partial M) &\cong& \mathbb{Z} \\ \mathrm{H}_1(\partial M) &\cong& \mathbb{Z}^6 \\ \mathrm{H}_0(\partial M) &\cong& \mathbb{Z} \\ \Longrightarrow \chi(\partial M) &=& -4 \end{array}$$

Proof. Let M be a compact topological manifold of dimension $2n+1$. Then

$$\chi(M) \;=\; \sum_{i=0}^{2n+1}(-1)^i\cdot\dim_{\mathbb{F}_2}(\mathrm{H}_i(M;\mathbb{F}_2)) \;\underset{\uparrow}{=}\; \sum_{i=0}^{2n+1}(-1)^i\cdot\dim_{\mathbb{F}_2}(\mathrm{H}_{2n+1-i}(M,\partial M;\mathbb{F}_2))$$

by the Homology Symmetry Theorem 79.4

$$\underset{\uparrow}{=}\; -\sum_{j=0}^{2n+1}(-1)^j\cdot\dim_{\mathbb{F}_2}(\mathrm{H}_j(M,\partial M;\mathbb{F}_2)) \;=\; -\chi(M,\partial M) \;\underset{\uparrow}{=}\; -\chi(M)+\chi(\partial M).$$

follows from the substitution $j = 2n+1-i$ together with the fact that $(-1)^{2n+1} = -1$

follows from the Relative-χ-Lemma 48.5

Summarizing we have shown that $\chi(M) = -\chi(M) + \chi(\partial M)$. The desired equality follows from solving for $\chi(M)$. ■

Using the Odd-Dimensional Manifold-χ-Proposition 79.6 we easily obtain the following surprising (?) corollary.

Corollary 79.7. (Non-Boundary Corollary)

(1) If W is a closed even-dimensional topological manifold that is homeomorphic to the boundary of a compact topological manifold, then $\chi(W)$ is even.
(2) The topological manifolds $\mathbb{R}\mathrm{P}^{2n}$ and $\mathbb{C}\mathrm{P}^{2n}$ are not homeomorphic to the boundary of a compact topological manifold.

Proof.

(1) Let W be a closed even-dimensional topological manifold that is homeomorphic to the boundary of a compact topological manifold M. Evidently M is odd-dimensional. Therefore we obtain from the Odd-Dimensional Manifold-χ-Proposition 79.6 that $\chi(W) = \chi(\partial M) = 2\cdot\chi(M) \in 2\cdot\mathbb{Z}$.
(2) This statement follows from Statement (1) together with the fact, shown on page 606, that $\chi(\mathbb{R}\mathrm{P}^{2n}) = \frac{1}{2}(1+(-1)^{2n}) = 1$ and that $\chi(\mathbb{C}\mathrm{P}^{2n}) = 2n+1$. ■

79.6. **The umkehr map.** The Poincaré Duality Theorem 79.1 allows us to introduce the weirdly named "umkehr map":

Definition. Let $f\colon M \to N$ be a continuous map between closed oriented n-dimensional topological manifolds. We consider the following diagram of maps:

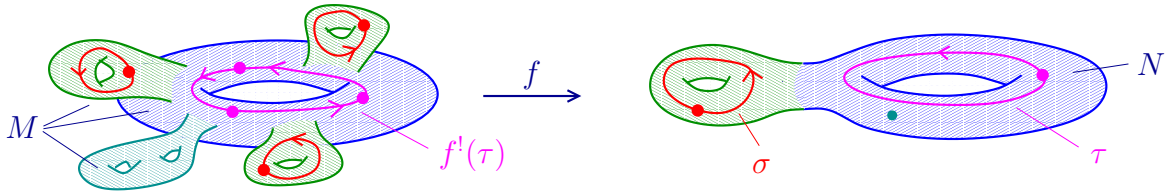
$$\begin{array}{ccc} \mathrm{H}_k(M;\mathbb{Z}) & \underset{\cap[M]}{\overset{\mathrm{PD}_M}{\rightleftarrows}} & \mathrm{H}^{n-k}(M;\mathbb{Z}) \\ {\scriptstyle f_*}\downarrow & & \uparrow{\scriptstyle f^*} \\ \mathrm{H}_k(N;\mathbb{Z}) & \underset{\cap[N]}{\overset{\mathrm{PD}_N}{\rightleftarrows}} & \mathrm{H}^{n-k}(N;\mathbb{Z}). \end{array}$$

Here the horizontal maps are the Poincaré Duality isomorphisms provided by the Poincaré Duality Theorem 79.1. We refer to

$$\begin{array}{rccl} f^!\colon \mathrm{H}_k(N;\mathbb{Z}) & \to & \mathrm{H}_k(M;\mathbb{Z}) \\ \sigma & \mapsto & f^*(\mathrm{PD}_N(\sigma)) \cap [M] \end{array}$$

as the umkehr map of f.[620] [621]

Example. In the figure below we show a continuous map $f\colon M \to N$ which is given by first collapsing the turquoise bit to a point, followed by a 3-fold covering map.



$f^!(\sigma)$ is the sum of the red terms

With some effort one can show that the umkehr map is of the form as described in the example.

Example. It is an entertaining exercise to show that if $f\colon M \to N$ is a finite covering, then the umkehr map $f^!$ equals the transfer map f^* that we defined on page 721. The details are provided in [**Fri24**].

To formulate our main result on umkehr maps let us recall the following definition from page 842:

Definition. Given a continuous map $f\colon M \to N$ between closed oriented connected non-empty n-dimensional topological manifolds we define the degree of f to be the unique integer $\deg(f) \in \mathbb{Z}$ with $f_*([M]) = \deg(f) \cdot [N]$.

Proposition 79.8. (Umkehr Map Proposition) Let $f\colon M \to N$ be a continuous map between closed oriented connected non-empty n-dimensional topological manifolds. Let

[620] This map is indeed called "umkehr map" in English. The name stems from the fact that it is a map on homology that "goes the other way".

[621] Wikipedia writes: "In category theory [...] certain unusual functors are denoted $f^!$ and $f_!$, with the exclamation mark used to indicate that they are exceptional in some way. They are thus accordingly sometimes called shriek maps, with "shriek" being slang for an exclamation mark [...]."

$k \in \mathbb{N}_0$. We consider the following two maps:

$$\mathrm{H}_k(N;\mathbb{Z}) \xrightarrow{\text{umkehr map } f^!} \mathrm{H}_k(M;\mathbb{Z}) \xrightarrow{\text{induced map } f_*} \mathrm{H}_k(N;\mathbb{Z}).$$

The following statements hold:

(1) The map $f_* \circ f^!$ is multiplication by $\deg(f)$.
(2) If $\deg(f) = \pm 1$, then there exists an isomorphism $\mathrm{H}_k(M;\mathbb{Z}) \cong \mathrm{H}_k(N;\mathbb{Z}) \oplus Q$ for some finitely generated abelian group Q.

Proof. Let $f\colon M \to N$ be a continuous map between closed oriented connected non-empty n-dimensional topological manifolds.

(1) Let $\sigma \in \mathrm{H}_k(N;\mathbb{Z})$. We perform the following calculation in $\mathrm{H}_k(N;\mathbb{Z})$:

$$\begin{array}{lcccl} & & \overset{\text{definition of } f^!}{\downarrow} & & \overset{\text{by the Cap Product-Naturality Lemma 75.8}}{\downarrow} \\ (f_* \circ f^!)(\sigma) & = & f_*(f^*(\mathrm{PD}_N(\sigma)) \cap [M]) & = & \mathrm{PD}_N(\sigma) \cap f_*([M]) \\ & = & \deg(f) \cdot \mathrm{PD}_N(\sigma) \cap [N] & = & \deg(f) \cdot \sigma. \\ & & \underset{\text{by definition of } \deg(f)}{\uparrow} & & \underset{\text{definition of } \mathrm{PD}_N}{\uparrow} \end{array}$$

(2) We now assume that $\deg(f) = \pm 1$. We consider the sequence

$$0 \to \mathrm{H}_k(N;\mathbb{Z}) \xrightarrow{f^!} \mathrm{H}_k(M;\mathbb{Z}) \to \mathrm{coker}(f^!) \to 0.$$

By (1) and our hypothesis we know that $f_* \circ f^!$ is an isomorphism. This shows that the umkehr map $f^!$ admits a left-inverse. In particular the left map is a monomorphism. It is now clear that the sequence is exact. Finally note that it follows from the Splitting Lemma 37.2 (2)⇒(3) that $\mathrm{H}_k(M;\mathbb{Z}) \cong \mathrm{H}_k(N;\mathbb{Z}) \oplus \mathrm{coker}(f^!)$. ■

On several occasions we were wondering whether there exists a continuous map of degree one between two given topological manifolds. We had obtained the following results:

(1) In the Degree Realization Proposition 65.4 we showed that for any closed oriented non-empty n-dimensional topological manifold M there exists a degree-one map $M \to S^n$.
(2) In the Degree One-Epimorphism Proposition 65.7 we saw that for a continuous map $f\colon M \to N$ of degree ± 1 the induced map $f_*\colon \pi_1(M) \to \pi_1(N)$ is an epimorphism.

For example, it follows from remark (2) above that there is no degree-one map from $\mathbb{R}\mathrm{P}^3$ to $S^1 \times S^2$, since evidently there is no epimorphism from $\pi_1(\mathbb{R}\mathrm{P}^3) \cong \mathbb{Z}_2$ onto $\pi_1(S^1 \times S^2) \cong \mathbb{Z}$. In Question 65.10 we had asked whether there exists a degree-one map from $S^1 \times S^2$ to $\mathbb{R}\mathrm{P}^3$. The above remark (2) has no objections to the existence of such a map. But now we can give a negative answer:

Corollary 79.9. (No Degree-1-Map Corollary) There is no degree-one map from $S^1 \times S^2$ to $\mathbb{R}\mathrm{P}^3$.

Proof. The corollary follows immediately from the Umkehr Map Proposition 79.8 and the observation that the group $\mathrm{H}_1(\mathbb{R}\mathrm{P}^3;\mathbb{Z}) \cong \mathbb{Z}_2$ is evidently not a subsummand of the group $\mathrm{H}_1(S^1 \times S^2;\mathbb{Z}) \cong \mathbb{Z}$. ■

CHAPTER 80

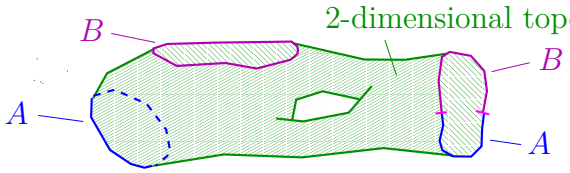
Proof of the Poincaré Duality Theorem 79.1

In the last chapter we formulated the following theorem:

Theorem 79.1. (Poincaré Duality Theorem) Let R be a commutative ring, let M be a compact R-oriented n-dimensional topological manifold and suppose that we are given a decomposition $\partial M = A \cup B$ where A and B are compact $(n-1)$-dimensional submanifolds of ∂M such that $A \cap B = \partial A = \partial B$. For each $k \in \mathbb{N}_0$ the map

$$\begin{array}{ccc} \mathrm{H}^k(M, A; R) & \xrightarrow{\cong} & \mathrm{H}_{n-k}(M, B; R) \\ \varphi & \mapsto & \varphi \cap [M] \end{array}$$

is an isomorphism.



for all commutative rings R we have
$\mathrm{H}^k(M, A; R) \xrightarrow{\cong} \mathrm{H}_{2-k}(M, B; R)$

In the last chapter we had a lot of fun applying the Poincaré Duality Theorem 79.1 to various settings. We obtained many interesting, sometimes perhaps also surprising, results. But now party time is over, we have to provide a proof of the theorem.

In fact the whole chapter is completely dedicated to the proof of the theorem. The structure of this chapter is as follows:

(1) In Section 80.1 we will use cohomology with compact support to formulate the Poincaré Duality–Non-Compact Theorem 80.2 which is a version of the Poincaré Duality Theorem 79.1 for topological manifolds with empty boundary but which are possibly non-compact.
(2) In Section 80.2 we provide the rather lengthy proof of the Poincaré Duality–Non-Compact Theorem 80.2. This gives us in particular a proof of the Poincaré Duality Theorem 79.1 for compact topological manifolds with empty boundary. This is certainly the key section of the whole chapter.
(3) In Section 80.3 we will use the Poincaré Duality–Non-Compact Theorem 80.2, and some formal tricks, to provide a proof of the Poincaré Duality Theorem 79.1 for the general case.

80.1. Poincaré Duality for non-compact manifolds with empty boundary.

In this section we will formulate a version of Poincaré Duality for every oriented topological manifold (not necessarily compact) with empty boundary.

Why do we do this? Here are a few answers:

(1) The key idea for the proof of Poincaré Duality Theorem 79.1 for *closed* topological manifolds is somewhat similar to the strategy of the proof of the Homology-Section Theorem 63.6: Using Mayer–Vietoris arguments and using limit arguments we want to reduce the proof eventually to the fairly trivial case of an open ball in $\mathbb{R}^n$.
(2) To make sense of the approach in (1) we need to allow in between topological manifolds that are non-compact. Thus we need to make sense of Poincaré Duality for non-compact manifolds.
(3) Later, when we study Poincaré Duality for topological manifolds we will see that relative cohomology $\mathrm{H}^k(M, \partial M; R)$ is actually isomorphic to $\mathrm{H}^k_c(M \setminus \partial M; R)$. Thus Poincaré Duality for non-compact manifolds will also allow us to deal with some cases of Poincaré Duality for compact manifolds with non-empty boundary.

The formulation of Poincaré Duality for (possibly non-compact) topological manifolds with empty boundary requires some preparations. First we recall some definitions and results from earlier on:

Notation. Let R be a commutative ring.

(1) Given a topological space X we denote by $\mathcal{K}(X)$ the set of all compact subsets of X. By the H^n_c-via-Direct Colimits Proposition 78.5 we have for every $k \in \mathbb{N}_0$ a natural isomorphism
$$\operatorname*{colim}_{K \in \mathcal{K}(X)} \mathrm{H}^k(X, X \setminus K; R) \xrightarrow{\cong} \mathrm{H}^k_c(X; R)$$
which we will use to identify these two groups. In fact in this chapter we will only work with the group on the left and we will not make use at all of the original definition of cohomology with compact support.
(2) Let M be an n-dimensional topological manifold with empty boundary. If M is equipped with an R-orientation $\{\alpha_x \in \mathrm{H}_n(M, M \setminus \{x\}; R)\}_{x \in M}$, then we saw in the Homology-Section Theorem 63.6 (i) that given any compact subset K of M there exists a unique class $\mu^M_K \in \mathrm{H}_n(M, M \setminus K; R)$ such that for every $x \in K$ the image of μ^M_K under the natural homomorphism $\mathrm{H}_n(M, M \setminus K; R) \to \mathrm{H}_n(M, M \setminus \{x\}; R)$ equals the given orientation α_x at x. If M is understood, then we drop it from the notation, i.e. we just write μ_K instead of μ^M_K.



With this notation we can now formulate the following lemma:

Lemma 80.1. (Cap–Non Compact Manifold Lemma) Let R be a commutative ring and let M be an R-oriented n-dimensional topological manifold with empty boundary.

(1) Let $K \subset L \subset M$ be compact subsets. We denote by $f_{KL} \colon (M, M \setminus L) \to (M, M \setminus K)$ the inclusion map of pairs.
   (a) The map $f_{KL*} \colon \mathrm{H}_n(M, M \setminus L; R) \to \mathrm{H}_n(M, M \setminus K; R)$ sends μ_L to μ_K.

(b) For every $k \in \mathbb{N}_0$ the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}^k(M, M \setminus L; R) & \xrightarrow{\cap \mu_L} & \mathrm{H}_{n-k}(M; R). \\ \uparrow{\scriptstyle f_{KL}^*} & \nearrow{\scriptstyle \cap \mu_K} & \\ \mathrm{H}^k(M, M \setminus K; R) & & \end{array}$$

(2) For every $k \in \mathbb{N}_0$ there exists a unique homomorphism

$$- \cap M \colon \mathrm{H}_{\mathrm{c}}^k(M; R) \to \mathrm{H}_{n-k}(M; R)$$

such that for each compact subset $K \subset M$ the following diagram commutes:[622]

$$\begin{array}{ccc} \mathrm{H}^k(M, M \setminus K; R) & \xrightarrow{\cap \mu_K} & \mathrm{H}_{n-k}(M; R). \\ & \searrow \quad \nearrow{\scriptstyle \cap M} & \\ & \mathrm{H}_{\mathrm{c}}^k(M; R) & \end{array}$$

Proof. Let M be an n-dimensional topological manifold with empty boundary that is equipped with an R-orientation $\{\alpha_x\}_{x \in M}$.

(1) (a) We need to show that $f_{KL*}(\mu_L)$ has the defining property of μ_K. But this is indeed the case since for every $x \in K$ we have, by functoriality of homology, the following commutative diagram:

$$\begin{array}{ccc} \mathrm{H}_n(M, M \setminus L; R) & \xrightarrow{f_{KL*}} & \mathrm{H}_n(M, M \setminus K; R) \\ & {\scriptstyle \mu_L \mapsto \alpha_x} \searrow \quad \swarrow & \\ & \mathrm{H}_n(M, M \setminus \{x\}; R). & \end{array}$$

(b) Let $\varphi \in \mathrm{H}^k(M, M \setminus K; R)$. We have the following equality in $\mathrm{H}_{n-k}(M; R)$:

$$\varphi \cap \mu_K \underset{\text{part (1a)}}{=} \varphi \cap f_{KL*}(\mu_L) \underset{\text{by the Cap Product-Naturality Lemma 75.8}}{=} f_{KL}^*(\varphi) \cap \mu_L.$$

But this means precisely that the diagram under consideration commutes.

(2) We consider the following diagram:

$$\begin{array}{ccccc} \mathrm{H}^n(M, M \setminus L; R) & & \xrightarrow{\cap \mu_L} & & \\ \uparrow{\scriptstyle f_{KL}^*} & \searrow & & & \\ & & \mathrm{H}_{\mathrm{c}}^n(M; R) = \operatorname*{colim}_{K \in \mathcal{K}(M)} \mathrm{H}^n(M, M \setminus K; R) & \xrightarrow[\cap M]{\exists!} & \mathrm{H}_{n-k}(M; R). \\ \mathrm{H}^n(M, M \setminus K; R) & \nearrow & & & \\ & & \xrightarrow{\cap \mu_K} & & \end{array}$$

The outer diagram commutes by (1b). It now follows from the universal property of the direct colimit that we get a uniquely defined homomorphism $- \cap M$ which makes the diagram commute. ∎

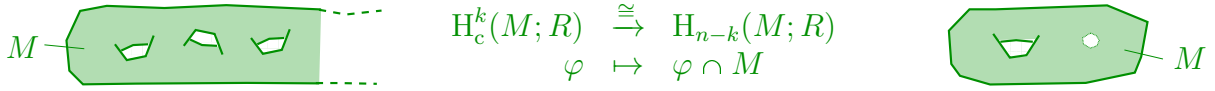
We can now formulate the promised result on Poincaré Duality for possibly non-compact topological manifolds:

[622] Note that for a class $\varphi \in \mathrm{H}^k(M, M \setminus K; R)$ and $\mu_K \in \mathrm{H}_n(M, M \setminus K; R)$ the cap product $\varphi \cap \mu_K$ does indeed lie in $\mathrm{H}_{n-k}(M; R)$.

Theorem 80.2. (Poincaré Duality–Non-Compact Theorem) Let R be a commutative ring and let M be an R-oriented n-dimensional topological manifold with empty boundary. For each $k \in \mathbb{N}_0$ the map

$$\begin{array}{rccc} - \cap M \colon & \mathrm{H}^k_{\mathrm{c}}(M;R) & \xrightarrow{\cong} & \mathrm{H}_{n-k}(M;R) \\ & \varphi & \mapsto & \varphi \cap M \end{array}$$

from the Cap–Non Compact Manifold Lemma 80.1 (2) is an isomorphism.



Proof. We provide the proof of the theorem in Section 80.2. ■

Example. Let $n \in \mathbb{N}_0$.

(1) Let M be an oriented connected non-empty n-dimensional topological manifold with empty boundary. Then

$$\begin{array}{ccccc} \mathrm{H}_n(M;\mathbb{Z}) & \xleftarrow[\uparrow]{\cong} & \mathrm{H}^0_{\mathrm{c}}(M;\mathbb{Z}) & \underset{\uparrow}{\cong} & \begin{cases} \mathbb{Z}, & \text{if } M \text{ is compact,} \\ 0, & \text{if } M \text{ is non-compact.} \end{cases} \\ \text{by the Poincaré Duality} & & & \text{see page 1001} & \\ \text{Non-Compact Theorem 80.2} & & & & \end{array}$$

This calculation is consistent with the results obtained in the Orientation-Top Homology-Theorem 63.3 and the Manifold-Top-H_*-Theorem 63.1.

(2) We note that

$$\begin{array}{ccccc} \mathrm{H}^n_{\mathrm{c}}(\mathbb{R}^n;\mathbb{Z}) & \xrightarrow[\uparrow]{\cong} & \mathrm{H}_0(\mathbb{R}^n;\mathbb{Z}) & \underset{\uparrow}{\cong} & \mathbb{Z}. \\ \text{by the Poincaré Duality} & & & \text{by the } \mathrm{H}_0\text{-Proposition 35.11} & \\ \text{Non-Compact Theorem 80.2} & & & & \end{array}$$

This is consistent with the calculation on page 1010. □

On a few occasions we will need a naturality statement regarding the map in the above Poincaré Duality–Non-Compact Theorem 80.2. To formulate the lemma let us recall the following definition from page 1011:

Notation. Let X be a Hausdorff space, let U be an open subset of X, let $k \in \mathbb{N}_0$ and let G be an abelian group. As on page 1011, we denote by $\iota_* = \iota_{XU*} \colon \mathrm{H}^k_{\mathrm{c}}(U;G) \to \mathrm{H}^k_{\mathrm{c}}(X;G)$ the homomorphism that is defined by

$$\mathrm{H}^k_{\mathrm{c}}(U) =_{\mathrm{i}} \varinjlim_{K \in \mathcal{K}(U)} \mathrm{H}^k(U, U \setminus K) \xleftarrow{\cong} \varinjlim_{K \in \mathcal{K}(U)} \mathrm{H}^k(X, X \setminus K) \to \varinjlim_{K \in \mathcal{K}(X)} \mathrm{H}^k(X, X \setminus K) =_{\mathrm{i}} \mathrm{H}^k_{\mathrm{c}}(X).$$

(We dropped "G" for space reasons.)

Now we are ready to formulate the promised naturality statement:

Lemma 80.3. (Poincaré Duality–Open Subset Lemma) Let R be a commutative ring, let M be an R-oriented n-dimensional topological manifold with empty boundary and let U be an open subset of M. We denote by $\iota \colon U \to M$ the inclusion. For every

$k \in \mathbb{N}_0$ the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}_c^k(U;R) & \xrightarrow{\iota_*} & \mathrm{H}_c^k(M;R) \\ \downarrow \cap U & & \downarrow \cap M \\ \mathrm{H}_{n-k}(U;R) & \xrightarrow{\iota_*} & \mathrm{H}_{n-k}(M;R). \end{array}$$

Proof. Let $U \subset M$ be an open subset.

Claim. For every compact subset $K \subset U$ the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}^k(U, U \setminus K;R) & \xrightarrow[\cong]{(\iota^*)^{-1}} & \mathrm{H}^k(M, M \setminus K;R) \\ \downarrow \cap \mu_K^U & & \downarrow \cap \mu_K^M \\ \mathrm{H}_k(U;R) & \xrightarrow{\iota_*} & \mathrm{H}_{n-k}(M;R). \end{array}$$

Proof. Given any $\phi \in \mathrm{H}^k(M, M \setminus K;R)$ we calculate that

$$\begin{array}{ccccc} \iota_*(\iota^*(\phi) \cap \mu_K^U) & = & \phi \cap \iota_*(\mu_K^U) & = & \phi \cap \mu_K^M. \\ \uparrow & & \uparrow & & \\ \text{by the Cap Product Naturality Lemma 75.8} & & \text{by the functoriality of homology groups and the uniqueness statement of the Homology-Section Theorem 63.6 we have } \iota_*(\mu_K^U) = \mu_K^M & & \end{array}$$

⊞

Next we consider the following diagram (we drop the R-coefficients for space reasons):

$$\begin{array}{ccccccccc} \mathrm{H}_c^k(U;\mathbb{Z}) & \xleftarrow{\cong} & \varinjlim_{K \in \mathcal{K}(U)} \mathrm{H}^k(U, U \setminus K) & \xrightarrow[\cong]{(\iota^*)^{-1}} & \varinjlim_{K \in \mathcal{K}(U)} \mathrm{H}^k(M, M \setminus K) & \longrightarrow & \varinjlim_{K \in \mathcal{K}(M)} \mathrm{H}^k(M, M \setminus K) & \xrightarrow{\cong} & \mathrm{H}_c^k(M) \\ \downarrow \cap U & & \downarrow & & \downarrow & & \downarrow & & \downarrow \cap M \\ \mathrm{H}_{n-k}(U) & \xrightarrow{\mathrm{id}} & \mathrm{H}_{n-k}(U) & \xrightarrow{\iota_*} & \mathrm{H}_{n-k}(M) & \xrightarrow{\mathrm{id}} & \mathrm{H}_{n-k}(M) & \xrightarrow{\mathrm{id}} & \mathrm{H}_{n-k}(M). \end{array}$$

(Here the unnamed vertical maps are induced by capping with μ_K^U and μ_K^M.) The squares to the left and right commute by the definition of the maps "$\cap U$" and "$\cap M$". It follows from the claim that the second square commutes. Finally note that it follows easily from the universal property of direct colimits that the third square commutes. ■

80.2. **Proof of the Poincaré Duality–Non-Compact Theorem 80.2.** The proof of the Poincaré Duality–Non-Compact Theorem 80.2 takes place in the following three subsections:

(1) In Subsection 80.2.1 we prove the desired statement for $M = B^n$.
(2) In Subsection 80.2.2 we will state and prove the Poincaré Duality–Mayer Vietoris Proposition 80.5 which allows us to deal with unions of open subsets which satisfy Poincaré Duality. This subsection contains technically the hardest argument for the proof of the Poincaré Duality–Non-Compact Theorem 80.2.
(3) In Subsection 80.2.3 we will use (1) and (2) to conclude the proof of the Poincaré Duality–Non-Compact Theorem 80.2. In contrast to the proof of (2), this argument is again quite pleasant to read.

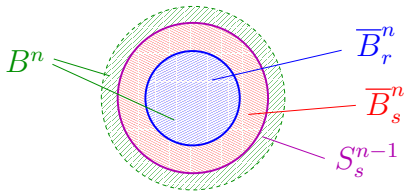
80.2.1. **Poincaré Duality for Balls.** As a warm up for the proof of the Poincaré Duality–Non-Compact Theorem 80.2 we will deal with the case $M = B^n$ by hand. More precisely, we will prove the following lemma:

Lemma 80.4. (PD-for-Ball Lemma) Let R be a commutative ring. For each $k \in \mathbb{N}_0$ the map[623]

$$\begin{array}{ccc} \mathrm{H}^k_c(B^n;R) & \to & \mathrm{H}_{n-k}(B^n;R) \\ \varphi & \mapsto & \varphi \cap B^n \end{array}$$

from the Cap–Non Compact Manifold Lemma 80.1 (2) is an isomorphism.

Proof. Let $k \in \mathbb{N}_0$. Given any $s \in (0,1)$ we write $\overline{B}^n_s := \{x \in \mathbb{R}^n \mid \|x\| \leq s\}$ and we also write $S^{n-1}_s := \partial \overline{B}^n_s$. For the remainder of the proof we fix $0 < r < s < 1$. We denote by $i\colon (\overline{B}^n_s, S^{n-1}_s) \to (B^n, B^n \setminus \overline{B}^n_r)$ the inclusion map of pairs.



We consider the following diagram:

$$\begin{array}{ccccc} \mathrm{H}^k(\overline{B}^n_s, S^{n-1}_s;R) & \xleftarrow{i^*} & \mathrm{H}^k(B^n, B^n \setminus \overline{B}^n_r;R) & \longrightarrow & \mathrm{H}^k_c(B^n;R) \\ \downarrow{\scriptstyle \cap [\overline{B}^n_s]} & & \downarrow{\scriptstyle \cap \mu_{\overline{B}^n_r}} & & \downarrow{\scriptstyle \cap B^n} \\ \mathrm{H}_{n-k}(\overline{B}^n_s;R) & \xrightarrow{i_*} & \mathrm{H}_{n-k}(B^n;R) & \xrightarrow{\mathrm{id}} & \mathrm{H}_{n-k}(B^n;R). \end{array}$$

We need to show that the vertical map to the right is an isomorphism. The lemma is an immediate consequence of the following claim:

Claim.

(1) The diagram commutes.
(2) The left vertical map is an isomorphism.
(3) All horizontal maps are isomorphisms.

Proof.

(1) The square on the right commutes by definition of the map $- \cap B^n$. So it remains to show that the square on the left commutes. Thus let $\varphi \in \mathrm{H}^k(B^n, B^n \setminus \overline{B}^n_r;R)$. We have

$$i_*(i^*\varphi \cap [\overline{B}^n_s]) \underset{\uparrow}{=} \varphi \cap i_*([\overline{B}^n_s]) \underset{\uparrow}{=} \varphi \cap \mu_{\overline{B}^n_r} \in \mathrm{H}_{n-k}(B^n;R).$$

by the Cap Product-Naturality Lemma 75.8

we have $i_*([\overline{B}^n_s]) = \mu_{\overline{B}^n}$ since for each $x \in \overline{B}^n_r$ the image of $i_*([\overline{B}^n_s])$ in $\mathrm{H}_n(B^n, B^n \setminus \{x\})$ equals the generator given by the orientation

(2) We consider the left vertical map of the diagram. Clearly both groups are zero for $k \neq n$. So it remains to consider the case $k = n$. It follows from the Cap Product-Kronecker Lemma 75.7 together with the discussion on page 661 that it remains to show that the map

$$\begin{array}{ccc} \mathrm{H}^n(\overline{B}^n_s, S^{n-1}_s;R) & \to & R \\ \varphi & \mapsto & \langle \varphi, [\overline{B}^n_s] \rangle = \underset{\uparrow}{\epsilon_{\overline{B}^n_s}}(\varphi \cap [\overline{B}^n_s]) \end{array}$$

augmentation homomorphism

[623] We use the convention that any n-dimensional submanifold of $\mathbb{R}^n$ is equipped with the orientation which is provided by the Standard Euclidean Orientation Lemma 61.10.

is an isomorphism. But this statement is an immediate consequence of the Manifold-Top-H*-Theorem 70.16 together with the Ring–Kronecker Pairing Lemma 71.1 (2).

(3) It follows easily from the Homotopy Equivalence-Relative-H$_*$-Corollary 38.4 (1) and the Homotopy Equivalence-H*-Corollary 67.8 (2) (say using the intermediate pair (B^n, S_s^{n-1})) that the left horizontal maps are isomorphisms. It remains to show that the top right horizontal map is an isomorphism. We consider the following diagram:

$$\begin{array}{ccc} \mathrm{H}^k(B^n, B^n \setminus \overline{B}_r^n; R) & \longrightarrow & \varinjlim_{K\in\mathcal{K}(B^n)} \mathrm{H}^k(B^n, B^n \setminus K; R) =_{\mathrm{i}} \mathrm{H}_{\mathrm{c}}^k(B^n; R). \\ & \searrow \quad \nearrow & \\ & \varinjlim_{t\in(0,1)} \mathrm{H}^k(B^n, B^n \setminus \overline{B}_t^n; R) & \end{array}$$

We make the following three observations:

(a) It follows easily from the definitions that the diagram commutes.

(b) Every compact subset of B^n is contained in a closed ball $\overline{B}_t^n$ with $t \in (0,1)$. Therefore it follows from the Cofinal-Isomorphism Lemma 78.4 that the right diagonal map is an isomorphism.

(c) It follows easily from the Homotopy Equivalence-H*-Corollary 67.8 (1) that for each $t > r$ the inclusion-induced homomorphism $\mathrm{H}^k(B^n, B^n \setminus \overline{B}_r^n; R) \to \mathrm{H}^k(B^n, B^n \setminus \overline{B}_t^n; R)$ is an isomorphism. It follows from this observation together with the Direct Colimit–Stabilization Lemma 25.1 that the left diagonal map is an isomorphism.

These two observations together imply that the horizontal map in the above triangular diagram is also an isomorphism. ■

80.2.2. **Poincaré Duality for unions.** The key to the proof of the Poincaré Duality–Non-Compact Theorem 80.2 is the following proposition:

Proposition 80.5. (Poincaré Duality–Mayer Vietoris Proposition) Let R be a commutative ring and let M be an R-oriented n-dimensional topological manifold with empty boundary that is the union of two open subsets U and V. Then there exists a diagram of homomorphisms (where we drop the "R" for space reasons)

$$\begin{array}{ccccccccc} \dots \longrightarrow & \mathrm{H}_{\mathrm{c}}^k(U\cap V) & \xrightarrow{(\iota^*,-\iota^*)} & \mathrm{H}_{\mathrm{c}}^k(U)\oplus\mathrm{H}_{\mathrm{c}}^k(V) & \xrightarrow{\iota^*\oplus\iota^*} & \mathrm{H}_{\mathrm{c}}^k(M) & \longrightarrow & \mathrm{H}_{\mathrm{c}}^{k+1}(U\cap V) & \longrightarrow \dots \\ & \downarrow{\scriptstyle \cap(U\cap V)} & & \downarrow{\scriptstyle \cap U\oplus\cap V} & & \downarrow{\scriptstyle \cap M} & & \downarrow{\scriptstyle \cap(U\cap V)} & \\ \dots \to & \mathrm{H}_{n-k}(U\cap V) & \xrightarrow{(\iota_*,-\iota_*)} & \mathrm{H}_{n-k}(U)\oplus\mathrm{H}_{n-k}(V) & \xrightarrow{\iota_*\oplus\iota_*} & \mathrm{H}_{n-k}(M) & \to & \mathrm{H}_{n-k-1}(U\cap V) & \to \dots \end{array}$$

with the following properties:

(1) The first two squares commute, the third square commutes up to the sign $(-1)^{k+1}$.

(2) The lower exact sequence is the long exact Mayer–Vietoris sequence for $M = U \cup V$.

(3) The upper horizontal sequence is also exact.

The remainder of this subsection is concerned with the proof of the Poincaré Duality–Mayer Vietoris Proposition 80.5. This proof is rather long and technical and, frankly, not particularly enlightening. For most readers the best thing to do is to jump straight to Subsection 80.2.3 where we conclude the proof of the Poincaré Duality–Non-Compact Theorem 80.2.

Since you decided to stick around for the proof of the Poincaré Duality–Mayer Vietoris Proposition 80.5, let us get started. Among many other things we will have to show that the top horizontal sequence is exact. The key tool is an "inverted" Mayer–Vietoris Theorem, which is very similar to the Inverted Mayer–Vietoris Theorem 63.8.

Before we can state this result we need to recall the following notation from page 940:

Notation. Let (X, A, B) be a triad of topological spaces. For each $k \in \mathbb{N}_0$ we write

$$\mathrm{C}_k(X, \{A, B\}) \;:=\; \frac{\mathrm{C}_*(X)}{\mathrm{C}_k^{\{A,B\}}(A \cup B)} \;=\; \frac{\mathrm{C}_*(X)}{\mathrm{C}_k(A) + \mathrm{C}_k(B)}.$$

Given any abelian group G we obtain the corresponding cochain complex $\mathrm{C}^*(X, \{A, B\}; G)$ and the corresponding cohomology groups $\mathrm{H}^k(X, \{A, B\}; G)$.

We also make use of the following convention:

Convention. Given a topological space X, given subspaces $A \subset B \subset X$ and given $\varphi \in \mathrm{C}^k(X, B; G)$ we denote by φ also the image of φ under the natural homomorphism $\mathrm{C}^k(X, B; G) \to \mathrm{C}^k(X, A; G)$.

Theorem 80.6. (Inverted Mayer–Vietoris Theorem) Let X be a topological space, let A and B be open subsets and let G be an abelian group.

(1) For every $n \in \mathbb{N}_0$ and every cochain $\varphi \in \mathrm{C}^n(X, A \cap B; G)$ there exist $\varphi_A \in \mathrm{C}^n(X, A; G)$ and $\varphi_B \in \mathrm{C}^n(X, B; G)$ such that[624] $\varphi_A + \varphi_B = \varphi$.

(2) For all $n \in \mathbb{N}_0$ there exists a unique homomorphism

$$\delta_n \colon \mathrm{H}^n(X, A \cap B; G) \;\to\; \mathrm{H}^{n+1}(X, \{A, B\}; G)$$

which has the following property: For every $\varphi_A \in \mathrm{C}^n(X, A; G)$ and $\varphi_B \in \mathrm{C}^n(X, B; G)$ such that $\varphi_A + \varphi_B$ is a cocycle in $\mathrm{C}^n(X, A \cap B; G)$ we have

$$\delta_n([\varphi_A + \varphi_B]) \;=\; [\delta_n(\varphi_A)] \;=\; -[\delta_n(\varphi_B)] \;\in\; \mathrm{H}^{n+1}(X, \{A, B\}; G).$$

(3) The natural map $\xi \colon \mathrm{H}^k(X, A \cup B; G) \to \mathrm{H}^k(X, \{A, B\}; G)$ is an isomorphism.

(4) The sequence

$$\cdots \to \mathrm{H}^n(X, A \cup B; G) \xrightarrow{(i^*, -i^*)} \mathrm{H}^n(X, A; G) \oplus \mathrm{H}^n(X, B; G) \xrightarrow{i^* \oplus i^*} \mathrm{H}^n(X, A \cap B; G) \xrightarrow{\xi^{-1} \circ \delta_n} \mathrm{H}^{n+1}(X, A \cup B; G) \to \dots$$

is exact.

Sketch of proof. First note that it is pretty clear that the following sequence of chain maps between chain complexes is exact:

$$0 \;\to\; \mathrm{C}_*(X, A \cap B) \xrightarrow{(i_*, i_*)} \mathrm{C}_*(X, A) \oplus \mathrm{C}_*(X, B) \xrightarrow{i_* \oplus (-i_*)} \mathrm{C}_*(X, \{A, B\}) \;\to\; 0.$$

Note that the group to the right is free abelian. It follows from the Free Abelian Group–Splitting Lemma 18.5 together with Hom-Split-SES Lemma 66.4 that the following induced sequence of cochain maps between cochain complexes is also exact:

$$0 \;\to\; \mathrm{C}^*(X, \{A, B\}; G) \xrightarrow{(i^*, -i^*)} \mathrm{C}^*(X, A; G) \oplus \mathrm{C}^*(X, B; G) \xrightarrow{i^* \oplus i^*} \mathrm{C}^*(X, A \cap B; G) \;\to\; 0.$$

We now turn to the proof of the four statements:

(1) This statement follows immediately from the above short exact sequence.

[624] Here we make use of the above convention.

(2) This statement follows from the above short exact sequence and the explicit description of the connecting homomorphism on page 483.
(3) This statement follows from the Excisive Triad–Invariants Lemma 73.1 (2b).
(4) As in the LES-Proposition 37.4 and in the Cohomology–LES Proposition 66.7 we get a long exact sequence in cohomology involving the group $\mathrm{H}^*(X;\{A,B\};G)$. Using (3) we obtain the desired long exact sequence. ■

In the proof of the Poincaré Duality–Non-Compact Theorem 80.2 we will also need the following two lemmas.

Lemma 80.7. (Compact Subset Decomposition Lemma) Let M be a topological manifold and let U, V be open subsets such that $M = U \cup V$. Then given any compact subset $C \subset M$ there exist compact subsets $K \subset U$ and $L \subset V$ such that $C = K \cup L$.



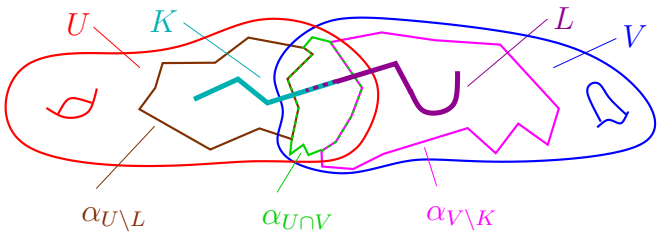
Proof of Lemma 80.7. Let M be an n-dimensional topological manifold and let U, V be open subsets with $M = U \cup V$. Furthermore let C be a compact subset of M. Using charts one can easily show that given any point $x \in C$ there exists a compact set A_x with the following properties:

(1) The point x lies in the interior of A_x.
(2) If $x \in U$ then the set A_x is contained in U.
(3) If $x \in V$ then the set A_x is contained in V.

It follows from (1), the fact that the sets A_x cover all of C together with the hypothesis that C is compact that there exist finitely many $y_1, \dots, y_r \in C$ with $C \subset \mathring{A}_{y_1} \cup \cdots \cup \mathring{A}_{y_r}$. One can now easily verify that $K := \bigcup_{y_i \in U} (A_{y_i} \cap C)$ and $L := \bigcup_{y_i \in V} (A_{y_i} \cap C)$ have the desired properties. ■

Lemma 80.8. (Fundamental Cycle Decomposition Lemma) Let R be a commutative ring, let M be a compact R-oriented n-dimensional topological manifold, let U, V be open subsets of M such that $M = U \cup V$ and let $K \subset U$ and $L \subset V$ be compact subsets. Then there exist singular n-chains $\alpha_{U\setminus L}$, $\alpha_{U\cap V}$ and $\alpha_{V\setminus K}$ in $U \setminus L$, $U \cap V$ and $V \setminus K$ such that the following three equalities hold:

(a)	$\mu^M_{K\cup L}$	$=$	$[\alpha_{U\setminus L} + \alpha_{U\cap V} + \alpha_{V\setminus K}]$	in $\mathrm{H}_n(M, M \setminus (K \cup L); R)$
(b)	$\mu^{U\cap V}_{K\cap L}$	$=$	$[\alpha_{U\cap V}]$	in $\mathrm{H}_n(U \cap V, (U \cap V) \setminus (K \cap L); R)$
(c)	μ^U_K	$=$	$[\alpha_{U\setminus L} + \alpha_{U\cap V}]$	in $\mathrm{H}_n(U, U \setminus K; R)$.



Proof of Lemma 80.8. We pick a representative $\alpha \in \mathrm{C}_n(M;R)$ for $\mu^M_{K\cup L}$. Note that the open subsets $U\setminus L, U\cap V$ and $V\setminus K$ cover M. It follows easily from the Iterated Subdivision Lemma 40.11 and the Simplex–Reduction Lemma 40.14 that there exist singular chains $\alpha_{U\setminus L}$, $\alpha_{U\cap V}$ and $\alpha_{V\setminus K}$ in $U\setminus L$, $U\cap V$ and $V\setminus K$ such that α is homologous in $\mathrm{C}_k(M;R)$ to $\alpha_{U\setminus L}+\alpha_{U\cap V}+\alpha_{V\setminus K}$. The chains are in particular homologous in $\mathrm{C}_k(M, M\setminus(K\cup L);R)$ which implies that (a) holds. Next note that we have

$$\mu^M_{K\cap L} \underset{\uparrow}{=} \text{image of } \mu^M_{K\cup L} = [\alpha_{U\setminus L}+\alpha_{U\cap V}+\alpha_{V\setminus K}] \underset{\uparrow}{=} [\alpha_{U\cap V}] \in \mathrm{H}_n(M, M\setminus(K\cap L);R).$$

by the Cap–Non Compact Manifold Lemma 80.1 (1)

since $\alpha_{U\setminus L}, \alpha_{V\setminus K}$ lie in $\mathrm{C}_n(M\setminus(K\cap L);R)$

But $\alpha_{U\cap V}$ lies in $U\cap V$, therefore it follows from functoriality of homology groups and the uniqueness statement of the Homology-Section Theorem 63.6 that $\mu^{U\cap V}_{K\cap L} = [\alpha_{U\cap V}]$ in $\mathrm{H}_n(U\cap V, (U\cap V)\setminus(K\cap L);R)$. We have thus verified (b).

Finally the proof of equality (c) is basically the same as the proof of equality (b). ∎

Now we are finally in a position to prove the Poincaré Duality–Mayer Vietoris Proposition 80.5:

Proof of the Poincaré Duality–Mayer Vietoris Proposition 80.5. Let R be a commutative ring and let M be an R-oriented n-dimensional topological manifold with empty boundary. For space reasons we will drop the "R" throughout the proof of this proposition.

Given any two subsets $X\subset Y$ of M and given $i\in\mathbb{N}_0$ we write, as on page 813, $\mathrm{H}^i(Y|X) := \mathrm{H}^i(Y, Y\setminus X)$. We note that if U is open and $K\subset U$ is compact, then we obtain a homomorphism

$$-\cap\mu^U_K\colon \mathrm{H}^k(U|K) \to \mathrm{H}_{n-k}(U).$$

Now let U and V be open subsets with $M=U\cup V$. Furthermore let $K\subset U$ and $L\subset V$ be compact subsets. We consider the following diagram:

$$\begin{array}{ccccccccc}
\dots\to \mathrm{H}^k(M|K\cap L) & \to & \mathrm{H}^k(M|K)\oplus\mathrm{H}^k(M|L) & \longrightarrow & \mathrm{H}^k(M|K\cup L) & \xrightarrow{\delta_k} & \mathrm{H}^{k+1}(M|K\cap L) & \to\dots \\
\downarrow\cong & & \downarrow\cong & & & & \downarrow\cong & \\
\mathrm{H}^k(U\cap V|K\cap L) & & \mathrm{H}^k(U|K)\oplus\mathrm{H}^k(V|L) & & \downarrow\cap\mu^M_{K\cup L} & & \mathrm{H}^{k+1}(U\cap V|K\cap L) & \\
\downarrow\cap\mu^{U\cap V}_{K\cap L} & & \downarrow\cap\mu^U_K\oplus\cap\mu^V_L & & & & \downarrow\cap\mu^{U\cap V}_{K\cap L} & \\
\dots\to \mathrm{H}_{n-k}(U\cap V) & \longrightarrow & \mathrm{H}_{n-k}(U)\oplus\mathrm{H}_{n-k}(V) & \longrightarrow & \mathrm{H}_{n-k}(M) & \xrightarrow{\partial_{n-k}} & \mathrm{H}_{n-k-1}(U\cap V) & \to\dots
\end{array}$$

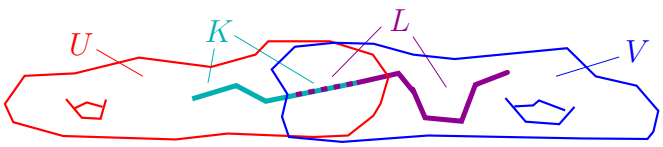
which we now explain in more detail:

- We write $A=M\setminus K$ and $B=M\setminus L$. The upper horizontal sequence is the inverted Mayer–Vietoris Sequence for cohomology groups from the Inverted Mayer–Vietoris Theorem 80.6.
- The lower horizontal sequence is the Mayer-Vietoris sequence coming from the Mayer–Vietoris Theorem 45.1 (1), corresponding to the decomposition $M=U\cup V$.
- The three vertical maps on the top are induced by inclusions of pairs of topological spaces, they are isomorphisms by the Excision-in-H*-Theorem 69.1 (1a).

The following is the key claim of the proof of the proposition:

Claim. The first two rectangles commute and the third rectangle commutes up to the sign $(-1)^{k+1}$.

$M = U \cup V$



Before we prove the claim we show that it implies our lemma. For each group appearing in the above diagram we take the direct colimit over the directed set[625]

$$\mathcal{K}(U,V) \coloneqq \{(K,L) \mid K \text{ compact subset of } U \text{ and } L \text{ compact subset of } V\}$$

and we consider the induced maps of direct colimits that were introduced on page 344. We make the following remarks regarding this new diagram:

(1) Using the Cofinal-Isomorphism Lemma 78.4 one can show that the groups that we obtain taking the direct colimit over $\mathcal{K}(U,V)$ are naturally isomorphic to the corresponding cohomology groups with compact support. We carry out the argument for two groups:

$$\underset{(K,L)\in\mathcal{K}(U,V)}{\operatorname{colim}} \mathrm{H}^k(U\cap V|K\cap L) \;=\; \underset{C\in\mathcal{K}(U\cap V)}{\operatorname{colim}} \mathrm{H}^k(U\cap V|C) \;=\; \mathrm{H}^k_{\mathrm{c}}(U\cap V).$$

we apply the Cofinal-Isomorphism Lemma 78.4 to the cofinal morphism of directed sets that is given by $(K,L)\mapsto K\cap L$

and

$$\underset{(K,L)\in\mathcal{K}(U,V)}{\operatorname{colim}} \mathrm{H}^k(M|K\cup L) \;=\; \underset{C\in\mathcal{K}(M)}{\operatorname{colim}} \mathrm{H}^k(M|C) \;=\; \mathrm{H}^k_{\mathrm{c}}(M;\mathbb{Z}).$$

we apply the Cofinal-Isomorphism Lemma 78.4 to the morphism of directed sets that is given by $(K,L)\mapsto K\cup L$, this morphism is cofinal by the Compact Subset Decomposition Lemma 80.7

All other cases are treated in a very similar fashion.

(2) By the functoriality of direct colimits, and since the signs in the commutativity do not depend on K and L, the new diagram also commutes (up to the same signs).

(3) Using (2) and the definition of the "PD-maps" in the Cap–Non Compact Manifold Lemma 80.1 (2) one can easily verify that the direct colimit of the diagram does indeed give rise to the diagram specified in the statement of the lemma.

(4) The Direct Colimit-Exactness Proposition 46.3 says that the direct colimit of exact sequences is again exact.

All these remarks put together show that the direct colimit of the above diagram gives rise precisely to the diagram that we claimed exists and that all claimed statements are satisfied. This concludes the proof that the claim implies the proposition.

Proof. Now we turn to the actual proof of the claim. We show in three separate subclaims that the three rectangles commute. The proof of the first two subclaims consists of a clever usage of the Cap Product-Naturality Lemma 75.8. The proof of the third subclaim is much more sophisticated since it relies on understanding the connecting homomorphisms in the Mayer-Vietoris sequences in (co-) homology.

[625] We view $\mathcal{K}(U,V)$ as a directed set given by the relation $(K,L)\leq(K',L')$ if $K\subset K'$ and $L\subset L'$. It is clear that each of the groups in the diagram forms a direct system with respect to this directed set.

Subclaim 1. The first rectangle of the diagram on page 1029 commutes.

So we need to show that the following diagram commutes:

$$\begin{array}{ccc}
\mathrm{H}^k(M, M\setminus(K\cap L)) & \xrightarrow{(i^*,-i^*)} & \mathrm{H}^k(M, M\setminus K)\oplus \mathrm{H}^k(M, M\setminus L;\mathbb{Z}) \\
\downarrow\cong & \searrow^{(i^*,-i^*)} & \downarrow\cong \\
\mathrm{H}^k(U\cap V, (U\cap V)\setminus(K\cap L)) & & \mathrm{H}^k(U, U\setminus K)\oplus \mathrm{H}^k(V, V\setminus L) \\
\downarrow\cap\mu_{K\cap L}^{U\cap V} & & \downarrow\cap\mu_K^U\oplus\cap\mu_L^V \\
\mathrm{H}_{n-k}(U\cap V) & \xrightarrow{(i_*,-i_*)} & \mathrm{H}_{n-k}(U)\oplus \mathrm{H}_{n-k}(V).
\end{array}$$

Evidently the triangle commutes. We note that the remaining pentagon breaks up into the "direct sum" of two diagrams. By symmetry reasons it suffices to consider the left-hand summands. So let $\phi\in \mathrm{H}^k(M, M\setminus(K\cap L))$. We consider the following diagram of continuous maps between pairs of topological spaces

$$\begin{array}{ccc}
(U\cap V, (U\cap V)\setminus(K\cap L)) & \xrightarrow{\alpha} & (U, U\setminus(K\cap L)) \\
\downarrow f\circ\alpha & \swarrow_{f} & \uparrow\beta \\
(M, M\setminus(K\cap L)) & \xleftarrow[f\circ\beta]{} & (U, U\setminus K).
\end{array}$$

With this notation we need to show that the following equality holds in $\mathrm{H}_{n-k}(U)$:

$$\alpha_*((f\circ\alpha)^*(\phi)\cap\mu_{K\cap L}^{U\cap V}) \;=\; (f\circ\beta)^*(\phi)\cap\mu_K^U.$$

But this is indeed the case, since

$$\begin{array}{ccccc}
\alpha_*(\underbrace{(f\circ\alpha)^*}_{=\alpha^*\circ f^*}(\phi)\cap\mu_{K\cap L}^{U\cap V}) & \overset{\downarrow}{=} & f^*(\phi)\cap\alpha_*(\mu_{K\cap L}^{U\cap V}) & = & f^*(\phi)\cap\mu_{K\cap L}^U \\
 & & & & \uparrow \\
 & & = \beta_*((f\circ\beta)^*(\phi)\cap\mu_K^U) & = & (f\circ\beta)^*(\phi)\cap\mu_K^U \\
 & & \uparrow & & \uparrow
\end{array}$$

by the Cap Product-Naturality Lemma 75.8 applied to the map $\alpha\colon (U\cap V, (U\cap V)\setminus(K\cap L), \varnothing)\to(U, U\setminus(K\cap L), \varnothing)$ of triples of topological spaces

by the functoriality of homology groups and the uniqueness statement of the Homology-Section Theorem 63.6 we have $\alpha_*(\mu_{K\cap L}^{U\cap V}) = \mu_{K\cap L}^U$

same argument backwards with the roles of α and β exchanged

since β is on the first coordinate the identity

□

Subclaim 2. The second rectangle of the diagram on page 1029 commutes.

The proof of that subclaim is very similar to the proof of the previous subclaim. Thus we leave it to the reader to fill in the details. □

Subclaim 3. The third rectangle of the diagram on page 1029 commutes up to the sign $(-1)^{k+1}$.

As a reminder, we need to show that the following diagram commutes up to the sign $(-1)^{k+1}$:

$$\begin{array}{ccccc}
\mathrm{H}^k(M, M\setminus(K\cup L)) & \xrightarrow{\delta_k} & \mathrm{H}^{k+1}(M, M\setminus(K\cap L)) & \xrightarrow{\cong} & \mathrm{H}^{k+1}(U\cap V, (U\cap V)\setminus(K\cap L)) \\
\downarrow\cap\mu_{K\cup L} & & & & \downarrow\cap\mu_{K\cap L} \\
\mathrm{H}_{n-k}(M) & & \xrightarrow{\qquad\qquad\partial_{n-k}\qquad\qquad} & & \mathrm{H}_{n-k-1}(U\cap V).
\end{array}$$

So let $[\varphi] \in \mathrm{H}^k(M, M \setminus (K \cup L)) = \mathrm{H}^k(M, A \cap B)$. We start out with the following preparations:

(1) By the Inverted Mayer–Vietoris Theorem 80.6 (1) we can find $\varphi_A \in \mathrm{C}^k(M, A)$ and $\varphi_B \in \mathrm{C}^k(M, B)$ such that $\varphi = \varphi_A + \varphi_B$.

(2) By the Fundamental Cycle Decomposition Lemma 80.8 there exist singular chains $\alpha_{U\setminus L}$, $\alpha_{U\cap V}$ and $\alpha_{V\setminus K}$ in $U \setminus L$, $U \cap V$ and $V \setminus K$ such that the following three equalities hold:

$$\begin{array}{lrcll} \text{(2a)} & \mu_{K\cup L} & = & [\alpha_{U\setminus L} + \alpha_{U\cap V} + \alpha_{V\setminus K}] & \text{in } \mathrm{H}_n(M, M \setminus (K \cup L)) \\ \text{(2b)} & \mu_{K\cap L}^{U\cap V} & = & [\alpha_{U\cap V}] & \text{in } \mathrm{H}_n(U \cap V, (U \cap V) \setminus (K \cap L)) \\ \text{(2c)} & \mu_K^U & = & [\alpha_{U\setminus L} + \alpha_{U\cap V}] & \text{in } \mathrm{H}_n(U, U \setminus K). \end{array}$$

After these preparations we obtain the following equalities in $\mathrm{H}_{n-k-1}(U \cap V)$:[626]

$$\begin{array}{l} \qquad\qquad\qquad\

We start out with the following claim:

Claim 1. Let W be a manifold.

(A) If Poincaré Duality holds for W equipped with an orientation, then it also holds for any other orientation of W.

(B) If W is homeomorphic to a manifold for which Poincaré Duality holds, then Poincaré Duality also holds for W.

(C) Suppose that W is the union of two open subsets U and V. If Poincaré Duality holds for U, V and $U \cap V$, then it also holds for W.

(D) If W is the union of a nested sequence $U_1 \subset U_2 \subset \dots$ of open subsets such that Poincaré Duality holds for each U_i, then it also holds for W.

Proof. So let W be a manifold.

(A) Clearly it suffices to consider the case that W is path-connected. By the Number-of-Orientations Lemma 61.9 the path-oriented connected topological manifold W admits only one other orientation, namely the reverse orientation. Evidently Poincaré Duality also holds for that orientation.

(B) If we have an orientation-preserving homeomorphism, then this statement is obvious. For a non-orientation-preserving homeomorphism this statement follows from (A).

(C) We consider the following commutative diagram from the Poincaré Duality–Mayer Vietoris Proposition 80.5:

$$\begin{array}{ccccccccc}
\dots \longrightarrow & \mathrm{H}_{\mathrm{c}}^k(U\cap V) & \xrightarrow{(\iota_*,-\iota_*)} & \mathrm{H}_{\mathrm{c}}^k(U)\oplus \mathrm{H}_{\mathrm{c}}^k(V) & \xrightarrow{\iota_*\oplus\iota_*} & \mathrm{H}_{\mathrm{c}}^k(W) & \longrightarrow & \mathrm{H}_{\mathrm{c}}^{k+1}(U\cap V) & \longrightarrow \dots \\
& \downarrow{\scriptstyle \cap(U\cap V)} & & \downarrow{\scriptstyle \cap U\oplus\cap V} & & \downarrow{\scriptstyle \cap W} & & \downarrow{\scriptstyle \cap(U\cap V)} & \\
\dots \to & \mathrm{H}_{n-k}(U\cap V) & \xrightarrow{(\iota_*,-\iota_*)} & \mathrm{H}_{n-k}(U)\oplus \mathrm{H}_{n-k}(V) & \xrightarrow{\iota_*\oplus\iota_*} & \mathrm{H}_{n-k}(W) & \to & \mathrm{H}_{n-k-1}(U\cap V) & \to \dots
\end{array}$$

(Here we again drop the "R" for space reasons.) If Poincaré Duality holds for U, V and $U \cap V$, then in the above commutative diagram the corresponding vertical maps are isomorphisms. Since the horizontal sequences are exact it follows from the Five Lemma 37.5 that the vertical maps corresponding to W are also isomorphisms, i.e. Poincaré Duality holds for $W = U \cup V$.

(D) Let $k \in \mathbb{N}_0$. We suppose that for every $i \in \mathbb{N}$ the map $\cap U_i \colon \mathrm{H}_{\mathrm{c}}^k(U_i; R) \to \mathrm{H}_{n-k}(U_i; R)$ is an isomorphism. We first point out that it follows from the Poincaré Duality–Open Subset Lemma 80.3 that for any $i \leq j$ the following diagram commutes:

$$\begin{array}{ccc}
\mathrm{H}_{\mathrm{c}}^k(U_i; R) & \xrightarrow{\cap U_i} & \mathrm{H}_{n-k}(U_i; R) \\
\downarrow{\scriptstyle \iota_*} & & \downarrow{\scriptstyle \iota_*} \\
\mathrm{H}_{\mathrm{c}}^k(U_j; R) & \xrightarrow{\cap U_j} & \mathrm{H}_{n-k}(U_j; R).
\end{array}$$

We obtain isomorphisms

$$\mathrm{H}_{\mathrm{c}}^k(W; R) \xrightarrow{\cong} \varinjlim \mathrm{H}_{\mathrm{c}}^k(U_i; R) \xrightarrow{\varinjlim(\cap U_i)} \varinjlim \mathrm{H}_{n-k}(U_i; R) \xleftarrow{\cong} \mathrm{H}_{n-k}(W; R).$$

isomorphism by the Exhaustion-$\mathrm{H}_{\mathrm{c}}^*$ Proposition 78.7

isomorphism by our hypothesis and since the above diagram commutes

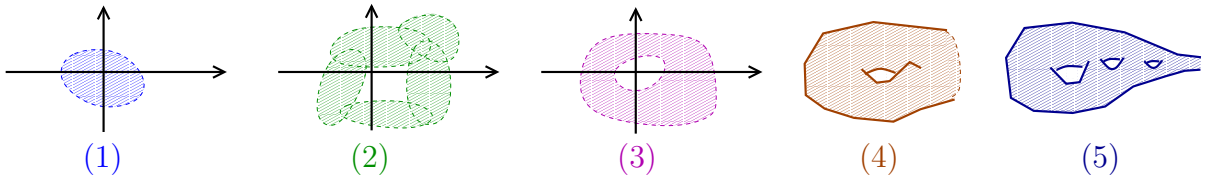
isomorphism by the Exhaustions-H_* Proposition 46.2

It follows easily from the definitions that for each compact subset $K \subset W$ the corresponding map $\mathrm{H}^k(W, W \setminus K; R) \to \mathrm{H}^k_c(W; R) \to \mathrm{H}_{n-k}(W; R)$ is given by capping with μ^W_K. It follows that the above map $\mathrm{H}^k_c(W; R) \to \mathrm{H}_{n-k}(W; R)$ is indeed given by $\cap W$.⊞

The following claim completes the proof of the theorem.

Claim 2. [627]

(1) Poincaré Duality holds for every open bounded convex subset of $\mathbb{R}^n$.
(2) For every $m \in \mathbb{N}$ Poincaré Duality holds for the union of m open bounded convex subsets of $\mathbb{R}^n$.
(3) Poincaré Duality holds for every open subset of $\mathbb{R}^n$.
(4) For every $m \in \mathbb{N}$ Poincaré Duality holds for all manifolds that admit an atlas with m charts.
(5) Poincaré Duality holds for all manifolds.



Proof.

(1) Let M be an open bounded convex subset of $\mathbb{R}^n$. It $M = \varnothing$, then Poincaré Duality holds for trivial reasons. If $M \neq \varnothing$, then it follows from the Convex-to-Ball Proposition 2.12 (1) there exists a homeomorphism $M \to B^n$. It follows from the PD-for-Ball Lemma 80.4 and (B) that Poincaré Duality also holds for M.
(2) We prove the claim by induction on m. By (1) we know that the statement holds for $m = 1$. Now suppose that it holds for all unions of $m-1$ open bounded convex subsets of $\mathbb{R}^n$. Let M be the union of m open bounded convex subsets $U_1, \dots, U_m$. We set $V = U_1 \cup \dots \cup U_{m-1}$. By our hypothesis and by induction the statement holds for U_m, for V and also for $V \cap U_m = (U_1 \cap U_m) \cup \dots \cup (U_{m-1} \cap U_m)$, which is the union of $(m-1)$ open bounded convex subsets.[628] Hence by (C) the statement also holds for

$$M \;=\; V \cup U_m \;=\; (U_1 \cup \dots \cup U_{m-1}) \cup U_m.$$

(3) Let M be an arbitrary open subset of $\mathbb{R}^n$. We can write M as the union of countably many open balls $B_i, i \in \mathbb{N}$. For $i \in \mathbb{N}$ we set $U_i := \bigcup_{j \leq i} B_j$. By (2) Poincaré Duality holds for each $U_i \in \mathbb{N}$. But then it follows from (D) that Poincaré Duality also holds for M.
(4) We prove this statement by induction on m. The case $m = 1$ is taken care of by (3) and (B). Now suppose Poincaré Duality holds for all manifolds that admit an atlas with $m-1$ charts. Let M be a manifold that admits an atlas $\{\phi_i \colon U_i \to V_i\}_{i=1,\dots,m}$ with m charts. We write $U = U_1 \cup \dots \cup U_{m-1}$ and $V = U_m$. Both are open subsets of a manifold, hence they are also manifolds. Furthermore U and V are manifolds that admit an atlas with $m-1$ charts respectively one chart. Similarly $U \cap V = (U_1 \cap U_m) \cup \dots \cup (U_{m-1} \cap U_m)$

[627] The five steps in the claim correspond precisely to the five steps in the proof of the Homology-Section Theorem 63.6.
[628] Here we once again use the convenient fact that the intersection of two convex sets is convex.

is a manifold that admits an atlas with $m-1$ charts. By induction Poincaré Duality holds for U, V and $U \cap V$. Thus by (C) Poincaré Duality also holds for M itself.

(5) Finally let M be any manifold. Using the fact that M is by definition second-countable one can easily show that M admits a countable atlas $\{\Phi_j \colon U_j \to V_j\}_{j\in\mathbb{N}}$. For $i \in \mathbb{N}$ we set $X_i := \bigcup_{j<i} U_j$. By (4) Poincaré Duality holds for each X_i. It follows from (D) that Poincaré Duality also holds for M. ■

80.3. The proof of Poincaré Duality for topological manifolds with boundary.

Now we will finally prove the general statement of the Poincaré Duality Theorem 79.1. For the reader's convenience we recall the statement.

Theorem 79.1. (Poincaré Duality Theorem) Let R be a commutative ring, let M be a compact R-oriented n-dimensional topological manifold and suppose that we are given a decomposition $\partial M = A \cup B$ where A and B are compact $(n-1)$-dimensional submanifolds of ∂M such that $A \cap B = \partial A = \partial B$. For each $k \in \mathbb{N}_0$ the map

$$\begin{array}{ccc} \mathrm{H}^k(M, A; R) & \xrightarrow{\cong} & \mathrm{H}_{n-k}(M, B; R) \\ \varphi & \mapsto & \varphi \cap [M] \end{array}$$

is an isomorphism.

In the following we first consider an important special case of the Poincaré Duality Theorem 79.1, namely the case $A = \partial M$ and $B = \varnothing$. Immediately afterwards we will use this special case to deal with the general case.

Proof of Theorem 79.1 for $A = \partial M$ and $B = \varnothing$. In this proof we consider the case that $A = \partial M$ and $B = \varnothing$. We need to show that for each $k \in \mathbb{N}_0$ the map

$$\begin{array}{ccc} \mathrm{H}^k(M, \partial M; R) & \xrightarrow{\cong} & \mathrm{H}_{n-k}(M; R) \\ \varphi & \mapsto & \varphi \cap [M] \end{array}$$

is an isomorphism. We introduce the following objects:

(i) Recall that from the Collar Neighborhood Theorem 23.2 we obtain an embedding $\chi \colon [-1,0] \times \partial M \to M$ such that $\chi(0, P) = P$ for all $P \in \partial M$. We identify $[-1,0] \times \partial M$ with its image under χ. We refer to $[-1,0] \times \partial M \subset M$ as the "internal collar" of ∂M.

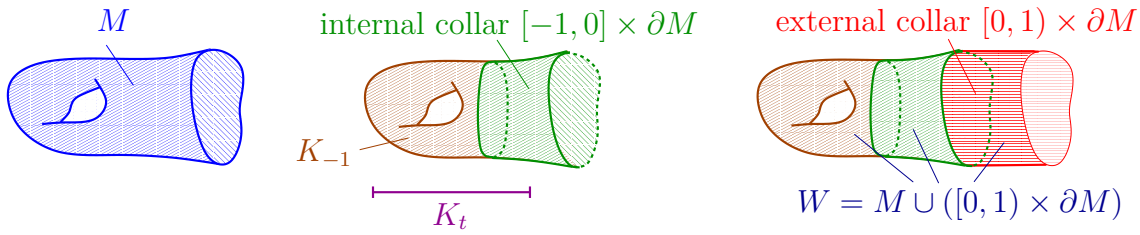
(ii) We set $$W := M \cup ([0,1) \times \partial M)/\sim \qquad \text{with } x \sim (x,0) \text{ for every } x \in \partial M.$$ Sometimes $[0,1) \times \partial M$ is called an *external collar of* M. Furthermore, given any $t \in [-1,1)$ we write $$K_t := (M \setminus ([-1,0] \times \partial M)) \cup ([-1,t] \times \partial M)/\sim .$$

(iii) We denote by $i \colon M \to W$ the obvious map. By the Topological Pushout-Maps Lemma 6.14 (3) this map is a closed embedding. We will use this to view M as a closed subset of W.

By elementary arguments one can easily show the following statements:

(A) W is an n-dimensional topological manifold with empty boundary.

(B) W admits an orientation such that the inclusion $M \to W$ is orientation-preserving.



We denote by $i\colon (M,\partial M) \to (W, W\setminus K_{-1})$ the obvious inclusion map. We obtain the following diagram:

$$\begin{array}{ccccc} \mathrm{H}^k(M,\partial M;R) & \xleftarrow{i^*} & \mathrm{H}^k(W, W\setminus K_{-1};R) & \longrightarrow & \mathrm{H}^k_{\mathrm{c}}(W;R) \\ \downarrow{\scriptstyle \cap[M]} & & \downarrow{\scriptstyle \cap\mu_{K_t}} & & \downarrow{\scriptstyle \cap W} \\ \mathrm{H}_{n-k}(M;R) & \xrightarrow{i_*} & \mathrm{H}_{n-k}(W;R) & \xrightarrow{\mathrm{id}} & \mathrm{H}_{n-k}(W;R). \end{array}$$

We want to show that the vertical map to the left is an isomorphism. It remains to prove the following claim which resembles closely the claim in the proof of the PD-for-Ball Lemma 80.4 on page 1025.

Claim. The following three statements hold:
(1) The diagram commutes.
(2) The right vertical map is an isomorphism.
(3) All horizontal maps are isomorphisms.

Proof.

(1) The right-hand square commutes by definition of $\cap W$. Using (B) we can apply the same argument as in the proof of the claim in the PD-for-Ball Lemma 80.4 to show that the left-hand square commutes.
(2) The Poincaré Duality–Non-Compact Theorem 80.2 implies that the right vertical map is an isomorphism.
(3) First note that $M = i(M)$ is a deformation retract of W. Next note that it follows easily from this discussion together with the Homotopy Equivalence-Relative-H_*-Corollary 38.4 and the Homotopy Equivalence-H^*-Corollary 67.8 (1) that the left horizontal maps are isomorphisms. Note that the bottom right horizontal map is the identity. Thus it remains to show that the top right horizontal map is an isomorphism. Similar to the proof of the PD-for-Ball Lemma 80.4 we consider the following commutative diagram:

$$\begin{array}{ccc} \mathrm{H}^k(W, W\setminus K_{-1};R) & \longrightarrow & \operatorname*{colim}_{K\in\mathcal{K}(W)} \mathrm{H}^k(W, W\setminus K;R) =_{\mathrm{i}} \mathrm{H}^k_{\mathrm{c}}(W;R). \\ & \searrow \quad \nearrow & \\ & \operatorname*{colim}_{t\in[-1,1)} \mathrm{H}^k(W, W\setminus K_t;R) & \end{array}$$

We make the following two observations:

(a) One can easily verify that every compact subset of W is contained in some K_t. Therefore it follows from the Cofinal-Isomorphism Lemma 78.4 that the right diagonal map is an isomorphism.

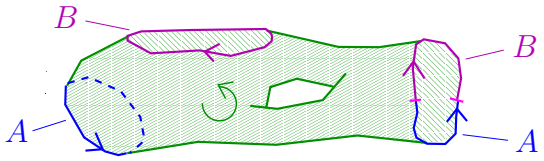
(b) Note that it follows easily from the Homotopy Equivalence-H^*-Corollary 67.8 (1) that for each $t \in [-1,1)$ the inclusion-induced map $\mathrm{H}^k(W, W\setminus K_{-1}; R) \to \mathrm{H}^k(W, W\setminus K_t; R)$ is an isomorphism. It follows from this observation and the Direct Colimit–Stabilization Lemma 25.1 that the left diagonal map is an isomorphism.

These two observations together imply that the horizontal map is also an isomorphism.

This concludes the proof of the claim and it concludes the proof of the Poincaré Duality Theorem 79.1 in the special case that $A = \partial M$ and $B = \varnothing$. ■

Now we finally consider the general case of the Poincaré Duality Theorem Theorem 79.1:

Proof of the Poincaré Duality Theorem Theorem 79.1 for the general case. Suppose that we are given a decomposition $\partial M = A \cup B$ where A and B are compact $(n-1)$-dimensional submanifolds of ∂M such that $A \cap B = \partial A = \partial B$. We recall that by the Excisive Triad-Proposition 50.11 (2) we know that the triad (M, A, B) is excisive.



we need to show that the map

$$\begin{array}{ccc} \mathrm{H}^k(M, A; R) & \xrightarrow{\cong} & \mathrm{H}_{n-k}(M, B; R) \\ \varphi & \mapsto & \varphi \cap [M] \end{array}$$

is an isomorphism

We equip ∂M and then also B with the orientations coming from the Boundary Orientation Proposition 62.3 and the Topological Manifold-Induced Orientation Lemma 62.1. We consider the following diagram:

$$\begin{array}{ccccccccc}
\ldots \longrightarrow & \mathrm{H}^k(M, \partial M; R) & \longrightarrow & \mathrm{H}^k(M, A; R) & \longrightarrow & \mathrm{H}^k(\partial M, A; R) & \xrightarrow{\delta_k} & \mathrm{H}^{k+1}(M, \partial M; R) & \longrightarrow \ldots \\
& \cong \downarrow -\cap[M] & & \downarrow -\cap[M] & & \cong \downarrow j^* & & \cong \downarrow -\cap[M] & \\
& & & & & \mathrm{H}^k(B, \partial B; R) & & & \\
& & & & & \cong \downarrow -\cap[B] & & & \\
\ldots \longrightarrow & \mathrm{H}_{n-k}(M; R) & \longrightarrow & \mathrm{H}_{n-k}(M, B; R) & \xrightarrow{\partial_{n-k}} & \mathrm{H}_{n-k-1}(B; R) & \longrightarrow & \mathrm{H}_{n-k-1}(M; R) & \longrightarrow \ldots
\end{array}$$

We make the following clarifications and observations:

(1) The top horizontal sequence is the long exact sequence in cohomology of the triple $(M, \partial M, A)$ provided by the LES-of-Triple Proposition 67.9.

(2) The bottom horizontal sequence is the long exact sequence in homology of the pair (M, B) provided by the LES-of-Pair Proposition 38.3.

(3) We denote by $j\colon (B, \partial B) \to (\partial M, A)$ the inclusion. We obtain the vertical map $j^*\colon \mathrm{H}^k(\partial M, A; R) \to \mathrm{H}^k(B, \partial B; R)$. Note that it follows from the Excisive Triad-Proposition 50.11 (2) together with the Excisive Characterization Lemma 50.10 (1)⇒(3) and the Also-Iso-on-Cohomology Proposition 69.2 that this map j^* is an isomorphism.

(4) The other maps decorated with "$\cong$" are isomorphisms by the special case of the Poincaré Duality Theorem 79.1 that we have already dealt with on page 1035.

By the Five Lemma 37.5 it remains to prove the following claim:

Claim. The above diagram commutes up to signs, more precisely:
(1) The first rectangle commutes.
(2) The second rectangle commutes up to the sign $(-1)^k$.
(3) The third rectangle commutes.

Proof. We consider the three rectangles separately.

(1) We consider the following diagram:

$$\begin{array}{ccc}
\mathrm{H}^k(M,\partial M;R)\times\mathrm{H}_n(M,\partial M;R) & \xrightarrow{\cap} & \mathrm{H}_{n-k}(M;R) \\
{\scriptstyle\mathrm{id}}\uparrow \quad \downarrow{\scriptstyle\mathrm{id}} & & \downarrow \\
\mathrm{H}^k(M,\partial M;R)\times\mathrm{H}_n(M,\partial M;R) & \xrightarrow{\cap} & \mathrm{H}_{n-k}(M,B;R) \\
\downarrow \quad \uparrow{\scriptstyle\mathrm{id}} & & {\scriptstyle\mathrm{id}}\uparrow \\
\mathrm{H}^k(M,A;R)\times\mathrm{H}_n(M,\partial M;R) & \xrightarrow{\cap} & \mathrm{H}_{n-k}(M,B;R).
\end{array}$$

Note that it follows from the Cap Product-Naturality Lemma 75.8, applied to the map $(M,\partial M,\varnothing)\to(M,\partial M,B)$ and to the map $(M,A,B)\to(M,\partial M,B)$ of excisive triads, that the diagram commutes. It follows easily from this observation that the first rectangle of the claim commutes.

Before we continue with the discussion of the other two rectangles we need to make one remark. We denote by $\partial_n\colon \mathrm{H}_n(M,\partial M;R)\to\mathrm{H}_{n-1}(\partial M,A;R)$ the connecting homomorphism in the long exact sequence of the triple $(M,\partial M,A)$. It follows from the Fundamental Class-of-∂-Proposition 64.7, the Codimension Zero-Fundamental Class Lemma 64.5 and the naturality of the connecting homomorphism that the following equality holds

$$(*)\qquad j_*([B]) \;=\; \partial_n([M]) \;\in\; \mathrm{H}_{n-1}(\partial M,A;R).$$

Now we continue with the proof of the claim.

(2) We apply the Cap Product-Triad-Connecting Homomorphism Lemma 75.13 to the excisive triad (M,A,B). (Recall that $A\cup B=\partial M$ and $A\cap B=\partial B$.) We obtain that for any $k\in\mathbb{N}_0$ the following diagram commutes up to the sign $(-1)^k$:

$$\begin{array}{ccc}
\mathrm{H}^k(M,A;R)\times\mathrm{H}_n(M,\partial M;R) & \xrightarrow{\cap} & \mathrm{H}_{n-k}(M,B;R) \\
{\scriptstyle i^*}\downarrow \quad \downarrow{\scriptstyle\partial_n} & & \\
\mathrm{H}^k(\partial M,A;R)\times\mathrm{H}_{n-1}(\partial M,A;R) & & \Big\downarrow{\scriptstyle\partial_{n-k}} \\
{\scriptstyle j^*}\downarrow{\scriptstyle\cong} \quad {\scriptstyle(j_*)^{-1}}\downarrow{\scriptstyle\cong} & & \\
\mathrm{H}^k(B,\partial B;R)\times\mathrm{H}_{n-1}(B,\partial B;R) & \xrightarrow{\cap} & \mathrm{H}_{n-k-1}(B;R).
\end{array}$$

Together with $(*)$ we obtain that the second rectangle of the claim does indeed commute up to the sign $(-1)^k$.

(3) We consider the following diagram:

$$\begin{array}{ccccc}
\mathrm{H}^k(B,\partial B;R) \times \mathrm{H}_{n-1}(B,\partial B;R) & \xrightarrow{\cap} & \mathrm{H}_{n-k-1}(B;R) \\
{\scriptstyle j^*}\uparrow \quad\quad \downarrow{\scriptstyle j_*} & & \downarrow{\scriptstyle j_*} \\
\mathrm{H}^k(\partial M,A;R) \times \mathrm{H}_{n-1}(\partial M,A;R) & \xrightarrow{\cap} & \mathrm{H}_{n-k-1}(\partial M;R) \\
{\scriptstyle \delta_k}\downarrow \quad\quad \uparrow{\scriptstyle \partial_n} & & \downarrow{\scriptstyle i_*} \\
\mathrm{H}^{k+1}(M,\partial M;R) \times \mathrm{H}_n(M,\partial M;R) & \xrightarrow{\cap} & \mathrm{H}_{n-k-1}(M;R).
\end{array}$$

The upper part of this diagram commutes by the Cap Product-Naturality Lemma 75.8 applied to the map $(B,\partial B,\varnothing) \to (\partial M,A,\varnothing)$ between excisive triads. The lower part commutes by the Cap Product-Triple-Connecting Homomorphism Lemma 75.12 applied to the triple $(M,\partial M,A)$. This discussion, together with $(*)$, implies that the third rectangle of the diagram in the claim commutes.

This concludes the proof of the claim and thus also of the Poincaré Duality Theorem 79.1. ■

Part X

Applications of Poincaré Duality

In this part of the lecture notes we explore the full power of the Poincaré Duality Theorem 79.1. This part consists of the following chapters:

- In Chapter 81 we use the Poincaré Duality Theorem 79.1 to calculate the cup products on $\mathrm{H}^*(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$ and $\mathrm{H}^*(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2)$.
- In Chapter 82 we introduce division algebras. Obvious examples of division algebras are given by $\mathbb{R}$ and $\mathbb{C} \cong \mathbb{R}^2$. We also introduce two more interesting division algebras, namely the quaternions $\mathbb{H} \cong \mathbb{R}^4$ and the octonions $\mathbb{O} \cong \mathbb{R}^8$. The main result of the chapter is the Hopf–Division Algebra Dimension Theorem 82.7 which says that if $\mathbb{R}^n$ admits the structure of a division algebra, then n is a power of 2.
- In Chapter 83 we introduce for each $n \in \mathbb{N}$ the quaternionic projective space $\mathbb{H}\mathrm{P}^n$ and we introduce the octonionic projective spaces $\mathbb{O}\mathrm{P}^1$ and $\mathbb{O}\mathrm{P}^2$. We also compute the cohomology rings of these projective spaces.
- In Chapter 84 we use the cup product to introduce the Hopf invariant which is a homomorphism $\pi_{4n-1}(S^n) \to \mathbb{Z}$. Using the calculations of the cup products of $\mathbb{C}\mathrm{P}^2$, $\mathbb{H}\mathrm{P}^2$ and $\mathbb{O}\mathrm{P}^2$ we see that this homomorphism is an epimorphism for $n = 2, 4, 8$. Furthermore in the Bidegree–Realization Proposition 84.15 we will show that the homomorphism is non-trivial for all even n.
- In Chapter 85 we will state and prove the Basic Sphere–Alexander Duality Theorem 85.5 which says, in a simplified version, that for any compact submanifold $K \subset S^n$ and any $j \in \mathbb{N}_0$ we have an isomorphism
$$\widetilde{\mathrm{H}}_j(S^n \setminus K) \;\cong\; \mathrm{H}^{n-j-1}(K;\mathbb{Z}).$$
- In Chapter 86 we will state and prove the General Alexander Duality Theorem 86.1 which says, again in a simplified version, that for any compact subspace $K \subset S^n$ which is "weakly locally contractible" and any $j \in \mathbb{N}_0$ we have a natural isomorphism
$$\widetilde{\mathrm{H}}_j(S^n \setminus K) \;\xrightarrow{\cong}\; \mathrm{H}^{n-j-1}(K;\mathbb{Z}).$$
- In Chapter 87 we will state and prove two fun "real life"-applications of cohomology. First of all, in the Table Theorem **??** we will show that, under suitable hypothesis, we can always arrange a table to be horizontal. Secondly, we will use the General Alexander Duality Theorem 86.1 to prove the Inscribed Rectangle Proposition 87.2 which says that every Jordan curve admits an inscribed rectangle.
- In Chapter 88, given a closed smooth manifold M, we will (re-) introduce the algebraic intersection number of two transverse submanifolds of M of opposite dimension. Furthermore, in the Cup Product–Intersection Number Theorem 88.4 we will relate the algebraic intersection number to the cup product. We will see that if two smooth manifolds have non-zero algebraic intersection number, then they cannot be made disjoint.
- In Chapter **??** we will prove the Whitney Trick-Theorem **??** which shows that if two submanifolds have zero algebraic intersection number then, under suitable hypotheses, they can actually be made disjoint.
- In Chapter **??** we generalize some results and concepts from Chapter 88. More precisely, we will introduce the notion of transverse cycles, we will introduce the algebraic intersection number of transverse cycles and in the Maps-Cup Product–Intersection Number Theorem **??** we will relate this algebraic intersection number to the cup product.

- In Chapter **??** we will use the Cup Product–Intersection Number Theorem 88.4 to prove the Lefschetz-Hopf Theorem **??** and we will use the more powerful Maps-Cup Product Intersection Number **??** to prove the more powerful Lefschetz-Hopf Theorem **??**. Both Lefschetz-Hopf Theorems relate, given a suitable self-map $f\colon X \to X$ of a topological space, the Lefschetz number of f to the sum of the indices of the fixed points of X.
- In Chapter **??** we will use the Lefschetz-Hopf Theorem **??** to prove the Poincaré-Hopf Theorem **??** which relates, given a smooth manifold with a vector field v, the Euler characteristic of M to the sum of the indices of the zeros of the vector field.

CHAPTER 81

The cup product of projective spaces

In this chapter we will use the Poincaré Duality Theorem 79.1 to determine the cup product on $\mathrm{H}^*(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$ and on $\mathrm{H}^*(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2)$. These calculations will allow us to prove several interesting consequences which at first glance have very little to do with cup products.

81.1. **Non-singular pairings and the cup product.** Before we start drawing more conclusions from the Poincaré Duality Theorem 79.1 we need to introduce some purely algebraic definitions:

Definition. Let R be a commutative ring and let V, W and S be R-modules.

(1) A pairing on $V \times W$ is a map
$$\varphi\colon V \times W \to S$$
such that for all $v, v' \in V$ and $w, w' \in W$ we have $\varphi(v+v', w) = \varphi(v,w) + \varphi(v',w)$ and $\varphi(v, w+w') = \varphi(v,w) + \varphi(v,w')$.

(2) A pairing is called R-bilinear if it is R-linear in V and if it is R-linear in W.[629]

(3) We say that an R-bilinear pairing $\langle\ ,\ \rangle\colon V \times W \to S$ is non-singular if the induced maps
$$\begin{array}{ccc} V & \to & \mathrm{Hom}_R(W,S) \\ v & \mapsto & \begin{pmatrix} W & \to & S \\ w & \mapsto & \langle v,w\rangle \end{pmatrix} \end{array} \quad \text{and} \quad \begin{array}{ccc} W & \to & \mathrm{Hom}_R(V,S) \\ w & \mapsto & \begin{pmatrix} V & \to & S \\ v & \mapsto & \langle v,w\rangle \end{pmatrix} \end{array}$$
are isomorphisms. Otherwise we say that the R-bilinear pairing is singular.

We start out with the following elementary lemma:

Lemma 81.1. (Non-Singularity Lemma) Let R be a commutative ring. Let V and W be R-modules and let
$$\langle\ ,\ \rangle\colon V \times W \to R$$
be an R-bilinear pairing.

(1) We suppose that V and W are finitely generated free R-modules. Let $v_1, \dots, v_m$ be a basis for V and let $w_1, \dots, w_n$ be a basis for W. Then the following statement holds:[630]
$$\text{The pairing } \langle\ ,\ \rangle \text{ is non-singular} \iff \begin{array}{ll} \text{(i)} & m = n \text{ and} \\ \text{(ii)} & \det(\langle v_i, w_j\rangle)_{i,j=1,\dots,n}) \text{ is a unit of } R. \end{array}$$

(2) If V and W are free R-modules of finite rank, then the pairing is non-singular if and only if *one* of the two maps $V \to \mathrm{Hom}(W,R)$ or $W \to \mathrm{Hom}(V,R)$ is an isomorphism.

[629] Note that a pairing is by definition always $\mathbb{Z}$-bilinear.

[630] Here is a useful special case. Suppose that V and W are of rank one. If v and w are generators of V respectively W, then the following statement holds:
$$\text{the pairing } \langle\ ,\ \rangle \text{ is non-singular} \iff \langle v,w\rangle \text{ is a unit of the ring } R.$$

Proof.

(1) Let $v_1, \dots, v_m$ be a basis for V and let $w_1, \dots, w_n$ be a basis for W. We denote by $v_1^*, \dots, v_m^* \in \operatorname{Hom}(V, R)$ and $w_1^*, \dots, w_n^* \in \operatorname{Hom}(W, R)$ the dual bases. We consider the maps

$$\begin{array}{rcl} V & \to & \operatorname{Hom}_R(W, R) \\ v & \mapsto & \begin{pmatrix} W & \to & R \\ w & \mapsto & \langle v, w \rangle \end{pmatrix} \end{array} \quad \text{and} \quad \begin{array}{rcl} W & \to & \operatorname{Hom}_R(V, R) \\ w & \mapsto & \begin{pmatrix} V & \to & R \\ v & \mapsto & \langle v, w \rangle \end{pmatrix}. \end{array}$$

Since $\langle\, , \rangle$ is R-bilinear, both maps are R-homomorphisms. Furthermore, it is straightforward to show that with respect to these bases the R-homomorphisms are represented by the (transpose of the) matrix $(\langle v_i, w_j \rangle_{i=1,\dots,m, j=1,\dots,n})$. The promised statement is a straightforward consequence of these observations and the fact that for a commutative ring a matrix is invertible only if its determinant is a unit.[631]

(2) This statement follows immediately from Statement (1) and the fact that a square matrix and its transpose have the same determinant. ∎

Now we return to topology:

Proposition 81.2. (Manifold-Non Singularity Proposition) Let M be a closed n-dimensional topological manifold and let $k \in \mathbb{N}_0$.

(1) If M is oriented and if $\mathrm{H}^k(M; \mathbb{Z})$ and $\mathrm{H}^{n-k}(M; \mathbb{Z})$ are torsion-free, then the pairing

$$\begin{array}{rcl} \mathrm{H}^k(M; \mathbb{Z}) \times \mathrm{H}^{n-k}(M; \mathbb{Z}) & \to & \mathbb{Z} \\ (\varphi, \psi) & \mapsto & \langle \varphi \cup \psi, [M] \rangle \end{array}$$

is non-singular.

(2) Let $\mathbb{F}$ be a field. If M is $\mathbb{F}$-oriented, then the pairing

$$\begin{array}{rcl} \mathrm{H}^k(M; \mathbb{F}) \times \mathrm{H}^{n-k}(M; \mathbb{F}) & \to & \mathbb{F} \\ (\varphi, \psi) & \mapsto & \langle \varphi \cup \psi, [M]_{\mathbb{F}} \rangle \end{array}$$

is non-singular.

Proof. Let M be a closed n-dimensional topological manifold and let $k \in \mathbb{N}_0$.

(1) We suppose that M is oriented and that $\mathrm{H}^k(M; \mathbb{Z})$ and $\mathrm{H}^{n-k}(M; \mathbb{Z})$ are torsion-free. By the Non-Singularity Lemma 81.1 (2) it suffices to show that the map

$$\begin{array}{rcl} \Omega \colon \mathrm{H}^{n-k}(M; \mathbb{Z}) & \to & \operatorname{Hom}(\mathrm{H}^k(M; \mathbb{Z}), \mathbb{Z}) \\ \psi & \mapsto & (\varphi \mapsto \langle \varphi \cup \psi, [M] \rangle) \end{array}$$

is an isomorphism. We start out with the following slightly confusing claim:

Claim. The following diagram commutes:[632]

$$\begin{array}{ccc} \mathrm{H}^{n-k}(M; \mathbb{Z}) & \xrightarrow{\Omega} & \operatorname{Hom}(\mathrm{H}^k(M; \mathbb{Z}), \mathbb{Z}). \\ & {\scriptstyle \mathrm{ev}} \searrow \quad \nearrow {\scriptstyle (\cap [M])^*} & \\ & \operatorname{Hom}_{\mathbb{Z}}(\mathrm{H}_{n-k}(M), \mathbb{Z}) & \end{array}$$

[631] If R is a domain, then this fact can be seen by going from R to its quotient field. For an arbitrary non-zero commutative ring R this fact can be proved by picking a maximal ideal $\mathfrak{m}$ and considering the field $R/\mathfrak{m}$. We refer to [**Hun80**, Corollary IV.2.12] or alternatively to [**Eis95**, Corollary 4.4] for details.

Proof. Note that for any $\psi \in \mathrm{H}^{n-k}(M;\mathbb{Z})$ and any $\varphi \in \mathrm{H}^k(M;\mathbb{Z})$ the following equalities hold in $\mathbb{Z}$:[633]

$$\Omega(\psi)(\varphi) \underset{\uparrow}{=} \epsilon_M((\varphi\cup\psi)\cap[M]) \overset{\downarrow}{=} \epsilon_M(\psi\cap(\varphi\cap[M])) \underset{\uparrow}{=} \langle\psi,\varphi\cap[M]\rangle = \mathrm{ev}(\psi)(\varphi\cap[M]).$$

by definition of Ω and by the Cap Product-Kronecker Lemma 75.7 — by the Cup-Cap Lemma 75.9 — by the Cap Product-Kronecker Lemma 75.7

But this observation immediately shows that the diagram commutes. ⊞

We need to show that in the above diagram the horizontal map is an isomorphism. By the claim it suffices to show that the two diagonal maps are isomorphisms:

- By the Universal Coefficient-H^*-Theorem 70.14 we have a short exact sequence

$$0 \to \underbrace{\mathrm{Ext}(\mathrm{H}_{n-k-1}(M),\mathbb{Z})}_{\substack{\text{torsion by the Ext-Properties Lemma 70.12}\\ \text{and the } \mathbb{Z}_k\text{-Ext-Lemma 70.11}}} \to \underbrace{\mathrm{H}^{n-k}(M;\mathbb{Z})}_{\substack{\text{torsion-free by}\\ \text{hypothesis}}} \xrightarrow{\mathrm{ev}} \mathrm{Hom}_{\mathbb{Z}}(\mathrm{H}_{n-k}(M),\mathbb{Z}) \to 0.$$

  It follows from the above data that the left homomorphism is trivial, thus the right homomorphism is, as desired, an isomorphism.
- The diagonal map to the right is an isomorphism by the Poincaré Duality Theorem 79.1.

(2) Now let $\mathbb{F}$ be a field. We suppose that M is $\mathbb{F}$-oriented. It follows from the Ring–Kronecker Pairing Lemma 71.1 and the Universal Coefficient-H^*–Field Theorem 71.2 that the map

$$\begin{array}{rcl} \mathrm{ev}\colon \mathrm{H}^k(M;\mathbb{F}) & \to & \mathrm{Hom}_{\mathbb{F}}(\mathrm{H}_k(M;\mathbb{F}),\mathbb{F}) \\ \varphi & \mapsto & \left(\begin{array}{rcl} \mathrm{H}_k(M;\mathbb{F}) & \to & \mathbb{F} \\ \sigma & \mapsto & \langle\varphi,\sigma\rangle \end{array}\right) \end{array}$$

is well-defined and that it is a natural isomorphism of $\mathbb{F}$-vector spaces. The proof of (2) is now basically the same as the proof of (1). ■

Example. Let $n \in \mathbb{N}$. We want to determine the pairing

$$\begin{array}{rcl} \mathrm{H}^n(S^n\times S^n;\mathbb{Z}) \times \mathrm{H}^n(S^n\times S^n;\mathbb{Z}) & \to & \mathbb{Z} \\ (\varphi,\psi) & \mapsto & \langle\varphi\cup\psi,[S^n\times S^n]\rangle. \end{array}$$

We start out with the following preparations:

(1) We denote by $p\colon S^n\times S^n \to S^n$ and by $q\colon S^n\times S^n\to S^n$ the two natural projections.

(2) By the Product-of-Spheres–Cup Product Lemma 76.3 we know that two pullbacks $p^*([S^n]^*), q^*([S^n]^*)$ form is a basis of $\mathrm{H}^n(S^n\times S^n;\mathbb{Z})$.

(3) We have

$$p^*([S^n]^*)\cup p^*([S^n]^*) \underset{\uparrow}{=} p^*([S^n]^*\cup[S^n]^*) \underset{\uparrow}{=} 0.$$

by the Cup Product-Naturality Lemma 72.2 — since $[S^n]^*\cup[S^n]^*\in\mathrm{H}^{2n}(S^n;\mathbb{Z})=0$

The same argument shows of course that $q^*([S^n]^*)\cup q^*([S^n]^*)=0$.

(4) It follows from the Cup Product-Commutativity Proposition 72.8 that the above pairing is symmetric if n is even and it is antisymmetric if n is odd.

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[632] The right diagonal map is the dual map corresponding to the map $\mathrm{H}^k(M;\mathbb{Z}) \xrightarrow{\cap[M]} \mathrm{H}_{n-k}(M)$.

[633] Here $\epsilon_M\colon \mathrm{H}_0(M)\to\mathbb{Z}$ is the augmentation homomorphism from page 464.

It follows from the above discussion together with the Non-Singularity Lemma 81.1 that for some $\epsilon \in \mathbb{Z}$ we obtain the following table:

$\langle \varphi \cup \psi, [S^n \times S^n] \rangle$	$p^*([S^n]^*)$	$q^*([S^n]^*)$
$p^*([S^n]^*)$	0	ϵ
$q^*([S^n]^*)$	$(-1)^n \cdot \epsilon$	0.

Finally note that by the Manifold-Non Singularity Proposition 81.2 we know that the pairing is non-singular. By the Non-Singularity Lemma 81.1 this implies that the above matrix is invertible, i.e. we see that $\epsilon \in \{-1, 1\}$.

This result recovers (up to the sign indeterminacy) the calculation of the non-trivial cup product on $S^n \times S^n$ that we had initially performed in the Product-of-Spheres–Cup Product Lemma 76.3 using the somewhat mysterious Product Theorem 76.1. □

81.2. **The cup product of $\mathbb{C}\mathrm{P}^n$ and applications.** In this section we apply our acquired wisdom to the complex projective spaces $\mathbb{C}\mathrm{P}^n$.

Definition.

(1) Given $m \leq n$ we consider the embedding[634]

$$\begin{array}{rcl} i\colon \mathbb{C}\mathrm{P}^m & \to & \mathbb{C}\mathrm{P}^n \\ [z_0 : \cdots : z_m] & \mapsto & [z_0 : \cdots : z_m : 0 : \cdots : 0]. \end{array}$$

(2) We set $\mathbb{C}\mathrm{P}^\infty := \operatorname{colim}_{\longrightarrow} \mathbb{C}\mathrm{P}^n$.

We prepare the ground for calculating the cup product of projective spaces with the following elementary lemma:

Lemma 81.3. ($\mathbb{C}\mathrm{P}^n$-H^*-Lemma)

(1) For every $n \in \mathbb{N} \cup \{\infty\}$ and every $k \in \mathbb{N}_0$ we have

$$\mathrm{H}^k(\mathbb{C}\mathrm{P}^n; \mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & \text{if } k \in \{0, 2, 4, \ldots, 2n\}, \\ 0, & \text{otherwise.} \end{cases}$$

(2) Let $m \leq n \leq \infty$. We denote by $i\colon \mathbb{C}\mathrm{P}^m \to \mathbb{C}\mathrm{P}^n$ the inclusion map. For every $k \in \mathbb{N}_0$ with $k \leq 2m$ the induced map $i^*\colon \mathrm{H}^k(\mathbb{C}\mathrm{P}^n; \mathbb{Z}) \to \mathrm{H}^k(\mathbb{C}\mathrm{P}^m; \mathbb{Z})$ is an isomorphism.

$$\begin{array}{ccc} 0 & & 0 \\ 0 & & \mathrm{H}^6(\mathbb{C}\mathrm{P}^3; \mathbb{Z}) \cong \mathbb{Z} \\ \mathrm{H}^4(\mathbb{C}\mathrm{P}^2; \mathbb{Z}) \cong \mathbb{Z} & \xleftarrow[\cong]{i^*} & \mathrm{H}^4(\mathbb{C}\mathrm{P}^3; \mathbb{Z}) \cong \mathbb{Z} \\ \mathrm{H}^2(\mathbb{C}\mathrm{P}^2; \mathbb{Z}) \cong \mathbb{Z} & \xleftarrow[\cong]{i^*} & \mathrm{H}^2(\mathbb{C}\mathrm{P}^3; \mathbb{Z}) \cong \mathbb{Z} \end{array}$$

Proof. We first consider the analogous statements for homology groups:

(1) Let $n \in \mathbb{N}_0 \cup \{\infty\}$. In the $\mathbb{C}\mathrm{P}^n$-H_*-Proposition 47.4 we showed that for any $k \in \mathbb{N}_0$ we have

$$\mathrm{H}_k(\mathbb{C}\mathrm{P}^n) \cong \begin{cases} \mathbb{Z}, & \text{if } k = 0, 2, 4, \ldots, 2n, \\ 0, & \text{otherwise.} \end{cases}$$

(2) Recall that on pages 410 we showed that $\mathbb{C}\mathrm{P}^n$ admits a CW-structure with one cell in all even dimensions up to $2n$ and with no cells in any other dimension. Furthermore we saw that the CW-structure can be chosen in such a way that for $k \in \{0, \ldots, n\}$ the

[634] On page 137 we showed that the given map $i\colon \mathbb{C}\mathrm{P}^m \to \mathbb{C}\mathrm{P}^n$ is indeed an embedding.

$(2k+1)$-skeleton on $\mathbb{C}\mathrm{P}^n$ equals $i(\mathbb{C}\mathrm{P}^k)$. It follows from the Skeleton-H_*-Lemma 47.1 (3) that for any $k \in \mathbb{N}_0$ with $k \leq 2m$ the induced map $i_*\colon \mathrm{H}_k(\mathbb{C}\mathrm{P}^m) \to \mathrm{H}_k(\mathbb{C}\mathrm{P}^n)$ is an isomorphism.

The statements regarding cohomology groups follow from the above discussion of homology groups together with the observation that from the Universal Coefficient-H^*-Theorem 70.14 and the Ext-Properties Lemma 70.12 we obtain for any $m \leq n \leq \infty$ the following commutative diagram with vertical isomorphisms:

$$\begin{array}{ccc} \mathrm{H}^k(\mathbb{C}\mathrm{P}^n;\mathbb{Z}) & \xrightarrow{\quad i^* \quad} & \mathrm{H}^k(\mathbb{C}\mathrm{P}^m;\mathbb{Z}) \\ {\scriptstyle \mathrm{ev}}\downarrow{\scriptstyle \cong} & & {\scriptstyle \mathrm{ev}}\downarrow{\scriptstyle \cong} \\ \mathrm{Hom}(\mathrm{H}_k(\mathbb{C}\mathrm{P}^n),\mathbb{Z}) & \xrightarrow[\substack{\text{isomorphism by the above,}\\ \text{for all } k\in\mathbb{N}_0 \text{ with } k\leq 2m}]{\quad i^* \quad} & \mathrm{Hom}(\mathrm{H}_k(\mathbb{C}\mathrm{P}^m),\mathbb{Z}). \end{array}$$

■

We also introduce the following notation:

Notation.

(1) Let $n \in \mathbb{N}_0$. Note that $\mathbb{C}\mathrm{P}^n$ is a closed $2n$-dimensional smooth manifold that comes with a natural orientation. Thus we will always view $\mathbb{C}\mathrm{P}^n$ as an *oriented* smooth manifold. In particular $\mathbb{C}\mathrm{P}^n$ comes with a canonical fundamental class $[\mathbb{C}\mathrm{P}^n] \in \mathrm{H}_{2n}(\mathbb{C}\mathrm{P}^n)$ and it comes with a canonical dual fundamental class $[\mathbb{C}\mathrm{P}^n]^* \in \mathrm{H}^{2n}(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$.

(2) For any $n \in \mathbb{N} \cup \{\infty\}$ we denote by $i\colon \mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^n$ the above embedding. By the $\mathbb{C}\mathrm{P}^n$-H^*-Lemma 81.3 we can define $x := (i^*)^{-1}([\mathbb{C}\mathrm{P}^1]^*) \in \mathrm{H}^2(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$. We refer to $x \in \mathrm{H}^2(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$ as the standard generator.

Now we have the tools to determine the cup product of all complex projective spaces:

Proposition 81.4. (Cup Product-$\mathbb{C}\mathrm{P}^n$-Proposition)

(1) Let $n \in \mathbb{N}_0 \cup \{\infty\}$. For every $k \in \mathbb{N}_0$ with $k \leq n$ the k-th power[635] x^k is a generator of $\mathrm{H}^{2k}(\mathbb{C}\mathrm{P}^n;\mathbb{Z}) \cong \mathbb{Z}$.

(2) (a) Let $n \in \mathbb{N}_0$. If we view $\mathbb{Z}[x]/(x^{n+1})$ as a superalgebra by equipping the ring with the grading determined by $\deg(x) = 2$, then the obvious map

$$\mathbb{Z}[x]/(x^{n+1}) \;\to\; (\mathrm{H}^*(\mathbb{C}\mathrm{P}^n;\mathbb{Z}),\cup)$$

is an isomorphism of superalgebras.

(b) If we view $\mathbb{Z}[x]$ as a superalgebra by equipping the ring with the grading determined by $\deg(x) = 2$, then the obvious map

$$\mathbb{Z}[x] \;\to\; (\mathrm{H}^*(\mathbb{C}\mathrm{P}^\infty;\mathbb{Z}),\cup)$$

is an isomorphism of superalgebras.

(c) Let $m \in \mathbb{N}$. Given $i \in \{1,\ldots,m\}$ we denote by $p_i\colon (\mathbb{C}\mathrm{P}^\infty)^m \to \mathbb{C}\mathrm{P}^\infty$ the i-th projection. If we view $\mathbb{Z}[x_1,\ldots,x_m]$ as a superalgebra by equipping the ring with the grading determined by $\deg(x_i) = 2$, then the map

$$\mathbb{Z}[x_1,\ldots,x_m] \;\to\; (\mathrm{H}^*(\underbrace{\mathbb{C}\mathrm{P}^\infty\times\ldots\times\mathbb{C}\mathrm{P}^\infty}_{m \text{ copies}};\mathbb{Z}),\cup)$$

which sends x_i to $p_i^*(x)$ is an isomorphism of superalgebras. We have obvious variations if we take finite products of arbitrary complex projective spaces.

Proof.

(1) (a) We first deal with the case $n \in \mathbb{N}_0$. We prove this statement by induction on n. Evidently the statement holds for $n = 0$. Now suppose that it holds for some $n-1$ with $n \geq 2$. We need to show that for every $k \in \{0, \dots, n\}$ the k-th power $x^k \in \mathrm{H}^{2k}(\mathbb{C}\mathrm{P}^n; \mathbb{Z}) \cong \mathbb{Z}$ is a generator. In the following two claims we deal with the cases $k \in \{0, \dots, n-1\}$ and $k = n$ separately:

Claim 1. For $k \in \{0, \dots, n-1\}$ the k-th power $x^k \in \mathrm{H}^{2k}(\mathbb{C}\mathrm{P}^n; \mathbb{Z})$ is a generator.

Proof. Let $i \colon \mathbb{C}\mathrm{P}^{n-1} \to \mathbb{C}\mathrm{P}^n$ be the usual embedding and let $\widetilde{x} \in \mathrm{H}^2(\mathbb{C}\mathrm{P}^{n-1}; \mathbb{Z})$ and $x \in \mathrm{H}^2(\mathbb{C}\mathrm{P}^n; \mathbb{Z})$ be the standard generators. It follows easily from the functoriality of cohomology that we have $i^*(x) = \widetilde{x}$. We consider the following diagram:

$$\begin{array}{ccc} \mathrm{H}^{2k}(\mathbb{C}\mathrm{P}^{n-1}; \mathbb{Z}) & \xleftarrow[\cong]{i^*} & \mathrm{H}^{2k}(\mathbb{C}\mathrm{P}^n; \mathbb{Z}) \\ {\scriptstyle \varphi \mapsto \varphi^k}\uparrow & & \uparrow{\scriptstyle \varphi \mapsto \varphi^k} \\ \mathrm{H}^2(\mathbb{C}\mathrm{P}^{n-1}; \mathbb{Z}) & \xleftarrow[\cong]{i^*} & \mathrm{H}^2(\mathbb{C}\mathrm{P}^n; \mathbb{Z}). \end{array}$$

It follows from the Cup Product-Naturality Lemma 72.2 that the diagram commutes and it follows from the $\mathbb{C}\mathrm{P}^n$-H^*-Lemma 81.3 (2) that the horizontal maps are isomorphisms. Since $k \leq n-1$ we obtain that $\widetilde{x}^k$ is by induction hypothesis a generator of $\mathrm{H}^{2k}(\mathbb{C}\mathrm{P}^{n-1}; \mathbb{Z})$ we deduce that x^k is a generator of $\mathrm{H}^{2k}(\mathbb{C}\mathrm{P}^n; \mathbb{Z})$. ⊞

It remains to prove the following claim:

Claim 2. The n-th power $x^n \in \mathrm{H}^{2n}(\mathbb{C}\mathrm{P}^n; \mathbb{Z})$ is a generator.

Proof. We consider the pairing

$$\begin{array}{rcl} \overbrace{\mathrm{H}^2(\mathbb{C}\mathrm{P}^n; \mathbb{Z})}^{=\mathbb{Z}\cdot x} \times \overbrace{\mathrm{H}^{2n-2}(\mathbb{C}\mathrm{P}^n; \mathbb{Z})}^{=\mathbb{Z}\cdot x^{n-1}} & \to & \mathbb{Z} \\ (\varphi, \psi) & \mapsto & \langle \varphi \cup \psi, [\mathbb{C}\mathrm{P}^n] \rangle. \end{array}$$

As we discussed above, $\mathbb{C}\mathrm{P}^n$ is a closed orientable $2n$-dimensional topological manifold. It thus follows from the Manifold-Non Singularity Proposition 81.2 (1) that the above pairing is non-singular. It follows from this observation, the fact that $x \in \mathrm{H}^2(\mathbb{C}\mathrm{P}^n; \mathbb{Z})$ and $x^{n-1} \in \mathrm{H}^{2n-2}(\mathbb{C}\mathrm{P}^n; \mathbb{Z})$ are basis elements and the Non-Singularity Lemma 81.1 (1) that $\langle x \cup x^{n-1}, [\mathbb{C}\mathrm{P}^n] \rangle = \pm 1$. By the Top Cohomology Theorem 70.16 this implies that $x \cup x^{n-1} = \pm [\mathbb{C}\mathrm{P}^n]^*$. ⊞

(b) Now we consider the case $n = \infty$. Let $k \in \mathbb{N}_0$. By (1) we know that x^k is a generator of $\mathrm{H}^{2k}(\mathbb{C}\mathrm{P}^k; \mathbb{Z})$. As in the proof of Claim 1 above we obtain from the $\mathbb{C}\mathrm{P}^n$-H^*-Lemma 81.3 (2) together with the Cup Product-Naturality Lemma 72.2 that x^k is a generator of $\mathrm{H}^{2k}(\mathbb{C}\mathrm{P}^\infty; \mathbb{Z})$.

[635] Let X be a topological space, let R be a commutative ring and let $\varphi \in \mathrm{H}^d(X; R)$. As in any ring we define $\varphi^0 = 1_X$ and given any $k \in \mathbb{N}$ we define $\varphi^k := \underbrace{\varphi \cup \cdots \cup \varphi}_{k \text{ times}} \in \mathrm{H}^{kd}(X; R)$.

(2) First note that it follows immediately from the definitions and the Cup Product-Commutativity Proposition 72.8 that the maps in (2) are homomorphisms of super-algebras. This observation, together with the $\mathbb{C}\mathrm{P}^n$-H^*-Lemma 81.3, shows that (2a) and (2b) are a consequence of (1).

Finally note that Statement (2c) follows from (2b) and from iteratively applying the Cohomology Ring-of-Product Proposition 77.8 (which we can apply by the CW–Chain Complex Condition Lemma 77.2 since $\mathbb{C}\mathrm{P}^\infty$ has a CW-structure with at most one cell in each dimension) to the product $\mathbb{C}\mathrm{P}^\infty \times (\mathbb{C}\mathrm{P}^\infty \times (\cdots \times \mathbb{C}\mathrm{P}^\infty)\dots)$ and from the discussion of tensor products of polynomial rings on page 993. ■

Using the Cup Product-$\mathbb{C}\mathrm{P}^n$-Proposition 81.4 we can now distinguish pairs of topological spaces which had so far eluded all our attempts to tell them apart. For the purpose of this discussion we introduce the following definition:

Definition. We say that two path-connected non-empty topological spaces X and X' are doppelgänger of one another if the fundamental groups $\pi_1(X)$ and $\pi_1(X')$ are isomorphic and if for all $k \in \mathbb{N}_0$ the homology groups $\mathrm{H}_k(X)$ and $\mathrm{H}_k(X')$ are isomorphic.[636]

Pairs of doppelgänger are notoriously hard to distinguish. The following corollary reminds us of two doppelgänger and it shows, using the cup product, that they are not homotopy equivalent. The corollary gives in particular an answer to Question 56.4.

Corollary 81.5. (Doppelgänger Corollary)
(1) $S^2 \vee S^4$ and $\mathbb{C}\mathrm{P}^2$ are doppelgänger but they are not homotopy equivalent.
(2) $S^2 \times S^4$ and $\mathbb{C}\mathrm{P}^3$ are doppelgänger but they are not homotopy equivalent.

Proof.

(1) (a) It follows from the Sphere-π_1-Proposition 11.15, the Wedge-π_1-Proposition 20.6 and the Projective Space–π_1-Proposition 29.3 (2) that $S^2 \vee S^4$ and $\mathbb{C}\mathrm{P}^2$ are both simply connected. Furthermore it follows from the Sphere-H_*-Proposition 41.2, that for any $k \in \mathbb{N}_0$ we have

$$\mathrm{H}_k(S^2 \vee S^4; \mathbb{Z}) \cong \mathrm{H}_k(\mathbb{C}\mathrm{P}^2; \mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & \text{if } k = 0, 2, 4, \\ 0, & \text{otherwise.} \end{cases}$$

(b) In the Wedge-of-Spheres–Cup Product Lemma 72.12 we saw that the cup product

$$\cup\colon \mathrm{H}^2(S^2 \vee S^4; \mathbb{Z}) \times \mathrm{H}^2(S^2 \vee S^4; \mathbb{Z}) \to \mathrm{H}^4(S^2 \vee S^4; \mathbb{Z})$$

is the zero-map. On the other hand we saw in the Cup Product-$\mathbb{C}\mathrm{P}^n$-Proposition 81.4 (1) that the corresponding cup product for $\mathbb{C}\mathrm{P}^2$ is non-trivial. It follows from these observations and the Homotopy Equivalence-H^*-Corollary 67.8 (2) together with the Cup Product-Ring Proposition 72.9 (2) that $S^2 \vee S^4$ and X are not homotopy equivalent.

(2) (a) On page 706 we already pointed out that $S^2 \times S^4$ and $\mathbb{C}\mathrm{P}^3$ cannot be distinguished using the fundamental group and using homology groups. In other words, we showed that they are doppelgänger. As on page 1046 it follows from the Cup

[636] It follows from the Universal Coefficient-H^*-Theorem 70.14 that all cohomology groups are also isomorphic.

Product-Naturality Lemma 72.2 that the cup product

$$\underbrace{\mathrm{H}^2(S^2\times S^4;\mathbb{Z})}_{=p^*([S^2]^*}\times\underbrace{\mathrm{H}^2(S^2\times S^4;\mathbb{Z})}_{=p^*([S^2]^*}\ \to\ \mathrm{H}^4(S^2\times S^4;\mathbb{Z})$$

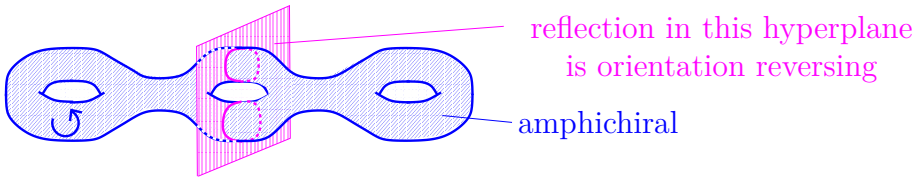
is zero. On the other hand we saw in the Cup Product-$\mathbb{C}\mathrm{P}^n$-Proposition 81.4 that the cup product

$$\mathrm{H}^2(\mathbb{C}\mathrm{P}^3;\mathbb{Z})\times\mathrm{H}^2(\mathbb{C}\mathrm{P}^3;\mathbb{Z})\ \to\ \mathrm{H}^4(\mathbb{C}\mathrm{P}^3;\mathbb{Z})$$

is non-zero. As in (1b) it follows that $S^2\times S^4$ and $\mathbb{C}\mathrm{P}^3$ are not homotopy equivalent. ■

Next we introduce the following natural definition:

Definition. Let M be a compact oriented topological manifold. We say that M is amphichiral if M admits an orientation-reversing self-homeomorphism, otherwise we say that M is chiral.



Many smooth manifolds are amphichiral. For example one can easily show that all compact oriented 2-dimensional topological manifolds are amphichiral. In particular $\mathbb{C}\mathrm{P}^1 =_{\mathrm{i}} S^2$ is amphichiral. In the following corollary we determine which complex projective spaces are amphichiral.

Corollary 81.6. ($\mathbb{C}\mathrm{P}^n$-Amphichiral Corollary) Given any $n\in\mathbb{N}$ we have

$$\text{the smooth manifold } \mathbb{C}\mathrm{P}^n \text{ is amphichiral} \iff n \text{ is odd.}$$

Proof. First we prove the "⇒"-direction. Suppose that there exists an orientation-reversing self-homeomorphism $f\colon \mathbb{C}\mathrm{P}^n\to\mathbb{C}\mathrm{P}^n$. We make a few observations and preparations:

(1) It follows from the discussion on page 842 and the Dual Fundamental Class–Degree Lemma 70.17 that $f^*([\mathbb{C}\mathrm{P}^n]^*) = -[\mathbb{C}\mathrm{P}^n]^*\in\mathrm{H}^{2n}(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$.

(2) As usual let $x\in\mathrm{H}^2(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$ be the standard generator. Since f is a homeomorphism we see that $f^*(x)\in\mathrm{H}^2(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$ is a generator. This implies that there exists an $\eta\in\{-1,1\}$ with $f^*(x)=\eta\cdot x$.

(3) By the Cup Product-$\mathbb{C}\mathrm{P}^n$-Proposition 81.4 we know that $x^n\in\mathrm{H}^{2n}(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$ is a generator. This implies that $x^n=\epsilon\cdot[\mathbb{C}\mathrm{P}^n]^*$ for some $\epsilon\in\{-1,1\}$.

Now we see that

$$-[\mathbb{C}\mathrm{P}^n]^* = f^*([\mathbb{C}\mathrm{P}^n]^*) = \epsilon\cdot f^*(x^n) \underset{\uparrow}{=} \epsilon\cdot(f^*(x))^n = \epsilon\cdot(\eta\cdot x)^n \underset{\uparrow}{=} \epsilon\cdot\eta^n\cdot x^n = \eta^n\cdot[\mathbb{C}\mathrm{P}^n]^*.$$

by the Cup Product-Naturality Lemma 72.2 — since $\cup$ is bilinear

Thus we have shown that for some $\eta\in\{-1,1\}$ we have $\eta^n=-1$. But this is only possible if n is odd.

Now we prove the "⇒"-direction. It remains to prove the following claim:

Claim. For any odd $n \in \mathbb{N}$ the map

$$\begin{array}{rcl} f\colon \mathbb{C}\mathrm{P}^n & \to & \mathbb{C}\mathrm{P}^n \\ [z_0 : \cdots : z_n] & \mapsto & [\overline{z}_0 : \cdots : \overline{z}_n] \end{array}$$

is an orientation-reversing diffeomorphism.

Proof. First note that this map is indeed well-defined and that it is its own inverse. One can easily verify that f is smooth, thus f is in fact a diffeomorphism. Next we consider the chart

$$\begin{array}{rcl} \Phi\colon \{[z_0 : z_1 : \cdots : z_n] \in \mathbb{C}\mathrm{P}^n \mid z_0 \neq 0\} & \to & \mathbb{C}^n =_{\mathrm{i}} \mathbb{R}^{2n} \\ [z_0 : z_1 : \cdots : z_n] & \mapsto & \left(\frac{z_1}{z_0}, \ldots, \frac{z_n}{z_0}\right). \end{array}$$

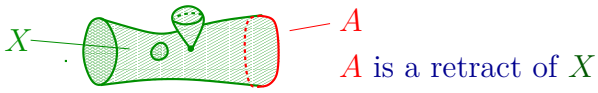
Note that we have the following commutative diagram:

$$\begin{array}{ccc} \mathbb{C}^n & \xrightarrow{(w_1,\ldots,w_n)\mapsto(\overline{w}_1,\ldots,\overline{w}_n)} & \mathbb{C}^n \\ {\scriptstyle\Phi^{-1}}\big\downarrow & & \big\downarrow{\scriptstyle\Phi^{-1}} \\ \mathbb{C}\mathrm{P}^n & \xrightarrow{\quad f \quad} & \mathbb{C}\mathrm{P}^n. \end{array}$$

Since n is odd one sees that the top horizontal map is orientation-reversing. But this implies that the bottom horizontal map is also orientation-reversing. ∎

Towards the end of this section we recall the definition of a retract.

Definition. Let A be a subspace of a topological space X. We denote by $i\colon A \to X$ the inclusion map. We say that A is a retract of X if there exists a continuous map $r\colon X \to A$ such that $r \circ i = \mathrm{id}_A$. (We recall that it follows immediately from the functoriality of homology groups and the Splitting Lemma 37.2 (3)⇒(2) that for each k the group $\mathrm{H}_k(A)$ is a subsummand of $\mathrm{H}_k(X)$.)



Let $m < n$. In Question 59.5 we had asked whether $i(\mathbb{C}\mathrm{P}^m) \subset \mathbb{C}\mathrm{P}^n$ is a retract of $\mathbb{C}\mathrm{P}^n$. Note that the $\mathbb{C}\mathrm{P}^n$-H_*-Proposition 47.4 shows that a study of homology groups does not preclude that for $m < n$ the complex projective space $\mathbb{C}\mathrm{P}^m$ is a retract of $\mathbb{C}\mathrm{P}^n$. Nonetheless, the final result of the section gives a negative answer to Question 59.5, i.e. we show that $i(\mathbb{C}\mathrm{P}^m)$ is not a retract of $\mathbb{C}\mathrm{P}^n$.

Proposition 81.7. ($\mathbb{C}\mathrm{P}^n$-Non-Retraction Proposition) If $1 \leq m < n \leq \infty$, then $i(\mathbb{C}\mathrm{P}^m) \subset \mathbb{C}\mathrm{P}^n$ is not a retract of $\mathbb{C}\mathrm{P}^n$.

Proof. Let $1 \leq m < n \leq \infty$. We denote by $i\colon \mathbb{C}\mathrm{P}^m \to \mathbb{C}\mathrm{P}^n$ the inclusion map. Suppose that there exists a retraction $r\colon \mathbb{C}\mathrm{P}^n \to \mathbb{C}\mathrm{P}^m$. We denote by $x \in \mathrm{H}^2(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$ the standard generator. We have

$$x^n \underset{\uparrow}{=} (r^*(i^*(x)))^n \overset{\downarrow}{=} r^*(i^*(x)^n) \underset{\uparrow}{=} r^*(0) = 0 \in \mathrm{H}^{2n}(\mathbb{C}\mathrm{P}^n;\mathbb{Z}).$$

by the Cup Product-Naturality Lemma 72.2 applied to r

by the $\mathbb{C}\mathrm{P}^n$-H^*-Lemma 81.3 the induced map $i^*\colon \mathrm{H}^2(\mathbb{C}\mathrm{P}^n;\mathbb{Z}) \to \mathrm{H}^2(\mathbb{C}\mathrm{P}^m;\mathbb{Z})$ is an isomorphism, since $i^* \circ r^* = (r \circ i)^* = \mathrm{id}$ we obtain that $r^* \circ i^* = \mathrm{id}$

note that $n > m$ implies that $i^*(x)^n$ lies in the trivial group $\mathrm{H}^{2n}(\mathbb{C}\mathrm{P}^m;\mathbb{Z}) = 0$

But this calculation contradicts the fact shown in the Cup Product-$\mathbb{C}\mathrm{P}^n$-Proposition 81.4 that $x^n \neq 0 \in \mathrm{H}^{2n}(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$. ∎

$$\begin{array}{lcccl} i^*(x)^3=0 & & 0 & \xrightarrow{\quad r^* \quad} & \mathrm{H}^6(\mathbb{C}\mathrm{P}^3;\mathbb{Z}) \quad x^3=(r^*(i^*(x)))^3=r^*(i^*(x)^3) \\ \uparrow & & \mathrm{H}^4(\mathbb{C}\mathrm{P}^2;\mathbb{Z}) & \underset{r^*}{\overset{i^* \ \cong}{\rightleftarrows}} & \mathrm{H}^4(\mathbb{C}\mathrm{P}^3;\mathbb{Z}) \quad \uparrow \\ i^*(x) & & \mathrm{H}^2(\mathbb{C}\mathrm{P}^2;\mathbb{Z}) & & \mathrm{H}^2(\mathbb{C}\mathrm{P}^3;\mathbb{Z}) \quad x=r^*(i^*(x)) \end{array}$$

81.3. **The cup product of $\mathbb{R}\mathrm{P}^n$ and applications.** In this section we intend to determine the cup product of the real projective space $\mathbb{R}\mathrm{P}^n$ with $\mathbb{F}_2$-coefficients. Evidently the approach will be very similar to the one taken in the previous section.

Definition.

(1) Given $m \leq n$ we consider the embedding[637]

$$\begin{array}{rcl} i\colon \mathbb{R}\mathrm{P}^m & \to & \mathbb{R}\mathrm{P}^n \\ [x_0:\cdots:x_m] & \mapsto & [x_0:\cdots:x_m:0:\cdots:0]. \end{array}$$

(2) We set $\mathbb{R}\mathrm{P}^\infty \coloneqq \operatorname{colim}_{\longrightarrow} \mathbb{R}\mathrm{P}^n$.

The following lemma plays the role of the $\mathbb{C}\mathrm{P}^n$-H^*-Lemma 81.3:

Lemma 81.8. ($\mathbb{R}\mathrm{P}^n$-H^*-Lemma)

(1) For every $n \in \mathbb{N} \cup \{\infty\}$ and every $k \in \mathbb{N}_0$ we have

$$\mathrm{H}^k(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2) \ \cong \ \begin{cases} \mathbb{F}_2, & \text{if } k=0,1,\ldots,n, \\ 0, & \text{otherwise.} \end{cases}$$

(2) Let $m \leq n \leq \infty$. For any $k \leq m$ the induced map $i^*\colon \mathrm{H}^k(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2) \to \mathrm{H}^k(\mathbb{R}\mathrm{P}^m;\mathbb{F}_2)$ is an isomorphism.

Proof. The proof is very similar to the proof of $\mathbb{C}\mathrm{P}^n$-H^*-Lemma 81.3. We first show, by basically the same approach, that the analogous statements hold for homology with $\mathbb{F}_2$-coefficients. The statements regarding cohomology then follow from the Universal Coefficient-H^*–Field Theorem 71.2. We leave it to the reader to fill in the details. ■

Notation. Given $n \in \mathbb{N} \cup \{\infty\}$ let $x \in \mathrm{H}^1(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2)$ be the unique non-trivial element.

The following proposition is the analogue of the Cup Product-$\mathbb{C}\mathrm{P}^n$ Proposition 81.4:

Proposition 81.9. (Cup Product-$\mathbb{R}\mathrm{P}^n$-Proposition)

(1) Let $n \in \mathbb{N}_0 \cup \{\infty\}$. For every $k \in \mathbb{N}_0$ with $k \leq n$ the k-th power x^k is the unique non-trivial element of $\mathrm{H}^k(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2) \cong \mathbb{F}_2$.

(2) (a) Let $n \in \mathbb{N}_0$. If we view $\mathbb{F}_2[x]/(x^{n+1})$ as a superalgebra by equipping the ring with the grading determined by $\deg(x)=1$, then the obvious map

$$\mathbb{F}_2[x]/(x^{n+1}) \ \to \ (\mathrm{H}^*(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2),\cup)$$

is an isomorphism of superalgebras.

(b) If we view $\mathbb{F}_2[x]$ as a superalgebra by equipping the ring with the grading determined by $\deg(x)=1$, then the obvious map

$$\mathbb{F}_2[x] \ \to \ (\mathrm{H}^*(\mathbb{R}\mathrm{P}^\infty;\mathbb{F}_2),\cup)$$

[637] On page 137 we showed that the given map $i\colon \mathbb{R}\mathrm{P}^m \to \mathbb{R}\mathrm{P}^n$ is indeed an embedding.

is an isomorphism of superalgebras.

(c) Let $m \in \mathbb{N}$. Given $i \in \{1, \dots, m\}$ we denote by $p_i \colon (\mathbb{R}\mathrm{P}^\infty)^m \to \mathbb{R}\mathrm{P}^\infty$ the i-th projection. If we view $\mathbb{F}_2[x_1, \dots, x_m]$ as a superalgebra by equipping the ring with the grading determined by $\deg(x_i) = 1$, then the map

$$\mathbb{F}_2[x_1, \dots, x_k] \ \to \ (\mathrm{H}^*(\underbrace{\mathbb{R}\mathrm{P}^\infty \times \ldots \times \mathbb{R}\mathrm{P}^\infty}_{m \text{ copies}}; \mathbb{F}_2), \cup)$$

which sends x_i to $p_i^*(x)$ is an isomorphism of superalgebras. We have obvious variations if we take finite products of arbitrary real projective spaces.

Proof.

(1) Note that the real projective space $\mathbb{R}\mathrm{P}^n$ is a closed n-dimensional smooth manifold. Next recall that the $\mathbb{F}_2$-Orientability Proposition 61.13 implies that any topological manifold, in particular $\mathbb{R}\mathrm{P}^n$, is $\mathbb{F}_2$-orientable. With this observation the proof of the proposition is basically identical to the proof of the Cup Product-$\mathbb{C}\mathrm{P}^n$-Proposition 81.4 (1). We just need to replace the Manifold-Non Singularity Proposition 81.2 (1) by the corresponding "part (2)-statement". We leave it to the reader to fill in the details.

(2) Note that it follows immediately from the definitions and the Cup Product-Commutativity Proposition 72.8 that the maps in (2) are homomorphisms of superalgebras. This observation, together with the $\mathbb{R}\mathrm{P}^n$-H^*-Lemma 81.8, shows that (2a) and (2b) are a consequence of (1).

Finally note that Statement (2c) follows from (2b) and from iteratively applying the the $\mathbb{F}_2$-analogue of the Cohomology Ring-of-Product Proposition 77.8 (which we can apply by the CW–Chain Complex Condition Lemma 77.2 since $\mathbb{R}\mathrm{P}^\infty$ has a CW-structure with at most one cell in each dimension) to the product $\mathbb{R}\mathrm{P}^\infty \times (\mathbb{R}\mathrm{P}^\infty \times (\cdots \times \mathbb{R}\mathrm{P}^\infty) \dots)$ and from the discussion of tensor products of polynomial rings on page 993. ■

For completeness' sake we point out that we can now give a new proof of the following proposition, which is the "real" analogue of the $\mathbb{C}\mathrm{P}^n$-Non-Retraction Proposition 81.7:

Proposition 59.4 ($\mathbb{R}\mathrm{P}^n$-Non-Retraction Proposition) If $1 \leq m < n \leq \infty$, then the subspace $i(\mathbb{R}\mathrm{P}^m) \subset \mathbb{R}\mathrm{P}^n$ is not a retract of $\mathbb{R}\mathrm{P}^n$.

Proof. The proof is totally analogous to the proof of the $\mathbb{C}\mathrm{P}^n$-Non-Retraction Proposition 81.7. In the proof we just need to replace $\mathbb{Z}$-coefficients by $\mathbb{F}_2$-coefficients and we need to replace the $\mathbb{C}\mathrm{P}^n$-H^*-Lemma 81.3 by the $\mathbb{R}\mathrm{P}^n$-H^*-Lemma 81.8. We leave it to the reader to fill in the details. ■

Finally recall that on page 950 we introduced the parametrization number $\mathrm{param}(M)$ of a smooth manifold and that on page 952 we introduced the Lusternik-Schnirelmann category $\mathrm{cat}(X)$ of a topological space X. One of the many consequences of our calculations from this chapter is that we can now calculate the parametrization number and the Lusternik-Schnirelmann category for $\mathbb{R}\mathrm{P}^n$ and $\mathbb{C}\mathrm{P}^n$. More precisely, we have the following proposition which gives a complete answer to Question 74.1.

Proposition 81.10. (LS Category–Projective Space Proposition) Let $n \in \mathbb{N}_0$.

(1) We have $\mathrm{param}(\mathbb{R}\mathrm{P}^n) = \mathrm{param}(\mathbb{C}\mathrm{P}^n) = n + 1$.

(2) We have $\mathrm{cat}(\mathbb{R}\mathrm{P}^n) = \mathrm{cat}(\mathbb{C}\mathrm{P}^n) = n$.

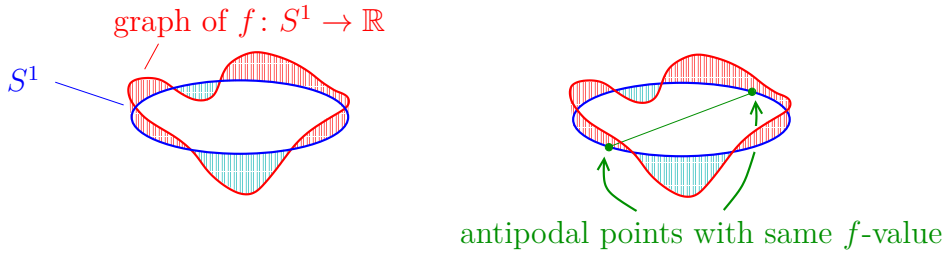
Proof. Given any $n \in \mathbb{N}_0$ we have

$$n \underset{\text{see page 951}}{\geq} \operatorname{param}(\mathbb{R}\mathrm{P}^n) - 1 \underset{\text{by the Parametrization LS Category Lemma 74.2}}{\geq} \operatorname{cat}(\mathbb{R}\mathrm{P}^n) \underset{\text{by the Cup Length–LS Category Proposition 74.3}}{\geq} \overbrace{\text{cup length cl}(\mathbb{R}\mathrm{P}^n)}^{\text{see page 953 for the definition}} \underset{\text{by the Cup Product-}\mathbb{R}\mathrm{P}^n \text{ Proposition 81.9}}{\geq} n.$$

The same argument also works for $\mathbb{C}\mathrm{P}^n$, we just need to replace the Cup Product-$\mathbb{R}\mathrm{P}^n$-Proposition 81.9, by the Cup Product-$\mathbb{C}\mathrm{P}^n$-Proposition 81.4. ■

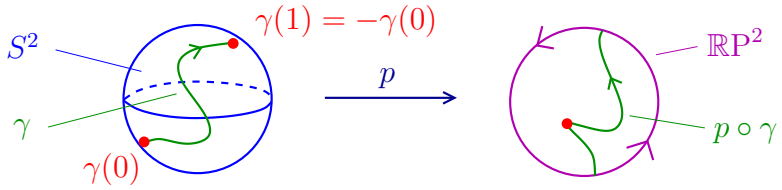
81.4. **Appendix: The Borsuk-Ulam Theorem via cup products.** In this appendix we will give an alternative proof for the Borsuk-Ulam Theorem. First we recall the statement of the Borsuk-Ulam Theorem.

Theorem 57.4. (Borsuk-Ulam Theorem) For every continuous map $f\colon S^n \to \mathbb{R}^n$ there exists a pair of antipodal points x and $-x$ on S^n with $f(x) = f(-x)$.



In the new proof of the Borsuk-Ulam Theorem below we will need two elementary lemmas.

Lemma 81.11. (Loop-in-$\mathbb{R}\mathrm{P}^n$-Lemma) Let $k \in \mathbb{N}$. We denote by $p\colon S^k \to \mathbb{R}\mathrm{P}^k$ the natural projection. Let $\gamma\colon [0,1] \to S^k$ be a path with $\gamma(1) = -\gamma(0)$.
(1) If $k = 1$, then $p \circ \gamma$ represent an odd multiple of a generator of $\pi_1(\mathbb{R}\mathrm{P}^1) \cong \mathbb{Z}$.
(2) If $k \geq 2$, then $p \circ \gamma$ represents the unique non-trivial element in[638] $\pi_1(\mathbb{R}\mathrm{P}^k) \cong \mathbb{Z}_2$.



Proof of Lemma 81.11. Let $\gamma\colon [0,1] \to S^k$ be a path with $\gamma(1) = -\gamma(0)$. Note that $p \circ \gamma$ is a loop in $(\mathbb{R}\mathrm{P}^k, \gamma(0))$ and that γ is a lift of this loop to the covering $p\colon S^k \to \mathbb{R}\mathrm{P}^k$. Since the lift γ of $p \circ \gamma$ is not a loop we obtain from the Loop–Lifting Proposition 14.12 that $[p \circ \gamma] \notin p_*(\pi_1(S^k), \gamma(0))$. This observation already deals with the case $k \geq 2$. If $k = 1$, then the desired statement follows from the following observation: Using the explicit homeomorphism $S^1 \to \mathbb{R}\mathrm{P}^1$ on page 137 and using the Circle-π_1-Theorem 11.13 one can easily show that the image $p_*(\pi_1(S^1), \gamma(0))$ is given by the even elements of $\pi_1(\mathbb{R}\mathrm{P}^1, [\gamma(0)]) \cong \mathbb{Z}$. ■

[638] Recall that for $k \geq 2$ we know from the Real Projective Space–π_1-Proposition 15.3 that $\pi_1(\mathbb{R}\mathrm{P}^k) \cong \mathbb{Z}_2$.

Lemma 81.12. (Isomorphism-on-H^1-Lemma) Let $k, l \in \mathbb{N}$, let $h\colon \mathbb{R}\mathrm{P}^k \to \mathbb{R}\mathrm{P}^l$ be a continuous map and let $x_0 \in \mathbb{R}\mathrm{P}^k$. We make the following assumption:

(1) If $l = 1$, then we assume that the image of $h_*\colon \pi_1(\mathbb{R}\mathrm{P}^k, x_0) \to \pi_1(\mathbb{R}\mathrm{P}^l, h(x_0)) \cong \mathbb{Z}$ contains an odd element.

(2) If $l \geq 2$, then we assume that $h_*\colon \pi_1(\mathbb{R}\mathrm{P}^k, x_0) \to \pi_1(\mathbb{R}\mathrm{P}^l, h(x_0)) \cong \mathbb{Z}_2$ has non-trivial image.

Then $h^*\colon \mathrm{H}^1(\mathbb{R}\mathrm{P}^l; \mathbb{F}_2) \to \mathrm{H}^1(\mathbb{R}\mathrm{P}^k; \mathbb{F}_2)$ is an isomorphism.

Proof of Lemma 81.12. By the H^1-Hom-Proposition 70.15 we have a commutative diagram with vertical isomorphisms of the following form:

$$\begin{array}{ccc} \mathrm{Hom}(\pi_1(\mathbb{R}\mathrm{P}^l, h(x_0)), \mathbb{Z}_2) & \xrightarrow{h^*} & \mathrm{Hom}(\pi_1(\mathbb{R}\mathrm{P}^k, x_0), \mathbb{Z}_2) \\ \downarrow\cong & & \downarrow\cong \\ \mathrm{H}^1(\mathbb{R}\mathrm{P}^l; \mathbb{Z}_2) & \xrightarrow{\quad h^* \quad} & \mathrm{H}^1(\mathbb{R}\mathrm{P}^k; \mathbb{Z}_2). \end{array}$$

It follows easily from our hypothesis that the top horizontal map has non-trivial image. But since the vertical maps are isomorphisms and since all four groups in the diagram are isomorphic to $\mathbb{Z}_2$ we see that the top and bottom horizontal maps are isomorphisms. ∎

Proof of the Borsuk-Ulam Theorem 57.4. In the following argument we deal with the case that $n \geq 3$. The case $n = 2$ can be proved the same way, we just need to replace the Loop-in-$\mathbb{R}\mathrm{P}^n$-Lemma 81.11 (2) and the Isomorphism-on-H^1-Lemma 81.12 (2) by the corresponding "(1)-Statements". Finally we leave it to the reader to modify the argument below to deal with the case $n = 1$.

Let us suppose that there exists a continuous map $f\colon S^n \to \mathbb{R}^n$ such that for any $x \in S^n$ we have that $f(x) \neq f(-x)$. This hypothesis allows us to consider the map

$$\begin{array}{rccc} g\colon & S^n & \to & S^{n-1} \\ & x & \mapsto & \frac{f(x)-f(-x)}{\|f(x)-f(-x)\|}. \end{array}$$

It is clear that $g(-x) = -g(x)$ for all $x \in S^n$. Therefore this map g descends to a map

$$\begin{array}{rccc} h\colon & \mathbb{R}\mathrm{P}^n & \to & \mathbb{R}\mathrm{P}^{n-1} \\ & [x] & \mapsto & [g(x)]. \end{array}$$

Note that it follows from the Twice Quotient Lemma 6.3 that the map h is also continuous. In summary we get the following commutative diagram:

$$\begin{array}{ccc} S^n & \xrightarrow{\quad g \quad} & S^{n-1} \\ \downarrow p & & q\downarrow \\ \mathbb{R}\mathrm{P}^n & \xrightarrow{[x]\mapsto h([x]):=[g(x)]} & \mathbb{R}\mathrm{P}^{n-1} \end{array}$$

where p and q denote the natural projections.

Claim. The map $h^*\colon \mathrm{H}^1(\mathbb{R}\mathrm{P}^{n-1}; \mathbb{F}_2) \to \mathrm{H}^1(\mathbb{R}\mathrm{P}^n; \mathbb{F}_2)$ is an isomorphism.

Proof. Since $n \geq 3$ it follows from the Isomorphism-on-H^1-Lemma 81.12 (1) that we only have to show that $h_*\colon \pi_1(\mathbb{R}\mathrm{P}^n) \to \pi_1(\mathbb{R}\mathrm{P}^{n-1})$ has non-trivial image. Let $\gamma\colon [0,1] \to S^n$ be

a path with $\gamma(1) = -\gamma(0)$. Note that $p \circ \gamma \colon [0,1] \to \mathbb{R}\mathrm{P}^n$ is a loop. We calculate that

$$h_*([p \circ \gamma]) \quad = \quad [h \circ p \circ \gamma] \quad \overset{\text{by the above commutative diagram}}{\overset{\downarrow}{=}} \quad [q \circ (g \circ \gamma)] \quad \underset{\text{by the Loop-in-}\mathbb{R}\mathrm{P}^n\text{-Lemma 81.11 (2) since } (g \circ \gamma)(1) = g(\gamma(1)) = g(-\gamma(0)) = -g(\gamma(0)) = -(g \circ \gamma)(0)}{\overset{\uparrow}{\neq}} \quad e \quad \in \pi_1(\mathbb{R}\mathrm{P}^{n-1}).$$

⊞

We denote by $y \in \mathrm{H}^1(\mathbb{R}\mathrm{P}^{n-1}; \mathbb{F}_2)$ and $x \in \mathrm{H}^1(\mathbb{R}\mathrm{P}^n; \mathbb{F}_2)$ the unique non-trivial elements. We have the following equality in $\mathrm{H}^n(\mathbb{R}\mathrm{P}^n; \mathbb{F}_2)$:[639]

$$0 \quad \underset{\text{by the Cup Product-}\mathbb{R}\mathrm{P}^n\text{-Proposition 81.9}}{\overset{\downarrow}{\neq}} \quad x^n \quad \underset{\text{by the claim we have } h^*(y) = x}{\overset{\uparrow}{=}} \quad (h^*(y))^n \quad \overset{\text{by the Cup Product-Naturality Lemma 72.2}}{\overset{\downarrow}{=}} \quad h^*(y^n) \quad \underset{\text{applied to } \mathbb{R}\mathrm{P}^{n-1}}{\overset{\uparrow}{=}} \quad h^*(0) \quad = \quad 0.$$

We have thus obtained the desired contradiction. ■

[639] The argument is very similar to the argument in the proof of the $\mathbb{C}\mathrm{P}^n$-Non-Retraction Proposition 81.7.

CHAPTER 82

Quaternions, octonions and Division algebras

In this chapter we do the following:

(1) We introduce the Cayley-Dickson construction which turns an $\mathbb{R}$-algebra with conjugation into a new $\mathbb{R}$-algebra, with double the dimension, with conjugation.
(2) The Cayley-Dickson construction turns $\mathbb{R}$ into the complex numbers $\mathbb{C}$, it turns $\mathbb{C}$ into the quaternions $\mathbb{H}$, it turns $\mathbb{H}$ into the octonions $\mathbb{O}$ and it turns $\mathbb{O}$ into the utterly useless sedenions $\mathbb{S}$.
(3) We will discuss the quaternions $\mathbb{H}$ and octonions $\mathbb{O}$ in detail and we will see that they are division algebras.
(4) We use cup products to prove the Hopf–Division Algebra Dimension Theorem 82.7 which says that if $\mathbb{R}^n$ admits the structure of division algebras, then n is a power of 2.

In the next Chapter 83 we will use the quaternions $\mathbb{H}$ and octonions $\mathbb{O}$ to introduce for each $n \in \mathbb{N}_0$ the quaternionic projective space $\mathbb{H}\mathrm{P}^n$ and to introduce the octonionic projective spaces $\mathbb{O}\mathrm{P}^1$ and $\mathbb{O}\mathrm{P}^2$.

82.1. **Real algebras.** We start out with recalling some definitions from algebra:

Definition. Let $\mathbb{F}$ be a field. An $\mathbb{F}$-algebra is an $\mathbb{F}$-vector space A together with an $\mathbb{F}$-bilinear map

$$\begin{array}{ccc} A \times A & \to & A \\ (v, w) & \mapsto & v \cdot w. \end{array}$$

(1) We say that the algebra is commutative if for all $a, b \in A$ we have $a \cdot b = b \cdot a$.
(2) (a) We say that the algebra is associative if for all $a, b, c \in A$ we have $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.
    (b) We say that the algebra is alternative if given any $a, b \in A$ we have the equalities $(a \cdot b) \cdot b = a \cdot (b \cdot b)$ and $(a \cdot a) \cdot b = a \cdot (a \cdot b)$.
(3) We say that the algebra has a multiplicatively neutral element if there exists an element $1_A \in A$ with $1_A \neq 0$ such that $1_A \cdot a = a \cdot 1_A = a$ for all $a \in A$.
(4) If A is an algebra with a multiplicatively neutral element 1_A, then given $a \in A$ we say that $b \in A$ is an inverse of a if $a \cdot b = b \cdot a = 1_A$.
(5) A division algebra is an algebra such that for all $a \neq 0$ and $b \in A$ there exists an $x \in A$ with $a \cdot x = b$ and a $y \in A$ with $y \cdot a = b$.

Example. The real numbers $\mathbb{R}$ are evidently an $\mathbb{R}$-algebra that is commutative and associative, it has a multiplicatively neutral element, and it is a division algebra.

Most of the above definitions are expected and standard, except perhaps for "alternative'. The following theorem gives an important property of alternative algebras:

Theorem 82.1. (Alternative Algebra Theorem) Let $\mathbb{F}$ be a field and let A be an $\mathbb{F}$-algebra. If A is alternative, then given any two $x, y \in A$ the algebra generated by 1, x and y is associative.[640]

Proof. This theorem is due to Emil Artin. Textbooks proofs are given in [**Sch66**, Theorem 3.1] and [**ZSSS82**, p. 36]. ■

We move on to the concept of a conjugation on an algebra:

Definition. Let $\mathbb{F}$ be a field and let A be an $\mathbb{F}$-algebra. A conjugation on A is an $\mathbb{F}$-linear map

$$\begin{array}{ccc} A & \to & A \\ a & \mapsto & \overline{a} \end{array}$$

with the following two properties:

(1) For every $a \in A$ we have $\overline{\overline{a}} = a$.
(2) For every $a, b \in A$ we have $\overline{a \cdot b} = \overline{b} \cdot \overline{a}$.

As always there are various adjectives that one can apply to a new mathematical object:

Definition. Let A be an $\mathbb{R}$-algebra with a multiplicatively neutral element 1_A and with conjugation.

(1) We say that the conjugation is imaginary if $\overline{1}_A = 1_A$ and if for every $a \in A$ we have $a + \overline{a} \in \mathbb{R} \cdot 1_A$.
(2) (a) We say that the conjugation is positive if for every non-zero $a \in A$ we have $a \cdot \overline{a} \in \mathbb{R}_{>0} \cdot 1_A$.
    (b) If the conjugation is positive then we denote by $|a|$ the unique non-negative real number with
$$a \cdot \overline{a} \ = \ |a|^2 \cdot 1_A.$$
(3) We say that the conjugation is symmetric if for every $a \in A$ we have $a \cdot \overline{a} = \overline{a} \cdot a$.

The following lemma shows why positive symmetric conjugations are interesting and what the property of being "alternative" is good for:

Lemma 82.2. (Involution-Norm-Algebra Lemma) Let A be an $\mathbb{R}$-algebra with a multiplicatively neutral element 1_A. We assume that A is equipped with a conjugation that is positive and symmetric.

(1) Given any $a \in A \setminus \{0\}$ there exists a multiplicative inverse, namely $\frac{1}{|a|^2} \cdot \overline{a}$.
(2) If A is alternative (in particular if A is associative) and if the conjugation is imaginary, then for any $a, b \in A$ we have $|a \cdot b| = |a| \cdot |b|$.

Proof.

(1) Let $a \in A \setminus \{0\}$. We calculate that
$$a \cdot \left(\tfrac{1}{|a|^2} \cdot \overline{a}\right) \ = \ \tfrac{1}{|a|^2} \cdot a \cdot \overline{a} \ = \ \tfrac{1}{|a|^2} \cdot |a|^2 \cdot 1_A \ = \ 1_A.$$
Since the conjugation is symmetric we have $a \cdot \overline{a} = \overline{a} \cdot a$. It follows that we also have $(\frac{1}{|a|^2} \cdot \overline{a}) \cdot a = 1_A$.
(2) We now assume that A is alternative and that the conjugation is imaginary. Let $a, b \in A$.

[640] Our main application of the Alternative Algebra Theorem 82.1 will be the octonions. For octonions an alternative proof is given in [**CS03**, p. 76].

Claim. The elements $a, \overline{a}, b, \overline{b}$ are contained in an associative subalgebra.

Proof. We define $\mathrm{Re}(a) := \frac{1}{2}(a + \overline{a})$ and $\mathrm{Im}(a) := \frac{1}{2}(a - \overline{a})$. Similarly we define $\mathrm{Re}(b)$ and $\mathrm{Im}(b)$. Since the conjugation is imaginary we see that $\mathrm{Re}(a), \mathrm{Re}(b) \in \mathbb{R} \cdot 1_A$. Thus we see that $a, \overline{a}, b, \overline{b}$ are contained in the subalgebra that is generated by $1, \mathrm{Im}(a)$ and $\mathrm{Im}(b)$. It follows from our hypothesis that A is alternative and the Alternative Algebra Theorem 82.1 that this subalgebra is associative. ⊞

For any $a, b \in A$ we calculate that

$$\begin{array}{rcccccl} |a \cdot b|^2 \cdot 1_A & = & (a \cdot b) \cdot \overline{(a \cdot b)} \cdot 1_A & = & (a \cdot b) \cdot (\overline{b} \cdot \overline{a}) \cdot 1_A & \overset{\text{by the above claim}}{=} & a \cdot (b \cdot \overline{b}) \cdot \overline{a} \cdot 1_A \\ & \overset{\text{since the conjugation is positive}}{=} & a \cdot (|b|^2 \cdot 1_A) \cdot \overline{a} \cdot 1_A & \overset{\text{since multiplication on } A \text{ is } \mathbb{R}\text{-bilinear}}{=} & |b|^2 \cdot (a \cdot \overline{a}) \cdot 1_A & = & |a|^2 \cdot |b|^2 \cdot 1_A. \end{array}$$

It follows that $|a \cdot b|^2 = |a|^2 \cdot |b|^2$ which implies that $|a \cdot b| = |a| \cdot |b|$. ■

82.2. **The Cayley-Dickson construction.** In this section we discuss the Cayley-Dickson construction which, as we already mentioned, iteratively produces a long string of (increasingly less interesting) $\mathbb{R}$-algebras.

Lemma 82.3. (Cayley-Dickson Construction Lemma) Let V be an $\mathbb{R}$-algebra with conjugation. We consider the $\mathbb{R}$-vector space $V \oplus V$ with the multiplication that is given by

$$\begin{array}{rcl} (V \oplus V) \times (V \oplus V) & \to & V \oplus V \\ ((v_1, v_2), (w_1, w_2)) & \mapsto & (v_1 \cdot w_1 - \overline{w_2} \cdot v_2, v_2 \cdot \overline{w_1} + w_2 \cdot v_1) \end{array}$$

and the conjugation

$$\begin{array}{rcl} V \oplus V & \to & V \oplus V \\ (v, w) & \mapsto & \overline{(v, w)} := (\overline{v}, -w). \end{array}$$

The following statements hold:

(1) The multiplication is $\mathbb{R}$-bilinear. In particular $(V \oplus V, +, \cdot)$ is an algebra over $\mathbb{R}$.
(2) (a) If V has a multiplicatively neutral element 1_V then $(1_V, 0) \in V \oplus V$ is a multiplicatively neutral element.
   (b) If V is commutative and if the conjugation on V is trivial, then the multiplication on $V \oplus V$ is commutative.
   (c) If V is commutative and associative, then $V \oplus V$ is associative.
(3) (a) The above conjugation on $V \oplus V$ is indeed a conjugation.
   (b) For any $(v, w) \in V \oplus V$ we have $(v, w) + \overline{(v, w)} = (v + \overline{v}, 0)$.
   (c) For any $(v, w) \in V \oplus V$ we have $(v, w) \cdot \overline{(v, w)} = (v \cdot \overline{v} + \overline{w} \cdot w, 0)$.
(4) (a) If the conjugation on V is imaginary, then so is the conjugation on $V \oplus V$.
   (b) If the conjugation on V is positive, then so is the conjugation on $V \oplus V$.
   (c) If the conjugation on V is symmetric, then so is the conjugation on $V \oplus V$.
(5) The map

$$\begin{array}{rcl} V & \to & V \oplus V \\ v & \mapsto & (v, 0) \end{array}$$

is a monomorphism of $\mathbb{R}$-algebras that preserves the multiplicatively neutral element and that preserves the conjugation.

Definition. Let V be an $\mathbb{R}$-algebra with conjugation. We say that the above $\mathbb{R}$-algebra $V \oplus V$ is obtained from V via the Cayley-Dickson construction.

Proof.

(1) This statement follows easily from the definitions.

(2) (a) This statement also follows immediately from the definitions.

(b) Fortunately this statement also follows immediately from the definitions.

(c) We assume that V is commutative and associative. Let $u = (u_1, u_2)$, $v = (v_1, v_2)$, $w = (w_1, w_2)$ be three elements of $V \oplus V$. Using the hypothesis that V is associative we see that

$$\begin{aligned}(u \cdot v) \cdot w &= (u_1 \cdot v_1 - \overline{v_2} \cdot u_2, u_2 \cdot \overline{v_1} + v_2 \cdot u_1) \cdot (w_1, w_2)\\ &= (u_1 \cdot v_1 \cdot w_1 - \overline{v_2} \cdot u_2 \cdot w_1 - \overline{w_2} \cdot u_2 \cdot \overline{v_1} - \overline{w_2} \cdot v_2 \cdot u_1,\\ &\quad u_2 \cdot \overline{v_1} \cdot \overline{w_1} + v_2 \cdot u_1 \cdot \overline{w_1} + w_2 \cdot u_1 \cdot v_1 - w_2 \cdot \overline{v_2} \cdot u_2)\\ &= (u_1 \cdot v_1 \cdot w_1 - u_1 \cdot \overline{w_2} \cdot v_2 - w_1 \cdot \overline{v_2} \cdot u_2 - \overline{v_1} \cdot \overline{w_2} \cdot u_2,\\ &\quad \uparrow \quad u_2 \cdot \overline{w_1} \cdot \overline{v_1} - u_2 \cdot \overline{v_2} \cdot w_2 + v_2 \cdot \overline{w_1} \cdot u_1 + w_2 \cdot v_1 \cdot u_1)\end{aligned}$$

since V is commutative

$$= (u_1, u_2) \cdot (v_1 \cdot w_1 - \overline{w_2} \cdot v_2, v_2 \cdot \overline{w_1} + w_2 \cdot v_1) = u \cdot (v \cdot w).$$

(3) (a) It is clear that the map $(v, w) \mapsto \overline{(v, w)} := (\overline{v}, -w)$ is $\mathbb{R}$-linear and it is clear that for any $(v, w) \in V \oplus V$ we have $\overline{(v, w)} = (\overline{v}, -w) = (v, w)$. Finally let $v = (v_1, v_2), w = (w_1, w_2) \in V \oplus V$. We see that

$$\begin{aligned}\overline{v \cdot w} &= \overline{(v_1 \cdot w_1 - \overline{w}_2 \cdot v_2, v_2 \cdot \overline{w}_1 + w_2 \cdot v_1)} &&= (\overline{v_1 \cdot w_1 - \overline{w}_2 \cdot v_2}, -v_2 \cdot \overline{w}_1 - w_2 \cdot v_1)\\ &= (\overline{w_1} \cdot \overline{v_1} - \overline{v}_2 \cdot w_2, -v_2 \cdot \overline{w}_1 - w_2 \cdot v_1) &&= (\overline{w_1} \cdot \overline{v_1} - \overline{v}_2 \cdot w_2, -w_2 \cdot v_1 - v_2 \cdot \overline{w}_1)\\ &= (\overline{w}_1, -w_2) \cdot (\overline{v}_1, -v_2) &&= \overline{w} \cdot \overline{v}\end{aligned}$$

(b) This statement follows immediately from the definitions.

(c) This statement also follows immediately from the definitions.

(4) These three statements can be verified easily using (3b) and (3c).

(5) This statement also follows immediately from the definitions. ∎

Example.

(1) We apply the Cayley-Dickson construction to $\mathbb{R}$, viewed as an $\mathbb{R}$-algebra in the obvious way, with the trivial conjugation and we obtain an algebra $\mathbb{R} \oplus \mathbb{R}$ with conjugation. Clearly the map

$$\begin{aligned}\mathbb{R} \oplus \mathbb{R} &\to \mathbb{C} = \mathbb{R} \oplus \mathbb{R} \cdot \mathrm{i}\\ (v, w) &\mapsto v + w \cdot \mathrm{i}\end{aligned}$$

is an isomorphism of $\mathbb{R}$-algebras with involution. Note that it follows from the Cayley-Dickson Construction Lemma 82.3 that $\mathbb{C}$ is commutative and associative.

(2) We apply the Cayley-Dickson construction to the $\mathbb{R}$-algebra $\mathbb{C}$ and we obtain the quaternions $\mathbb{H} = \mathbb{C} \oplus \mathbb{C} = \mathbb{R}^4$. We will study these in greater detail in Section 82.3. We will see that the quaternions are a division algebra that is non-commutative but associative.

(3) We apply the Cayley-Dickson construction to the $\mathbb{R}$-algebra $\mathbb{H}$ and we obtain the octonions $\mathbb{O} = \mathbb{H} \oplus \mathbb{H} = \mathbb{R}^8$. We will study these in greater detail in Section 82.4. We will see that the octonions are a division algebra that is non-commutative and non-associative, but it is at least alternative.

(4) If we apply the Cayley-Dickson construction to the $\mathbb{R}$-algebra $\mathbb{O}$, then we obtain the sedenion $\mathbb{S} = \mathbb{O} \oplus \mathbb{O} = \mathbb{R}^{16}$. One can easily show that the sedenions are *not* a division algebra.

(5) We can iterate this procedure and we obtain algebras which are creatively called pathions (32-dimensional), chingons (64-dimensional), routons (128-dimensional), voudons (256-dimensional). We challenge the reader to come up with the explanations for these names. Alternatively the reader can look at [**dM02**, p. 7]. We refer to [**Bae01**, Section 2.2] for a more detailed discussion of the Cayley-Dickson construction and we refer to [**Mor97**] for more information on the failure of the sedenions to be a division algebra.

In the following table we summarize the key properties of $\mathbb{R}$, the complex numbers $\mathbb{C}$, the quaternions $\mathbb{H}$, the octonions $\mathbb{O}$ and the sedenions $\mathbb{S}$:

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$	$\mathbb{S}$
conjugation is trivial	✓	✗	✗	✗	✗
algebra is commutative	✓	✓	✗	✗	✗
algebra is associative	✓	✓	✓	✗	✗
algebra is alternative	✓	✓	✓	✓	✗
non-zero elements have inverses	✓	✓	✓	✓	✓
division algebra	✓	✓	✓	✓	✗

In the following sections we explain where the ✓ and ✗ come from.

82.3. **The quaternions.** As we already mentioned in the previous section, we can now introduce the quaternions using the Cayley-Dickson construction:

Definition. We define the quaternions $\mathbb{H} = \mathbb{C} \oplus \mathbb{C}$ to be the $\mathbb{R}$-algebra with conjugation that we obtain from applying the Cayley-Dickson construction to the $\mathbb{R}$-algebra $\mathbb{C}$ (with the usual conjugation). Recall that this means that the multiplication on $\mathbb{H} = \mathbb{C} \oplus \mathbb{C}$ is given by

$$(z_1, z_2) \underset{\mathbb{H}}{\cdot} (w_1, w_2) := (z_1 \underset{\mathbb{C}}{\cdot} w_1 - \overline{w}_2 \underset{\mathbb{C}}{\cdot} z_2,\ z_2 \underset{\mathbb{C}}{\cdot} \overline{w}_1 + w_2 \underset{\mathbb{C}}{\cdot} z_1)$$

and the conjugation by

$$\overline{(w, z)} := (\overline{w}, -z).$$

Before we discuss the properties of quaternions let us remind the reader of an alternative definition of quaternions:

Definition. We define

$$\mathbb{H} := \mathbb{R} \cdot 1 \oplus \mathbb{R} \cdot \mathrm{i} \oplus \mathbb{R} \cdot \mathrm{j} \oplus \mathbb{R} \cdot \mathrm{k}$$

with the $\mathbb{R}$-bilinear multiplication that is determined by

(1) $\mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = -1$,
(2) $1 \cdot \mathrm{i} = \mathrm{i} \cdot 1 = \mathrm{i}$, $1 \cdot \mathrm{j} = \mathrm{j} \cdot 1 = \mathrm{j}$, $1 \cdot \mathrm{k} = \mathrm{k} \cdot 1 = \mathrm{k}$, and
(3) $\mathrm{i} \cdot \mathrm{j} = k$, $\mathrm{j} \cdot \mathrm{k} = \mathrm{i}$, $\mathrm{k} \cdot \mathrm{i} = \mathrm{j}$ and $\mathrm{j} \cdot \mathrm{i} = -\mathrm{k}$, $\mathrm{k} \cdot \mathrm{j} = -\mathrm{i}$, $\mathrm{i} \cdot \mathrm{k} = -\mathrm{j}$.

For a quaternion

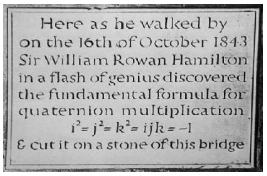
$$z := a \cdot 1 + b \cdot \mathrm{i} + c \cdot \mathrm{j} + d \cdot \mathrm{k} \in \mathbb{H}$$

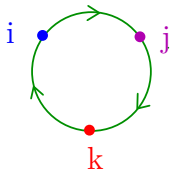
we define the conjugate

$$\overline{z} := a \cdot 1 - b \cdot \mathrm{i} - c \cdot \mathrm{j} - d \cdot \mathrm{k} \in \mathbb{H}.$$

Remark. The multiplication table for $\mathbb{H}$ is be summarized in the following figure:
More precisely, we show the three generators i, j and k. We have $\mathrm{i} \cdot \mathrm{j} = \mathrm{k}$ and the analogous statement holds for any *cyclic* permutation of $\{\mathrm{i}, \mathrm{j}, \mathrm{k}\}$. On the other hand for an anti-cyclic permutation we pick up a minus sign, e.g. we have $\mathrm{j} \cdot \mathrm{i} = -\mathrm{k}$. □







$$\begin{aligned} \mathrm{i}\cdot\mathrm{j} &= \mathrm{k} \\ \mathrm{j}\cdot\mathrm{k} &= \mathrm{i} \\ \mathrm{k}\cdot\mathrm{i} &= \mathrm{j} \end{aligned}$$

In the following we will use the following lemma to go back and forth between the two definitions of quaternions:

Lemma 82.4. (Equivalence-of-Quaternions Lemma) The map

$$\begin{array}{rcl} \mathbb{C}\oplus\mathbb{C} & \to & \mathbb{R}\oplus\mathbb{R}\cdot\mathrm{i}\oplus\mathbb{R}\cdot\mathrm{j}\oplus\mathbb{R}\cdot\mathrm{k} \\ ((a+b\cdot\mathrm{i}),(c+d\cdot\mathrm{i})) & \mapsto & a+b\cdot\mathrm{i}+\underbrace{c\cdot\mathrm{j}+d\cdot\mathrm{k}}_{=(c+d\cdot\mathrm{i})\cdot\mathrm{j}} \end{array}$$

is an isomorphism of $\mathbb{R}$-algebras that preserves the conjugation.

Proof. The lemma follows immediately from the definitions. We leave the details to the reader. ∎

Now we can summarize the key properties of quaternions:

Lemma 82.5. (Quaternion Lemma)

(1) The quaternions $\mathbb{H} = \mathbb{C}\oplus\mathbb{C}$ are a non-commutative $\mathbb{R}$-algebra with multiplicatively neutral element $1_{\mathbb{H}}$.
(2) The quaternions are associative.
(3) The map $(z,w)\mapsto\overline{(z,w)} := (\overline{z},-w)$ is a conjugation.
(4) The conjugation is positive, symmetric and imaginary.

It follows from (4) and the definition on page 1059 that for each $z\in\mathbb{H}$ there exists a unique $|z|\in\mathbb{R}_{\geq 0}$ which satisfies $z\cdot\overline{z} = |z|^2\cdot 1_{\mathbb{H}}$.

(5) If we make the obvious identification $\mathbb{H} = \mathbb{R}^4$, then the norm $|-|$ on $\mathbb{H}$ agrees with the Euclidean norm $\|-\|$ on $\mathbb{R}^4$.[641]
(6) For every $z\neq 0$ a multiplicative inverse is given by $z^{-1} = \frac{\overline{z}}{|z|^2}$.
(7) For all $z,w\in\mathbb{H}$ we have $|z\cdot w| = |z|\cdot|w|$.
(8) The quaternions are a division algebra.

Proof.

(1) It follows from the Cayley-Dickson Construction Lemma 82.3 (1) and (2a) that $\mathbb{H}$ is an $\mathbb{R}$-algebra with multiplicatively neutral element. One can easily verify by example that $\mathbb{H}$ is not commutative, for example we already saw that $\mathrm{i}\cdot\mathrm{j} = \mathrm{k} \neq -\mathrm{k} = \mathrm{j}\cdot\mathrm{i}$.

[641] This shows in particular that the quaternion norm is indeed a norm.

(2) This statement follows from the Cayley-Dickson Construction Lemma 82.3 (2c) and the fact that $\mathbb{C}$ is commutative.
(3) This statement follows from the Cayley-Dickson Construction Lemma 82.3 (3).
(4) This statement follows from the Cayley-Dickson Construction Lemma 82.3 (4).
(5) This statement follows easily from the Cayley-Dickson Construction Lemma 82.3 (3c).
(6) This statement follows from (4) and the Involution-Norm-Algebra Lemma 82.2 (1).
(7) This statement follows from (2) and the Involution-Norm-Algebra Lemma 82.2 (2).
(8) Let $a \neq 0$ and b be two elements in $\mathbb{H}$. We set $y := b \cdot a^{-1}$. Then

$$y \cdot a \;=\; (b \cdot a^{-1}) \cdot a \;\underset{\uparrow}{=}\; b \cdot (a^{-1} \cdot a) \;=\; b \cdot 1 \;=\; b.$$

since by (2) we know that $\mathbb{H}$ is associative

The same way we show that there exists an $x \in \mathbb{H}$ with $a \cdot x = b$. ■

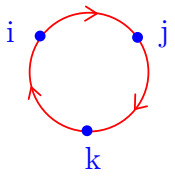
82.4. **The octonions.** Since applying the Cayley-Dickson construction to $\mathbb{C}$ was so much fun, it is natural to iterate this procedure and to apply the Cayley-Dickson construction to the quaternions $\mathbb{H}$. This leads us to the following definition:

Definition. We define the Cayley octonions $\mathbb{O}$ (or usually just octonions) to be the $\mathbb{R}$-algebra with conjugation that we obtain from applying the Cayley-Dickson construction to the $\mathbb{R}$-algebra $\mathbb{H}$ (with the above conjugation). Recall that this means that the multiplication on $\mathbb{O} = \mathbb{H} \oplus \mathbb{H}$ is given by

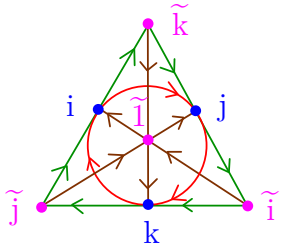
$$(z_1, z_2) \underset{\mathbb{O}}{\cdot} (w_1, w_2) \;:=\; (z_1 \underset{\mathbb{H}}{\cdot} w_1 - \overline{w}_2 \underset{\mathbb{H}}{\cdot} z_2,\; z_2 \underset{\mathbb{H}}{\cdot} \overline{w}_1 + w_2 \underset{\mathbb{H}}{\cdot} z_1)$$

and the conjugation by $\overline{(w, z)} \;:=\; (\overline{w}, -z)$.

Remark. We equip the 8-dimensional real vector space $\mathbb{O} = \mathbb{H} \oplus \mathbb{H} = \mathbb{R}^4 \oplus \mathbb{R}^4$ with the basis that is given by $1 = (1, 0)$, $\mathrm{i} = (\mathrm{i}, 0)$, $\mathrm{j} = (\mathrm{j}, 0)$, $\mathrm{k} = (\mathrm{k}, 0)$ and the almost identical twins $\widetilde{1} = (0, 1)$, $\widetilde{\mathrm{i}} = (0, \mathrm{i})$, $\widetilde{\mathrm{j}} = (0, \mathrm{j})$, $\widetilde{\mathrm{k}} = (0, \mathrm{k})$ and $\widetilde{1} = (0, 1)$. With respect to this basis octonionic multiplication is summarized the right-part of the figure below.



quaternionic multiplication



octonionic multiplication

To the right of the figure below we show the *Fano plane*, which is a diagram with 7 points and 7 directed "lines". Similar to the discussion on page 1062 it encodes the multiplication where the product of two generators equals the third generator on the line with the positive sign if they are cyclic permutation of the given ordering and with a negative sign otherwise. For example we have

$$(\widetilde{\mathrm{j}} \cdot \mathrm{i}) \cdot \mathrm{j} \;=\; \widetilde{\mathrm{k}} \cdot \mathrm{j} \;=\; \widetilde{\mathrm{i}} \qquad \text{and} \qquad \widetilde{\mathrm{j}} \cdot (\mathrm{i} \cdot \mathrm{j}) \;=\; \widetilde{\mathrm{j}} \cdot \mathrm{k} \;=\; -\widetilde{\mathrm{i}}.$$

Note that this calculation also shows that the octonions are not associative. We refer to [**Bae01**, Chapter 2.1] for a more detailed discussion of the role of the Fano plane. □

The following lemma is the analogue of the Quaternion Lemma 82.5.

Lemma 82.6. (Octonion Lemma)

(1) The octonions $\mathbb{O} = \mathbb{H} \oplus \mathbb{H}$ are a non-commutative $\mathbb{R}$-algebra with multiplicatively neutral element $1_{\mathbb{O}}$.
(2) The octonions are not associative but they are alternative, recall that this means that given any $x, y \in \mathbb{O}$ we have $(x \cdot y) \cdot y = x \cdot (y \cdot y)$ and $(x \cdot x) \cdot y = x \cdot (x \cdot y)$.
(3) The map $(z, w) \mapsto \overline{(z, w)} := (\overline{z}, -w)$ is a conjugation.
(4) The conjugation is positive, symmetric and imaginary.

It follows from (4) and the definition on page 1059 that for each $z \in \mathbb{O}$ there exists a unique $|z| \in \mathbb{R}_{\geq 0}$ which satisfies $z \cdot \overline{z} = |z|^2 \cdot 1_{\mathbb{O}}$.

(5) If we make the obvious identification $\mathbb{O} = \mathbb{R}^8$, then the norm $|-|$ on $\mathbb{O}$ agrees with the Euclidean norm $\|-\|$ on $\mathbb{R}^4$.
(6) For every $z \neq 0$ a multiplicative inverse is given by $z^{-1} = \frac{1}{|z|^2} \cdot \overline{z}$.
(7) For all $z, w \in \mathbb{O}$ we have $|z \cdot w| = |z| \cdot |w|$.
(8) The octonions are a division algebra.

Proof. As we will see, the proof has many similarities with the proof of the Quaternion Lemma 82.5. But as we will see, there are also some important differences. In particular we will prove the statements in a different order:

(1) It follows from the Cayley-Dickson Construction Lemma 82.3 (1) and (2a) that $\mathbb{H}$ is an $\mathbb{R}$-algebra with multiplicatively neutral element. Since $\mathbb{H}$ is not commutative one sees easily that $\mathbb{O}$ is not commutative either.
(3) This statement follows from the Cayley-Dickson Construction Lemma 82.3 (3).
(4) This statement follows from the Cayley-Dickson Construction Lemma 82.3 (4).
(2) In the previous remark we saw that the octonions are not associative. But we will now see that the octonions are alternative: Let $x, y \in \mathbb{O}$. We start out with the following claim:

Claim. We have the equalities $(x \cdot y) \cdot \overline{y} = x \cdot (y \cdot \overline{y})$ and $(x \cdot \overline{x}) \cdot y = x \cdot (\overline{x} \cdot y)$.

Proof. Let us first prove the first equality. We write $x = (x_1, x_2)$, $y = (y_1, y_2)$ with $x_1, x_2, y_1, y_2 \in \mathbb{H}$. Heroically we calculate that

$$\begin{aligned}
(x \cdot y) \cdot \overline{y} &= ((x_1, x_2)(y_1, y_2)) \cdot (\overline{y}_1, -y_2) \;=\; (x_1 y_1 - \overline{y}_2 x_2, y_2 x_1 + x_2 \overline{y}_1) \cdot (\overline{y}_1, -y_2) \\
&= ((x_1 y_1 - \overline{y}_2 x_2)\overline{y}_1 + \overline{y}_2 (y_2 x_1 + x_2 \overline{y}_1), (-y_2)(x_1 y_1 - \overline{y}_2 x_2) + (y_2 x_1 + x_2 \overline{y}_1) y_1) \\
&= ((|y_1|^2 + |y_2|^2) x_1, (|y_1|^2 + |y_2|^2) x_2) \;=\; (|y_1|^2 + |y_2|^2) \cdot (x_1, x_2) \\
&\quad \uparrow \\
&\quad \text{since } \mathbb{H} \text{ is associative we have } (\overline{y}_2 x_2)\overline{y}_1 = \overline{y}_2 (x_2 \overline{y}_1) \text{ and } y_2 (x_1 y_1) = (y_2 x_1) y_1 \\
&\quad \text{we also use that } \overline{y}_1 y_1 = |y_1|^2 \cdot 1 \in \mathbb{R}, \text{ hence } \overline{y}_1 y_1 \text{ commutes with any } z \in \mathbb{H} \\
&= |y|^2 \cdot x \;=\; x \cdot |y|^2 \;=\; x \cdot (y \cdot \overline{y}).
\end{aligned}$$

The proof of the second equality is basically the same. ⊞

By (4) we know that the involution is imaginary. This implies that $x + \overline{x} \in \mathbb{R} \cdot 1$ and $y + \overline{y} \in \mathbb{R} \cdot 1$. In other words, there exist $a, b \in \mathbb{R}$ with $\overline{x} = -x + 2a \cdot 1$ and $\overline{y} = -y + 2b \cdot 1$. We plug these equalities into the equalities of the claim and, since we are dealing with an algebra over the real numbers, we obtain the desired equalities $(x \cdot y) \cdot y = x \cdot (y \cdot y)$ and $(x \cdot x) \cdot y = x \cdot (x \cdot y)$.

(5) This statement follows easily from the Cayley-Dickson Construction Lemma 82.3 (3c).

(6) This statement follows from (4) and the Involution-Norm-Algebra Lemma 82.2 (1).
(7) This statement follows from (2) and the Involution-Norm-Algebra Lemma 82.2 (2).
(8) Finally we want to prove that $\mathbb{O}$ is in fact a division algebra. Since the octonions are not associative we cannot copy-paste the proof of the Quaternion Lemma 82.5 (8), since the latter proof used the associativity of the quaternions. Fortunately the spartan associativity result given in (2) is enough to give us the desired result. Let $a, b \in \mathbb{O}$ with $a \neq 0$. We need to show that there exists an $x \in \mathbb{O}$ with $a \cdot x = b$ and a $y \in \mathbb{O}$ with $y \cdot a = b$. We set $x := a^{-1} \cdot b$. We calculate that

$$a \cdot x = a \cdot (a^{-1} \cdot b) \underset{\uparrow}{=} a \cdot \left(\tfrac{1}{|a|^2} \cdot \overline{a} \cdot b\right) = \tfrac{1}{|a|^2} \cdot a \cdot (\overline{a} \cdot b) \underset{\uparrow}{=} \tfrac{1}{|a|^2} \cdot (a \cdot \overline{a}) \cdot b = b.$$

by (6) — by the claim in the proof of (2)

Basically the same calculation shows that we have $y \cdot a = b$. ∎

82.5. **Dimensions of division algebras.** We are now aware of the division algebras $\mathbb{R}, \mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$, i.e. we are aware of division algebras that are defined over $\mathbb{R}^n$ with $n = 1, 2, 4$ and 8. These dimensions are powers of 2 and the following theorem, which was first proved by Heinz Hopf [**Hop**] in 1940, shows that this is not a coincidence:

Theorem 82.7. (Hopf–Division Algebra Dimension Theorem) If $\mathbb{R}^n$ admits the structure of a real division algebra, then n is a power of 2.

In the proof of the Hopf–Division Algebra Dimension Theorem 82.7 we will need the following lemma:

Lemma (No Zero Divisor Lemma) Let A be a *finite-dimensional* algebra over a field $\mathbb{F}$. The following two statements are equivalent:
(1) A is a division algebra,
(2) A has no zero-divisors, i.e. for any $a, b \in A$ with $a \cdot b = 0$ we have $a = 0$ or $b = 0$.

Proof of the No Zero Divisor Lemma . Let A be a finite-dimensional algebra over a field $\mathbb{F}$. Let $a \in A$ be non-zero. Then the following holds:

$$\begin{aligned} \text{for any } y \in A \text{ there exists } x \in A \text{ with } ax = y &\Leftrightarrow \text{the map } A \to A,\ x \mapsto ax \text{ is surjective} \\ &\underset{\uparrow}{\Leftrightarrow} \text{the map } A \to A,\ x \mapsto ax \text{ is injective} \end{aligned}$$

since $x \mapsto ax$ is a homomorphism of finite-dimensional $\mathbb{F}$-vector spaces

$$\Leftrightarrow \text{for any } b \neq 0 \text{ we have } ab \neq 0.$$

The same statement holds for left-multiplication. The equivalence of (1) and (2) now follows easily from this observation and the definitions. ∎

Proof of Theorem 82.7. Let "$*$" be a real division algebra structure on $\mathbb{R}^n$. We want to show that n is a power of 2. Since 1 and 2 are powers of 2 we can assume that $n \geq 3$. We start out with the following claim:

Claim 1. The map

$$\begin{aligned} \widetilde{f}\colon S^{n-1} &\to S^{n-1} \\ x &\mapsto \widetilde{f}(x) := \frac{x * x}{\|x * x\|} \end{aligned}$$

is well-defined and continuous.

Proof. This map is indeed well-defined, since for $x \neq 0$ it follows from our hypothesis on $(\mathbb{R}^n, *)$ and from the No Zero Divisor Lemma that $x * x \neq 0$. The map $\widetilde{f}$ is continuous since the multiplication map $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is bilinear, hence continuous.[642] ⊞

Claim 2.
(1) For every $x, y \in S^{n-1}$ we have $\widetilde{f}(-x, y) = -\widetilde{f}(x, y)$ and $\widetilde{f}(x, -y) = -\widetilde{f}(x, y)$.
(2) The map
$$\begin{array}{rccc} f\colon & \mathbb{RP}^{n-1} \times \mathbb{RP}^{n-1} & \to & \mathbb{RP}^{n-1} \\ & ([x],[y]) & \mapsto & [\widetilde{f}(x,y)]. \end{array}$$
is well-defined and continuous.

Proof.

(1) This statement follows immediately from the fact that "$*$" defines an algebra.
(2) It follows from (1) that the map f is well-defined. It remains to show that f is continuous. This requires a little bit of thought. We consider the following diagram:

$$\begin{array}{ccc} \mathbb{RP}^{n-1} \times \mathbb{RP}^{n-1} & \xrightarrow[f]{([x],[y]) \mapsto [\widetilde{f}(x,y)]} & \mathbb{RP}^{n-1} \\ {\scriptstyle [(x,y)] \mapsto ([x],[y])} \uparrow \cong & \nearrow_{[(x,y)] \mapsto [\widetilde{f}(x,y)]} & \uparrow {\scriptstyle \mathrm{id}} \\ (S^{n-1} \times S^{n-1})/\sim & \longrightarrow & S^{n-1}/\sim \end{array}$$

We make the following clarifications and observations:
(a) Let $\sim$ be the equivalence relation on $S^{n-1} \times S^{n-1}$ that is given by $(x, y) \sim (\pm x, \pm y)$.
(b) One can easily verify that the rectangle commutes. The diagonal map is defined in such a way that the two triangles commute.
(c) It follows immediately from (b) and the Twice Quotient Lemma 6.3 that in the above diagram the diagonal map $(S^{n-1} \times S^{n-1})/\sim \to S^n/\sim = \mathbb{RP}^{n-1}$ is continuous.
(d) It follows from the Topological-Quotient Proposition 6.1 (1a) and (1b) and the Topological-Product Proposition 5.1 (1a) and (1b) (applied in the correct order) that the left vertical map is continuous. The left vertical map is clearly a bijection.
(e) It follows from (d) and the Projective Spaces-Compact+Hausdorff Proposition 6.10 together with the Compact-Hausdorff Proposition 2.10 (3) that the left vertical map is a homeomorphism.

It follows from this discussion that the top horizontal map f is indeed continuous. ⊞

Next we pick $\widetilde{s}_0 \in S^{n-1}$. We write $s_0 := [\widetilde{s}_0] \in \mathbb{RP}^{n-1}$. We consider the embeddings
$$\begin{array}{rccc} i\colon & \mathbb{RP}^{n-1} & \to & \mathbb{RP}^{n-1} \times \mathbb{RP}^{n-1} \\ & x & \mapsto & (x, s_0) \end{array} \quad \text{and} \quad \begin{array}{rccc} j\colon & \mathbb{RP}^{n-1} & \to & \mathbb{RP}^{n-1} \times \mathbb{RP}^{n-1} \\ & x & \mapsto & (s_0, x). \end{array}$$

Claim 3. The map $(f \circ i)_*\colon \pi_1(\mathbb{RP}^{n-1}, s_0) \to \pi_1(\mathbb{RP}^{n-1}, s_0))$ is an isomorphism. The same conclusion holds with i replaced by j.

Proof. We provide the proof for Statement (1). The proof with i replaced by j is basically the same. Since $\pi_1(\mathbb{RP}^{n-1}) \cong \mathbb{Z}_2$ it suffices to show that the map
$$(f \circ i)_*\colon \pi_1(\mathbb{RP}^{n-1}, s_0) \to \pi_1(\mathbb{RP}^{n-1}, s_0)$$

[642] Indeed, the fact that the map is bilinear implies that there exists a real $n \times n$–matrix P such that for any $v, w \in \mathbb{R}^n$ we have $v * w = v^T P w$, hence the map is continuous.

has non-trivial image. We introduce the following notation:

(1) We denote by $\pi\colon S^{n-1} \to \mathbb{R}\mathrm{P}^{n-1}$ the natural projection.
(2) We pick a path $\gamma\colon [0,1] \to S^{n-1}$ with $\gamma(0) = \widetilde{s}_0$ and with $\gamma(1) = -\gamma(0) = -\widetilde{s}_0$. Note that $\pi \circ \gamma\colon [0,1] \to \mathbb{R}\mathrm{P}^{n-1}$ is a loop in s_0.
(3) We denote by $\widetilde{i}$ the embedding $S^{n-1} \to S^{n-1} \times S^{n-1}$ that is given by $\widetilde{i}(x) = (x, \widetilde{s}_0)$.

Note that we get the following commutative diagram:

$$\begin{array}{ccccccc}
[0,1] & \xrightarrow{\gamma} & S^{n-1} & \xrightarrow{\widetilde{i}} & S^{n-1} \times S^{n-1} & \xrightarrow{\widetilde{f}} & S^{n-1} \\
 & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi\times\pi} & & \downarrow{\scriptstyle \pi} \\
 & & \mathbb{R}\mathrm{P}^{n-1} & \xrightarrow{i} & \mathbb{R}\mathrm{P}^{n-1} \times \mathbb{R}\mathrm{P}^{n-1} & \xrightarrow{f} & \mathbb{R}\mathrm{P}^{n-1}.
\end{array}$$

Finally we note that

$$(f \circ i)_*([\pi \circ \gamma]) \;=\; [f \circ i \circ \pi \circ \gamma] \overset{\text{by the above commutative diagram}}{=} [\pi \circ (\widetilde{f} \circ \widetilde{i} \circ \gamma)] \neq e \in \pi_1(\mathbb{R}\mathrm{P}^{n-1}, s_0).$$

this follows from the Loop-in-$\mathbb{R}\mathrm{P}^n$-Lemma 81.11 since the following equality holds:
$(\widetilde{f} \circ \widetilde{i} \circ \gamma)(1) = \widetilde{f}(\gamma(1), \widetilde{s}_0) = \widetilde{f}(-\gamma(0), \widetilde{s}_0) = -\widetilde{f}(\gamma(0), \widetilde{s}_0) = -(\widetilde{f} \circ \widetilde{i} \circ \gamma)(0)$ ⊞

We denote by $\varphi \in \mathrm{H}^1(\mathbb{R}\mathrm{P}^{n-1}; \mathbb{F}_2)$ the unique non-trivial element. Furthermore, as so often we denote by p and q the two natural projections $\mathbb{R}\mathrm{P}^{n-1} \times \mathbb{R}\mathrm{P}^{n-1} \to \mathbb{R}\mathrm{P}^{n-1}$. Finally we write $\alpha \coloneqq p^*(\varphi)$ and $\beta \coloneqq q^*(\varphi) \in \mathrm{H}^1(\mathbb{R}\mathrm{P}^{n-1}; \mathbb{F}_2)$.

Claim 4. The following equality holds

$$f^*(\varphi) \;=\; \alpha + \beta \;\in\; \mathrm{H}^1(\mathbb{R}\mathrm{P}^{n-1} \times \mathbb{R}\mathrm{P}^{n-1}; \mathbb{F}_2).$$

Proof. We start out the proof of the claim with the following three observations:

(1) By the H^1-Hom-Proposition 70.15 we have for each $t_0 \in \mathbb{R}\mathrm{P}^{n-1}$ a natural isomorphism
$$\mathrm{ev}\colon \mathrm{H}^1(\mathbb{R}\mathrm{P}^{n-1}; \mathbb{F}_2) \;\to\; \mathrm{Hom}(\pi_1(\mathbb{R}\mathrm{P}^{n-1}, t_0), \mathbb{F}_2).$$
(2) Let $\gamma \in \pi_1(\mathbb{R}\mathrm{P}^{n-1}, s_0)$ be the unique non-trivial element. It follows from the Product-π_1-Proposition 13.7 that $\pi_1(\mathbb{R}\mathrm{P}^{n-1} \times \mathbb{R}\mathrm{P}^{n-1}, (s_0, s_0))$ is generated by $i_*(\gamma)$ and $j_*(\gamma)$.
(3) Since $\mathrm{H}^1(\mathbb{R}\mathrm{P}^{n-1}; \mathbb{F}_2)$ and $\pi_1(\mathbb{R}\mathrm{P}^{n-1})$ both contain a unique non-trivial element, namely φ and γ, we see that $\mathrm{ev}(\varphi)(\gamma) = 1 \in \mathbb{F}_2$.

It follows from (1) and (2) that it suffices to show that $\mathrm{ev}(f^*(\varphi))$ and $\mathrm{ev}(\alpha + \beta)$ agree on $i_*(\gamma)$ and $j_*(\gamma)$. In fact we have

$$\mathrm{ev}(f^*(\varphi))(i_*(\gamma)) \overset{\text{by the naturality of ev}}{=} \mathrm{ev}(\varphi)(f_*(i_*(\gamma))) = \mathrm{ev}(\varphi)((f \circ i)_*(\gamma))) \overset{\text{by Claim 3}}{=} \mathrm{ev}(\varphi)(\gamma) \overset{\text{by (3)}}{=} 1.$$

We also see that

$$\begin{aligned}
\mathrm{ev}(\alpha + \beta)(i_*(\gamma)) &\overset{\text{by definition of } \alpha \text{ and } \beta}{=} \mathrm{ev}(p^*(\varphi) + q^*(\varphi))(i_*(\gamma)) \overset{\text{since ev is a homomorphism}}{=} \mathrm{ev}(p^*(\varphi))(i_*(\gamma)) + \mathrm{ev}(q^*(\varphi))(i_*(\gamma)) \\
&\overset{\text{by the naturality of ev}}{=} \mathrm{ev}(\varphi)((p \circ i)_*(\gamma)) + \mathrm{ev}(\varphi)((q \circ i)_*(\gamma)) \overset{\substack{\text{since } p \circ i = \mathrm{id} \text{ and} \\ \text{since } q \circ i \text{ is a constant map}}}{=} \mathrm{ev}(\varphi)(\gamma) \overset{\text{by (3)}}{=} 1.
\end{aligned}$$

We have thus shown that $\mathrm{ev}(f^*(\varphi))$ and $\mathrm{ev}(\alpha + \beta)$ agree on $i_*(\gamma)$. The same argument also works with $i_*(\gamma)$ replaced by $j_*(y)$. ⊞

Next we make the following observations:

(1) We note that

$$0 = f^*(0) \underset{\uparrow}{=} f^*(\varphi^n) \overset{\downarrow}{=} (f^*(\varphi))^n \underset{\uparrow}{=} (\alpha+\beta)^n = \sum_{k=0}^{n} \binom{n}{k} \cdot \alpha^k \cdot \beta^{n-k} \underset{\uparrow}{=} \sum_{k=1}^{n-1} \binom{n}{k} \cdot \alpha^k \cdot \beta^{n-k}.$$

by the Cup Product-Naturality Lemma 72.2

follows from $\varphi \in \mathrm{H}^1(\mathbb{R}\mathrm{P}^{n-1};\mathbb{F}_2)$ and the Cup Product-$\mathbb{R}\mathrm{P}^n$-Proposition 81.9

by Claim 4

since $\alpha^n = \beta^n = 0$

(2) By the Cup Product-$\mathbb{R}\mathrm{P}^n$-Proposition 81.9 (3) the obvious map

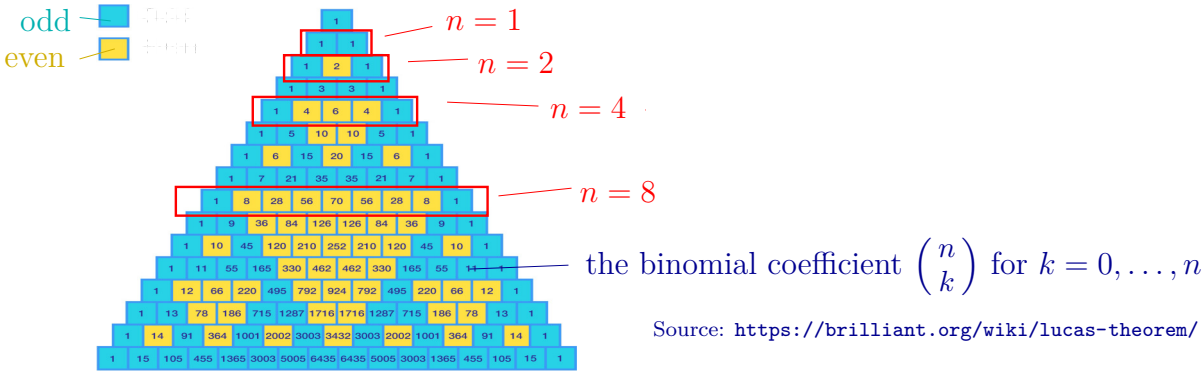
$$\mathbb{F}_2[\alpha,\beta]/(\alpha^n,\beta^n) \;\to\; \mathrm{H}^*(\mathbb{R}\mathrm{P}^{n-1} \times \mathbb{R}\mathrm{P}^{n-1};\mathbb{F}_2)$$

is an isomorphism.

(3) In the polynomial ring $\mathbb{F}_2[\alpha,\beta]/(\alpha^n,\beta^n)$ the $n-1$ monomials $\alpha\beta^{n-1}, \alpha^2\beta^{n-2}, \ldots, \alpha^{n-1}\beta$ are linearly independent over the field $\mathbb{F}_2$.

It follows from these three observations that for $k = 1, \ldots, n-1$ we have $\binom{n}{k} \equiv 0 \bmod 2$. Thus it remains to prove the following purely algebraic claim:

Claim 5. Let $n \in \mathbb{N}$. If $\binom{n}{k} \equiv 0 \bmod 2$ for $k = 1, \ldots, n-1$, then n is a power of 2..



Proof. We can and will write n as a sum $n = n_1 + \cdots + n_r$ where each n_i is a power of 2 and with $n_1 < \cdots < n_r$. In the polynomial ring $\mathbb{F}_2[x]$ we obtain that

$$1 + x^n \overset{\downarrow}{=} \sum_{k=0}^{n} \binom{n}{k} \cdot x^k \overset{\downarrow}{=} (1+x)^n \overset{\downarrow}{=} (1+x)^{n_1} \cdot \ldots \cdot (1+x)^{n_r}$$
$$= \underset{\uparrow}{(1+x^{n_1}) \cdot \ldots \cdot (1+x^{n_r})} \underset{\uparrow}{=} 1 + x^{n_1} + \text{higher powers}.$$

follows from our hypothesis

by the generalized binomial theorem

since $n = n_1 + \cdots + n_r$

since each n_i is a power of 2 and since in $\mathbb{F}_2[x]$ we have $(a+b)^2 = a^2 + b^2$

since $n_1 < \cdots < n_r$

But this is only possible if $n = n_1$, i.e. if n itself is a power of 2. ∎

In fact, with more fancy topological methods one can actually strengthen the previous theorem to obtain the following theorem. Unfortunately the proof is well beyond the scope of these modest lecture notes.

Theorem 82.8. (Division Algebra–Dimensions Theorem) Every finite-dimensional division algebra over $\mathbb{R}$ is of dimension 1,2, 4 or 8.

Proof. The theorem was first proved by Michel Kervaire [**Ker58**] and John Milnor [**Mil58b**, Corollary 1], building on the "Bott Periodicity Theorem" proved by Raoul Bott [**Bot57, Bot59**]. Full details for the proof can also be found in [**Hat**, Theorem 2.16]. ■

CHAPTER 83

Quaternionic and octonionic projective spaces

In Chapter 82 we just learned about the quaternions $\mathbb{H}$ and the rather mysterious octonions $\mathbb{O}$, which can both be viewed as generalizations of the complex numbers.

In this section we introduce the $4n$-dimensional quaternionic projective spaces $\mathbb{H}\mathrm{P}^n$ and we introduce the significantly more subtle 16-dimensional octonionic projective space $\mathbb{O}\mathrm{P}^2$. These are closed orientable smooth manifolds which now get added to our ever expanding list of amusing topological spaces. Note though that in the octonionic setting we fail at producing higher dimensional projective spaces.

In this chapter we will also calculate the cohomology rings of the quaternionic and octonionic projective spaces and we will see that these are again polynomial rings. These examples raise several interesting questions which we discuss at the end of the chapter and which we will partially answer later in the lecture notes. The calculation of the cup product on $\mathbb{H}\mathrm{P}^2$ and $\mathbb{O}\mathrm{P}^2$ will also be useful in the next chapter where we will use these calculations to show that the homotopy group $\pi_7(S^4)$ and $\pi_{15}(S^8)$ are non-zero.

83.1. **Quaternionic projective spaces.** On page 400, given $n \in \mathbb{N}_0$, we introduced the real projective space $\mathbb{R}\mathrm{P}^n$ and the complex projective spaces $\mathbb{C}\mathrm{P}^n$. In the following we will introduce the quaternionic projective spaces $\mathbb{H}\mathrm{P}^n$. Since the quaternions $\mathbb{H}$ are non-commutative we better do the constructions carefully.

Lemma 83.1. (Quaternion Equivalence Relation Lemma) If we are given two points $v, w \in \mathbb{H}^{n+1} \setminus \{(0,\dots,0)\}$, then we write $v \sim w$ if there exists a non-zero $h \in \mathbb{H}$ with $h \cdot v = w$. This defines an equivalence relation on $\mathbb{H}^{n+1} \setminus \{(0,\dots,0)\}$.

Proof.

(1) It follows from the fact that $\mathbb{H}$ contains a multiplicatively neutral element that the relation $\sim$ is reflexive.

(2) We want to show that $\sim$ is symmetric. Thus let $v, w \in \mathbb{H}^{n+1} \setminus \{(0,\dots,0)\}$ with $v \sim w$. By definition this means that there exists a non-zero $h \in \mathbb{H}$ with $h \cdot v = w$. Since $\mathbb{H}$ is a division algebra we can consider the multiplicative inverse h^{-1} and since $\mathbb{H}$ is associative we see that $h^{-1} \cdot w = h^{-1} \cdot (h \cdot v) = (h^{-1} \cdot h) \cdot v = 1_{\mathbb{H}} \cdot v = v$. Thus we have shown that $w \sim v$.

(3) Finally suppose that $u \sim v$ and $v \sim w$. Thus there exist $g, h \in \mathbb{H} \setminus \{0\}$ with $g \cdot u = v$ and $h \cdot v = w$. Since $\mathbb{H}$ is associative we see that $(h \cdot g) \cdot u = h \cdot (g \cdot u) = h \cdot v = w$. ■

Now we can define the quaternionic projective spaces:

Definition. Let $n \in \mathbb{N}_0$.

(1) We denote by "$\sim$" the equivalence relation on $\mathbb{H}^{n+1} \setminus \{(0,\dots,0)\}$ from the Quaternion Equivalence Relation Lemma 83.1.

(2) For $(h_0, \ldots, h_n) \in \mathbb{H}^{n+1} \setminus \{0\}$ let $[h_0 : \ldots : h_n]$ be the corresponding equivalence class.
(3) Given $n \in \mathbb{N}$ we define the quaternionic projective space

$$\mathbb{H}\mathrm{P}^n \;:=\; (\mathbb{H}^{n+1} \setminus \{(0, \ldots, 0)\})/\sim .$$

As on page 137 we see that the maps $[z] \mapsto [(z, 0)]$ define embeddings $\mathbb{H}\mathrm{P}^n \to \mathbb{H}\mathrm{P}^{n+1}$. Sometimes, given $m \leq n$, we use these embeddings to view $\mathbb{H}\mathrm{P}^m$ as a subspace of $\mathbb{H}\mathrm{P}^n$.
(4) We can use the embeddings from (3) to form the corresponding direct colimit

$$\mathbb{H}\mathrm{P}^\infty \;:=\; \operatorname*{colim}_{\longrightarrow} \mathbb{H}\mathrm{P}^n.$$

We refer to $\mathbb{H}\mathrm{P}^\infty$ as the infinite-dimensional quaternionic projective space.[643]

The following lemma summarizes some key properties of quaternionic projective spaces:

Lemma 83.2. (Quaternion-Projective Space Lemma)
(1) The map

$$\begin{array}{rcl} \mathbb{H}\mathrm{P}^1 & \to & \mathbb{H} \cup \{\infty\} = \mathbb{R}^4 \cup \{\infty\} \overset{\text{by the Stereographic Projection Lemma 2.11}}{\underset{\text{i}}{=}} S^4 \\ [z_0 : z_1] & \mapsto & \begin{cases} z_0 \cdot z_1^{-1}, & \text{if } z_1 \neq 0, \\ \infty, & \text{if } [z_0 : z_1] = [1 : 0] \end{cases} \end{array}$$

is a homeomorphism.
(2) For each $n \in \mathbb{N}_0$ the quaternionic projective space $\mathbb{H}\mathrm{P}^n$ is a closed orientable $4n$-dimensional smooth manifold.
(3) (a) Let $n \in \mathbb{N}_0$. The filtration $\varnothing \subset \mathbb{H}\mathrm{P}^0 \subset \mathbb{H}\mathrm{P}^1 \subset \cdots \subset \mathbb{H}\mathrm{P}^n$ is a CW-structure which has exactly one cell in the dimensions $0, 4, 8, \ldots, 4n$ and which has no other cells.
(b) We can equip $\mathbb{H}\mathrm{P}^\infty$ with a CW-structure that has exactly one cell in the dimensions $0, 4, 8, \ldots$ and which has no other cells.

Sketch of proof. We start out with the following claim:

Claim. Let $n \in \mathbb{N}_0$. It follows from the Quaternion Lemma 82.5 (5)–(7) that quaternion multiplication turns the sphere $S^3 = \{v \in \mathbb{H} \,|\, |z| = 1\}$ into a group and that this group acts continuously on $S^{4n+3} \subset \mathbb{H}^{4n+4}$ by left multiplication.
(a) We have a homeomorphism $S^{4n+3}/S^3 \to \mathbb{H}\mathrm{P}^n$.
(b) $\mathbb{H}\mathrm{P}^n$ is compact and Hausdorff.

Proof. As in the Projective Space-Sphere Quotient Lemma 6.9 we see that the maps

$$\begin{array}{rcl} S^{4n+3}/S^3 & \to & \mathbb{H}\mathrm{P}^n \\ [z] & \mapsto & [z] \end{array} \qquad \text{and} \qquad \begin{array}{rcl} \mathbb{H}\mathrm{P}^n & \to & S^{4n+3}/S^3 \\ [z_1 : \ldots : z_{n+1}] & \mapsto & \left[\frac{(z_1, \ldots, z_{n+1})}{\sqrt{|z_1|^2 + \ldots + |z_{n+1}|^2}}\right] \end{array}$$

are continuous and that they are inverses of one another. This shows that $\mathbb{H}\mathrm{P}^n$ is homeomorphic to S^{4n+3}/S^3. As in the Projective Spaces-Compact+Hausdorff Proposition 6.10 one can now show that $\mathbb{H}\mathrm{P}^n \cong S^{4n+3}/S^3$ is compact and Hausdorff. ⊞

[643] It follows quite easily from the Direct Colimit–Action Lemma 25.10 that we have a homeomorphism $\operatorname*{colim}_{\longrightarrow} \mathbb{H}\mathrm{P}^n \cong (\mathbb{H}^\infty \setminus \{0\})/\sim$.

Now we turn to the actual statements of the lemma.

(1) Now that we know that $\mathbb{H}\mathrm{P}^1$ is compact we can mostly copy-paste the proof of the Projective Space-Avatar Lemma 6.11 (1) and we can show, using the Compact-Hausdorff Proposition 2.10 (3), that the given map $\mathbb{H}\mathrm{P}^1 \to \mathbb{H} \cup \{\infty\}$ is a homeomorphism.

(2) By the claim we already know that $\mathbb{H}\mathrm{P}^n \cong S^{4n+3}/S^3$ is compact and Hausdorff. Next note that one can easily verify that for $i = 0, \dots, n$ the map

$$\begin{array}{rcl} \Phi_i \colon \{[z_0 : \cdots : z_n] \in \mathbb{H}\mathrm{P}^n \mid z_i \neq 0\} & \to & \mathbb{H}^n =_{\mathrm{i}} \mathbb{R}^{4n} \\ [z_0 : \cdots : z_n] & \mapsto & \left(z_i^{-1} \cdot z_0, \dots, z_i^{-1} \cdot z_{i-1}, z_i^{-1} \cdot z_{i+1}, \dots, z_i^{-1} \cdot z_n\right) \end{array}$$

is a homeomorphism and that the transition maps are orientation-preserving diffeomorphisms. It follows from this discussion that $\mathbb{H}\mathrm{P}^n$ is a topological manifold, that the above maps define a smooth atlas and that the above atlas endows $\mathbb{H}\mathrm{P}^n$ with an orientation.[644]

(3) First let $n \in \mathbb{N}_0$. As in the Projective Space–Cell Attachment Lemma 29.2 we consider the continuous map

$$\begin{array}{rcl} \varphi \colon S^{4n-1} & \to & \mathbb{H}\mathrm{P}^{n-1} \\ (z_1, \dots, z_n) & \mapsto & [z_1 : \dots : z_n] \end{array}$$

and we consider the map

$$\begin{array}{rcl} \Theta \colon \mathbb{H}\mathrm{P}^{n-1} \cup_\varphi \overline{B}^{4n} & \to & \mathbb{H}\mathrm{P}^n \\ [P] & \mapsto & \begin{cases} [z_1 : \cdots : z_n : 0], & \text{if } P = [z_1 : \cdots : z_n] \in \mathbb{H}\mathrm{P}^{n-1}, \\ [P : \sqrt{1 - \|P\|^2}], & \text{if } P \in \overline{B}^{4n}. \end{cases} \end{array}$$

Using the fact that $\mathbb{H}\mathrm{P}^n$ is Hausdorff and using that $\mathbb{H}\mathrm{P}^{n-1}$ is compact we can show again that Θ is a homeomorphism.

The statements about the CW-structure of $\mathbb{H}\mathrm{P}^n$ and $\mathbb{H}\mathrm{P}^\infty$ now follow, as on page 410, from the above observations and the definitions. ∎

We conclude this section with the calculation of the cup product on quaternion projective spaces. The statement is of course, in light of the Cup Product-$\mathbb{C}\mathrm{P}^n$-Proposition 81.4, entirely predictable.

Proposition 83.3. (Cup Product-$\mathbb{H}\mathrm{P}^n$-Proposition)

(1) Given any $n \in \mathbb{N}_0$ there exists an isomorphism

$$\mathbb{Z}[z]/(z^{n+1}) \to (\mathrm{H}^*(\mathbb{H}\mathrm{P}^n; \mathbb{Z}), \cup)$$

of superalgebras, where z is an element of degree 4.

(2) There exists an isomorphism

$$\mathbb{Z}[z] \to (\mathrm{H}^*(\mathbb{H}\mathrm{P}^\infty; \mathbb{Z}), \cup)$$

of superalgebras, where z is an element of degree 4.

Proof.

(1) Let $n \in \mathbb{N}_0$. By the above Quaternion-Projective Space Lemma 83.2 (2) and (3a) we know that $\mathbb{H}\mathrm{P}^n$ is a $4n$-dimensional closed orientable smooth manifold which admits a CW-structure that has exactly one cell in the dimensions $0, 4, 8, \dots, 4n$ and such that for $k = 0, \dots, n$ the $(4k+1)$-skeleton of $\mathbb{H}\mathrm{P}^n$ is given by $\mathbb{H}\mathrm{P}^k$. Using these observations

[644] Note though that $\mathbb{H}\mathrm{P}^n$ does in general not have a complex structure, for example $\mathbb{H}\mathrm{P}^1 = S^4$ does not have a complex structure, see [**Ste51**, Corollary 41.20] and [**Hsi95**, p. 233].

the proof of the Cup Product-$\mathbb{C}\mathrm{P}^n$-Proposition 81.4 can be used, with very minor modifications, to prove the desired result.

(2) It should by now be clear to the reader how to prove (2). ■

83.2. **Octonionic projective spaces.** Since the octonions $\mathbb{O}$ are not associative, the definition of projective spaces on pages 400 and 1072 cannot be applied to the octonions. But there is some hope in low dimensions. More precisely, as we will see shortly, one can use the fact that octonions are alternative to construct smooth manifolds that have all the properties that one would naively expect $\mathbb{O}\mathrm{P}^1$ and $\mathbb{O}\mathrm{P}^2$ to have.

Definition.

(1) We equip[645]
$$T_2 := \mathbb{O}^2 \setminus \{(0,0)\}$$
with the equivalence relation[646] [647]
$$(x_1,y_1) \sim (x_2,y_2) \iff \begin{array}{l}\text{there exists a } \lambda \in \mathbb{R}_{>0} \text{ such that}\\ y_1\overline{y_1} = \lambda \cdot y_2\overline{y_2} \text{ and } x_1\overline{y_1} = \lambda \cdot x_2\overline{y_2}.\end{array}$$
We set
$$\mathbb{O}\mathrm{P}^1 := T_2/\sim$$
and we refer to $\mathbb{O}\mathrm{P}^1$ as the octonionic projective line.

(2) We equip
$$T_3 := \{(x,y,z) \in \mathbb{O}^3 \setminus \{(0,0,0)\} \mid 1,x,y,z \text{ generate an associative algebra}\}$$
with the equivalence relation[648]
$$(x_1,y_1,z_1) \sim (x_2,y_2,z_2) \iff \begin{array}{l}\text{there exists a } \lambda \in \mathbb{R}_{>0} \text{ such that}\\ x_1\overline{x_1} = \lambda\cdot x_2\overline{x_2}, \quad x_1\overline{y_1} = \lambda\cdot x_2\overline{y_2}, \quad x_1\overline{z_1} = \lambda\cdot x_2\overline{z_2},\\ y_1\overline{y_1} = \lambda\cdot y_2\overline{y_2}, \quad y_1\overline{z_1} = \lambda\cdot y_2\overline{z_2}, \quad z_1\overline{z_1} = \lambda\cdot z_2\overline{z_2}.\end{array}$$
We set[649]
$$\mathbb{O}\mathrm{P}^2 := T_3/\sim$$
and we refer to $\mathbb{O}\mathrm{P}^2$ as the octonionic projective plane. In the literature the octonionic projective plane is often also called the Cayley projective plane.

(3) We denote the equivalence class of a point $(x,y) \in T_2$ by $[x:y]$ and we denote the equivalence class of a point $(x,y,z) \in T_3$ by $[x:y:z]$.

The following proposition hopefully convinces the reader that the octonion projective line $\mathbb{O}\mathrm{P}^1$ and the octonion projective plane $\mathbb{O}\mathrm{P}^2$ deserve their names.

---

[645] What happens if we use these definitions of $\mathbb{O}\mathrm{P}^1$ and $\mathbb{O}\mathrm{P}^2$, but if we replace the octonions by our more familiar friends $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$?

[646] Note that this is obviously an equivalence relation.

[647] The definition of the equivalence relation is a priori asymmetric in the first and the second variable. But this asymmetry is an illusion, in fact if $y_1\overline{y_1} = \lambda y_2\overline{y_2}$ and $x_1\overline{y_1} = \lambda x_2\overline{y_2}$, then using the Alternative Algebra Theorem 82.1 and the Octonion Lemma 82.6 (2) and (4) one can argue that the following calculation makes sense:
$$x_1\overline{x}_1 = x_1\underbrace{((\overline{y}_1\overline{y}_1^{-1})(y_1^{-1}y_1))}_{=1}\overline{x}_1 = x_1\overline{y}_1(y_1\overline{y}_1)^{-1}\overline{x_1\overline{y}_1} = \lambda x_2\overline{y}_2(\lambda y_2\overline{y}_2)^{-1}\overline{\lambda x_2\overline{y}_2} = \lambda x_2\overline{y}_2\overline{y}_2^{-1}y_2^{-1}y_2\overline{x}_2 = \lambda x_2\overline{x}_2.$$
We will provide the details in Exercise **??**.

[648] As for $\sim$ on T_2, this is basically by definition an equivalence relation.

[649] We equip $\mathbb{O} = \mathbb{R}^8$ with the usual topology, equip $\mathbb{O}^3$ with the product topology, we equip T_3 with the subspace topology and we equip $\mathbb{O}\mathrm{P}^2$ with the quotient topology.

Proposition 83.4. (Octonion Plane Proposition)

(1) The map

$$\begin{array}{rcl} \mathbb{O}\mathrm{P}^1 & \to & \mathbb{O}\cup\{\infty\} = \mathbb{R}^8\cup\{\infty\} \overset{\text{by the Stereographic Projection Lemma 2.11}}{\underset{\downarrow}{=_{\mathrm{i}}}} S^8 \\ [x:y] & \mapsto & \begin{cases} x\cdot y^{-1}, & \text{if } y\neq 0, \\ \infty, & \text{if } [x:y]=[1:0] \end{cases} \end{array}$$

is well-defined and it is a homeomorphism.

(2) The octonionic projective plane $\mathbb{O}\mathrm{P}^2$ is a closed orientable 16-dimensional smooth manifold.

(3) (a) The map

$$\begin{array}{rcl} \mathbb{O}\mathrm{P}^1 & \to & \mathbb{O}\mathrm{P}^2 \\ [x:y] & \mapsto & [x:y:0] \end{array}$$

is well-defined and it is an embedding. We use it to view $\mathbb{O}\mathrm{P}^1$ as a subspace of $\mathbb{O}\mathrm{P}^2$.

(b) The octonionic projective plane $\mathbb{O}\mathrm{P}^2$ admits a CW-structure with precisely one cell in dimensions 0, 8 and 16, and which has no other cells. Furthermore this CW-structure can be chosen such that the 8-skeleton is precisely $\mathbb{O}\mathrm{P}^1$.

Proof. We start out with the following claim:

Claim 1. The octonionic projective line $\mathbb{O}\mathrm{P}^1$ and the octonionic projective plane $\mathbb{O}\mathrm{P}^2$ are both compact and Hausdorff.

Proof. It is straightforward to define equivalence relations on S^{15} and on S^{23} such that the octonionic projective line $\mathbb{O}\mathrm{P}^1$ is homeomorphic to $S^{15}/\sim$ and such that the octonionic projective plane $\mathbb{O}\mathrm{P}^2$ is homeomorphic to $S^{23}/\sim$. It follows from the Compact Image Lemma 2.8 that $\mathbb{O}\mathrm{P}^1$ and $\mathbb{O}\mathrm{P}^2$ are both compact. Furthermore, using the Quotient Topology-Properties Proposition 6.2 (3) one can show fairly easily that $\mathbb{O}\mathrm{P}^1$ and $\mathbb{O}\mathrm{P}^2$ are both Hausdorff. We leave the details to the reader.[650] ⊞

Now we turn to the proofs of the actual statements we are interested in:

(1) First we need to show that the given map $\Theta\colon \mathbb{O}\mathrm{P}^1 \to \mathbb{O}\cup\{\infty\}$ is well-defined. Thus suppose that we are given two points (x_1,y_1) and (x_2,y_2) in $\mathbb{O}^2\setminus\{(0,0)\}$ that are equivalent. One can verify easily that $y_1\neq 0$ if and only if $y_2\neq 0$. By definition of Θ we only need to worry about the case that $y_1,y_2\neq 0$. In this case we do indeed have

$$x_1\cdot y_1^{-1} \underset{\uparrow}{=} x_1\cdot\frac{1}{|y_1|^2}\cdot\overline{y}_1 \overset{\text{by definition of } |y_1|}{\underset{\downarrow}{=}} \frac{1}{y_1\cdot\overline{y}_1}\cdot(x_1\cdot\overline{y}_1) \underset{\uparrow}{=} \frac{1}{y_2\cdot\overline{y}_2}\cdot(x_2\cdot\overline{y}_2) \overset{\text{same steps backwards}}{\underset{\downarrow}{=}} \ldots = x_2\cdot y_2^{-1}.$$

by the Octonion Lemma 82.6 (6) — by definition of "$\sim$" we have $y_1\cdot\overline{y_1}=\lambda\cdot y_2\cdot\overline{y_2}$ and $x_1\cdot\overline{y_1}=\lambda\cdot x_2\cdot\overline{y_2}$

We have thus shown that Θ is well-defined. It is clear that the map Θ is a surjection. We leave it to the reader to show that Θ is also an injection. Next note that the argument in the proof of the Projective Space-Avatar Lemma 6.11 (2) shows that Θ is continuous. We showed in the above claim that $\mathbb{O}\mathrm{P}^1$ is compact. Evidently $\mathbb{O}\cup\{\infty\} =_{\mathrm{i}} S^8$ is Hausdorff. In summary we obtain from the Compact-Hausdorff Proposition 2.10 (3) that $\Theta\colon \mathbb{O}\mathrm{P}^1 \to \mathbb{O}\cup\{\infty\}$ is a homeomorphism.

[650] An arguably more elegant proof that $\mathbb{O}\mathrm{P}^2$ is Hausdorff is given in [**Lac12**].

(2) We start out with the following claim:

Claim 2. The map
$$\begin{array}{rccc} \Phi_z\colon & \{[x:y:z]\in\mathbb{O}\mathrm{P}^2 \,|\, z\neq 0\} & \to & \mathbb{O}^2 \\ & [x:y:z] & \mapsto & (xz^{-1}, yz^{-1}) \end{array}$$
is a homeomorphism between an open subset of $\mathbb{O}\mathrm{P}^2$ and $\mathbb{O}^2=\mathbb{R}^{16}$.

Proof. We prove the claim in three mostly elementary steps.

(a) Basically the same argument as in (1) shows that the map Φ_z is well-defined.

(b) It follows easily from the definition of the quotient topology that the domain of the map Φ_z is indeed an open subset of $\mathbb{O}\mathrm{P}^2$.

(c) We consider the map
$$\begin{array}{rccc} \Psi_z\colon & \mathbb{O}^2 & \to & \{[x:y:z]\in\mathbb{O}\mathrm{P}^2 \,|\, z\neq 0\} \\ & (x,y) & \mapsto & [x:y:1]. \end{array}$$
We first have to argue that this map makes sense, namely that for any $x,y\in\mathbb{O}$ we have $(x,y,1)\in T_3$. To do so, recall that by the Octonion Lemma 82.6 (2) we know that $\mathbb{O}$ is alternative. By the Alternative Algebra Theorem 82.1 this implies the algebra generated by $1, x$ and y is associative. In other words, we have indeed $(x,y,1)\in T_3$.
Furthermore note that it follows from the Topological-Quotient Proposition 6.1 (1b) that Ψ_z is continuous. It is clear that $\Phi_z\circ\Psi_z=\mathrm{id}$. Finally note that given $[x:y:z]\in\mathbb{O}\mathrm{P}^2$ with $z\neq 0$ we have
$$\Psi_z(\Phi_z([x:y:z])) = [x\cdot z^{-1}:y\cdot z^{-1}:1] \underset{\uparrow}{=} [x:y:z]$$
follows with $\lambda=|z|^2$ and from the observation that the hypothesis $(x,y,z)\in T_3$ together with the Octonion Lemma 82.6 (4) (which implies that we always have $w+\overline{w}\in\mathbb{R}\cdot 1_{\mathbb{O}}$) implies that the elements $x,y,z,\overline{x},\overline{y},\overline{z}$ generate an associative algebra

We have thus shown that Ψ_z is an inverse to Φ_z. This implies that Φ_z is indeed a homeomorphism. ⊞

Similarly we define charts Φ_x and Φ_y. It is straightforward to verify that the transition maps between the three charts are smooth and orientation-preserving. In particular these three charts define a smooth atlas for $\mathbb{O}\mathrm{P}^2$. In Claim 1 we showed that $\mathbb{O}\mathrm{P}^2$ is compact and Hausdorff. The combination of all of the above shows that $\mathbb{O}\mathrm{P}^2$ is indeed a closed 16-dimensional smooth manifold. Finally, since the transition maps between the three charts are orientation-preserving we obtain that $\mathbb{O}\mathrm{P}^2$ is actually orientable.

(3) (a) As in the proof of (2) we use the non-trivial fact that for any $x,y\in\mathbb{O}$ we have $(1,x,y)\in T_3$. This implies that the map
$$\begin{array}{ccc} \mathbb{O}\mathrm{P}^1 & \to & \mathbb{O}\mathrm{P}^2 \\ [x:y] & \mapsto & [x:y:0] \end{array}$$
is well-defined. It follows easily from the Twice Quotient Lemma 6.3 that the map is continuous. It follows basically immediately from the discussion in Footnote 647 that the given map $\mathbb{O}\mathrm{P}^1\to\mathbb{O}\mathrm{P}^2$ is an injection. By Claim 1 we know that $\mathbb{O}\mathrm{P}^1$ is compact and that $\mathbb{O}\mathrm{P}^2$ is Hausdorff. Thus it follows from the Compact-Hausdorff Proposition 2.10 (2) that the given map $\mathbb{O}\mathrm{P}^1\to\mathbb{O}\mathrm{P}^2$ is an embedding.

(b) We view $\mathbb{O}\mathrm{P}^1 = S^8$ as a CW-complex with one 0-cell and one 8-cell in the obvious way. Next we consider the attaching map

$$\begin{array}{rcl} \varphi\colon S^{15} = \{(x_1, x_2) \in \mathbb{O}^2 \,|\, |x_1|^2 + |x_2|^2 = 1\} & \to & \mathbb{O}\mathrm{P}^1 \\ (x_1, x_2) & \mapsto & [x_1 : x_2]. \end{array}$$

Furthermore we consider the map

$$\begin{array}{rcl} \Theta\colon \mathbb{O}\mathrm{P}^1 \cup_\varphi \overline{B}^{16} & \to & \mathbb{O}\mathrm{P}^2 \\ [P] & \mapsto & \begin{cases} [x_1 : x_2 : 0], & \text{if } P = [x_1 : x_2] \in \mathbb{O}\mathrm{P}^1, \\ [P : 1 - \|P\|], & \text{if } P \in \overline{B}^{16}. \end{cases} \end{array}$$

As before we note that it follows from the Octonion Lemma 82.6 (2) together with the Alternative Algebra Theorem 82.1 that the map actually does take values in $\mathbb{O}\mathrm{P}^2$. We leave the slightly tedious task of verifying that the map is well-defined and a bijection to the reader. It follows easily from Claim 1 that the left-hand side is compact and it follows from Claim 1 that the right-hand side is Hausdorff. Thus it follows from the Compact-Hausdorff Proposition 2.10 (3) that the map is a homeomorphism. But this implies that $\mathbb{O}\mathrm{P}^2$ admits a CW-structure with precisely three cells, one in dimensions 0, 8 and 16. ■

We continue with the following predictable analogue of the Cup Product-$\mathbb{C}\mathrm{P}^n$-Proposition 81.4 and the Cup Product-$\mathbb{H}\mathrm{P}^n$-Proposition 83.3:

Proposition 83.5. (Cup Product-$\mathbb{O}\mathrm{P}^2$-Proposition) There exists an isomorphism

$$\mathbb{Z}[z]/(z^3) \;\to\; (\mathrm{H}^*(\mathbb{O}\mathrm{P}^2; \mathbb{Z}), \cup)$$

of superalgebras, where z is an element of degree 8.

Proof. In the Octonion Plane Proposition 83.4 (2) we showed that the octonionic projective plane $\mathbb{O}\mathrm{P}^2$ is a closed orientable 16-dimensional smooth manifold which admits a CW-structure with precisely one cell in dimensions 0, 8 and 16, and which has no other cells. Once again the proof of the Cup Product-$\mathbb{C}\mathrm{P}^n$-Proposition 81.4 clinches the deal. ■

CHAPTER 84

The Hopf maps and higher homotopy groups

In previous chapters we have seen that the calculation of the cup product, in particular on projective spaces, allowed us to answer many questions on topological spaces that we could not deal with before. In this chapter we want to use the cup product to address the following question that we formulated a long time ago:

Question 28.12. Does the Hopf map $H\colon S^3 \to S^2$ represent a non-trivial element in the homotopy group $\pi_3(S^2)$?

In this chapter we will use the calculation of the cup product of $\mathbb{C}\mathrm{P}^2$ to give an affirmative answer to this question. Furthermore we will use the quaternions and octonions to define higher dimensional analogues of the Hopf map which, as we will see, also represent non-trivial elements in homotopy groups of spheres.

First we recall the definition of the Hopf map from page 389:

Definition. We refer to the map[651]

see page 139

$$\begin{array}{rcl} H\colon S^3 = \{(z_1,z_2) \in \mathbb{C}^2 \,|\, |z_1|^2 + |z_2|^2 = 1\} & \to & S^2 \overset{\downarrow}{=}_{\mathrm{i}} \mathbb{C}\mathrm{P}^1 \\ (z_1,z_2) & \mapsto & [z_1 : z_2] \end{array}$$

as the Hopf map.

Since the Hopf map is so much fun we will introduce two higher-dimensional analogues of the Hopf map. More precisely, we will repeat the trick with $\mathbb{C}$ replaced by the quaternions $\mathbb{H}$ and the octonions $\mathbb{O}$. First we point out that we can use the homeomorphisms from the Quaternion-Projective Space Lemma 83.2 (1) and the Octonion Plane Proposition 83.4 (1) to make the identifications $\mathbb{H}\mathrm{P}^1 = S^4$ and $\mathbb{O}\mathrm{P}^1 = S^8$. Now we can introduce the quaternionic and octonionic Hopf maps.

Definition. We refer to

$$\begin{array}{rcl} H_{\mathbb{H}}\colon S^7 = \{(z_1,z_2) \in \mathbb{H}^2 \,|\, |z_1|^2 + |z_2|^2 = 1\} & \to & S^4 =_{\mathrm{i}} \mathbb{H}\mathrm{P}^1 \\ (z_1,z_2) & \mapsto & [z_1 : z_2] \end{array}$$

as the quaternionic Hopf map and we refer to

$$\begin{array}{rcl} H_{\mathbb{O}}\colon S^{15} = \{(z_1,z_2) \in \mathbb{O}^2 \,|\, |z_1|^2 + |z_2|^2 = 1\} & \to & S^8 =_{\mathrm{i}} \mathbb{O}\mathrm{P}^1 \\ (z_1,z_2) & \mapsto & [z_1 : z_2] \end{array}$$

as the octonionic Hopf map.

The idea for answering Question 28.12 is to translate this question about maps to a question about topological spaces for which we can then bring our new invariants to bear. To carry out this approach we need the following notions:

[651] See also the figure on page 390 for an illustration of the map.

Definition.

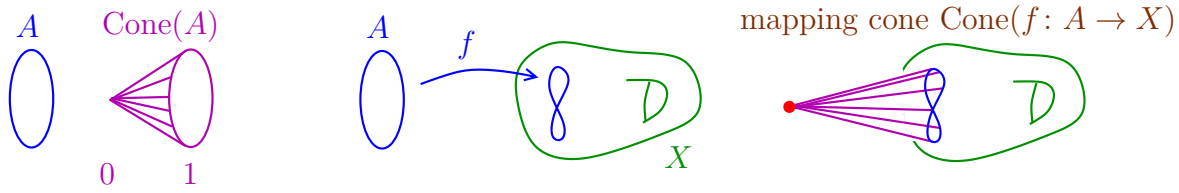
(1) The cone of a topological space A is defined as the topological space

$$\mathrm{Cone}(A) \;:=\; (A \times [0,1])/(A \times \{0\}).$$

(2) The mapping cone of a continuous map $f\colon A \to X$ between topological spaces as

$$\mathrm{Cone}(f\colon A \to X) \;:=\; (\mathrm{Cone}(A) \cup X)/\sim$$

where $(a,1) \sim f(a)$ for all $a \in A$.



Example. It follows easily from the Compact-Hausdorff Proposition 2.10 (3) and the Topological-Quotient Proposition 6.1 (1b) that the map

$$\begin{array}{rcl} \mathrm{Cone}(S^{n-1}) & \to & \overline{B}^n \\ [(P,t)] & \mapsto & t \cdot P \end{array}$$

is a homeomorphism. If X is a topological space and if $f\colon S^{n-1} \to X$ is a continuous map, then we can use the above homeomorphism to get a homeomorphism

$$\mathrm{Cone}(f\colon S^{n-1} \to X) \;=\; \overline{B}^n \cup_f X.$$

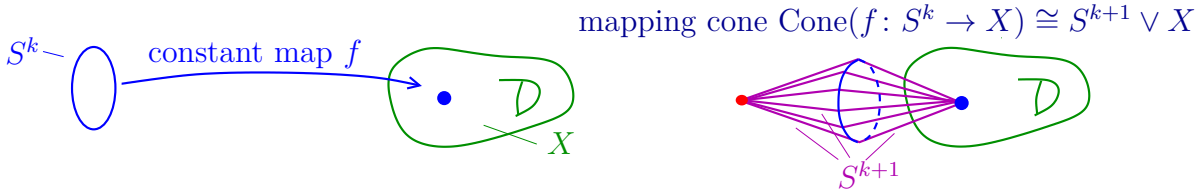
It follows immediately from this observation that any finite CW-complex can be viewed as an iterated mapping cone (where we start out with the empty topological space).

The next lemma gives us another elementary but oddly useful example.

Lemma 84.1. (Mapping Cone-of-Constant Map Lemma) If X is a topological space and if $f\colon S^k \to X$ is a constant map, then there exists a homeomorphism

$$\mathrm{Cone}(f\colon S^k \to X) \;\xrightarrow{\cong}\; S^{k+1} \vee X,$$

where we perform the wedge for suitable points in S^{k+1} and X.



Proof. The lemma is a consequence of the general observation that if $f\colon A \to X$ is a constant map between topological spaces, then there exists a homeomorphism

$$\mathrm{Cone}(f\colon A \to X) \;\xrightarrow{\cong}\; \Sigma(A) \vee X.$$

The current lemma follows from this general fact and the observation that the suspension of S^k is homeomorphic to S^{k+1}. ∎

The following lemma gives particularly interesting examples of mapping cones.

Lemma 84.2. (Mapping Cone-of-Hopf Map Lemma) There exist homeomorphisms

$$\begin{array}{lrcl} (1) & \mathrm{Cone}(H\colon S^3 \to S^2) & \cong & \mathbb{C}\mathrm{P}^2 \\ (2) & \mathrm{Cone}(H_{\mathbb{H}}\colon S^7 \to S^4) & \cong & \mathbb{H}\mathrm{P}^2 \\ (3) & \mathrm{Cone}(H_{\mathbb{O}}\colon S^{15} \to S^8) & \cong & \mathbb{O}\mathrm{P}^2. \end{array}$$

Proof.

(1) As we discussed in the Projective Space-Cell Attachment Lemma 29.2 on page 410 we can view $\mathbb{C}\mathrm{P}^2$ as a CW-complex that is obtained from the CW-complex $\mathbb{C}\mathrm{P}^1 =_{\mathrm{i}} S^2$ by attaching a 4-cell such that the attaching map $S^3 \to \mathbb{C}\mathrm{P}^1$ of the 4-cell is precisely the Hopf map. This observation, together with the discussion on page 1079, already implies the statement.For completeness' sake we give the argument more explicitly and in more detail. We consider the maps[652]

$$\begin{array}{rcl} f\colon \mathrm{Cone}(S^3) & \to & \mathbb{C}\mathrm{P}^2 \\ [(z_0,z_1),r] & \mapsto & [r\cdot z_0 : r\cdot z_1 : 1-r] \end{array} \quad \text{and} \quad \begin{array}{rcl} g\colon \mathbb{C}\mathrm{P}^1 & \to & \mathbb{C}\mathrm{P}^2 \\ [z_0:z_1] & \mapsto & [z_0:z_1:0]. \end{array}$$

It follows immediately from the definitions together with the Topological-Quotient Proposition 6.1 (1b) that these two maps define a continuous map

$$\underbrace{\mathrm{Cone}(H\colon S^3 \to S^2)}_{=(\mathrm{Cone}(S^3)\cup\mathbb{C}\mathrm{P}^1)/\sim} \to \mathbb{C}\mathrm{P}^2.$$

One can easily verify that this map is a bijection. Furthermore, using the Compact-Hausdorff Proposition 2.10 (3) and the Projective Spaces-Compact+Hausdorff Proposition 6.10 one sees that the map is in fact a homeomorphism.

(2) In the proof of Quaternion-Projective Space Lemma 83.2 we showed that $\mathbb{H}\mathrm{P}^2$ is homeomorphic to the result of attaching an 8-cell to $\mathbb{H}\mathrm{P}^1 = S^4$ via the quaternion Hopf map $H_{\mathbb{H}}\colon S^7 \to S^4$. With this observation the proof of (2) is basically identical to the proof of (1).

(3) Implicitly we mostly proved this statement in the proof of the Octonion Plane Proposition 83.4. Alternatively we also refer to [**Lac12**, p. 4] for details. ■

Now we can finally answer Question 28.12 in the affirmative. More precisely we have the following theorem:

Theorem 84.3. (Hopf Map–Homotopy Theorem)[653]

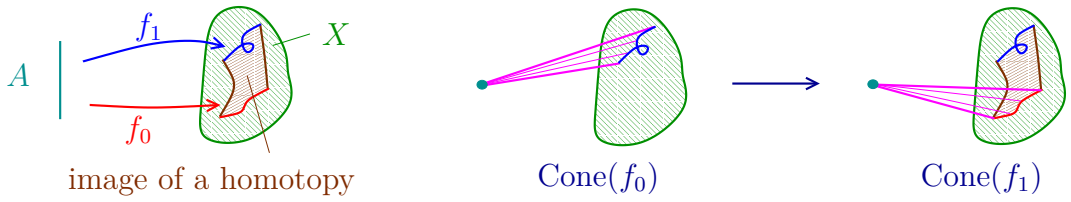
(1) The Hopf map $H\colon S^3 \to S^2$ represents a non-trivial element in $\pi_3(S^2)$, i.e. we have $[H] \neq 0 \in \pi_3(S^2)$.
(2) We have $[H_{\mathbb{H}}] \neq 0 \in \pi_7(S^4)$.
(3) We have $[H_{\mathbb{O}}] \neq 0 \in \pi_{15}(S^8)$.

The proof of the Hopf Map–Homotopy Theorem 84.3 builds on the following proposition.

Proposition 84.4. (Mapping Cone Homotopy Equivalent Proposition) Suppose we are given continuous maps $f_0, f_1\colon A \to X$ between topological spaces. If f_0, f_1 are homotopic, then $\mathrm{Cone}(f_0)$ and $\mathrm{Cone}(f_1)$ are homotopy equivalent.

[652] It follows from the Topological-Quotient Proposition 6.1 (1b) and the Topological-Quotient Proposition 6.1 (1a) that these two maps are continuous.

[653] The statement regarding the "classical" Hopf map was first proved by Heinz Hopf [**Hop31**] in 1931.



Proof of Proposition 84.4. We outsource the proof of this utterly believable statement to [**Fri24**]. Alternatively we leave it as an amusing exercise to the reader to provide a proof. ∎

Proof of the Hopf Map–Homotopy Theorem 84.3. First we prove (1). Recall that for two topological spaces X and Y we write $X \simeq Y$ if they are homotopy equivalent. We now see that

$$\begin{array}{ccccccc} \text{Cone(Hopf map } H) & \underset{\uparrow}{\cong} & \mathbb{C}\mathrm{P}^2 & \underset{\downarrow}{\not\simeq} & S^2 \vee S^4 & \underset{\uparrow}{\cong} & \text{Cone(constant map).} \\ \text{by the Mapping Cone-of-Hopf Map Lemma 84.2 (1)} & & & \text{by the Doppelgänger Corollary 81.5} & & \text{by the Mapping Cone-of-Constant Map Lemma 84.1} & \end{array}$$

But by the Mapping Cone Homotopy Equivalent Proposition 84.4 this means that the Hopf map is not homotopic to the constant map. In particular the Hopf map $H\colon S^3 \to S^2$ represents a non-trivial element in the homotopy group $\pi_3(S^2)$.

The proof of the other two statements is quite similar. Again we use the Mapping Cone-of-Hopf Map Lemma 84.2 and we use the calculations of the cup products of $\mathbb{H}\mathrm{P}^2$ and $\mathbb{O}\mathrm{P}^2$, see the Cup Product-$\mathbb{H}\mathrm{P}^n$-Proposition 83.3 and the Cup Product-$\mathbb{O}\mathrm{P}^2$-Proposition 83.5, to conclude that the mapping cones of the quaternionic and octonionic Hopf maps are not homotopy equivalent to a wedge of spheres. ∎

84.1. **The Hopf invariant.** It is natural to ask how far we can push the methods of the Hopf Map–Homotopy Theorem 84.3 to gain more insight into homotopy groups of spheres. In this section, given $n \geq 2$ we will introduce the Hopf invariant

$$\mathrm{Hopf}\colon \pi_{2n-1}(S^n) \;\to\; \mathbb{Z}$$

which satisfies $\mathrm{Hopf}(H) = \pm 1$, $\mathrm{Hopf}(H_{\mathbb{H}}) = \pm 1$ and $\mathrm{Hopf}(H_{\mathbb{O}}) = \pm 1$. Furthermore we will show that the Hopf invariant is in fact a homomorphism.

The idea behind the definition of the Hopf invariant is straightforward. Namely, given $n \in \mathbb{N}_{\geq 2}$ and given a continuous map $f\colon S^{2n-1} \to S^n$ we consider again the corresponding mapping cone $\mathrm{Cone}(f\colon S^{2n-1} \to S^n)$. We will see in the Mapping Cone-of-Spheres-H*-Lemma 84.6 that it follows from the fact that $n \geq 2$ that we have $\mathrm{H}^n(\mathrm{Cone}(f);\mathbb{Z}) \cong \mathbb{Z}$ and $\mathrm{H}^{2n}(\mathrm{Cone}(f);\mathbb{Z}) \cong \mathbb{Z}$. We will equip $\mathrm{H}^{2n}(\mathrm{Cone}(f);\mathbb{Z})$ with a "preferred generator". We pick a generator $\varphi \in \mathrm{H}^n(\mathrm{Cone}(f);\mathbb{Z})$ and we define $\mathrm{Hopf}(f)$ to be the unique integer that satisfies

$$\varphi \cup \varphi \;=\; \mathrm{Hopf}(f) \cdot \text{preferred generator} \;\in\; \mathrm{H}^{2n}(\mathrm{Cone}(f);\mathbb{Z}).$$

To carry out this program we need to carefully study the cohomology groups of a mapping cone. Recall that in the Mapping Cone-H$_*$-Lemma **??** we saw that the homology groups of a mapping cone fit naturally into a long exact sequence of homology groups. The following lemma shows that the analogous statement also holds for the cohomology groups of a mapping cone.

Lemma 84.5. (Mapping Cone-H*-Lemma) Let G be an abelian group.

(1) Let $f\colon A \to X$ be a continuous map between topological spaces.
   (a) We denote by $j(f)\colon X \to \mathrm{Cone}(f) = (\mathrm{Cone}(A) \sqcup X)/\sim$ the obvious inclusion map.
   (b) We denote by U the image of $A \times [0, \frac{3}{4})$ in $\mathrm{Cone}(f)$ and we denote by V the complement of the image of $A \times [0, \frac{1}{4}]$. (See also the figure below.) We consider the following diagram:

$$\begin{array}{ccccccc} \ldots \longrightarrow & \mathrm{H}^{k-1}(U \cap V; G) & \xrightarrow{\delta_k} & \mathrm{H}^k(\mathrm{Cone}(f); G) & \longrightarrow & \mathrm{H}^k(U; G) \oplus \mathrm{H}^k(V; G) & \longrightarrow \ldots \\ & \downarrow \cong & \nearrow_{=:\delta_{k-1}(f)} & & & & \\ & \mathrm{H}^{k-1}(A; G). & & & & & \end{array}$$

   Here the top sequence is the long exact sequence arising from the Mayer–Vietoris-H*-Theorem 69.5 (1)[654] and the vertical map comes from the natural isomorphisms $\mathrm{H}^{k-1}(U \cap V; G) \xleftarrow{\cong} \mathrm{H}^{k-1}(A \times \frac{1}{2}; G) \xrightarrow{\cong} \mathrm{H}^{k-1}(A; G)$. From the above diagram we obtain the map

$$\delta_{k-1}(f)\colon \mathrm{H}^{k-1}(A; G) \;\to\; \mathrm{H}^k(\mathrm{Cone}(f); G).$$

   With this notation the following sequence is exact

$$\ldots \longrightarrow \widetilde{\mathrm{H}}^k(\mathrm{Cone}(f); G) \xrightarrow{j(f)^*} \mathrm{H}^k(X; G) \xrightarrow{f^*} \mathrm{H}^k(A; G) \xrightarrow{\delta_k(f)} \widetilde{\mathrm{H}}^{k+1}(\mathrm{Cone}(f); G) \longrightarrow \ldots$$

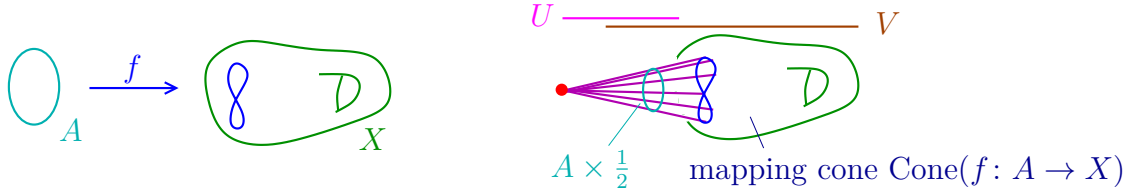
(2) The connecting homomorphism in (1) is natural. In particular given any commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\psi} \\ \widetilde{A} & \xrightarrow{\tilde{f}} & \widetilde{X} \end{array}$$

of continuous maps between topological spaces the following diagram commutes:

$$\begin{array}{ccccccccc} \ldots \longrightarrow & \widetilde{\mathrm{H}}^k(\mathrm{Cone}(f); G) & \xrightarrow{j(f)^*} & \mathrm{H}^k(X; G) & \xrightarrow{f^*} & \mathrm{H}^k(A; G) & \xrightarrow{\delta_k(f)} & \widetilde{\mathrm{H}}^{k+1}(\mathrm{Cone}(f); G) & \longrightarrow \ldots \\ & \uparrow & & \uparrow{\scriptstyle\psi^*} & & \uparrow{\scriptstyle\varphi^*} & & \uparrow & \\ \ldots \longrightarrow & \widetilde{\mathrm{H}}^k(\mathrm{Cone}(\tilde{f}); G) & \xrightarrow{j(\tilde{f})^*} & \mathrm{H}^k(\widetilde{X}; G) & \xrightarrow{\tilde{f}^*} & \mathrm{H}^k(\widetilde{A}; G) & \xrightarrow{\delta_k(\tilde{f})} & \widetilde{\mathrm{H}}^{k+1}(\mathrm{Cone}(\tilde{f}); G) & \longrightarrow \ldots \end{array}$$

   Here the vertical maps between the cohomology groups of the mapping cones are induced by the maps from the Mapping Cone–Functor Lemma **??** (3).



Proof. The proof is basically identical to the proof of the Mapping Cone-H$_*$-Lemma **??**. The meticulous reader who filled in the details of the proof of the Mapping Cone-H$_*$-Lemma **??** will have no troubles working out the details of the present proof. ∎

[654] Note that U and V are open subsets of $\mathrm{Cone}(f)$.

Notation. Throughout this section we use the notation that we had already introduced in the formulation of the Mapping Cone-H*-Lemma 84.5. More precisely, given a continuous map $f\colon A \to X$ between two topological spaces we denote by

$$j(f)\colon X \;\to\; \mathrm{Cone}(f) = (\mathrm{Cone}(A) \cup X)/\sim$$

the natural embedding and furthermore we denote by

$$\delta_{k-1}(f)\colon \mathrm{H}^{k-1}(A;\mathbb{Z}) \;\to\; \mathrm{H}^k(\mathrm{Cone}(f);\mathbb{Z})$$

the coboundary map in the long exact sequence in cohomology of the Mapping Cone-H*-Lemma 84.5.

In the following we consider a special case of the Mapping Cone-H*-Lemma 84.5 which will be particularly interesting for us:

Lemma 84.6. (Mapping Cone-of-Spheres-H*-Lemma) Let $f\colon S^m \to S^n$ be a continuous map. If $m \geq n+1$ and $n \geq 1$, then the maps

$$\mathrm{H}^n(\mathrm{Cone}(f);\mathbb{Z}) \xrightarrow{j(f)^*} \mathrm{H}^n(S^n;\mathbb{Z})$$

and

$$\mathrm{H}^m(S^m;\mathbb{Z}) \xrightarrow{\delta_m(f)} \mathrm{H}^{m+1}(\mathrm{Cone}(f);\mathbb{Z})$$

are isomorphisms and all other reduced cohomology groups of $\mathrm{Cone}(f)$ are zero. In particular we have isomorphisms

$$\mathrm{H}^k(\mathrm{Cone}(f\colon S^m \to S^n);\mathbb{Z}) \;\cong\; \begin{cases} \mathbb{Z}, & \text{if } k = 0, n, m+1, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. The lemma is an almost immediate consequence of the Mapping Cone-H*-Lemma 84.5 and the calculation of the cohomology groups of spheres, see page 880. ■

Now we are finally in a position to turn the idea from page 1081 into a proper definition.

Definition. Let $n \in \mathbb{N}_{\geq 2}$ and let $f\colon S^{2n-1} \to S^n$ be a continuous map. The Hopf invariant $\mathrm{Hopf}(f)$ is defined to be the unique integer that satisfies the following equality in $\mathrm{H}^{2n}(\mathrm{Cone}(f);\mathbb{Z}) \cong \mathbb{Z}$:[655] [656]

$$\underbrace{(j(f)^*)^{-1}([S^n]^*)}_{\substack{\text{generator of}\\ \mathrm{H}^n(\mathrm{Cone}(f);\mathbb{Z})\\ \text{by the above lemma}}} \cup \underbrace{(j(f)^*)^{-1}([S^n]^*)}_{\substack{\text{generator of}\\ \mathrm{H}^n(\mathrm{Cone}(f);\mathbb{Z})\\ \text{by the above lemma}}} = \mathrm{Hopf}(f) \cdot \underbrace{\delta_{2n-1}(f)([S^{2n-1}]^*)}_{\substack{\text{generator of}\\ \mathrm{H}^{2n}(\mathrm{Cone}(f);\mathbb{Z})\\ \text{by the above lemma}}}$$

Remark. Note that it follows immediately from the (anti-) commutativity of the cup product, see the Cup Product-Commutativity Proposition 72.8, that the Hopf invariant is zero for all odd $n \in \mathbb{N}$. □

We formulate our first example as a lemma:

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[655] Here we implicitly use that in the Mapping Cone-of-Spheres-H*-Lemma 84.6 we showed that $j(f)^*$ and $\delta_{2n-1}(f)$ are isomorphisms.

[656] Instead of the generator $(j(f)^*)^{-1}([S^n]^*) \in$ we could also have used the other generator of $\mathrm{H}^n(\mathrm{Cone}(f);\mathbb{Z})$, since the sign ambiguity goes away with the square.

Lemma 84.7. (Hopf Map-Hopf Invariant Lemma) The Hopf maps $H\colon S^3 \to S^2$, $H_{\mathbb{H}}\colon S^7 \to S^4$ and $H_{\mathbb{O}}\colon S^{15} \to S^8$ all have Hopf invariant ± 1.[657]

Proof. In the Mapping Cone-of-Hopf Map Lemma 84.2 we showed that the mapping cone $\mathrm{Cone}(H)$ is homeomorphic to $\mathbb{C}\mathrm{P}^2$. By the Mapping Cone-of-Spheres-H^*-Lemma 84.6 we know that $(j(H)^*)^{-1}([S^2]^*) \in \mathrm{H}^2(\mathbb{C}\mathrm{P}^2;\mathbb{Z})$ and $\delta_3(f)([S^3]^*) \in \mathrm{H}^4(\mathbb{C}\mathrm{P}^2;\mathbb{Z})$ are generators. It follows from the calculation of the cup product on $\mathbb{C}\mathrm{P}^2$, see the Cup Product-$\mathbb{C}\mathrm{P}^n$-Proposition 81.4, that $\mathrm{Hopf}(H) = -1$ or $\mathrm{Hopf}(H) = 1$. By the same argument it follows from the Cup Product-$\mathbb{H}\mathrm{P}^n$-Proposition 83.3 and the Cup Product-$\mathbb{O}\mathrm{P}^2$-Proposition 83.5 that the Hopf invariants of $H_{\mathbb{H}}\colon S^7 \to S^4$ and $H_{\mathbb{O}}\colon S^{15} \to S^8$ are ± 1. ∎

Lemma 84.8. (Hopf Invariant–Homotopy Lemma) Let $n \geq 2$. Suppose we are given two continuous maps $f_0, f_1\colon S^{2n-1} \to S^n$. If the two maps f_0 and f_1 are homotopic, then $\mathrm{Hopf}(f_0) = \mathrm{Hopf}(f_1)$.

Proof. Let $H\colon S^{2n-1} \times [0,1] \to S^n$ be a homotopy between f_0 and f_1. We denote by $i_0, i_1\colon S^{2n-1} \to S^{2n-1} \times [0,1]$ the two obvious inclusion maps. We obtain the following two commutative diagrams

$$\begin{array}{ccc} S^{2n-1} & \xrightarrow{f_0} & S^n \\ {\scriptstyle i_0}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ S^{2n-1}\times[0,1] & \xrightarrow{H} & S^n \end{array} \qquad \text{and} \qquad \begin{array}{ccc} S^{2n-1} & \xrightarrow{f_1} & S^n \\ {\scriptstyle i_1}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ S^{2n-1}\times[0,1] & \xrightarrow{H} & S^n. \end{array}$$

Note that by the Mapping Cone–Functor Lemma **??** (3) the diagrams give rise to embeddings $\Xi_0\colon \mathrm{Cone}(f_0) \to \mathrm{Cone}(H)$ and $\Xi_1\colon \mathrm{Cone}(f_1) \to \mathrm{Cone}(H)$. From the Mapping Cone-$\mathrm{H}^*$-Lemma 84.5 we obtain the following commutative diagram of long exact sequences (where we drop the $\mathbb{Z}$-coefficients for space reasons):

$$\begin{array}{ccccccccc}
\ldots \to \widetilde{\mathrm{H}}^k(\mathrm{Cone}(f_0)) & \xrightarrow{j(f_0)^*} & \mathrm{H}^k(S^n) & \xrightarrow{f_0^*} & \mathrm{H}^k(S^{2n-1}) & \xrightarrow{\delta} & \widetilde{\mathrm{H}}^{k+1}(\mathrm{Cone}(f_0)) \to \ldots \\
{\scriptstyle \Xi_0^*}\uparrow & & {\scriptstyle \mathrm{id}}\uparrow & & {\scriptstyle \cong}\uparrow{\scriptstyle i_0^*} & & {\scriptstyle \Xi_0^*}\uparrow \\
\ldots \to \widetilde{\mathrm{H}}^k(\mathrm{Cone}(H)) & \xrightarrow{j(H)^*} & \mathrm{H}^k(S^n) & \xrightarrow{H^*} & \mathrm{H}^k(S^{2n-1}\times[0,1]) & \xrightarrow{\delta} & \widetilde{\mathrm{H}}^{k+1}(\mathrm{Cone}(H)) \to \ldots \\
{\scriptstyle \Xi_1^*}\downarrow & & {\scriptstyle \mathrm{id}}\downarrow & & {\scriptstyle \cong}\downarrow{\scriptstyle i_1^*} & & {\scriptstyle \Xi_1^*}\downarrow \\
\ldots \to \widetilde{\mathrm{H}}^k(\mathrm{Cone}(f_1)) & \xrightarrow{j(f_1)^*} & \mathrm{H}^k(S^n) & \xrightarrow{f_1^*} & \mathrm{H}^k(S^{2n-1}) & \xrightarrow{\delta} & \widetilde{\mathrm{H}}^{k+1}(\mathrm{Cone}(f_1)) \to \ldots
\end{array}$$

Note that the vertical maps between the cohomology groups of the mapping cones are isomorphisms. This can be shown using the Five Lemma 37.5, or alternatively, in a more fancy way, using the Mapping Cone Inclusion Lemma **??**.

Next we consider the generator $\Theta := (i_0^*)^{-1}([S^{2n-1}]^*) = (i_1^*)^{-1}([S^{2n-1}]^*)$ of the group $\mathrm{H}^{2n-1}(S^{2n-1} \times [0,1];\mathbb{Z})$. We define a number $\mathrm{Hopf}(H)$ via the equality

$$(j(H)^*)^{-1}([S^n]^*) \cup (j(H)^*)^{-1}([S^n]^*) \;=\; \mathrm{Hopf}(H) \cdot \delta_{2n-1}(H)(\Theta)$$

[657] With more care one can show that the Hopf invariant in each case is $+1$. But this extra information is irrelevant for our purpose.

in $\mathrm{H}^{2n}(\mathrm{Cone}(H);\mathbb{Z}) \cong \mathbb{Z}$. Finally we apply the above commutative diagram and we apply the Cup Product-Naturality Lemma 72.2 to the maps $\Xi_0\colon \mathrm{Cone}(f_0) \to \mathrm{Cone}(H)$ and $\Xi_1\colon \mathrm{Cone}(f_1) \to \mathrm{Cone}(H)$ and we see that $\mathrm{Hopf}(f_0) = \mathrm{Hopf}(H) = \mathrm{Hopf}(f_1)$. ■

Definition. It follows from the Hopf Invariant–Homotopy Lemma 84.8 that for each $n \geq 2$ the map

$$\begin{array}{rcl} \mathrm{Hopf}\colon \pi_{2n-1}(S^n,*) & \to & \mathbb{Z} \\ [f] & \mapsto & \mathrm{Hopf}(f) \end{array}$$

is well-defined. This map is called the Hopf invariant.

Our next proposition says that the Hopf invariant is a homomorphism. This is arguably somewhat surprising since at first glance the definition of the Hopf variant, given by squaring an element, does not look particularly like a linear map.

Proposition 84.9. (Hopf Invariant–Homomorphism Proposition) For any $n \geq 2$ the Hopf invariant $\mathrm{Hopf}\colon \pi_{2n-1}(S^n,*) \to \mathbb{Z}$ is a homomorphism.

Before we talk about the proof of the Hopf Invariant–Homomorphism Proposition 84.9 let us formulate the following immediate corollary:

Corollary 84.10. (Hopf Invariant One-Corollary) The Hopf invariants

$$\begin{array}{rcl} (1) \quad \mathrm{Hopf}\colon \pi_3(S^2) & \to & \mathbb{Z} \\ (2) \quad \mathrm{Hopf}\colon \pi_7(S^4) & \to & \mathbb{Z} \\ (3) \quad \mathrm{Hopf}\colon \pi_{15}(S^8) & \to & \mathbb{Z} \end{array}$$

are epimorphisms. In particular all three homotopy groups to the left are infinite.

Proof. The corollary follows immediately from the Hopf Invariant–Homomorphism Proposition 84.9 and the calculation of the Hopf invariant for the Hopf maps H, $H_{\mathbb{H}}$ and $H_{\mathbb{O}}$ in the Hopf Map-Hopf Invariant Lemma 84.7. ■

The technically slightly dicey proof of the Hopf Invariant–Homomorphism Proposition 84.9 will require the remainder of this section. The reader is cordially invited to skip the argument and to jump to the much more entertaining content of the next Section 84.2.

Our proof of the Hopf Invariant–Homomorphism Proposition 84.9 will require a few preparations. First of all, on several occasions we will need the following construction.

Construction 84.11. Suppose we are given the following commutative diagram of continuous maps between topological spaces:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\psi} \\ \widetilde{X} & \xrightarrow{\widetilde{f}} & \widetilde{Y}. \end{array}$$

Then, as discussed in the Mapping Cone–Functor Lemma **??** (3), we obtain an induced continuous map

$$\begin{array}{rcl} \mathrm{Cone}(f\colon X \to Y) & \to & \mathrm{Cone}(f\colon \widetilde{X} \to \widetilde{Y}) \\ [P] & \mapsto & \begin{cases} [(\varphi(x),t)], & \text{if } P = (x,t) \text{ with } x \in X,\ t \in [0,1], \\ [\psi(y)], & \text{if } P = y \text{ with } y \in Y \end{cases} \end{array}$$

By the Mapping Cone–Functor Lemma **??** (4) the map fits into the following commutative diagram:

$$\begin{array}{ccccc} Y & \xhookrightarrow{j} & \mathrm{Cone}(f\colon X \to Y) & \xrightarrow{p} & \Sigma(X) \\ {\scriptstyle\psi}\downarrow & & \downarrow & & \downarrow{\scriptstyle\Sigma(\varphi)} \\ \widetilde{Y} & \xhookrightarrow{j} & \mathrm{Cone}(\widetilde{f}\colon \widetilde{X} \to \widetilde{Y}) & \xrightarrow{p} & \Sigma(\widetilde{X}) \end{array}$$

where the horizontal maps to the left are the natural embeddings i that we introduced in the Mapping Cone Basics Lemma **??** (2) and where the horizontal maps p on the right are given by the natural projection and the natural homeomorphism from the Mapping Cone Basics Lemma **??** (4).

We will also need the following slight reinterpretation of the Hopf invariant.

Lemma 84.12. (Hopf Invariant–Reinterpretation Lemma) Let $f\colon S^{2n-1} \to S^n$ be a continuous map. We denote by $j\colon S^n \to \mathrm{Cone}(f)$ the natural embedding and we denote by $p\colon \mathrm{Cone}(f) \to \mathrm{Cone}(f)/S^n = \Sigma(S^{2n-1})$ the natural map that is given by the projection and homeomorphism from the Mapping Cone Basics Lemma **??** (4). Then

$$(j^*)^{-1}([S^n]^*) \cup (j^*)^{-1}([S^n]^*) \;=\; \mathrm{Hopf}(f) \cdot p^*(\underset{\substack{\uparrow \\ \text{isomorphism provided by} \\ \text{the Suspension-H}^*\text{-Lemma } ??}}{\Sigma([S^{2n-1}]^*)}) \;\in\; \mathrm{H}^{2n}(\mathrm{Cone}(f);\mathbb{Z}).$$

Proof of Lemma 84.12. We start out with the following claim:

Claim. Let $n \in \mathbb{N}$. Let $\Sigma(S^n) \to S^{n+1}$ be the homeomorphism from page **??**. We have

$$\Sigma_{S^n}([S^n]^*) \;=\; (-1)^{n+1} \cdot [S^{n+1}]^* \;\in\; \mathrm{H}^{n+1}(S^{n+1};\mathbb{Z}).$$

Proof. Recall that on page **??** we saw that $\Sigma_{S^n}([S^n]) = (-1)^{n+1} \cdot [S^{n+1}] \in \mathrm{H}_{n+1}(S^{n+1})$. The claim now follows from Statement (2) of the Suspension-H*-Lemma **??**. ⊞

The statement follows basically immediately from the claim[658] and the cohomological analogue of the Cone–Connecting Homomorphism Lemma **??**. ■

Now we are finally in a position to prove the Hopf Invariant–Homomorphism Proposition 84.9:

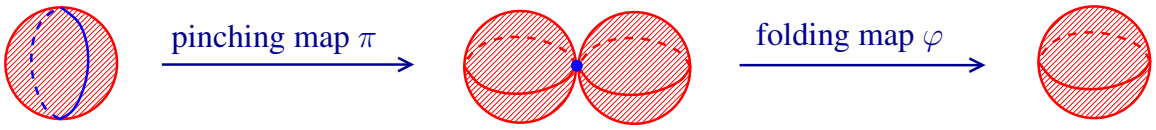
Proof of Proposition 84.9. We start out with a few preparations.

(1) Given a pointed topological space (X, x_0) we define $X \vee X$ by identifying two copies of X along the base point. We refer to the map $\varphi\colon X \vee X \to X$ which is given by the identity on both copies as the *folding map*.

(2) Let $k \in \mathbb{N}_{\geq 2}$. On page 382 we introduced a pinching map $\pi\colon S^k \to S^k \vee S^k$. The precise definition is rather irrelevant to us right now. What matters is that we saw on page 382 that given any pointed topological space (X, x_0) and given any two continuous maps $f_1, f_2\colon (S^k, *) \to (X, x_0)$ we have

$$[f_1 * f_2] \;=\; [\, S^k \xrightarrow{\pi} S^k \vee S^k \xrightarrow{f_1 \vee f_2} X \vee X \xrightarrow{\varphi} X \,] \;\in\; \pi_k(X, x_0).$$

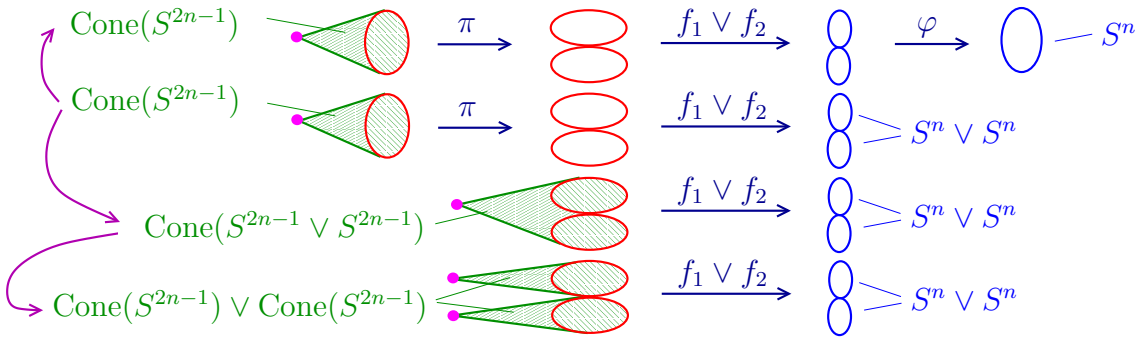
[658] Note that here we use the convenient fact that $(-1)^{(2n-1)+1} = +1$. In reality the sign is largely irrelevant, as long as we can be sure that the sign depends only on n.



Now we turn to our actual situation. Let $n \in \mathbb{N}_{\geq 2}$. Suppose we are given two continuous maps $f_1, f_2 \colon (S^{2n-1}, *) \to (S^n, *)$. For $i = 1, 2$ we consider the following diagram:

$$\begin{array}{ccccc}
S^n & \xrightarrow{j} & \mathrm{Cone}(\varphi \circ (f_1 \vee f_2) \circ \pi \colon S^{2n-1} \to S^n) & \xrightarrow{p} & \Sigma S^{2n-1} \\
{\scriptstyle\varphi}\uparrow & & \uparrow{\scriptstyle\varphi} & & \uparrow{\scriptstyle\mathrm{id}} \\
S^n \vee S^n & \xrightarrow{j} & \mathrm{Cone}((f_1 \vee f_2) \circ \pi \colon S^{2n-1} \to S^n \vee S^n) & \xrightarrow{p} & \Sigma S^{2n-1} \\
{\scriptstyle\mathrm{id}}\downarrow & & \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\Sigma(\pi)} \\
S^n \vee S^n & \xrightarrow{j} & \mathrm{Cone}(f_1 \vee f_2 \colon S^{2n-1} \vee S^{2n-1} \to S^n \vee S^n) & \xrightarrow{p} & \Sigma(S^{2n-1} \vee S^{2n-1}) \\
{\scriptstyle\mathrm{id}}\uparrow & & \uparrow{\scriptstyle\Xi} & & \uparrow{\scriptstyle\Omega} \\
S^n \vee S^n & \xrightarrow{j \vee j} & \mathrm{Cone}(f_1 \colon S^{2n-1} \to S^n) \vee \mathrm{Cone}(f_2 \colon S^{2n-1} \to S^n) & \xrightarrow{p \vee p} & \Sigma S^{2n-1} \vee \Sigma S^{2n-1} \\
{\scriptstyle q_i}\downarrow & & \downarrow{\scriptstyle q_i} & & \downarrow{\scriptstyle q_i} \\
S^n & \xrightarrow{j} & \mathrm{Cone}(f_i \colon S^{2n-1} \to S^n) & \xrightarrow{p} & \Sigma S^{2n-1}.
\end{array}$$

The top three vertical maps are illustrated in the figure below. We make the following



clarifications:

(1) The horizontal maps to the left are the inclusions maps j introduced in the Mapping Cone Basics Lemma **??** (2).

(2) In the Mapping Cone Basics Lemma **??** (4) we showed that for any mapping cone $\mathrm{Cone}(g \colon A \to X)$ we have a natural homeomorphism $\mathrm{Cone}(g \colon A \to X)/X \to \Sigma(A)$. In the above diagram the horizontal maps p on the right are given by the composition of a projection and the homeomorphism we just mentioned.

(3) The vertical maps between the first and the second and between the second and the third row are obtained from Construction 84.11 applied to the following two commutative diagrams

$$\begin{array}{ccc}
S^{2n-1} & \xrightarrow{(f_1 \vee f_2) \circ \pi} & S^n \vee S^n \\
{\scriptstyle\mathrm{id}}\downarrow & & \downarrow{\scriptstyle\varphi} \\
S^{2n-1} & \xrightarrow{\varphi \circ (f_1 \vee f_2) \circ \pi} & S^n
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
S^{2n-1} & \xrightarrow{(f_1 \vee f_2) \circ \pi} & S^n \vee S^n \\
{\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\mathrm{id}} \\
S^{2n-1} \vee S^{2n-1} & \xrightarrow{f_1 \vee f_2} & S^n \vee S^n.
\end{array}$$

In particular we see that the four upper squares commute.

(4) For $i = 1, 2$ we consider the commutative diagram

$$\begin{array}{ccc} S^{2n-1} & \xrightarrow{\quad f_i \quad} & S^n \\ \downarrow & & \downarrow \\ S^{2n-1} \vee S^{2n-1} & \xrightarrow{f_1 \vee f_2} & S^n \vee S^n \end{array}$$

which is given by the embedding into the first respectively second factor. By Construction 84.11 these diagrams give rise to two sequences of maps. We "wedge" the two sequences and we obtain the vertical maps from the fourth to the third row.

(5) At the bottom, in each column the map q_i is the projections onto the i-th factor.

Next we apply the cohomology functor $X \mapsto \mathrm{H}^k(X;\mathbb{Z})$ with $k = n$ and $k = 2n$ to the above diagram (except that at the bottom we consider both maps q_1 and q_2). We obtain the following commutative diagram (for space reasons we dropped the $\mathbb{Z}$-coefficients):

$$\begin{array}{ccccc}
\mathrm{H}^k(S^n) & \xleftarrow[\cong]{j^*} & \mathrm{H}^k(\mathrm{Cone}(\varphi\circ(f_1\vee f_2)\circ\pi : S^{2n-1} \to S^n)) & \xleftarrow[\cong]{p^*} & \mathrm{H}^k(\Sigma S^{2n-1}) \\
{\scriptstyle\varphi^*}\uparrow & & \downarrow{\scriptstyle\varphi^*} & & \downarrow{\scriptstyle \mathrm{id}} \\
\mathrm{H}^k(S^n\vee S^n) & \xleftarrow[\cong]{j^*} & \mathrm{H}^k(\mathrm{Cone}((f_1\vee f_2)\circ\pi : S^{2n-1} \to S^n\vee S^n)) & \xleftarrow[\cong]{p^*} & \mathrm{H}^k(\Sigma S^{2n-1}) \\
{\scriptstyle\mathrm{id}}\uparrow & & \uparrow{\scriptstyle\psi^*} & & \Uparrow{\scriptstyle\Sigma(\pi)^*} \\
\mathrm{H}^k(S^n\vee S^n) & \xleftarrow[\cong]{j^*} & \mathrm{H}^k(\mathrm{Cone}(f_1\vee f_2 : S^{2n-1}\vee S^{2n-1} \to S^n\vee S^n)) & \xleftarrow[\cong]{p^*} & \mathrm{H}^k(\Sigma(S^{2n-1}\vee S^{2n-1})) \\
{\scriptstyle\mathrm{id}}\downarrow & & \cong\downarrow{\scriptstyle\Xi^*} & & \cong\downarrow{\scriptstyle\Omega^*} \\
\mathrm{H}^k(S^n\vee S^n) & \xleftarrow[\cong]{(j\vee j)^*} & \mathrm{H}^k(\mathrm{Cone}(f_1)\vee\mathrm{Cone}(f_2)) & \xleftarrow[\cong]{(p\vee p)^*} & \mathrm{H}^k(\Sigma S^{2n-1}\vee\Sigma S^{2n-1}) \\
{\scriptstyle q_1^*\oplus q_2^*}\uparrow\cong & & \cong\uparrow{\scriptstyle q_1^*\oplus q_2^*} & & \cong\uparrow{\scriptstyle q_1^*\oplus q_2^*} \\
\mathrm{H}^k(S^n)\oplus\mathrm{H}^k(S^n) & \xleftarrow[\cong]{j_1^*\oplus j_2^*} & \mathrm{H}^k(\mathrm{Cone}(f_1))\oplus\mathrm{H}^k(\mathrm{Cone}(f_2)) & \xleftarrow[\cong]{p_1^*\oplus p_2^*} & \mathrm{H}^k(\Sigma S^{2n-1})\oplus\mathrm{H}^k(\Sigma S^{2n-1}).
\end{array}$$

We make the following observations:

(1) It follows almost immediately from the Mapping Cone-H*-Lemma 84.5 that the horizontal maps to the left are isomorphisms in degree n. Since these isomorphisms are only in degree n they are colored in red.

(2) It follows easily from the Mapping Cone-H*-Lemma 84.5 and the cohomological analogue of the Cone–Connecting Homomorphism Lemma **??** that the horizontal maps to the right are isomorphisms for $k = 2n$. Since these isomorphisms are only in degree $2n$ they are colored in green.

(3) We leave it to the reader to show that the map Ω^* is an isomorphism.

(4) It follows from (1), (2) and (3) that the map Ξ^* is an isomorphism in degrees n and $2n$, thus in all dimensions we care about.

(5) The vertical maps at the very bottom are induced by the natural projections. The maps are isomorphisms by the Wedge-H*-Proposition 69.4.

(6) Note that the folding map $\varphi\colon S^n \vee S^n \to S^n$ has a right-inverse given by any one of the two obvious embeddings $S^n \to S^n \vee S^n$. It follows from this observation and the functoriality of cohomology groups that the induced maps $\varphi^*\colon \mathrm{H}^k(S^n;\mathbb{Z}) \to \mathrm{H}^k(S^n \vee S^n;\mathbb{Z})$ are monomorphisms.

Next we denote by $\alpha := [S^n]^* \in \mathrm{H}^n(S^n;\mathbb{Z})$ the dual fundamental class. Furthermore we write $\beta := [\Sigma([S^{2n-1}]^*)] \in \mathrm{H}^{2n}(\Sigma S^{2n-1};\mathbb{Z})$.

Claim 1. We have $\varphi^*(\alpha) = q_1^*(\alpha) + q_2^*(\alpha)$ and $(\Sigma(\pi)^* \circ (\Omega^*)^{-1})(q_1^*(\beta) + q_2^*(\beta)) = \beta$.

Proof. We leave it to the reader to verify this claim.

The key calculation takes place in the fourth row. More precisely, we have the following claim:

Claim 2. The following equality holds in $\mathrm{H}^{2n}(\mathrm{Cone}(f_1) \vee \mathrm{Cone}(f_2); \mathbb{Z})$:

$$(((j \vee j)^*)^{-1}(q_1^*(\alpha) + q_2^*(\alpha)))^2 \;=\; (\mathrm{Hopf}(f_1) + \mathrm{Hopf}(f_2)) \cdot (p \vee p)^*(q_1^*(\beta) + q_2^*(\beta)).$$

Proof. We note that

$$\begin{aligned}
&((j \vee j)^{*^{-1}}(q_1^*\alpha + q_2^*\alpha))^2 \overset{\text{since } j_i \circ q_i = q_i \circ (j \vee j)\colon S^n \vee S^n \to \mathrm{Cone}(f_i\colon S^{2n-1} \to S^n)}{=} (q_1^*(j_1^{*^{-1}}\alpha) + q_2^*(j_2^{*^{-1}}\alpha)) \cup (q_1^*(j_1^{*^{-1}}\alpha) + q_2^*(j_2^{*^{-1}}\alpha)) \\
&= \underset{\text{by the Suspension-Cup Product Lemma 73.5 the "mixed" terms are zero}}{q_1^*(j_1^{*^{-1}}\alpha) \cup q_1^*(j_1^{*^{-1}}\alpha)} + q_2^*(j_2^{*^{-1}}\alpha) \cup q_2^*(j_2^{*^{-1}}\alpha) \underset{\text{by the Cup Product-Naturality Lemma 72.2}}{=} q_1^*(j_1^{*^{-1}}\alpha \cup j_1^{*^{-1}}\alpha) + q_2^*(j_2^{*^{-1}}\alpha \cup j_2^{*^{-1}}\alpha) \\
&= \underset{\text{by the reinterpretation of } \mathrm{Hopf}(f_1) \text{ and } \mathrm{Hopf}(f_2) \text{ given in the Hopf Invariant–Reinterpretation Lemma 84.12}}{q_1^*(\mathrm{Hopf}(f_1) \cdot p_1^*(\beta))} + q_2^*(\mathrm{Hopf}(f_2) \cdot p_2^*(\beta)) \underset{\text{since } p_i \circ q_i = q_i \circ (p \vee p)\colon\ \mathrm{Cone}(f_1) \vee \mathrm{Cone}(f_2) \to \Sigma S^{2n-1}}{=} (\mathrm{Hopf}(f_1) + \mathrm{Hopf}(f_2)) \cdot (p \vee p)^*(q_1^*(\beta) + q_2^*(\beta)).
\end{aligned}$$

⊞

It follows from $[f_1 * f_2] = [\varphi \circ (f_1 \vee f_2) \circ \pi]$, the definition of α and β and the reinterpretation of the Hopf invariant given in the Hopf Invariant–Reinterpretation Lemma 84.12, that it remains to prove the following claim:

Claim 3. In the first row of the above big diagram we have the following equality:

$$((j^*)^{-1}(\alpha))^2 \;=\; (\mathrm{Hopf}(f_1) + \mathrm{Hopf}(f_2)) \cdot p^*(\beta).$$

Proof. The idea is of course to push the equality from Claim 2 through the above commutative diagram towards the top. The key tool hereby is the naturality of the cup product, see the Cup Product-Naturality Lemma 72.2. First we note that it follows from the Cup Product-Naturality Lemma 72.2 and the above observation that the map φ^* is a monomorphism that it remains to prove that the following equality holds in the second row:

$$(*) \qquad (j^*)^{-1}(\varphi^*(\alpha))^2 \;=\; (\mathrm{Hopf}(f_1) + \mathrm{Hopf}(f_2)) \cdot p^*(\beta).$$

To prove this equality we note that it follows from Claims 1 and 2 together with the Cup Product-Naturality Lemma 72.2 that the following equality holds in the third row:

$$((j^*)^{-1}(\varphi^*(\alpha)))^2 \;=\; (\mathrm{Hopf}(f_1) + \mathrm{Hopf}(f_2)) \cdot p^*((\Omega^*)^{-1}(q_1^*(\beta) + q_2^*(\beta))).$$

Again, using the Cup Product-Naturality Lemma 72.2 we see that the following equality holds in the second row:

$$((j^*)^{-1}(\varphi^*(\alpha)))^2 \;=\; (\mathrm{Hopf}(f_1) + \mathrm{Hopf}(f_2)) \cdot p^*(\underbrace{\Sigma(\pi)^*((\Omega^*)^{-1}(q_1^*(\beta) + q_2^*(\beta)))}_{=\beta \text{ by Claim 1}}).$$

Using Claim 1 we see that this is exactly the desired equality that we formulated in $(*)$. ■

84.2. **The Hopf invariant and multiplication maps.** In this section we will see that any continuous map $f\colon S^k \times S^k \to S^k$ give rises to a potentially interesting continuous map $\Theta_f\colon S^{2k+1} \to S^k$. In the key calculation of this section we show how we can determine the Hopf invariant $\mathrm{Hopf}(\Theta_f)$ directly in terms of homological data of the original map f.

Definition. Let $n \in \mathbb{N}$ and let $f\colon S^n \times S^n \to S^n$ be a continuous map. We pick $x_0, y_0 \in S^n$. We define the bidegree of f as[659]

$$\operatorname{bideg}(f) \coloneqq (p,q) \text{ where } p \coloneqq \deg\begin{pmatrix} S^n & \to & S^n \\ x & \mapsto & f(x,y_0) \end{pmatrix} \text{ and } q \coloneqq \deg\begin{pmatrix} S^n & \to & S^n \\ y & \mapsto & f(x_0,y) \end{pmatrix}.$$

Examples.

(1) The map $f\colon S^1 \times S^1 \to S^1$ that is given by complex multiplication has bidegree $(1,1)$. This can be seen as follows: Taking $x_0 = y_0 = 1$, we see that the two maps $S^1 \to S^1$ given by $z \mapsto f(z,1)$ and $z \mapsto f(1,z)$ are in fact the identity.

(2) The same argument as in (1) shows that the quaternion multiplication $S^3 \times S^3 \to S^3$ is a map of bidegree $(1,1)$ and that the octonion multiplication $S^7 \times S^7 \to S^7$ is also a map of bidegree $(1,1)$.[660]

(3) Let $n \in \mathbb{N}$ and let $f\colon S^n \times S^n \to S^n$ be a continuous map of bidegree (p,q). Furthermore let $\varphi, \psi\colon S^n \to S^n$ be two continuous maps. It follows almost immediately from the definitions that the bidegree of the continuous map $(x,y) \mapsto f(\varphi(x), \psi(y))$ is given by $(p \cdot \deg(\varphi), q \cdot \deg(\psi))$. □

The previous example shows that the division algebras $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ give rise to maps of bidegree $(1,1)$. For general division algebras we have the following closely related non-triviality result:

Lemma 84.13. (Division Algebra–Bidegree-Lemma) Let $n \in \mathbb{N}_{\geq 2}$. If $\mathbb{R}^n$ admits the structure of a division algebra[661], then there exists a continuous map $S^{n-1} \times S^{n-1} \to S^{n-1}$ of bidegree (p,q) where both p and q are odd.

Proof. Let $*$ be the multiplication map of a division algebra on $\mathbb{R}^n$. As in the proof of the Hopf–Division Algebra Dimension Theorem 82.7 we consider the map

$$\begin{array}{rcl} f\colon S^{n-1} \times S^{n-1} & \to & S^{n-1} \\ (x,y) & \mapsto & f(x,y) \coloneqq \dfrac{x*y}{\|x*y\|}. \end{array}$$

In the proof of the Commutative Division Algebra Theorem **??** we already showed, using the hypothesis that "$*$" defines a division algebra, that this map is well-defined and that it is continuous. Now we pick a point $\star \in S^{n-1}$ and we consider the maps

$$\begin{array}{rcl} \varphi\colon S^{n-1} & \to & S^{n-1} \\ x & \mapsto & f(x,\star) \end{array} \quad \text{and} \quad \begin{array}{rcl} \psi\colon S^{n-1} & \to & S^{n-1} \\ y & \mapsto & f(\star,y). \end{array}$$

It follows easily from the fact that $*$ is the multiplication of a real algebra that for all $z \in S^{n-1}$ we have $\varphi(-z) = -\varphi(z)$ and $\psi(-z) = -\psi(z)$. By the Odd Map–Odd Degree Proposition 57.5 this implies that $\deg(\varphi)$ and $\deg(\psi)$ are odd. Conveniently enough that is exactly what we had set out to show. ■

[659] One can easily show that the definition does not depend on the choice of x_0 and y_0.

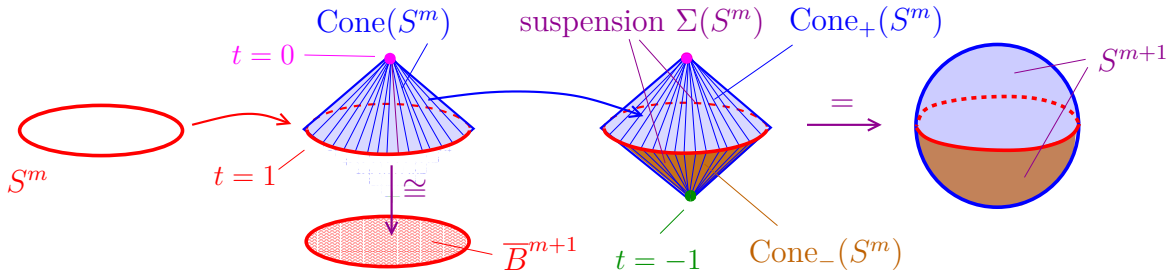
[660] Here we used implicitly the Quaternion Lemma 82.5 (7) and the Octonion Lemma 82.6 (7) to ensure that these maps do indeed take values in S^3 and S^7.

[661] Recall that by the definition on page **??** a division algebra is an algebra that has left and right inverses. We do not assume that division algebras are associative and we do not assume that they preserve the scalar product.

Before we continue we recall a few conventions that we had introduced along the way. Most of them are also illustrated in the figure below.

(1) Given a topological space we consider the cone $\mathrm{Cone}(X)$, the suspension $\Sigma(X)$ and we consider the two copies $\mathrm{Cone}_\pm(X)$ of the cone in the suspension.[662]
(2) Let $m \in \mathbb{N}_0$. Note that by the discussion on page **??** we can make the identification $\mathrm{Cone}(S^m) =_{\mathrm{i}} \overline{B}^{m+1}$ and by the discussion on page **??** we can make the identification $\Sigma(S^m) =_{\mathrm{i}} S^{m+1}$.
(3) Let $m, n \in \mathbb{N}_0$. By the discussion on page **??** we can make the identification

$$(S^m \times \overline{B}^{n+1}) \cup_{\mathrm{id}_{S^m \times S^n}} (\overline{B}^{m+1} \times S^n) \xrightarrow{=} S^{m+n+1}.$$



Now we can move on to the promised construction of potentially interesting maps between spheres.

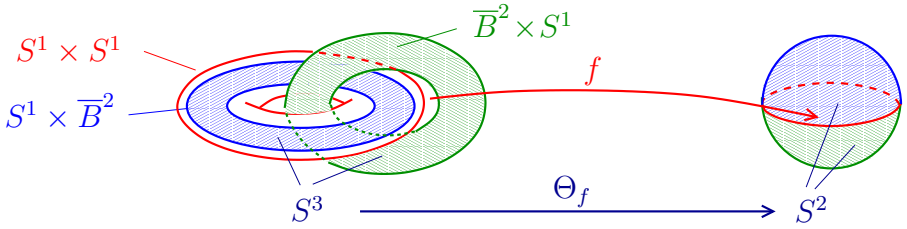
Definition. Let $f\colon S^p \times S^q \to S^r$ be a continuous map. We consider the maps

$$\begin{array}{rcl} S^p \times \mathrm{Cone}(S^q) & \to & \mathrm{Cone}_+(S^r) \\ (P, [(Q,t)]) & \mapsto & [(f(P,Q), 1-t)] \end{array} \quad \text{and} \quad \begin{array}{rcl} \mathrm{Cone}(S^p) \times S^q & \to & \mathrm{Cone}_-(S^r) \\ ([(P,t)], Q) & \mapsto & [(f(P,Q), t-1)]. \end{array}$$

We leave the task of showing that these two maps are continuous to the reader. Evidently the two maps agree on the overlap $S^p \times S^q$. Thus by the Topological Pushout Proposition 6.12 (1b) these two maps define a continuous map

$$\Theta_f\colon \underbrace{(S^p \times \mathrm{Cone}(S^q)) \cup_{S^p \times S^q} (\mathrm{Cone}(S^p) \times S^q) = S^{p+q+1}}_{=(S^p \times \overline{B}^{q+1}) \cup_{S^p \times S^q} (\overline{B}^{p+1} \times S^q)} \to \underbrace{\Sigma(S^r)}_{=S^{r+1}}.$$

We refer to the figure below for a moderately successful attempt at illustrating the map Θ_f.



Example.

(1) Again we consider the map $f\colon S^1 \times S^1 \to S^1$ that is given by complex multiplication. We obtain by the above construction a continuous map $\Theta_f\colon S^3 \to S^2$. The heroically

[662] Here we implicitly use the Cone-Suspension Basics Lemma **??**.

inclined reader will surely rise to the challenge of showing that this map is homotopic to the Hopf map.[663]

(2) Similarly to (1) one can show that the quaternion multiplication $S^3 \times S^3 \to S^3$ gives rise to the quaternionic Hopf map $S^7 \to S^4$ and that the octonion multiplication $S^7 \times S^7 \to S^7$ gives rise to the quaternionic Hopf map $S^{15} \to S^8$. □

Given a continuous map $f\colon S^{n-1} \times S^{n-1} \to S^{n-1}$ the next proposition computes the Hopf invariant of the corresponding map $\Theta_f\colon S^{2n-1} \to S^n$ in terms of the bidegree of f.

Proposition 84.14. (Bidegree–Hopf Invariant Proposition) Let $n \in \mathbb{N}_{\geq 2}$. Furthermore let $f\colon S^{n-1} \times S^{n-1} \to S^{n-1}$ be a continuous map of bidegree (r, s). The induced map $\Theta_f\colon S^{2n-1} \to S^n$ has Hopf invariant $\pm r \cdot s$.[664]

The proof of the Bidegree–Hopf Invariant Proposition 84.14 will require the remainder of this section. In the last Section 84.3 we will go back to examples and applications to homotopy groups.

Proof. We make the following preparations:

(1) To simplify the notation we write $\mathrm{C}(\Theta_f)$ instead of $\mathrm{Cone}(\Theta_f)$.

(2) Recall that by the Mapping Cylinder Basics Lemma **??** we know that the inclusion $S^n \to \mathrm{C}(\Theta_f\colon S^{2n-1} \to S^n)$ is actually an embedding. Thus we can identify S^n with its image in $\mathrm{C}(\Theta_f)$.

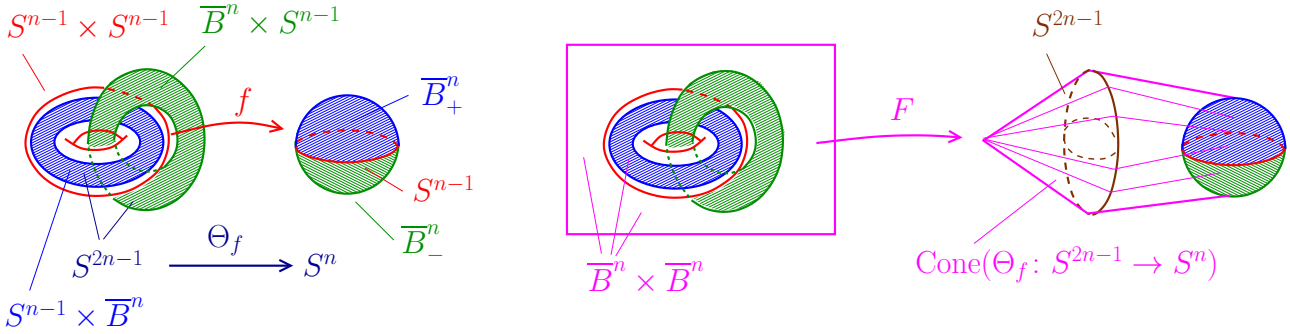
(3) We denote by $\overline{B}^n_+$ the upper hemisphere of S^n and we denote by $\overline{B}^n_-$ the lower hemisphere of S^n.

(4) By the discussion on page **??** we can make the identification

$$\overline{B}^n \times \overline{B}^n \;=\; \overline{B}^{2n} \;=\; \mathrm{Cone}\Big(\underbrace{(S^{n-1} \times \overline{B}^n) \cup_{S^{n-1}\times S^{n-1}} (\overline{B}^n \times S^{n-1})}_{=S^{2n-1}}\Big).$$

Let $$F\colon \overline{B}^n \times \overline{B}^n \;\to\; \mathrm{C}(\Theta_f\colon S^{2n-1} \to S^n) \;=\; \underbrace{\mathrm{Cone}(S^{2n-1})}_{=\overline{B}^n\times\overline{B}^n=\overline{B}^{2n}} \cup_{\Theta_f} S^n$$

the obvious map which is given by the identification $\overline{B}^n \times \overline{B}^n = \mathrm{Cone}(S^{2n-1})$ followed by the projection.



[663] To be on the safe side, let us more cautiously say that the map Θ_f is either homotopic to the Hopf map or that it is homotopic to the composition of the Hopf map with a reflection in a hyperplane.

[664] We leave it to the sign aficionados to determine the correct sign.

Next we pick generators for various cohomology groups:

(1) We denote by $x \in \mathrm{H}^n(\mathrm{C}(\Theta_f);\mathbb{Z})$ and $z \in \mathrm{H}^{2n}(\mathrm{C}(\Theta_f);\mathbb{Z})$ the generators that are given in the definition of the Hopf invariant, see page 1083.

(2) The long exact sequence in cohomology of the pair $(\mathrm{C}(\Theta_f), \overline{B}^n_\pm)$ shows that the maps $\text{я}\colon \mathrm{H}^n(\mathrm{C}(\Theta_f), \overline{B}^n_\pm;\mathbb{Z}) \to \mathrm{H}^n(\mathrm{C}(\Theta_f);\mathbb{Z})$ are isomorphisms. Let $x_\pm \in \mathrm{H}^n(\mathrm{C}(\Theta_f), \overline{B}^n_\pm;\mathbb{Z})$ be the generators that correspond to x under the given isomorphisms.

(3) We denote by $[S^{n-1}]^* \in \mathrm{H}^{n-1}(S^{n-1};\mathbb{Z})$ and $[\overline{B}^n]^* \in \mathrm{H}^n(\overline{B}^n, S^{n-1};\mathbb{Z})$ the standard dual generators. For $i = 1, 2$ we denote by $p_i\colon \overline{B}^n \times \overline{B}^n \to \overline{B}^n$ the projection onto the first respectively second coordinate. We write

$$\begin{aligned} y_1 &:= p_1^*([\overline{B}^n]^*) \in \mathrm{H}^n(\overline{B}^n \times \overline{B}^n, S^{n-1} \times \overline{B}^n;\mathbb{Z}) \quad \text{and} \\ y_2 &:= p_2^*([\overline{B}^n]^*) \in \mathrm{H}^n(\overline{B}^n \times \overline{B}^n, \overline{B}^n \times S^{n-1};\mathbb{Z}). \end{aligned}$$

The following claim contains the key calculation of the proof.

Claim. The map F induces the following maps: on relative cohomology:

$$\begin{array}{lccc} \text{(i)} & \mathrm{H}^n(\mathrm{C}(\Theta_f), \overline{B}^n_+;\mathbb{Z}) & \xrightarrow{F^*} & \mathrm{H}^n(\overline{B}^n \times \overline{B}^n, S^{n-1} \times \overline{B}^n;\mathbb{Z}) \\ & x_+ & \mapsto & \pm r \cdot y_1 \\ \text{(ii)} & \mathrm{H}^n(\mathrm{C}(\Theta_f), \overline{B}^n_-;\mathbb{Z}) & \xrightarrow{F^*} & \mathrm{H}^n(\overline{B}^n \times \overline{B}^n, \overline{B}^n \times S^{n-1};\mathbb{Z}) \\ & x_- & \mapsto & \pm s \cdot y_2. \end{array}$$

Proof. Before we turn to the actual calculation we make the following two preparations:

(a) The restrictions of F to various subspaces gives us the following commutative diagram:

$$\begin{array}{ccc} \overline{B}^n \times \overline{B}^n & \xrightarrow{F} & \mathrm{C}(\Theta_f) \\ \uparrow & & \uparrow \\ \overline{B}^n \times S^{n-1} & \xrightarrow{\Theta_f} & \overline{B}^n_- \\ \uparrow & & \uparrow \\ S^{n-1} \times S^{n-1} & \xrightarrow{f} & S^{n-1}. \end{array}$$

(b) We pick a point $* \in S^{n-1}$.

To provide the actual proof of the claim we consider the following diagram (where once again we drop the $\mathbb{Z}$-coefficients for space reasons):

$$\begin{array}{ccccc} \mathrm{H}^n(\mathrm{C}(\Theta_f)) & \xleftarrow{\cong} & \mathrm{H}^n(\mathrm{C}(\Theta_f), \overline{B}^n_+) & \xrightarrow{F^*} & \mathrm{H}^n(\overline{B}^n \times \overline{B}^n, S^{n-1} \times \overline{B}^n) \\ \downarrow \cong & & \downarrow & & \downarrow \cong \\ \mathrm{H}^n(S^n) & \xleftarrow{\cong} & \mathrm{H}^n(S^n, \overline{B}^n_+) \xrightarrow[\text{exc.}]{\cong} & \mathrm{H}^n(\overline{B}^n_-, S^{n-1}) & \xrightarrow{\Theta_f^*} \mathrm{H}^n(\overline{B}^n \times *, S^{n-1} \times *) \\ & & & \cong \uparrow \delta & \qquad \delta \uparrow \cong \\ & & & \mathrm{H}^{n-1}(S^{n-1}) & \xrightarrow[[S^{n-1}]^* \mapsto r \cdot [S^{n-1} \times *]^*]{f^*} \mathrm{H}^{n-1}(S^{n-1} \times *). \end{array}$$

We make the following clarifications and observations:

(1) The maps that are not specified are induced by the obvious inclusions of (pairs) of topological spaces.

(2) The inclusion-induced maps that are decorated by "$\cong$" are isomorphisms for the following reasons:

(a) by the long exact sequence in cohomology for the corresponding pair,

(b) the maps are induced by a homotopy equivalence,
(c) by the long exact sequence provided by the Mapping Cone-H*-Lemma 84.5,
(d) by the Excision-in-H*-Theorem 69.1 (2).

(3) The quadrilateral to the left commutes since all maps are induced by inclusions.
(4) It follows from (2) and (3) that the diagonal map is an isomorphism.
(5) The triangle towards the center commutes since all maps are induced by inclusions.
(6) It follows from (2), (4) and (5) that the middle vertical map is an isomorphism.
(7) The squares to the right commute by the commutative diagram shown in (a) and by the naturality of the connecting homomorphisms.
(8) The connecting homomorphisms δ are easily seen to be isomorphisms since the other groups in the corresponding long exact sequence are zero.
(9) Basically by the definition of the bidegree of the map f, the bottom horizontal map is given by $[S^{n-1}]^* \mapsto r \cdot [S^{n-1} \times *]^*$.

The above arguments show that the vertical maps of the blue part of the diagram are isomorphisms. Since the bottom map is given by multiplication by r we see that the top map, with whatever choice of generators, is also given by multiplication by $\pm r$. This concludes the proof of part (i) of the claim. The proof of part (ii) is evidently basically the same. ⊞

Finally we consider the following diagram (again we drop the $\mathbb{Z}$-coefficients for space reasons):

$$\begin{array}{ccccc}
\begin{array}{c}\mathrm{H}^n(\mathrm{C}(\Theta_f)) \\ \times \\ \mathrm{H}^n(\mathrm{C}(\Theta_f))\end{array} & \xleftarrow[\cong]{\substack{x \hookleftarrow x_- \\ x \hookleftarrow x_+}} & \begin{array}{c}\mathrm{H}^n(\mathrm{C}(\Theta_f), \overline{B}^n_-) \\ \times \\ \mathrm{H}^n(\mathrm{C}(\Theta_f), \overline{B}^n_+)\end{array} & \xrightarrow[F^*]{\substack{x_- \mapsto \pm r \cdot y_1 \\ x_+ \mapsto \pm s \cdot y_2}} & \begin{array}{c}\mathrm{H}^n(\overline{B}^n \times \overline{B}^n, S^{n-1} \times \overline{B}^n) \\ \times \\ \mathrm{H}^n(\overline{B}^n \times \overline{B}^n, \overline{B}^n \times S^{n-1})\end{array} \\
\cup \downarrow {\scriptstyle (x,x) \mapsto x^2 = \mathrm{Hopf}(\Theta_f) \cdot z} & & \cup \downarrow & & \cup \downarrow {\scriptstyle (y_1, y_2) \mapsto y_1 \cup y_2 = [\overline{B}^n \times \overline{B}^n]^*} \\
\mathrm{H}^{2n}(\mathrm{C}(\Theta_f)) & \xleftarrow[\cong]{} & \mathrm{H}^{2n}(\mathrm{C}(\Theta_f), S^n) & \xrightarrow[\cong]{F^*} & \mathrm{H}^{2n}(\overline{B}^n \times \overline{B}^n, \overline{B}^n \times S^{n-1} \cup S^{n-1} \times \overline{B}^n).
\end{array}$$

$$\pm z \hookleftarrow [\overline{B}^n \times \overline{B}^n]^*$$

We make the following comments:

(1) It follows easily from the Excisive Triad-Proposition 50.11 (2) that $(\mathrm{C}(\Theta_f), \overline{B}^n_-, \overline{B}^n_+)$ and $(\overline{B}^n \times \overline{B}^n, S^{n-1} \times \overline{B}^n, \overline{B}^n \times S^{n-1})$ are excisive triads. Thus we see that the relative cup product in the center and to the right are actually defined.
(2) It follows from the naturality of the cup product, see the Cup Product-Naturality Lemma 73.3, that the two squares commute.
(3) By the Fundamental Class-Product Proposition 76.2 (1) we have the following equality: $y_1 \cup y_2 = p_1^*([\overline{B}^n]) \cup p_2^*([\overline{B}^n]) = [\overline{B}^n \times \overline{B}^n]^*$.
(4) By the above claim we know that the top right horizontal map is in fact given by $x_- \mapsto \pm r \cdot y_1$ and $x_+ \mapsto \pm s \mapsto s \cdot y_2$.
(5) Since the bottom horizontal maps are isomorphisms we see that going right to left the generator $[\overline{B}^n \times \overline{B}^n]^*$ gets sent to $\pm z$.

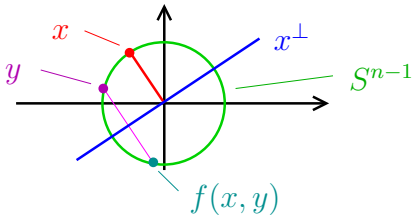
It follows immediately from the above comments that $\mathrm{Hopf}(\Theta_f) = \pm r \cdot s$. ■

84.3. **Examples of non-trivial bidegrees.** In light of the Hopf Invariant–Homomorphism Proposition 84.9 and the Bidegree–Hopf Invariant Proposition 84.14 we are now very eager to find maps $S^{n-1} \times S^{n-1} \to S^{n-1}$ with bidegree (r, s) where $r \cdot s \neq 0$. Perhaps somewhat surprisingly it is not that hard to find such maps:

Proposition 84.15. (Bidegree–Realization Proposition) Given any *even* $n \in \mathbb{N}$ the map

$$\begin{array}{rcl} f\colon S^{n-1} \times S^{n-1} & \to & S^{n-1} \\ (x, y) & \mapsto & y - 2 \cdot \langle x, y \rangle \cdot x \ = \text{ reflection of } y \text{ in the hyperplane } x^{\perp} \end{array}$$

has bidegree $(2, -1)$.



Proof. We need to show that the given map $f\colon S^{n-1} \times S^{n-1} \to S^{n-1}$ has bidegree $(2, -1)$. The proof of this statement naturally breaks up into two parts.

(1) We take $y_0 = (1, 0, \ldots, 0)$. We consider the map $\varphi\colon S^{n-1} \to S^{n-1}$ that is given by $x \mapsto f(x, y_0)$. Note that $\varphi^{-1}((1, 0, \ldots, 0)) = (\pm 1, 0, \ldots, 0)$. Using this observation together with the Degree-via-Local Orientations Proposition 44.7 one can easily show that the degree of the map $x \mapsto (x, y_0)$ equals $1 + (-1)^{n-2}$. Since n is by hypothesis even we see that the degree equals two.

(2) We take $x_0 = (1, 0, \ldots, 0)$. The corresponding map $S^{n-1} \to S^{n-1}$, $y \mapsto f(x_0, y)$ is the reflection in a hyperplane, hence it follows from the Sphere-Degree-Matrix Lemma 43.4 that the map has degree -1. ∎

We can now easily prove the following amazing theorem:

Theorem 84.16. ($\pi_{4n-1}(S^{2n})$–Infinite Theorem) Given any $n \in \mathbb{N}$ there exists an epimorphism $\pi_{4n-1}(S^{2n}) \to \mathbb{Z}$.

Proof. It follows from the Hopf Invariant–Homomorphism Proposition 84.9 together with the Bidegree–Realization Proposition 84.15 that the Hopf map defines a homomorphism $\pi_{4n-1}(S^{2n}) \to \mathbb{Z}$ with non-zero image. Since every non-zero subgroup of $\mathbb{Z}$ is isomorphic to $\mathbb{Z}$ this implies that there exists an epimorphism $\pi_{4n-1}(S^{2n}) \to \mathbb{Z}$. ∎

As always in mathematics, every result opens up lots of new questions. Here are a few:

Question 84.17. By the Hopf Invariant One-Corollary 84.10 the Hopf maps

$$\begin{array}{rcl} \text{(1)} \quad \text{Hopf}\colon \pi_3(S^2) & \to & \mathbb{Z} \\ \text{(2)} \quad \text{Hopf}\colon \pi_7(S^4) & \to & \mathbb{Z} \\ \text{(3)} \quad \text{Hopf}\colon \pi_{15}(S^8) & \to & \mathbb{Z} \end{array}$$

are epimorphisms. Are they also isomorphisms?

We will provide the answer to Question 84.17 (1) in the $\pi_n(S^3) = \pi_n(S^2)$–Theorem **??** we will see the answer to Question 84.17 (2) in the $\pi_{n+3}(S^n)$-Theorem **??** (1) and we will see the answer to (3) on page **??**.

Question 84.17 points to a problem: We have now developed some tools for showing that certain homotopy groups of spheres are non-trivial, but we have no tools yet for computing any interesting homotopy groups. We will partially rectify this problem in Chapter **??**.

Here is another question:

Question 84.18. Given any $n \in \mathbb{N}$ we know by the Hurewicz Sphere Epimorphism Lemma 59.3 and the $\pi_{4n-1}(S^{2n})$–Infinite Theorem 84.16 that $\pi_n(S^n)$ and $\pi_{4n-1}(S^{2n})$ admit an epimorphism onto $\mathbb{Z}$. Are there are any other homotopy groups of spheres that admit an epimorphism onto $\mathbb{Z}$?

We delay the answer to the $\pi_*(S^n)$-Finiteness Theorem **??**.

In light of the Division Algebra–Bidegree-Lemma 84.13 and the Bidegree–Realization Proposition 84.15 the following question becomes interesting:

Question 84.19. For which $n \in \mathbb{N}_{\geq 2}$ does there exist a continuous map $f\colon S^{2n-1} \to S^n$ such that $\mathrm{Hopf}(f) = 1$?[665]

We will see the complete answer to Question 84.19 in the Hopf Invariant–Power-of-2 Theorem **??** and the Adams Theorem **??**.

Initially one might not appreciate Question 84.19. But rather mysteriously it is not that hard to come up with maps of Hopf degree 2, but it is strikingly difficult to come up with maps of Hopf degree 1. To underline our point we conclude this chapter with another construction of a continuous map of Hopf degree 2.

Example. Let $n \in \mathbb{N}$. We start out with the following two reminders:

(1) We can view S^n as a CW-complex with one 0-cell and one n-cell. In particular, if we denote by S^n_a and S^n_b two copies of S^n then we view S^n_a as a CW-complex with one 0-cell P and one n-cell that we denote by S^n_a. Similarly we view S^n_b as a CW-complex with one 0-cell Q and one n-cell S^n_b.

(2) Given two finite CW-complexes X and Y we introduced in the CW-Complex Product Proposition 33.9 together with the CW-Category–Product Proposition 33.7 the product CW-structure on $X \times Y$. Recall that the n-cells of $X \times Y$ correspond precisely to products $e_x \times f_y$ where e_x is a p-cell of X and f_y is an $(n-p)$-cell of Y.

It follows from (1) and (2) that we can view $S^n_a \times S^n_b$ as a CW-complex with one 0-cell (P, Q), two n-cells $S^n_a \times \{Q\}$ and $\{P\} \times S^n_b$ and one $2n$-cell $S^n_a \times S^n_b$. We consider the maps

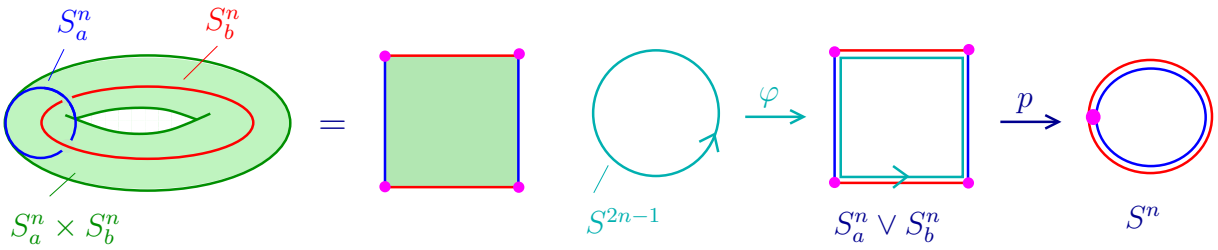
$$S^{2n-1} \xrightarrow{\varphi} \overbrace{(S^n_a \times \{Q\}) \cup (\{P\} \times S^n_b)}^{=S^n_a \vee S^n_b} \xrightarrow{p} S^n.$$

φ: attaching map of the $2n$-cell to the $(2n-1)$-skeleton of $S^n_a \times S^n_b$ which is precisely $S^n_a \times \{Q\} \cup \{P\} \times S^n_b$

p: obvious projection on $S^n_a \times \{Q\}$ and $\{P\} \times S^n_b$

The composition of the maps φ and p defines a continuous map $f\colon S^{2n-1} \to S^n$.
It is clear from the figure above that this map is not very interesting for $n = 1$. But using fairly elementary arguments one can show that for all even n we have $\mathrm{Hopf}(f) = \pm 2$. We

[665] Note that the Hopf Invariant–Homomorphism Proposition 84.9 and the Bidegree–Realization Proposition 84.15 tell us that this question is equivalent to the question, for which $n \in \mathbb{N}$ does there exist a continuous map $f\colon S^{2n-1} \to S^n$ such that $\mathrm{Hopf}(f)$ is odd?



refer to [**Hat02**, p. 428] or [**Vic94**, p. 138] for details. Alternatively we leave it as a brain teaser for the reader to perform this calculation. □

CHAPTER 85

The Basic Sphere–Alexander Duality Theorem

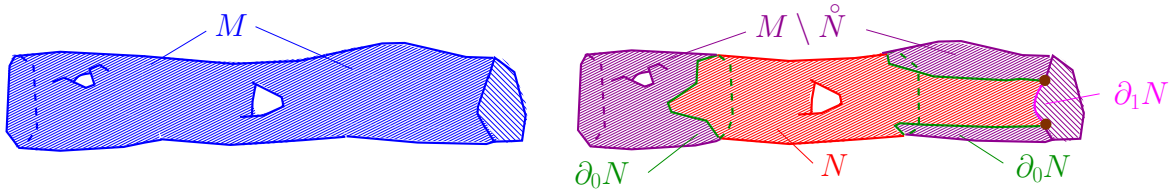
In this and the following chapter we will study two versions of the Sphere–Alexander Duality Theorem which both, given a "reasonable" subspace X, of S^n relate the homology groups of $S^n \setminus X$ to the cohomology groups of X. More precisely, we will prove the following results:

(1) In this chapter we will prove the "basic" Sphere–Alexander Duality Theorem 85.5 which says the following: If K is a subspace of S^n and if K admits a "regular neighborhood" (e.g. if K is a closed smooth submanifold of S^n), then for every $i \in \mathbb{N}_0$ there exists an isomorphism
$$\widetilde{\mathrm{H}}_i(S^n \setminus K;\mathbb{Z}) \;\cong\; \widetilde{\mathrm{H}}^{n-i-1}(K;\mathbb{Z}).$$
As we will see, once one has digested the definition of a regular neighborhood the above isomorphism follows from a short and pleasant argument involving the Poincaré Duality Theorem 79.1.

(2) In the following chapter we will prove the General Alexander Duality Theorem 86.1 which says that if $K \subset S^n$ is a closed subset that is "weakly locally contractible", then there exists a *natural* isomorphism
$$\widetilde{\mathrm{H}}_i(S^n \setminus K;\mathbb{Z}) \;\xrightarrow{\cong}\; \widetilde{\mathrm{H}}^{n-i-1}(K;\mathbb{Z}).$$
This statement is stronger than the statement in (1), but at the price, that the proof is much more involved. In particular it will build on Borsuk's Theorem 60.4.

85.1. **Definition and properties of regular neighborhoods.** In this section we will introduce the notion of a regular neighborhood of a compact subspace of a topological manifold. This notion is a variation on the notion of a tubular neighborhood of a smooth submanifold, that some readers might be familiar with.

Notation. Let M be an m-dimensional topological manifold and let $N \subset M$ be a codimension-zero submanifold of N with corner.[666] We write
$$\partial_0 N \;\coloneqq\; N \cap (\overline{M \setminus N}) \quad \text{and} \quad \partial_1 N \;\coloneqq\; N \cap \partial M.$$



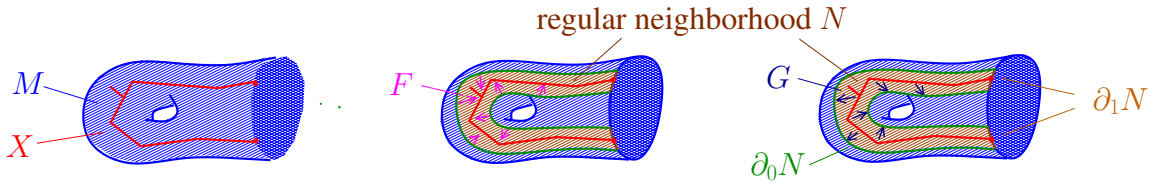
Now we can provide the definition of a regular neighborhood:

[666] We refer to [**Fri24**] for the definition of a submanifold with corner. For the most part a naive understanding of the concept will be more than enough.

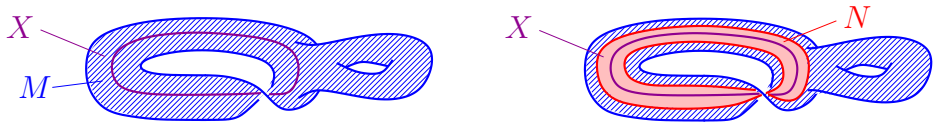
Definition. Let M be an m-dimensional topological manifold and let X be a compact subspace. A regular neighborhood of X is a compact codimension-zero submanifold N with corner of M with the following properties:[667] [668]

(1) X is contained in $N \setminus \partial_0 N$.
(2) There exists a deformation retraction $F\colon N \times [0,1] \to N$ from N to X such that $F(\partial_1 N \times [0,1]) \subset \partial_1 N$.
(3) There exists a deformation retraction $G\colon (N \setminus X) \times [0,1] \to N \setminus X$ from $N \setminus X$ to $\partial_0 N$ such that $G((\partial_1 N \setminus X) \times [0,1]) \subset \partial_1 N \setminus X$.

We refer to $N \setminus \partial_0 N$ as an open regular neighborhood of X.



Proposition 85.1. (Tubular–Regular Neighborhood Proposition) Suppose that M is a smooth manifold. Every compact proper smooth submanifold X of M admits a "tubular neighborhood" (i.e. a neighborhood which is a linear disk bundle), and every "tubular neighborhood N" of X is a regular neighborhood of X.



Proof. This theorem is proved in [**Fri24**]. ∎

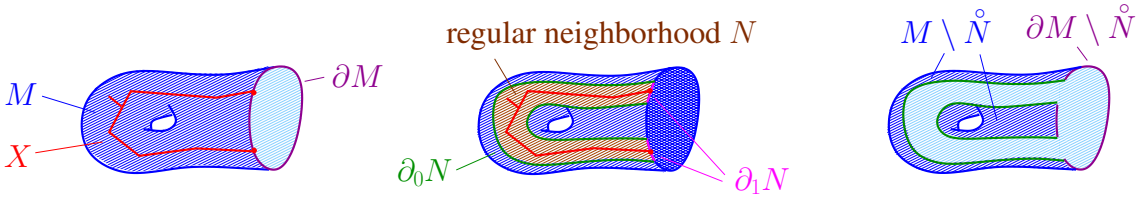
Let M be a topological manifold. Note that in the definition of a regular neighborhood we did not demand that the subspace $X \subset M$ is a submanifold, in particular X could a priori be any (closed) subspace. It will surely not come as a surprise that not even every compact subspace admits a regular neighborhood. For example one can easily show that the compact subspace $\{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\}$ of $\mathbb{R}$ does not admit a regular neighborhood. But there are many interesting examples of subspaces, which are not submanifolds, but which nonetheless admit regular neighborhoods. For example we will show in the Spatial Graph–Regular Neighborhood Proposition 85.4 that every "spatial graph" in $\mathbb{R}^3$ admits a regular neighborhood.

Lemma 85.2. (Basics-of-Regular Neighborhood Lemma) Let M be a topological manifold, let X be a compact subspace and let N be a regular neighborhood of X. As always we set $\partial_0 N := N \cap (\overline{M \setminus N})$. The following statements hold:

(1) (a) $\mathring{N} := N \setminus \partial_0 N$ is the interior of $N \subset M$.
    (b) N is a neighborhood of $X \subset M$.

[667] We will see in the Basics-of-Regular Neighborhood Lemma 85.2 that a regular neighborhood is indeed a neighborhood of X.
[668] Very loosely speaking a regular neighborhood can be thought of as a codimension-zero thickening of X.

(2) (a) $M \setminus \mathring{N}$ is a codimension-zero submanifold with corner of M where the boundary is given by $\partial(M \setminus \mathring{N}) = \partial_0 N \cup (\partial M \setminus \mathring{N})$.
(b) $\partial_0 N$ and $\partial M \setminus \mathring{N}$ are codimension-zero submanifolds of the topological manifold $\partial(M \setminus \mathring{N})$ and they are closed subsets of $\partial(M \setminus \mathring{N})$.
(3) (a) $M \setminus \mathring{N}$ is a deformation retract of $M \setminus X$ and $\partial M \setminus \mathring{N}$ is a deformation retract of $\partial M \setminus X$.
(b) The inclusions $M \setminus \mathring{N} \to M \setminus X$ and $\partial M \setminus \mathring{N} \to \partial M \setminus X$ are homotopy equivalences.



Proof.

(1) This statement is proved in [**Fri24**].
(2) This statement is proved in [**Fri24**].
(3) (a) First we recall that by definition of a regular neighborhood there exists a deformation retraction $G\colon (N \setminus X) \times [0,1] \to N \setminus X$ from $N \setminus X$ to $\partial_0 N$ such that $G((\partial_1 N \setminus X) \times [0,1]) \subset \partial_1 N \setminus X$. We extend this deformation retraction to all of $M \setminus X$ in the obvious way. More precisely, we consider the map

$$\begin{array}{rcl} G\colon (M \setminus X) \times [0,1] & \to & M \setminus X \\ (P,t) & \mapsto & \begin{cases} G(P,t), & \text{if } P \in N \setminus X, \\ P, & \text{if } P \notin \mathring{N}. \end{cases} \end{array}$$

It follows from (1a) and the Compact-Closed Lemma 1.9 that $N \setminus X$ and $M \setminus \mathring{N}$ are closed subsets of $M \setminus X$. It thus follows from the Homotopy Combination Lemma 8.2 (1) that this map G is indeed continuous. Once continuity is verified it is straightforward to show that G is indeed a deformation retraction from $M \setminus X$ to $M \setminus \mathring{N}$ and that it restricts to a deformation retraction from $\partial M \setminus X$ to $\partial M \setminus \mathring{N}$.
(b) It follows from (a) and the Deformation Retract–Homotopy Equivalence Lemma 9.3 that the inclusions $M \setminus X \to M \setminus \mathring{N}$ and $\partial M \setminus \mathring{N} \to \partial M \setminus X$ are indeed homotopy equivalences. ■

The following corollary, which in the literature is often used implicitly or explicitly, gives one application of the theory of regular neighborhoods.

Lemma 85.3. (Submanifold Complement–Invariants Lemma) Let M be a compact topological manifold and let $X \subset M$ be a compact subspace. If X admits a regular neighborhood, then the following two statements hold:

(1) The fundamental group $\pi_1(M \setminus X)$ is finitely presented.
(2) For each $k \in \mathbb{N}_0$ the homology group $\mathrm{H}_k(M \setminus X)$ is finitely generated.

Proof. Let N be a regular neighborhood of X. The desired statements follow from the following observations:

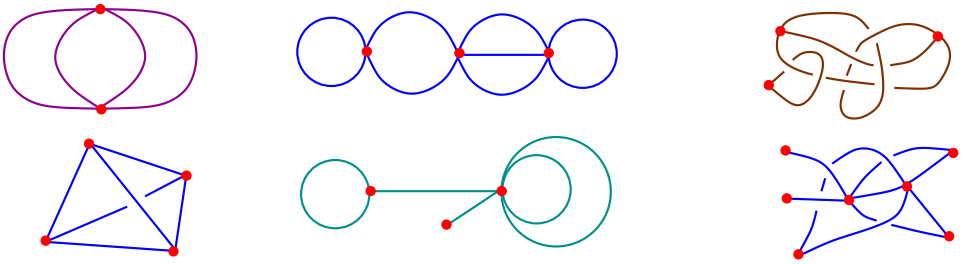
(a) By the Basics-of-Regular Neighborhood Lemma 85.2 the topological space $M \setminus X$ is homotopy equivalent to $M \setminus \mathring{N}$.

(b) Using the fact that N is compact and using the fact that N is a compact codimension-zero submanifold with corner one can easily show that that $M \setminus \overset{\circ}{N}$ is a compact n-dimensional topological manifold. Details are provided in [**Fri24**].

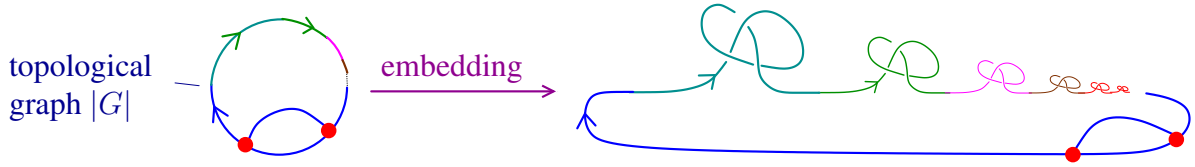
(c) By (b) and the Topological Manifolds-Invariants Proposition 60.10 we know that $\pi_1(M \setminus \overset{\circ}{N})$ is finitely presented and we know that the homology groups $\mathrm{H}_k(M \setminus \overset{\circ}{N})$ are finitely generated.

(d) It follows from (a), the Homotopy-π_1-Proposition 13.4 (2) and the Homotopy Equivalence-H_*-Corollary 36.7 that the invariants of $M \setminus \overset{\circ}{N}$ are isomorphic to the invariants of $M \setminus X$. ■

85.2. **Spatial graphs.** On several occasions we already considered knots and links in the 3-dimensional sphere $S^3 =_{\mathrm{i}} \mathbb{R}^3 \cup \{\infty\}$, i.e. we considered closed 1-dimensional smooth submanifolds of S^3. Playing with pen and paper one quickly ends up with generalizations of this concept, for example with the following "spatial graphs" in $\mathbb{R}^3 \subset S^3$:



We now want to give a formal definition of a "spatial graph" which captures the above examples. A first attempt might be to attempt to define "a spatial graph" as the image of an embedding of a (say finite) topological graph into $\mathbb{R}^3$. But below we show a "wild" spatial graph, which is perhaps not what we had initially in mind:



The last example shows that we need some control over such embeddings to end up with reasonable objects.

Some leisurely thinking leads us to the following definition:

Definition. A spatial graph is a pair $(G = (V, E, i, t), \Theta\colon |G| \to \mathbb{R}^3)$ where the following holds:

(1) The quadruple $G = (V, E, i\colon E \to V, t\colon E \to V)$ is a finite abstract graph.

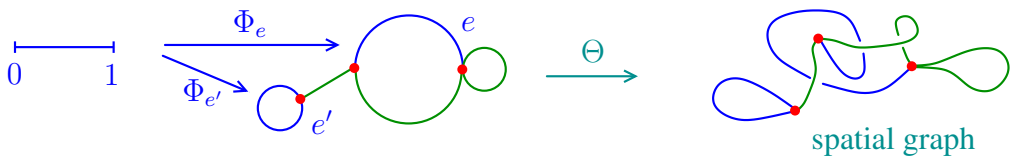
(2) The map[669] $\Theta\colon |G| \to \mathbb{R}^3$ is an embedding of the topological realization $|G|$ into $\mathbb{R}^3$.

(3) For every edge $e \in E$ with characteristic map $\Phi_e\colon [0,1] \to |G|$ the following statements hold:

(a) The map $f_e := \Theta \circ \Phi_e\colon [0,1] \to \mathbb{R}^3$ is smooth.

(b) The first derivative $f_e'(t)$ is non-zero for all $t \in [0,1]$,

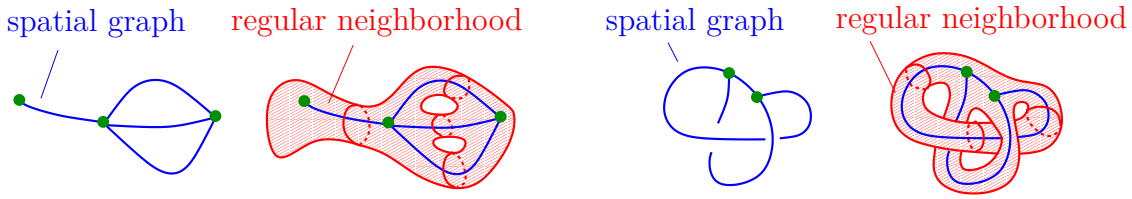
Sometimes we just refer to $\Theta(|G|)$ as a "spatial graph" and we ignore the extra data involved in the proper definition of a spatial graph.



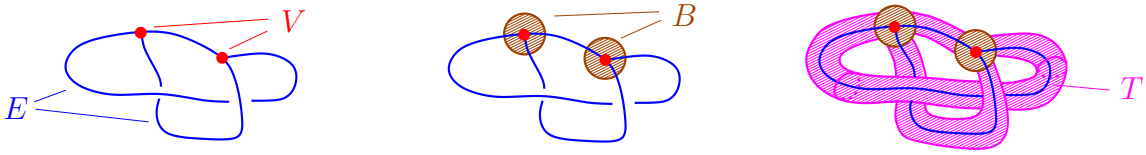
Example. It should hopefully be clear that the examples sketched in the figure on page 1101 are spatial graphs. □

Now we can formulate the following proposition:

Proposition 85.4. (Spatial Graph–Regular Neighborhood Proposition) Every spatial graph G in $\mathbb{R}^3$ admits a regular neighborhood.



Sketch of proof. Let $G = (V, E) \subset \mathbb{R}^3$ be a spatial graph. The idea of the proof is quite straightforward: First one shows that one can pick $\epsilon > 0$ with the following properties:
(1) For each $v \in V$ and each $t \in (0, \epsilon]$ the sphere $S^2_t(v)$ intersects each edge transversally.[670]
(2) The closed balls $\{\overline{B}^3_\epsilon(v)\}_{v\in V}$ are disjoint.
We set $B \coloneqq \bigcup_{v\in V} \overline{B}^3_\epsilon(v)$. It follows from (1) that $E \cap (\mathbb{R}^3 \setminus \mathring{B})$ is a proper 1-dimensional smooth submanifold of $\mathbb{R}^3 \setminus \mathring{B}$. Thus it follows from the Tubular Neighborhood Theorem [**Fri24**] that it admits a tubular neighborhood $T \subset \mathbb{R}^3 \setminus \mathring{B}$.



Then $N \coloneqq B \cup T$ has (after rounding corners) the desired properties. The details are worked out in [**FH18, FH19**]. ■

85.3. **The Basic Sphere–Alexander Duality Theorem.** In this and the following chapter we will prove several theorems which are called "Sphere–Alexander Duality Theorem". We start out with the most basic Sphere–Alexander Duality Theorem. This version is surprisingly easy to prove and it is already quite helpful:

Theorem 85.5. (Sphere–Alexander Duality Theorem) Let $n \in \mathbb{N}$ and let $X \subsetneq S^n$ be a compact subspace of S^n that admits a regular neighborhood in the sense of the definition on page 1099. For every $i \in \mathbb{N}_0$ there exists an isomorphism

$$\widetilde{\mathrm{H}}_i(S^n \setminus X) \;\cong\; \widetilde{\mathrm{H}}^{n-i-1}(X; \mathbb{Z}).$$

[670] Note that to arrange this conclusion one needs to make use of the fact that the derivatives of the maps f_e are non-zero also at the endpoints.

(By the Tubular–Regular Neighborhood Theorem 85.1 this conclusion holds for any closed submanifold $X \subsetneq S^n$.)

Proof. In the following let $X \subsetneq S^n$ be a compact subspace of S^n that admits a regular neighborhood N.



The following claim contains the part of the argument, which does not really rely on the fact that we work with the sphere S^n:

Claim 1. For every $i \in \mathbb{N}_0$ there exists an isomorphism

$$\mathrm{H}_i(S^n \setminus X) \;\cong\; \mathrm{H}^{n-i}(S^n, X; \mathbb{Z}).$$

Proof. We have the following isomorphisms:

by the Basics-of-Regular Neighborhood Lemma 85.2 (1) and (2), and the Topological Manifold-Induced Orientation Lemma 62.1 we know that $S^n \setminus \mathring{N}$ is a compact orientable n-dimensional topological manifolds, thus the isomorphism follows from by the Poincaré Duality Theorem 79.1 applied to $S^n \setminus \mathring{N}$

$$\mathrm{H}_i(S^n \setminus X) \xrightarrow[\uparrow]{\cong} \mathrm{H}_i(S^n \setminus X; \mathbb{Z}) \xleftarrow[\uparrow]{\cong} \mathrm{H}_i(S^n \setminus \mathring{N}; \mathbb{Z}) \xleftarrow[\cong]{\downarrow} \mathrm{H}^{n-i}(S^n \setminus \mathring{N}, \partial N; \mathbb{Z})$$

see page 659 — by the Basics-of-Regular Neighborhood Lemma 85.2 (3b) and the Homotopy-H_*-Proposition 36.5

$$\xleftarrow[\uparrow]{\cong} \mathrm{H}^{n-i}(S^n, N; \mathbb{Z}) \xleftarrow[\uparrow]{\cong} \mathrm{H}^{n-i}(S^n, X; \mathbb{Z}).$$

by the Excision-in-H^*-Theorem 69.1 (2) — by the Homotopy Equivalence-Relative-H^*-Corollary 67.10 since X is a deformation retract of N ⊞

The following claim now gives us the desired result:

Claim 2. For every $i \in \mathbb{N}_0$ there exists an isomorphism

$$\widetilde{\mathrm{H}}_i(S^n \setminus X) \;\cong\; \widetilde{\mathrm{H}}^{n-i-1}(X; \mathbb{Z}).$$

Proof. We denote by $\iota\colon X \to S^n$ the inclusion map. We consider the long exact sequence in reduced cohomology of the pair (S^n, X) that is provided by the Reduced LES-of-Pair Proposition 67.14:[671]

$$\underbrace{\widetilde{\mathrm{H}}^{n-i-1}(S^n; \mathbb{Z})}_{=0 \text{ for } i \neq -1} \to \widetilde{\mathrm{H}}^{n-i-1}(X; \mathbb{Z}) \xrightarrow{\delta} \underbrace{\mathrm{H}^{n-i}(S^n, X; \mathbb{Z})}_{\cong \mathrm{H}_i(S^n \setminus X), \text{ by Claim 1}} \to \underbrace{\widetilde{\mathrm{H}}^{n-i}(S^n; \mathbb{Z})}_{=0 \text{ for } i \neq 0} \xrightarrow{\iota^*} \widetilde{\mathrm{H}}^{n-i}(X; \mathbb{Z}) \to \dots$$

The case $i \neq 0$ of the claim follows from the above and the Reduced Homology Lemma 39.2 (1). The case $i = 0$ requires slightly more thought. In that case we have the following exact

[671] In the argument below we make use of the fact that by the calculation of $\widetilde{\mathrm{H}}^*(S^n; \mathbb{Z})$ on page 880 we know that $\widetilde{\mathrm{H}}^i(S^n; \mathbb{Z}) = 0$ for all groups except for $i = n$ and that $\widetilde{\mathrm{H}}^i(S^n; \mathbb{Z}) \cong \mathbb{Z}$.

sequence:

$$0 \to \widetilde{\mathrm{H}}^{n-1}(X;\mathbb{Z}) \xrightarrow{\delta_{n-1}} \underbrace{\mathrm{H}^n(S^n,X;\mathbb{Z})}_{\cong \mathrm{H}_0(S^n\setminus X)} \to \underbrace{\widetilde{\mathrm{H}}^n(S^n;\mathbb{Z})}_{\cong\,\mathbb{Z}} \xrightarrow{\iota^*} \widetilde{\mathrm{H}}^n(X;\mathbb{Z}) \to \mathit{dots}$$

We make a few observations:

(1) Recall that we assume that $X \subsetneq S^n$ is a proper subset of S^n. This means that there exists a $P \in S^n \setminus X$. Thus the inclusion $\iota\colon X \to S^n$ factors through $S^n \setminus \{P\}$. In particular $\iota^*\colon \widetilde{\mathrm{H}}^n(S^n;\mathbb{Z}) \to \widetilde{\mathrm{H}}^n(X;\mathbb{Z})$ factors through $\widetilde{\mathrm{H}}^n(S^n \setminus \{P\};\mathbb{Z})$, which is zero, since $S^n \setminus \{P\}$ is homeomorphic to $\mathbb{R}^n$.

(2) It follows from the above long exact sequence, (1), the Free Abelian Group–Splitting Lemma 18.5 and the Splitting Lemma 37.2 that $\mathrm{H}_0(S^n \setminus X) \cong \widetilde{\mathrm{H}}^{n-1}(X;\mathbb{Z}) \oplus \mathbb{Z}$.

(3) Since we assume that $S^n \setminus X \neq \varnothing$ we obtain from the Reduced Homology Lemma 39.2 that $\mathrm{H}_0(S^n \setminus X) \cong \widetilde{\mathrm{H}}_0(S^n \setminus X) \oplus \mathbb{Z}$.

(4) Finally note that it follows from (2), (3) and the cancellation property of finitely generated abelian groups (which we can apply by the Submanifold Complement–Invariants Lemma 85.3) that $\widetilde{\mathrm{H}}_0(S^n \setminus X) \cong \widetilde{\mathrm{H}}^{n-1}(X;\mathbb{Z})$. ■

In practice we often like to consider objects that lie in $\mathbb{R}^n$ instead of S^n. The following variation on the Sphere–Alexander Duality Theorem 85.5 deals with this situation.

Theorem 85.6. (Sphere–Alexander Duality-in-$\mathbb{R}^n$-Theorem) Let $n \in \mathbb{N}_0$ and let X be a compact non-empty subspace of $\mathbb{R}^n$. If X admits a regular neighborhood, then for any $i \in \mathbb{N}_0$ we have an isomorphism[672]

$$\widetilde{\mathrm{H}}_i(\mathbb{R}^n \setminus X) \;\cong\; \mathrm{H}^{n-i-1}(X;\mathbb{Z}).$$

(By the Tubular–Regular Neighborhood Theorem 85.1 this conclusion holds for any closed non-empty submanifold $X \subset \mathbb{R}^n$.)

The proof of the Sphere–Alexander Duality-in-$\mathbb{R}^n$-Theorem 85.6 rests on the following lemma:

Lemma 85.7. (Complement-in-$\mathbb{R}^n$-S^n-Lemma) Let $X \subset \mathbb{R}^n$ be a non-empty compact subspace. We also view X as a subspace of $\mathbb{R}^n \cup \{\infty\} =_{\mathrm{i}} S^n$. For any $i \in \mathbb{N}$ we have an isomorphism

$$\widetilde{\mathrm{H}}_i(\mathbb{R}^n \setminus X) \;\cong\; \begin{cases} \widetilde{\mathrm{H}}_i(S^n \setminus X), & \text{if } i \neq n-1, \\ \widetilde{\mathrm{H}}_i(S^n \setminus X) \oplus \mathbb{Z}, & \text{if } i = n-1. \end{cases}$$

The analogous statements also hold if we replace reduced homology by ordinary homology.

Proof. The lemma follows from a pleasant Mayer-Vietoris sequence argument. We leave the details to the reader. ■

Proof of the Sphere–Alexander Duality-in-$\mathbb{R}^n$-Theorem 85.6. Let X be a compact non-empty subspace of $\mathbb{R}^n$ that admits a regular neighborhood N in $\mathbb{R}^n$. Recall that N is by definition compact. It follows easily that N is also a regular neighborhood for X in

[672] Note that the effect of replacing S^n by $\mathbb{R}^n$ is to replace reduced cohomology by unreduced cohomology.

$S^n =_{\mathrm{i}} \mathbb{R}^n \cup \{\infty\}$. Now let $i \in \mathbb{N}_0$. Then

$$\mathrm{H}^{n-i-1}(X;\mathbb{Z}) \underset{\uparrow}{\cong} \begin{cases} \widetilde{\mathrm{H}}_i(S^n \setminus X), & \text{if } i \neq n-1, \\ \widetilde{\mathrm{H}}_{n-1}(S^n \setminus X) \oplus \mathbb{Z}, & \text{if } i = n-1 \end{cases} \underset{\uparrow}{\cong} \widetilde{\mathrm{H}}_i(\mathbb{R}^n \setminus X).$$

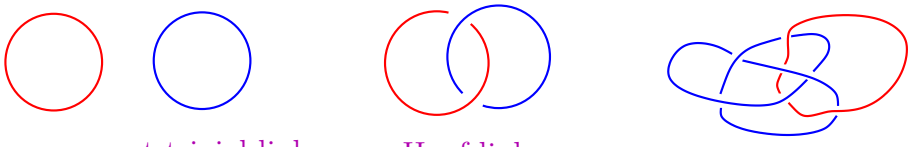
follows from the Reduced Cohomology Lemma 67.12 and the Sphere–Alexander Duality Theorem 85.5 which we can apply since X admits a regular neighborhood in S^n and since $X \subset \mathbb{R}^n \subsetneq S^n$

by the Complement-in-$\mathbb{R}^n$-S^n-Lemma 85.7, here we use that $X \neq \varnothing$

■

In the remainder of this section we discuss a few applications of the Sphere–Alexander Duality Theorem 85.5.

Definition. Let $m \in \mathbb{N}_0$. An m-component link is a 1-dimensional submanifold of S^3 that is diffeomorphic to the disjoint union of m copies of S^1.



2-component trivial link

Hopf link

The following lemma computes the homology groups of a link complement.

Proposition 85.8. (Link Complement-H_*-Proposition) Let $m \in \mathbb{N}$. For any m-component link L in S^3 we have

$$\mathrm{H}_i(S^3 \setminus L) \cong \begin{cases} \mathbb{Z}^{m-1}, & \text{if } i = 2, \\ \mathbb{Z}^m, & \text{if } i = 1, \\ \mathbb{Z}, & \text{if } i = 0, \\ 0, & \text{otherwise.} \end{cases}$$

For the homology groups $\mathrm{H}_i(\mathbb{R}^3 \setminus L)$ we get the same result, except for an extra $\mathbb{Z}$-summand in degree 2.

Proof. Let L be an m-component link. We calculate that

$$\widetilde{\mathrm{H}}_i(S^3 \setminus L) \underset{\uparrow}{\cong} \widetilde{\mathrm{H}}^{3-1-i}(L;\mathbb{Z}) \cong \widetilde{\mathrm{H}}^{2-i}(\underbrace{S^1 \sqcup \cdots \sqcup S^1}_{m\text{-copies}};\mathbb{Z}) \underset{\uparrow}{\cong} \begin{cases} \mathbb{Z}^{m-1}, & \text{if } i = 2, \\ \mathbb{Z}^m, & \text{if } i = 1, \\ 0, & \text{otherwise.} \end{cases}$$

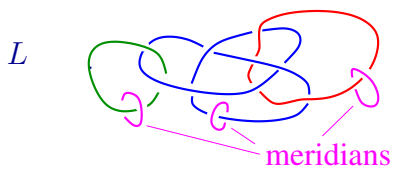
by the Sphere–Alexander Duality Theorem 85.5, since L is a smooth submanifold

by the discussion on page 880, the Path-Component-H^*-Lemma 67.4, and the Reduced Cohomology Lemma 67.12

The statement about the homology groups of $S^3 \setminus L$ follows from this calculation together with the Reduced Homology Lemma 39.2.

Finally note that we obtain the homology groups $\mathrm{H}_i(\mathbb{R}^3 \setminus L)$ by applying the Tubular–Regular Neighborhood Theorem 85.1 together with the Sphere–Alexander Duality-in-$\mathbb{R}^n$-Theorem 85.6, or alternatively, by cutting out the middle man, by applying the Complement-in-$\mathbb{R}^n$-S^n-Lemma 85.7 to the above calculation. ■

Example. For the three 3-component link L shown below we see that $\mathrm{H}_1(S^3 \setminus L) \cong \mathbb{Z}^3$ and $\mathrm{H}_2(S^3 \setminus L) \cong \mathbb{Z}^2$.
The problem with this calculation is that we do not obtain any understanding of what the homology classes "look like". It seems reasonable to guess that the first homology group is



$$\begin{aligned} \mathrm{H}_2(S^3 \setminus L) &\cong \mathbb{Z}^2, \\ \mathrm{H}_1(S^3 \setminus L) &\cong \mathbb{Z}^3, \\ \mathrm{H}_0(S^3 \setminus L) &\cong \mathbb{Z} \end{aligned}$$

generated by the three meridians. But (a), we do not know how to prove it, and (b), we have no idea what a generator of the second homology group looks like. □

Proposition 85.9. (Spatial Graph Complement-H$_*$-Proposition) Let G be an a finite abstract graph with $m \in \mathbb{N}$ components and let $\Theta\colon |G| \to \mathbb{R}^3$ be an embedding such that $\Theta(|G|)$ is a spatial graph. We also view $\Theta(|G|)$ as a subspace of $\mathbb{R}^3 \cup \{\infty\} =_{\mathrm{i}} S^3$. Then

$$\mathrm{H}_i(S^3 \setminus \Theta(|G|)) \cong \begin{cases} \mathbb{Z}^{m-1}, & \text{if } i = 2, \\ \mathbb{Z}^{m-\chi(G)}, & \text{if } i = 1, \\ \mathbb{Z}, & \text{if } i = 0, \\ 0, & \text{otherwise.} \end{cases}$$

For the homology groups $\mathrm{H}_i(\mathbb{R}^3 \setminus \Theta(|G|))$ we get the same result, except for an extra $\mathbb{Z}$-summand in degree 2.

Proof. We have the following isomorphisms:

$$\widetilde{\mathrm{H}}_i(S^3 \setminus \Theta(|G|)) \underset{\uparrow}{\cong} \widetilde{\mathrm{H}}^{2-i}(\Theta(|G|); \mathbb{Z}) \cong \widetilde{\mathrm{H}}^{2-i}(|G|; \mathbb{Z}) \underset{\uparrow}{\cong} \begin{cases} \mathbb{Z}^{m-1}, & \text{if } i = 2, \\ \mathbb{Z}^{m-\chi(G)}, & \text{if } i = 1, \\ 0, & \text{otherwise.} \end{cases}$$

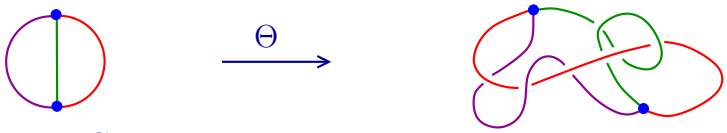
by the Spatial Graph–Regular Neighborhood Proposition 85.4 we can apply Sphere–Alexander Duality-in-$\mathbb{R}^n$-Theorem 85.6

by the discussion on page 880 together with the Path-Component-H*-Lemma 67.4 and the Reduced Cohomology Lemma 67.12

Again the statement about the homology groups of $S^3 \setminus \Theta(|G|)$ follows from this calculation together with the Reduced Homology Lemma 39.2.

Finally note that once again we obtain the homology groups $\mathrm{H}_i(\mathbb{R}^3 \setminus \Theta(|G|))$ by applying the Sphere–Alexander Duality-in-$\mathbb{R}^n$-Theorem 85.6, or alternatively by applying the Complement-in-$\mathbb{R}^n$-S^n-Lemma 85.7. ■

Example. We consider the connected abstract graph G with two vertices and three edges that is illustrated in the figure below on the left. On the right of the figure we show a corresponding spatial graph $\Theta(|G|)$ in $S^3 =_{\mathrm{i}} \mathbb{R}^3 \cup \{\infty\}$ that is homeomorphic to G.



Note that $m = 1$ and $\chi(G) = -1$. From the Spatial Graph Complement-H$_*$-Proposition 85.9 we thus obtain that

$$\mathrm{H}_i(\mathbb{R}^3 \setminus \Theta(|G|)) \cong \begin{cases} \mathbb{Z}, & \text{if } i = 2, \\ \mathbb{Z}^2, & \text{if } i = 1, \\ \mathbb{Z}, & \text{if } i = 0, \\ 0, & \text{otherwise.} \end{cases}$$

In this case it is also rather unclear what a basis of $\mathrm{H}_1(S^3 \setminus G)$ should look like. □

85.4. **Immersions and smooth embeddings of smooth manifolds into $\mathbb{R}^n$.** We now have the tool to prove the following pretty proposition:

Proposition 85.10. (Smooth Codimension-One Orientability Proposition) Let N be a closed n-dimensional smooth manifold. If N is non-orientable, then there is no smooth embedding of N into $\mathbb{R}^{n+1}$.

Proof. Clearly it suffices to deal with the case that N is connected. Thus let N be a closed *non-orientable* connected n-dimensional smooth manifold. Suppose that N admits a smooth embedding $N \to \mathbb{R}^{n+1}$. It follows that we can view N as a submanifold of $\mathbb{R}^{n+1}$. We obtain that

$$\mathbb{Z}_2 \overset{\text{by the Manifold-Top-H}^*\text{-Theorem 70.16 (2)}}{\cong} \mathrm{H}^n(N;\mathbb{Z}) \underset{\substack{\text{by the Sphere–Alexander Duality-in-}\mathbb{R}^n\text{-Theorem 85.6 which we can apply since } N \text{ is a non-empty closed smooth submanifold of } \mathbb{R}^{n+1}}}{\cong} \widetilde{\mathrm{H}}_0(\mathbb{R}^{n+1} \setminus N;\mathbb{Z}) \overset{\substack{\text{by the } \mathrm{H}_0\text{-Proposition 35.11 together with the Reduced Homology Lemma 39.2 (0)}}}{\cong} \mathbb{Z}^{\text{number of path-components of } \mathbb{R}^{n+1} \setminus N \text{ minus } 1}.$$

But this is absurd. We have thus obtained the desired contradiction. ■

It is worth pointing out the following special case:

Corollary 85.11. (Surface-No Smooth Embedding Corollary) Neither $\mathbb{R}\mathrm{P}^2$ nor the Klein bottle admit a smooth embedding into $\mathbb{R}^3$.

Proof. This corollary follows immediately from the Smooth Codimension-One Orientability Proposition 85.10 since $\mathbb{R}\mathrm{P}^2$ and the Klein bottle are both closed non-orientable 2-dimensional smooth manifolds. ■

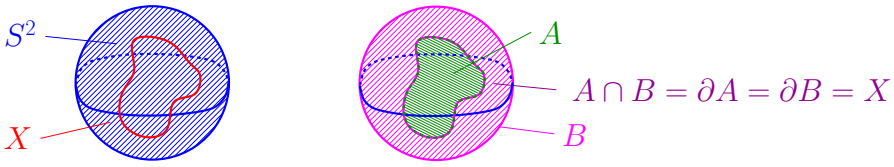
Note that it follows from the Orientation-Top Homology-Theorem 63.3 and the calculation of the homology groups of $\mathbb{R}\mathrm{P}^n$ in the $\mathbb{R}\mathrm{P}^n$-H_*-Proposition 47.10, that $\mathbb{R}\mathrm{P}^n$ is orientable if and only if n is odd. In particular it follows from the Smooth Codimension-One Orientability Proposition 85.10 that for n even it is not possible to smoothly embed $\mathbb{R}\mathrm{P}^n$ into S^{n+1}. But this raises the following question: Given n odd, can one find a smooth embedding of $\mathbb{R}\mathrm{P}^n$ into S^{n+1}? For $n = 1$ we can do so: First note that $\mathbb{R}\mathrm{P}^1$ is diffeomorphic to[673] S^1 and clearly we can find a smooth embedding of S^1 into S^2. But for odd $n \geq 3$ we can answer this question in the negative:

Proposition 85.12. ($\mathbb{R}\mathrm{P}^n$–No Smooth Embedding Proposition) If $n \in \mathbb{N}_{\geq 2}$, then there is no smooth embedding from $\mathbb{R}\mathrm{P}^n$ into S^{n+1}.

[673] In the Projective Space-Avatar Lemma 6.11 we showed that $\mathbb{R}\mathrm{P}^1$ is homeomorphic to S^1 and one can easily show that $\mathbb{R}\mathrm{P}^1$ is actually diffeomorphic to S^1.

In the proof of the $\mathbb{R}\mathrm{P}^n$–No Smooth Embedding Proposition 85.12 we will need the following lemma.

Lemma 85.13. (Smooth Sphere-Splitting Lemma) Let $n \in \mathbb{N}_{\geq 2}$. If X is a closed orientable connected non-empty $(n-1)$-dimensional smooth submanifold of the smooth manifold S^n, then there exist two compact connected codimension-zero smooth submanifolds A and B of S^n such that $A \cap B = \partial A = \partial B = X$.



Proof. First we calculate that

$$\mathrm{H}_0(S^n \setminus X) \underset{\uparrow}{\cong} \widetilde{\mathrm{H}}_0(S^n \setminus X) \oplus \mathbb{Z} \underset{\uparrow}{\cong} \mathrm{H}^{n-1}(X;\mathbb{Z}) \oplus \mathbb{Z} \underset{\uparrow}{\cong} \mathbb{Z} \oplus \mathbb{Z}.$$

by the Reduced Homology Lemma 39.2 (0b), since $X \subsetneq S^n$

by the Sphere–Alexander Duality Theorem 85.5

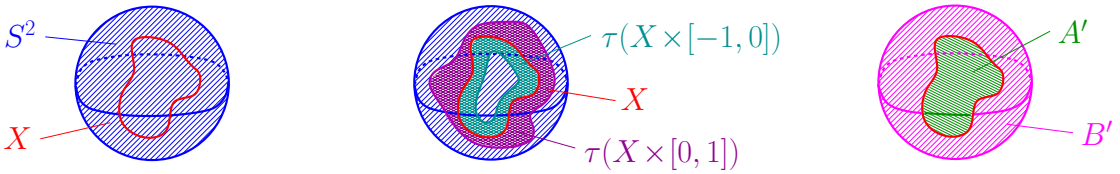
by the Orientation-Top Homology Theorem 63.3 and since X is closed, orientable connected, non-empty and $(n-1)$-dimensional

It follows from this calculation and the H_0-Proposition 35.11 that $S^n \setminus X$ has precisely two path-components A' and B'.

Claim 1. The subsets A' and B' of S^n are open.

Proof. Since $X \subset S^n$ is compact we know by the Compact-Closed Lemma 1.9 that X is a closed subset of S^n. Thus $S^n \setminus X$ is an open subset of S^n. Next note that it follows from the fact that topological manifolds are locally path-connected that A' and B' are open subsets of $S^n \setminus X$. It follows from the Open-Closed Inclusion Lemma 2.7 that A' and B' are also open subsets of S^n. ⊞

Next note that it follows from the Tubular Neighborhood Theorem [**Fri24**] that the submanifold X admits a tubular map $\tau\colon [-1,1] \times X \to S^n$. Also note that it follows from the Product Topology-Properties Proposition 5.4 (2) that $\tau((-1,0) \times X)$ and $\tau((0,1) \times X)$ are also path-connected.



Since $\tau((-1,0) \times X)$ is path-connected and since it is contained in $S^n \setminus X = A' \cup B'$ it follows from the Path-Component Lemma 2.21 (4b) that $\tau((-1,0) \times X)$ is contained in A' or it is contained in B'. Without loss of generality we can assume that $\tau((-1,0)) \times X$ is contained in A'.

Claim 2. The subspace $\tau((0,1) \times X)$ is contained in B'.

Proof. We consider the set $A'' := A' \cup \tau((-1,1) \times X)$. Note that $S^n = A'' \cup B'$. It follows from Claim 1 that both A'' and B' are open subsets of S^n. Since S^n is connected it follows that $A'' \cap B' \neq \varnothing$. Since $(A' \cup X) \cap B' = \varnothing$ we see that this implies that

$\tau((0,1)\times X)\cap B'\neq\varnothing$. But since $(0,1)\times X$ is connected and contained in $S^n\setminus X = A'\cup B'$ we obtain that $\tau((0,1)\times X)$ is contained in B'. ⊞

We set $A := A'\cup X$ and $B := B'\cup X$.

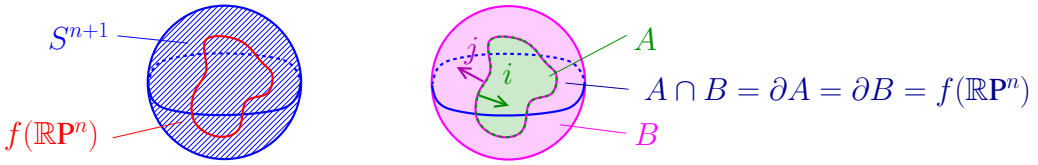
Claim 3. The subspaces A and B are compact connected codimension-zero submanifolds of S^n such that $A\cap B = \partial A = \partial B = X$.

Proof. First note that it follows from the above discussion that $S^n\setminus A = B'$ and $S^n\setminus B = A'$. It follows from Claim 1 that A and B are closed subsets of S^n. It follows from the Compact-Closed Lemma 1.9 that A and B are compact. Next note that $A = A'\cup\tau((-1,0]\times X)$ and $B = B'\cup\tau([0,1)\times X)$. It follows from this observation and the Union Connected Lemma 2.19 that A and B are connected.

Finally we need to find submanifold charts for A and B. Since A' and B' are open subsets of S^n we obtain for the points in A' and B' submanifold charts of type (α). Finally, for points in X we have the tubular neighborhood $\tau((-1,1)\times X)$, which gives rise to submanifold charts of type (γ). Using these two types of charts we see that A and B are codimension-zero submanifolds with $A\cap B = \partial A = \partial B = X$. ■

Proof of Proposition 85.12. As we had discussed just before the proposition, it remains to deal with the case that $n\geq 2$ is odd. So suppose that $n\geq 3$ is odd and that there exists a smooth embedding $f\colon \mathbb{R}\mathrm{P}^n\to S^{n+1}$. First note that $f(\mathbb{R}\mathrm{P}^n)$ is a submanifold of S^{n+1}. Next note that it follows from $n\geq 3$ and the discussion on page 880 that $\mathrm{H}^1(S^n;\mathbb{F}_2) = \mathrm{H}^2(S^n;\mathbb{F}_2) = 0$.

Since n is odd we know that $\mathbb{R}\mathrm{P}^n$ is orientable. Thus we can apply the Smooth Sphere-Splitting Lemma 85.13 which tells us that there exist two compact connected $(n+1)$-dimensional smooth submanifolds A and B of S^{n+1} with $A\cap B = \partial A = \partial B = f(\mathbb{R}\mathrm{P}^n)$. We denote by $i\colon f(\mathbb{R}\mathrm{P}^n)\to A$ and by $j\colon f(\mathbb{R}\mathrm{P}^n)\to B$ the inclusion maps. By the discussion on page 881 we know that $\mathrm{H}^1(\mathbb{R}\mathrm{P}^n;\mathbb{F}_2)\cong\mathbb{F}_2$. As usual we denote by $x\in\mathrm{H}^1(f(\mathbb{R}\mathrm{P}^n);\mathbb{F}_2)\cong\mathbb{F}_2$ the unique non-zero element.



Since A and B are compact codimension-zero smooth submanifolds with $A\cap B = \partial A = \partial B$ we can apply the Mayer–Vietoris-H^*-Theorem 69.5 (3) to $S^{n+1} = A\cup B$ and we obtain the long exact sequence

$$\cdots\to\underbrace{\mathrm{H}^1(S^{n+1};\mathbb{F}_2)}_{=0\text{ since }n\geq 3}\to\mathrm{H}^1(A;\mathbb{F}_2)\oplus\mathrm{H}^1(B;\mathbb{F}_2)\xrightarrow{(i^*,-j^*)}\underbrace{\mathrm{H}^1(f(\mathbb{R}\mathrm{P}^n);\mathbb{F}_2)}_{=\mathbb{F}_2\cdot x}\to\underbrace{\mathrm{H}^2(S^{n+1};\mathbb{F}_2)}_{=0\text{ since }n\geq 3}\to\dots$$

Thus we see that the map $i^*\oplus j^*\colon \mathrm{H}^1(A;\mathbb{F}_2)\oplus\mathrm{H}^1(B;\mathbb{F}_2)\to\mathrm{H}^1(f(\mathbb{R}\mathrm{P}^n);\mathbb{F}_2) = \mathbb{F}_2\cdot x$ is an isomorphism. Without loss of generality we can assume that i^* is non-trivial which means that there exists an $a\in\mathrm{H}^1(A;\mathbb{F}_2)$ with $i^*(a) = x$. Next note that

$$i^*(a^n) \underset{\text{by the Cup Product-Naturality Lemma 72.2}}{\overset{\uparrow}{=}} (i^*(a))^n = x^n \underset{\text{by the Cup Product-}\mathbb{R}\mathrm{P}^n\text{-Proposition 81.9}}{\overset{\uparrow}{\neq}} 0 \in \mathrm{H}^n(f(\mathbb{R}\mathrm{P}^n);\mathbb{F}_2).$$

In particular the map $i^*\colon \mathrm{H}^n(A;\mathbb{F}_2) \to \mathrm{H}^n(f(\mathbb{R}\mathrm{P}^n);\mathbb{F}_2)$ is non-zero. Now we consider the other end of the above Mayer-Vietoris sequence and we obtain the exact sequence

$$\mathrm{H}^n(A;\mathbb{F}_2)\oplus\mathrm{H}^n(B;\mathbb{F}_2) \xrightarrow{(i^*,-j^*)} \overbrace{\mathrm{H}^n(f(\mathbb{R}\mathrm{P}^n);\mathbb{F}_2)}^{=\mathbb{F}_2\cdot x^n} \to \overbrace{\mathrm{H}^{n+1}(S^{n+1};\mathbb{F}_2)}^{\cong\mathbb{F}_2} \to \underbrace{\mathrm{H}^{n+1}(A;\mathbb{F}_2)\oplus\mathrm{H}^{n+1}(B;\mathbb{F}_2)}_{\text{both are zero by the Top Cohomology Theorem 70.16 (2a), since } A \text{ and } B \text{ are compact, connected } (n+1)\text{-dimensional smooth manifolds with non-empty boundary}}.$$

$$(a^n, 0) \qquad \mapsto \quad x^n$$

Thus we see that the map on the left is an epimorphism and that the group on the right is trivial. But then the fact that $\mathrm{H}^{n+1}(S^{n+1};\mathbb{F}_2) \neq 0$ (which we obtained on page 880) shows that the sequence cannot possibly be exact. We have thus obtained the desired contradiction. ∎

85.5. **Alexander duality for smooth manifolds with boundary.** The following theorem generalizes the Sphere–Alexander Duality Theorem 85.5:

Theorem 85.14. (Alexander Duality–Boundary Theorem) Let $n \in \mathbb{N}$, let Σ be a compact orientable n-dimensional smooth manifold and let $\partial\Sigma = A \cup B$ be a decomposition of the boundary $\partial\Sigma$ into unions of components. Let X be a compact subspace of Σ that admits a regular neighborhood in the sense of the definition on page 1099.

(1) Let $j \in \mathbb{N}_0$ such that $\mathrm{H}_j(\Sigma, A) = \mathrm{H}_{j+1}(\Sigma, A) = 0$. Any choice of a regular neighborhood of $X \subset \Sigma$ defines an isomorphism

$$\mathrm{H}_j(\Sigma \setminus X, A \setminus X) \xrightarrow{\cong} \mathrm{H}^{n-j-1}(X, X \cap B; \mathbb{Z}).$$

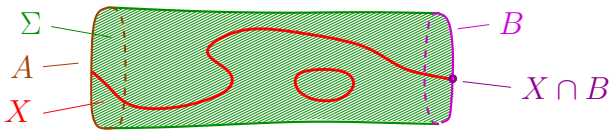
(2) We have two extra results for the "extremes" $j = 0$, $A = \varnothing$ and $j = n-1$, $B = \varnothing$:[674]

(a) If $X \subsetneq \Sigma$ and if $\widetilde{\mathrm{H}}_0(\Sigma) = \mathrm{H}_1(\Sigma) = 0$, then we obtain an isomorphism

$$\widetilde{\mathrm{H}}_0(\Sigma \setminus X) \xrightarrow{\cong} \mathrm{H}^{n-1}(X, X \cap \partial\Sigma; \mathbb{Z}).$$

(b) If $X \subsetneq \Sigma$, if $X \neq \varnothing$, if Σ is connected and if $\mathrm{H}_{n-1}(\Sigma, \partial\Sigma) = 0$, then there exists an isomorphism

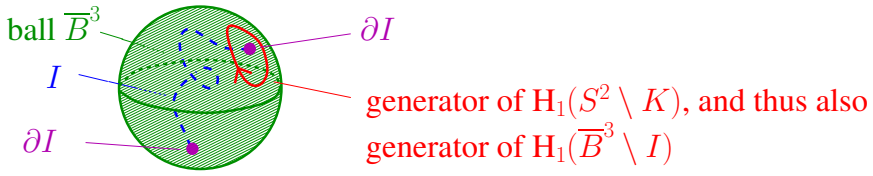
$$\mathrm{H}_{n-1}(\Sigma \setminus X, \partial\Sigma \setminus X) \xrightarrow{\cong} \widetilde{\mathrm{H}}^0(X; \mathbb{Z}).$$



we have $\mathrm{H}_*(\Sigma, A) = 0$ since A is a deformation retract of Σ

Example. Let $I \subset \overline{B}^3$ be a proper 1-dimensional smooth submanifold that is diffeomorphic to an interval.

[674] These two special cases deal with the nuisance that for a compact connected orientable non-empty smooth manifold Σ we have $\mathrm{H}_0(\Sigma) \cong \mathbb{Z}$ and we have $\mathrm{H}_n(\Sigma, \partial\Sigma) \cong \mathbb{Z}$. For $j = 0$, $A = \varnothing$ and $j = n-1$ and $B = \varnothing$ this prevents us from applying (1) directly.



Note that for any $j \in \mathbb{N}_0$ we have

$$\widetilde{\mathrm{H}}_j(\overline{B}^3 \setminus I) \xrightarrow{\cong} \mathrm{H}^{3-j-1}(I, I \cap S^2; \mathbb{Z}) \xrightarrow{\cong} \mathrm{H}^{2-j}([0,1], \{0,1\}; \mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & \text{if } j = 1, \\ 0, & \text{otherwise.} \end{cases}$$

by the Alexander Duality–Boundary Theorem 85.14 (1) (2a), applied with $A = \varnothing$ and with $B = S^2$, note that we use that the reduced homology groups of $\overline{B}^3$ are zero

since I is a properly embedded interval

this follows for example from the long exact sequence in cohomology of the pair $([0,1], \{0,1\})$

Note that these groups are isomorphic to the corresponding homology groups of $S^2 \setminus \partial I$. So what can we see about the inclusion-induced map $i_*\colon \mathrm{H}_1(S^2 \setminus \partial I) \to \mathrm{H}_1(\overline{B}^3 \setminus I)$? To answer this question we consider the long exact sequence of the pair $(\overline{B}^3 \setminus I, S^2 \setminus \partial I)$:

$$\mathrm{H}_2(\overline{B}^3 \setminus I, S^2 \setminus \partial I) \xrightarrow{\partial_2} \mathrm{H}_1(S^2 \setminus \partial I) \xrightarrow{i_*} \mathrm{H}_1(\overline{B}^3 \setminus I) \to \mathrm{H}_1(\overline{B}^3 \setminus I, S^2 \setminus \partial I).$$

To understand the middle map we need to understand the groups to the left and right. We now see that for $j = 1, 2$ we have

$$\mathrm{H}_j(\overline{B}^3 \setminus I, S^2 \setminus \partial I) \xrightarrow{\cong} \widetilde{\mathrm{H}}^{3-j-1}(I; \mathbb{Z}) \xrightarrow{\cong} \widetilde{\mathrm{H}}^{2-j}([0,1]; \mathbb{Z}) = 0.$$

by the Alexander Duality–Boundary Theorem 85.14 (1) and (2b), applied with $A = S^2$ and with $B = \varnothing$, note that we use that the homology groups of $(\overline{B}^3, S^2)$ are zero in dimensions $1, 2$

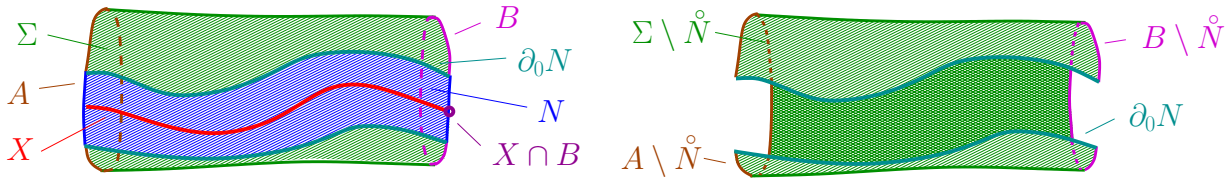
since I is a properly embedded interval

since I is contractible

It follows that in the above long exact sequence the middle map $i_*\colon \mathrm{H}_1(S^2 \setminus \partial I) \to \mathrm{H}_1(\overline{B}^3 \setminus I)$ is an isomorphism. □

The proof of the Alexander Duality–Boundary Theorem 85.14 will require the remainder of this section. The impatient reader can also jump to Chapter 86 where we will prove a generalization of the Sphere–Alexander Duality Theorem 85.5.

Proof of the Alexander Duality–Boundary Theorem 85.14 (1). Let $n \in \mathbb{N}$, let Σ be a compact orientable n-dimensional smooth manifold and let $\partial\Sigma = A \cup B$ be a decomposition of the boundary $\partial\Sigma$ into unions of components. Let X be a compact subspace of Σ that admits a regular neighborhood N. Finally let $j \in \mathbb{N}_0$ such that $\mathrm{H}_j(\Sigma, A) = \mathrm{H}_{j+1}(\Sigma, A) = 0$.



Rather similar to the proof of the Sphere–Alexander Duality Theorem 85.5 on page 1103 we have the following isomorphisms:

$$\begin{array}{lll}
\mathrm{H}_j(\Sigma \setminus X, A \setminus X) & \xleftarrow[(0)]{\cong} \mathrm{H}_j(\Sigma \setminus \mathring{N}, A \setminus \mathring{N}) & \xrightarrow[(1)]{\cong} \mathrm{H}_j(\Sigma \setminus \mathring{N}, A \setminus \mathring{N}; \mathbb{Z}) \\
& \xleftarrow[(2)]{\cong} \mathrm{H}^{n-j}(\Sigma \setminus \mathring{N}, \partial_0 N \cup (B \setminus \mathring{N}); \mathbb{Z}) & \xleftarrow[(3)]{\cong} \mathrm{H}^{n-j}(\Sigma, N \cup B; \mathbb{Z}) \\
& \xleftarrow[(4)]{\cong} \mathrm{H}^{n-j-1}(N \cup B, B; \mathbb{Z}) & \xrightarrow[(5)]{\cong} \mathrm{H}^{n-j-1}(N, N \cap B; \mathbb{Z}) \\
& \xrightarrow[(6)]{\cong} \mathrm{H}^{n-j-1}(X, X \cap B; \mathbb{Z}). &
\end{array}$$

All that remains to do, is to explain, where the maps come from, and why they are isomorphisms. Here is the executive summary:

(0) This holds since the topological spaces are homotopy equivalent.
(1) This holds since ordinary homology is isomorphic to homology with $\mathbb{Z}$-coefficients.
(2) This follows from the Poincaré Duality Theorem 79.1.
(3) This follows from an excision argument.
(4) This follows from our hypothesis on the homology groups of (Σ, A), Poincaré Duality and a long exact sequence argument.
(5) This follows again from an excision style argument.
(6) This holds since the topological spaces are homotopy equivalent.

Here are the gory details:

(0) It follows easily from the Basics-of-Regular Neighborhood Lemma 85.2 that the inclusions $\Sigma \setminus \mathring{N} \to \Sigma \setminus X$ and $A \setminus \mathring{N} \to A \setminus X$ are homotopy equivalences. It thus follows from the Homotopy Equivalence-Relative-H_*-Corollary 38.4 that the inclusion-induced map $\mathrm{H}_j(\Sigma \setminus \mathring{N}, A \setminus \mathring{N}) \to \mathrm{H}_j(\Sigma \setminus X, A \setminus X)$ is an isomorphism.
(1) It follows from the discussion on page 659 of homology with $\mathbb{Z}$-coefficients that we have an isomorphism $\mathrm{H}_j(\Sigma \setminus \mathring{N}, A \setminus \mathring{N}) \to \mathrm{H}_j(\Sigma \setminus \mathring{N}, A \setminus \mathring{N}; \mathbb{Z})$.
(2) By the Basics-of-Regular Neighborhood Lemma 85.2 (1) and (2), and the Topological Manifold-Induced Orientation Lemma 62.1 we know that $\Sigma \setminus \mathring{N}$ is a compact orientable n-dimensional topological manifold. Furthermore we know by the Basics-of-Regular Neighborhood Lemma 85.2 (2b) that $\partial(\Sigma \setminus \mathring{N}) = (A \setminus \mathring{N}) \cup (\partial_0 N \cup (B \setminus \mathring{N}))$ is a decomposition into two compact codimension-zero submanifolds that intersect in the common boundary. It thus follows from the Poincaré Duality Theorem 79.1 applied to $\Sigma \setminus \mathring{N}$ (together with the decomposition $\partial(\Sigma \setminus \mathring{N}) = (A \setminus \mathring{N}) \cup (\partial_0 N \cup (B \setminus \mathring{N}))$), that we have an isomorphism $\mathrm{H}^{n-j}(\Sigma \setminus \mathring{N}, \partial_0 N \cup (B \setminus \mathring{N}); \mathbb{Z}) \to \mathrm{H}_j(\Sigma \setminus \mathring{N}, A \setminus \mathring{N}; \mathbb{Z})$.
(3) By definition of a regular neighborhood we know that N is a smooth submanifold with corner of N. It thus follows from a slight generalization of the Excision-in-H^*-Theorem 69.1 to codimension-zero smooth submanifolds with corner that the inclusion-induced map

$$\mathrm{H}^{n-j}(\Sigma, N \cup B; \mathbb{Z}) \;\to\; \mathrm{H}^{n-j}(\Sigma \setminus \mathring{N}, \partial_0 N \cup (B \setminus \mathring{N}); \mathbb{Z})$$

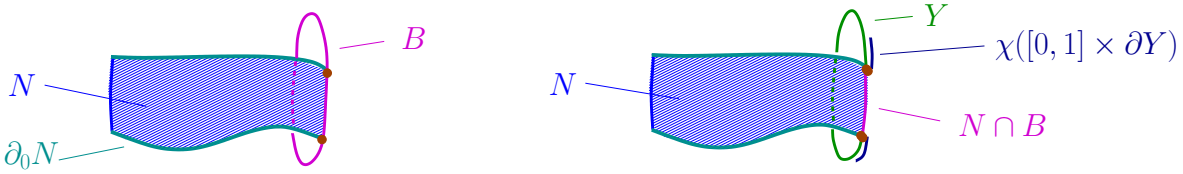
is an isomorphism.

(4) From the LES-of-Triple Proposition 67.9 we obtain the following exact sequence:

$$\underbrace{\mathrm{H}^{n-j-1}(\Sigma, B; \mathbb{Z})}_{\cong \mathrm{H}_{j+1}(\Sigma, A)=0} \to \mathrm{H}^{n-j-1}(N \cup B, B; \mathbb{Z}) \xrightarrow{\delta} \mathrm{H}^{n-j}(\Sigma, N \cup B; \mathbb{Z}) \to \underbrace{\mathrm{H}^{n-j}(\Sigma, B; \mathbb{Z})}_{\cong \mathrm{H}_j(\Sigma, A)=0} \to$$

It follows from the Poincaré Duality Theorem 79.1 and our hypothesis that the groups to the left and right are zero. Thus the homomorphism we care about is indeed an isomorphism.

(5) One can easily verify that $N \cap B$ is a compact codimension-zero submanifold of B with boundary $(\partial_0 N) \cap B$. We set $Y := B \setminus \mathring{N}$. Note that Y is a codimension-zero submanifold of B. By the Smooth Collar Neighborhood Theorem **??** we can pick a collar $\chi\colon [0,1] \times \partial Y \to Y$.



For any $k \in \mathbb{N}_0$ we now consider the following diagram:

$$\begin{array}{ccc} \mathrm{H}^k(N \cup B, B; \mathbb{Z}) & \longrightarrow & \mathrm{H}^k(N, N \cap B; \mathbb{Z}) \\ & \searrow \quad \nearrow & \\ & \mathrm{H}^k(N \cup \chi([0,1] \times \partial Y), (N \cap B) \cup \chi([0,1] \times \partial Y); \mathbb{Z}) & \end{array}$$

We make the following clarifications and observations:

(a) All maps are induced by the inclusions of pairs of topological spaces.

(b) The diagram commutes by the functoriality of cohomology.

(c) One can easily verify that $Y \setminus \chi([0,1] \times \partial Y)$ lies in the interior of $B \subset N \cup B$. Thus it follows from the Excision-in-H^*-Theorem 69.1 (1) that the left diagonal map is an isomorphism.

(d) A straightforward argument shows that N and $\chi([0,1] \times \partial Y)$ are closed subsets of $N \cup \chi([0,1] \times \partial Y)$. It follows from the Homotopy Combination Lemma 8.2 that the constant homotopy and the obvious deformation retraction from $\chi([0,1] \times \partial Y)$ to ∂Y define a deformation retraction from $N \cup \chi([0,1] \times \partial Y)$ to N. It thus follows from the Homotopy Equivalence-Relative-H^*-Corollary 67.10 that the right diagonal map is an isomorphism.

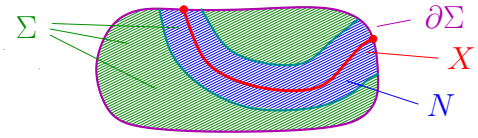
Finally note that it follows from the above discussion that the top horizontal map $\mathrm{H}^k(N \cup B, B; \mathbb{Z}) \to \mathrm{H}^k(N, N \cap B; \mathbb{Z})$ is, as desired, an isomorphism.

(6) By the definition of a regular neighborhood on page 1099 we know that there exists a deformation retraction $F\colon N \times [0,1] \to N$ from N to X such that $F(\partial N \times [0,1]) \subset \partial N$. This implies that X is a deformation retract of N and that $X \cap B$ is a deformation retract of $N \cap B$. It thus follows from the Homotopy Equivalence-Relative-H^*-Corollary 67.10 that $\mathrm{H}^{n-j-1}(N, N \cap B; \mathbb{Z}) \to \mathrm{H}^{n-j-1}(X, X \cap B; \mathbb{Z})$ is an isomorphism. ■

Proof of the Alexander Duality–Boundary Theorem 85.14 (2a). Let Σ be a compact orientable n-dimensional smooth manifold such that $\widetilde{\mathrm{H}}_0(\Sigma) = \mathrm{H}_1(\Sigma) = 0$. Let $X \subsetneq \Sigma$ be a compact subspace of Σ that admits a regular neighborhood N in the sense of the

definition on page 1099. We need to show that there exists an isomorphism

$$\widetilde{\mathrm{H}}_0(\Sigma \setminus X) \xrightarrow{\cong} \mathrm{H}^{n-1}(X, X \cap \partial\Sigma; \mathbb{Z}).$$



We start out with the following technical claim:

Claim. We have $N \neq \Sigma$.

Proof. We prove the claim by contradiction. So assume that $\Sigma = N$. Clearly we only have to deal with the case that Σ is connected. Since $X \subsetneq \Sigma$ there exists a $P \in \Sigma \setminus X$. We consider the following diagram:

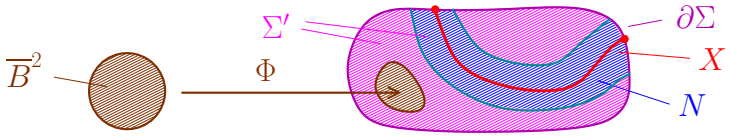
$$\begin{array}{ccc} \mathrm{H}_n(X, X \cap \partial\Sigma) & \longrightarrow & \overbrace{\mathrm{H}_n(\Sigma, \partial\Sigma)}^{\cong \mathbb{Z}} \\ & \searrow \quad \nearrow & \\ & \underbrace{\mathrm{H}_n(\Sigma \setminus \{P\}, \partial\Sigma \setminus \{P\})}_{=0}. & \end{array}$$

We make the following clarifications and observations:

(1) All maps are induced by inclusions. Thus the diagram commutes.
(2) It follows from the Homotopy Equivalence-Relative-H_*-Corollary 38.4, Condition (2) of the definition of a regular neighborhood and the assumption that Σ is a regular neighborhood of X that the top horizontal map is an isomorphism.
(3) Since Σ is compact orientable connected and non-empty it follows from the Manifold-Top-H_*-Theorem 63.1 that the top right group is isomorphic to $\mathbb{Z}$.
(4) If $n = 0$, then $\Sigma \setminus \{P\} = \varnothing$ and the bottom group is zero. If $n > 0$, then one can easily show that each component of $\Sigma \setminus \{P\}$ is non-compact. Thus we obtain from the Manifold-Top-H_*-Theorem 63.1 that the bottom group is zero.

The above four statements are inconsistent. We have thus obtained the desired contradiction. ⊞

Since N is compact we obtain from the Compact-Closed Lemma 1.9 and the claim that $\Sigma \setminus N$ is an non-empty open subset of Σ. Note that $\Sigma \setminus \partial\Sigma$ is an open dense subset of Σ. It follows from this discussion that we can pick a smooth embedding $\Phi\colon \overline{B}^n \to \Sigma \setminus (N \cup \partial\Sigma)$ such that $\Phi(\overline{B}^n)$ is a codimension-zero smooth submanifold. We set $\Sigma' := \Sigma \setminus \Phi(B^n)$. Note that Σ' is a compact orientable n-dimensional smooth manifold.



We obtain the following isomorphisms:

$$\begin{array}{ccccc}
\widetilde{\mathrm{H}}_0(\Sigma \setminus X) & \xrightarrow{\cong} & \mathrm{H}_0(\Sigma \setminus X, \Phi(\{0\})) & \xrightarrow{\cong} & \mathrm{H}_0(\Sigma \setminus X, \Phi(\overline{B}^n)) \\
& \xleftarrow{\cong} & \widetilde{\mathrm{H}}_0(\Sigma' \setminus X, \Phi(S^{n-1})) & \cong & \mathrm{H}^{n-1}(X, X \cap \partial\Sigma; \mathbb{Z}).
\end{array}$$

isomorphism from the Reduced-to-Relative-H$_*$-Lemma 39.7

this follows from the Homotopy Equivalence-Relative-H$_*$-Corollary 38.4, since $\Phi(\{0\}))$ is a deformation retract of $\Phi(\overline{B}^n)$

isomorphism by the Manifold–Excision Theorem 40.3

follows from (1) applied to Σ' with $A = \Phi(S^{n-1})$ and $B = \partial\Sigma$, here we use that N is also a regular neighborhood in Σ' ■

Proof of the Alexander Duality–Boundary Theorem 85.14 (2b). We now assume that $X \subsetneq \Sigma$, that $X \neq \varnothing$, that Σ is connected and that $\mathrm{H}_{n-1}(\Sigma, \partial\Sigma) = 0$. We need to show that there exists an isomorphism

$$\mathrm{H}_{n-1}(\Sigma \setminus X, \partial\Sigma \setminus X) \xrightarrow{\cong} \widetilde{\mathrm{H}}^0(X; \mathbb{Z}).$$

In the proof of (2a) we saw that we can pick an embedding $\Phi\colon \overline{B}^n \to \Sigma \setminus (N \cup \partial\Sigma)$ such that $\Phi(\overline{B}^n)$ is a codimension-zero submanifold. We set again $\Sigma' \coloneqq \Sigma \setminus \Phi(B^n)$. First note that

$$\mathrm{H}_{n-1}(\Sigma' \setminus X, \partial\Sigma \setminus X) \cong \mathrm{H}^0(X; \mathbb{Z}).$$

this follows from (1) applied to Σ' with $A = \partial\Sigma$ and $B = \Phi(S^{n-1})$, here we use that N is also a regular neighborhood in Σ'

Next note that we have a long exact sequence

$$\to \underbrace{\mathrm{H}_n(\Sigma \setminus X, \Sigma' \setminus X)}_{\cong \mathrm{H}_n(\overline{B}^n, S^{n-1}) \cong \mathbb{Z}} \xrightarrow{\partial_n} \mathrm{H}_{n-1}(\Sigma' \setminus X, \partial\Sigma \setminus X) \to \mathrm{H}_{n-1}(\Sigma \setminus X, \partial\Sigma \setminus X) \to \underbrace{\mathrm{H}_j(\Sigma \setminus X, \Sigma' \setminus X)}_{\cong \mathrm{H}_{n-1}(\overline{B}^n, S^{n-1}) = 0}$$

We make the following observations:

(1) For $j \in \mathbb{N}_0$ we have

$$\mathrm{H}_j(\Sigma \setminus X, \Sigma' \setminus X) \cong \mathrm{H}_j(\overline{B}^n, S^{n-1}) \cong \begin{cases} \mathbb{Z}, & \text{if } j = n, \\ 0, & \text{else.} \end{cases}$$

follows from the Manifold Excision Theorem 40.3

follows from the Reduced LES-of-Pair Proposition 67.14 and the discussion on pages 872 and 880

(2) First recall that on page 1112 we showed that $\mathrm{H}_{n-1}(\Sigma \setminus X, \partial\Sigma \setminus X)$ is isomorphic to $\mathrm{H}_{n-1}(\Sigma \setminus \mathring{N}, A \setminus \mathring{N})$. Next note that it follows from the Homology-rel-Partial Boundary Corollary 79.3 that the latter group is free abelian.

(3) It follows from the Manifold-Top-H$_*$-Theorem 63.1 and $X \neq \varnothing$ that $\mathrm{H}_n(\Sigma \setminus X, \partial\Sigma \setminus X) = 0$.

(4) It follows from the above long exact sequence, (1), (2), (3), the Free Abelian Group–Splitting Lemma 18.5 and the Splitting Lemma 37.2 that there exists an isomorphism

$$\mathrm{H}_{n-1}(\Sigma \setminus X, \partial\Sigma \setminus X) \cong \mathrm{H}_{n-1}(\Sigma \setminus X, \partial\Sigma \setminus X) \oplus \mathbb{Z}.$$

(5) Since we assume that $X \neq \varnothing$ we obtain from the Reduced Cohomology Lemma 67.12 (0) that $\mathrm{H}^0(X; \mathbb{Z}) \cong \widetilde{\mathrm{H}}^0(X; \mathbb{Z}) \oplus \mathbb{Z}$.

(6) It follows from (4), (5) and the cancellation property of finitely generated abelian groups (which we can apply by the Submanifold Complement–Invariants Lemma 85.3) that there exists an isomorphism $\mathrm{H}_{n-1}(\Sigma \setminus X, \partial\Sigma \setminus X) \cong \widetilde{\mathrm{H}}^0(X; \mathbb{Z})$. ■

CHAPTER 86

The General Alexander Duality Theorem

In the last chapter we proved the Sphere–Alexander Duality Theorem 85.5 which says that for any compact subspace $X \subsetneq S^n$ of S^n that admits a regular neighborhood (in the sense of the definition on page 1099) and any $i \in \mathbb{N}_0$ there exists an isomorphism

$$\widetilde{\mathrm{H}}_i(S^n \setminus X) \;\cong\; \widetilde{\mathrm{H}}^{n-i-1}(X;\mathbb{Z}).$$

This result had many applications, for example we used it to show that there is no smooth embedding of $\mathbb{R}\mathrm{P}^2$ as a smooth manifold into $\mathbb{R}^3$.

In this chapter we will formulate and prove a significant generalization of the Sphere–Alexander Duality Theorem 85.5. In the proof of the more general General Alexander Duality Theorem we will not make use of regular neighborhoods, instead we will make use of the Manifold–Neighborhood Retract Theorem 60.5 (2) (which is a consequence of the tricky Borsuk Theorem 60.4) on weakly locally contractible subspaces of topological manifolds. One of the many applications of the General Alexander Duality Theorem is that there is no embedding of $\mathbb{R}\mathrm{P}^2$ as a topological space into $\mathbb{R}^3$.

86.1. **The General Alexander Duality Theorem.** As we just mentioned, by the Sphere–Alexander Duality Theorem 85.5 we know that for any compact subspace $X \subsetneq S^n$ of S^n that admits a regular neighborhood and any $i \in \mathbb{N}_0$ there exists an isomorphism

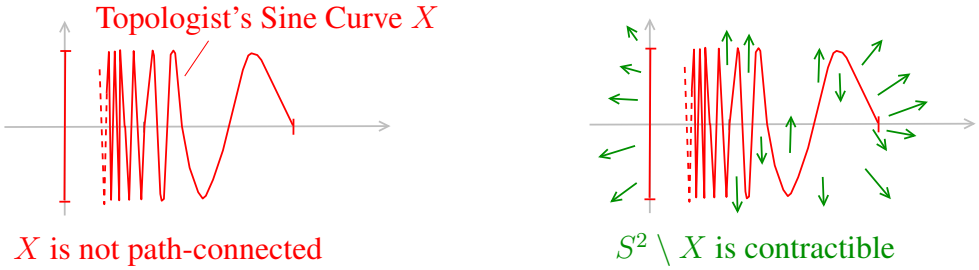
$$\widetilde{\mathrm{H}}_i(S^n \setminus X) \;\cong\; \widetilde{\mathrm{H}}^{n-i-1}(X;\mathbb{Z}).$$

In a bout of optimism one might hope that the statement of the Sphere–Alexander Duality Theorem 85.5 holds for any (compact) subspace of S^n. But as the following example shows, this is not the case:

Example. We consider again the Topologist's Sine Curve

$$X \;=\; \{(0,y)\,|\,y\in[-1,1]\} \cup \{(x,\sin(\tfrac{1}{x}))\,|\,x\in(0,\tfrac{1}{\pi}]\} \;\subset\; \mathbb{R}^2$$

that we had initially introduced on page 80.



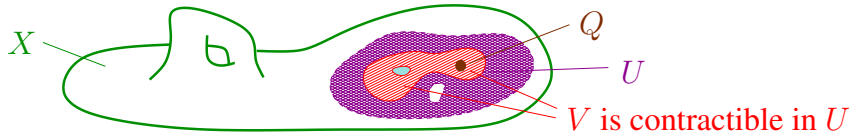
It is fairly straightforward to see that X is a closed subset of $\mathbb{R}^2$. Evidently X is bounded. Thus the Heine-Borel Theorem 1.12 implies that X is in fact a compact subspace of $\mathbb{R}^2$ and thus of $S^2 =_{\mathrm{i}} \mathbb{R}^2 \cup \{\infty\}$. Back on page 80 we saw that X is not path-connected,

which by the H^0-Proposition 67.5 and the Path-Component-H^*-Lemma 67.4 implies that $\widetilde{H}^0(X;\mathbb{Z}) \neq 0$.

On the other hand, with some effort one can show that there exists a deformation retraction from $S^2 \setminus X = (\mathbb{R}^2 \setminus X) \cup \{\infty\}$ to ∞, which implies that $S^2 \setminus X$ is contractible, which in turn shows (by the Contractible-H_*-Corollary 36.8) that $H_1(S^2 \setminus X) = 0$. In summary we have shown that $\widetilde{H}_1(S^2 \setminus X) \ncong \widetilde{H}^{2-1-1}(X;\mathbb{Z})$.

This example shows that we cannot hope to generalize the Sphere–Alexander Duality Theorem 85.5 to all compact subspaces of S^n. But as we will see shortly, we can extend it to compact subspaces which are weakly locally contractible. Just in case the reader has only a faint memory of the definition, let us recall the definition of a weakly locally contractible space from page 764:

Definition. Let X be a topological space. We say that X is weakly locally contractible if given any $Q \in X$ and given any open neighborhood U of Q there exists an open neighborhood V of Q that is contained in U such that the inclusion $V \to U$ is contractible in U, i.e. such that the inclusion is homotopic in U to a constant map.



X is weakly locally contractible

Example.

(1) By the discussion on page 764 we know that topological manifolds and CW-complexes are weakly locally contractible.

(2) It is an amusing little exercise to show that the Topologist's Sine Curve, which we discussed above, is not weakly locally contractible. □

Now we can formulate our main theorem of the chapter:

Theorem 86.1. (General Alexander Duality Theorem) Let Σ be a closed oriented n-dimensional topological manifold and let $X \subset \Sigma$ be a weakly locally contractible, compact subspace of Σ.

(1) Let $j \in \mathbb{N}_0$ such that $H_j(\Sigma) = H_{j+1}(\Sigma) = 0$. There exists a natural isomorphism

$$AD_X \colon H_j(\Sigma \setminus X) \xrightarrow{\cong} H^{n-j-1}(X;\mathbb{Z}).$$

Here "natural" means, that if $X \subset Y \subset \Sigma$ are two weakly locally contractible, compact subspaces of Σ, then we have the following commutative diagram:

$$\begin{array}{ccc} H_j(\Sigma \setminus Y) & \xrightarrow[\cong]{AD_Y} & H^{n-j-1}(Y;\mathbb{Z}) \\ \downarrow & & \downarrow \\ H_j(\Sigma \setminus X) & \xrightarrow[\cong]{AD_X} & H^{n-j-1}(X;\mathbb{Z}), \end{array}$$

here the left vertical map is induced by the inclusion $\Sigma \setminus Y \to \Sigma \setminus X$ and the right vertical map is induced by the inclusion $X \to Y$.

(2) We now assume that Σ is connected.[675]
  (a) If $\mathrm{H}_1(\Sigma) = 0$ and if $X \subsetneq \Sigma$, then there is a natural isomorphism
$$\mathrm{AD}_X \colon \widetilde{\mathrm{H}}_0(\Sigma \setminus X) \xrightarrow{\cong} \mathrm{H}^{n-1}(X;\mathbb{Z}).$$
  (b) If $\mathrm{H}_{n-1}(\Sigma) = 0$ and if $X \neq \varnothing$, then there is a natural isomorphism
$$\mathrm{AD}_X \colon \mathrm{H}_{n-1}(\Sigma \setminus X) \xrightarrow{\cong} \widetilde{\mathrm{H}}^0(X;\mathbb{Z}).$$

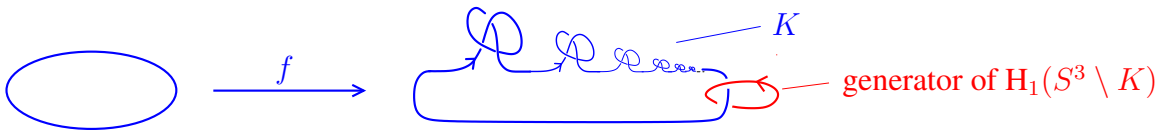
Notation. We continue with the notation from the General Alexander Duality Theorem 86.1. We refer to the isomorphism
$$\mathrm{AD}_X \colon \mathrm{H}_j(\Sigma \setminus X) \xrightarrow{\cong} \mathrm{H}^{n-j-1}(X;\mathbb{Z})$$
(and the reduced versions) which we will completely specify on pages 1121, 1126 and 1126, as the Alexander Duality Isomorphism.

We proceed as follows:

(1) In the remainder of this section we discuss one application of the General Alexander Duality Theorem 86.1.
(2) In Section 86.2 we will give the explicit definition of the map AD_X and we will prove the General Alexander Duality Theorem 86.1.
(3) In Section 86.3 we will state and discuss an important special case of the General Alexander Duality Theorem 86.1, namely the Sphere–Alexander Duality Theorem 86.3.
(4) Section 86.4 we will use the General Alexander Duality Theorem 86.1 to show that there exist "wild knots", i.e. there exist embeddings $w \colon S^1 \to S^3$ such that $w(S^1) \subset S^3$ is not a topological manifold.

For better readability we formulate our first example as a lemma:

Lemma 86.2. (Wild Knot Complement-H_*-Lemma)
(1) Let $f \colon S^1 \to S^3$ be any embedding. We set $K := f(S^1)$. Given $j \in \mathbb{N}_0$ we have
$$\widetilde{\mathrm{H}}_j(S^3 \setminus K) \cong \begin{cases} \mathbb{Z}, & \text{if } j = 1, \\ 0, & \text{otherwise.} \end{cases}$$
(2) Let $K \subset S^3$ be the subspace that is shown in the figure below. The curve shown in the figure below represents a generator of $\mathrm{H}_1(S^3 \setminus K)$.



Proof.

(1) Since $f \colon S^1 \to S^3$ is an embedding we know that $K = f(S^1)$ is homeomorphic to S^1. This shows in particular that K is weakly locally contractible and we see that the

[675]Note that if Σ is non-empty, then $\mathrm{H}_0(\Sigma) \neq 0$ and $\mathrm{H}_n(\Sigma) \neq 0$. Thus (1) cannot be applied for $j = 0$ and $j = n-1, j = n$. The case $j = n$ is almost trivial. But we have to modify the statements slightly to get some useful isomorphisms for $j = 0$ and $j = n-1$.

General Alexander Duality Theorem 86.1 applies. Thus we obtain that

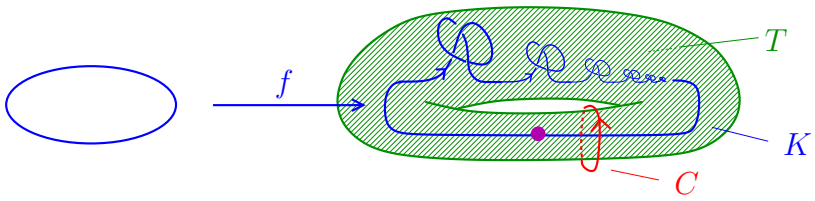
$$\widetilde{\mathrm{H}}_j(S^3 \setminus K) \overset{\cong}{\underset{\uparrow}{\to}} \widetilde{\mathrm{H}}^{2-j}(K;\mathbb{Z}) \cong \widetilde{\mathrm{H}}^{2-j}(S^1;\mathbb{Z}) \overset{\cong}{\underset{\uparrow}{\to}} \begin{cases} \mathbb{Z}, & \text{if } j=1, \\ 0, & \text{otherwise.} \end{cases}$$

isomorphism AD_K from the General Alexander Duality Theorem 86.1

by the calculation on page 880 together with the Reduced Cohomology Lemma 67.12

(2) Using the Compact-Hausdorff Proposition 2.10 (3) one can show that the subspace K in the above figure is indeed homeomorphic to S^1.[676] Thus it follows from (1) that $\mathrm{H}_1(S^3 \setminus K) \cong \mathbb{Z}$. But the calculation in (1) does not specify an explicit generator for $\mathrm{H}_1(S^3 \setminus K)$. As we will now see, one way to pin down a generator is use the naturality of the Alexander Duality isomorphism AD_K.

In the following we consider the "standard closed solid torus" $T \subset S^3$ that is shown in the figure below:



Note that that $S^3 \setminus T$ is diffeomorphic to $S^1 \times B^2$, where $S^1 \times \{0\}$ corresponds to the red curve C shown in the figure. Thus C represents a generator of $\mathrm{H}_1(S^3 \setminus T)$. We want to show that C also represents a generator of $\mathrm{H}_1(S^3 \setminus K)$. We denote by $i\colon K \to T$ the inclusion map. By the above discussion it now suffices to show that the map $\mathrm{H}_1(S^3 \setminus T) \to \mathrm{H}_1(S^3 \setminus K)$ is an isomorphism.

To do so we consider the following diagram:

$$\begin{array}{ccccc} \mathrm{H}_1(S^3 \setminus T) & \xrightarrow[\cong]{\mathrm{AD}_T} & \mathrm{H}^{3-1-1}(T;\mathbb{Z}) & \xrightarrow[\cong]{\mathrm{ev}} & \mathrm{Hom}(\mathrm{H}_1(T),\mathbb{Z}) \\ \downarrow & & \downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle (i_*)^*} \\ \mathrm{H}_1(S^3 \setminus K) & \xrightarrow[\cong]{\mathrm{AD}_K} & \mathrm{H}^{3-1-1}(K;\mathbb{Z}) & \xrightarrow[\cong]{\mathrm{ev}} & \mathrm{Hom}(\mathrm{H}_1(K),\mathbb{Z}). \end{array}$$

We make the following observations:

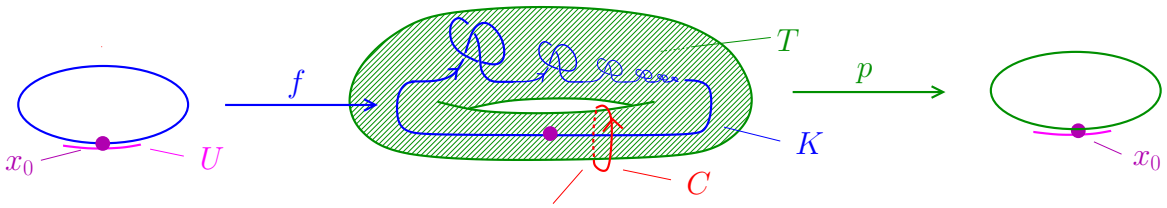
- It follows from the existence and the naturality of the isomorphism of the General Alexander Duality Theorem 86.1 that the left horizontal maps are isomorphisms and that the rectangle to the left commutes.
- The H^1-Hom-Proposition 70.15 tells us that the right horizontal maps are isomorphisms and that the rectangle to the right commutes.

We see that it now remains to prove the following claim:

Claim. The map $i_*\colon \mathrm{H}_1(K) \to \mathrm{H}_1(T)$ is an isomorphism.

Proof. Let $f\colon S^1 \to K$ be a homeomorphism and let $p\colon T \to S^1$ be the obvious homotopy equivalence. We can pick f and p and a point $x_0 \in S^1$ such that $p \circ f$ is the identity in a neighborhood U of x_0 and such that $(p \circ f)^{-1}(x_0) = \{x_0\}$.

[676] On page 1131 we will give a more precise description of K, which then also makes the claim, that K is homeomorphic to S^1, more meaningful.



We consider the following little diagram:

$$\begin{array}{ccccc} & & \mathrm{H}_1(T) & & \\ & & \uparrow{\scriptstyle i_*} & \searrow^{p_*} & \\ \mathrm{H}_1(S^1) & \xrightarrow{f_*} & \mathrm{H}_1(K) & \xrightarrow{(p|_K)_*} & \mathrm{H}_1(S^1). \\ & \searrow_{(p|_K \circ f)_*} & & \nearrow & \end{array}$$

We make the following observations:

(a) The diagram commutes by the functoriality of homology.

(b) The diagonal map is an isomorphism since $p\colon T \to S^1$ is a homotopy equivalence.

(c) It follows from the S^n-Local-Global Degree Proposition 44.2 and the Degree-via-Local Orientations Proposition 44.7 that the composition of the bottom two horizontal maps, i.e. the map $(p|_K \circ f)_*\colon \mathrm{H}_1(S^1) \to \mathrm{H}_1(S^1)$, is an isomorphism.

It follows from the above discussion that the vertical map $i_*\colon \mathrm{H}_*(K) \to \mathrm{H}_*(T)$ is indeed an isomorphism. ■

86.2. **Proof of the General Alexander Duality Theorem 86.1.** The definition of the Alexander duality isomorphism requires some preparations. We start out with the following notation:

Notation. We recall the following definition from page 1004:

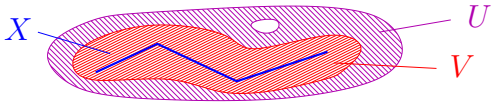
(1) Given a topological space Y we denote by $\mathcal{K}(Y)$ the set of all compact subspaces of Y. Given $K, L \in \mathcal{K}(Y)$ we write $K \leq L$ if $K \subset L$. The set $\mathcal{K}(Y)$ together with the relation "$\leq$" is a directed set[677] since the union of two compact subspaces is again compact.

We also introduce a new definition and notation:

(2) Let X be a subspace of a topological space Y. We denote by $\mathcal{N}(X)$ the set of all open neighborhoods of X in Y. Given $U, V \in \mathcal{N}(X)$ we write $U \leq V$ if $V \subset U$. (Note the reversal of V and U.) The set $\mathcal{N}(X)$ together with the relation "$\leq$" is a directed set since the intersection of two open neighborhoods of X is again an open neighborhood of X.[678]

(3) Given any two subspaces $A \subset B$ of Σ we denote by $i_{A,B}\colon A \to B$ the inclusion map.

(4) It follows from our definition of "$\leq$" on $\mathcal{N}(X)$ that for any $m \in \mathbb{N}_0$ the cohomology groups $\{\mathrm{H}^m(U;\mathbb{Z})\}_{U \in \mathcal{N}(X)}$ together with the inclusion-induced maps form a *direct* system[679] of abelian groups.



we have $U \leq V$ and $i^*_{V,U}\colon \widetilde{\mathrm{H}}^m(U;\mathbb{Z}) \to \widetilde{\mathrm{H}}^m(V;\mathbb{Z})$

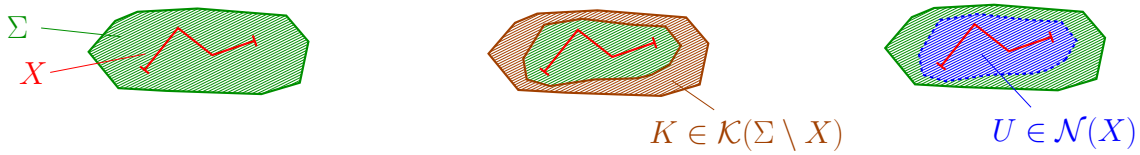
With this notation we can now finally define the Alexander duality isomorphisms:

Definition. Let $n \in \mathbb{N}$, let Σ be a closed oriented[680] n-dimensional topological manifold and let $X \subset \Sigma$ be a weakly locally contractible, compact subspace of Σ. Let $j \in \mathbb{N}_0$ such that $\mathrm{H}_j(\Sigma) = \mathrm{H}_{j+1}(\Sigma) = 0$. Rather similar to the discussions on pages 1103 and 1112 we consider the following sequence of homomorphisms:

$$\begin{aligned}
\mathrm{H}_j(\Sigma \setminus X) &\xrightarrow[(1)]{\cong} \mathrm{H}_j(\Sigma \setminus X;\mathbb{Z}) \xleftarrow[(2)]{\cong} \overbrace{\operatorname*{colim}_{K\in\mathcal{K}(\Sigma\setminus X)} \mathrm{H}^{n-j}(\Sigma \setminus X, \underbrace{(\Sigma \setminus X)\setminus K}_{=(\Sigma\setminus K)\setminus X};\mathbb{Z})}^{\text{naturally isomorphic to cohomology } \mathrm{H}_c^{n-j}(\Sigma\setminus X;\mathbb{Z}) \text{ with compact support of } \Sigma\setminus X} \\
&\xleftarrow[(3)]{\cong} \operatorname*{colim}_{U\in\mathcal{N}(X)} \mathrm{H}^{n-j}(\Sigma \setminus X, U \setminus X;\mathbb{Z}) \\
&\xleftarrow[(4)]{\cong} \operatorname*{colim}_{U\in\mathcal{N}(X)} \mathrm{H}^{n-j}(\Sigma, U;\mathbb{Z}) \xleftarrow[(5)]{\cong} \operatorname*{colim}_{U\in\mathcal{N}(X)} \mathrm{H}^{n-j-1}(U;\mathbb{Z}) \xrightarrow[(6)]{\cong} \mathrm{H}^{n-j-1}(X;\mathbb{Z}).
\end{aligned}$$

Below we will see that all maps are isomorphisms. Thus it makes sense to consider the composition and we refer to this composition as the Alexander duality isomorphism

$$\mathrm{AD}_X\colon \mathrm{H}_j(\Sigma \setminus X) \to \mathrm{H}^{n-j-1}(X;\mathbb{Z}).$$



Proof of the General Alexander Duality Theorem 86.1 (1). Let $n \in \mathbb{N}$, let Σ be a closed oriented n-dimensional topological manifold and let $X \subset \Sigma$ be a weakly locally contractible, compact subspace of Σ. Let $j \in \mathbb{N}_0$ such that $\mathrm{H}_j(\Sigma) = \mathrm{H}_{j+1}(\Sigma) = 0$. We need to prove the following statements:

(a) All six homomorphisms in the above definition are isomorphisms.
(b) All six homomorphisms are natural.

The fact that the homomorphisms are natural follows, as we will see, from formal reasons. The proof that the six homomorphisms are isomorphisms can be summarized as follows:

(1) This isomorphism reflects that ordinary homology is isomorphic to homology with $\mathbb{Z}$-coefficients.
(2) This map is an isomorphism by the Poincaré Duality–Non-Compact Theorem 80.2.
(3) This map is basically an identity.

[677] Recall that according to the definition on page 338, a directed set is a set X together with a partial order $\leq$ that has the extra property, that for any $x, y \in X$ there exists a $z \in X$ with $x \leq z$ and $y \leq z$.
[678] Here the idea is that the neighborhoods "approximate" X, in particular $U \leq V$ means that V is a "better approximation of X", which just means that $V \subset U$.
[679] We refer to page 339 for the definition of a direct system.

(4) It follows from excision that this map is an isomorphism.
(5) It follows from long exact sequences and our information on the homology of Σ that this map is an isomorphism.
(6) We will use Borsuk's Theorem 60.4, and our hypothesis that X is weakly locally contractible, to show that this map is an isomorphism.

We now discuss the six homomorphisms in detail:

(1) This isomorphism comes from the discussion on page 659.
(2) (a) Since X is compact, we know from the Compact-Closed Lemma 1.9 (2) that $\Sigma \setminus X$ is an open subset of $\Sigma \setminus X$. In particular $\Sigma \setminus X$ is also an n-dimensional topological manifold and via the Topological Manifold-Induced Orientation Lemma 62.1 it inherits an orientation from Σ. Also note that it follows from our hypothesis that $\partial\Sigma = \varnothing$ and the Topological Manifold–Boundary Proposition 42.5 that $\Sigma \setminus X$ has no boundary. After this preamble the first isomorphism is the Poincaré Duality isomorphism from the Poincaré Duality–Non-Compact Theorem 80.2 corresponding to the oriented topological manifold $\Sigma \setminus X$. [681]
(b) As we discussed in the (proof of) the Poincaré Duality–Open Subset Lemma 80.3, it follows from the Cap Product-Naturality Lemma 75.8 and the universal properties of direct colimits that this isomorphism is natural.
(3) (a) Since Σ is compact it follows immediately from the Compact-Closed Lemma 1.9 that a subset $U \subset \Sigma$ is open if and only if $\Sigma \setminus U$ is compact. This observation shows that the map

$$\begin{array}{ccc} \mathcal{N}(X) & \to & \mathcal{K}(\Sigma \setminus X) \\ U & \mapsto & \Sigma \setminus U \end{array}$$

is an isomorphism of directed sets. From this observation we obtain an isomorphism of direct systems. It is now clear that the second homomorphism is an isomorphism.
(b) It follows from the universal properties of direct colimits that this map is natural.
(4) (a) For any two open neighborhoods U and V of X with $U \leq V$, i.e. with $V \subset U$, we have the diagram

$$\begin{array}{ccc} \mathrm{H}^{n-j}(\Sigma \setminus X, V \setminus X; \mathbb{Z}) & \longleftarrow & \mathrm{H}^{n-j}(\Sigma, V; \mathbb{Z}) \\ \uparrow & & \uparrow{\scriptstyle i^*_{V,U}} \\ \mathrm{H}^{n-j}(\Sigma \setminus X, U \setminus X; \mathbb{Z}) & \longleftarrow & \mathrm{H}^{n-j}(\Sigma, U; \mathbb{Z}) \end{array}$$

where all the maps are induced by the obvious inclusions of pairs of topological spaces. The diagram commutes by the functoriality of cohomology. Since X is closed and since U and V are open we see that the we obtain from the Excision-in-H*-Theorem 69.1 (1a) that the horizontal maps are isomorphisms. The desired isomorphism of direct colimits is now an immediate consequence of the Cofinal-Isomorphism Lemma 78.4.
(b) It follows from the universal properties of direct colimits that this map is natural.

[681] Note that we formulated the Poincaré Duality–Non-Compact Theorem 80.2 in terms of cohomology with compact support. But we really proved it in terms of the colimit and we used the notation for "cohomology with compact support" just to abbreviate the formulation.)

(5) (a) For any open neighborhoods U and V of X with $U \leq V$, i.e. with $V \subset U$, we have a commutative diagram

$$\begin{array}{ccccccc}
\overbrace{\mathrm{H}^{n-j}(\Sigma;\mathbb{Z})}^{\cong \mathrm{H}_j(\Sigma)} & \longleftarrow & \mathrm{H}^{n-j}(\Sigma, V;\mathbb{Z}) & \xleftarrow{\delta_{n-j-1}} & \mathrm{H}^{n-j-1}(V;\mathbb{Z}) & \longleftarrow & \overbrace{\mathrm{H}^{n-j-1}(\Sigma;\mathbb{Z})}^{\cong \mathrm{H}_{j+1}(\Sigma)} \\
\uparrow{\scriptstyle \mathrm{id}} & & \uparrow & & \uparrow{\scriptstyle i^*_{V,U}} & & \uparrow{\scriptstyle \mathrm{id}} \\
\underbrace{\mathrm{H}^{n-j}(\Sigma;\mathbb{Z})}_{\cong \mathrm{H}_j(\Sigma)} & \longleftarrow & \mathrm{H}^{n-j}(\Sigma, U;\mathbb{Z}) & \xleftarrow{\delta_{n-j-1}} & \mathrm{H}^{n-j-1}(U;\mathbb{Z}) & \longleftarrow & \underbrace{\mathrm{H}^{n-j-1}(\Sigma;\mathbb{Z})}_{\cong \mathrm{H}_{j+1}(\Sigma)}
\end{array}$$

where the vertical maps are induced by the obvious inclusions and where the horizontal maps are the connecting homomorphisms of the long exact sequences in cohomology provided by the LES-of-Triple Proposition 67.9. Since $j \geq 1$ we obtain from these long exact sequences, our hypotheses on the homology groups of Σ and the Poincaré Duality Theorem 79.1 that the middle horizontal maps are in fact isomorphisms. As in (4) we appeal to the Cofinal-Isomorphism Lemma 78.4 to obtain the desired isomorphism of direct colimits.

(b) It follows from the universal properties of direct colimits that this map is natural.

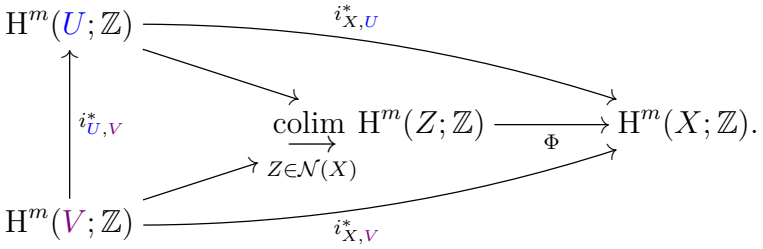
(6) (a) We need to prove the following claim:

Claim. For every $m \in \mathbb{N}_0$ the map

$$\Phi\colon \operatorname*{colim}_{U \in \mathcal{N}(X)} \mathrm{H}^m(U;\mathbb{Z}) \;\to\; \mathrm{H}^m(X;\mathbb{Z}),$$

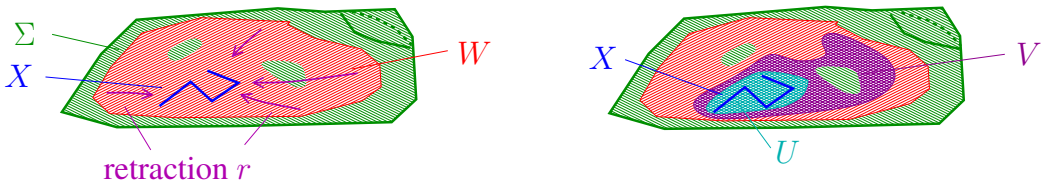
that is induced by the restriction maps $i^*_{X,U}\colon \mathrm{H}^m(U;\mathbb{Z}) \to \mathrm{H}^m(X;\mathbb{Z})$, is an isomorphism.

Proof. For orientation purposes we now state the commutative diagram which comes with the definition of a direct colimit. More precisely, for open neighborhoods $X \subset U \subset V \subset \Sigma$ we have the following commutative diagram:



Now let us start with the actual proof of the claim. By our hypothesis we know that X is compact and weakly locally contractible. Since Σ is a topological manifold we know by the Manifold–Neighborhood Retract Theorem 60.5 (2) (which is a consequence of the tricky Borsuk Theorem 60.4) that there exists an open neighborhood $W \subset \Sigma$ of $X \subset \Sigma$ and a retraction $r\colon W \to X$ such that furthermore the following condition is satisfied:

(∗) Given any neighborhood V of X there exists an open neighborhood U with $X \subset U \subset V \cap W$ such that the inclusion map $i_{U,V}\colon U \to V$ and the retraction $r|_U\colon U \to X \subset V$ are homotopic maps from U to V.



Next we consider the following diagram:

$$\begin{array}{ccc} \operatorname{colim}_{Z\in\mathcal{N}(X)} \mathrm{H}^m(Z;\mathbb{Z}) & \xrightarrow{\Phi} & \mathrm{H}^m(X;\mathbb{Z}). \\ \uparrow & \nearrow_{i^*_{X,W}} & \\ \mathrm{H}^m(W;\mathbb{Z}) & \xleftarrow{r^*} & \end{array}$$

First note that the outer triangle commutes by the universal property of direct colimits. Next note that we have $i^*_{X,W} \circ r^* = (r \circ i_{X,W})^* = (\mathrm{id}_X)^* = \mathrm{id}_{\mathrm{H}^m(X;\mathbb{Z})}$. We deduce that the map $i^*_{X,W}$ is an epimorphism. It follows that the horizontal map Φ is also an epimorphism. It remains to show that Φ is a monomorphism.

By the Elements-in-Direct Colimit Lemma 25.4 (3) it remains to show that for every $V \in \mathcal{N}(X)$ there exists a $U \in \mathcal{N}(X)$ with $V \leq U$, i.e. with $U \subset V$, such that

$$\ker(i^*_{X,V} \colon \mathrm{H}^m(V;\mathbb{Z}) \to \mathrm{H}^m(X;\mathbb{Z})\} \ \subset \ \ker(i^*_{U,V} \colon \mathrm{H}^m(V;\mathbb{Z}) \to \mathrm{H}^m(U;\mathbb{Z})).$$

So let $V \in \mathcal{N}(X)$. By $(*)$ there exists an open neighborhood U with $X \subset U \subset V \cap W$ such that the inclusion $i_{U,V} \colon U \to V$ and the retraction $r|_U \colon U \to X \subset V$ are homotopic maps from U to V.

We consider the following diagram:

$$\begin{array}{ccc} \mathrm{H}^m(U;\mathbb{Z}) & \xleftarrow{(r|_U)^*} & \\ \uparrow{\scriptstyle (r|_U)^*} \ \downarrow{\scriptstyle i^*_{U,V}} & & \mathrm{H}^m(X;\mathbb{Z}) \\ \mathrm{H}^m(V;\mathbb{Z}). & \xrightarrow{i^*_{X,V}} & \end{array}$$

Since the two maps $i_{U,V} \colon U \to V$ and $r|_U \colon U \to X \subset V$ are homotopic it follows from the Homotopy-H*-Proposition 67.7 (1) that the two maps $i^*_{U,V}$ and r^* from $\mathrm{H}^m(V;\mathbb{Z})$ to $\mathrm{H}^m(U;\mathbb{Z})$ are actually the same. Since r is the identity on X we see that $r \colon V \to X \subset U$ equals the map $i_{X,V} \circ r \colon V \to X$. This shows that the diagram commutes. It follows immediately from these observations that U has the desired property. ⊞

(b) It follows from the universal properties of direct colimits that this map is natural. ■

Proof of the General Alexander Duality Theorem 86.1 (2a). Let Σ be a closed oriented n-dimensional topological manifold and let $X \subset \Sigma$ be a weakly locally contractible, compact subspace of Σ. We now assume that Σ is connected and that $\mathrm{H}_1(\Sigma) = 0$. As we will see, the above steps are unchanged, except for Step (5) on page 1122. To deal with our new setting we start out with the following claim:

Claim. Given any compact weakly locally contractible subspace $Y \subset \Sigma$ we have an exact sequence

$$\mathrm{H}^n(Y;\mathbb{Z}) \xleftarrow{i^*_{Y,\Sigma}} \mathrm{H}^n(\Sigma;\mathbb{Z}) \leftarrow \mathrm{H}_0(\Sigma \setminus Y) \leftarrow \mathrm{H}^{n-1}(Y;\mathbb{Z}) \leftarrow 0,$$

where the left hand map is induced by the inclusion $Y \to \Sigma$ and where all maps are natural in Y.

Proof. First note that, similar to the proof of (5) on page 1122, we have the following commutative diagram of exact sequences:

$$\begin{array}{ccccccccc}
\mathrm{H}^n(V;\mathbb{Z}) & \longleftarrow & \mathrm{H}^n(\Sigma;\mathbb{Z}) & \xleftarrow{\text{я}} & \mathrm{H}^n(\Sigma,V;\mathbb{Z}) & \xleftarrow{\delta_{n-1}} & \mathrm{H}^{n-1}(V;\mathbb{Z}) & \longleftarrow & \overbrace{\mathrm{H}^{n-1}(\Sigma;\mathbb{Z})}^{\cong \mathrm{H}_1(\Sigma)=0} \\
\uparrow{\scriptstyle i^*_{V,U}} & & \uparrow{\scriptstyle \mathrm{id}} & & \uparrow & & \uparrow{\scriptstyle i^*_{V,U}} & & \uparrow{\scriptstyle \mathrm{id}} \\
\mathrm{H}^n(U;\mathbb{Z}) & \longleftarrow & \mathrm{H}^n(\Sigma;\mathbb{Z}) & \xleftarrow{\text{я}} & \mathrm{H}^n(\Sigma,U;\mathbb{Z}) & \xleftarrow{\delta_{n-1}} & \mathrm{H}^{n-1}(U;\mathbb{Z}) & \longleftarrow & \underbrace{\mathrm{H}^{n-1}(\Sigma;\mathbb{Z})}_{\cong \mathrm{H}_1(\Sigma)=0}.
\end{array}$$

Next note that it follows from the Direct Colimit-H_*-Lemma 46.5 that direct colimits preserve exactness. Thus we obtain the following exact sequence:

$$\underbrace{\varinjlim_{U\in\mathcal{N}(Y)} \mathrm{H}^n(U;\mathbb{Z})}_{\substack{\text{by (6) naturally}\\ \text{isomorphic}\\ \text{to } \mathrm{H}^n(Y;\mathbb{Z})}} \leftarrow \underbrace{\varinjlim_{U\in\mathcal{N}(Y)} \mathrm{H}^n(\Sigma;\mathbb{Z})}_{\cong \mathrm{H}^n(\Sigma;\mathbb{Z})} \leftarrow \underbrace{\varinjlim_{U\in\mathcal{N}(Y)} \mathrm{H}^n(\Sigma,U;\mathbb{Z})}_{\substack{\text{by (1), (2), (3), (4) naturally}\\ \text{isomorphic to } \mathrm{H}_0(\Sigma\setminus Y)}} \leftarrow \underbrace{\varinjlim_{U\in\mathcal{N}(Y)} \mathrm{H}^{n-1}(U;\mathbb{Z})}_{\substack{\text{by (6) naturally}\\ \text{isomorphic}\\ \text{to } \mathrm{H}^{n-1}(Y;\mathbb{Z})}} \leftarrow 0$$

where all maps are natural in Y. Using the explicit natural isomorphisms (1), (2), (3), (4) and (6) from above (which we still have, since for these we did not use any hypothesis on the homology groups of Σ), we obtain the promised natural maps and exact sequence. ⊞

Now let $X \subsetneq \Sigma$ be a compact, weakly locally contractible subspace. We apply the claim to $Y = X$ and $Y = \varnothing$. By naturality we obtain the following commutative diagram:

$$\begin{array}{ccccccccc}
\mathrm{H}^n(X;\mathbb{Z}) & \xleftarrow{0} & \mathrm{H}^n(\Sigma;\mathbb{Z}) & \longleftarrow & \mathrm{H}_0(\Sigma\setminus X) & \longleftarrow & \mathrm{H}^{n-1}(X;\mathbb{Z}) & \longleftarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle i_*} & & \downarrow & & \\
\underbrace{\mathrm{H}^n(\varnothing;\mathbb{Z})}_{=0} & \xleftarrow{0} & \mathrm{H}^n(\Sigma;\mathbb{Z}) & \xleftarrow{\cong} & \mathrm{H}_0(\Sigma\setminus\varnothing) & \longleftarrow & \underbrace{\mathrm{H}^{n-1}(\varnothing;\mathbb{Z})}_{=0} & \longleftarrow & 0.
\end{array}$$

We make the following observations:

(1) Since $X \subsetneq \Sigma$ there exists a point $P \in \Sigma \setminus X$. Thus the horizontal maps to the top left factors through $\mathrm{H}^n(\Sigma\setminus\{P\};\mathbb{Z})$, but by the Manifold-Top-H^*-Theorem 70.16 we know that that group is trivial. We thus see that the horizontal maps to the top left is the zero map.

(2) Since $\mathrm{H}^{n-1}(\Sigma;\mathbb{Z}) \cong \mathrm{H}_1(\Sigma) = 0$ we see that the second map to the left, in the bottom row, is an isomorphism.

We now consider the following maps:

$$\mathrm{H}^{n-1}(X;\mathbb{Z}) \xrightarrow[\uparrow]{\cong} \ker\{i_*\colon \mathrm{H}_0(\Sigma\setminus X)\to \mathrm{H}_0(\Sigma)\} \xleftarrow[\uparrow]{\cong} \widetilde{\mathrm{H}}_0(\Sigma\setminus X).$$

it follows from the above discussion that we have such a natural isomorphism

it follows from the Reduced Homology Lemma 39.2 (0b), and the hypotheses that $\Sigma\setminus X\neq\varnothing$ and that Σ is connected, that the natural map ж from the $\widetilde{\mathrm{H}}_n$–Functor Lemma 39.1 is an isomorphism

We have thus shown that there exists a natural isomorphism

$$\mathrm{AD}_X\colon \widetilde{\mathrm{H}}_0(\Sigma\setminus X) \xrightarrow{\cong} \mathrm{H}^{n-1}(X;\mathbb{Z}).$$ ■

Proof of the General Alexander Duality Theorem 86.1 (2b). Let Σ be a closed oriented n-dimensional topological manifold and let $X\subset\Sigma$ be a weakly locally contractible, compact subspace of Σ. We now assume that Σ is connected, that $\mathrm{H}_{n-1}(\Sigma)=0$ and that X is non-empty.

Claim. We have an exact sequence

$$0 \leftarrow \mathrm{H}_{n-1}(\Sigma\setminus X) \leftarrow \mathrm{H}^0(X;\mathbb{Z}) \leftarrow \mathrm{H}^0(\Sigma;\mathbb{Z})$$

where the right hand map is induced by the inclusion $X\to\Sigma$ and where all maps are natural in X.

Proof. Similar to the proof of (5) on page 1122 we have the following commutative diagram of exact sequences:

$$\begin{array}{ccccccc}
\overbrace{\mathrm{H}^1(\Sigma;\mathbb{Z})}^{\cong \mathrm{H}_{n-1}(\Sigma)=0} & \xleftarrow{я} & \mathrm{H}^1(\Sigma,V;\mathbb{Z}) & \xleftarrow{\delta_0} & \mathrm{H}^0(V;\mathbb{Z}) & \longleftarrow & \mathrm{H}^0(\Sigma;\mathbb{Z}) \\
\uparrow{\scriptstyle \mathrm{id}} & & \uparrow & & \uparrow{\scriptstyle i^*_{V,U}} & & \uparrow{\scriptstyle \mathrm{id}} \\
\underbrace{\mathrm{H}^1(\Sigma;\mathbb{Z})}_{\cong \mathrm{H}_{n-1}(\Sigma)=0} & \xleftarrow{я} & \mathrm{H}^1(\Sigma,U;\mathbb{Z}) & \xleftarrow{\delta_0} & \mathrm{H}^0(U;\mathbb{Z}) & \longleftarrow & \mathrm{H}^0(\Sigma;\mathbb{Z})
\end{array}$$

As in the proof of (2a) above, the claim now follows from the Direct Colimit-H_*-Lemma 46.5 and the proofs of (1),(2),(3),(4) and (6). ⊞

We now consider the following maps:

$$\mathrm{H}_{n-1}(\Sigma\setminus X) \xleftarrow[\uparrow]{\cong} \mathrm{coker}\{i_*\colon \mathrm{H}^0(\Sigma;\mathbb{Z})\to \mathrm{H}^0(X;\mathbb{Z})\} \xrightarrow[\uparrow]{\cong} \widetilde{\mathrm{H}}^0(X;\mathbb{Z}).$$

it follows from the above claim that we have such a natural isomorphism

it follows from the Reduced Cohomology Lemma 67.12 (0b), and the hypotheses that $X\neq\varnothing$ and that Σ is connected, that the natural map ж from the $\widetilde{\mathrm{H}}^n$–Functor Lemma 67.11 is an isomorphism

We have thus shown that there exists a natural isomorphism

$$\mathrm{AD}_X\colon \mathrm{H}_{n-1}(\Sigma\setminus X) \xrightarrow{\cong} \widetilde{\mathrm{H}}^0(X;\mathbb{Z}).$$ ■

86.3. **The Sphere–Alexander Duality Theorems.** We start out with the following immediate consequence of the General Alexander Duality Theorem 86.1:

Theorem 86.3. (Sphere–Alexander Duality Theorem) Let $X\subsetneq S^n$ be a compact non-empty subspace of S^n. If X is weakly locally contractible, then for every $j\in\mathbb{N}_0$ the

General Alexander Duality Theorem 86.1 gives us an isomorphism

$$\mathrm{AD}_X\colon \widetilde{\mathrm{H}}_j(S^n \setminus X) \xrightarrow{\cong} \widetilde{\mathrm{H}}^{n-j-1}(S^n;\mathbb{Z})$$

which is natural in the sense of the General Alexander Duality Theorem 86.1.

Proof. It is clear that the hypotheses of the General Alexander Duality Theorem 86.1 (1), (2a) and (2b) are satisfied. Thus we obtain the desired result for free. ■

For most applications of Alexander Duality we will actually need the following straightforward consequence of the above theorem.

Theorem 86.4. (Sphere–Alexander Duality Theorem) Let $n \in \mathbb{N}_0$ and let X be a compact, non-empty topological manifold of dimension $< n$. If $f\colon X \to S^n$ is an injective continuous map, then for any $j \in \mathbb{N}_0$ we have a natural isomorphism

$$\mathrm{AD}_X\colon \widetilde{\mathrm{H}}_j(S^n \setminus f(X)) \xrightarrow{\cong} \widetilde{\mathrm{H}}^{n-j-1}(X;\mathbb{Z})$$

which is natural in the sense of the General Alexander Duality Theorem 86.1.

Proof. Let $k < n$ and let $f\colon X \to S^n$ be an injective continuous map from a compact, non-empty k-dimensional topological manifold to S^n. It is a consequence of the Compact-Hausdorff Proposition 2.10 (3) that $f\colon X \to f(X)$ is a homeomorphism. From $k < n$ it follows that $f(X) \neq S^n$. It follows from this discussion that we have the following isomorphisms:

$$\widetilde{\mathrm{H}}_j(S^n \setminus f(X)) \underset{\uparrow}{\xrightarrow{\cong}} \widetilde{\mathrm{H}}^{n-j-1}(f(X);\mathbb{Z}) \xrightarrow{\cong} \widetilde{\mathrm{H}}^{n-j-1}(X;\mathbb{Z}).$$

we can apply the General Alexander Duality Theorem 86.1 since the topological manifold $f(X)$ is weakly locally contractible, and since $X \neq \varnothing$ and since $f(X) \neq S^n$

■

Examples. Let $n \in \mathbb{N}_0$.

(1) Let $k < n$ and let $f\colon S^k \to S^n$ be an injective continuous map. We obtain that

$$\widetilde{\mathrm{H}}_j(S^n \setminus f(S^k)) \underset{\uparrow}{\cong} \widetilde{\mathrm{H}}^{n-j-1}(S^k;\mathbb{Z}) \underset{\uparrow}{\cong} \begin{cases} \mathbb{Z}, & \text{if } j = n-k-1, \\ 0, & \text{otherwise.} \end{cases}$$

by the Sphere–Alexander Duality Theorem 86.4

see page 880 and the Reduced Cohomology Lemma 67.12

(2) Let X be a compact connected non-empty $(n-1)$-dimensional topological manifold with $\partial X \neq \varnothing$ and let $f\colon X \to S^n =_{\mathrm{i}} \mathbb{R}^n \cup \{\infty\}$ be an injective continuous map.



We see that

$$\widetilde{\mathrm{H}}_0(S^n \setminus f(X)) \underset{\uparrow}{\cong} \widetilde{\mathrm{H}}^{n-1}(X;\mathbb{Z}) \underset{\uparrow}{\cong} \mathrm{H}^{n-1}(X;\mathbb{Z}) \underset{\uparrow}{=} 0.$$

by the Sphere–Alexander Duality Theorem 86.4

since $n \geq 2$

by the Manifold-Top-H*-Theorem 70.16 (2a) since X is connected and since $\partial X \neq \varnothing$

It follows from this calculation together with the Reduced Homology Lemma 39.2 (0c) that $S^n \setminus f(X)$ is path-connected. □

The following proposition is the topological analogue of the Smooth Codimension-One Orientability Proposition 85.10.

Proposition 86.5. (Topological Codimension-One Orientability Proposition) Let X be a closed n-dimensional topological manifold. If X is non-orientable, then there is no *topological* embedding of X into $\mathbb{R}^{n+1}$.

Proof. [682]As we will see, the proof of the Topological Codimension-One Orientability Proposition 86.5 is structurally the same as the proof of the Smooth Codimension-One Orientability Proposition 85.10. Let X be a closed n-dimensional non-orientable topological manifold. Clearly it suffices to deal with the case that X is connected. Suppose that X admits an embedding $f\colon X \to \mathbb{R}^{n+1}$. As always we view $\mathbb{R}^{n+1}$ as a subspace of $\mathbb{R}^{n+1} \cup \{\infty\} =_{\mathrm{i}} S^{n+1}$.

We obtain that

$$\begin{array}{ccccccc} & & \text{by the Manifold-Top-H}^*\text{-Theorem 70.16 (2),} & & & & \text{by the Reduced Homology} \\ & & \text{since } N \text{ is non-orientable and connected} & & & & \text{Lemma 39.2 (0c)} \\ & & \downarrow & & & & \downarrow \\ \mathbb{Z}_2 & \cong & \mathrm{H}^n(X;\mathbb{Z}) & \cong & \widetilde{\mathrm{H}}_0(S^{n+1}\setminus X) & \cong & \mathbb{Z}^{b_0(S^{n+1}\setminus X)-1}. \\ & & & & \uparrow & & \\ & & & & \text{by the Sphere–Alexander Duality Theorem 86.4} & & \end{array}$$

But this is absurd. We have thus obtained the desired contradiction. ■

The following corollary is a strengthening of the Surface-No Smooth Embedding Corollary 85.11:

Corollary 86.6. (Surface-No Topological Embedding-Corollary)

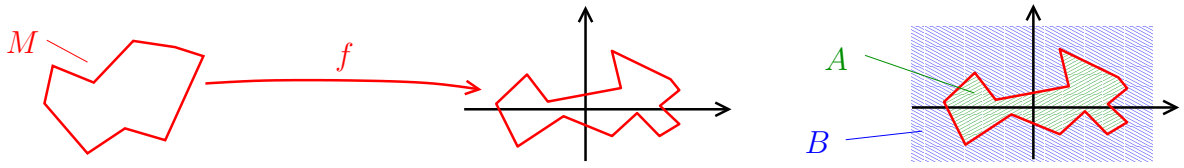
(1) For $k \in \mathbb{N}$ the non-orientable surface of genus k does not admit an embedding into $\mathbb{R}^3$.
(2) Neither $\mathbb{R}\mathrm{P}^2$ nor the Klein bottle admit an embedding into $\mathbb{R}^3$.

Proof. Since $N_1 = \mathbb{R}\mathrm{P}^2$ and N_2 is homeomorphic to the Klein bottle it remains to prove Statement (1). Let $k \in \mathbb{N}$. Note that N_k is non-orientable as a smooth manifold. It follows from the Smooth-Topological Orientation Proposition 61.15 that N_k is also non-orientable as a topological manifold. Since N_k is a closed 2-dimensional topological manifold it follows from the Topological Codimension-One Orientability Proposition 86.5 that N_k does not admit an embedding into $\mathbb{R}^3$. ■

We continue with another major result:

Theorem 86.7. (Generalized Jordan-Brouwer Separation Theorem) Let $n \in \mathbb{N}_{\geq 2}$, let M be an $(n-1)$-dimensional closed connected non-empty topological manifold and let $f\colon M \to \mathbb{R}^n$ be an injective continuous map.

(1) The complement $\mathbb{R}^n \setminus f(M)$ has precisely two path-components, one of which is bounded and one which is unbounded.
(2) If A and B are these path-components, then $\partial A = \partial B = \overline{A} \cap \overline{B} = f(M)$.



Proof. Let $n \in \mathbb{N}_{\geq 2}$, let M be an $(n-1)$-dimensional closed connected non-empty topological manifold and let $f\colon M \to \mathbb{R}^n$ be an injective continuous map. By the Topological Codimension-One Orientability Proposition 86.5 we know that M is orientable.

(1) We use the embedding $\mathbb{R}^n \to \mathbb{R}^n \cup \{\infty\} =_{\mathrm{i}} S^n$ to view $\mathbb{R}^n$ as a subspace of S^n. Now we see that

$$\widetilde{\mathrm{H}}_0(\mathbb{R}^n \setminus f(M)) \overset{\text{by the Complement-in-}\mathbb{R}^n\text{-}S^n\text{-Lemma 85.7}}{\cong} \widetilde{\mathrm{H}}_0(S^n \setminus f(M)) \overset{\text{by the Sphere–Alexander Duality Theorem 86.4}}{\cong} \widetilde{\mathrm{H}}^{n-1}(M;\mathbb{Z}) \overset{\text{since } n \geq 2}{\cong} \mathrm{H}^{n-1}(M;\mathbb{Z}) \overset{\text{by the Top Cohomology Theorem 70.16 (2a), since } M \text{ is closed, connected, non-empty and } (n-1)\text{-dimensional}}{\cong} \mathbb{Z}.$$

It follows from this calculation and the Reduced Homology Lemma 39.2 (0c) that $\mathbb{R}^n \setminus f(M)$ has precisely two path-components. It remains to show that precisely one of them is unbounded.

Since M is compact there exists an $r \in \mathbb{R}$ such that $f(M) \subset B^n_r$. Since $n \geq 2$ it follows from an elementary argument that $\mathbb{R}^n \setminus B^n_r$ is path-connected. It follows from the Path-Component Lemma 2.21 (4b) that $\mathbb{R}^n \setminus B^n_r$ is contained in one of the two path-components. This shows that precisely one path-component is unbounded.

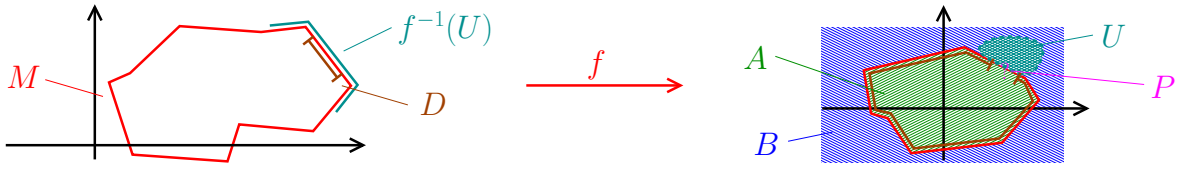
(2) We start out with the following claim:

Claim 1. A and B are open subsets of $\mathbb{R}^n$.

Proof. First note that it follows from the Compact Image Lemma 2.8 and the Compact-Closed Lemma 1.9 that $\mathbb{R}^{n+1} \setminus f(M)$ is open. In particular, as we remarked on page 85, $\mathbb{R}^{n+1} \setminus f(M)$ is locally path-connected. It follows from the Locally Path-Connected Lemma 2.24 that A and B are both open subsets of $\mathbb{R}^n$. ⊞

Claim 2. We have $\overline{A} = A \cup f(M)$ and $\overline{B} = B \cup f(M)$.

Proof. In the following we show that $\overline{A} = A \cup f(M)$. The proof that $\overline{B} = B \cup f(M)$ is basically identical. Since B is open we know that $A \cup f(M) = \mathbb{R}^n \setminus B$ is closed. By definition of $\overline{A}$ this implies that $\overline{A} \subset A \cup f(M)$. Now let $P \in A \cup f(M)$. We need to show that $P \in \overline{A}$. Suppose that this is not the case. By the Interior-Closure Lemma 1.6 we know that $\overline{A}$ is closed. Thus our assumption that $P \notin \overline{A}$ implies that there exists an open subset $U \subset \mathbb{R}^n$ with $P \in U \subset \mathbb{R}^n \setminus \overline{A}$. One can easily show that there exists a compact codimension-zero submanifold $D \subset M$ which is homeomorphic to $\overline{B}^n$, with $D \subset f^{-1}(U)$ and such that $E := M \setminus \mathring{D}$ is also a compact $(n-1)$-dimensional topological manifold. From $n \geq 2$ it follows that E is connected with non-empty boundary.



We now obtain the following equalities:

$$\underbrace{\mathbb{R}^n \setminus f(E)}_{\substack{\text{path-connected by the discussion on page 1127 and the Complement-in-}\mathbb{R}^n\text{-}S^n\text{-Lemma 85.7 (here we use that } \partial E \neq \varnothing \text{ and that } E \text{ is connected)}}} \overset{\text{since } \mathbb{R}^{n+1} = A \sqcup B \sqcup f(M)}{\overset{\downarrow}{=}} A \sqcup B \sqcup (f(M) \setminus f(E)) \overset{\text{since } f(M) \setminus f(E) = f(M \setminus E) = f(D) \subset U}{\overset{\downarrow}{=}} A \sqcup (B \cup (U \cap \mathbb{R}^n \setminus f(E))).$$

$$\uparrow$$

disjoint union since $U \cap A = \varnothing$

We have thus written the path-connected space $\mathbb{R}^n \setminus f(E)$ as the disjoint union of the two open non-empty sets A and $B \cup (U \cap \mathbb{R}^n \setminus f(E))$. But by the Path-Connected-Implies-Connected Corollary 2.15 this is not possible. We have thus obtained a contradiction. ⊞

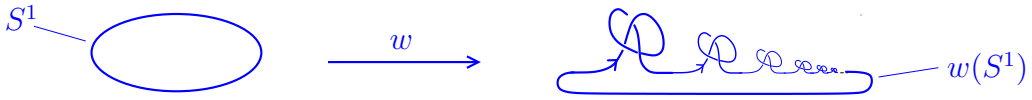
We now see that

$$\partial A \underset{\uparrow}{=} \overline{A} \setminus \mathring{A} \underset{\uparrow}{=} (A \cup f(M)) \setminus A \underset{\uparrow}{=} f(M).$$

by the definition of ∂A, see page 59 — by Claim 1 and 2 — since $A \cap f(M) = \varnothing$

The same way we see that $\partial B = f(M)$. This gives us the desired result. ■

86.4. **Wild knots.** In these lectures note we considered already on several occasions the embedding $w\colon S^1 \to \mathbb{R}^3 \subset \mathbb{R}^3 \cup \{\infty\} =_{\text{i}} S^3$ that is shown in the figure below.



We are aware that the image is "wild", but it is hard to pin this down. In this section we will prove the following theorem, which now makes it precise that $w(S^1)$ is indeed "wild".

Theorem 86.8. (Wild Knot Existence Theorem) There exist embeddings $w\colon S^1 \to S^3$ (e.g. the one shown above) such that $w(S^1) \subset S^3$ is not a topological manifold.

In the following we will give a very precise construction of such a map $w\colon S^1 \to S^3$. The reader who is comfortable with the above pictorial description can also jump straight to the Wild Embedding Existence Proposition 86.10 for the essential part of the proof of the Wild Knot Existence Theorem 86.8.

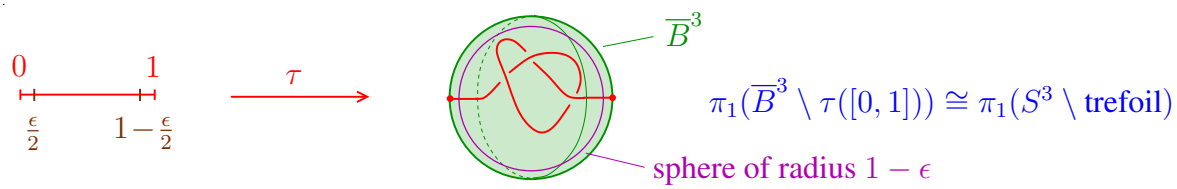
The precise construction of a "wild" embedding $w\colon S^1 \to S^3$ requires some preparations. We start out with the following lemma:

Lemma 86.9. (Disk Knot–Non-Abelian Lemma) There exists a proper smooth embedding $\tau\colon [0,1] \to \overline{B}^3$ with the following two properties:

(1) There exists an $\epsilon > 0$ such that $\tau(t) = (2t-1, 0, 0)$ for $t \in [0, \frac{\epsilon}{2}]$ and $t \in [1 - \frac{\epsilon}{2}, 1]$ and such that $\tau^{-1}(\{x \in \overline{B}^3 \mid \|x\| > 1 - \epsilon\}) = [0, \frac{\epsilon}{2}] \cup [1 - \frac{\epsilon}{2}, 1]$.
(2) $\pi_1(\overline{B}^3 \setminus \tau([0,1]))$ is non-abelian.

Proof. Such a construction is illustrated in the figure below:
More details can be found in [**Fri24**]. ■

Our goal is to construct an embedding $f\colon S^1 \to S^3$ such that $f(S^1)$ is not a topological submanifold of S^3. To simplify the notation we will use the homeomorphisms from the Ball



Quotient-Sphere Lemma 6.5 and pages 144 to make the identifications $[0,2]/0 \sim 2 =_{\mathrm{i}} S^1$ and $\overline{B}^3_1 \cup_{S^2_1 = S^2_2} \overline{B}^3_2 =_{\mathrm{i}} S^3$.

Notation.

(1) For the remainder of this section we fix a proper smooth embedding $\tau\colon [0,1] \to \overline{B}^3$ as in the Disk Knot–Non-Abelian Lemma 86.9.

(2) Given $n \in \mathbb{N}$ we consider the smooth embedding

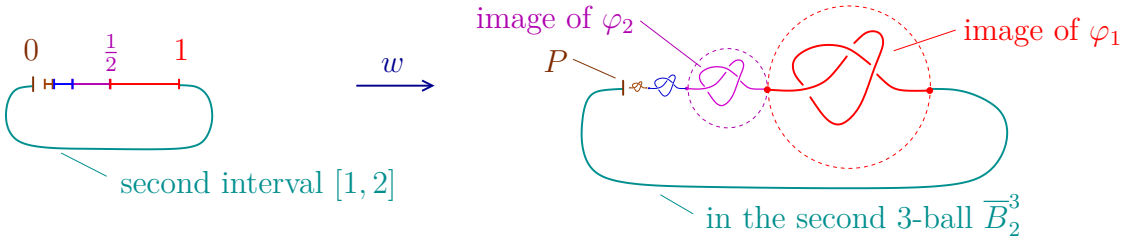
$$\begin{aligned} \varphi_n\colon \overline{B}^3 &\to B^3 \subset \overline{B}^3 \subset S^3 \\ x &\mapsto (-1 + \tfrac{3}{2^n}, 0, 0) + \tfrac{1}{2^n} \cdot x. \end{aligned}$$

(3) We consider the map

$$w\colon \overbrace{[0,2]/0 \sim 2}^{=_{\mathrm{i}} S^1} \to \overbrace{\overline{B}^3_1 \cup_{S^2_1 = S^2_2} \overline{B}^3_2}^{=_{\mathrm{i}} S^3}$$

$$[t] \mapsto \begin{cases} (-1,0,0) \in \overline{B}^3_1, & \text{if } t = 0, \\ \varphi_n(\tau(\underbrace{2^{n-1} \cdot (t - \frac{1}{2^n})}_{\in (0,1]})) \in \overline{B}^3_1, & \text{if } t \in (\frac{1}{2^n}, \frac{1}{2^{n-1}}] \text{ for some } n \in \mathbb{N}, \\ (3 - 2t, 0, 0) \in \overline{B}^3_2, & \text{if } t \in [1,2]. \end{cases}$$

(4) We set $W := w(S^1)$ and we write $P := w([0])$.



The following proposition immediately implies the Wild Knot Existence Theorem 86.8.

Proposition 86.10. (Wild Embedding Existence Proposition) Let $w\colon S^1 \to S^3$ be the above map.

(1) The map $w\colon S^1 \to S^3$ is an embedding.

(2) There is no topological submanifold chart for the subspace $W := w(S^1) \subset S^3$ around $P = w([0])$. In particular $W = w(S^1)$ is not a topological submanifold of S^3.

Our proof of the Wild Embedding Existence Proposition 86.10 crucially relies on the following lemma, which is a descendant of the General Alexander Duality Theorem 86.1.

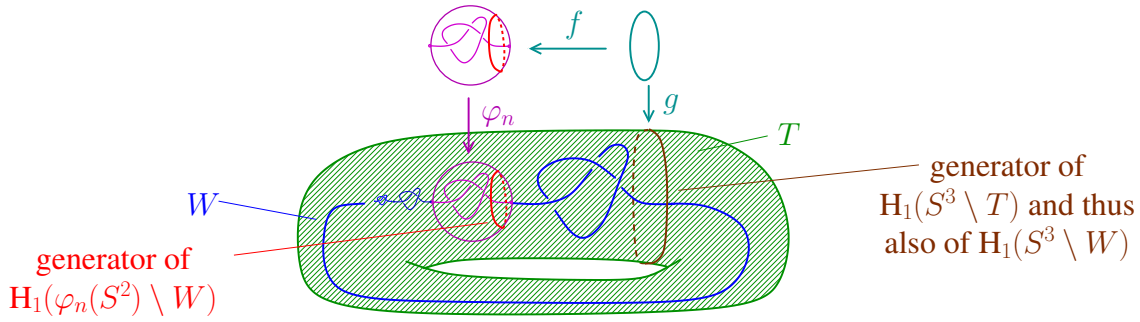
Auxiliary Lemma 86.11. For any $n \in \mathbb{N}$ the inclusion-induced map

$$\mathrm{H}_1(\varphi_n(S^2) \setminus W) \;\to\; \mathrm{H}_1(S^3 \setminus W)$$

is an isomorphism.

Proof. Let $n \in \mathbb{N}$. We make the following preparations and observations:

(1) We denote by $i\colon \varphi_n(S^2) \setminus W \to S^3 \setminus W$ the inclusion.
(2) We denote by $T =_{\mathrm{i}} \overline{B}^2 \times S^1$ the "standard solid torus" in $S^3 = \overline{B}^3_1 \cup_{S^2_1 = S^2_2} \overline{B}^3_2$ that is given by the points $(x, y, z) \in \overline{B}_i$ in each ball with $y^2 + z^2 \leq \frac{1}{2}$. We denote by $j\colon W \to T$ the inclusion map.
(3) Note that $(\varphi_n|_{S^2})^{-1}(W) = \{(\pm 1, 0, 0)\}$.
(4) (a) Let $f\colon S^1 \to S^2 \setminus \{(\pm 1, 0, 0)\}$ be the map that is given by $(x, y) \mapsto (0, x, y)$. Note that f is an embedding and note that $S^2 \setminus f(S^1)$ is a deformation retract of $S^2 \setminus \{(\pm 1, 0, 0)\}$. It follows that f is a homotopy equivalence.
(b) Let $g\colon S^1 \to \partial T = \partial(\overline{B}^2 \times S^1)$ be the map that is given by $z \mapsto (z, w_0)$ for some fixed $w_0 \in S^1$.
(c) The two maps $\varphi_n \circ f, g\colon S^1 \to S^3 \setminus W$ are clearly homotopic.



We consider the following diagram:

$$\begin{array}{ccc} \mathrm{H}_1(S^1) & \xrightarrow{\quad g_* \quad} & \mathrm{H}_1(S^3 \setminus T) \\ {\scriptstyle (\varphi_n \circ f)_*}\downarrow & & \downarrow \\ \mathrm{H}_1(\varphi_n(S^2) \setminus W) & \longrightarrow & \mathrm{H}_1(S^3 \setminus W). \end{array}$$

We make a few more observations:

(5) It follows from the Wild Knot Complement-H_*-Lemma 86.2 that the right vertical map is an isomorphism.
(6) (a) It follows from (4a) and the Homotopy Equivalence-H_*-Corollary 36.7 that the left vertical map is an isomorphism.
(b) It follows from (4b) that the top horizontal map is an isomorphism.
(c) It follows from (4c) and the Homotopy-H_*-Proposition 36.5 that the diagram commutes.

It follows from (5) and (6) that the bottom horizontal map is indeed an isomorphism. ∎

Our proof of the Wild Embedding Existence Proposition 86.10 relies also on the following purely algebraic proposition.

Proposition 86.12. (Amalgamated Product-Monomorphism Proposition) Suppose we are given two group homomorphisms $\alpha\colon G \to A$ and $\beta\colon G \to B$. If α and β are both monomorphisms, then the natural homomorphisms $A \to A *_G B$ and $B \to A *_G B$ are also both monomorphisms.

Proof of Proposition 86.12. The proposition is proved in [**Ser80**, p. 3]. ■

Proof of Proposition 86.10. Let $w\colon S^1 \to S^3$ be the map from page 1131. Recall that we need to prove the following two statements:

(1) The map $w\colon S^1 \to S^3$ is an embedding.
(2) There is no topological submanifold chart for the subspace $W := w(S^1) \subset S^3$ around $P = w([0])$. In particular $W = w(S^1)$ is not a topological submanifold of S^3.

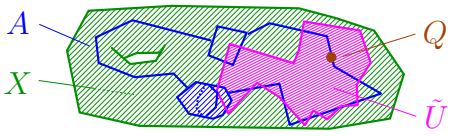
First we provide the proof of Statement (1). Using the ϵ-δ-criterion from the Metric Continuity Proposition 2.5 one can easily show that the map $[0,1] \to S^3$, induced by w, is continuous. It is clear that the corresponding map $[1,2] \to S^3$ is continuous. It follows from the Pasting Proposition 2.4 together with the Topological-Quotient Proposition 6.1 (1b) that the given map $f\colon S^1 \to S^3$ is continuous. It now follows from the Compact-Hausdorff Proposition 2.10 (2) that $f\colon S^1 \to S^3$ is actually an embedding.

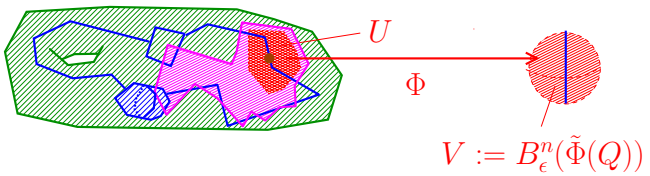
Now we turn to the proof of Statement (2). We start out with the following harmless general claim:

Claim 1. Let X be an n-dimensional topological manifold, let $A \subset X$ be a subspace and let $Q \in A$. If there exists an $(n-2)$-dimensional submanifold chart of type (α) for $A \subset X$ and $Q \in A$, then there exists an $(n-2)$-dimensional submanifold chart $\Phi\colon U \to V$ such that $U \setminus A$ is path-connected and such that for every $x_0 \in U \setminus A$ the group $\pi_1(U \setminus A, x_0)$ is isomorphic to $\mathbb{Z}$.[683]

Proof. Let $\widetilde{\Phi}\colon \widetilde{U} \to \widetilde{V}$ be an $(n-2)$-dimensional submanifold chart for $A \subset X$ and $Q \in A$. Since $\widetilde{\Phi}\colon \widetilde{U} \to \widetilde{V}$ is an $(n-2)$-dimensional submanifold chart of type (α) we know that $\widetilde{\Phi}(A \cap \widetilde{U}) = (\{0\} \times \mathbb{R}^{n-2}) \cap \widetilde{V}$.

Since $\widetilde{V} \subset \mathbb{R}^n$ is open there exists an $\epsilon > 0$ such that $V := B^n_\epsilon(\widetilde{\Phi}(Q)) \subset \widetilde{V}$. We now set $U := \widetilde{\Phi}^{-1}(V)$ and $\Phi := \widetilde{\Phi}|_U$.





From the above it follows that $\Phi(A \cap U) = \{0\} \times B^{n-2} \subset B^n$. We now see that

$$\pi_1(U \setminus A, x_0) \underset{\text{since } \Phi|_U \text{ is a homeomorphism}}{\cong} \pi_1(B^n \setminus (\{0\} \times B^{n-2})) \underset{\text{since } B^n \setminus (\{0\} \times B^{n-2}) \text{ admits a deformation retraction to the circle } S^1_{\frac{1}{2}} \times \{0\}}{\cong} \pi_1(S^1) \underset{\text{by the Circle-}\pi_1 \text{ Theorem 11.13}}{\cong} \mathbb{Z}.$$

⊞

It follows from this discussion, together with Claim 1, that it suffices to prove the following claim:

[683] Since $U \setminus A$ is path-connected we know by the Base Point Independence Corollary 11.7 that the isomorphism type of the fundamental group does not depend on the choice of the base point.

Claim 2. For every open neighborhood U of $P \in S^3$ such that $U \setminus W$ is path-connected, the fundamental group $\pi_1(U \setminus W)$ is non-abelian.

Proof. Let U be an open neighborhood of $P \in S^3$ such that $U \setminus W$ is path-connected. Since U is a neighborhood of $P = w([0])$ there exists an $n \in \mathbb{N}$ such that $\varphi_n(\overline{B}^3) \subset U$. We pick an $\epsilon > 0$ as in the Disk Knot–Non-Abelian Lemma 86.9.

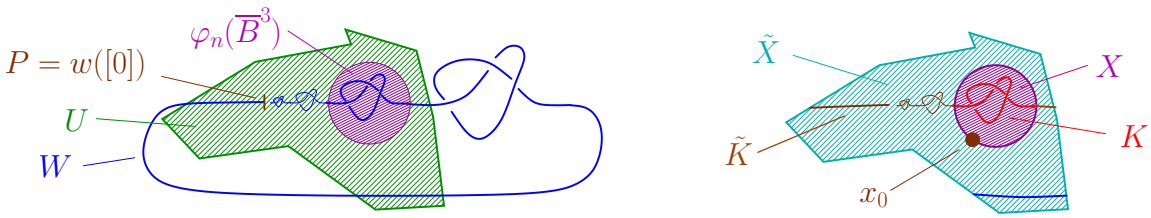
(a) Basically by construction we know that $\varphi_n(B^3)$ is an open subset of S^3.
(b) Note that $S^2_{1-\epsilon}$ is a submanifold of B^3 that intersects $\tau([0,1])$ transversally in two points.
(c) Note that $\overline{B}^3_{1-\epsilon}$ is a codimension-zero submanifold of B^3.
(d) It follows from (a) and (c) that $\varphi_n(\overline{B}^3_{1-\epsilon})$ is a codimension-zero submanifold of S^3.
(e) It follows from the Topological Submanifold Proposition 42.6 and (d) that $S^3 \setminus \varphi_n(\overline{B}^3_{1-\epsilon})$ is also a codimension-zero submanifold of S^3.

We introduce the following notation:

- We write $X \coloneqq \varphi_n(\overline{B}^3_{1-\epsilon})$, $\widetilde{X} \coloneqq U \setminus \varphi_n(B^3_{1-\epsilon})$ and $Y \coloneqq \varphi_n(S^2_{1-\epsilon})$,
- We write $K \coloneqq W \cap X$, $\widetilde{K} \coloneqq K \cap \widetilde{X}$ and $L \coloneqq K \cap Y$.

We now see that for any $x_0 \in Y \setminus L$ we have

$$\pi_1(U \setminus W, x_0) \xleftarrow[\cong]{\text{follows from the Manifold–Seifert-van Kampen Theorem 23.1}} \underbrace{\pi_1(X \setminus K, x_0)}_{\substack{\text{non-abelian} \\ \text{by choice of } W}} *_{\pi_1(Y \setminus L, x_0)} \pi_1(\widetilde{X} \setminus \widetilde{K}, x_0).$$



We know that $\pi_1(X \setminus K)$ is non-abelian. To conclude that $\pi_1(U \setminus W)$ is non-abelian, it suffices by the Amalgamated Product-Monomorphism Proposition 86.12 to prove the following subclaim:

Subclaim. The two inclusion-induced homomorphisms $\pi_1(Y \setminus L, x_0) \to \pi_1(X \setminus K, x_0)$ and $\pi_1(Y \setminus L, x_0) \to \pi_1(\widetilde{X} \setminus \widetilde{K}, x_0)$ are monomorphisms.

Proof of subclaim. We consider the following diagram:

$$\begin{array}{ccc} \pi_1(Y \setminus L, x_0) & \longrightarrow & \pi_1(X \setminus K, x_0) \\ \downarrow & \searrow & \downarrow \\ \pi_1(\widetilde{X} \setminus \widetilde{K}, x_0) & \longrightarrow & \pi_1(S^3 \setminus W, x_0). \end{array}$$

Here all maps are induced by inclusions. By the functoriality of fundamental groups the diagram commutes. We see that it suffices to prove that the diagonal map is a monomorphism. For that we consider the following diagram:

$$\begin{array}{ccc}
\pi_1(Y \setminus L, x_0) & \longrightarrow & \pi_1(S^3 \setminus W, x_0) \\
\downarrow \cong & & \downarrow \\
\pi_1(Y \setminus L, x_0)_{\mathrm{ab}} & \longrightarrow & \pi_1(S^3 \setminus W, x_0)_{\mathrm{ab}} \\
\downarrow \cong & & \downarrow \cong \\
\mathrm{H}_1(Y \setminus L) & \xrightarrow{\cong} & \mathrm{H}_1(S^3 \setminus W).
\end{array}$$

We make the following observations and clarifications:

(i) The top vertical maps are the natural maps from page 273. Since $\pi_1(Y \setminus L, x_0) \cong \mathbb{Z}$ is abelian we know that the top left vertical map is an isomorphism.

(ii) The bottom vertical maps are are the natural Hurewicz homomorphisms from the Hurewicz Homomorphism Proposition 58.1. By the Hurewicz Theorem 58.5 we know that these are isomorphisms.

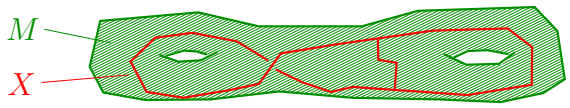
(iii) By Lemma 86.11 we know that the bottom horizontal map is an isomorphism.

By naturality the diagram commutes. It thus follows from the above discussion that the top horizontal map is, as promised, a monomorphism. ■

86.5. **Invariants of complements.** Let us first recall a result about smooth manifolds and smooth embeddings: Let M be a compact smooth manifold, let X be a compact smooth manifold and let $f\colon X \to M$ be a *smooth* embedding. Since X admits a tubular neighborhood we see that $M \setminus f(X)$ is homotopy equivalent to a compact smooth manifold. This implies that that $\pi_1(M \setminus f(X))$ is finitely presented and that the homology groups $\mathrm{H}_k(M \setminus f(X))$ are finitely generated.

We now want to consider what happens, if we consider topological manifolds instead of smooth manifolds. The following proposition shows that when it comes to homology groups, we can generalize the above statement to the more general topological setting:

Proposition 86.13. (Complement-of-Subspace-H_*-Proposition) Let M be a compact orientable topological manifold and let $X \subset M$ be a weakly locally contractible, compact subspace such that $X \cap \partial M$ is weakly locally contractible. Then the homology groups $\mathrm{H}_k(M \setminus X)$ are also finitely generated.



each $\mathrm{H}_k(M \setminus X)$ is finitely generated

The proof of the Complement-of-Subspace-H_*-Proposition 86.13 relies on the following lemma:

Lemma 86.14. (Finite-H_*-H^*-Lemma) Let M be a compact orientable topological manifold and let $X \subset M$ be a weakly locally contractible, compact subspace of M. The homology groups $\mathrm{H}_i(X)$ and the cohomology groups $\mathrm{H}^i(X;\mathbb{Z})$ of X are finitely generated.

Proof of Lemma 86.14. Since X is a compact subspace of a topological manifold and since X is weakly locally contractible we know by the Euclidean Embedding Theorem 60.6,

Borsuk's Theorem 60.4 and the Simplicial Complex Retract Proposition 60.8 that X is homeomorphic to a retract of a finite simplicial complex.

By the Simplicial Complex-Retract-Invariants Lemma 60.11 (2b) we know that the homology groups of X are finitely generated. It thus follows from the Universal Coefficient-H^*-Theorem 70.14 and basic facts about Hom-groups and Ext-groups that the cohomology groups of X are also finitely generated. ■

Proof of the Complement-of-Subspace-H_*-Proposition 86.13. Let M be a compact orientable topological manifold and let $X \subset M$ be a weakly locally contractible, compact subspace. We first consider the case M that is actually *closed*. In this setting we consider the following maps:

$$\begin{array}{ccccccccc} & & & & \mathrm{H}_k(M \setminus X) & & & & \\ & & & & \cong \downarrow & \text{(1)–(4)} & & & \\ \mathrm{H}^{n-k}(M;\mathbb{Z}) & \leftarrow & \underset{U\in\mathcal{N}(X)}{\operatorname{colim}} \mathrm{H}^{n-k}(M,U;\mathbb{Z}) & \xleftarrow{(5)} & \underset{U\in\mathcal{N}(X)}{\operatorname{colim}} \mathrm{H}^{n-k-1}(U;\mathbb{Z}) & \leftarrow & \mathrm{H}^{n-k-1}(M;\mathbb{Z}). \\ & & & & \cong \downarrow & \text{(6)} & & & \\ & & & & \mathrm{H}^{n-k-1}(X;\mathbb{Z}) & & & & \end{array}$$

We make the following clarifications and observations:

(a) On page 1121 we wrote down six homomorphisms. In the proof of the General Alexander Duality Theorem 86.1 (1) we showed that, regardless of any homological information, the maps (1)–(4) and (6) are isomorphisms. These give us the vertical isomorphisms.
(b) In the proof of the General Alexander Duality Theorem 86.1 (1) we also discussed the homomorphism (5). In our setting this is not an isomorphism, but we saw that it fits into the horizontal sequence and that this sequence is exact.
(c) Since X and M are compact it follows from the Finite-H_*-H^*-Lemma 86.14 that the groups $\mathrm{H}^{n-k}(M;\mathbb{Z})$ and $\mathrm{H}^{n-k-1}(X;\mathbb{Z})$ are finitely generated.
(d) It follows from the above discussion together with the Finite Generation Lemma 18.8 that $\mathrm{H}_k(M \setminus X)$ is finitely generated.

The case $\partial M \neq \varnothing$ is dealt with via a doubling argument. The details can be found in [**Fri24**]. ■

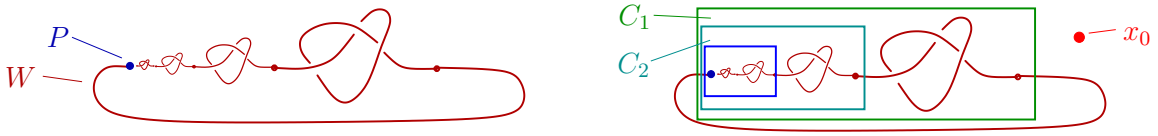
The Complement-of-Subspace-H_*-Proposition 86.13 raise the question, whether we can also control the size of the *fundamental groups* of complements of weakly locally contractible, compact subspaces. But the following proposition shows that this is not the case.

Proposition 86.15. (Wild Knot-π_1-Proposition) There is an embedding $w\colon S^1 \to S^3$ such that $\pi_1(S^3 \setminus w(S^1))$ is not finitely generated.

Sketch of proof. We consider the "wild knot" $W \coloneqq w(S^1) \subset \mathbb{R}^3 \subset \mathbb{R}^3 \cup \{\infty\} =_{\mathrm{i}} S^3$ from page 1131. Our goal is to show that $\pi_1(S^3 \setminus W)$ is not finitely generated. We introduce the following notation:

(1) For $n \in \mathbb{N}$ we denote by $C_n \subset \mathbb{R}^3$ the compact cuboid that is shown in the figure below. Note that $\bigcap_{n\in\mathbb{N}} C_n = \{P\}$. We set $U_n \coloneqq (S^3 \setminus C_n) \setminus W$. Note that the following statements hold:
- Each U_n is an open subset of $S^3 \setminus W$.
- For each $n \in \mathbb{N}$ we have $U_n \subset U_{n+1}$.
- We have $\bigcup_{n\in\mathbb{N}} U_n = S^3 \setminus W$.

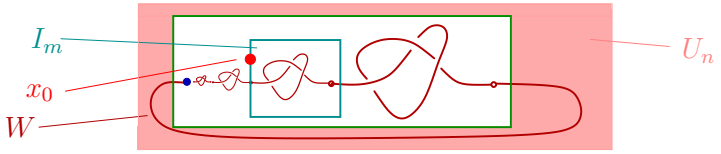
(2) We pick $x_0 \in U_1$.



We continue with the following crucial observation: If $\pi_1(S^3 \setminus W, x_0)$ was finitely generated, then it would follow from an elementary compactness argument, as in the proof of the Exhaustion-π_1-Proposition 26.2, that there exists an $n \in \mathbb{N}$ such that the inclusion-induced map $\pi_1(U_n, x_0) \to \pi_1(S^3 \setminus W, x_0)$ is an epimorphism. Thus it remains to prove the following claim:

Claim. There is no $n \in \mathbb{N}$ such that the inclusion-induced map $\pi_1(U_n, x_0) \to \pi_1(S^3 \setminus W, x_0)$ is an epimorphism.

Proof. Let $n \in \mathbb{N}$. We pick an $m > n$. We denote by I_m the m-th cube in the construction of the "wild knot" W. We consider the two inclusion maps $i\colon \partial(I_m \setminus W) \to I_m \setminus W$ and $j\colon \partial(I_m \setminus W) \to (S^3 \setminus \mathring{I}_m) \setminus W$. We pick $x_0 \in \partial(I_m \setminus W)$ and we immediately erase it from the notation in the discussion below.



Next we consider the following two diagrams:

$$\begin{array}{ccc}
\overbrace{\pi_1(\partial(I_m\setminus W))}^{\cong\mathbb{Z}} & \xrightarrow{j_*} & \pi_1((S^3\setminus \mathring{I}_m)\setminus W) \\
\cong\downarrow & & \downarrow \\
\pi_1(\partial(I_m\setminus W))_{\mathrm{ab}} & \xrightarrow{j_*} & \pi_1((S^3\setminus \mathring{I}_m)\setminus W)_{\mathrm{ab}} \\
\cong\downarrow & & \downarrow \\
\mathrm{H}_1(\partial(I_m\setminus W)) & \xrightarrow{j_*} & \mathrm{H}_1((S^3\setminus \mathring{I}_m)\setminus W) \\
\mathrm{id}\downarrow & & \downarrow \\
\mathrm{H}_1(\partial(I_m\setminus W)) & \xrightarrow[\cong]{j_*} & \mathrm{H}_1(S^3\setminus W).
\end{array}$$

$$\begin{array}{ccc}
 & & \pi_1(U_n) \\
 & \overset{\rho}{\leftarrow} & \downarrow \\
\pi_1(\partial(I_m\setminus W)) & \xrightarrow[j_*]{} & \pi_1((S^3\setminus \mathring{I}_m)\setminus W) \\
i_*\downarrow & & \downarrow \\
\pi_1(I_m\setminus W) & \longrightarrow & \pi_1(S^3\setminus W)
\end{array}$$

$i_*\circ\rho$ (curved arrow from $\pi_1(U_n)$ to $\pi_1(I_m\setminus W)$); $=:\Theta$ (dashed arrow from $\pi_1(S^3\setminus W)$ to $\pi_1(I_m\setminus W)$); id (curved arrow from $\pi_1(I_m\setminus W)$ to $\pi_1(I_m\setminus W)$).

We make the following observations and clarifications:

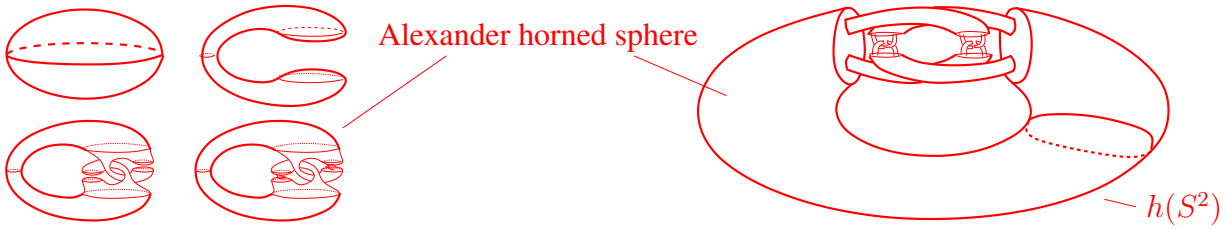
(1) First we consider the diagram to the left. This diagram is very similar to the diagram on page 1135. In fact we again see that the left vertical maps and the bottom horizontal maps are isomorphisms. It follows from this discussion that there exists a left-inverse $\rho\colon \pi_1((S^3 \setminus \mathring{I}_m) \setminus W) \to \pi_1(\partial(I_m \setminus W))$ to $j_*\colon \pi_1(\partial(I_m \setminus W)) \to \pi_1((S^3 \setminus \mathring{I}_m) \setminus W)$.

(2) In the diagram to the right we show the pushout diagram that we obtain from the Manifold–Seifert-van Kampen Theorem 23.1. By the universal property of pushouts and the construction of ρ we get an induced homomorphism $\Theta\colon \pi_1(S^3 \setminus W) \to \pi_1(I_m \setminus W)$ which makes the diagram commute.

(3) By construction the of W on page 1131 we know that $\pi_1(I_m \setminus W)$ is non-abelian.

In summary we have found a homomorphism $\Theta\colon \pi_1(S^3 \setminus W) \to \pi_1(I_m \setminus W)$ with non-abelian image but such that the image of $\pi_1((S^3 \setminus \mathring{I}_m) \setminus W)$ is abelian. From $m > n$ we obtain that $U_n \subset (S^3 \setminus \mathring{I}_m) \setminus W$. Thus the image of $\pi_1(U_m)$ is also abelian. It follows that the inclusion-induced map $\pi_1(U_n) \to \pi_1(S^3 \setminus W)$ is not an epimorphism. ■

Example. We conclude this discussion on "wild" subspaces with an example of a "wild sphere". In the figure below we show an embedding $h\colon S^2 \to \mathbb{R}^3$ which was introduced by James Alexander [**Ale24**]. The image $h(S^2)$ is called the *Alexander horned sphere*.



A rigorous description is given in [**Ale24**] and alternatively in [**Hat02**, p. 170] and in [**DV09**, p. 46]. Furthermore in these references it is shown that one of the components of $\mathbb{R}^3 \setminus h(S^2)$ is not simply connected. □

CHAPTER 87

The square peg problem

In this chapter we will introduce the square peg problem and we will see how the General Alexander Duality Theorem 86.1 can be used to prove an interesting variation of the square peg problem.

To formulate the problem we need to introduce the following two definitions:

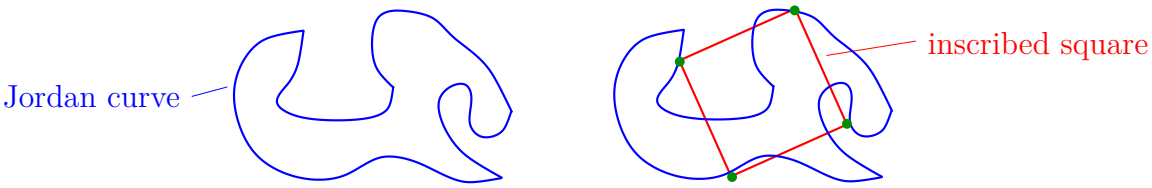
Definition.

(1) The image of an injective continuous map $S^1 \to \mathbb{R}^2$ is called a Jordan curve.

(2) Given a Jordan curve C we say that a rectangle R is inscribed in C if all vertices of R lie on C.

The following question was asked by Otto Toeplitz [**Toe11**] in 1911.

Question 87.1. (Square Peg Problem) Does every Jordan curve admit an inscribed square?



This question is known as the "inscribed square problem" or the "square-peg problem". Lev Schnirelmann [**Sch44**] (see also [**Str89**]) gave a positive answer if the closed curve is smooth, in fact he gave a proof if the closed curve is C^2, i.e. if the map $S^1 \to \mathbb{R}^2$ is twice continuously differentiable. But the general case is still open.



The following proposition is the best result that we can prove in the context of the square peg problem:

Proposition 87.2. (Inscribed Rectangle Proposition) Every Jordan curve admits an inscribed rectangle.

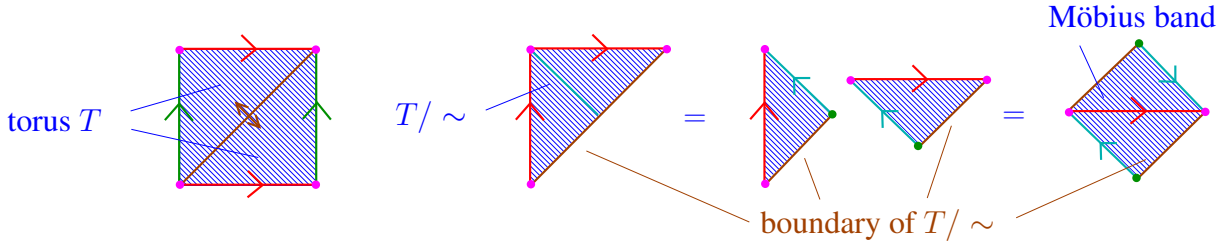
In preparation for the proof we need the following elementary lemma:

Lemma 87.3. (Torus Quotient–Möbius Band Lemma) Let $T = S^1 \times S^1$ be the torus. Let $\sim$ be the equivalence relation on T generated by $(z, w) \sim (w, z)$. The quotient

space $T/\sim$ is homeomorphic to the Möbius band and[684]

$$\partial(T/\sim) \;=\; \{[(z,z)]\,|\,z\in S^1\}.$$

Proof. The homeomorphism is illustrated in the figure below.



We leave it to the reader to fill in the details. ■

Now we are already ready to prove the Inscribed Rectangle Proposition 87.2.

Proof of the Inscribed Rectangle Proposition 87.2. Let $\gamma\colon S^1\to\mathbb{R}^2$ be an injective continuous map. We have to show that $\gamma(S^1)$ admits an inscribed rectangle. We consider the map

$$\begin{aligned}\Phi\colon T:=S^1\times S^1 &\to \mathbb{R}^2\times\mathbb{R}_{\geq 0}\\ (z,w) &\mapsto \Big(\tfrac{\gamma(z)+\gamma(w)}{2},\|\gamma(z)-\gamma(w)\|\Big).\end{aligned}$$

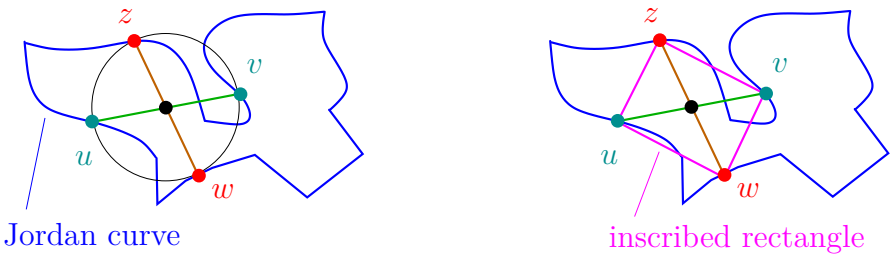
Clearly this map is continuous. Next note that for all $z,w\in S^1$ we have the equality $\Phi(z,w)=\Phi(w,z)$. Thus we obtain from the Topological-Quotient Proposition 6.1 (1b) that Φ descends to a continuous map

$$\Psi\colon T/\sim\;\to\;\mathbb{R}^3$$

where $\sim$ denotes of course the equivalence relation generated by $(z,w)\sim(w,z)$.

Claim. The map $\Psi\colon T/\sim\to\mathbb{R}^2\times\mathbb{R}_{\geq 0}$ is not injective.

We first show that the claim implies the proposition. So now that we know that Ψ is not injective we know that there exist distinct pairs of points $\{z,w\}$ and $\{u,v\}$ on S^1 such that $\Phi(z,w)=\Phi(u,v)$. We consider the quadrilateral given by the points z,u,v,w.

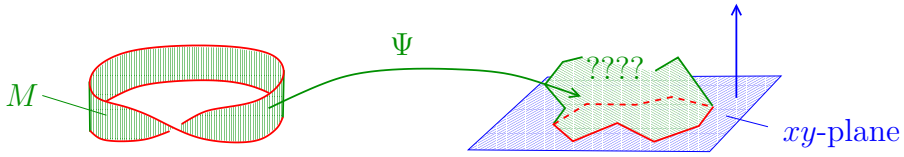


It follows from $\Phi(z,w)=\Phi(u,v)$ that the diagonals bisect each other and that the diagonals have the same length. An elementary argument (see Exercise **??**) shows that such a quadrilateral is necessarily a rectangle.

Proof of claim. We will prove the claim by contradiction. Thus let us now assume that the map $\Psi\colon T/\sim\to H:=\mathbb{R}^2\times\mathbb{R}_{\geq 0}$ is injective. By the Torus Quotient–Möbius Band Lemma 87.3 the quotient $M:=T/\sim$ is homeomorphic to the Möbius band and the boundary

[684]Here by "$\partial(T/\sim)$" we mean the boundary of the topological manifold $T/\sim$.

∂M is given by $\{[(z,z)]\,|\,z \in S^1\}$. It follows easily from the definition of Ψ and the injectivity of γ that for a point $[(z,w)] \in M = T/\!\sim$ we have $\Psi([(z,w)]) \in \partial H = \mathbb{R}\times\mathbb{R}\times\{0\}$ if and only if $[(z,w)] \in \partial M$.



We denote by M_1 and M_2 two copies of $M$and we denote by

$$\mathrm{D}M \;=\; M_1 \cup_{\partial M_1 = \partial M_2} M_2 \;=\; (M_1 \sqcup M_2)/(P,1) \sim (P,2) \quad \text{for } P \in \partial M$$

the double of M. Note that $\mathrm{D}M$ is in fact homeomorphic to the Klein bottle, in particular $\mathrm{D}M$ is a closed, non-orientable 2-dimensional topological manifold.
We denote by $\rho\colon \mathbb{R}^3 \to \mathbb{R}^3$ the reflection in the xy-plane, i.e. ρ is the map that is given by $\rho(x,y,z) = (x,y,-z)$. We consider the map

$$\begin{array}{rcl} \mathrm{D}M = M_1 \cup_{\partial M_1 = \partial M_2} M_2 & \to & \mathbb{R}^3 \\ [P] & \mapsto & \begin{cases} \Psi(P), & \text{if } P \in M_1, \\ \rho(\Psi(P)), & \text{if } P \in M_2. \end{cases} \end{array}$$

It follows immediately from the above discussion that this map is well-defined and injective. Furthermore it follows from the Topological-Quotient Proposition 6.1 (1b) that the map is in fact continuous. But now we have constructed an injective continuous map from the Klein bottle to $\mathbb{R}^3$. Since the Klein bottle is compact we obtain from the Compact-Hausdorff Proposition 2.10 that such a map is in fact an embedding. But recently we proved in the Surface-No Topological Embedding-Corollary 86.6 that such an embedding cannot exist. ■

CHAPTER 88

The cup product and algebraic intersection numbers

In this chapter we will introduce the "algebraic intersection number" of two "complementary" oriented submanifolds. This concept is very geometric and in practice it is usually easy and fun to calculate algebraic intersection numbers. The main result of this chapter shows that the algebraic intersection number is related to the cup product. This results cuts in two ways: We can use it to calculate cup products and we can use it to prove results on algebraic intersection numbers.

88.1. **Algebraic intersection numbers of submanifolds.** Before we can introduce the algebraic intersection number we need to introduce a few general concepts:

Definition. Let M be a smooth manifold.

(1) A submanifold $X \subset M$ is called proper, if X is a closed subset and if $\partial X = X \cap \partial M$.
(2) We say that two oriented proper submanifolds Z_0 and Z_1 of M are properly smoothly isotopic if there exists a proper smooth isotopy between Z_0 and Z_1, i.e. if there exists a smooth isotopy $F\colon Z_0 \times [0,1] \to M$ such that $F_0 = \mathrm{id}_{Z_0}$, such that each $F_t\colon Z_0 \to M$ is a proper embedding, and such that $F_1\colon Z_0 \to Z_1$ is an orientation-preserving diffeomorphism.
(3) Let X and Y be two oriented proper submanifolds of M. We say that X and Y can be made disjoint if there exist oriented proper disjoint submanifolds $\widetilde{X}$ and $\widetilde{Y}$ of M such that X and $\widetilde{X}$ and also Y and $\widetilde{Y}$ are properly smoothly isotopic.

Example. We consider the three submanifolds X, Y and Z of the torus that are illustrated in the figure below on the left.



As is shown in the same figure on the right, one can make X and Y disjoint. □

It does not look like we can make the submanifolds X and Z disjoint. But how can we prove it? We will record this question for later:

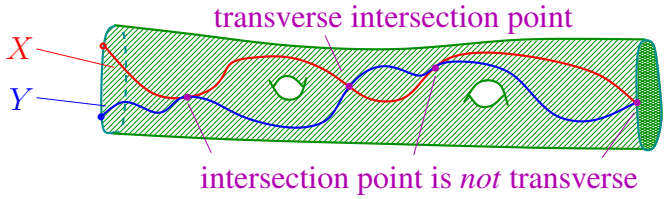
Question 88.1. Can we make the submanifolds X and Z, that are illustrated in the figure above, disjoint?

Our approach to addressing Question 88.1 is to introduce the "algebraic intersection number". As a first step we introduce the following definition:

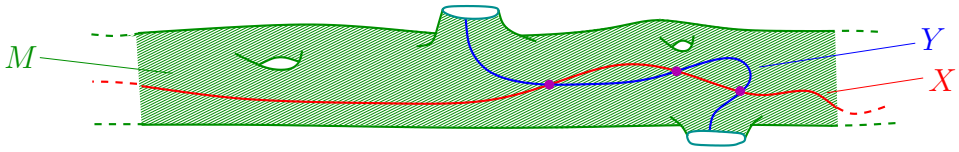
Definition. Let M be a smooth manifold and let X and Y be proper submanifolds of M with $\dim(X) + \dim(Y) = \dim(M)$.

(1) Let $P \in X \cap Y$. We say that P is a transverse intersection point if $P \notin \partial M$ and if X and Y intersect transversally in P, i.e. if the equality[685] $\mathrm{T}_P X + \mathrm{T}_P Y = \mathrm{T}_P M$ holds.

(2) We say that X and Y intersect transversally if each intersection point is a transverse intersection point.



Lemma 88.2. (Finite Transverse Intersection Lemma) Let M be a smooth manifold and let X and Y be proper submanifolds of M with $\dim(X) + \dim(Y) = \dim(M)$. If X and Y intersect transversally and if at least one of X or Y is compact, then the intersection $X \cap Y$ consists of finitely many points.



Proof. Without loss of generality we can now assume that Y is compact. Since X and Y intersect transversally it follows from an elementary argument that $X \cap Y$ is a 0-dimensional submanifold of M. It follows easily from the definition of a 0-dimensional submanifold that $X \cap Y$ is a discrete subset of M. Since Y is compact and since X is in particular a closed subset of M we obtain from the Compact-Closed Lemma 1.9 (1) that $X \cap Y$ is also compact. It now follows from the Compact-Discrete Lemma 1.10 that $X \cap Y$ is finite. ■

Definition. Let $m \in \mathbb{N}$ and let $k \in \{1, \ldots, m-1\}$. Furthermore let M be an m-dimensional smooth manifold, let $X \subset M$ be a compact proper submanifold of dimension k and let $Y \subset M$ be a compact proper submanifold of the complementary dimension $m-k$. Suppose that X and Y intersect transversally.

(1) We define[686]

$$\text{geometric intersection number of } X \text{ and } Y \;:=\; \#(X \cap Y).$$

(2) We now assume that X, Y and M are oriented.

(a) Let P be an intersection point of X and Y. We set $\mathrm{sign}(P) := 1$, if

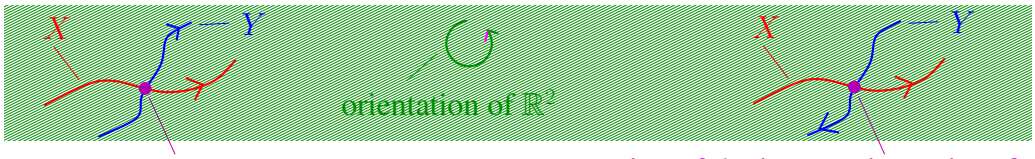
$$(\text{positive basis of } \mathrm{T}_P X, \text{positive basis of } \mathrm{T}_P Y) \;=\; \text{positive basis of } \mathrm{T}_P M.$$

Otherwise we set $\mathrm{sign}(P) := -1$.[687]

(b) We define

$$\text{algebraic intersection number of } X \text{ and } Y \;:=\; X \cdot Y \;:=\; \sum_{P \in X \cap Y} \mathrm{sign}(P) \in \mathbb{Z}.$$

[685] Since here we assume that $\dim(X) + \dim(Y) = \dim(M)$ we see that we have $\mathrm{T}_P X + \mathrm{T}_P Y = \mathrm{T}_P M$ if and only if $\mathrm{T}_P X \oplus \mathrm{T}_P Y = \mathrm{T}_P M$.

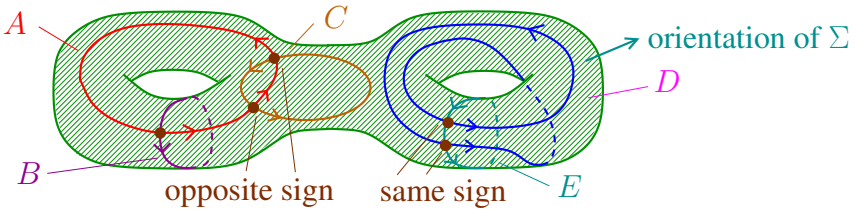


sign of the intersection point of the submanifolds X and Y in $\mathbb{R}^2$ equals $+1$

sign of the intersection point of the submanifolds X and Y in $\mathbb{R}^2$ equals -1

Examples.

(1) We consider the surface of genus 2 with the orientation where the normal vector "sticks outward". Furthermore we consider the oriented submanifolds A, B, C, D and E that are illustrated in the figure below:



For the submanifolds A and B we have

$$\text{geometric intersection number of } A \text{ and } B \;=\; 1,$$

the sign of the intersection point is negative, hence

$$\text{algebraic intersection number of } A \text{ and } B \;=\; -1.$$

For the submanifolds A and C we have

$$\text{geometric intersection number of } A \text{ and } C \;=\; 2,$$

the intersection points have opposite signs, therefore

$$\text{algebraic intersection number of } A \text{ and } C \;=\; 0.$$

Finally for the submanifolds D and E we have

$$\text{geometric intersection number of } D \text{ and } E \;=\; 2,$$

here both intersection points have negative sign, thus

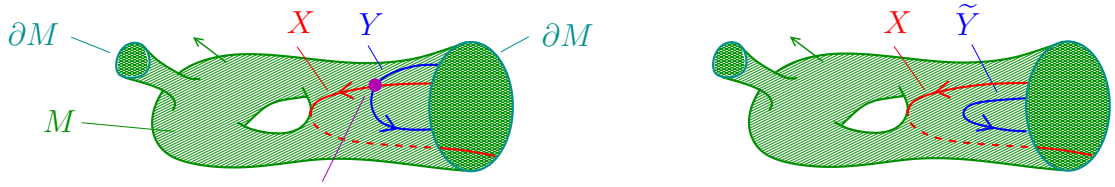
$$\text{algebraic intersection number of } D \text{ and } E \;=\; -2.$$

(2) The above example (1) suggests that if two submanifolds have non-zero algebraic intersection number, then they cannot be made disjoint. As we will see in the Intersection Number–Homotopy Invariance Theorem 88.5, this hunch is basically correct. But if we deal with submanifolds with boundary one needs to think carefully about the correct setting. For example in the figure below we show on the left a smooth manifold M and we show two proper submanifolds X and Y such that the boundaries ∂X and ∂Y lie

[686] It follows from the Finite Transverse Intersection Lemma 88.2 that the geometric intersection number is finite and that the algebraic intersection number is given by a finite sum.

[687] It is a straightforward exercise in linear algebra to show that the definition of $\operatorname{sign}(P)$ does not depend on the choice of the positive bases of $\mathrm{T}_P X$ and $\mathrm{T}_P Y$.

on the same component of ∂M. The algebraic intersection number is $+1$, but as we see in the figure below on the right, these two submanifolds can be made disjoint.



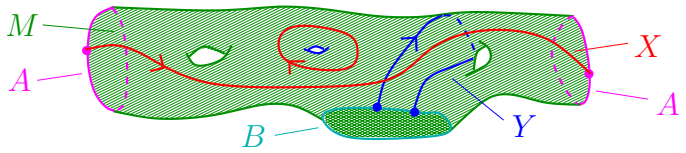
the algebraic intersection number of X and Y equals $+1$

Therefore in the following discussion we will always assume that the boundary components of X and Y lie in different boundary components of M. □

Before we continue, and in light of the last example, we introduce the following definition:

Definition. Let M be an oriented smooth manifold.

(1) A boundary decomposition for M is a decomposition $\partial M = A \cup B$ where A and B are unions of components of ∂M and $A \cap B = \varnothing$.
(2) Let X and Y be two submanifolds of M. We say that X and Y are complementary if the following five conditions are satisfied:
   (a) X and Y are proper submanifolds.
   (b) There exists a boundary decomposition $\partial M = A \cup B$ with $\partial X \subset A$ and $\partial Y \subset B$.
   (c) Both X and Y are oriented.
   (d) We have $\dim(X) + \dim(Y) = \dim(M)$ and we have $\dim(X), \dim(Y) \geq 1$.
   (e) The submanifolds X and Y intersect transversally.

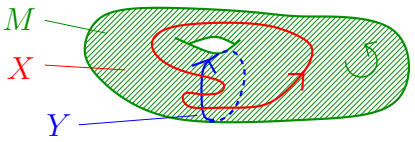


X and Y are complementary

We conclude this introduction to algebraic intersection numbers with the following elementary lemma

Lemma 88.3. (Intersection Number–Symmetry Lemma) Let M be a compact oriented smooth manifold. If X and Y are complementary submanifolds of M, then

$$Y \cdot X \;=\; (-1)^{\dim(X)\cdot\dim(Y)} \cdot X \cdot Y.$$



$$X \cdot Y = +1 - 1 + 1 \;=\; +1$$
$$Y \cdot X = -1 + 1 - 1 \;=\; -1$$

Proof. We set $k = \dim(X)$ and $l = \dim(Y)$. It follows easily from the definitions that at each $P \in X \cap Y = Y \cap X$ the signs of the intersection points differ by the sign of the determinant of the following matrix:

$$\begin{pmatrix} 0_{l\times k} & \mathrm{id}_l \\ \mathrm{id}_k & 0_{k\times l} \end{pmatrix}.$$

An linear algebra elementary argument shows that the determinant of this matrix equals $(-1)^{k\cdot l}$. ∎

88.2. **The cup product and algebraic intersection numbers.** We recall the following notation:

Notation.

(1) Given a compact oriented l-dimensional smooth manifold W we denote as usual by $[W] \in \mathrm{H}_l(W, \partial W)$ the fundamental class that we introduced on page 832. If W is a proper submanifold of some smooth manifold M, then sometimes we denote by $[W] \in \mathrm{H}_l(M, \partial M)$ the image of $[W]$ under the inclusion-induced homomorphism $\mathrm{H}_l(W, \partial W) \to \mathrm{H}_l(M, \partial M)$.

(2) Let M be a compact oriented m-dimensional smooth manifold and suppose that we are given a boundary decomposition $\partial M = A \cup B$. By the Poincaré Duality Theorem 79.1 we know that for each $k \in \mathbb{N}_0$ the map

$$\mathrm{H}^k(M, A; \mathbb{Z}) \xrightarrow{\cap [M]} \mathrm{H}_{m-k}(M, B)$$

is an isomorphism. As on page 1014 we denote by

$$\mathrm{PD}_M \colon \mathrm{H}_{m-k}(M, B) \longrightarrow \mathrm{H}^k(M, A; \mathbb{Z})$$

its inverse. If M is understood from the context then often we just write PD instead of PD_M.

The following theorem relates the cup product on a smooth manifold to algebraic intersection numbers. Or conversely, it expresses the algebraic intersection number of submanifolds as a cup product.

Theorem 88.4. (Cup Product–Intersection Number Theorem) Let M be a compact oriented m-dimensional smooth manifold together with a boundary decomposition $\partial M = A \cup B$. Let X and Y be a complementary pair of submanifolds of M with $\partial X \subset A$ and $\partial Y \subset B$. We write $k = \dim(X)$ and we denote by $i \colon (X, A) \to (M, \partial M)$ and $j \colon (Y, B) \to (M, \partial M)$ the obvious inclusion maps. Furthermore we denote by $[X] \in \mathrm{H}_k(X, \partial X)$ and $[Y] \in \mathrm{H}_{m-k}(Y, \partial Y)$ the fundamental classes of X and Y. Then[688]

$$\underbrace{X \cdot Y}_{\substack{\text{defined}\\ \text{on page 1143}}} = \Big\langle \overbrace{\mathrm{PD}_M\big(\underbrace{i_*([X])}_{\in \mathrm{H}_k(M,A)}\big)}^{\in \mathrm{H}^{m-k}(M,B;\mathbb{Z})} \cup \overbrace{\mathrm{PD}_M\big(\underbrace{j_*([Y])}_{\in \mathrm{H}_{m-k}(M,B)}\big)}^{\in \mathrm{H}^{k}(M,A;\mathbb{Z})}, \overbrace{[M]}^{\in \mathrm{H}_m(M,\partial M)} \Big\rangle \in \mathbb{Z}.$$

In particular with the abbreviation $[X] \coloneqq i_*([X])$ and $[Y] \coloneqq j_*([Y])$ we have the slightly more readable equality

$$X \cdot Y = \langle \mathrm{PD}_M([X]) \cup \mathrm{PD}_M([Y]), [M] \rangle.$$

[688] It follows from the Excisive Triad-Proposition 50.11 that (M, A, B) is an excisive triad. Thus the cup product is actually defined.

Remark. The following citation from [**Kir89**, p. 21] encapsulates the roles of algebraic intersection numbers and cup products:

"Think with [algebraic] intersection [numbers], prove with cup products."

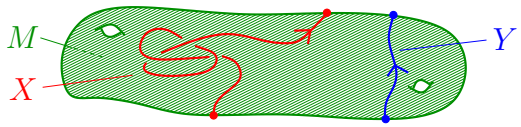
More precisely, algebraic intersection numbers are very intuitive and often they can be used to guess a statement. But when it comes to proving statements, e.g. why does the intersection number only depend on the smooth isotopy type of a submanifold?, it is usually much easier to work with cup products. □

The remainder of this chapter is structured as follows:

(1) In the following we discuss many applications of the Cup Product–Intersection Number Theorem 88.4, to convince ourselves that proving the theorem is worth the effort.
(2) In Section 88.3 we provide the lengthy proof of the Cup Product–Intersection Number Theorem 88.4.
(3) In Section88.4.1 we discuss algebraic intersection numbers of complex submanifolds.
(4) In Section 88.4.2 we use Cup Product–Intersection Number Theorem 88.4 to discuss the cup product on complex projective spaces. In particular we will see that we can dispose of a sign indeterminacy, which had been bothering us.

As a first major application of the Cup Product–Intersection Number we will now show that algebraic intersection numbers are preserved under "proper smooth homotopies". To make sense of this result, we need to define what we mean by such homotopies:

Definition. Let M be a smooth manifold and let $X, Y \subset M$ be two oriented proper submanifold. We say that X and Y are properly homotopic if there exists a homotopy $F\colon X \times [0,1] \to M$ with the following three properties:
(1) We have $F_0 = \mathrm{id}$.
(2) The map F_1 is an orientation-preserving diffeomorphism $X \to Y$.
(3) For each $t \in [0,1]$ we have $F_t(\partial X) \subset \partial M$.

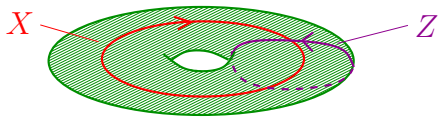


X and Y are properly homotopic
but they are not properly isotopic

Now we can state the promised theorem:

Theorem 88.5. (Intersection Number–Homotopy Invariance Theorem) Let M be a compact oriented smooth manifold.
(1) Let $\{X, Y\}$ and $\{X', Y'\}$ be pairs of complementary submanifolds of M. If X, X' are properly homotopic and if Y, Y' are also properly homotopic, then $X \cdot Y = X' \cdot Y'$.
(2) If X and Y are two complementary submanifolds of M that can be made disjoint, then $X \cdot Y = 0$.

Example. Now we can answer Question 88.1.



It follows from the Intersection Number–Homotopy Invariance Theorem 88.5 that the submanifolds X and Z, that are illustrated above and in the figure on page 1142, cannot be made disjoint using proper homotopies (let alone through proper smooth isotopies) since their algebraic intersection number is non-zero. □

Proof. Let M be a compact oriented smooth manifold. We start out with the following claim:

Claim. Let Z and Z' be oriented proper s-dimensional submanifolds of M. If Z and Z' are properly homotopic, then $[Z] = [Z'] \in \mathrm{H}_s(M, \partial M)$.

Proof. We denote by $i\colon Z \to M$ and $i'\colon Z' \to M$ the inclusion maps. Since Z and Z' are properly homotopic there exists in particular an orientation-preserving diffeomorphism $f\colon Z \to Z'$ such that the maps $i, i' \circ f\colon (Z, \partial Z) \to (M, \partial M)$ are homotopic. We obtain that

$$[Z] \;=\; i_*([Z]) \underset{\uparrow}{=} (i' \circ f)_*([Z]) = \; i'_*(f_*([Z])) \underset{\uparrow}{=} i'_*([Z']) \;=\; [Z'] \;\in\; \mathrm{H}_s(M, \partial M).$$

by the Homotopy-H_*-Proposition 38.6 — since f is an orientation-preserving homeomorphism ⊞

Now let $\{X, Y\}$ and $\{X', Y'\}$ be two pairs of complementary submanifolds of M. We assume that the proper submanifolds X, X' are properly homotopic and that the smooth manifolds Y, Y' are homotopic. We obtain that

$$X \cdot Y \underset{\uparrow}{=} \langle \mathrm{PD}_M([X]), \mathrm{PD}_M([Y]), [M] \rangle \underset{\uparrow}{=} \langle \mathrm{PD}_M([X']), \mathrm{PD}_M([Y']), [M] \rangle \underset{\uparrow}{=} X' \cdot Y'.$$

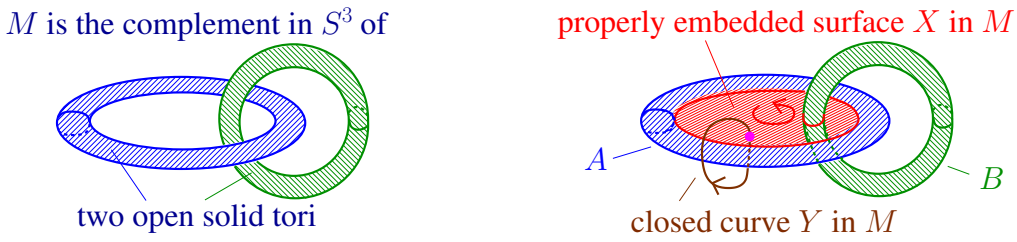
by the Cup Product–Intersection Number Theorem 88.4 — by the claim — by the Cup Product–Intersection Number Theorem 88.4

This concludes the proof of Statement (1). Statement (2) is an immediate consequence of Statement (1) and the observation that the algebraic intersection number of disjoint submanifolds is of course zero. ■

Now we will use the Cup Product–Intersection Number Theorem 88.4 "the other way", i.e. we will apply the theorem to use the algebraic intersection number to draw conclusions on (co-) homology classes and to determine cup products using our geometric understanding of submanifolds.

Corollary 88.6. (Complementary Pair Corollary) Let M be a compact oriented m-dimensional smooth manifold together with a boundary decomposition $\partial M = A \cup B$. Let X be an oriented proper k-dimensional submanifold of M with $\partial X \subset A$. If there exists an oriented proper submanifold Y with $\partial Y \subset B$ such that X and Y form a complementary pair and if $X \cdot Y = \pm 1$, then $[X] \in \mathrm{H}_k(M, A)$ and $[Y] \in \mathrm{H}_{m-k}(M, B)$ generate infinite cyclic summands.

Example. Let M be the complement in S^3 of the two open 3-dimensional tori shown in the figure below on the left. On the right of that figure we see an oriented properly embedded annulus X in M and we also see a closed oriented 1-dimensional submanifold Y of M. The two submanifolds X and Y are clearly transverse and they intersect in one point. So with whatever orientation, we have $X \cdot Y = \pm 1$. The Complementary Pair Corollary 88.6 implies that $[X] \in \mathrm{H}_2(M, \partial M)$ and $[Y] \in \mathrm{H}_1(M)$ generate infinite cyclic summands. □



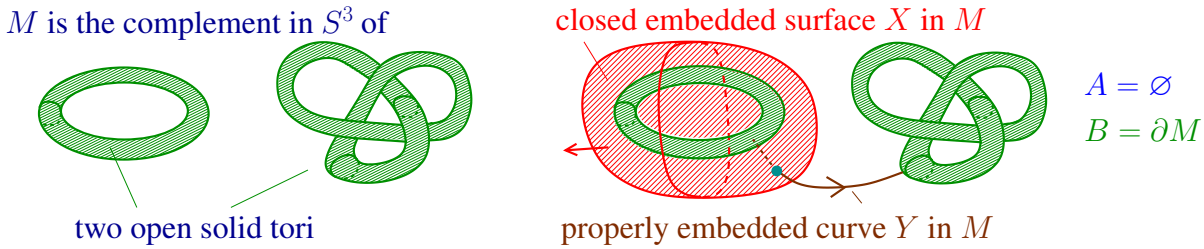
Proof. We consider the homomorphism

$$\begin{aligned}\Theta\colon \mathrm{H}_k(M,A) &\to \mathbb{Z}\\ \varphi &\mapsto \langle \mathrm{PD}_M(\varphi)\cup \mathrm{PD}_M([Y]),[M]\rangle.\end{aligned}$$

Note that $\Theta([X]) \underset{\text{by definition of }\Theta}{=} \langle \mathrm{PD}_M([X])\cup \mathrm{PD}_M([Y]),[M]\rangle \underset{\text{by the Cup Product–Intersection Number Theorem 88.4}}{=} X\cdot Y \underset{\text{by hypothesis}}{=} \pm 1.$

It now follows from the Free Abelian Group–Splitting Lemma 18.5 that $[X]\in \mathrm{H}_k(M,A)$ generates an infinite cyclic summand. The same argument, with the roles of X and Y reversed, also shows that $[Y]\in \mathrm{H}_{m-k}(M,B)$ generates an infinite cyclic summand. ■

Example. This time let M be the complement in S^3 of the two open 3-dimensional tori that are shown in the figure below on the left.



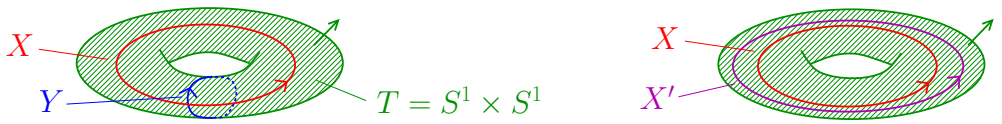
We consider the oriented proper submanifolds X and Y that are shown in the figure. It follows again from the Complementary Pair Corollary 88.6 that $[X]\in \mathrm{H}_2(M)$ and $[Y]\in \mathrm{H}_1(M,\partial M)$ generate infinite cyclic summands. □

In the next example we will show how to use the Cup Product–Intersection Number Theorem 88.4 to compute cup products.

Example. We consider the torus $T=S^1\times S^1$ together with the submanifolds $X=S^1\times\{*\}$ and $Y=\{*\}\times S^1$ which we endow with the usual orientations. By now we have many ways for showing that $\mathrm{PD}([X])$ and $\mathrm{PD}([Y])$ form a basis for $\mathrm{H}^1(T;\mathbb{Z})$. Thus if we want to understand the cup product on the torus, it suffices to understand the cup products involving these two cohomology classes. We calculate that

$$-\langle \mathrm{PD}([Y])\cup \mathrm{PD}([X]),[S^1\times S^1]\rangle \underset{\text{by the Cup Product-Commutativity Proposition 72.8}}{=} \langle \mathrm{PD}([X])\cup \mathrm{PD}([Y]),[S^1\times S^1]\rangle \underset{\text{by the Cup Product–Intersection Number Theorem 88.4}}{=} X\cdot Y \underset{\text{see the figure below}}{=} +1.$$

Next we want to calculate the "square" $\mathrm{PD}([X])\cup \mathrm{PD}([X])$. This requires a little trick since evidently X is not transverse to itself. To deal with this issue we consider a "parallel copy" of X, namely we consider $X'=S^1\times\{z\}$ with $z\neq 1$.



We calculate that

$$\langle \mathrm{PD}([X]) \cup \mathrm{PD}([X]), [S^1 \times S^1] \rangle \underset{\uparrow}{=} \langle \mathrm{PD}([X]) \cup \mathrm{PD}([X']), [S^1 \times S^1] \rangle \underset{\uparrow}{=} X \cdot X' = 0.$$

by the claim in the proof of the Intersection Number Homotopy Invariance Theorem 88.5

by the Cup Product–Intersection Number Theorem 88.4, since X and X' are disjoint, in particular transverse

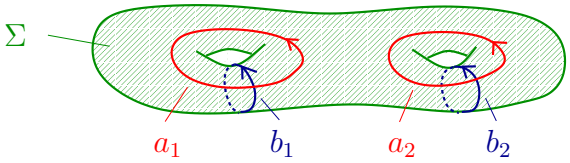
and with the same argument we show that $\langle \mathrm{PD}([Y]) \cup \mathrm{PD}([Y]), [S^1 \times S^1] \rangle = 0$. □

Since this was so much fun, let us consider one more example:

Example. Let Σ be the surface of genus two. We consider the following bilinear pairing:

$$\begin{aligned} Q\colon \mathrm{H}_1(\Sigma) \times \mathrm{H}_1(\Sigma) &\to \mathbb{Z} \\ (\sigma, \tau) &\mapsto \langle \mathrm{PD}_\Sigma(\sigma) \cup \mathrm{PD}_\Sigma(\tau), [\Sigma] \rangle. \end{aligned}$$

Recall that by the Surface-H_*-Proposition 47.9 we know that $\mathrm{H}_1(\Sigma) \cong \mathbb{Z}^4$. Next we consider the closed oriented curves a_1, b_1, a_2, b_2 that are shown in the figure below.



It follows easily from the Cup Product–Intersection Number Theorem 88.4 that[689]

$$\begin{pmatrix} Q([a_1],[a_1]) & Q([a_1],[b_1]) & Q([a_1],[a_2]) & Q([a_1],[b_2]) \\ Q([b_1],[a_1]) & Q([b_1],[b_1]) & Q([b_1],[a_2]) & Q([b_1],[b_2]) \\ Q([a_2],[a_1]) & Q([a_2],[b_1]) & Q([a_2],[a_2]) & Q([a_2],[b_2]) \\ Q([b_2],[a_1]) & Q([b_2],[b_1]) & Q([b_2],[a_2]) & Q([b_2],[b_2]) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

□

We will do one more calculation of the cup product in Section 88.4.1, right after the proof of the Cup Product–Intersection Number Theorem 88.4.

88.3. **Proof of the Cup Product–Intersection Number Theorem 88.4.** In this section we will prove the Cup Product–Intersection Number Theorem 88.4. The following lemma says that given a compact oriented topological manifold M and given a homology class in a codimension-zero submanifold W of we M, can apply Poincaré Duality in M or W and essentially we end up with the same result:

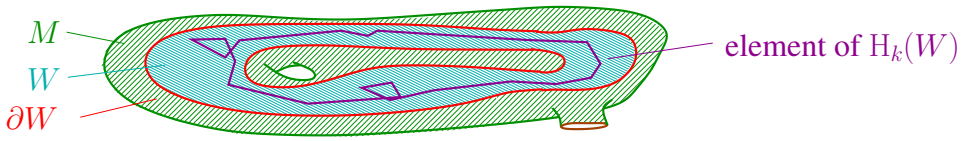
Lemma 88.7. (Small H_*-to-Large H^*-PD Lemma) Let M be a compact oriented m-dimensional topological manifold and let W be a compact codimension-zero submanifold with $W \cap \partial M = \varnothing$. We write $\mathring{W} = W \setminus \partial W$. We fix the following notation:

(1) We denote by w the inclusion map $W \to M$ and also the inclusion map of the pairs $(W, \partial W) \to (M, M \setminus \mathring{W})$.

[689] For the diagonal terms we use the "little trick" from the previous example, namely we use a "parallel copy" of the closed curves a_1, b_1, a_2, b_2.

(2) We denote by $p\colon (M,\partial M)\to(M,M\setminus\mathring{W})$ the inclusion map of pairs.
For every $k\in\mathbb{N}_0$ the following diagram commutes:[690]

$$\begin{array}{ccccc}
\mathrm{H}_k(W) & \xrightarrow{\mathrm{PD}_W} & \mathrm{H}^{m-k}(W,\partial W;\mathbb{Z}) & \xrightarrow[\cong]{(w^*)^{-1}} & \mathrm{H}^{m-k}(M,M\setminus\mathring{W};\mathbb{Z}) \\
\downarrow{\scriptstyle w_*} & & & & \downarrow{\scriptstyle p^*} \\
\mathrm{H}_k(M) & & \xrightarrow{\qquad\mathrm{PD}_M\qquad} & & \mathrm{H}^{m-k}(M,\partial M;\mathbb{Z}).
\end{array}$$



Proof. By definition of the isomorphisms PD_W and PD_M, proving the lemma is equivalent to proving that the following diagram commutes:

$$\begin{array}{ccccc}
\mathrm{H}_k(W) & \xleftarrow[\cong]{\cap[W]} & \mathrm{H}^{n-k}(W,\partial W;\mathbb{Z}) & \xleftarrow[w^*]{\cong} & \mathrm{H}^{n-k}(M,M\setminus\mathring{W};\mathbb{Z}) \\
\downarrow{\scriptstyle w_*} & & & & \downarrow{\scriptstyle p^*} \\
\mathrm{H}_k(M) & & \xleftarrow[\cong]{\qquad\cap[M]\qquad} & & \mathrm{H}^{n-k}(M,\partial M;\mathbb{Z}).
\end{array}$$

Now let $\beta\in\mathrm{H}^{n-k}(M,M\setminus\mathring{W};\mathbb{Z})$. We have the following equality in $\mathrm{H}_k(M)$:

$$w_*(w^*(\beta)\cap[W]) \underset{\text{by the Cap Product-Naturality Lemma 75.8 applied to the map } w\colon (W,\partial_0 W,\partial W)\to(M,M\setminus\mathring{W},M\setminus\mathring{W})}{=} \beta\cap w_*([W]) \underset{\text{by the Codimension Zero-Fundamental Class Lemma 64.5 we have } p_*([M])=w_*([W])\in\mathrm{H}_m(M,M\setminus\mathring{W})}{=} \beta\cap p_*([M]) \underset{\text{by the Cap Product-Naturality Lemma 75.8 applied to the map } p\colon (M,\partial M,\varnothing)\to(M,M\setminus\mathring{W},\varnothing)}{=} p_*(\overbrace{p^*(\beta)\cap[M]}^{\in\mathrm{H}_k(M)}) \underset{\text{since } p\colon(M)\to(M) \text{ is id we have } p_*=\mathrm{id}_*=\mathrm{id}}{=} p^*(\beta)\cap[M].$$

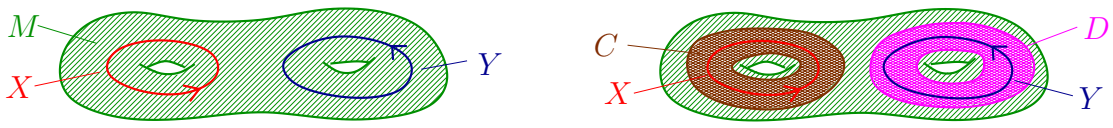
We have thus shown that the above diagram does indeed commute. ∎

We take a short rest from more technicalities and enjoy the fact that we can already prove the Cup Product–Intersection Number Theorem 88.4 in the special case that M is closed and that the submanifolds *do not intersect*.

Proof of Theorem 88.4 for disjoint submanifolds. Let M be a closed oriented m-dimensional smooth manifold and let X and Y be a complementary pair of *disjoint* submanifolds of M. Since the submanifolds are disjoint it suffices to show that the cup product of the Poincaré duals of $[X]$ and $[Y]$ is also zero.

We write $k=\dim(X)$. We apply the Tubular Neighborhood Theorem [**Fri24**] to the submanifold $X\cup Y$ of M. We obtain *disjoint* tubular neighborhoods C for X and D for Y. We denote by $p\colon(M,\varnothing)\to(M,M\setminus\mathring{C})$ and $p\colon(M,\varnothing)\to(M,M\setminus\mathring{D})$ the obvious inclusion maps of pairs of topological spaces.
We apply the Small H_*-to-Large H^*-PD Lemma 88.7 to the codimension-zero submanifold C and the homology class $[X]\in\mathrm{H}_k(C)$. We obtain an $\alpha\in\mathrm{H}^{m-k}(M,M\setminus\mathring{C};\mathbb{Z})$ with

[690] Here we use that by the Excision-in-H^*-Theorem 69.1 we know that $w\colon(W,\partial W)\to(M,M\setminus\mathring{W})$ induces an isomorphism on cohomology groups.



$p^*(\alpha) = \mathrm{PD}_M([X])$. Similarly we see that there exists a $\beta \in \mathrm{H}^k(M, M \setminus \mathring{D}; \mathbb{Z})$ such that $p^*(\beta) = \mathrm{PD}_M([Y])$.

We consider the following diagram:

$$\begin{array}{ccc} \mathrm{H}^{m-k}(M;\mathbb{Z}) \times \mathrm{H}^k(M;\mathbb{Z}) & \xrightarrow{\ \cup\ } & \mathrm{H}^m(M;\mathbb{Z}) \\ \uparrow{\scriptstyle p^*\times p^*} & & \uparrow \\ \mathrm{H}^{m-k}(M, M\setminus\mathring{C};\mathbb{Z}) \times \mathrm{H}^k(M, M\setminus\mathring{D};\mathbb{Z}) & \xrightarrow{\ \cup\ } & \mathrm{H}^m(M, \underbrace{(M\setminus\mathring{C}) \cup (M\setminus\mathring{D})}_{=M,\text{ since } C\cap D=\varnothing};\mathbb{Z}). \end{array}$$

$$(\mathrm{PD}_M([X]), \mathrm{PD}_M([Y])) \updownarrow (\alpha, \beta)$$

The diagram commutes by the Cup Product-Naturality Lemma 73.3. It follows from the fact that the lower right group is zero, that the cup product of $\mathrm{PD}_M([X])$ and $\mathrm{PD}_M([Y])$ is zero. ■

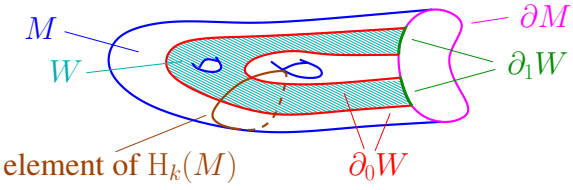
After this refreshing proof of the special case we continue with trying to prove the general case of the Cup Product–Intersection Number Theorem 88.4. We will need the following variation on the Small H_*-to-Large H^*-PD Lemma 88.7.

Lemma (Large H_*-to-Small H^*-PD Lemma) Let M be a compact oriented m-dimensional smooth manifold and let W be a compact codimension-zero submanifold with corner. We fix the following notation:

(1) We write $\partial_1 W = W \cap \partial M$ and $\partial_0 W = \overline{\partial W \setminus \partial_1 W}$.
(2) We write $\mathring{W} = W \setminus \partial_0 W$.
(3) The inclusion maps $(W, \partial_0 W) \to (M, M \setminus \mathring{W})$ and $(W, \partial_1 W) \to (M, \partial M)$ are denoted by w.
(4) Finally we denote by $p\colon (M, \varnothing) \to (M, M \setminus \mathring{W})$ the obvious map.

For every $k \in \mathbb{N}_0$ the following diagram commutes:

$$\begin{array}{ccccc} \mathrm{H}_k(M) & & \xrightarrow{\ \mathrm{PD}_M\ } & & \mathrm{H}^{m-k}(M, \partial M;\mathbb{Z}) \\ \downarrow{\scriptstyle p_*} & & & & \downarrow{\scriptstyle w^*} \\ \mathrm{H}_k(M, M\setminus\mathring{W}) & \xrightarrow[\cong]{(w_*)^{-1}} & \mathrm{H}_k(W, \partial_0 W) & \xrightarrow{\ \mathrm{PD}_W\ } & \mathrm{H}^{m-k}(W, \partial_1 W;\mathbb{Z}). \end{array}$$



Proof. By definition of the isomorphisms PD_W and PD_M, proving the lemma is equivalent to proving that the following diagram commutes:

$$\begin{array}{ccccc} \mathrm{H}_k(M) & \xleftarrow[\cong]{\cap[M]} & & & \mathrm{H}^{n-k}(M,\partial M;\mathbb{Z}) \\ \downarrow{\scriptstyle p_*} & & & & \downarrow{\scriptstyle w^*} \\ \mathrm{H}_k(M, M\setminus \mathring{W}) & \xleftarrow[\cong]{w_*} & \mathrm{H}_k(W,\partial_0 W) & \xleftarrow{\cap[W]} & \mathrm{H}^{n-k}(W,\partial_1 W;\mathbb{Z}). \end{array}$$

So let $\beta \in \mathrm{H}^{n-k}(M,\partial M;\mathbb{Z})$. We have the following equalities in $\mathrm{H}_k(M, M\setminus \mathring{W})$:

$$\begin{array}{ccccccc} p_*(\beta\cap[M]) & = & \beta \cap p_*([M]) & = & \beta \cap w_*([W]) & = & w_*(w^*(\beta)\cap[W]). \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ \text{by the Cap Product-Naturality Lemma 75.8 applied to the map } p\colon (M,\partial M,\varnothing)\to(M,\partial M,M\setminus\widehat{W}) \text{ of excisive triads} & & \text{by the Codimension Zero-Fundamental Class Lemma 64.5 we have } p_*([M]) = w_*([W]) \in \mathrm{H}_n(M,(M\setminus\mathring{W})\cup\partial M) & & & & \text{by the Cap Product-Naturality Lemma 75.8 applied to the map } w\colon (W,\partial_1 W,\partial_0 W)\to(M,\partial M,M\setminus\widehat{W}) \text{ of excisive triads} \end{array}$$

■

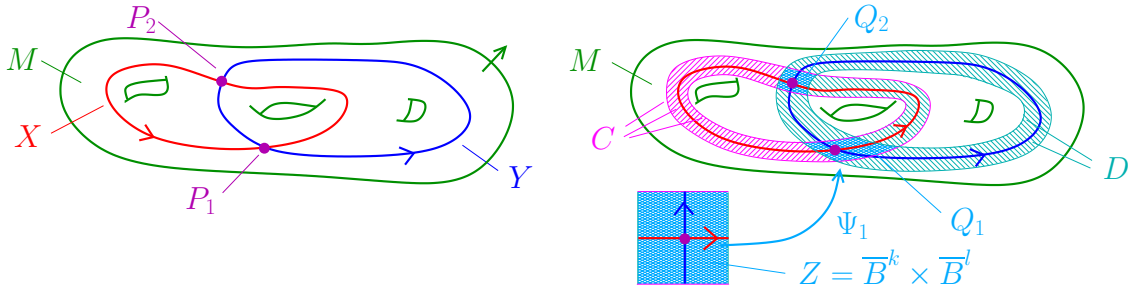
In the proof of the Cup Product–Intersection Number Theorem 88.4 we will also need the following proposition:

Proposition 88.8. (Transverse Neighborhood Proposition) Let M be a compact oriented smooth manifold and let X and Y be two complementary proper submanifolds of dimensions k and l. We denote by $P_1,\dots,P_s$ the intersection points of X and Y. Then there exist

(1) compact submanifolds with corner C and D such that C is a neighborhood for X and D is a neighborhood for Y, and
(2) embeddings $\Psi_i\colon \overline{B}^k\times\overline{B}^l\to M$, $i=1,\dots,s$

such that the following statements hold:

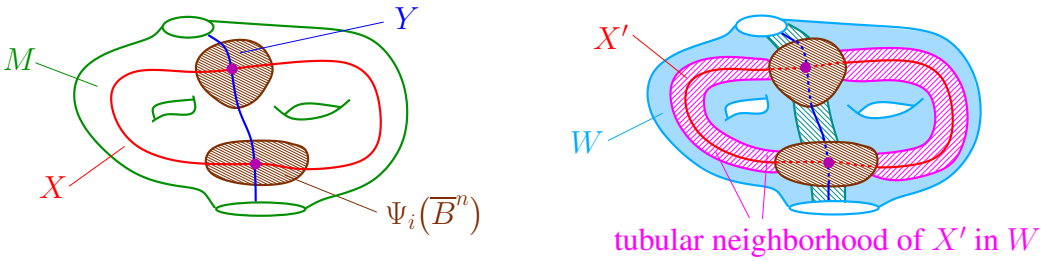
(a) For each $i\in\{1,\dots,s\}$ we have $\Psi_i((0,0)) = P_i$.
(b) The images $Q_i := \Psi_i\big(\overline{B}^k\times\overline{B}^l\big)$, $i=1,\dots,s$ are disjoint.
(c) We have $Q_1\cup\dots\cup Q_s = C\cap D$.
(d) For each $i\in\{1,\dots,s\}$ the following statements hold:
  (d1) The restriction of Ψ_i to $B^k\times B^l$ is a diffeomorphism onto its image.
  (d2) We have $Q_i\cap X = \Psi_i(\overline{B}^k\times\{0\})$ and the map $\Psi_i\colon \overline{B}^k\times\{0\}\to X$ is an orientation-preserving smooth embedding.
  (d3) We have $Q_i\cap Y = \Psi_i(\{0\}\times\overline{B}^l)$ and the map $\Psi_i\colon \{0\}\times\overline{B}^l\to Y$ is an orientation-preserving smooth embedding.



Sketch of a proof. Let M be a compact oriented n-dimensional smooth manifold and let X and Y be two complementary proper submanifolds of dimensions k and l. We denote by $P_1, \dots, P_s$ the intersection points of X and Y. For $i = 1, \dots, s$ we can and will pick smooth embeddings $\Psi_i \colon \overline{B}^n \to M \setminus \partial M$ such that the following conditions are satisfied:

(1) The images are disjoint.
(2) For each $i \in \{1, \dots, s\}$ the map Ψ_i restricts to orientation-preserving diffeomorphisms $B^k \times \{0\} \to \Psi_i(B^n) \cap X$ and $\{0\} \times B^l \to \Psi_i(B^n) \cap Y$.

We consider $W \coloneqq M \setminus \overset{s}{\underset{i=1}{\cup}} \Psi_i(B^n)$. Note that $X' \coloneqq W \cap X$ and $Y' \coloneqq W \cap Y$ are proper submanifolds of the compact smooth manifold W.



Thus we can apply the Tubular Neighborhood Theorem [**Fri24**] to obtain tubular neighborhoods for X' and Y'. We can arrange the intersections of the tubular neighborhoods with $\Psi_i(S^{n-1})$ to be "standard $(n-1)$-dimensional balls". At this stage it is fairly straightforward to extend the tubular neighborhoods to the interiors of the $\Psi_i(B^n)$ and to obtain all the desired objects. We have faith in the unwavering enthusiasm of our readers for filling in the details. ■

Given the situation of the Transverse Neighborhood Proposition 88.8 we can make one more statement.

Lemma 88.9. (Sign-Q_i-Lemma) We use the notation of the Transverse Neighborhood Proposition 88.8. Let $i \in \{1, \dots, s\}$. We equip the topological manifold Q_i, which is a codimension-zero submanifold of M, with the induced orientation coming from the Topological Manifold-Induced Orientation Lemma 62.1. Furthermore we equip the topological manifold $\overline{B}^k \times \overline{B}^l$ with the orientation coming from the standard orientation of $\mathbb{R}^k \times \mathbb{R}^l = \mathbb{R}^{k+l}$. With these choices the following statement holds:

(e) If we denote by σ_i the sign of the intersection point P_i, then

$$\Psi_{i*}([\overline{B}^k \times \overline{B}^l]) \;=\; \sigma_i \cdot [Q_i].$$

Proof. Let $i \in \{1, \dots, s\}$. It follows from (d2) and (d3) that at the point $(0,0)$ the diffeomorphism $\Psi_i \colon B^k \times B^l \to \mathring{Q}_i$ is orientation-preserving if and only if $\sigma_i = 1$. Now the statement follows easily from the definition of σ_i, the Orientation Preserving–Characterisation Proposition **??** and the Orientations–Preserved Lemma 61.16. ■

The last preparation we need before we can provide the actual proof of the Cup Product–Intersection Number Theorem 88.4 is the following deep theorem:

Proposition 88.10. (Poincaré Duality-Products Proposition) Let X be a compact oriented connected m-dimensional topological manifold and let Y be a compact oriented

connected n-dimensional topological manifold. Let $p\colon X\times Y\to X$ and $q\colon X\times Y\to Y$ be the natural projections. We equip $X\times Y$ with the product orientation introduced in the Product Orientation Proposition 62.7 (1). Given any points $P\in X$ and $Q\in Y$ we have the following equality in $\mathrm{H}^{m+n}(X\times Y,\partial(X\times Y);\mathbb{Z})$:[691]

$$\mathrm{PD}_{X\times Y}([X\times\{Q\}])\cup\mathrm{PD}_{X\times Y}([\{P\}\times Y])\;=\;[X\times Y]^*.$$

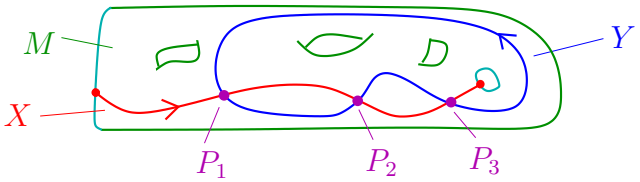
Proof. This proposition follows almost immediately from the Fundamental Class-Product Proposition 76.2. ■

We turn to the actual proof of the Cup Product–Intersection Number Theorem 88.4:

Proof of the Cup Product–Intersection Number Theorem 88.4. Let M be a compact oriented connected m-dimensional smooth manifold. To simplify the notation we assume that M is closed. The case that M has non-empty boundary is proved in a rather similar fashion.

Let X and Y be a complementary pair of submanifolds of M. We introduce the following notation:

(1) We write $k=\dim(X)$ and $l=\dim(Y)$. Note that $l=m-k$.
(2) As usual we denote by $[X]\in\mathrm{H}_k(X)$ and $[Y]\in\mathrm{H}_l(Y)$ the fundamental classes of X and Y. By a slight abuse of notation we denote by $[X]$ respectively $[Y]$ also the images of these homology classes in any subspace of M that contains X respectively Y.
(3) We denote the intersection points of X and Y by $P_1,\dots,P_s$ and we denote the corresponding signs by $\sigma_1,\dots,\sigma_s$.
(4) As usual, given a topological space X we denote by $\epsilon_X=\epsilon\colon\mathrm{H}_0(X)\to\mathbb{Z}$ the augmentation map.



With the notation that we just introduced, and keeping in mind the Cap Product-Kronecker Lemma 75.7, the statement of the Cup Product–Intersection Number Theorem 88.4 can be reformulated as the following statement:

Statement.
$$\epsilon_M((\mathrm{PD}_M([X])\cup\mathrm{PD}_M([Y]))\cap[M])\;=\;\sum_{i=1}^{s}\sigma_i.$$

We introduce many more objects, notations and conventions:

(5) We apply the Transverse Neighborhood Proposition 88.8 to the submanifolds X and Y. In the remainder of the proof we use the notation that we introduced in that proposition.
(6) We write $\mathring{C}=C\setminus\partial C$ and $\mathring{D}=D\setminus\partial D$.
(7) We denote all of the inclusion maps $(M,\varnothing)\to(M,M\setminus\mathring{C})$, $(M,\varnothing)\to(M,M\setminus\mathring{D})$ and $(M,\varnothing)\to(M,M\setminus(\mathring{C}\cap\mathring{D}))$ by p.

[691]Recall that for a compact oriented connected non-empty n-dimensional topological manifold M we introduced on page 909 the dual fundamental class $[M]^*\in\mathrm{H}^n(M,\partial M;\mathbb{Z})$, which is characterized by $\langle[M]^*,[M]\rangle=1$.

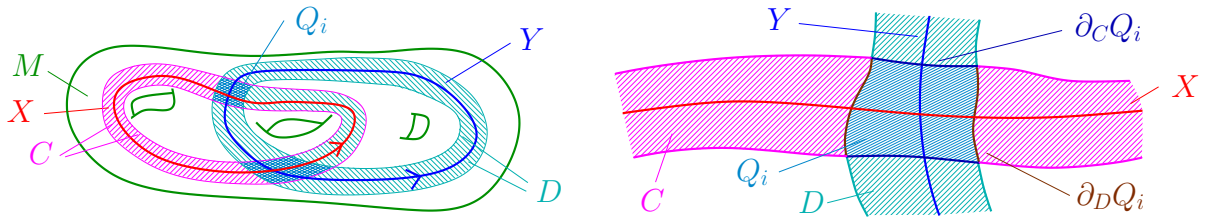
(8) We write $E := C \cap D = Q_1 \cup \cdots \cup Q_s$. Furthermore we write $\partial_C E = \partial C \cap D$ and $\partial_D E = C \cap \partial D$. Note that $\partial E = \partial_C E \cup \partial_D E$. Also note that $(M \setminus \mathring{C}) \cup (M \setminus \mathring{D}) = M \setminus \mathring{E}$.

(9) We denote by $e\colon E \to M$ the inclusion map and we denote by c also all the corresponding inclusion maps of pairs where the first terms are E and M.

(10) Given $i \in \{1, \ldots, s\}$ we write $\partial_C Q_i := \partial_C E \cap Q_i$ and $\partial_D Q_i := \partial_D E \cap Q_i$. Note that $\partial Q_i = \partial_C Q_i \cup \partial_D Q_i$.

(11) Given $i \in \{1, \ldots, s\}$ we denote by $q_i\colon Q_i \to E$ the inclusion map and we denote by q_i also all the corresponding inclusion maps of pairs where the first terms are Q_i and E.

(12) It follows from the descriptions in the Transverse Neighborhood Proposition 88.8 that E and each Q_i are codimension-zero submanifolds of M. We equip these submanifolds with the corresponding orientation introduced in the Topological Manifold-Induced Orientation Lemma 62.1.



Now we consider the following diagram:[692][693]

$$
\begin{array}{ccccccc}
\mathrm{H}^l(M) \times \mathrm{H}^k(M) & \xrightarrow{\cup} & \mathrm{H}^m(M) & \xrightarrow{\cap[M]} & \mathrm{H}_0(M) & \xrightarrow{\epsilon_M} & \mathbb{Z} \\
\uparrow{\scriptstyle p^*\times p^*} & & \uparrow{\scriptstyle p^*} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle =} \\
\mathrm{H}^l(M, M\setminus\mathring{C}) \times \mathrm{H}^k(M, M\setminus\mathring{D}) & \xrightarrow{\cup} & \mathrm{H}^m(M, M\setminus\mathring{E}) & \xrightarrow{\cap p_*([M])} & \mathrm{H}_0(M) & \xrightarrow{\epsilon_M} & \mathbb{Z} \\
\downarrow{\scriptstyle e^*\times e^*} & & \downarrow{\scriptstyle e^*} & & \uparrow{\scriptstyle e_*} & & \uparrow{\scriptstyle \mathrm{id}} \\
\mathrm{H}^l(E, \partial_C E) \times \mathrm{H}^k(E, \partial_D E) & \xrightarrow{\cup} & \mathrm{H}^m(E, \partial E) & \xrightarrow{\cap[E]} & \mathrm{H}_0(E) & \xrightarrow{\epsilon_E} & \mathbb{Z} \\
\downarrow{\scriptstyle \bigoplus_{i=1}^s q_i^*\times q_i^*} & & \downarrow{\scriptstyle \bigoplus_{i=1}^s q_i^*} & & \uparrow{\scriptstyle \sum_{i=1}^s q_{i*}} & & \uparrow{\scriptstyle \mathrm{id}} \\
\bigoplus_{i=1}^s \mathrm{H}^l(Q_i, \partial_C Q_i) \times \mathrm{H}^k(Q_i, \partial_D Q_i) & \xrightarrow{\oplus\cup} & \bigoplus_{i=1}^s \mathrm{H}^m(Q_i, \partial Q_i) & \xrightarrow{\bigoplus_{i=1}^s \cap[Q_i]} & \bigoplus_{i=1}^s \mathrm{H}_0(Q_i) & \xrightarrow{\sum \epsilon_{Q_i}} & \mathbb{Z}.
\end{array}
$$

Claim 1. The above diagram commutes.

Proof. In the following we deal with all of the nine squares of the diagram.

(a) The left squares in the first and second row commute by the Cup Product-Naturality Lemma 73.3.

(b) It follows easily from the aforementioned the Cup Product-Naturality Lemma 73.3 that the left square in the third row commutes.

(c) The middle square in the first row commutes by the Cap Product-Naturality Lemma 75.8.

[692] For space reasons we suppressed the $\mathbb{Z}$-coefficients from the cohomology groups.

[693] Many of the vertical maps in the diagram are isomorphisms by Excision-in-H^*-Theorem 69.1. But this is in fact irrelevant for our purpose.

(d) It follows from our orientation conventions and the Codimension Zero-Fundamental Class Lemma 64.5 that $e_*([E]) = p_*([M])$. It follows from the Cap Product-Naturality Lemma 75.8 that the middle square in the second row commutes.

(e) First note that it follows easily from the definitions and the Path-Component-H_*-Lemma 35.8 that $\sum_{i=1}^{s} q_{i*}([Q_i]) = [E]$. Now we obtain from the Cap Product-Naturality Lemma 75.8 that the middle square in the third row commutes.

(f) In the Augmentation Lemma 35.10 (3) we saw that the squares that involve the augmentation maps commute. ⊞

We introduce a few last pieces of notation:

(13) We denote by $c\colon (C, \partial C) \to (M, M \setminus \mathring{C})$ and $d\colon (D, \partial D) \to (M, M \setminus \mathring{D})$ the inclusion maps. Note that it follows from the Excision-in-H^*-Theorem 69.1 (2) that the maps a and b induce isomorphisms on cohomology.

(14) We write

$$\begin{aligned} \alpha &= (c^*)^{-1}(\overbrace{\mathrm{PD}_C([X])}^{\in \mathrm{H}^l(C,\partial C;\mathbb{Z})}) && \in \mathrm{H}^l(M, M \setminus \mathring{C}; \mathbb{Z}) \\ \beta &= (d^*)^{-1}(\underbrace{\mathrm{PD}_D([Y])}_{\in \mathrm{H}^k(D,\partial D;\mathbb{Z})}) && \in \mathrm{H}^k(M, M \setminus \mathring{D}; \mathbb{Z}). \end{aligned}$$

(15) Given $i \in \{1, \dots, s\}$ we write $X_i := X \cap Q_i$ and $Y_i := Y \cap Q_i$. It follows from the Transverse Neighborhood Proposition 88.8 (e) and (f) that X_i and Y_i are codimension zero submanifolds of X and Y. Since X and Y are oriented we can view X_i and Y_i also as oriented topological manifolds. As usual we denote by $[X_i] \in \mathrm{H}_k(X_i, \partial X_i)$ and $[Y_i] \in \mathrm{H}_k(Y_i, \partial Y_i)$ the corresponding fundamental classes. With the same convention as before we denote by $[X_i]$ also the image under the inclusion-induced homomorphism $\mathrm{H}_k(X_i, \partial X_i) \to \mathrm{H}_k(Q_i, \partial_D Q_i)$ and similarly we denote by $[Y_i]$ also the image under the inclusion-induced homomorphism $\mathrm{H}_l(Y_i, \partial Y_i) \to \mathrm{H}_l(Q_i, \partial_C Q_i)$.

Claim 2. We have

(a) We have $p^*(\alpha) = \mathrm{PD}_M([X]) \in \mathrm{H}^l(M; \mathbb{Z})$ and $p^*(\beta) = \mathrm{PD}_M([Y]) \in \mathrm{H}^k(M; \mathbb{Z})$.

(b) For any $i \in \{1, \dots, s\}$ we have the following two equalities:

$$\begin{aligned} (q_i^* \circ e^*)(\alpha) &= \mathrm{PD}_{Q_i}([X_i]) && \in \mathrm{H}^l(Q_i, \partial_C Q_i; \mathbb{Z}) \\ (q_i^* \circ e^*)(\beta) &= \mathrm{PD}_{Q_i}([Y_i]) && \in \mathrm{H}^k(Q_i, \partial_D Q_i; \mathbb{Z}). \end{aligned}$$

(c) For any $i \in \{1, \dots, s\}$ we have

$$\epsilon_{Q_i}((\mathrm{PD}_{Q_i}([X_i]) \cup \mathrm{PD}_{Q_i}([Y_i])) \cap [Q_i]) = \sigma_i.$$

Proof.

(a) By the Small H_*-to-Large H^*-PD Lemma 88.7 the following diagram commutes:

$$\begin{array}{ccccc} & & \overset{[X]\mapsto\alpha}{\frown} & & \\ \mathrm{H}_k(C) & \xrightarrow[\mathrm{PD}_C]{} & \mathrm{H}^l(C, \partial C; \mathbb{Z}) & \xrightarrow[(c^*)^{-1}]{\cong} & \mathrm{H}^l(M, M \setminus \mathring{C}; \mathbb{Z}) \\ {\scriptstyle [X]}\downarrow{\scriptstyle [X]} & & & & \downarrow{\scriptstyle p^*} \\ \mathrm{H}_k(M) & & \xrightarrow{\qquad\qquad \mathrm{PD}_M \qquad\qquad} & & \mathrm{H}^l(M; \mathbb{Z}). \end{array}$$

We apply this commutative diagram to $[X] \in \mathrm{H}_k(C)$ and we obtain immediately that $p^*(\alpha) = \mathrm{PD}_M([X])$. The same way one shows that $p^*(\beta) = \mathrm{PD}_M([Y])$.

(b) Let $i \in \{1, \dots, s\}$. We denote by $r_i \colon Q_i \to C$ the inclusion map. We consider the following diagram:

$$\begin{array}{ccccc}
\overbrace{\mathrm{H}_k(C)}^{[X]} & \xrightarrow{\mathrm{PD}_C} & \mathrm{H}^l(C, \partial C; \mathbb{Z}) & \xleftarrow[\cong]{c^*} & \overbrace{\mathrm{H}^l(M, M \setminus \mathring{C}; \mathbb{Z})}^{\alpha} \\
\downarrow & & \downarrow{\scriptstyle r_i^*} & \swarrow{\scriptstyle (e \circ q_i)^*} & \\
\mathrm{H}_k(C, C \setminus \mathring{Q}_i) & \xleftarrow[\cong]{r_{i*}} \underbrace{\mathrm{H}_k(Q_i, \partial_D Q_i)}_{[X_i]} & \xrightarrow{\mathrm{PD}_{Q_i}} & \mathrm{H}^l(Q_i, \partial_C Q_i; \mathbb{Z}). &
\end{array}$$

We make the following observations:

(α) The triangle on the right commutes since $c \circ r_i = e \circ q_i \colon Q_i \to M$ are both the inclusion map.

(β) It follows from the Large H_*-to-Small H^*-PD Lemma , applied to the codimension-zero submanifold Q_i of C, that the rectangle commutes.[694]

(γ) From the Codimension Zero-Fundamental Class Lemma 64.5 we obtain the equality $r_{i*}([X_i]) = [X] \in \mathrm{H}_k(C, C \setminus \mathring{Q}_i)$.

(δ) It follows from a slight generalization of the Excision-in-H^*-Theorem 69.1 (2) that the induced homomorphism $r_{i*} \colon \mathrm{H}_k(Q_i, \partial_D Q_i) \to \mathrm{H}_k(C, C \setminus \mathring{Q}_i)$ is an isomorphism.

The desired equality $(q_i^* \circ e^*)(\alpha) = \mathrm{PD}_{Q_i}([X_i])$ follows from the above commutative diagram applied to $[X] \in \mathrm{H}_k(C)$. Furthermore, exactly the same way we can also show that $(q_i^* \circ e^*)(\beta) = \mathrm{PD}_{Q_i}([Y_i])$.

(c) Let $i \in \{1, \dots, s\}$. We set $Z = \overline{B}^k \times \overline{B}^l$. We consider the following diagram:

$$\begin{array}{ccccccc}
\mathrm{H}^l(Q_i, \partial_C Q_i; \mathbb{Z}) \times \mathrm{H}^k(Q_i, \partial_D Q_i; \mathbb{Z}) & \xrightarrow{\cup} & \mathrm{H}^m(Q_i, \partial Q_i; \mathbb{Z}) & \xrightarrow{\cap [Q_i]} & \mathrm{H}_0(Q_i) & \xrightarrow{\epsilon_{Q_i}} & \mathbb{Z} \\
\downarrow{\scriptstyle \Psi_i^* \times \Psi_i^*} & & \downarrow{\scriptstyle \Psi_i^*} & & \uparrow{\scriptstyle \Psi_i^*} & & \uparrow{\scriptstyle \mathrm{id}} \\
\mathrm{H}^l(Z, \overline{B}^k \times \partial \overline{B}^l; \mathbb{Z}) \times \mathrm{H}^k(Z, \partial \overline{B}^k \times \overline{B}^l; \mathbb{Z}) & \xrightarrow{\cup} & \mathrm{H}^m(Z, \partial Z; \mathbb{Z}) & \xrightarrow{\cap \sigma_i \cdot [Z]} & \mathrm{H}_0(Z) & \xrightarrow{\epsilon_Z} & \mathbb{Z}.
\end{array}$$

As in the big diagram on page 1156, the squares to the left and right commute. Next note that the middle square is slightly more interesting: The Sign-Q_i-Lemma 88.9 says that $\Psi_{i*}([Q_i]) = \sigma_i \cdot [Z]$. Using this observation we obtain once again from the Cap Product-Naturality Lemma 75.8 that the middle square commutes.

We calculate that

[694] One can easily convince oneself that $\partial_C Q_i = \partial_1 Q_i$ and $\partial_D Q_i = \partial_0 Q_i$ for Q_i viewed as a submanifold with corner of C.

$$\epsilon_{Q_i}((\mathrm{PD}_{Q_i}([X_i])\cup \mathrm{PD}_{Q_i}([Y_i]))\cap[Q_i]) \overset{\text{by the above commutative diagram}}{=} \sigma_i\cdot\epsilon_Z((\Psi_i^*(\mathrm{PD}_{Q_i}([X_i]))\cup\Psi_i^*(\mathrm{PD}_{Q_i}([Y_i])))\cap[Z])$$
$$= \sigma_i\cdot\epsilon_Z((\mathrm{PD}_Z([\overline{B}^k\times\{0\}])\cup\mathrm{PD}_Z([\{0\}\times\overline{B}^l]))\cap[Z])$$
$$\overset{\text{by the Transverse Neighborhood Proposition 88.8 (d)}}{=} \sigma_i\cdot\epsilon_{\overline{B}^k\times\overline{B}^l}([\overline{B}^k\times\overline{B}^l]^*\cap[\overline{B}^k\times\overline{B}^l]) \overset{\text{definition of the dual fundamental class}}{=} \sigma_i.$$

the Poincaré Duality-Products Proposition 88.10

After all these preparations we can prove the desired equality:

$$\epsilon_M((\mathrm{PD}_M([X])\cup\mathrm{PD}_M([Y]))\cap[M]) \overset{\text{by the first row of the commutative diagram from Claim 1 and by Claim 2 (a)}}{=} \epsilon_M((\alpha\cup\beta)\cap p_*([M]))$$
$$\overset{\text{by the second and third row of commutative diagram from Claim 1 and by Claim 2 (b)}}{=} \sum_{i=1}^s \epsilon_{Q_i}((\mathrm{PD}_{Q_i}([X_i])\cup\mathrm{PD}_{Q_i}([Y_i]))\cap[Q_i]) \overset{\text{by Claim 2 (c)}}{=} \sum_{i=1}^s \sigma_i.$$

This concludes the proof of the theorem. ■

88.4. **Cup product on complex manifolds.** This section has three subsections:

(1) In Subsection 88.4.1 we study intersection numbers of complex submanifolds.
(2) In Subsection 88.4.2 we specialize to complex projective spaces and compute their cup products without any sign ambiguities.
(3) In Subsection 88.4.3 we will show that the tubular neighborhood of $\mathbb{C}\mathrm{P}^1\subset\mathbb{C}\mathrm{P}^2$ is not a product.

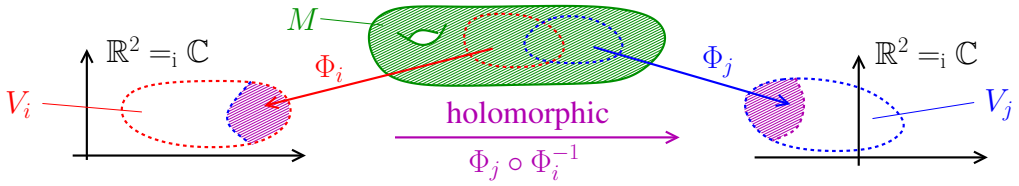
88.4.1. **Algebraic intersection numbers of complex submanifolds.** In this subsection we will study the algebraic intersection numbers of complex submanifolds of any complex manifold.

Definition.

(1) Let M be a $2n$-dimensional topological manifold M. An atlas $\{\Phi_i\colon U_i\to V_i\}_{i\in I}$ of charts of type (i), i.e. of charts where each V_i is an open subset of $\mathbb{R}^{2n}$, for M is called holomorphic if for all $i,j\in I$ the transition map

$$\Phi_j\circ\Phi_i^{-1}\colon \underbrace{\Phi_i(U_i\cap U_j)}_{\text{open in }\mathbb{R}^{2n}=\mathbb{C}^n} \to \underbrace{\Phi_j(U_i\cap U_j)}_{\subset\mathbb{R}^{2n}=\mathbb{C}^n}$$

is holomorphic.



Definition.

(2) An n-dimensional complex manifold is a pair $(M, \mathcal{A})$ consisting of a $2n$-dimensional topological manifold M, together with a maximal holomorphic atlas for M.

Definition. Let M be a complex manifold. We say that $W \subset M$ is a k-dimensional complex submanifold if for every point $P \in W$ there exists a chart $\Phi\colon U \to V$ from the maximal holomorphic atlas with $P \in U$ and which satisfies $\Phi(U \cap W) \subset \mathbb{C}^k \times \{0\}$.

Remark.

(1) Let M be a complex manifold. It follows easily from the definitions that every complex submanifold of M is a complex manifold in its own right.
(2) Let V be an n-dimensional complex vector space. Basic linear algebra implies the following: If $(v_1, \dots, v_n)$ is a basis for V as a complex vector space, then the ordered basis $(v_1, \mathrm{i}v_1, \dots, v_n, \mathrm{i}v_n)$ gives an orientation of the $2n$-dimensional real vector space V which does *not* depend on the choice of the basis $v_1, \dots, v_n$. This orientation of the real vector space V is called the *natural orientation*.
(3) Let M be an n-dimensional complex manifold. One can easily show that $\mathrm{T}_P M$ is naturally a complex vector space and we showed that the corresponding natural orientations from (2) define a natural orientation of M, viewed as a $2n$-dimensional smooth manifold. □

Now we can formulate the following elementary but rather useful lemma:

Lemma 88.11. (Complex Submanifolds–Intersection Number Lemma) Let M be a closed complex manifold and let X and Y be two closed complex submanifolds of M with $\dim(X) + \dim(Y) = \dim(M)$. Then the following hold:

(1) The sign at every transverse intersection point of X and Y is equal to $+1$.
(2) If X and Y intersect transversally, then the algebraic intersection number equals the geometric intersection number.

Proof.

(1) We set $k = \dim(X)$ and $l = \dim(Y)$. Suppose that P is a transverse intersection point of the submanifolds X and Y. We pick a basis $(v_1, \dots, v_k)$ for the complex vector space $\mathrm{T}_P X$ and we pick a basis $(w_1, \dots, w_l)$ for the complex vector space $\mathrm{T}_P Y$. Note that the vectors $(v_1, \dots, v_k, w_1, \dots, w_l)$ evidently form a basis for the complex vector space $\mathrm{T}_P X \oplus \mathrm{T}_P Y =_{\mathrm{i}} \mathrm{T}_P M$. We have

$$\underbrace{v_1, \mathrm{i}v_1, \dots, v_k, \mathrm{i}v_k}_{\substack{\text{positive basis for } \mathrm{T}_P X, \\ \text{viewed as an } \mathbb{R}\text{-vector space}}} \quad , \quad \underbrace{w_1, \mathrm{i}w_1, \dots, w_l, \mathrm{i}w_l}_{\substack{\text{positive basis for } \mathrm{T}_P Y, \\ \text{viewed as an } \mathbb{R}\text{-vector space}}} \quad = \quad \underbrace{v_1, \mathrm{i}v_1, \dots, v_k, \mathrm{i}v_k, w_1, \mathrm{i}w_1, \dots, w_l, \mathrm{i}w_l}_{\substack{\text{positive basis for } \mathrm{T}_P M, \\ \text{viewed as an } \mathbb{R}\text{-vector space}}}.$$

This shows that $\mathrm{sign}(P) = +1$.
(2) Statement (2) is an immediate consequence of Statement (1). ■

Example. We consider the complex projective space $\mathbb{C}\mathrm{P}^n$. In the $\mathbb{C}\mathrm{P}^n$–Complex Manifold Lemma **??** we saw that $\mathbb{C}\mathrm{P}^n$ is a complex manifold of complex dimension n. One can easily

show that for any $k \leq n$ the subspace

$$X_k \;:=\; \{[z_0 : \ldots : z_k : 0 : \ldots : 0] \mid (z_0, \ldots, z_k) \in \mathbb{C}^{k+1} \setminus \{0\}\}$$

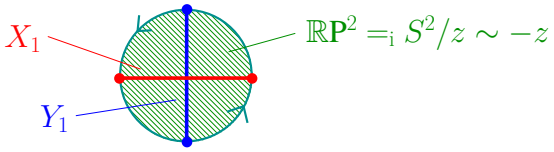
is a k-dimensional complex submanifold which is biholomorphic to $\mathbb{C}\mathrm{P}^k$. Similarly we see that for any $l \leq n$ the subspace

$$Y_l \;:=\; \{[0 : \ldots : 0 : z_{n-l} : \ldots : z_n] \mid (z_{n-l}, \ldots, z_n) \in \mathbb{C}^{l+1} \setminus \{0\}\}$$

is an l-dimensional complex submanifold which is biholomorphic to $\mathbb{C}\mathrm{P}^l$. Note that given any $k \leq n$ we have

$$X_k \cap Y_{n-k} \;=\; \{[0 : \ldots : 0 : \underset{\substack{\uparrow \\ (k+1)\text{-st entry}}}{1} : 0 : \ldots : 0]\}$$

consists of a single point P. We leave it to the reader to verify that X_k and Y_{n-k} intersect transversally in P. For illustration purposes we show the corresponding submanifolds in the real projective space $\mathbb{R}\mathrm{P}^2$:



Finally note that it follows from the Complex Submanifolds–Intersection Number Lemma 88.11 that $X_k \cdot Y_{n-k} = 1$ and that $Y_k \cdot X_{n-k} = 1$. □

88.4.2. **The cup product of complex projective spaces.** In this subsection we will use the results from Subsection 88.4.1 to calculate the cup product on complex projective spaces. The following notation will come in handy:

Notation. Let $n \in \mathbb{N}$ and let $m \in \{0, \ldots, n\}$.

(1) We consider the smooth embedding

$$\begin{array}{rcl} i_{m,n} \colon \mathbb{C}\mathrm{P}^m & \to & \mathbb{C}\mathrm{P}^n \\ [z_0 : \ldots : z_m] & \mapsto & [z_0 : \ldots : z_m : 0 : \cdots : 0]. \end{array}$$

When m and n are understood we write i instead of $i_{m,n}$. Recall that in the proof of the $\mathbb{C}\mathrm{P}^n$-H_*-Proposition 47.4 we showed that for any $k \leq 2m$ the induced maps $i_* \colon \mathrm{H}_k(\mathbb{C}\mathrm{P}^m; \mathbb{Z}) \to \mathrm{H}_k(\mathbb{C}\mathrm{P}^n; \mathbb{Z})$ and $i^* \colon \mathrm{H}^k(\mathbb{C}\mathrm{P}^n; \mathbb{Z}) \to \mathrm{H}^k(\mathbb{C}\mathrm{P}^m; \mathbb{Z})$ are isomorphisms.

(2) By a slight abuse of notation we abbreviate $(i_{m,n})_*([\mathbb{C}\mathrm{P}^m])$ to $[\mathbb{C}\mathrm{P}^m] \in \mathrm{H}_{2m}(\mathbb{C}\mathrm{P}^n; \mathbb{Z})$. Furthermore, again by a slight abuse of notation, we write $[\mathbb{C}\mathrm{P}^m]^* \in \mathrm{H}^{2m}(\mathbb{C}\mathrm{P}^n; \mathbb{Z})$ for the unique element with $\langle [\mathbb{C}\mathrm{P}^m]^*, [\mathbb{C}\mathrm{P}^m] \rangle = 1$. Finally we write $x := [\mathbb{C}\mathrm{P}^1]^*$.

Now we can formulate the main result of this subsection:

Proposition 88.12. (Cup Product-$\mathbb{C}\mathrm{P}^n$-Proposition) Let $n \in \mathbb{N}$. For every choice of $k \in \{0, \ldots, n\}$ we have

$$(\underbrace{[\mathbb{C}\mathrm{P}^1]^*}_{=x})^k \;=\; [\mathbb{C}\mathrm{P}^k]^* \;\in\; \mathrm{H}^{2k}(\mathbb{C}\mathrm{P}^n; \mathbb{Z}).$$

Remark. Recall that in the Cup Product-$\mathbb{C}\mathrm{P}^n$-Proposition 81.4 we used the Poincaré Duality Theorem 79.1 to show that for given $n \in \mathbb{N}$ and $k \in \{0, \ldots, n\}$ the k-th power $x^k \in \mathrm{H}^{2k}(\mathbb{C}\mathrm{P}^n; \mathbb{Z}) \cong \mathbb{Z}$ is a generator. In other words, we had proved that $x^k = \pm[\mathbb{C}\mathrm{P}^k]^*$.

The Cup Product-$\mathbb{C}\mathrm{P}^n$-Proposition 88.12 is a much better result since it allows us to nail down the cohomology class x^k, without any sign ambiguity. □

The key to the proof of the Cup Product-$\mathbb{C}\mathrm{P}^n$-Proposition 88.12 is the following lemma:

Lemma 88.13. (Cup Product-of-Duals-in-$\mathbb{C}\mathrm{P}^n$-Lemma) Let $n \in \mathbb{N}_0$. Given any $k \in \{0, \dots, n\}$ we have

$$\mathrm{PD}_{\mathbb{C}\mathrm{P}^n}(\underbrace{[\mathbb{C}\mathrm{P}^k]}_{\in \mathrm{H}_{2k}(\mathbb{C}\mathrm{P}^n)}) \cup \mathrm{PD}_{\mathbb{C}\mathrm{P}^n}(\underbrace{[\mathbb{C}\mathrm{P}^{n-k}]}_{\in \mathrm{H}_{2n-2k}(\mathbb{C}\mathrm{P}^n)}) = [\mathbb{C}\mathrm{P}^n]^* \in \mathrm{H}^{2n}(\mathbb{C}\mathrm{P}^n; \mathbb{Z}).$$

Proof. Let $n \in \mathbb{N}$. For $k = 0$ and $k = n$ the statement follows easily from the definitions. We now consider the case that $k \in \{1, \dots, n-1\}$. We consider the smooth embedding

$$\begin{array}{rcl} t_{k,n} \colon \mathbb{C}\mathrm{P}^k & \to & \mathbb{C}\mathrm{P}^n \\ [z_0 : \dots : z_k] & \mapsto & [0 : \dots : 0 : z_0 : \dots : z_k]. \end{array}$$

Claim. There exists a homotopy from $i_{k,n} \colon \mathbb{C}\mathrm{P}^k \to \mathbb{C}\mathrm{P}^n$ to $t_{k,n} \colon \mathbb{C}\mathrm{P}^k \to \mathbb{C}\mathrm{P}^n$.

Proof. We consider the following map:

$$\begin{array}{rcl} \Phi \colon \mathbb{C}^{k+1} \times [0,1] & \to & \mathbb{C}^{n+1} \\ (z,t) & \mapsto & \underbrace{(z \cdot (1-t), 0)}_{\in \mathbb{C}^{n+1}} + \underbrace{(0, z \cdot t)}_{\in \mathbb{C}^{n+1}}. \end{array}$$

We make the following observations:

(1) For any $z \neq 0$ and any $t \in [0,1]$ we have $\Phi(z,t) \neq 0$. This can be seen as follows: If $k = n$ or if $t = 1$, then the statement is clear. If $k \neq n$ and $t \in [0,1)$, then we consider the first coordinate of z which is non-zero. We see that for $t \neq 1$ this coordinate of $\Phi(z,t)$ is also non-zero.

(2) It is clear that this map descends to a map $\mathbb{C}\mathrm{P}^k \times [0,1] \to \mathbb{C}\mathrm{P}^n$. It follows fairly easily from the Homotopy Quotient Theorem 8.5 that this map is continuous. Clearly this is now the desired homotopy. ⊞

Using this claim we can perform the following calculation:

$$\begin{array}{rcl} \langle \mathrm{PD}([\mathbb{C}\mathrm{P}^k]) \cup \mathrm{PD}([\mathbb{C}\mathrm{P}^{n-k}]), [\mathbb{C}\mathrm{P}^n] \rangle & = & \langle \mathrm{PD}(i_{k,n*}([\mathbb{C}\mathrm{P}^k])) \cup \mathrm{PD}(i_{n-k,n*}([\mathbb{C}\mathrm{P}^{n-k}])), [\mathbb{C}\mathrm{P}^n] \rangle \\ & = & \langle \mathrm{PD}(t_{k,n*}([\mathbb{C}\mathrm{P}^k])) \cup \mathrm{PD}(i_{n-k,n*}([\mathbb{C}\mathrm{P}^{n-k}])), [\mathbb{C}\mathrm{P}^n] \rangle \\ & & \uparrow \\ & & \text{by the Homotopy-H}_*\text{-Proposition 36.5, since} \\ & & \text{since by the claim we know that } i_{k,n} \text{ and } t_{k,n} \text{ are homotopic} \\ & = & \langle \mathrm{PD}([t_{k,n}(\mathbb{C}\mathrm{P}^k)]) \cup \mathrm{PD}([i_{n-k,n}(\mathbb{C}\mathrm{P}^{n-k})]), [\mathbb{C}\mathrm{P}^n] \rangle \\ & = & t_{k,n}(\mathbb{C}\mathrm{P}^k) \cdot i_{n-k,n}(\mathbb{C}\mathrm{P}^{n-k}) \\ & & \uparrow \\ & & \text{by the Cup Product–Intersection Number Theorem 88.4 since} \\ & & \text{the submanifolds } t_{k,n}(\mathbb{C}\mathrm{P}^k) \text{ and } i_{n-k,n}(\mathbb{C}\mathrm{P}^{n-k}) \text{ are transverse} \end{array} \quad = \quad 1.$$

see page 1161 for details

By definition of the dual fundamental class $[\mathbb{C}\mathrm{P}^n]^*$ this implies the desired result. ■

Proof of the Cup Product-$\mathbb{C}\mathrm{P}^n$-Proposition 88.12. We start out with the following claim:

Claim 1. Let $n \in \mathbb{N}$. For any $k \in \{0, \dots, n\}$ we have

$$\mathrm{PD}_{\mathbb{C}\mathrm{P}^n}(\underbrace{[\mathbb{C}\mathrm{P}^k]}_{\in \mathrm{H}_{2k}(\mathbb{C}\mathrm{P}^n)}) = [\mathbb{C}\mathrm{P}^{n-k}]^* \in \mathrm{H}^{2n-2k}(\mathbb{C}\mathrm{P}^n; \mathbb{Z}).$$

Proof. Let $n \in \mathbb{N}$ and let $k \in \{0, \dots, n\}$. Then

$$\begin{aligned} \langle \mathrm{PD}_{\mathbb{C}\mathrm{P}^n}([\mathbb{C}\mathrm{P}^k]), [\mathbb{C}\mathrm{P}^{n-k}] \rangle &\overset{\text{by definition of } \mathrm{PD}_{\mathbb{C}\mathrm{P}^n}([\mathbb{C}\mathrm{P}^{n-k}])}{=} \langle \mathrm{PD}_{\mathbb{C}\mathrm{P}^n}([\mathbb{C}\mathrm{P}^k]), \mathrm{PD}_{\mathbb{C}\mathrm{P}^n}([\mathbb{C}\mathrm{P}^{n-k}]) \cap [\mathbb{C}\mathrm{P}^n] \rangle \\ &\overset{\text{by the Cup-Cap Lemma 75.9 and the Cup Product-Commutativity Proposition 72.8}}{=} \langle \mathrm{PD}_{\mathbb{C}\mathrm{P}^n}([\mathbb{C}\mathrm{P}^k]) \cup \mathrm{PD}_{\mathbb{C}\mathrm{P}^n}([\mathbb{C}\mathrm{P}^{n-k}]), [\mathbb{C}\mathrm{P}^n] \rangle \overset{\text{by the Cup Product-of-Duals-in-}\mathbb{C}\mathrm{P}^n \text{ Lemma 88.13}}{=} 1. \end{aligned}$$

By definition of the dual fundamental class this implies the desired result. ⊞

We now start with the actual proof. Recall that we need to show that for every $n \in \mathbb{N}$ and every $k \in \{0, \dots, n\}$ we have $([\mathbb{C}\mathrm{P}^1]^*)^k = [\mathbb{C}\mathrm{P}^k]^* \in \mathrm{H}^{2k}(\mathbb{C}\mathrm{P}^n; \mathbb{Z})$. We start out with the following observation that follows immediately from the discussion on page 1161 and the Cup Product-Naturality Lemma 72.2:

Observation. Let $m \in \mathbb{N}$ and let $k \in \{1, \dots, m\}$. If $([\mathbb{C}\mathrm{P}^1]^*)^k = [\mathbb{C}\mathrm{P}^k]^* \in \mathrm{H}^{2k}(\mathbb{C}\mathrm{P}^m; \mathbb{Z})$, then for any $n \geq m$ we also have $([\mathbb{C}\mathrm{P}^1]^*)^k = [\mathbb{C}\mathrm{P}^k]^* \in \mathrm{H}^{2k}(\mathbb{C}\mathrm{P}^n; \mathbb{Z})$

By the observation it suffices to prove the following claim:

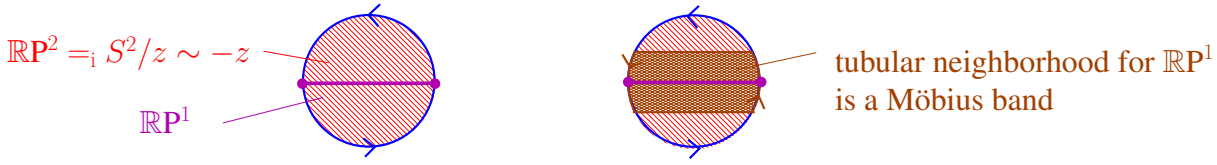
Claim 2. For any $k \in \mathbb{N}$ we have $([\mathbb{C}\mathrm{P}^1]^*)^k = [\mathbb{C}\mathrm{P}^k]^* \in \mathrm{H}^{2k}(\mathbb{C}\mathrm{P}^k; \mathbb{Z})$.

Proof. We prove the claim by induction on k. For $k = 1$ the statement holds by definition. So suppose that the statement holds for some $k-1$ with $k \geq 2$. Then the following equality holds in $\mathrm{H}^{2k}(\mathbb{C}\mathrm{P}^k; \mathbb{Z})$:

$$([\mathbb{C}\mathrm{P}^1]^*)^k = ([\mathbb{C}\mathrm{P}^1]^*)^{k-1} \cup [\mathbb{C}\mathrm{P}^1]^* \overset{\text{by induction and the observation}}{=} [\mathbb{C}\mathrm{P}^{k-1}]^* \cup [\mathbb{C}\mathrm{P}^1]^* \overset{\text{by Claim 1}}{=} \mathrm{PD}([\mathbb{C}\mathrm{P}^1]) \cup \mathrm{PD}([\mathbb{C}\mathrm{P}^{k-1}]) \overset{\text{by the Cup Product-of-Duals-in-}\mathbb{C}\mathrm{P}^k\text{-Lemma 88.13}}{=} [\mathbb{C}\mathrm{P}^k]^*.$$

This concludes the proof of the claim and thus of the proposition. ■

88.4.3. **Tubular neighborhoods in complex projective spaces.** We conclude this section with a short discussion of tubular neighborhoods. By the Tubular Neighborhood Theorem [**Fri24**] we know that every proper submanifold N of a given smooth manifold M admits a tubular neighborhood, which is, loosely speaking a neighborhood of N which is also a $B^{\dim(M)-\dim(N)}$-bundle over N. In the Product Tubular Neighborhood Theorem **??** we showed that in many cases the tubular neighborhood is a product. But we also saw some examples where the tubular neighborhood is not a product. For example the figure below shows convincingly that the tubular neighborhood of $\mathbb{R}\mathrm{P}^1$ in $\mathbb{R}\mathrm{P}^2$ is a Möbius band, and thus not a product.



Similarly, if N is a non-orientable submanifold of an orientable smooth manifold, then it follows from orientability considerations that the tubular neighborhood of N in M cannot be a product.

The following proposition shows that, for better or worse, the tubular neighborhood of an orientable submanifold of an orientable smooth manifold is not necessarily a product either:

Proposition 88.14. (Non-Product Tubular Neighborhood Proposition) No tubular neighborhood of $\mathbb{C}\mathrm{P}^1 = i_{1,2}(\mathbb{C}\mathrm{P}^1)$ in $\mathbb{C}\mathrm{P}^2$ is a product.

Proof. We now suppose that $\mathbb{C}\mathrm{P}^1$ admits a product tubular neighborhood in $\mathbb{C}\mathrm{P}^2$. By definition this would mean that there exists a smooth embedding $\tau\colon \mathbb{C}\mathrm{P}^1 \times \overline{B}^2 \to \mathbb{C}\mathrm{P}^2$ with $\tau(P,(0,0)) = P$ for all $P \in \mathbb{C}\mathrm{P}^1$. We denote by $i = i_{1,2}\colon \mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^2$ the usual smooth embedding and we denote by $f\colon \mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^2$ the smooth embedding that is given by $P \mapsto \tau(P,(1,0))$. Then

$$\pm 1 \underset{\substack{\uparrow\\ \text{by the Cup Product-}\mathbb{C}\mathrm{P}^n\text{-Proposition 88.12 or}\\ \text{by the Cup Product-}\mathbb{C}\mathrm{P}^n\text{-Proposition 81.4}}}{=} \langle \mathrm{PD}(i_*([\mathbb{C}\mathrm{P}^1]) \cup \mathrm{PD}(i_*([\mathbb{C}\mathrm{P}^1])), [\mathbb{C}\mathrm{P}^2]\rangle \underset{\substack{\uparrow\\ \text{by the Homotopy-H}_*\text{-Proposition 36.5}\\ \text{since } i, f\colon \mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^2 \text{ are homotopic}\\ \text{via the homotopy } (P,t) \mapsto \tau(P,(t,0))}}{=} \langle \mathrm{PD}(i_*([\mathbb{C}\mathrm{P}^1]) \cup \mathrm{PD}(f_*([\mathbb{C}\mathrm{P}^1])), [\mathbb{C}\mathrm{P}^2]\rangle$$

$$\underset{\substack{\uparrow\\ \text{by the Cup Product–Intersection}\\ \text{Number Theorem 88.4}}}{=} i(\mathbb{C}\mathrm{P}^1) \cdot f(\mathbb{C}\mathrm{P}^1) \underset{\substack{\uparrow\\ \text{since } i(\mathbb{C}\mathrm{P}^1) \text{ and } f(\mathbb{C}\mathrm{P}^1)\\ \text{are disjoint}}}{=} 0.$$

We have thus obtained a contradiction. ∎

CHAPTER 89

The intersection form: Definition and calculations

In this chapter we will introduce and study the intersection form of compact oriented even-dimensional topological manifolds. We will calculate the intersection form of various topological manifolds, e.g. of the complex projective spaces and products $S^n \times S^n$. Furthermore we will study the behavior of the intersection form under the connected sum operation.

This will lead to some interesting algebraic problems, which we will deal with in Chapter 90.

89.1. **Pairings, forms and matrices.** Before we turn to topology it is convenient to remind ourselves of a few basic notions and results in algebra. On page 1044 we introduced the notion of a pairing over a commutative ring R. We recall the definition and we introduce some new notions:

Definition. Let R be a commutative ring and let V, W and S be R-modules.

(1) A pairing on $V \times W$ is a map

$$\varphi\colon V \times W \ \to \ S$$

such that for all $v, v' \in V$ and $w, w' \in W$ we have $\varphi(v + v', w) = \varphi(v, w) + \varphi(v', w)$ and $\varphi(v, w + w') = \varphi(v, w) + \varphi(v, w')$.

(2) A pairing is called R-bilinear if it is R-linear in V and if it is R-linear in W.

(3) Let $\varphi\colon V \times V \to S$ be a pairing.
- We say that φ is symmetric if $\varphi(v, w) = \varphi(w, v)$ for all $v, w \in V$.
- We say that φ is anti-symmetric if $\varphi(v, w) = -\varphi(w, v)$ for all $v, w \in V$.

(4) Let (V, φ) and (W, ψ) be two R-bilinear pairings.
   (a) An isometry between (V, φ) and (W, ψ) is an isomorphism $f\colon V \to W$ of R-modules such that $\varphi(v_1, v_2) = \psi(f(v_1), f(v_2))$ for all $v_1, v_2 \in V$.
   (b) We say that (V, φ) and (W, ψ) are isometric, written $(V, \varphi) \cong (W, \psi)$, if an isometry exists.

In the discussions below we will often treat abelian groups as $\mathbb{Z}$-modules in the obvious way.

In this chapter we will mostly be interested in the following special type of an R-bilinear pairing:

Definition. Let R be a commutative ring. A form is a finitely generated free R-module V together with an R-bilinear pairing $\varphi\colon V \times V \to R$ which is either symmetric or anti-symmetric. We refer to the rank of V also as the rank of the form.

For the reader's convenience we recall the following definition from page 1044:

Definition. Let R be a commutative ring and let $\varphi\colon V \times V \to R$ be a form. The form is is called non-singular if the map

$$\begin{array}{ccc} V & \to & \mathrm{Hom}_R(V,R) \\ v & \mapsto & \begin{pmatrix} V & \to & R \\ w & \mapsto & \varphi(v,w) \end{pmatrix} \end{array}$$

is an isomorphism.[695]

In the following discussion we will study the relationship between forms and matrices. We start out with the following definition:

Definition. Let R be a commutative ring.

(1) Let A be an (anti-) symmetric[696] $n \times n$–matrix over R. We consider the map

$$\begin{array}{rccc} \varphi(A) := \varphi_R(A)\colon & R^n \times R^n & \to & R \\ & (v,w) & \mapsto & v^T A w. \end{array}$$

It is elementary to verify that $\varphi_R(A)$ is an (anti-) symmetric form over R.[697]

(2) Let $\varphi\colon V \times V \to R$ be a form. Given any basis $\{v_1, \dots, v_k\}$ of the free R-module V we refer to the $(k \times k)$–matrix

$$(\varphi(v_i, v_j))_{i,j=1,\dots,k} \in \mathrm{M}(k \times k, R)$$

as a matrix representing the form φ.

The following lemma shows in particular that isometry classes of (anti-) symmetric (non-singular) forms over a commutative ring R are in bijection to congruence classes of (anti-) symmetric (invertible) matrices over R:

Lemma 89.1. (Form–Matrix-Lemma) Let R be a commutative ring.

(1) (a) Let (V, ψ) be an (anti-) symmetric form over R of rank n. If A is a matrix that represents ψ, then $(V, \psi) \cong (R^n, \varphi_R(A))$. In particular every (anti-) symmetric form over R is isometric to $\varphi_R(A)$ for some (anti-) symmetric $n \times n$–matrix A over R.

(b) Let A and B be two (anti-) symmetric matrices over R. The forms $\varphi_R(A)$ and $\varphi_R(B)$ are isometric if and only if there exists a $P \in \mathrm{GL}(n, R)$ with $B = P^T \cdot A \cdot P$.

(2) The form $\varphi_R(A)$ is non-singular, in the sense of the definition on page 1166, if and only if A is invertible, i.e. if $\det(A)$ is a unit in R.

Convention. The Form–Matrix-Lemma 89.1 allows us to go back and forth between (anti-) symmetric forms and (anti-) symmetric matrices. Therefore in the following discussion

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[695] Note that a form is by definition (anti-) symmetric. It follows from this fact that the given condition is satisfied if and only if the condition with the two entries of φ swapped is satisfied. Thus the definition we give here is indeed consistent with the definition from page 1044.

[696] Recall that a matrix A is called *anti-symmetric* if $A^T = -A$.

[697] Indeed, for any $v, w \in R^n$ we have

$$\varphi_R(A)(w,v) = w^T A v = \underset{\substack{\uparrow \\ \text{for a } (1\times 1)\text{–matrix } (x) \text{ we have } (x)^T = (x)}}{(w^T A^T v)^T} = v^T A^T w = \varphi_R(A^T)(v,w) = \pm \varphi_R(A)(v,w).$$

we will blur the distinction between these two concepts. For example, given an (anti-) symmetric matrix A over a commutative ring we will often just refer to the (anti-) symmetric form $\varphi_R(A)$ as the form A.

Proof.

(1) (a) Let $\psi\colon V \times V \to R$ be an (anti-) symmetric form over R. Let $v_1, \dots, v_n$ be a basis of V. We denote by $A = (\psi_R(v_i, v_j))_{i,j=1,\dots,n}$ the corresponding matrix representing the form ψ. We consider the isomorphism

$$\begin{array}{rcl} \Theta\colon R^n & \to & V \\ (\lambda_1, \dots, \lambda_n) & \mapsto & \lambda_1 \cdot v_1 + \dots + \lambda_n \cdot v_n. \end{array}$$

We claim that Θ is an isometry from the form $(R^n, \varphi_R(A))$ to the form (V, ψ). Indeed, if we denote by $e_1, \dots, e_n$ the standard basis for R^n, then for any choice of $i, j \in \{1, \dots, n\}$ we have

$$\psi_R(\Theta(e_i), \Theta(e_j)) \;=\; \psi_R(v_i, v_j) \;=\; ij\text{-entry of } A \;=\; e_i^T A e_j \;=\; \varphi_R(A)(e_i, e_j).$$

By the bilinearity of forms we obtain the corresponding statement for all pairs of elements in R^n.

(b) Let A and B be two (anti-) symmetric matrices over R. First we prove the "if"-direction. Thus we assume that there exists a $P \in \mathrm{GL}(n, R)$ with $B = P^T \cdot A \cdot P$. We claim that the isomorphism $v \mapsto P \cdot v$ defines an isometry from the form $(R^n, \varphi_R(P^T A P))$ to the form $(R^n, \varphi_R(A))$. Indeed, for any $v, w \in R^n$ we have

$$\varphi_R(A)(Pv, Pw) \;=\; (Pv)^T A P w \;=\; v^T (P^T A P) w \;=\; \varphi_R(P^T A P)(v, w).$$

We turn to the proof of the "only if"-direction. Thus we assume that there exists an isometry $f\colon (R^n, \varphi_R(A)) \to (R^n, \varphi_R(B))$. Let P be the matrix representing f with respect to the standard basis of R^n. Since f is an isomorphism we know that $P \in \mathrm{GL}(n, R)$. Finally the above calculation shows that $B = P^T A P$.

(2) This statement is basically the content of the Non-Singularity Lemma 81.1. ∎

We continue with a few basic constructions of R-bilinear pairings:

Definition. Let R be a commutative ring.

(1) Given an R-bilinear pairing $\varphi\colon V \times V \to S$ we denote by $-\varphi$ the R-bilinear pairing that is given by $(-\varphi)(v, w) := -\varphi(v, w)$ for all $v, w \in V$.

(2) Given two R-bilinear pairings $\varphi\colon V \times V \to S$ and $\psi\colon W \times W \to S$ we define the direct sum of φ and ψ to be the R-bilinear pairing

$$\begin{array}{rcl} \varphi \oplus \psi\colon (V \oplus W) \times (V \oplus W) & \to & S \\ ((v, w), (v', w')) & \mapsto & \varphi(v, v') + \psi(w, w'). \end{array}$$

(3) Given an R-bilinear pairing (V, φ) and $n \in \mathbb{Z}$ we define

$$n \cdot (V, \varphi) \;:=\; \begin{cases} \overbrace{(V, \varphi) \oplus \dots \oplus (V, \varphi)}^{n\text{-summands}}, & \text{if } n \geq 0, \\ \underbrace{(V, -\varphi) \oplus \dots \oplus (V, -\varphi)}_{|n|\text{-summands}}, & \text{if } n \leq 0. \end{cases}$$

Lemma 89.2. (Matrix–Form Operations Lemma) Let R be a commutative ring, let $\varphi\colon V \times V \to R$ be a form and let $\psi\colon W \times W \to R$ be a form.

(1) If A is a matrix that represents φ, then the matrix $-A$ represents the form $-\varphi$.

(2) If A is a matrix that represents φ and if B is a matrix that represents ψ, then the matrix $A \oplus B \coloneqq \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ represents the form $\varphi \oplus \psi$.

(3) Let $n \in \mathbb{N}_0$. If A is a matrix that represents φ, then the matrix $n \cdot A \coloneqq \underbrace{A \oplus \cdots \oplus A}_{n\text{-times}}$ represents the form $n \cdot \varphi$.

Proof. The lemma follows immediately from the definitions. ∎

89.2. Maximal free abelian quotients.

Definition. Given a finitely generated abelian group H we refer to

$$\mathrm{F}H \;\coloneqq\; H/\operatorname{Tor}(H) \quad \text{where } \operatorname{Tor}(H) \text{ denotes the torsion subgroup of } H$$

as the maximal torsion-free quotient of H.

Convention. On many occasions we do not distinguish in the notation between an element in an abelian group H and the element it represents in the maximal torsion-free quotient $\mathrm{F}H$. Furthermore, given a homomorphism $\alpha\colon A \to B$ between two finitely generated abelian groups we often denote the induced map $\mathrm{F}A \to \mathrm{F}B$ by α as well.

89.3. The definition of the intersection form of topological manifolds.

Let M be a compact oriented m-dimensional topological manifold. Recall that by the Poincaré Duality Theorem 79.1 we know that for any $k \in \mathbb{N}_0$ the map

$$\begin{array}{rcl} \mathrm{H}^{m-k}(M, \partial M; \mathbb{Z}) & \to & \mathrm{H}_k(M) \\ \varphi & \mapsto & \varphi \cap [M] \end{array}$$

is an isomorphism. As on page 1014 we denote by

$$\mathrm{PD}_M\colon \mathrm{H}_k(M) \;\to\; \mathrm{H}^{m-k}(M, \partial M; \mathbb{Z})$$

the inverse of this isomorphism. The Cup Product–Intersection Number Theorem 88.4 motivates the following definition:

Definition. Let M be a compact oriented topological manifold of even dimension $2n$. We refer to the map[698 699]

$$\begin{array}{rcl} Q_M\colon \mathrm{FH}_n(M) \times \mathrm{FH}_n(M) & \to & \mathbb{Z} \\ (a, b) & \mapsto & \Big\langle \underbrace{\underbrace{\mathrm{PD}_M(a)}_{\in \mathrm{FH}^n(M,\partial M;\mathbb{Z})} \cup \underbrace{\mathrm{PD}_M(b)}_{\in \mathrm{FH}^n(M,\partial M;\mathbb{Z})}}_{\in \mathrm{FH}^{2n}(M,\partial M;\mathbb{Z})}, [M] \Big\rangle \end{array}$$

as the intersection form of M.

Remark. Since $\mathrm{PD}_M\colon \mathrm{H}_n(M) \to \mathrm{H}^n(M,\partial M;\mathbb{Z})$ is an isomorphism and since by the Manifold-Top-H*-Theorem 70.16 we know that the cohomology group $\mathrm{H}^{2n}(M,\partial M;\mathbb{Z})$ is torsion-free it is clear that the intersection form is basically just a reformulation of the cup product pairing

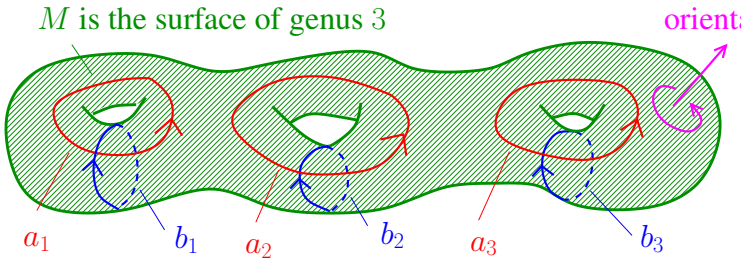
$$\begin{array}{rcl} \mathrm{H}^n(M,\partial M;\mathbb{Z}) \times \mathrm{H}^n(M,\partial M;\mathbb{Z}) & \to & \mathrm{H}^{2n}(M,\partial M;\mathbb{Z}) \\ (\alpha,\beta) & \mapsto & \alpha \cup \beta \end{array}$$

on cohomology groups in the middle dimension that we introduced in the Manifold-Non Singularity Proposition 81.2. The intersection form has two advantages:

(1) The big psychological advantage of homology is that very often one can "see homology classes". For example, in many cases homology classes are represented by submanifolds, this allows us to use the Cup Product–Intersection Number Theorem 88.4 to compute the intersection form of a smooth manifold.
(2) The intersection form always has the target $\mathbb{Z}$, so it makes sense to say that two intersection forms are isometric. □

Examples.

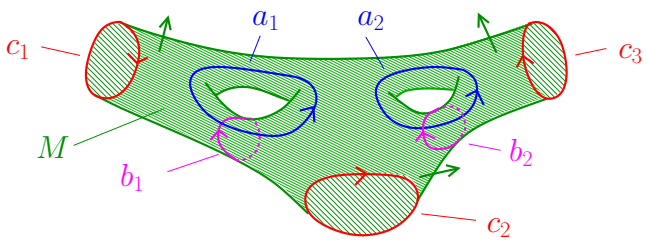
(1) First we consider the surface M of genus 3. Note that the closed oriented curves $a_1, b_1, a_2, b_2, a_3, b_3$ illustrated in the figure below represent a basis for $\mathrm{H}_1(M)$. It follows easily from the Cup Product–Intersection Number Theorem 88.4 that with respect to this basis the intersection form is represented by the matrix shown in the figure below.
An analogous statement holds of course for any surface of genus $g \geq 1$.



$$Q_M = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

(2) Next we let M be the surface of genus two minus three open disks that is shown in the figure below.
Note that a basis for $\mathrm{H}_1(M)$ is represented by the closed curves a_1, b_1, a_2, b_2 together with two out of the three boundary curves c_1, c_2, c_3. It follows from the Cup Product–Intersection Number Theorem 88.4 that with respect to this basis the intersection form is represented by the matrix given in the figure above.
(3) Next let us consider the $2n$-dimensional smooth manifold $M = S^n \times S^n$. By the Künneth Formula 56.3 we know that a basis for $\mathrm{H}_n(S^n \times S^n)$ is given by $a = [S^n \times \{*\}]$

[698]Note that the homology group $\mathrm{H}_n(M)$ in the middle dimension is not necessarily torsion-free. For example $M = S^1 \times \mathbb{R}\mathrm{P}^3$ is orientable, but it follows easily from the Topological Künneth Formula 56.3 and the calculation of the homology groups of $\mathbb{R}\mathrm{P}^3$ in the $\mathbb{R}\mathrm{P}^n$-H$_*$-Proposition 47.10 that $\mathrm{H}_2(S^1 \times \mathbb{R}\mathrm{P}^3) \cong \mathbb{Z}_2$.
[699]As mentioned above, in the notation usually we do not distinguish between an element in $\mathrm{H}_n(M)$ and the element it represents in $\mathrm{FH}_n(M)$.



$$Q_M = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and $b = [\{*\} \times S^n]$. We have two approaches for determining the intersection form with respect to this basis:

(a) We can use the Product Theorem 76.1 and its corollaries. More precisely, we first determine the Poincaré duals of $[S^n \times \{*\}]$ and $[\{*\} \times S^n]$ as in the Fundamental Class-Product Proposition 76.2 (2) and then we can apply the Product-of-Spheres–Cup Product Lemma 76.3.

(b) As on page 1149 we can use algebraic intersection numbers and the Cup Product–Intersection Number Theorem 88.4 to determine the intersection form.

In both cases we see that with respect to this basis the intersection form of $S^n \times S^n$ is represented by the matrix

$$\begin{pmatrix} Q_M(a,a) & Q_M(a,b) \\ Q_M(b,a) & Q_M(b,b) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ (-1)^n & 0 \end{pmatrix}.$$

(4) We consider the complex manifold $\mathbb{C}\mathrm{P}^2$. As always we equip $\mathbb{C}\mathrm{P}^2$ with the natural orientation coming from the complex structure. It follows from the Cup Product-$\mathbb{C}\mathrm{P}^n$-Proposition 88.12 that the intersection form

$$Q_{\mathbb{C}\mathrm{P}^2} \colon \mathrm{H}_2(\mathbb{C}\mathrm{P}^2) \times \mathrm{H}_2(\mathbb{C}\mathrm{P}^2) \to \mathbb{Z}$$

is represented by the 1×1–matrix (1).

(5) Let M be a closed oriented $2n$-dimensional topological manifold. As usual we denote by $-M$ the same topological manifold with the opposite orientation. It follows immediately from the definitions together with the observation that $[-M] = -[M]$ that $Q_{-M} = -Q_M$.

(6) For the complex projective space $\mathbb{C}\mathrm{P}^2$ we use the common notation that we write $\overline{\mathbb{C}\mathrm{P}}^2$ instead of $-\mathbb{C}\mathrm{P}^2$. With this notation it follows from (4) and (5) that the intersection form

$$Q_{\overline{\mathbb{C}\mathrm{P}}^2} \colon \mathrm{H}_2(\overline{\mathbb{C}\mathrm{P}}^2) \times \mathrm{H}_2(\overline{\mathbb{C}\mathrm{P}}^2) \to \mathbb{Z}$$

is represented by the 1×1–matrix (-1). □

89.4. **Properties of the intersection form.** In the following proposition we summarize some of the key properties of the intersection form:

Proposition 89.3. (Intersection Form Proposition) Let M be a compact oriented $2n$-dimensional topological manifold. Then the following hold:[700]

(1) The group $\mathrm{FH}_n(M)$ is a finitely generated free abelian group.
(2) If n is even, then the intersection form Q_M is symmetric, otherwise it is anti-symmetric.
(3) If M is closed, then the intersection form is non-singular.[701]

Proof.

(1) By the Topological Manifolds-Invariants Proposition 60.10 we know that $\mathrm{H}_n(M)$ is finitely generated. This implies of course that $\mathrm{FH}_n(M)$ is also finitely generated. It follows from the Free Abelian Quotient Lemma **??** (or more directly from the Abelian Group-Classification Theorem 18.11) that $\mathrm{FH}_n(M)$ is a free abelian group.

(2) Given $v, w \in \mathrm{FH}_n(M)$ we have the following equality:

$$\overbrace{\Big\langle \underbrace{\mathrm{PD}_M(v)}_{\in \mathrm{H}^n(M,\partial M;\mathbb{Z})} \cup \underbrace{\mathrm{PD}_M(w)}_{\in \mathrm{H}^n(M,\partial M;\mathbb{Z})}, [M] \Big\rangle}^{=Q_M(v,w)} = (-1)^{n\cdot n} \cdot \overbrace{\Big\langle \underbrace{\mathrm{PD}_M(w)}_{\in \mathrm{H}^n(M,\partial M;\mathbb{Z})} \cup \underbrace{\mathrm{PD}_M(v)}_{\in \mathrm{H}^n(M,\partial M;\mathbb{Z})}, [M] \Big\rangle}^{=Q_M(w,v)}.$$

by the Cup Product-Commutativity Proposition 73.4

The statement follows from the observation that $(-1)^{n\cdot n} = (-1)^n$.

(3) Now suppose that M is closed. We need to show that the intersection form Q_M is non-singular. Evidently this is equivalent to showing that the form

$$\begin{array}{rcl} \mathrm{FH}^n(M;\mathbb{Z}) \times \mathrm{FH}^n(M;\mathbb{Z}) & \to & \mathbb{Z} \\ (\varphi, \psi) & \mapsto & \langle \varphi \cup \psi, [M] \rangle \end{array}$$

is non-singular. We dealt with the case that the cohomology groups are torsion-free in the Manifold-Non Singularity Proposition 81.2. The proof of the general case is essentially the same. For the reader's convenience we provide the argument. It follows from the Cup Product-Commutativity Proposition 72.8 that it suffices to show that the map

$$\begin{array}{rcl} \Omega\colon \mathrm{FH}^n(M;\mathbb{Z}) & \to & \mathrm{Hom}_{\mathbb{Z}}(\mathrm{FH}^n(M;\mathbb{Z}),\mathbb{Z}) \\ \psi & \mapsto & (\varphi \mapsto \langle \varphi \cup \psi, [M] \rangle) \end{array}$$

is an isomorphism. As on page 1045 we start out with the following claim:

Claim. The following diagram commutes:

$$\begin{array}{ccc} \mathrm{FH}^n(\overset{\psi}{M};\mathbb{Z}) & \xrightarrow{\quad\Omega\quad} & \mathrm{Hom}_{\mathbb{Z}}(\mathrm{FH}^n(\overset{\varphi}{M};\mathbb{Z}),\mathbb{Z}). \\ & {\scriptstyle \mathrm{ev}} \searrow \quad \nearrow {\scriptstyle (\cap[M])^*} & \\ & \mathrm{Hom}_{\mathbb{Z}}(\mathrm{FH}_n(M),\mathbb{Z}) & \end{array}$$

Proof. Note that for any $\psi \in \mathrm{FH}^n(M;\mathbb{Z})$ and any $\varphi \in \mathrm{FH}^n(M;\mathbb{Z})$ the following equalities hold in $\mathbb{Z}$:[702]

[700] The proposition shows in particular that the intersection form is indeed a form in the sense of the definition on page 1165.

[701] This statement is a slight generalization of the Manifold-Non Singularity Proposition 81.2.

[701] The right diagonal is the dual map corresponding to the map $\mathrm{FH}^n(M;\mathbb{Z}) \xrightarrow{\cap[M]} \mathrm{FH}_n(M)$.

[702] Here $\epsilon_M\colon \mathrm{H}_0(M) \to \mathbb{Z}$ is the augmentation homomorphism from page 464.

$$\Omega(\psi)(\varphi) \underset{\uparrow}{=} \epsilon_M((\varphi\cup\psi)\cap[M]) \overset{\downarrow}{=} \epsilon_M(\psi\cap(\varphi\cap[M])) \underset{\uparrow}{=} \langle\psi,\varphi\cap[M]\rangle = \mathrm{ev}(\psi)(\varphi\cap[M]).$$

by the Cup-Cap Lemma 75.9

by definition of Ω and by the Cap Product-Kronecker Lemma 75.7

by the Cap Product-Kronecker Lemma 75.7

But this observation immediately shows that the diagram commutes. ⊞

We need to show that in the above diagram the horizontal map is an isomorphism. By the claim it suffices to show that the two diagonal maps are isomorphisms:

- By the Universal Coefficient-H^*-Theorem 70.14 we have a short exact sequence

$$0 \to \underbrace{\mathrm{Ext}(\mathrm{H}_{n-1}(M),\mathbb{Z})}_{\text{torsion by the Ext-Properties Lemma 70.12 and the } \mathbb{Z}_k\text{-Ext-Lemma 70.11}} \to \mathrm{H}^n(M;\mathbb{Z}) \xrightarrow{\mathrm{ev}} \mathrm{Hom}_{\mathbb{Z}}(\mathrm{H}_n(M),\mathbb{Z}) \to 0.$$

  It follows from the above data that the group to the left is torsion. It follows easily that the homomorphism $\mathrm{ev}\colon \mathrm{FH}^n(M;\mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(\mathrm{FH}_n(M),\mathbb{Z})$ is, as desired, an isomorphism.
- The diagonal map to the right is an isomorphism by the Poincaré Duality Theorem 79.1. ■

In the next lemma we want to see to what degree the isometry type of the intersection form is an invariant of the homeomorphism or homotopy type of the topological manifold:

Lemma 89.4. (Intersection Form–Homeomorphism Lemma) Let M and N be two compact oriented connected non-empty $2n$-dimensional topological manifolds. Furthermore let $f\colon (M,\partial M) \to (N,\partial N)$ be a continuous map such that $f\colon M \to N$ and $f\colon \partial M \to \partial N$ are homotopy equivalences.

(1) There exists an $\varepsilon \in \{-1,1\}$ which satisfies $f_*([M]) = \varepsilon\cdot[N]$. If f is an orientation-preserving homeomorphism, then $\varepsilon = 1$, and if f is an orientation-reversing homeomorphism, then $\varepsilon = -1$.

(2) (a) If $f_*([M]) = \ [N]$, then f induces an isometry $Q_M \cong \ Q_N$.
(b) If $f_*([M]) = -[N]$, then f induces an isometry $Q_M \cong -Q_N$.

Proof. [703] Let $f\colon (M,\partial M) \to (N,\partial N)$ be a continuous map such that $f\colon M \to N$ and $f\colon \partial M \to \partial N$ are homotopy equivalences. It follows from the Homotopy Equivalence-Relative-H_*-Corollary 38.4 together with the Fundamental Class-Proposition 64.1 (2) that there exists an $\varepsilon \in \{-1,1\}$ which satisfies $f_*([M]) = \varepsilon\cdot[N]$. If f is a homeomorphism, then we obtain ε from the Local Degree Lemma 65.2.

Claim. For any $c \in \mathrm{H}_n(M)$ we have $\mathrm{PD}_N(f_*(c)) = \varepsilon\cdot(f^*)^{-1}(\mathrm{PD}_M(c)) \in \mathrm{H}^n(N,\partial N;\mathbb{Z})$.

Proof. So let $c \in \mathrm{H}_n(M)$. We have

$$(f^*)^{-1}(\mathrm{PD}_M(c))\cap[N] \underset{\uparrow}{=} (f^*)^{-1}(\mathrm{PD}_M(c))\cap\varepsilon\cdot f_*([M]) \underset{\uparrow}{=} \varepsilon\cdot f_*(\mathrm{PD}_M(c)\cap[M]) \underset{\uparrow}{=} \varepsilon\cdot f_*(c).$$

by the definition of ε

by the Cap Product-Naturality Lemma 75.8

definition of PD_M

[703] The proof is perhaps slightly longer than one might expect. In a sense the statement of the lemma is basically trivial, but we have to pay a price for using the slightly awkward Poincaré Duality isomorphism $\mathrm{PD}_M\colon \mathrm{H}_n(M) \to \mathrm{H}^{2n-n}(M,\partial M;\mathbb{Z})$.

It follows from the definition of PD_N that we get the desired equality. ⊞

Now let $a, b \in \mathrm{FH}_n(M)$. Then we see that

$$\begin{aligned} Q_N(f_*(a), f_*(b)) &\overset{\text{by definition of } Q_N}{=} \langle \mathrm{PD}_N(f_*(a)) \cup \mathrm{PD}_N(f_*(b)), [N]\rangle \overset{\text{by the claim}}{=} \\ &= \langle \varepsilon \cdot (f^*)^{-1}(\mathrm{PD}_M(a)) \cup \varepsilon \cdot (f^*)^{-1}(\mathrm{PD}_M(b)), \varepsilon \cdot f_*([M])\rangle \\ &= \varepsilon \cdot \langle (f^*)^{-1}(\mathrm{PD}_M(a) \cup \mathrm{PD}_M(b)), f_*([M])\rangle \overset{\text{by the Kronecker Pairing Lemma 68.3 (2) and the definition of } Q_M}{=} \varepsilon \cdot Q_M(a,b). \end{aligned}$$

by the Cup Product-Naturality Lemma 73.3 and since $\varepsilon^3 = \varepsilon$

The lemma is now a consequence of the following observation: It follows from the hypothesis that f is a homotopy equivalence together with the Homotopy Equivalence-H_*-Corollary 36.7 (1a) that $f_*\colon \mathrm{FH}_n(M) \to \mathrm{FH}_n(N)$ is an isomorphism. ■

89.5. **The intersection form and degree-one maps.** As we remarked on page 1169, the intersection form of a compact oriented $2n$-dimensional topological manifold contains basically the same information as the cup product pairing

$$\begin{aligned} \mathrm{H}^n(M, \partial M; \mathbb{Z}) \times \mathrm{H}^n(M, \partial M; \mathbb{Z}) &\to \mathrm{H}^{2n}(M, \partial M; \mathbb{Z}) \\ (\alpha, \beta) &\mapsto \alpha \cup \beta. \end{aligned}$$

Given a continuous map $f\colon X \to Y$ between topological spaces it follows from the Cup Product-Naturality Lemma 72.2 that the cup product behaves well under the induced maps f^* on cohomology groups. It stands to reason that the intersection form should also behave well with respect to induced maps on homology.

So suppose that we are given two closed oriented connected non-empty $2n$-dimensional topological manifolds and let $f\colon X \to Y$ be a continuous map of degree 1, i.e. such that $f_*([X]) = [Y]$. By the aforementioned Cup Product-Naturality Lemma 72.2 we know that for any $\alpha \in \mathrm{FH}^i(Y;\mathbb{Z})$ and $\beta \in \mathrm{FH}^j(Y;\mathbb{Z})$ we have $f^*(\alpha) \cup f^*(\beta) = f^*(\alpha \cup \beta)$. The following question arises: Given $\alpha \in \mathrm{H}_n(X)$ and $\beta \in \mathrm{H}_n(X)$, does the following equality hold:

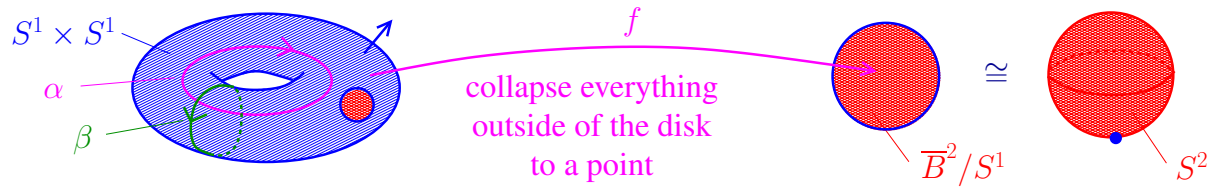
$$(*) \qquad \text{¿¿ } Q_X(\alpha, \beta) \;=\; Q_Y(f_*(\alpha), f_*(\beta)) \text{ ??}$$

In fact here is a diagram that looks awfully relevant:

$$\begin{array}{ccccccc} \mathrm{H}_n(X) \times \mathrm{H}_n(X) & \xrightarrow[\cong]{\mathrm{PD}_X \times \mathrm{PD}_X} & \mathrm{H}^n(X;\mathbb{Z}) \times \mathrm{H}^n(X;\mathbb{Z}) & \xrightarrow{\cup} & \mathrm{H}^{2n}(X;\mathbb{Z}) & \xrightarrow{\langle -, [X]\rangle} & \mathbb{Z} \\ \downarrow{\scriptstyle f_* \times f_*} & & {\scriptstyle f^* \times f^*}\uparrow & & {\scriptstyle f^*}\uparrow{\scriptstyle \cong} & & \mathrm{id}\downarrow \\ \mathrm{H}_n(Y) \times \mathrm{H}_n(Y) & \xrightarrow[\cong]{\mathrm{PD}_Y \times \mathrm{PD}_Y} & \mathrm{H}^n(Y;\mathbb{Z}) \times \mathrm{H}^n(Y;\mathbb{Z}) & \xrightarrow{\cup} & \mathrm{H}^{2n}(Y;\mathbb{Z}) & \xrightarrow{\langle -, [Y]\rangle} & \mathbb{Z}. \end{array}$$

If this diagram did commute, then (∗) would hold. One could now waste a lot of time trying to convince oneself that the diagram commutes. Or alternatively one might notice, as we do in the following example, that (∗) in general just does not hold.

Example. We consider the map $f\colon S^1 \times S^1 \to S^2$ that is illustrated in the figure below. It follows easily from the Local-Global Degree Proposition 65.3 that this is a degree-one map. We consider the two homology classes α and β on the torus that are sketched in the figure below. First note that it follows easily from the Cup Product–Intersection Number Theorem 88.4 that $Q_{S^1 \times S^1}(\alpha, \beta) = +1$. On the other hand note that we have



$f_*(\alpha) = f_*(\beta) = 0 \in \mathrm{H}_1(S^2)$, which evidently implies that $Q_{S^2}(f_*(\alpha), f_*(\beta)) = 0$. Thus in this case the equality (∗) does not hold.[704] □

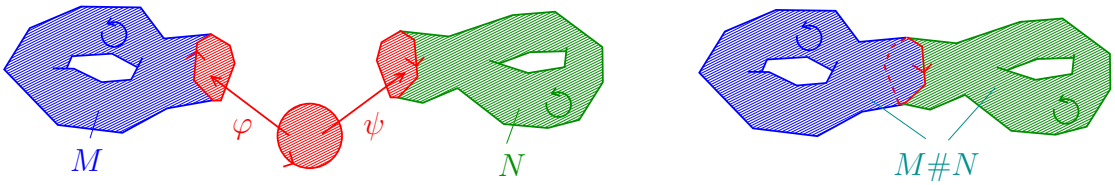
89.6. **The intersection form and the connected sum operation.** To get more examples of interesting intersection forms we want to study the intersection forms of the connected sum of smooth and topological manifolds. Recall that on page **??** we introduced the connected sum of two *smooth* manifolds. We now generalize this definition to topological manifolds:

Definition. Let $n \in \mathbb{N}$. Let M and N be two closed oriented connected non-empty n-dimensional topological manifolds. By the discussion on page **??** we can pick

- an orientation-preserving embedding $\varphi\colon \overline{B}^n \to M$ and
- an orientation-reversing embedding $\varphi\colon \overline{B}^n \to N$

such that the images are codimension-zero submanifolds. We define the connected sum of M and N as

$$M\#N := (M \setminus \varphi(B^n)) \sqcup (N \setminus \psi(B^n))/\sim \qquad \text{where } \varphi(P) \sim \psi(P) \text{ for all } P \in S^{n-1}.$$



In [**Fri24**] we prove the following proposition:

Proposition 89.5. (Topological Connected Sum-Proposition) Let $n \in \mathbb{N}$ and let M and N be two closed oriented oriented connected n-dimensional topological manifolds. Furthermore

- let $\varphi\colon \overline{B}^n \to M$ be an an orientation-preserving embedding and
- let $\psi\colon \overline{B}^n \to N$ be an orientation-reversing embedding

such that the images are codimension-zero submanifolds. The following four statements hold:[705]

(1) The resulting connected sum $M\#N$ is an n-dimensional topological manifold. The homeomorphism type of $M\#N$ does not depend on the choice of φ and ψ.
(2) $M\#N$ is also closed and connected.
(3) $M \setminus \varphi(B^n)$ and $N \setminus \psi(B^n)$ are closed subsets of $M\#N$ and they are both codimension-zero submanifolds of $M\#N$.

[704] This still begs the question, in the above diagram, which of the squares does not commute?

(4) The topological manifold $M\#N$ admits a unique orientation that restricts to the given orientations of $M \setminus \varphi(B^n)$ and $N \setminus \psi(B^n)$.

Before we turn to the study of intersection forms of connected sums, let us first see how the homology groups and fundamental groups behave under the connected sum operation:

Proposition 89.6. (Connected Sum-Invariants Proposition) Let $n \in \mathbb{N}$ and let M and N be two closed oriented connected n-dimensional topological manifolds. Furthermore

- let $\varphi\colon \overline{B}^n \to M$ be an an orientation-preserving embedding and
- let $\psi\colon \overline{B}^n \to N$ be an orientation-reversing embedding

such that the images are codimension-zero submanifolds. Then the following two statements hold:

(1) If $n \geq 3$, then the following inclusion-induced maps are isomorphisms:

$$\pi_1(M\#N) \xleftarrow{\cong} \pi_1(M \setminus \varphi(B^n)) * \pi_1(N \setminus \psi(B^n)) \xrightarrow{\cong} \pi_1(M) * \pi_1(N).$$

(2) For any $i \neq 0, n$ the following inclusion-induced maps are isomorphisms

$$\mathrm{H}_i(M\#N) \xleftarrow{\cong} \mathrm{H}_i(M\setminus\varphi(B^n)) \oplus \mathrm{H}_i(N\setminus\psi(B^n)) \xrightarrow{\cong} \mathrm{H}_i(M) \oplus \mathrm{H}_i(N).$$

An analogous statement holds for homology groups with coefficients in any abelian group.

Proof. In the case of smooth manifolds we proved the statements in the Connected Sum-π_1-Proposition 23.3 and the Connected Sum-H_*-Proposition **??**. The proof for topological manifolds is exactly the same. (Note that the Manifold–Seifert-van Kampen Theorem 23.1 and the Mayer–Vietoris Theorem 45.1 (3) apply to smooth *and* topological manifolds). ■

The next proposition says that the isomorphism of homology groups from the Connected Sum-Invariants Proposition 89.6 (2) induces an isometry of intersection forms:

Proposition 89.7. (Intersection Form-Additivity Proposition) Let $n \in \mathbb{N}$ and let M and N be two closed oriented connected $2n$-dimensional topological manifolds. Furthermore

- let $\varphi\colon \overline{B}^{2n} \to M$ be an an orientation-preserving embedding and
- let $\psi\colon \overline{B}^{2n} \to N$ be an orientation-reversing embedding

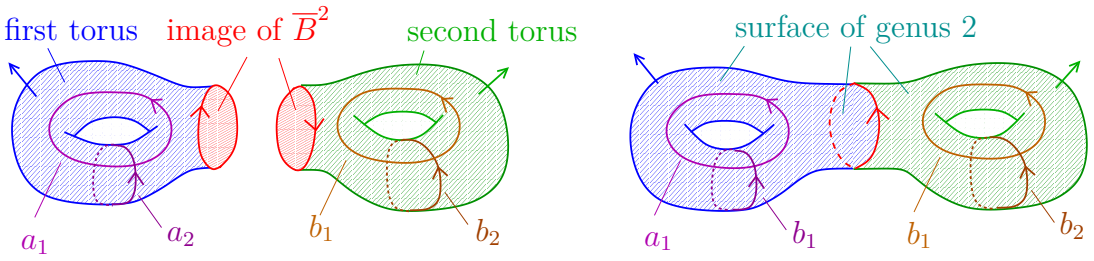
such that the images of both maps are codimension-zero submanifolds. The isomorphism $\mathrm{H}_n(M\#N) \xrightarrow{\cong} \mathrm{H}_n(M) \oplus \mathrm{H}_n(N)$ from the Connected Sum-Invariants Proposition 89.6 induces an isometry

$$Q_{M\#N} \xrightarrow{\cong} Q_M \oplus Q_N$$

of forms.

Example. We consider the surface Σ of genus two. In the figure below we recall that we can view Σ as the connected sum of two tori.
It thus follows from the Intersection Form-Additivity Proposition 89.7, the discussion on page 1170 and the Matrix–Form Operations Lemma 89.2 (2) that the intersection form on

[705] The statements are mostly straightforward, except for the fact that the homeomorphism type of $M\#N$ does not depend on the choice of φ and ψ.

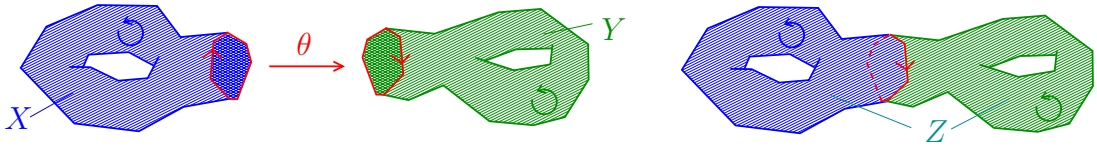


$\mathrm{H}_1(\Sigma) \cong \mathbb{Z}^4$ is represented by the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Not surprisingly this recovers the result from page 1169. □

Proof of the Intersection Form-Additivity Proposition 89.7. Throughout the proof let $n \geq 1$. We start out with a lengthy preparation: Thus let X and Y be two oriented connected $2n$-dimensional topological manifolds such that ∂X and ∂Y are both homeomorphic to S^{2n-1}. We equip ∂X and ∂Y with the orientation coming from the Boundary Orientation Proposition 62.3. Let $\theta\colon \partial X \to \partial Y$ be an orientation-reversing homeomorphism. We write $Z = X \cup_\theta Y$. In the following we view X and Y as subspaces of Z. By the Gluing-Topological Manifolds-Proposition 62.4 we know that X and Y are in fact codimension-zero submanifolds of Z and that we can equip Z with an orientation such that the induced orientations on X and Y are the given orientations.



Claim. The inclusions $i\colon X \to Z$ and $j\colon Y \to Z$ induce an isometry

$$Q_X \oplus Q_Y \xrightarrow{\cong} Q_Z.$$

Proof. As in the proof of the Connected Sum-H_*-Proposition **??** we see that the two inclusion maps i and j induce an isomorphism $\mathrm{H}_n(X) \oplus \mathrm{H}_n(Y) \to \mathrm{H}_n(Z)$. Evidently this isomorphism descends to an isomorphism $\mathrm{FH}_n(X) \oplus \mathrm{FH}_n(Y) \to \mathrm{FH}_n(Z)$. Therefore it suffices to prove the following three statements.

(1) For $a, b \in \mathrm{FH}_n(X)$ we have $Q_X(a,b) = Q_Z(i_*(a), i_*(b))$.
(2) For $a, b \in \mathrm{FH}_n(Y)$ we have $Q_Y(a,b) = Q_Z(j_*(a), j_*(b))$.
(3) For $a \in \mathrm{FH}_n(X)$ and $b \in \mathrm{FH}_n(Y)$ we have $Q_Z(i_*(a), j_*(b)) = 0$.

We start out with the proof of Statement (1): Thus let $a, b \in \mathrm{FH}_n(X)$. We write $\mathring{X} = X \setminus \partial X$. We denote by $p\colon (Z, \varnothing) \to (Z, Z \setminus \mathring{X})$ and $i\colon (X, \partial X) \to (Z, Z \setminus \mathring{X})$ the two obvious inclusion maps of pairs of topological spaces. It is a consequence of the Excision-in-H^*-Theorem 69.1 (2) that $i_*\colon \mathrm{FH}_k(X, \partial X) \to \mathrm{FH}_k(Z, Z \setminus \mathring{X})$ is an isomorphism for every

$k \in \mathbb{N}_0$. Then

$$Q_X(a,b) \underset{\uparrow}{=} \langle \mathrm{PD}_X(a) \cup \mathrm{PD}_X(b), [X] \rangle \underset{\uparrow}{=} \big\langle (i^*)^{-1}(\mathrm{PD}_X(a)) \cup (i^*)^{-1}(\mathrm{PD}_X(b)), i_*([X]) \big\rangle$$

by definition of Q_X

by the Kronecker Pairing Lemma 68.3 (2) applied to $i\colon (X, \partial X) \to (Z, Z \setminus \mathring{X})$, together with the Cup Product-Naturality Lemma 73.3

$$\underset{\uparrow}{=} \big\langle p^*((i^*)^{-1}(\mathrm{PD}_X(a))) \cup p^*((i^*)^{-1}(\mathrm{PD}_X(b))), [Z] \big\rangle$$

first note that by the Codimension Zero-Fundamental Class Lemma 64.5 we have, $i_*([X]) = p_*([Z])$,
next apply the Kronecker Pairing Lemma 68.3 (2) to p,
finally apply the Cup Product-Naturality Lemma 73.3 to p

$$\underset{\uparrow}{=} \big\langle \mathrm{PD}_Z(i_*(a)) \cup \mathrm{PD}_Z(i_*(b)), [Z] \big\rangle \underset{\uparrow}{=} Q_Z(i_*(a), i_*(b)).$$

by the Small H_*-to-Large H^*-PD Lemma 88.7 we know that $p^*(i^*)^{-1}(\mathrm{PD}_X(c)) = \mathrm{PD}_Z(i_*(c))$

by definition of Q_Z

Statement (2) is of course proved the same way. We turn to the proof of Statement (3). So let $a \in \mathrm{FH}_n(X)$ and $b \in \mathrm{FH}_n(Y)$. The above argument shows that we can compute $Q_Z(i_*(a), j_*(b))$ in terms of the cup product in Z of an element in the image $\mathrm{H}^n(Z, Z \setminus \mathring{X}; \mathbb{Z}) \to \mathrm{H}^n_{}(Z; \mathbb{Z})$ and an element in the image of $\mathrm{H}^n(Z, Z \setminus \mathring{Y}; \mathbb{Z}) \to \mathrm{H}^n(Z; \mathbb{Z})$. But $(Z \setminus \mathring{X}) \cup (Z \setminus \mathring{Y}) = Z$. So it follows from the Cup Product-Naturality Lemma 73.3 that this cup product is zero. ⊞

After these preparations it is very easy to prove the actual statement of the proposition. Thus let M and N be closed oriented $2n$-dimensional topological manifolds. Furthermore

- let $\varphi\colon \overline{B}^n \to M$ be an an orientation-preserving embedding and
- let $\psi\colon \overline{B}^n \to N$ be an orientation-reversing embedding

such that the images are codimension-zero submanifolds. We write $X = M \setminus \varphi(B^{2n})$ and $Y = N \setminus \psi(B^{2n})$. It follows from the Topological Submanifold Proposition 42.6 that X and Y are compact topological manifolds. With this notation we can write $M = X \cup \overline{B}^{2n}$ and $N = Y \cup \overline{B}^{2n}$. It follows that the inclusions induce isometries

$$Q_{M\#N} \underset{\uparrow}{\xleftarrow{\cong}} Q_X \oplus Q_Y \underset{\uparrow}{\xrightarrow{\cong}} Q_M \oplus Q_N.$$

isometry by the claim

isometry by the claim since $\mathrm{H}_n(\overline{B}^{2n}) = 0$

We have thus proved the desired statement. ■

CHAPTER 90

The classification of forms

Before we do anything else, let us recall the following notation from page 1166:

Definition. Let R be a commutative ring and let A be an (anti-) symmetric $n \times n$–matrix over R. We introduce the following (anti-) symmetric form over R:

$$\begin{array}{rccc} \theta(A) \ := \ \theta_R(A) \colon & R^n \times R^n & \to & R \\ & (v, w) & \mapsto & v^T A w. \end{array}$$

Sometimes, for brevity, we just say "the form A", instead of "the form $\theta(A)$". The Form–Matrix-Lemma 89.1 allows us to go back and forth between isometry classes of (anti-) symmetric forms and congruence classes of (anti-) symmetric matrices over R.

From our earlier discussions we obtain the following table:

manifold	π_1	H_1	H_0	H_2	H_3	H_4	intersection form
$S^2 \times S^2$	$\{e\}$	$\mathbb{Z}$	0	$\mathbb{Z}^2$	0	$\mathbb{Z}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
$\mathbb{C}P^2 \# \mathbb{C}P^2$	$\{e\}$	$\mathbb{Z}$	0	$\mathbb{Z}^2$	0	$\mathbb{Z}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
$\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$	$\{e\}$	$\mathbb{Z}$	0	$\mathbb{Z}^2$	0	$\mathbb{Z}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

So if we want to distinguish $S^2 \times S^2$, $\mathbb{C}P^2 \# \mathbb{C}P^2$ and $\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$ our only hope lies in the intersection form.

In this chapter we will try our best to classify (anti-) symmetric non-singular forms over $\mathbb{Z}$ and over $\mathbb{R}$. Along the way we will in particular develop the tools to distinguish the above three intersection forms.

This chapter is organized as follows:

(1) In Section 90.1 we will classify non-singular anti-symmetric forms over $\mathbb{Z}$.
(2) In Section 90.2 we will classify symmetric forms over $\mathbb{R}$.
(3) In Section 90.3 we introduce the E_8-form in a convoluted but fun way.
(4) In Section 90.4 we will study symmetric forms over $\mathbb{Z}$.

90.1. **Non-singular anti-symmetric forms.** In this section we will do the following:

- First we completely classify non-singular anti-symmetric forms over $\mathbb{R}$ and over $\mathbb{Z}$.
- We will apply this algebraic result to topology, namely we will determine which integers can arise as Euler characteristics of closed orientable topological manifolds.

90.1.1. **Non-singular anti-symmetric forms: Algebra.** First we consider the matrix

$$S \ := \ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

It is anti-symmetric and it has determinant 1. From the definition on page 1166 and 1178 we obtain an anti-symmetric form $\theta(S)$ over any commutative ring. The following proposition shows that S is the mother of all non-singular anti-symmetric forms over $\mathbb{R}$ or over $\mathbb{Z}$:

Proposition 90.1. (Anti-Symmetric Form Proposition) Let $R = \mathbb{R}$ or $R = \mathbb{Z}$. If (V, φ) is a non-singular anti-symmetric form over R, then there exists a $k \in \mathbb{N}_0$ and an isometry[706] $(V, \varphi) \cong k \cdot \theta_R(S)$. In other words, φ is represented by the $2k \times 2k$–matrix

$$\underbrace{\begin{pmatrix} 0 & 1 & \dots & 0 & 0 \\ -1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & -1 & 0 \end{pmatrix}}_{k \text{ diagonal blocks of } 2 \times 2\text{-matrices}}.$$

In particular the rank of V is even.

Sketch of proof. First we deal with the case $R = \mathbb{R}$. Thus let (V, φ) be a non-singular anti-symmetric form over $\mathbb{R}$ of rank n. We pick an $(n \times n)$–matrix A that represents φ. By the Form–Matrix-Lemma 89.1 it suffices to prove the following claim:

Claim. The rank n is even and A is congruent to the matrix $\frac{n}{2} \cdot S$.

Proof. In the following we consider the usual three types of $(n \times n)$-elementary matrices:

$$D_\lambda := \begin{pmatrix} \mathrm{id} & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mathrm{id} \end{pmatrix}, \lambda \neq 0, \quad P := \begin{pmatrix} \mathrm{id} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \mathrm{id} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathrm{id} \end{pmatrix} \quad \text{and} \quad Q_\lambda := \begin{pmatrix} \mathrm{id} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \lambda & 0 \\ 0 & 0 & \mathrm{id} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \mathrm{id} \end{pmatrix}.$$

We make the following elementary observations:

(i) Replacing a matrix B by $D_\lambda^T B D_\lambda$ means that first we multiply a column by λ, and then we multiply the corresponding row by λ.

(ii) Replacing a matrix B by $P^T B P$ means that first we exchange two columns, and then we exchange the corresponding two rows.

(iii) Replacing a matrix B by $Q_\lambda^T B Q_\lambda$ means that first we add a λ-multiple of one column to another column, and then we do the same operation for the corresponding rows.

Now we can perform the following sequence of operations on A:

$$A \overset{(1)}{=} \begin{pmatrix} 0 & * & * & \dots & * \\ * & 0 & * & \dots & * \\ * & * & 0 & * & * \\ \vdots & \vdots & * & \ddots & * \\ * & * & * & * & 0 \end{pmatrix} \overset{(2)}{\to} \begin{pmatrix} 0 & 1 & * & \dots & * \\ -1 & 0 & * & \dots & * \\ * & * & 0 & * & * \\ \vdots & \vdots & * & \ddots & * \\ * & * & * & * & 0 \end{pmatrix} \overset{(3)}{\to} \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ -1 & 0 & * & \dots & * \\ 0 & * & 0 & * & * \\ \vdots & \vdots & * & \ddots & * \\ 0 & * & * & * & 0 \end{pmatrix} \overset{(4)}{\to} \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ -1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & * & * \\ \vdots & \vdots & * & \ddots & * \\ 0 & 0 & * & * & 0 \end{pmatrix}.$$

More precisely we perform the following steps:

(1) We write out A. Since φ is anti-symmetric we see that all terms of A on the diagonal are zero. Here and throughout the steps we use the convention that $*$ can mean any real number, the only condition being that all matrices are anti-symmetric.

[706] Here, when we write $k \cdot \theta(S)$ we use the definition from page 1167.

(2) Since $\det(A) \neq 0$ there exists at least one entry on the first row which is non-zero. After operations of type (ii) and (i) we can arrange that the $(1,2)$-entry equals 1. By the anti-symmetry we see that the $(2,1)$-entry now equals -1.
(3) Using operations of type (iii) we can "clean up" the remaining terms on the first row and column.
(4) Similarly, we can use operations of type (iii) to "clean up" the remaining terms on the second row and column.

At this stage we can continue with the same process, applied to the $(n-2)\times(n-2)$–matrix at the right bottom. Eventually we will have shown that n is even and we will have turned A into the matrix $\frac{n}{2}\cdot S$. ⊞

We leave it to the reader, say using the concept of the greatest common divisor, to modify the above argument to deal with the case $R = \mathbb{Z}$. ■

90.1.2. **Non-singular anti-symmetric forms: Topology.** In a sense the Anti-Symmetric Form Proposition 90.1 is rather disappointing: We had hoped that intersection forms will give us new meaningful invariants. But the Anti-Symmetric Form Proposition 90.1 together with the Intersection Form Proposition 89.3 shows that intersection forms of closed oriented $(4k+2)$-dimensional topological manifolds do not give us any extra information that would allow us to distinguish manifolds.

Nonetheless, the Anti-Symmetric Form Proposition 90.1 has just enough content to answer an old question about Euler characteristics. More precisely, recall that in the Odd-Dimensional Manifold-χ-Proposition 79.6 we showed that the Euler characteristic of every closed odd-dimensional topological manifold is necessarily zero. This observation gave rise to the following question: Which integers appear as Euler characteristics of closed orientable connected even-dimensional topological manifolds? Now we can provide a complete answer to this question:

Proposition 90.2. (Euler Characteristic–Realisation Proposition)
(1) Let $k \in \mathbb{N}$. Given any $n \in \mathbb{Z}$ there exists a closed orientable connected $4k$-dimensional smooth manifold with Euler characteristic n.
(2) (a) The Euler characteristic of any closed orientable connected topological manifold of dimension 2 is even and ≤ 2. Conversely, any even integer ≤ 2 is the Euler characteristic of such a smooth manifold.
(b) Let $k \in \mathbb{N}$. The Euler characteristic of any closed orientable connected topological manifold of dimension $4k+2$ is even. Conversely, any even integer is the Euler characteristic of such a smooth manifold.

Proof. We start out with two claims that give us the stated restrictions on Euler characteristics of closed orientable topological manifolds.

Claim 1. Let $k \in \mathbb{N}_0$. The Euler characteristic of every closed orientable topological manifold M of dimension $4k+2$ is even.

Proof. We have

$$\chi(M) \underset{\uparrow}{=} \sum_{i=0}^{4k+2}(-1)^i \cdot \operatorname{rank}(\mathrm{H}_i(M)) \underset{\uparrow}{=} -\operatorname{rank}(\mathrm{H}_{2k+1}(M)) + \sum_{i=0}^{2k}(-1)^i \cdot \overbrace{2 \cdot \operatorname{rank}(\mathrm{H}_i(M))}^{\text{even}}$$

by the Manifold-χ Proposition ?? (5)

by the Betti Number Symmetry Corollary 79.5 we have $\operatorname{rank}(\mathrm{H}_i(M)) = \operatorname{rank}(\mathrm{H}_{4k+2-i}(M))$

$$\underset{\uparrow}{\equiv} 0 \mod 2.$$

by the Intersection Form Proposition 89.3 and the facts that M is closed and that $2k+1$ is odd we know that the intersection form on $\mathrm{FH}_{2k+1}(M)$ is anti-symmetric and non-singular, by the Anti-Symmetric Form Proposition 90.1 we thus know that $\operatorname{rank}(\mathrm{H}_{2k+1}(M))$ is even ⊞

Claim 2. The Euler characteristic of every every closed orientable connected 2-dimensional topological manifold M is ≤ 2.

Proof. Of course this statement follows from the Surface Classification Theorem 22.8 together with the calculation of the Euler characteristics on page 605. But one can prove the claim directly: First note that the case that $M = \varnothing$ is trivial. Now assume that M is non-empty. In this case we have the following (in-) equalities:

$$\chi(M) \underset{\uparrow}{=} \sum_{i\in\mathbb{N}_0}(-1)^i \cdot \operatorname{rank}(\mathrm{H}_i(M)) \underset{\uparrow}{=} \overbrace{\operatorname{rank}(\mathrm{H}_0(M))}^{=1 \text{ by the } \mathrm{H}_0\text{-Proposition 35.11}} - \operatorname{rank}(\mathrm{H}_1(M)) + \overbrace{\operatorname{rank}(\mathrm{H}_2(M))}^{=1 \text{ by the Orientation-Top Homology Theorem 63.3}} \leq 2.$$

by the Euler Characteristic-H_* Proposition 48.1

by the Manifold-Top-H_*-Theorem 63.1 we know that $\mathrm{H}_k(M) = 0$ for $k \geq 3$ ⊞

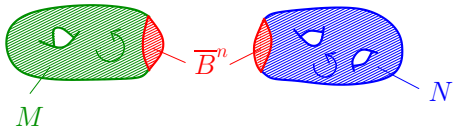
It remains to show that besides these two restrictions from Claim 1 and 2 on the possible values of Euler characteristics, all other values can be realized by closed orientable connected smooth manifolds. In other words, it remains to prove the following claim:

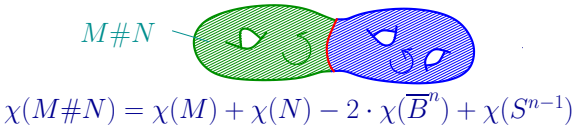
Claim 3.

(1) Let $m \in \mathbb{N}$ be even. Given any $n \in \mathbb{Z}$ there exists a closed orientable connected $2m$-dimensional smooth manifold with Euler characteristic n.
(2) (a) Every even number ≤ 2 is the Euler characteristic of a closed orientable connected smooth manifold of dimension 2.
    (b) Let $m \in \mathbb{N}$ be odd. Every even number is the Euler characteristic of a closed orientable connected smooth manifold of dimension $2m+2$.

Proof. We start out with the following observations:

(i) On page 606 we saw that for every $m \in \mathbb{N}_0$ we have $\chi(\mathbb{C}\mathrm{P}^m) = m+1$.
(ii) There is any number of reasons, e.g. we could use the Product-χ-Proposition 48.9, why given any $d \in \mathbb{N}$ the Euler characteristic of the d-dimensional torus $(S^1)^d$ is zero.
(iii) The Connected Sum-χ-Lemma ?? says that if M and N are two compact oriented connected *even-dimensional* smooth manifolds, then $\chi(M\#N) = \chi(M) + \chi(N) - 2$.





Note that it follows from these observations that for any $m, r, s \in \mathbb{N}_0$ we have

$$(*) \quad \chi(r \cdot \mathbb{C}\mathrm{P}^m \# s \cdot (S^1)^{2m}) \;=\; r \cdot \underbrace{(m+1)}_{=\chi(\mathbb{C}\mathrm{P}^m)} - 2 \cdot \underbrace{(r+s-1)}_{\substack{\text{number of} \\ \text{connected sum} \\ \text{operations}}} \;=\; r \cdot (m-1) - 2 \cdot (s-1).$$

Now we turn to the actual proof of the claim:

(1) Let $m \in 2\mathbb{N}$ and let $n \in \mathbb{Z}$. Since m is even we see that $s := \frac{|n| \cdot (m-1) - n}{2} + 1$ is a natural number. It follows from (∗) with $r = |n|$ and the above s that $\chi(r \cdot \mathbb{C}\mathrm{P}^m \# s \cdot (S^1)^{2m}) = n$.

(2) (a) Let $n \in \mathbb{Z}_{\leq 2}$ be even. It follows from (∗) that $\chi((1 - \frac{n}{2}) \cdot (S^1 \times S^1)) = n$.[707]

(b) Let $m \in \mathbb{N}$ be odd and let n be an even integer. Since n is even we see that $s := \frac{|n| \cdot (m-1) - n}{2} + 1$ is a natural number. It follows again from (∗) with $r = |n|$ and the above s that $\chi(r \cdot \mathbb{C}\mathrm{P}^m \# s \cdot (S^1)^{2m}) = n$. ■

90.2. **Symmetric forms over $\mathbb{R}$.** In Section 90.1 we saw that non-singular *anti-symmetric* forms are almost depressingly boring mathematical objects. In this and the following sections we will consider non-singular *symmetric* forms. These will, fortunately, turn out to be much more interesting animals. The current section is organized as follows:

- First we classify symmetric forms over $\mathbb{R}$.
- Afterwards we apply the classification to topological manifolds, in particular we introduce the signature of a compact oriented topological manifold and study some of its properties. The signature will allow us to show that $\mathbb{C}\mathrm{P}^2 \# \mathbb{C}\mathrm{P}^2$ is neither homotopy equivalent to $\mathbb{C}\mathrm{P}^2 \# \mathbb{C}\mathrm{P}^2$ nor to $S^2 \times S^2$.

90.2.1. **Symmetric forms over $\mathbb{R}$: Algebra.** It is natural to ask whether there is some type of analogue of the Anti-Symmetric Form Proposition 90.1 for symmetric forms. In fact using the methods of the proof of the Anti-Symmetric Form Proposition 90.1 one can prove the following statement:

Proposition 90.3. (Symmetric Forms-over-$\mathbb{R}$-Proposition) Let (V, φ) be a symmetric form over $\mathbb{R}$. There exist $n, k_+, k_- \in \mathbb{N}_0$ such that

$$(V, \varphi) \;\cong\; \underbrace{n \cdot (0) \,\oplus\, k_+ \cdot (+1) \,\oplus\, k_- \cdot (-1)}_{\substack{\text{here we use the notation from the} \\ \text{Matrix–Form Operations Lemma 89.2} \\ \text{i.e. we consider the diagonal matrix} \\ \text{where } n \text{ entries are zero, } k_\pm \text{ entries are } \pm 1}}.$$

Remark. Recall that the Anti-Symmetric Form Proposition 90.1 dealt with anti-symmetric forms over $\mathbb{R}$ and also with anti-symmetric forms over $\mathbb{Z}$. In contrast the Symmetric Forms-over-$\mathbb{R}$-Proposition 90.3 only deals with symmetric forms over $\mathbb{R}$. It is a very instructive exercise to run through the proof of the Symmetric Forms-over-$\mathbb{R}$-Proposition 90.3 and to determine where it breaks down if we work over $\mathbb{Z}$. □

Proof. Let (V, φ) be a non-singular symmetric form over $\mathbb{R}$ of rank r. We pick an $(r \times r)$–matrix A that represents φ. By the Form–Matrix-Lemma 89.1 it suffices to prove the following claim:

[707] Of course that is old news: We already saw on page 605 that for the surface Σ_g of genus g we have $\chi(\Sigma_g) = 2 - 2g$.

Claim. There exist $n, k_+, k_- \in \mathbb{N}_0$ such that the matrix A is congruent to the diagonal matrix $n \cdot (0) \oplus k_+ \cdot (+1) \oplus k_- \cdot (-1)$.

Proof. In the subsequent arguments below we use the conventions and notions we introduced in the proof of the Anti-Symmetric Form Proposition 90.1 on page 1179. In particular we will also use the operations of type (i), (ii) and (iii) that we introduced in the proof of the Anti-Symmetric Form Proposition 90.1. We perform the following sequence of operations on A:

$$A \overset{(1)}{=} \begin{pmatrix} * & * & \dots & * \\ * & * & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \dots & * \end{pmatrix} \xrightarrow{(2a)} \begin{pmatrix} \epsilon_1 & * & \dots & * \\ * & * & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \dots & * \end{pmatrix} \xrightarrow{(2b)} \begin{pmatrix} \epsilon_1 & 0 & \dots & 0 \\ 0 & * & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & * & \dots & * \end{pmatrix}.$$

More precisely, we perform the following steps:

(1) We write out A. Here and throughout the steps we use the convention that $*$ can mean any real number, the only condition being that all matrices are symmetric.

(2)
- If the first row consists entirely of zeros, then so does (by symmetry) the first column.
- We now assume that the first row has at least one entry which is non-zero.
  - (a) After operations of type (iii) and (i) we can arrange that the $(1,1)$-entry equals some $\epsilon_1 \in \{-1, 1\}$.
  - (b) Using operations of type (iii) we can "clean up" the remaining terms on the first row and column.

At this stage we can continue with the same process, applied to the $(r-1) \times (r-1)$–matrix at the right bottom. Eventually we will have turned A into a diagonal matrix where all entries lie in $\{-1, 0, 1\}$. After operations of type (ii) we can arrange that the first diagonal entries are equal to 0, the following entries are equal to $+1$ and the final ones are equal to -1. ∎

Let $n \in \mathbb{N}_0$. The Symmetric Forms-over-$\mathbb{R}$-Proposition 90.3 raises the question whether or not the symmetric forms defined by $n \cdot (0) \oplus k_+ \cdot (+1) \oplus k_+ \cdot (-1)$ are isometric for different values of n, k_-, k_+. Put differently, the question is, can we read off n, k_-, k_+ from a given symmetric form over $\mathbb{R}$?

To answer this question it is helpful to introduce the following definition:

Definition. Let (V, φ) be a form over a field $\mathbb{F}$.

(1) The vector subspace $\operatorname{rad}(\varphi) := \{v \in V \mid \theta(v, w) = 0 \text{ for all } w \in V\}$ is called the radical of the form φ.

(2) We refer to
$$\nu(\varphi) \;:=\; \dim_{\mathbb{F}}(\operatorname{rad}(\varphi)).$$
as the nullity $\nu(\varphi)$.

Examples.

(1) In the setting of the Symmetric Forms-over-$\mathbb{R}$-Proposition 90.3 we clearly have an inequality $\nu(\varphi) \geq n$. In the Signature-Eigenvalues Proposition 90.6 we will see that this inequality is in fact an equality.

(2) It follows easily from the Non-Singularity Lemma 81.1 that a symmetric form (V, φ) over $\mathbb{R}$ has nullity zero if and only if (V, φ) is non-singular. □

Definition. Let R be a subring of $\mathbb{R}$ and let (V,φ) be a symmetric form over $\mathbb{R}$.
(1) We say that φ is positive-definite if $\theta(v,v) > 0$ for all $v \neq 0$.
(2) We say that φ is negative-definite if $\theta(v,v) < 0$ for all $v \neq 0$.
(3) We say that φ is indefinite if φ is neither positive-definite nor negative-definite.

Examples. Let R be a subring of $\mathbb{R}$.
(1) The form $\theta_R(+1)$ is positive-definite and the form $\theta_R(-1)$ is negative-definite.
(2) We consider the matrix $H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The corresponding form $\theta_R(H)$ is indefinite, indeed, we have

$$\theta_R(H)\left(\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}\right) = (1\ \ 1)\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = 2$$

and

$$\theta_R(H)\left(\begin{pmatrix}1\\-1\end{pmatrix},\begin{pmatrix}1\\-1\end{pmatrix}\right) = (1\ \ -1)\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix} = -2.$$

□

Notation. Let R be a commutative ring. Given a form $(V,\varphi\colon V\times V\to R)$ and a free R-submodule U of V we denote by $\varphi|_U$ the form that is given by restricting the map $\varphi\colon V\times V\to R$ to $U\times U\to R$.

Definition. Let (V,φ) be a symmetric form over $\mathbb{R}$. We define

$b^+(\varphi) :=$ maximal dimension of a vector subspace U of V s. th. $\varphi|_U$ is positive-definite,
$b^-(\varphi) :=$ maximal dimension of a vector subspace U of V s. th. $\varphi|_U$ is negative-definite.

Furthermore we define the signature $\operatorname{sign}(\varphi)$ as

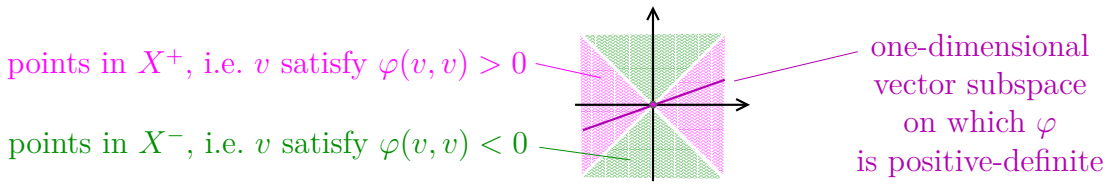
$$\operatorname{sign}(\varphi) := b^+(\varphi) - b^-(\varphi).$$

Example. We consider the symmetric form

$$\begin{array}{rcl}\varphi\colon \mathbb{R}^2\times\mathbb{R}^2 & \to & \mathbb{R} \\ \left(\begin{pmatrix}v_1\\v_2\end{pmatrix},\begin{pmatrix}w_1\\w_2\end{pmatrix}\right) & \mapsto & (v_1\ \ v_2)\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}w_1\\w_2\end{pmatrix}.\end{array}$$

In the figure below we show the subsets

$$X^+ := \{v\in\mathbb{R}^2 \mid \theta(v,v) > 0\}\cup\{0\} \quad \text{and} \quad X^- := \{v\in\mathbb{R}^2 \mid \theta(v,v) < 0\}\cup\{0\}.$$



These subsets X^+ and X^- are evidently *not* vector subspaces of $V = \mathbb{R}^2$. The maximal dimension of a vector subspace contained in X^+ and X^- is in both cases equal to one. Thus we see that $b^+(\varphi) = b^-(\varphi) = 1$. In particular we see that $\operatorname{sign}(\varphi) = 0$. □

In the following two lemmas we summarize very elementary statements about these invariants of symmetric forms over $\mathbb{R}$:

Lemma 90.4. (Isometry Invariants Lemma) Let (V, φ) and (W, ψ) be two symmetric forms over $\mathbb{R}$. If (V, φ) and (W, ψ) are isometric, then

$$\begin{aligned} \dim_{\mathbb{R}}(V) &= \dim_{\mathbb{R}}(W), \\ \nu(\varphi) &= \nu(\psi), \\ b^{\pm}(\varphi) &= b^{\pm}(\psi), \\ \operatorname{sign}(\varphi) &= \operatorname{sign}(\psi). \end{aligned}$$

Proof. The lemma is an immediate consequence of the definitions. ■

Lemma 90.5. (Signature-Sign Lemma) Given any symmetric form (V, φ) over $\mathbb{R}$ we have

$$\begin{aligned} \nu(-\varphi) &= \nu(\varphi), \\ b^{\pm}(-\varphi) &= b^{\mp}(\varphi), \\ \operatorname{sign}(-\varphi) &= -\operatorname{sign}(\varphi). \end{aligned}$$

Proof. This lemma also follows easily from the definitions. ■

In the following proposition we determine the nullity ν, the invariants $b^{\pm}$ and the signature for a symmetric form in terms of the eigenvalues of any matrix representing the form:

Proposition 90.6. (Signature-Eigenvalues Proposition) Let A be a symmetric real matrix. Then

$$\begin{aligned} \nu(\theta_{\mathbb{R}}(A)) &= \text{number of zero eigenvalues of } A \text{, counted with multiplicities,} \\ b^{+}(\theta_{\mathbb{R}}(A)) &= \text{number of positive eigenvalues of } A \text{, counted with multiplicities,} \\ b^{-}(\theta_{\mathbb{R}}(A)) &= \text{number of negative eigenvalues of } A \text{, counted with multiplicities.} \end{aligned}$$

In particular

$$\operatorname{sign}(\theta_{\mathbb{R}}(A)) = \begin{array}{l} \text{number of positive eigenvalues of } A \\ - \text{ number of negative eigenvalues of } A. \end{array}$$

Example. The following calculation will be helpful on several occasions:

$$\operatorname{sign}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \underset{\text{by the Forms-Matrices Lemma 89.1}}{\uparrow} = \operatorname{sign}\left(\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\right) = \operatorname{sign}\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \underset{\text{Signature-Eigenvalues Proposition 90.6}}{\uparrow} = 0.$$

□

Proof. Let A be a symmetric $(r \times r)$-matrix over $\mathbb{R}$. We write

$$\begin{aligned} n &:= \text{number of zero eigenvalues of } A \text{, counted with multiplicities,} \\ k^{+} &:= \text{number of positive eigenvalues of } A \text{, counted with multiplicities,} \\ k^{-} &:= \text{number of negative eigenvalues of } A \text{, counted with multiplicities.} \end{aligned}$$

By the Spectral Theorem [**Axl24**, Theorem 7.13] we know that there exists an orthogonal matrix $P \in \mathrm{O}(r)$ such that $P^{-1} \cdot A \cdot P =: D$ is a diagonal matrix, such that the first n entries of D are zero, the next k_{+} entries of D are positive and such that the remaining k_{-} entries are negative.

Since P is orthogonal we know that $P^{-1} = P^{T}$. Thus it follows from the Form–Matrix-Lemma 89.1 that $\theta(A)$ is isometric to $\theta(D)$. It follows from this observation and the Isometry Invariants Lemma 90.4 that it suffices to prove the proposition for the diagonal matrix D.

We write

$$\begin{aligned} V^0 &:= \{(x_1,\dots,x_n,0,\dots,0) \in \mathbb{R}^r \mid x_1,\dots,x_n \in \mathbb{R}\}, \\ V^+ &:= \{(0,\dots,0,y_1,\dots,y_{k_+},0,\dots,0) \in \mathbb{R}^r \mid y_1,\dots,y_{k_+} \in \mathbb{R}\}, \\ V^- &:= \{(0,\dots,0,z_1,\dots,z_{k_-}) \in \mathbb{R}^r \mid z_1,\dots,z_{k_-} \in \mathbb{R}\}. \end{aligned}$$

We make the following observations:

- It is clear that V^0 is contained in the radical of φ. Hence $\nu(\theta(D)) \geq n$.
- It is clear that $\theta(D)|_{V^+}$ is positive-definite. Hence $b^+(\theta(D)) \geq k_+$.
- It is clear that $\theta(D)|_{V^-}$ is negative-definite. Hence $b^-(\theta(D)) \geq k_-$.

If remains to prove the following claim:

Claim. For any symmetric form (W,ψ) over $\mathbb{R}$ we have $\nu(\psi)+b^+(\psi)+b^-(\psi) \leq \dim_{\mathbb{R}}(W)$.

Proof. We assume that we have the following:

- W^0 is vector subspace of W which is contained in the radical of φ.
- W^+ is a positive-definite vector subspace of W.
- W^- is a negative-definite vector subspace of W.

We need to show that the sum of the dimensions of these subspaces is less or equal than the dimension of W. We make the following observations:

(1) W^+ and W^- have trivial intersection: Indeed, $W^+ \cap W^-$ is a vector subspace that is positive-definite and negative-definite. So for any non-zero vector $v \in W^+ \cap W^-$ we have $\psi(v,v) > 0$ and $\psi(v,v) < 0$. This is only possible if no such v exists, i.e. we have shown that the intersection $W^+ \cap W^- = \{0\}$.

(2) W^0 and $W^+ \oplus W^-$ have trivial intersection: Let $w^+ \in W^+$ and $w^- \in W^-$ such that $w^+ + w^-$ is non-zero. We see that

$$\theta(w^+ + w^-, w^+ - w^-) \underset{\substack{\uparrow \\ \text{since } \varphi \text{ is bilinear} \\ \text{and symmetric}}}{=} \theta(w^+,w^+) - \theta(w^-,w^-) \underset{\substack{\uparrow \\ \text{since } w^+ \in W^+ \text{ and } w^- \in W^- \\ \text{and at least one is non-zero}}}{>} 0.$$

This shows that $w^+ + w^- \notin W^0$.

Now we see that

$$\dim(W^0)+\dim(W^+)+\dim(W^-) \underset{\substack{\uparrow \\ \text{this follows from elementary linear algebra} \\ \text{since } W^- \cap W^+ = \{0\} \text{ and } W^0 \cap (W^+ + W^-) = \{0\}}}{=} \dim(W^0+W^+ + W^-) + \underset{\substack{\uparrow \\ \text{since } W^0+W^+ + W^- \subset W}}{\leq} \dim(W).$$

■

The following theorem (which gives us in particular a converse to the Isometry Invariants Lemma 90.4) says that the dimension, the nullity and the signature are a complete invariant for isometry types of symmetric forms over $\mathbb{R}$.

Theorem 90.7. (Sylvester's Theorem a. k. a. Sylvester's law of inertia)

(1) Let (V,φ) be a symmetric form over $\mathbb{R}$. We have an isometry[708]

$$(V,\varphi) \cong \nu(\varphi)\cdot(0) \oplus b^+(\varphi)\cdot(+1) \oplus b^-(\varphi)\cdot(-1).$$

In particular:

$$\dim_{\mathbb{R}}(V) = \nu(\varphi)+b^+(\varphi)+b^-(\varphi).$$

(2) Let (V,φ) and (W,ψ) be two symmetric forms over $\mathbb{R}$. Then

$$(V,\varphi) \text{ and } (W,\psi) \text{ are isometric} \iff \begin{array}{rcl} \dim_{\mathbb{R}}(V) & = & \dim_{\mathbb{R}}(W), \\ \nu(\varphi) & = & \nu(\psi), \\ \operatorname{sign}(\varphi) & = & \operatorname{sign}(\psi). \end{array}$$

Proof.

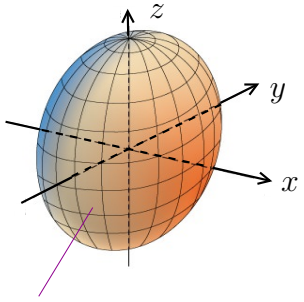
(1) By the Symmetric Forms-over-$\mathbb{R}$-Proposition 90.3 and the Isometry Invariants Lemma 90.4 it suffices to prove the statement for symmetric forms that are represented by diagonal matrices. But for those forms the statement follows immediately from the Signature-Eigenvalues Proposition 90.6.

(2) The "$\Rightarrow$"-direction is precisely the content of the Isometry Invariants Lemma 90.4. It remains to prove the "$\Leftarrow$"-direction. So suppose that we are given two symmetric forms (V,φ) and (W,ψ) over $\mathbb{R}$ with same dimension, the same nullity and the same signature. It follows from our hypothesis together with (1) and the definition of the signature $\operatorname{sign}(\varphi) = b^+(\varphi) - b^-(\varphi)$ that there exist $n, k_+, k_- \in \mathbb{N}_0$ such that (V,φ) and (W,ψ) are both isometric to $n \cdot (0) \oplus k_+ \cdot (+1) \oplus k_- \cdot (-1)$. But this implies that (V,φ) and (W,ψ) are themselves isometric. ■

Example. Given a symmetric form φ on $\mathbb{R}^3$ one can visualize it by considering the "unit-sphere"

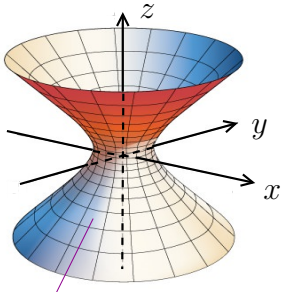
$$S(\varphi) \;:=\; \{v \in \mathbb{R}^3 \,|\, \theta(v,v) = 1\}.$$

(a)



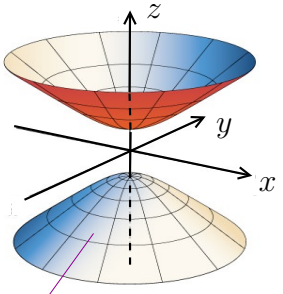
$x^2 + y^2 + z^2 = 1$

(b)



$x^2 + y^2 - z^2 = 1$

(c)



$-x^2 - y^2 + z^2 = 1$

It follows from Sylvester's Theorem 90.7, the Isometry Invariants Lemma 90.4 and the Signature-Eigenvalues Proposition 90.6 that every non-singular symmetric form on $\mathbb{R}^3$ is isometric to precisely one of the following four forms:

(a) $3 \cdot (1)$, (b) $2 \cdot (1) \oplus (-1)$
(c) $2 \cdot (-1) \oplus (1)$ (d) $3 \cdot (-1)$.

In the last case the unit-sphere is the empty set. In the other cases the unit-spheres are sketched in the figure above. □

We conclude this subsection with the following proposition, which says that the signature is additive under the direct sum operation:

[708] If (V,φ) is non-singular, then we have $\nu(\varphi) = 0$. Thus we obtain that $b^+(\varphi) + b^-(\varphi) = \dim_{\mathbb{R}}(V)$. Together with $\operatorname{sign}(\varphi) = b^+(\varphi) - b^-(\varphi)$ we see that in this case any two out of $\dim(V)$, $\operatorname{sign}(\varphi)$, $b^+(\varphi)$ and $b^-(\varphi)$ determine the other invariants.

Proposition 90.8. (Signature Additivity Proposition) If (V, φ) and (W, ψ) are two symmetric forms over $\mathbb{R}$, then

$$\begin{array}{rcl} \nu(\varphi \oplus \psi) & = & \nu(\varphi) + \nu(\psi), \\ b^{\pm}(\varphi \oplus \psi) & = & b^{\pm}(\varphi) + b^{\pm}(\psi), \\ \operatorname{sign}(\varphi \oplus \psi) & = & \operatorname{sign}(\varphi) + \operatorname{sign}(\psi). \end{array}$$

Proof. This statement follows immediately from the Symmetric Forms-over-$\mathbb{R}$-Proposition 90.3 applied to φ and ψ, the Matrix-Form Operations Lemma 89.2 (2) and the Signature-Eigenvalues Proposition 90.6. ■

We conclude this section with the following definition, which shows how we can associate to a symmetric from over $\mathbb{Z}$ a symmetric form over $\mathbb{R}$:

Definition. Let (V, φ) be a symmetric form over $\mathbb{Z}$. It follows easily from the Tensor Product–Universal Property Lemma 51.1 and the Tensor-with-Ring Lemma 51.7 that the following map is well-defined:

$$\begin{array}{rcl} \varphi \otimes \mathbb{R} \colon V \otimes \mathbb{R} \times V \otimes \mathbb{R} & \to & \mathbb{R} \\ (v \otimes r, w \otimes s) & \mapsto & r \cdot \theta(v, w) \cdot s, \end{array}$$

and that this map defines a symmetric form over $\mathbb{R}$. This allows us to define

$$\operatorname{sign}(\varphi) \coloneqq \operatorname{sign}(\varphi \otimes \mathbb{R}).$$

90.2.2. **Symmetric forms over $\mathbb{R}$: Topology.** Now it is time to apply our recently acquired knowledge to topology:

Definition. Let $k \geq 1$ and let M be a compact oriented $4k$-dimensional topological manifold. We define the signature of M to be[709]

$$\operatorname{sign}(M) \coloneqq \operatorname{sign}(Q_M).$$

↑ on page ?? we introduced the intersection from $Q_M \colon \mathrm{FH}_{2k}(M) \times \mathrm{FH}_{2k}(M) \to \mathbb{Z}$

Remark. The signature of a manifold was first introduced in 1923 by Hermann Weyl [**Wey23**, p. 432] in a little known paper written in Spanish. As is explained in [**Eck06**], Hermann Weyl deliberately chose this obscurity since at this point, according to Hermann Weyl, algebraic topology was not considered "a respectable field of mathematics". □

Examples. Let $k \in \mathbb{N}$.

(a) Note that

$$\operatorname{sign}(\mathbb{C}\mathrm{P}^{2k}) \underset{\uparrow}{=} \operatorname{sign}(1) = 1.$$

↑ follows from the Cup Product-$\mathbb{C}\mathrm{P}^n$-Proposition 88.12

(b) Note that

$$\operatorname{sign}(S^{2k} \times S^{2k}) \underset{\uparrow}{=} \operatorname{sign} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \underset{\uparrow}{=} 0.$$

↑ see page 1170 ↑ see page 1185

□

[709] Recall that by the Intersection Form Proposition 89.3 we know that the intersection form Q_M is indeed symmetric, so it makes sense to consider its signature.

Lemma 90.9. (Manifold-Signature Lemma) Let $k \in \mathbb{N}$ and let M and N be two compact oriented $4k$-dimensional topological manifolds.

(1)
- If there exists an orientation-preserving homeomorphism $M \to N$, then we have $\operatorname{sign}(M) = \operatorname{sign}(N)$.
- If there exists an orientation-reversing homeomorphism $M \to N$, then we have $\operatorname{sign}(M) = -\operatorname{sign}(N)$.

(2) If there exists a homotopy equivalence $f\colon M \to N$ that restricts to a homotopy equivalence $f|_{\partial M}\colon \partial M \to \partial N$, then we have the equality $\operatorname{sign}(M) = \pm\operatorname{sign}(N)$.

Proof. These two statements are an immediate consequence of the Intersection Form–Homeomorphism Lemma 89.4, the Isometry Invariants Lemma 90.4 and the Signature-Sign Lemma 90.5. ■

Lemma 90.10. (Manifold Signature Additivity Lemma) The signature is additive under the connected sum operation and under disjoint union.

Proof. It is elementary to see that the signature is additive under disjoint union. If we are given compact oriented topological manifolds, then it follows immediately from the straightforward generalization of the Intersection Form-Additivity Proposition 89.7 to compact topological manifolds (with possibly non-empty boundary) and the Signature Additivity Proposition 90.8 that the signature is additive under the connected sum operation. ■

Example. Note that

$$\begin{array}{ccccccccccc}
\operatorname{sign}(\mathbb{C}P^2\#\mathbb{C}P^2) & = & \operatorname{sign}(\mathbb{C}P^2)+\operatorname{sign}(\mathbb{C}P^2) & = & \operatorname{sign}(1)+\operatorname{sign}(1) & = & 1+1 & = & 2 \\
\uparrow & & \uparrow & & & & & & \\
\text{by Manifold Signature Additivity Lemma 90.10} & & \text{see page 1188} & & & & & & \\
\text{and the Manifold-Signature Lemma 90.9 (1)} & & & & & & & & \\
\downarrow & & \downarrow & & & & & & \\
\operatorname{sign}(\mathbb{C}P^2\#\overline{\mathbb{C}P}^2) & = & \operatorname{sign}(\mathbb{C}P^2)-\operatorname{sign}(\mathbb{C}P^2) & = & \operatorname{sign}(1)-\operatorname{sign}(1) & = & 1-1 & = & 0.
\end{array}$$ □

Proposition 90.11. ($\mathbb{C}P^2\#\mathbb{C}P^2$-Homotopy Type Proposition) The topological space $\mathbb{C}P^2\#\mathbb{C}P^2$ is neither homotopy equivalent (in particular it is not homeomorphic and certainly not diffeomorphic) to $\mathbb{C}P^2\#\overline{\mathbb{C}P}^2$ nor to $S^2 \times S^2$.

Proof. In the above examples we just saw that $\operatorname{sign}(\mathbb{C}P^2\#\mathbb{C}P^2) = 2$ and we also showed that $\operatorname{sign}(\mathbb{C}P^2\#\overline{\mathbb{C}P}^2) = \operatorname{sign}(S^2 \times S^2) = 0$. It follows from the Intersection Form–Homeomorphism Lemma 89.4 and the Manifold-Signature Lemma 90.9 (2) that $\mathbb{C}P^2\#\mathbb{C}P^2$ is neither homotopy equivalent to $\mathbb{C}P^2\#\overline{\mathbb{C}P}^2$ nor to $S^2 \times S^2$. ■

The following corollary shows in particular that in the definition of the connected sum of two orientable (topological) smooth manifolds on pages **??** and **??** we needed to work with *oriented* (topological) smooth manifolds.

Proposition 90.12. (Connected Sum–Orientation Dependence Proposition) There exist closed oriented smooth manifolds M and N such that $M\#N$ is not homotopy equivalent (let alone homeomorphic or diffeomorphic) to $M\#(-N)$.

Proof. We just saw in the $\mathbb{C}P^2\#\mathbb{C}P^2$-Homotopy Type Proposition 90.11 that the closed oriented smooth manifolds $M = N = \mathbb{C}P^2$ have this property. ■

Unfortunately we are not yet able to distinguish the two smooth manifolds $S^2 \times S^2$ and $\mathbb{C}P^2\#\overline{\mathbb{C}P}^2$. In fact Sylvester's Theorem 90.7 might suggest that we are again at a dead

end. But this is not the case since Sylvester's Theorem 90.7 is a statement about symmetric forms over $\mathbb{R}$, but we are really dealing with symmetric forms defined over $\mathbb{Z}$. We will exploit this extra piece of information in Section 90.4.

90.3. **Interlude: Sphere-packing.** Before we turn to the study of symmetric forms over $\mathbb{Z}$ we take a short break, and we consider sphere packings. As we will see, the relationship with our actual goal of dealing with symmetric forms is arguably tenuous, but nonetheless fascinating.

Definition. Let $v_1, \dots, v_n$ be linearly independent vectors in $\mathbb{R}^n$. We refer to the group

$$\mathbb{Z} \cdot v_1 + \cdots + \mathbb{Z} \cdot v_n \;=\; \left\{ \sum_{i=1}^{n} a_i \cdot v_i \,\middle|\, a_1, \dots, a_n \in \mathbb{Z} \right\}$$

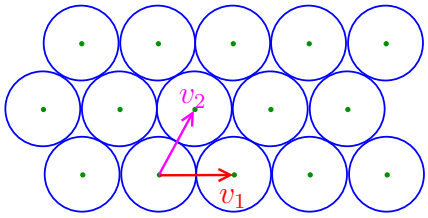
as the lattice spanned by $v_1, \dots, v_n$.[710]

We start out with the following theorem:

Theorem 90.13. (Densest Packing Theorem)

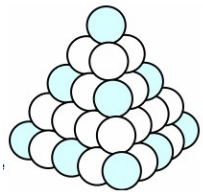
(1) A densest packing[711] of disks of radius 1 in $\mathbb{R}^2$ is given by setting the centers on the "hexagonal" lattice spanned by $v_1 = (2, 0)$ and $v_2 = (1, \sqrt{3})$ shown in the figure below on the left. The density equals $\frac{\pi}{\sqrt{12}}$.

(2) A densest packing of balls of the same radius in $\mathbb{R}^3$ is given by the "face-centered cubic packing" shown in the figure below in the middle and on the right. The density equals $\frac{\pi}{\sqrt{18}}$.



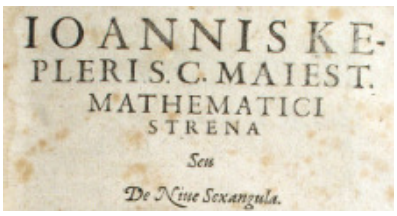
densest disk packing

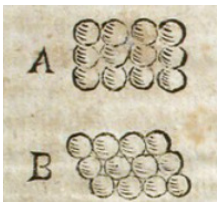


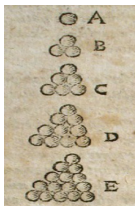


ball packings of highest density

Proof. Statement (1) was proved by L. Fejes Tóth [**Tót43, Tót72**] in 1943. It is much harder to prove Statement (2). Statement (2) had been conjectured by Johannes Kepler [**Kep11**] in 1611 and was known as the Kepler Problem.







It was finally proved by Thomas Hales [**Hal00, Hal05, ea17**] in 2005. ■

For a mathematician it is natural to wonder what are the highest densities of packings in higher dimensions. Not surprisingly this is a very hard problem. Astonishingly the next

[710] This is perhaps not the standard definition of a lattice, but who cares?

[711] The fact that strictly speaking we did not define this term will not bother us.

case that was dealt with is the 8-dimensional setting. More precisely, the following theorem was proved by Maryna Viazovska [**Via17**] in 2016:[712] [713]

Theorem 90.14. (E_8-Densest Packing Theorem) We consider the following matrix:

$$E_8 \;:=\; \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \end{pmatrix}.$$

A densest packing of balls of radius 1 in $\mathbb{R}^8$ is given by the lattice spanned by the eight column vectors of the E_8–matrix. Furthermore, this packing has density $\frac{\pi^4}{384}$.

Proof. The original proof can be found in [**Via17**]. An exposition of the proof is also given in [**Coh23**]. ■

The E_8-matrix from the E_8-Densest Packing Theorem 90.14 looks interesting. Let us explore its properties:

Lemma 90.15. (E_8-Lemma) The form

$$\theta_{\mathbb{Z}}(E_8)\colon \mathbb{Z}^8 \times \mathbb{Z}^8 \;\to\; \mathbb{Z}$$

that is defined by the E_8-matrix is non-singular and positive-definite.

Proof. As in the proof of the Symmetric Forms-over-$\mathbb{R}$-Proposition 90.3 on page 1182 we perform simultaneous row and column operations of type (iii) (which correspond to matrices in $\mathrm{GL}(8,\mathbb{R})$ of determinant $+1$) that we introduced in the proof of the Anti-Symmetric Form Proposition 90.1. This leads us to the following sequence of matrices:[714]

$$\underbrace{\begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \end{pmatrix}}_{=E_8} \rightsquigarrow \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \end{pmatrix} \rightsquigarrow \cdots \rightsquigarrow \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{4}{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{5}{4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{7}{10} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{4}{7} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

This shows that there exists a matrix $P \in \mathrm{GL}(8,\mathbb{R})$ *of determinant* $+1$, such that $P^T \cdot E_8 \cdot P$ is a diagonal matrix where the diagonal entries are all positive and such that the product of the diagonal entries is $+1$. It follows from the Signature-Eigenvalues Proposition 90.6, the

[712] In 2022 Maryna Viazovska was awarded a Fields medal for this work.

[713] Shortly afterwards the 24-dimensional case was settled in [**CKM+17**], in this case a densest packing is given by the "Leech lattice".

[714] The missing inbetween matrices are provided in [**Sco05**, p. 126].

Isometry Invariants Lemma 90.4 and the Form–Matrix-Lemma 89.1 that the form $\theta_{\mathbb{Z}}(E_8)$ is non-singular and positive-definite.[715] ■

OK, that was fun, but now let us return to our actual problem, and let us see whether this section was of any use.

90.4. **Non-singular symmetric forms over $\mathbb{Z}$.** In this section we will turn to the study of non-singular symmetric forms over $\mathbb{Z}$. As we will see, these forms contain much more information than their real cousins.

Examples. We list some symmetric integral matrices with determinant equal to ± 1:

(1) The 1×1-matrices $(+1)$ and (-1).
(2) By the E_8-Lemma 90.15 we know that the slightly mysterious 8×8–matrix E_8 from the E_8-Densest Packing Theorem 90.14 has determinant one and that the corresponding form is positive-definite.
(3) The matrix
$$H := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
is symmetric and it has determinant -1.

By the Form–Matrix-Lemma 89.1 (2) these matrices define in particular non-singular forms over $\mathbb{R}$ and also over $\mathbb{Z}$.

In the following example we will see that forms over $\mathbb{Z}$ are more subtle than forms over $\mathbb{R}$:

Example. We consider the matrices
$$H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and } D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$
It follows from Sylvester's Theorem 90.7 that the matrices give rise to *isometric forms over* $\mathbb{R}$. The question arises, whether the forms on $\mathbb{Z}^2$ defined by these matrices are isometric. To get an idea let us consider what we can say about the "squares $\theta(v,v)$" defined by these forms. Note that given any $v = \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{Z}^2$ we have

$$\theta(H)(v,v) = (a\ b)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 2ab \quad \text{and} \quad \theta(H)(v,v) = (a\ b)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = a^2 - b^2.$$
We see that for the form to the left we only get even integers whereas to the right we obtain odd and even integers. This already shows that the two forms are not isometric over $\mathbb{Z}$. □

This encouraging discussion leads us to the following definition:

Definition. A symmetric form (V, φ) over $\mathbb{Z}$ is called even if
$$\theta(v,v) \equiv 0 \mod 2 \quad \text{for all } v \in V,$$
otherwise the i form is called odd. We refer to the property of φ being even or odd as the parity of the form.

The following lemma summarizes some basic facts about the parity of symmetric forms:

Lemma 90.16. (Parity Lemma)
(1) Isometric symmetric forms have the same parity.
(2) For a symmetric form (V, φ) the parities of (V, φ) and $(V, -\varphi)$ agree.

[715] A more conceptual proof of the lemma is given in [**HGK10**, Theorem 1.16.14], [**EH18**, p. 194] and [**Bar15**, p. 140].

(3) The direct sum of an odd form with any other symmetric form is also odd.
(4) If A is a symmetric square matrix over $\mathbb{Z}$, then

$$\theta(A) \text{ is even} \iff \text{all diagonal entries of } A \text{ are even.}$$

Example. It follows from the Parity Lemma 90.16 (4) and earlier results that we have the following table:

Form	rank	signature	parity
(± 1)	1	± 1	odd
$H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	2	0	even
E_8	8	8	even

□

Proof. The only statement which requires a little thought is the statement that if $A = (a_{ij})$ is a symmetric $(n \times n)$–matrix with even diagonal entries, then the form $\theta(A)$ is even. Thus let $v = (v_1, \dots, v_n) \in \mathbb{Z}^n$. We calculate that

$$\theta(A)(v,v) \;=\; v^T A v \;=\; \sum_{i=1}^{n}\sum_{j=1}^{n} v_i \cdot v_j \cdot a_{ij} \;\underset{\text{since } A \text{ is symmetric}}{\uparrow}\; = \; 2\sum_{i=1}^{n}\sum_{j=1}^{i-1} v_i \cdot v_j \cdot a_{ij} + \sum_{i=1}^{n} v_i^2 \cdot a_{ii} \;\underset{\text{since the } a_{ii} \text{ are even}}{\uparrow}\; \equiv 0 \bmod 2.$$

■

The above example now lets us distinguish the remaining near identical twins from our original triplet:

Proposition 90.17. ($\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$-$S^2 \times S^2$-Proposition) The topological space $S^2 \times S^2$ is not homotopy equivalent (in particular it is not homeomorphic) to $\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$.

Proof. In the above example we just showed that the forms

$$H \;=\; \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and } D \;=\; \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are not isometric. The same argument also shows that H is not isometric to $-D$. Recall that it follows from the calculation of the intersection forms on page **??** that H and D are precisely the intersection forms of $S^2 \times S^2$ and $\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$. Thus it it follows from the Intersection Form–Homeomorphism Lemma 89.4 that $S^2 \times S^2$ and $\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$ are not homotopy equivalent. ■

90.4.1. **Indefinite forms over $\mathbb{Z}$.** The following is the key theorem in the classification of non-singular *indefinite* symmetric forms over $\mathbb{Z}$:

Theorem 90.18. (Indefinite Forms Proposition) If (V_1, θ_1) and (V_2, θ_2) are two non-singular *indefinite* symmetric forms over $\mathbb{Z}$, then the following holds:

$$\theta_1 \text{ and } \theta_2 \text{ are isometric} \iff \theta_1 \text{ and } \theta_2 \text{ have the same rank, signature and parity.}$$

Proof. The proof of the theorem is provided in [**MH73**, Theorem II.5.3] (see also [**GS99**, Theorem 1.2.14] for an outline of the proof). The proof relies crucially on a significant input from algebraic number theory, namely the Hasse-Minkowski theorem: This theorem says that given $a_1, \dots, a_n \in \mathbb{Q} \setminus \{0\}$ the equation

$$x_1^2 \cdot a_1 + \cdots + x_n^2 \cdot a_n \;=\; 0$$

has a non-trivial integral solution if and only if it has a non-trivial real solution and if for each prime p the equation has also has a non-trivial solution over the p-adic numbers $\mathbb{Q}_p$. A

proof of the Hasse-Minkowski theorem is given in [**Ser73**, Chapter IV Theorem 8], [**BS66**, Section 7.1] or [**Neu99**, p. 385]. ■

Example. Using the parity we had just seen that the intersection forms of the closed oriented 4-dimensional smooth manifolds $M = S^2 \times S^2$ and $N = \mathbb{C}\mathrm{P}^2 \# \overline{\mathbb{C}\mathrm{P}}^2$ are not isometric, even up to sign. Now we consider the closed oriented 4-dimensional smooth manifolds that we obtain from M and N by performing the connected sum with $\mathbb{C}\mathrm{P}^2$. In other words, we are considering $M \# \mathbb{C}\mathrm{P}^2 = (S^2 \times S^2) \# \mathbb{C}\mathrm{P}^2$ and $N \# \mathbb{C}\mathrm{P}^2 = \mathbb{C}\mathrm{P}^2 \# \overline{\mathbb{C}\mathrm{P}}^2 \# \mathbb{C}\mathrm{P}^2$. It follows from the Intersection Form-Additivity Proposition 89.7 that the intersection forms of these two smooth manifolds are represented by the matrices

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

These two forms are isometric. This can be seen as follows:

- Either through a direct calculation.
- One can notice that both forms are non-singular, of signature $+1$ and odd. Thus it follows from the Indefinite Forms Theorem 90.18 that the forms are isometric.

Note that this discussion shows that we cannot distinguish $M \# \mathbb{C}\mathrm{P}^2 = (S^2 \times S^2) \# \mathbb{C}\mathrm{P}^2$ and $N \# \mathbb{C}\mathrm{P}^2 = \mathbb{C}\mathrm{P}^2 \# \overline{\mathbb{C}\mathrm{P}}^2 \# \mathbb{C}\mathrm{P}^2$ using their intersection forms. In fact, it turns out that there is a good reason why we have a hard time distinguishing them: As is shown in [**GS99**, p. 151] the two smooth manifolds $(S^2 \times S^2) \# \mathbb{C}\mathrm{P}^2$ and $\mathbb{C}\mathrm{P}^2 \# \overline{\mathbb{C}\mathrm{P}}^2 \# \mathbb{C}\mathrm{P}^2$ are in fact diffeomorphic, in fact with good eyes one can write down an explicit diffeomorphism. □

As an immediate corollary to the Indefinite Forms Theorem 90.18 we obtain the classification of non-singular *indefinite odd* forms over $\mathbb{Z}$. More precisely, we have the following corollary:

Corollary 90.19. (Indefinite Odd Forms Corollary) If φ is a non-singular *indefinite odd* symmetric form over $\mathbb{Z}$, then

$$\varphi \cong b^+(\varphi) \cdot (1) \oplus b^-(\varphi) \cdot (-1).$$

Proof. Let φ be a non-singular indefinite odd symmetric form. It follows from the Signature-Eigenvalues Proposition 90.6 that the rank, signature and parity of the two non-singular indefinite odd forms in the statement of the corollary agree. So these two forms are isometric by the Indefinite Forms Theorem 90.18. ■

Now we turn to the study of *even* forms over $\mathbb{Z}$. We have the following, perhaps somewhat surprising proposition:

Proposition 90.20. (Even Forms Proposition) Let φ be a non-singular symmetric form over $\mathbb{Z}$. If φ is even, then $\mathrm{sign}(\varphi) \equiv 0 \bmod 8$.

Example. As a reality check, the two non-singular even symmetric forms that we are acquainted with are $H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and E_8. The signatures are 0 and 8. These calculations are consistent with the Even Forms Proposition 90.20. □

Proof. Given a non-singular symmetric form (W, ψ) over $\mathbb{Z}$ we say that an element $v \in W$ is *characteristic* if for all $w \in W$ we have

$$\psi(v, w) \equiv \psi(w, w) \mod 2.$$

We make the following observations:

(1) It follows immediately from the definitions that (W, ψ) is even if and only if 0 is a characteristic element.
(2) If $\psi = k \cdot (1) \oplus l \cdot (-1)$, then it follows from a straightforward calculation, as in the proof of the Parity Lemma 90.16, that an element $u \in \mathbb{Z}^{k+l}$ is characteristic if and only if all its entries are odd.[716]
(3) If v is a characteristic vector for φ and v' is a characteristic element for φ', then (v, v') is a characteristic element for $\varphi \oplus \varphi'$.
(4) Isometries preserve characteristic vectors.

It follows from Observation (1) that the following claim implies in particular the desired statement of the Even Forms Proposition 90.20:

Claim. Let (V, φ) be a non-singular symmetric form over $\mathbb{Z}$. If $v \in V$ is a characteristic element, then
$$\theta(v, v) \equiv \operatorname{sign}(\varphi) \mod 8.$$

Proof. Let $v \in V$ be a characteristic element of (V, φ). It follows easily from Observation (3) that $(v, 1, 1)$ is a characteristic element of the form $(V \oplus \mathbb{Z} \oplus \mathbb{Z}, \varphi \oplus (1) \oplus (-1))$. Note that, regardless of φ, the form $\varphi \oplus (1) \oplus (-1)$ is indefinite and odd. Thus it follows from the Indefinite Odd Forms Corollary 90.19 together with the Signature Additivity Proposition 90.8 that there exists an isometry[717]
$$\Phi\colon \varphi \oplus (1) \oplus (-1) \xrightarrow{\cong} (b^+(\varphi) + 1) \cdot (1) \oplus (b^-(\varphi) + 1) \cdot (-1).$$
We write $u \coloneqq \Phi(v, 1, 1)$. Since Φ is an isometry we obtain from Observation (4) that u is a characteristic element for the form on the right-hand side. The above Observation (2) tells us that each entry of u is an odd integer. Now we calculate that

$$\begin{aligned}
\theta(v, v) &\overset{\text{straightforward calculation}}{=} (\varphi \oplus (1) \oplus (-1))((v, 1, 1), (v, 1, 1)) \overset{\text{since } \Phi \text{ is an isometry}}{=} \\
&= ((b^+(\varphi) + 1) \cdot (1) \oplus (b^-(\varphi) + 1) \cdot (-1))(u, u) \\
&= \sum_{i=1}^{b^+(\varphi)+1} (\underbrace{i\text{-th entry of } u}_{\text{odd integer}})^2 + \sum_{i=b^+(\varphi)+2}^{b^+(\varphi)+b^-(\varphi)+2} (-1) \cdot (\underbrace{i\text{-th entry of } u}_{\text{odd integer}})^2 \\
&\equiv \sum_{i=1}^{b^+(\varphi)+1} 1 + \sum_{i=b^+(\varphi)+2}^{b^+(\varphi)+b^-(\varphi)+2} (-1) = (b^+(\varphi) + 1) - (b^-(\varphi) + 1) = \operatorname{sign}(\varphi) \mod 8.
\end{aligned}$$

this equality holds modulo 8, since for each odd number k we have $k^2 \equiv 1 \mod 8$[718]

■

Now we can provide the classification of non-singular *indefinite even* symmetric forms.

Corollary 90.21. (Indefinite Even Forms Corollary) If φ is a non-singular *indefinite even* symmetric form over $\mathbb{Z}$, then
$$\varphi \cong \frac{\operatorname{sign}(\varphi)}{8} \cdot E_8 \oplus \frac{\operatorname{rank}(\varphi) - |\operatorname{sign}(\varphi)|}{2} \cdot H.$$

[716] The key observation is that for $v \in \mathbb{Z}$ we have $v \cdot w \equiv \pm w \cdot w \mod 2$ for all $w \in \mathbb{Z}$.
[717] Note that both sides are non-singular indefinite odd forms, regardless of what non-singular symmetric form φ we actually consider.
[718] Why is it true that for each odd number k we have $k^2 \equiv 1 \mod 8$?

Proof. First let φ be a non-singular even symmetric form over $\mathbb{Z}$. Note that it follows from the Even Forms Proposition 90.20 that $\operatorname{sign}(\varphi)$ is indeed divisible by 8. Furthermore it follows from the equalities

$$\begin{aligned} \operatorname{rank}(\varphi) &= b^+(\varphi) + b^-(\varphi) && \text{(from Sylvester's Theorem 90.7 (1),} \\ \operatorname{sign}(\varphi) &= b^+(\varphi) - b^-(\varphi) && \text{(from the definition)} \end{aligned}$$

that $\operatorname{rank}(\varphi) - |\operatorname{sign}(\varphi)|$ is divisible by 2. So it makes sense to write down the right-hand side.

Next we recall that the two non-singular symmetric forms E_8 and H are even, that by the E_8-Lemma 90.15 we know that $\operatorname{rank}(E_8) = \operatorname{sign}(E_8) = 8$ and that $\operatorname{rank}(H) = 2$ and $\operatorname{sign}(H) = 0$. Now assume that φ is furthermore indefinite. It follows from the above observation regarding E_8 and H together with the Signature Additivity Proposition 90.8 and the Parity Lemma 90.16 (3) that the rank, signature and parity of the two non-singular indefinite even forms in the statement of the corollary agree. So these two forms are isometric by the Indefinite Forms Theorem 90.18. ■

90.4.2. **Definite forms over $\mathbb{Z}$.** Now we turn to the study of non-singular *definite* symmetric forms over $\mathbb{Z}$. We start out with some more examples of such forms:

Lemma 90.22. (Γ-Forms Lemma) We denote by $e_1, \dots, e_{4m}$ the standard basis for $\mathbb{Z}^{4m}$. We denote by Γ_{4m} the submodule of $\mathbb{R}^{4m}$ spanned by the vectors $\{e_i + e_j\}_{i,j=1,\dots,4m}$ and $\frac{1}{2}(e_1 + \dots + e_{4m})$.

(1) Γ_{4m} is a free abelian group of rank $4m$.
(2) The restriction of the usual form $\langle\ ,\ \rangle$ on $\mathbb{Z}^{4m}$ to Γ_{4m} is a non-singular positive-definite symmetric form.
(3) The form Γ_{4m} is odd if and only if m is odd.
(4) The form Γ_8 is isometric to E_8.
(5) The form Γ_{16} is not isometric to $E_8 \oplus E_8$.

Proof. The lemma is proved in [**MH73**, Lemmas II.6.1 and 6.2] and [**Ser73**, p. 51]. ■

Example. It turns out that there are many more *definite* symmetric forms than indefinite symmetric forms and that there is no known classification of such forms. For example, by [**MH73**, p. 28] and [**Ser73**, p. 55] we have the following table:[719]

dimension	8	16	24	32	40
number of isometry types of non-singular positive-definite *even* symmetric forms	1	2	24	$\geq 10^7$	$\geq 10^{51}$

Here in dimension 8, up to isometry, the unique example is $\Gamma_8 = E_8$ whereas in dimension 16 the unique examples are $\Gamma_8 \oplus \Gamma_8 = E_8 \oplus E_8$ and Γ_{16}. □

Finally we turn to non-singular positive-definite odd symmetric forms:

Example. A form $\varphi\colon V \times V \to \mathbb{Z}$ is called *standard* if it can be represented by the identity matrix. In Exercise **??** we will show that for every $n \in \mathbb{N}$ the form $E_8 \oplus n \cdot (1)$ is not

[719] For a non-singular positive-definite form over $\mathbb{Z}$ we have $b^+(\varphi) = \operatorname{rank}(\varphi)$ and $b^-(\varphi) = 0$, thus $\operatorname{rank}(\varphi) = \operatorname{sign}(\varphi)$. If φ is even, then it follows from the Even Forms Proposition 90.20 that this rank is divisible by 8.

standard. The non-singular positive-definite odd non-standard form of lowest rank, which does not decompose as a direct sum, is given by the following (12×12)-matrix:

$$D_{12}^{+} \coloneqq \begin{pmatrix} 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix}.$$

We refer to [**CS82**, Table 1], [**CS99**, p. 416], [**Kin03**, Table 2] and

`http://www.math.rwth-aachen.de/~Gabriele.Nebe/LATTICES/D12p.html`

for details and for lists of isometry classes of non-singular positive-definite *odd* symmetric forms of a given rank. □

CHAPTER 91

Signatures: Multiplicativity and additivity

Let $k \geq 1$ and let M be a compact oriented $2k$-dimensional topological manifold. Recall that on page 1168 we introduced the intersection form

$$\begin{aligned} Q_M \colon \mathrm{FH}_k(M) \times \mathrm{FH}_k(M) &\to \mathbb{Z} \\ (a,b) &\mapsto \Big\langle \underbrace{\underbrace{\mathrm{PD}_M(a)}_{\in \mathrm{FH}^k(M,\partial M;\mathbb{Z})} \cup \underbrace{\mathrm{PD}_M(b)}_{\in \mathrm{FH}^k(M,\partial M;\mathbb{Z})}}_{\in \mathrm{FH}^{4k}(M,\partial M;\mathbb{Z})}, [M] \Big\rangle \end{aligned}$$

of M, which is $(-1)^k$-symmetric. In particular, if M is actually $4k$-dimensional, then the intersection form is symmetric. This allowed us on page 1188 to introduce the signature

$$\mathrm{sign}(M) \coloneqq \mathrm{sign}(Q_M \otimes \mathbb{R}) \in \mathbb{Z}$$

of the manifold.

The signature turns out to be one of the most central invariants of compact oriented $4k$-dimensional smooth and topological manifolds. In particular, rather weirdly, it shares many properties with the Euler characteristic. In this chapter we will prove the following two theorems:

(1) In the Signature–Product Theorem 91.2 we will see that (similar to the Product-χ-Proposition 48.9) the signature is multiplicative under products.
(2) In the Novikov Additivity Theorem 91.7 we will see that (similar to the Union-χ-Proposition 48.7) signatures are additive under gluings.

Later we will also add the following theorem to our collection:

(3) In the Signature–Covering Theorem **??** we will see that (similar to the Covering-χ-Proposition 48.10) for *closed* smooth manifolds the signature is multiplicative under finite coverings.

By definition the signature of a compact oriented $4k$-dimensional topological manifold is defined in terms of the intersection form on homology. But as we saw above, if one strips away the Poincaré duality isomorphism, the intersection form is defined via the cup product on cohomology.

For the purpose of this chapter it is convenient to completely bypass homology (despite our emotional attachment to it) and to work with cohomology throughout. This leads us to the following notation:

Notation. Given a compact oriented $2m$-dimensional topological manifold M we consider the form

$$\begin{aligned} I_M \colon \mathrm{H}^m(M,\partial M;\mathbb{R}) \times \mathrm{H}^m(M,\partial M;\mathbb{R}) &\to \mathbb{R} \\ (\varphi,\psi) &\mapsto \langle \varphi \cup \psi, [M] \rangle. \end{aligned}$$

The following lemma shows that this new form I_M is isometric to $Q_M \otimes \mathbb{R}$:

Lemma 91.1. (Intersection Form Reinterpretation Lemma) Let M be a compact oriented $2m$-dimensional topological manifold. The form $Q_M \otimes \mathbb{R}$ is isometric to the form

$$\begin{array}{rcl} I_M \colon \mathrm{H}^m(M, \partial M; \mathbb{R}) \times \mathrm{H}^m(M, \partial M; \mathbb{R}) & \to & \mathbb{R} \\ (\varphi, \psi) & \mapsto & \langle \varphi \cup \psi, [M] \rangle. \end{array}$$

Proof. Let M be a compact oriented $2m$-dimensional topological manifold. We consider the following diagram:

$$\begin{array}{ccc} \mathrm{H}_m(M) \otimes \mathbb{R} \times \mathrm{H}_m(M) \otimes \mathbb{R} & \xrightarrow{Q_M \otimes \mathbb{R}} & \mathbb{R} \\ \cong \big\downarrow \mu \times \mu & & \\ \mathrm{H}_m(M; \mathbb{R}) \times \mathrm{H}_m(M; \mathbb{R}) & & \big\downarrow \mathrm{id} \\ \cong \big\downarrow \mathrm{PD}_M \times \mathrm{PD}_M & & \\ \mathrm{H}^m(M; \mathbb{R}) \times \mathrm{H}^m(M; \mathbb{R}) & \xrightarrow{(\varphi,\psi) \mapsto \langle \varphi \cup \psi, [M] \rangle} & \mathbb{R}. \end{array}$$

We make the following clarifications and observations:

(1) The top left vertical maps are the isomorphisms μ given by the Torsion–Free Coefficients Proposition 53.7.
(2) The bottom left vertical maps are the isomorphisms given by the Poincaré Duality Theorem 79.1
(3) It follows easily from the definition of $Q_M \otimes \mathbb{R}$ and the Cup Product-Coefficients Lemma **??** that the diagram commutes.

It thus follows that the form at the top is isometric to the form at the bottom. ■

91.1. **Multiplicativity of the signature.** Recall that we introduced the signature of a compact oriented $4k$-dimensional topological manifold. We now extend this definition to topological manifolds of other dimensions in the most unimaginative way possible:

Definition. Let X be a compact oriented topological manifold. If $\dim(X)$ is not divisible by 4, then we set $\mathrm{sign}(X) := 0$.

With this definition we can formulate the first of the two main theorems of this chapter:

Theorem 91.2. (Signature–Product Theorem) For any two compact oriented topological manifolds M and N we have

$$\mathrm{sign}(M \times N) \;=\; \mathrm{sign}(M) \cdot \mathrm{sign}(N).$$

Example.

(1) Note that for any $k \in \mathbb{N}$ we have

$$\mathrm{sign}(S^{2k} \times S^{2k}) \underset{\uparrow}{=} \mathrm{sign} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \underset{\uparrow}{=} 0 = 0 \cdot 0 = \mathrm{sign}(S^{2k}) \cdot \mathrm{sign}(S^{2k}).$$

see page 1170 see page 1189

This calculation is consistent with the Signature–Product Theorem 91.2.

(2) Let $g \in \mathbb{N}$. We denote by Σ_g the surface of genus g. We consider the closed oriented 4-dimensional smooth manifolds $M = \Sigma_g \times \Sigma_g$ and $N = 2g \cdot (S^1 \times S^3) \# 4g \cdot \mathbb{C}\mathrm{P}^2$.

Using the $\mathbb{C}P^n$-H_*-Proposition 47.4, the Surface-H_*-Proposition 47.9, the Connected Sum-Invariants Proposition 89.6 and the Künneth Theorem 56.3 one can easily verify that the (co-) homology groups of M and N are all isomorphic. But it follows from the Signature–Product Theorem 91.2 that $\operatorname{sign}(M) = 0$ whereas it follows from the discussion on page 1188 and the Manifold Signature Additivity Lemma 90.10 that $\operatorname{sign}(N) = 4g$. □

Before we can turn to the proof of the Signature–Product Theorem 91.2 we need to upgrade our algebraic understanding of the signature.

Definition. Let R be a commutative ring and let $\varphi\colon V \times V \to R$ be a symmetric form over R.

(1) We say that φ is metabolic if there exists a decomposition $V = V' \oplus V''$ such that $\varphi|_{V'} \equiv 0$ and such that V' is isomorphic to V''.
(2) We say that φ is hyperbolic if there exists a decomposition $V = V' \oplus V''$ such that $\varphi|_{V'} \equiv 0$, $\varphi|_{V''} \equiv 0$ and such that V' is isomorphic to V''.

Example.

(1) Let $\mathbb{F}$ be a field and let $\varphi\colon V \times V \to \mathbb{F}$ be a symmetric form over $\mathbb{F}$. It follows immediately from the definitions that

$$\varphi \text{ is metabolic} \iff \varphi \text{ is represented by a matrix of the form } \begin{pmatrix} 0 & P \\ P^T & Q \end{pmatrix}$$
$$\varphi \text{ is hyperbolic} \iff \varphi \text{ is represented by a matrix of the form } \begin{pmatrix} 0 & P \\ P^T & 0 \end{pmatrix}$$

where P and Q are square matrices.
(2) Clearly every hyperbolic form is metabolic. But the converse does not hold. For example the matrix $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ represents a form over $\mathbb{R}$ which is clearly metabolic. But one can show easily, e.g. using the Vanishing Signature Lemma 91.3 (2), that it is not hyperbolic. □

Lemma 91.3. (Vanishing Signature Lemma) Let $\varphi\colon V \times V \to \mathbb{R}$ be a symmetric form over $\mathbb{R}$.

(1) If φ is metabolic and non-singular, then $\operatorname{sign}(\varphi) = 0$.
(2) If φ is hyperbolic, then $\operatorname{sign}(\varphi) = 0$.

Proof. Let $\varphi\colon V \times V \to \mathbb{R}$ be a symmetric form over $\mathbb{R}$.

(1) We assume that φ is metabolic and non-singular. As we mentioned above, this implies that φ can be a represented by a matrix of the form

$$\begin{pmatrix} 0 & P \\ P^T & Q \end{pmatrix}$$

where P and Q are square matrices and where Q is a symmetric matrix. It follows from the hypothesis that φ is non-singular, together with the Non-Singularity Lemma 81.1,

that $\det(P) \neq 0$. Now we consider the following three transformations

$$\begin{pmatrix} P^{-1} & 0 \\ 0 & \mathrm{id} \end{pmatrix} \begin{pmatrix} 0 & P \\ P^T & Q \end{pmatrix} \begin{pmatrix} (P^T)^{-1} & 0 \\ 0 & \mathrm{id} \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{id} \\ \mathrm{id} & Q \end{pmatrix},$$
$$\begin{pmatrix} \mathrm{id} & 0 \\ -\frac{1}{2}Q & \mathrm{id} \end{pmatrix} \begin{pmatrix} 0 & \mathrm{id} \\ \mathrm{id} & Q \end{pmatrix} \begin{pmatrix} \mathrm{id} & -\frac{1}{2}Q \\ 0 & \mathrm{id} \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{id} \\ \mathrm{id} & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} \mathrm{id} & \mathrm{id} \\ \mathrm{id} & -\mathrm{id} \end{pmatrix} \begin{pmatrix} 0 & \mathrm{id} \\ \mathrm{id} & 0 \end{pmatrix} \begin{pmatrix} \mathrm{id} & \mathrm{id} \\ \mathrm{id} & -\mathrm{id} \end{pmatrix} = \begin{pmatrix} 2\,\mathrm{id} & 0 \\ 0 & -2\,\mathrm{id} \end{pmatrix}.$$

It follows from the Form–Matrix-Lemma 89.1 (1b) that the three matrices to the right all define isometric forms. In particular they are all isometric to φ. Thus it follows from the Isometry Invariants Lemma 90.4 that the signatures agree. Considering the last matrix we obtain from the Signature-Eigenvalues Proposition 90.6 that the signature is zero.

(2) We assume that φ is hyperbolic. As we mentioned above, this implies that φ can be a represented by a matrix of the form

$$\begin{pmatrix} 0 & P \\ P^T & 0 \end{pmatrix}$$

where P is a square matrix. Elementary linear algebraic shows that there exist invertible square matrices A and B such that $A \cdot P \cdot B =: D$ is a diagonal matrix. Now we consider the following two transformations

$$\begin{pmatrix} 0 & B^T \\ A & 0 \end{pmatrix} \begin{pmatrix} 0 & P \\ P^T & 0 \end{pmatrix} \begin{pmatrix} 0 & A^T \\ B & 0 \end{pmatrix} = \begin{pmatrix} 0 & D \\ D & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} \mathrm{id} & \mathrm{id} \\ \mathrm{id} & -\mathrm{id} \end{pmatrix} \begin{pmatrix} 0 & D \\ D & 0 \end{pmatrix} \begin{pmatrix} \mathrm{id} & \mathrm{id} \\ \mathrm{id} & -\mathrm{id} \end{pmatrix} = \begin{pmatrix} 2D & 0 \\ 0 & -2D \end{pmatrix}.$$

It follows from the Form–Matrix-Lemma 89.1 (1b) that the two matrices to the right define isometric forms. Considering the last matrix we obtain from the Signature-Eigenvalues Proposition 90.6 that the signature is zero. ∎

Next we recall the following definition from page 1183:

Definition. Let (V, φ) be a form over a field $\mathbb{F}$. The vector subspace

$$\mathrm{rad}(\varphi) \coloneqq \{v \in V \mid \varphi(v, w) = 0 \text{ for all } w \in V\}$$

is called the radical of the form φ.

Lemma 91.4. (Radical Summand Lemma) Let $\mathbb{F}$ be a field and let $\varphi\colon V \times V \to \mathbb{F}$ be a form. There exists a non-singular form (W, ψ) over $\mathbb{F}$ and an isometry

$$(\mathrm{rad}(\varphi), 0) \oplus (W, \psi) \cong (V, \varphi).$$

Sketch of proof. We write $R \coloneqq \mathrm{rad}(\varphi)$. First note that the map

$$\begin{array}{rcl} \overline{\varphi}\colon V/R \times V/R & \to & \mathbb{F} \\ ([v], [w]) & \mapsto & \varphi(v, w) \end{array}$$

is a well-defined form. One can easily verify that this form is non-singular. Since we are dealing with vector spaces we know that the projection $V \to V/R$ admits a right-inverse

$\Theta\colon V/R \to V$. Next note that, as in the proof of the Splitting Lemma 37.2 (1)⇒(2), one can easily verify that the map

$$\begin{array}{rcl} R \oplus V/R & \to & V \\ (r, w) & \mapsto & r + \Theta(w) \end{array}$$

is an isomorphism. Finally note that it follows easily from the definitions that this map actually defines an isometry $(\mathrm{rad}(\varphi), 0) \oplus (V/R, \overline{\varphi}) \to (V, \varphi)$. ■

Next we introduce the tensor product of two forms over $\mathbb{R}$:

Lemma 91.5. (Form–Tensor Product Lemma) Let $\varphi\colon V \times V \to \mathbb{R}$ be a μ-symmetric form over $\mathbb{R}$ and let $\psi\colon W \times W \to \mathbb{R}$ be a ν-symmetric form over $\mathbb{R}$. There exists a unique $\mathbb{R}$-bilinear map

$$\varphi \otimes \psi\colon V \otimes_{\mathbb{R}} W \times V \otimes_{\mathbb{R}} W \ \to \ \mathbb{R}$$

such that for every $v, v' \in V$ and $w, w' \in W$ we have

$$((v \otimes w), (v' \otimes w')) \ \mapsto \ \varphi(v, v') \cdot \psi(w, w').$$

Furthermore, this map has the following properties:

(1) $\varphi \otimes \psi$ is $\mu \cdot \nu$-symmetric.
(2) If φ and ψ are non-singular, then so is $\varphi \otimes \psi$.
(3) (a) If $\mu = \nu = -1$, then we have $\mathrm{sign}(\varphi \otimes \psi) = 0$.
    (b) If $\mu = \nu = 1$, then we have $\mathrm{sign}(\varphi \otimes \psi) = \mathrm{sign}(\varphi) \cdot \mathrm{sign}(\psi)$.

Sketch of proof. First note that it follows from the analogue of the Tensor Product-Universal Property Lemma 51.1 for tensor products of $\mathbb{R}$-vector spaces that there does exist a unique $\mathbb{R}$-bilinear map

$$\varphi \otimes \psi\colon V \otimes_{\mathbb{R}} W \times V \otimes_{\mathbb{R}} W \ \to \ \mathbb{R}$$

with the given property. We start out with the following observations:

(i) It follows from the Radical Summand Lemma 91.4 together with the Anti-Symmetric Form Proposition 90.1 that every anti-symmetric form over $\mathbb{R}$ is the direct sum of a zero form and of copies of the form that is represented by the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

(ii) It follows from the Symmetric Forms-over-$\mathbb{R}$-Proposition 90.3 that every symmetric form over $\mathbb{R}$ is the direct sum of a zero form and of forms that are represented by (1×1)-matrices (ϵ_i) with $\epsilon_i \in \{\pm 1\}$.

(iii) Let (V_i, φ_i), $i = 1, \dots, k$ be μ-symmetric forms over $\mathbb{R}$ and let (W_i, ψ_i), $i = 1, \dots, l$ be ν-symmetric forms over $\mathbb{R}$ such that we have isometries $(V, \varphi) \cong \bigoplus_{i=1}^{k} (V_i, \varphi_i)$ and $(W, \psi) \cong \bigoplus_{i=1}^{l} (W_i, \psi_i)$. It follows from the real analogue of the Tensor-Direct Sum Lemma 51.6 that we have a natural isomorphism

$$\bigoplus_{i=1}^{k} \bigoplus_{j=1}^{l} V \otimes W_j \ \to \ V \otimes W.$$

Furthermore it follows easily from the definition of the direct sum of forms, see page 1167, that this isomorphism induces an isometry

$$\bigoplus_{i=1}^{k}\bigoplus_{j=1}^{l}\varphi_i\otimes\psi_j \;\to\; \varphi\otimes\psi.$$

(iv) It follows easily from the definitions that we have an isometry $(\epsilon)\otimes(\eta)\cong(\epsilon\cdot\eta)$ and, more interestingly, that we have the following isometry:

$$\begin{array}{c} \\ a_1\\ a_2 \end{array}\!\!\begin{array}{c} \begin{array}{cc} a_1 & a_2 \end{array}\\ \begin{pmatrix} 0 & 1\\ -1 & 0 \end{pmatrix} \end{array} \otimes \begin{array}{c} \\ b_1\\ b_2 \end{array}\!\!\begin{array}{c} \begin{array}{cc} b_1 & b_2 \end{array}\\ \begin{pmatrix} 0 & 1\\ -1 & 0 \end{pmatrix} \end{array} \cong \begin{array}{c} \\ a_1\otimes b_1\\ a_1\otimes b_2\\ a_2\otimes b_1\\ a_2\otimes b_2 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} a_1\otimes b_1 & a_1\otimes b_2 & a_2\otimes b_1 & a_2\otimes b_2 \end{array}\\ \begin{pmatrix} 0 & 0 & 0 & 1\\ 0 & 0 & -1 & 0\\ 0 & -1 & 0 & 0\\ 1 & 0 & 0 & 0 \end{pmatrix} \end{array}.$$

Next note that the tensor product of a zero-form with any other form is clearly again a zero-form. Finally note that one can easily verify that we have an isometry

$$(\epsilon)\otimes\begin{pmatrix} 0 & 1\\ -1 & 0 \end{pmatrix} \cong \begin{pmatrix} 0 & \epsilon\\ -\epsilon & 0 \end{pmatrix}.$$

We turn to the proofs of the statements:

(1) The symmetry statement is essentially clear.

(2) If φ and ψ are non-singular, then it follows easily from (i),(iii),(iv) that $\varphi\otimes\psi$ is also non-singular.

(3) (a) We now assume that $\mu=\nu=-1$. It follows easily from (i)-(iv) together with the Vanishing Signature Lemma 91.3 that $\operatorname{sign}(\varphi\otimes\psi)=0$.

(b) We now assume that $\mu=\nu=1$. It follows easily from (ii)-(iv) and the Signature-Eigenvalues Proposition 90.6 that

$$\begin{aligned} b^+(\varphi\otimes\psi) &= b^+(\varphi)\cdot b^+(\psi)+b^-(\varphi)\cdot b^-(\psi),\\ b^-(\varphi\otimes\psi) &= b^+(\varphi)\cdot b^-(\psi)+b^-(\varphi)\cdot b^+(\psi). \end{aligned}$$

It follows that from this discussion and the definitions that

$$\begin{aligned} \operatorname{sign}(\varphi\otimes\psi) &= b^+(\varphi\otimes\psi)-b^-(\varphi\otimes\psi)\\ &= b^+(\varphi)\cdot b^+(\psi)+b^-(\varphi)\cdot b^-(\psi)-b^+(\varphi)\cdot b^-(\psi)-b^-(\varphi)\cdot b^+(\psi)\\ &= (b^+(\varphi)-b^-(\varphi))\cdot(b^+(\psi)-b^-(\psi)) \;=\; \operatorname{sign}(\varphi)\cdot\operatorname{sign}(\psi). \end{aligned}$$

■

We also need one major topological input in the proof of the Signature–Product Theorem 91.2, namely we will need the following special case of the Cross-Cup-Cap Theorem 76.1:

Theorem 91.6. (Manifold Cross-Cup-Cap Theorem) Let M and N be topological manifolds. We denote by $p\colon M\times N\to M$ and $q\colon M\times N\to N$ the natural projections. For any $\alpha\in\mathrm{H}^k(M,\partial M;\mathbb{Z})$, $\mu\in\mathrm{H}_m(M,\partial M;\mathbb{Z})$, $\beta\in\mathrm{H}^l(N,\partial N;\mathbb{Z})$ and $\nu\in\mathrm{H}_n(N,\partial N;\mathbb{Z})$ we have the equality[720]

$$(p^*(\alpha)\cup q^*(\beta))\cap(\mu\times\nu) \;=\; (-1)^{\deg(\alpha)\cdot\deg(\beta)+\deg(\beta)\cdot\deg(\mu)}\cdot(\alpha\cap\mu)\times(\beta\cap\nu)$$

in $\mathrm{H}_{m+n-k-l}(M\times N;\mathbb{Z})$.

[720]Here "$\times$" denotes the cross product

$$\times\colon \mathrm{H}_r(M;\mathbb{Z})\times\mathrm{H}_s(N;\mathbb{Z}) \;\to\; \mathrm{H}_{r+s}(M\times N;\mathbb{Z})$$

that we introduced on page 712.

Proof. First note that, as we discussed on page 970, it follows from the Topological Manifold Product Proposition 62.6, the Topological Manifold–Boundary Proposition 42.5 and the Excisive Triad-Proposition 50.11 (2) that the triad $(M \times N, \partial M \times N, M \times \partial N)$ is excisive. Therefore the promised equality follows directly from the Cross-Cup-Cap Theorem 76.1. ■

Now we are finally ready to prove the Signature–Product Theorem 91.2:

Proof of the Signature–Product Theorem 91.2. Let M be a compact oriented m-dimensional topological manifold and let N be a compact oriented n-dimensional topological manifold. If $\dim(M) + \dim(N)$ is not divisible by 4, then also at least one of $\dim(M)$, $\dim(N)$ is not divisible by 4. Thus in this case the statement of the theorem is trivial. So we now assume that $m + n = \dim(M) + \dim(N)$ is divisible by 4, i.e. we can assume that there exists a $d \in \mathbb{N}_0$ with $m + n = 4d$. We adopt the following convention:

> Throughout this proof we work with real coefficients. We will generously reinterpret earlier results on $\mathbb{Z}$-coefficients in terms of $\mathbb{R}$-coefficients.

Throughout the proof we will need the following notation:

- Given a compact oriented k-dimensional topological manifold W and given $r, s \in \mathbb{N}_0$ with $r + s = k$ we consider the $\mathbb{R}$-bilinear pairing

$$\begin{aligned} I_W^{r,s}\colon \mathrm{H}^r(W, \partial W) \times \mathrm{H}^s(W, \partial W) &\to \mathbb{R} \\ (\varphi, \psi) &\mapsto \langle \varphi \cup \psi, [W] \rangle \underset{\uparrow}{=} \epsilon_W((\varphi \cup \psi) \cap [W]). \end{aligned}$$

  by the Cap Product-Kronecker Lemma 75.7, here $\epsilon_W\colon \mathrm{H}_0(W; \mathbb{R}) \to \mathbb{R}$ is the augmentation map from page 660

  Note that by the Intersection Form Reinterpretation Lemma 91.1 we know that for any $k \in \mathbb{N}$ and any compact oriented $2k$-dimensional topological manifold W the form $Q_W \otimes \mathbb{R}$ is isometric to the form

$$\begin{aligned} I_W^{k,k}\colon \mathrm{H}^k(W, \partial W) \times \mathrm{H}^k(W, \partial W) &\to \mathbb{R} \\ (\varphi, \psi) &\mapsto \langle \varphi \cup \psi, [W] \rangle = \epsilon_W((\varphi \cup \psi) \cap [W]). \end{aligned}$$

  Thus our goal will be to calculate $\mathrm{sign}(I_{M\times N}^{2d,2d})$.
- We denote by $p\colon M \times N \to M$ and $q\colon M \times N \to N$ the natural projections. Since we work with real coefficients we obtain by the same argument as in the Cohomological Künneth Theorem 77.1[721] and the Universal Coefficient-H*–Field Theorem 71.2 an isomorphism

$$\begin{aligned} \Theta\colon \bigoplus_{k+l=2d} \mathrm{H}^k(M, \partial M) \otimes \mathrm{H}^l(N, \partial N) &\to \mathrm{H}^{2d}(M \times N, \overbrace{(\partial M \times N) \cup (M \times \partial N)}^{=\partial(M\times N)}) \\ \varphi \otimes \psi &\mapsto p^*(\varphi) \cup q^*(\psi). \end{aligned}$$

Claim. Let $\alpha_i \in \mathrm{H}^{k_i}(M, \partial M)$, $i = 1, 2$ and let $\beta_i \in \mathrm{H}^{l_i}(N, \partial N)$, $i = 1, 2$ such that we have the equalities $k_1 + l_1 = 2d$ and $k_2 + l_2 = 2d$. If if[722] $k_1 + k_2 = m$, then

$$I_{M\times N}^{2d,2d}(\Theta(\alpha_1 \otimes \beta_1), \Theta(\alpha_2 \otimes \beta_2)) \;=\; (-1)^{k_2 \cdot l_1} \cdot (I_M^{k_1,k_2} \otimes I_N^{l_1,l_2})(\alpha_1 \otimes \beta_1, \alpha_2 \otimes \beta_2).$$

[721] It follows from the Topological Manifolds-Invariants Proposition 60.10 that the condition from the Algebraic–Chain Complex Condition Lemma 77.3 is satisfied.

Otherwise $$I^{2d,2d}_{M\times N}(\Theta(\alpha_1\otimes\beta_1),\Theta(\alpha_2\otimes\beta_2)) \;=\; 0.$$

Given $k,l\in\mathbb{N}_0$ we write $A_k := \mathrm{H}^k(M,\partial M)$ and $B_l := \mathrm{H}^l(N,\partial N)$. With this notation we can illustrate the statement of the claim for $\dim(M)=\dim(N)=4$ as follows: The map Θ induces an isometry

$$I^{4,4}_{M\times N} \;\cong\; \begin{array}{c|ccccc} & A_0\otimes B_4 & A_1\otimes B_3 & A_2\otimes B_2 & A_3\otimes B_1 & A_4\otimes B_1 \\ A_0\otimes B_4 & 0 & 0 & 0 & 0 & I^{0,4}_M\otimes I^{4,0}_N \\ A_1\otimes B_3 & 0 & 0 & 0 & -I^{1,3}_M\otimes I^{3,1}_N & 0 \\ A_2\otimes B_2 & 0 & 0 & I^{2,2}_M\otimes I^{2,2}_N & 0 & 0 \\ A_3\otimes B_1 & 0 & -I^{3,1}_M\otimes I^{1,3}_N & 0 & 0 & 0 \\ A_4\otimes B_0 & I^{4,0}_M\otimes I^{0,4}_N & 0 & 0 & 0 & 0 \end{array}.$$

Proof. Note that

$$\begin{aligned} I^{2d,2d}_{M\times N}(\Theta(\alpha_1\otimes\beta_1),\Theta(\alpha_2\otimes\beta_2)) &\overset{\text{by the definition of }\Theta\text{ and }I^{2d,2d}_{M\times N}\text{ and since by the Fundamental Class-of-Product Proposition 64.10 we know that }[M\times N]=[M]\times[N]}{=} \\ &= \epsilon_{M\times N}((\underbrace{p^*(\alpha_1)\cup q^*(\beta_1)}_{=\Theta(\alpha_1,\beta_1)}\cup\underbrace{p^*(\alpha_2)\cup q^*(\beta_2)}_{=\Theta(\alpha_2,\beta_2)})\cap[M\times N]) \\ &= (-1)^{k_2\cdot l_1}\cdot\epsilon_{M\times N}((p^*(\underbrace{\alpha_1\cup\alpha_2}_{\in\mathrm{H}^{k_1+k_2}(M,\partial M)})\cup q^*(\underbrace{\beta_1\cup\beta_2}_{\in\mathrm{H}^{l_1+l_2}(N,\partial N)}))\cap([M]\times[N])) \;=\; (*). \\ &\overset{\text{by the up Product-Commutativity Proposition 73.4 together with the Cup Product-Naturality Lemma 73.3}}{\uparrow} \end{aligned}$$

Next note that it follows from the Top Cohomology Theorem 70.16 (2a) that the first cup product $p^*(\alpha_1\cup\alpha_2)\in\mathrm{H}^{k_1+k_2}(M,\partial M)$ is zero if $k_1+k_2>m$ and that the second cup product $q^*(\beta_1\cup\beta_2)\in\mathrm{H}^{l_1+l_2}(N,\partial N)$ is zero if $l_1+l_2>n$. Since $k_1+k_2+l_1+l_2=m+n$ we see that the cup product is zero, unless $k_1+k_2=m$ and $l_1+l_2=n$. Thus we henceforth assume that $k_1+k_2=m$ and that $l_1+l_2=n$. We continue with the above calculation and we see that

$$\begin{aligned} (*) &= (-1)^{k_2\cdot l_1}\cdot\epsilon_{M\times N}((p^*(\alpha_1\cup\alpha_2)\cup q^*(\beta_1\cup\beta_2))\cap([M]\times[N])) \\ &= (-1)^{k_2\cdot l_1}\cdot\epsilon_{M\times N}(((\underbrace{\alpha_1\cup\alpha_2}_{\in\mathrm{H}^m(M,\partial M)})\cap[M])\times((\underbrace{\beta_1\cup\beta_2}_{\in\mathrm{H}^n(N,\partial N)})\cap[N])) \\ &\overset{\text{follows from the Manifold Cross-Cup-Cap Theorem 91.6 (note that the sign in the theorem equals }(-1)^{k_2\cdot l_1+n\cdot m}=1)}{\uparrow} \\ &= (-1)^{k_2\cdot l_1}\cdot\langle\alpha_1\cup\alpha_2,[M]\rangle\cdot\langle\beta_1\cup\beta_2,[N]\rangle \\ &\overset{\text{follows from the definition of }I^{k_1,k_2}_M\text{ and }I^{l_1,l_2}_N\text{, the fact that }\epsilon_{M\times N}(\{P\}\times\{Q\})=\epsilon_M(P)\cdot\epsilon_N(Q)\text{ and the Cap Product-Kronecker Lemma 75.7}}{\uparrow} \\ &= (-1)^{k_2\cdot l_1}\cdot(I^{k_1,k_2}_M\otimes I^{l_1,l_2}_N)(\alpha_1\otimes\beta_1,\alpha_2\otimes\beta_2). \\ &\overset{\text{by the definition of }I_M\otimes I_N\text{ in the Form–Tensor Product Lemma 91.5}}{\uparrow} \end{aligned}$$

⊞

[722] Note that from $k_1+l_1=2d$ and $k_2+l_2=2d$ and $m+n=4d$ it follows that $k_1+k_2=m$ is equivalent to $l_1+l_2=n$.

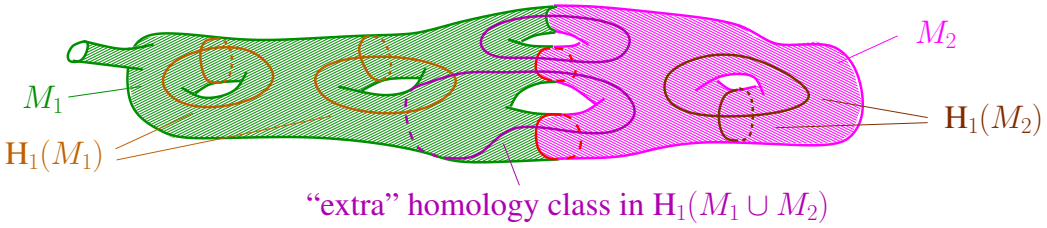
We now see that

$$\begin{array}{l}
\text{sign}(M\times N) \overset{\text{by the Intersection Form Reinterpretation Lemma 91.1}}{=} \text{sign}(I^{2d,2d}_{M\times N}) \overset{\text{follows from the claim together with the Signature Additivity Proposition 90.8}}{=} \\
\quad = \sum_{k=0}^{\lfloor\frac{m-1}{2}\rfloor} \pm \underbrace{\text{sign}\begin{pmatrix} 0 & I^{k,m-k}_M \otimes I^{2d-k,n+k-2d}_N \\ I^{m-k,k}_M \otimes I^{n+k-2d,2d-k}_N & 0 \end{pmatrix}}_{=0 \text{ by the Vanishing Signature Lemma 91.3 (2)}} + \text{sign}(\underbrace{I^{\frac{m}{2},\frac{m}{2}}_M \otimes I^{\frac{n}{2},\frac{n}{2}}_N}_{=0,\text{ unless } m \text{ and } n \text{ are even}}) \\
\quad = \begin{cases} \text{sign}(I^{\frac{m}{2},\frac{m}{2}}_M \otimes I^{\frac{n}{2},\frac{n}{2}}_N), & \text{if } m,n \text{ even}, \\ 0, & \text{otherwise} \end{cases} \\
\quad \overset{\uparrow}{=} \begin{cases} \text{sign}(I^{\frac{m}{2},\frac{m}{2}}_M)\cdot\text{sign}(I^{\frac{n}{2},\frac{n}{2}}_N), & \text{if } m \text{ and } n \text{ are divisible by 4}, \\ 0, & \text{otherwise} \end{cases} \\
\qquad \text{by the Form–Tensor Product Lemma 91.5} \\
\quad = \text{sign}(M)\cdot\text{sign}(N). \\
\qquad \overset{\uparrow}{} \\
\qquad \text{by the Intersection Form Reinterpretation Lemma 91.1}
\end{array}$$

■

91.2. **Statement of the Novikov Additivity Theorem.** In the Intersection Form-Additivity Proposition 89.7 we saw that for closed topological manifolds the intersection form is additive under the connected sum operation.[723] In the Manifold Signature Additivity Lemma 90.10 we pointed out that this implies that the signature is additive under the connected sum operation.

As we can see in the example below, if we glue two compact oriented $4k$-dimensional topological manifolds M_1 and M_2 along arbitrary boundary components, then the middle homology is in general not additive. In particular the intersection form is certainly in no way additive.



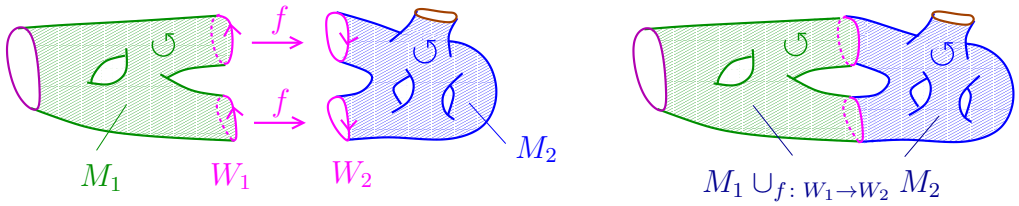
Rather surprisingly, the following theorem, which was first stated by Sergej Novikov [**Nov70**, p. 154] in the mid 1960s, shows that the signature is additive under gluing along boundary components:

Theorem 91.7. (Novikov Additivity Theorem) Let M_1 and M_2 be two compact oriented $4k$-dimensional topological manifolds, let W_1 be a union of components of ∂M_1

[723] In fact the same argument also shows that the intersection form is additive under the connected sum of compact oriented topological manifolds with non-empty boundary.

and let W_2 be a union of components of ∂M_2. Finally let $f\colon W_1 \to W_2$ be an orientation-reversing homeomorphism. Then

$$\operatorname{sign}\Big(\underbrace{M_1 \cup_{f\colon W_1 \to W_2} M_2}_{\substack{\text{by the Gluing-Topological Manifolds-Proposition 62.4 this is} \\ \text{an oriented topological manifold}}}\Big) = \operatorname{sign}(M_1) + \operatorname{sign}(M_2).$$



Proof. This theorem is proved in [**Fri24**]. ■

Remark.

(1) If one glues two $4k$-manifolds along submanifolds of the boundary that are not unions of components, then in general the conclusion of the Novikov Additivity Theorem 91.7 no longer holds.

For example, by the discussion on page 1170 we know that $\operatorname{sign}(\mathbb{C}\mathrm{P}^2) = 1$. On the other hand the Handle Decomposition Theorem **??** says that every closed smooth manifold, in particular $\mathbb{C}\mathrm{P}^2$, can be obtained iteratively by handle attachments.[724] Each handle has vanishing second homology, in particular vanishing intersection form and thus signature zero.

(2) In this more general setting, where we glue smooth manifolds along submanifolds of the boundary, C. T. C. Wall [**Wal69**] proved a refinement of the Novikov Additivity Theorem 91.7. This refinement is usually referred to as the Wall Non-Additivity Theorem. A detailed discussion is given in [**Bru05**, Chapter 5]. □

Before we move on to the long and exhausting proof of the Novikov Additivity Theorem 91.7 let us state the following theorem for general culture. It says that the signature is essentially the only invariant of compact oriented smooth manifolds that has the additivity property stated in the Novikov Additivity Theorem 91.7.

Theorem 91.8. (Jänich Theorem) Let $k \in \mathbb{N}$ and let φ be a real valued orientation-preserving diffeomorphism[725] invariant for compact oriented $4k$-dimensional smooth manifolds. We assume that φ satisfies the following two conditions:

(1) The map φ satisfies the additivity property stated in the Novikov Additivity Theorem 91.7.

(2) For any compact oriented $4k$-dimensional smooth manifold M we have the equality $\varphi(-M) = -\varphi(M)$.

Then there exists a $C \in \mathbb{R}$ such that for every closed oriented $4k$-dimensional smooth manifold we have $\varphi(M) = C \cdot \operatorname{sign}(M)$.

[724] It follows from Exercise **??** that $\mathbb{C}\mathrm{P}^2$ has a handle decomposition consisting of a single 0-handle, a single 2-handle and a single 4-handle.

[725] It is not clear to me whether the analogous statement also holds for topological manifolds.

Proof. The theorem was proved by Klaus Jänich [**Jän68, Jän69**] in 1968. A generalization of the theorem is provided by [**KKNO73**, Theorem 1.3a]. ■

CHAPTER 92

The intersection form of 4-dimensional manifolds

We have now introduced lots of invariants of topological spaces, namely the fundamental group, higher homotopy groups, homology groups and the cohomology groups together with the cup product. In particular, for a compact oriented $2n$-dimensional topological manifold M we introduced on page 1168 the intersection form

$$\begin{array}{rcl} Q_M\colon \mathrm{FH}_n(M)\times \mathrm{FH}_n(M) & \to & \mathbb{Z} \\ (a,b) & \mapsto & \big\langle \underbrace{\underbrace{\mathrm{PD}_M(a)}_{\in \mathrm{FH}^n(M,\partial M;\mathbb{Z})} \cup \underbrace{\mathrm{PD}_M(b)}_{\in \mathrm{FH}^n(M,\partial M;\mathbb{Z})}}_{\in \mathrm{FH}^{2n}(M,\partial M;\mathbb{Z})}, [M] \big\rangle \end{array}$$

The question arises, whether we are done at some point, or whether this list of ever new invariants continues forever. Later in Part **??** we will introduce twisted (co-) homology groups and in Part **??** we will introduce Reidemeister torsion, which give us both extra invariants of non-simply connected topological spaces. But for simply connected topological spaces we are running out of ideas for producing new invariants.

Fortunately we have achieved some successes. For example in the Surface Classification Theorem 22.8 we saw that our invariants are strong enough to classify compact 2-dimensional topological and smooth manifolds. We also saw that the intersection form allows us distinguish $S^2 \times S^2$ from $\mathbb{C}\mathrm{P}^2\#\mathbb{C}\mathrm{P}^2$ and $\mathbb{C}\mathrm{P}^2\#\overline{\mathbb{C}\mathrm{P}^2}$, even though the homology groups of these simply connected 4-dimensional smooth manifolds are the same.

In this chapter we want to consider closed oriented simply connected 4-dimensional (topological) smooth manifolds and their intersection forms. What follows is a long survey on some amazing results. Unfortunately the proofs of any of these statements goes well beyond what we can cover in these introductory notes.

The following question will guide us through the chapter:

Question 92.1.

(1) Is the intersection form a complete invariant of closed oriented simply connected 4-dimensional (topological) smooth manifolds?[726]

(2) Which isometry types of non-singular symmetric forms over $\mathbb{Z}$ are realized by closed oriented simply connected 4-dimensional (topological) smooth manifolds?

[726] First note that for a simply connected 4-dimensional topological manifold M we obtain from the Hurewicz Theorem 58.5 that $\mathrm{H}_1(M) = 0$ Next note that it follows from $\mathrm{H}_1(M) = 0$ and the Universal Coefficient-H^*-Theorem 70.14 that $\mathrm{H}^2(M;\mathbb{Z})$ is torsion-free. We now assume that M is closed and orientable. It follows from the above and the Poincaré Duality Theorem 79.1 $\mathrm{H}_2(M) \cong \mathrm{H}^2(M;\mathbb{Z})$ is torsion-free. Furthermore it follows from the above and the Poincaré Duality Theorem 79.1 that $\mathrm{H}^3(M;\mathbb{Z}) \cong \mathrm{H}_1(M) = 0$. It follows from $\mathrm{H}^3(M;\mathbb{Z}) = 0$, the fact that $\mathrm{H}_2(M)$ is torsion-free and the

92.1. **1910–1958: The K3-surface.** In the following we will give a mostly chronological account of the developments regarding the two (or possibly four?) questions posed in Question 92.1.

On page 1191 we introduced the initially rather unexpected form E_8. One might wonder whether a form as strange as E_8 can ever appear "in nature". As we will now see, somewhat surprisingly this is indeed the case.

Definition. We refer to

$$\mathrm{K3} \;:=\; \left\{[z_0 : z_1 : z_2 : z_3] \in \mathbb{C}\mathrm{P}^3 \;\middle|\; z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0\right\}$$

as the K3-surface. Sometimes the K3-surface is also called the Kummer surface.

The following proposition summarizes some of the key aspects of the K3-surface:

Proposition 92.2. (K3-Intersection Form Proposition) The K3-surface is a closed simply connected complex manifold of complex dimension 2 or equivalently of real dimension 4. Furthermore we have $\mathrm{H}_2(\mathrm{K3}) \cong \mathbb{Z}^{22}$ and [727]

$$Q_{\mathrm{K3}} \;\cong\; 2 \cdot E_8 \oplus 3 \cdot H \qquad \text{where } H \;=\; \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

In particular we have $\mathrm{sign}(\mathrm{K3}) = 16$.

Proof. Using the Complex Regular Value Theorem it is elementary to show that the K3-surface is a closed complex manifold of complex dimension 2. Using the Lefschetz Hyperplane Theorem one can show that the K3-surface is simply connected. The second homology group and the intersection form are calculated in [**GS99**, Theorem 1.3.8], or alternatively in [**MS16**, p. 176] and in [**Huy16**, Proposition 3.5].[728]

The statement about the signature then follows from the E_8-Lemma 90.15 and the Signature Additivity Proposition 90.8. ∎

In the following we want to discuss one particular aspect of the K3-surface, namely: The calculation of the intersection form of the K3-surface, of $Q_{S^2\times S^2} \cong H$ and the Intersection Form-Additivity Proposition 89.7 raises the question whether one can split off copies of $S^2 \times S^2$ from the K3-surface. More precisely the following question arose almost immediately after the introduction of the K3-surface:

Question 92.3.

(1) Does there exist a closed oriented 4-dimensional topological manifold M such that the K3-surface is homeomorphic to $M\#(S^2 \times S^2)$?
(2) Does there exist a closed oriented 4-dimensional smooth manifold M such that the K3-surface is diffeomorphic to $M\#(S^2 \times S^2)$?

We will address Question 92.3 in the following sections.

Universal Coefficient-H*-Theorem 70.14 that $\mathrm{H}_3(M) = 0$. Thus we see that for closed oriented simply connected 4-dimensional topological manifolds the intersection form contains all the (co-) homological data we could ask for.

[727] Since the K3-surface is a complex manifold we know that it has a natural orientation.

[728] The calculation of the intersection form, perhaps somewhat disappointingly, makes use of the Indefinite Even Forms Corollary 90.21.

92.2. **1952: Rokhlin's Theorem.** We will return to Question 92.3 shortly. But before we do so, we continue with our chronological discussion. Next in line is a result proved by Vladimir Rokhlin in 1952. To formulate Rokhlin's Theorem we need the notion of a "spin structure". Unfortunately we have to give a "black box" definition:

Definition. A smooth manifold M is called spin if its tangent bundle admits a spin structure in the sense of the definition in [**GS99**, Chapter 1.4.2].

At this stage the definition of a "spin structure" is of no concern to us and we will use this notion as a black box. For the discussion of this chapter it suffices to be aware of the following proposition about the (non-) existence of spin structures in our context:

Proposition 92.4. (Spin–Intersection Form Proposition) Let X be a compact oriented 4-dimensional smooth manifold.

(1) If X is spin, then the intersection form Q_X is even.
(2) If the intersection form Q_X is even and if $\mathrm{H}_1(X)$ has no 2-torsion (by the Hurewicz Theorem 58.5 this condition is satisfied if X is simply connected), then the smooth manifold X is spin.

Proof. The proposition is the content of [**GS99**, Corollary 5.7.6], which in turn is proved using the "Wu formula". The proposition is also discussed in [**Sco05**, p. 163]. ■

Example. It follows from the Spin–Intersection Form Proposition 92.4 (2), the K3-Intersection Form Proposition 92.2 and familiar facts about $S^2 \times S^2$ that the K3-surface and $S^2 \times S^2$ are both spin.

Let X be a closed oriented 4-dimensional smooth manifold. If X is spin, then we know by the Spin–Intersection Form Proposition 92.4 (1) that the intersection from Q_X is even. It follows from the Even Forms Proposition 90.20, for purely algebraic reasons, that the signature of X is divisible by 8. Rather mysteriously, Rokhlin's theorem from 1952 gives us a refinement of this statement:

Theorem 92.5. (Rokhlin Theorem) Let X be a closed oriented 4-dimensional smooth manifold. If X is spin, then $\mathrm{sign}(X)$ is divisible by 16.

Proof. This theorem was proved by Vladimir Rokhlin [**Rok52**][**Rok86**, p. 21] in 1952. More modern proofs of Rokhlin's Theorem are given in [**FK78**], [**LM89**, Theorem II.2.13 and Corollary IV.1.2] and [**Sav02**, Theorem 2.1]. This theorem was generalized to closed oriented $(4+8k)$-dimensional spin manifolds by Ochanine [**Och81**]. ■

We conclude this section with the following variation on Rokhlin's Theorem 92.5, which was historically the first result to address Question 92.1 (2):

Theorem 92.6. (Rokhlin Theorem II) Let X be a closed oriented 4-dimensional smooth manifold. If the intersection form Q_X is even and if $\mathrm{H}_1(X)$ has no 2-torsion (by the Hurewicz Theorem 58.5 this condition is satisfied if X is simply connected), then $\mathrm{sign}(X)$ is divisible by 16.

Proof. This theorem follows immediately from the Spin–Intersection Form Proposition 92.4 together with Rokhlin's Theorem 92.5. ■

92.3. **1958: The homotopy type.** We just learned in the Rokhlin Theorem 92.6 that not every non-singular symmetric form over $\mathbb{Z}$ can appear as the intersection form of a closed oriented 4-dimensional smooth manifolds. In the opposite direction, the following theorem was the first noteworthy result towards an affirmative answer to Question 92.1 (1):

Theorem 92.7. (Homotopy Type–4D-Theorem) Let X and Y be two closed oriented simply connected 4-dimensional topological manifolds. Then

$$X \text{ and } Y \text{ are homotopy equivalent} \iff Q_X \text{ is isometric to } \pm Q_Y.$$

Proof. The straightforward "$\Rightarrow$"-direction is the content of the Intersection Form–Homeomorphism Lemma 89.4. The much more difficult "$\Leftarrow$"-direction was proved for smooth manifolds by John Milnor [**Mil58a**, Theorem 3] in 1958, building on work of John Whitehead [**Whi49**]. This proof is also written up in [**MH73**, Theorem V.1.5]. Furthermore [**GS99**, Theorem 1.2.25] and [**Kir89**, Theorem 2.1] sketch a proof that also works for topological manifolds. ■

92.4. **1977: The work of Rob Kirby and Larry Siebenmann.** Unfortunately we need to start out again with a "black box" definition:

Definition.

(1) Given any compact topological manifold X we denote by $\mathrm{ks}(X) \in \mathrm{H}^4(X;\mathbb{Z}_2)$ the corresponding Kirby-Siebenmann invariant which is defined in [**KS77**, p. 318] or alternatively in [**FQ90**, Section 10.2B] and [**Rud16**, Chapter 3.4].

(2) If X is a closed oriented connected non-empty 4-dimensional topological manifold, then we know by the Manifold-Top-H*-Theorem 70.16 (2) that we have $\mathrm{H}^4(X;\mathbb{Z}_2) \cong \mathbb{Z}_2$. Thus in that case we think of the Kirby-Siebenmann invariant as an element of $\mathbb{Z}_2$.

The following theorem summarizes a few key properties of the Kirby-Siebenmann invariant:

Proposition 92.8. (Kirby-Siebenmann Invariant Proposition)

(1) Let X be a compact topological manifold.
  (a) If X admits a PL-structure[729] or a smooth structure, then $\mathrm{ks}(X) = 0$.
  (b) If $\mathrm{ks}(X) = 0$ and if $\dim(X) \geq 5$, then X admits a PL-structure.

(2) If $f\colon X \to Y$ is a local homeomorphism, then $\mathrm{ks}(X) = f^*(\mathrm{ks}(Y)) \in \mathrm{H}^4(X;\mathbb{Z}_2)$.[730]

(3) (a) The Kirby-Siebenmann invariant of closed oriented connected non-empty 4-dimensional topological manifolds is additive under connected sums.
  (b) If X is a closed oriented simply connected non-empty 4-dimensional topological manifold with even intersection form, then[731] [732] $\mathrm{ks}(X) \equiv \frac{1}{8} \cdot \mathrm{sign}(X) \bmod 2$.

Proof.

(1) (a) By [**Rud16**, Theorem 3.4.3] the Kirby-Siebenmann invariant vanishes if X admits a PL-structure. Furthermore note that in the Smooth-implies-PL Theorem **??** we saw

[729] We refer to page **??** for the definition of a PL-structure.

[730] It follows in particular that for dimensions ≥ 5 the Kirby-Siebenmann invariant, whatever it might be, is a complete obstruction to the existence of a PL-structure on a topological manifold.

[731] It follows from the Even Forms Proposition 90.20 that $\mathrm{sign}(X)$ is divisible by 8.

[732] For topological manifolds there is also a notion of being "spin" which will not get into. With this notion one can replace the hypothesis "simply connected and even intersection form" by "spin". We refer to [**FNOP19**, Theorem 9.2] for details.

that every smooth manifold admits a PL-structure. In particular we see that the Kirby-Siebenmann invariant vanishes for smooth manifolds. We refer to [**Cer68**, p. IX] and [**Sco05**, p. 220] for more details.

(b) If $\mathrm{ks}(X) = 0$ and if $\dim(X) \geq 5$, then it follows again from [**Rud16**, Theorem 3.4.3] that X admits a PL-structure.

(2) This statement follows easily from the definition of the Kirby-Siebenmann invariant in [**Rud16**, Definition 3.4.2].

(3) These two statements are proved in [**FNOP19**, Theorem 9.2]. ■

A priori it is not clear whether there exist compact topological manifolds with non-zero Kirby-Siebenmann invariant. But in the next section we will see that they do exist in the 4-dimensional setup.

92.5. **1981: The work of Michael Freedman.** After the somewhat preliminary results by Vladimir Rokhlin and John Milnor in the 1950, the early 1980s saw two major breakthroughs in the 4-dimensional setting, due to Michael Freedman and Simon Donaldson. We will explain these amazing results in the following two sections.

First, in 1981 Michael Freedman [**Fre82**][733] gave a complete answer to Question 92.1 in the setting of topological manifolds:

Theorem 92.9. (Freedman Classification Theorem)

(1) Let Q be a non-singular symmetric form over $\mathbb{Z}$ and let $k \in \mathbb{Z}_2$. (If Q is even, then we assume that $k \equiv \frac{1}{8} \cdot \mathrm{sign}(Q) \bmod 2$.) There exists a closed oriented simply connected 4-dimensional topological manifold X whose intersection form is isometric to Q and with $\mathrm{ks}(X) = k \in \mathbb{Z}_2$.

(2) Let X and Y be two closed oriented simply connected 4-dimensional topological manifolds with $\mathrm{ks}(X) = \mathrm{ks}(Y) \in \mathbb{Z}_2$. If $\varphi\colon (\mathrm{H}_2(X), Q_X) \to (\mathrm{H}_2(Y), Q_Y)$ is an isometry of the intersection forms, then there exists an orientation-preserving homeomorphism $f\colon X \to Y$ such that $f_* = \varphi$. Furthermore f is unique up to isotopy.

Proof. The theorem is proved in [**Fre82**, Theorem 1.5] and [**FQ90**, p. 161]. (The observation that in Statement (2) we can find an *orientation-preserving* homeomorphism is not explicitly mentioned in [**FQ90**, p. 161], but it does follow from the proof.) Some aspects of the proof are dealt with in much greater detail in [**BKK$^+$21**]. The last statement, that f is unique up to isotopy, was proved by Frank Quinn in [**Qui86**, Theorem 1.1] (see also [**GGH$^+$23**, Section 1.3]). ■

The Freedman Classification Theorem 92.9 says in particular the following:

- Every non-singular symmetric even form over $\mathbb{Z}$ is realized by a closed oriented simply connected 4-dimensional topological manifold that is unique up to homeomorphism.
- Every non-singular symmetric odd form over $\mathbb{Z}$ is realized by two closed oriented simply connected 4-dimensional topological manifolds that are distinguished by the Kirby-Siebenmann invariant and that are unique up to homeomorphism.

In the following we will discuss several consequences of the Freedman Classification Theorem 92.9.

[733] Michael Freedman (*1951) is an American mathematician, in 1986 he was awarded a Fields medal for the proof of the Freedman Classification Theorem 92.9.

Definition. Let M be a closed connected non-empty n-dimensional topological manifold. If $\pi_k(M) = 0$ for $k = 1, \dots, \lfloor \frac{n}{2} \rfloor$, then we call M a homotopy sphere.

Theorem 92.10. (4D-Topological Poincaré Theorem) Let X be a 4-dimensional topological manifold. If X is a homotopy sphere, then X is homeomorphic to S^4.

Proof. Let X be a closed connected 4-dimensional topological manifold which is a homotopy sphere. By definition this means that $\pi_1(X) = 0$ and $\pi_2(X) = 0$. Since $\pi_1(X) = 0$ we obtain from the Hurewicz Theorem 59.6 that the Hurewicz homomorphism $\pi_2(X) \to \mathrm{H}_2(X)$ is an isomorphism. It follows that $\mathrm{H}_2(X) = 0$. This implies in particular, by the Kirby-Siebenmann Invariant Proposition 92.8 (3b), that $\mathrm{ks}(X) = \mathrm{ks}(S^4) = 0$. It now follows from the Freedman Classification Theorem 92.9 (2) that X is homeomorphic to S^4. ■

The Freedman Classification Theorem 92.9 gives us two unusual, deeply mysterious, closed oriented 4-dimensional topological manifolds:

Definition.

(1) By the Freedman Classification Theorem 92.9 there exists a closed oriented simply connected 4-dimensional topological manifold X with $Q_X \cong E_8$, which is unique up to an orientation-preserving[734] homeomorphism. We refer to this topological manifold as the E_8-manifold and sometimes we denote it by E_8 as well.

(2) We consider the form that is represented by the (1×1)–matrix (1). By the Freedman Classification Theorem 92.9 (2) there exist, up to homeomorphism, precisely two closed oriented simply connected 4-dimensional topological manifolds with this intersection form, and these two topological manifolds are distinguished by the Kirby-Siebenmann invariant.
- By the discussion on page 1170 and the Kirby-Siebenmann Invariant Proposition 92.8 (1) we know that the closed oriented simply connected smooth manifold $\mathbb{C}\mathrm{P}^2$ corresponds to the case of vanishing Kirby-Siebenmann invariant.
- The closed oriented simply connected 4-dimensional topological manifold with intersection form (1) and non-zero Kirby-Siebenmann invariant is usually denoted by $*\mathbb{C}\mathrm{P}^2$.

Note that the topological manifolds $\mathbb{C}\mathrm{P}^2$ and $*\mathbb{C}\mathrm{P}^2$ are not homeomorphic since the Kirby-Siebenmann invariants differ. But also note that it follows from the Homotopy Type–4D-Theorem 92.7 that the topological manifolds $\mathbb{C}\mathrm{P}^2$ and $*\mathbb{C}\mathrm{P}^2$ are homotopy equivalent.

Using the Freedman Classification Theorem 92.9 (1) and (2) we can give an affirmative answer to Question 92.3 (1):

Theorem 92.11. (K3-Topological Splitting Theorem) The K3-surface is homeomorphic to $2 \cdot E_8 \# 3 \cdot (S^2 \times S^2)$.

[734] If X and Y are two closed oriented connected 4-dimensional topological manifolds with positive signature, then it follows from the Manifold-Signature Lemma 90.9 that any homeomorphism $f\colon X \to Y$ that induces an isometry of the intersection forms is necessarily orientation-preserving. By the E_8-Lemma 90.15 we know that $\mathrm{sign}(E_8) > 0$. It follows from this discussion that the E_8-manifold is indeed unique up to an orientation-preserving homeomorphism.

Proof. It follows from the Connected Sum-Invariants Proposition 89.6 and the Intersection Form-Additivity Proposition 89.7, together with the uniqueness statement of the Freedman Classification Theorem 92.9 (2), that there exists an orientation-preserving homeomorphism of the K3-surface to $2 \cdot X \# 3 \cdot (S^2 \times S^2)$. ■

Among many other consequences of the Freedman Classification Theorem 92.9 we obtain the following rather surprising result:

Theorem 92.12. (No Smooth Structure-4D-Theorem) There exist closed orientable 4-dimensional topological manifolds that do not admit a smooth structure.

Proof. At this point we can combine the above results to give two types of examples of such topological manifolds:

(1) It follows from Rokhlin's Theorem 92.6 (or alternatively from the Kirby-Siebenmann Invariant Proposition 92.8) that the E_8-manifold cannot admit a smooth structure.
(2) We consider $*\mathbb{C}\mathrm{P}^2$. Since by construction we have $\mathrm{ks}(*\mathbb{C}\mathrm{P}^2) = 1$ it follows from the Kirby-Siebenmann Invariant Proposition 92.8 that $*\mathbb{C}\mathrm{P}^2$ cannot admit a smooth structure. ■

We conclude this section with a remark:

Remark. It is natural to ask to what degree the Freedman Classification Theorem 92.9 generalizes to the non-simply connected case. In [**FQ90**, Theorem 10.7A] a generalization is proved for the case $\pi_1 = \mathbb{Z}$, where one needs to replace the ordinary intersection form by the twisted intersection pairings. A few more (partial) classification results are summarized in [**FNOP19**, Chapter 12]. But for most fundamental groups the classification of closed oriented 4-dimensional topological manifolds is still completely out of reach.

92.6. **1982: The work of Simon Donaldson.** In a remarkable twist, shortly after Freedman proved his results on the intersection forms of 4-dimensional topological manifolds, Simon Donaldson[735], proved the following result regarding intersection forms of 4-dimensional smooth manifolds.

Theorem 92.13. (Donaldson Diagonalization Theorem) Let X be a closed oriented connected 4-dimensional smooth manifold. If Q_X is positive-definite, then Q_X is represented by the identity matrix.[736]

Proof. The simply connected case was first proved by Simon Donaldson in [**Don83**, Theorem 1], see also [**Don86**, Theorem A]. Later, in [**Don87**, Theorem 1], the general case (without any restrictions on the fundamental group) was proved. An exposition of the proof is also given in [**Law85**]. A different proof (of the general statement), using Seiberg-Witten invariants is given in [**Nic00**, Theorem 2.4.18] and a proof using Heegaard Floer homology is given in [**OS03**, Theorem 9.1]. ■

[735]Simon Donaldson (*1957) is an English mathematician. He proved Theorem 92.13 when he was a second year PhD student. The Fields medalist Michael Atiyah described in [**Ati87**] the work of Simon Donaldson as follows: "In 1982, when he was a second-year graduate student, Simon Donaldson proved a result that stunned the mathematical world." Simon Donaldson was awarded a Fields medal in 1986.

[736]If Q_X is negative-definite, then we apply the theorem to $-X$, and we see that Q_X is represented by $-\mathrm{id}$.

Example. We consider the positive-definite symmetric form $E_8 \oplus E_8$. It has signature 16, hence Rokhlin's Theorem 92.6 has no objections to this form being realized by a closed oriented 4-dimensional smooth manifold. But $E_8 \oplus E_8$ is an even form. It follows from the Parity Lemma 90.16 that it cannot be represented by the identity matrix. Thus the Donaldson Diagonalization Theorem 92.13 implies that $E_8 \oplus E_8$ cannot be the intersection form of a closed oriented connected 4-dimensional smooth manifold. □

To understand the significance of the Donaldson Diagonalization Theorem 92.13, often referred to as *Donaldson Theorem A*, recall that on page 1196 we saw that there are more than 10^{51} isometry classes of non-singular positive-definite symmetric forms of rank 40 over $\mathbb{Z}$. By the Freedman Classification Theorem 92.9 all of them arise as intersection forms of closed oriented simply connected 4-dimensional topological manifolds. By the Donaldson Diagonalization Theorem 92.13 the only one which appears as the intersection form of a closed oriented connected 4-dimensional smooth manifold is the one represented by the identity matrix.[737]



Remark. It is difficult to exaggerate the importance of the Donaldson Diagonalization Theorem 92.13 in mathematics.[738] In fact the Donaldson Diagonalization Theorem 92.13 can appear in totally unexpected situations. For example in 2017 Duncan McCoy [**McC17**] used the Donaldson Diagonalization Theorem 92.13, and many other ingredients, to show that if K is an alternating knot with unknotting number one[739], then such a crossing change can already be performed in an alternating diagram. For example, according to [**LM20**] the three alternating knots shown in the figure below have unknotting number one.



the knot $11a_{12}$ | the knot $11a_{41}$ | the knot $11a_{56}$

[737] The identity matrix represents, by the Intersection Form-Additivity Proposition 89.7 and the discussion on page 1170, the intersection form of the connected sum of 40 copies of $\mathbb{C}P^2$.

[738] Admittedly it is also hard to underestimate the role of the Donaldson Diagonalization Theorem 92.13 outside of mathematics and perhaps physics.

[739] We recall the following definitions:

- A knot is called *alternating* if it admits a diagram that is alternating, i.e. which has the property that each undercrossing is followed by an overcrossing and vice versa.
- The *unknotting number* of a knot K is the minimal number of crossing changes needed to turn K into the trivial knot.

By the aforementioned result we now know that we can turn these three knots into the trivial knot by changing a crossing in the alternating diagrams shown above. Which crossings do you need to change? □

92.7. **1996: The 10/8-Theorem and the 11/8-Conjecture.** The Donaldson Diagonalization Theorem 92.13 gives us a complete answer to the question, which non-singular *positive-definite* symmetric forms can arise as the intersection form of a closed oriented 4-dimensional smooth manifold: Only the forms that are represented by the identity matrix.

Now the question arises, what can we say about *indefinite* intersection forms of closed oriented 4-dimensional smooth manifolds? First recall that in the Indefinite Odd Forms Corollary 90.19 we saw that every non-singular indefinite odd symmetric form over $\mathbb{Z}$ is isometric to $m \cdot (1) \oplus n \cdot (-1)$ for some $m, n \in \mathbb{N}_0$. Next note that according to the Intersection Form-Additivity Proposition 89.7 and the discussion on page 1170 every such form is realized by the smooth manifold $m \cdot \mathbb{C}\mathrm{P}^2 \# n \cdot \overline{\mathbb{C}\mathrm{P}}^2$.

Therefore, in the following we only need to discuss the realizability of non-singular indefinite *even* symmetric forms. Recall that in the Indefinite Even Forms Corollary 90.21 we showed that every such form is isometric to $m \cdot E_8 \oplus n \cdot H$ for some $m \in \mathbb{Z}$ and $n \in \mathbb{N}_0$. Which values of m and n can we realize by closed oriented connected smooth manifolds? We can certainly take connected sums of k copies of the K3-surface and l copies of $S^2 \times S^2$. It follows from the Intersection Form-Additivity Proposition 89.7, the calculation of the intersection forms of the K3-surface, see the K3-Intersection Form Proposition 92.2, and of the intersection form of $S^2 \times S^2$ that

$$Q_{k \cdot \mathrm{K3} \,\#\, l \cdot (S^2 \times S^2)} \;\cong\; 2k \cdot E_8 \oplus (l + 3k) \cdot H.$$

Recall that it follows from the E_8-Lemma 90.15 that $\operatorname{sign}(E_8) = 8$ and that on page 1185 we saw that $\operatorname{sign}(H) = 0$. Some 8th grade mathematics now shows that

$$b_2(k \cdot \mathrm{K3} \,\#\, l \cdot (S^2 \times S^2)) \;\geq\; \tfrac{11}{8} \cdot |\operatorname{sign}(k \cdot \mathrm{K3} \,\#\, l \cdot (S^2 \times S^2))|,$$

and that equality holds precisely if $l = 0$.

Note that by definition we have for any closed oriented 4-dimensional topological manifold the trivial inequality $b_2(M) \geq |\operatorname{sign}(M)|$. So the question that arises is, whether for closed oriented smooth manifolds we can do better than the inequality given by the above examples. The following theorem gives us some restrictions:

Theorem 92.14. (Furuta's 10/8-Theorem) If M is a closed oriented spin 4-dimensional smooth manifold with indefinite intersection form, then

$$b_2(M) \;\geq\; \tfrac{10}{8} \cdot |\operatorname{sign}(M)| + 2.$$

In particular $Q_M \cong m \cdot E_8 \oplus n \cdot H$ with $n \geq |m| + 1$.

Proof. Let M be a closed oriented spin 4-dimensional smooth manifold with indefinite intersection form. Mikio Furuta [**Fur01**] (see also [**Akb96**, Theorem (Furuta)] proved in 1996 the inequality

$$b_2(M) \;\geq\; \tfrac{10}{8} \cdot \operatorname{sign}(M) + 2.$$

It follows from the Indefinite Even Forms Corollary 90.21 and the calculation of the rank and signature of H and E_8 that $Q_M \cong m \cdot H \oplus n \cdot E_8$ with $m \geq |n| + 1$. ■

Furuta's 10/8 Theorem 92.14 is not yet optimal since it does not quite close the gap between the forms which we can realize by closed oriented smooth manifolds and the forms which we can exclude. The following conjecture (which was first formulated in [**Mat82**, p. 289]) predicts that we cannot do better than the above examples.

Conjecture 92.15. (11/8-Conjecture) If M is a closed oriented spin 4-dimensional smooth manifold with indefinite intersection form, then

$$b_2(M) \geq \tfrac{11}{8} \cdot |\operatorname{sign}(M)|.$$

In particular the conjecture predicts, that in this setting $Q_M \cong m \cdot E_8 \oplus n \cdot H$ with $n \geq \frac{3}{2} \cdot |m|$.[740]

Remark. For simply connected smooth manifolds currently the best result towards the 11/8-Conjecture 92.15 is [**HLSX18**, Corollary 1.13] which says that if M is a closed oriented spin simply connected 4-dimensional smooth manifold that is not homeomorphic to S^4, $S^2 \times S^2$ or the K3-surface and whose intersection form is indefinite, then $b_2(M) \geq \frac{10}{8} \cdot |\operatorname{sign}(M)| + 4$. □

Furuta's 10/8-Theorem 92.14 allows us to answer Question 92.3 (2) in the negative. More precisely, we have the following theorem in the smooth setting, which differs dramatically from the K3-Topological Splitting Theorem 92.11 in the topological setting:

Theorem 92.16. (K3-Smooth Non-Splitting Theorem) If there exist closed oriented smooth manifolds M and N such that K3 is homeomorphic[741] to $M\#N$, then one of M or N is homeomorphic to S^4.

Proof. Suppose that we have closed oriented smooth manifolds M and N such that K3 is homeomorphic to $M\#N$. We make the following four observations:

(1) Without loss of generality we have $\operatorname{sign}(M) \geq \operatorname{sign}(N)$.
(2) (a) It follows from the K3-Intersection Form Proposition 92.2, the Parity Lemma 90.16 (3) and the Intersection Form-Additivity Proposition 89.7 that Q_M and Q_N are even.
    (b) It follows from the K3-Intersection Form Proposition 92.2, the Hurewicz Theorem 58.5 and the Connected Sum-H_*-Proposition **??** that $H_1(M) = H_1(N) = 0$.
(3) It follows from (2) and the Spin–Intersection Form Proposition 92.4 that M and N are spin.
(4) From the Manifold Signature Additivity Lemma 90.10 and the K3-Intersection Form Proposition 92.2 we obtain that $\operatorname{sign}(M) + \operatorname{sign}(N) = 16$.
(5) By the K3-Intersection Form Proposition 92.2 and the Connected Sum-Invariants Proposition 89.6 we know that $b_2(M) + b_2(N) = 22$.

Since we assume that M is smooth it follows from these facts together with Rokhlin's Theorem 92.6 that $\operatorname{sign}(M) = 16$. Next note that Furuta's 10/8-Theorem 92.14 implies that $b_2(M) \geq \frac{10}{8} \cdot 16 + 2 = 22$. By the K3-Intersection Form Proposition 92.2 this equals $b_2(\text{K3})$. Thus we obtain from the Connected Sum-Invariants Proposition 89.6 that $b_2(N) = 0$.

[740] As before, it follows from the Indefinite Even Forms Corollary 90.21 and the calculation of the rank and signature of H and E_8 that these two statements are equivalent.
[741] Note that "smooth manifold" and "homeomorphic" are not typos.

Next note that it follows from the Connected Sum-Invariants Proposition 89.6 and the K3-Intersection Form Proposition 92.2 we that $\pi_1(N) = 0$. Recall that by the Hurewicz Theorem 59.6 we know that if $\pi_1(N) = 0$ and $\mathrm{H}_2(N) = 0$, then $\pi_2(N) = 0$. We have thus shown that N is a homotopy 4-sphere. Thus it follows from the 4D-Topological Poincaré Theorem 92.10 that N is homeomorphic to S^4. ■

92.8. **Non-diffeomorphic smooth structures.** The other question which arises in this context is the following: If two closed orientable 4-dimensional smooth manifolds are homeomorphic, are they also necessarily diffeomorphic?

The following elementary lemma let's us rephrase the question:

Lemma 92.17. (Inequivalent Smooth Structure Lemma) Let X be a topological manifold and let I be a set. The following two statements are equivalent:

(a) There exists a family $\{X_i\}_{i\in I}$ of smooth manifolds such that each X_i is homeomorphic to X and such that the X_i are pairwise non-diffeomorphic.
(b) There exists a family $\{\mathcal{A}_i\}_{i\in I}$ of smooth structures on X such that the smooth manifolds $(X, \mathcal{A}_i)$ are pairwise non-diffeomorphic.

Proof. The (b)⇒(a)-direction is obvious. We turn to the proof of the (a)⇒(b)-direction. Let $\{(X_i, \mathcal{A}_i)\}_{i\in I}$ be a family of smooth manifolds such that for each $i \in I$ there exists a homeomorphism $f\colon X \to X_i$ and such that the X_i are pairwise non-diffeomorphic. The homeomorphisms allow us to define smooth structures $f^*(\mathcal{A}_i)$ on X. These have the desired properties. ■

We continue with the following semi-mathematical definition:

Definition.

(1) Let X be a topological manifold. We say that two smooth structures $\mathcal{A}$ und $\mathcal{B}$ are non-diffeomorphic if the smooth manifolds $(X, \mathcal{A})$ and $(X, \mathcal{B})$ are not diffeomorphic.
(2) If X is a smooth manifold that is equipped with a "standard smooth structure $\mathcal{S}$", then every smooth structure $\mathcal{A}$ which is not diffeomorphic to $\mathcal{S}$ is called exotic.

We can now answer the question, which we posed in the beginning of this section, in the negative. For example the following theorem was proved in 1998 by Ron Fintushel and Ron Stern [**FS98**]:

Theorem 92.18. (Exotic K3-Theorem) The K3-surface admits infinitely many pairwise non-diffeomorphic smooth structures.

Similar results have been obtained for many other 4-dimensional smooth manifolds, e.g. for $\mathbb{C}\mathrm{P}^2\#k\overline{\mathbb{C}\mathrm{P}}^2$ with $k = 9$ by Simon Donaldson [**Don87**] in 1987, $k = 8$ by Dieter Kotschick [**Kot89**] in 1989 and $k = 7$ by Jongil Park [**Par05**] in 2005, and since then also for $k = 6, 5, 4, 3, 2$, see e.g. [**AP08, AP10, BK08**] for more details on the history of this problem. It is unknown whether $\mathbb{C}\mathrm{P}^2\#\overline{\mathbb{C}\mathrm{P}}^2$ or $S^2 \times S^2$ admit an exotic smooth structure.

The following question is surely one of the most interesting problem in the theory of 4-dimensional smooth manifolds:

Question 92.19. Does every closed oriented connected non-empty 4-dimensional smooth manifold M admit infinitely many pairwise non-diffeomorphic smooth structures?

The following question is an important special case of Question 92.19:

Question 92.20. (Smooth Poincaré Conjecture in Dimension 4) Does S^4 admit an exotic smooth structure?[742]

So far we discussed closed manifolds. How about non-compact manifolds? In 1961 John Stallings [**Sta61**, p. 481] wrote:

Euclidean space $\mathbb{R}^4$ ought to have [...] a unique differentiable structure.

It turns out that this is indeed true for $n \leq 3$ by work of Christos Papakyriakopoulos [**Pap46**], Edwin Moise [**Moi52**, Theorem 4] and James Munkres [**Mun60**, Corollary 6.6]. Furthermore John Stallings [**Sta61**, Corollary 5.2] showed that the above statement is also true for $n \geq 5$.

It turns out that the situation is dramatically different in dimension 4. Again the Donaldson Diagonalization Theorem 92.13 together with the Freedman Classification Theorem 92.9 are at the heart of the proof of the following theorem:

Theorem 92.21. ($\mathbb{R}^4$-Smooth Structure Theorem) There are uncountably many pairwise non-diffeomorphic smooth structures on $\mathbb{R}^4$.

Proof. The first examples of exotic smooth structures on $\mathbb{R}^4$ were given by Michael Freedman (see [**Gom83**, Chapter 2]). Later infinitely, and even uncountably many pairwise non-diffeomorphic smooth structures were found by Robert Gompf [**Gom85**, Theorem 1.3] and Cliff Taubes [**Tau87**, Theorem 1.1]. ■

92.9. **Stabilizations and diffeomorphisms.** We conclude the discussion of the relationship between smooth and topological aspects of 4-dimensional smooth manifolds with the following theorem, which says that smooth manifolds that are homeomorphic are in fact "stably diffeomorphic".

Theorem 92.22. (Stable Diffeomorphism Theorem) Let M and N be two compact 4-dimensional smooth manifolds that are homeomorphic. If the smooth manifolds M and N are orientable, then there exists a $k \in \mathbb{N}$ such that $M \# k \cdot (S^2 \times S^2)$ and $N \# k \cdot (S^2 \times S^2)$ are diffeomorphic.

Proof. The theorem was first proved by Robert Gompf [**Gom84**, p. 116] in 1984. An alternative proof, using Matthias Kreck's modified surgery [**Kre99**], is given in [**Tei92**, Theorem 5.1.1] and [**FNOP19**, Theorem 12.2]. Also note that the special case that the smooth manifolds are closed and simply connected was first proved by C. T. C. Wall [**Wal64**, Theorem 3] in 1964 (see also [**Sco05**, p. 149] and [**Qui83**, Theorem 1.1]). ■

This raises the question, whether in the above theorem $k = 1$ suffices. More precisely the following question is still open:

Question 92.23. Let M and N be two closed oriented 4-dimensional smooth manifolds that are homeomorphic. Is $M \# (S^2 \times S^2)$ diffeomorphic to $N \# (S^2 \times S^2)$?

There are some indications, see [**JZ18**], that the answer to the above question should be no.

[742] An answer to this question would lead, assuming the mathematician is under 40, straight to a Fields medal.

Part XI

Cohomology theories

In this part of the lecture notes we will study the question, which properties characterize the theory of real cohomology groups of smooth manifolds? In the Homotopy-H^*-Proposition 67.7 we showed that singular cohomology is a homotopy invariant and in the Mayer–Vietoris-H^*-Theorem 69.5 (1) we showed that singular cohomology gives rise to Mayer–Vietoris sequences.

It turns out that there is a third property one needs to consider to get a full characterization, namely one needs to consider cohomology groups of unions of open subsets of a topological space or a smooth manifold.

The formulation of that result requires the notion of an inverse limit. Thus in this part of the lecture notes we first study inverse limits, then we study cohomology groups of ascending sequences of subspaces, before we characterize cohomology on smooth manifolds. This characterization will then allow us to show that for smooth manifolds de Rham cohomology is naturally isomorphic to singular cohomology with real coefficients.

This part of the lecture notes is organized as follows:

- In Chapter 93 we will introduce the very general notion of an inverse limit of an inverse system.
- In Chapter 94 we will see that if we can write a topological space X as the ascending union of a sequence X_i of open subsets, then the cohomology groups of X agree with the inverse limit of the inverse sequence of cohomology groups of the X_i.
- In Chapter 95 we will introduce the notion of a "smooth cohomology theory". We will see that real singular cohomology and de Rham cohomology both define a "smooth cohomology theory". Furthermore we will see that any two smooth cohomology theories are isomorphic if they agree for the 0-dimensional smooth manifold given by a point and if there exists a natural transformation from one to the other. In the De Rham Isomorphism-Theorem 95.8 we will use this result to show that for smooth manifolds with empty boundary real singular cohomology is naturally isomorphic to de Rham cohomology.
- In Chapter 96 we will show in the De Rham–Ring Isomorphism Proposition 96.1 that the isomorphism from the De Rham Isomorphism-Theorem 95.8 is actually an isomorphism of rings.
- In Chapter 97 we will use the De Rham Isomorphism-Theorem 95.8 to prove the Maps-to-S^1-Proposition 97.2 which says that for a smooth manifold M homotopy classes of smooth maps $M \to S^1$ are in bijection to cohomology classes in $\mathrm{H}^1(M;\mathbb{Z})$.
- In Chapter **??** we will discuss which homology classes can be represented by submanifolds. For example, we will use the Poincaré Duality Theorem 79.1 and the Maps-to-S^1-Proposition 97.2 to prove the Codimension-One Submanifold Existence Proposition **??** which says that, given a compact orientable n-dimensional smooth manifold, every class in $\mathrm{H}_{n-1}(M)$ can be represented by a submanifold.

CHAPTER 93

Inverse limits

In Chapter 25 we discussed in great detail the direct colimit of a direct system. In this chapter we now study the "contravariant" analogue of direct colimits, namely inverse limits. These will play an important role in the following two chapters.

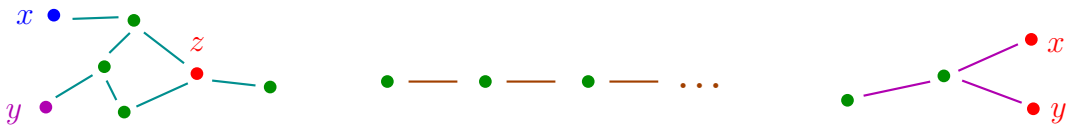
93.1. **Inverse systems and inverse limits.** First we recall the following definitions from pages 37 and 338.

Definition.

(1) Let X be a set and let $\leq$ be a relation on X.
   (a) The relation is called reflexive if for every $x \in X$ we have $x \leq x$.
   (b) The relation is called transitive if for every $x, y, z \in X$ we have
   $$x \leq y \text{ and } y \leq z \implies x \leq z.$$
   (c) The relation is called antisymmetric if $x \leq y$ and $y \leq x$ implies $x = y$.
   (d) A relation that is reflexive, transitive and antisymmetric is called a partial order.
(2) A directed set is a set X together with a partial order $\leq$ that has the extra property, that for any $x, y \in X$ there exists a $z \in X$ with $x \leq z$ and $y \leq z$.

Examples.

(1) We consider the three partial orders that are illustrated in the figure below. Here it is understood that $P \leq Q$ if P is connected to Q via a path that goes throughout from left to right. We see that the two partial orders to the left form directed sets whereas the one on the right does not form a directed set.



directed set directed set not a directed set

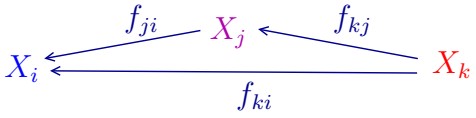
(2) Let X be a subset of $\mathbb{R}$. If we denote by $\leq$ the usual "less or equal" relation on the real numbers, then $(X, \leq)$ is a directed set. In particular $(\mathbb{N}, \leq)$ is a directed set.
(3) We let $I = \mathbb{N}$, but this time we use the partial order given by divisibility, i.e. we write $k \leq l$ if and only if $k \,|\, l$. This turns $\mathbb{N}$ into a direct set, since given $x, y \in \mathbb{N}$ we have $x \,|\, xy$ and $y \,|\, xy$. □

The following definition is basically the same as the definition of a direct system on page 339, except that all arrows get reversed. We will make use of this concept only rarely, mostly when we study cohomology groups.

Definition. Let $(I, \leq)$ be a directed set and let $\mathcal{C}$ be a category. An inverse system in the category $\mathcal{C}$ over I is a diagram in the opposite category $\mathcal{C}^{\mathrm{op}}$ over the partially ordered set $(I, \leq)$ in the sense of the definition on page 106. In other words, an inverse system in the category $\mathcal{C}$ over I is a family of objects $\{X_i\}_{i \in I}$ in $\mathcal{C}$, together with a family of morphisms $\{f_{ji} \colon X_j \to X_i\}$ for all $i, j \in I$ with $i \leq j$[743] such that the following two conditions are satisfied:

(1) $f_{ii} = \mathrm{id}_{X_i}$ for all $i \in I$,

(2) $f_{ki} = f_{ji} \circ f_{kj}$ for all $i, j, k \in I$ with $i \leq j \leq k$.



Example.

(A) We let $I = \mathbb{N}$. As above we use the partial order given by divisibility, i.e. we write $k \leq l$ if and only if $k \,|\, l$. For each $k \in \mathbb{N}$ we define $X_k := \mathbb{Z}_k$ and for $k \,|\, l$ we denote by

$$\begin{array}{rcl} f_{lk} \colon \mathbb{Z}_l & \to & \mathbb{Z}_k \\ n + l \cdot \mathbb{Z} & \mapsto & n + k \cdot \mathbb{Z} \end{array}$$

the natural projection. Clearly $(\{\mathbb{Z}_k\}_{k \in \mathbb{N}}, \{f_{lk}\}_{k \,|\, l})$ forms an inverse system in the category *Ring* of rings. □

When we studied direct systems we saw that most direct systems we were interested in arose from "direct sequences". Similarly we will see that most inverse systems we are interested will arise from "inverse sequences". The following definition of an "inverse sequence" is almost identical to the definition of a "direct sequence" on page 339, we just need to "invert the arrows":

Definition. Let $\mathcal{C}$ be a category and let $w \in \mathbb{Z}$. A inverse sequence in the category $\mathcal{C}$ consists of a family $\{X_i\}_{i \in \mathbb{Z}_{\geq w}}$ of objects and a family of morphisms $\{\varphi_i \colon X_i \to X_{i-1}\}_{i \in \mathbb{Z}_{\geq w+1}}$. Usually we write an inverse sequence as follows:

$$X_w \xleftarrow{\varphi_{w+1}} X_{w+1} \xleftarrow{\varphi_{w+2}} X_{w+2} \xleftarrow{\varphi_{w+3}} \dots$$

An inverse sequence as above gives rise to an inverse system over the directed set $(\mathbb{Z}_{\geq w}, \leq)$ by considering for $i \leq j$ the morphism $f_{ji} := \varphi_{i+1} \circ \dots \circ \varphi_{j-1} \circ \varphi_j \colon X_j \to X_i$.

Examples.

(B) The following is an inverse sequence in the category *AbGroup* of abelian groups:

$$\mathbb{Z} \leftarrow \mathbb{Z}^2 \leftarrow \mathbb{Z}^3 \leftarrow \dots.$$

Here all homomorphisms are the projections onto the first coordinates:

$$\begin{array}{rcl} \mathbb{Z}^n & \leftarrow & \mathbb{Z}^{n+1} \\ (x_1, \dots, x_n) & \leftarrow\!\shortmid & (x_1, \dots, x_n, x_{n+1}). \end{array}$$

□

(C) Let p be a prime. The following is an inverse sequence in the category *Ring* of rings:

$$\mathbb{Z}/p \cdot \mathbb{Z} \leftarrow \mathbb{Z}/p^2 \cdot \mathbb{Z} \leftarrow \mathbb{Z}/p^3 \cdot \mathbb{Z} \leftarrow \dots.$$

[743] So here for $i \leq j$ we have a morphism $X_j \to X_i$ instead of $X_i \to X_j$ in a direct system.

Here all homomorphisms are the obvious epimorphisms

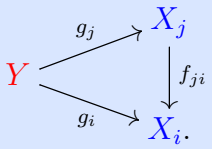
$$\begin{array}{ccc} \mathbb{Z}/p^i\cdot\mathbb{Z} & \leftarrow & \mathbb{Z}/p^{i+1}\cdot\mathbb{Z} \\ n+p^i\cdot\mathbb{Z} & \leftarrow\!\shortmid & n+p^{i+1}\cdot\mathbb{Z}. \end{array}$$

□

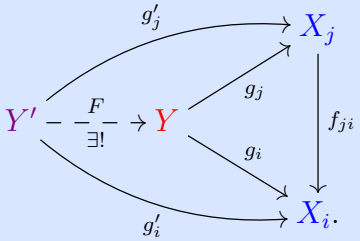
The following definition is the "mirror" image of the definition of the direct colimit of a direct system which we gave on page 339:

Definition. Let $(I,\leq)$ be a directed set and let $\mathcal{C}$ be a category. Suppose we are given an inverse system $(\{X_i\}_{i\in I},\{f_{ji}\colon X_j\to X_i\}_{i\leq j})$ in the category $\mathcal{C}$. An inverse limit[744] of the inverse system is a limit, in the sense of the definition on page 107 of the inverse system, viewed as a diagram in the opposite category $\mathcal{C}^{\mathrm{op}}$. In other words, an inverse limit is an object Y in $\mathcal{C}$ together with morphisms $g_i\colon Y\to X_i$, $i\in I$, in the category $\mathcal{C}$ such that the following two conditions are satisfied:

(1) For all $i\leq j$ we have $f_{ji}\circ g_j = g_i\colon Y\to X_i$, i.e. for all $i\leq j$ the following diagram commutes:



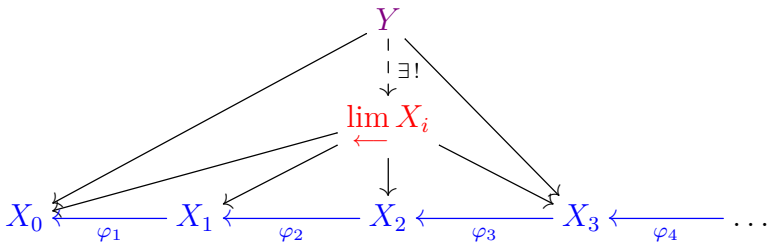
(2) If we are given another object Y' and morphisms $g'_i\colon Y'\to X_i$, $i\in I$, that satisfy (1), then there exists a unique morphism $F\colon Y'\to Y$ such that for all $i\in I$ we have $g'_i = g_i\circ F$. In particular for all $i\leq j$ the following diagram commutes:



By the Limit Uniqueness Lemma 4.5 we know, that if an inverse limit exists, then it is unique in the usual sense. We denote the inverse limit, if it exists, by $\varprojlim X_i$ or sometimes by $\varprojlim_I X_i$.

Remark.

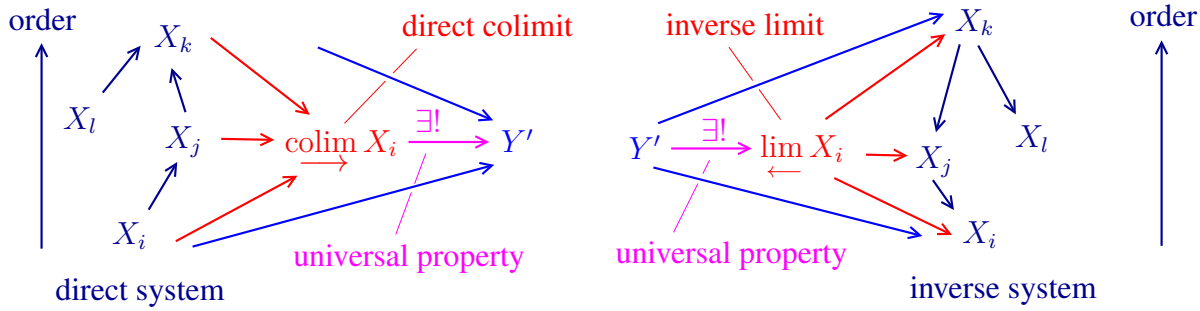
(1) In the case of an inverse sequence $X_0 \xleftarrow{\varphi_1} X_1 \xleftarrow{\varphi_2} X_2 \xleftarrow{\varphi_3} \dots$ of morphisms the notion of an inverse limit can be summarized in the following diagram:



[744] Somewhat confusingly an "inverse limit" is often also called an "inductive limit".

More precisely, an inverse limit $\varprojlim X_i$, if it exists, comes with morphisms $\varprojlim X_i \to X_i$ which form commutative diagrams, furthermore given morphisms $Y \to X_i$ which form commutative diagrams, we get a unique morphism $Y \to \varprojlim X_i$ which makes everything commute.

(2) In the following diagram we compare schematically the definition of the direct colimit of a direct system to the inverse limit of an inverse system:



The difference between a direct colimit and an inverse limit is not the direction of the arrows between the X_i. In fact given a partial order "$\leq$" one could define a partial order "$\widetilde{\leq}$" via $a \mathrel{\widetilde{\leq}} b$ if and only if $b \leq a$, hence a direct system over $(I, \leq)$ is an inverse system over $(I, \widetilde{\leq})$. The real difference is whether the X_i map to the colimit, or whether the limit maps to the X_i. □

93.2. **Existence of inverse limits.** We saw in the Sets–Direct Colimit Existence Proposition 25.2, the Algebraic–Direct Colimit Existence Proposition 25.3 and the Topological-Direct Colimit Proposition 25.5 that in categories we cared about the direct colimit could be viewed as a quotient of a coproduct. We will now see that the inverse limit can be viewed as a "subobject" of a product. Since in all categories we care about, the product has the same underlying set, we can give a uniform description of inverse limits:

Proposition 93.1. (Inverse Limit Existence Proposition) Suppose we are dealing with one of the following three categories:

(a) the category *Set* of sets,
(b) the category *Top* of topological spaces,
(c) the category *Group* of groups.
(d) the category *AbGroup* of abelian groups,
(e) the category *Ring* of rings.

In these categories we can explicitly construct inverse limits:

(1) Let $(I, \leq)$ be a directed set and let $(\{X_i\}_{i\in I}, \{f_{ji}\colon X_j \to X_i\}_{i\leq j})$ be an inverse system in one of the above categories. We set

$$\varprojlim X_i \;:=\; \Big\{(x_i)_{i\in I} \in \prod_{i\in I} X_i \,\Big|\, f_{ji}(x_j) = x_i \text{ for all } i \leq j\Big\}.$$

With the appropriate structures (topology, multiplication, etc.) this set, together with the obvious projections, become the inverse limit of the inverse system.

(2) Suppose we are given an inverse sequence

$$X_1 \xleftarrow{\varphi_2} X_2 \xleftarrow{\varphi_3} X_3 \xleftarrow{\varphi_4} \dots$$

As discussed on page 1224 such an inverse sequence gives rise to an inverse system. In this setting we can simplify the description of the set in (1) and we have

$$\varprojlim X_i = \big\{(x_1, x_2, x_3, \dots) \mid x_n \in X_n \text{ for all } n \in \mathbb{N} \text{ and } \varphi_n(x_n) = x_{n-1} \text{ for all } n \in \mathbb{N}_{\geq 2}\big\}.$$

Remark. Let

$$X_1 \xleftarrow{\varphi_2} X_2 \xleftarrow{\varphi_3} X_3 \xleftarrow{\varphi_4} \dots$$

be an inverse sequence in one of the three categories that we considered in the Inverse Limit Existence Proposition 93.1. We can visualize the inverse limit, as described in Inverse Limit Existence Proposition 93.1 (2), as the set (topological space, group, etc.) of all infinite sequences

$$\begin{array}{ccccccccc} X_1 & \xleftarrow{\varphi_2} & X_2 & \xleftarrow{\varphi_3} & X_3 & \xleftarrow{\varphi_4} & X_4 & \xleftarrow{\varphi_5} & \dots \\ x_1 & \longmapsto & x_2 & \longmapsto & x_3 & \longmapsto & x_4 & \longmapsto & \dots \end{array}$$

□

Proof. First we prove Statement (1). Thus let $(I, \leq)$ be a directed set and let $\mathcal{C}$ be one of the three given categories. Furthermore let $(\{X_i\}_{i\in I}, \{f_{ji} \colon X_j \to X_i\}_{i\leq j})$ be an inverse system in the category $\mathcal{C}$. We set

$$\varprojlim X_i := \Big\{(x_i)_{i\in I} \in \prod_{i\in I} X_i \,\Big|\, f_{ji}(x_j) = x_i \text{ for all } i \leq j\Big\}.$$

Furthermore we denote by $g_i \colon \varprojlim X_i \to X_i$ the natural projections. It follows immediately from the definitions that for any $i \leq j$ we have $f_{ji} \circ g_j = g_i$. Furthermore, given an object Y' and morphisms $g'_i \colon Y' \to X_i$, $i \in I$ which satisfy the condition $f_{ji} \circ g'_j = g'_i$ for all $i \in I$ we set

$$\begin{array}{rcl} F \colon Y' & \to & \varprojlim X_i \\ y & \mapsto & \begin{pmatrix} I & \to & \prod_{i\in I} X_i \\ i & \mapsto & g'_i(y) \end{pmatrix}. \end{array}$$

It is straightforward to verify that $F(Y')$ lies in $\varprojlim X_i$ and that F is the unique map with the desired property. This shows that the inverse limit exists in the category of sets.[745]

For the remaining four categories we will use the same definition of the direct colimit as a set and the same maps, but we now have to verify that we have objects and morphisms in the given category.

(b) Suppose that we work in the category of topological spaces. We equip $\prod_{i\in I} X_i$ with the product topology that we defined in the Topological-Product Proposition 5.1. We equip $\varprojlim X_i \subset \prod_{i\in I} X_i$ with the subspace topology. It follows from the Topological-Product Proposition 5.1 (1a) and the Basics-of-Continuity Lemma 2.1 (3) that the maps $g_i \colon \varprojlim X_i \to X_i$ are continuous and it follows from the Topological-Product Proposition 5.1 (1b) together with the Basics-of-Continuity Lemma 2.1 (4) that the map $F \colon Y \to \varprojlim X_i$ is continuous.

(c) In the category *Group* of groups we view $\prod_{i\in I} X_i$ as a group via component-wise multiplication. Since the f_{ij} are group homomorphisms we see that $\varprojlim X_i$ is a

[745] Now we will see that we can use the "same" inverse limit for all the categories. This is very different from the construction of the direct colimit for the same categories. There, in the proof of the Direct Algebraic–Limit Existence Proposition 25.3 and the Topological-Direct Colimit Proposition 25.5, we had to use several different construction.

subgroup of $\prod_{i\in I} X_i$. It is straightforward to verify that the above maps are all morphisms in the category *Group* of groups.

(d),(e) These categories are dealt with the same way as (c).

Finally note that Statement (2) is an elementary consequence of the definitions. ■

Now we return to our initial examples of inverse systems and their inverse limits. We will deal with them in a slightly modified order:

Examples.

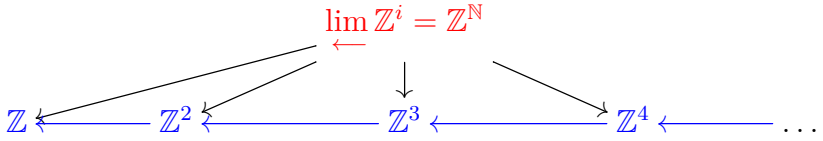
(B) As on page 1224 we consider the inverse sequence

$$\mathbb{Z} \leftarrow \mathbb{Z}^2 \leftarrow \mathbb{Z}^3 \leftarrow \dots$$

of abelian groups that is given by the projections onto the first coordinates. One can show quite easily, either by hand or using the Inverse Limit Existence Proposition 93.1 (2) that[746]

$$\varprojlim_{i\in\mathbb{N}} \mathbb{Z}^i \;=\; \{(x_1, x_2, x_3, \dots) \mid x_j \in \mathbb{Z}\} \;\cong\; \mathbb{Z}^{\mathbb{N}}$$

and where the maps from $\mathbb{Z}^{\mathbb{N}} \to \mathbb{Z}^n$ are the projections onto the first n coordinates. We can summarize the situation in the following diagram:



(C) Let p be a prime. The inverse limit $\widehat{\mathbb{Z}}_p := \varprojlim \mathbb{Z}/p^i \cdot \mathbb{Z}$ of the rings $\mathbb{Z}/p^i\mathbb{Z}$ is called the *ring of the p-adic numbers.*[747] The p-adic numbers play a very prominent role in many branches of pure mathematics.

(A) The inverse limit $\widehat{\mathbb{Z}} := \varprojlim \mathbb{Z}_n$ of the rings $\mathbb{Z}_n$, $n \in \mathbb{N}$, with the partial order on $\mathbb{N}$ defined by the divisibility relation $k \mid l$, is called the *ring of the profinite integers.* Using the Chinese Remainder Theorem 18.2 one can show that there exists a ring isomorphism

$$\widehat{\mathbb{Z}} \;\cong\; \prod_{\text{prime } p} \widehat{\mathbb{Z}}_p.$$

□

Inverse limits will play a minor role in these lecture notes. They will mostly arise through the following lemma, which plays the role of the Tensor–Direct Colimit Lemma **??**.

Lemma 93.2. (Hom-of-Direct Colimit Lemma) Let $(I, \leq)$ be a directed set, let $(\{A_i\}_{i\in I}, \{\varphi_{ij} \colon A_i \to A_j\}_{i\leq j})$ be a direct system of abelian groups and let G be an abelian group. We can consider the corresponding direct colimit $\varinjlim A_i$ as defined on page 339. The natural homomorphisms $A_j \to \varinjlim A_i$ induce a natural isomorphism

$$\varprojlim \operatorname{Hom}(A_i, G) \;\xrightarrow{\cong}\; \operatorname{Hom}\left(\varinjlim A_i, G\right)$$

of abelian groups (or G-modules if G is a commutative ring).

[746] Note that $\mathbb{Z}^{\mathbb{N}}$ is an uncountable group.

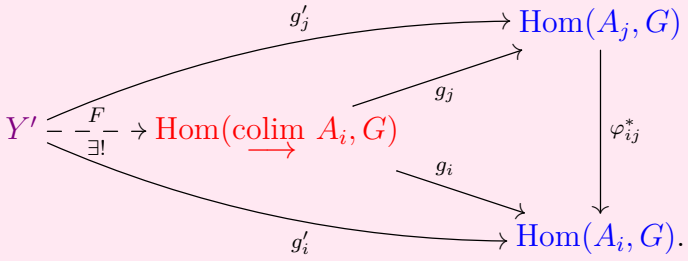
[747] In number theory one often uses a different convention, namely one writes explicitly $\mathbb{Z}/p\mathbb{Z}$ for the quotient ring and one denotes the p-adic numbers by $\mathbb{Z}_p$. If the expression "$\mathbb{Z}_p$" appears in a paper or a book it is a good idea to check the convention used by the author.

Proof. Let $(I, \leq)$ be a directed set, let $(\{A_i\}_{i\in I}, \{\varphi_{ij}\colon A_i \to A_j\}_{i\leq j})$ be a direct system of abelian groups and let G be an abelian group. It follows easily from the definitions and the elementary Hom–Contravariant Functor Lemma 66.1 that

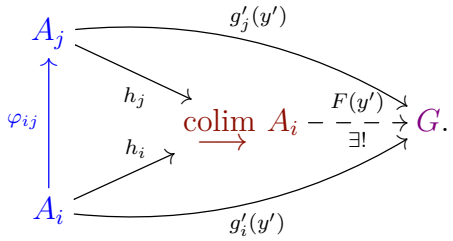
$$(\{\mathrm{Hom}(A_i, G)\}_{i\in I}, \{\varphi_{ij}^*\colon \mathrm{Hom}(A_j, G) \to \mathrm{Hom}(A_i, G)\}_{i\leq j})$$

is an inverse system of abelian groups. The natural homomorphisms $h_j\colon A_j \to \varinjlim A_i$ induce homomorphisms $g_j\colon \mathrm{Hom}(\varinjlim A_i, G) \to \mathrm{Hom}(A_j, G)$. By the uniqueness of inverse limits it remains to show that $\mathrm{Hom}(\varinjlim A_i, G)$ together with the homomorphisms g_j has the universal property an inverse limit that we introduced on page 1225. Put differently, we need to prove the following claim:

Claim. Let Y' be an abelian group and let $g_i'\colon Y' \to \mathrm{Hom}(A_i, G)$ be homomorphisms such that $\varphi_{ij}^* \circ g_j' = g_i'\colon Y' \to \mathrm{Hom}(A_i, G)$ for all $i \leq j$. Then there exists a unique homomorphism $F\colon Y' \to \mathrm{Hom}(\varinjlim A_i, G)$ such that the following diagram commutes:



Proof. Let Y' be an abelian group and let $g_i'\colon Y' \to \mathrm{Hom}(A_i, G)$ be homomorphisms as above. Let $y' \in Y'$. By the definition on page 339 of a direct colimit we obtain a unique homomorphism $F(y')\colon \varinjlim A_i \to G$ such that the following diagram commutes:



This discussion defines a map

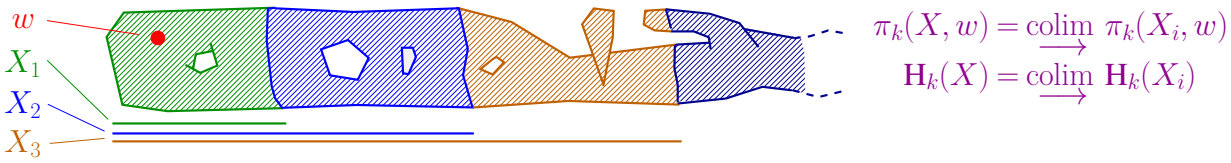
$$\begin{aligned} F\colon Y' &\to \mathrm{Hom}\left(\varinjlim A_i, G\right) \\ y' &\mapsto F(y'). \end{aligned}$$

Using similar arguments as above one shows that this map is a homomorphism and that it is the unique homomorphism that makes the diagram of the claim commute. ∎

CHAPTER 94

Cohomology groups and limits

In the Exhaustion-π_1-Proposition 26.2 and the Direct Colimit-H_*-Proposition 46.2 we saw that in favorable situations the fundamental group and the homology groups of an ascending union of topological spaces are the direct colimit of the corresponding fundamental groups and homology groups.



In this chapter we will try to extend these results to cohomology groups. The results of this chapter will not only be useful for computing cohomology groups, but they will also play an essential role in many proofs later on. For example the proof that de Rham cohomology is isomorphic to singular cohomology with real coefficients will build on the somewhat technical results of this chapter.

94.1. **Homology of ascending unions.** For the reader's convenience we recall the Direct Colimit-H_*-Proposition 46.2 which we already alluded to in the above introduction:

Proposition 46.2. (Exhaustions-H_*-Proposition) Let X be a topological space and let $X_0 \subset X_1 \subset X_2 \subset \dots$ be a sequence of subspaces such that $X = \bigcup_{i\in\mathbb{N}_0} X_i$ and such that one of the following conditions holds:

(1) each X_i is contained in the interior of X_{i+1}, or
(2) X is a CW-complex and each X_i is a subcomplex of X,

then for any $n \in \mathbb{N}_0$ the inclusion-induced maps $\mathrm{H}_n(X_i) \to \mathrm{H}_n(X)$ induce an isomorphism[748]

$$\operatorname{colim} \mathrm{H}_n(X_i) \xrightarrow{\cong} \mathrm{H}_n(X).$$

In the following we outline the proof of the Exhaustions-H_*-Proposition 46.2. First note that singular homology of a topological space is the composition of the following two covariant functors:

$$\begin{array}{rcll} X & \mapsto & (\mathrm{C}_*(X), \partial_*) & \text{(singular chain complex functor)} \\ (C_*, \partial_*) & \mapsto & \mathrm{H}_k(C_*, \partial_*) & \text{(homology functor)} \end{array}$$

Therefore, to prove Exhaustions-H_*-Proposition 46.2 it made sense to consider the two functors separately. We did so with the following two lemmas. The first lemma is of topological nature:

Lemma 46.1. (Exhaustion-C_*-Lemma) Let X be a topological space. Furthermore let $X_1 \subset X_2 \subset X_3 \subset \ldots$ be a sequence of subspaces of X with $\bigcup_{i\in\mathbb{N}_0} X_i = X$. Then the chain complexes $\mathrm{C}_*(X_i), i \in \mathbb{N}$ together with the inclusion-induced chain maps $\mathrm{C}_*(X_i) \to \mathrm{C}_*(X_j)$ for $i \leq j$ form a direct system of chain complexes. We now suppose that one of the following conditions is satisfied:

(1) each X_i is contained in the interior of X_{i+1}, or
(2) X is a CW-complex and each X_i is a subcomplex of X.

Then the inclusion-induced maps $\mathrm{C}_*(X_i) \to \mathrm{C}_*(X)$ induce an isomorphism

$$\operatorname*{colim}_{\longrightarrow} \mathrm{C}_*(X_i) \xrightarrow{\cong} \mathrm{C}_*(X)$$

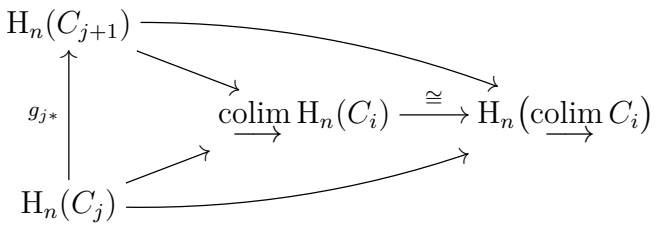
of chain complexes.

The second lemma is purely algebraic:

Lemma 46.5. (Direct Colimit-H_*-Lemma) Let $(\{C_i\}_{i\in\mathbb{N}}, \{g_i \colon C_i \to C_{i+1}\}_{i\in\mathbb{N}})$ be a direct sequence of chain complexes. Then for each $n \in \mathbb{N}_0$ we obtain an induced direct system $(\{\mathrm{H}_n(C_i)\}_{i\in\mathbb{N}}, \{g_{i*}\}_{i\in\mathbb{N}})$ of abelian groups. Furthermore, given any $n \in \mathbb{N}_0$ the maps $\mathrm{H}_n(C_i) \to \mathrm{H}_n\big(\operatorname*{colim}_{\longrightarrow} C_i\big)$ induce an isomorphism

$$\operatorname*{colim}_{\longrightarrow} \mathrm{H}_n(C_i) \xrightarrow{\cong} \mathrm{H}_n\big(\operatorname*{colim}_{\longrightarrow} C_i\big).$$

Let $(\{C_i\}_{i\in\mathbb{N}}, \{g_i \colon C_i \to C_{i+1}\}_{i\in\mathbb{N}})$ be a direct sequence of chain complexes. The Direct Colimit-H_*-Lemma 46.5 can be summarized as follows: For every $j \in \mathbb{N}$ and any $n \in \mathbb{N}_0$ we have the following diagram:



Finally note that the Exhaustions-H_*-Proposition 46.2 is an almost immediate consequence of the Exhaustion-C_*-Lemma 46.1 and the Direct Colimit-H_*-Lemma 46.5.

94.2. **The $\lim^1$-functor and the Mittag-Leffler condition.** Singular cohomology of a topological space with G-coefficients is the composition of three functors:

$$\begin{array}{rcll} X & \mapsto & (\mathrm{C}_*(X), \partial_*) & \text{(singular chain complex functor)} \\ (C_*, \partial_*) & \mapsto & (\mathrm{Hom}(C_*, G), \partial^*) & \text{(hom-functor)} \\ (D^*, \delta^*) & \mapsto & \mathrm{H}^k(D^*, \delta^*) & \text{(cohomology functor)} \end{array}$$

So if we want to understand cohomology of an ascending union we should again consider the three functors separately. Fortunately we already studied the first functor in the above Exhaustions-H_*-Proposition 46.2. We also already considered the second functor in the following lemma:

Lemma 93.2. (Hom-of-Direct Colimit Lemma) Let $(I, \leq)$ be a directed set, let $(\{A_i\}_{i\in I}, \{\varphi_{ij} \colon A_i \to A_j\}_{i\leq j})$ be a direct system of abelian groups and let G be an abelian

group. We can consider the corresponding direct colimit $\operatorname{colim}_{\longrightarrow} A_i$ as defined on page 339. The natural homomorphisms $A_j \to \operatorname{colim}_{\longrightarrow} A_i$ induce a natural isomorphism

$$\lim_{\longleftarrow} \operatorname{Hom}(A_i, G) \xrightarrow{\cong} \operatorname{Hom}\left(\operatorname{colim}_{\longrightarrow} A_i, G\right)$$

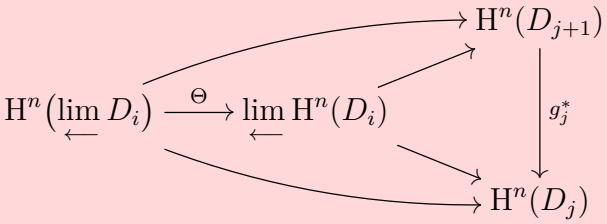
of abelian groups (or G-modules if G is a commutative ring).

The composition of the Exhaustion-C_*-Lemma 46.1 and the Hom-of-Direct Colimit Lemma 93.2 together with the Dual Cochain Map Lemma 66.9 leads us to the following question:

Question 94.1. Let $(\{D_i\}_{i\in\mathbb{N}}, \{g_i\colon D_{i+1} \to D_i\}_{i\in\mathbb{N}})$ be an inverse sequence of cochain complexes and let $n \in \mathbb{N}_0$. Note that the natural cochain maps $\lim_{\longleftarrow} D_i \to D_i$ induce homomorphisms $\lim_{\longleftarrow} \mathrm{H}^n(D_i) \to \mathrm{H}^n(D_i)$ which, by the universal property of the inverse limit $\lim_{\longleftarrow} \mathrm{H}^n(D_i)$, induce a unique homomorphism

$$\Theta\colon \mathrm{H}^n(D_i) \to \mathrm{H}^n\left(\lim_{\longleftarrow} D_i\right)$$

such that the following diagram commutes:



The question that arises is the following: Is the above map

$$\Theta\colon \mathrm{H}^n\left(\lim_{\longleftarrow} D_i\right) \to \lim_{\longleftarrow} \mathrm{H}^n(D_i)$$

always an isomorphism?

It turns out that, in contrast to the Direct Colimit-H_*-Lemma 46.5, this is not necessarily the case. To formulate the relationship between the two sides of the above question it is useful to introduce a piece of notation and to rewrite the inverse limit from the Inverse Limit Existence Proposition 93.1 (1):

Notation. Given an inverse sequence $(\{D_i\}_{i\in\mathbb{N}}, \{g_i\colon D_{i+1} \to D_i\}_{i\in\mathbb{N}})$ of abelian groups we consider the homomorphism

$$\begin{array}{rccc} \mu_{D_i}\colon & \prod_{i\in\mathbb{N}} D_i & \to & \prod_{i\in\mathbb{N}} D_i \\ & (x_i)_{i\in\mathbb{N}} & \mapsto & (x_i - g_i(x_{i+1}))_{i\in\mathbb{N}}. \end{array}$$

Remark. With the above notation the Inverse Limit Existence Proposition 93.1 can be reformulated as saying that $\lim_{\longleftarrow} D_i = \ker(\mu_{D_i})$. □

This leads us to the following definition:

Definition. Let $(\{D_i\}_{i\in\mathbb{N}}, \{g_i\colon D_{i+1} \to D_i\}_{i\in\mathbb{N}})$ be an inverse sequence of abelian groups (or of modules over a commutative ring R). Using the above notation we define[749]

$$\lim_{\longleftarrow}{}^1 D_i := \operatorname{coker}(\mu_{D_i}).$$

Example. We consider the inverse sequence

$$D_*: \quad 0 \xleftarrow{0} \mathbb{Z} \xleftarrow{\cdot 2} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{\cdot 2} \mathbb{Z} \leftarrow \dots.$$

We invite the reader to show that $\varprojlim D_* = 0$ and $\varprojlim^1 D_* = 0$. We also consider the inverse sequence

$$E_*: \quad 0 \xleftarrow{\cdot 2} \mathbb{Z} \xleftarrow{\cdot 3} \mathbb{Z} \xleftarrow{4} \mathbb{Z} \xleftarrow{\cdot 5} \mathbb{Z} \leftarrow \dots.$$

We also invite the reader to show that $\varprojlim^1 E_* \neq 0$. □

We continue with the following definition:

Definition. Let $\mathcal{C}$ be a category such that limits always exist and let $(X_i, f_i)_{i\in\mathbb{N}}$ and $(Y_i, g_i)_{i\in\mathbb{N}}$ be two inverse sequences in $\mathcal{C}$.

(1) A morphism of inverse sequences from (X_i, f_i) to (Y_i, g_i) consists of a sequence of morphisms $u_i \colon X_i \to Y_i$, $i \in \mathbb{N}$ such that for each i we have the equality $g_i \circ u_i = u_{i-1} \circ f_i$.

(2) Given a morphism of inverse sequences it follows from the definition of an inverse limit that the morphisms $\varprojlim X_i \to X_i \to Y_i$ induce a morphism

$$u_* \colon \varprojlim X_i \ \to \ \varprojlim Y_i.$$

(3) We now work in the category of abelian groups. Let $u \colon (X_i, f_i) \to (Y_i, g_i)$ be a morphism between inverse sequences. It follows easily from the definitions that the homomorphism $u \colon \prod_{i\in\mathbb{N}} X_i \to \prod_{i\in\mathbb{N}} Y_i$ restricts to a homomorphism

$$u_* \colon \varprojlim{}^1 X_i \ \to \ \varprojlim{}^1 Y_i.$$

In the following we will at times implicitly make use of the following lemma:

Lemma 94.2. (Inverse Limit Functor Lemma) The above notion of morphisms allows us to consider the category *InvSeq* of inverse sequences of abelian groups. The maps

$$(X_i, f_i)_{i\in\mathbb{N}} \ \mapsto \ \varprojlim (X_i, f_i) \qquad \text{and} \qquad (X_i, f_i)_{i\in\mathbb{N}} \ \mapsto \ \varprojlim{}^1 (X_i, f_i)$$

both define covariant functors from the category *InvSeq* to the category *AbGroup* of abelian groups. An analogous statement holds if, given a commutative ring R, we work with the category of R-modules instead of the category of abelian groups.

Proof. The lemma follows easily from the definitions. ■

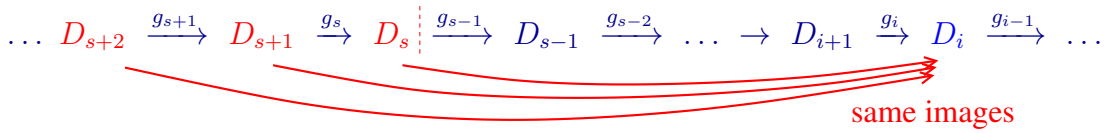
We cannot proceed without introducing the following somewhat technical definition.

Definition. Let $(\{D_i\}_{i\in\mathbb{N}}, \{g_i \colon D_{i+1} \to D_i\}_{i\in\mathbb{N}})$ be an inverse sequence of abelian groups. We say the inverse sequence satisfies the Mittag-Leffler condition[750] [751] if for every $i \in \mathbb{N}$ there exists an $n \geq i$ such that for all $s \geq n$ we have

$$\operatorname{im}(g_i \circ \dots \circ g_{s-1} \colon D_s \to D_i) \ = \ \operatorname{im}(g_i \circ \dots \circ g_{n-1} \colon D_n \to D_i).$$

.

[749] If R is a commutative ring and if $(\{D_i\}_{i\in\mathbb{N}}, \{g_i \colon D_{i+1} \to D_i\}_{i\in\mathbb{N}})$ is an inverse sequence of R-modules, then $\varprojlim^1 D_i$ is also an R-module in an obvious way.



Example. It is straightforward to verify that the above inverse system D_* satisfies the Mittag-Leffler condition whereas the above inverse system E_* does not satisfy the Mittag-Leffler condition. □

In many situations one can ignore the precise definition of the Mittag-Leffler condition and instead one can appeal to the following lemma:

Lemma 94.3. (Mittag-Leffler Epimorphism Lemma) Suppose that we are given an inverse sequence $(\{D_i\}_{i\in\mathbb{N}}, \{g_i\colon D_{i+1} \to D_i\}_{i\in\mathbb{N}})$ of abelian groups. If there exists an $m \in \mathbb{N}$ such that for all $i \geq m$ the map $g_i\colon D_{i+1} \to D_i$ is an epimorphism, then the inverse sequence satisfies the Mittag-Leffler condition.

Proof. The lemma follows almost immediately from the definitions. ■

Our main interest in the Mittag-Leffler condition arises from the fact that if an inverse sequence satisfies this condition, then the pesky $\varprojlim^1$-limit vanishes:

Lemma 94.4. (Mittag-Leffler Limit Lemma) Let $(\{D_i\}_{i\in\mathbb{N}}, \{g_i\colon D_{i+1} \to D_i\}_{i\in\mathbb{N}})$ be an inverse sequence of abelian groups. If this inverse sequence satisfies the Mittag-Leffler condition, then

$$\varprojlim{}^1 D_i \;=\; 0.$$

Proof. Let $(\{D_i\}_{i\in\mathbb{N}}, \{g_i\colon D_{i+1} \to D_i\}_{i\in\mathbb{N}})$ be an inverse sequence of abelian groups that satisfies the Mittag-Leffler condition. This means that given given any $i \in \mathbb{N}$ there exists an $n(i) \geq i$ such that for all $s \geq n(i)$ we have

$$\operatorname{im}(g_i \circ \cdots \circ g_{s-1}\colon D_s \to D_i) \;=\; \operatorname{im}(g_i \circ \cdots \circ g_{n(i)-1}\colon D_{n(i)} \to D_i).$$

Without loss of generality we can assume that $n(i)$ is monotonously increasing in i. We have to show that the map

$$\begin{array}{rcl} \mu_{D_i}\colon \prod\limits_{i\in\mathbb{N}} D_i & \to & \prod\limits_{i\in\mathbb{N}} D_i \\ (x_i)_{i\in\mathbb{N}} & \mapsto & (x_i - g_i(x_{i+1}))_{i\in\mathbb{N}} \end{array}$$

is an epimorphism.

Given $j \geq i$ we now write $f_{ji} \coloneqq g_i \circ \cdots \circ g_{j-1}\colon D_j \to D_i$. First we consider $(a_i)_{i\in\mathbb{N}} \in \prod\limits_{i\in\mathbb{N}} D_i$ with the property that for all $i \in \mathbb{N}$ we have $a_i \in \operatorname{im}(f_{n(i),i}\colon D_{n(i)} \to D_i)$. To show that $\{a_i\}_{i\in\mathbb{N}}$ lies in the image of μ_{D_i} it suffices to prove the following claim:

Claim 1. There exist $b_i \in \operatorname{im}(f_{n(i),i}\colon D_{n(i)} \to D_i)$ with $i \in \mathbb{N}$ such that for each i we have $a_i = b_i - g_i(b_{i+1})$.

[750]The name "Mittag-Leffler condition" comes from the observation, that it is in a rather roundabout way connected to the classical Mittag-Leffler theorem from complex analysis. We refer to [**Bou98**, Chapter 3.5] and [**Run05**, Theorem 2.4.14] for a few more details.

[751]For $n = i$ the map $g_i \circ \cdots \circ g_{n-1}\colon D_n \to D_i$ is a homomorphism $D_n \to D_n$ which happens to the composition of the empty set of homomorphisms. The composition of the empty set of morphisms is always understood to be the identity.

Proof. We define the b_i inductively. We set $b_0 = a_0$. Suppose that $b_0, \dots, b_k$ are already defined with the desired properties. By our hypothesis we have $a_k \in \operatorname{im}(f_{n(k),k})$ and by our choice of b_k we have $b_k \in \operatorname{im}(f_{n(k),k})$. By the Mittag-Leffler condition and since we arranged that $n(k+1) \geq n(k)$ we have $\operatorname{im}(f_{n(k+1),k}) = \operatorname{im}(f_{n(k),k})$. Thus we can pick $c \in D_{n(k+1)}$ with $f_{n(k+1),k}(c) = a_k - b_k$. Now we put

$$b_{k+1} \;:=\; -f_{n(k+1),k+1}(c).$$

Then

$$b_k - g_{k+1}(b_{k+1}) = b_k + g_{k+1}(f_{n(k+1),k+1}(c)) = b_k + \underbrace{(g_{k+1} \circ f_{n(k+1),k+1})}_{=f_{(n(k+1),k)}}(c) = b_k + (a_k - b_k) = a_k.$$

We have thus shown that $b_0, \dots, b_{k+1}$ have the desired properties. ⊞

Now let $(b_i)_{i\in\mathbb{N}} \in \prod\limits_{i\in\mathbb{N}} D_i$ be an arbitrary element. For each $i \in \mathbb{N}$ we set

$$c_i \;:=\; b_i + \sum_{j=i+1}^{n(i)} f_{j,i}(b_j).$$

Claim 2. For each $i \in \mathbb{N}$ we have $b_i - (c_i - g_i(c_{i+1})) \in \operatorname{im}(f_{n(i),i})$.

Proof. Let $i \in \mathbb{N}$. We have

$$\begin{aligned} b_i - (c_i - g_i(c_{i+1})) \;&=\; b_i - b_i - \sum_{j=i+1}^{n(i)} f_{j,i}(b_j) + g_i\Big(b_{i+1} + \sum_{j=i+2}^{n(i+1)} f_{j,i+1}(b_j)\Big) \\ &=\; -\sum_{j=i+1}^{n(i)} f_{j,i}(b_j) + f_{i+1,i}(b_{i+1}) + \sum_{j=i+2}^{n(i+1)} f_{j,i}(b_j) \;=\; \sum_{j=n(i)+1}^{n(i+1)} f_{j,i}(b_j). \end{aligned}$$

But by definition of $n(i)$ we know that for each $j \in \{n(i)+1, \dots, n(i+1)\}$ we have $f_{j,i}(b_j) \in \operatorname{im}(f_{n(i),i})$. ⊞

It follows from Claim 2 and the above discussion that $(b_i - (c_i - g_i(c_{i+1})))_{i\in\mathbb{N}}$ lies in the image of μ_{D_i}. Evidently $(c_i - g_i(c_{i+1}))_{i\in\mathbb{N}}$ lies in the image of μ_{D_i}. Since μ_{D_i} is a homomorphism we see that $(b_i)_{i\in\mathbb{N}}$ also lies in the image of μ_{D_i}. ■

94.3. **Cohomology groups and inverse limits.** Now we can formulate the following proposition, which is structurally similar to the Algebraic Universal Coefficient Theorem 54.8 and the Algebraic Universal Coefficient-H*-Theorem 70.13:

Proposition 94.5. (Inverse Limit-H*-Proposition) Suppose that we are given an inverse sequence $(\{D_i^*\}_{i\in\mathbb{N}}, \{f_i \colon D_{i+1}^* \to D_i^*\}_{i\in\mathbb{N}})$ of cochain complexes. If for each $k \in \mathbb{N}_0$ the inverse sequence of cochain groups $\{D_i^k\}_{i\in\mathbb{N}}$ satisfies the Mittag-Leffler condition[752], then there exists a natural[753] homomorphism

$$\tau \colon \varprojlim{}^1 \mathrm{H}^{n-1}(D_i^*) \;\to\; \mathrm{H}^n\big(\varprojlim D_i^*\big)$$

such that the following sequence is exact:[754]

$$0 \;\to\; \varprojlim{}^1 \mathrm{H}^{n-1}(D_i^*) \;\xrightarrow{\tau}\; \mathrm{H}^n\big(\varprojlim D_i^*\big) \;\xrightarrow{\Theta}\; \varprojlim \mathrm{H}^n(D_i^*) \;\to\; 0$$

If R is a commutative ring and if the D_i^* are cochain complexes of R-modules, then the map τ is a natural R-module homomorphism.

The proof of the Inverse Limit-H*-Proposition 94.5 makes use of the following lemma:

Lemma 94.6. (Inverse Limit–Snake Lemma) Given any short exact sequence

$$0 \;\to\; (X_i, f_i) \;\xrightarrow{u_i}\; (Y_i, g_i) \;\xrightarrow{v_i}\; (Z_i, h_i) \;\to\; 0$$

of inverse sequences of abelian groups there exists a natural homomorphism

$$\delta\colon \varprojlim Z_i \;\to\; \varprojlim{}^1 X_i$$

such that the following sequence is exact:

$$0 \to \varprojlim X_i \xrightarrow{u_*} \varprojlim Y_i \xrightarrow{v_*} \varprojlim Z_i \xrightarrow{\delta} \varprojlim{}^1 X_i \xrightarrow{u_*} \varprojlim{}^1 Y_i \xrightarrow{v_*} \varprojlim{}^1 Z_i \to 0.$$

If R is a commutative ring and if all inverse sequences and all maps are in the category of R-modules, then the maps in the short exact sequence are natural R-module homomorphisms.

Proof of Lemma 94.6. We consider the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \prod_{i\in\mathbb{N}} X_i & \xrightarrow{u} & \prod_{i\in\mathbb{N}} Y_i & \xrightarrow{v} & \prod_{i\in\mathbb{N}} Z_i & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mu_{X_i}} & & \downarrow{\scriptstyle \mu_{Y_i}} & & \downarrow{\scriptstyle \mu_{Z_i}} & & \\
0 & \longrightarrow & \prod_{i\in\mathbb{N}} X_i & \xrightarrow{u} & \prod_{i\in\mathbb{N}} Y_i & \xrightarrow{v} & \prod_{i\in\mathbb{N}} Z_i & \longrightarrow & 0.
\end{array}$$

It follows almost immediately from the definitions that the diagram commutes and that the horizontal sequences are exact. We can view the rows as chain complexes. The desired natural exact sequence now follows from the long exact sequence of homology groups, the description of inverse limits on page 1232 and the definition of $\varprojlim{}^1$. ∎

Proof of Proposition 94.5. Let $(\{D_i^*\}_{i\in\mathbb{N}}, \{f_i\}_{i\in\mathbb{N}})$ be an inverse sequence of cochain complexes such that for each $k \in \mathbb{N}_0$ the inverse sequence of cochain groups $\{D_i^k\}_{i\in\mathbb{N}}$ satisfies the Mittag-Leffler condition. In the following, given $m \in \mathbb{N}_0$ and $i \in \mathbb{N}$, we denote by $B_i^m := \operatorname{im}(D_i^{m-1} \to D_i^m)$ the image of the $(m-1)$-st coboundary map and we denote by $Z_i^m := \ker(D_i^m \to D_i^{m+1})$ the kernel of the m-th coboundary map. These give rise to the following two short exact sequences of inverse sequences

$$\begin{array}{lccccccccc}
\text{(i)} & 0 & \to & B_i^m & \to & Z_i^m & \to & \mathrm{H}^m(D_i) & \to & 0 \\
\text{(ii)} & 0 & \to & Z_i^m & \to & D_i^m & \to & B_i^{m+1} & \to & 0.
\end{array}$$

Let $m \in \mathbb{N}_0$. By our hypothesis the inverse sequences $\{D_i^m\}_{i\in\mathbb{N}}$ satisfy the Mittag-Leffler condition. It is straightforward to see that then the inverse sequence $\{B_i^m\}_{i\in\mathbb{N}}$ also satisfies the Mittag-Leffler condition. It follows from the Mittag-Leffler Limit Lemma 94.4 that

$$\varprojlim{}^1 B_i^{m+1} \;=\; 0 \quad \text{and} \quad \varprojlim{}^1 D_i^m \;=\; 0.$$

---

[752] By the Mittag-Leffler Epimorphism Lemma 94.3 this condition is satisfied if the f_i are all epimorphisms. That is the only setting we will ever make use of.

[753] Using the category *InvSeq* of inverse sequences and using the Inverse Limit Functor Lemma 94.2 one can make precise what "natural" is actually supposed to mean.

[754] The map to the right is given as follows: By definition an inverse limit of cochain complexes comes with cochain maps $\varprojlim D_i^* \to D_i^*$. Since cohomology of cochain complexes is covariant in cochain maps we get induced maps $\mathrm{H}^n(\varprojlim D_i) \to \mathrm{H}^n(D_i^*)$ which then, by definition of the inverse limit, give rise to a map $\mathrm{H}^n(\varprojlim D_i) \to \varprojlim \mathrm{H}^n(D_i^*)$.

(Here and throughout the proof all limits are taken over $i \in \mathbb{N}$.) Using this data we obtain from the Inverse Limit–Snake Lemma 94.6, applied to the two short exact sequences (i) and (ii) of inverse sequences above, the following two exact sequences:

$$0 \to \varprojlim B_i^m \to \varprojlim Z_i^m \to \varprojlim \mathrm{H}^m(D_i) \xrightarrow{\delta} \underbrace{\varprojlim{}^1 B_i^m}_{=0} \to \varprojlim{}^1 Z_i^m \to \varprojlim{}^1 \mathrm{H}^m(D_i) \to 0$$

$$0 \to \varprojlim Z_i^m \to \varprojlim D_i^m \to \varprojlim B_i^{m+1} \xrightarrow{\delta} \varprojlim{}^1 Z_i^m \to \underbrace{\varprojlim{}^1 D_i^m}_{=0} \to \varprojlim{}^1 B_i^{m+1} \to 0.$$

From these sequence we deduce that the maps

$$\begin{array}{ll} \text{(i)} & \varprojlim{}^1 Z_i^m \to \varprojlim{}^1 \mathrm{H}^m(D_i) \\ \text{(ii)} & \varprojlim Z_i^m \to Z^m\big(\varprojlim D_i\big) := \ker\big(\varprojlim D_i^m \to \varprojlim B_i^{m+1}\big) \end{array}$$

are isomorphisms and that the following sequences are exact

$$\begin{array}{lccccccccc} \text{(i')} & 0 & \to & \varprojlim B_i^m & \to & \varprojlim Z_i^m & \to & \varprojlim \mathrm{H}^m(D_i) & \to & 0 \\ \text{(ii')} & 0 & \to & \underbrace{B^m\big(\varprojlim D_i\big)}_{:=\mathrm{im}(\varprojlim D_i^{m-1} \to \varprojlim B_i^m)} & \to & \varprojlim B_i^m & \to & \varprojlim{}^1 Z_i^{m-1} & \to & 0. \end{array}$$

From (ii') and the isomorphism (ii) we obtain the inclusions

$$B^m\big(\varprojlim D_i\big) \subset \varprojlim B_i^m \subset \varprojlim Z_i^m = Z^m\big(\varprojlim D_i\big)$$

which gives rise to the short exact sequence[755]

$$0 \to \underbrace{\frac{\varprojlim B_i^m}{B^m(\varprojlim D_i)}}_{=\varprojlim{}^1 \mathrm{H}^{m-1}(D_i) \text{ by (ii') and (ii)}} \to \underbrace{\frac{Z^m(\varprojlim D_i)}{B^m(\varprojlim D_i)}}_{=\mathrm{H}^m(\varprojlim D_i) \text{ by definition}} \to \underbrace{\frac{\varprojlim Z_i^m}{\varprojlim B_i^m}}_{=\varprojlim \mathrm{H}^m(D_i) \text{ by (i')}} \to 0.$$

■

94.4. **The cohomology group of an ascending sequence of topological spaces.** The following proposition can be viewed as the analogue of the Exhaustions-H_*-Proposition 46.2 for cohomology groups:

Proposition 94.7. (Exhaustions-H^*-Proposition) Let X be a topological space. Furthermore let $X_0 \subset X_1 \subset X_2 \subset X_3 \subset \dots$ be a sequence of subspaces of X with $\bigcup_{i \in \mathbb{N}_0} X_i = X$.

We suppose that one of the following conditions is satisfied:

(1) each X_i is contained in the interior of X_{i+1}, or
(2) X is a CW-complex and each X_i is a subcomplex of X.

Then for any $n \in \mathbb{N}_0$ and any abelian group G there exists a natural homomorphism

$$\tau\colon \varprojlim{}^1 \mathrm{H}^{n-1}(X_i; G) \to \mathrm{H}^n(X; G)$$

[755]Here we use that a filtration $P \subset Q \subset R$ of three abelian groups induces a short exact sequence $0 \to Q/P \to R/P \to R/Q \to 0$.

such that the following sequence is exact:

$$0 \to \varprojlim{}^1 \mathrm{H}^{n-1}(X_i; G) \xrightarrow{\tau} \mathrm{H}^n(X; G) \xrightarrow{\Theta} \varprojlim \mathrm{H}^n(X_i; G) \to 0,$$

where the right-hand map is induced by the restriction maps $\mathrm{H}^n(X; G) \to \mathrm{H}^n(X_i; G)$. If G is a commutative ring, then all maps are G-module homomorphisms.

Proof. Let X be a topological space and let $X_0 \subset X_1 \subset X_2 \subset X_3 \subset \dots$ be a sequence of subspaces of X with $\bigcup_{i \in \mathbb{N}_0} X_i = X$ such that one of the following conditions is satisfied:

(1) each X_i is contained in the interior of X_{i+1}, or
(2) X is a CW-complex and each X_i is a subcomplex of X.

Furthermore let G be an abelian group. For $i \in \mathbb{N}_0$ we consider the cochain complex $D_i^* := \mathrm{Hom}(\mathrm{C}_*(X_i), G)$ and for $i \in \mathbb{N}_0$ we denote by $f_i \colon D_{i+1}^* \to D_i^*$ the map that is induced by the inclusion $X_i \to X_{i+1}$.

Claim. For every choice of $k \in \mathbb{N}$ the inverse sequence of cochain groups that is given by $(\{D_i^k\}_{i \in \mathbb{N}}, \{f_i \colon D_{i+1}^k \to D_i^k\}_{i \in \mathbb{N}_0})$ satisfies the Mittag-Leffler condition.

Proof. Let $k \in \mathbb{N}_0$. By the Mittag-Leffler Epimorphism Lemma 94.3 it suffices to show that for every $i \in \mathbb{N}_0$ the map

$$D_{i+1}^k = \mathrm{Hom}(\mathrm{C}_n(X_{i+1}); G) \xrightarrow{f_i} \mathrm{Hom}(\mathrm{C}_n(X_i); G) = D_i^k$$

is an epimorphism. Since $X_i \subset X_{i+1}$ we know that the set of singular chains in X_i is a subset of the set of singular chains in X_{i+1}. It follows from the universal properties of free abelian groups that the map $\mathrm{Hom}(\mathrm{C}_n(X_{i+1}); G) \to \mathrm{Hom}(\mathrm{C}_n(X_i); G)$ is indeed an epimorphism.[756] ⊞

Let $n \in \mathbb{N}_0$. From the Inverse Limit-H^*-Proposition 94.5 we obtain a natural homomorphism

$$\tau \colon \varprojlim{}^1 \mathrm{H}^{n-1}(D_i^*) \to \mathrm{H}^n\big(\varprojlim D_i^*\big)$$

such that the following sequence is exact:

$$0 \to \varprojlim{}^1 \mathrm{H}^{n-1}(D_i^*) \xrightarrow{\tau} \mathrm{H}^n\big(\varprojlim D_i^*\big) \xrightarrow{\Theta} \varprojlim \mathrm{H}^n(D_i^*) \to 0$$

Let $i \in \mathbb{N}_0$. We make the following two observations:

(1) By definition we have $\mathrm{H}^k(D_i^*) = \mathrm{H}^k(X_i; G)$ for every $k \in \mathbb{N}_0$.
(2) We have natural isomorphisms

$$\varprojlim D_i^* \underset{\text{by definition of the } D_i}{\overset{\cong}{\uparrow}} \varprojlim \mathrm{Hom}(\mathrm{C}_*(X_i); G) \underset{\text{by the Hom-of-Direct Colimit Lemma 93.2}}{=} \uparrow \mathrm{Hom}\big(\varinjlim \mathrm{C}_*(X_i), G\big) \underset{\text{by the Exhaustion-C}_*\text{-Lemma 46.1 and our hypothesis (1) or (2)}}{\overset{\cong}{\uparrow}} \mathrm{Hom}(\mathrm{C}_*(X), G).$$

Combining all of the above we obtain our desired short exact sequence

$$0 \to \varprojlim{}^1 \mathrm{H}^{n-1}(X_i; G) \to \mathrm{H}^n(X; G) \to \varprojlim \mathrm{H}^n(X_i; G) \to 0.$$

If G is a commutative ring, then one can easily verify that τ and all the other maps are G-module homomorphisms. ■

[756] Here we use the following observation: If S is a subset of a set T, then we have a natural inclusion $i \colon \mathbb{Z}^{(S)} \to \mathbb{Z}^{(T)}$ and it follows from the Abelian Group-Homomorphism Lemma 66.2 (2) that the map $i^* \colon \mathrm{Hom}(\mathbb{Z}^{(T)}, G) \to \mathrm{Hom}(\mathbb{Z}^{(S)}, G)$ is an epimorphism. Indeed, we just extend the map $S \to G$ to a map $T \to G$ by assigning the trivial element in G to all elements in $T \setminus S$.

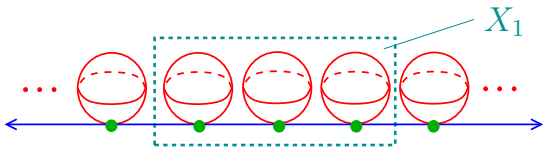
We provide a few examples for the above proposition.

Example. We consider the real line with infinitely many 2-dimensional spheres attached, i.e. we consider

$$X \;:=\; (\mathbb{R} \cup (S^2 \times \mathbb{Z}))/\sim$$

where given $i \in \mathbb{Z}$ we identify $i \in \mathbb{R}$ with $((0,0,-1),i) \in S^2 \times \mathbb{Z}$. Furthermore, given $k \in \mathbb{N}$ we consider the open subset

$$X_k \;:=\; ((-k-\tfrac{1}{2}, k+\tfrac{1}{2}) \cup (S^2 \times \{-k,\dots,k\}))/\sim .$$

$X =$ 

$$\begin{aligned} \mathrm{H}_2(X) &\cong \mathbb{Z}^{(\mathbb{N})} \\ \mathrm{H}^2(X;\mathbb{Z}) &\cong \mathbb{Z}^{\mathbb{N}} \end{aligned}$$

We leave it to the reader to verify that we can make the identifications[757]

$$\mathrm{H}^n(X_k;\mathbb{Z}) \;=\; \begin{cases} \mathbb{Z}, & \text{if } n=0, \\ \mathbb{Z}^{2k+1}, & \text{if } n=2, \\ 0, & \text{else.} \end{cases}$$

such that for each $k \in \mathbb{N}$ the inclusion maps $X_k \to X_{k+1}$ induce the identity map on $\mathbb{Z} = \mathrm{H}^0(X_{k+1};\mathbb{Z}) \to \mathrm{H}^0(X_k;\mathbb{Z}) = \mathbb{Z}$ and such that the inclusions induce the map

$$\begin{array}{rcl} \mathbb{Z}^{2k+3} = \mathrm{H}^2(X_{k+1};\mathbb{Z}) & \to & \mathrm{H}^2(X_k;\mathbb{Z}) = \mathbb{Z}^{2k+1} \\ (a_1,\dots,a_{2k+3}) & \mapsto & (a_2,\dots,a_{2k+2}) \end{array}$$

It follows from the Mittag-Leffler Epimorphism Lemma 94.3 and the above calculations that for each $n \in \mathbb{N}_0$ the inverse sequence of cohomology groups $\{\mathrm{H}^n(X_k;\mathbb{Z})\}_{k\in\mathbb{N}}$ satisfies the Mittag-Leffler condition. We obtain that

$$\mathrm{H}^n(X;\mathbb{Z}) \;\underset{\uparrow}{=}\; \varprojlim \mathrm{H}^n(X_k;\mathbb{Z}) \;=\; \varprojlim \begin{cases} \mathbb{Z}, & \text{if } n=0, \\ \mathbb{Z}^{2k+1}, & \text{if } n=2, \\ 0, & \text{else} \end{cases} \;\underset{\uparrow}{=}\; \begin{cases} \mathbb{Z}, & \text{if } n=0, \\ \mathbb{Z}^{\mathbb{N}}, & \text{if } n=2, \\ 0, & \text{else.} \end{cases}$$

by the Exhaustions-H$_*$-Proposition 94.7 and the Mittag-Leffler Limit Lemma 94.4

see page 1228

94.5. **Appendix: De Rham cohomology and exhaustions.** For fans of de Rham cohomology we now prove an analogue of the Exhaustions-H*-Proposition 94.7:

Proposition 94.8. (Exhaustions-H$^*_{\mathrm{dR}}$-Proposition) Let M be a smooth manifold and let $\{U_i\}_{i\in\mathbb{N}}$ be a sequence of open subsets of M that satisfy the following three conditions:
(1) for each $i \in \mathbb{N}$ the closure $\overline{U}_i$ of U_i is compact,
(2) for each $i \in \mathbb{N}$ we have $U_i \subset U_{i+1}$, and
(3) $M = \bigcup_{i\in\mathbb{N}} U_i$.
Then for every $n \in \mathbb{N}_0$ there exists a natural homomorphism

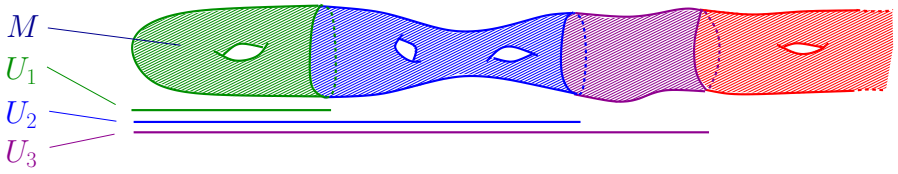
$$\tau\colon \varprojlim{}^1 \mathrm{H}^{n-1}_{\mathrm{dR}}(U_i) \;\to\; \mathrm{H}^n_{\mathrm{dR}}(M)$$

[757] One possible approach to showing these statements is to use the Mayer–Vietoris-H*-Theorem 69.5 (1).

such that the following sequence is exact:

$$0 \;\to\; \varprojlim{}^1 \mathrm{H}^{n-1}_{\mathrm{dR}}(U_i) \;\xrightarrow{\tau}\; \mathrm{H}^n_{\mathrm{dR}}(M) \;\xrightarrow{\Theta}\; \varprojlim \mathrm{H}^n_{\mathrm{dR}}(U_i) \;\to\; 0.$$

(Here the right-hand map Θ is induced by the obvious maps $\mathrm{H}^n_{\mathrm{dR}}(M) \to \mathrm{H}^n_{\mathrm{dR}}(U_i)$.)



Proof. First recall that given a smooth manifold W we denote, as defined on page **??**, by $\Omega^k(W)$ the real vector space of all smooth k-forms on W. In our setting we consider for each $k \in \mathbb{N}_0$ the inverse sequence

$$\Omega^k(U_1) \;\leftarrow\; \Omega^k(U_2) \;\leftarrow\; \Omega^k(U_3) \;\leftarrow\; \dots$$

where all the maps are induced by the inclusion, put differently, they are all restriction maps.

Claim. The inverse sequence satisfies the Mittag-Leffler condition from page 1233.

Proof. Given any $j \geq i$ we denote by $\rho_{ji}\colon \Omega^k(U_j) \to \Omega^k(U_i)$ the restriction map. Now let $i \in \mathbb{N}$. By definition of the Mittag-Leffler condition from page 1233 we have to show that there exists an $n \geq i$ such that for all $r \geq n$ we have

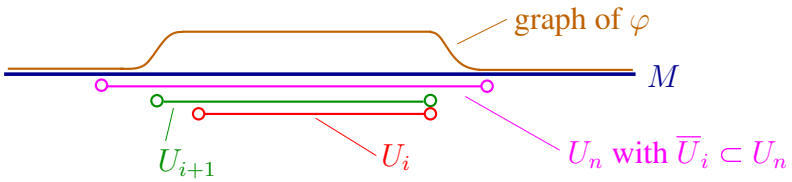
$$\mathrm{im}(\rho_{ri}\colon \Omega^k(U_r) \to \Omega^k(U_i)) \;=\; \mathrm{im}(\rho_{ni}\colon \Omega^k(U_n) \to \Omega^k(U_i)).$$

Since $\overline{U}_i$ is compact, since the U_j are nested and since $\overline{U}_i$ is contained in the union of all U_j there exists an $n \in \mathbb{N}$ such that $\overline{U}_i \subset U_n$.

We claim that that n has the desired property. So let $r \geq n$. We have to show that

$$\mathrm{im}(\rho_{ri}\colon \Omega^k(U_r) \to \Omega^k(U_i)) \;=\; \mathrm{im}(\rho_{ni}\colon \Omega^k(U_n) \to \Omega^k(U_i)).$$

Clearly we have the inclusion "$\subset$". It remains to prove the reverse inclusion. So suppose $\eta \in \Omega^k(U_i)$ is the restriction of some $\mu \in \Omega^k(U_n)$. Note that U_n and $M \setminus \overline{U}_i$ form an open cover for M. It follows immediately from the Manifold–Partitions-of-Unity Theorem **??** that there exists a smooth function $\varphi\colon M \to \mathbb{R}$ with $\varphi|_{\overline{U}_i} \equiv 1$ and such that $\mathrm{supp}(\varphi) \subset U_n$. Let $r \geq n$.



As in the Extend Smooth Forms Lemma **??** we denote by $\epsilon(\varphi \cdot \mu, U_r)$ the smooth k-form that is given by $\varphi \cdot \mu$ on U_r and that is extended to the zero k-form outside of U_r. We see that

$$\eta \;=\; \mu|_{U_i} \;\underset{\substack{\uparrow\\ \text{since } \varphi \equiv 1 \text{ on } U_i}}{=}\; (\varphi \cdot \mu)|_{U_i} \;=\; \epsilon(\varphi \cdot \mu, U_r)|_{U_i}.$$

We have thus shown that η is also the restriction of some smooth form on U_r. ⊞

By the claim we can apply the Inverse Limit-H^*-Proposition 94.5 to the inverse sequence $(\{\Omega^*(U_i)\}_{i\in\mathbb{N}}, \{\rho_{ij}\}_{i\geq j})$ of cochain complexes over $\mathbb{N}$ and we obtain the natural homomorphism

$$\tau\colon \varprojlim{}^1 \mathrm{H}^{n-1}_{\mathrm{dR}}(U_i) \;\to\; \mathrm{H}^n_{\mathrm{dR}}(M)$$

such that the following sequence is exact:

$$0 \;\to\; \varprojlim_{i\in\mathbb{N}}{}^1 \underbrace{\mathrm{H}^{n-1}(\Omega^*(U_i))}_{=\mathrm{H}^{n-1}_{\mathrm{dR}}(U_i)} \;\to\; \underbrace{\mathrm{H}^n\Big(\overbrace{\varprojlim_{i\in\mathbb{N}} \Omega^*(U_i)}^{=\Omega^*(M)}\Big)}_{=\mathrm{H}^n_{\mathrm{dR}}(M)} \;\to\; \varprojlim_{i\in\mathbb{N}} \underbrace{\mathrm{H}^n(\Omega^*(U_i))}_{=\mathrm{H}^n_{\mathrm{dR}}(U_i)} \;\to\; 0.$$

■

CHAPTER 95

De Rham cohomology and singular cohomology

In this chapter we will show that, given a smooth manifold M, there exists a natural isomorphism between the de Rham cohomology groups $\mathrm{H}^i_{\mathrm{dR}}(M)$ (that we introduced in Chapter **??**) and the singular cohomology groups $\mathrm{H}^i(M;\mathbb{R})$ (that we introduced in Chapter 67).

95.1. **Smooth cohomology theories.** To a smooth manifold M we can associate the singular cohomology groups $\mathrm{H}^i(M;\mathbb{R})$ and also the de Rham cohomology groups $\mathrm{H}^i_{\mathrm{dR}}(M)$. Rereading the chapters on these cohomology groups one notices many common features, for example they both satisfy a Mayer–Vietoris type theorem. Now we want to put both cohomologies in a common framework. To build this common framework it is helpful to recall and introduce some notation:

Notation.

(1) As on page **??** we define

$$\mathcal{SmMfd} \;:=\; \text{category of smooth manifolds}$$

where objects are smooth manifolds and morphisms are smooth maps.

(2) Let $\mathbb{K}$ be a field.

- An $\mathbb{N}_0$-graded $\mathbb{K}$-vector space is a family $\{V_n\}_{n\in\mathbb{N}_0}$ of $\mathbb{K}$-vector spaces. We often write an $\mathbb{N}_0$-graded $\mathbb{K}$-vector space as a direct sum $\bigoplus\limits_{n\in\mathbb{N}_0} V_n$.
- A morphism between $\mathbb{N}_0$-graded $\mathbb{K}$-vector spaces $\{V_n\}_{n\in\mathbb{N}_0}$ and $\{W_n\}_{n\in\mathbb{N}_0}$ is a family $\{\varphi_n\colon V_n \to W_n\}_{n\in\mathbb{N}_0}$ of $\mathbb{K}$-linear homomorphisms.
- We denote by $\mathcal{Vec}_{\mathbb{K}}^{\mathbb{N}_0}$ the category of $\mathbb{N}_0$-graded $\mathbb{K}$-vector spaces.

This notation lets us formulate the common framework for the different types of cohomology groups:

Definition. A smooth cohomology theory is a contravariant functor[758]

$$\begin{array}{rccl} \mathcal{H}^*\colon & \text{category } \mathcal{SmMfd} \text{ of smooth manifolds} & \to & \text{category } \mathcal{Vec}_{\mathbb{R}}^{\mathbb{N}_0} \text{ of } \mathbb{N}_0\text{-graded real vector spaces} \\ & M & \mapsto & \mathcal{H}^*(M) = \bigoplus\limits_{n\in\mathbb{N}_0} \mathcal{H}^n(M) \\ & (f\colon M\to N) & \mapsto & \{f^* = \mathcal{H}^n(f)\colon \mathcal{H}^n(N) \to \mathcal{H}^n(M)\}_{n\in\mathbb{N}_0} \end{array}$$

together with natural homomorphisms $\delta_{\mathcal{H}}$ and $\tau_{\mathcal{H}}$ (described below) such that the following four axioms are satisfied:

(1) For every $n \in \mathbb{N}_0$ we have $\mathcal{H}^n(\varnothing) = 0$.

(2) (Smooth homotopy invariance) Let $f, g\colon M \to N$ be smooth maps between smooth manifolds. If f and g are smoothly homotopic[759], then for all $n \in \mathbb{N}_0$ we have
$$f^* = g^*\colon \mathcal{H}^n(N) \to \mathcal{H}^n(M).$$
(3) (Mayer–Vietoris sequence) If M is a smooth manifold and if U and V are open subsets of M such that $M = U \cup V$, then there exists a family of natural homomorphisms
$$\delta_{\mathcal{H}}\colon \mathcal{H}^n(U \cap V) \to \mathcal{H}^{n+1}(M)$$
with $n \in \mathbb{N}_0$ such that the following sequence is exact:
$$\ldots \to \mathcal{H}^n(M) \xrightarrow{j_U^* \oplus j_V^*} \mathcal{H}^n(U) \oplus \mathcal{H}^n(V) \xrightarrow{(i_U^*, -i_V^*)} \mathcal{H}^n(U \cap V) \xrightarrow{\delta_{\mathcal{H}}} \mathcal{H}^{n+1}(M) \to \ldots$$
(Here $j_U\colon U \to M$, $j_V\colon V \to M$, $i_U\colon U \cap V \to U$ and $i_V\colon U \cap V \to V$ denote the inclusion maps.)
(4) (Limit property) Let M be a smooth manifold and let $\{U_i\}_{i\in\mathbb{N}}$ be a sequence of open subsets of M that satisfy the following three conditions:
(a) for each $i \in \mathbb{N}$ the closure $\overline{U}_i$ of U_i is compact,
(b) for each $i \in \mathbb{N}$ we have $U_i \subset U_{i+1}$, and
(c) $M = \bigcup\limits_{i\in\mathbb{N}} U_i$.
Then for every $n \in \mathbb{N}_0$ there exists a natural homomorphism
$$\tau_{\mathcal{H}}\colon \varprojlim{}^1 \mathcal{H}^{n-1}(U_i) \to \mathcal{H}^n(M)$$
such that the following sequence is exact:
$$0 \to \varprojlim{}^1 \mathcal{H}^{n-1}(U_i) \xrightarrow{\tau_{\mathcal{H}}} \mathcal{H}^n(M) \xrightarrow{\Theta} \varprojlim \mathcal{H}^n(U_i) \to 0.$$
(Here the right-hand map Θ is induced by the maps $\mathcal{H}^n(M) \to \mathcal{H}^n(U_i)$ which in turn are induced by the inclusions $U_i \to M$.)

This time it is easy to give examples for our new definition:

Proposition 95.1. (Smooth-$\mathcal{H}^*$-Theories Proposition)
(1) Singular cohomology with real coefficients (together with the natural maps from the Mayer–Vietoris-H*-Theorem 69.5 (1) and the Exhaustions-H*-Proposition 94.7) is a smooth cohomology theory.
(2) De Rham cohomology (together with the natural maps from the Mayer–Vietoris-$\mathrm{H}^*_{\mathrm{dR}}$-Theorem **??** and the Exhaustions-$\mathrm{H}^*_{\mathrm{dR}}$-Proposition 94.8) is a smooth cohomology theory.

Proof.

(1) It follows from the Empty Space-H*-Lemma 67.3, the Homotopy-H*-Proposition 67.7, the Mayer–Vietoris-H*-Theorem 69.5 (1) and the Exhaustions-H*-Proposition 94.7 (1) that singular cohomology with real coefficients is a smooth cohomology theory.

[758]Given a smooth map $f\colon M \to N$ the contravariant functor provides for each $n \in \mathbb{N}_0$ a morphism $\mathcal{H}^n(f)\colon \mathcal{H}^n(N) \to \mathcal{H}^n(M)$. As usual, when there is no danger of confusion we just write f^* instead of $\mathcal{H}^n(f)$.
[759]Since we assume that $\partial M = \varnothing$ we obtain from the Replace-by-Smooth Homotopy Proposition **??** that two smooth maps $f, g\colon M \to N$ are smoothly homotopic if and only if they are homotopic.

(2) It follows from the Empty Manifold-$\mathrm{H}^*_{\mathrm{dR}}$-Lemma **??**, the Homotopy-$\mathrm{H}^n_{\mathrm{dR}}$-Proposition **??**, the Mayer–Vietoris-$\mathrm{H}^*_{\mathrm{dR}}$-Theorem **??** and the Exhaustions-$\mathrm{H}^*_{\mathrm{dR}}$-Proposition 94.8 that de Rham cohomology is a smooth cohomology theory. ■

95.2. **Isomorphisms between smooth cohomology theories.** One might now be optimistic and hope that smooth cohomology theories are necessarily isomorphic. Unfortunately, as it stands, this thought is too optimistic. For example three other smooth cohomology theories are given by

$$\begin{aligned}\mathcal{H}^n(M) &:= 0,\\ \mathcal{H}^n(M) &:= \mathrm{H}^{n+5}(M;\mathbb{R}),\\ \mathcal{H}^n(M) &:= \mathrm{H}^n(M;\mathbb{R})\oplus\mathrm{H}^n(M;\mathbb{R}).\end{aligned}$$

This shows that if we want to prove that smooth cohomology theories are isomorphic, we have to add some extra hypotheses. For example we could demand that the 0-th cohomology groups of a point are isomorphic. This thought motivates the following notation:

Notation. In the following $\star$ denotes the topological space consisting of a single point. Evidently $\star$ is a 0-dimensional smooth manifold.

On page 98 we introduced the notion of a natural transformation between two contravariant functors. For the reader's convenience we recall the definition, since it will play an essential role in the following discussion:

Definition. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories and let $F,G\colon\mathcal{C}\to\mathcal{D}$ be two contravariant functors. A natural transformation between the contravariant functors F and G assigns to each object $X\in\mathrm{Ob}(\mathcal{C})$ a morphism $\Phi_X\colon F(X)\to G(X)$ in $\mathcal{D}$ such that for each morphism $\varphi\colon X\to Y$ in $\mathcal{C}$ the following diagram commutes:

$$\begin{array}{ccc} F(Y) & \xrightarrow{F(\varphi)} & F(X)\\ {\scriptstyle\Phi_Y}\downarrow & & \downarrow{\scriptstyle\Phi_X}\\ G(Y) & \xrightarrow{G(\varphi)} & G(X).\end{array}$$

If all the maps Φ_X are isomorphisms we also say that it is a natural isomorphism.

With this notation and this definition we can now formulate the following theorem:

Theorem 95.2. (Smooth-$\mathcal{H}^*$-Theory Isomorphism Theorem) Let $\mathcal{H}^*$ and $\mathcal{K}^*$ be smooth cohomology theories and let $t\colon\mathcal{H}^*\to\mathcal{K}^*$ be a natural transformation. Suppose that t has the following three properties:

(i) The natural transformation t is compatible with the connecting homomorphisms in the Mayer-Vietoris sequence, i.e. with the obvious notation we always have a commutative diagram

$$\begin{array}{ccccccccccc}
\ldots\longrightarrow\mathcal{H}^n(M) & \xrightarrow{j_U^*\oplus j_V^*} & \mathcal{H}^n(U)\oplus\mathcal{H}^n(V) & \xrightarrow{(i_U^*,-i_V^*)} & \mathcal{H}^n(U\cap V) & \xrightarrow{\delta_{\mathcal{H}}} & \mathcal{H}^{n+1}(M)\longrightarrow\ldots\\
\downarrow{\scriptstyle t^n} & & \downarrow{\scriptstyle t^n\oplus t^n} & & \downarrow{\scriptstyle t^n} & & \downarrow{\scriptstyle t^{n+1}}\\
\ldots\longrightarrow\mathcal{K}^n(M) & \xrightarrow{j_U^*\oplus j_V^*} & \mathcal{K}^n(U)\oplus\mathcal{K}^n(V) & \xrightarrow{(i_U^*,-i_V^*)} & \mathcal{K}^n(U\cap V) & \xrightarrow{\delta_{\mathcal{K}}} & \mathcal{K}^{n+1}(M)\longrightarrow\ldots
\end{array}$$

(ii) The natural transformation t is compatible with the map τ appearing in the limit property, i.e. with the obvious notation we always have a commutative diagram[760]

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \varprojlim{}^1 \mathcal{H}^{n-1}(U_i) & \xrightarrow{\tau_{\mathcal{H}}} & \mathcal{H}^n(M) & \xrightarrow{\Theta} & \varprojlim \mathcal{H}^n(U_i) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle t^{n-1}} & & \downarrow{\scriptstyle t^n} & & \downarrow{\scriptstyle t^n} & & \\ 0 & \longrightarrow & \varprojlim{}^1 \mathcal{K}^{n-1}(U_i) & \xrightarrow{\tau_{\mathcal{K}}} & \mathcal{K}^n(M) & \xrightarrow{\Theta} & \varprojlim \mathcal{K}^n(U_i) & \longrightarrow & 0. \end{array}$$

(iii) For all $n \in \mathbb{N}_0$ the map $t^n_\star \colon \mathcal{H}^n(\star) \to \mathcal{K}^n(\star)$ is an isomorphism.

Then for every smooth manifold M and every $n \in \mathbb{N}_0$ the map

$$t^n \colon \mathcal{H}^n(M) \ \to \ \mathcal{K}^n(M)$$

is an isomorphism.

The proof of the Smooth-$\mathcal{H}^*$-Theory Isomorphism Theorem 95.2 rests on the following useful little lemma:

Lemma 95.3. (Rational Open Balls Lemma)

(1) Every open subset U of $\mathbb{R}^m$ is the union of open balls of the form $B^m_\epsilon(x)$ where ϵ is rational, $x \in \mathbb{Q}^m$ and with $\overline{B}^m_\epsilon(x) \subset U$.

(2) Every open subset U of $\mathbb{H}^m$ is the union of open subsets of the form $B^m_\epsilon(x) \cap \mathbb{H}^m$ where ϵ is rational, $x \in \mathbb{Q}^m$ and with $\overline{B}^m_\epsilon(x) \subset U$.

Proof of Lemma 95.3.

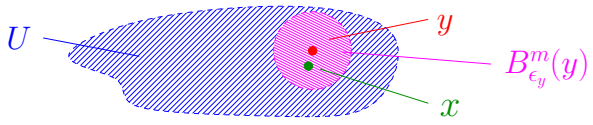
(1) Let U be an open subset of $\mathbb{R}^m$. For $x \in U$ we set

$$\mu_x \ := \ \sup\{\mu \in [0,1] \,|\, B^m_\mu(x) \subset U\}.$$

The hypothesis that U is open implies that $\mu_x > 0$. We now pick a rational number $\epsilon_x \in (\frac{1}{2}\mu_x, \frac{3}{4}\mu_x)$. Note that $\overline{B^m_{\epsilon_x}(x)} \subset \overline{B}^m_{\frac{3}{4}\mu_x}(x) \subset U$. It remains to prove the following claim:

Claim. The subset U is the union of all $B^m_{\epsilon_x}(x)$ with $x \in U \cap \mathbb{Q}^m$.

Proof. Let $y \in U$. Since $\mathbb{Q}^m$ is dense in $\mathbb{R}^m$ we can pick $x \in U \cap \mathbb{Q}^m$ with $\|x-y\| < \frac{1}{4}\epsilon_y$. Note that this is equivalent to saying that $y \in B^m_{\frac{1}{4}\epsilon_y}(x)$. Thus it remains to prove that $B^m_{\frac{1}{4}\epsilon_y}(x) \subset B^m_{\epsilon_x}(x)$.



Recall that above we saw that $B^m_{\epsilon_y}(y) \subset U$. It follows from the inequality $\|x-y\| < \frac{1}{4}\epsilon_y$ and the triangle inequality that we have an inclusion $B^m_{\frac{3}{4}\epsilon_y}(x) \subset B^m_{\epsilon_y}(y)$. It follows that $B^m_{\frac{3}{4}\epsilon_y}(x) \subset U$. By definition of μ_x this implies that $\frac{3}{4}\epsilon_y \leq \mu_x$. Hence we have $\frac{1}{4}\epsilon_y \leq \frac{3}{8}\epsilon_y \leq \frac{1}{2}\mu_x \leq \epsilon_x$. But then we also have $B^m_{\frac{1}{4}\epsilon_y}(x) \subset B^m_{\epsilon_x}(x)$, as desired.

(2) Let U be an open subset of $\mathbb{H}^m$. By definition of the subspace topology there exists an open subset $V \subset \mathbb{R}^m$ with $U = \mathbb{H}^m \cap V$. We apply (1) to V and we obtain the desired statement. ∎

[760] Note that the left and right vertical maps are given by the definition on page 1233.

Proof of Theorem 95.2. Let $\mathcal{H}^*$ and $\mathcal{K}^*$ be smooth cohomology theories, which means, by the definition on page 1242, in a nutshell that they both have the following properties:

(1) the invariants of the empty manifold are zero,
(2) smooth homotopy invariance,
(3) the Mayer-Vietoris sequence,
(4) the limit property.

Furthermore let $t\colon \mathcal{H}^* \to \mathcal{K}^*$ be a natural transformation that has the properties (i), (ii) and (iii) from the statement of the theorem on page 1244. We need to prove the following statement:

Statement. Let $m \in \mathbb{N}_0$. For every m-dimensional smooth manifold M and every $n \in \mathbb{N}_0$ the map

$$t^n_M\colon \mathcal{H}^n(M) \;\to\; \mathcal{K}^n(M)$$

is an isomorphism.

We start out with the following two observations:

(A) Suppose that M is an m-dimensional smooth manifold and that U and V are open subsets with $M = U \cup V$. Let $n \in \mathbb{N}_0$. If t^n and t^{n+1} are isomorphisms for U, V and $U \cap V$, then it follows from (3), (i) and the Five Lemma 37.5 that t^n is an isomorphism for M.

(B) Let M be an m-dimensional smooth manifold and let $\{U_i\}_{i\in\mathbb{N}}$ be a sequence of open subsets of M that satisfy the following three conditions:
  (a) for each $i \in \mathbb{N}$ the closure $\overline{U}_i$ of U_i is compact,
  (b) for each $i \in \mathbb{N}$ we have $U_i \subset U_{i+1}$, and
  (c) $M = \bigcup\limits_{i\in\mathbb{N}} U_i$.

  Let $n \in \mathbb{N}_0$. If t^{n-1} and t^n are isomorphisms for all U_i, then it follows, as in (A), from (4) and the Five Lemma 37.5 that t^n is an isomorphism for M.

After these preparations we will prove the desired statement in five easy steps, which are reminiscent of the five steps in the proof of the Homology-Section Theorem 63.6 and the five steps in the proof of the Poincaré Duality Theorem 79.1 on page 1034:

Claim 2.

(α) Every t^n is an isomorphism for every open bounded convex subset of $\mathbb{H}^m$.
(β) Every t^n is an isomorphism for the union of k open bounded convex subsets of $\mathbb{H}^m$.
(γ) Every t^n is an isomorphism for every open subset of $\mathbb{H}^m$.

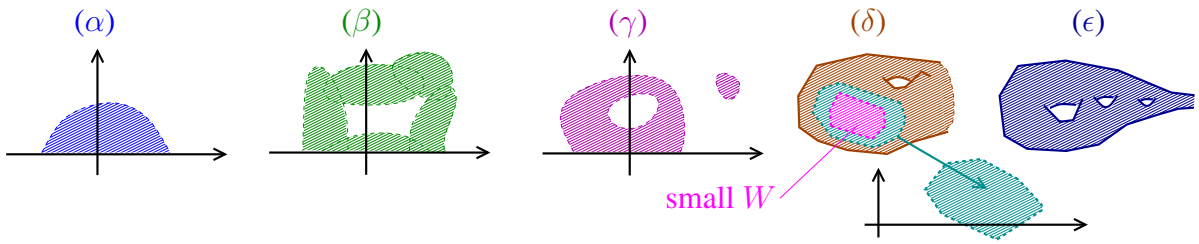
To formulate the next statement of the claim we need some extra language: We say that a subset W of a smooth manifold M is small if it is open and if W is contained in the domain of some chart for M where the target of the chart is an open subset of $\mathbb{H}^m$.[760]

(δ) Every t^n is an isomorphism for all smooth manifolds that are covered by m small subsets.
(ϵ) Every t^n is an isomorphism for all m-dimensional smooth manifolds.

Proof.

(α) Let M be an open convex subset of $\mathbb{H}^m$. If M is empty, then the statement follows from (1). Now assume that M is non-empty. We pick an inclusion $f\colon \star \to M$. Since

[760] In other words, in the language of page **??** we demand that W is precompact.



M is convex we have an obvious smooth deformation retraction from M to $\star$. Given $n \in \mathbb{N}_0$ we consider the diagram

$$\begin{array}{ccc} \mathcal{H}^n(M) & \xrightarrow{f^*} & \mathcal{H}^n(\star) \\ {\scriptstyle t}\downarrow & & \cong\downarrow{\scriptstyle t} \\ \mathcal{K}^n(M) & \xrightarrow{f^*} & \mathcal{K}^n(\star). \end{array}$$

The diagram commutes since t is a natural transformation. It follows from (2), by the same argument as in the Homotopy Equivalence-H$_*$-Corollary 36.7, that the horizontal maps are isomorphisms. By (iii) the right vertical map is also an isomorphism. It follows that the left vertical map is, as desired, an isomorphism

(β) We claim that t is an isomorphism for every subset M of $\mathbb{H}^m$ that is the union of k open convex subsets of $\mathbb{H}^m$. We prove this by induction on the number k of open convex subsets:
- (a) If $k = 1$, then M itself is an open convex subset of $\mathbb{H}^m$, hence the statement follows from (α).
- (b) Suppose the statement holds whenever M is the union of $k-1$ open convex subsets with $k \geq 2$. Now let M be the union of k open convex subsets $U_1, \ldots, U_k$ of $\mathbb{H}^m$. We write $V = U_1 \cup \cdots \cup U_{k-1}$. By induction the statement holds for V and also for U_k. Furthermore since the intersection of two convex open subsets is again open and convex we see that $V \cap U_k = (U_1 \cap U_k) \cup \cdots \cup (U_{k-1} \cap U_k)$ is the union of $k-1$ open convex subsets, i.e. by induction the statement holds for $V \cap U_k$. But then it follows from (A) that the statement also holds for M.

(γ) Let M be an open subset of $\mathbb{H}^m$. By the Rational Open Balls Lemma 95.3 (2) we can write M as the union of countably many open convex subsets $B_i, i \in \mathbb{N}$ of $\mathbb{H}^m$ such that each closure $\overline{B}_i \subset \mathbb{H}^m$ is contained in M. We obtain from this information and the elementary argument of the Compact Exhaustion Lemma **??** a sequence U_i of open subsets of M such that the following three conditions are satisfied:
- (a) For each $i \in \mathbb{N}$ the subset U_i is the union of finitely many open balls and the closure $\overline{U}_i$ of U_i in M is compact,
- (b) For each $i \in \mathbb{N}$ we have $U_i \subset U_{i+1}$.
- (c) $M = \bigcup_{i \in \mathbb{N}} U_i$.

By (β) the statement holds for each U_i, but then it follows from (B) that the statement holds for M.

(δ) We claim that t is an isomorphism for every m-dimensional smooth manifold M that is the union of k small subsets.
- (a) If $k = 1$, then it follows from the fact that $\mathcal{H}$ and $\mathcal{K}$ are diffeomorphism invariants and from (γ) that the statement holds for M.
- (b) The intersection of two small subsets is clearly small again. With this simple observation the same induction argument as in (β) shows that the statement holds whenever M is covered by finitely many small subsets.

(ϵ) Finally let M be any m-dimensional smooth manifold. Note that it follows easily from the fact that the smooth manifold M is second-countable, the existence of charts around every point and the Small Basis Lemma **??** that there exist small sets $W_i, i \in \mathbb{N}$ which cover M and such that each closure $\overline{W}_i$ is compact. As in (γ) we obtain from the argument of the Compact Exhaustion Lemma **??** a sequence U_i of open subsets of M such that the following three conditions are satisfied:
- (a) For each $i \in \mathbb{N}$ the subset U_i is the union of finitely many small subsets and the closure $\overline{U}_i$ of U_i is compact,
- (b) For each $i \in \mathbb{N}$ we have $U_i \subset U_{i+1}$.
- (c) $M = \bigcup_{i \in \mathbb{N}} U_i$.

We had just seen in (δ) that the statement holds for all the U_i. But then it follows from (B) that the statement holds for M. ∎

95.3. **Smooth singular cohomology.** One might think that it follows immediately from the Smooth-$\mathcal{H}^*$-Theories Proposition 95.1 and the Smooth-$\mathcal{H}^*$-Theory Isomorphism Theorem 95.2 (together with the calculations of the cohomology groups of a point in the Singleton-$\mathrm{H}^*_{\mathrm{dR}}$-Lemma **??** and the Singleton-H^*-Lemma 67.6) that the singular real valued cohomology groups and the de Rham cohomology groups of smooth manifolds are always isomorphic.

But recall that to apply the Smooth-$\mathcal{H}^*$-Theory Isomorphism Theorem 95.2 we need to find a natural transformation

$$t\colon \text{de Rham cohomology} \quad \to \quad \text{real valued singular cohomology}$$

that has the property that

$$\mathrm{H}^n_{\mathrm{dR}}(\star) \quad \xrightarrow{\cong} \quad \mathrm{H}^n(\star;\mathbb{R}) \qquad \text{for all } n \in \mathbb{N}_0.$$

Here is an idea: In Section **??** we had seen that given a compact oriented n-dimensional smooth manifold W and a smooth differential n-form ω on W there exists an integral

$$\int_W \omega \ \in \ \mathbb{R}$$

that has many good properties. So given some smooth manifold M (not necessarily of dimension n) we could try to define a map

$$\begin{array}{rcl} t\colon \Omega^n(M) & \to & \mathrm{Hom}(\mathrm{C}_n(M),\mathbb{R}) \\ \omega & \mapsto & \left(\begin{array}{rcl} \mathrm{C}_n(M) & \to & \mathbb{R} \\ \sum_{i=1}^k a_i \cdot (f_i\colon \Delta^n \to M) & \mapsto & \sum_{i=1}^k a_i \cdot \int_{\Delta^n} f_i^*(\omega) \end{array} \right). \end{array}$$

Unfortunately, there are several problems with this "definition":

(1) The first question that arises is the following: What does it mean to integrate a form over Δ^n? After all, it is not clear in what sense Δ^n is a smooth manifold. With a little bit of luck, that should not be too much of an issue, since Δ^n is clearly pretty close to being a smooth manifold.
(2) Let us suppose that we have resolved (1). Now let $f\colon \Delta^n \to M$ be a singular n-chain. If the map f is not smooth, then $f^*(\omega)$ just does not make any sense.[761]
(3) After fixing the previous issues, why should t be a cochain map?

Arguably the most serious issue is (2). The only way around it is, to only work with smooth singular simplices. Let us recall the following definitions:

Definition. Let $n \in \mathbb{N}_0$.

(1) As in the Δ^n-Homeomorphism Lemma 35.3 (1) we consider the planar simplex

$$\triangle_n := \left\{(t_1,\dots,t_n) \in \mathbb{R}^n_{\geq 0} \,\middle|\, \sum_{i=1}^n t_i \in [0,1]\right\}.$$

(2) In the Δ^n-Homeomorphism Lemma 35.3 (1) we saw that the map[762]

$$\begin{aligned} \Theta_n\colon \triangle_n &\to \Delta^n \\ (t_1,\dots,t_n) &\mapsto \left(t_1,\dots,t_n,1-\sum_{i=1}^n t_i\right) \end{aligned}$$

is a homeomorphism.



Next we turn to smooth manifolds. Continuing with the above definition we can now introduce the notion of a smooth singular simplex:

Definition. Let M be a smooth manifold.

(3) Recall that by the definition on page **??** a map $f\colon \triangle_n \to M$ is smooth if given any $P \in \triangle_n$ there exists an open neighborhood U in $\mathbb{R}^n$ of P and a smooth map $\widetilde{f}\colon U \to M$ which coincides with f on $\triangle_n \cap U$. We define the differential of f at $P \in \triangle_n$ to be the differential of $\widetilde{f}$.[763]
(4) We say that a singular n-simplex $\sigma\colon \Delta^n \to M$ is smooth if the map $\sigma \circ \Theta_n\colon \triangle_n \to M$ is smooth.

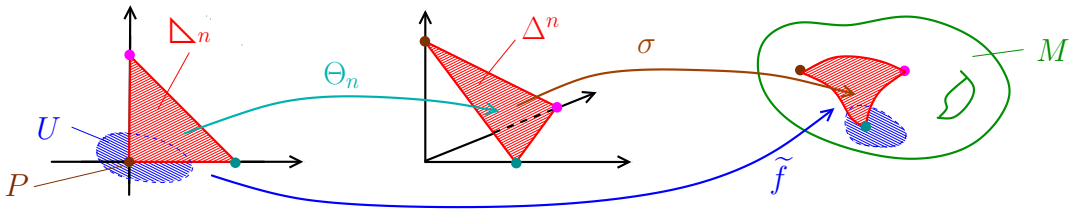
Definition.

(5) We write

[761] Recall that if $f\colon N \to M$ is a smooth map between smooth manifolds and if ω is a smooth differential n-form on M, then on page **??** we defined $f^*(\omega)$ to be the differential n-form on N that for $v_1,\dots,v_n \in \mathrm{T}_P N$ has the value $(f^*(\omega))_P(v_1,\dots,v_n) = \omega_{f(P)}(\mathrm{D}f_P(v_1),\dots,\mathrm{D}f_P(v_n))$. But if f is not differentiable, then there is no differential $\mathrm{D}f_P\colon \mathrm{T}_P N \to \mathrm{T}_{f(P)} M$.

[762] Put differently, Θ_n is the inverse of the obvious projection map $\Delta^n \to \triangle_n$.

[763] It follows easily from the Regular Closed–∂_*-Lemma **??** that this definition does not depend on the choice of U and $\widetilde{f}$. This is the reason why we work with $\triangle_n$ instead of Δ^n.



$$\mathrm{C}_n^{\mathrm{sm}}(M) \;:=\; \begin{array}{c}\text{the free abelian group generated by the}\\ \text{set of smooth singular } n\text{-simplices in } M.\end{array}$$

This is a subgroup of $\mathrm{C}_n(M)$ and it is clear that the boundary map

$$\begin{aligned}\partial_n\colon \mathrm{C}_n(M) &\to \mathrm{C}_{n-1}(M)\\ \sigma &\mapsto \sum_{j=0}^{n}(-1)^j\cdot(\sigma\circ \underset{\substack{\uparrow\\ \text{linear (thus smooth)}\\ \text{map } \Delta^{n-1}\to\Delta^n}}{i_j^n}).\end{aligned}$$

restricts to a map $\partial_n\colon \mathrm{C}_n^{\mathrm{sm}}(M)\to \mathrm{C}_{n-1}^{\mathrm{sm}}(M)$.

(6) Given an abelian group G we write

$$\mathrm{C}^n_{\mathrm{sm}}(M;G) \;:=\; \mathrm{Hom}\left(\mathrm{C}_n^{\mathrm{sm}}(M), G\right)$$

and we define $\delta_n\colon \mathrm{C}^n_{\mathrm{sm}}(M;G)\to \mathrm{C}^{n+1}_{\mathrm{sm}}(M;G)$ as the map $\delta_n := \partial^*_{n+1}$.

(7) We define the n-th smooth singular cohomology group of M with G-coefficients as

$$\mathrm{H}^n_{\mathrm{sm}}(M;G) \;:=\; \frac{\ker\left(\delta_n\colon \mathrm{C}^n_{\mathrm{sm}}(M;G)\to \mathrm{C}^{n+1}_{\mathrm{sm}}(M;G)\right)}{\mathrm{im}\left(\delta_{n-1}\colon \mathrm{C}^{n-1}_{\mathrm{sm}}(M;G)\to \mathrm{C}^{n}_{\mathrm{sm}}(M;G)\right)}.$$

These definitions lead us to the following unsurprising proposition:

Proposition 95.4. (Smooth Singular Cohomology Proposition) Smooth singular cohomology with real coefficients is a smooth cohomology theory.

Sketch of proof. We need to verify that smooth singular cohomology with real coefficients satisfies the properties of a smooth cohomology theory from page 1242:

(0) First we introduce the induced maps: Thus let $f\colon M\to N$ be a *smooth* map between two smooth manifolds. Evidently the induced map $f_*\colon \mathrm{C}_n(M)\to \mathrm{C}_n(N)$ restricts to a chain map $f_*\colon \mathrm{C}_n^{\mathrm{sm}}(M)\to \mathrm{C}_n^{\mathrm{sm}}(N)$. In particular we get an induced chain cochain map $f^*\colon \mathrm{C}^n_{\mathrm{sm}}(N;\mathbb{R})\to \mathrm{C}^n_{\mathrm{sm}}(M;\mathbb{R})$. Thus we obtain for each $n\in\mathbb{N}_0$ an induced map

$$f^*\colon \mathrm{H}^n_{\mathrm{sm}}(N;\mathbb{R}) \;\to\; \mathrm{H}^n_{\mathrm{sm}}(M;\mathbb{R}).$$

It is now pretty clear that smooth singular cohomology with real coefficients is a contravariant functor from the category $\mathcal{SmMfd}$ of smooth manifolds to the category $\mathcal{Vec}_{\mathbb{K}}^{\mathbb{N}_0}$ of $\mathbb{N}_0$-graded real vector spaces.

(1) It is clear that for all $n\in\mathbb{N}_0$ we have $\mathrm{H}^n_{\mathrm{sm}}(\varnothing;\mathbb{R})=0$.

(2) Let $f,g\colon M\to N$ be two smooth maps between two smooth manifolds. Furthermore let $F\colon M\times[0,1]\to N$ be a *smooth* homotopy between f and g. Using the notation on

page 475 we introduce for each $n \in \mathbb{N}_0$ the singular chain

$$\Omega_n \;:=\; \sum_{j=0}^{n} (-1)^j \cdot \underbrace{[v_0, \ldots, v_j, w_j, \ldots, w_n]}_{\substack{\text{linear (in particular smooth)} \\ \text{map } \Delta^{n+1} \to \Delta^n \times [0,1]}} \;\in\; \mathrm{C}^{\mathrm{sm}}_{n+1}(\Delta^n \times [0,1]).$$

As on page 477 we introduce the homomorphisms

$$\begin{array}{rcl} P_n \colon \mathrm{C}_n(M) & \to & \mathrm{C}_{n+1}(N) \\ (\sigma \colon \Delta^n \to M) & \mapsto & (F \circ (\sigma \times \mathrm{id}_{[0,1]}))_*(\Omega_n). \end{array}$$

On page 477 we saw that these maps form a chain homotopy $P_n \colon \mathrm{C}_n(M) \to \mathrm{C}_{n+1}(N)$. It follows immediately from the definitions that these maps restrict to a chain homotopy $P_n \colon \mathrm{C}^{\mathrm{sm}}_n(M) \to \mathrm{C}^{\mathrm{sm}}_{n+1}(N)$. Now we can once again appeal to the Dual Cochain Map Lemma 66.9 to conclude that the induced maps on smooth singular cohomology are the same.

(3) Let M be a smooth manifold and let U and V be open subsets with $M = U \cup V$. We gave two different proofs of the Mayer–Vietoris Theorem 45.1. We will explain how to modify the second proof: In Subsection 45.1.2 the key to proving the existence of the Mayer–Vietoris sequence for ordinary singular cohomology was to show that the inclusion maps

$$\underbrace{\{\alpha + \beta \,|\, \alpha \in \mathrm{C}_k(U), \beta \in \mathrm{C}_k(V)\}}_{=\mathrm{C}_k^{\{U,V\}}(M)} \;\to\; \mathrm{C}_k(M)$$

define a chain homotopy equivalence. This in turn we showed using the Comfortable Cover Proposition 40.7. As part of the proof of the Comfortable Cover Proposition 40.7 we introduced on page 511 the subdivision maps

$$u_n \colon \mathrm{C}_n(M) \;\to\; \mathrm{C}_n(M), \quad n \in \mathbb{N}_0.$$

It follows immediately from the explicit definitions that these maps restrict to maps

$$u_n \colon \mathrm{C}^{\mathrm{sm}}_n(M) \;\to\; \mathrm{C}^{\mathrm{sm}}_n(M), \quad n \in \mathbb{N}_0.$$

Using this observation the proof of the existence of a natural connecting homomorphism in smooth singular cohomology that gives rise to a long exact sequence is otherwise identical to the proof for ordinary singular cohomology.

(4) The limit property is proved the same way as the Exhaustions-H*-Proposition 94.7 (1). We only have to replace the fact, proved in the Exhaustion-C$_*$-Lemma 46.1 that $\mathrm{C}_*(M) = \operatorname{colim} \mathrm{C}_*(U_i)$ by the observation that the same argument as in the Exhaustion-C$_*$-Lemma 46.1 shows that $\mathrm{C}^{\mathrm{sm}}_*(M) = \operatorname{colim} \mathrm{C}^{\mathrm{sm}}_*(U_i)$. ∎

95.4. **Relating the three smooth cohomology theories.** In this section we will show that for any smooth manifold M there exist explicit natural isomorphisms

$$\mathrm{H}^n(M;\mathbb{R}) \xrightarrow[\text{Proposition 95.5}]{\cong} \mathrm{H}^n_{\mathrm{sm}}(M;\mathbb{R}) \xleftarrow[\text{Proposition 95.6}]{\cong} \mathrm{H}^n_{\mathrm{dR}}(M).$$

These two natural isomorphisms give us in particular a natural isomorphism between singular cohomology and de Rham cohomology. We will discuss this isomorphism in greater detail in the follow-up Section 95.5.

In the following proposition we provide the first of the two natural isomorphisms:

Proposition 95.5. (Smooth Singular-H*-Isomorphism Proposition) For any smooth manifold M and any $n \in \mathbb{N}_0$ we have a natural isomorphism

$$s\colon \mathrm{H}^n(M;\mathbb{R}) \xrightarrow{\cong} \mathrm{H}^n_{\mathrm{sm}}(M;\mathbb{R}).$$

Proof. For any smooth manifold M we have the natural inclusion

$$\mathrm{C}^{\mathrm{sm}}_*(M) \to \mathrm{C}_*(M)$$

which evidently respects the boundary maps. These maps thus give rise to natural cochain maps

$$\mathrm{C}^*(M;\mathbb{R}) \to \mathrm{C}^*_{\mathrm{sm}}(M;\mathbb{R}).$$

Thus we obtain a natural homomorphism

$$s_M\colon \mathrm{H}^*(M;\mathbb{R}) \to \mathrm{H}^*_{\mathrm{sm}}(M;\mathbb{R}).$$

The proposition follows immediately from the Smooth-$\mathcal{H}^*$-Theory Isomorphism Theorem 95.2 and the following three observations:

(i) Since the natural transformation $s_M\colon \mathrm{H}^*(M;\mathbb{R}) \to \mathrm{H}^*_{\mathrm{sm}}(M;\mathbb{R})$ is induced by the inclusion map it follows easily from the Connecting Homomorphism Lemma 37.3 (2) that the natural transformation s is compatible with the connecting homomorphism in the Mayer-Vietoris sequences.

(ii) It follows easily from the naturality of the Inverse Limit-H*-Proposition 94.5 that the natural transformation s is compatible with the maps τ appearing in the limit property.

(iii) Note that every map $\Delta^n \to \star$ is smooth. Thus there is no difference between ordinary singular cohomology and smooth singular cohomology for the smooth manifold $\star$. It follows that the map $s_\star\colon \mathrm{H}^*(\star;\mathbb{R}) \to \mathrm{H}^*_{\mathrm{sm}}(\star;\mathbb{R})$ is an isomorphism. ■

Now we also want to relate smooth singular cohomology to de Rham cohomology. The discussion on page 1248 suggests that the key for doing so is the following definition:

Definition.

(1) Given $x \in \triangle_n$ we define $\widetilde{\mathrm{T}}_x\triangle_n := \mathbb{R}^n$. Using this definition it makes sense to talk of smooth n-forms on $\triangle_n$.

(2) Given a smooth n-form α on $\triangle_n$ we define[764]

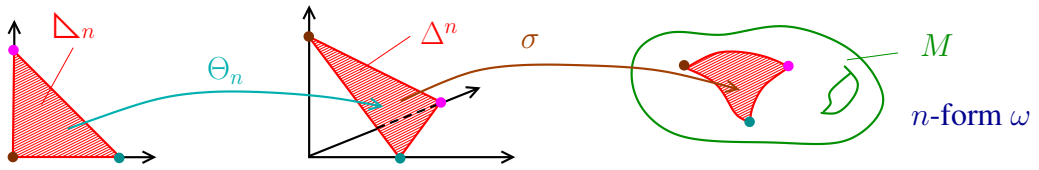
$$\int_{\triangle_n} \alpha := \int_{x\in\triangle_n} \alpha_x(e_1,\dots,e_n)\, d\lambda.$$

(3) Let M be a smooth manifold and let $f\colon \triangle_n \to M$ be a smooth map. Since $\triangle_n$ is regular closed, i.e. since it equals the closure of its interior, we obtain from the argument of the Regular Closed–∂_*-Lemma **??** that it makes sense to consider the map $\mathrm{D}f_x\colon \mathbb{R}^n \to \mathrm{T}_{f(x)}M$ for all $x \in \triangle_n$.

(4) Given a smooth manifold M, given a smooth differential n-form ω on M and given a smooth singular n-simplex $\sigma\colon \Delta^n \to M$ we use (1), (2) and (3) to define[765]

$$\int_\sigma \omega := (-1)^n \cdot \int_{\triangle_n} (\sigma\circ\Theta_n)^*(\omega).$$

We extend this definition linearly to define the integral of the smooth differential n-form ω over a smooth singular n-chain $\sum_{i=1}^k a_i\cdot\sigma_i \in \mathrm{C}^{\mathrm{sm}}_n(M)$.



Proposition 95.6. (De Rham–Smooth-H*-Isomorphism Theorem) For any smooth manifold M and any $k \in \mathbb{N}_0$ we have, by the above discussion, a homomorphism

$$\begin{array}{rcl} \Omega^k(M) & \to & \mathrm{C}^k_{\mathrm{sm}}(M;\mathbb{R}) \\ \omega & \mapsto & \begin{pmatrix} \mathrm{C}^{\mathrm{sm}}_k(M) & \to & \mathbb{R} \\ \sigma & \mapsto & \int\limits_\sigma \omega \end{pmatrix}. \end{array}$$

This map descends to a natural isomorphism

$$t_M \colon \mathrm{H}^k_{\mathrm{dR}}(M) \to \mathrm{H}^k_{\mathrm{sm}}(M;\mathbb{R}).$$

The proof of the De Rham–Smooth-H*-Isomorphism Proposition 95.6 relies on the following proposition:

Proposition 95.7. (Stokes-for-Simplices Proposition) Let M be a smooth manifold, let ω be a smooth differential $(n-1)$-form on M and let $\sigma \colon \Delta^n \to M$ be a smooth singular n-simplex. Then[766] [767]

$$\int\limits_\sigma d\omega = \int\limits_{\partial_n(\sigma)} \omega.$$

Proof of Proposition 95.7. The proposition is evidently just a variation on Stokes' Theorem **??**. In fact, as we will see, we can prove the proposition using a pretty elementary and direct calculation, which unfortunately turns out to be somewhat lengthy. To preserve the flow of the current discussion we outsource the proof of the proposition to Appendix 95.6. ∎

Proof of Proposition 95.6. First note that it follows basically immediately from the definitions that the map

$$\begin{array}{rcl} \Omega^k(M) & \to & \mathrm{C}^k_{\mathrm{sm}}(M;\mathbb{R}) \\ \omega & \mapsto & \begin{pmatrix} \mathrm{C}^{\mathrm{sm}}_k(M) & \to & \mathbb{R} \\ \sigma & \mapsto & \int\limits_\sigma \omega \end{pmatrix} \end{array}$$

is natural. Next note that it is an immediate consequence of the Stokes-for-Simplices Proposition 95.7 that these maps together form a cochain map $\Omega^*(M) \to \mathrm{C}^*_{\mathrm{sm}}(M;\mathbb{R})$.

---

[764] It follows from the Borel σ-Algebra Proposition **??** (1), the Lebesgue Integrability Proposition **??**, the observation that $\triangle_n$ is a compact subset of $\mathbb{R}^n$ and the observation that the integrand is continuous that the integral of the map $x \mapsto \alpha_x(e_1,\dots,e_n)$ over $\triangle_n$ is defined.

[765] Recall that on page 457 we equipped $\overset{\circ}{\Delta}{}^n$ with an orientation. The factor $(-1)^n$ is motivated by the fact that the restriction of the map $\Theta_n \colon \triangle_n \to \Delta^n$ to the interiors is orientation-preserving if and only if n is even.

[766] Here $d\omega$ denotes the differential of the smooth n-form ω, as defined on page **??**.

[767] Note that on the left we integrate a smooth differential n-form over the smooth singular n-simplex σ and that on the right we integrate a smooth $(n-1)$-form over the smooth singular $(n-1)$-chain $\partial_n(\sigma) \in \mathrm{C}^{\mathrm{sm}}_{n-1}(M)$.

Summarizing we see that we have constructed a natural map

$$t_M \colon \mathrm{H}^*_{\mathrm{dR}}(M) \ \to \ \mathrm{H}^*_{\mathrm{sm}}(M;\mathbb{R}).$$

The proposition follows immediately from the Smooth-$\mathcal{H}^*$-Theory Isomorphism Theorem 95.2 and the following three observations:

(i) We need to prove the following claim:

Claim 1. This natural transformation $t_M \colon \mathrm{H}^*_{\mathrm{dR}}(M) \to \mathrm{H}^*_{\mathrm{sm}}(M;\mathbb{R})$ is compatible with the connecting homomorphism in the Mayer-Vietoris sequences.

Proof. Let M be a smooth manifold and let U and V be open subsets of M such that $M = U \cup V$. We denote by $j_U \colon U \to M$, $j_V \colon V \to M$, $i_U \colon U \cap V \to U$ and $i_V \colon U \cap V \to V$ the inclusion maps. We consider the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega^*(M) & \xrightarrow{j_U^* \oplus j_V^*} & \Omega^*(U) \oplus \Omega^*(V) & \xrightarrow{(i_U^*, -i_V^*)} & \Omega^*(U \cap V) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle t_M} & & & & & & \\
 & & \mathrm{Hom}(\mathrm{C}^{\mathrm{sm}}_*(M), \mathbb{R}) & & \downarrow{\scriptstyle t_U \oplus t_V} & & \downarrow{\scriptstyle t_{U\cap V}} & & \\
 & & \downarrow{\scriptstyle \rho} & & & & & & \\
0 & \to & \mathrm{Hom}(\mathrm{C}^{\mathrm{sm}}_n(U) + \mathrm{C}^{\mathrm{sm}}_n(V), \mathbb{R}) & \xrightarrow{j_U^* \oplus j_V^*} & \mathrm{C}^*_{\mathrm{sm}}(U,\mathbb{R}) \oplus \mathrm{C}^*_{\mathrm{sm}}(V,\mathbb{R}) & \xrightarrow{(i_U^*, -i_V^*)} & \mathrm{C}^*_{\mathrm{sm}}(U \cap V, \mathbb{R}) & \to & 0.
\end{array}$$

We make the following clarifications and observations:

- We denote by $\mathrm{C}^{\mathrm{sm}}_n(U) + \mathrm{C}^{\mathrm{sm}}_n(V) \subset \mathrm{C}^{\mathrm{sm}}_n(M)$ the vector subspace of $\mathrm{C}^{\mathrm{sm}}_n(M)$ that is spanned by $\mathrm{C}^{\mathrm{sm}}_n(U)$ and $\mathrm{C}^{\mathrm{sm}}_n(V)$.
- The vertical homomorphism ρ is defined as the restriction map that is induced by the inclusion $\mathrm{C}^{\mathrm{sm}}_n(U) + \mathrm{C}^{\mathrm{sm}}_n(V) \subset \mathrm{C}^{\mathrm{sm}}_n(M)$.
- It follows easily from the naturality of t and the definition of the restriction map that the diagram commutes.
- By the Mayer–Vietoris-$\mathrm{H}^*_{\mathrm{dR}}$-Theorem **??** we know that the top sequence is exact and that it gives rise to the connecting homomorphism and the long exact sequence in de Rham cohomology.
- As in the second proof of the Mayer–Vietoris-H^*-Theorem 69.5 (which builds on the Mayer–Vietoris Theorem 45.1 from Subsection 45.1.2) we see that the bottom horizontal sequence is exact and we see that, together with the natural isomorphism $\rho^* \colon \mathrm{H}^n(M;\mathbb{R}) \ \to \ \mathrm{H}^n(\mathrm{Hom}(\mathrm{C}^{\mathrm{sm}}_n(U) + \mathrm{C}^{\mathrm{sm}}_n(V), \mathbb{R}))$, it gives rise to the connecting homomorphism and the long exact sequence in smooth singular cohomology.
- It follows from the above and the Connecting Homomorphism Lemma 37.3 that t commutes with the connecting homomorphisms. ⊞

(ii) We need to prove the following claim:

Claim 2. This natural transformation $t_M \colon \mathrm{H}^*_{\mathrm{dR}}(M) \to \mathrm{H}^*_{\mathrm{sm}}(M;\mathbb{R})$ is compatible with the map τ appearing in the limit property.

Proof. Let M be a smooth manifold and let $\{U_i\}_{i\in\mathbb{N}}$ be a sequence of open subsets of M that satisfy the following three conditions:
(a) for each $i \in \mathbb{N}$ the closure $\overline{U}_i$ of U_i is compact,
(b) for each $i \in \mathbb{N}$ we have $U_i \subset U_{i+1}$, and

(c) $M = \bigcup_{i\in\mathbb{N}} U_i$.

We consider the following maps

$$\begin{array}{ccccccc}
\Omega^k(U_1) & \longleftarrow & \Omega^k(U_2) & \longleftarrow & \Omega^k(U_3) & \longleftarrow & \dots \\
\downarrow{\scriptstyle t_{U_1}} & & \downarrow{\scriptstyle t_{U_2}} & & \downarrow{\scriptstyle t_{U_3}} & & \\
\mathrm{C}^k_{\mathrm{sm}}(U_1) & \longleftarrow & \mathrm{C}^k_{\mathrm{sm}}(U_2) & \longleftarrow & \mathrm{C}^k_{\mathrm{sm}}(U_3) & \longleftarrow & \dots
\end{array}$$

where the horizontal maps are induced by inclusions. Note that the diagram commutes by the naturality of t. The claim now follows from the Inverse Limit-H^*-Proposition 94.5. ⊞

(iii) Finally note that in the special case of $M = \star$ the map $t_\star\colon \mathrm{H}^*_{\mathrm{dR}}(\star) \to \mathrm{H}^*_{\mathrm{sm}}(\star;\mathbb{R})$ is easily seen to be an isomorphism. ■

95.5. **The isomorphism between singular and de Rham cohomology.** After the preparations in the previous sections we can now easily prove the theorem that we had been aiming for.

Theorem 95.8. (De Rham-Isomorphism Theorem) For any smooth manifold M and any $k \in \mathbb{N}_0$ we have a natural isomorphism[768]

$$u_M\colon \mathrm{H}^k_{\mathrm{dR}}(M) \xrightarrow{\cong} \mathrm{H}^k(M;\mathbb{R})$$

which has the property that for any closed differential k-form ω and any smooth singular k-chain $\sigma \in \mathrm{C}^{\mathrm{sm}}_k(M)$ that is a cycle we have

$$\big\langle \underbrace{u_M([\omega])}_{\in \mathrm{H}^k(M;\mathbb{R})}, \underbrace{[\sigma]}_{\in \mathrm{H}_k(M)} \big\rangle_M = \int_\sigma \omega.$$

Definition. Given a smooth manifold M we refer to the natural isomorphism

$$u_M\colon \mathrm{H}^k_{\mathrm{dR}}(M) \xrightarrow{\cong} \mathrm{H}^k(M;\mathbb{R})$$

from the De Rham-Isomorphism Theorem 95.8 and its proof as the de Rham isomorphism.

Proof. We consider the natural isomorphisms

$$\begin{array}{c} s\colon \mathrm{H}^k(M;\mathbb{R}) \to \mathrm{H}^k_{\mathrm{sm}}(M;\mathbb{R}) \\ [\varphi\colon \mathrm{C}_k(M) \to \mathbb{R}] \mapsto [\varphi|_{\mathrm{C}^{\mathrm{sm}}_k(M)}\colon \mathrm{C}^{\mathrm{sm}}_k(M) \to \mathbb{R}] \end{array} \quad \text{and} \quad \begin{array}{c} t\colon \mathrm{H}^k_{\mathrm{dR}}(M) \to \mathrm{H}^k_{\mathrm{sm}}(M;\mathbb{R}) \\ [\omega] \mapsto \left[\begin{array}{c} \mathrm{C}^{\mathrm{sm}}_k(M) \to \mathbb{R} \\ \sigma \mapsto \int_\sigma \omega \end{array}\right] \end{array}$$

from the Smooth Singular-H^*-Isomorphism Proposition 95.5 and the De Rham–Smooth-H^*-Isomorphism Proposition 95.6. It is clear that $u := s^{-1} \circ t\colon \mathrm{H}^k_{\mathrm{dR}}(M) \to \mathrm{H}^k(M;\mathbb{R})$ is a natural isomorphism. Finally note that for any closed differential k-form ω and any smooth singular k-chain σ that is a cycle we have

$$\langle u([\omega]),[\sigma]\rangle_M = \langle (s^{-1}\circ t)([\omega]),[\sigma]\rangle_M \underset{\text{since } \sigma \text{ is smooth}}{\uparrow} = \langle t([\omega]),[\sigma]\rangle_M \underset{\text{by definition of } t}{\uparrow} = \int_\sigma \omega.$$

■

[768] In the De Rham–Ring Isomorphism Proposition 96.1 we will see that this isomorphism is actually an isomorphism of rings.

The following proposition describes the natural isomorphism $u_M\colon \mathrm{H}^n_{\mathrm{dR}}(M) \to \mathrm{H}^n(M;\mathbb{R})$ from the above de Rham-Isomorphism Theorem 95.8 in the top dimension:

Proposition 95.9. (Top–de Rham Isomorphism Proposition) Let M be a closed oriented connected non-empty n-dimensional smooth manifold. Furthermore let $\omega \in \Omega^n(M)$ be a smooth differential n-form.

(1) We have $$\langle u_M([\omega]), [M]\rangle_M = \int\limits_M \omega.$$

(2) (a) If $\int_M \omega = 1$, then $u_M([\omega]) = [M]^* \in \mathrm{H}^n(M;\mathbb{R})$ where $[M]^*$ denotes the dual fundamental class that we introduced on page 909.

(b) If $[M]^*_{\mathrm{dR}} \in \mathrm{H}^n_{\mathrm{dR}}(M)$ denotes the de Rham fundamental class that we introduced on page **??**, then $u_M([M]^*_{\mathrm{dR}}) = [M]^*$.

Proof.

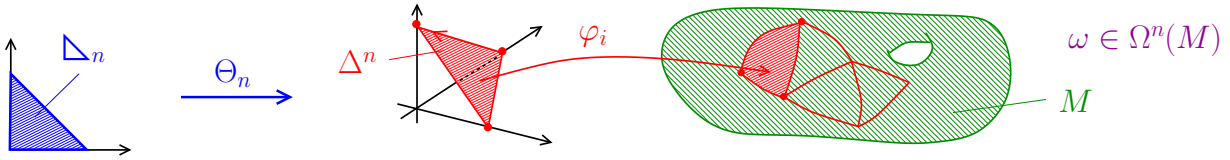
(1) Let M be a closed oriented connected non-empty n-dimensional smooth manifold and let $\omega \in \Omega^n(M)$ be a smooth differential n-form. As we explained in the Fundamental Class Algorithm **??**, there exist smooth singular simplices $\varphi_1, \dots, \varphi_k\colon \Delta^n \to M$ with the following properties:

(i) Each $\varphi_i\colon \Delta^n \to M$ is a smooth embedding in the sense of the definition on page **??**.

(ii) We have $\bigcup\limits_{i=1}^{k} \varphi_i(\Delta^n) = M$.

(iii) For any $i \neq j \in \{1, \dots, k\}$ we have $\varphi_i(\Delta^n) \cap \varphi_j(\Delta^n) = \varphi_i(\partial\Delta^n) \cap \varphi_j(\partial\Delta^n)$.

(iv) If for each $\in \{1, \dots, k\}$ we set $\epsilon_i := +1$ if φ_i is orientation-preserving and $\epsilon_i := -1$ otherwise, then the singular chain $\sigma := \epsilon_1 \cdot \varphi_1 + \dots + \epsilon_k \cdot \varphi_k$ is a singular cycle that represents the fundamental class $[M]$.



As above we denote by $\Theta_n\colon \Delta_n \to \Delta^n$ the inverse of the obvious projection map. We calculate that

$$\int\limits_M \omega \;\overset{\text{(1)}}{=}\; \sum_{i=1}^{k} (-1)^n \cdot \epsilon_i \cdot \int\limits_{\Delta_n} (\varphi_i \circ \Theta_n)^*(\omega) \;\overset{\text{(2)}}{=}\; \int\limits_\sigma \omega \;\overset{\text{(3)}}{=}\; \langle u_M([\omega]), [M]\rangle_M.$$

(1) follows from (i)–(iv), our orientation conventions the Lebesgue Integral Proposition **??** (2a), (3b), the Form Integration Proposition **??** (5) and the Measure-Zero Properties Proposition **??** (4)

(2) see definition on page 1252

(3) by the De Rham-Isomorphism Theorem 95.8

(2) (a) This statement follows immediately from (1) and the definition of the dual fundamental class $[M]^* \in \mathrm{H}^n(M;\mathbb{R})$ on page 909.

(b) This statement follows immediately from (a) and the definition of the de Rham fundamental class $[M]^*_{\mathrm{dR}} \in \mathrm{H}^n_{\mathrm{dR}}(M)$ on page **??**. ∎

Next we turn to the ubiquitous notion of the degree of a map between smooth manifolds. More precisely, let $f\colon M \to N$ be a smooth map between closed oriented connected non-empty n-dimensional smooth manifolds. On numerous occasions we introduced the degree $\deg(f\colon M \to N)$ of f:

(1) On page **??** we defined the degree using regular values and local degrees.
(2) On page **??** we defined the degree using de Rham cohomology.
(3) On page **??** we defined the degree using the smooth fundamental class.
(4) On page 842 we defined the degree using the topological fundamental class.

We can now show that all the definitions agree:

Proposition 95.10. (Degrees-Agree Proposition) Let $f\colon M \to N$ be a smooth map between closed oriented connected non-empty n-dimensional smooth manifolds.

(1) The definitions of the degree given on pages **??**, **??**, **??** and 842 agree.
(2) If ω is a smooth differential n-form on N with $\int_N \omega = 1$, then we have the following equality:
$$\int_M f^*(\omega) \;=\; \deg(f\colon M \to N).$$

Proof. Let $f\colon M \to N$ be a smooth map between closed oriented connected non-empty n-dimensional smooth manifolds.

(1) It follows from the Local-Global Degree Proposition 65.3 together with the Local Degree Lemma 65.2 (3) that the definitions on pages **??** and 842 agree. Furthermore it follows from the Smooth-Topological Fundamental Class Lemma **??** that the definitions on pages **??** and 842 agree. Finally note that it follows immediately from the Dual Fundamental Class–Degree Lemma 70.17, the Top–de Rham Isomorphism Proposition 95.10 and the naturality of u that the definitions on pages **??** and 842 agree.
(2) Let ω be a smooth differential n-form on N with $\int_N \omega = 1$. We have the following equalities:[769]

$$\begin{array}{ccccc}
\int_M f^*(\omega) & \overset{\text{by the Top–de Rham Isomorphism Isomorphism Proposition 95.10}}{\downarrow =} & \langle u_M(f^*([\omega])), [M]\rangle_M & \overset{\text{since } u \text{ is a } \textit{natural} \text{ isomorphism}}{\downarrow =} & \langle f^*(u_N([\omega])), [M]\rangle_M \\
& \overset{\text{by the definition of } \deg(f) \text{ on page 842}}{\uparrow =} & \deg(f) \cdot \langle u_N([\omega]), [N]\rangle_N & \overset{\text{by the Top–de Rham Isomorphism Proposition 95.10}}{\uparrow =} & \deg(f) \cdot \int_N \omega
\end{array}$$

$$\begin{array}{c}
\overset{\text{by the Kronecker Pairing Lemma 68.3 (2)}}{\downarrow =} \\
\langle u_N([\omega]), f_*([M])\rangle_N \\
\overset{\text{by choice of } \omega}{\uparrow =} \\
\deg(f).
\end{array}$$

■

The de Rham Isomorphism-Theorem 95.8 allows us to transport information from de Rham cohomology to real-valued singular cohomology and vice-versa. For example the many computational techniques that we had developed, e.g. cellular cohomology, and the results, e.g. the Manifold-Top-H*-Theorem 70.16, for singular cohomology can now be used to calculate isomorphism types of de Rham cohomology groups. In particular we can easily give a new proof of the following corollary:

[769] In the Degree-Integration Lemma **??** we gave a slightly different proof of this statement.

Corollary ?? (Top Dimension-H^n_{dR}-Isomorphism Corollary) If M is a closed orientable connected non-empty n-dimensional smooth manifold, then the homomorphism

$$\begin{array}{rcl} H^n_{dR}(M) & \to & \mathbb{R} \\ [\eta] & \mapsto & \int\limits_M \eta \end{array}$$

from the Top Dimension-H^n_{dR}-Integration Proposition **??** is an isomorphism.

Proof. By the Top Dimension-H^n_{dR}-Integration Proposition **??** the map is well-defined and it is an epimorphism. On the other hand, by the Manifold-Top-H^*-Theorem 70.16 (2a) and the De Rham-Isomorphism Theorem 95.8 we know that $H^n_{dR}(M) \cong \mathbb{R}$, which implies that the map is an isomorphism. ■

We can now also translate many earlier calculations of singular (co-) homology groups to de Rham cohomology. In particular we can now easily calculate the de Rham cohomology groups of complex projective spaces:

Corollary 95.11. ($\mathbb{C}P^n$-H^*_{dR}-Corollary) For any $n \in \mathbb{N}_0$ and any $k \in \mathbb{N}_0$ we have

$$H^k_{dR}(\mathbb{C}P^n) \cong \begin{cases} \mathbb{R}, & \text{if } k \in \{0, 2, 4, \dots, 2n\}, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Given $n \in \mathbb{N}_0$ and given $k \in \mathbb{N}_0$ we have the following isomorphisms:

$$\begin{array}{ccccc} H^k_{dR}(\mathbb{C}P^n) & \cong & H^k(\mathbb{C}P^n;\mathbb{R}) & \cong & H^k(\mathbb{C}P^n;\mathbb{Z}) \otimes \mathbb{R} \\ \uparrow & & \uparrow & & \uparrow \\ \text{De Rham-Isomorphism Theorem 95.8} & & \text{H}^*\text{–Change-of-Coefficients Theorem ??} & & \text{see page 881} \end{array} \quad \cong \quad \begin{cases} \mathbb{R}, & \text{if } k \in \{0, 2, 4, \dots, 2n\}, \\ 0, & \text{otherwise.} \end{cases}$$

■

95.6. **Appendix: Proof of the Stokes-for-Simplices Proposition 95.7.** In this appendix we will finally provide the proof of the following technical but essential proposition:

Proposition 95.7. (Stokes-for-Simplices Proposition) Let M be a smooth manifold, let ω be a smooth differential $(n-1)$-form on M and let $\sigma\colon \Delta^n \to M$ be a smooth singular n-simplex. Then

$$\int\limits_\sigma d\omega = \int\limits_{\partial_n(\sigma)} \omega.$$

All the hard work for the proof of the Stokes-for-Simplices Proposition 95.7 is done in the following lemma:

Lemma 95.12. (Stokes-on-$\triangle_n$-Lemma) For $i = 1, \dots, n$ we consider the map

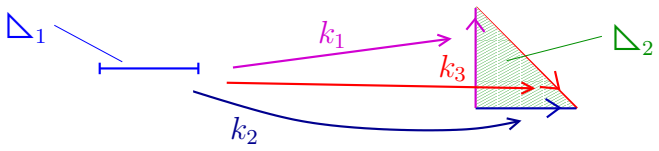
$$\begin{array}{rcl} k_i\colon \triangle_{n-1} & \to & \triangle_n \\ (x_1, \dots, x_{n-1}) & \to & (x_1, \dots, x_{i-1}, 0, x_i, \dots, x_{n-1}). \end{array}$$

Furthermore we consider the map

$$\begin{array}{rcl} k_{n+1}\colon \triangle_{n-1} & \to & \triangle_n \\ (x_1, \dots, x_{n-1}) & \to & (x_1, \dots, x_{n-1}, 1 - x_1 - \cdots - x_{n-1}). \end{array}$$

Given any smooth $(n-1)$-form α on $\triangle_n$ we have

$$\int\limits_{\triangle_n} d\alpha = \sum_{i=1}^{n} (-1)^i \cdot \int\limits_{\triangle_{n-1}} k_i^*\alpha + (-1)^{n+1} \cdot \int\limits_{\triangle_{n-1}} k_{n+1}^*\alpha.$$



Proof. Let α be a smooth $(n-1)$-form on $\triangle_n$ By the Alternating Forms-Basis Proposition **??** we can write

$$\alpha \;=\; \sum_{j=1}^{n} f_j \cdot dx^1 \wedge \cdots \wedge \widehat{dx^j} \wedge \cdots \wedge dx^n$$

where $f_1, \ldots, f_n \colon \triangle_n \to \mathbb{R}$ are smooth functions. First let us calculate the differential of the n-form $d\alpha$.

Claim 1. For each $x \in \triangle_n$ we have

$$(d\alpha)_x(e_1, \ldots, e_n) \;=\; \sum_{i=1}^{n} (-1)^{i-1} \cdot \frac{\partial f_i}{\partial x_i}(x).$$

Proof. We performed the calculation already in the proof of Stokes' Theorem **??**. But there is no harm done in doing so again. So let us do so:

$$d\alpha \overset{\text{see definition on page ??}}{\underset{\downarrow}{=}} \sum_{j=1}^{n} df_j \wedge dx^1 \wedge \cdots \wedge \widehat{dx^j} \wedge \cdots \wedge dx^n \overset{\text{by the 1-Form-via-}dx^i\text{-Lemma ??}}{\underset{\downarrow}{=}} \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial f_j}{\partial x_i} \cdot dx^i \wedge dx^1 \wedge \cdots \wedge \widehat{dx^j} \wedge \cdots \wedge dx^n$$

$$= \sum_{j=1}^{n} (-1)^{i-1} \cdot \frac{\partial f_j}{\partial x_j} \cdot dx^1 \wedge \cdots \wedge dx^j \wedge \cdots \wedge dx^n$$

$\uparrow$ follows from $dx^i \wedge dx^i = 0$ and $dx^i \wedge dx^j = -dx^j \wedge dx^i$

Thus for each $x \in \triangle_n$ we have

$$(d\alpha)_x(e_1, \ldots, e_n) \;=\; \sum_{i=1}^{n} (-1)^{i-1} \cdot \frac{\partial f_i}{\partial x_i}(x) \cdot \underbrace{(dx^1 \wedge \cdots \wedge dx^n)(e_1, \ldots, e_n)}_{=1} \;=\; \sum_{i=1}^{n} (-1)^{i-1} \cdot \frac{\partial f_i}{\partial x_i}(x).$$

⊞

We adopt the following notation: Given $i \in \{1, \ldots, n\}$ and given $x \in \mathbb{R}^n$ we denote by $\varphi_i(x) \in \mathbb{R}^{i-1}$ the first $i-1$ coordinates and we denote by $\psi_i(x) \in \mathbb{R}^{n-i}$ the last $n-i$ coordinates. Furthermore we denote by $|x|$ the sum of the coordinates of x. Now we see that

$$\int_{\triangle_n} d\alpha \;=\; \int_{x \in \triangle_n} (d\alpha)_x(e_1, \ldots, e_n) \overset{\text{by Claim 1}}{\underset{\downarrow}{=}} \sum_{i=1}^{n} (-1)^{i-1} \cdot \int_{x \in \triangle_n} \frac{\partial f_i}{\partial x_i}(x)\, d\lambda \overset{\text{Fubini Theorem ??}}{\underset{\downarrow}{=}}$$

$$= \sum_{i=1}^{n} (-1)^{i-1} \cdot \int_{s \in \triangle_{n-1}} \underbrace{\int_{x_i=0}^{x_i=1-|s|} \frac{\partial f_i}{\partial x_i}(\varphi_i(s), x_i, \psi_i(s))\, dx_i}_{\substack{\text{can be calculated using the} \\ \text{Fundamental Theorem of Calculus ??}}} d\lambda$$

$$= \sum_{i=1}^{n} (-1)^{i-1} \cdot \int_{s \in \triangle_{n-1}} (f_i(\varphi_i(s), 1-|s|, \psi_i(s)) - f_i(\varphi_i(s), 0, \psi_i(s))\, d\lambda$$

$$= \sum_{i=1}^{n} (-1)^{i-1} \cdot \int_{s \in \triangle_{n-1}} \underbrace{f_i(\varphi_i(s), 0, \psi_i(s))}_{=k_i(s)}\, d\lambda + \sum_{i=1}^{n} (-1)^{i-1} \cdot \int_{s \in \triangle_{n-1}} (f_i(\varphi_i(s), 1-|s|, \psi_i(s))\, d\lambda.$$

It remains to recognize the terms. We will do so in the following two claims.

Claim 2. Given any $i \in \{1, \dots, n\}$ we have

$$\int\limits_{s \in \triangle_{n-1}} f_i(k_i(s))\, d\lambda \;=\; \int\limits_{\triangle_{n-1}} k_i^* \alpha.$$

Proof. We calculate that

$$\begin{aligned}
\int\limits_{\triangle_{n-1}} k_i^* \alpha &= \int\limits_{s \in \triangle_{n-1}} (k_i^* \alpha)_s(e_1, \dots, e_{n-1})\, d\lambda \overset{\text{definition of } k_i^*}{=} \int\limits_{s \in \triangle_{n-1}} \alpha_{k_i(s)}(k_{i*}(e_1), \dots, k_{i*}(e_{n-1}))\, d\lambda \overset{\text{definition of } \alpha}{=} \\
&= \int\limits_{s \in \triangle_{n-1}} \sum_{j=1}^{n} f_j(k_i(s)) \cdot \underbrace{(dx^1 \wedge \dots \wedge \widehat{dx^j} \wedge \dots \wedge dx^n)(e_1, \dots, \widehat{e_i}, \dots, e_{n-1})}_{= \delta_{ij}, \text{ where } \delta_{ij} \text{ denotes the Kronecker delta}}\, d\lambda \\
&= \int\limits_{s \in \triangle_{n-1}} f_i(k_i(s))\, d\lambda.
\end{aligned}$$

⊞

Claim 3.

$$\sum_{i=1}^{n} \int\limits_{s \in \triangle_{n-1}} (-1)^{i-1} \cdot f_i(\varphi_i(s), 1 - |s|, \psi_i(s))\, d\lambda \;=\; (-1)^{n+1} \cdot \int\limits_{\triangle_{n-1}} k_{n+1}^* \alpha.$$

Proof. We calculate that

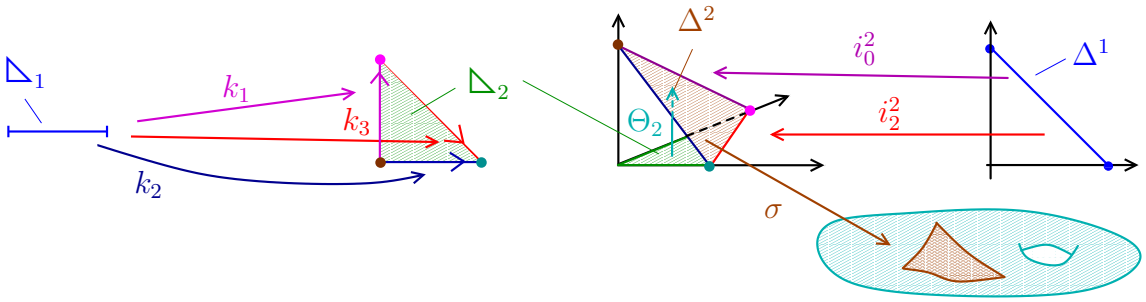
$$\begin{aligned}
\int\limits_{\triangle_{n-1}} k_{n+1}^* \alpha &= \int\limits_{s \in \triangle_{n-1}} (k_{n+1}^* \alpha)_s(e_1, \dots, e_{n-1})\, d\lambda \overset{\text{definition of } k_i^*}{=} \int\limits_{s \in \triangle_{n-1}} \alpha_{k_{n+1}(s)}(k_{n+1,*}(e_1), \dots, k_{n+1,*}(e_{n-1}))\, d\lambda \overset{\text{definition of } \alpha}{=} \\
&= \int\limits_{s \in \triangle_{n-1}} \sum_{j=1}^{n} f_j(k_{n+1}(s)) \cdot \underbrace{(dx^1 \wedge \dots \wedge \widehat{dx^j} \wedge \dots \wedge dx^n)(e_1 - e_n, \dots, e_{n-1} - e_n)}_{= (-1)^{n-j}}\, d\lambda \\
&= \sum_{j=1}^{n} \int\limits_{s \in \triangle_{n-1}} (-1)^{n-j} \cdot f_j(k_{n+1}(s))\, d\lambda \;=\; \sum_{j=1}^{n} \int\limits_{s \in \triangle_{n-1}} (-1)^{n-j} \cdot f_j\underbrace{(\varphi_n(s), 1 - |s|, \psi_n(s))}_{= k_{n+1}(s)}\, d\lambda \\
&= \sum_{j=1}^{n} \int\limits_{s \in \triangle_{n-1}} (-1)^{n-j} \cdot f_j(\varphi_j(s), 1 - |s|, \psi_j(s))\, d\lambda.
\end{aligned}$$

↑ follows from the Lebesgue Transformation Theorem **??** applied to the diffeomorphism that swaps the j-th and the n-th coordinate

This calculation of course implies the claim. ■

Proof of the Stokes-for-Simplices Proposition 95.7. Let M be a smooth manifold, let ω be a smooth differential $(n-1)$-form on M and let $\sigma \colon \Delta^n \to M$ be a smooth singular n-simplex. For $j \in \{0, \dots, n\}$ we consider as usual the map

$$\begin{aligned}
i_j^n \colon \Delta^{n-1} &\to \Delta^n \\
(t_0, \dots, t_{n-1}) &\mapsto (t_0, \dots, t_{j-1}, 0, t_j, \dots, t_{n-1}).
\end{aligned}$$



The proposition follows from the following calculation:

$$\int_\sigma d\omega \overset{\text{by the definition on page 1252}}{=} (-1)^n \cdot \int_{\Delta_n} (\sigma\circ\Theta_n)^*(d\omega) \overset{\text{by the Differential-Natural Proposition ??}}{=} (-1)^n \cdot \int_{\Delta_n} d((\sigma\circ\Theta_n)^*(\omega)) \overset{\text{by the Stokes-on-}\Delta_n\text{-Lemma 95.12 applied to } \alpha=(\sigma\circ\Theta_n)^*(d\omega)}{=}$$

$$= \sum_{j=1}^{n} (-1)^{n+i} \cdot \int_{\Delta_{n-1}} \underbrace{k_j^*((\sigma\circ\Theta_n)^*(\omega))}_{(\sigma\circ\Theta_n\circ k_j)^*(\omega)} + (-1)^{2n+1} \cdot \int_{\Delta_{n-1}} \underbrace{k_{n+1}^*((\sigma\circ\Theta_n)^*(\omega))}_{(\sigma\circ\Theta_n\circ k_j)^*(\omega)}$$

$$= (-1)^{n-1} \cdot \sum_{l=0}^{n} (-1)^l \cdot \int_{\Delta_{n-1}} (\sigma\circ i_l^n\circ\Theta_{n-1})^*(\omega) = (-1)^{n-1} \cdot \sum_{l=0}^{n} (-1)^l \cdot \int_{\sigma\circ i_l^n} \omega = \int_{\partial\sigma} \omega.$$

since $\Theta_n\circ k_j = i_{j+1}^{n-1}\circ\Theta_{n-1}\colon \Delta_{n-1}\to\Delta^n$ (by the definition on page 1252) (by definition of $\partial\sigma$) ■

CHAPTER 96

The wedge product and the cup product

In the previous chapter we saw in the De Rham Isomorphism-Theorem 95.8 that for a smooth manifold M the de Rham cohomology vector spaces $\mathrm{H}^k_{\mathrm{dR}}(M)$ are naturally isomorphic to the singular cohomology vector spaces $\mathrm{H}^k(M;\mathbb{R})$. De Rham cohomology and singular cohomology each carry an extra structure which turns de Rham cohomology and singular cohomology into rings. In this fairly short chapter we will see that the above de Rham isomorphism $\mathrm{H}^*_{\mathrm{dR}}(M) \to \mathrm{H}^*(M;\mathbb{R})$ is actually a ring isomorphism.

96.1. **The wedge product of differential forms.** Recall that given a smooth manifold M we introduced in the Wedge-on-de Rham Cohomology Proposition **??** the wedge product

$$\begin{array}{rcl} \wedge\colon \mathrm{H}^k_{\mathrm{dR}}(M) \times \mathrm{H}^l_{\mathrm{dR}}(M) & \to & \mathrm{H}^{k+l}_{\mathrm{dR}}(M) \\ ([\omega],[\sigma]) & \mapsto & [\omega]\wedge[\sigma] \coloneqq [\omega\wedge\sigma]. \end{array}$$

Furthermore recall that given any topological space X and given any commutative ring R we introduced on page 919 the cup product

$$\begin{array}{rcl} \cup\colon \mathrm{H}^k(X;R) \times \mathrm{H}^l(X;R) & \to & \mathrm{H}^{k+l}(X;R) \\ ([\varphi],[\psi]) & \mapsto & [\varphi]\cup[\psi] \coloneqq [\varphi\cup\psi]. \end{array}$$

The following proposition shows that the natural isomorphism $u_M\colon \mathrm{H}^k_{\mathrm{dR}}(M) \to \mathrm{H}^k(M;\mathbb{R})$ from the De Rham Isomorphism-Theorem 95.8 preserves these structures:

Proposition 96.1. (De Rham–Ring Isomorphism Proposition) Let M be a smooth manifold and let

$$u_M\colon \mathrm{H}^k_{\mathrm{dR}}(M) \to \mathrm{H}^k(M;\mathbb{R})$$

be the de Rham isomorphism from the De Rham Isomorphism-Theorem 95.8. Given any $\varphi \in \mathrm{H}^k_{\mathrm{dR}}(M)$ and given any $\psi \in \mathrm{H}^l_{\mathrm{dR}}(M)$ we have

$$u_M(\varphi\wedge\psi) = u_M(\varphi)\cup u_M(\psi) \in \mathrm{H}^{k+l}(M;\mathbb{R}).$$

Proof. We postpone the proof of the De Rham–Ring Isomorphism Proposition 96.1 to Section 96.2. ∎

Example. As usual we denote by dt on $\mathbb{R}$ the 1-form that for every $v \in \mathrm{T}_P\mathbb{R} = \mathbb{R}$ is given by $dt(v) = v$. We denote by $\pi\colon \mathbb{R} \to \mathbb{R}/\mathbb{Z} =_{\mathrm{i}} S^1$ the natural projection. By the G-Invariant Form Lemma **??** together with the Pullback-Closed-Exact Lemma **??** (3) the closed 1-form dt descends to a closed 1-form on S^1, i.e. there exists a unique 1-form $\overline{dt}$ on $\mathbb{R}/\mathbb{Z} =_{\mathrm{i}} S^1$ whose pullback equals dt. Let $\sigma\colon \Delta^1 \to \mathbb{R}/\mathbb{Z} =_{\mathrm{i}} S^1$ be a smooth singular 1-simplex. Since Δ^1 is simply connected we can lift this singular 1-simplex, i.e. there exists a continuous

map (which is in fact smooth) $\widetilde{\sigma}\colon \Delta^1 \to \mathbb{R}$ with $\pi \circ \widetilde{\sigma} = \sigma$. We have

$$\int_\sigma \overline{dt} \underset{\uparrow}{=} -\int_{s=0}^{s=1} \overline{dt}(\sigma'(t,1-t))ds \underset{\uparrow}{=} -\int_{s=0}^{s=1} \widetilde{\sigma}'(t,1-t)ds \underset{\uparrow}{=} -\widetilde{\sigma}(1,0)+\widetilde{\sigma}(0,1) \underset{\uparrow}{=} \theta_{\mathbb{R}}(\sigma).$$

see definition on page 1252 — definition of $\overline{dt}$ — Fundamental Theorem of Calculus — see page 866 for for the definition of the 1-cochain $\theta_{\mathbb{R}}$

This little calculation shows that under the natural isomorphisms from the De Rham-Isomorphism Theorem 95.8 the class $[\overline{dt}] \in \mathrm{H}^1_{\mathrm{dR}}(M)$ gets sent to our favorite generator $[\theta_{\mathbb{R}}] \in \mathrm{H}^1(M;\mathbb{R})$, i.e. we have $u([\overline{dt}]) = [\theta_{\mathbb{R}}]$.

Next we consider the torus $M = S^1 \times S^1$. As before we denote $p\colon S^1 \times S^1 \to S^1$ respectively $q\colon S^1 \times S^1 \to S^1$ the projection onto the first, respectively the second factor. On page **??** we showed that, with the usual product orientation on the torus $M = S^1 \times S^1$ we have

$$\int_{S^1 \times S^1} p^*(\overline{dt}) \wedge q^*(\overline{dt}) \;=\; +1.$$

On the other hand, by the naturality of u we have $u(p^*([\overline{dt}])) = p^*(u([\overline{dt}])) = p^*([\theta_{\mathbb{R}}])$ and similarly we also have $u(q^*([\overline{dt}]) = q^*([\theta_{\mathbb{R}}])$. The calculation of the above integral is now consistent with the calculation in the Torus–Cup Product Lemma 72.11 where we had, implicitly, shown that $\langle p^*([\theta_{\mathbb{R}}]) \cup q^*([\theta_{\mathbb{R}}]), [S^1 \times S^1]\rangle = +1$. □

We can now use the De Rham–Ring Isomorphism Proposition 96.1 together with results on singular cohomology groups to give a new proof of the following result:

Theorem ?? (De Rham Poincaré Duality Theorem For any closed oriented n-dimensional smooth manifold M the following $\mathbb{R}$-bilinear pairing is non-singular:

$$\begin{array}{rcl} \mathrm{H}^k_{\mathrm{dR}}(M) \times \mathrm{H}^{n-k}_{\mathrm{dR}}(M) & \to & \mathbb{R} \\ ([\omega],[\eta]) & \mapsto & \displaystyle\int_M \omega \wedge \eta. \end{array}$$

Proof. The De Rham–Ring Isomorphism Proposition 96.1 and the Top–de Rham Isomorphism Proposition 95.10 give us the following commutative diagram with vertical isomorphisms:

$$\begin{array}{ccc} \mathrm{H}^k_{\mathrm{dR}}(M) \times \mathrm{H}^{n-k}_{\mathrm{dR}}(M) & \xrightarrow{(\varphi,\psi)\mapsto \int_M \varphi\wedge\psi} & \mathbb{R} \\ \cong\downarrow u \quad u\downarrow\cong & & \downarrow \mathrm{id} \\ \mathrm{H}^k(M;\mathbb{R}) \times \mathrm{H}^{n-k}(M;\mathbb{R}) & \xrightarrow{(\varphi,\psi)\mapsto \langle \varphi\cup\psi,[M]\rangle} & \mathbb{R}. \end{array}$$

Since M is closed we obtain from the Manifold-Non Singularity Proposition 81.2 (2) that the $\mathbb{R}$-bilinear pairing on the bottom is non-singular. Thus the $\mathbb{R}$-bilinear pairing on the top is also non-singular. ■

The name "De Rham–Ring Isomorphism Proposition 96.1" hints at the fact, that the statement of the proposition should be phrased in terms of rings. To make this precise it is helpful to recall the following definition from page 929:

Definition.

(1) A graded ring is a ring R together with a direct sum decomposition[770] $R = \bigoplus_{n\in\mathbb{N}_0} R_n$ into subgroups such that for any $m, n \in \mathbb{N}_0$ the multiplication map restricts to a bilinear map
$$R_m \times R_n \to R_{m+n}.$$
(2) Let $R = \bigoplus_{n\in\mathbb{N}_0} R_n$ and $S = \bigoplus_{n\in\mathbb{N}_0} S_n$ be two graded rings. A morphism between the two graded rings is a ring homomorphism $\varphi\colon R \to S$ such that for any $n \in \mathbb{N}_0$ we have $\varphi(R_n) \subset S_n$.
(3) We define the category $\mathcal{GrRing}$ of graded rings to be the category where the objects are graded rings and the morphisms are morphisms of graded rings.

Example. De Rham cohomology and singular cohomology with real coefficients both give rise to graded rings:

(1) Let M be a smooth manifold. We write $\mathrm{H}^*_{\mathrm{dR}}(M) \coloneqq \bigoplus_{i=0}^{\infty} \mathrm{H}^i_{\mathrm{dR}}(M)$. We consider the map
$$\begin{aligned} \wedge\colon \mathrm{H}^*_{\mathrm{dR}}(M) \times \mathrm{H}^*_{\mathrm{dR}}(M) &\to \mathrm{H}^*_{\mathrm{dR}}(M) \\ \left(\sum_{i=0}^{\infty} x_i, \sum_{i=0}^{\infty} y_i\right) &\mapsto \sum_{i,j=0}^{\infty} \varphi_i \wedge \psi_j \end{aligned}$$
where it is understood that each x_i and y_i is homogeneous of degree i. In the De Rham-$\mathrm{H}^*_{\mathrm{dR}}$-Ring Functor Proposition **??** we showed that the maps
$$\begin{aligned} M &\mapsto (\mathrm{H}^k_{\mathrm{dR}}(M), \cup) \\ (f\colon M \to N) &\mapsto (f^*\colon \mathrm{H}^k_{\mathrm{dR}}(N) \to \mathrm{H}^k_{\mathrm{dR}}(M)) \end{aligned}$$
define a contravariant functor from the category $\mathcal{SmMfd}$ of smooth manifolds to the category of graded rings $\mathcal{GrRing}$.

(2) Let X be a topological space. We write $\mathrm{H}^*(X;\mathbb{R}) \coloneqq \bigoplus_{i=0}^{\infty} \mathrm{H}^i(X;\mathbb{R})$. We consider the map
$$\begin{aligned} \cup\colon \mathrm{H}^*(X;\mathbb{R}) \times \mathrm{H}^*(X;\mathbb{R}) &\to \mathrm{H}^*(X;\mathbb{R}) \\ \left(\sum_{i=0}^{\infty} \varphi_i, \sum_{i=0}^{\infty} \psi_i\right) &\mapsto \sum_{i,j=0}^{\infty} x_i \cup y_j. \end{aligned}$$
where again it is understood that each φ_i and ψ_i is homogeneous of degree i. In the Cup Product-Ring Proposition 72.9 (3) we showed that the maps
$$\begin{aligned} X &\mapsto (\mathrm{H}^k(X;\mathbb{R}), \cup) \\ (f\colon X \to Y) &\mapsto (f^*\colon \mathrm{H}^*(Y;\mathbb{R}) \to \mathrm{H}^*(X;\mathbb{R})) \end{aligned}$$
define a contravariant functor from the category $\mathcal{Top}$ of topological spaces to the category of graded rings $\mathcal{GrRing}$. □

With this language we can now formulate the following theorem:

Theorem 96.2. The natural isomorphisms u from the De Rham-Isomorphism Theorem 95.8 are in fact ring homomorphisms, i.e. they define a natural isomorphism from the functor
$$\begin{aligned} \mathcal{SmMfd} &\to \mathcal{GrRing} \\ M &\mapsto (\mathrm{H}^*_{\mathrm{dR}}(M), \wedge) \end{aligned}$$
to the functor
$$\begin{aligned} \mathcal{SmMfd} &\to \mathcal{GrRing} \\ M &\mapsto (\mathrm{H}^*(M;\mathbb{R}), \cup). \end{aligned}$$

Proof. The theorem is an immediate consequence of the De Rham-Isomorphism Theorem 95.8 together with the De Rham–Ring Isomorphism Proposition 96.1. ■

[770] In particular, R viewed as an abelian group is $\mathbb{N}_0$-graded.

96.2. **Proof of the De Rham–Ring Isomorphism Proposition 96.1.** In this, the second and final section of the chapter, we provide the proof of the De Rham–Ring Isomorphism Proposition 96.1. In the proof of the De Rham–Ring Isomorphism Proposition 96.1 we will need the shuffle map which we first introduced on page 718:

Definition. Let $k, l \in \mathbb{N}_0$.

(1) An (k,l)-shuffle is a pair (μ, ν) of disjoint sets of integers

$$1 \leq \mu_1 < \mu_2 < \cdots < \mu_k \leq k+l \quad \text{and} \quad 1 \leq \nu_1 < \nu_2 < \cdots < \nu_l \leq k+l.$$

(2) We denote by $S_{k,l}$ the set of all (k,l)-shuffles.

(3) We denote by $\operatorname{sign}(\mu, \nu)$ the sign of the permutation $(\mu_1, \ldots, \mu_k, \nu_1, \ldots, \nu_l)$.

(4) Given a sequence $1 \leq \mu_1 < \mu_2 < \cdots < \mu_k \leq k+l$ we denote by $\eta^\mu \colon \mathbb{R}^{k+l+1} \to \mathbb{R}^{k+1}$ the unique linear map which satisfies $\eta^\mu(\mathfrak{v}_i) = \mathfrak{v}_j$ for $\mu_j \leq i < \mu_{j+1}$. One can easily verify that η^μ restricts to a map $\Delta^{k+l} \to \Delta^k$ which we also denote by η^μ.

(5) Given two topological spaces X and Y we consider the homomorphism

$$\begin{aligned} \nabla := \nabla_{k,l} \colon \mathrm{C}_k(X) \otimes \mathrm{C}_l(Y) &\to \mathrm{C}_{k+l}(X \times Y) \\ (\sigma \otimes \tau) &\mapsto \sum_{(\mu,\nu) \in S_{k,l}} \operatorname{sign}(\mu,\nu) \cdot \Big(\underbrace{\overbrace{(\sigma \times \tau)}^{\Delta^k \times \Delta^l \to X \times Y} \circ \overbrace{(\eta^\mu, \eta^\nu)}^{\Delta^{k+l} \to \Delta^k \times \Delta^l}}_{\text{map } \Delta^{k+l} \to X \times Y} \Big). \end{aligned}$$

The following lemma will play an essential role in the proof of the De Rham–Ring Isomorphism Proposition 96.1:

Lemma 96.3. (Shuffle–Decomposition Lemma) Let $k, l \in \mathbb{N}_0$. The following statements hold:

(1) The images of the maps $(\eta^\mu, \eta^\nu) \colon \Delta^{k+l} \to \Delta^k \times \Delta^l$ with $(\mu, \nu) \in S_{m,n}$ cover $\Delta^k \times \Delta^l$ and the images overlap only in subsets of measure zero.

(2) For each $(\mu, \nu) \in S_{m,n}$ the map $(\eta^\mu, \eta^\nu) \colon \Delta^{k+l} \to \Delta^k \times \Delta^l$ is orientation-preserving if and only if $\operatorname{sign}(\mu, \nu) = 1$.

Sketch of proof of Lemma 96.3. The lemma follows easily from the Simplicial Complex-Product Lemma **??**. ∎

Proof of the De Rham–Ring Isomorphism Proposition 96.1. Let M be a smooth manifold. We start out with the following claim:

Claim 1. We consider the following diagram:

$$\begin{array}{ccccccc}
 & & & \overset{\wedge}{\frown} & & & \\
\mathrm{H}^k_{\mathrm{dR}}(M) \times \mathrm{H}^l_{\mathrm{dR}}(M) & & \xrightarrow{([\varphi],[\psi]) \mapsto [p_1^*(\varphi) \wedge p_2^*(\psi)]} & & \mathrm{H}^{k+l}_{\mathrm{dR}}(M \times M) & \xrightarrow{d_M^*} & \mathrm{H}^{k+l}_{\mathrm{dR}}(M) \\
\downarrow s \cong \quad \cong \downarrow s & & & & \cong \downarrow s & & \cong \downarrow s \\
\mathrm{H}^k_{\mathrm{sm}}(M;\mathbb{R}) \times \mathrm{H}^l_{\mathrm{sm}}(M;\mathbb{R}) & \xrightarrow{\Xi_{k,l}} & \mathrm{H}^{k+l}(\mathrm{C}^{\mathrm{sm}}_*(M) \otimes \mathrm{C}^{\mathrm{sm}}_*(M);\mathbb{R}) & \xleftarrow[\nabla^*]{\cong} & \mathrm{H}^{k+l}_{\mathrm{sm}}(M \times M;\mathbb{R}) & \xrightarrow{d_M^*} & \mathrm{H}^k_{\mathrm{sm}}(M;\mathbb{R}) \\
\uparrow t \cong \quad \cong \uparrow t & & \cong \uparrow t & & \cong \uparrow t & & \cong \uparrow t \\
\mathrm{H}^k(M;\mathbb{R}) \times \mathrm{H}^l(M;\mathbb{R}) & \xrightarrow{\Xi_{k,l}} & \mathrm{H}^{k+l}(\mathrm{C}_*(M) \otimes \mathrm{C}_*(M);\mathbb{R}) & \xleftarrow[\nabla^*]{\cong} & \mathrm{H}^{k+l}(M \times M;\mathbb{R}) & \xrightarrow{d_M^*} & \mathrm{H}^k(M;\mathbb{R}). \\
 & & & \underset{\cup}{\smile} & & &
\end{array}$$

(Some of the maps are explained in the proof of the claim.) Eventually we need to show that the diagram commutes. In this claim we prove the following statements:

(1) All parts of the diagram, except possibly the big top left rectangle, commute.
(2) All maps decorated by "$\cong$" are indeed isomorphisms.

Proof. The claim follows from the following clarifications and observations:

(a) For $i = 1, 2$ we denote by $p_i \colon M \times M \to M$ the projection onto the i-th factor.
(b) $d_M \colon M \to M \times M$ is the diagonal map.
(c) It follows from $p_1 \circ d_M = p_2 \circ d_M = \mathrm{id}_M$ and the Alternating Form-Functoriality Lemma **??** that the top horizontal maps define the wedge product on de Rham cohomology.
(d) We denote by $s_N \colon \mathrm{H}^j_{\mathrm{dR}}(N) \to \mathrm{H}^j_{\mathrm{sm}}(N;\mathbb{R})$ the natural isomorphisms from the Smooth Singular-H*-Isomorphism Proposition 95.5 and we denote by $t_N \colon \mathrm{H}^j(N;\mathbb{R}) \to \mathrm{H}^j_{\mathrm{sm}}(N;\mathbb{R})$ the natural isomorphisms from the De Rham–Smooth-H*-Isomorphism Theorem 95.6.
(e) We introduce the following maps:
- We denote by $\mathrm{C}_*(M) \otimes \mathrm{C}_*(M)$ the tensor product of chain complexes, as defined on page 693. Recall that on the chain level we have by definition
$$(\mathrm{C}_*(M) \otimes \mathrm{C}_*(M))_d \;=\; \bigoplus_{r+s=d} \mathrm{C}_r(M) \otimes \mathrm{C}_s(M).$$
- We consider the map $\Pi_{k,l} \colon (\mathrm{C}_*(M) \otimes \mathrm{C}_*(M))_{k+l} \to \mathrm{C}_k(M) \otimes \mathrm{C}_l(M)$ that is given by projection onto the corresponding summand.
- We denote by $\mu_{\mathbb{R}} \colon \mathbb{R} \otimes \mathbb{R} \to \mathbb{R}$ the map that is induced by $x \otimes y \mapsto x \cdot y$.

It follows from the Cocycle Tensor Lemma 72.1 that the map
$$\begin{aligned} \Xi_{k,l} \colon \mathrm{H}^k(M;\mathbb{R}) \times \mathrm{H}^l(M;\mathbb{R}) &\to \mathrm{H}^{k+l}(\mathrm{C}_*(M) \otimes \mathrm{C}_*(M), \mathbb{R}) \\ ([\varphi], [\psi]) &\mapsto [\mu_{\mathbb{R}} \circ (\varphi \otimes \psi) \circ \Pi_{k,l}] \end{aligned}$$
is well-defined. Clearly this map also works for smooth simplices and we get a corresponding map on smooth (co-) chain complexes and cohomology groups.
(f) It follows from the definition on page 919 and the Shuffle Proposition 56.11 that the composition of the bottom horizontal maps gives us the cup product.
(g) Note that the right hand square commutes by the naturality of u.
(h) Given any two topological spaces X and Y we denote by
$$\begin{aligned} \nabla \colon \mathrm{C}_m(X) \otimes \mathrm{C}_n(Y) &\to \mathrm{C}_{m+n}(X \times Y) \\ (\sigma \otimes \tau) &\mapsto \sum_{(\mu,\nu) \in S_{m,n}} \mathrm{sign}(\mu,\nu) \cdot ((\sigma \times \tau) \circ (\eta^\mu, \eta^\nu) \end{aligned}$$
the shuffle map that we first introduced on page 718.
(i) Finally note that it follows from the Shuffle Proposition 56.11 together with the Eilenberg-Zilber Theorem 56.1 and the Cochain Homotopy-H*-Lemma 66.6 that the map $\nabla^* \colon \mathrm{H}^{k+l}(\mathrm{C}_*(M \times M);\mathbb{R}) \to \mathrm{H}^{k+l}(\mathrm{C}_*(M) \otimes \mathrm{C}_*(M);\mathbb{R})$ is an isomorphism. ⊞

It follows from Claim 1 that it remains to show that big rectangle in the statement of the claim commutes, i.e. we need to show the two maps from $\mathrm{H}^k_{\mathrm{dR}}(M) \times \mathrm{H}^l_{\mathrm{dR}}(M)$ to $\mathrm{H}^{k+l}(\mathrm{C}^{\mathrm{sm}}_*(M) \otimes \mathrm{C}^{\mathrm{sm}}_*(M);\mathbb{R})$ agree. In other words, it suffice to prove the following claim:

Claim 2. The following diagram commutes:

$$\begin{array}{ccccc}
\Omega^k(M) \times \Omega^l(M) & & \xrightarrow{(\varphi,\psi)\mapsto p_1^*(\varphi)\wedge p_2^*(\psi)} & & \Omega^{k+l}(M\times M) \\
\downarrow s & & & & \downarrow s \\
\mathrm{C}^k_{\mathrm{sm}}(M;\mathbb{R}) \times \mathrm{C}^l_{\mathrm{sm}}(M;\mathbb{R}) & \xrightarrow{\Xi_{k,l}} & \mathrm{Hom}((\mathrm{C}^{\mathrm{sm}}_*(M)\otimes \mathrm{C}^{\mathrm{sm}}_*(M))_{k+l};\mathbb{R}) & \xleftarrow{\nabla^*} & \mathrm{C}^{k+l}_{\mathrm{sm}}(M\times M;\mathbb{R})
\end{array}$$

Proof. Let $\varphi \in \Omega^k(M)$ and $\psi \in \Omega^l(M)$ be two smooth forms. We need to show that the two maps from $\Omega^k(M)\times\Omega^l(M)$ to $\mathrm{Hom}((\mathrm{C}^{\mathrm{sm}}_*(M)\otimes\mathrm{C}^{\mathrm{sm}}_*(M))_{k+l};\mathbb{R})$ send (φ,ψ) to the same homomorphism $(\mathrm{C}^{\mathrm{sm}}_*(M)\otimes\mathrm{C}^{\mathrm{sm}}_*(M))_{k+l}\to\mathbb{R}$. We only have to show that the two resulting homomorphisms agree on a generating set of $\mathrm{C}^{\mathrm{sm}}_*(M)\otimes\mathrm{C}^{\mathrm{sm}}_*(M))_{k+l}$. Thus let $r,s\in\mathbb{N}_0$ with $r+s=k+l$ and let $\varphi\colon\Delta^r\to M$ and $\tau\colon\Delta^s\to M$ be two smooth simplices.

(1) First we assume that $r=k$ (and thus also $s=l$). Note that

$$\begin{aligned}
(\nabla^*\circ s)(p_1^*(\varphi)\wedge p_2^*(\psi)))(\sigma\otimes\tau) &\overset{\text{by definition of the dual map}}{=} s(p_1^*(\varphi)\wedge p_2^*(\psi))(\nabla(\sigma\otimes\tau)) \\
&\overset{\text{by definition of } s}{=} \int_{\nabla(\sigma\otimes\tau)} p_1^*(\varphi)\wedge p_2^*(\psi) \\
&\overset{\text{by definition of } \nabla}{=} \sum_{(\mu,\nu)\in S_{k,l}} \int_{(\sigma\times\tau)\circ(\eta^\mu,\eta^\nu)\colon \Delta^{k+l}\to M\times N} \operatorname{sign}(\mu,\nu)\cdot p_1^*(\varphi)\wedge p_2^*(\psi) \\
&= \sum_{(\mu,\nu)\in S_{k,l}} \int_{(\eta_\mu,\eta_\nu)\colon \Delta^{k+l}\to\Delta^k\times\Delta^l} \operatorname{sign}(\mu,\nu)\cdot((\sigma\times\tau)^*(p_1^*(\varphi)\wedge p_2^*(\psi))) \\
&= \sum_{(\mu,\nu)\in S_{k,l}} \int_{(\eta_\mu,\eta_\nu)(\Delta^{k+l})} \operatorname{sign}(\mu,\nu)\cdot((\sigma\times\tau)^*(p_1^*(\varphi)\wedge p_2^*(\psi))) \\
&\overset{\text{by the Form Integration Proposition ?? (4)}}{=} \int_{\Delta^k\times\Delta^l} (\sigma\times\tau)^*(p_1^*(\varphi)\wedge p_2^*(\psi)) \\
&\overset{\substack{\text{follows from the Shuffle–Decomposition Lemma 96.3}\\ \text{and the Lebesgue Integral Proposition ?? (2a), (3b)}}}{=} \int_{\Delta^k}\sigma^*(\varphi)\cdot\int_{\Delta^l}\tau^*(\psi) \\
&\overset{\substack{\text{follows, as in the Product-Pullback-Integral}\\ \text{Proposition ?? (1), from the Fubini Theorem ??}\\ \text{(note that here we use that } r=k \text{ and } s=l)}}{=} (\Xi_{k,l}\circ(s(\varphi)\times s(\psi))(\sigma\otimes\tau).
\end{aligned}$$

$$\int_{\Delta^k\times\Delta^l} p_1^*(\sigma^*(\varphi))\wedge p_2^*(\tau^*(\psi)) \overset{\substack{\text{follows from } p_1\circ(\sigma\times\tau)=\sigma\\ \text{and } p_2\circ(\sigma\times\tau)=\tau}}{=} (\mu_{\mathbb{R}}\circ(u(\varphi)\otimes u(\psi))\circ\Pi_{k,l})(\sigma\otimes\tau)$$

follows from the definitions (here we also use that $r=k$ and $s=l$)

(2) Now we assume that $r\neq k$ (which is equivalent to $s\neq l$). The argument is almost identical to the above. Except that now instead of the product of two integrals we obtain in both cases 0. This is a consequence of a (slight variation on) the Product-Pullback-Integral Proposition ?? (2) and the definition of $\Pi_{k,l}$. ∎

CHAPTER 97

Realizing cohomology classes by maps to S^1

In this relatively short chapter we will use the de Rham Isomorphism-Theorem 95.8 to prove the following two results:

(1) In the Maps-to-S^1-Proposition 97.2 we will show that for a smooth manifold M there exists a natural bijection between $\mathrm{H}^1(M;\mathbb{Z})$ and the set of homotopy classes of smooth maps $M \to S^1$.

(2) In the Product-as-Mapping Torus Proposition 97.3 we will use the Maps-to-S^1-Proposition 97.2 to show that for a smooth manifold M the product $M \times S^1$ can often be rewritten as non-trivial mapping tori.

97.1. **Realizing cohomology classes by maps to S^1.** We start out this section with the following definition:

Definition. Let (X, x_0) and (Y, y_0) be pointed topological spaces. We say that a homomorphism $\varphi\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is realized by a continuous map $g\colon X \to Y$ if $g(x_0) = y_0$ and if $g_* = \varphi$. If there is no danger of confusion we will drop the base points from the notation.

In the following we will mostly discuss the case $(Y, y_0) = (S^1, 1)$. Before we can continue we introduce the following conventions:

Convention.

(1) By the Circle-π_1-Theorem 11.13 we have an isomorphism $\mathbb{Z} \to \pi_1(S^1, 1)$ given by sending 1 to the equivalence class of the loop $t \mapsto \exp(2\pi\,\mathrm{i}t)$. We use this isomorphism to make the identification $\pi_1(S^1, 1) =_{\mathrm{i}} \mathbb{Z}$.

(2) Let X be a path-connected topological space and let $x_0 \in X$. As shown in the H^1-Hom-Proposition 70.15 we can use the Hurewicz homomorphism and the evaluation homomorphism to make the identifications

$$\mathrm{H}^1(X;\mathbb{Z}) \;=_{\mathrm{i}}\; \mathrm{Hom}(\mathrm{H}_1(X);\mathbb{Z}) \;=_{\mathrm{i}}\; \mathrm{Hom}(\pi_1(X, x_0), \mathbb{Z}).$$

Now we can formulate the following question:

Question 97.1. Let X be a path-connected topological space that is "reasonable", e.g. that is a smooth manifold or a CW-complex. Furthermore let $x_0 \in X$ be a base point. Can every homomorphism $\varphi\colon \pi_1(X, x_0) \to \mathbb{Z} =_{\mathrm{i}} \pi_1(S^1, 1)$ be realized by a continuous map $(X, x_0) \to (S^1, 1)$?

The following proposition gives an affirmative answer if X is a smooth manifold:

Proposition 97.2. (Maps-to-S^1-Proposition) Let M be a connected smooth manifold and let $P \in M$ be a base point.

(1) Every class $\phi \in \mathrm{H}^1(M;\mathbb{Z}) =_{\mathrm{i}} \mathrm{Hom}(\pi_1(M,P),\mathbb{Z})$ can be realized by a smooth map $g\colon M \to S^1$ with $g(P) = 1$.
(2) Let $f, g\colon M \to S^1$ be two smooth maps with $f(P) = g(P) = 1$. If the induced maps $f_* = g_*\colon \pi_1(M,P) \to \pi_1(S^1,1)$ agree, then f and g are homotopic.
(3) Let $\phi \in \mathrm{H}^1(M;\mathbb{Z})$ and let $P \in \partial M$. If $f\colon \partial M \to S^1$ is a smooth map that realizes $\phi|_{\partial M}$, then there exists a smooth map $g\colon M \to S^1$ with $g|_{\partial M} = f$ that realizes ϕ.

Remark.

(1) Using the H^1-Hom-Proposition 70.15 we can reformulate the Maps-to-S^1-Proposition 97.2 as follows: Let M be a connected smooth manifold and let $P \in M$ be a base point. Given any $\phi \in \mathrm{H}^1(M;\mathbb{Z})$ there exists a smooth map $g\colon M \to S^1$ with $\phi = g^*([S^1]^*)$ where $[S^1]^* \in \mathrm{H}^1(S^1;\mathbb{Z})$ is the standard dual fundamental class.
(2) The Maps-to-S^1-Proposition 97.2 (2) can be viewed as a partial converse to the Homotopy-π_1-Proposition 13.4.
(3) Later on, in the Maps-to-K(π,n)-Proposition **??** we will prove an analogue of the Maps-to-S^1-Proposition 97.2 for CW-complexes. We will use this proposition, in the Map-out-of-CW Proposition **??**, to give a new proof of the Maps-to-S^1-Proposition 97.2. □

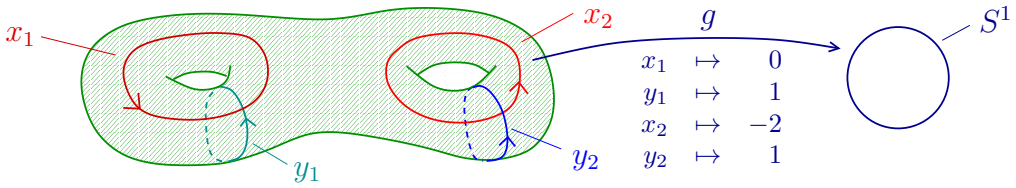
Examples.

(1) We consider the torus $T = S^1 \times S^1 = \mathbb{R}^2/\mathbb{Z}^2$ and we make the usual identification $\mathrm{H}_1(T) =_{\mathrm{i}} \mathbb{Z}^2$ coming from the Künneth Theorem 56.3. We consider the cohomology classes in $\mathrm{H}^1(T;\mathbb{Z}) =_{\mathrm{i}} \mathrm{Hom}(\mathrm{H}_1(T),\mathbb{Z})$ that are given by

$$\begin{array}{rccc} \phi\colon \mathrm{H}_1(T) =_{\mathrm{i}} \mathbb{Z}^2 & \to & \mathbb{Z} \\ (m,n) & \mapsto & m \end{array} \qquad \text{and} \qquad \begin{array}{rccc} \psi\colon \mathrm{H}_1(T) =_{\mathrm{i}} \mathbb{Z}^2 & \to & \mathbb{Z} \\ (m,n) & \mapsto & n. \end{array}$$

These two cohomology classes are evidently realized by the projection $S^1 \times S^1 \to S^1$ onto the first, respectively the second factor. We challenge the reader to provide an explicit continuous map $g\colon T \to S^1$ that realizes the homomorphism $\varphi\colon \mathrm{H}_1(T) =_{\mathrm{i}} \mathbb{Z}^2 \to \mathbb{Z}$ given by $\varphi(m,n) = 6m + 4n$.

(2) In the figure below we consider the surface Σ of genus 2:



As we saw on page **??**, the oriented curves x_1, y_1, x_2, y_2 represent a basis for $\mathrm{H}_1(\Sigma)$. We consider some random epimorphism $\phi\colon \mathrm{H}_1(\Sigma) \to \mathbb{Z}$, e.g. we could consider the epimorphism $\phi\colon \mathrm{H}_1(\Sigma) \to \mathbb{Z}$ that is given by $\phi(x_1) = 0$, $\phi(y_1) = 1$, $\phi(x_2) = -2$ and $\phi(y_2) = 1$. By the Maps-to-S^1-Proposition 97.2 we know that ϕ can be realized by some smooth map $g\colon \Sigma \to S^1$. But can one "visualize" such a map? We will return to this question on page **??**. □

We break the proof of the Maps-to-S^1-Proposition 97.2 into three parts.

Proof of the Maps-to-S^1-Proposition 97.2 (1). Let M be a connected n-dimensional smooth manifold, let $P \in M$ and let $\phi \in \mathrm{H}^1(M;\mathbb{Z}) =_{\mathrm{i}} \mathrm{Hom}(\pi_1(M,P),\mathbb{Z})$. We have to show that ϕ can be realized by a smooth map $g\colon M \to S^1$ with $g(P) = 1$. We start out with two preparations:

(1) Let ω be a smooth differential 1-form on M and let $\gamma\colon [a,b] \to M$ be a piecewise smooth path. We pick $a = t_0 < t_1 < \cdots < t_n = b$ such that the restriction of γ to each subinterval $[t_i, t_{i+1}]$ is smooth. We set

$$\int\limits_{[a,b]} \gamma^*(\omega) \;:=\; \sum_{i=0}^{n-1} \underbrace{\int\limits_{[t_i,t_{i+1}]} \gamma^*(\omega)}_{\substack{\text{defined in the}\\ \text{1-Form Integration}\\ \text{Proposition ??}}}.$$

It follows almost immediately from basic facts on Riemann-integrals that this definition does not depend on the choice of the subdivision.

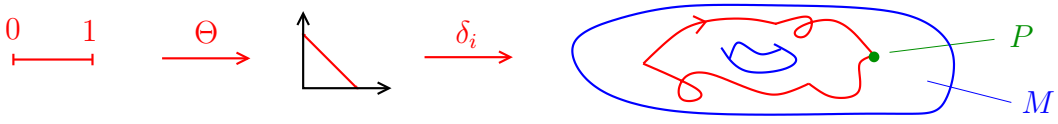
(2) We denote by $u\colon \mathrm{H}^1_{\mathrm{dR}}(M) \to \mathrm{H}^1(M;\mathbb{R})$ the natural isomorphism from the de Rham Isomorphism-Theorem 95.8.

We now start with the actual proof of the proposition. We pick a smooth closed differential 1-form ω on M with $u([\omega]) = \phi \in \mathrm{Hom}(\pi_1(M,P);\mathbb{Z}) \subset \mathrm{Hom}(\pi_1(M,P);\mathbb{R}) =_{\mathrm{i}} \mathrm{H}^1(M;\mathbb{R})$.

Claim 1. Let $\delta\colon [0,1] \to M$ be a piecewise smooth loop in P. Then

$$\int\limits_{[0,1]} \delta^*(\omega) \;=\; \phi\Big(\underbrace{[\delta]}_{\in \pi_1(M,P)}\Big) \;\in\; \mathbb{R}.$$

Proof. Let $\delta\colon [0,1] \to M$ be a piecewise smooth loop in P. One can show easily that, after possibly performing an orientation-preserving reparametrization, we can write $\delta = \delta_1 * \cdots * \delta_k$ as a concatenation of smooth paths $\delta_i\colon [0,1] \to M$, $i = 1,\ldots,k$. Let $\Theta\colon [0,1] \to \Delta^1$ be the homeomorphism that is given by $\Theta(t) = (1-t,t)$. Note that with this definition the singular chain $\delta_1 \circ \Theta^{-1} + \cdots + \delta_k \circ \Theta^{-1}$ lies in $\mathrm{C}_1^{\mathrm{sm}}(M)$.



We see that:

$$\int\limits_{[0,1]} \delta^*(\omega) \;\overset{\text{(1) and (6)}}{=}\; \int\limits_{\delta_1\circ\Theta^{-1}+\cdots+\delta_k\circ\Theta^{-1}} \omega \;\overset{\text{Theorem 95.8}}{=}\; \langle \phi, [\delta_1 \circ \Theta^{-1} + \cdots + \delta_k \circ \Theta^{-1}] \rangle \;=\; \phi([\delta]).$$

(1) and (6): by the definitions and by the Form Integration Proposition ??
Theorem 95.8: by the De Rham-Isomorphism Theorem 95.8, since $u([\omega]) = \phi$
$=$: via the identification $\mathrm{H}^1(M;\mathbb{Z}) =_{\mathrm{i}} \mathrm{Hom}(\pi_1(M,P),\mathbb{Z})$ ⊞

We define

$$\begin{aligned} \widetilde{g}\colon M &\to \mathbb{R}/\mathbb{Z} \\ Q &\mapsto \Big[\int\limits_{[0,1]} \gamma^*(\omega)\Big] \end{aligned}$$

where γ is a smooth path[771] from P to Q

and

$$\begin{aligned} g\colon M &\to S^1 \\ Q &\mapsto \exp(2\pi\,\mathrm{i}\cdot\widetilde{g}(Q)). \end{aligned}$$

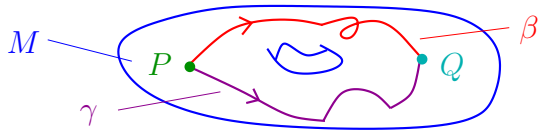
The following claim shows that the map g has all the required properties:

Claim 2.

(i) The map $\widetilde{g}$ (and thus the map g) is well-defined.
(ii) We have $g(P) = 1 \in S^1$.
(iii) The map g is smooth.
(iv) We have $g_* = \phi\colon \pi_1(M,P) \to \pi_1(S^1,1) =_{\mathrm{i}} \mathbb{Z}$.

Proof. Let us prove these four statements:

(i) Let Q in M. Suppose we are given two piecewise smooth paths β and γ from the base point P to Q.



Then $\int_{[0,1]} \beta^*(\omega) - \int_{[0,1]} \gamma^*(\omega) = \int_{[0,1]} (\beta * \overline{\gamma})^*(\omega) = \phi([\beta * \overline{\gamma}]) \in \mathbb{Z}$.

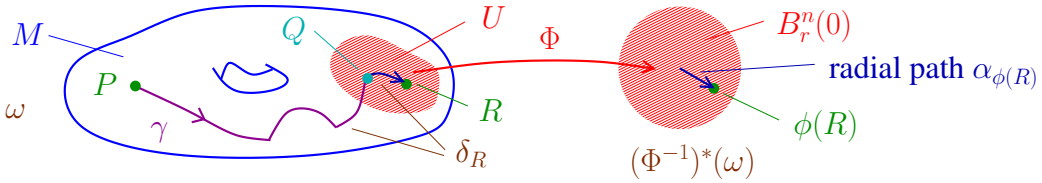
The first $=$ follows from the definitions and the Form Integration Proposition ?? (1) and (6); the second $=$ is by Claim 1, since $\beta * \overline{\gamma}$ is a piecewise smooth loop.

This shows that the map $g\colon M \to \mathbb{R}/\mathbb{Z}$ is well-defined.

(ii) We determine $g(P)$ by taking the constant path from P to P. It follows immediately from the definition of g that $g(P) = 1 \in S^1$.

(iii) Let $Q \in M$. We need to show that g is smooth in an open neighborhood of Q. We pick a path γ from P to Q. If $P \in M \setminus \partial M$, then we pick a chart $\Phi\colon U \to B_r^n(0)$ around Q with $\Phi(Q) = 0$. If $P \in \partial M$, then we pick a chart $\Phi\colon U \to B_r^n(0) \cap \mathbb{H}^n$ around Q with $\Phi(Q) = 0$. Either way, given a point $R \in U$ we denote by δ_R the piecewise smooth path that is given by the concatenation of the following two paths:

(a) The path γ from P to Q.
(b) The "radial path" from Q to R that corresponds, under the diffeomorphism Φ, to the radial path $\alpha_{\phi(R)}$ from 0 to $\Phi(R) \in B_r^n(0)$.



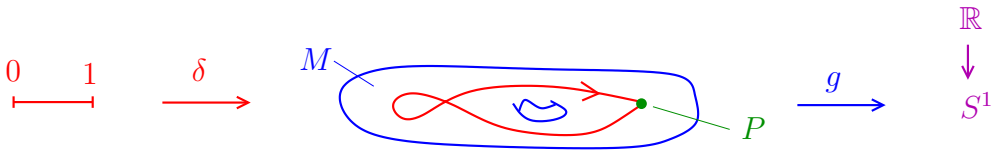
It now suffices to show that the map $U \to \mathbb{R}$ that is given by $R \mapsto \int_{[0,1]} \delta_R^*(\omega)$ is smooth.

But this map is indeed smooth, since for any $R \in U$ we have

$$\int_{[0,1]} \delta_R^*(\omega) = \int_{[0,1]} \delta_Q^*(\omega) + \int_{[0,1]} \alpha_{\phi(R)}^*((\Phi^{-1})^*(\omega)) = \int_{[0,1]} \delta_Q^*(\omega) + \underbrace{\int_{t=0}^{t=1} \overbrace{(\Phi^{-1})^*(\omega)}^{\text{smooth } n\text{-form on } B_r^n(0)}\, {}_{t\cdot\phi(R)}(\phi(R))}_{\text{smooth in } R}.$$

[771] By the Smooth Path-Connectivity Proposition ?? (2) any two points on a connected smooth manifold are connected by a smooth path.

(iv) Suppose we are given an element in $\pi_1(M,P)$. By the Smooth Path-Connectivity Proposition **??** (1) we can represent this element by a smooth loop $\delta\colon [0,1]\to M$.



We have the following equalities:

$$\begin{aligned} g_*([\delta]) &\overset{\text{by the Action–}\pi_1\text{-Theorem 15.1}}{=} \text{endpoint of the lift of } g\circ\delta\colon[0,1]\to\mathbb{R} \text{ to the universal covering } \mathbb{R} \text{ of } S^1 \\ &= \text{endpoint of the lift of the path } t\mapsto \exp\Big(2\pi\,\mathrm{i}\int\limits_{[0,t]}\delta^*(\omega)\Big) \text{ to the universal covering } \mathbb{R} \\ &= \text{endpoint of the path } t\mapsto \int\limits_{[0,t]}\delta^*(\omega) \;=\; \text{the point } \int\limits_{[0,1]}\delta^*(\omega) \overset{\text{by Claim 1}}{=} \phi([\delta]). \end{aligned}$$

We have thus shown that $g_* = \phi\colon \pi_1(M,P)\to\pi_1(S^1,1) =_{\mathrm{i}} \mathbb{Z}$. ■

Proof of the Maps-to-S^1-Proposition 97.2 (2). Let $f,g\colon M\to S^1$ be two smooth maps with $g(P)=1$, such that we have the equality $f_* = g_*\colon \pi_1(M,P)\to\pi_1(S^1,1)$. We need to show that f and g are homotopic. Let ω be any closed differential 1-form on S^1 such that $u_M([\omega])$ represents a generator of $\mathrm{Hom}(\pi_1(S^1,1),\mathbb{Z})\subset\mathrm{Hom}(\pi_1(S^1,1),\mathbb{R}) =_{\mathrm{i}} \mathrm{H}^1(S^1;\mathbb{R})$. We set $\alpha := f^*(\omega)$ and $\beta := g^*(\omega)$.

Claim. Let $p\colon M\times[0,1]\to M$ be the natural projection. There exists a closed smooth differential 1-form ψ on $M\times[0,1]$ such that the restriction to $M\times\{0\}$ equals $p^*(\alpha)|_{M\times\{0\}}$ and such that the restriction to $M\times\{1\}$ equals $p^*(\beta)|_{M\times\{1\}}$.

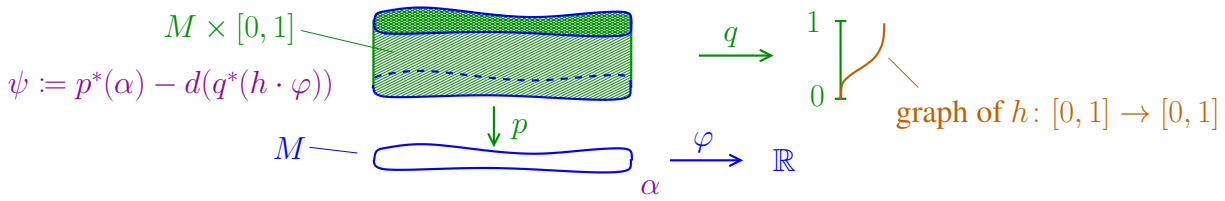
Proof. We start out with a few preparations:

- First note that it follows easily from the hypothesis that $f_* = g_*$ and from the natural isomorphisms $\mathrm{H}^1_{\mathrm{dR}}(M) = \mathrm{H}^1(M;\mathbb{R}) =_{\mathrm{i}} \mathrm{Hom}(\pi_1(M,P),\mathbb{R})$ that we have the equalities $[\alpha] = f^*([\omega]) = g^*([\omega]) = [\beta] \in \mathrm{H}^1_{\mathrm{dR}}(M)$. By definition of de Rham cohomology this means that there exists a smooth 0-form φ on M with $d\varphi = \alpha - \beta$.
- Let $q\colon M\times[0,1]\to[0,1]$ be the natural projection.
- It follows from the Smooth Transition Function Lemma **??** that there exists a smooth map $h\colon[0,1]\to[0,1]$ that is $\equiv 0$ close to 0 and that is $\equiv 1$ close to 1.

On $M\times[0,1]$ we consider the 1-form that is given by

$$\psi \;:=\; p^*(\alpha) - d(q^*(h\cdot\varphi)).$$

We claim that ψ has the desired properties.



First note that

$$d(\psi) \;=\; d(p^*(\alpha) - d(q^*(h\cdot\varphi))) \underset{\text{by the Differential-Natural Proposition ??}}{\overset{\uparrow}{=}} p^*(d(\alpha)) - d(d(q^*(h\cdot\varphi))) \underset{\text{by the Differentials-on-Manifolds Proposition ?? (4)}}{\overset{\uparrow}{=}} p^*(\overbrace{d(\alpha)}^{=0}) \;=\; 0.$$

In other words, we have just shown that ψ is closed. Finally we calculate that

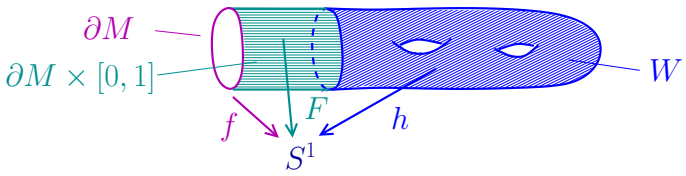
$$\begin{aligned}\psi \;&=\; p^*(\alpha) - d(q^*(h\cdot\varphi)) \overset{\text{by the Differential-Natural Proposition ?? and the Differentials-on-Manifold Proposition ?? (3)}}{\overset{\downarrow}{=}} p^*(\alpha) - (q^*(dh)\cdot\varphi + q^*(h)\cdot d\varphi)\\ &= \underbrace{p^*(\alpha) - q^*(dh)\cdot\varphi - q^*(h)\cdot d\varphi}_{\substack{\text{on } M\times\{0\} \text{ this equals } p^*(\alpha)-0-0=p^*(\alpha)\\ \text{on } M\times\{1\} \text{ this equals } p^*(\alpha)-0-(p^*(\alpha)-p^*(\beta))=p^*(\beta)}}.\end{aligned}$$

⊞

Next note that, as in the proof of Statement (1), the closed 1-form ψ on $M\times[0,1]$ gives rise, by integrating the 1-form ψ along smooth paths emanating from $(P,0)$, to a smooth map $F\colon M\times[0,1]\to S^1$. We set $w := F(P,1)$. It follows fairly easily from the definitions that $F_0 = f\colon M\to S^1$ and that $F_1 = \rho_w\circ g\colon M\to S^1$ where $\rho_w\colon S^1\to S^1$ is given by multiplication by w. This shows that $F_0 = f$ and $F_1 = \rho_w\circ g$ are homotopic. But clearly ρ_w is homotopic to the identity. This implies that $\rho_w\circ g$ is homotopic to g. Since being homotopic is an equivalence relation we see that f is homotopic to g. ■

Proof of the Maps-to-S^1-Proposition 97.2 (3). Let $P\in\partial M$. Furthermore let $\phi\in \mathrm{H}^1(M;\mathbb{Z}) =_{\mathrm{i}} \mathrm{Hom}(\pi_1(M,P),\mathbb{Z})$, and let $f\colon \partial M\to S^1$ be a smooth map that realizes $\phi|_{\partial M}$. We need to show that there exists a smooth map $g\colon M\to S^1$ with $g|_{\partial M} = f$ that realizes ϕ.

First note that it follows from the Smooth Collar Neighborhood Theorem **??** that we can pick a collar neighborhood $[0,1]\times\partial M$. We set $W := M\setminus([0,1)\times\partial M)$ and we equip W with the base point $(P,1)$. By the Basics-of-Collars Lemma **??** (1) we know that W is an n-dimensional smooth manifold with $\partial W = \{1\}\times M$. By the Maps-to-$S^1$-Proposition 97.2 (1) there exists a smooth map $h\colon W\to S^1$ which realizes $\phi|_W$. It is now straightforward to see that under the obvious homeomorphism $\{0\}\times M\to\{1\}\times M$ the maps $f_*\colon \mathrm{H}_1(M)\to\mathrm{H}_1(S^1)$ and $h_*\colon \mathrm{H}_1(M)\to\mathrm{H}_1(S^1)$ agree. At this point we can conclude from the Maps-to-S^1-Proposition 97.2 (2) that there exists a smooth map $F\colon [0,1]\times\partial M\to S^1$ such that $F_0 = f$ and $F_1 = h|_{\{1\}\times M}$.



Note that by definition of a collar neighborhood and by the Basics-of-Collars Lemma **??** we know that $[0,1]\times\partial M$ and W are closed subsets of M. Thus we obtain from the Pasting Proposition 2.4 (2) that the two maps F and g combine to a continuous map $h\colon M\to S^1$. Since h is smooth on $[0,1]\times\partial M$ we obtain from the Whitney Approximation Theorem **??** that h is homotopic rel $\{0\}\times\partial M$ to a smooth map $g\colon M\to S^1$. This map now has all the desired properties. ■

97.2. **Rewriting products $M \times S^1$.** The Maps-to-S^1-Proposition 97.2 (1) has many consequences. For example in the Codimension-One Submanifold Existence Proposition **??** we will use it to show that for a given compact orientable n-dimensional smooth manifold M any class in $\mathrm{H}_{n-1}(M)$ can be represented by a compact oriented proper submanifold. In this section we will give a different application, namely we will see that in many situations one can find interesting ways to rewrite a product $M \times S^1$ where M is a closed connected smooth manifold.

To formulate that application it is convenient to recall the following definition from page **??**:

Definition. Let X be a topological space and let $f\colon X \to X$ be a continuous map. We refer to

$$\mathrm{Tor}(X,f) \;:=\; (X \times [0,1])\,/\sim \qquad \text{where } (x,0) \sim (f(x),1) \text{ for all } x \in X,$$

as the mapping torus of (X,f). We refer to the continuous map

$$\begin{array}{rccl} q\colon \mathrm{Tor}(X,f) = (X\times[0,1])/(x,0)\sim(f(x),1) & \to & S^1 \\ [(x,t)] & \mapsto & \exp(2\pi\,\mathrm{i}t) \end{array}$$

as the natural projection onto S^1.

Throughout this section we also use the following notation:

Notation. We denote by $\Theta\colon \mathrm{H}_1(S^1) \to \mathbb{Z}$ the isomorphism given by $\Theta([S^1]) = 1$. (Here $[S^1] \in \mathrm{H}_1(S^1)$ is the standard generator.)

Now we can state the main result of this section:

Proposition 97.3. (Product-as-Mapping Torus Proposition) Let M be a smooth manifold and let $\phi\colon \mathrm{H}_1(M\times S^1) \to \mathbb{Z}$ be an epimorphism. We set $k = k(\phi) := \phi([\{*\}\times S^1])$. If $k \neq 0$, then there exists a $|k|$-fold covering $p\colon \widetilde{M} \to M$ and a deck transformation $f\colon \widetilde{M} \to \widetilde{M}$ together with a diffeomorphism[772] $\Xi\colon \mathrm{Tor}(\widetilde{M},f) \to M \times S^1$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}_1(\mathrm{Tor}(\widetilde{M},f)) & \xrightarrow{\quad\Xi_*\quad} & \mathrm{H}_1(M\times S^1) \\ & {\scriptstyle \Theta\circ q_*}\searrow \quad \swarrow{\scriptstyle \phi} & \\ & \mathbb{Z}. & \end{array}$$

Examples.

(1) Let M be a smooth manifold and let $\psi\colon \mathrm{H}_1(M) \to \mathbb{Z}$ be a homomorphism. We apply the Product-as-Mapping Torus Proposition 97.3 to the epimorphism

$$\begin{array}{rccl} \phi\colon \mathrm{H}_1(M\times S^1) \overset{\cong}{\leftarrow} \mathrm{H}_1(M)\oplus\mathrm{H}_1(S^1) & \to & \mathbb{Z} \\ (\sigma, n\cdot[S^1]) & \mapsto & \psi(\sigma)+n. \end{array}$$

(by the Künneth Theorem 56.9)

Note that we have $k = k(\phi) = 1$. This implies that the map $p\colon \widetilde{M} \to M$ is a covering of degree 1. It follows from the Covering Unique Existence Theorem 27.5 that we can now

[772] By the Manifold-Mapping Torus Lemma **??** the mapping torus $\mathrm{Tor}(\widetilde{M},f)$ naturally has the structure of a smooth manifold.

assume that $\widetilde{M} = M$ and that $p = \mathrm{id}$. Note that the only deck transformation of the 1-fold covering $\mathrm{id}\colon M \to M$ is the identity. Thus we obtain from the Product-as-Mapping Torus Proposition 97.3 that there exists a diffeomorphism $\Xi\colon M \times S^1 \to M \times S^1$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{H}_1(M \times S^1) & \xrightarrow{\Xi_*} & \mathrm{H}_1(M \times S^1) \\ & {\scriptstyle \Theta \circ q_*} \searrow \quad \swarrow {\scriptstyle \phi} & \\ & \mathbb{Z}, & \end{array}$$

where $q\colon M \times S^1 \to S^1$ denotes the natural projection. Note that if $\psi\colon \mathrm{H}_1(M) \to \mathbb{Z}$ is non-zero, then Ξ_* is not the identity. Summarizing, we have now shown that if $\mathrm{H}^1(M;\mathbb{Z}) = \mathrm{Hom}(\mathrm{H}_1(M),\mathbb{Z}) \neq 0$, then $M \times S^1$ can be decomposed as a product in more than one way.

(2) Let Σ be the surface of some genus $g \in \mathbb{N}_{\geq 2}$ and let $\phi\colon \mathrm{H}_1(\Sigma \times S^1) \to \mathbb{Z}$ be an epimorphism such that $k = k(\phi) \neq 0$. It follows from the Product-as-Mapping Torus Proposition 97.3, together with the Subgroups-of-Surface Groups Proposition **??** (1) or alternatively with the Genus-of-Covering Lemma 48.11, that $\Sigma \times S^1$ is diffeomorphic to a mapping torus of the form $\mathrm{Tor}(\widetilde{\Sigma}, f)$ where $\widetilde{\Sigma}$ is a surface of genus $|k| \cdot (g-1)+1$. In particular we obtain the slightly mind boggling statement that the product $\Sigma \times S^1$ can also be viewed as a mapping torus of a self-diffeomorphism of a surface of arbitrarily high genus. □

Proof. By considering the components of M individually we can assume that M is actually connected and non-empty. In this setting we pick points $* \in M$ and $* \in S^1$. We denote by $j_1\colon M \xrightarrow{x \mapsto (x,*)} M \times S^1$ and $j_2\colon S^1 \xrightarrow{z \mapsto (*,z)} M \times S^1$ the corresponding embeddings. Note that by the Maps-to-S^1-Proposition 97.2 (1) there exists a smooth map $g\colon M \to S^1$ such that the map $\mathrm{H}_1(M) \xrightarrow{g_*} \mathrm{H}_1(S^1) \xrightarrow{\Theta} \mathbb{Z}$ equals $\phi \circ j_{1*}\colon \mathrm{H}_1(M) \to \mathbb{Z}$. Now let us consider the following pullback diagram:

$$\begin{array}{ccc} \widetilde{M} := \{(x,z) \in M \times S^1 \mid g(x) = z^k\} & \xrightarrow{(x,z) \mapsto z} & S^1 \\ \downarrow{\scriptstyle (x,z) \mapsto x} & & \downarrow{\scriptstyle z \mapsto z^k} \\ M & \xrightarrow{\quad g \quad} & S^1. \end{array}$$

By the Pullback-Covering Lemma 14.15 we know that $p\colon \widetilde{M} \to M$ is a k-fold covering. The map

$$\begin{array}{rcl} f\colon \widetilde{M} & \to & \widetilde{M} \\ (x,z) & \mapsto & (x, \exp(-2\pi\,\mathrm{i}/k) \cdot z) \end{array}$$

is evidently a deck transformation. Next we consider the following diagram:

$$\begin{array}{ccc} \widetilde{M} \times S^1 & \xrightarrow{((x,z),w) \mapsto ((x,z),z^{-1} \cdot w)} & \widetilde{M} \times S^1 \\ \downarrow{\scriptstyle ((x,z),\exp(2\pi\,\mathrm{i}t)) \mapsto [((x,z),k \cdot t \bmod \mathbb{Z})]} & & \downarrow{\scriptstyle ((x,z),w) \mapsto (x,w)} \\ \mathrm{Tor}(\widetilde{M}, f) & \xrightarrow[\Xi]{[((x,z),t)] \mapsto (x, z^{-1} \cdot \exp(2\pi\,\mathrm{i}t/k))} & M \times S^1. \end{array}$$

We make the following observations:

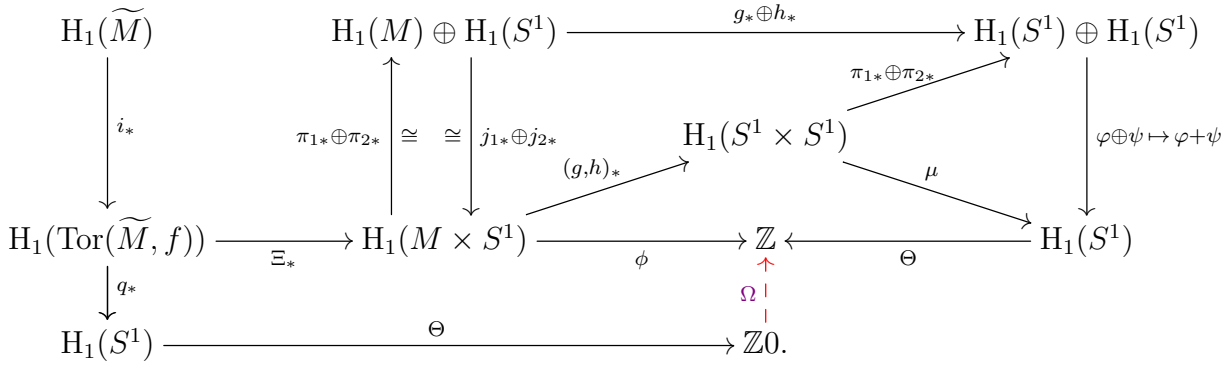
(1) One can easily verify that the bottom map is in fact well-defined and that the diagram commutes.

(2) It is pretty clear that the top map is a diffeomorphism.[773]
(3) It is straightforward to show that the vertical maps are local diffeomorphisms and that they are degree k coverings.

It follows from the above together that the bottom map is also a diffeomorphism. It remains to prove the following claim:

Claim. We have $\phi \circ \Xi_* = \Theta \circ q_* \colon \mathrm{H}_1(\mathrm{Tor}(\widetilde{M}, f)) \to \mathbb{Z}$.

Proof. We consider the following diagram:



We make the following clarifications and observations:

(1) The map $i \colon \widetilde{M} \to \mathrm{Tor}(\widetilde{M}, f))$ is the natural embedding given by $x \mapsto [(x, 0)]$.
(2) The map $\mu \colon S^1 \times S^1 \to S^1$ is the multiplication map.
(3) In both settings we denote by π_1 and π_2 the projection onto the first respectively second factor.
(4) The map $h \colon S^1 \to S^1$ is given by $z \mapsto z^k$. Note that by the Degree-of-Power Lemma 43.2 we know that $h_* \colon \mathrm{H}_1(S^1) \to \mathrm{H}_1(S^1)$ is given by multiplication by k.
(5) It follows from the discussion on page **??** that the vertical maps in the middle are isomorphisms and inverses of one another.
(6) The upper left triangle commutes by the functoriality of homology groups.
(7) We leave it as an exercise to the reader to show that the triangle to the right commutes.
(8) The big rectangle to the right commutes by the choice of g and h.
(9) It follows from (5), (6), (7) and (8) that the triangle at the bottom commutes.
(10) Recall that any point $(x, z) \in \widetilde{M}$ satisfies by definition $g(x) = z^k$. It now follows immediately that the map $(g, h) \circ \mu \circ \Xi \circ i \colon \widetilde{M} \to S^1$ is the constant map.
(11) It follows from (9) and (10) that $\phi \circ \Xi_* \circ i_* \colon \mathrm{H}_1(\widetilde{M}) \to \mathbb{Z}$ is the zero map.
(12) By the Mapping Torus-H_1-Lemma **??** we know that the vertical sequence to the left is exact.
(13) It follows from (11) and (12) that the vertical map Ω exists.
(14) We pick $(x, z) \in \widetilde{M}$. We use the homeomorphism from the Mapping Torus-of-Homeomorphism Lemma **??** to make the identification

$$\mathrm{Tor}(\widetilde{M}, f) \ =_{\mathrm{i}} \ (\widetilde{M} \times \mathbb{R})/(w, t) \sim (f(w), t + 1).$$

[773]The reader might note that the top map is the restriction of the map $(M \times S^1) \times S^1 \to (M \times S^1) \times S^1$ given by the same formula, which is clearly a diffeomorphism, to the submanifold $\widetilde{M} \times S^1$ of $(M \times S^1) \times S^1$.

Now we consider the singular 1-cycle

$$\begin{array}{rcl} \sigma\colon \Delta^1 & \to & \mathrm{Tor}(\widetilde{M}, f) = (\widetilde{M} \times \mathbb{R})/(w,t) \sim (f(w), t+1) \\ (1-t,t) & \mapsto & [((x,z),kt)]. \end{array}$$

It follows immediately from the definitions that $(\phi \circ \Xi_*)([\sigma]) = k$ and $(\Theta \circ q_*)([\sigma]) = k$. Thus we have $\Omega(k) = k$. Since $k \neq 0$ we see that $\Omega = \mathrm{id}$. ■